

MAT1002 Final Exam Reference Solution

1. (i) $\frac{\partial w}{\partial \theta} = -f_x \cdot \rho \sin \phi \sin \theta + f_y \cdot \rho \sin \phi \cos \theta$

(ii) $\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$

(iii) $\nabla f(1,1,1) = \langle 2, 1, 2 \rangle$, $\vec{u} = \frac{1}{\sqrt{5}} \langle 0, 2, -1 \rangle$

$$D_{\vec{u}} f(1,1,1) = \langle 2, 1, 2 \rangle \cdot \frac{1}{\sqrt{5}} \langle 0, 2, -1 \rangle = 0$$

(iv) $Q(x,y) = 2 - x^2 - y^2$.

(v) F

(vi) $\int_{-1}^0 \int_0^{y^2} \int_0^1 dz dx dy$.

(vii) 2

$$\begin{aligned} \text{(viii)} \quad \frac{x}{\sqrt[3]{1+x}} &= x(1+x)^{-\frac{1}{3}} = \binom{-\frac{1}{3}}{0} x + \binom{-\frac{1}{3}}{1} x^2 + \binom{-\frac{1}{3}}{2} x^3 + \binom{-\frac{1}{3}}{3} x^4 \\ &= x - \frac{1}{3} x^2 + \frac{2}{9} x^3 - \frac{14}{81} x^4 \end{aligned}$$

(ix) F

2. (i) As $(x,y) \rightarrow (0,0)$, $\sin x + \cos y \rightarrow 1$ but $xy \rightarrow 0$,
 so limit does not exist (as a real number).

$$(ii) \left| \frac{x^3 - y^3}{x^2 + y^2} \right| = \left| \frac{r^3(\cos^3 \theta - \sin^3 \theta)}{r^2} \right| = r |\cos^3 \theta - \sin^3 \theta| \leq 2r \quad \forall \theta.$$

$$\Rightarrow -2r \leq \frac{x^3 - y^3}{x^2 + y^2} \leq 2r.$$

As $(x,y) \rightarrow (0,0)$, $r \rightarrow 0$, so $2r \rightarrow 0$. By squeeze, limit = 0.

(Alternatively, $\left| \frac{x^3}{x^2 + y^2} \right| \leq |x| \rightarrow 0$ as $(x,y) \rightarrow (0,0)$ and

$\left| \frac{y^3}{x^2 + y^2} \right| \leq |y| \rightarrow 0$ as $(x,y) \rightarrow (0,0)$, so

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} = 0 \quad \text{and} \quad \lim_{(x,y) \rightarrow (0,0)} \frac{y^3}{x^2 + y^2} = 0.$$

This means $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2} = 0 + 0 = 0.$)

(iii) Since $\lim_{\substack{(r,\theta) \rightarrow (1,0) \\ \text{along } \theta=0}} \frac{r \sin \theta}{r^2 - 1} = 0$ and

$$\lim_{\substack{(r,\theta) \rightarrow (1,0) \\ \text{along } r=1+\theta}} \frac{r \sin \theta}{r^2 - 1} = \lim_{\theta \rightarrow 0} \frac{(1+\theta) \sin \theta}{(\theta+2)\theta} = \lim_{\theta \rightarrow 0} \left(\frac{1+\theta}{2+\theta} \right) \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \frac{1}{2},$$

limit D.N.E. by the two-path test.

$$3. \quad f_x(x,y) = 0 \quad \text{on } \mathbb{R}^2.$$

$$\Rightarrow f_{xx}(x,y) = 0, \quad D = \mathbb{R}^2.$$

$$\& f_{xy}(x,y) = 0, \quad D = \mathbb{R}^2.$$

$$f_y(x,y) = \begin{cases} 3y^2, & y \geq 0 \\ -2y, & y < 0 \end{cases} \quad \text{on } \mathbb{R}^2$$

$$\Rightarrow f_{yy}(x,y) = \begin{cases} 6y, & y > 0 \\ -2, & y < 0 \end{cases}, \quad D = \mathbb{R}^2 \setminus \{(x,y) : y=0\}$$

(or xy -plane with x -axis removed)

$$4. \quad \nabla g = \langle 2x, 4y, 2z \rangle \Rightarrow \nabla g(1,1,1) = \langle 2, 4, 2 \rangle$$

$$L: \begin{cases} x = 1 + 2t \\ y = 1 + 4t \\ z = 1 + 2t \end{cases}, \quad t \in \mathbb{R}$$

$$M: 2(x-1) + 4(y-1) + 2(z-1) = 0$$

$$\text{or } 2x + 4y + 2z = 8.$$

5. Method 1 : Minimize $x^2 + y^2 + z^2$
 s.t. $2x + y - z = 4$.

$$\text{Solve } \begin{cases} \nabla f = \lambda \nabla g \\ 2x + y - z = 4 \end{cases} \Leftrightarrow \begin{cases} 2x = \lambda \cdot 2 \\ 2y = \lambda \\ 2z = -\lambda \\ 2x + y - z = 4 \end{cases}$$

$$\Rightarrow 2\lambda + \frac{\lambda}{2} + \frac{\lambda}{2} = 4 \Rightarrow \lambda = \frac{4}{3} \Rightarrow \begin{cases} x = 4/3 \\ y = 2/3 \\ z = -2/3 \end{cases}$$

By geometric property, $(x, y, z) = (\frac{4}{3}, \frac{2}{3}, -\frac{2}{3})$ gives the desired minimum distance.

Method 2 : Define $h(x, y) = x^2 + y^2 + (2x + y - 4)^2$.

$$\begin{aligned} \text{Then } \nabla h &= \langle 2x + 2(2x + y - 4) \cdot 2, 2y + 2(2x + y - 4) \rangle \\ &= \langle 10x + 4y - 16, 4x + 4y - 8 \rangle, \end{aligned}$$

$$\nabla h = \langle 0, 0 \rangle \Leftrightarrow (x, y) = (\frac{4}{3}, \frac{2}{3}).$$

$$\text{Since } h_{xx} = 10, h_{xy} = 4 = h_{yx}, h_{yy} = 4,$$

$$h_{xx}h_{yy} - h_{xy}^2 = 24 > 0, h_{xx} > 0, (x, y) = (\frac{4}{3}, \frac{2}{3}) \text{ gives}$$

a local min. Since it is the only local min, it is the global min, so $(x, y, z) = (\frac{4}{3}, \frac{2}{3}, -\frac{2}{3})$ gives the desired minimum.

6. Minimize and maximize xyz^2 cts
 s.t. $x^2 + y^2 + z^2 = 4$. closed and bdd

$$\text{Solve } \begin{cases} \nabla f = \lambda \nabla g \\ x^2 + y^2 + z^2 = 4 \end{cases} \Leftrightarrow \begin{cases} yz^2 = 2x\lambda & \textcircled{1} \\ xz^2 = 2y\lambda & \textcircled{2} \\ 2xyz = 2z\lambda & \textcircled{3} \\ x^2 + y^2 + z^2 = 4 & \textcircled{4} \end{cases}$$

First, note that any solution with $x=0$, $y=0$, or $z=0$ must give $T(x,y,z)=0$. Consider other solutions. Then $\textcircled{3} \Rightarrow xy = \lambda$.

$$\begin{aligned} \text{Now } \textcircled{1} \cdot \textcircled{2} &\Rightarrow z^4 = 4\lambda^2 \Rightarrow z^4 = 4x^2y^2 & y=x \\ &\Rightarrow z^2 = \pm 2xy & \textcircled{5} \end{aligned}$$

$$\begin{aligned} \text{and } \textcircled{1} \div \textcircled{2} &\Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow y^2 = x^2 \\ &\Rightarrow y = \pm x & \textcircled{6} \end{aligned}$$

$$\text{Now } \textcircled{4}, \textcircled{5}, \textcircled{6} \Rightarrow (x,y,z) \in \{(1,1,\pm\sqrt{2}), (-1,-1,\pm\sqrt{2}), (1,-1,\pm\sqrt{2}), (-1,1,\pm\sqrt{2})\}.$$

Points with highest temperature: $(1,1,\pm\sqrt{2}), (-1,-1,\pm\sqrt{2})$, $T=2$;

Points with lowest temperature: $(1,-1,\pm\sqrt{2}), (-1,1,\pm\sqrt{2})$, $T=-2$.

$$7. (i) \iint_R (2x-1-y) dA$$

$$= \int_0^1 \int_{4x-2}^{2\sqrt{x}} (2x-1-y) dy dx$$

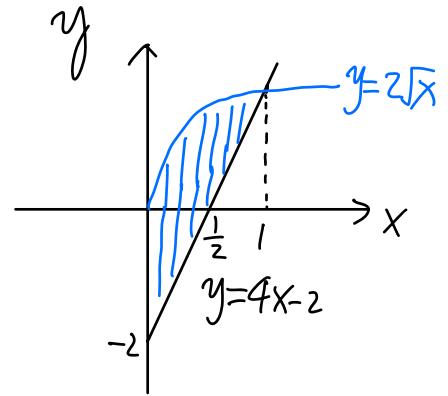
$$= \int_0^1 \left(2xy - y - \frac{1}{2}y^2 \right) \Big|_{y=4x-2}^{2\sqrt{x}} dx$$

$$= \int_0^1 \left(4x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - 2x - 2x(4x-2) + 4x-2 + \frac{1}{2}(4x-2)^2 \right) dx$$

$$= \int_0^1 \left(4x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - 2x - 8x^2 + 4x + 4x - 2 + 8x^2 + 2 - 8x \right) dx$$

$$= \int_0^1 \left(4x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - 2x \right) dx = \left. \frac{8}{5}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} - x^2 \right|_{x=0}^1$$

$$= -\frac{11}{15}.$$



$$(ii) \int_0^\infty \frac{e^{-x} - e^{-2x}}{x} dx = \int_0^\infty \int_1^2 e^{-xy} dy dx$$

$$= \lim_{b \rightarrow \infty} \int_0^b \int_1^2 e^{-xy} dy dx = \lim_{b \rightarrow \infty} \int_1^2 \int_0^b e^{-xy} dx dy$$

$$= \lim_{b \rightarrow \infty} \int_1^2 \frac{1 - e^{-by}}{y} dy = \int_1^2 \frac{1}{y} dy = \ln 2.$$

$$8. (i) \quad x^2 + y^2 + (z-1)^2 = 1$$

$$\Leftrightarrow \rho^2 - 2\rho \cos\phi = 0 \quad \Leftrightarrow \rho = 2 \cos\phi$$

$$z = \sqrt{x^2 + y^2} \quad \Leftrightarrow \rho \cos\phi = \rho \sin\phi \quad \Leftrightarrow \phi = \frac{\pi}{4}.$$

$$V(E) = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$(ii) \quad V(E) = 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{3} \sin\phi \rho^3 \right) \Big|_{\rho=0}^{2\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 8 \sin\phi \cos^3\phi \, d\phi$$

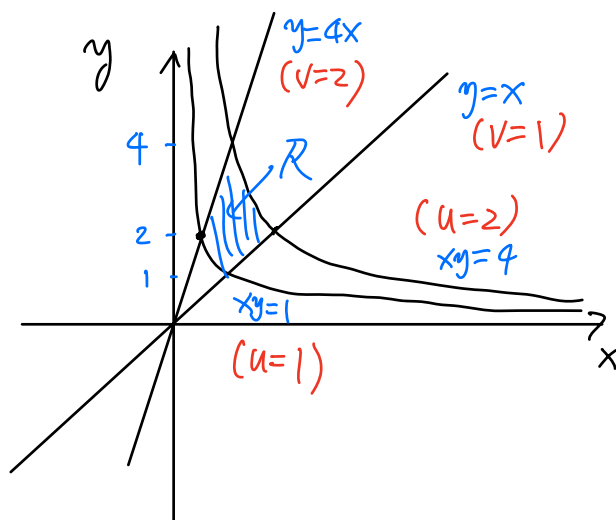
$$= \frac{16\pi}{3} \int_{\frac{\sqrt{2}}{2}}^0 u^3 (-du)$$

$$= \frac{16\pi}{3} \left(\frac{1}{4} u^4 \Big|_0^{\frac{\sqrt{2}}{2}} \right) = \frac{\pi}{3}.$$

9. The integral sum is

$$\iint_R (x^2 + y^2) dA,$$

where R is as shown on the right.



With $x = \frac{u}{v}$ & $y = uv$,

We have
$$\begin{cases} u^2 = xy \\ v^2 = y/x \end{cases}.$$

One possible transformation is to consider u & v to be both positive; then the corresponding uv -region is

$$D: \{(u, v), 1 \leq u \leq 2, 1 \leq v \leq 2\}.$$

Now

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{u}{v} + \frac{u}{v} = \frac{2u}{v} > 0,$$

$$\text{So } \iint_R (x^2 + y^2) dA = \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2 v^2 \right) \frac{2u}{v} du dv.$$

10. (i)

$$M_y = N_x \Rightarrow -e^x \sin y = -ae^{ax} \sin y$$

$$\Rightarrow a=1$$

$$M_z = P_x \Rightarrow \frac{1}{z} = \frac{b}{z} \Rightarrow b=1$$

M N P

x y z

$$(ii) \text{ Let } \vec{F} = \vec{F} = \langle e^x \cos y + \ln z, -e^x \sin y, \frac{x}{z} \rangle.$$

$$f_x = e^x \cos y + \ln z \Rightarrow f = e^x \cos y + x \ln z + K(y, z) \quad (1)$$

$$\Rightarrow f_y = -e^x \sin y + K_y(y, z)$$

$$\text{But } f_y = N = -e^x \sin y \Rightarrow K_y(y, z) = 0 \Rightarrow K(y, z) = L(z).$$

$$\text{Now } (1) \Rightarrow f_z = \frac{x}{z} + L'(z).$$

$$\text{But } f_z = P = \frac{x}{z} \Rightarrow L'(z) = 0 \Rightarrow L(z) = C.$$

$$\text{Now } (1) \Rightarrow f = e^x \cos y + x \ln z + C, \text{ where } C \text{ is any constant.}$$

(iii) Since \vec{F} is conservative,

$$\begin{aligned} \text{Work} &= \int_{(2, \pi, 1)}^{(0, 3, 2)} \vec{F} \cdot d\vec{r} = f(0, 3, 2) - f(2, \pi, 1) \\ &= \cos 3 - (e^2(-1)) = \cos 3 + e^2. \end{aligned}$$

$$11. \text{ Circulation} = \oint_C \vec{F} \cdot d\vec{r} = \iint_R (N_x - M_y) dA = \iint_R 0 dA = 0.$$

$$\text{Flux} = \oint_C \vec{F} \cdot \vec{n} ds = \iint_R (M_x + N_y) dA = \iint_R (2x + 2y) dA$$

$$= 2 \int_0^\pi \int_0^a r(\cos\theta + \sin\theta) r dr d\theta$$

$$= 2 \left(\int_0^\pi (\cos\theta + \sin\theta) d\theta \right) \left(\int_0^a r^2 dr \right)$$

$$= 2 (\sin\theta - \cos\theta) \Big|_0^\pi \frac{1}{3} a^3$$

$$= \frac{2}{3} a^3 (1+1) = \frac{4}{3} a^3.$$

$$12. \text{ Flux} = \iint_{\substack{x^2+y^2+z^2=1 \\ z \geq 0}} \text{curl}(\vec{A} \cdot \vec{n}) \, d\sigma$$

$$\text{Stokes' } \iint_C \vec{A} \cdot d\vec{r} = \iint_C Mdx + Ndy + Pdz \quad \left(C: \begin{array}{l} x = \cos t \\ y = \sin t, \\ z = 0 \\ 0 \leq t \leq 2\pi \end{array} \right)$$

$$= \int_0^{2\pi} \sin t (-\sin t \, dt) + e^0 \cos t \, dt$$

$$= \int_0^{2\pi} (\cos t - \sin^2 t) \, dt$$

$$= \int_0^{2\pi} \left(\cos t - \frac{1 - \cos 2t}{2} \right) \, dt$$

$$= \left(\sin t - \frac{1}{2}t + \frac{1}{4}\sin 2t \right) \Big|_{t=0}^{2\pi}$$

$$= -\pi$$

$$13. \text{ Flux} = \iint_S \vec{F} \cdot \vec{n} \, d\sigma$$

$$\stackrel{\text{div.}}{=} \iint_E \underbrace{\text{div } \vec{F}}_{6x^2+3y^2+3y^2} \, dV$$

$$E: \quad 0 \leq z \leq 1-x^2-y^2, \\ x^2+y^2 \leq 1.$$

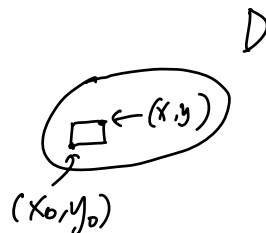
$$= \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6r^2 \cdot r \, dz \, dr \, d\theta$$

$$= 2\pi \cdot \int_0^1 (1-r^2) 6r^3 \, dr$$

$$= 2\pi \left(\frac{6}{4} r^4 - r^6 \right) \Big|_{r=0}^1 = 2\pi \frac{1}{2} = \pi.$$

14. Proof: Fix any $(x, y) \in D$, and let $(x_0, y_0) \in D$ satisfy $x_0 < x$ and $y_0 < y$. Define F near (x, y) by

$$F(x', y') := \int_{\substack{x_0 \leq u \leq x' \\ y_0 \leq v \leq y'}} f(u, v) dA$$



Fubini's
thm

$$\begin{cases} = \int_{x_0}^{x'} \int_{y_0}^{y'} f(u, v) dv du \\ = \int_{y_0}^{y'} \int_{x_0}^{x'} f(u, v) du dv \end{cases}$$

By assumption, $F \equiv 0$ near (x, y) , so all partial derivatives of $F = 0$ at (x, y) . (*)

On the other hand,

$$\frac{\partial}{\partial x} F(x, y) = \frac{\partial}{\partial x} \int_{x_0}^x \int_{y_0}^y f(u, v) dv du \stackrel{\text{FTC}}{=} \int_{y_0}^y f(x, v) dv$$

$$\Rightarrow \frac{\partial}{\partial y} \frac{\partial}{\partial x} F(x, y) \stackrel{\text{FTC}}{=} f(x, y). \quad (**)$$

By (*) & (**), $f(x, y) = 0$.

