MAT1002 Lecture 16, Tuesday, Mar/21/2023

Outline

- Gradient vectors (14.5)
  - ↳ Geometric meanings
- Tangent planes and linear approximation (14.6)
  - ↳ Tangent planes to level surfaces
  - ↳ Linear approximation
  - ↳ Total differentials

Geometric Meanings of Gradient Vectors

Q: When climbing mountain at the horizontal position (x_0, y_0), which direction should you climb in order to go up the fastest? To go down the fastest?

Geometric Meaning 1: direction of fastest increase and decrease.

Theorem

Let f be a differentiable function (with multiple variables), and let $\vec{x_0}$ represent an interior point of the domain of f. Then, among all directional derivatives $D_{\vec{u}}f(\vec{x_0})$:

- *The maximum value of $D_{\vec{u}}f(\vec{x_0})$ is $|\nabla f(\vec{x_0})|$, and it occurs when $\vec{u}$ is the direction of $\nabla f(\vec{x_0})$.*
- *The minimum value of $D_{\vec{u}}f(\vec{x_0})$ is $-|\nabla f(\vec{x_0})|$, and it occurs when $\vec{u}$ is the direction of $-\nabla f(\vec{x_0})$.*

If $\nabla f(\vec{x_0}) \neq \vec{0}$ and $\vec{u}$ is orthogonal to $\nabla f(\vec{x_0})$, then $D_{\vec{u}}f(\vec{x_0}) = 0$.

Remark

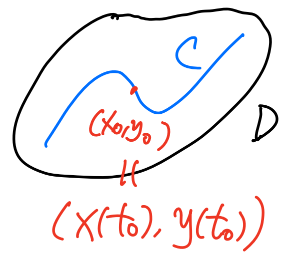
The theorem implies that at any point, a differentiable function increases the fastest in the direction of ∇f, and decreases the fastest in the direction of $-\nabla f$.

Exercise

Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where temperature is measured in degrees Celsius and distance is measured in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

Sol:
- $\nabla T(1,1,2) = \frac{5}{8} \langle -1, -2, 6 \rangle$.
- $\nearrow$ fastest in the direction $\vec{u} = \nabla T(1,1,2) / |\nabla T(1,1,2)|$.
- Max $D_{\vec{u}} T(1,1,2) = |\nabla T(1,1,2)| = \frac{5}{8}\sqrt{41}$ $(^\circ C/m)$

Let $f(x,y)$ be differentiable at (x_0, y_0). Consider a curve C in the domain passing through (x_0, y_0), and let $\vec{r}(t) = \langle x(t), y(t) \rangle$ be a smooth parametrization of C. Consider the "height" $z = f(x,y)$, and restrict it to C:



$$z = f(x(t), y(t)) = f(\vec{r}(t)).$$

How fast is height changing at time t_0 along the curve? By chain rule,

$$\left.\frac{dz}{dt}\right|_{t=t_0} = f_x(x(t_0), y(t_0))\, x'(t_0) + f_y(x(t_0), y(t_0))\, y'(t_0),$$

i.e.,

$$(f \circ \vec{r})'(t_0) = \nabla f(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0). \qquad ①$$

This gives a "familiar" form of the chain rule.

Geometric Meaning 2: relation to level curves.

In particular, if C is a level curve given by $f(x,y) = K$ and is parametrize by $\vec{r}(t)$, then $\frac{dz}{dt} \equiv 0 \;\; \forall t$, and ① says that

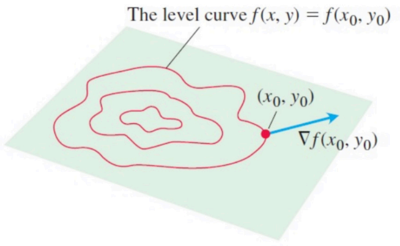
$$\nabla f(x_0, y_0) \cdot \vec{r}\,'\big|_{(x_0,y_0)} = 0.$$

$\forall$ points (x_0, y_0) on C.

This gives a second geometric significance of ∇f.

Theorem

At any point (x_0, y_0) in the domain of a differentiable function f, if C is the level curve through (x_0, y_0), then the gradient of f at (x_0, y_0) is perpendicular to the tangent vector to C at (x_0, y_0).



This geometric fact allows us to describe the tangent line to the level curve of f at a point (x_0, y_0) using the equation

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0.$$

Example

Find an equation for the tangent line to the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$ at the point $(-2, 1)$.

Sol: · $f(x,y) = \frac{x^2}{8} + \frac{y^2}{2}$; $f_x = \frac{x}{4}$, $f_y = y$.

· $f_x(-2,1) = -\frac{1}{2}$, $f_y(-2,1) = 1$

· Tangent is $-\frac{1}{2}(x+2) + 1(y-1) = 0$.

$(y = \frac{1}{2}x + 2.)$

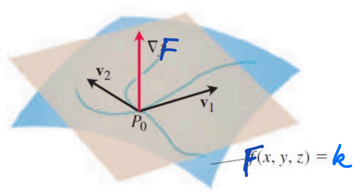
Properties of Gradients

Algebra Rules for Gradients

1. *Sum Rule:*	$\nabla(f+g) = \nabla f + \nabla g$	
2. *Difference Rule:*	$\nabla(f-g) = \nabla f - \nabla g$	
3. *Constant Multiple Rule:*	$\nabla(kf) = k\nabla f$ (any number k)	
4. *Product Rule:*	$\nabla(fg) = f\nabla g + g\nabla f$	Scalar multipliers on left of gradients
5. *Quotient Rule:*	$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$	

Tangent Planes and Linear Approximation

- Let $\mathcal{S}$ be a surface in the domain of F described by $F(x, y, z) = k$, where F is differentiable.
- Let (x_0, y_0, z_0) be a point in $\mathcal{S}$.
- Take any curve $\mathcal{C}$ on $\mathcal{S}$ passing through (x_0, y_0, z_0).
- Just like what happened to level curves, $\nabla F(x_0, y_0, z_0)$ is perpendicular to any tangent vector of $\mathcal{C}$ at (x_0, y_0, z_0).



By considering different curves passing through the point, we have the picture above.

Def: Given any level surface S given by $F(x,y,z)=k$ of a differentiable function F, the tangent plane at a point $P_0 \in S$ to S is the plane through P_0 with normal $\nabla F(P_0)$, and the normal line of the surface at P_0 is the line through P_0 in the direction of $\nabla F(P_0)$.

The tangent plane to a level surface of F at (x_0, y_0, z_0) is described by

$$F_x(x_0, y_0, z_0)(x-x_0)+F_y(x_0, y_0, z_0)(y-y_0)+F_z(x_0, y_0, z_0)(z-z_0) = 0.$$

In particular, for graph of $f(x,y)$ (surface $z=f(x,y)$), the tangent plane to it at $P_0=(x_0, y_0, f(x_0,y_0))$ is

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-f(x_0,y_0)) = 0$$

or equivalently,

$$\boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).}$$

The function L whose graph is the tangent plane above, is given by

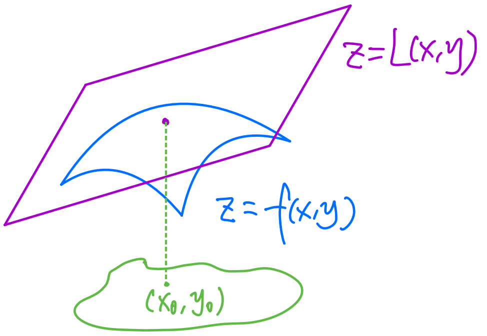
$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The function L is called the linearization of f at (x_0, y_0). The approximation

$$f(x, y) \approx L(x, y),$$

is referred to as the (standard) linear approximation or the tangent plane approximation of f at (x_0, y_0).

This approximation is "good" for differentiable functions, whose graphs locally look like a plane.



Error of Approximation

Now fix a starting point (x_0, y_0) and consider the linear approximation.
The error of the approximation is $E(x,y) := f(x,y) - L(x,y)$.

Suppose:

- $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ are all continuous on an open region D.
- In some rectangle R centred at (x_0, y_0) $\left(\text{where } R \subseteq D\right)$, $\exists M$ such that $|f_{xx}|$, $|f_{yy}|$ and $|f_{xy}|$ $(= |f_{yx}|)$ are all bounded above by (the same) M.

Then

$$\boxed{|E(x,y)| \leq \frac{1}{2} M \left(|x-x_0| + |y-y_0|\right)^2.}$$

for all $(x,y) \in R$.

We will postpone the proof to a later section.

Differentials

Analogous to the one-variable case $y=f(x)$, we define differentials for a differentiable function $z=f(x,y)$ similarly. Given any point (x_0,y_0) in the domain of f:

- Introduce independent variables Δx and Δy.
- Also introduce dx and dy ($dx := \Delta x$, $dy := \Delta y$).
- $\Delta z = \Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$. (Change in f)
- $$dz = df = L(x_0+dx, y_0+dy) - L(x_0,y_0)$$
  $$= f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy.$$ (Change in L)

This dz is called the total differential. Note that it is a function of x_0, y_0, dx and dy. The total differential can be used to approximate the sensitivity of the function value subject to a small change in values of the independent variables.

e.g. A right circular cylindrical storage has height $h=2.5\,m$ and base radius $r=0.5\,m$ (which you don't know yet). You try to measure the h and r of the storage. Which parameter should you be more careful with the measurement in order to avoid big measurement error in volume?

Functions with Three Variables

For $w = f(x,y,z)$, differentiable, "base point" $P_0 = (x_0, y_0, z_0)$:

- Linearization of f at P_0:

$$L(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0)$$

- Error: If all second-order partials are cts on an open D and their absolute values are bounded by M on $R \subseteq D$, rectangular solid. then

$$|E(x,y,z)| \le \frac{1}{2} M \left(|x-x_0| + |y-y_0| + |z-z_0| \right)^2.$$

for all $(x,y,z) \in R$.

- Total differential: $dw = df = f_x(P_0)\,dx + f_y(P_0)\,dy + f_z(P_0)\,dz$.

e.g. For $f(x,y,z) = x^2 - xy + 3\sin z$, consider the standard linear approximation at $P_0 = (2,1,0)$ on

$$R = [1.99, 2.01] \times [0.98, 1.02] \times [-0.01, 0.01]$$
$$= \{ (x,y,z) : x \in [1.99, 2.01],\ y \in [0.98, 1.02],\ z \in [-0.01, 0.01] \}.$$

- $f(P_0) = 2$, $f_x(P_0) = 3$, $f_y(P_0) = -2$, $f_z(P_0) = 3$, so

$$L(x,y,z) = 2 + 3(x-2) - 2(y-1) + 3z = 3x - 2y + 3z - 2.$$

- Since $f_{xx} = 2$, $f_{yy} = 0$, $f_{zz} = -3\sin z$, $f_{xy} = -1$, $f_{xz} = 0$, $f_{yz} = 0$, all their absolute values ≤ 2.

- Hence, for all (x, y, z) in R,

$$|E(x,y,z)| \leq \frac{1}{2} \cdot 2 \left(|x-2| + |y-1| + |z-0| \right)^2$$
$$\leq (0.01 + 0.02 + 0.01)^2 = 0.0016.$$