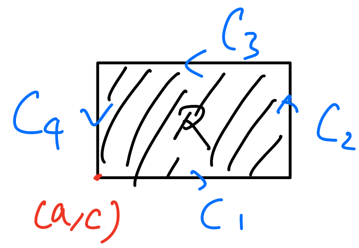
MAT1002 Lecture 24, Tuesday, Apr/18/2023

Outline

- Green's theorem (16.4)
  - ↳ Circulation and flux
  - ↳ Intuition
  - ↳ More general regions
- Surfaces in $\mathbb{R}^3$ (16.5)

Green's Theorem



- Given the velocity field $\vec{F}$ of some fluid, consider the counterclockwise circulation along the boundary C of rectangle $R := [a,b]\times[c,d]$.

- $\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + \oint_C N\,dy$

- $\oint_C M\,dx = \int_{C_1} M\,dx + \underbrace{\int_{C_2} M\,dx}_{=0} + \int_{C_3} M\,dx + \underbrace{\int_{C_4} M\,dx}_{=0}$ since $dx = 0$

$= \int_{C_1} M\,dx - \int_{-C_3} M\,dx$

$(C_1: \vec{r}_1(t) = \langle t, c\rangle,\ a \le t \le b;\ -C_3: \vec{r}_3(t) = \langle t, d\rangle,\ a \le t \le b.)$

$= \int_a^b M(t,c)\,dt - \int_a^b M(t,d)\,dt$

$= \int_a^b [M(t,c) - M(t,d)]\,dt \overset{FTC}{=} \int_a^b \int_d^c \frac{\partial M}{\partial y}(t,y)\,dy\,dt$

$= \int_a^b \int_d^c \frac{\partial M}{\partial y}(x,y)\,dy\,dx$ (Rename t to x)

$= -\int_a^b \int_c^d \frac{\partial M}{\partial y}(x,y)\,dy\,dx$

$= -\iint_R \frac{\partial M}{\partial y}(x,y)\,dA$ ①

- Similarly,

$\oint_C N\,dy = \int_{C_2} N\,dy - \int_{-C_4} N\,dy$

$C_2: \vec{r}_2(t) = \langle b, t\rangle,\ c \le t \le d$
$-C_4: \vec{r}_4(t) = \langle a, t\rangle,\ c \le t \le d$

$= \int_c^d N(b,t)\,dt - \int_c^d N(a,t)\,dt$

$$\overset{FTC}{=} \int_c^d \int_a^b \frac{\partial}{\partial x} N(x,t)\, dx\, dt = \int_c^d \int_a^b \frac{\partial}{\partial x} N(x,y)\, dx\, dy$$

$$= \iint_R \frac{\partial N}{\partial x}(x,y)\, dA. \qquad ②$$

- Combining ① & ②, we have

$$\oint_C \vec{F}\cdot\vec{T}\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA. \qquad ③$$

- You can use the same technique to show that ③ also holds if R is a region that is both of type-I and type-II. In fact, the result holds for any bounded region R that has a simple closed boundary, as stated formally below.

Theorem (Green's Theorem) (Circulation Version)

Let C be a ~~*positively oriented,*~~ counterclockwise *piecewise-smooth, simple closed curve in the xy-plane, and let R be the region bounded by C. Assume that M* ~~*ad*~~ and *N have continuous partial derivatives on an open region containing R. Then*

required for FTC

"circulation density"

$$\oint_C \vec{F}\cdot\vec{T}\, ds = \oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

A result about Counterclockwise Circulation.

Key: Green's theorem (above) relates a double integral $\iint_R$ with a circulation integral along its counterclockwise BOUNDARY C.

e.g. Find the clockwise circulation of $\vec{F} := \langle x^4, xy \rangle$ along the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$.

Ans : $-\frac{1}{6}$.

Example

Evaluate

$$\oint_C (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4 + 1})\, dy,$$

Ans : 36π

where C is the circle $x^2 + y^2 = 9$, traversed ~~clockwise~~. counterclockwise

Exercise

Evaluate

$$\oint_C y^2\, dx + 3xy\, dy,$$

where C is the positively oriented boundary of the region R given by $1 \le x^2 + y^2 \le 4$, $y \ge 0$.

= Counterclockwise

Ans : $14/3$.

Question

- Consider $\mathbf{F}(x, y) := \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$
- Let C be the unit circle $x^2 + y^2 = 1$, counter clockwise.
- We showed that $\oint_C \vec{F} \cdot \vec{T}\, ds = 2\pi$ by direct computation.
- On the other hand, one can check that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ always holds.
- By Green's theorem, $\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0$.
- Therefore, $2\pi = 0$.

Recall the component test.

Theorem (Component Test for Conservative Fields) (in $\mathbb{R}^2$)

Let $\mathbf{F}(x,y) := \langle M(x,y), N(x,y)\rangle$ *be a vector field on an open simply connected domain* D*, such that* M *and* N *have continuous partial derivatives. Then* $\mathbf{F}$ *is* conservative *on* D *if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

on D.

We can now prove it.

Proof: "$\Rightarrow$": Assume that $\vec{F}$ is conservative on D. Then $\exists f$ s.t. $\vec{F} = \nabla f = \langle f_x, f_y \rangle$ on D. Now

$$\frac{\partial M}{\partial y} = (f_x)_y = f_{xy} = f_{yx} = (f_y)_x = \frac{\partial N}{\partial x}.$$

both are cts by assumption; mixed derivative theorem.

"$\Leftarrow$": Assume $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ on D.

- Let C be any closed curve in D. First assume C is simple.
- Since D is simply connected, the region R bounded by C lies entirely in D.
- By Green's theorem, $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\equiv 0 \text{ on } R \subseteq D} dA = 0$
- If C is a closed curve which is not simple, decompose it into simple closed curves: $C = C_1 \cup C_2 \cup \dots \cup C_k$. Then $\oint_C \vec{F}\cdot\vec{T}\,ds = \sum_{i=1}^{k} \int_{C_i} \vec{F}\cdot\vec{T}\,ds$.

- Hence, $\oint_C \vec{F}\cdot\vec{T}\,ds = 0$, $\forall$ closed curve C in D.
- By the loop property, $\vec{F}$ is conservative on D. $\square$

We also have Green's theorem for flux (in place of circulations).

- Consider the flux across the boundary of a rectangle R.
- From last time, flux $= \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx$.
- Using the reasoning on Page 2, we can show that

$$\oint_C M\,dy = \iint_R \frac{\partial M}{\partial x}\,dA \qquad \& \qquad \oint_C N\,dx = -\iint_R \frac{\partial N}{\partial y}\,dA$$

- Hence, flux $= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$.

The general result is given by the following theorem.

Theorem (Green's Theorem, Flux Version)

Under the assumptions for Green's theorem (circulation version), except orientation (direction) of C is no longer required,

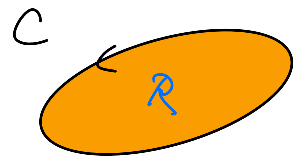
$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{\text{``flux density''}} dA.$$

Green's Theorem : Intuition

In short : $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

(Assume: M, N have cts partials)



Here : $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$: "circulation density"

$$\approx \frac{\text{circulation along a small simple closed curve around a point}}{\text{area of bounded region}}$$

$\operatorname{div}\vec{F}$

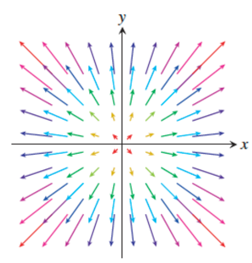
$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$: "(outward) flux density" or "divergence"

$\operatorname{div}\vec{F}(x,y) > 0$: expanding

< 0 : shrinking / compressing.

$$\approx \frac{\text{outward flux across a small simple closed curve around a point}}{\text{area of bounded region}}$$

e.g. (a) $\vec{F} = \langle cx, cy \rangle$ (picture shows for $c>0$)

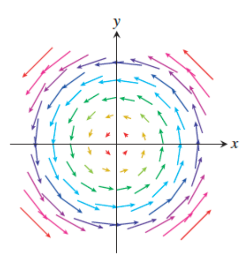


(a)

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$: no rotation at very small scale.

$\operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 2c \begin{cases} >0, & \text{if } c>0 \to \text{diverges/expands} \\ <0, & \text{if } c<0 \to \text{compresses} \end{cases}$

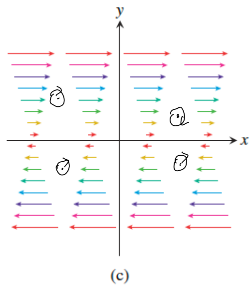
(b) $\vec{F} = \langle -cy, cx \rangle$ (picture shows for $c>0$)



(b)

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = c + c = 2c \begin{cases} >0, & \text{if } c>0 \to \circlearrowleft \\ <0, & \text{if } c<0 \to \circlearrowright \end{cases}$

$\operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$: no expansion

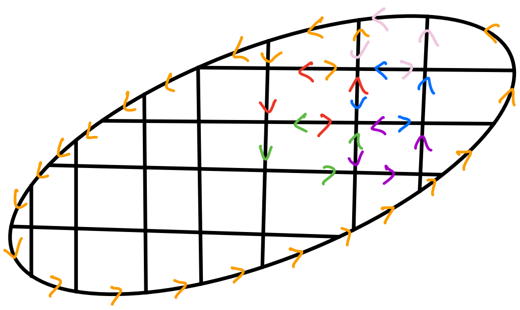


(c) $\vec{F} = \langle cy, 0 \rangle$ (picture shows for $c>0$)

- $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -c$ $\begin{cases} \text{clockwise rotation at small scale, if } c>0 \\ \text{counterclockwise " " " ", if } c<0 \end{cases}$

- $\mathrm{div}(\vec{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$: no expansion.

Green's Theorem in a Photo (Circulation)

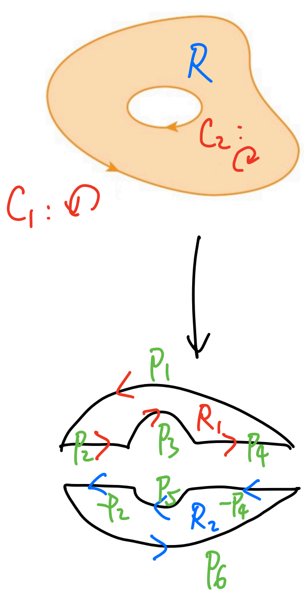


"Microscopic Circulation"

- Sum up all circulations around small regions $\left(\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \right)$
- Adjacent "side" cancelled
- Leaving only the circulation along boundary of the whole region R.

(Idea for flux version is the same.)

More General Versions



Consider the region R on the left.

- Boundary of R is not a simple closed curve — it is the union of two such curves.
- Previous version of Green's theorem cannot be used directly.
- $\iint_R (N_x - M_y)\, dA = \iint_{R_1} + \iint_{R_2}$

$$\overset{\text{Green's}}{=\!=} \left(\int_{P_1} \vec{F}\cdot d\vec{r} + \int_{P_2} + \int_{P_3} + \int_{P_4}\right) + \left(\int_{-P_4} + \int_{P_5} + \int_{-P_2} + \int_{P_6}\right)$$

$$= \left(\int_{P_1} + \int_{P_6}\right) + \left(\int_{P_3} + \int_{P_5}\right) = \oint_{C_1} + \oint_{C_2} = \int_{\substack{\text{Boundary} \\ \text{of } R}} \vec{F} \cdot d\vec{r}$$

Green's Theorem (General Version for circulations, R may not be simply connected)

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{\substack{\text{positively oriented} \\ \text{boundary } C \text{ of } R}} \vec{F} \cdot d\vec{r}$$

C is the union of one or more closed curves

(Here, C is positively oriented if R is always on the left of C as C being traversed.)

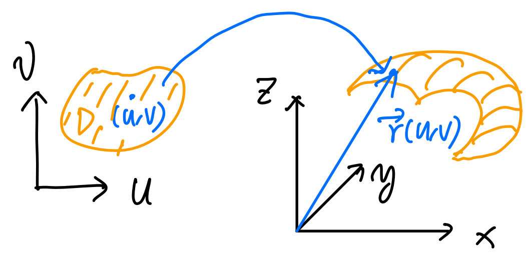
e.g. Consider $\mathbf{F}(x, y) := \left\langle \dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2} \right\rangle$. Show that $\oint_C \vec{F} \cdot d\vec{r} = 2\pi$ for every simple closed curve C enclosing the origin.

Surfaces in $\mathbb{R}^3$

Def: A surface (in $\mathbb{R}^3$) is the range of a continuous function $\vec{r}: D \to \mathbb{R}^3$ where D is a 2-dimensional connected subset of $\mathbb{R}^2$:
"has area > 0"

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \quad (u,v) \in D.$$

- $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$ are parametric equations of the surface.
- D is called the parameter domain.
- $\vec{r}$ is often one-to-one in the interior of D.



Three Basic Representations

1. Parametric form : $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, $(u,v) \in D$
2. Explicit form : $z = f(x,y)$
3. Implicit form : $F(x,y,z) = K$.

Example

(a) Find two different parametric representations for the cone given by

$$z = \sqrt{x^2 + y^2}.$$

(b) Find a parametric representation for the sphere given by

$$x^2 + y^2 + z^2 = a^2,$$

where $a > 0$.

(c) Do the same for the cylinder $x^2 + (y-3)^2 = 9$, $0 \le z \le 5$.