STA2001 Assignment 9

1. (5.3-11). Let X_1, X_2, X_3 be three independent random variables with binomial distributions $b(4, 1/2)$, $b(6, 1/3)$, and $b(12, 1/6)$, respectively. Find

   (a) $P(X_1 = 2, X_2 = 2, X_3 = 5)$
   (b) $E(X_1 X_2 X_3)$
   (c) The mean and the variance of $Y = X_1 + X_2 + X_3$.

   Solution:

   (a) By independence, we can write

$$\begin{aligned} P(X_1 = 2, X_2 = 2, X_3 = 5) &= P(X_1 = 2)P(X_2 = 2)P(X_3 = 5) \\ &= \binom{4}{2}(0.5)^2(0.5)^2 \times \binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 \times \binom{12}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^7 \\ &\approx 0.0035 \end{aligned}$$

   (b) By independence, we can write

$$E(X_1 X_2 X_3) = E(X_1) \cdot E(X_2) \cdot E(X_3) = \left(4 \times \frac{1}{2}\right) \cdot \left(6 \times \frac{1}{3}\right) \cdot \left(12 \times \frac{1}{6}\right) = 8$$

   (c)

$$E(Y) = E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3) = 2 + 2 + 2 = 6$$

   By independence, we can write

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3) \\ &= 4 \times 0.5 \times 0.5 + 6 \times \frac{1}{3} \times \frac{2}{3} + 12 \times \frac{1}{6} \times \frac{5}{6} \\ &= 4 \end{aligned}$$

2. (5.3-19). Two components operate in parallel in a device, so the device fails when and only when both components fail. The lifetimes, X_1 and X_2, of the respective components are independent and identically distributed with an exponential distribution with $\theta = 2$. The cost of operating the device is $Z = 2Y_1 + Y_2$, where $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$. Compute $E(Z)$.

   Solution:

   Note that the X_1 and X_2 are i.i.d and follow exponential distributions, so their pdf and cdf are given by (they have the same expression of the cdf and pdf)

$$\begin{aligned} f(x) &= \frac{1}{\theta}e^{-x/\theta} = \frac{1}{2}e^{-x/2}, \quad 0 < x < \infty \\ F(x) &= 1 - e^{-x/\theta} = 1 - e^{-x/2}, \quad 0 < x < \infty \end{aligned}$$

   Therefore, the cdf of Y_1 is obtained by

$$\begin{aligned} G_1(y) &= P(Y_1 \le y) \\ &= 1 - P(Y_1 > y) \\ &= 1 - P(X_1 > y, X_2 > y) \\ &= 1 - P(X_1 > y)P(X_2 > y) \\ &= 1 - (1 - F(y))^2 \end{aligned}$$

$$
\begin{aligned}
&= 1 - \left(e^{-y/2}\right)^2 \\
&= 1 - e^{-y}
\end{aligned}
$$

Differentiating it we obtain the pdf of Y_2,

$$g_1(y) = \frac{d}{dy} G_1(y) = e^{-y}$$

The mean of Y_1 is computed by

$$E(Y_1) = \int_0^{+\infty} y e^{-y}\, dy = 1$$

where we obtain the final result using integration by parts (check the textbook or lecture note).

Similarly, the cdf of Y_2 is obtained by

$$
\begin{aligned}
G_2(y) &= P(Y_2 \leq y) \\
&= P(X_1 \leq y, X_2 \leq y) \\
&= P(X_1 \leq y)\, P(X_2 \leq y) \\
&= \left(1 - e^{-y/2}\right)^2
\end{aligned}
$$

Differentiating it we obtain the pdf of Y_2,

$$g_2(y) = \frac{d}{dy} G_2(y) = 2\left(1 - e^{-y/2}\right)\left(\frac{1}{2} e^{-y/2}\right) = e^{-y/2} - e^{y}$$

The mean of Y_2 is computed by

$$
\begin{aligned}
E(Y_2) &= \int_0^{+\infty} y\left(e^{-y/2} - e^{y}\right) dy \\
&= \int_0^{+\infty} y e^{-y/2}\, dy - \int_0^{+\infty} y e^{-y}\, dy \\
&= 2 \times \int_0^{+\infty} \frac{y}{2} e^{-y/2}\, dy - \int_0^{+\infty} y e^{-y}\, dy \\
&= 2 \times 2 - 1 \\
&= 3
\end{aligned}
$$

where we obtain the final results of the integrals using integration by parts (check the textbook or lecture note).

Hence,

$$E(Z) = 2E(Y_1) + E(Y_2) = 5$$

3. (5.3-21). Flip $n = 8$ fair coins and remove all that came up heads. Flip the remaining coins (that came up tails) and remove the heads again. Continue flipping the remaining coins until each has come up heads. We shall find the pmf of Y, the number of trials needed. Let X_i equal the number of flips required to observe heads on coin i, $i = 1, 2, \ldots, 8$. Then $Y = \max(X_1, X_2, \ldots, X_8)$.

   (a) Show that $P(Y \leq y) = [1 - (1/2)^y]^8$.

   (b) Show that the pmf of Y is defined by $P(Y = y) = [1 - (1/2)^y]^8 - \left[1 - (1/2)^{y-1}\right]^8$, $y = 1, 2, \ldots$.

Solution:

(a) As X_i is the number of flips required to observe heads on coin i and the coin tossing results for each coin are independent, they are i.i.d. We can write the pmf of X_i, for $i = 1, 2, \ldots, 8$

$$P\left(X_i = k\right) = \left(\frac{1}{2}\right)^k, \quad k = 1, 2, \ldots$$

Then the cdf is given by

$$P(X_i \le k) = \sum_{n=1}^{k} \left(\frac{1}{2}\right)^n = \frac{1}{2} \cdot \frac{1 - (1/2)^k}{1 - 1/2} = 1 - \frac{1}{2^k}, \quad k = 1, 2, \ldots$$

The cdf of Y is given by

$$\begin{aligned} P(Y \le y) &= P\left(\max\left(X_1, X_2, \ldots, X_8\right) \le y\right) \\ &= P\left(X_1 \le y, X_2 \le y, \ldots, X_8 \le y\right) \\ &= P(X_1 \le y) \cdot P(X_2 \le y) \cdots P(X_8 \le y) \\ &= \left[1 - (1/2)^y\right]^8, \quad y = 1, 2, \ldots \end{aligned}$$

where the 3rd equality is due to the independence.

(b) From the answer to (a), we know that Y is a discrete random variable with support $S_Y = \{1, 2, \ldots\}$. Therefore, the pmf of Y can be obtained by

$$\begin{aligned} P(Y = y) &= P(Y \le y) - P(Y \le y - 1) \\ &= \left[1 - (1/2)^y\right]^8 - \left[1 - (1/2)^{y-1}\right]^8, \quad y = 1, 2, \ldots \end{aligned}$$

(c) Omitted.

(d) Omitted.

4. (5.4-3). Let X_1, X_2, X_3 be mutually independent random variables with Poisson distributions having means 2, 1, and 4, respectively.
(a) Find the mgf of the sum $Y = X_1 + X_2 + X_3$.
(b) How is Y distributed?
(c) Compute $P(3 \le Y \le 9)$.

Solution:

(a) Recall that for a random variable X following Poisson distribution with mean λ, the corresponding mgf is given by

$$M(t) = \exp\left(\lambda\left(e^t - 1\right)\right), \quad -\infty < t < \infty$$

Therefore, we have the mgfs for X_1, X_2 and X_3 by replacing λ with respective values.

Then,

$$\begin{aligned} M_Y(t) = E(e^{tY}) &= E(e^{tX_1 + tX_2 + tX_3}) \\ &= E(e^{tX_1}) \cdot E(e^{tX_2}) \cdot E(e^{tX_3}) \\ &= e^{2\left(e^t - 1\right)} \cdot e^{1\left(e^t - 1\right)} \cdot e^{4\left(e^t - 1\right)} \\ &= e^{7\left(e^t - 1\right)}, \quad -\infty < t < \infty \end{aligned}$$

where we break the expectation at the 2nd equality due to the independence.

(b) From the answer to (a), we know that Y has a Poisson distribution with mean $\lambda = 7$ as its mgf takes the same form.

(c)

$$P(3 \le Y \le 9) = \sum_{k=3}^{9} P(Y = k) = \sum_{k=3}^{9} \frac{7^k e^{-7}}{k!} = 0.8$$

5. (5.4-8). Let $W = X_1 + X_2 + \cdots + X_h$, a sum of h mutually independent and identically distributed exponential random variables with mean θ. Show that W has a gamma distribution with parameters $\alpha = h$ and θ, respectively.

   Solution:

   Recall that for an exponential distribution with mean θ, its mgf is given by $\frac{1}{(1-\theta t)}$ for $t < 1/\theta$.

   We find the distribution of W by checking its moment generating function,

$$\begin{aligned} M(t) &= E\left[e^{tW}\right] \\ &= E\left[e^{t(X_1+X_2+\cdots+X_h)}\right] \\ &= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] \cdots E\left[e^{X_h}\right] \\ &= \frac{1}{(1-\theta t)^h}, \quad t < 1/\theta \end{aligned}$$

   where we break the expectation at the 3rd equality due to the independence.

   The final result shows that W follows a gamma distribution, as a given mgf uniquely determines a distribution. The parameters are h and θ.

6. (5.4-9). Let X and Y, with respective pmfs $f(x)$ and $g(y)$, be independent discrete random variables, each of whose support is a subset of the nonnegative integers $0, 1, 2, \ldots$ Show that the pmf of $W = X + Y$ is given by the **convolution formula**

$$h(w) = \sum_{x=0}^{w} f(x)g(w-x), \quad w = 0, 1, 2, \ldots$$

   Hint: Argue that $h(w) = P(W = w)$ is the probability of the $w + 1$ mutually exclusive events $(x, y = w - x), x = 0, 1, \ldots, w$.

   Solution:

   Let S_X, S_Y and S_W be the supports.

   In the following, if there is some $x \notin S_X$ then we let $P(X = x) = 0$, and similarly for $\mathbb{P}(X = x, W = w)$. Therefore we may include some zeros in the summation (you shall see in the following).

   Note that $X + Y = W$, then the pmf is given by

$$\begin{aligned} h(w) &= \mathbb{P}(W = w) \\ &= \sum_{x=0}^{w} \mathbb{P}(X = x, W = w) \\ &= \sum_{x=0}^{w} \mathbb{P}(X = x, Y = w - x) \\ &= \sum_{x=0}^{w} \mathbb{P}(X = x)\mathbb{P}(Y = w - x) \\ &= \sum_{x=0}^{w} f(x)g(w - x) \end{aligned}$$

where the 2nd equality is due to total probability formula and the 4th equality is due to the independence.

7. (a) (5.4-6). Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a geometric distribution with $p = 1/3$. Find the mgf of $Y = X_1 + X_2 + X_3 + X_4 + X_5$. How is Y distributed?

   (b) (5.4-7). Let X_1, X_2, X_3 denote a random sample of size 3 from a gamma distribution with $\alpha = 7$ and $\theta = 5$. Find the mgf of $Y = X_1 + X_2 + X_3$. How is Y distributed?

   Solution:

   (a) The mgf of the geometric distribution is

$$M_X(t) = \frac{pe^t}{1-(1-p)e^t} \tag{1}$$

   and thus

$$M_Y(t) = E[e^{(X_1+X_2+\cdots+X_5)t}] = E[e^{X_1 t}]E[e^{X_2 t}]\cdots E[e^{X_5 t}] = (\frac{pe^t}{1-(1-p)e^t})^5$$

   which implies that Y is a negative binomial with $r = 5, p = 1/3$

   (b) The mgf of the gamma distribution is

$$M_X(t) = (\frac{\theta}{\theta - t})^\alpha \tag{2}$$

   and thus

$$M_Y(t) = E[e^{(X_1+X_2+X_3)t}] = E[e^{X_1 t}]E[e^{X_2 t}]E[e^{X_3 t}] = (\frac{\theta}{\theta - t})^{3\alpha}$$

   which implies that Y is a Gamma distribution with $\alpha = 21, \theta = 5$

8. (5.4-16). The number X of sick days taken during a year by an employee follows a Poisson distribution with mean 2. Let us observe four such employees. Assuming independence, compute the probability that their total number of sick days exceeds 2.

   Solution: The mgf of Poisson distribution is

$$M_X(t) = e^{\lambda(e^t-1)}$$

   where $\lambda = 2$. If we have four employees, then the total occurrence $Y = X_1 + X_2 + X_3 + X_4$ has mgf

$$M_Y(t) = E[e^{(X_1+X_2+X_3+X_4)t}] = E[e^{X_1 t}]E[e^{X_2 t}]E[e^{X_3 t}]E[e^{X_4 t}] = (e^{\lambda(e^t-1)})^4 = e^{4\lambda(e^t-1)}$$

   implies that Y is also Poisson distribution with mean $\lambda' = 4\lambda = 8$. Then

$$P(Y > 2) = 1 - P(Y \le 2) = 1 - \sum_{x=0}^{2} \frac{8^x e^{-8}}{x!} = 0.0138$$

9. Consider a group of 4 people. Each person randomly sends a like to one of the rest 3 people (i.e.

with probability $\frac{1}{3}$ each). We say a person is popular if the person received at least 2 likes. Let N denote the number of popular people among the four.

(a) Find $E[N]$

(b) Find $Var(N)$

(c) Find $P(N = 0)$

Hint: Let X_i denote the event that i is a popular person, $i = 1, 2, 3, 4$. Let $I_i \triangleq I(X_i)$ denote the indicator random variable that equals 1 if X_i is true and 0 otherwise. By definition, $N = I_1 + I_2 + I_3 + I_4$.

Solution:

(a) (4 points) Therefore,

$$\begin{aligned} E[N] &= E[I_1 + I_2 + I_3 + I_4] \\ &= E[I_1] + E[I_2] + E[I_3] + E[I_4] \\ &= P(X_1) + P(X_2) + P(X_3) + P(X_4). \end{aligned}$$

We compute $P(X_1)$. X_1 is true $\Leftrightarrow$ Person 1 received at least two likes $\Leftrightarrow$ Person 1 received likes from $(2,3),(2,4),(3,4)$ or $(2,3,4)$. $(2,3)$ likes 1 and 4 does not like 1 happens with probability $\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$. This is the same for $(2,4)$ and $(3,4)$. $(2,3,4)$ is with probability $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$. Therefore, $P(X_1) = \frac{7}{27}$. Since X_i are symmetric across $i = 1,2,3,4$, we thus have $E[N] = 4 \times \frac{7}{27} = \frac{28}{27}$.

(b) (4 points) $Var(N) = E[N^2] - (E[N])^2$. Let's compute $E[N^2]$. In particular,

$$\begin{aligned} E[N^2] &= E[(I_1 + I_2 + I_3 + I_4)^2] \\ &= E[I_1^2] + E[I_2^2] + E[I_3^2] + E[I_4^2] + 2 \sum_{1 \le i < j \le 4} E[I_i \times I_j] \\ &= E[I_1] + E[I_2] + E[I_3] + E[I_4] + 2 \sum_{1 \le i < j \le 4} E[I_i \times I_j] \\ &= E[N] + 12 E[I_1 \times I_2] \end{aligned}$$

where the last equality is due to symmetry. Observe that $I_1 \times I_2 \Leftrightarrow I(X_1 \cap X_2)$. Therefore $E[I_1 \times I_2] = P(X_1 \cap X_2)$. Both Person 1 and 2 are popular, it must be the two of them received all 4 likes. In particular, Person 1 and person 2 liked each other, and person 3 and person 4 liked person 1 and person 2 (or person 2 and person 1), respectively. Thus $P(X_1 \cap X_2) = \frac{2}{81}$. Combining the above, $E[N^2] = \frac{28}{27} + 12 \times \frac{2}{81} = \frac{4}{3}$. And $Var(N) = \frac{4}{3} - (\frac{28}{27})^2 = \frac{188}{729}$

(c) (2 points) $N = 0$ if and only if every one received exactly one like. We count the number of samples that satisfies this. There are two cases. Either there's only one cycle, in which case we have $(1 \to 2 \to 3 \to 4)$ type of patterns. There are in total $3 \times 2 \times 1 = 6$ possibilities. Or there are two cycles, in which case we have $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$ type of patterns. There are in total $\binom{4}{2}/2 = 3$ possibilities. In all there are 9 samples out of 81 that nobody is popular. Thus $P(N = 0) = \frac{9}{81} = \frac{1}{9}$.

10. This question centers around a technique known as importance sampling. Suppose that X and Y are continuous random variables with probability density function (pdf) denoted by f_X and f_Y respectively on a common sample space $\overline{S}$. We wish to estimate

$$I = E(g(X)) = \int_{\overline{S}} g(x) f_X(x)\, dx,$$

where g is a function such that $\mathrm{Var}(g(X))$ is finite. At times it maybe difficult to sample from f_X. However it is perhaps easier to sample from f_Y. With importance sampling, we can sample from f_Y

and estimate I. Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed (i.i.d) random variables with pdf f_Y, and define

$$w(y) = \begin{cases} \dfrac{f_X(y)}{f_Y(y)}, & \text{if } f_Y(y) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$J_n = \frac{1}{n} \sum_{i=1}^{n} g(Y_i) w(Y_i).$$

Prove that

(a) $E(J_n) = E(g(Y)w(Y)) = I$.

(b) $\text{Var}(J_n) = \dfrac{1}{n} \left(E\left(g^2(X)w(X)\right) - I^2 \right)$.

Solution:

(a) (2 points) As $Y_1, \ldots, Y_n$ are i.i.d, $g(Y_1)w(Y_1), \ldots, g(Y_n)w(Y_n)$ are i.i.d. To see the first equality, we note that

$$E(J_n) = \frac{1}{n} \sum_{i=1}^{n} E(g(Y_i)w(Y_i)) = E(g(Y)w(Y)).$$

To see the second equality, we write the expectation in integral form:

$$E(g(Y)w(Y)) = \int_{\overline{S}} g(y) \frac{f_X(y)}{f_Y(y)} f_Y(y)\, dy = \int_{\overline{S}} g(y) f_X(y)\, dy = E(g(X)) = I.$$

(b) (2 points) As in part (a), $g(Y_1)w(Y_1), \ldots, g(Y_n)w(Y_n)$ are i.i.d. By book page 185 (or lecture slides), we have

$$\text{Var}(J_n) = \frac{1}{n} \text{Var}(g(Y)w(Y)) = \frac{1}{n} \left(E(g^2(Y)w^2(Y)) - I^2 \right),$$

and so it remains to prove

$$\begin{aligned} E(g^2(Y)w^2(Y)) = \int_{\overline{S}} g^2(y) \frac{f_X^2(y)}{f_Y^2(y)} f_Y(y)\, dy &= \int_{\overline{S}} g^2(y) w(y) f_X(y)\, dy \\ &= E\left(g^2(X)w(X)\right). \end{aligned}$$