Rearrangements of Absolutely Convergent Series

Theorem (3.55)

Let $\sum_n a_n$ *be an absolutely convergent series. If* $\sum_k b_k$ *is a rearrangement of* $\sum_n a_n$*, then* $\sum_k b_k = \sum_n a_n$*. In other words, all rearrangements of an absolutely convergent series converge to the same number.*

(Optional) Proof Outline for Theorem 3.55.

1. Let f be a permutation of $\mathbb{N}$ such that $b_k := a_{f(k)}$. Let A_i and B_i be the partial sums of $\sum_n a_n$ and $\sum_k b_k$, respectively.

2. Since $-|A_i - B_i| \leq A_i - B_i \leq |A_i - B_i|$, it suffices to show that $|A_i - B_i| \to 0$, because then the sandwich theorem allows us to conclude that $\lim_{i\to\infty} A_i = \lim_{i\to\infty} B_i$.

3. Let $\epsilon > 0$. Since (A_i) converges absolutely, there exists N such that for all m, n greater than or equal to N, we have $\sum_{i=m}^{n} |a_i| < \epsilon$.

4. Pick N_0 such that

$$N_0 \geq \max\{f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(N)\},$$

where $f^{-1}(n) := k$ if and only if $f(k) = n$. Note that if $i > N_0$, then $A_i - B_i$ is a finite sum containing only terms after a_N, with no repeated terms due to cancellation. Step 3 and triangle inequality then guarantee $|A_i - B_i| < \epsilon$. This shows that $|A_i - B_i| \to 0$. □

Remark:

- If a series only converges conditionally, then by a theorem of Riemann, it can be made convergent to any real number or even divergent by rearranging its terms suitably. (This is a not an elementary result and we omit it in this course.)
- The concept of absolute convergence will appear frequently when we study power series later.