STA2001 Tutorial 13

1. 5.7-13. Let $X_1, X_2, \cdots, X_{36}$ be a random sample of size 36 from the geometric distribution with pmf $f(x) = (1/4)^{x-1}(3/4), x = 1, 2, 3, \cdots$. Approximate

   (a) $P(46 \leq \sum_{i=1}^{36} X_i \leq 49)$.

   (b) $P(1.25 \leq \bar{X} \leq 1.50)$.

   Hint: Observe that the distribution of the sum is of the discrete type.

   Solution:

   Let $Y = \sum_{i=1}^{36} X_i$.

   This question is similar as the previous one, so we have two ways to derive the expectation and variance of Y. The first way is to prove that Y follows a negative binomial distribution with parameters $p = 3/4$ and $r = 36$ (see Exercise 5.4-6), so that $E(Y) = \frac{36}{3/4} = 48$, $\text{Var}(Y) = \frac{36\cdot(1-3/4)}{(3/4)^2} = 16$.

   The second way, which is more general, starts from the expectation and variance of X_i and makes use of theorem 5.3-2

$$\begin{aligned}
E(X_i) &= \frac{1}{3/4} = \frac{4}{3} \\
\text{Var}(X_i) &= \frac{1-3/4}{(3/4)^2} = \frac{4}{9} \\
E(Y) = E\left(\sum_{i=1}^{36} X_i\right) &= \frac{4}{3} \cdot 36 = 48 \\
\text{Var}(Y) = \text{Var}\left(\sum_{i=1}^{36} X_i\right) &= \frac{4}{9} \cdot 36 = 16
\end{aligned}$$

   Next, we use $N(0,1)$ to approximate the distribution of Y according to the CLT.

   (a) Since Y takes value in $36, 37, 38, \cdots$, it has a discrete distribution.

$$\begin{aligned}
P(46 \leq Y \leq 49) &\approx P\left(\frac{46-48-0.5}{4} \leq Z \leq \frac{49-48+0.5}{4}\right) \\
&= \Phi(0.375) - \Phi(-0.625) \\
&= 0.3802
\end{aligned}$$

   where the half-unit correction is used above.

   (b) Since $\bar{X} = \frac{Y}{36}$ and Y is a discrete distribution, $\bar{X}$ is a discrete distribution as well.

We transform the probability related to $\bar{X}$ into the probability related to Y, and then we can use the same method in question (a).

$$\begin{aligned}
P(1.25 \le \bar{X} \le 1.50) &= P(1.25 \cdot 36 \le 36 \cdot \bar{X} \le 1.5 \cdot 36) \\
&= P(45 \le Y \le 54) \\
&\approx P\left(\frac{45 - 48 - 0.5}{4} \le Z \le \frac{54 - 48 + 0.5}{4}\right) \\
&= \Phi(1.625) - \Phi(-0.875) \\
&= 0.7571
\end{aligned}$$

2. 5.9-3. Let S^2 be the sample variance of a random sample of size n from $N(\mu, \sigma^2)$. Show that the limit, as $n \to \infty$, of the mgf of S^2 is $e^{\sigma^2 t}$.

Solution:

Since S^2 is the sample variance of a random sample of size n from $N(\mu, \sigma^2)$, we know that $W = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ according to Theorem 5.5-2. a

Thus, the mgf of W is given by

$$M_W(t) = E(e^{tW}) = (1-2t)^{-(n-1)/2}, \qquad t < \frac{1}{2}$$

Note that $S^2 = \frac{\sigma^2}{n-1}W$, by a simple transformation the mgf of S^2 can be written as

$$M_{S^2}(t) = M_W\left(\frac{\sigma^2}{n-1} \cdot t\right) = \left(1 - \frac{\sigma^2 t}{(n-1)/2}\right)^{-(n-1)/2}$$

Take the limit for both sides and by the definition of e^x (actually, one of the definitions of exponential function), we have

$$\lim_{n\to\infty} M_{S^2}(t) = \lim_{n\to\infty}\left(1 - \frac{\sigma^2 t}{(n-1)/2}\right)^{-(n-1)/2} = \lim_{n\to\infty}\left(1 + \frac{\sigma^2 t}{-(n-1)/2}\right)^{-(n-1)/2} = e^{\sigma^2 t}$$

3. Let $X_n \xrightarrow{d} X$ where $X \equiv x$ is a constant random variable. Prove that $X_n \xrightarrow{p} X$.

   Note that $\xrightarrow{d}$ is the convergence in distribution and $\xrightarrow{p}$ is the convergence in probability.

   Solution: Fix $\varepsilon > 0$. We have

$$\begin{aligned} \Pr\left(|X_n - X| \le \varepsilon\right) &= \Pr\left(|X_n - x| \le \varepsilon\right) \\ &= \Pr(x - \varepsilon \le X \le x + \varepsilon) \\ &\ge \Pr(x - \varepsilon < X \le x + \varepsilon) \\ &= F_{X_n}(x + \varepsilon) - F_{X_n}(x - \varepsilon) \\ &\overset{n\to\infty}{\longrightarrow} F_X(x + \varepsilon) - F_X(x - \varepsilon) \\ &= \Pr(x - \varepsilon < X \le x + \varepsilon) = 1 \end{aligned}$$

   Hence we have

$$\lim_{n\to\infty} \Pr\left(|X_n - X| \le \varepsilon\right) = 1$$

   for any $\varepsilon > 0$, proving the result.