

MAT1001 Midterm Examination

Saturday, October 30, 2021

Time: 9:30 - 11:30 AM

Notes and Instructions

1. *No books, no notes, no dictionaries, and no calculators.*
2. *The total score of this examination is 140.*
3. *There are **11** questions (with parts) in total.*
4. *The symbol $[N]$ at the beginning of a question indicates that the question is worth N points.*
5. *Answer all questions on the **answer book**.*
6. *Show your intermediate steps **except Questions 1, 2 and 3** — answers without intermediate steps will receive minimal (or even no) marks.*

MAT1001 Midterm Questions

1. [15] Multiple Choice. No explanation is required.

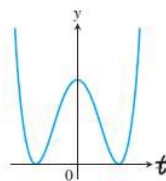
(i) $\lim_{x \rightarrow 1} 2^{\frac{3}{x-1}} = \underline{\hspace{2cm}}.$

- A) 0
- B) 1
- C) ∞
- D) None of the above

(i) $\lim_{x \rightarrow \infty} \frac{4x^2 - 3x + 2}{x^4 - 2x^2 + x - 5} = \underline{\hspace{2cm}}.$

- A) 0
- B) 4
- C) ∞
- D) $-\infty$

(iii) Given the graph of the velocity of a particle moving along a horizontal line, which of the following could be the position function $s = f(t)$ for the particle?



- A) $s = \sin |t|$
- B) $s = \frac{t^4}{4} - 2t^2 + 4$
- C) $s = \frac{3}{4}(t^2 - 1)^{\frac{2}{3}}$
- D) $s = \frac{1}{5}t^5 - \frac{8}{3}t^3 + 16t$

(iv) For $x \neq 0$, we have $\frac{d}{dx} \left(\sqrt{|x|} \right) = \underline{\hspace{2cm}}.$

- A) $\frac{1}{2\sqrt{|x|}}$
- B) $\frac{x}{2\sqrt{|x|}}$
- C) $\frac{\sqrt{|x|}}{2x}$
- D) $\frac{\sqrt{|x|}}{-2x}$

(v) What is the normal line of $y = \sqrt{1-x}$ through the point $(-3, 2)$?

- A) $x + 4y - 5 = 0$
- B) $4x - y - 14 = 0$
- C) $x + 4y + 5 = 0$
- D) $4x - y + 14 = 0$

2. [10] True or False (in general)? No explanation is required.

- (i) Let f, g be functions defined for all real numbers and both not continuous at $x = 0$. Then $f + g$ must be non-differentiable at $x = 0$.
- (ii) Let $y = f(x)$ be defined for all real numbers such that the left-hand derivative and the right-hand derivative at $x = a$ both exist. Then $y = f(x)$ is differentiable at $x = a$.
- (iii) Let f and g be functions having the same domain D . If $f'(x) = g'(x)$ for all $x \in D$, then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in D$.
- (iv) If $\lim_{x \rightarrow a} (f(x) - g(x))$ exists and $\lim_{x \rightarrow a} f(x)$ exists, then $\lim_{x \rightarrow a} g(x)$ exists.
- (v) Let f be differentiable on the interval $(0, 6)$. Then $\lim_{x \rightarrow 3} f'(x)$ must exist.

3. [21] Short questions: no explanation is required.

(i) Find the values of a and b that make f continuous on \mathbb{R} , where

$$f(x) = \begin{cases} x + 2, & \text{if } x < 2 \\ ax^2 - bx + 3, & \text{if } 2 \leq x < 3 \\ 2x - a + b, & \text{if } x \geq 3 \end{cases}.$$

- (ii) If $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 4$, find $\lim_{x \rightarrow 2} f(x)$.
- (iii) Let $f(x) = x^3 + 2x - 4$. Starting with $x_0 = 2$, find x_1 using Newton's method.
- (iv) Estimate the area under the curve $y = x^2 + (1/x)$ for x from 1 to 5 using the midpoint sum S with two subintervals.

- (v) In (iv) above, is S greater than, smaller than, or equal to the exact area under the curve?
- (vi) Find the function $y = f(x)$ that satisfies $y' = \sqrt{x} + \frac{2}{x^4} - \sin(\pi x)$ and $y(1) = \pi$.
- (vii) Let f and g be differentiable functions such that

$$f(3) = 3, \quad g(3) = -4, \quad f'(3) = 2\pi \quad \text{and} \quad g'(3) = 5.$$

Find the derivative of $\sqrt{(f(x))^2 + (g(x))^2} + 5^\pi$ at $x = 3$.

4. [24] Evaluate the following limits, or explain why they do not exist.

- (i) $\lim_{x \rightarrow -6} \frac{2x + 12}{|x + 6|}$
- (ii) $\lim_{x \rightarrow \infty} \sqrt{4x^2 + 3x} + 2x$
- (iii) $\lim_{x \rightarrow 0} \frac{|2x - 1| - |2x + 1|}{x}$
- (iv) $\lim_{x \rightarrow 0} \frac{\sin(1 - \cos x)}{x}$

5. [6] A curve is given by $y^3 - 4 \sin(xy) = 8$. Find the tangent line to the curve at $x = 0$.

6. [2+6+6=14] Consider the function rule $f(x) = 3x^2 + \frac{x^2 - 1}{(x - 1)(x - \sin x)}$.

- (i) Find its natural domain D (that is, the biggest domain in \mathbb{R}).
- (ii) Extend the function to have domain \mathbb{R} by giving the function some values at the points missing in D (in part (i)). At which of these points can the function be extended continuously?
- (iii) Find all asymptotes (horizontal/vertical/oblique) for $y = f(x)$.

7. [6+6+4+4=20] Consider the function $f(x) = x^{2/3}(6-x)^{1/3}$ defined on \mathbb{R} .

- (i) Find all intervals on which the function is increasing/decreasing.
- (ii) Find all intervals on which the function is concave up/concave down.
- (iii) Find all inflection points. (State their x -coordinates.)
- (iv) Find all local extrema and global extrema by stating their x -coordinates, or explain why they do not exist.

8. [6] A projectile is fired from a canon over horizontal ground and lands a distance s away from the canon, where s is given by the equation

$$s = \frac{v_0^2}{9.8} \sin 2\alpha,$$

where v_0 is the initial velocity of the projectile when it is fired, and α is the angle to the horizontal at which it is fired. At what angle should the canon be fired to maximize the distance travelled by the projectile?

9. [8] Let

$$f(x) = \begin{cases} x \cos \frac{\pi}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

Show that $f(x)$ is continuous but not differentiable at $x = 0$.

10. [8] Prove that $\sin x < x$ for all $x \in (0, 2\pi]$.

11. [8] A Ferris wheel with radius 10 m is rotating at a rate of one cycle every 2 minutes. During the rising process, how fast is the rider rising when her seat is 16 m above the ground? You may assume that the base of the wheel is just touching the ground level.