MAT1002 Lecture 10, Tuesday, Feb/28/2023

Outline

- Dot products (12.3)
  - ↳ Geometric meaning
  - ↳ Vector projection
- Cross products (12.4)
  - ↳ Geometric properties and remarks
  - ↳ Algebraic formula and determinants
  - ↳ Torque
- Lines and planes in $\mathbb{R}^3$ (12.5)

Dot Products

Definition
Given two vectors $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$, the dot product of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u} \cdot \vec{v}$, is defined by

$$\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example

(a) $\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3)$
$= -6 - 4 + 3 = -7$

(b) $\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$

Properties of the Dot Product

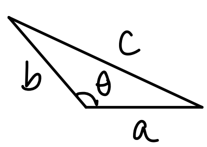
If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0$.

Proofs are straightforward verifications ; e.g.,

4. $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + \ldots + u_n^2 = \left(\sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}\right)^2 = |\vec{u}|^2$.

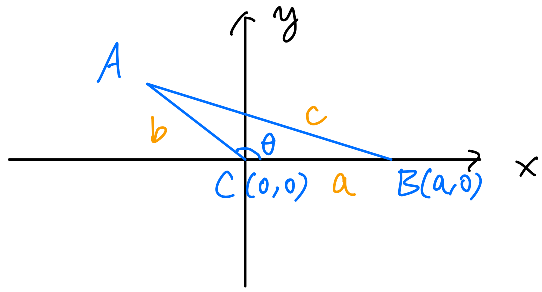
Law of Cosine

Consider the follow triangle : 

$(\theta \in [0, \pi])$.

The law of cosine states that $c^2 = a^2 + b^2 - 2ab\cos\theta$.

To see this, construct

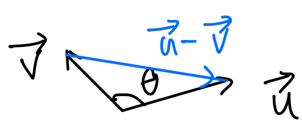


Then $A = (b\cos\theta, b\sin\theta)$, and

$$c^2 = |\vec{AB}|^2 = (a - b\cos\theta)^2 + b^2\sin^2\theta = a^2 - 2ab\cos\theta + b^2.$$

Dot Products & Angles in $\mathbb{R}^2$ & $\mathbb{R}^3$

Consider $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ (or $\mathbb{R}^3$), with angle θ made between them ($0 \le \theta \le \pi$). Then by the law of cosine,



$$\underbrace{|\vec{u} - \vec{v}|^2}_{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})} = |\vec{u}|^2 + |\vec{v}|^2 - 2\cos\theta\, |\vec{u}||\vec{v}|$$

$$\Rightarrow \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\cos\theta\, |\vec{u}||\vec{v}|$$

$$\Rightarrow \boxed{\vec{u}\cdot\vec{v} = \cos\theta\, |\vec{u}||\vec{v}|.}$$

Remarks

- If $\vec{u}$ and $\vec{v}$ are nonzero, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} \Rightarrow \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right).$$

- If $\vec{u}$ and $\vec{v}$ are unit vectors, then

$$\cos\theta = \vec{u}\cdot\vec{v} \implies \theta = \arccos(\vec{u}\cdot\vec{v})$$

- $\vec{u}\cdot\vec{v} = 0 \iff \left(\vec{u}=\vec{0} \text{ or } \vec{v}=\vec{0} \text{ or } \theta = \frac{\pi}{2}\right).$

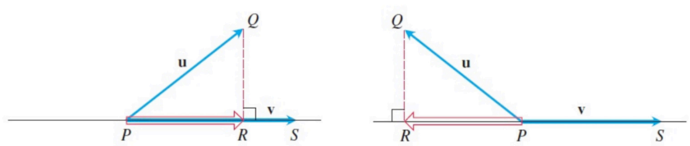
Generalizing from $\mathbb{R}^2$ and $\mathbb{R}^3$:

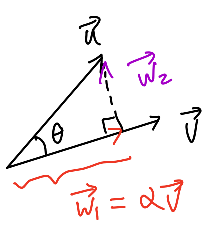
<u>Def</u>: For $\vec{u}, \vec{v}$ in $\mathbb{R}^n$:

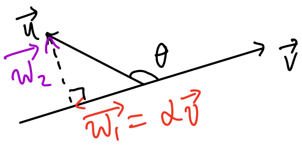
- If $\vec{u}\neq\vec{0}$ and $\vec{v}\neq\vec{0}$, then the angle made by $\vec{u}$ and $\vec{v}$ is $\theta := \arccos\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$.
- $\vec{u}$ and $\vec{v}$ are orthogonal (or perpendicular) if $\vec{u}\cdot\vec{v} = 0$ (note that $\vec{u}$ and $\vec{v}$ can be zero).

Vector Projections

Consider the following figure. Geometrically, the vector projection of $\vec{u} = \overrightarrow{PQ}$ onto a nonzero vector $\vec{v} = \overrightarrow{PS}$, denoted by $\text{proj}_{\vec{v}}\vec{u}$, is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS.







$$\vec{u} = \alpha\vec{v} + \vec{w}_2 \ ; \ \vec{w}_2 \perp \vec{v}$$

$$\Rightarrow \vec{u}\cdot\vec{v} = \alpha\vec{v}\cdot\vec{v} + \cancel{\vec{w}_2\cdot\vec{v}}\ 0 = \alpha|\vec{v}|^2$$

$$\Rightarrow \alpha = (\vec{u}\cdot\vec{v})/|\vec{v}|^2.$$

Definition: If $\vec{v} \neq \vec{0}$ and θ is the angle between $\vec{u}$ and $\vec{v}$, then the vector projection of $\vec{u}$ onto $\vec{v}$ is defined by

$$\mathrm{proj}_{\vec{v}}\vec{u} := \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}.$$

The number $(\vec{u}\cdot\vec{v})/|\vec{v}|$ is called the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

$$\mathrm{proj}_{\vec{v}}\vec{u} = \underbrace{\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)}_{\text{Scalar component} = (\cos\theta)|\vec{u}|}\underbrace{\left(\frac{\vec{v}}{|\vec{v}|}\right)}_{\text{Direction of } \vec{v}.}$$

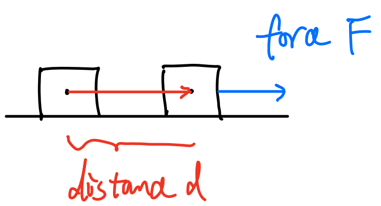
EXAMPLE Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$.

Sol: $\vec{u}\cdot\vec{v} = 6 - 3\cdot 2 - 2\cdot 2 = -4$; $|\vec{v}| = 3$

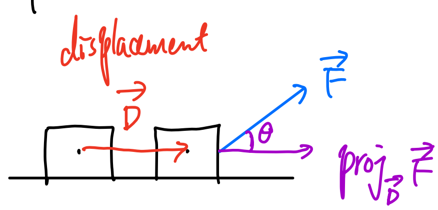
$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{-4}{9}\langle 1, -2, -2\rangle.$$

$$\text{Scalar component} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = -\frac{4}{3}.$$

e.g. Work done by constant force $\vec{F}$.



$W = Fd.$



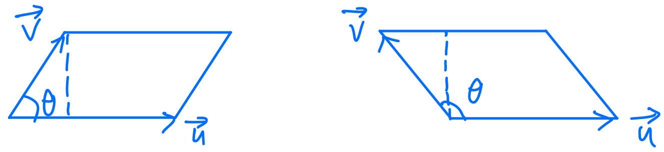
$$W = |\mathrm{proj}_{\vec{D}}\vec{F}||\vec{D}| = \frac{\vec{F}\cdot\vec{D}}{|\vec{D}|}\cdot|\vec{D}| = \vec{F}\cdot\vec{D}$$

$$W = \vec{F}\cdot\vec{D}.$$

Cross Products

$\vec{u}, \vec{v}$ collinear $\Leftrightarrow$ parallel $\overset{\text{def}}{\Longleftrightarrow}$ $\vec{u} = \lambda\vec{v}$, $\lambda \neq 0$

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^3$ that are not collinear, and let θ be the angle between them. Then the parallelogram spanned by them has area $|\vec{u}||\vec{v}|\sin\theta$.



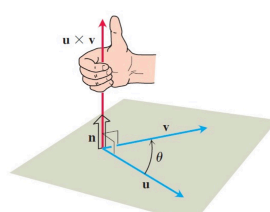
In this section, we assume that the ordered triple $(\vec{i}, \vec{j}, \vec{k})$ is arranged to satisfy the right-hand rule (default arrangement).

Def: Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$. The **cross product** $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ defined by the following:

- If $\vec{u} = 0$, $\vec{v} = 0$, or $\vec{u}$ and $\vec{v}$ are collinear, then $\vec{u} \times \vec{v} := \vec{0}$.
- Otherwise, $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$ and points to a direction such that the ordered triple $(\vec{u}, \vec{v}, \vec{u} \times \vec{v})$ satisfies the right-hand rule.
- The length of $\vec{u} \times \vec{v}$ is defined to be the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$, i.e.,

$$|\vec{u} \times \vec{v}| := |\vec{u}||\vec{v}|\sin\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.



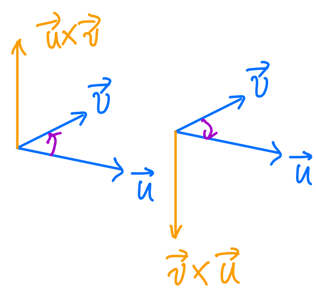
Geometric Properties and Remarks

1. By definition, we have

$$\vec{i}\times\vec{j}=\vec{k},\quad \vec{j}\times\vec{k}=\vec{i},\quad \vec{k}\times\vec{i}=\vec{j},$$
$$\vec{j}\times\vec{i}=-\vec{k},\quad \vec{k}\times\vec{j}=-\vec{i},\quad \vec{i}\times\vec{k}=-\vec{j}.$$

2. More generally, for any $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$,

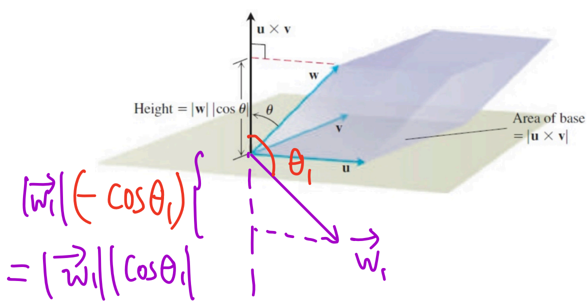
$$\vec{u}\times\vec{v}=-(\vec{v}\times\vec{u})$$



3. Let $\vec{u}$ and $\vec{v}$ be non-parallel. Since

Triple scalar product $\underline{(\vec{u}\times\vec{v})\cdot\vec{w}} = |\vec{u}\times\vec{v}||\vec{w}|\cos\theta$, ← θ is angle between $\vec{u}\times\vec{v}$ and $\vec{w}$.

the quantity $|(\vec{u}\times\vec{v})\cdot\vec{w}| = \underbrace{|\vec{u}\times\vec{v}|}_{\text{Base area}}\underbrace{|\vec{w}||\cos\theta|}_{\text{height}}$ measures the volume of the parallelopiped spanned by $\vec{u}$, $\vec{v}$ and $\vec{w}$.



4. In point 3 above,

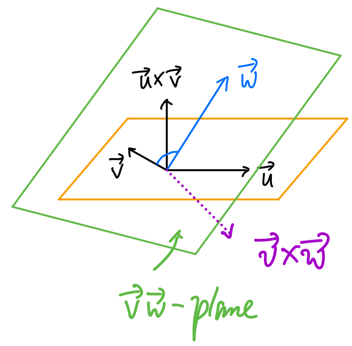
- $(\vec{u}\times\vec{v})\cdot\vec{w}>0 \iff \cos\theta>0 \iff \theta\in[0,\frac{\pi}{2})$

$\iff$ $\vec{w}$ lies in the positive half-space given by the $\vec{u}\vec{v}$-plane (right-hand rule).

- $(\vec{u}\times\vec{v})\cdot\vec{w}<0 \iff \cos\theta<0 \iff \theta\in(\frac{\pi}{2},\pi]$
  $\iff \vec{w}$ lies in the negative half-space given by the $\vec{u}\vec{v}$-plane (right-hand rule).

5. Suppose $(\vec{u}\times\vec{v})\cdot\vec{w}>0$. By point 4 above, $\vec{w}$ lies in the positive half-space determined by the $\vec{u}\vec{v}$-plane. Note that this would force $\vec{u}$ to lie in the positive half-space given by the $\vec{v}\vec{w}$-plane (with right-hand rule). By point 4, $(\vec{v}\times\vec{w})\cdot\vec{u}>0$. By point 3, we have

$$(\vec{u}\times\vec{v})\cdot\vec{w} = (\vec{v}\times\vec{w})\cdot\vec{u}.$$



This equation also holds even if $(\vec{u}\times\vec{v})\cdot\vec{w}\leq 0$ (by a similar argument). Applying this equation two times shows that

$$\boxed{(\vec{u}\times\vec{v})\cdot\vec{w} = (\vec{v}\times\vec{w})\cdot\vec{u} = (\vec{w}\times\vec{u})\cdot\vec{v}.} \quad (5.1)$$

6.

$\forall \vec{u},\vec{v},\vec{w}\in\mathbb{R}^3$,

$$(6.1)\quad \vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v}+\vec{u}\times\vec{w};$$
$$(6.2)\quad (\vec{v}+\vec{w})\times\vec{u} = \vec{v}\times\vec{u}+\vec{w}\times\vec{u}.$$

We show (6.1); then (6.2) follows from (6.1) and point 2. We show that $(\vec{u}\times(\vec{v}+\vec{w}))\cdot\vec{y} = (\vec{u}\times\vec{v}+\vec{u}\times\vec{w})\cdot\vec{y}$, $\forall\vec{y}\in\mathbb{R}^3$; then by plugging in $\vec{y}=\vec{i},\vec{j}$, and $\vec{k}$, we get (6.1). Now

$$(\vec{u}\times(\vec{v}+\vec{w}))\cdot\vec{y} = (\vec{y}\times\vec{u})\cdot(\vec{v}+\vec{w}) \qquad \text{by } (5.1)$$

$$= (\vec{y} \times \vec{u}) \cdot \vec{v} + (\vec{y} \times \vec{u}) \cdot \vec{w}$$ by dot product property 3

$$= (\vec{u} \times \vec{v}) \cdot \vec{y} + (\vec{u} \times \vec{w}) \cdot \vec{y}$$ by (5.1)

$$= (\vec{u} \times \vec{v} + \vec{u} \times \vec{w}) \cdot \vec{y},$$ by dot product property

as desired.

Algebraic Formula

The algebraic formula for $\vec{u} \times \vec{v}$ can be obtained from the distributive laws (6.1) and (6.2):

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\boxed{\mathbf{i} \times \mathbf{i}} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} \\
&\quad + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\boxed{\mathbf{j} \times \mathbf{j}} + u_2v_3\mathbf{j} \times \mathbf{k} \\
&\quad + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\boxed{\mathbf{k} \times \mathbf{k}} \quad \vec{0} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.
\end{aligned}$$

In other words:

$$\boxed{\vec{u} \times \vec{v} := \langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1 \rangle.}$$

Determinants

For each $n \times n$ matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$, there is a special number $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$ called the determinant. Determinants for can be computed as follows:

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} := \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} := ad - bc.$

- $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} := a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

  $\qquad\qquad\qquad\qquad (+) \qquad\qquad (-) \qquad\qquad (+)$

A note about computing $n \times n$ determinants will be posted on BB.

In determinant notation, $\vec{u} \times \vec{v}$ can be memorized as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example

If $\vec{u} = \langle 2, 1, 1 \rangle$ and $\vec{v} = \langle -4, 3, 1 \rangle$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \mathbf{k}$$
$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

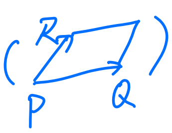
Example

Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$.

- $\overrightarrow{PQ} = \langle 1, 2, -1 \rangle$; $\overrightarrow{PR} = \langle -2, 2, 2 \rangle$

Solution

The area is $3\sqrt{2}$.

- $\text{Area}(\triangle) = \frac{1}{2}\,\text{Area}($  $)$

$= \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| = \frac{1}{2} \left| \langle 6, 0, 6 \rangle \right| = 3\sqrt{2}.$

Properties of Cross Products

Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

All can be verified algebraically using the formula of a cross product.

Note that cross product is not associative: $(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0}$,

but $\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$.

Determinant Form of $(\vec{u} \times \vec{v}) \cdot \vec{w}$

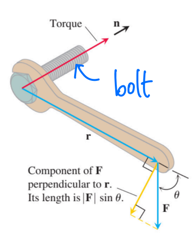
It can be checked by applying the cross product formula that the triple scalar product can be computed as a 3x3 determinant:

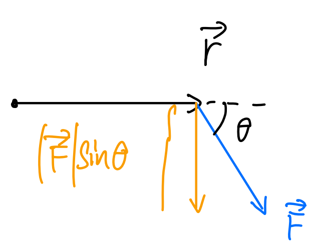
$$\boxed{(\vec{u} \times \vec{v}) \cdot \vec{w} = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.}$$

See 12.4.6 for an example.

Torque

A force $\vec{F}$ is applied to a wrench to turn the bolt, as indicated below.





The torque is the force (vector) defined by $\vec{\tau} := \vec{r} \times \vec{F}$.

↑ Greek letter "Tau"

Note that:

- The direction of $\vec{\tau}$ is perpendicular to the plane of rotation.
- $|\vec{\tau}| = |\vec{r}||\vec{F}| \sin\theta$, which measures how much the bolt moves.

See 12.4.5 for an example.

Lines in $\mathbb{R}^3$

A vector equation for the line L through a point $\underbrace{(x_0, y_0, z_0)}^{P_0}$ parallel to $\vec{v}$ is

$$\vec{r}(t) = \vec{r_0} + t\vec{v}, \quad t \in \mathbb{R},$$

where $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$. The corresponding parametric equations for L is

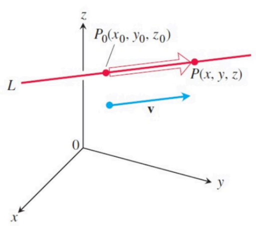
position vector $\vec{OP_0}$ of the point P_0.

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad t \in \mathbb{R},$$

where $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Notation:

- $P = (x_0, y_0, z_0)$: point
- $\langle x_0, y_0, z_0 \rangle = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$: position vector of P.



Example

The line through $(5, -2, 0)$ and $(-1, 3, 2)$ has parametric equations

$$x = 5 + 6t, \quad y = -2 - 5t, \quad z = -2t, \quad t \in \mathbb{R}.$$

Alternatively, $x = -1 + 12t$, $y = 3 - 10t$, $z = 2 - 4t$, $t \in \mathbb{R}$ also work.

Remark: • A line has many different vector equations

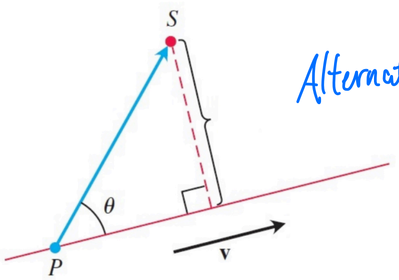
$$\vec{r}(t) = \vec{r_0} + t\vec{v},$$

since there are different choices for $\vec{r_0}$ and $\vec{v}$.

Distance from a Point to a Line in the Space

Let L be a line (in the space) through a point P parallel to $\vec{v}$, and let S be a point. Then the distance d from S to L can be computed by

$$d = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|}.$$



Alternatively, $d = \left|\overrightarrow{PS} - \mathrm{proj}_{\vec{v}}\overrightarrow{PS}\right|$

See 12.5.5 for an example.

Planes in $\mathbb{R}^3$

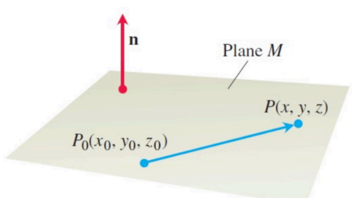
Each plane M in the space has a vector $\vec{n}$ that is perpendicular to the plane; such vector is called a normal vector to the plane. If $P_0(x_0, y_0, z_0)$ is a point in M, then M consists of points $P(x, y, z)$ such that

$$\overrightarrow{P_0P} \cdot \vec{n} = 0.$$ Vector equation of M

In particular, if $\vec{n} = \langle A, B, C\rangle$, then

Component equation of M

$$M = \{(x, y, z) : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\}.$$



It can also be described by

$\vec{n} \cdot \overrightarrow{OP} = \vec{n} \cdot \overrightarrow{OP_0}$

$$Ax + By + Cz = D,$$

where $D = \underbrace{Ax_0 + By_0 + Cz_0}_{\vec{n} \cdot \overrightarrow{OP_0}}$ and $\underbrace{\langle A, B, C\rangle}_{\vec{n}}$ is a normal vector of the plane.

Example

Find an equation of the plane passing through the points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$.

Sol (Outline): • $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 12, 20, 14 \rangle$. Can pick this as $\vec{n}$.

• $\vec{n} \cdot \overrightarrow{OP} = 100$ $(= \vec{n} \cdot \overrightarrow{OQ} = \vec{n} \cdot \overrightarrow{OR})$

• Equation: $12x + 20y + 14z = 100$.

Distance from a Point to a Plane

Let P be a point on a plane with a normal vector $\vec{n}$, and let S be any point. By considering vector projection ~~(Page 17)~~, the distance from S to the plane is

$$\left| \frac{\overrightarrow{PS} \cdot \vec{n}}{|\vec{n}|} \right|.$$

Scalar component of $\overrightarrow{PS}$ in the direction of $\vec{n}$.

Q: How to pick a point on the plane $Ax + By + Cz = D$?

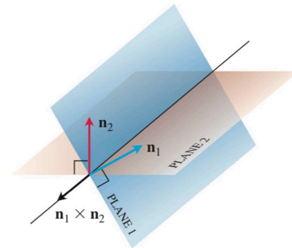
e.g. Find the distance from $S(1,1,3)$ to $\mathcal{M}$ given by $3x + 2y + 6z = 6$.

Sol (Outline): • $P(0,0,1)$ is on $\mathcal{M}$, and $\overrightarrow{PS} = \langle 1, 1, 2 \rangle$.

• $\vec{n} = \langle 3, 2, 6 \rangle$.

• Distance $= \left| \frac{\overrightarrow{PS} \cdot \vec{n}}{|\vec{n}|} \right| = \frac{17}{7}$.

The intersection of two nonparallel planes is a line.



Exercise

Find parametric equations for the line which is the intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Sol (Outline):

- Any vector that is parallel to the line is $\perp$ to both normals

  $$\vec{n}_1 = \langle 3, -6, -2 \rangle \quad \text{and} \quad \vec{n}_2 = \langle 2, 1, -2 \rangle,$$

  and vice versa.
- Can take $\vec{n}_1 \times \vec{n}_2 = \langle 14, 2, 15 \rangle$ to indicate line's direction.
- Find a point on the line (how?) : $(10, 0, \frac{15}{2})$ is on the line.
- $x = 10 + 14t, \quad y = 2t, \quad z = \frac{15}{2} + 15t$.

More Terminology

Def:

- The angle between two planes is the angle θ, with $\theta \in [0, \frac{\pi}{2}]$, made by their normal vectors
- Two planes are parallel if their normal vectors are parallel.
- Two lines (in $\mathbb{R}^3$) are skew if they do not intersect and are not parallel.

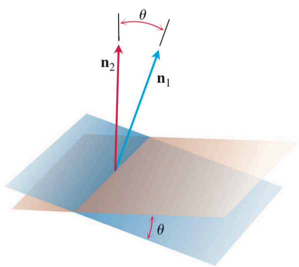
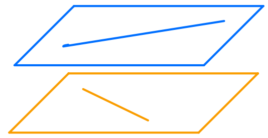


FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.



Two skew lines