MAT1002 Lecture 6, Tuesday, Feb/14/2023

Outline

- Taylor polynomials and Taylor's theorem
- Taylor's series with big- and little-oh
- Applications in finding limits (of indeterminate forms)
- Proof of Taylor's theorem

Taylor Polynomials

Definition

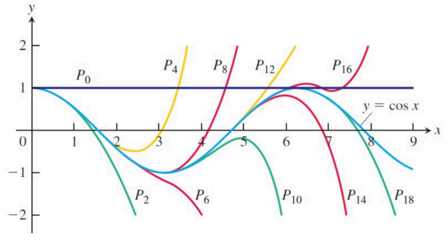
Let f be a function such that for all $n \in \{0, 1, \ldots, N\}$, $f^{(n)}$ exists on some open interval containing a. The Taylor polynomial of f (of order n) centered at a is the polynomial

$$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Remarks:

- For $f(x) = \cos x$ and $a = 0$, since $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$, we have $P_5(x) = P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$: $P_n(x)$ may have degree $< n$, in general.

  (order $\neq$ degree)

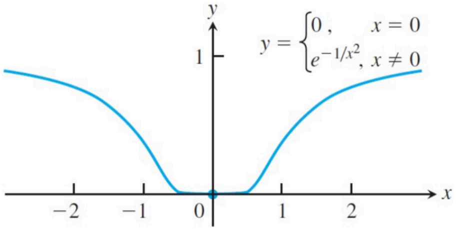


- The Taylor series of f always converges to $f(a)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(a-a)^n = \frac{f^{(0)}(a)}{0!} = f(a).$$

  However, in general, $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ may fail to converge to $f(x)$ for all other values of x.

  e.g. $f(x) := \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.



It can be shown (for the f above) that:

- f is infinitely differentiable
- $f^{(n)}(0)=0,\ \forall n\geq 0$.
- So its Maclaurin Series is the zero function.
- But $f(x)\neq 0$ unless $x=0$, so the Maclaurin series of f does not equal to f on ANY open interval centered at 0.

Details omitted; see the optional note on Blackboard for a proof outline.

Q: When can f be represented as its Taylor series on an neighbourhood of the center a?

open interval

A: Taylor's theorem.

Taylor's Theorem

Suppose:

- $f, f', \ldots, f^{(n)}$ are continuous on $[a,x]$ (or $[x,a]$);
- $f^{(n+1)}$ exists on (a,x) (or (x,a)).

Then there exists $c\in(a,x)$ (or $c\in(x,a)$) such that

$$f(x)=\underbrace{\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k}_{P_n(x)}+\underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(x)\text{ : Remainder.}}.$$

Remarks

- By taking $n=0$, we get $f(x) = f(a) + f'(c)(x-a)$, the MVT, as a special case.
- For a fixed x $(\neq a)$, if $R_n(x) \to 0$ as $n \to \infty$, then

$$f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} P_n(x) + \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Hence, $R_n(x) \to 0$ gives a sufficient condition for the Taylor series of f to converge to f (at this x).

In particular, for a given x, if there is a positive constant M such that $\underbrace{|f^{(n+1)}(t)|}_{\forall \text{ big enough } n} \leq M$ for all t between a and x, then

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!},$$

Recall: $\lim_{n\to\infty} \frac{A^n}{n!} = 0$ for constant A.

so $R_n(x) \to 0$ as $n \to \infty$. In this case,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example

(a) Show that the Maclaurin series of $f(x) := e^x$ converges to $f(x)$ for all $x \in \mathbb{R}$. (e.g. 10.9.1)

(b) Show that the Maclaurin series of $f(x) := \sin x$ converges to $f(x)$ for all $x \in \mathbb{R}$. (e.g. 10.9.2)

The proof of Taylor's theorem will be given at the end.

Big-O and Little-O (as $x \to a$)

Def: For functions f and g, we write

- $f(x) = o(g(x))$ as $x \to a$, if $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.

  Little-o

- $f(x) = O(g(x))$ as $x \to a$, if $\exists M \in \mathbb{R}_{\geq 0}$ and $\exists \delta > 0$ s.t.

  Big-O

  $\forall x \in (a-\delta, a+\delta) \setminus \{a\}$, $|f(x)| \leq M|g(x)|$.

$\Leftrightarrow \left|\frac{f(x)}{g(x)}\right| \leq M$ if g is never zero

Suppose that $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ on some neighbourhood I of a.

For any fixed $n \in \mathbb{Z}_{\geq 0}$ and $x \in I$, $f(x) = P_n(x) + R_n(x)$,

where $R_n(x) = \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$. Fixing n, consider $\frac{R_n(x)}{(x-a)^{n+1}}$:

$$\lim_{x\to a} \frac{R_n(x)}{(x-a)^{n+1}} = \lim_{x\to a} \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^{k-n-1}$$

$$= \sum_{k=n+1}^{\infty} \lim_{x\to a} \frac{f^{(k)}(a)}{k!}(x-a)^{k-n-1}$$

(an un-proven property of power series)

$$= \lim_{x\to a} \frac{f^{(n+1)}(a)}{(n+1)!} = \frac{f^{(n+1)}(a)}{(n+1)!} \quad (=: L)$$

This means that $\left|\frac{R_n(x)}{(x-a)^{n+1}}\right|$ is bounded on some $(a-\delta, a+\delta) \setminus \{a\}$, so

$R_n(x) = O((x-a)^{n+1})$ as $x \to a$. On the other hand, if we consider $\frac{R_n(x)}{(x-a)^n}$ as $x \to a$, then

$$\lim_{x\to a} \frac{R_n(x)}{(x-a)^n} = \lim_{x\to a} \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^{k-n}$$

$$= \sum_{k=n+1}^{\infty} \lim_{x\to a} \frac{f^{(k)}(a)}{k!}(x-a)^{\underbrace{k-n}_{\geq 1}}$$

(an un-proven property of power series)

$$= 0,$$

So $R_n(x) = o((x-a)^n)$ as $x \to a$.

Hence, a Taylor series can be written with big- and little-oh notation.

Theorem If f can be represented by its Taylor series with center $x=a$ on some neighbourhood I of a, then for any fixed $n \in \mathbb{Z}_{\geq 0}$, as $x \to a$,

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{P_n(x)} + O((x-a)^{n+1}) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + o((x-a)^n).$$

e.g. • $\frac{1}{1-x} = 1+x+x^2+\ldots = 1+O(x) = 1+x+O(x^2) = 1+x+x^2+O(x^3)$

and $\frac{1}{1-x} = 1+x+x^2+\ldots = 1+o(1) = 1+x+o(x) = 1+x+x^2+o(x^2)$

as $x \to 0$.

$$\lim_{x\to 0} \frac{x^3+x^4+x^5+\ldots}{x^2}$$

$$= \lim_{x\to 0} x+x^2+x^3+\ldots$$

$$= 0$$

Similarly,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = x - \frac{x^3}{3!} + O(x^5) = x + O(x^3);$$

$$\sin x = x - \frac{x^3}{3!} + o(x^4) = x + o(x^2), \quad \text{as } x \to 0.$$

Remarks

(1) If $m > 0$ and $f(x) = O((x-a)^m)$ as $x \to a$, then $\lim_{x \to a} f(x) = 0$.

Proof: $f(x) = O((x-a)^m)$ as $x \to a$

$$\Rightarrow |f(x)| \le M|(x-a)^m|, \quad \forall x \in (a-\delta, a+\delta) \setminus \{a\}$$

$$\Rightarrow -M\underbrace{|x-a|^m}_{\to 0} \le f(x) \le M\underbrace{|x-a|^m}_{\to 0}, \quad \forall x \in (a-\delta, a+\delta) \setminus \{a\}$$

as $x \to a$

$$\Rightarrow \lim_{x \to a} f(x) = 0 \text{ by the squeeze theorem}$$

(2) If $f(x) = O((x-a)^m)$ and $g(x) = O((x-a)^n)$ as $x \to a$,

then $f(x)g(x) = O((x-a)^{m+n})$ as $x \to a$. In short,

$$O((x-a)^m)\,O((x-a)^n) = O((x-a)^{m+n}) \text{ as } x \to a.$$

In particular, $O(x^m)O(x^n) = O(x^{m+n})$ as $x \to 0$.

Taylor series together with big-Oh notation may be useful for finding limits involving indeterminate forms.

e.g. $\displaystyle \lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\sin x} = \lim_{x\to 0}\frac{x-(x+O(x^3))}{x(x+O(x^3))}$ as $x\to 0$

$$= \lim_{x\to 0}\frac{O(x^3)}{x^2+O(x^4)} = \lim_{x\to 0}\frac{x^{-2}O(x^3)}{1+x^{-2}O(x^4)} \overset{\text{Remark (2)}}{=} \lim_{x\to 0}\frac{O(x)}{1+O(x^2)}$$

$$\overset{\text{Remark (1)}}{=} \frac{0}{1+0} = 0.$$

e.g. $\displaystyle \lim_{x\to 0}\frac{\cos x - 1 + \frac{1}{2}x\sin x}{[\ln(1+x)]^4}$

$$= \lim_{x\to 0}\frac{\left(1-\frac{1}{2}x^2+\frac{1}{4!}x^4+O(x^6)\right)-1+\frac{1}{2}x\left(x-\frac{x^3}{3!}+O(x^5)\right)}{(x+O(x^2))^4}$$

$$= \lim_{x\to 0}\frac{\frac{1}{4!}x^4-\frac{1}{3!\cdot 2}x^4+\underbrace{\frac{1}{2}xO(x^5)}_{O(x^6)}+O(x^6)}{(x+O(x^2))^4}$$

$$= \lim_{x\to 0}\frac{-\frac{1}{24}x^4+O(x^6)+O(x^6)}{(x+O(x^2))^4} \overset{\text{divide top \& bottom by } x^4}{=} \lim_{x\to 0}\frac{-\frac{1}{24}+O(x^2)+O(x^2)}{(1+O(x))^4}$$

$$= \frac{-\frac{1}{24}+0+0}{(1+0)^4} = -\frac{1}{24}.$$

Proof of Taylor's Theorem

Before proving Taylor's theorem, let us observe a property of the Taylor polynomial $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ of f centered at a:

$P_n(x)$ has the same derivatives as $f(x)$ at $x=a$ up to the n^{th} order:

$$P_n'(a) = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} k(a-a)^{k-1} = f'(a)$$

$$P_n''(a) = \sum_{k=2}^{n} \frac{f^{(k)}(a)}{k!} k(k-1)(a-a)^{k-2} = f''(a)$$

$$\vdots$$

$$P_n^{(i)}(a) = \sum_{k=i}^{n} \frac{f^{(k)}(a)}{k!} k(k-1)\cdots(k-i+1)(a-a)^{k-i} = f^{(i)}(a).$$

Taylor's Theorem

Suppose:

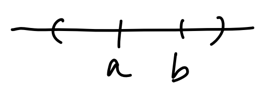
- $f, f', \ldots, f^{(n)}$ are continuous on $[a,b]$ (or $[b,a]$);
- $f^{(n+1)}$ exists on (a,b) (or (b,a)).

Then there exists $c \in (a,b)$ (or $c \in (b,a)$) such that

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Proof: • We show for $b > a$. It is similar for $b < a$.

• Define $P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ on $[a,b]$.



- Define $F(x) := f(x) - \left(P_n(x) + K(x-a)^{n+1}\right)$, where K is chosen so that $F(b) = 0$, i.e., $K = \dfrac{f(b) - P_n(b)}{(b-a)^{n+1}}$.

- Note that for $k \in \{0, 1, \ldots, n\}$,

$$F^{(k)}(a) = f^{(k)}(a) - P_n^{(k)}(a) = 0.$$

- Since $F(a) = F(b)$, F cts on $[a,b]$, diff'able on (a,b), by Rolle's theorem, $\exists\, c_1 \in (a,b)$ s.t. $F'(c_1) = 0$.
- Since $F'(a) = F'(c_1) = 0$, F' cts on $[a, c_1]$ and diff'able on (a, c_1), $\exists\, c_2 \in (a, c_1)$ s.t. $F''(c_2) = 0$.
- Continued with the same logic, inductively,

$$\exists\, c_{n+1} \in (a, c_n) \subseteq (a,b) \text{ s.t. } F^{(n+1)}(c_{n+1}) = 0.$$

- Note that $F^{(n+1)}(x) = f^{(n+1)}(x) - \underbrace{P_n^{(n+1)}(x)}_{=0} - K(n+1)!$

$$= f^{(n+1)}(x) - K(n+1)!.$$

Substituting $x = c := c_{n+1}$ yields

$$0 = f^{(n+1)}(c) - K(n+1)!,$$

i.e.,

$$K = \frac{f^{(n+1)}(c)}{(n+1)!}.$$

- Since $F(b) = 0$, we have

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}, \text{ as desired.}$$

$\square$