MAT1002 Lecture 8, Tuesday, Feb/21/2023

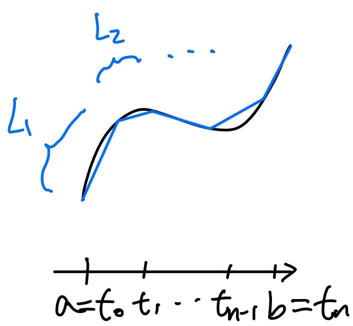
Outline

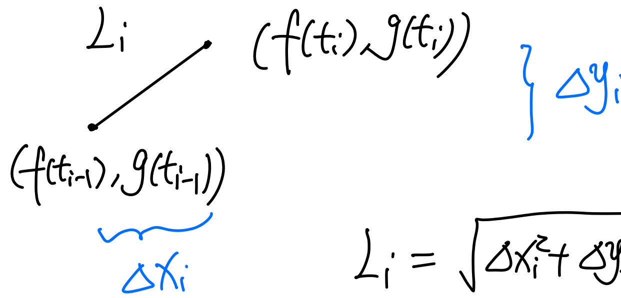
- Calculus with parametric curves (11.2)
  - ↳ Arc length
  - ↳ Area under a curve
  - ↳ Arc length differentials and surfaces of revolution
- Polar coordinates and polar curves (11.3)
- Sketching polar curves (11.4)
- Calculus with polar curves (11.5)

Arc Lengths

f and g are continuously differentiable.

Suppose C is given by $x = f(t)$, $y = g(t)$, $t \in [a,b]$. Assume that f' and g' are continuous, and C is traversed exactly once. We may approximate the length of C, as follows.





$$L_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$\Delta x_i = f(t_i) - f(t_{i-1}) \overset{MVT}{=} f'(c_i)\Delta t_i \text{ for some } c_i \in (t_{i-1}, t_i).$$

Similarly, $\Delta y_i = g'(d_i)\Delta t_i$ for some $d_i \in (t_{i-1}, t_i)$.

Length of C: $L \approx \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} \sqrt{f'(c_i)^2 + g'(d_i)^2}\,\Delta t_i$.

It can be shown formally using upper and lower sums that (if f' and g' are cts)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{f'(c_i)^2 + g'(d_i)^2}\,\Delta t_i = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt.$$

This motivates the following definition

Definition

Let C be a curve given by $x = f(t)$, $y = g(t)$ and $t \in [a, b]$, where f' and g' are continuous on $[a, b]$. If C is traversed exactly once as t increases from a to b (except that $t = a$ and $t = b$ may give the same point), then the length L of C is defined by

$$L := \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

Example

Find the length of the parametric curve given by

$$x = \sin 2t, \quad y = \cos 2t, \quad 0 \le t \le 2\pi.$$

Sol: The curve is the unit circle, but traced twice. It is traced exactly once when t goes from 0 to π. So

$$L = \int_0^\pi \sqrt{f'(t)^2 + g'(t)^2}\, dt = \int_0^\pi \sqrt{4\cos^2(2t) + 4\sin^2(2t)}\, dt$$

$$= 2\int_0^\pi dt = 2\pi.$$

Area under a Curve

We know that the area under a the graph of a nonnegative function $y = h(x)$ from $x = \alpha$ to $x = \beta$ is given by $A = \int_\alpha^\beta h(x)\,dx$. If the graph is traced out exactly once by the parametric equations

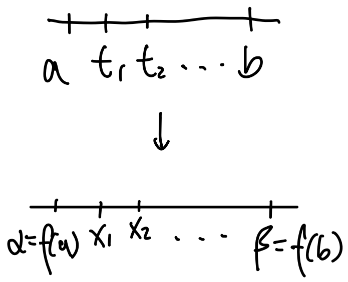
$$x = f(t), \quad y = g(t), \quad a \le t \le b,$$

where f' and g' are continuous, then by substitution, we have

$$A = \int_a^b g(t) f'(t)\,dt \quad \left(\text{or } A = \int_b^a g(t) f'(t)\,dt\right).$$

if $x \nearrow$ as $t \nearrow$ if $x \searrow$ as $t \nearrow$

Idea:



$x_k = f(t_k)$

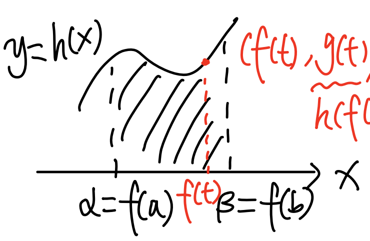
- $\Delta x_k = x_k - x_{k-1} = f(t_k) - f(t_{k-1}) = f'(t_k^*)\,\Delta t_k$
- Choose $x_k^* \in (x_k, x_{k-1})$, where $x_k^* = f(t_k^*)$.

  Then the sample height is

  $$y_k^* := h(x_k^*) = g(t_k^*).$$

- $A \approx \sum_{k=1}^{n} y_k^* \Delta x_k = \sum_{k=1}^{n} g(t_k^*) f'(t_k^*)\,\Delta t_k.$

- Taking limit as $n \to \infty$ gives $\int_a^b g(t) f'(t)\,dt$.

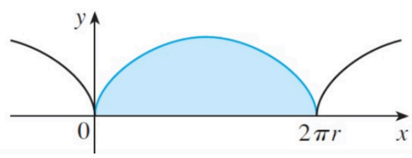


$$A = \int_\alpha^\beta h(x)\,dx = \int_a^b h(f(t)) f'(t)\,dt$$

$$= \int_a^b g(t) f'(t)\,dt.$$

Example

Find the area under one arch of the cycloid.



$x = r(\theta - \sin\theta)$
$y = r(1 - \cos\theta)$
$\theta \in [0, 2\pi]$

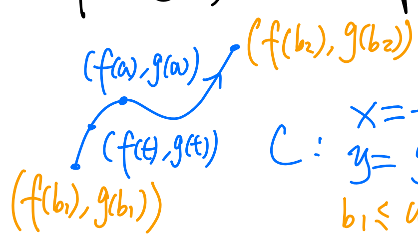
$$A = \int_0^{2\pi} r(1-\cos\theta)\, r(1-\cos\theta)\, d\theta = \text{exercise} = 3\pi r^2.$$

Arc Length Differentials

Given a parametric curve $(x,y) = (f(u), g(u))$, $u \in I$, fix $a \in I$, and define

$$S(t) := \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2}\, du,$$

the signed (can be negative) distance from the base point $(f(a), g(a))$ to a point $(f(t), g(t))$ along the curve (in the direction of increasing parameter).



$C:$ $x = f(u)$, $y = g(u)$, $b_1 \leq u \leq b_2$

By FTC, $\dfrac{ds}{dt} = \sqrt{f'(t)^2 + g'(t)^2} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$

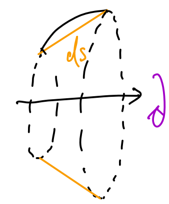
The differential

$$ds = \sqrt{f'(t)^2 + g'(t)^2}\, dt \leftarrow \text{" A small section of the curve"}$$

is called the arc length differential.

Surfaces of Revolution

In MAT1001, we saw that the area of the surface generated by revolving the graph $y = h(x)$ $(\geq 0, \forall x)$ about the x-axis for $x \in [\alpha, \beta]$ is

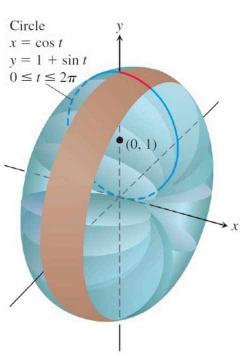


$$S = \int_\alpha^\beta 2\pi y \overbrace{\sqrt{1 + h'(x)^2}\, dx}^{ds}.$$

More generally, if the surface is generated by revolving about the x-axis a curve given by $x = f(t)$, $y = g(t)$ (≥ 0) for $a \leq t \leq b$, its area is

$$S = \int_a^b 2\pi \underbrace{g(t)}_{y} \underbrace{\sqrt{f'(t)^2 + g'(t)^2}\, dt}_{ds}.$$

Using the formula, the surface area drawn in the following figure can be computed with a straightforward integral. (This is Example 9 in Chapter 11.2 of the book.)

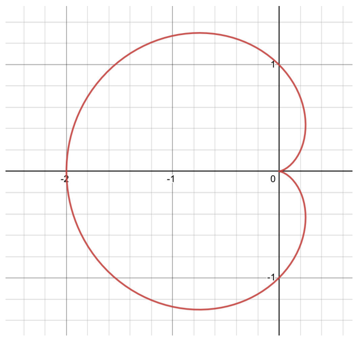


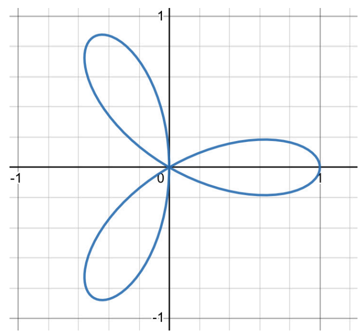
$$S = \int_0^{2\pi} 2\pi(1+\sin t)\sqrt{(-\sin t)^2 + \cos^2 t}\, dt$$

$$= 2\pi \int_0^{2\pi} (1+\sin t)\, dt = 4\pi^2.$$

Polar Coordinates

Polar curves form a special type of parametric curves that could be useful in science and engineering (e.g., describing directional microphone pickup patterns). The following are two polar curves.





We first introduce polar coordinates.

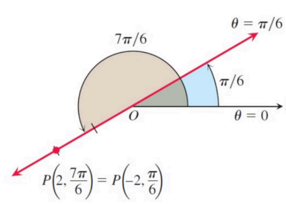
Definition

For a point $(x, y) \in \mathbb{R}^2$ on the xy-plane in Cartesian coordinates:

- let r be the length of the line segment L joining the points $(0, 0)$ and (x, y), and;
- let θ be the angle made by L and the positive x-axis, with the positive sign indicating the counterclockwise measurement.

Then the point (r, θ) is called a polar coordinate of the point (x, y).

Sometimes, we extend the definition by allowing "negative distance": if a point has polar coordinate (r, θ) where $r < 0$, we can also represent it by $(-r, \theta + \pi)$, e.g.



For points on the xy-plane:

(x,y) : Cartesian coordinate, unique.

(r,θ) : Polar coordinate, not unique.

↳ e.g. $(1, \frac{\pi}{4}) = (1, \frac{9\pi}{4}) = (-1, \frac{5\pi}{4})$

Conversion between Cartesian and Polar Coordinates

Given (r,θ) : (x,y) is uniquely given by

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Given (x,y) : one (r,θ) is given by

$$r^2 = x^2+y^2, \quad \tan\theta = \frac{y}{x} \quad (\text{if } x \neq 0).$$

θ is chosen in the correct quadrant depending on the signs of x and y.

If $x=0$, then $\theta = \frac{\pi}{2}$ if $y>0$, and $\theta = \frac{3\pi}{2}$ if $y<0$.

(The origin can be given by any angle.)

A polar curve $F(r,\theta)=0$ is a the set of all points in the xy-plane whose polar coordinates satisfy the equation. A special case is given by $F(r,\theta) = r - f(\theta)$, i.e., $r = f(\theta)$.

For the polar curve $r = \frac{4}{2\cos\theta - \sin\theta}$, when $\theta = \frac{\pi}{4}$, $r = \frac{4}{2\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}} = 4\sqrt{2}$, so the point $(x,y) = (r\cos\theta, r\sin\theta) = (4\sqrt{2}\frac{1}{\sqrt{2}}, 4\sqrt{2}\frac{1}{\sqrt{2}}) = (4,4)$ is on the curve.

For each polar angle θ, draw a point with distance r away from the origin.

e.g. Express the curve $\underbrace{r = \frac{4}{2\cos\theta - \sin\theta}}$ ① with a Cartesian equation.

a polar curve: all points in the xy-plane whose polar coordinates satisfy the equation.

Sol: ① $\Leftrightarrow$ $2r\cos\theta - r\sin\theta = 4$

$\Leftrightarrow$ $2x - y = 4$ $\Leftrightarrow$ $\underbrace{y = 2x - 4}_{\text{Answer.}}$.

e.g. Express the Curve $x^2 + xy + y^2 = 1$ ② with a polar equation.

Sol: ② $\Leftrightarrow$ $r^2 + r^2\cos\theta\sin\theta = 1$

$\Leftrightarrow$ $r^2(1 + \cos\theta\sin\theta) = 1$

$\Leftrightarrow$ $r^2 = \frac{1}{1 + \cos\theta\sin\theta}$ Ans. (or $r = \pm\sqrt{\frac{1}{1+\cos\theta\sin\theta}}$)

Sketch Polar Curves

e.g. Sketch the polar curve $r = 1 - \cos\theta$ (Cardioid).

- May use a computer.
- Or you can have a quick sketch to have some idea on its approximated shape.

Symmetry

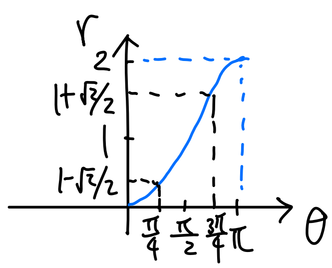
Symmetry about x-axis:
(r,θ) on C $\Rightarrow$ $(r,-\theta)$ or $(-r,-\theta+\pi)$ on C

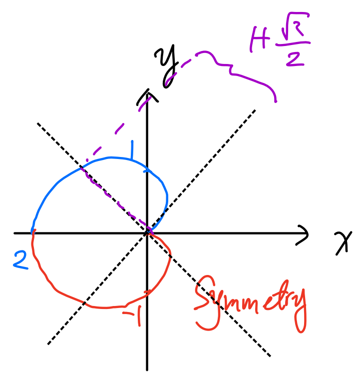
Symmetry about y-axis:
(r,θ) on C $\Rightarrow$ $(r,\pi-\theta)$ or $(-r,-\theta)$ on C

Symmetry about origin by rotation of 180°:
(r,θ) on C $\Rightarrow$ $(r,\theta+\pi)$ or $(-r,\theta)$ on C

e.g. Sketch $r = \underbrace{1-\cos\theta}_{f(\theta)}$ (cardioid) on the xy-plane.

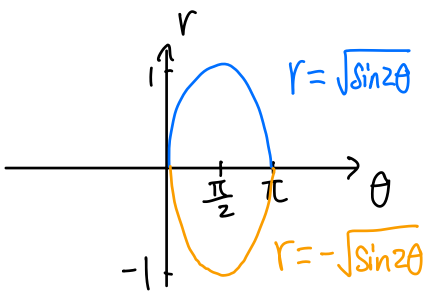
Sol: • Since $\cos(-\theta) = \cos\theta$, $f(-\theta) = f(\theta)$, so curve is symmetric about the x-axis —— only need to investigate $\theta \in [0,\pi]$.

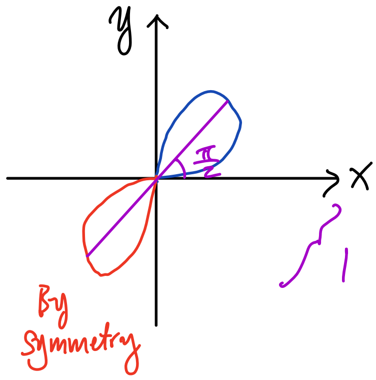


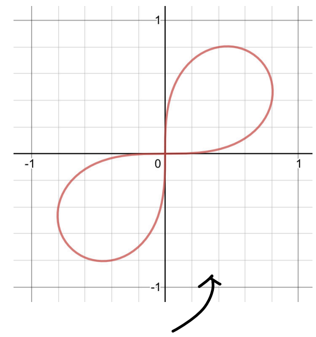


e.g. How does the polar curve $r^2 = \sin 2\theta$ look like?

- Since $(-r)^2 = r^2$, curve is symmetric about the origin.
- For $\theta \in [0,2\pi]$, $\sin 2\theta \geq 0$ $\Leftrightarrow$ $\theta \in [0,\frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$.
- By symmetry, suffices to investigate $\theta \in [0,\frac{\pi}{2}]$.







By Desmos

Calculus with Polar Curves

For a polar curve $r = f(\theta)$, the xy-positions of it points are

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta, \qquad (3)$$

so it is a parametric curve with parameter θ. The theory in 11.1 allows one to do calculus with polar curves.

Slope

The slope at a point of a polar curve given by $\theta = \theta_0$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\theta_0} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\theta_0} = \left.\frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}\right|_{\theta=\theta_0},$$

given that the denominator $\neq 0$.

Arc Lengths

Consider a curve on the xy-plane given in polar coordinates by

$$r = f(\theta), \ a \le \theta \le b,$$

where f' is continuous. If the curve is traversed exactly once, then by the arc length formula ~~on Page 8~~ in 11.2, its length L is given by

$$L = \int_a^b \sqrt{(f(\theta))^2 + f'(\theta))^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

To see this, since $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$,

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_a^b \left((f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2 \right)^{\frac{1}{2}} d\theta$$

$$= \int_a^b \left(f'(\theta)^2\cos^2\theta + f(\theta)^2\sin^2\theta + f'(\theta)^2\sin^2\theta + f(\theta)^2\cos^2\theta \right)^{\frac{1}{2}} d\theta$$

$$= \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

e.g. Find the arc length of the polar curve $r = 1 - \cos\theta$.

Sol. Curve is traced out exactly once for $0 \le \theta \le 2\pi$.

$$L = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos\theta}\, d\theta = \sqrt{2} \int_0^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}}\, d\theta \qquad \left(\cos 2x = 1 - 2\sin^2 x \right)$$

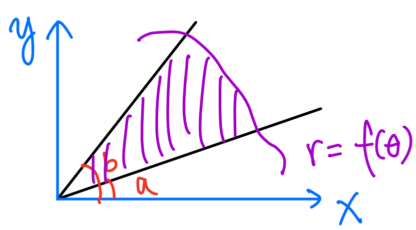
$$= \sqrt{2}\sqrt{2}\int_0^{2\pi} \sin\frac{\theta}{2}\,d\theta = 2\left(2\cos\frac{\theta}{2}\right)\Big|_{2\pi}^{0} = 4(1+1) = 8.$$

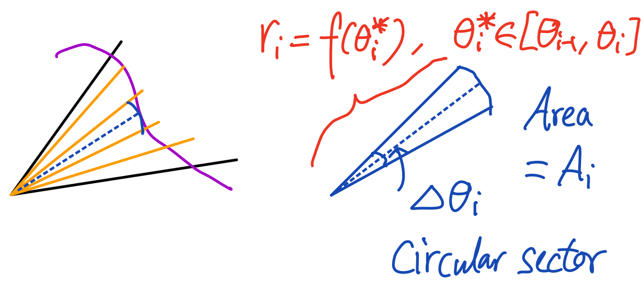
Areas: Fan-Shaped Regions

Consider the region on the xy-plane whose polar coordinates satisfy

$$0 \le r \le f(\theta), \quad a \le \theta \le b \quad (b-a \le 2\pi).$$

This is a fan-shaped region. What is its area A?



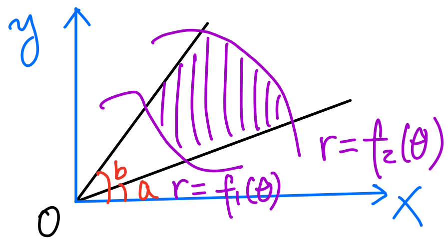
- Partition the θ-interval $[a,b]$.
- 

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} A_i = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\Delta\theta_i}{2\pi}\pi f(\theta_i^*)^2 = \lim_{n\to\infty}\frac{1}{2}\sum_{i=1}^{\infty} f(\theta_i^*)^2\Delta\theta_i$$

$$\boxed{A = \frac{1}{2}\int_a^b f(\theta)^2\,d\theta.}$$

More generally, for the xy-plane region whose polar coordinates satisfy

$$0 \le f_1(\theta) \le r \le f_2(\theta), \quad a \le \theta \le b \quad (b-a \le 2\pi),$$



its area is given by

$$\boxed{A=\frac{1}{2}\int_a^b \left(f_2(\theta)^2-f_1(\theta)^2\right)d\theta.}$$

Example

(a) Find the area of the region enclosed by the curve

$$r = 2(1+\cos\theta),\ 0 \le \theta \le 2\pi.$$

(b) Consider the region S enclosed by the curve

$$r = 1-\cos\theta,\ 0 \le \theta \le 2\pi.$$

Find the area of the region that lies outside S and inside the circle $r=1$.

Solution

(a) 6π.

(b) $2-(\pi/4)$.

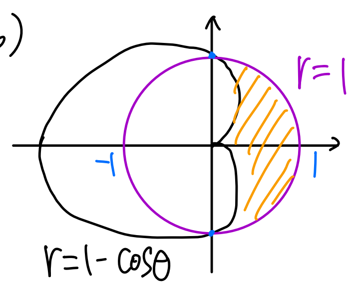
Sol: (a) (Idea)

- Realize that the region is a fan-shaped region

$$0 \le r \le 2(1+\cos\theta), \quad 0\le\theta\le 2\pi.$$

- $A=\int_0^{2\pi}\frac{1}{2}2^2(1+\cos\theta)^2\,d\theta = 2\int_0^{2\pi}(1+\cos\theta)^2\,d\theta = \ldots = 6\pi.$

(b)



$$1-\cos\theta \le r \le 1, \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$

$1-\cos\theta=1 \iff \theta=\pm\frac{\pi}{2}$ (in $[-\pi,\pi]$)

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(1^2 - (1-\cos\theta)^2\right) d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(2\cos\theta - \cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(2\cos\theta - \frac{\cos(2\theta)+1}{2}\right) d\theta$$

$$= \int_{0}^{\pi/2} \left(2\cos\theta - \frac{\cos(2\theta)+1}{2}\right) d\theta \qquad \text{even integrand}$$

$$= 2\sin\theta - \frac{1}{4}\sin(2\theta) - \frac{1}{2}\theta \Big|_{\theta=0}^{\pi/2}$$

$$= 2 - \pi/4.$$