CHAPTER 6 APPLICATIONS OF DEFINITE INTEGRALS

6.1 VOLUMES USING CROSS-SECTIONS

1. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left(\sqrt{x}-\left(-\sqrt{x}\right)\right)^2}{2} = 2x;\ \ a = 0, b = 4;\ V = \int_a^b A(x)\,dx = \int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16$

2. $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi\left[\left(2-x^2\right)-x^2\right]^2}{4} = \frac{\pi\left[2\left(1-x^2\right)\right]^2}{4} = \pi\left(1-2x^2+x^4\right);\quad a = -1, b = 1;$

   $V = \int_a^b A(x)\,dx = \int_{-1}^1 \pi\left(1-2x^2+x^4\right)dx = \pi\left[x - \frac{2}{3}x^3 + \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{16\pi}{15}$

3. $A(x) = (\text{edge})^2 = \left[\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right]^2 = \left(2\sqrt{1-x^2}\right)^2 = 4\left(1-x^2\right);\quad a = -1, b = 1;$

   $V = \int_a^b A(x)\,dx = \int_{-1}^1 4\left(1-x^2\right)dx = 4\left[x - \frac{x^3}{3}\right]_{-1}^1 = 8\left(1-\frac{1}{3}\right) = \frac{16}{3}$

4. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left[\sqrt{1-x^2}-\left(-\sqrt{1-x^2}\right)\right]^2}{2} = 2\frac{\left(2\sqrt{1-x^2}\right)^2}{2} = 2\left(1-x^2\right);\quad a = -1, b = 1;$

   $V = \int_a^b A(x)\,dx = 2\int_{-1}^1 \left(1-x^2\right)dx = 2\left[x - \frac{x^3}{3}\right]_{-1}^1 = 4\left(1-\frac{1}{3}\right) = \frac{8}{3}$

5. (a) STEP 1) $A(x) = \frac{1}{2}(\text{side})\cdot(\text{side})\cdot\left(\sin\frac{\pi}{3}\right) = \frac{1}{2}\cdot\left(2\sqrt{\sin x}\right)\cdot\left(2\sqrt{\sin x}\right)\left(\sin\frac{\pi}{3}\right) = \sqrt{3}\sin x$

   STEP 2) $a = 0, b = \pi$

   STEP 3) $V = \int_a^b A(x)\,dx = \sqrt{3}\int_0^\pi \sin x\,dx = \left[-\sqrt{3}\cos x\right]_0^\pi = \sqrt{3}(1+1) = 2\sqrt{3}$

   (b) STEP 1) $A(x) = (\text{side})^2 = \left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right) = 4\sin x$

   STEP 2) $a = 0, b = \pi$

   STEP 3) $V = \int_a^b A(x)\,dx = \int_0^\pi 4\sin x\,dx = \left[-4\cos x\right]_0^\pi = 8$

6. (a) STEP 1) $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi}{4}(\sec x - \tan x)^2 = \frac{\pi}{4}\left(\sec^2 x + \tan^2 x - 2\sec x\tan x\right)$

   $= \frac{\pi}{4}\left[\sec^2 x + \left(\sec^2 x - 1\right) - 2\frac{\sin x}{\cos^2 x}\right]$

   STEP 2) $a = -\frac{\pi}{3}, b = \frac{\pi}{3}$

   STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3} \frac{\pi}{4}\left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = \frac{\pi}{4}\left[2\tan x - x + 2\left(-\frac{1}{\cos x}\right)\right]_{-\pi/3}^{\pi/3}$

   $= \frac{\pi}{4}\left[2\sqrt{3} - \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right) - \left(-2\sqrt{3} + \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right)\right)\right] = \frac{\pi}{4}\left(4\sqrt{3} - \frac{2\pi}{3}\right)$

(b) STEP 1) $A(x) = (\text{edge})^2 = (\sec x - \tan x)^2 = \left(2\sec^2 x - 1 - 2\frac{\sin x}{\cos^2 x}\right)$

STEP 2) $a = -\frac{\pi}{3}, b = \frac{\pi}{3}$

STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3}\left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = 2\left(2\sqrt{3} - \frac{\pi}{3}\right) = 4\sqrt{3} - \frac{2\pi}{3}$

7. (a) STEP 1) $A(x) = (\text{length})\cdot(\text{height}) = (6-3x)\cdot(10) = 60 - 30x$

STEP 2) $a = 0, b = 2$

STEP 3) $V = \int_a^b A(x)\,dx = \int_0^2 (60-30x)\,dx = \left[60x - 15x^2\right]_0^2 = (120-60) - 0 = 60$

(b) STEP 1) $A(x) = (\text{length})\cdot(\text{height}) = (6-3x)\cdot\left(\frac{20-2(6-3x)}{2}\right) = (6-3x)(4+3x) = 24 + 6x - 9x^2$

STEP 2) $a = 0, b = 2$

STEP 3) $V = \int_a^b A(x)dx = \int_0^2\left(24 + 6x + 9x^2\right)dx = \left[24x + 3x^2 - 3x^3\right]_0^2 = (48 + 12 - 24) - 0 = 36$

8. (a) STEP 1) $A(x) = \frac{1}{2}(\text{base})\cdot(\text{height}) = \left(\sqrt{x} - \frac{x}{2}\right)\cdot(6) = 6\sqrt{x} - 3x$

STEP 2) $a = 0, b = 4$

STEP 3) $V = \int_a^b A(x)\,dx = \int_0^4\left(6x^{1/2} - 3x\right)dx = \left[4x^{3/2} - \frac{3}{2}x^2\right]_0^4 = (32-24) - 0 = 8$

(b) STEP 1) $A(x) = \frac{1}{2}\cdot\pi\left(\frac{\text{diameter}}{2}\right)^2 = \frac{1}{2}\cdot\pi\left(\frac{\sqrt{x} - \frac{x}{2}}{2}\right)^2 = \frac{\pi}{2}\cdot\frac{x - x^{3/2} + \frac{1}{4}x^2}{4} = \frac{\pi}{8}\left(x - x^{3/2} + \frac{1}{4}x^2\right)$

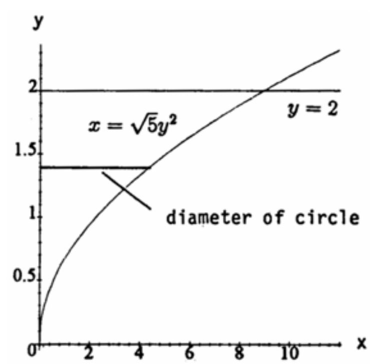
STEP 2) $a = 0, b = 4$

STEP 3) $V = \int_a^b A(x)\,dx = \frac{\pi}{8}\int_0^4\left(x - x^{3/2} + \frac{1}{4}x^2\right)dx = \left[\frac{1}{2}x^2 - \frac{2}{5}x^{5/2} + \frac{1}{12}x^3\right]_0^4 = \frac{\pi}{8}\left(8 - \frac{64}{5} + \frac{16}{3}\right) - \frac{\pi}{8}(0) = \frac{\pi}{15}$

9. $A(y) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{5}y^2 - 0\right)^2 = \frac{5\pi}{4}y^4$;

$c = 0, d = 2; V = \int_c^d A(y)\,dy$

$= \int_0^2 \frac{5\pi}{4}y^4\,dy = \left[\left(\frac{5\pi}{4}\right)\left(\frac{y^5}{5}\right)\right]_0^2 = \frac{\pi}{4}\left(2^5 - 0\right) = 8\pi$



10. $A(y) = \frac{1}{2}(\text{leg})(\text{leg}) = \frac{1}{2}\left[\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\right]^2 = \frac{1}{2}\left(2\sqrt{1-y^2}\right)^2 = 2\left(1-y^2\right)$; $c = -1, d = 1$;

$V = \int_c^d A(y)\,dy = \int_{-1}^1 2\left(1-y^2\right)dy = 2\left[y - \frac{y^3}{3}\right]_{-1}^1 = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$

11. The slices perpendicular to the edge labeled 5 are triangles, and by similar triangles we have $\frac{b}{h} = \frac{4}{3} \Rightarrow h = \frac{3}{4}b$. The equation of the line through $(5, 0)$ and $(0, 4)$ is $y = -\frac{4}{5}x + 4$, thus the length of the base $= -\frac{4}{5}x + 4$ and the height $= \frac{3}{4}\left(-\frac{4}{5}x + 4\right) = -\frac{3}{5}x + 3$. Thus $A(x) = \frac{1}{2}(\text{base}) \cdot (\text{height}) = \frac{1}{2}\left(-\frac{4}{5}x + 4\right) \cdot \left(-\frac{3}{5}x + 3\right)$ $= \frac{6}{25}x^2 - \frac{12}{5}x + 6$ and $V = \int_a^b A(x)\,dx = \int_0^5 \left(\frac{6}{25}x^2 - \frac{12}{5}x + 6\right) dx = \left[\frac{2}{25}x^3 - \frac{6}{5}x^2 + 6x\right]_0^5 = (10 - 30 + 30) - 0 = 10$

12. The slices parallel to the base are squares. The cross section of the pyramid is a triangle, and by similar triangles we have $\frac{b}{h} = \frac{3}{5} \Rightarrow b = \frac{3}{5}h$. Thus $A(y) = (\text{base})^2 = \left(\frac{3}{5}y\right)^2 = \frac{9}{25}y^2 \Rightarrow V = \int_c^d A(y)\,dy = \int_0^5 \frac{9}{25}y^2 dy$ $= \left[\frac{3}{25}y^3\right]_0^5 = 15 - 0 = 15$

13. (a) It follows from Cavalieri's Principle that the volume of a column is the same as the volume of a right prism with a square base of side length s and altitude h. Thus,

STEP 1) $A(x) = (\text{sidelength})^2 = s^2$;

STEP 2) $a = 0, b = h$;

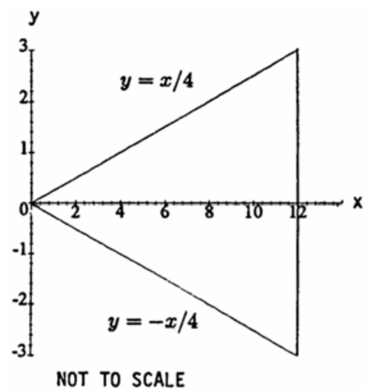
STEP 3) $V = \int_a^b A(x)\,dx = \int_0^h s^2\,dx = s^2 h$

(b) From Cavalieri's Principle we conclude that the volume of the column is the same as the volume of the prism described above, regardless of the number of turns $\Rightarrow V = s^2 h$

14. 1) The solid and the cone have the same altitude of 12.

2) The cross sections of the solid are disks of diameter $x - \left(\frac{x}{2}\right) = \frac{x}{2}$. If we place the vertex of the cone at the origin of the coordinate system and make its axis or symmetry coincide with the x-axis then the cone's cross sections will be circular disks of diameter $\frac{x}{4} - \left(-\frac{x}{4}\right) = \frac{x}{2}$ (see accompanying figure).

3) The solid and the cone have equal altitudes and identical parallel cross sections. From Cavalier's Principle we conclude that the solid and the cone have the same volume.



15. $R(x) = y = 1 - \frac{x}{2} \Rightarrow V = \int_0^2 \pi[R(x)]^2 dx = \pi\int_0^2 \left(1 - \frac{x}{2}\right)^2 dx = \pi\int_0^2 \left(1 - x + \frac{x^2}{4}\right) dx = \pi\left[x - \frac{x^2}{2} + \frac{x^3}{12}\right]_0^2$ $= \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$

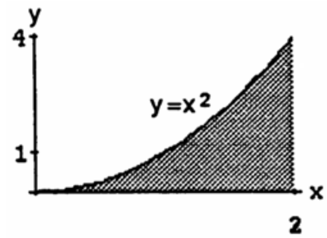
16. $R(y) = x = \frac{3y}{2} \Rightarrow V = \int_0^2 \pi[R(y)]^2 dy = \pi\int_0^2 \left(\frac{3y}{2}\right)^2 dy = \pi\int_0^2 \frac{9}{4}y^2 dy = \pi\left[\frac{3}{4}y^3\right]_0^2 = \pi \cdot \frac{3}{4} \cdot 8 = 6\pi$

17. $R(y) = \tan\left(\frac{\pi}{4}y\right)$; $u = \frac{\pi}{4}y \Rightarrow du = \frac{\pi}{4}dy \Rightarrow 4\,du = \pi\,dy$; $y = 0 \Rightarrow u = 0$, $y = 1 \Rightarrow u = \frac{\pi}{4}$;

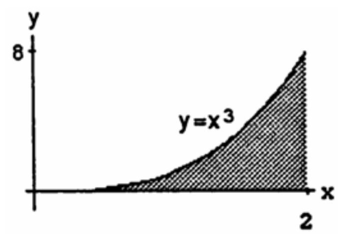
$V = \int_0^1 \pi[R(y)]^2 dy = \pi\int_0^1 \left[\tan\left(\frac{\pi}{4}y\right)\right]^2 dy = 4\int_0^{\pi/4} \tan^2 u\,du = 4\int_0^{\pi/4} \left(-1 + \sec^2 u\right) du = 4[-u + \tan u]_0^{\pi/4}$ $= 4\left(-\frac{\pi}{4} + 1 - 0\right) = 4 - \pi$

18. $R(x) = \sin x \cos x$; $R(x) = 0 \Rightarrow a = 0$ and $b = \frac{\pi}{2}$ are the limits of integration;
$V = \int_0^{\pi/2} \pi [R(x)]^2 dx = \pi \int_0^{\pi/2} (\sin x \cos x)^2 dx = \pi \int_0^{\pi/2} \frac{(\sin 2x)^2}{4} dx$; $\left[u = 2x \Rightarrow du = 2\,dx \Rightarrow \frac{du}{8} = \frac{dx}{4};\right.$
$\left. x = 0 \Rightarrow u = 0,\ x = \frac{\pi}{2} \Rightarrow u = \pi \right] \to V = \pi \int_0^{\pi} \frac{1}{8} \sin^2 u\, du = \frac{\pi}{8}\left[\frac{u}{2} - \frac{1}{4}\sin 2u\right]_0^{\pi} = \frac{\pi}{8}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{\pi^2}{16}$

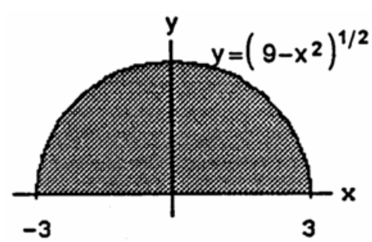
19. $R(x) = x^2 \Rightarrow V = \int_0^2 \pi [R(x)]^2 dx$
$= \pi \int_0^2 \left(x^2\right)^2 dx = \pi \int_0^2 x^4 dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$



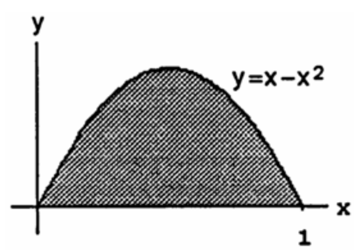
20. $R(x) = x^3 \Rightarrow V = \int_0^2 \pi [R(x)]^2 dx$
$= \pi \int_0^2 \left(x^3\right)^2 dx = \pi \int_0^2 x^6 dx = \pi \left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$



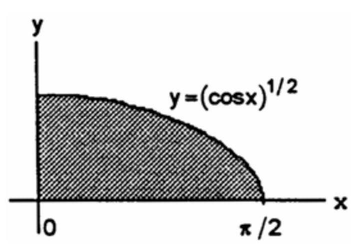
21. $R(x) = \sqrt{9 - x^2} \Rightarrow V = \int_{-3}^{3} \pi [R(x)]^2 dx$
$= \pi \int_{-3}^{3} \left(9 - x^2\right) dx = \pi \left[9x - \frac{x^3}{3}\right]_{-3}^{3}$
$= 2\pi \left[9(3) - \frac{27}{3}\right] = 2 \cdot \pi \cdot 18 = 36\pi$



22. $R(x) = x - x^2 \Rightarrow V = \int_0^1 \pi [R(x)]^2 dx$
$= \pi \int_0^1 \left(x - x^2\right)^2 dx = \pi \int_0^1 \left(x^2 - 2x^3 + x^4\right) dx$
$= \pi \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1 = \pi \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$
$= \frac{\pi}{30}(10 - 15 + 6) = \frac{\pi}{30}$

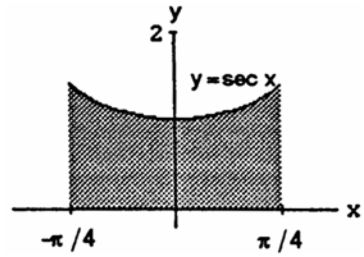


23. $R(x) = \sqrt{\cos x} \Rightarrow V = \int_0^{\pi/2} \pi [R(x)]^2 dx$
$= \pi \int_0^{\pi/2} \cos x\, dx = \pi [\sin x]_0^{\pi/2} = \pi(1 - 0) = \pi$



24. $R(x) = \sec x \Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi\left[R(x)\right]^2 dx$

$= \pi\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \pi\left[\tan x\right]_{-\pi/4}^{\pi/4} = \pi[1-(-1)] = 2\pi$



25. $R(x) = \sqrt{2} - \sec x \tan x \Rightarrow V = \int_0^{\pi/4} \pi\left[R(x)\right]^2 dx$

$= \pi\int_0^{\pi/4}\left(\sqrt{2} - \sec x \tan x\right)^2 dx$

$= \pi\int_0^{\pi/4}\left(2 - 2\sqrt{2}\sec x \tan x + \sec^2 x \tan^2 x\right) dx$

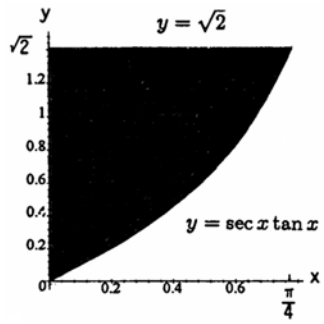
$= \pi\left(\int_0^{\pi/4} 2\, dx - 2\sqrt{2}\int_0^{\pi/4} \sec x \tan x\, dx\right.$

$\left.+ \int_0^{\pi/4} (\tan x)^2 \sec^2 x\, dx\right)$

$= \pi\left(\left[2x\right]_0^{\pi/4} - 2\sqrt{2}\left[\sec x\right]_0^{\pi/4} + \left[\frac{\tan^3 x}{3}\right]_0^{\pi/4}\right)$

$= \pi\left[\left(\frac{\pi}{2} - 0\right) - 2\sqrt{2}\left(\sqrt{2} - 1\right) + \frac{1}{3}(1^3 - 0)\right]$

$= \pi\left(\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3}\right)$



26. $R(x) = 2 - 2\sin x = 2(1 - \sin x) \Rightarrow V = \int_0^{\pi/2} \pi\left[R(x)\right]^2 dx$

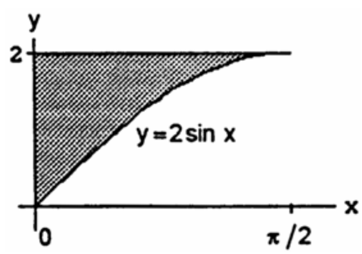
$= \pi\int_0^{\pi/2} 4(1-\sin x)^2 dx = 4\pi\int_0^{\pi/2}\left(1 + \sin^2 x - 2\sin x\right) dx$

$= 4\pi\int_0^{\pi/2}\left[1 + \frac{1}{2}(1 - \cos 2x) - 2\sin x\right] dx$

$= 4\pi\int_0^{\pi/2}\left(\frac{3}{2} - \frac{\cos 2x}{2} - 2\sin x\right)$

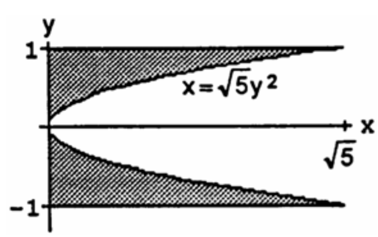
$= 4\pi\left[\frac{3}{2}x - \frac{\sin 2x}{4} + 2\cos x\right]_0^{\pi/2}$

$= 4\pi\left[\left(\frac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 2)\right] = \pi(3\pi - 8)$



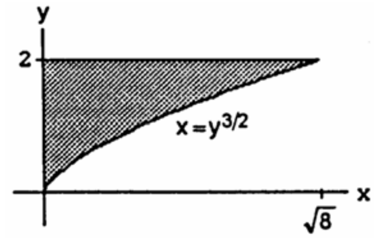
27. $R(y) = \sqrt{5}y^2 \Rightarrow V = \int_{-1}^{1} \pi\left[R(y)\right]^2 dy = \pi\int_{-1}^{1} 5y^4\, dy$

$= \pi\left[y^5\right]_{-1}^{1} = \pi[1-(-1)] = 2\pi$



28. $R(y) = y^{3/2} \Rightarrow V = \int_0^2 \pi [R(y)]^2 dy = \pi \int_0^2 y^3 dy$

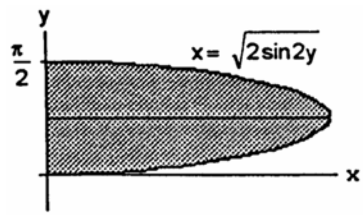
$= \pi \left[\frac{y^4}{4}\right]_0^2 = 4\pi$



29. $R(y) = \sqrt{2 \sin 2y} \Rightarrow V = \int_0^{\pi/2} \pi [R(y)]^2 \, dy$

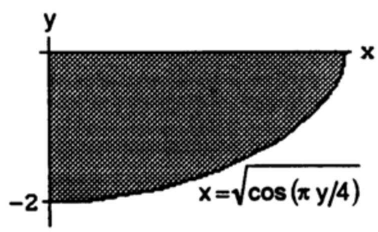
$= \pi \int_0^{\pi/2} 2 \sin 2y \, dy = \pi [-\cos 2y]_0^{\pi/2}$

$= \pi[1 - (-1)] = 2\pi$



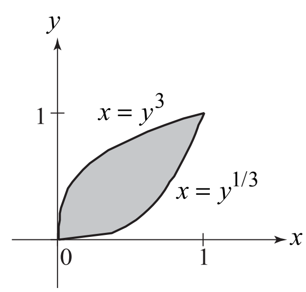
30. $R(y) = \sqrt{\cos \frac{\pi y}{4}} \Rightarrow V = \int_{-2}^0 \pi [R(y)]^2 dy$

$= \pi \int_{-2}^0 \cos\left(\frac{\pi y}{4}\right) dy = 4\left[\sin \frac{\pi y}{4}\right]_{-2}^0 = 4[0 - (-1)] = 4$



31. $R_1(y) = y^3, R_2(y) = y^{1/3} \Rightarrow V = \int_0^1 \pi \left(R_2{}^2 - R_1{}^2\right) dy$

$= \int_0^1 \pi \left(y^{2/3} - y^6\right) dy = \pi \left(\frac{3}{5} y^{5/3} - \frac{1}{7} y^7\right)\Big|_0^1 = \frac{16\pi}{35}$

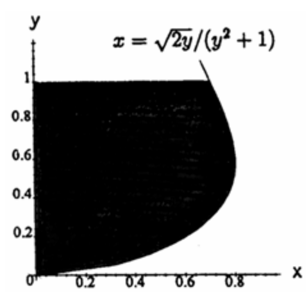


32. $R(y) = \sqrt{\frac{2y}{y^2+1}} \Rightarrow V = \int_0^1 \pi [R(y)]^2 dy$

$= \pi \int_0^1 2y\left(y^2 + 1\right)^{-2} dy;$ $[u = y^2 + 1 \Rightarrow du = 2y \, dy;$

$y = 0 \Rightarrow u = 1, \, y = 1 \Rightarrow u = 2]$

$\rightarrow V = \pi \int_1^2 u^{-2} du = \pi \left[-\frac{1}{u}\right]_1^2 = \pi\left[-\frac{1}{2} - (-1)\right] = \frac{\pi}{2}$



33. For the sketch given, $a = -\frac{\pi}{2}, b = \frac{\pi}{2}; R(x) = 1, r(x) = \sqrt{\cos x};$ $V = \int_a^b \pi \left([R(x)]^2 - [r(x)]^2\right) dx$

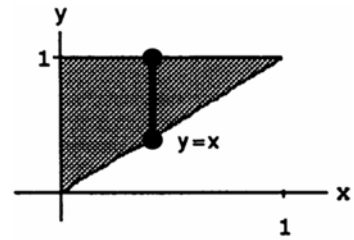
$= \int_{-\pi/2}^{\pi/2} \pi(1 - \cos x) \, dx = 2\pi \int_0^{\pi/2} (1 - \cos x) \, dx = 2\pi [x - \sin x]_0^{\pi/2} = 2\pi\left(\frac{\pi}{2} - 1\right) = \pi^2 - 2\pi$

34. For the sketch given, $c = 0$, $d = \frac{\pi}{4}$; $R(y) = 1$, $r(y) = \tan y$; $V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi\int_0^{\pi/4}\left(1 - \tan^2 y\right) dy = \pi\int_0^{\pi/4}\left(2 - \sec^2 y\right) dy = \pi[2y - \tan y]_0^{\pi/4} = \pi\left(\frac{\pi}{2} - 1\right) = \frac{\pi^2}{2} - \pi$

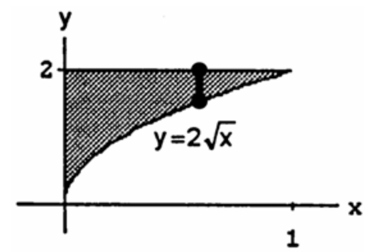
35. $r(x) = x$ and $R(x) = 1 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \int_0^1 \pi\left(1 - x^2\right) dx = \pi\left[x - \frac{x^3}{3}\right]_0^1 = \pi\left[\left(1 - \frac{1}{3}\right) - 0\right] = \frac{2\pi}{3}$



36. $r(x) = 2\sqrt{x}$ and $R(x) = 2 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi\int_0^1 (4 - 4x)\, dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1 = 4\pi\left(1 - \frac{1}{2}\right) = 2\pi$



37. $r(x) = x^2 + 1$ and $R(x) = x + 3$

$\Rightarrow V = \int_{-1}^2 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

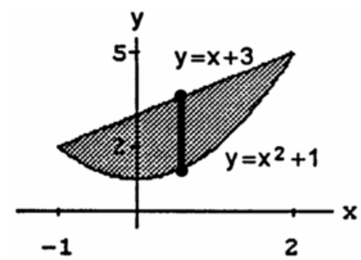
$= \pi\int_{-1}^2 \left[(x+3)^2 - \left(x^2+1\right)^2\right] dx$

$= \pi\int_{-1}^2 \left[\left(x^2 + 6x + 9\right) - \left(x^4 + 2x^2 + 1\right)\right] dx$

$= \pi\int_{-1}^2 \left(-x^4 - x^2 + 6x + 8\right) dx = \pi\left[-\frac{x^5}{5} - \frac{x^3}{3} + \frac{6x^2}{2} + 8x\right]_{-1}^2$

$= \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + \frac{24}{2} + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + \frac{6}{2} - 8\right)\right]$

$= \pi\left(-\frac{33}{5} - 3 + 28 - 3 + 8\right) = \pi\left(\frac{5\cdot 30 - 33}{5}\right) = \frac{117\pi}{5}$



38. $r(x) = 2 - x$ and $R(x) = 4 - x^2$

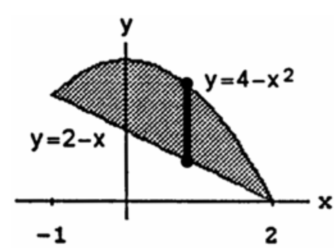
$\Rightarrow V = \int_{-1}^2 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi\int_{-1}^2 \left[\left(4 - x^2\right)^2 - (2-x)^2\right] dx$

$= \pi\int_{-1}^2 \left[\left(16 - 8x^2 + x^4\right) - \left(4 - 4x + x^2\right)\right] dx$

$= \pi\int_{-1}^2 \left(12 + 4x - 9x^2 + x^4\right) dx = \pi\left[12x + 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{-1}^2$

$= \pi\left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right] = \pi\left(15 + \frac{33}{5}\right) = \frac{108\pi}{5}$

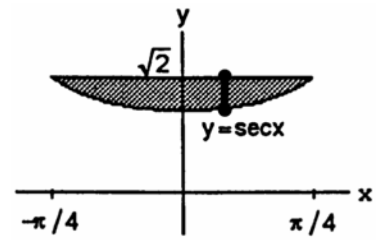


39. $r(x) = \sec x$ and $R(x) = \sqrt{2}$

$\Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right) dx$

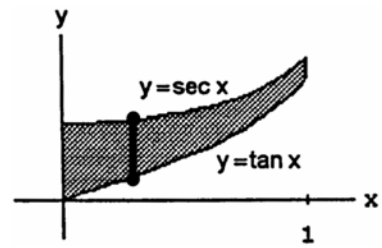
$= \pi\int_{-\pi/4}^{\pi/4}\left(2 - \sec^2 x\right) dx = \pi\left[2x - \tan x\right]_{-\pi/4}^{\pi/4}$

$= \pi\left[\left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)\right] = \pi(\pi - 2)$



40. $R(x) = \sec x$ and $r(x) = \tan x \Rightarrow V = \int_0^1 \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right) dx$

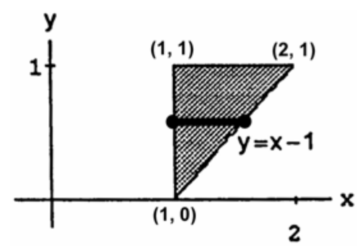
$= \pi\int_0^1\left(\sec^2 x - \tan^2 x\right) dx = \pi\int_0^1 1\, dx = \pi\left[x\right]_0^1 = \pi$



41. $r(y) = 1$ and $R(y) = 1 + y \Rightarrow V = \int_0^1 \pi\left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy$

$= \pi\int_0^1\left[(1+y)^2 - 1\right] dy = \pi\int_0^1\left(1 + 2y + y^2 - 1\right) dy$

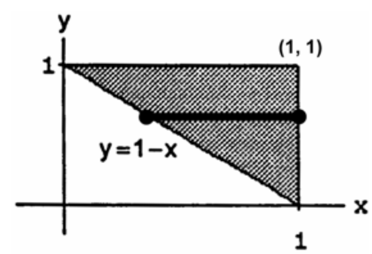
$= \pi\int_0^1\left(2y + y^2\right) dy = \pi\left[y^2 + \frac{y^3}{3}\right]_0^1 = \pi\left(1 + \frac{1}{3}\right) = \frac{4\pi}{3}$



42. $R(y) = 1$ and $r(y) = 1 - y \Rightarrow V = \int_0^1 \pi\left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy$

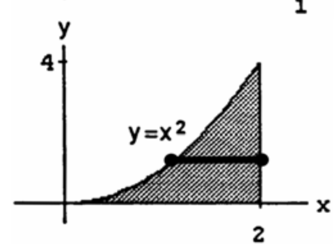
$= \pi\int_0^1\left[1 - (1-y)^2\right] dy = \pi\int_0^1\left[1 - \left(1 - 2y + y^2\right)\right] dy$

$= \pi\int_0^1\left(2y - y^2\right) dy = \pi\left[y^2 - \frac{y^3}{3}\right]_0^1 = \pi\left(1 - \frac{1}{3}\right) = \frac{2\pi}{3}$



43. $R(y) = 2$ and $r(y) = \sqrt{y} \Rightarrow V = \int_0^4 \pi\left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy$

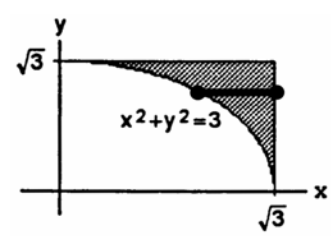
$= \pi\int_0^4 (4 - y)\, dy = \pi\left[4y - \frac{y^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$



44. $R(y) = \sqrt{3}$ and $r(y) = \sqrt{3 - y^2}$

$\Rightarrow V = \int_0^{\sqrt{3}} \pi\left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy$

$= \pi\int_0^{\sqrt{3}}\left[3 - \left(3 - y^2\right)\right] dy = \pi\int_0^{\sqrt{3}} y^2\, dy = \pi\left[\frac{y^3}{3}\right]_0^{\sqrt{3}} = \pi\sqrt{3}$

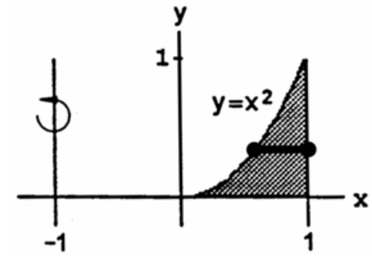


45. $R(y) = 2$ and $r(y) = 1 + \sqrt{y} \Rightarrow V = \int_0^1 \pi\left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy$

$= \pi\int_0^1\left[4 - \left(1+\sqrt{y}\right)^2\right] dy = \pi\int_0^1\left(4 - 1 - 2\sqrt{y} - y\right) dy$

$= \pi\int_0^1\left(3 - 2\sqrt{y} - y\right) dy = \pi\left[3y - \frac{4}{3}y^{3/2} - \frac{y^2}{2}\right]_0^1$

$= \pi\left(3 - \frac{4}{3} - \frac{1}{2}\right) = \pi\left(\frac{18-8-3}{6}\right) = \frac{7\pi}{6}$

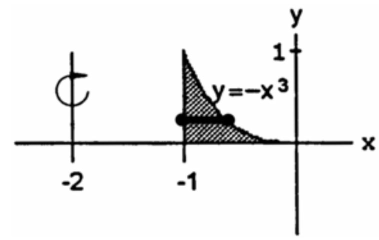


46. $R(y) = 2 - y^{1/3}$ and $r(y) = 1 \Rightarrow V = \int_0^1 \pi\left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy$

$= \pi\int_0^1\left[\left(2 - y^{1/3}\right)^2 - 1\right] dy = \pi\int_0^1\left(4 - 4y^{1/3} + y^{2/3} - 1\right) dy$

$= \pi\int_0^1\left(3 - 4y^{1/3} + y^{2/3}\right) dy = \pi\left[3y - 3y^{4/3} + \frac{3y^{5/3}}{5}\right]_0^1$

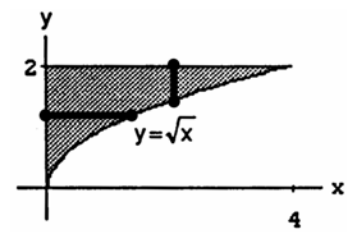
$= \pi\left(3 - 3 + \frac{3}{5}\right) = \frac{3\pi}{5}$



47. (a) $r(x) = \sqrt{x}$ and $R(x) = 2$

$\Rightarrow V = \int_0^4 \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right) dx$

$= \pi\int_0^4 (4 - x)\, dx = \pi\left[4x - \frac{x^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$



(b) $r(y) = 0$ and $R(y) = y^2 \Rightarrow V = \int_0^2 \pi\left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy = \pi\int_0^2 y^4 dy = \pi\left[\frac{y^5}{5}\right]_0^2 = \frac{32\pi}{5}$

(c) $r(x) = 0$ and $R(x) = 2 - \sqrt{x} \Rightarrow V = \int_0^4 \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right) dx = \pi\int_0^4\left(2 - \sqrt{x}\right)^2 dx$

$= \pi\int_0^4\left(4 - 4\sqrt{x} + x\right) dx = \pi\left[4x - \frac{8x^{3/2}}{3} + \frac{x^2}{2}\right]_0^4 = \pi\left(16 - \frac{64}{3} + \frac{16}{2}\right) = \frac{8\pi}{3}$

(d) $r(y) = 4 - y^2$ and $R(y) = 4 \Rightarrow V = \int_0^2 \pi\left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy = \pi\int_0^2\left[16 - \left(4 - y^2\right)^2\right] dy$

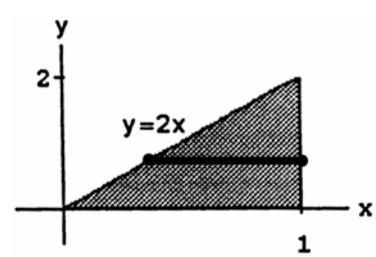
$= \pi\int_0^2\left(16 - 16 + 8y^2 - y^4\right) dy = \pi\int_0^2\left(8y^2 - y^4\right) dy = \pi\left[\frac{8}{3}y^3 - \frac{y^5}{5}\right]_0^2 = \pi\left(\frac{64}{3} - \frac{32}{5}\right) = \frac{224\pi}{15}$

48. (a) $r(y) = 0$ and $R(y) = 1 - \frac{y}{2}$

$\Rightarrow V = \int_0^2 \pi\left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy$

$= \pi\int_0^2\left(1 - \frac{y}{2}\right)^2 dy = \pi\int_0^2\left(1 - y + \frac{y^2}{4}\right) dy$

$= \pi\left[y - \frac{y^2}{2} + \frac{y^3}{12}\right]_0^2 = \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$



(b) $r(y)=1$ and $R(y)=2-\frac{y}{2}$

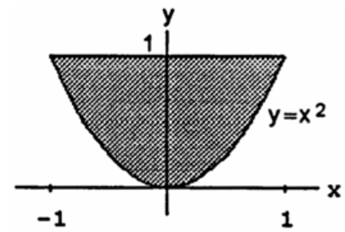
$\Rightarrow V=\int_0^2 \pi\left([R(y)]^2-[r(y)]^2\right)dy=\pi\int_0^2\left[\left(2-\frac{y}{2}\right)^2-1\right]dy=\pi\int_0^2\left(4-2y+\frac{y^2}{4}-1\right)dy$

$=\pi\int_0^2\left(3-2y+\frac{y^2}{4}\right)dy=\pi\left[3y-y^2+\frac{y^3}{12}\right]_0^2=\pi\left(6-4+\frac{8}{12}\right)=\pi\left(2+\frac{2}{3}\right)=\frac{8\pi}{3}$

49. (a) $r(x)=0$ and $R(x)=1-x^2 \Rightarrow V=\int_{-1}^1 \pi\left([R(x)]^2-[r(x)]^2\right)dx$

$=\pi\int_{-1}^1\left(1-x^2\right)^2dx=\pi\int_{-1}^1\left(1-2x^2+x^4\right)dx$

$=\pi\left[x-\frac{2x^3}{3}+\frac{x^5}{5}\right]_{-1}^1=2\pi\left(1-\frac{2}{3}+\frac{1}{5}\right)=2\pi\left(\frac{15-10+3}{15}\right)=\frac{16\pi}{15}$



(b) $r(x)=1$ and $R(x)=2-x^2 \Rightarrow V=\int_{-1}^1 \pi\left([R(x)]^2-[r(x)]^2\right)dx=\pi\int_{-1}^1\left[\left(2-x^2\right)^2-1\right]dx$

$=\pi\int_{-1}^1\left(4-4x^2+x^4-1\right)dx=\pi\int_{-1}^1\left(3-4x^2+x^4\right)dx=\pi\left[3x-\frac{4}{3}x^3+\frac{x^5}{5}\right]_{-1}^1=2\pi\left(3-\frac{4}{3}+\frac{1}{5}\right)$

$=\frac{2\pi}{15}(45-20+3)=\frac{56\pi}{15}$

(c) $r(x)=1+x^2$ and $R(x)=2 \Rightarrow V=\int_{-1}^1 \pi\left([R(x)]^2-[r(x)]^2\right)dx=\pi\int_{-1}^1\left[4-\left(1+x^2\right)^2\right]dx$

$=\pi\int_{-1}^1\left(4-1-2x^2-x^4\right)dx=\pi\int_{-1}^1\left(3-2x^2-x^4\right)dx=\pi\left[3x-\frac{2}{3}x^3-\frac{x^5}{5}\right]_{-1}^1=2\pi\left(3-\frac{2}{3}-\frac{1}{5}\right)$

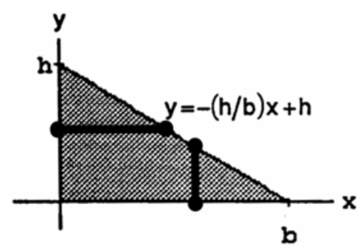
$=\frac{2\pi}{15}(45-10-3)=\frac{64\pi}{15}$

50. (a) $r(x)=0$ and $R(x)=-\frac{h}{b}x+h$

$\Rightarrow V=\int_0^b \pi\left([R(x)]^2-[r(x)]^2\right)dx$

$=\pi\int_0^b\left(-\frac{h}{b}x+h\right)^2dx=\pi\int_0^b\left(\frac{h^2}{b^2}x^2-\frac{2h^2}{b}x+h^2\right)dx$

$=\pi h^2\left[\frac{x^3}{3b^2}-\frac{x^2}{b}+x\right]_0^b=\pi h^2\left(\frac{b}{3}-b+b\right)=\frac{\pi h^2 b}{3}$



(b) $r(y)=0$ and $R(y)=b\left(1-\frac{y}{h}\right) \Rightarrow V=\int_0^h \pi\left([R(y)]^2-[r(y)]^2\right)dy=\pi b^2\int_0^h\left(1-\frac{y}{h}\right)^2dy$

$=\pi b^2\int_0^h\left(1-\frac{2y}{h}+\frac{y^2}{h^2}\right)dy=\pi b^2\left[y-\frac{y^2}{h}+\frac{y^3}{3h^2}\right]_0^h=\pi b^2\left(h-h+\frac{h}{3}\right)=\frac{\pi b^2 h}{3}$

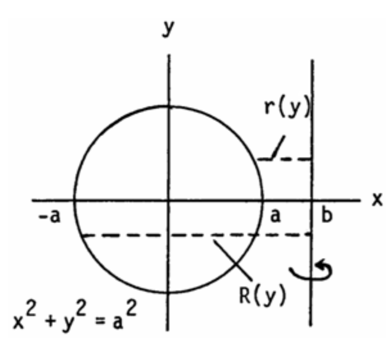
51. $R(y)=b+\sqrt{a^2-y^2}$ and $r(y)=b-\sqrt{a^2-y^2}$

$\Rightarrow V=\int_{-a}^a \pi\left([R(y)]^2-[r(y)]^2\right)dy$

$=\pi\int_{-a}^a\left[\left(b+\sqrt{a^2-y^2}\right)^2-\left(b-\sqrt{a^2-y^2}\right)^2\right]dy$

$=\pi\int_{-a}^a 4b\sqrt{a^2-y^2}\,dy=4b\pi\int_{-a}^a\sqrt{a^2-y^2}\,dy$

$=4b\pi\cdot$ area of semicircle of radius $a=4b\pi\cdot\frac{\pi a^2}{2}=2a^2b\pi^2$



52. (a) A cross section has radius $r = \sqrt{2y}$ and area $\pi r^2 = 2\pi y$. The volume is $\int_0^5 2\pi y\,dy = \pi\left[y^2\right]_0^5 = 25\pi$.

(b) $V(h) = \int A(h)dh$, so $\frac{dV}{dh} = A(h)$. Therefore $\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = A(h)\cdot\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{1}{A(h)}\cdot\frac{dV}{dt}$.

For $h = 4$, the area is $2\pi(4) = 8\pi$, so $\frac{dh}{dt} = \frac{1}{8\pi}\cdot 3\,\frac{\text{units}^3}{\text{s}} = \frac{3}{8\pi}\cdot\frac{\text{units}^3}{\text{s}}$.

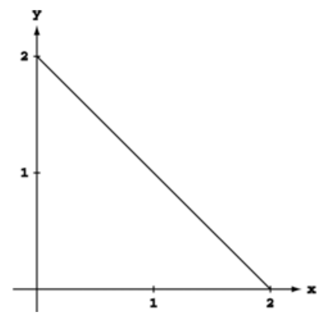
53. (a) $R(y) = \sqrt{a^2 - y^2} \Rightarrow V = \pi\int_{-a}^{h-a}\left(a^2 - y^2\right)dy = \pi\left[a^2 y - \frac{y^3}{3}\right]_{-a}^{h-a} = \pi\left[a^2 h - a^3 - \frac{(h-a)^3}{3} - \left(-a^3 + \frac{a^3}{3}\right)\right]$

$= \pi\left[a^2 h - \frac{1}{3}\left(h^3 - 3h^2 a + 3ha^2 - a^3\right) - \frac{a^3}{3}\right] = \pi\left(a^2 h - \frac{h^3}{3} + h^2 a - ha^2\right) = \frac{\pi h^2(3a-h)}{3}$

(b) Given $\frac{dV}{dt} = 0.2\ \text{m}^3/\text{s}$ and $a = 5$ m, find $\left.\frac{dh}{dt}\right|_{h=4}$. From part (a), $V(h) = \frac{\pi h^2(15-h)}{3} = 5\pi h^2 - \frac{\pi h^3}{3}$

$\Rightarrow \frac{dV}{dh} = 10\pi h - \pi h^2 \Rightarrow \frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = \pi h(10-h)\frac{dh}{dt} \Rightarrow \left.\frac{dh}{dt}\right|_{h=4} = \frac{0.2}{4\pi(10-4)} = \frac{1}{(20\pi)(6)} = \frac{1}{120\pi}$ m/s.

54. Suppose the solid is produced by revolving $y = 2 - x$ about the y-axis. Cast a shadow of the solid on a plane parallel to the xy-plane. Use an approximation such as the Trapezoid Rule, to estimate $\int_a^b \pi\left[R(y)\right]^2 dy \approx \sum_{k=1}^{n}\pi\left(\frac{d_{\hat{k}}}{2}\right)^2 \Delta y$.



55. The cross section of a solid right circular cylinder with a cone removed is a disk with radius *R* from which a disk of radius *h* has been removed. Thus its area is $A_1 = \pi R^2 - \pi h^2 = \pi(R^2 - h^2)$. The cross section of the hemisphere is a disk of radius $\sqrt{R^2 - h^2}$. Therefore its area is $A_2 = \pi\left(\sqrt{R^2 - h^2}\right)^2 = \pi\left(R^2 - h^2\right)$.
We can see that $A_1 = A_2$. The altitudes of both solids are *R*. Applying Cavalieri's Principle we find Volume of Hemisphere = (Volume of Cylinder) − (Volume of Cone) $= \left(\pi R^2\right)R - \frac{1}{3}\pi\left(R^2\right)R = \frac{2}{3}\pi R^3$.

56. $R(x) = \frac{x}{12}\sqrt{36 - x^2} \Rightarrow V = \int_0^6 \pi\left[R(x)\right]^2 dx = \pi\int_0^6 \frac{x^2}{144}\left(36 - x^2\right)dx = \frac{\pi}{144}\int_0^6\left(36x^2 - x^4\right)dx$

$= \frac{\pi}{144}\left[12x^3 - \frac{x^5}{5}\right]_0^6 = \frac{\pi}{144}\left(12\cdot 6^3 - \frac{6^5}{5}\right) = \frac{\pi\cdot 6^3}{144}\left(12 - \frac{36}{5}\right) = \left(\frac{196\pi}{144}\right)\left(\frac{60-36}{5}\right) = \frac{36\pi}{5}\ \text{cm}^3.$

The plumb bob will weigh about $W = (8.5)\left(\frac{36\pi}{5}\right) \approx 192$ g, to the nearest gram.

57. $R(y) = \sqrt{256 - y^2} \Rightarrow V = \int_{-16}^{-7}\pi\left[R(y)\right]^2 dy = \pi\int_{-16}^{-7}\left(256 - y^2\right)dy = \pi\left[256y - \frac{y^3}{3}\right]_{-16}^{-7}$

$= \pi\left[(256)(-7) + \frac{7^3}{3} - \left((256)(-16) + \frac{16^3}{3}\right)\right] = \pi\left(\frac{7^3}{3} + 256(16-7) - \frac{16^3}{3}\right) = 1053\pi\ \text{cm}^3 \approx 3308\ \text{cm}^3$

58. (a) $R(x) = |c - \sin x|$, so $V = \pi\int_0^{\pi}[R(x)]^2 dx = \pi\int_0^{\pi}(c-\sin x)^2\, dx = \pi\int_0^{\pi}\left(c^2 - 2c\sin x + \sin^2 x\right) dx$

$= \pi\int_0^{\pi}\left(c^2 - 2c\sin x + \frac{1-\cos 2x}{2}\right) dx = \pi\int_0^{\pi}\left(c^2 + \frac{1}{2} - 2c\sin x - \frac{\cos 2x}{2}\right) dx = \pi\left[\left(c^2+\frac{1}{2}\right)x + 2c\cos x - \frac{\sin 2x}{4}\right]_0^{\pi}$

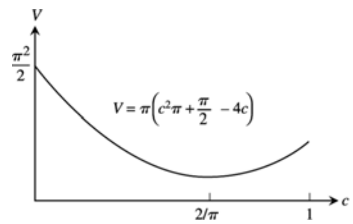
$= \pi\left[\left(c^2\pi + \frac{\pi}{2} - 2c - 0\right) - (0 + 2c - 0)\right] = \pi\left(c^2\pi + \frac{\pi}{2} - 4c\right)$. Let $V(c) = \pi\left(c^2\pi + \frac{\pi}{2} - 4c\right)$. We find the extreme values of $V(c)$: $\frac{dV}{dc} = \pi(2c\pi - 4) = 0 \Rightarrow c = \frac{2}{\pi}$ is a critical point, and $V\left(\frac{2}{\pi}\right) = \pi\left(\frac{4}{\pi} + \frac{\pi}{2} - \frac{8}{\pi}\right)$

$= \pi\left(\frac{\pi}{2} - \frac{4}{\pi}\right) = \frac{\pi^2}{2} - 4$; Evaluate V at the endpoints: $V(0) = \frac{\pi^2}{2}$ and $V(1) = \pi\left(\frac{3}{2}\pi - 4\right) = \frac{\pi^2}{2} - (4-\pi)\pi$.

Now we see that the function's absolute minimum value is $\frac{\pi^2}{2} - 4$, taken on at the critical point $c = \frac{2}{\pi}$. (See also the accompanying graph.)

(b) From the discussion in part (a) we conclude that the function's absolute maximum value is $\frac{\pi^2}{2}$, taken on at the endpoint $c = 0$.

(c) The graph of the solid's volume as a function of c for $0 \le c \le 1$ is given at the right. As c moves away from $[0, 1]$ the volume of the solid increases without bound. If we approximate the solid as a set of solid disks, we can see that the radius of a typical disk increases without bounds as c moves away from $[0, 1]$.



59. Volume of the solid generated by rotating the region bounded by the x-axis and $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is $V = \int_a^b \pi[f(x)]^2 dx = 4\pi$, and the volume of the solid generated by rotating the same region about the line $y = -1$ is $V = \int_a^b \pi[f(x)+1]^2 dx = 8\pi$. Thus $\int_a^b \pi[f(x)+1]^2 dx - \int_a^b \pi[f(x)]^2 dx = 8\pi - 4\pi$

$\Rightarrow \pi\int_a^b\left([f(x)]^2 + 2f(x) + 1 - [f(x)]^2\right) dx = 4\pi \Rightarrow \int_a^b (2f(x)+1)\, dx = 4 \Rightarrow 2\int_a^b f(x)\, dx + \int_a^b dx = 4$

$\Rightarrow \int_a^b f(x)\, dx + \frac{1}{2}(b-a) = 2 \Rightarrow \int_a^b f(x)\, dx = \frac{4-b+a}{2}$

60. Volume of the solid generated by rotating the region bounded by the x-axis and $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is $V = \int_a^b \pi[f(x)]^2 dx = 6\pi$, and the volume of the solid generated by rotating the same region about the line $y = -2$ is $V = \int_a^b \pi[f(x)+2]^2 dx = 10\pi$. Thus

$\int_a^b \pi[f(x)+2]^2\, dx - \int_a^b \pi[f(x)]^2 dx = 10\pi - 6\pi \Rightarrow \pi\int_a^b\left([f(x)]^2 + 4f(x) + 4 - [f(x)]^2\right) dx = 4\pi$

$\Rightarrow \int_a^b (4f(x)+4)\, dx = 4 \Rightarrow 4\int_a^b f(x)\, dx + 4\int_a^b dx = 4 \Rightarrow \int_a^b f(x)\, dx + (b-a) = 1 \Rightarrow \int_a^b f(x)\, dx = 1 - b + a$

6.2 VOLUMES USING CYLINDRICAL SHELLS

1. For the sketch given, $a = 0, b = 2$;

$V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi x\left(1 + \frac{x^2}{4}\right) dx = 2\pi\int_0^2\left(x + \frac{x^3}{4}\right) dx = 2\pi\left[\frac{x^2}{2} + \frac{x^4}{16}\right]_0^2 = 2\pi\left(\frac{4}{2} + \frac{16}{16}\right)$

$= 2\pi \cdot 3 = 6\pi$

2. For the sketch given, $a=0, b=2$;

$$V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^2 2\pi x\left(2-\frac{x^2}{4}\right)dx=2\pi\int_0^2\left(2x-\frac{x^3}{4}\right)dx=2\pi\left[x^2-\frac{x^4}{16}\right]_0^2=2\pi(4-1)=6\pi$$

3. For the sketch given, $c=0, d=\sqrt{2}$;

$$V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^{\sqrt{2}} 2\pi y\cdot\left(y^2\right)dy=2\pi\int_0^{\sqrt{2}} y^3dy=2\pi\left[\frac{y^4}{4}\right]_0^{\sqrt{2}}=2\pi$$

4. For the sketch given, $c=0, d=\sqrt{3}$;

$$V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^{\sqrt{3}} 2\pi y\cdot\left[3-\left(3-y^2\right)\right]dy=2\pi\int_0^{\sqrt{3}} y^3dy=2\pi\left[\frac{y^4}{4}\right]_0^{\sqrt{3}}=\frac{9\pi}{2}$$

5. For the sketch given, $a=0, b=\sqrt{3}$;

$$V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^{\sqrt{3}} 2\pi x\cdot\left(\sqrt{x^2+1}\right)dx;$$

$\left[u=x^2+1\Rightarrow du=2x\,dx;\ x=0\Rightarrow u=1,\ x=\sqrt{3}\Rightarrow u=4\right]$

$$\rightarrow V=\pi\int_1^4 u^{1/2}du=\pi\left[\frac{2}{3}u^{3/2}\right]_1^4=\frac{2\pi}{3}\left(4^{3/2}-1\right)=\left(\frac{2\pi}{3}\right)(8-1)=\frac{14\pi}{3}$$

6. For the sketch given, $a=0, b=3$;

$$V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^3 2\pi x\left(\frac{9x}{\sqrt{x^3+9}}\right)dx;$$

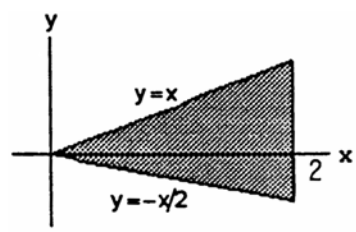
$[u=x^3+9\Rightarrow du=3x^2\ dx\Rightarrow 3\ du=9x^2\ dx;\ x=0\Rightarrow u=9,\ x=3\Rightarrow u=36]$

$$\rightarrow V=2\pi\int_9^{36} 3u^{-1/2}du=6\pi\left[2u^{1/2}\right]_9^{36}=12\pi\left(\sqrt{36}-\sqrt{9}\right)=36\pi$$

7. $a=0, b=2$;

$$V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^2 2\pi x\left[x-\left(-\frac{x}{2}\right)\right]dx$$

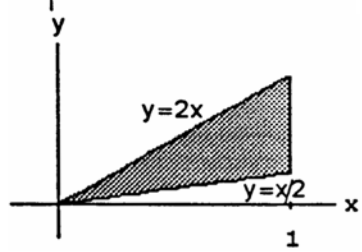
$$=\int_0^2 2\pi x^2\cdot\frac{3}{2}dx=\pi\int_0^2 3x^2dx=\pi\left[x^3\right]_0^2=8\pi$$



8. $a=0, b=1$;

$$V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^1 2\pi x\left(2x-\frac{x}{2}\right)dx$$

$$=\pi\int_0^1 2\left(\frac{3x^2}{2}\right)dx=\pi\int_0^1 3x^2dx=\pi\left[x^3\right]_0^1=\pi$$

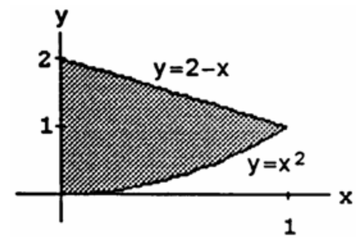


9. $a = 0, b = 1;$

$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x \left[(2-x) - x^2\right] dx$

$= 2\pi \int_0^1 \left(2x - x^2 - x^3\right) dx = 2\pi \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$

$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = 2\pi\left(\frac{12-4-3}{12}\right) = \frac{10\pi}{12} = \frac{5\pi}{6}$

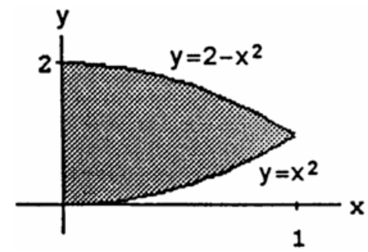


10. $a = 0, b = 1;$

$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x \left[\left(2 - x^2\right) - x^2\right] dx$

$= 2\pi \int_0^1 x\left(2 - 2x^2\right) dx = 4\pi \int_0^1 \left(x - x^3\right) dx$

$= 4\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 4\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \pi$

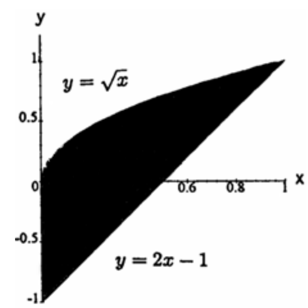


11. $a = 0, b = 1;$

$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x \left[\sqrt{x} - (2x-1)\right] dx$

$= 2\pi \int_0^1 \left(x^{3/2} - 2x^2 + x\right) dx = 2\pi \left[\frac{2}{5} x^{5/2} - \frac{2}{3} x^3 + \frac{1}{2} x^2\right]_0^1$

$= 2\pi\left(\frac{2}{5} - \frac{2}{3} + \frac{1}{2}\right) = 2\pi\left(\frac{12-20+15}{30}\right) = \frac{7\pi}{15}$

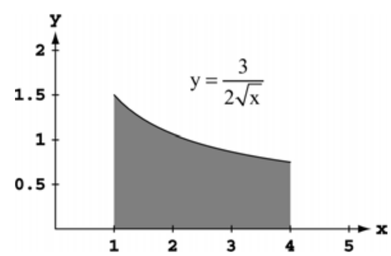


12. $a = 1, b = 4;$

$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_1^4 2\pi x \left(\frac{3}{2} x^{-1/2}\right) dx$

$= 3\pi \int_1^4 x^{1/2}\, dx = 3\pi \left[\frac{2}{3} x^{3/2}\right]_1^4 = 2\pi\left(4^{3/2} - 1\right)$

$= 2\pi(8-1) = 14\pi$



13. (a) $x f(x) = \begin{cases} x \cdot \frac{\sin x}{x}, & 0 < x \le \pi \\ x, & x = 0 \end{cases} \Rightarrow x f(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ 0, & x = 0 \end{cases}$; since $\sin 0 = 0$ we have

$x f(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ \sin x, & x = 0 \end{cases} \Rightarrow x f(x) = \sin x,\ 0 \le x \le \pi$

(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^\pi 2\pi x \cdot f(x)\, dx$ and $x \cdot f(x) = \sin x, 0 \le x \le \pi$ by part (a)

$\Rightarrow V = 2\pi \int_0^\pi \sin x\, dx = 2\pi\left[-\cos x\right]_0^\pi = 2\pi(-\cos \pi + \cos 0) = 4\pi$

14. (a) $xg(x) = \begin{cases} x \cdot \frac{\tan^2 x}{x}, & 0 < x \le \frac{\pi}{4} \\ x \cdot 0, & x = 0 \end{cases} \Rightarrow xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ 0, & x = 0 \end{cases}$; since $\tan 0 = 0$ we have

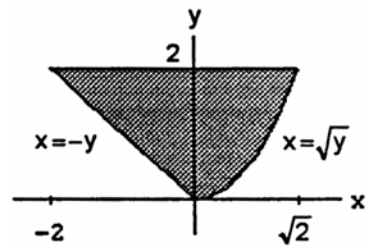
$$xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ \tan^2 x, & x = 0 \end{cases} \Rightarrow xg(x) = \tan^2 x, 0 \le x \le \pi/4$$

(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^{\pi/4} 2\pi x \cdot g(x)\, dx$ and $x \cdot g(x) = \tan^2 x, 0 \le x \le \pi/4$ by part (a)

$$\Rightarrow V = 2\pi \int_0^{\pi/4} \tan^2 x\, dx = 2\pi \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx = 2\pi \left[\tan x - x\right]_0^{\pi/4} = 2\pi\left(1 - \frac{\pi}{4}\right) = \frac{4\pi - \pi^2}{2}$$

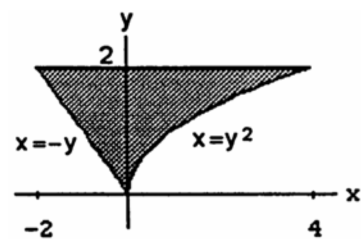
15. $c = 0, d = 2;$

$$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\left[\sqrt{y} - (-y)\right] dy$$
$$= 2\pi \int_0^2 \left(y^{3/2} + y^2\right) dy = 2\pi \left[\frac{2y^{5/2}}{5} + \frac{y^3}{3}\right]_0^2$$
$$= 2\pi \left[\frac{2}{5}\left(\sqrt{2}\right)^5 + \frac{2^3}{3}\right] = 2\pi\left(\frac{8\sqrt{2}}{5} + \frac{8}{3}\right) = 16\pi\left(\frac{\sqrt{2}}{5} + \frac{1}{3}\right)$$
$$= \frac{16\pi}{15}\left(3\sqrt{2} + 5\right)$$



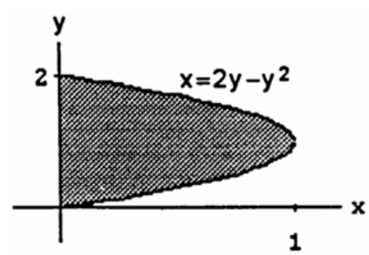
16. $c = 0, d = 2;$

$$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\left[y^2 - (-y)\right] dy$$
$$= 2\pi \int_0^2 \left(y^3 + y^2\right) dy = 2\pi \left[\frac{y^4}{4} + \frac{y^3}{3}\right]_0^2 = 16\pi\left(\frac{2}{4} + \frac{1}{3}\right)$$
$$= 16\pi\left(\frac{5}{6}\right) = \frac{40\pi}{3}$$



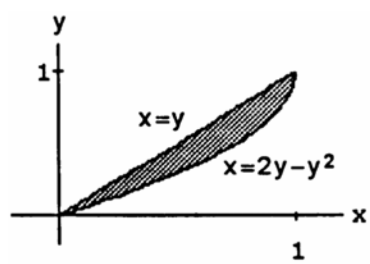
17. $c = 0, d = 2;$

$$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\left(2y - y^2\right) dy$$
$$= 2\pi \int_0^2 \left(2y^2 - y^3\right) dy = 2\pi \left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = 2\pi\left(\frac{16}{3} - \frac{16}{4}\right)$$
$$= 32\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{32\pi}{12} = \frac{8\pi}{3}$$



18. $c = 0, d = 1;$

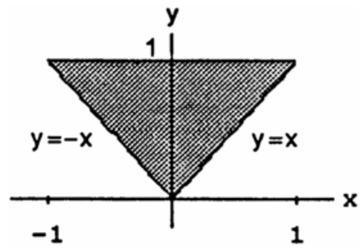
$$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^1 2\pi y\left(2y - y^2 - y\right) dy$$
$$= 2\pi \int_0^1 y\left(y - y^2\right) dy = 2\pi \int_0^1 \left(y^2 - y^3\right) dy$$
$$= 2\pi \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$$



19. $c=0, d=1;$

$V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy = 2\pi\int_0^1 y\left[y-(-y)\right]dy$

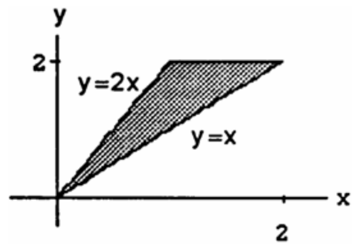
$=2\pi\int_0^1 2y^2\,dy=\frac{4\pi}{3}\left[y^3\right]_0^1=\frac{4\pi}{3}$



20. $c=0, d=2;$

$V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^2 2\pi\, y\left(y-\frac{y}{2}\right)dy$

$=2\pi\int_0^2 \frac{y^2}{2}\,dy=\frac{\pi}{3}\left[y^3\right]_0^2=\frac{8\pi}{3}$

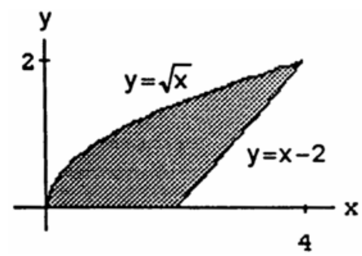


21. $c=0, d=2;$

$V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^2 2\pi y\left[(2+y)-y^2\right]dy$

$=2\pi\int_0^2\left(2y+y^2-y^3\right)dy=2\pi\left[y^2+\frac{y^3}{3}-\frac{y^4}{4}\right]_0^2$

$=2\pi\left(4+\frac{8}{3}-\frac{16}{4}\right)=\frac{\pi}{6}(48+32-48)=\frac{16\pi}{3}$

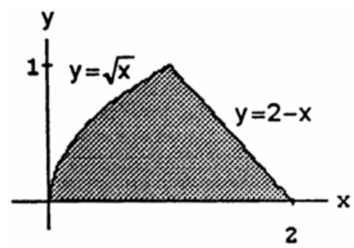


22. $c=0, d=1;$

$V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi y\left[(2-y)-y^2\right]dy$

$=2\pi\int_0^1\left(2y-y^2-y^3\right)dy=2\pi\left[y^2-\frac{y^3}{3}-\frac{y^4}{4}\right]_0^1$

$=2\pi\left(1-\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{6}(12-4-3)=\frac{5\pi}{6}$



23. (a) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^2 2\pi\, x\,(3x)dx=6\pi\int_0^2 x^2dx=2\pi\left[x^3\right]_0^2=16\pi$

(b) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^2 2\pi\,(4-x)\,(3x)dx=6\pi\int_0^2\left(4x-x^2\right)dx=6\pi\left[2x^2-\frac{1}{3}x^3\right]_0^2$

$=6\pi\left(8-\frac{8}{3}\right)=32\pi$

(c) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^2 2\pi(x+1)\,(3x)dx=6\pi\int_0^2\left(x^2+x\right)dx=6\pi\left[\frac{1}{3}x^3+\frac{1}{2}x^2\right]_0^2$

$=6\pi\left(\frac{8}{3}+2\right)=28\pi$

(d) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^6 2\pi\, y\left(2-\frac{1}{3}y\right)dy=2\pi\int_0^6\left(2y-\frac{1}{3}y^2\right)dy=2\pi\left[y^2-\frac{1}{9}y^3\right]_0^6$

$=2\pi(36-24)=24\pi$

(e) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^6 2\pi\,(7-y)\left(2-\frac{1}{3}y\right)dy=2\pi\int_0^6\left(14-\frac{13}{3}y+\frac{1}{3}y^2\right)dy$

$=2\pi\left[14y-\frac{13}{6}y^2+\frac{1}{9}y^3\right]_0^6=2\pi(84-78+24)=60\pi$

(f) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^6 2\pi\,(y+2)\left(2-\frac{1}{3}y\right)dy=2\pi\int_0^6\left(4+\frac{4}{3}y-\frac{1}{3}y^2\right)dy$

$=2\pi\left[4y+\frac{2}{3}y^2-\frac{1}{9}y^3\right]_0^6=2\pi(24+24-24)=48\pi$

24. (a) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^2 2\pi\,x\left(8-x^3\right)dx=2\pi\int_0^2\left(8x-x^4\right)dx=2\pi\left[4x^2-\frac{1}{5}x^5\right]_0^2$

$=2\pi\left(16-\frac{32}{5}\right)=\frac{96\pi}{5}$

(b) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^2 2\pi\,(3-x)\left(8-x^3\right)dx=2\pi\int_0^2\left(24-8x-3x^3+x^4\right)dx$

$=2\pi\left[24x-4x^2-\frac{3}{4}x^4+\frac{1}{5}x^5\right]_0^2=2\pi\left(48-16-12+\frac{32}{5}\right)=\frac{264\pi}{5}$

(c) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^2 2\pi\,(x+2)\left(8-x^3\right)dx=2\pi\int_0^2\left(16+8x-2x^3-x^4\right)dx$

$=2\pi\left[16x+4x^2-\frac{1}{2}x^4-\frac{1}{5}x^5\right]_0^2=2\pi\left(32+16-8-\frac{32}{5}\right)=\frac{336\pi}{5}$

(d) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^8 2\pi\,y\cdot y^{1/3}dy=2\pi\int_0^8 y^{4/3}dy=\frac{6\pi}{7}\left[y^{7/3}\right]_0^8=\frac{6\pi}{7}(128)=\frac{768\pi}{7}$

(e) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^8 2\pi\,(8-y)\,y^{1/3}dy=2\pi\int_0^8\left(8y^{1/3}-y^{4/3}\right)dy=2\pi\left[6y^{4/3}-\frac{3}{7}y^{7/3}\right]_0^8$

$=2\pi\left(96-\frac{384}{7}\right)=\frac{576\pi}{7}$

(f) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^8 2\pi\,(y+1)\,y^{1/3}dx=2\pi\int_0^8\left(y^{4/3}+y^{1/3}\right)dy=2\pi\left[\frac{3}{7}y^{7/3}+\frac{3}{4}y^{4/3}\right]_0^8$

$=2\pi\left(\frac{384}{7}+12\right)=\frac{936\pi}{7}$

25. (a) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_{-1}^2 2\pi\,(2-x)\left(x+2-x^2\right)dx=2\pi\int_{-1}^2\left(4-3x^2+x^3\right)dx$

$=2\pi\left[4x-x^3+\frac{1}{4}x^4\right]_{-1}^2=2\pi(8-8+4)-2\pi\left(-4+1+\frac{1}{4}\right)=\frac{27\pi}{2}$

(b) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_{-1}^2 2\pi\,(x+1)\left(x+2-x^2\right)dx=2\pi\int_{-1}^2\left(2+3x-x^3\right)dx$

$=2\pi\left[2x+\frac{3}{2}x^2-\frac{1}{4}x^4\right]_{-1}^2=2\pi\,(4+6-4)-2\pi\left(-2+\frac{3}{2}-\frac{1}{4}\right)=\frac{27\pi}{2}$

(c) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^1 2\pi\,y\left(\sqrt{y}-\left(-\sqrt{y}\right)\right)dy+\int_1^4 2\pi\,y\left(\sqrt{y}-(y-2)\right)dy$

$=4\pi\int_0^1 y^{3/2}dy+2\pi\int_1^4\left(y^{3/2}-y^2+2y\right)dy=\frac{8\pi}{5}\left[y^{5/2}\right]_0^1+2\pi\left[\frac{2}{5}y^{5/2}-\frac{1}{3}y^3+y^2\right]_1^4$

$=\frac{8\pi}{5}(1)+2\pi\left(\frac{64}{5}-\frac{64}{3}+16\right)-2\pi\left(\frac{2}{5}-\frac{1}{3}+1\right)=\frac{72\pi}{5}$

(d) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^1 2\pi\,(4-y)\left(\sqrt{y}-\left(-\sqrt{y}\right)\right)dy+\int_1^4 2\pi\,(4-y)\left(\sqrt{y}-(y-2)\right)dy$

$=4\pi\int_0^1\left(4\sqrt{y}-y^{3/2}\right)dy+2\pi\int_1^4\left(y^2-y^{3/2}-6y+4\sqrt{y}+8\right)dy$

$$= 4\pi\left[\tfrac{8}{3}y^{3/2} - \tfrac{2}{5}y^{5/2}\right]_0^1 + 2\pi\left[\tfrac{1}{3}y^3 - \tfrac{2}{5}y^{5/2} - 3y^2 + \tfrac{8}{3}y^{3/2} + 8y\right]_1^4$$

$$= 4\pi\left(\tfrac{8}{3} - \tfrac{2}{5}\right) + 2\pi\left(\tfrac{64}{3} - \tfrac{64}{5} - 48 + \tfrac{64}{3} + 32\right) - 2\pi\left(\tfrac{1}{3} - \tfrac{2}{5} - 3 + \tfrac{8}{3} + 8\right) = \tfrac{108\pi}{5}.$$

26. (a) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx = \int_{-1}^1 2\pi\,(1-x)\left(4-3x^2-x^4\right)dx = 2\pi\int_{-1}^1\left(x^5 - x^4 + 3x^3 - 3x^2 - 4x + 4\right)dx$

$= 2\pi\left[\tfrac{1}{6}x^6 - \tfrac{1}{5}x^5 + \tfrac{3}{4}x^4 - x^3 - 2x^2 + 4x\right]_{-1}^1 = 2\pi\left(\tfrac{1}{6} - \tfrac{1}{5} + \tfrac{3}{4} - 1 - 2 + 4\right) - 2\pi\left(\tfrac{1}{6} + \tfrac{1}{5} + \tfrac{3}{4} + 1 - 2 - 4\right) = \tfrac{56\pi}{5}$

(b) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^1 2\pi y\left(\sqrt[4]{y} - \left(-\sqrt[4]{y}\right)\right)dy + \int_1^4 2\pi\ y\left[\sqrt{\tfrac{4-y}{3}} - \left(-\sqrt{\tfrac{4-y}{3}}\right)\right]dy$

$= 4\pi\int_0^1 y^{5/4}dy + \tfrac{4\pi}{\sqrt{3}}\int_1^4 y\sqrt{4-y}\,dy \quad [u = 4-y \Rightarrow y = 4-u \Rightarrow du = -du;\ y = 1 \Rightarrow u = 3,\ y = 4 \Rightarrow u = 0]$

$= \tfrac{16\pi}{9}\left[y^{9/4}\right]_0^1 - \tfrac{4\pi}{\sqrt{3}}\int_3^0 (4-u)\sqrt{u}\,du = \tfrac{16\pi}{9}(1) + \tfrac{4\pi}{\sqrt{3}}\int_0^3\left(4\sqrt{u} - u^{3/2}\right)du = \tfrac{16\pi}{9} + \tfrac{4\pi}{\sqrt{3}}\left[\tfrac{8}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right]_0^3$

$= \tfrac{16\pi}{9} + \tfrac{4\pi}{\sqrt{3}}\left(8\sqrt{3} - \tfrac{18}{5}\sqrt{3}\right) = \tfrac{16\pi}{9} + \tfrac{88\pi}{5} = \tfrac{872\pi}{45}$

27. (a) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^1 2\pi y\cdot 12\left(y^2 - y^3\right)dy = 24\pi\int_0^1\left(y^3 - y^4\right)dy = 24\pi\left[\tfrac{y^4}{4} - \tfrac{y^5}{5}\right]_0^1$

$= 24\pi\left(\tfrac{1}{4} - \tfrac{1}{5}\right) = \tfrac{24\pi}{20} = \tfrac{6\pi}{5}$

(b) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^1 2\pi(1-y)\left[12\left(y^2 - y^3\right)\right]dy = 24\pi\int_0^1 (1-y)\left(y^2 - y^3\right)dy$

$= 24\pi\int_0^1\left(y^2 - 2y^3 + y^4\right)dy = 24\pi\left[\tfrac{y^3}{3} - \tfrac{y^4}{2} + \tfrac{y^5}{5}\right]_0^1 = 24\pi\left(\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5}\right) = 24\pi\left(\tfrac{1}{30}\right) = \tfrac{4\pi}{5}$

(c) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^1 2\pi\left(\tfrac{8}{5} - y\right)\left[12\left(y^2 - y^3\right)\right]dy = 24\pi\int_0^1\left(\tfrac{8}{5} - y\right)\left(y^2 - y^3\right)dy$

$= 24\pi\int_0^1\left(\tfrac{8}{5}y^2 - \tfrac{13}{5}y^3 + y^4\right)dy = 24\pi\left[\tfrac{8}{15}y^3 - \tfrac{13}{20}y^4 + \tfrac{y^5}{5}\right]_0^1 = 24\pi\left(\tfrac{8}{15} - \tfrac{13}{20} + \tfrac{1}{5}\right)$

$= \tfrac{24\pi}{60}(32 - 39 + 12) = \tfrac{24\pi}{12} = 2\pi$

(d) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^1 2\pi\left(y + \tfrac{2}{5}\right)\left[12\left(y^2 - y^3\right)\right]dy = 24\pi\int_0^1\left(y + \tfrac{2}{5}\right)\left(y^2 - y^3\right)dy$

$= 24\pi\int_0^1\left(y^3 - y^4 + \tfrac{2}{5}y^2 - \tfrac{2}{5}y^3\right)dy = 24\pi\int_0^1\left(\tfrac{2}{5}y^2 + \tfrac{3}{5}y^3 - y^4\right)dy = 24\pi\left[\tfrac{2}{15}y^3 + \tfrac{3}{20}y^4 - \tfrac{y^5}{5}\right]_0^1$

$= 24\pi\left(\tfrac{2}{15} + \tfrac{3}{20} - \tfrac{1}{5}\right) = \tfrac{24\pi}{60}(8 + 9 - 12) = \tfrac{24\pi}{12} = 2\pi$

28. (a) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^2 2\pi y\left[\tfrac{y^2}{2} - \left(\tfrac{y^4}{4} - \tfrac{y^2}{2}\right)\right]dy = \int_0^2 2\pi y\left(y^2 - \tfrac{y^4}{4}\right)dy = 2\pi\int_0^2\left(y^3 - \tfrac{y^5}{4}\right)dy$

$= 2\pi\left[\tfrac{y^4}{4} - \tfrac{y^6}{24}\right]_0^2 = 2\pi\left(\tfrac{2^4}{4} - \tfrac{2^6}{24}\right) = 32\pi\left(\tfrac{1}{4} - \tfrac{4}{24}\right) = 32\pi\left(\tfrac{1}{4} - \tfrac{1}{6}\right) = 32\pi\left(\tfrac{2}{24}\right) = \tfrac{8\pi}{3}$

(b) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^2 2\pi(2-y)\left[\tfrac{y^2}{2} - \left(\tfrac{y^4}{4} - \tfrac{y^2}{2}\right)\right]dy = \int_0^2 2\pi(2-y)\left(y^2 - \tfrac{y^4}{4}\right)dy$

$= 2\pi\int_0^2\left(2y^2 - \tfrac{y^4}{2} - y^3 + \tfrac{y^5}{4}\right)dy = 2\pi\left[\tfrac{2y^3}{3} - \tfrac{y^5}{10} - \tfrac{y^4}{4} + \tfrac{y^6}{24}\right]_0^2 = 2\pi\left(\tfrac{16}{3} - \tfrac{32}{10} - \tfrac{16}{4} + \tfrac{64}{24}\right) = \tfrac{8\pi}{5}$

(c) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^2 2\pi(5-y)\left[\frac{y^2}{2}-\left(\frac{y^4}{4}-\frac{y^2}{2}\right)\right]dy=\int_0^2 2\pi(5-y)\left(y^2-\frac{y^4}{4}\right)dy$

$=2\pi\int_0^2\left(5y^2-\frac{5}{4}y^4-y^3+\frac{y^5}{4}\right)dy=2\pi\left[\frac{5y^3}{3}-\frac{5y^5}{20}-\frac{y^4}{4}+\frac{y^6}{24}\right]_0^2=2\pi\left(\frac{40}{3}-\frac{160}{20}-\frac{16}{4}+\frac{64}{24}\right)=8\pi$

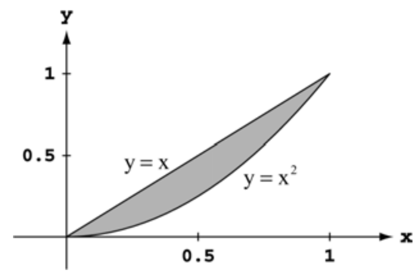
(d) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^2 2\pi\left(y+\frac{5}{8}\right)\left[\frac{y^2}{2}-\left(\frac{y^4}{4}-\frac{y^2}{2}\right)\right]dy=\int_0^2 2\pi\left(y+\frac{5}{8}\right)\left(y^2-\frac{y^4}{4}\right)dy$

$=2\pi\int_0^2\left(y^3-\frac{y^5}{4}+\frac{5}{8}y^2-\frac{5}{32}y^4\right)dy=2\pi\left[\frac{y^4}{4}-\frac{y^6}{24}+\frac{5y^3}{24}-\frac{5y^5}{160}\right]_0^2=2\pi\left(\frac{16}{4}-\frac{64}{24}+\frac{40}{24}-\frac{160}{160}\right)=4\pi$

29. (a) About x-axis: $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy$

$=\int_0^1 2\pi y\left(\sqrt{y}-y\right)dy=2\pi\int_0^1\left(y^{3/2}-y^2\right)dy$

$=2\pi\left[\frac{2}{5}y^{5/2}-\frac{1}{3}y^3\right]_0^1=2\pi\left(\frac{2}{5}-\frac{1}{3}\right)=\frac{2\pi}{15}$



About y-axis: $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx$

$=\int_0^1 2\pi x\left(x-x^2\right)dx=2\pi\int_0^1\left(x^2-x^3\right)dx$

$=2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=2\pi\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{6}$

(b) About x-axis: $R(x)=x$ and $r(x)=x^2\Rightarrow V=\int_a^b\pi\left([R(x)]^2-[r(x)]^2\right)dx=\int_0^1\pi\left(x^2-x^4\right)dx$

$=\pi\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1=\pi\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{2\pi}{15}$

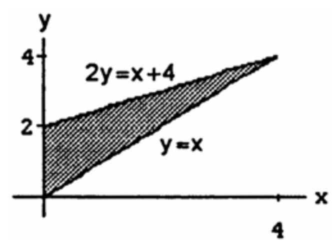
About y-axis: $R(y)=\sqrt{y}$ and $r(y)=y\Rightarrow V=\int_c^d\pi\left([R(y)]^2-[r(y)]^2\right)dy=\int_0^1\pi\left(y-y^2\right)dy$

$=\pi\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{\pi}{6}$

30. (a) $V=\int_a^b\pi\left([R(x)]^2-[r(x)]^2\right)dx=\pi\int_0^4\left[\left(\frac{x}{2}+2\right)^2-x^2\right]dx$

$=\pi\int_0^4\left(-\frac{3}{4}x^2+2x+4\right)dx=\pi\left[-\frac{x^3}{4}+x^2+4x\right]_0^4$

$=\pi(-16+16+16)=16\pi$



(b) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^4 2\pi x\left(\frac{x}{2}+2-x\right)dx=\int_0^4 2\pi x\left(2-\frac{x}{2}\right)dx=2\pi\int_0^4\left(2x-\frac{x^2}{2}\right)dx$

$=2\pi\left[x^2-\frac{x^3}{6}\right]_0^4=2\pi\left(16-\frac{64}{6}\right)=\frac{32\pi}{3}$

(c) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^4 2\pi(4-x)\left(\frac{x}{2}+2-x\right)dx=\int_0^4 2\pi(4-x)\left(2-\frac{x}{2}\right)dx=2\pi\int_0^4\left(8-4x-\frac{x^2}{2}\right)dx$

$=2\pi\left[8x-2x^2+\frac{x^3}{6}\right]_0^4=2\pi\left(32-32+\frac{64}{6}\right)=\frac{64\pi}{3}$

(d) $V=\int_a^b \pi\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx=\pi\int_0^4\left[(8-x)^2-\left(6-\frac{x}{2}\right)^2\right]dx=\pi\int_0^4\left[\left(64-16x+x^2\right)-\left(36-6x+\frac{x^2}{4}\right)\right]dx$

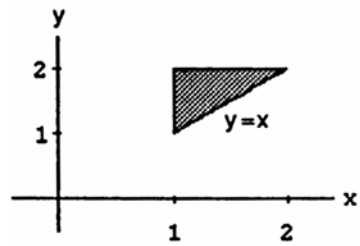
$\pi\int_0^4\left(\frac{3}{4}x^2-10x+28\right)dx=\pi\left[\frac{x^3}{4}-5x^2+28x\right]_0^4=\pi[16-(5)(16)+(7)(16)]=\pi(3)(16)=48\pi$

31. (a) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_1^2 2\pi y(y-1)\,dy$

$=2\pi\int_1^2\left(y^2-y\right)dy=2\pi\left[\frac{y^3}{3}-\frac{y^2}{2}\right]_1^2$

$=2\pi\left[\left(\frac{8}{3}-\frac{4}{2}\right)-\left(\frac{1}{3}-\frac{1}{2}\right)\right]$

$=2\pi\left(\frac{7}{3}-2+\frac{1}{2}\right)=\frac{\pi}{3}(14-12+3)=\frac{5\pi}{3}$



(b) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_1^2 2\pi x(2-x)\,dx=2\pi\int_1^2\left(2x-x^2\right)dx=2\pi\left[x^2-\frac{x^3}{3}\right]_1^2$

$=2\pi\left[\left(4-\frac{8}{3}\right)-\left(1-\frac{1}{3}\right)\right]=2\pi\left[\left(\frac{12-8}{3}\right)-\left(\frac{3-1}{3}\right)\right]=2\pi\left(\frac{4}{3}-\frac{2}{3}\right)=\frac{4\pi}{3}$

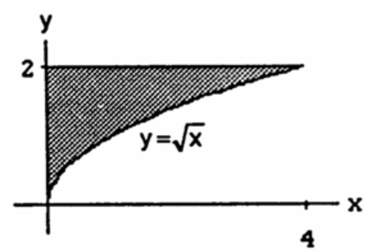
(c) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_1^2 2\pi\left(\frac{10}{3}-x\right)(2-x)\,dx=2\pi\int_1^2\left(\frac{20}{3}-\frac{16}{3}x+x^2\right)dx$

$=2\pi\left[\frac{20}{3}x-\frac{8}{3}x^2+\frac{1}{3}x^3\right]_1^2=2\pi\left[\left(\frac{40}{3}-\frac{32}{3}+\frac{8}{3}\right)-\left(\frac{20}{3}-\frac{8}{3}+\frac{1}{3}\right)\right]=2\pi\left(\frac{3}{3}\right)=2\pi$

(d) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_1^2 2\pi(y-1)(y-1)\,dy=2\pi\int_1^2(y-1)^2=2\pi\left[\frac{(y-1)^3}{3}\right]_1^2=\frac{2\pi}{3}$

32. (a) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^2 2\pi y\left(y^2-0\right)dy$

$=2\pi\int_0^2 y^3dy=2\pi\left[\frac{y^4}{4}\right]_0^2=2\pi\left(\frac{2^4}{4}\right)=8\pi$



(b) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx$

$=\int_0^4 2\pi x\left(2-\sqrt{x}\right)dx=2\pi\int_0^4\left(2x-x^{3/2}\right)dx$

$=2\pi\left[x^2-\frac{2}{5}x^{5/2}\right]_0^4=2\pi\left(16-\frac{2\cdot 2^5}{5}\right)$

$=2\pi\left(16-\frac{64}{5}\right)=\frac{2\pi}{5}(80-64)=\frac{32\pi}{5}$

(c) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^4 2\pi(4-x)\left(2-\sqrt{x}\right)dx=2\pi\int_0^4\left(8-4x^{1/2}-2x+x^{3/2}\right)dx$

$=2\pi\left[8x-\frac{8}{3}x^{3/2}-x^2+\frac{2}{5}x^{5/2}\right]_0^4=2\pi\left(32-\frac{64}{3}-16+\frac{64}{5}\right)=\frac{2\pi}{15}(240-320+192)=\frac{2\pi}{15}(112)=\frac{224\pi}{15}$

(d) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^2 2\pi(2-y)\left(y^2\right)dy=2\pi\int_0^2\left(2y^2-y^3\right)dy=2\pi\left[\frac{2}{3}y^3-\frac{y^4}{4}\right]_0^2$

$=2\pi\left(\frac{16}{3}-\frac{16}{4}\right)=\frac{32\pi}{12}(4-3)=\frac{8\pi}{3}$

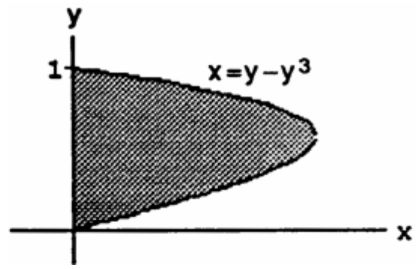
33. (a) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi y\left(y-y^3\right)dy$

$=\int_0^1 2\pi\left(y^2-y^4\right)dy=2\pi\left[\frac{y^3}{3}-\frac{y^5}{5}\right]_0^1=2\pi\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{4\pi}{15}$

(b) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi(1-y)\left(y-y^3\right)dy$

$=2\pi\int_0^1\left(y-y^2-y^3+y^4\right)dy=2\pi\left[\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}\right]_0^1$

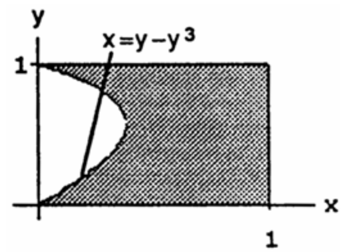
$=2\pi\left(\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\right)=\frac{2\pi}{60}(30-20-15+12)=\frac{7\pi}{30}$



34. (a) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi y\left[1-\left(y-y^3\right)\right]dy$

$=2\pi\int_0^1\left(y-y^2+y^4\right)dy=2\pi\left[\frac{y^2}{2}-\frac{y^3}{3}+\frac{y^5}{5}\right]_0^1$

$=2\pi\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{5}\right)=\frac{2\pi}{30}(15-10+6)=\frac{11\pi}{15}$



(b) Use the washer method:

$V=\int_c^d \pi\left([R(y)]^2-[r(y)]^2\right)dy=\int_0^1\pi\left[1^2-\left(y-y^3\right)^2\right]dy=\pi\int_0^1\left(1-y^2-y^6+2y^4\right)dy$

$=\pi\left[y-\frac{y^3}{3}-\frac{y^7}{7}+\frac{2y^5}{5}\right]_0^1=\pi\left(1-\frac{1}{3}-\frac{1}{7}+\frac{2}{5}\right)=\frac{\pi}{105}(105-35-15+42)=\frac{97\pi}{105}$

(c) Use the washer method:

$V=\int_c^d \pi\left([R(y)]^2-[r(y)]^2\right)dy=\int_0^1\pi\left[\left[1-\left(y-y^3\right)\right]^2-0\right]dy=\pi\int_0^1\left[1-2\left(y-y^3\right)+\left(y-y^3\right)^2\right]dy$

$=\pi\int_0^1\left(1+y^2+y^6-2y+2y^3-2y^4\right)dy=\pi\left[y+\frac{y^3}{3}+\frac{y^7}{7}-y^2+\frac{y^4}{2}-\frac{2y^5}{5}\right]_0^1$

$=\pi\left(1+\frac{1}{3}+\frac{1}{7}-1+\frac{1}{2}-\frac{2}{5}\right)=\frac{\pi}{210}(70+30+105-2\cdot 42)=\frac{121\pi}{210}$

(d) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi(1-y)\left[1-\left(y-y^3\right)\right]dy=2\pi\int_0^1(1-y)\left(1-y+y^3\right)dy$

$=2\pi\int_0^1\left(1-y+y^3-y+y^2-y^4\right)dy=2\pi\int_0^1\left(1-2y+y^2+y^3-y^4\right)dy=2\pi\left[y-y^2+\frac{y^3}{3}+\frac{y^4}{4}-\frac{y^5}{5}\right]_0^1$

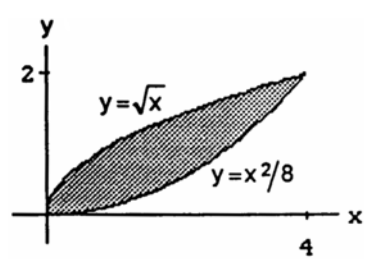
$=2\pi\left(1-1+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}\right)=\frac{2\pi}{60}(20+15-12)=\frac{23\pi}{30}$

35. (a) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi y\left(\sqrt{8y}-y^2\right)dy$

$=2\pi\int_0^2\left(2\sqrt{2}y^{3/2}-y^3\right)dy=2\pi\left[\frac{4\sqrt{2}}{5}y^{5/2}-\frac{y^4}{4}\right]_0^2$

$=2\pi\left(\frac{4\sqrt{2}\cdot\left(\sqrt{2}\right)^5}{5}-\frac{2^4}{4}\right)=2\pi\left(\frac{4\cdot 2^3}{5}-\frac{4\cdot 4}{4}\right)$

$=2\pi\cdot 4\left(\frac{8}{5}-1\right)=\frac{8\pi}{5}(8-5)=\frac{24\pi}{5}$



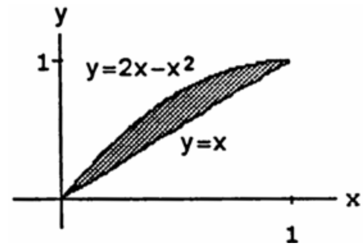
(b) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^4 2\pi x\left(\sqrt{x}-\frac{x^2}{8}\right)dx=2\pi\int_0^4\left(x^{3/2}-\frac{x^3}{8}\right)dx=2\pi\left[\frac{2}{5}x^{5/2}-\frac{x^4}{32}\right]_0^4$

$=2\pi\left(\frac{2\cdot 2^5}{5}-\frac{4^4}{32}\right)=2\pi\left(\frac{2^6}{5}-\frac{2^8}{32}\right)=\frac{\pi\cdot 2^7}{160}(32-20)=\frac{\pi\cdot 2^9\cdot 3}{160}=\frac{\pi\cdot 2^4\cdot 3}{5}=\frac{48\pi}{5}$

36. (a) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^1 2\pi x\left[\left(2x-x^2\right)-x\right]dx$

$=2\pi\int_0^1 x\left(x-x^2\right)dx=2\pi\int_0^1\left(x^2-x^3\right)dx$

$=2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=2\pi\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{6}$



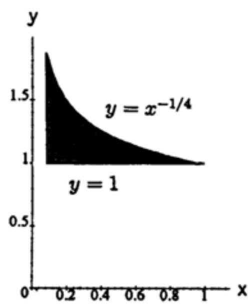
(b) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^1 2\pi(1-x)\left[\left(2x-x^2\right)-x\right]dx=2\pi\int_0^1(1-x)\left(x-x^2\right)dx$

$=2\pi\int_0^1\left(x-2x^2+x^3\right)dx=2\pi\left[\frac{x^2}{2}-\frac{2}{3}x^3+\frac{x^4}{4}\right]_0^1=2\pi\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)=\frac{2\pi}{12}(6-8+3)=\frac{\pi}{6}$

37. (a) $V=\int_a^b \pi\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx=\pi\int_{1/16}^1\left(x^{-1/2}-1\right)dx$

$=\pi\left[2x^{1/2}-x\right]_{1/16}^1=\pi\left[(2-1)-\left(2\cdot\frac{1}{4}-\frac{1}{16}\right)\right]$

$=\pi\left(1-\frac{7}{16}\right)=\frac{9\pi}{16}$



(b) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi y\left(\frac{1}{y^4}-\frac{1}{16}\right)dy$

$=2\pi\int_1^2\left(y^{-3}-\frac{y}{16}\right)dy=2\pi\left[-\frac{1}{2}y^{-2}-\frac{y^2}{32}\right]_1^2$

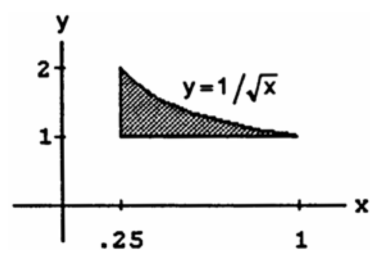
$=2\pi\left[\left(-\frac{1}{8}-\frac{1}{8}\right)-\left(-\frac{1}{2}-\frac{1}{32}\right)\right]=2\pi\left(\frac{1}{4}+\frac{1}{32}\right)$

$=\frac{2\pi}{32}(8+1)=\frac{9\pi}{16}$

38. (a) $V=\int_c^d \pi\left(\left[R(y)\right]^2-\left[r(y)\right]^2\right)dy=\int_1^2 \pi\left(\frac{1}{y^4}-\frac{1}{16}\right)dy$

$=\pi\left[-\frac{1}{3}y^{-3}-\frac{y}{16}\right]_1^2=\pi\left[\left(-\frac{1}{24}-\frac{1}{8}\right)-\left(-\frac{1}{3}-\frac{1}{16}\right)\right]$

$=\frac{\pi}{48}(-2-6+16+3)=\frac{11\pi}{48}$



(b) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_{1/4}^1 2\pi\left(\frac{1}{\sqrt{x}}-1\right)dx=2\pi\int_{1/4}^1\left(x^{1/2}-x\right)dx=2\pi\left[\frac{2}{3}x^{3/2}-\frac{x^2}{2}\right]_{1/4}^1$

$=2\pi\left[\left(\frac{2}{3}-\frac{1}{2}\right)-\left(\frac{2}{3}\cdot\frac{1}{8}-\frac{1}{32}\right)\right]=\pi\left(\frac{4}{3}-1-\frac{1}{6}+\frac{1}{16}\right)=\frac{\pi}{48}(4\cdot 16-48-8+3)=\frac{11\pi}{48}$

39. (a) *Disk*: $V = V_1 - V_2$

$V_1 = \int_{a_1}^{b_1} \pi\left[R_1(x)\right]^2 dx$ and $V_2 = \int_{a_2}^{b_2} \pi\left[R_2(x)\right]^2 dx$ with $R_1(x) = \sqrt{\frac{x+2}{3}}$ and $R_2(x) = \sqrt{x}$,

$a_1 = -2, b_1 = 1; a_2 = 0, b_2 = 1 \Rightarrow$ two integrals are required

(b) *Washer*: $V = V_1 + V_2$

$V_1 = \int_{a_1}^{b_1} \pi\left(\left[R_1(x)\right]^2 - \left[r_1(x)\right]^2\right) dx$ with $R_1(x) = \sqrt{\frac{x+2}{3}}$ and $r_1(x) = 0; a_1 = -2$ and $b_1 = 0$;

$V_2 = \int_{a_2}^{b_2} \pi\left(\left[R_2(x)\right]^2 - \left[r_2(x)\right]^2\right) dx$ with $R_2(x) = \sqrt{\frac{x+2}{3}}$ and $r_2(x) = \sqrt{x}; a_2 = 0$ and $b_2 = 1$

$\Rightarrow$ two integrals are required

(c) *Shell*: $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_c^d 2\pi y\binom{\text{shell}}{\text{height}} dy$ where shell height $= y^2 - \left(3y^2 - 2\right) = 2 - 2y^2$;

$c = 0$ and $d = 1$. Only *one* integral is required. It is, therefore preferable to use the *shell* method. However, whichever method you use, you will get $V = \pi$.

40. (a) *Disk*: $V = V_1 - V_2 - V_3$

$V_i = \int_{c_i}^{d_i} \pi\left[R_i(y)\right]^2 dy,\ \ i = 1, 2, 3$ with $R_1(y) = 1$ and $c_1 = -1, d_1 = 1; R_2(y) = \sqrt{y}$ and $c_2 = 0$ and $d_2 = 1$;

$R_3(y) = (-y)^{1/4}$ and $c_3 = -1, d_3 = 0 \Rightarrow$ three integrals are required

(b) *Washer*: $V = V_1 + V_2$

$V_i = \int_{c_i}^{d_i} \pi\left(\left[R_i(y)\right]^2 - \left[r_i(y)\right]^2\right) dy,\ \ i = 1, 2$ with $R_1(y) = 1, r_1(y) = \sqrt{y}, c_1 = 0$ and $d_1 = 1$;

$R_2(y) = 1,\ \ r_2(y) = (-y)^{1/4},\ \ c_2 = -1$ and $d_2 = 0 \Rightarrow$ two integrals are required

(c) *Shell*: $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_a^b 2\pi x\binom{\text{shell}}{\text{height}} dx$, where shell height $= x^2 - \left(-x^4\right) = x^2 + x^4,\ \ a = 0$

and $b = 1 \Rightarrow$ only one integral is required. It is, therefore preferable to use the *shell* method. However, whichever method you use, you will get $V = \frac{5\pi}{6}$.

41. (a) $V = \int_a^b \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right) dx = \int_{-4}^{4} \pi\left[\left(\sqrt{25 - x^2}\right)^2 - (3)^2\right] dx = \pi\int_{-4}^{4}\left(25 - x^2 - 9\right) dx = \pi\int_{-4}^{4}\left(16 - x^2\right) dx$

$= \pi\left[16x - \frac{1}{3}x^3\right]_{-4}^{4} = \pi\left(64 - \frac{64}{3}\right) - \pi\left(-64 + \frac{64}{3}\right) = \frac{256\pi}{3}$

(b) Volume of sphere $= \frac{4}{3}\pi(5)^3 = \frac{500\pi}{3} \Rightarrow$ Volume of portion removed $= \frac{500\pi}{3} - \frac{256\pi}{3} = \frac{244\pi}{3}$

42. $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_1^{\sqrt{1+\pi}} 2\pi\ x \sin\left(x^2 - 1\right) dx;\ \ [u = x^2 - 1 \Rightarrow du = 2x\ dx;$

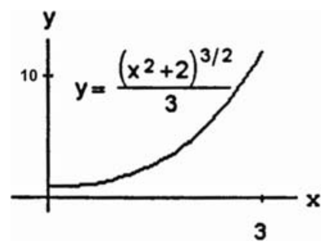
$x = 1 \Rightarrow u = 0, x = \sqrt{1+\pi} \Rightarrow u = \pi] \rightarrow \pi\int_0^{\pi} \sin u\ du = -\pi\left[\cos u\right]_0^{\pi} = -\pi(-1-1) = 2\pi$

43. $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^r 2\pi\, x\left(-\frac{h}{r}x+h\right)dx=2\pi\int_0^r\left(-\frac{h}{r}x^2+h\,x\right)dx=2\pi\left[-\frac{h}{3r}x^3+\frac{h}{2}x^2\right]_0^r$
$=2\pi\left(-\frac{r^2h}{3}+\frac{r^2h}{2}\right)=\frac{1}{3}\pi\, r^2h$

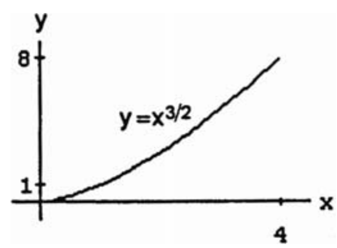
44. $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^r 2\pi\, y\left[\sqrt{r^2-y^2}-\left(-\sqrt{r^2-y^2}\right)\right]dy=4\pi\int_0^r y\sqrt{r^2-y^2}\,dy$
$[u=r^2-y^2\Rightarrow du=-2y\,dy;\ y=0\Rightarrow u=r^2,\ y=r\Rightarrow u=0]\to -2\pi\int_{r^2}^0\sqrt{u}\,du=2\pi\int_0^{r^2}u^{1/2}du=\frac{4\pi}{3}\left[u^{3/2}\right]_0^{r^2}$
$=\frac{4\pi}{3}r^3$

6.3 ARC LENGTH

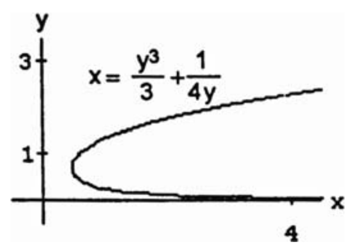
1. $\frac{dy}{dx}=\frac{1}{3}\cdot\frac{3}{2}\left(x^2+2\right)^{1/2}\cdot 2x=\sqrt{\left(x^2+2\right)}\cdot x$
$\Rightarrow L=\int_0^3\sqrt{1+\left(x^2+2\right)x^2}\,dx=\int_0^3\sqrt{1+2x^2+x^4}\,dx$
$=\int_0^3\sqrt{\left(1+x^2\right)^2}\,dx=\int_0^3\left(1+x^2\right)dx=\left[x+\frac{x^3}{3}\right]_0^3$
$=3+\frac{27}{3}=12$



2. $\frac{dy}{dx}=\frac{3}{2}\sqrt{x}\Rightarrow L=\int_0^4\sqrt{1+\frac{9}{4}x}\,dx;$
$\left[u=1+\frac{9}{4}x\Rightarrow du=\frac{9}{4}dx\Rightarrow\frac{4}{9}du=dx;\right.$
$x=0\Rightarrow u=1;\ x=4\Rightarrow u=10]$
$\to L=\int_1^{10}u^{1/2}\left(\frac{4}{9}du\right)=\frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10}=\frac{8}{27}\left(10\sqrt{10}-1\right)$



3. $\frac{dx}{dy}=y^2-\frac{1}{4y^2}\Rightarrow\left(\frac{dx}{dy}\right)^2=y^4-\frac{1}{2}+\frac{1}{16y^4}$
$\Rightarrow L=\int_1^3\sqrt{1+y^4-\frac{1}{2}+\frac{1}{16y^4}}\,dy=\int_1^3\sqrt{y^4+\frac{1}{2}+\frac{1}{16y^4}}\,dy$
$=\int_1^3\sqrt{\left(y^2+\frac{1}{4y^2}\right)^2}\,dy=\int_1^3\left(y^2+\frac{1}{4y^2}\right)dy$
$=\left[\frac{y^3}{3}-\frac{y^{-1}}{4}\right]_1^3=\left(\frac{27}{3}-\frac{1}{12}\right)-\left(\frac{1}{3}-\frac{1}{4}\right)=9-\frac{1}{12}-\frac{1}{3}+\frac{1}{4}$
$=9+\frac{(-1-4+3)}{12}=9+\frac{(-2)}{12}=\frac{53}{6}$



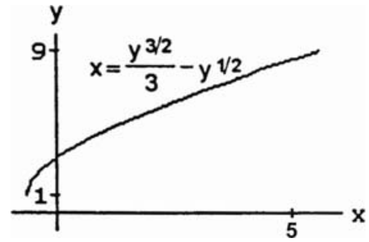
4. $\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)$

$\Rightarrow L = \int_1^9 \sqrt{1 + \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}\, dy = \int_1^9 \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}\, dy$

$= \int_1^9 \frac{1}{2}\sqrt{\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2}\, dy = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right) dy$

$= \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \left[\frac{y^{3/2}}{3} + y^{1/2}\right]_1^9$

$= \left(\frac{3^3}{3} + 3\right) - \left(\frac{1}{3} + 1\right) = 11 - \frac{1}{3} = \frac{32}{3}$



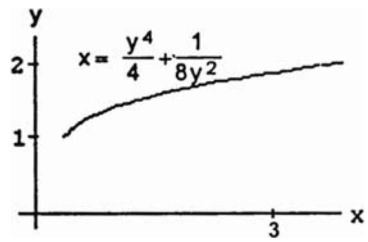
5. $\frac{dx}{dy} = y^3 - \frac{1}{4y^3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^6 - \frac{1}{2} + \frac{1}{16y^6}$

$\Rightarrow L = \int_1^2 \sqrt{1 + y^6 - \frac{1}{2} + \frac{1}{16y^6}}\, dy = \int_1^2 \sqrt{y^6 + \frac{1}{2} + \frac{1}{16y^6}}\, dy$

$= \int_1^2 \sqrt{\left(y^3 + \frac{y^{-3}}{4}\right)^2}\, dy = \int_1^2 \left(y^3 + \frac{y^{-3}}{4}\right) dy = \left[\frac{y^4}{4} - \frac{y^{-2}}{8}\right]_1^2$

$= \left(\frac{16}{4} - \frac{1}{(16)(2)}\right) - \left(\frac{1}{4} - \frac{1}{8}\right) = 4 - \frac{1}{32} - \frac{1}{4} + \frac{1}{8}$

$= \frac{128-1-8+4}{32} = \frac{123}{32}$



6. $\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y^4 - 2 + y^{-4}\right)$

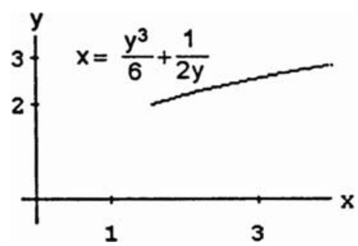
$\Rightarrow L = \int_2^3 \sqrt{1 + \frac{1}{4}\left(y^4 - 2 + y^{-4}\right)}\, dy$

$= \int_2^3 \sqrt{\frac{1}{4}\left(y^4 + 2 + y^{-4}\right)}\, dy$

$= \frac{1}{2}\int_2^3 \sqrt{\left(y^2 + y^{-2}\right)^2}\, dy = \frac{1}{2}\int_2^3 \left(y^2 + y^{-2}\right) dy$

$= \frac{1}{2}\left[\frac{y^3}{3} - y^{-1}\right]_2^3 = \frac{1}{2}\left[\left(\frac{27}{3} - \frac{1}{3}\right) - \left(\frac{8}{3} - \frac{1}{2}\right)\right]$

$= \frac{1}{2}\left(\frac{26}{3} - \frac{8}{3} + \frac{1}{2}\right) = \frac{1}{2}\left(6 + \frac{1}{2}\right) = \frac{13}{4}$



7. $\frac{dy}{dx} = x^{1/3} - \frac{1}{4}x^{-1/3} \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^{2/3} - \frac{1}{2} + \frac{x^{-2/3}}{16}$

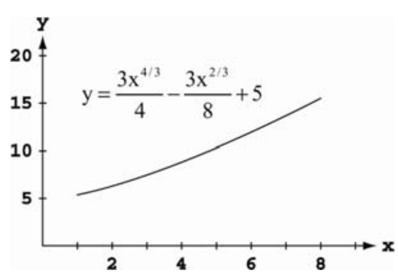
$\Rightarrow L = \int_1^8 \sqrt{1 + x^{2/3} - \frac{1}{2} + \frac{x^{-2/3}}{16}}\, dx$

$= \int_1^8 \sqrt{x^{2/3} + \frac{1}{2} + \frac{x^{-2/3}}{16}}\, dx = \int_1^8 \sqrt{\left(x^{1/3} + \frac{1}{4}x^{-1/3}\right)^2}\, dx$

$= \int_1^8 \left(x^{1/3} + \frac{1}{4}x^{-1/3}\right) dx = \left[\frac{3}{4}x^{4/3} + \frac{3}{8}x^{2/3}\right]_1^8$

$= \frac{3}{8}\left[2x^{4/3} + x^{2/3}\right]_1^8 = \frac{3}{8}\left[\left(2 \cdot 2^4 + 2^2\right) - (2+1)\right]$

$= \frac{3}{8}(32 + 4 - 3) = \frac{99}{8}$



8. $\frac{dy}{dx} = x^2 + 2x + 1 - \frac{4}{(4x+4)^2} = x^2 + 2x + 1 - \frac{1}{4}\frac{1}{(1+x)^2}$

$= (1+x)^2 - \frac{1}{4}\frac{1}{(1+x)^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = (1+x)^4 - \frac{1}{2} + \frac{1}{16(1+x)^4}$

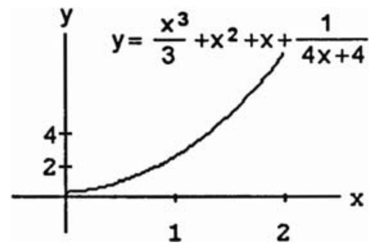
$\Rightarrow L = \int_0^2 \sqrt{1 + (1+x)^4 - \frac{1}{2} + \frac{(1+x)^{-4}}{16}}\, dx$

$= \int_0^2 \sqrt{(1+x)^4 + \frac{1}{2} + \frac{(1+x)^{-4}}{16}}\, dx$

$= \int_0^2 \sqrt{\left[(1+x)^2 + \frac{(1+x)^{-2}}{4}\right]^2}\, dx$

$= \int_0^2 \left[(1+x)^2 + \frac{(1+x)^{-2}}{4}\right] dx$; $[u = 1+x \Rightarrow du = dx;\ x = 0 \Rightarrow u = 1,\ x = 2 \Rightarrow u = 3] \rightarrow L = \int_1^3 \left(u^2 + \frac{1}{4}u^{-2}\right) du$

$= \left[\frac{u^3}{3} - \frac{1}{4}u^{-1}\right]_1^3 = \left(9 - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{108-1-4+3}{12} = \frac{106}{12} = \frac{53}{6}$

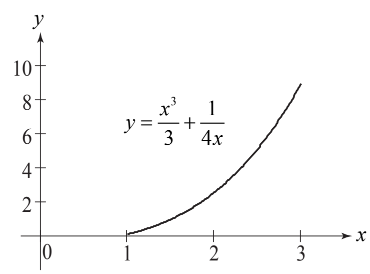


9. $\frac{dy}{dx} = x^2 - \frac{1}{4x^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(x^2 - \frac{1}{4x^2}\right)^2 = x^4 - \frac{1}{2} + \frac{1}{16x^4}$

$\Rightarrow L = \int_1^3 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}}\, dx =$

$\int_1^3 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\, dx = \int_1^3 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx =$

$\int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^3 = \left(9 + \frac{1}{12}\right) - \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{53}{6}$



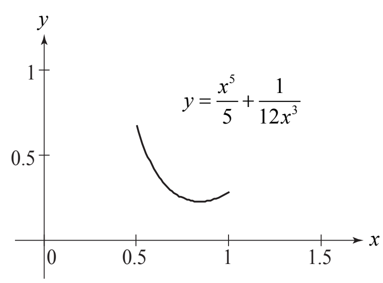
10. $\frac{dy}{dx} = x^4 - \frac{1}{4x^4} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(x^4 - \frac{1}{4x^4}\right)^2 = x^8 - \frac{1}{2} + \frac{1}{16x^8}$

$\Rightarrow L = \int_{1/2}^1 \sqrt{1 + x^8 - \frac{1}{2} + \frac{1}{16x^8}}\, dx =$

$\int_{1/2}^1 \sqrt{x^8 + \frac{1}{2} + \frac{1}{16x^8}}\, dx = \int_{1/2}^1 \sqrt{\left(x^4 + \frac{1}{4x^4}\right)^2}\, dx =$

$\int_{1/2}^1 \left(x^4 + \frac{1}{4x^4}\right) dx = \left[\frac{x^5}{5} - \frac{1}{12x^3}\right]_{1/2}^1 =$

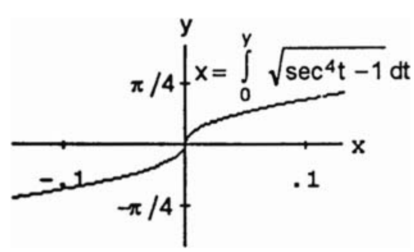
$\left(\frac{1}{5} - \frac{1}{12}\right) - \left(\frac{1}{160} - \frac{2}{3}\right) = \frac{373}{480}$



11. $\frac{dx}{dy} = \sqrt{\sec^4 y - 1} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^4 y - 1$

$\Rightarrow L = \int_{-\pi/4}^{\pi/4} \sqrt{1 + \left(\sec^4 y - 1\right)}\, dy = \int_{-\pi/4}^{\pi/4} \sec^2 y\, dy$

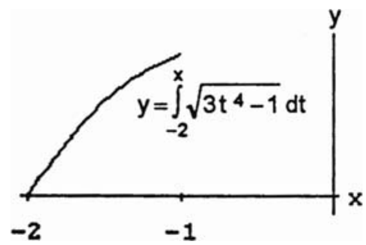
$= \left[\tan y\right]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$



12. $\frac{dy}{dx} = \sqrt{3x^4 - 1} \Rightarrow \left(\frac{dy}{dx}\right)^2 = 3x^4 - 1$

$\Rightarrow L = \int_{-2}^{-1} \sqrt{1 + \left(3x^4 - 1\right)}\, dx = \int_{-2}^{-1} \sqrt{3}\; x^2 dx$

$= \sqrt{3}\left[\frac{x^3}{3}\right]_{-2}^{-1} = \frac{\sqrt{3}}{3}\left[-1 - (-2)^3\right] = \frac{\sqrt{3}}{3}(-1 + 8) = \frac{7\sqrt{3}}{3}$



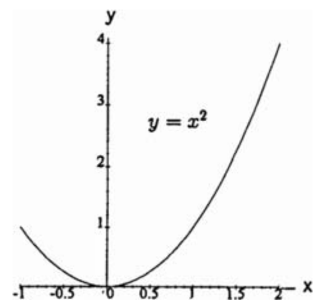
13. (a) $\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)^2 = 4x^2$

$\Rightarrow L = \int_{-1}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_{-1}^{2} \sqrt{1 + 4x^2}\, dx$

(c) $L \approx 6.13$

(b)

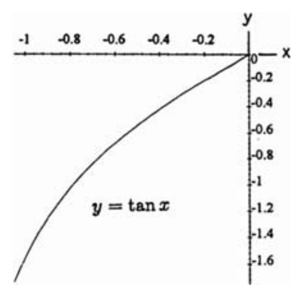


14. (a) $\frac{dy}{dx} = \sec^2 x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \sec^4 x$

$\Rightarrow L = \int_{-\pi/3}^{0} \sqrt{1 + \sec^4 x}\, dx$

(c) $L \approx 2.06$

(b)

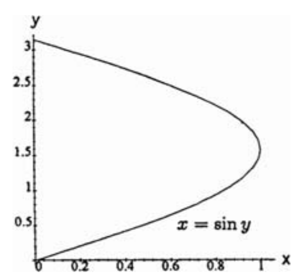


15. (a) $\frac{dx}{dy} = \cos y \Rightarrow \left(\frac{dx}{dy}\right)^2 = \cos^2 y$

$\Rightarrow L = \int_{0}^{\pi} \sqrt{1 + \cos^2 y}\, dy$

(c) $L \approx 3.82$

(b)



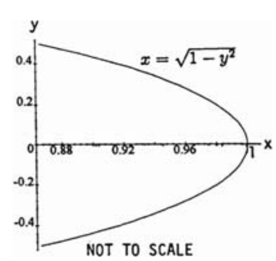
16. (a) $\frac{dx}{dy} = -\frac{y}{\sqrt{1-y^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{1-y^2}$

$\Rightarrow L = \int_{-1/2}^{1/2} \sqrt{1 + \frac{y^2}{1-y^2}}\, dy = \int_{-1/2}^{1/2} \sqrt{\frac{1}{1-y^2}}\, dy$

$= \int_{-1/2}^{1/2} \left(1 - y^2\right)^{-1/2} dy$

(c) $L \approx 1.05$

(b)

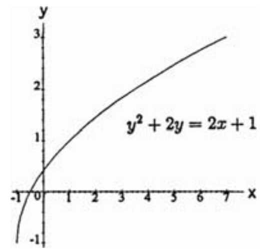


17. (a) $2y+2=2\frac{dx}{dy} \Rightarrow \left(\frac{dx}{dy}\right)^2=(y+1)^2$

$\Rightarrow L=\int_{-1}^{3}\sqrt{1+(y+1)^2}\,dy$

(c) $L \approx 9.29$

(b)

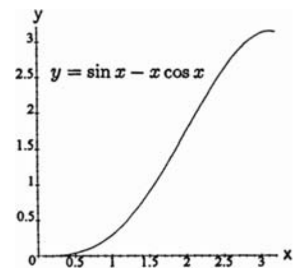


18. (a) $\frac{dy}{dx}=\cos x-\cos x+x\sin x \Rightarrow \left(\frac{dy}{dx}\right)^2=x^2\sin^2 x$

$\Rightarrow L=\int_0^{\pi}\sqrt{1+x^2\sin^2 x}\;dx$

(c) $L \approx 4.70$

(b)



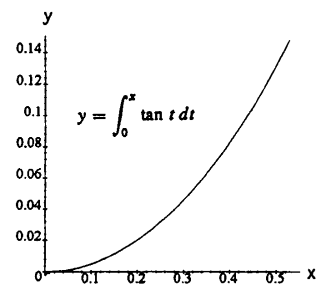
19. (a) $\frac{dy}{dx}=\tan x \Rightarrow \left(\frac{dy}{dx}\right)^2=\tan^2 x$

$\Rightarrow L=\int_0^{\pi/6}\sqrt{1+\tan^2 x}\;dx=\int_0^{\pi/6}\sqrt{\frac{\sin^2 x+\cos^2 x}{\cos^2 x}}\;dx$

$=\int_0^{\pi/6}\frac{dx}{\cos x}=\int_0^{\pi/6}\sec x\;dx$

(c) $L \approx 0.55$

(b)



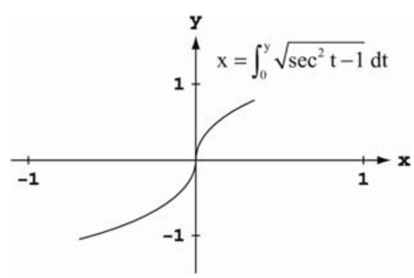
20. (a) $\frac{dx}{dy}=\sqrt{\sec^2 y-1} \Rightarrow \left(\frac{dx}{dy}\right)^2=\sec^2 y-1$

$\Rightarrow L=\int_{-\pi/3}^{\pi/4}\sqrt{1+\left(\sec^2 y-1\right)}\;dy=\int_{-\pi/3}^{\pi/4}|\sec y|\;dy$

$=\int_{-\pi/3}^{\pi/4}\sec y\;dy$

(c) $L \approx 2.20$

(b)



21. (a) $\left(\frac{dy}{dx}\right)^2$ corresponds to $\frac{1}{4x}$ here, so take $\frac{dy}{dx}$ as $\frac{1}{2\sqrt{x}}$. Then $y=\sqrt{x}+C$ and since $(1,1)$ lies on the curve, $C=0$. So $y=\sqrt{x}$ from $(1,1)$ to $(4,2)$.

(b) Only one. We know the derivative of the function and the value of the function at one value of x.

22. (a) $\left(\frac{dx}{dy}\right)^2$ corresponds to $\frac{1}{y^4}$ here, so take $\frac{dy}{dx}$ as $\frac{1}{y^2}$. Then $x=-\frac{1}{y}+C$ and, since $(0,1)$ lies on the curve, $C=1$. So $y=\frac{1}{1-x}$.

(b) Only one. We know the derivative of the function and the value of the function at one value of x.

23. $y=\int_0^x \sqrt{\cos 2t}\, dt \Rightarrow \frac{dy}{dx}=\sqrt{\cos 2x} \Rightarrow L=\int_0^{\pi/4}\sqrt{1+\left[\sqrt{\cos 2x}\right]^2}\, dx=\int_0^{\pi/4}\sqrt{1+\cos 2x}\, dx=\int_0^{\pi/4}\sqrt{2\cos^2 x}\, dx$
$=\int_0^{\pi/4}\sqrt{2}\cos x\, dx=\sqrt{2}\left[\sin x\right]_0^{\pi/4}=\sqrt{2}\sin\left(\frac{\pi}{4}\right)-\sqrt{2}\sin(0)=1$

24. $y=\left(1-x^{2/3}\right)^{3/2},\ \frac{\sqrt{2}}{4}\le x\le 1 \Rightarrow \frac{dy}{dx}=\frac{3}{2}\left(1-x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)=-\frac{\left(1-x^{2/3}\right)^{1/2}}{x^{1/3}} \Rightarrow L=\int_{\sqrt{2}/4}^{1}\sqrt{1+\left[-\frac{\left(1-x^{2/3}\right)^{1/2}}{x^{1/3}}\right]^2}\, dx$
$=\int_{\sqrt{2}/4}^{1}\sqrt{1+\frac{1-x^{2/3}}{x^{2/3}}}\, dx=\int_{\sqrt{2}/4}^{1}\sqrt{1+\frac{1}{x^{2/3}}-1}\, dx=\int_{\sqrt{2}/4}^{1}\sqrt{\frac{1}{x^{2/3}}}\, dx=\int_{\sqrt{2}/4}^{1}\frac{1}{x^{1/3}}\, dx=\int_{\sqrt{2}/4}^{1}x^{-1/3}dx=\frac{3}{2}\left[x^{2/3}\right]_{\sqrt{2}/4}^{1}$
$=\frac{3}{2}(1)^{2/3}-\frac{3}{2}\left(\frac{\sqrt{2}}{4}\right)^{2/3}=\frac{3}{2}-\frac{3}{2}\left(\frac{1}{2}\right)=\frac{3}{4} \Rightarrow$ total length $=8\left(\frac{3}{4}\right)=6$

25. $y=3-2x, 0\le x\le 2 \Rightarrow \frac{dy}{dx}=-2 \Rightarrow L=\int_0^2\sqrt{1+(-2)^2}dx=\int_0^2\sqrt{5}\, dx=\left[\sqrt{5}\, x\right]_0^2=2\sqrt{5}.$
$d=\sqrt{(2-0)^2+(3-(-1))^2}=2\sqrt{5}$

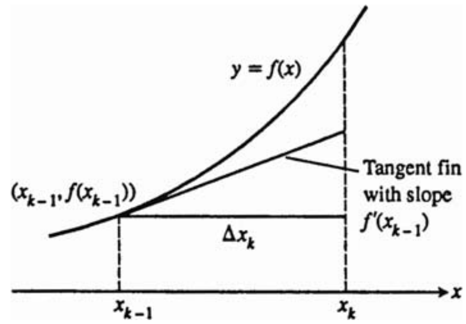
26. Consider the circle $x^2+y^2=r^2$, we will find the length of the portion in the first quadrant, and multiply our result by 4.
$y=\sqrt{r^2-x^2},\ 0\le x\le r \Rightarrow \frac{dy}{dx}=\frac{-x}{\sqrt{r^2-x^2}} \Rightarrow L=4\int_0^r\sqrt{1+\left[\frac{-x}{\sqrt{r^2-x^2}}\right]^2}dx=4\int_0^r\sqrt{1+\frac{x^2}{r^2-x^2}}\, dx=4\int_0^r\sqrt{\frac{r^2}{r^2-x^2}}\, dx$
$=4\int_0^r\frac{r}{\sqrt{r^2-x^2}}\, dx=4r\int_0^r\frac{dx}{\sqrt{r^2-x^2}}$

27. $9x^2=y(y-3)^2 \Rightarrow \frac{d}{dy}\left[9x^2\right]=\frac{d}{dy}\left[y(y-3)^2\right] \Rightarrow 18x\frac{dx}{dy}=2y(y-3)+(y-3)^2=3(y-3)(y-1)$
$\Rightarrow \frac{dx}{dy}=\frac{(y-3)(y-1)}{6x} \Rightarrow dx=\frac{(y-3)(y-1)}{6x}dy;\ \ ds^2=dx^2+dy^2=\left[\frac{(y-3)(y-1)}{6x}dy\right]^2+dy^2=\frac{(y-3)^2(y-1)^2}{36x^2}dy^2+dy^2$
$=\frac{(y-3)^2(y-1)^2}{4y(y-3)^2}dy^2+dy^2=\left[\frac{(y-1)^2}{4y}+1\right]dy^2=\frac{y^2-2y+1+4y}{4y}\, dy^2=\frac{(y+1)^2}{4y}\, dy^2$

28. $4x^2-y^2=64 \Rightarrow \frac{d}{dx}\left[4x^2-y^2\right]=\frac{d}{dx}[64] \Rightarrow 8x-2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{4x}{y} \Rightarrow dy=\frac{4x}{y}dx;$
$ds^2=dx^2+dy^2=dx^2+\left[\frac{4x}{y}dx\right]^2=dx^2+\frac{16x^2}{y^2}dx^2=\left(1+\frac{16x^2}{y^2}\right)dx^2=\frac{y^2+16x^2}{y^2}dx^2=\frac{4x^2-64+16x^2}{y^2}\, dx^2$
$=\frac{20x^2-64}{y^2}\, dx^2=\frac{4}{y^2}(5x^2-16)\, dx^2$

29. $\sqrt{2}\, x=\int_0^x\sqrt{1+\left(\frac{dy}{dt}\right)^2}\, dt, x\ge 0 \Rightarrow \sqrt{2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow \frac{dy}{dx}=\pm 1 \Rightarrow y=f(x)=\pm x+C$ where C is any real number.

30. (a) From the accompanying figure and definition of the differential (change along the tangent line) we see that $dy = f'(x_{k-1})\Delta x_k \Rightarrow$ length of kth tangent fin is $\sqrt{(\Delta x_k)^2 + (dy)^2} = \sqrt{(\Delta x_k)^2 + [f'(x_{k-1})\Delta x_k]^2}$.



(b) Length of curve $= \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n}$ (length of kth tangent fin) $= \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f'(x_{k-1})\Delta x_k]^2}$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1+[f'(x_{k-1})]^2}\ \Delta x_k = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

31. $x^2 + y^2 = 1 \Rightarrow y = \sqrt{1-x^2}$; $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\} \Rightarrow L \approx \sum\limits_{k=1}^{4} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$

$$= \sqrt{\left(\tfrac{1}{4}-0\right)^2 + \left(\tfrac{\sqrt{15}}{4}-1\right)^2} + \sqrt{\left(\tfrac{1}{2}-\tfrac{1}{4}\right)^2 + \left(\tfrac{\sqrt{3}}{2}-\tfrac{\sqrt{15}}{4}\right)^2} + \sqrt{\left(\tfrac{3}{4}-\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{7}}{4}-\tfrac{\sqrt{3}}{2}\right)^2} + \sqrt{\left(1-\tfrac{3}{4}\right)^2 + \left(0-\tfrac{\sqrt{7}}{4}\right)^2}$$

≈ 1.55225

32. Let (x_1, y_1) and (x_2, y_2), with $x_2 > x_1$, lie on $y = mx + b$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$, then $\frac{dy}{dx} = m$

$$\Rightarrow L = \int_{x_1}^{x_2} \sqrt{1+m^2}\,dx = \sqrt{1+m^2}\,[x]_{x_1}^{x_2} = \sqrt{1+m^2}\,(x_2 - x_1) = \sqrt{1+\left(\tfrac{y_2-y_1}{x_2-x_1}\right)^2}\,(x_2 - x_1)$$

$$= \sqrt{\frac{(x_2-x_1)^2 + (y_2-y_1)^2}{(x_2-x_1)^2}}\,(x_2 - x_1) = \frac{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}{(x_2-x_1)}\,(x_2 - x_1) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}.$$

33. $y = 2x^{3/2} \Rightarrow \frac{dy}{dx} = 3x^{1/2}$; $L(x) = \int_0^x \sqrt{1+\left(3t^{1/2}\right)^2}\,dt = \int_0^x \sqrt{1+9t}\,dt$;

$[u = 1+9t \Rightarrow du = 9dt;\ t = 0 \Rightarrow u = 1, t = x \Rightarrow u = 1+9x] \rightarrow \frac{1}{9}\int_1^{1+9x} \sqrt{u}\,du = \frac{2}{27}\left[u^{3/2}\right]_1^{1+9x} = \frac{2}{27}(1+9x)^{3/2} - \frac{2}{27}$;

$L(1) = \frac{2}{27}(10)^{3/2} - \frac{2}{27} = \frac{2(10\sqrt{10}-1)}{27}$

34. $y = \frac{x^3}{3} + x^2 + x + \frac{1}{4x+4} \Rightarrow \frac{dy}{dx} = x^2 + 2x + 1 - \frac{1}{4(x+1)^2} = (x+1)^2 - \frac{1}{4(x+1)^2}$;

$$L(x) = \int_0^x \sqrt{1+\left[(t+1)^2 - \tfrac{1}{4(t+1)^2}\right]^2}\,dt = \int_0^x \sqrt{1+\left[\tfrac{4(t+1)^4-1}{4(t+1)^2}\right]^2}\,dt = \int_0^x \sqrt{1+\tfrac{[4(t+1)^4-1]^2}{16(t+1)^4}}\,dt$$

$$= \int_0^x \sqrt{\tfrac{16(t+1)^4+16(t+1)^8-8(t+1)^4+1}{16(t+1)^4}}\,dt = \int_0^x \sqrt{\tfrac{16(t+1)^8+8(t+1)^4+1}{16(t+1)^4}}\,dt = \int_0^x \sqrt{\tfrac{[4(t+1)^4+1]^2}{16(t+1)^4}}\,dt = \int_0^x \tfrac{4(t+1)^4+1}{4(t+1)^2}\,dt$$

$$= \int_0^x \left[(t+1)^2 + \tfrac{1}{4(t+1)^2}\right]dt;\ [u = t+1 \Rightarrow du = dt;\ t = 0 \Rightarrow u = 1, t = x \Rightarrow u = x+1] \rightarrow \int_1^{x+1}\left[u^2 + \tfrac{1}{4}u^{-2}\right]du$$

$$= \left[\tfrac{1}{3}u^3 - \tfrac{1}{4}u^{-1}\right]_1^{x+1} = \left(\tfrac{1}{3}(x+1)^3 - \tfrac{1}{4(x+1)}\right) - \left(\tfrac{1}{3} - \tfrac{1}{4}\right) = \tfrac{1}{3}(x+1)^3 - \tfrac{1}{4(x+1)} - \tfrac{1}{12};\ L(1) = \tfrac{8}{3} - \tfrac{1}{8} - \tfrac{1}{12} = \tfrac{59}{24}$$

35-40. Example CAS commands:

Maple:

```
with( plots );
with( Student[Calculus1] );
with( student );
f := x -> sqrt(1-x^2);a := -1;
b := 1;
N := [2, 4, 8];
for n in N do
   xx := [seq( a+i*(b-a)/n, i=0..n )];
   pts := [seq([x, f(x)], x=xx)];
   L := simplify(add( distance(pts[i+1], pts[i]), i=1..n ));          # (b)
   T := sprintf("#35(a) (Section 6.3)\nn=%3d L=%8.5f \n", n, L );
   P[n] := plot( [f(x), pts], x=a..b, title=T ):                      # (a)
end do:
display( [seq(P[n], n=N)], insequence=true, scaling=constrained );
L := ArcLength( f(x), x=a..b, output=integral ):
L = evalf( L );                                                       # (c)
```

Mathematica: (assigned function and values for a, b, and n may vary)

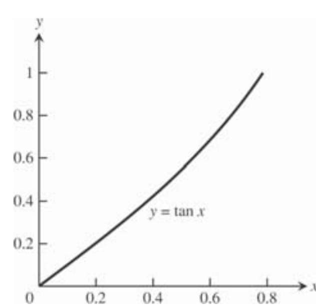
```
Clear[x, f]
{a, b} = {–1, 1}; f[x_] = Sqrt[1 – x^2]
p1 = Plot[f[x], {x, a, b}]
n = 8;
pts = Table[{xn, f[xn]}, {xn, a, b, (b – a)/n}]/ / N
Show[p1,Graphics[{Line[pts]}]]
Sum[ Sqrt[ (pts[[i+1, 1]] – pts[[i, 1]])^2 + (pts[[i + 1, 2]] – pts[[i, 2]])^2 ], {i, 1, n}]
NIntegrate[Sqrt[ 1 + f'[x]^2 ], {x, a, b}]
```

6.4 AREAS OF SURFACES OF REVOLUTION

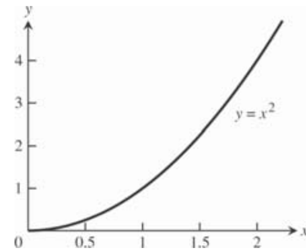
1. (a) $\frac{dy}{dx} = \sec^2 x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \sec^4 x$

$\Rightarrow S = 2\pi\int_0^{\pi/4} (\tan x)\sqrt{1+\sec^4 x}\, dx$

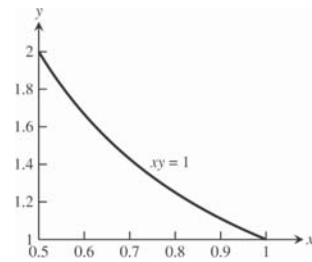
(b)



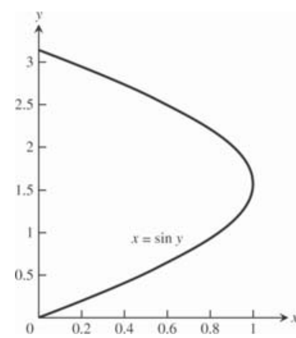
(c) $S \approx 3.84$

2. (a) $\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)^2 = 4x^2$
$\Rightarrow S = 2\pi\int_0^2 x^2\sqrt{1+4x^2}\,dx$
(b)


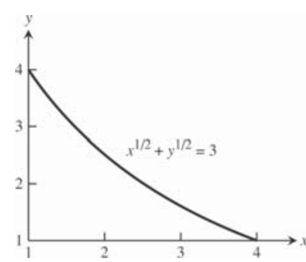
(c) $S \approx 53.23$

3. (a) $xy = 1 \Rightarrow x = \frac{1}{y} \Rightarrow \frac{dx}{dy} = -\frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{y^4}$
$\Rightarrow S = 2\pi\int_1^2 \frac{1}{y}\sqrt{1+y^{-4}}\,dy$
(b)


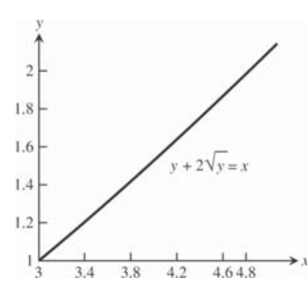
(c) $S \approx 5.02$

4. (a) $\frac{dx}{dy} = \cos y \Rightarrow \left(\frac{dx}{dy}\right)^2 = \cos^2 y$
$\Rightarrow S = 2\pi\int_0^{\pi} (\sin y)\sqrt{1+\cos^2 y}\,dy$
(b)


(c) $S \approx 14.42$

5. (a) $x^{1/2} + y^{1/2} = 3 \Rightarrow y = \left(3 - x^{1/2}\right)^2$
$\Rightarrow \frac{dy}{dx} = 2\left(3 - x^{1/2}\right)\left(-\frac{1}{2}x^{-1/2}\right)$
$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(1 - 3x^{-1/2}\right)^2$
$\Rightarrow S = 2\pi\int_1^4 \left(3 - x^{1/2}\right)^2\sqrt{1+\left(1-3x^{-1/2}\right)^2}\,dx$
(b)


(c) $S \approx 63.37$

6. (a) $\frac{dx}{dy} = 1 + y^{-1/2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \left(1 + y^{-1/2}\right)^2$
$\Rightarrow S = 2\pi\int_1^2 \left(y + 2\sqrt{y}\right)\sqrt{1+\left(1+y^{-1/2}\right)^2}\,dx$
(b)


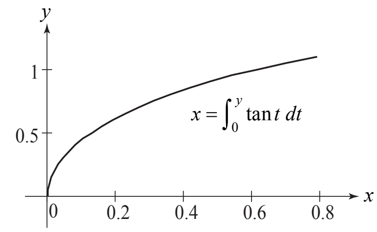
(c) $S \approx 51.33$

7. (a) $\frac{dx}{dy} = \tan y \Rightarrow \left(\frac{dx}{dy}\right)^2 = \tan^2 y$

$\Rightarrow S = 2\pi \int_0^{\pi/3} \left(\int_0^y \tan t \, dt \right) \sqrt{1 + \tan^2 y} \, dy$

$= 2\pi \int_0^{\pi/3} \left(\int_0^y \tan t \, dt \right) \sec y \, dy$

(b)



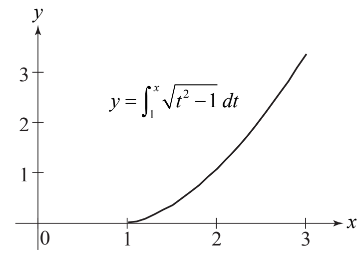
(c) $S \approx 2.08$

8. (a) $\frac{dy}{dx} = \sqrt{x^2 - 1} \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^2 - 1$

$\Rightarrow S = 2\pi \int_1^{\sqrt{5}} \left(\int_1^x \sqrt{t^2 - 1} \, dt \right) \sqrt{1 + \left(x^2 - 1\right)} \, dx$

$= 2\pi \int_1^{\sqrt{5}} \left(\int_1^x \sqrt{t^2 - 1} \, dt \right) x \, dx$

(b)



(c) $S \approx 8.55$

9. $y = \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$; $S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \Rightarrow S = \int_0^4 2\pi \left(\frac{x}{2}\right) \sqrt{1 + \frac{1}{4}} \, dx = \frac{\pi\sqrt{5}}{2} \int_0^4 x \, dx = \frac{\pi\sqrt{5}}{2} \left[\frac{x^2}{2} \right]_0^4 = 4\pi\sqrt{5}$;

Geometry formula: base circumference $= 2\pi(2)$, slant height $= \sqrt{4^2 + 2^2} = 2\sqrt{5}$

$\Rightarrow$ Lateral surface area $= \frac{1}{2}(4\pi)\left(2\sqrt{5}\right) = 4\pi\sqrt{5}$ in agreement with the integral value

10. $y = \frac{x}{2} \Rightarrow x = 2y \Rightarrow \frac{dx}{dy} = 2$; $S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_0^2 2\pi \cdot 2y \sqrt{1 + 2^2} \, dy = 4\pi\sqrt{5} \int_0^2 y \, dy = 2\pi\sqrt{5} \left[y^2 \right]_0^2$

$= 2\pi\sqrt{5} \cdot 4 = 8\pi\sqrt{5}$; Geometry formula: base circumference $= 2\pi(4)$, slant height $= \sqrt{4^2 + 2^2} = 2\sqrt{5}$

$\Rightarrow$ Lateral surface area $= \frac{1}{2}(8\pi)\left(2\sqrt{5}\right) = 8\pi\sqrt{5}$ in agreement with the integral value

11. $\frac{dx}{dy} = \frac{1}{2}$; $S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_1^3 2\pi \frac{(x+1)}{2} \sqrt{1 + \left(\frac{1}{2}\right)^2} \, dx = \frac{\pi\sqrt{5}}{2} \int_1^3 (x+1) \, dx = \frac{\pi\sqrt{5}}{2} \left[\frac{x^2}{2} + x \right]_1^3$

$= \frac{\pi\sqrt{5}}{2} \left[\left(\frac{9}{2} + 3\right) - \left(\frac{1}{2} + 1\right) \right] = \frac{\pi\sqrt{5}}{2}(4 + 2) = 3\pi\sqrt{5}$; Geometry formula: $r_1 = \frac{1}{2} + \frac{1}{2} = 1$, $r_2 = \frac{3}{2} + \frac{1}{2} = 2$, slant height $= \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{5}$ $\Rightarrow$ Frustum surface area $= \pi\left(r_1 + r_2\right) \times$ slant height $= \pi(1+2)\sqrt{5} = 3\pi\sqrt{5}$ in agreement with the integral value

12. $y = \frac{x}{2} + \frac{1}{2} \Rightarrow x = 2y - 1 \Rightarrow \frac{dx}{dy} = 2$; $S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_1^2 2\pi \, (2y-1)\sqrt{1+4} \, dy = 2\pi\sqrt{5} \int_1^2 (2y-1) \, dy$

$= 2\pi\sqrt{5} \left[y^2 - y \right]_1^2 = 2\pi\sqrt{5}\left[(4-2) - (1-1)\right] = 4\pi\sqrt{5}$; Geometry formula: $r_1 = 1, r_2 = 3$,

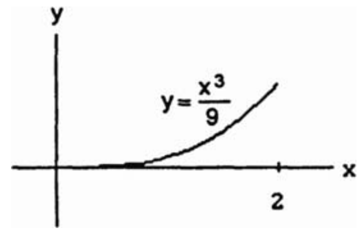
slant height $= \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{5} \Rightarrow$ Frustum surface area $= \pi(1+3)\sqrt{5} = 4\pi\sqrt{5}$ in agreement with the integral value

13. $\frac{dy}{dx} = \frac{x^2}{3} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^4}{9} \Rightarrow S = \int_0^2 \frac{2\pi x^3}{9}\sqrt{1+\frac{x^4}{9}}\,dx;$

$\Big[u = 1+\frac{x^4}{9} \Rightarrow du = \frac{4}{9}x^3 dx \Rightarrow \frac{1}{4}\,du = \frac{x^3}{9}\,dx;$

$x = 0 \Rightarrow u = 1,\ x = 2 \Rightarrow u = \frac{25}{9}\Big] \to S = 2\pi\int_1^{25/9} u^{1/2}\cdot\frac{1}{4}\,du$

$= \frac{\pi}{2}\left[\frac{2}{3}u^{3/2}\right]_1^{25/9} = \frac{\pi}{3}\left(\frac{125}{27}-1\right) = \frac{\pi}{3}\left(\frac{125-27}{27}\right) = \frac{98\pi}{81}$

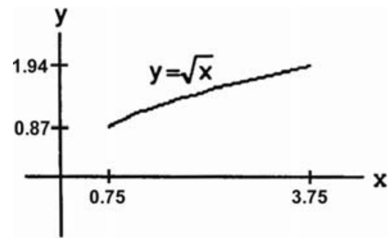


14. $\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4x}$

$\Rightarrow S = \int_{3/4}^{15/4} 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx = 2\pi\int_{3/4}^{15/4}\sqrt{x+\frac{1}{4}}\,dx$

$= 2\pi\left[\frac{2}{3}\left(x+\frac{1}{4}\right)^{3/2}\right]_{3/4}^{15/4} = \frac{4\pi}{3}\left[\left(\frac{15}{4}+\frac{1}{4}\right)^{3/2} - \left(\frac{3}{4}+\frac{1}{4}\right)^{3/2}\right]$

$= \frac{4\pi}{3}\left[\left(\frac{4}{2}\right)^3 - 1\right] = \frac{4\pi}{3}(8-1) = \frac{28\pi}{3}$

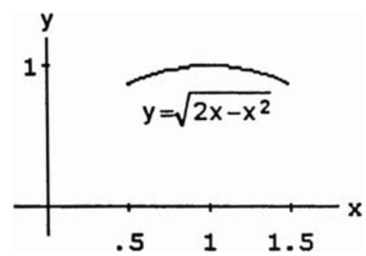


15. $\frac{dy}{dx} = \frac{1}{2}\frac{(2-2x)}{\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{2x-x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(1-x)^2}{2x-x^2}$

$\Rightarrow S = \int_{0.5}^{1.5} 2\pi\sqrt{2x-x^2}\sqrt{1+\frac{(1-x)^2}{2x-x^2}}\,dx$

$= 2\pi\int_{0.5}^{1.5}\sqrt{2x-x^2}\,\frac{\sqrt{2x-x^2+1-2x+x^2}}{\sqrt{2x-x^2}}\,dx$

$= 2\pi\int_{0.5}^{1.5} dx = 2\pi\left[x\right]_{0.5}^{1.5} = 2\pi$



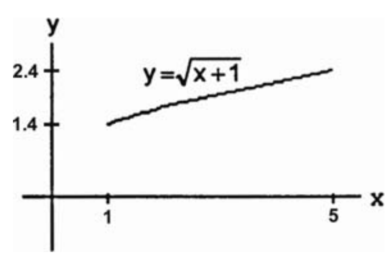
16. $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4(x+1)}$

$\Rightarrow S = \int_1^5 2\pi\sqrt{x+1}\sqrt{1+\frac{1}{4(x+1)}}\,dx = 2\pi\int_1^5\sqrt{(x+1)+\frac{1}{4}}\,dx$

$= 2\pi\int_1^5\sqrt{x+\frac{5}{4}}\,\mathrm{dx} = 2\pi\left[\frac{2}{3}\left(x+\frac{5}{4}\right)^{3/2}\right]_1^5$

$= \frac{4\pi}{3}\left[\left(5+\frac{5}{4}\right)^{3/2} - \left(1+\frac{5}{4}\right)^{3/2}\right] = \frac{4\pi}{3}\left[\left(\frac{25}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2}\right]$

$= \frac{4\pi}{3}\left(\frac{5^3}{2^3} - \frac{3^3}{2^3}\right) = \frac{\pi}{6}(125-27) = \frac{98\pi}{6} = \frac{49\pi}{3}$

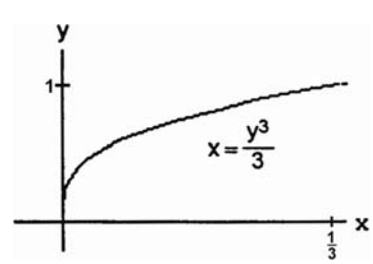


17. $\frac{dx}{dy} = y^2 \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4 \Rightarrow S = \int_0^1 \frac{2\pi y^3}{3}\sqrt{1+y^4}\,dy;$

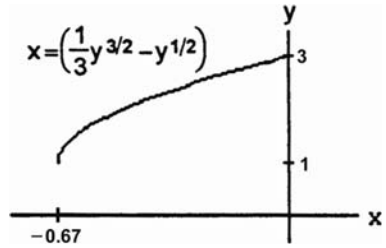
$\Big[u = 1+y^4 \Rightarrow du = 4y^3 dy \Rightarrow \frac{1}{4}\,du = y^3\,dy;$

$y = 0 \Rightarrow u = 1,\ y = 1 \Rightarrow u = 2\Big] \to S = \int_1^2 2\pi\left(\frac{1}{3}\right)u^{1/2}\left(\frac{1}{4}du\right)$

$= \frac{\pi}{6}\int_1^2 u^{1/2}du = \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]_1^2 = \frac{\pi}{9}(\sqrt{8}-1)$



18. $x=\left(\frac{1}{3}y^{3/2}-y^{1/2}\right)\le 0$, when $1\le y\le 3$. To get positive area, we take $x=-\left(\frac{1}{3}y^{3/2}-y^{1/2}\right)$



$$\Rightarrow \frac{dx}{dy}=-\frac{1}{2}\left(y^{1/2}-y^{-1/2}\right)\Rightarrow\left(\frac{dx}{dy}\right)^2=\frac{1}{4}\left(y-2+y^{-1}\right)$$

$$\Rightarrow S=-\int_1^3 2\pi\left(\frac{1}{3}y^{3/2}-y^{1/2}\right)\sqrt{1+\frac{1}{4}\left(y-2+y^{-1}\right)}\,dy$$

$$=-2\pi\int_1^3\left(\frac{1}{3}y^{3/2}-y^{1/2}\right)\sqrt{\frac{1}{4}\left(y+2+y^{-1}\right)}\,dy$$

$$=-2\pi\int_1^3\left(\frac{1}{3}y^{3/2}-y^{1/2}\right)\frac{\sqrt{\left(y^{1/2}+y^{-1/2}\right)^2}}{2}\,dy=-\pi\int_1^3 y^{1/2}\left(\frac{1}{3}y-1\right)\left(y^{1/2}+\frac{1}{y^{1/2}}\right)dy=-\pi\int_1^3\left(\frac{1}{3}y-1\right)(y+1)\,dy$$

$$=-\pi\int_1^3\left(\frac{1}{3}y^2-\frac{2}{3}y-1\right)dy=-\pi\left[\frac{y^3}{9}-\frac{y^3}{3}-y\right]_1^3=-\pi\left[\left(\frac{27}{9}-\frac{9}{3}-3\right)-\left(\frac{1}{9}-\frac{1}{3}-1\right)\right]=-\pi\left(-3-\frac{1}{9}+\frac{1}{3}+1\right)$$

$$=-\frac{\pi}{9}(-18-1+3)=\frac{16\pi}{9}$$

19. $\frac{dx}{dy}=\frac{-1}{\sqrt{4-y}}\Rightarrow\left(\frac{dx}{dy}\right)^2=\frac{1}{4-y}\Rightarrow S=\int_0^{15/4}2\pi\cdot 2\sqrt{4-y}\sqrt{1+\frac{1}{4-y}}\,dy=4\pi\int_0^{15/4}\sqrt{(4-y)+1}\,dy$

$$=4\pi\int_0^{15/4}\sqrt{5-y}\,dy=-4\pi\left[\frac{2}{3}(5-y)^{3/2}\right]_0^{15/4}=-\frac{8\pi}{3}\left[\left(5-\frac{15}{4}\right)^{3/2}-5^{3/2}\right]=-\frac{8\pi}{3}\left[\left(\frac{5}{4}\right)^{3/2}-5^{3/2}\right]$$

$$=\frac{8\pi}{3}\left(5\sqrt{5}-\frac{5\sqrt{5}}{8}\right)=\frac{8\pi}{3}\left(\frac{40\sqrt{5}-5\sqrt{5}}{8}\right)=\frac{35\pi\sqrt{5}}{3}$$

20. $\frac{dx}{dy}=\frac{1}{\sqrt{2y-1}}\Rightarrow\left(\frac{dx}{dy}\right)^2=\frac{1}{2y-1}\Rightarrow S=\int_{5/8}^1 2\pi\sqrt{2y-1}\sqrt{1+\frac{1}{2y-1}}\,dy=2\pi\int_{5/8}^1\sqrt{(2y-1)+1}\,dy=2\pi\int_{5/8}^1\sqrt{2}y^{1/2}dy$

$$=2\pi\sqrt{2}\left[\frac{2}{3}y^{3/2}\right]_{5/8}^1=\frac{4\pi\sqrt{2}}{3}\left[1^{3/2}-\left(\frac{5}{8}\right)^{3/2}\right]=\frac{4\pi\sqrt{2}}{3}\left(1-\frac{5\sqrt{5}}{8\sqrt{8}}\right)=\frac{4\pi\sqrt{2}}{3}\left(\frac{8\cdot 2\sqrt{2}-5\sqrt{5}}{8\cdot 2\sqrt{2}}\right)=\frac{\pi}{12}\left(16\sqrt{2}-5\sqrt{5}\right)$$

21. $\frac{dy}{dx}=x\Rightarrow\left(\frac{dy}{dx}\right)^2=x^2\Rightarrow S=\int_0^1 2\pi x\sqrt{1+x^2}\,dx=\frac{2\pi}{3}\left(1+x^2\right)^{3/2}\Big|_0^1=\frac{2\pi}{3}\left(2\sqrt{2}-1\right)$

22. $y=\frac{1}{3}\left(x^2+2\right)^{3/2}\Rightarrow dy=x\sqrt{x^2+2}\,dx\Rightarrow ds=\sqrt{1+\left(2x^2+x^4\right)}\,dx\Rightarrow S=2\pi\int_0^{\sqrt{2}}x\sqrt{1+2x^2+x^4}\,dx$

$$=2\pi\int_0^{\sqrt{2}}x\sqrt{\left(x^2+1\right)^2}\,dx=2\pi\int_0^{\sqrt{2}}x\left(x^2+1\right)dx=2\pi\int_0^{\sqrt{2}}\left(x^3+x\right)dx=2\pi\left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^{\sqrt{2}}=2\pi\left(\frac{4}{4}+\frac{2}{2}\right)=4\pi$$

23. $ds=\sqrt{dx^2+dy^2}=\sqrt{\left(y^3-\frac{1}{4y^3}\right)^2+1}\,dy=\sqrt{\left(y^6-\frac{1}{2}+\frac{1}{16y^6}\right)+1}\,dy=\sqrt{\left(y^6+\frac{1}{2}+\frac{1}{16y^6}\right)}\,dy$

$$=\sqrt{\left(y^3+\frac{1}{4y^3}\right)^2}\,dy=\left(y^3+\frac{1}{4y^3}\right)dy;\quad S=\int_1^2 2\pi y\,ds=2\pi\int_1^2 y\left(y^3+\frac{1}{4y^3}\right)dy=2\pi\int_1^2\left(y^4+\frac{1}{4}y^{-2}\right)dy$$

$$=2\pi\left[\frac{y^5}{5}-\frac{1}{4}y^{-1}\right]_1^2=2\pi\left[\left(\frac{32}{5}-\frac{1}{8}\right)-\left(\frac{1}{5}-\frac{1}{4}\right)\right]=2\pi\left(\frac{31}{5}+\frac{1}{8}\right)=\frac{2\pi}{40}(8\cdot 31+5)=\frac{253\pi}{20}$$

24. $y=\cos x \Rightarrow \frac{dy}{dx}=-\sin x \Rightarrow \left(\frac{dy}{dx}\right)^2=\sin^2 x \Rightarrow S=2\pi\int_{-\pi/2}^{\pi/2}(\cos x)\sqrt{1+\sin^2 x}\,dx$

25. $y=\sqrt{a^2-x^2} \Rightarrow \frac{dy}{dx}=\frac{1}{2}\left(a^2-x^2\right)^{-1/2}(-2x)=\frac{-x}{\sqrt{a^2-x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{x^2}{a^2-x^2}$
$\Rightarrow S=2\pi\int_{-a}^{a}\sqrt{a^2-x^2}\sqrt{1+\frac{x^2}{a^2-x^2}}\,dx=2\pi\int_{-a}^{a}\sqrt{\left(a^2-x^2\right)+x^2}\,dx=2\pi\int_{-a}^{a}a\,dx=2\pi a\left[x\right]_{-a}^{a}$
$=2\pi a[a-(-a)]=(2\pi a)(2a)=4\pi a^2$

26. $y=\frac{r}{h}x \Rightarrow \frac{dy}{dx}=\frac{r}{h} \Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{r^2}{h^2} \Rightarrow S=2\pi\int_0^h \frac{r}{h}x\sqrt{1+\frac{r^2}{h^2}}\,dx=2\pi\int_0^h \frac{r}{h}x\sqrt{\frac{h^2+r^2}{h^2}}\,dx=\frac{2\pi r}{h}\sqrt{\frac{h^2+r^2}{h^2}}\int_0^h x\,dx$
$=\frac{2\pi r}{h^2}\sqrt{h^2+r^2}\left[\frac{x^2}{2}\right]_0^h=\frac{2\pi r}{h^2}\sqrt{h^2+r^2}\left(\frac{h^2}{2}\right)=\pi r\sqrt{h^2+r^2}$

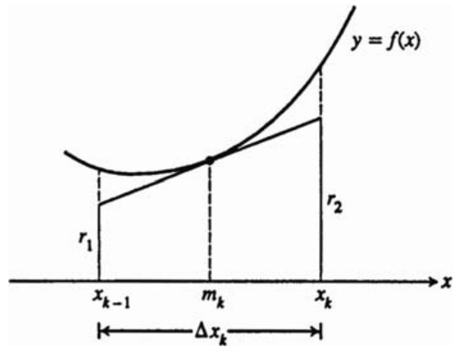
27. The area of the surface of one wok is $S=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$. Now, $x^2+y^2=16^2 \Rightarrow x=\sqrt{16^2-y^2}$
$\Rightarrow \frac{dx}{dy}=\frac{-y}{\sqrt{16^2-y^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2=\frac{y^2}{16^2-y^2}$; $S=\int_{-16}^{-7}2\pi\sqrt{16^2-y^2}\sqrt{1+\frac{y^2}{16^2-y^2}}\,dy=\int_{-16}^{-7}2\pi\sqrt{\left(16^2-y^2\right)+y^2}\,dy$
$=2\pi\int_{-16}^{-7}16\,dy=32\pi\cdot 9=288\pi\approx 904.78\text{ cm}^2$. The enamel needed to cover one surface of one wok is
$V=S\cdot 0.5\text{ mm}=S\cdot 0.05\text{ cm}=(904.78)(0.05)\text{ cm}^3=45.24\text{ cm}^3$. For 5000 woks, we need
$5000\cdot V=5000\cdot 45.24\text{ cm}^3=(5)(45.24)\,L=226.2L \Rightarrow 226.2$ liters of each color are needed.

28. $y=\sqrt{r^2-x^2} \Rightarrow \frac{dy}{dx}=-\frac{1}{2}\frac{2x}{\sqrt{r^2-x^2}}=\frac{-x}{\sqrt{r^2-x^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2=\frac{x^2}{r^2-x^2}$; $S=2\pi\int_a^{a+h}\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$
$=2\pi\int_a^{a+h}\sqrt{\left(r^2-x^2\right)+x^2}\,dx=2\pi r\int_a^{a+h}dx=2\pi rh$, which is independent of a.

29. $y=\sqrt{R^2-x^2} \Rightarrow \frac{dy}{dx}=-\frac{1}{2}\frac{2x}{\sqrt{R^2-x^2}}=\frac{-x}{\sqrt{R^2-x^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2=\frac{x^2}{R^2-x^2}$; $S=2\pi=\int_a^{a+h}\sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx$
$=2\pi\int_a^{a+h}\sqrt{\left(R^2-x^2\right)+x^2}\,dx=2\pi R\int_a^{a+h}dx=2\pi Rh$

30. (a) $x^2+y^2=15^2 \Rightarrow x=\sqrt{15^2-y^2} \Rightarrow \frac{dx}{dy}=\frac{-y}{\sqrt{15^2-y^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2=\frac{y^2}{15^2-y^2}$;
$S=\int_{-7.5}^{15}2\pi\sqrt{15^2-y^2}\sqrt{1+\frac{y^2}{15^2-y^2}}\,dy=2\pi\int_{-7.5}^{15}\sqrt{\left(15^2-y^2\right)+y^2}\,dy=2\pi\cdot 15\int_{-7.5}^{15}dy$
$=(2\pi)(15)(22.5)=675\pi$ square meters
(b) 2121 square meters

31. (a) An equation of the tangent line segment is (see figure) $y = f(m_k) + f'(m_k)(x - m_k)$. When $x = x_{k-1}$ we have
$r_1 = f(m_k) + f'(m_k)(x_{k-1} - m_k)$
$= f(m_k) + f'(m_k)\left(-\frac{\Delta x_k}{2}\right) = f(m_k) - f'(m_k)\frac{\Delta x_k}{2}$;
when $x = x_k$ we have
$r_2 = f(m_k) + f'(m_k)(x_k - m_k)$
$= f(m_k) + f'(m_k)\frac{\Delta x_k}{2}$;



(b) $L_k^2 = (\Delta x_k)^2 + (r_2 - r_1)^2 = (\Delta x_k)^2 + \left[f'(m_k)\frac{\Delta x_k}{2} - \left(-f'(m_k)\frac{\Delta x_k}{2}\right)\right]^2 = (\Delta x_k)^2 + [f'(m_k)\Delta x_k]^2$
$\Rightarrow L_k = \sqrt{(\Delta x_k)^2 + [f'(m_k)\Delta x_k]^2}$, as claimed

(c) From geometry it is a fact that the lateral surface area of the frustum obtained by revolving the tangent line segment about the x-axis is given by $\Delta S_k = \pi(r_1 + r_2)L_k = \pi[2f(m_k)]\sqrt{(\Delta x_k)^2 + [f'(m_k)\Delta x_k]^2}$ using parts (a) and (b) above. Thus, $\Delta S_k = 2\pi f(m_k)\sqrt{1 + [f'(m_k)]^2}\,\Delta x_k$.

(d) $S = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\Delta S_k = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} 2\pi f(m_k)\sqrt{1+[f'(m_k)]^2}\,\Delta x_k = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

32. $y = \left(1 - x^{2/3}\right)^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}\left(1 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -\frac{\left(1-x^{2/3}\right)^{1/2}}{x^{1/3}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1-x^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}} - 1$
$\Rightarrow S = 2\int_0^1 2\pi\left(1-x^{2/3}\right)^{3/2}\sqrt{1+\left(\frac{1}{x^{2/3}}-1\right)}\,dx = 4\pi\int_0^1\left(1-x^{2/3}\right)^{3/2}\sqrt{x^{-2/3}}\,dx = 4\pi\int_0^1\left(1-x^{2/3}\right)^{3/2}x^{-1/3}\,dx$;
$\left[u = 1 - x^{2/3} \Rightarrow du = -\frac{2}{3}x^{-1/3}dx \Rightarrow -\frac{3}{2}\,du = x^{-1/3}\,dx;\ \ x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 0\right]$
$\to S = 4\pi\int_1^0 u^{3/2}\left(-\frac{3}{2}du\right) = -6\pi\left[\frac{2}{5}u^{5/2}\right]_1^0 = -6\pi\left(0 - \frac{2}{5}\right) = \frac{12\pi}{5}$

6.5 WORK AND FLUID FORCES

1. The force required to stretch the spring from its natural length of 2 m to a length of 5 m is $F(x) = kx$. The work done by F is $W = \int_0^3 F(x)\,dx = k\int_0^3 x\,dx = \frac{k}{2}\left[x^2\right]_0^3 = \frac{9k}{2}$. This work is equal to 1800 J $\Rightarrow \frac{9}{2}k = 1800 \Rightarrow k = 400$ N/m

2. (a) We find the force constant from Hooke's Law: $F = kx \Rightarrow k = \frac{F}{x} \Rightarrow k = \frac{800}{4} = 200$ N/cm.

(b) The work done to stretch the spring 2 cm beyond its natural length is $W = \int_0^2 kx\,dx = 200\int_0^2 x\,dx$
$= 200\left[\frac{x^2}{2}\right]_0^2 = 200(2-0) = 400\ \text{cm}\cdot\text{N} = 4$ J

(c) We substitute $F = 1600$ into the equation $F = 200x$ to find $1600 = 200x \Rightarrow x = 8$ cm.

3. We find the force constant from Hooke's law: $F = kx$. A force of 2 N stretches the spring to 0.02 m $\Rightarrow 2 = k \cdot (0.02) \Rightarrow k = 100\,\frac{\text{N}}{\text{m}}$. The force of 4 N will stretch the rubber band y m, where $F = ky \Rightarrow y = \frac{F}{k}$ $\Rightarrow y = \frac{4\text{ N}}{100\frac{\text{N}}{\text{m}}} \Rightarrow y = 0.04$ m $= 4$ cm. The work done to stretch the rubber band 0.04 m is $W = \int_0^{0.04} kx\, dx$ $= 100\int_0^{0.04} x\, dx = 100\left[\frac{x^2}{2}\right]_0^{0.04} = \frac{(100)(0.04)^2}{2} = 0.08$ J

4. We find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} \Rightarrow k = \frac{90}{1} \Rightarrow k = 90\,\frac{\text{N}}{\text{m}}$. The work done to stretch the spring 5 m beyond its natural length is $W = \int_0^5 kx\, dx = 90\int_0^5 x\, dx = 90\left[\frac{x^2}{2}\right]_0^5 = (90)\left(\frac{25}{2}\right) = 1125$ J

5. (a) We find the spring's constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} = \frac{96{,}000}{20-12} = \frac{96{,}000}{8} \Rightarrow k = 12{,}000\,\frac{\text{N}}{\text{cm}}$

   (b) The work done to compress the assembly the first centimeter is $W = \int_0^1 kx\, dx = 12{,}000\int_0^1 x\, dx$ $= 12{,}000\left[\frac{x^2}{2}\right]_0^1 = (12{,}000)\frac{(1)^2}{2} = \frac{(12{,}000)(1)}{2} = 6{,}000\ \text{N}\cdot\text{cm}$. The work done to compress the assembly the second centimeter is:
   $W = \int_1^2 kx\, dx = 12{,}000\int_1^2 x\, dx = 12{,}000\left[\frac{x^2}{2}\right]_1^2 = \frac{12{,}000}{2}\left[4-1\right] = \frac{(12{,}000)(3)}{2} = 18{,}000\ \text{N}\cdot\text{cm}$

6. First, we find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} = \frac{(70)(9.8)}{1.5} = 457.3\,\frac{\text{N}}{\text{mm}}$. If someone compresses the scale $x = 3$ mm, he/she must weigh $F = kx = 457.3(3) = 1372$ N (140 kg). The work done to compress the scale this far is $W = \int_0^3 kx\, dx = 457.3\left[\frac{x^2}{2}\right]_0^3 = (457.3)(9) = 4116\ \text{N}\cdot\text{mm} = 4.116$ J

7. The force required to haul up the rope is equal to the rope's weight, which varies steadily and is proportional to x, the length of the rope still hanging: $F(x) = 0.624x$. The work done is: $W = \int_0^{50} F(x)\, dx = \int_0^{50} 0.624x\, dx$ $= 0.624\left[\frac{x^2}{2}\right]_0^{50} = 780$ J

8. The weight of sand decreases steadily by 300 N over the 6 m, at 50 N/m. So the weight of sand when the bag is x m off the ground is $F(x) = 600 - 50x$. The work done is: $W = \int_a^b F(x)dx = \int_0^6 (600-50x)\, dx$ $= \left[600x - 25x^2\right]_0^6 = 2700$ J

9. The force required to lift the cable is equal to the weight of the cable paid out: $F(x) = (60)(60-x)$ where x is the position of the car off the first floor. The work done is: $W = \int_0^{60} F(x)\, dx = 60\int_0^{60} (60-x)\, dx$ $= 60\left[60x - \frac{x^2}{2}\right]_0^{60} = 60\left(60^2 - \frac{60^2}{2}\right) = \frac{60\cdot 60^2}{2} = 108{,}000$ J

10. Since the force is acting <u>toward</u> the origin, it acts opposite to the positive x-direction. Thus $F(x) = -\frac{k}{x^2}$.

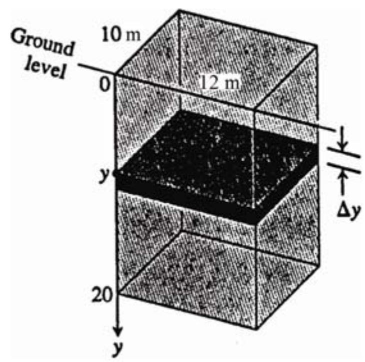
The work done is $W = \int_a^b -\frac{k}{k^2}\, dx = k\int_a^b -\frac{1}{x^2}\, dx = k\left[\frac{1}{x}\right]_a^b = k\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{k(a-b)}{ab}$

11. Let $r =$ the constant rate of leakage. Since the bucket is leaking at a constant rate and the bucket is rising at a constant rate, the amount of water in the bucket is proportional to $(6-x)$, the distance the bucket is being raised. The leakage rate of the water is 13 N/m raised and the weight of the water in the bucket is $F = 13(6-x)$. So: $W = \int_0^6 13\,(6-x)dx = 13\left[6x - \frac{x^2}{2}\right]_0^6 = 234$ J.

12. Let $r =$ the constant rate of leakage. Since the bucket is leaking at a constant rate and the bucket is rising at a constant rate, the amount of water in the bucket is proportional to $(6-x)$, the distance the bucket is being raised. The leakage rate of the water is 32.5 N/m raised and the weight of the water in the bucket is $F = 32.5(6-x)$. So: $W = \int_0^6 32.5(6-x)\, dx = 32.5\left[6x - \frac{x^2}{2}\right]_0^6 = 585$ J.

Note that since the force in Exercise 12 is 2.5 times the force in Exercise 11 at each elevation, the total work is also 2.5 times as great.

13. We will use the coordinate system given.



(a) The typical slab between the planes at y and $y + \Delta y$ has a volume of $\Delta V = (10)(12)\Delta y = 120\Delta y$ m^3. The force F required to lift the slab is equal to its weight: $F = 9800\,\Delta V = 9800 \cdot 120\Delta y$ N. The distance through which F must act is about y m, so the work done lifting the slab is about $\Delta W =$ force $\times$ distance $= 9800 \cdot 120 \cdot y \cdot \Delta y$ J. The work it takes to lift all the water is approximately

$$W \approx \sum_0^{20} \Delta W = \sum_0^{20} 9800 \cdot 120 y \cdot \Delta y \text{ J.}$$

This is a Riemann sum for the function $9800 \cdot 120y$ over the interval $0 \le y \le 20$. The work of pumping the tank empty is the limit of these sums:

$$W = \int_0^{20} 9800 \cdot 120 y\, dy = (9800)(120)\left[\frac{y^2}{2}\right]_0^{20} = (9800)(120)\left(\frac{400}{2}\right) = (9800)(120)(200) = 235{,}200{,}000 \text{ J}$$

(b) The time t it takes to empty the full tank with $5-$ hp motor is $t = \frac{W}{3678 \text{ J/s}} = \frac{235{,}200{,}000 \text{ J}}{3678 \text{ J/s}} \approx 63{,}948$ s $= 17.763$ h $\Rightarrow t \approx 17$ h and 46 min

(c) Following all the steps of part (a), we find that the work it takes to lower the water level 10 m is

$$W = \int_0^{10} 9800 \cdot 120 y\, dy = (9800)(120)\left[\frac{y^2}{2}\right]_0^{10} = (9800)(120)\left(\frac{100}{2}\right) = 58{,}800{,}000 \text{ J}$$ and the time is

$t = \frac{W}{3678 \text{ J/s}} = 15987$ s ≈ 266 min

(d) In a location where water weighs $9780\,\frac{\text{N}}{\text{m}^3}$:

a) $W = (9780)(24{,}000) = 234{,}720{,}000$ J .

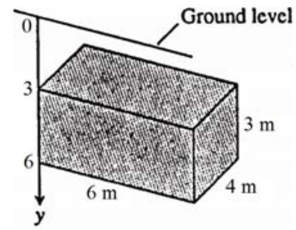
b) $t = \frac{234{,}720{,}000}{3678} \approx 63817$ s ≈ 17.727 h $\Rightarrow t \approx 17$ h and 44 min

In a location where water weighs $9820 \frac{\text{N}}{\text{m}^3}$

a) $W = (9820)(24,000) = 235,680,000$ J

b) $t = \frac{235,680,000}{3678} \approx 64,078 \text{ s} \approx 17.80 \text{ h} \Rightarrow t \approx 17 \text{ h and } 48 \text{ min}$

14. We will use the coordinate system given.



(a) The typical slab between the planes at y and $y + \Delta y$ has a volume of $\Delta V = (6)(4)\ \Delta y = 24\Delta y \text{ m}^3$. The force F required to lift the slab is equal to its weight: $F = 9800\Delta V = 9800 \cdot 24\Delta y$ N. The distance through which F must act is about y m, so the work done lifting the slab is about $\Delta W = \text{force} \times \text{distance}$ $= 9800 \cdot 24 \cdot y \cdot \Delta y$ J. The work it takes to lift all the water is approximately $W \approx \sum_3^6 \Delta W$ $= \sum_3^6 9800 \cdot 24 y \cdot \Delta y$ J. This is a Riemann sum for the function $9800 \cdot 24y$ over the interval $3 \le y \le 6$. The work it takes to empty the cistern is the limit of these sums:

$$W = \int_3^6 9800 \cdot 24 y\, dy = (9800)(24)\left[\frac{y^2}{2}\right]_3^6 = (9800)(24)(18 - 4.5) = (9800)(24)(13.5) = 3,175,200 \text{ J}$$

(b) $t = \frac{W}{370 \text{ J/s}} = \frac{3,175,200}{370} \approx 8581.6 \text{ s} \approx 2.38 \text{ hours} \approx 2 \text{ h and } 23 \text{ min}$

(c) Following all the steps of part (a), we find that the work it takes to empty the tank halfway is

$$W = \int_3^{4.5} 9800 \cdot 24 y\ dy = (9800)(24)\left[\frac{y^2}{2}\right]_3^{4.5} = (9800)(24)\left(\frac{9800}{2} - \frac{9}{2}\right) = (9800)(24)\left(\frac{11.25}{2}\right) = 1,323,000 \text{ J.}$$

Then the time is $t = \frac{W}{370} = \frac{1,323,000}{370} \approx 3575.68 \text{ s} \approx 59.5 \text{ min}$

(d) In a location where water weighs $9780 \frac{\text{N}}{\text{m}^3}$:

a) $W = (9780)(24)(13.5) = 3,168,720$ J.

b) $t = \frac{3,168,720}{370} = 8564.11 \text{ s} = 2.379 \text{ hours} \approx 2 \text{ h and } 22.7 \text{ min}$

c) $W = (9780)(24)\left(\frac{11.25}{2}\right) = 1,320,300$ J; $t = \frac{1,320,300}{370} = 3568.38 \text{ s} \approx 0.99 \text{ hours} \approx 59.5 \text{ min}$

In a location where water weighs $9820 \frac{\text{N}}{\text{m}^3}$:

a) $W = (9820)(24)(13.5) = 3,181,680$ J.

b) $t = \frac{3,181,680}{370} = 8599.14 \text{ s} = 2.389 \text{ hours} \approx 2 \text{ h and } 23.3 \text{ min}$

c) $W = (9820)(24)\left(\frac{11.25}{2}\right) = 1,325,700$ J; $t = \frac{1,325,700}{370} \approx 3583 \text{ s} \approx 0.995 \text{ hours} \approx 59.7 \text{ min}$

15. The slab is a disk of area $\pi x^2 = \pi\left(\frac{y}{2}\right)^2$, thickness Δy, and height below the top of the tank $(10 - y)$. So the work to pump the oil in this slab, ΔW, is $8820\ (10 - y)\pi\left(\frac{y}{2}\right)\Delta y$. The work to pump all the oil to top of the tank is $W = \int_0^{10} \frac{8820\pi}{4}\left(10y^2 - y^3\right) dy = \frac{8820\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^{10} = 1,837,500\pi \text{ J} \approx 5,772,676.5 \text{ J}$

16. Each slab of oil is to be pumped to a height of 11 m. So the work to pump a slab is $8820(11-y)(\pi)\left(\frac{y}{2}\right)^2$ and since the tank is half full and the volume of the original cone is $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(5^2\right)(10)=\frac{250\pi}{3}\text{ m}^3$, half the volume $=\frac{250\pi}{6}\text{ m}^3$, and with half the volume the cone is filled to a height y, $\frac{250\pi}{6}=\frac{1}{3}\pi\frac{y^2}{4}y \Rightarrow y=\sqrt[3]{500}$ m. So $W=\int_0^{\sqrt[3]{500}}\frac{8820\pi}{4}\left(11y^2-y^3\right)dy=\frac{8820\pi}{4}\left[\frac{11y^3}{3}-\frac{y^4}{4}\right]_0^{\sqrt[3]{500}} \approx 1{,}843{,}840$ J.

17. The typical slab between the planes at y and $y+\Delta y$ has a volume of $\Delta V=\pi(\text{radius})^2(\text{thickness})$ $=\pi\left(\frac{6}{2}\right)^2\Delta y=\pi\cdot 9\,\Delta y\text{ m}^3$. The force F required to lift the slab is equal to its weight: $F=7840\Delta V=7840\cdot 9\pi\,\Delta y\text{ N}\Rightarrow F=70{,}560\pi\,\Delta y$ N. The distance through which F must act is about $(9-y)$ m. The work it takes to lift all the kerosene is approximately $W\approx\sum_0^9\Delta W=\sum_0^9 70{,}560\pi(9-y)\Delta y$ J which is a Riemann sum. The work to pump the tank dry is the limit of these sums:
$W=\int_0^9 70{,}560\pi\,(9-y)\,dy=70{,}560\pi\left[9y-\frac{y^2}{2}\right]_0^9=70{,}560\pi\left(\frac{81}{2}\right)=(70{,}560)(40.5\pi)\approx 8{,}977{,}666$ J

18. (a) Follow all the steps of Example 5 but make the substitution of $10{,}100\frac{\text{N}}{\text{m}^3}$ for $8820\frac{\text{N}}{\text{m}^3}$. Then,
$W=\int_0^8\frac{10{,}100\pi}{4}(10-y)y^2\,dy=\frac{10{,}100\pi}{4}\left[\frac{10y^3}{3}-\frac{y^4}{4}\right]_0^8=\frac{10{,}100\pi}{4}\left(\frac{10\cdot 8^3}{3}-\frac{8^4}{4}\right)=\left(\frac{10{,}100\pi}{4}\right)\left(8^3\right)\left(\frac{10}{3}-2\right)=\frac{10{,}100\pi\cdot 8^3}{3}$
$\approx 5{,}415{,}268$ J

(b) Exactly as done in Example 5 but change the distance through which F acts to distance $\approx(11-y)$ m.
Then $W=\int_0^8\frac{8820\pi}{4}(11-y)y^2\,dy=\frac{8820\pi}{4}\left[\frac{11y^3}{3}-\frac{y^4}{4}\right]_0^8=\frac{8820\pi}{4}\left(\frac{11\cdot 8^3}{3}-\frac{8^4}{4}\right)=\left(\frac{8820\pi}{4}\right)\left(8^3\right)\left(\frac{11}{3}-2\right)$
$=\frac{8820\pi\cdot 8^3\cdot 5}{3\cdot 4}=(2940\pi)(8^2)(5)(2)\approx 5{,}911{,}220.7$ J

19. The typical slab between the planes at y and $y+\Delta y$ has a volume of about $\Delta V=\pi(\text{radius})^2(\text{thickness})$ $=\pi\left(\sqrt{y}\right)^2\Delta y\text{ m}^3$. The force $F(y)$ required to lift this slab is equal to its weight: $F(y)=11{,}500\cdot\Delta V$ $=11{,}500\pi\left(\sqrt{y}\right)^2\Delta y=11{,}500\pi y\,\Delta y$ m. The distance through which $F(y)$ must act to lift the slab to the top of the reservoir is about $(4-y)\text{ m}^3$, so the work done is approximately $\Delta W\approx 11{,}500\pi y(4-y)\Delta y$ J. The work done lifting all the slabs from $y=0$ m to $y=4$ m is approximately $W\approx\sum_{k=0}^{n}11{,}500\pi\,y_k\left(4-y_k\right)\Delta y$ J. Taking the limit of these Riemann sums as $n\to\infty$, we get $W=\int_0^4 11{,}500\pi\,y\,(4-y)\,dy=11{,}500\pi\int_0^4\left(4y-y^2\right)dy$ $=11{,}500\pi\left[2y^2-\frac{1}{3}y^3\right]_0^4=11{,}500\pi\left(32-\frac{64}{3}\right)=\frac{368{,}000\pi}{3}\text{ J}\approx 385{,}369$ J.

20. The typical slab between the planes at y and $y+\Delta y$ has volume of about $\Delta V=$ (length)(width)(thickness) $=\left(2\sqrt{2.25-y^2}\right)(3)\Delta y\text{ m}^3$. The force $F(y)$ required to lift this slab is equal to its weight:

$F(y) = 8300 \cdot \Delta V = 8300\left(2\sqrt{2.25-y^2}\right)(3)\ \Delta y = 49{,}800\sqrt{2.25-y^2}\ \Delta y$ N. The distance through which $F(y)$ must act to lift the slab to the level of 4.5 m above the top of the reservoir is about $(6-y)$ m, so the work done is approximately $\Delta W \approx 49{,}800\sqrt{2.25-y^2}\,(6-y)\Delta y$ J. The work done lifting all the slabs from $y=-1.5$ m to $y=1.5$ m is approximately $W \approx \sum_{k=0}^{n} 49{,}800\sqrt{2.25-y_k^2}\,(6-y_k)\Delta y$ J. Taking the limit of these Riemann sums as $n\to\infty$, we get $W = \int_{-1.5}^{1.5} 49{,}800\sqrt{2.25-y^2}\ (6-y)dy = 49{,}800\int_{-1.5}^{1.5}(6-y)\sqrt{2.25-y^2}\ dy$

$= 49{,}800\left[\int_{-1.5}^{1.5} 6\sqrt{2.25-y^2}\ dy - \int_{-1.5}^{1.5} y\sqrt{2.25-y^2}\ dy\right]$. To evaluate the first integral, we use we can interpret $\int_{-1.5}^{1.5}\sqrt{2.25-y^2}\,dy$ as the area of the semicircle whose radius is 1.5, thus $\int_{-1.5}^{1.5} 6\sqrt{2.25-y^2}\,dy$

$= 6\int_{-1.5}^{1.5}\sqrt{2.25-y^2}\,dy = 6\left[\frac{1}{2}\pi(1.5)^2\right] = 6.75\pi$. To evaluate the second integral let

$u = 2.25-y^2 \Rightarrow du = -2y\,dy;\ \ y=-1.5 \Rightarrow u=0,\ \ y=1.5 \Rightarrow u=0$, thus

$\int_{-1.5}^{1.5} y\sqrt{2.25-y^2}\ dy = -\frac{1}{2}\int_0^0 \sqrt{u}\ du = 0$. Thus, $49{,}800\left[\int_{-1.5}^{1.5} 6\sqrt{2.25-y^2}\,dy - \right.$

$\left. 49{,}800\left[\int_{-1.5}^{1.5} 6\sqrt{2.25-y^2}\,dy - \int_{-1.5}^{1.5} y\sqrt{2.25-y^2}\,dy\right] = 49{,}800(6.75\pi - 0) = 336{,}150\pi \approx 1{,}056{,}046\ \text{J}\right.$

21. The typical slab between the planes at y and $y+\Delta y$ has a volume of about $\Delta V = \pi\,(\text{radius})^2(\text{thickness})$ $= \pi\left(\sqrt{25-y^2}\right)^2\ \Delta\text{y m}^3$. The force $F(y)$ required to lift this slab is equal to its weight:

$F(y) = 9800 \cdot \Delta V = 9800\pi\left(\sqrt{25-y^2}\right)^2\ \Delta y = 9800\pi\left(25-y^2\right)\Delta y$ N. The distance through which $F(y)$ must act to lift the slab to the level of 4 m above the top of the reservoir is about $(4-y)$ m, so the work done is approximately $\Delta W \approx 9800\pi\left(25-y^2\right)(4-y)\Delta y\ \text{N}\cdot\text{m}$. The work done lifting all the slabs from $y=-5$ m to $y=0$ m is approximately $W \approx \sum_{-5}^{0} 9800\pi\left(25-y^2\right)(4-y)\Delta y\ \text{N}\cdot\text{m}$. Taking the limit of these Riemann sums, we get $W = \int_{-5}^{0} 9800\pi\left(25-y^2\right)(4-y)\,dy = 9800\pi\int_{-5}^{0}\left(100-25y-4y^2+y^3\right)dy$

$= 9800\pi\left[100y - \frac{25}{2}y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_{-5}^{0} = -9800\pi\left(-500 - \frac{25\cdot 25}{2} + \frac{4}{3}\cdot 125 + \frac{625}{4}\right) \approx\ 15{,}073{,}099.75$ J

22. The typical slab between the planes at y and $y+\Delta y$ has a volume of about $\Delta V = \pi\,(\text{radius})^2(\text{thickness})$ $= \pi\left(\sqrt{9-y^2}\right)^2\ \Delta y = \pi\left(9-y^2\right)\Delta y\ \text{m}^3$. The force is $F(y) = \frac{8800\ \text{N}}{\text{m}^3}\cdot\Delta V = 8800\pi\left(9-y^2\right)\Delta y$ N. The distance through which $F(y)$ must act to lift the slab to the level of 0.6 m above the top of the tank is about $(3.6-y)$ m, so the work done is $\Delta W \approx 8800\pi\left(9-y^2\right)(3.6-y)\Delta y$ J. The work done lifting all the slabs from $y=0$ m to $y=3$ m is approximately $W \approx \sum_{0}^{3} 8800\pi\left(9-y^2\right)(3.6-y)\Delta y$ J. Taking the limit of these

Riemann sums, we get $W = \int_0^3 8800\pi\left(9-y^2\right)(3.6-y)\,dy = 8800\pi\int_0^3\left(9-y^2\right)(3.6-y)\,dy$

$= 8800\pi\int_0^3\left(32.4-9y-3.6y^2+y^3\right)dy = 8800\pi\left[32.4y-\frac{9y^2}{2}-\frac{3.6y^3}{3}+\frac{y^4}{4}\right]_0^3$

$= 8800\pi\left(97.2-\frac{81}{2}-1.2\cdot 27+\frac{81}{4}\right) = (8800\pi)(44.55) \approx 1{,}231{,}630$ J. It would cost $(0.4)(1{,}231{,}630) = 492{,}652$¢ $= \$4926.52$. Yes, you can afford to hire the firm.

23. $F = m\frac{dv}{dt} = mv\frac{dv}{dx}$ by the chain rule $\Rightarrow W = \int_{x_1}^{x_2} mv\frac{dv}{dx}\,dx = m\int_{x_1}^{x_2}\left(v\frac{dv}{dx}\right)dx = m\left[\frac{1}{2}v^2(x)\right]_{x_1}^{x_2}$
$= \frac{1}{2}m\left[v^2(x_2)-v^2(x_1)\right] = \frac{1}{2}mv_2^2-\frac{1}{2}mv_1^2$, as claimed.

24. $W = (1/2)(0.06\text{ kg})(50\text{ m/s})^2 \approx 75$ J

25. 144 km/h $= \frac{90\text{ km}}{1\text{ h}}\cdot\frac{1\text{ h}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}\cdot\frac{1000\text{ m}}{1\text{ km}} = 40$ m/s; $m = 0.15$ kg;
$W = \left(\frac{1}{2}\right)(0.15\text{ kg})(40\text{ m/s})^2 \approx 120$ J

26. $W = (1/2)(0.05\text{ kg})(84\text{ m/s})^2 \approx 176.4$ J

27. $m = 0.06$ kg; $W = \frac{1}{2}(0.06\text{ kg})(68\text{ m/s})^2 = 138.72$ J

28. $m = 0.2$ kg; $W = \left(\frac{1}{2}\right)(0.2\text{ kg})(40\text{ m/s})^2 \approx 160$ J

29. We imagine the milkshake divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0, 18]$. The typical slab between the planes at y and $y+\Delta y$ has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{y+36}{12}\right)^2\Delta y\text{ cm}^3$. The force $F(y)$ required to lift this slab is equal to its weight: $F(y) = 0.8\cdot\frac{9.8}{1000}\Delta V = \frac{7.84\pi}{1000}\left(\frac{y+36}{12}\right)^2\Delta y$ N. The distance through which $F(y)$ must act to lift this slab to the level of 3 cm above the top is about $(21-y)$ cm. The work done lifting the slab is about $\Delta W = \left(\frac{7.84\pi}{1000}\right)\frac{(y+36)^2}{12^2}\frac{(21-y)}{100}\Delta y$ J. The work done lifting all the slabs from $y=0$ to $y=18$ is approximately $W = \sum_0^{18}\frac{7.84\pi}{10^5\cdot 12}(y+36)^2(21-y)\Delta y$ J which is a Riemann sum. The work is the limit of these sums as the norm of the partition goes to zero:

$W = \int_0^{18}\frac{7.84\pi}{10^5\cdot 12}(y+36)^2(21-y)\,dy = \frac{7.84\pi}{10^5\cdot 12}\int_0^{18}\left(27{,}216+216y-51y^2-y^3\right)dy$

$= \frac{7.84\pi}{10^5\cdot 12}\left[-\frac{y^4}{4}-17y^3+108y^2+27{,}216y\right]_0^{18} = \frac{7.84\pi}{10^5\cdot 12}\left[-\frac{18^4}{4}-17\cdot 18^3+108\cdot 18^2+27{,}216\cdot 18\right] \approx 2.175$ J

30. We fill the pipe and the tank. To find the work required to fill the tank note that radius = 3 m, then $\Delta V = \pi\cdot 9\Delta y\text{ m}^3$. The force required will be $F = 9800\cdot\Delta V = 9800\cdot 9\pi\Delta y = 88{,}200\pi\Delta y$ N. The distance through which F must act is y so the work done lifting the slab is about $\Delta W_1 = 88{,}200\pi\cdot y\cdot\Delta y$ N. The work it

takes to lift all the water into the tank is: $W_1 \approx \sum_{108}^{115.5} \Delta W_1 = \sum_{108}^{115.5} 88{,}200\pi \cdot y \cdot \Delta y$ J. Taking the limit we end up with $W_1 = \int_{108}^{115.5} 88{,}200\pi\, y\, dy = 88{,}200\pi \left[\frac{y^2}{2}\right]_{108}^{115.5} = \frac{88{,}200\pi}{2}[115.5^2 - 108^2] \approx 232{,}234{,}776$ J

To find the work required to fill the pipe, do as above, but take the radius to be $5 \text{ cm} = \frac{1}{20}$ m. Then $\Delta V = \pi \cdot \frac{1}{400} \Delta y \text{ m}^3$ and $F = 9800 \cdot \Delta V = \frac{9800\pi}{400} \Delta y$. Also take different limits of summation and integration:
$W_2 \approx \sum_{0}^{108} \Delta W_2 \Rightarrow W_2 = \int_0^{108} \frac{9800}{400} \pi\, y\, dy = \frac{9800\pi}{400}\left[\frac{y^2}{2}\right]_0^{108} = \left(\frac{9800\pi}{400}\right)\left(\frac{108^2}{2}\right) \approx 448{,}883$ J

The total work is $W = W_1 + W_2 \approx 232{,}234{,}776 + 448{,}883 \approx 232{,}683{,}659$ J. The time it takes to fill the tank and the pipe is $\text{Time} = \frac{W}{2000} \approx \frac{232{,}683{,}659}{2000} \approx 116{,}342 \text{ s} \approx 32$ h

31. Work $= \int_{6{,}370{,}000}^{35{,}780{,}000} \frac{1000\, MG}{r^2}\, dr = 1000\, MG \int_{6{,}370{,}000}^{35{,}780{,}000} \frac{dr}{r^2} = 1000\, MG\left[-\frac{1}{r}\right]_{6{,}370{,}000}^{35{,}780{,}000}$
$= (1000)\left(5.975 \times 10^{24}\right)\left(6.672 \times 10^{-11}\right)\left(\frac{1}{6{,}370{,}000} - \frac{1}{35{,}780{,}000}\right) \approx 5.144 \times 10^{10}$ J

32. (a) Let ρ be the x-coordinate of the second electron. Then $r^2 = (\rho - 1)^2$
$\Rightarrow W = \int_{-1}^{0} F(\rho)\, d\rho = \int_{-1}^{0} \frac{(23\times10^{-29})}{(\rho-1)^2}\, d\rho = -\left[\frac{23\times10^{-29}}{\rho-1}\right]_{-1}^{0} = (23\times10^{-29})\left(1 - \frac{1}{2}\right) = 11.5 \times 10^{-29}$

(b) $W = W_1 + W_2$ where W_1 is the work done against the field of the first electron and W_2 is the work done against the field of the second electron. Let ρ be the x-coordinate of the third electron. Then $r_1^2 = (\rho - 1)^2$ and $r_2^2 = (\rho + 1)^2$
$\Rightarrow W_1 = \int_3^5 \frac{23\times10^{-29}}{r_1^2}\, d\rho = \int_3^5 \frac{23\times10^{-29}}{(\rho-1)^2}\, d\rho = -23\times10^{-29}\left[\frac{1}{\rho-1}\right]_3^5 = (-23\times10^{-29})\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{23}{4}\times10^{-29}$, and
$W_2 = \int_3^5 \frac{23\times10^{-29}}{r_2^2}\, d\rho = \int_3^5 \frac{23\times10^{-29}}{(\rho+1)^2}\, d\rho = -23\times10^{-29}\left[\frac{1}{\rho+1}\right]_3^5 = (-23\times10^{-29})\left(\frac{1}{6} - \frac{1}{4}\right) = \frac{23\times10^{-29}}{12}(3-2)$
$= \frac{23}{12}\times10^{-29}$.
Therefore $W = W_1 + W_2 = \left(\frac{23}{4}\times10^{-29}\right) + \left(\frac{23}{12}\times10^{-29}\right) = \frac{23}{3}\times10^{-29} \approx 7.67\times10^{-29}$ J

33. To find the width of the plate at a typical depth y, we first find an equation for the line of the plate's right-hand edge: $y = x - 5$. If we let x denote the width of the right-hand half of the triangle at depth y, then $x = 5 + y$ and the total width is $L(y) = 2x = 2(5 + y)$. The depth of the strip is $(-y)$. The force exerted by the water against one side of the plate is therefore $F = \int_{-1.6}^{-0.6} w(-y) \cdot L(y)\, dy = \int_{-1.6}^{-0.6} 9800 \cdot (-y) \cdot 2(1.6 + y)\, dy$
$= 19{,}600 \int_{-1.6}^{-0.6} \left(-1.6y - y^2\right) dy = 19{,}600\left[-0.8y^2 - \frac{1}{3}y^3\right]_{-1.6}^{-0.6}$
$= 19{,}600\left[\left(-0.8 \cdot 0.36 + \frac{1}{3} \cdot 0.216\right) - \left(-0.8 \cdot 2.56 + \frac{1}{3} \cdot 4.096\right)\right] = (19{,}600)(0.6826667 - 0.216)$
$= (19{,}600)(0.466667) = 9{,}146.7$ N

34. An equation for the line of the plate's right-hand edge is $y = x \Rightarrow x = y$. Thus the total width is $L(y) = 2x = 2y$. The depth of the strip is $(0.6 - y)$. The force exerted by the water is

$$F = \int_{-1}^{0} w(0.6 - y)\,L(y)\,dy = \int_{-1}^{0} 9800 \cdot (0.6 - y) \cdot 2y\,dy = 19{,}600 \int_{-1}^{0} \left(0.6 - 0.4y - y^2\right) dy$$

$$= 19{,}600 \left[0.6y - 0.2y^2 - \tfrac{y^3}{3}\right]_{-1}^{0} = (-19{,}600)\left(-0.6 - 0.2 + \tfrac{1}{3}\right) = (-19{,}600)(0.46667) = 9{,}146.7 \text{ N}$$

35. (a) The width of the strip is $L(y) = 1.2$, the depth of the strip is $(3 - y) \Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right) F(y)\,dy$

$$= \int_0^{0.9} 9800(3 - y)(1.2)dy = 39{,}200 \int_0^{0.9} (3 - y)\,dy = 39{,}200 \left[3y - \tfrac{y^2}{2}\right]_0^{0.9} = 39{,}200\left(2.7 - \tfrac{0.81}{2}\right) = 89{,}964 \text{ N}$$

(b) The width of the strip is $L(y) = 0.9$, the depth of the strip is $(3 - y) \Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right) F(y)\,dy$

$$= \int_0^{1.2} 9800(3 - y)(0.9)\,dy = 29{,}400 \int_0^{1.2} (3 - y)dy = 29{,}400 \left[3y - \tfrac{y^2}{2}\right]_0^{1.2} = 29{,}400(3.6 - 0.72) = 84{,}672 \text{ N}$$

36. The width of the strip is $L(y) = 2\sqrt{25 - y^2}$, the depth of the strip is $(6 - y) \Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right) F(y)\,dy$

$$= \int_0^5 9800\,(6 - y)\left(2\sqrt{25 - y^2}\right) dy = 19{,}600 \int_0^5 (6 - y)\sqrt{25 - y^2}\,dy = 19{,}600 \left[\int_0^5 6\sqrt{25 - y^2}\,dy - \int_0^5 y\sqrt{25 - y^2}\,dy\right]$$

To evaluate the first integral, we use we can interpret $\int_0^5 \sqrt{25 - y^2}\,dy$ as the area of a quarter circle whose radius is 5, thus $\int_0^5 6\sqrt{25 - y^2}\,dy = 6\int_0^5 \sqrt{25 - y^2}\,dy = 6\left[\tfrac{1}{4}\pi(5)^2\right] = \tfrac{75\pi}{2}$. To evaluate the second integral let $u = 25 - y^2 \Rightarrow du = -2y\,dy$; $y = 0. \Rightarrow u = 25$, $y = 5 \Rightarrow u = 0$, thus $\int_0^5 y\sqrt{25 - y^2}\,dy = -\tfrac{1}{2}\int_{25}^{0} \sqrt{u}\,du$ $= \tfrac{1}{2}\int_0^{25} u^{1/2}du = \tfrac{1}{3}\left[u^{3/2}\right]_0^{25} = \tfrac{125}{3}$. Thus, $19{,}600\left[\int_0^5 6\sqrt{25 - y^2}\,dy - \int_0^5 y\sqrt{25 - y^2}\,dy\right] = 19{,}600\left(\tfrac{75\pi}{2} - \tfrac{125}{3}\right)$ $\approx 1{,}492{,}404$ N.

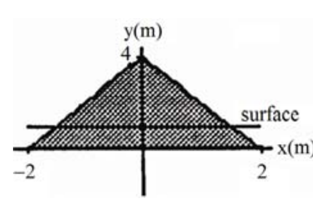
37. Using the coordinate system of Exercise 32, we find the equation for the line of the plate's right-hand edge to be $y = 2x - 4 \Rightarrow x = \frac{y+4}{2}$ and $L(y) = 2x = y + 4$. The depth of the strip is $(1 - y)$.

(a) $F = \int_{-4}^{0} w(1 - y)L(y)\,dy = \int_{-4}^{0} 9800 \cdot (1 - y)(y + 4)\,dy = 9800\int_{-4}^{0}\left(4 - 3y - y^2\right) dy = 9800\left[4y - \tfrac{3y^2}{2} - \tfrac{y^3}{3}\right]_{-4}^{0}$

$$= (-9800)\left[(-4)(4) - \tfrac{(3)(16)}{2} + \tfrac{64}{3}\right] = (-9800)(-16 - 24 + \tfrac{64}{3}) = \tfrac{(-9800)(-120+64)}{3} \approx 182{,}933 \text{ N}$$

(b) $F = (-10{,}050)\left[(-4)(4) - \tfrac{(3)(16)}{2} + \tfrac{64}{3}\right] = \tfrac{(-10{,}050)(-120+64)}{3} \approx 187{,}600$ N

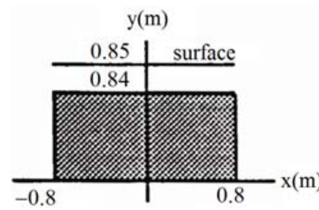
38. Using the coordinate system given, we find an equation for the line of the plate's right-hand edge to be $y = -2x + 4 \Rightarrow x = \frac{4-y}{2}$ and $L(y) = 2x = 4 - y$. The depth of the strip is $(1 - y)$ $\Rightarrow F = \int_0^1 w\,(1 - y)(4 - y)\,dy = 9800\int_0^1\left(y^2 - 5y + 4\right) dy$



$= 9800\left[\frac{y^3}{3} - \frac{5y^2}{2} + 4y\right]_0^1 = (9800)\left(\frac{1}{3} - \frac{5}{2} + 4\right) = (9800)\left(\frac{2-15+24}{6}\right)$

$= \frac{(9800)(11)}{6} \approx 17{,}967 \text{ N}$

39. Using the coordinate system given in the accompanying figure, we see that the total width is $L(y) = 1.6$ and the depth of the strip is $(0.85 - y) \Rightarrow F = \int_0^{0.84} w(0.85 - y)L(y)\,dy$

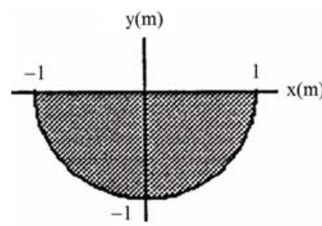


$= \int_0^{0.84} 10{,}050(0.85 - y)\cdot 1.6\,dy = (10{,}050)(1.6)\int_0^{0.84}(0.85 - y)\,dy$

$= (10{,}050)(1.6)\left[0.85y - \frac{y^2}{2}\right]_0^{0.84} = 16{,}080\left[(0.85)(0.84) - \frac{0.84}{2}\right]$

$= \frac{(16{,}080)(0.84)(1.7-0.84)}{(2)} = 5808 \text{ N}$

40. Using the coordinate system given in the accompanying figure, we see that the right-hand edge is $x = \sqrt{1-y^2}$ so the total width is $L(y) = 2x = 2\sqrt{1-y^2}$ and the depth of the strip is $(-y)$. The force exerted by the water is therefore



$F = \int_{-1}^{0} w\cdot(-y)\cdot 2\sqrt{1-y^2}\,dy$

$= 9800\int_{-1}^{0}\sqrt{1-y^2}\,(-2y)\,dy = 9800\left[\frac{2}{3}\left(1-y^2\right)^{3/2}\right]_{-1}^{0}$

$= (9800)\left(\frac{2}{3}\right)(1-0) = 6533 \text{ N}$

41. (a) $F = \left(9800\frac{\text{N}}{\text{m}^3}\right)(2 \text{ m})\left(1 \text{ m}^2\right) = 19{,}600 \text{ N}$

(b) The width of the strip is $L(y) = 1,$ the depth of the strip is $(2 - y) \Rightarrow F = \int_a^b w\cdot\left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right)F(y)\,dy$

$= \int_0^1 9800(2-y)(1)\,dy = 9800\int_0^1(2-y)\,dy = 9800\left[2y - \frac{y^2}{2}\right]_0^1 = 9800\left(2 - \frac{1}{2}\right) = 14{,}700 \text{ N}$

(c) The width of the strip is $L(y) = 1,$ the depth of the strip is $(2 - y),$ the height of the strip is $\sqrt{2}\,dy$

$\Rightarrow F = \int_a^b w\cdot\left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right)F(y)\,dy = \int_0^{1/\sqrt{2}} 9800\,(2-y)(1)\sqrt{2}\,dy = 9800\sqrt{2}\int_0^{1/\sqrt{2}}(2-y)\,dy$

$= 9800\sqrt{2}\left[2y - \frac{y^2}{2}\right]_0^{1/\sqrt{2}} = 9800\sqrt{2}\left(\frac{2}{\sqrt{2}} - \frac{1}{4}\right) \approx 16{,}135 \text{ N}$

42. The width of the strip is $L(y) = \frac{3}{4}\left(2\sqrt{3} - \text{y}\right),$ the depth of the strip is $(6 - y),$ the height of the strip is $\frac{2}{\sqrt{3}}dy$

$\Rightarrow F = \int_a^b w\cdot\left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right)F(y)\,dy = \int_0^{2\sqrt{3}} 9800(6-y)\cdot\frac{3}{4}\left(2\sqrt{3} - y\right)\frac{2}{\sqrt{3}}\,dy = \frac{14{,}700}{\sqrt{3}}\int_0^{2\sqrt{3}}\left(12\sqrt{3} - 6y - 2y\sqrt{3} + y^2\right)dy$

$= \frac{14{,}700}{\sqrt{3}}\left[12y\sqrt{3} - 3y^2 - y^2\sqrt{3} + \frac{y^3}{3}\right]_0^{2\sqrt{3}} = \frac{14{,}700}{\sqrt{3}}\left(72 - 36 - 12\sqrt{3} + 8\sqrt{3}\right) \approx 246{,}734 \text{ N}$

43. The coordinate system is given in the text. The right-hand edge is $x = \sqrt{y}$ and the total width is $L(y) = 2x = 2\sqrt{y}$.

(a) The depth of the strip is $(2-y)$ so the force exerted by the liquid on the gate is $F = \int_0^1 w(2-y)L(y)\,dy$

$= \int_0^1 8{,}000(2-y)\cdot 2\sqrt{y}\,dy = 16{,}000\int_0^1 (2-y)\sqrt{y}\,dy = 16{,}000\int_0^1 \left(2y^{1/2} - y^{3/2}\right)dy = 16{,}000\left[\frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1$

$= 16{,}000\left(\frac{4}{3} - \frac{2}{5}\right) = \left(\frac{16{,}000}{15}\right)(20-6) \approx 14{,}933$ N

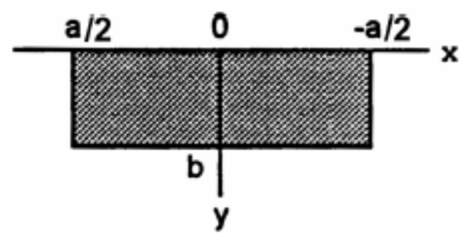
(b) We need to solve $25{,}000 = \int_0^1 w(H-y)\cdot 2\sqrt{y}\,dy$ for h. $25{,}000 = 16{,}000\left(\frac{2H}{3} - \frac{2}{5}\right) \Rightarrow H = 2.94$ m.

44. Suppose that h is the maximum height. Using the coordinate system given in the text, we find an equation for the line of the end plate's right-hand edge is $y = 3x \Rightarrow x = \frac{1}{3}y$. The total width is $L(y) = 2x = \frac{2}{3}y$ and the depth of the typical horizontal strip at level y is $(h-y)$. Then the force is $F = \int_0^h w(h-y)L(y)\,dy = F_{\max}$, where $F_{\max} = 25{,}000$ N. Hence,

$F_{\max} = w\int_0^h (h-y)\cdot\frac{2}{3}y\,dy = (9800)\left(\frac{2}{3}\right)\int_0^h \left(hy - y^2\right)dy = (9800)\left(\frac{2}{3}\right)\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h$

$= (9800)\left(\frac{2}{3}\right)\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = (9800)\left(\frac{2}{3}\right)\left(\frac{1}{6}\right)h^3 = \frac{9800}{9}h^3 \Rightarrow h = \sqrt[3]{9\left(\frac{F_{\max}}{9800}\right)} = \sqrt[3]{9\left(\frac{25{,}000}{9800}\right)} \approx 2.842$ m. The volume of water which the tank can hold is $V = \frac{1}{2}$ (Base)(Height) $\cdot 10$, where Height $= h$ and $\frac{1}{2}$ (Base) $= \frac{1}{3}h \Rightarrow V = \left(\frac{1}{3}h^2\right)(10) = \frac{10}{3}h^2 \approx \frac{10}{3}(2.842)^2 \approx 26.92$ m^3.

45. The pressure at level y is $p(y) = w\cdot y \Rightarrow$ the average pressure is $\overline{p} = \frac{1}{b}\int_0^b p(y)\,dy = \frac{1}{b}\int_0^b w\cdot y\,dy$ $= \frac{1}{b}w\left[\frac{y^2}{2}\right]_0^b = \left(\frac{w}{b}\right)\left(\frac{b^2}{2}\right) = \frac{wb}{2}$. This is the pressure at level $\frac{b}{2}$, which is the pressure at the middle of the plate.



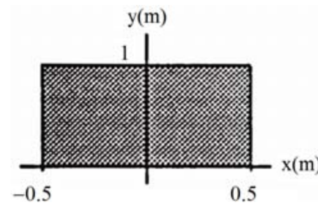
46. The force exerted by the fluid is $F = \int_0^b w(\text{depth})(\text{length})\,dy = \int_0^b w\cdot y\cdot a\,dy = (w\cdot a)\int_0^b y\,dy = (w\cdot a)\left[\frac{y^2}{2}\right]_0^b$

$= w\left(\frac{ab^2}{2}\right) = \left(\frac{wb}{2}\right)(ab) = \overline{p}\cdot$ Area, where $\overline{p}$ is the average value of the pressure.

47. When the water reaches the top of the tank the force on the movable side is $\int_{-1}^{0}(9800)\left(2\sqrt{1-y^2}\right)(-y)dy$

$= (9800)\int_{-1}^{0}\left(1-y^2\right)^{1/2}(-2y)\,dy = (9800)\left[\frac{2}{3}\left(1-y^2\right)^{3/2}\right]_{-1}^{0} = (9800)\left(\frac{2}{3}\right)(1) = 6{,}533$ J. The force compressing the spring is $F = 3{,}000x$ so when the tank is full we have $6{,}533 = 3{,}000x \Rightarrow x \approx 2.18$ m. Therefore the movable end does not reach the required 2.5 m to allow drainage $\Rightarrow$ the tank will overflow.

48. (a) Using the given coordinate system we see that the total width $L(y) = 1$ and the depth of the strip is $(1-y)$.



Thus, $F = \int_0^1 w(1-y)L(y)\,dy = \int_0^1 (9800)(1-y)\,dy$

$= (9800)\int_0^1 (1-y)\,dy = (9800)\left[y - \frac{y^2}{2}\right]_0^1$

$= (9800)\left(1-\frac{1}{2}\right) = (9800)\left(\frac{1}{2}\right) = 4900$ N

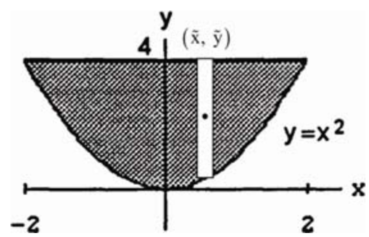
(b) Find a new water level Y such that $F_Y = (0.75)(4900\text{ N}) = 3675$ N. The new depth of the strip is $(Y-y)$ and Y is the new upper limit of integration. Thus, $F_Y = \int_0^Y w(Y-y)L(y)\,dy = 9800\int_0^Y (Y-y)\,dy$

$= (9800)\int_0^Y (Y-y)\,dy = (9800)\left[Yy - \frac{y^2}{2}\right]_0^Y = (9800)\left(Y^2 - \frac{Y^2}{2}\right) = (9800)\left(\frac{Y^2}{2}\right)$. Therefore,

$Y = \sqrt{\frac{2F_Y}{(9800)}} = \sqrt{\frac{7350}{9800}} = \sqrt{0.75} \approx 0.866$ m. So, $\Delta Y = 1 - Y \approx 1 - 0.866 \approx 0.134$ m ≈ 13.4 cm

6.6 MOMENTS AND CENTERS OF MASS

1. Since the plate is symmetric about the y-axis and its density is constant, the distribution of mass is symmetric about the y-axis and the center of mass lies on the y-axis. This means that $\bar{x} = 0$. It remains to find $\bar{y} = \frac{M_x}{M}$. We model the distribution of mass with *vertical* strips. The typical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{x^2+4}{2}\right)$, length: $4 - x^2$ width: dx,



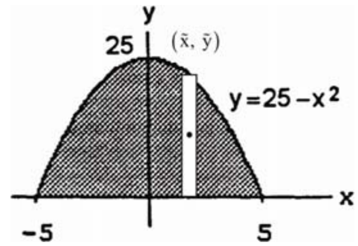
area: $dA = \left(4-x^2\right)dx$, mass: $dm = \delta dA = \delta\left(4-x^2\right)dx$

The moment of the strip about the x-axis is $\tilde{y}\,dm = \left(\frac{x^2+4}{2}\right)\delta\left(4-x^2\right)dx = \frac{\delta}{2}\left(16-x^4\right)dx$. The moment of the plate about the x-axis is $M_x = \int \tilde{y}\,dm = \int_{-2}^{2}\frac{\delta}{2}\left(16-x^4\right)dx = \frac{\delta}{2}\left[16x - \frac{x^5}{5}\right]_{-2}^{2} = \frac{\delta}{2}\left[\left(16\cdot 2 - \frac{2^5}{5}\right) - \left(-16\cdot 2 + \frac{2^5}{5}\right)\right]$

$= \frac{\delta\cdot 2}{2}\left(32 - \frac{32}{5}\right) = \frac{128\delta}{5}$. The mass of the plate is $M = \int \delta\,(4-x^2)\,dx = \delta\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 2\delta\left(8-\frac{8}{3}\right) = \frac{32\delta}{3}$.

Therefore $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{128\delta}{5}\right)}{\left(\frac{32\delta}{3}\right)} = \frac{12}{5}$ The plate's center of mass is the point $(\bar{x}, \bar{y}) = \left(0, \frac{12}{5}\right)$.

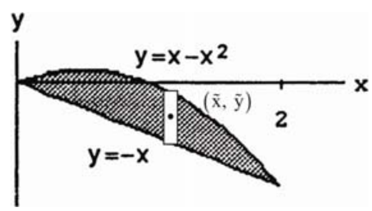
2. Applying the symmetry argument analogous to the one in Exercise 1, we find $\overline{x}=0$. To find $\overline{y}=\frac{Mx}{M}$, we use the *vertical* strips technique. The typical strip has center of mass: $(\tilde{x},\tilde{y})=\left(x,\frac{25-x^2}{2}\right)$, length: $25-x^2$, width: dx, area: $dA=\left(25-x^2\right)dx$, mass: $dm=\delta\, dA=\delta\left(25-x^2\right)dx$.



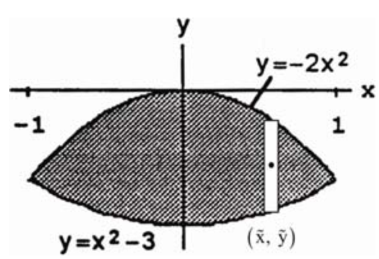
The moment of the strip about the x-axis is $\tilde{y}\,dm=\left(\frac{25-x^2}{2}\right)\delta\left(25-x^2\right)dx=\frac{\delta}{2}\left(25-x^2\right)^2 dx$.

The moment of the plate about the x-axis is $M_x=\int\tilde{y}\,dm=\int_{-5}^{5}\frac{\delta}{2}\left(25-x^2\right)^2 dx=\frac{\delta}{2}\int_{-5}^{5}\left(625-50x^2+x^4\right)dx$ $=\frac{\delta}{2}\left[625x-\frac{50}{3}x^3+\frac{x^5}{5}\right]_{-5}^{5}=2\cdot\frac{\delta}{2}\left(625\cdot5-\frac{50}{3}\cdot5^3+\frac{5^5}{5}\right)=\delta\cdot625\left(5-\frac{10}{3}+1\right)=\delta\cdot625\cdot\left(\frac{8}{3}\right)$. The mass of the plate is $M=\int dm=\int_{-5}^{5}\delta\left(25-x^2\right)dx=\delta\left[25x-\frac{x^3}{3}\right]_{-5}^{5}=2\delta\left(5^3-\frac{5^3}{3}\right)=\frac{4}{3}\delta\cdot5^3$. Therefore $\overline{y}=\frac{M_x}{M}=\frac{\delta\cdot5^4\cdot\left(\frac{8}{3}\right)}{\delta\cdot5^3\cdot\left(\frac{4}{3}\right)}=10$. The plate's center of mass is the point $(\overline{x},\overline{y})=(0,10)$.

3. Intersection points: $x-x^2=-x\Rightarrow 2x-x^2=0$ $\Rightarrow x(2-x)=0\Rightarrow x=0$ or $x=2$. The typical *vertical* strip has center of mass: $(\tilde{x},\tilde{y})=\left(x,\frac{\left(x-x^2\right)+(-x)}{2}\right)$ $=\left(x,-\frac{x^2}{2}\right)$, length: $\left(x-x^2\right)-(-x)=2x-x^2$, width: dx, area: $dA=\left(2x-x^2\right)dx$, mass: $dm=\delta\,dA$ $=\delta\left(2x-x^2\right)dx$. The moment of the strip about the x-axis is $\tilde{y}\,dm=\left(-\frac{x^2}{2}\right)\delta\left(2x-x^2\right)dx$; about the y-axis it is $\tilde{x}\,dm=x\cdot\delta\,(2x-x^2)\,dx$. Thus, $M_x=\int\tilde{y}\,dm=-\int_0^2\left(\frac{\delta}{2}x^2\right)\left(2x-x^2\right)dx=-\frac{\delta}{2}\int_0^2\left(2x^3-x^4\right)dx$ $=-\frac{\delta}{2}\left[\frac{x^4}{2}-\frac{x^5}{5}\right]_0^2=-\frac{\delta}{2}\left(2^3-\frac{2^5}{5}\right)=-\frac{\delta}{2}\cdot2^3\left(1-\frac{4}{5}\right)=-\frac{4\delta}{5}$; $M_y=\int\tilde{x}\,dm=\int_0^2 x\cdot\delta\left(2x-x^2\right)dx$ $=\delta\int_0^2\left(2x^2-x^3\right)dx=\delta\left[\frac{2}{3}x^3-\frac{x^4}{4}\right]_0^2=\delta\left(\frac{2}{3}\cdot2^3-\frac{2^4}{4}\right)=\frac{\delta\cdot2^4}{12}=\frac{4\delta}{3}$; $M=\int dm=\int_0^2\delta\left(2x-x^2\right)dx$ $=\delta\int_0^2\left(2x-x^2\right)dx=\delta\left[x^2-\frac{x^3}{3}\right]_0^2=\delta\left(4-\frac{8}{3}\right)=\frac{4\delta}{3}$. Therefore, $\overline{x}=\frac{M_y}{M}=\left(\frac{4\delta}{3}\right)\left(\frac{3}{4\delta}\right)=1$ and $\overline{y}=\frac{M_x}{M}$ $=\left(-\frac{4\delta}{5}\right)\left(\frac{3}{4\delta}\right)=-\frac{3}{5}\Rightarrow(\overline{x},\overline{y})=\left(1,-\frac{3}{5}\right)$ is the center of mass.



4. Intersection points: $x^2-3=-2x^2\Rightarrow 3x^2-3=0$ $\Rightarrow 3(x-1)(x+1)=0\Rightarrow x=-1.$ or $x=1$ Applying the symmetry argument analogous to the one in Exercise 1, we find $\overline{x}=0$ The typical *vertical* strip has center of mass: $(\tilde{x},\tilde{y})=\left(x,\frac{-2x^2+\left(x^2-3\right)}{2}\right)=\left(x,\frac{-x^2-3}{2}\right)$, length: $-2x^2-\left(x^2-3\right)$ $=3\left(1-x^2\right)$, width: dx, area: $dA=3\left(1-x^2\right)dx$,



mass: $dm = \delta\, dA = 3\delta\left(1-x^2\right) dx$. The moment of the strip about the x-axis is

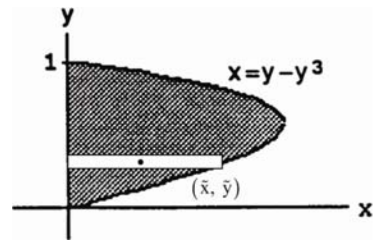
$\tilde{y}\, dm = \frac{3}{2}\delta\left(-x^2-3\right)\left(1-x^2\right) dx = \frac{3}{2}\delta\left(x^4+3x^2-x^2-3\right) dx = \frac{3}{2}\delta\left(x^4+2x^2-3\right) dx;$

$M_x = \int \tilde{y}\, dm = \frac{3}{2}\delta\int_{-1}^{1}\left(x^4+2x^2-3\right) dx = \frac{3}{2}\delta\left[\frac{x^5}{5}+\frac{2x^3}{3}-3x\right]_{-1}^{1} = \frac{3}{2}\cdot\delta\cdot 2\left(\frac{1}{5}+\frac{2}{3}-3\right) = 3\delta\left(\frac{3+10-45}{15}\right) = -\frac{32\delta}{5};$

$M = \int dm = 3\delta\int_{-1}^{1}\left(1-x^2\right) dx = 3\delta\left[x-\frac{x^3}{3}\right]_{-1}^{1} = 3\delta\cdot 2\left(1-\frac{1}{3}\right) = 4\delta.$ Therefore, $\overline{y} = \frac{M_x}{M} = -\frac{\delta\cdot 32}{5\cdot\delta\cdot 4} = -\frac{8}{5}$

$\Rightarrow (\overline{x}, \overline{y}) = \left(0, -\frac{8}{5}\right)$ is the center of mass.

5. The typical *horizontal* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{y-y^3}{2}\right)$, length: $y-y^3$, width: dy, area: $dA = \left(y-y^3\right) dy$, mass: $dm = \delta\, dA = \delta\left(y-y^3\right) dy$. The moment of the strip about the y-axis is $\tilde{x}\, dm = \delta\left(\frac{y-y^3}{2}\right)\left(y-y^3\right) dy = \frac{\delta}{2}\left(y-y^3\right)^2 dy$



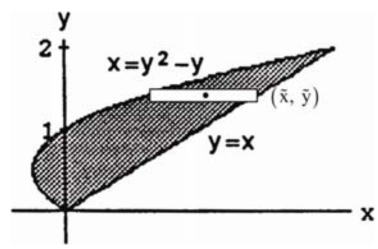
$= \frac{\delta}{2}\left(y^2-2y^4+y^6\right) dy$; the moment about the x-axis is $\tilde{y}\, dm = \delta y\left(y-y^3\right) dy = \delta\left(y^2-y^4\right) dy$. Thus,

$M_x = \int \tilde{y}\, dm = \delta\int_0^1\left(y^2-y^4\right) dy = \delta\left[\frac{y^3}{3}-\frac{y^5}{5}\right]_0^1 = \delta\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{2\delta}{15};\ M_y = \int \tilde{x}\, dm = \frac{\delta}{2}\int_0^1\left(y^2-2y^4+y^6\right) dy$

$= \frac{\delta}{2}\left[\frac{y^3}{3}-\frac{2y^5}{5}+\frac{y^7}{7}\right]_0^1 = \frac{\delta}{2}\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right) = \frac{\delta}{2}\left(\frac{35-42+15}{3\cdot5\cdot7}\right) = \frac{4\delta}{105};\ M = \int dm = \delta\int_0^1 (y-y)^3\, dy = \delta\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1$

$= \delta\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\delta}{4}$. Therefore, $\overline{x} = \frac{M_y}{M} = \left(\frac{4\delta}{105}\right)\left(\frac{4}{\delta}\right) = \frac{16}{105}$ and $\overline{y} = \frac{M_x}{M} = \left(\frac{2\delta}{15}\right)\left(\frac{4}{\delta}\right) = \frac{8}{15} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{16}{105}, \frac{8}{15}\right)$ is the center of mass.

6. Intersection points: $y = y^2-y \Rightarrow y^2-2y = 0$ $\Rightarrow y(y-2) = 0 \Rightarrow y = 0$ or $y = 2$ The typical *horizontal* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{\left(y^2-y\right)+y}{2}, y\right) = \left(\frac{y^2}{2}, y\right)$, length: $y-\left(y^2-y\right) = 2y-y^2$, width: dy, area: $dA = \left(2y-y^2\right) dy$, mass: $dm = \delta dA = \delta\left(2y-y^2\right) dy$.



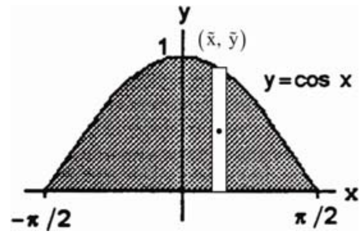
The moment about the y-axis is $\tilde{x}\, dm = \frac{\delta}{2}\cdot y^2\left(2y-y^2\right) dy = \frac{\delta}{2}\left(2y^3-y^4\right) dy$; the moment about the x-axis is $\tilde{y}\, dm = \delta y\left(2y-y^2\right) dy = \delta\left(2y^2-y^3\right) dy$. Thus, $M_x = \int \tilde{y}\, dm = \delta\int_0^2\left(2y^2-y^3\right) dy = \delta\left[\frac{2y^3}{3}-\frac{y^4}{4}\right]_0^2$

$= \delta\left(\frac{16}{3}-\frac{16}{4}\right) = \frac{16\delta}{12}(4-3) = \frac{4\delta}{3};\ M_y = \int \tilde{x}\, dm = \int_0^2 \frac{\delta}{2}\left(2y^3-y^4\right) dy = \frac{\delta}{2}\left[\frac{y^4}{2}-\frac{y^5}{5}\right]_0^2 = \frac{\delta}{2}\left(8-\frac{32}{5}\right)$

$= \frac{\delta}{2}\left(\frac{40-32}{5}\right) = \frac{4\delta}{5};\ M = \int dm = \int_0^2 \delta\left(2y-y^2\right) dy = \delta\left[y^2-\frac{y^3}{3}\right]_0^2 = \delta\left(4-\frac{8}{3}\right) = \frac{4\delta}{3}$. Therefore,

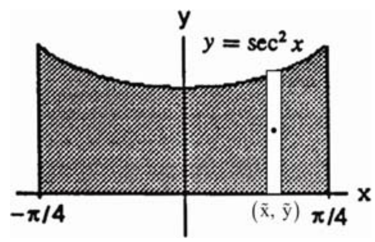
$\overline{x} = \frac{M_y}{M} = \left(\frac{4\delta}{5}\right)\left(\frac{3}{4\delta}\right) = \frac{3}{5}$ and $\overline{y} = \frac{M_x}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{3}{4\delta}\right) = 1 \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{3}{5}, 1\right)$ is the center of mass.

7. Applying the symmetry argument analogous to the one used in Exercise 1, we find $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\cos x}{2}\right)$, length: $\cos x$, width: dx, area: $dA = \cos x\, dx$, mass: $dm = \delta dA = \delta \cos x\, dx$. The moment of the strip about the x-axis is $\tilde{y}\, dm = \delta \cdot \frac{\cos x}{2} \cdot \cos x\, dx$ $= \frac{\delta}{2}\cos^2 x\, dx = \frac{\delta}{2}\left(\frac{1+\cos 2x}{2}\right) dx = \frac{\delta}{4}(1+\cos 2x)\, dx$; thus,



$M_x = \int \tilde{y}\, dm = \int_{-\pi/2}^{\pi/2} \frac{\delta}{4}(1+\cos 2x)\, dx = \frac{\delta}{4}\left[x + \frac{\sin 2x}{2}\right]_{-\pi/2}^{\pi/2} = \frac{\delta}{4}\left[\left(\frac{\pi}{2}+0\right) - \left(-\frac{\pi}{2}\right)\right] = \frac{6\delta}{4}$; $M = \int dm$ $= \delta \int_{-\pi/2}^{\pi/2} \cos x\, dx = \delta[\sin x]_{-\pi/2}^{\pi/2} = 2\delta$. Therefore, $\bar{y} = \frac{M_x}{M} = \frac{\delta\pi}{4 \cdot 2\delta} = \frac{\pi}{8} \Rightarrow (\bar{x}, \bar{y}) = \left(0, \frac{\pi}{8}\right)$ is the center of mass.

8. Applying the symmetry argument analogous to the one used in Exercise 1, we find $\bar{x} = 0$. The typical vertical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sec^2 x}{2}\right)$, length: $\sec^2 x$, width: dx, area: $dA = \sec^2 x\, dx$, mass: $dm = \delta\, dA = \delta \sec^2 x\, dx$. The moment about the x-axis is $\tilde{y}\, dm = \left(\frac{\sec^2 x}{2}\right)(\delta \sec^2 x)\, dx$



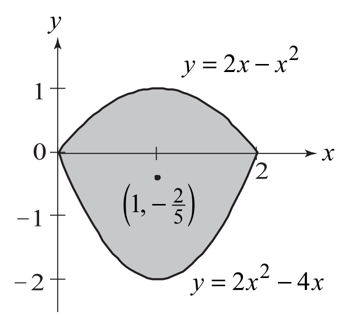
$= \frac{\delta}{2}\sec^4 x\, dx$. $M_x = \int_{-\pi/4}^{\pi/4} \tilde{y}\, dm = \frac{\delta}{2}\int_{-\pi/4}^{\pi/4} \sec^4 x\, dx = \frac{\delta}{2}\int_{-\pi/4}^{\pi/4} \left(\tan^2 x + 1\right)\left(\sec^2 x\right) dx$

$= \frac{\delta}{2}\int_{-\pi/4}^{\pi/4} (\tan x)^2 \left(\sec^2 x\right) dx + \frac{\delta}{2}\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \frac{\delta}{2}\left[\frac{(\tan x)^3}{3}\right]_{-\pi/4}^{\pi/4} + \frac{\delta}{2}[\tan x]_{-\pi/4}^{\pi/4}$

$= \frac{\delta}{2}\left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] + \frac{\delta}{2}[1-(-1)] = \frac{\delta}{3} + \delta = \frac{4\delta}{3}$; $M = \int dm = \delta \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \delta[\tan x]_{-\pi/4}^{\pi/4} = \delta[1-(-1)] = 2\delta$.
Therefore, $\bar{y} = \frac{M_x}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{1}{2\delta}\right) = \frac{2}{3} \Rightarrow (\bar{x}, \bar{y}) = \left(0, \frac{2}{3}\right)$ is the center of mass.

9. By symmetry, $\bar{x} = 1$.

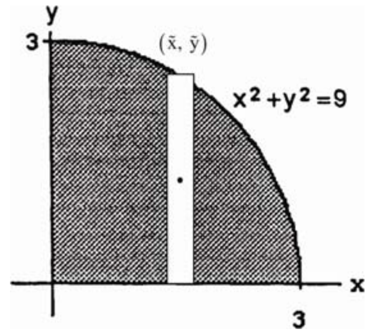
$$M_x = \delta \int_0^2 \frac{1}{2}\left[(2x - x^2)^2 - (2x^2 - 4x)^2\right] dx$$
$$= \delta \int_0^2 \left(-\frac{3}{2}x^4 + 6x^3 - 6x^2\right) dx$$
$$= \delta\left(-\frac{3}{10}x^5 + \frac{3}{2}x^4 - 2x^3\right)\Big|_0^2 = -\frac{8}{5}\delta$$



$$M = \delta \int_0^2 \left[(2x - x^2) - (2x^2 - 4x)\right] dx$$
$$= \delta \int_0^2 \left(-3x^2 + 4x + 2\right) dx$$
$$= \delta\left(-x^3 + 2x^2 + 2x\right)\Big|_0^2 = 4\delta$$

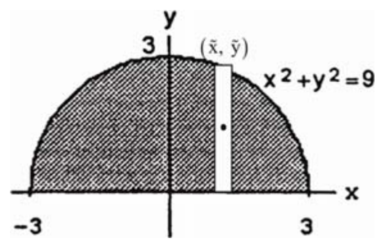
$$\bar{y} = \frac{M_x}{M} = \frac{-\frac{8}{5}\delta}{4\delta} = -\frac{2}{5}$$

10. (a) Since the plate is symmetric about the line $x = y$ and its density is constant, the distribution of mass is symmetric about this line. This means that $\bar{x} = \bar{y}$ The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{9-x^2}}{2}\right)$, length: $\sqrt{9-x^2}$ width: dx, area: $dA = \sqrt{9-x^2}\ dx$, mass: $dm = \delta\ dA = \delta\sqrt{9-x^2}\ dx$. The moment about the x-axis is $\tilde{y}\ dm = \delta\left(\frac{\sqrt{9-x^2}}{2}\right)\sqrt{9-x^2}\ dx = \frac{\delta}{2}\left(9-x^2\right)dx$



Thus, $M_x = \int \tilde{y}\ dm = \int_0^3 \frac{\delta}{2}\left(9-x^2\right)dx = \frac{\delta}{2}\left[9x - \frac{x^3}{3}\right]_0^3 = \frac{\delta}{2}(27-9) = 9\delta$; $M = \int dm = \int \delta\ dA = \delta\int dA$ $= \delta$ (Area of a quarter of a circle of radius 3) $= \delta\left(\frac{9\pi}{4}\right) = \frac{9\pi\delta}{4}$. Therefore, $\bar{y} = \frac{Mx}{M} = (9\delta)\left(\frac{4}{9\pi\delta}\right) = \frac{4}{\pi}$ $\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{4}{\pi}, \frac{4}{\pi}\right)$ is the center of mass.

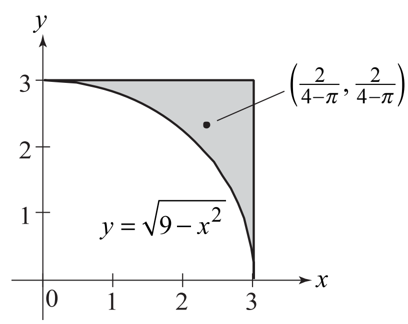
(b) Applying the symmetry argument analogous to the one used in Exercise 1, we find that $\bar{x} = 0$. The typical vertical strip has the same parameters as in part (a). Thus, $M_x = \int \tilde{y}\ dm = \int_{-3}^3 \frac{\delta}{2}\left(9-x^2\right)dx$ $= 2\int_0^3 \frac{\delta}{2}\left(9-x^2\right)dx = 2(9\delta) = 18\delta$; $M = \int dm = \int \delta\ dA$



$= \delta\int dA = \delta$ (Area of a semi-circle of radius 3) $= \delta\left(\frac{9\pi}{2}\right) = \frac{9\pi\delta}{2}$. Therefore, $\bar{y} = \frac{M_x}{M} = (18\delta)\left(\frac{2}{9\pi\delta}\right) = \frac{4}{\pi}$, the same $\bar{y}$ as in part (a) $\Rightarrow (\bar{x}, \bar{y}) = \left(0, \frac{4}{\pi}\right)$ is the center of mass.

11. By symmetry, $\bar{x} = \bar{y}$.

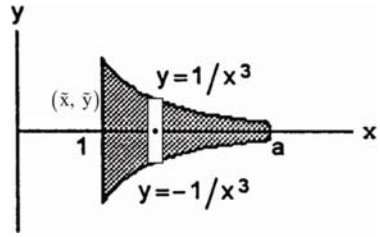
$$M_y = \delta\int_0^3 x\left(3-\sqrt{9-x^2}\right)dx$$
$$= \delta\int_0^3 \left(3x - x\sqrt{9-x^2}\right)dx$$
$$= \delta\left(\frac{3}{2}x^2 - \frac{1}{3}(9-x^2)^{3/2}\right)\Bigg|_0^3 = \frac{9}{2}\delta$$



To compute M, we multiply the area of the "triangle" by δ, obtaining $\delta\left(9 - \frac{9\pi}{4}\right)$.

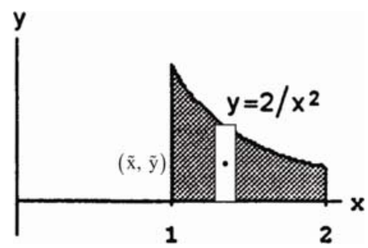
$$\bar{y} = \bar{x} = \frac{M_y}{M} = \frac{\frac{9}{2}\delta}{\delta\left(9-\frac{9\pi}{4}\right)} = \frac{2}{4-\pi}.$$

12. Applying the symmetry argument analogous to the one used in Exercise 1, we find that $\bar{y}=0$. The typical *vertical* strip has center of mass: $(\tilde{x},\tilde{y})=\left(x,\frac{\frac{1}{x^3}-\frac{1}{x^3}}{2}\right)=(x,0)$, length: $\frac{1}{x^3}-\left(-\frac{1}{x^3}\right)=\frac{2}{x^3}$, width: dx,



area: $dA=\frac{2}{x^3}dx$, mass: $dm=\delta\,dA=\frac{2\delta}{x^3}dx$. The moment about the y-axis is $\tilde{x}\,dm=x\cdot\frac{2\delta}{x^3}dx=\frac{2\delta}{x^2}dx$. Thus, $M_y=\int\tilde{x}\,dm=\int_1^a\frac{2\delta}{x^2}dx=2\delta\left[-\frac{1}{x}\right]_1^a=2\delta\left(-\frac{1}{a}+1\right)=\frac{2\delta(a-1)}{a}$; $M=\int dm=\int_1^a\frac{2\delta}{x^3}dx=\delta\left[-\frac{1}{x^2}\right]_1^a=\delta\left(-\frac{1}{a^2}+1\right)$ $=\frac{\delta(a^2-1)}{a^2}$. Therefore, $\bar{x}=\frac{M_y}{M}=\left(\frac{2\delta(a-1)}{a}\right)\left(\frac{a^2}{\delta(a^2-1)}\right)=\frac{2a}{a+1}\Rightarrow(\bar{x},\bar{y})=\left(\frac{2a}{a+1},0\right)$. Also, $\lim_{a\to\infty}\bar{x}=2$.

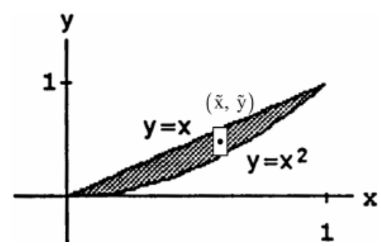
13. $M_x=\int\tilde{y}\,dm=\int_1^2\frac{\left(\frac{2}{x^2}\right)}{2}\cdot\delta\cdot\left(\frac{2}{x^2}\right)dx=\int_1^2\left(\frac{1}{x^2}\right)\left(x^2\right)\left(\frac{2}{x^2}\right)dx$ $=\int_1^2\frac{2}{x^2}\,dx=2\int_1^2x^{-2}dx=2\left[-x^{-1}\right]_1^2=2\left[\left(-\frac{1}{2}\right)-(-1)\right]=2\left(\frac{1}{2}\right)=1$; $M_y=\int\tilde{x}\,dm=\int_1^2x\cdot\delta\cdot\left(\frac{2}{x^2}\right)dx$



$=\int_1^2x\left(x^2\right)\left(\frac{2}{x^2}\right)dx=2\int_1^2x\,dx=2\left[\frac{x^2}{2}\right]_1^2=2\left(2-\frac{1}{2}\right)=4-1=3$; $M=\int dm=\int_1^2\delta\left(\frac{2}{x^2}\right)dx=\int_1^2x^2\left(\frac{2}{x^2}\right)dx$ $=2\int_1^2dx=2[x]_1^2=2(2-1)=2$. So $\bar{x}=\frac{M_y}{M}=\frac{3}{2}$ and $\bar{y}=\frac{M_x}{M}=\frac{1}{2}\Rightarrow(\bar{x},\bar{y})=\left(\frac{3}{2},\frac{1}{2}\right)$ is the center of mass.

14. We use the *vertical* strip approach:

$M_x=\int\tilde{y}\,dm=\int_0^1\frac{\left(x+x^2\right)}{2}\left(x-x^2\right)\cdot\delta\,dx$
$=\frac{1}{2}\int_0^1\left(x^2-x^4\right)\cdot12x\,dx=6\int_0^1\left(x^3-x^5\right)dx$
$=6\left[\frac{x^4}{4}-\frac{x^6}{6}\right]_0^1=6\left(\frac{1}{4}-\frac{1}{6}\right)=\frac{6}{4}-1=\frac{1}{2}$;



$M_y=\int\tilde{x}\,dm=\int_0^1x\left(x-x^2\right)\cdot\delta\,dx=\int_0^1\left(x^2-x^3\right)\cdot12x\,dx=12\int_0^1\left(x^3-x^4\right)dx=12\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_0^1=12\left(\frac{1}{4}-\frac{1}{5}\right)=\frac{12}{20}=\frac{3}{5}$;
$M=\int dm=\int_0^1\left(x-x^2\right)\cdot\delta\,dx=12\int_0^1\left(x^2-x^3\right)dx=12\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=12\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{12}{12}=1$. So $\bar{x}=\frac{M_y}{M}=\frac{3}{5}$ and $\bar{y}=\frac{M_x}{M}=\frac{1}{2}\Rightarrow\left(\frac{3}{5},\frac{1}{2}\right)$ is the center of mass.

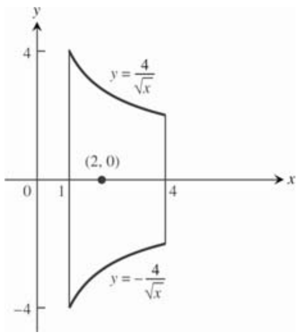
15. (a) We use the shell method: $V=\int_a^b2\pi\left(\begin{smallmatrix}\text{shell}\\\text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\\text{height}\end{smallmatrix}\right)dx=\int_1^42\pi x\left[\frac{4}{\sqrt{x}}-\left(-\frac{4}{\sqrt{x}}\right)\right]dx=16\pi\int_1^4\frac{x}{\sqrt{x}}dx$ $=16\pi\int_1^4x^{1/2}dx=16\pi\left[\frac{2}{3}x^{3/2}\right]_1^4=16\pi\left(\frac{2}{3}\cdot8-\frac{2}{3}\right)=\frac{32\pi}{3}(8-1)=\frac{224\pi}{3}$

(b) Since the plate is symmetric about the x-axis and its density $\delta(x)=\frac{1}{x}$ is a function of x alone, the distribution of its mass is symmetric about the x-axis. This means that $\bar{y}=0$. We use the vertical strip approach to find $\bar{x}$: $M_y=\int\tilde{x}\,dm=\int_1^4x\cdot\left[\frac{4}{\sqrt{x}}-\left(-\frac{4}{\sqrt{x}}\right)\right]\cdot\delta\,dx=\int_1^4x\cdot\frac{8}{\sqrt{x}}\cdot\frac{1}{x}\,dx=8\int_1^4x^{-1/2}\,dx=8\left[2x^{1/2}\right]_1^4$

$= 8(2\cdot 2-2)=16;\; M=\int dm=\int_1^4\left[\frac{4}{\sqrt{x}}-\left(\frac{-4}{\sqrt{x}}\right)\right]\cdot\delta\,dx=8\int_1^4\left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{x}\right)dx=8\int_1^4 x^{-3/2}\,dx=8\left[-2x^{-1/2}\right]_1^4$

$=8[-1-(-2)]=8.$ So $\bar{x}=\frac{M_y}{M}=\frac{16}{8}=2\Rightarrow(\bar{x},\bar{y})=(2,0)$ is the center of mass.

(c)



16. (a) We use the disk method: $V=\int_a^b\pi[R(x)]^2\,dx=\int_1^4\pi\left(\frac{4}{x^2}\right)dx=4\pi\int_1^4 x^{-2}dx=4\pi\left[-\frac{1}{x}\right]_1^4=4\pi\left[\frac{-1}{4}-(-1)\right]$

$=\pi[-1+4]=3\pi$

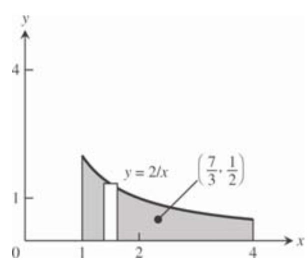
(b) We model the distribution of mass with vertical strips: $M_x=\int\tilde{y}\,dm=\int_1^4\frac{\left(\frac{2}{x}\right)}{2}\cdot\left(\frac{2}{x}\right)\cdot\delta\,dx=\int_1^4\frac{2}{x^2}\cdot\sqrt{x}\,dx$

$=2\int_1^4 x^{-3/2}dx=2\left[\frac{-2}{\sqrt{x}}\right]_1^4=2[-1-(-2)]=2;\; M_y=\int\tilde{x}\,dm=\int_1^4 x\cdot\frac{2}{x}\cdot\delta\,dx=2\int_1^4 x^{1/2}dx=2\left[\frac{2x^{3/2}}{3}\right]_1^4$

$=2\left[\frac{16}{3}-\frac{2}{3}\right]=\frac{28}{3};\; M=\int dm=\int_1^4\frac{2}{x}\cdot\delta\,dx=2\int_1^4\frac{\sqrt{x}}{x}\,dx=2\int_1^4 x^{-1/2}dx=2\left[2x^{1/2}\right]_1^4=2(4-2)=4.$

So $\bar{x}=\frac{M_y}{M}=\frac{\left(\frac{28}{3}\right)}{4}=\frac{7}{3}$ and $\bar{y}=\frac{M_x}{M}=\frac{2}{4}=\frac{1}{2}\Rightarrow(\bar{x},\bar{y})=\left(\frac{7}{3},\frac{1}{2}\right)$ is the center of mass.

(c)



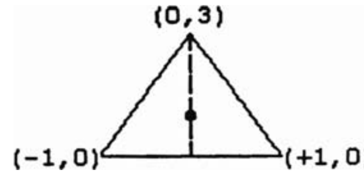
17. The mass of a horizontal strip is $dm=\delta\,dA=\delta L\,dy$, where L is the width of the triangle at a distance of y above its base on the x-axis as shown in the figure in the text. Also, by similar triangles we have $\frac{L}{b}=\frac{h-y}{h}$

$\Rightarrow L=\frac{b}{h}(h-y).$ Thus, $M_x=\int\tilde{y}\,dm=\int_0^h\delta y\left(\frac{b}{h}\right)(h-y)\,dy=\frac{\delta b}{h}\int_0^h\left(hy-y^2\right)dy=\frac{\delta b}{h}\left[\frac{hy^2}{2}-\frac{y^3}{3}\right]_0^h=\frac{\delta b}{h}\left(\frac{h^3}{2}-\frac{h^3}{3}\right)$

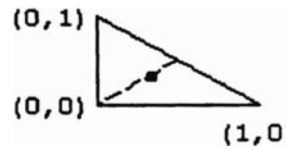
$=\delta bh^2\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{\delta bh^2}{6};\; M=\int dm=\int_0^h\delta\left(\frac{b}{h}\right)(h-y)\,dy=\frac{\delta b}{h}\int_0^h(h-y)\,dy=\frac{\delta b}{h}\left[hy-\frac{y^2}{2}\right]_0^h=\frac{\delta b}{h}\left(h^2-\frac{h^2}{2}\right)=\frac{\delta bh}{2}.$ So

$\bar{y}=\frac{M_x}{M}=\left(\frac{\delta bh^2}{6}\right)\left(\frac{2}{\delta bh}\right)=\frac{h}{3}\Rightarrow$ the center of mass lies above the base of the triangle one-third of the way toward the opposite vertex. Similarly the other two sides of the triangle can be placed on the x-axis and the same results will occur. Therefore the centroid does lie at the intersection of the medians, as claimed.

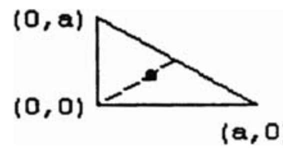
18. From the symmetry about the y-axis it follows that $\bar{x} = 0$. It also follows that the line through the points $(0, 0)$ and $(0, 3)$ is a median $\Rightarrow \bar{y} = \frac{1}{3}(3-0) = 1 \Rightarrow (\bar{x}, \bar{y}) = (0, 1)$.



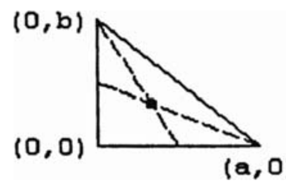
19. From the symmetry about the line $x = y$ it follows that $\bar{x} = \bar{y}$. It also follows that the line through the points $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a median $\Rightarrow \bar{y} = \bar{x} = \frac{2}{3} \cdot \left(\frac{1}{2} - 0\right) = \frac{1}{3} \Rightarrow (\bar{x}, \bar{y}) = \left(\frac{1}{3}, \frac{1}{3}\right)$.



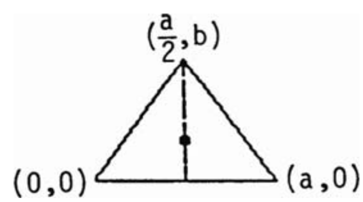
20. From the symmetry about the line $x = y$ it follows that $\bar{x} = \bar{y}$. It also follows that the line through the point $(0, 0)$ and $\left(\frac{a}{2}, \frac{a}{2}\right)$ is a median $\Rightarrow \bar{y} = \bar{x} = \frac{2}{3}\left(\frac{a}{2} - 0\right) = \frac{1}{3}a \Rightarrow (\bar{x}, \bar{y}) = \left(\frac{a}{3}, \frac{a}{3}\right)$.



21. The point of intersection of the median from the vertex $(0, b)$ to the opposite side has coordinates $\left(0, \frac{a}{2}\right) \Rightarrow \bar{y} = (b-0) \cdot \frac{1}{3} = \frac{b}{3}$ and $\bar{x} = \left(\frac{a}{2} - 0\right) \cdot \frac{2}{3} = \frac{a}{3} \Rightarrow (\bar{x}, \bar{y}) = \left(\frac{a}{3}, \frac{b}{3}\right)$.



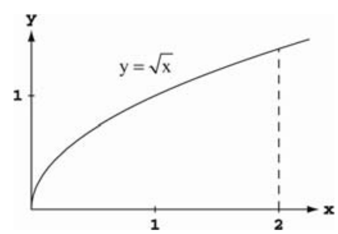
22. From the symmetry about the line $x = \frac{a}{2}$ it follows that $\bar{x} = \frac{a}{2}$. It also follows that the line through the points $\left(\frac{a}{2}, 0\right)$ and $\left(\frac{a}{2}, b\right)$ is a median $\Rightarrow \bar{y} = \frac{1}{3}(b-0) = \frac{b}{3} \Rightarrow (\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{b}{3}\right)$.



23. $y = x^{1/2} \Rightarrow dy = \frac{1}{2}x^{-1/2}dx \Rightarrow ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \frac{1}{4x}}\,dx$;

$$M_x = \delta\int_0^2 \sqrt{x}\sqrt{1 + \frac{1}{4x}}\,dx = \delta\int_0^2 \sqrt{x + \frac{1}{4}}\,dx = \frac{2\delta}{3}\left[\left(x + \frac{1}{4}\right)^{3/2}\right]_0^2$$

$$= \frac{2\delta}{3}\left[\left(2 + \frac{1}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right] = \frac{2\delta}{3}\left[\left(\frac{9}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right] = \frac{2\delta}{3}\left(\frac{27}{8} - \frac{1}{8}\right) = \frac{13\delta}{6}$$

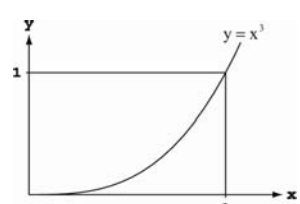


24. $y = x^3 \Rightarrow dy = 3x^2\,dx \Rightarrow dx = \sqrt{(dx)^2 + \left(3x^2 dx\right)^2} = \sqrt{1 + 9x^4}\,dx$;

$M_x = \delta\int_0^1 x^3\sqrt{1 + 9x^4}\,dx$;

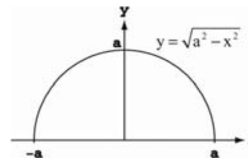
$[u = 1 + 9x^4 \Rightarrow du = 36x^3 dx \Rightarrow \frac{1}{36}du = x^3 dx$;

$x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 10] \rightarrow M_x = \delta\int_1^{10} \frac{1}{36}u^{1/2}du = \frac{\delta}{36}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{\delta}{54}\left(10^{3/2} - 1\right)$



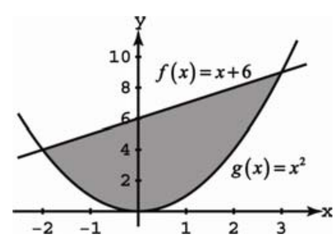
25. From Example 4 we have $M_x = \int_0^{\pi} a(a\sin\theta)(k\sin\theta)d\theta = a^2k\int_0^{\pi}\sin^2\theta\, d\theta = \frac{a^2k}{2}\int_0^{\pi}(1-\cos 2\theta)\, d\theta$
$= \frac{a^2k}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{a^2k\pi}{2}$; $M_y = \int_0^{\pi} a(a\cos\theta)(k\sin\theta)\, d\theta = a^2k\int_0^{\pi}\sin\theta\cos\theta\, d\theta = \frac{a^2k}{2}\left[\sin^2\theta\right]_0^{\pi} = 0$;
$M = \int_0^{\pi} ak\sin\theta\, d\theta = ak\left[-\cos\theta\right]_0^{\pi} = 2ak$. Therefore, $\overline{x} = \frac{M_y}{M} = 0$ and $\overline{y} = \frac{M_x}{M} = \left(\frac{a^2k\pi}{2}\right)\left(\frac{1}{2ak}\right) = \frac{a\pi}{4} \Rightarrow \left(0, \frac{a\pi}{4}\right)$ is the center of mass.

26. $M_x = \int \tilde{y}\, dm = \int_0^{\pi}(a\sin\theta)\cdot\delta\cdot a\, d\theta = \int_0^{\pi}\left(a^2\sin\theta\right)\left(1+k\left|\cos\theta\right|\right)d\theta$



$= a^2\int_0^{\pi/2}(\sin\theta)(1+k\cos\theta)\, d\theta + a^2\int_{\pi/2}^{\pi}(\sin\theta)(1-k\cos\theta)\, d\theta$
$= a^2\int_0^{\pi/2}\sin\theta\, d\theta + a^2k\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta + a^2\int_{\pi/2}^{\pi}\sin\theta\, d\theta - a^2k\int_{\pi/2}^{\pi}\sin\theta\cos\theta\, d\theta$
$= a^2\left[-\cos\theta\right]_0^{\pi/2} + a^2k\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} + a^2\left[-\cos\theta\right]_{\pi/2}^{\pi} - a^2k\left[\frac{\sin^2\theta}{2}\right]_{\pi/2}^{\pi}$
$= a^2\left[0-(-1)\right] + a^2k\left(\frac{1}{2}-0\right) + a^2 + \left[-(-1)-0\right] - a^2k\left(0-\frac{1}{2}\right) = a^2 + \frac{a^2k}{2} + a^2 + \frac{a^2k}{2} = 2a^2 + a^2k = a^2(2+k)$;
$M_y = \int \tilde{x}\, dm = \int_0^{\pi}(a\cos\theta)\cdot\delta\cdot \mathrm{a}\, \mathrm{d}\theta = \int_0^{\pi}\left(a^2\cos\theta\right)\left(1+k\left|\cos\theta\right|\right)d\theta$
$= a^2\int_0^{\pi/2}(\cos\theta)(1+k\cos\theta)\, d\theta + a^2\int_{\pi/2}^{\pi}(\cos\theta)(1-k\cos\theta)d\theta$
$= a^2\int_0^{\pi/2}\cos\theta\, d\theta + a^2k\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta + a^2\int_{\pi/2}^{\pi}\cos\theta\, d\theta - a^2k\int_{\pi/2}^{\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$
$= a^2\left[\sin\theta\right]_0^{\pi/2} + \frac{a^2k}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} + a^2\left[\sin\theta\right]_{\pi/2}^{\pi} - \frac{a^2k}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{\pi}$
$= a^2(1-0) + \frac{a^2k}{2}\left[\left(\frac{\pi}{2}-0\right)-(0+0)\right] + a^2(0-1) - \frac{a^2k}{2}\left[(\pi+0)-\left(\frac{\pi}{2}+0\right)\right] = a^2 + \frac{a^2k\pi}{4} - a^2 - \frac{a^2k\pi}{4} = 0$;
$M = \int_0^{\pi}\delta\cdot a\, d\theta = a\int_0^{\pi}\left(1+k\left|\cos\theta\right|\right)d\theta = a\int_0^{\pi/2}(1+k\cos\theta)\, d\theta + a\int_{\pi/2}^{\pi}(1-k\cos\theta)\, d\theta$
$= a\left[\theta + k\sin\theta\right]_0^{\pi/2} + a\left[\theta - k\sin\theta\right]_{\pi/2}^{\pi} = \left[\left(\frac{\pi}{2}+k\right)-0\right] + a\left[(\pi+0)-\left(\frac{\pi}{2}-k\right)\right] = \frac{a\pi}{2} + ak + a\left(\frac{\pi}{2}+k\right) = a\pi + 2ak$
$= a(\pi+2k)$. So $\overline{x} = \frac{M_y}{M} = 0$ and $\overline{y} = \frac{M_x}{M} = \frac{a^2(2+k)}{a(\pi+2k)} = \frac{a(2+k)}{\pi+2k} \Rightarrow \left(0, \frac{2a+ka}{\pi+2k}\right)$ is the center of mass.

27. $f(x) = x+6$, $g(x) = x^2$, $f(x) = g(x) \Rightarrow x+6 = x^2$
$\Rightarrow x^2 - x - 6 = 0 \Rightarrow x = 3, x = -2$; $\delta = 1$
$M = \int_{-2}^{3}\left[(x+6)-x^2\right]dx = \left[\frac{1}{2}x^2 + 6x - \frac{1}{3}x^3\right]_{-2}^{3}$
$= \left(\frac{9}{2}+18-9\right) - \left(2-12+\frac{8}{3}\right) = \frac{125}{6}$

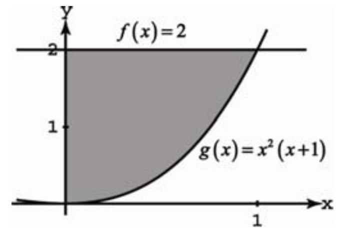


$\overline{x} = \frac{1}{125/6}\int_{-2}^{3}\left[x(x+6)-x^2\right]dx = \frac{6}{125}\int_{-2}^{3}\left[x^2+6x-x^3\right]dx = \frac{6}{125}\left[\frac{1}{3}x^3 + 3x^2 - \frac{1}{4}x^4\right]_{-2}^{3}$
$= \frac{6}{125}\left(9+27-\frac{81}{4}\right) - \frac{6}{125}\left(-\frac{8}{3}+12-4\right) = \frac{1}{2}$;

$\overline{y} = \frac{1}{125/6}\int_{-2}^{3}\frac{1}{2}\left[(x+6)^2-\left(x^2\right)^2\right]dx = \frac{3}{125}\int_{-2}^{3}\left[x^2+12x+36-x^4\right]dx = \frac{3}{125}\left[\frac{1}{3}x^3+6x^2+36x-\frac{1}{5}x^5\right]_{-2}^{3}$

$= \frac{3}{125}\left(9+54+108-\frac{243}{5}\right)-\frac{3}{125}\left(-\frac{8}{3}+24-72+\frac{32}{5}\right)=4 \Rightarrow \left(\frac{1}{2},4\right)$ is the center of mass.

28. $f(x)=2,\ \ g(x)=x^2(x+1),\ \ f(x)=g(x) \Rightarrow 2=x^2(x+1)$



$\Rightarrow x^3+x^2-2=0 \Rightarrow x=1; \delta=1$

$M=\int_0^1\left[2-x^2(x+1)\right]dx=\int_0^1\left[2-x^3-x^2\right]dx$

$=\left[2x-\frac{1}{4}x^4-\frac{1}{3}x^3\right]_0^1=\left(2-\frac{1}{4}-\frac{1}{3}\right)-0=\frac{17}{12};$

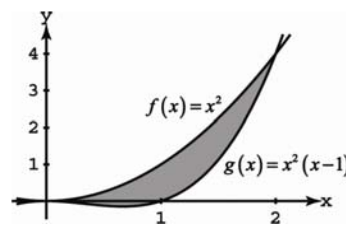
$\overline{x}=\frac{1}{17/12}\int_0^1 x\left[2-x^2(x+1)\right]dx=\frac{12}{17}\int_0^1\left[2x-x^4-x^3\right]dx$

$=\frac{12}{17}\left[x^2-\frac{1}{5}x^5-\frac{1}{4}x^4\right]_0^1=\frac{12}{17}\left(1-\frac{1}{5}-\frac{1}{4}\right)-0=\frac{33}{85};$

$\overline{y}=\frac{1}{17/12}\int_0^1\frac{1}{2}\left[2^2-\left(x^2(x+1)\right)^2\right]dx=\frac{6}{17}\int_0^1\left[4-x^6-2x^5-x^4\right]dx=\frac{6}{17}\left[4x-\frac{1}{7}x^7-\frac{1}{3}x^6-\frac{1}{5}x^5\right]_0^1$

$=\frac{6}{17}\left(4-\frac{1}{7}-\frac{1}{3}-\frac{1}{5}\right)-0=\frac{698}{595}\Rightarrow\left(\frac{33}{85},\frac{698}{595}\right)$ is the center of mass.

29. $f(x)=x^2,\ \ g(x)=x^2(x-1), f(x)=g(x)$



$\Rightarrow x^2=x^2(x-1)\Rightarrow x^3-2x^2=0\Rightarrow x=0, x=2;\ \delta=1$

$M=\int_0^2\left[x^2-x^2(x-1)\right]dx=\int_0^2\left[2x^2-x^3\right]dx$

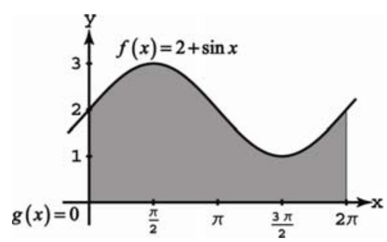
$=\left[\frac{2}{3}x^3-\frac{1}{4}x^4\right]_0^2=\left(\frac{16}{3}-4\right)-0=\frac{4}{3};$

$\overline{x}=\frac{1}{4/3}\int_0^2 x\left[x^2-x^2(x-1)\right]dx=\frac{3}{4}\int_0^2\left[2x^3-x^4\right]dx$

$=\frac{3}{4}\left[\frac{1}{2}x^4-\frac{1}{5}x^5\right]_0^2=\frac{3}{4}\left(8-\frac{32}{5}\right)-0=\frac{6}{5};$

$\overline{y}=\frac{1}{4/3}\int_0^2\frac{1}{2}\left[\left(x^2\right)^2-\left(x^2(x-1)\right)^2\right]dx=\frac{3}{8}\int_0^2\left[2x^5-x^6\right]dx=\frac{3}{8}\left[\frac{1}{3}x^6-\frac{1}{7}x^7\right]_0^2=\frac{3}{8}\left(\frac{64}{3}-\frac{128}{7}\right)-0=\frac{8}{7}\Rightarrow\left(\frac{6}{5},\frac{8}{7}\right)$ is the center of mass.

30. $f(x)=2+\sin x,\ \ g(x)=0, x=0, x=2\pi;\ \delta=1;$



$M=\int_0^{2\pi}\left[2+\sin x\right]dx=\left[2x-\cos x\right]_0^{2\pi}=(4\pi-1)-(0-1)=4\pi;$

$\overline{x}=\frac{1}{4\pi}\int_0^{2\pi}x\left[2+\sin x-0\right]dx=\frac{1}{4\pi}\int_0^{2\pi}\left[2x+x\sin x\right]dx$

$=\frac{1}{4\pi}\int_0^{2\pi}2x\,dx+\frac{1}{4\pi}\int_0^{2\pi}x\sin x\,dx=\frac{1}{4\pi}\left[x^2\right]_0^{2\pi}+\frac{1}{4\pi}\left[\sin x-x\cos x\right]_0^{2\pi}$

$=\frac{1}{4\pi}\left(4\pi^2\right)-0+\frac{1}{4\pi}(0-2\pi)-0=\frac{2\pi-1}{2};$

$\overline{y}=\frac{1}{4\pi}\int_0^{2\pi}\frac{1}{2}\left[(2+\sin x)^2-(0)^2\right]dx$

$= \frac{1}{8\pi}\left[\int_0^{2\pi} 4 + 4\sin x + \sin^2 x\right]dx = \frac{1}{8\pi}\int_0^{2\pi}[4 + 4\sin x]dx + \frac{1}{8\pi}\int_0^{2\pi}\left[\sin^2 x\right]dx$

$= \frac{1}{8\pi}\int_0^{2\pi}[4 + 4\sin x]dx + \frac{1}{8\pi}\int_0^{2\pi}\left[\frac{1-\cos 2x}{2}\right]dx = \frac{1}{8\pi}[4x - 4\cos x]_0^{2\pi} + \frac{1}{16\pi}\int_0^{2\pi}dx - \frac{1}{16\pi}\int_0^{2\pi}\cos 2x\,dx$

$[u = 2x \Rightarrow du = 2dx, x = 0 \Rightarrow u = 0, x = 2\pi \Rightarrow u = 4\pi] \to \frac{1}{8\pi}[4x - 4\cos x]_0^{2\pi} + \frac{1}{16\pi}[x]_0^{2\pi} - \frac{1}{32\pi}\int_0^{4\pi}\cos u\,du$

$= \frac{1}{8\pi}[4x - 4\cos x]_0^{2\pi} + \frac{1}{16\pi}[x]_0^{2\pi} - \frac{1}{32\pi}[\sin u]_0^{4\pi} = \frac{1}{8\pi}(8\pi - 4) - \frac{1}{8\pi}(0 - 4) + \frac{1}{16\pi}(2\pi) - 0 - 0 = \frac{9}{8} \Rightarrow \left(\frac{2\pi-1}{2}, \frac{9}{8}\right)$ is the center of mass.

31. Consider the curve as an infinite number of line segments joined together. From the derivation of arc length we have that the length of a particular segment is $ds = \sqrt{(dx)^2 + (dy)^2}$. This implies that $M_x = \int \delta y\,ds$, $M_y = \int \delta x\,ds$ and $M = \int \delta\,ds$. If δ is constant, then $\overline{x} = \frac{M_y}{M} = \frac{\int x\,ds}{\int ds} = \frac{\int x\,ds}{\text{length}}$ and $\overline{y} = \frac{M_x}{M} = \frac{\int y\,ds}{\int ds} = \frac{\int y\,ds}{\text{length}}$.

32. Applying the symmetry argument analogous to the one used in Exercise 1, we find that $\overline{x} = 0$. The typical vertical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{a + \frac{x^2}{4p}}{2}\right)$, length: $a - \frac{x^2}{4p}$, width: dx, area : $dA = \left(a - \frac{x^2}{4p}\right)dx$,

mass: $dm = \delta\,dA = \delta\left(a - \frac{x^2}{4p}\right)dx$. Thus, $M_x = \int \tilde{y}\,dm = \int_{-2\sqrt{pa}}^{2\sqrt{pa}} \frac{1}{2}\left(a + \frac{x^2}{4p}\right)\left(a - \frac{x^2}{4p}\right)\delta dx$

$= \frac{\delta}{2}\int_{-2\sqrt{pa}}^{2\sqrt{pa}}\left(a^2 - \frac{x^4}{16p^2}\right)dx = \frac{\delta}{2}\left[a^2 x - \frac{x^5}{80p^2}\right]_{-2\sqrt{pa}}^{2\sqrt{pa}} = 2\cdot\frac{\delta}{2}\left[a^2 x - \frac{x^5}{80p^2}\right]_0^{2\sqrt{pa}} = \delta\left(2a^2\sqrt{pa} - \frac{2^5 p^2 a^2\sqrt{pa}}{80p^2}\right)$

$= 2a^2\delta\sqrt{pa}\left(1 - \frac{16}{80}\right) = 2a^2\delta\sqrt{pa}\left(\frac{80-16}{80}\right) = 2a^2\delta\sqrt{pa}\left(\frac{64}{80}\right) = \frac{8a^2\delta\sqrt{pa}}{5}$; $M = \int dm = \int_{-2\sqrt{pa}}^{2\sqrt{pa}}\delta\left(a - \frac{x^2}{4p}\right)dx$

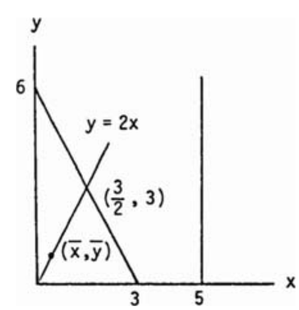
$= \delta\left[ax - \frac{x^3}{12p}\right]_{-2\sqrt{pa}}^{2\sqrt{pa}} = 2\cdot\delta\left[ax - \frac{x^3}{12p}\right]_0^{2\sqrt{pa}} = 2\delta\left(2a\sqrt{pa} - \frac{2^3 pa\sqrt{pa}}{12p}\right) = 4a\delta\sqrt{pa}\left(1 - \frac{4}{12}\right) = 4a\delta\sqrt{pa}\left(\frac{12-4}{12}\right)$

$= \frac{8a\delta\sqrt{pa}}{3}$. So $\overline{y} = \frac{M_x}{M} = \left(\frac{8a^2\delta\sqrt{pa}}{5}\right)\left(\frac{3}{8a\delta\sqrt{pa}}\right) = \frac{3}{5}a$, as claimed

33. The centroid of the square is located at $(2, 2)$. The volume is $V = (2\pi)(\overline{y})(A) = (2\pi)(2)(8) = 32\pi$ and the surface area is $S = (2\pi)(\overline{y})(L) = (2\pi)(2)\left(4\sqrt{8}\right) = 32\sqrt{2}\pi$ (where $\sqrt{8}$ is the length of a side).

34. The midpoint of the hypotenuse of the triangle is $\left(\frac{3}{2}, 3\right) \Rightarrow y = 2x$ is an equation of the median $\Rightarrow$ the line $y = 2x$ contains the centroid. The point $\left(\frac{3}{2}, 3\right)$ is $\frac{3\sqrt{5}}{2}$ units from the origin $\Rightarrow$ the x-coordinate of the centroid solves the equation $\sqrt{\left(x - \frac{3}{2}\right)^2 + (2x - 3)^2} = \frac{\sqrt{5}}{2}$

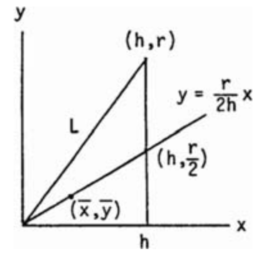
$\Rightarrow \left(x^2 - 3x + \frac{9}{4}\right) + \left(4x^2 - 12x + 9\right) = \frac{5}{4}$



$\Rightarrow 5x^2 - 15x + 9 = -1 \Rightarrow x^2 - 3x + 2 = (x-2)(x-1) = 0 \Rightarrow \overline{x} = 1$ since the centroid must lie inside the triangle $\Rightarrow \overline{y} = 2$. By the Theorem of Pappus, the volume is $V =$ (distance traveled by the centroid)(area of the region) $= 2\pi(5 - \overline{x})\left[\frac{1}{2}(3)(6)\right] = (2\pi)(4)(9) = 72\pi$

35. The centroid is located at $(2, 0) \Rightarrow V = (2\pi)(\overline{x})(A) = (2\pi)(2)(\pi) = 4\pi^2$

36. We create the cone by revolving the triangle with vertices $(0, 0), (h, r)$ and $(h, 0)$ about the x-axis (see the accompanying figure).Thus, the cone has height h and base radius r. By Theorem of Pappus, the lateral surface area swept out by the hypotenuse L is given by $S = 2\pi\bar{y}L = 2\pi\left(\frac{r}{2}\right)\sqrt{h^2+r^2} = \pi r\sqrt{r^2+h^2}$. To calculate the volume we need the position of the centroid of the triangle. From the diagram we see that the centroid lies on the



line $y = \frac{r}{2h}x$. The x-coordinate of the centroid solves the equation $\sqrt{(x-h)^2 + \left(\frac{r}{2h}x - \frac{r}{2}\right)^2} = \frac{1}{3}\sqrt{h^2 + \frac{r^2}{4}}$

$\Rightarrow \left(\frac{4h^2+r^2}{4h^2}\right)x^2 - \left(\frac{4h^2+r^2}{2h}\right)x + \frac{r^2}{4} + \frac{2\left(r^2+4h^2\right)}{9} = 0 \Rightarrow x = \frac{2h}{3}$ or $\frac{4h}{3} \Rightarrow \bar{x} = \frac{2h}{3}$, since the centroid must lie inside the triangle $\Rightarrow \bar{y} = \frac{r}{2h}\bar{x} = \frac{r}{3}$. By the Theorem of Pappus, $V = \left[2\pi\left(\frac{r}{3}\right)\right]\left(\frac{1}{2}hr\right) = \frac{1}{3}\pi r^2 h.$

37. $S = 2\pi\,\bar{y}\,L \Rightarrow 4\pi a^2 = (2\pi\bar{y})(\pi a) \Rightarrow \bar{y} = \frac{2a}{\pi}$, and by symmetry $\bar{x} = 0$

38. $S = 2\pi\rho L \Rightarrow \left[2\pi\left(a - \frac{2a}{\pi}\right)\right](\pi a) = 2\pi a^2(\pi - 2)$

39. $V = 2\pi\,\bar{y}\,A \Rightarrow \frac{4}{3}\pi ab^2 = (2\pi\bar{y})\left(\frac{\pi ab}{2}\right) \Rightarrow \bar{y} = \frac{4b}{3\pi}$ and by symmetry $\bar{x} = 0$

40. $V = 2\pi\rho A \Rightarrow V = \left[2\pi\left(a + \frac{4a}{3\pi}\right)\right]\left(\frac{\pi a^2}{2}\right) = \frac{\pi a^3(3\pi+4)}{3}$

41. $V = 2\pi\rho A = (2\pi)$ (area of the region) (distance from the centroid to the line $y = x - a$). We must find the distance from $\left(0, \frac{4a}{3\pi}\right)$ to $y = x - a$. The line containing the centroid and perpendicular to $y = x - a$ has slope -1 and contains the point $\left(0, \frac{4a}{3\pi}\right)$. This line is $y = -x + \frac{4a}{3\pi}$. The intersection of $y = x - a$ and $y = -x + \frac{4a}{3\pi}$ is the point $\left(\frac{4a+3a\pi}{6\pi}, \frac{4a-3a\pi}{6\pi}\right)$. Thus, the distance from the centroid to the line $y = x - a$ is

$\sqrt{\left(\frac{4a+3a\pi}{6\pi}\right)^2 + \left(\frac{4a}{3\pi} - \frac{4a}{6\pi} + \frac{3a\pi}{6\pi}\right)^2} = \frac{\sqrt{2}(4a+3a\pi)}{6\pi} \Rightarrow V = (2\pi)\left(\frac{\sqrt{2}(4a+3a\pi)}{6\pi}\right)\left(\frac{\pi a^2}{2}\right) = \frac{\sqrt{2}\pi a^3(4+3\pi)}{6}$

42. The line perpendicular to $y = x - a$ and passing through the centroid $\left(0, \frac{2a}{\pi}\right)$ has equation $y = -x + \frac{2a}{\pi}$. The intersection of the two perpendicular lines occurs when $x - a = -x + \frac{2a}{\pi} \Rightarrow x = \frac{2a+a\pi}{2\pi} \Rightarrow x = \frac{2a+a\pi}{2\pi}$

$\Rightarrow y = \frac{2a-a\pi}{2\pi}$. Thus the distance from the centroid to the line $y = x - a$ is $\sqrt{\left(\frac{2a+\pi a}{2} - 0\right)^2 + \left(\frac{2a-\pi a}{2} - \frac{2a}{2}\right)^2}$

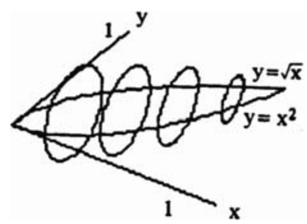
$= \frac{a(2+\pi)}{\sqrt{2}\pi}$. Therefore, by the Theorem of Pappus the surface area is $S = 2\pi\left[\frac{a(2+\pi)}{\sqrt{2}\pi}\right](\pi a) = \sqrt{2}\pi a^2(2+\pi)$.

43. If we revolve the region about the y-axis: $r = a, h = b \Rightarrow A = \frac{1}{2}ab, V = \frac{1}{3}\pi a^2 b$, and $\rho = \bar{x}$. By the Theorem of Pappus: $\frac{1}{3}\pi a^2 b = 2\pi\,\bar{x}\left(\frac{1}{2}ab\right) \Rightarrow \bar{x} = \frac{a}{3}$; If we revolve the region about the x-axis: $r = b, h = a \Rightarrow A = \frac{1}{2}ab,\ V = \frac{1}{3}\pi\, b^2 a$, and $\rho = \bar{y}$. By the Theorem of Pappus: $\frac{1}{3}\pi b^2 a = 2\pi\,\bar{y}\left(\frac{1}{2}ab\right) \Rightarrow \bar{y} = \frac{b}{3} \Rightarrow \left(\frac{a}{3}, \frac{b}{3}\right)$ is the center of mass.

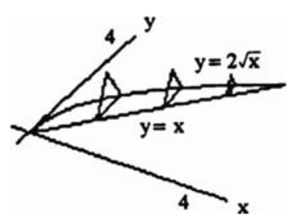
44. Let $O(0, 0), P(a, c),$ and $Q(a, b)$ be the vertices of the given triangle. If we revolve the region about the x-axis: Let R be the point $R(a, 0)$. The volume is given by the volume of the outer cone, radius $= RP = c$, minus the volume of the inner cone, radius $= RQ = b$, thus $V = \frac{1}{3}\pi c^2 a - \frac{1}{3}\pi b^2 a = \frac{1}{3}\pi a\left(c^2 - b^2\right)$, the area is given by the area of triangle OPR minus area of triangle OQR, $A = \frac{1}{2}ac - \frac{1}{2}ab = \frac{1}{2}a(c-b)$, and $\rho = \overline{y}$. By the Theorem of Pappus: $\frac{1}{3}\pi\, a\left(c^2 - b^2\right) = 2\pi\,\overline{y}\left[\frac{1}{2}a(c-b)\right] \Rightarrow \overline{y} = \frac{c+b}{3}$; If we revolve the region about the y-axis: Let S and T be the points $S(0, c)$ and $T(0, b)$, respectively. Then the volume is the volume of the cylinder with radius $OR = a$ and height $RP = c$, minus the sum of the volumes of the cone with radius $= SP = a$ and height $= OS = c$ and the portion of the cylinder with height $= OT = b$ and radius $= TQ = a$ with a cone of height $= OT = b$ and radius $= TQ = a$ removed. Thus $V = \pi a^2 c - \left[\frac{1}{3}\pi a^2 c + \left(\pi a^2 b - \frac{1}{3}\pi a^2 b\right)\right] = \frac{2}{3}\pi a^2 c - \frac{2}{3}\pi a^2 b = \frac{2}{3}\pi a^2 (a-b)$. The area of the triangle is the same as before, $A = \frac{1}{2}ac - \frac{1}{2}ab = \frac{1}{2}a(c-b)$, and $\rho = \overline{x}$. By the Theorem of Pappus: $\frac{2}{3}\pi a^2(a-b) = 2\pi\,\overline{x}\left[\frac{1}{2}a(c-b)\right] \Rightarrow \overline{x} = \frac{2a(a-b)}{3(c-b)} \Rightarrow \left(\frac{2a(a-b)}{3(c-b)}, \frac{c+b}{2}\right)$ is the center of mass.

CHAPTER 6 PRACTICE EXERCISES

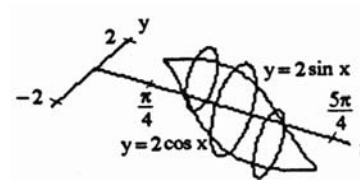
1. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{x} - x^2\right)^2$
$= \frac{\pi}{4}\left(x - 2\sqrt{x}\cdot x^2 + x^4\right)$; $a = 0, b = 1$
$\Rightarrow V = \int_a^b A(x)dx = \frac{\pi}{4}\int_0^1\left[x - 2x^{5/2} + x^4\right]dx$
$= \frac{\pi}{4}\left[\frac{x^2}{2} - \frac{4}{7}x^{7/2} + \frac{x^5}{5}\right]_0^1 = \frac{\pi}{4}\left(\frac{1}{2} - \frac{4}{7} + \frac{1}{5}\right)$
$\frac{\pi}{4\cdot 70}(35 - 40 + 14) = \frac{9\pi}{280}$



2. $A(x) = \frac{1}{2}(\text{side})^2\left(\sin\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}\left(2\sqrt{x} - x\right)^2$
$= \frac{\sqrt{3}}{4}\left(4x - 4x\sqrt{x} + x^2\right)$; $a = 0, b = 4$
$\Rightarrow V = \int_a^b A(x)dx = \frac{\sqrt{3}}{4}\int_0^4\left(4x - 4x^{3/2} + x^2\right)dx$
$= \frac{\sqrt{3}}{4}\left[2x^2 - \frac{8}{5}x^{5/2} + \frac{x^3}{3}\right]_0^4 = \frac{\sqrt{3}}{4}\left(32 - \frac{8\cdot 32}{5} + \frac{64}{3}\right)$
$= \frac{32\sqrt{3}}{4}\left(1 - \frac{8}{5} + \frac{2}{3}\right) = \frac{8\sqrt{3}}{15}(15 - 24 + 10) = \frac{8\sqrt{3}}{15}$



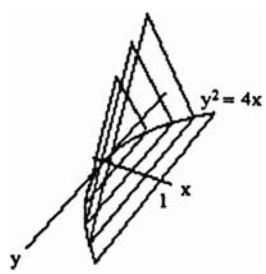
3. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}(2\sin x - 2\cos x)^2$
$= \frac{\pi}{4}\cdot 4\left(\sin^2 x - 2\sin x\cos x + \cos^2 x\right) = \pi(1-\sin 2x);$
$a = \frac{\pi}{4}, b = \frac{5\pi}{4} \Rightarrow V = \int_a^b A(x)\,dx$
$= \pi\int_{\pi/4}^{5\pi/4}(1-\sin 2x)\,dx = \pi\left[x + \frac{\cos 2x}{2}\right]_{\pi/4}^{5\pi/4}$
$= \pi\left[\left(\frac{5\pi}{4} + \frac{\cos\frac{5x}{2}}{2}\right) - \left(\frac{\pi}{4} - \frac{\cos\frac{\pi}{2}}{2}\right)\right] = \pi^2$



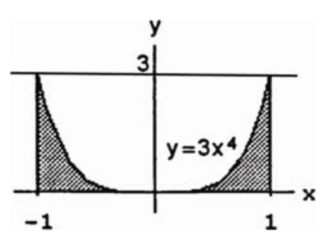
4. $A(x) = (\text{edge})^2 = \left(\left(\sqrt{6} - \sqrt{x}\right)^2 - 0\right)^2 = \left(\sqrt{6} - \sqrt{x}\right)^4 = 36 - 24\sqrt{6}\sqrt{x} + 36x - 4\sqrt{6}x^{3/2} + x^2;\ \ a = 0, b = 6 \Rightarrow V$
$= \int_a^b A(x)\,dx = \int_0^6\left(36 - 24\sqrt{6}\sqrt{x} + 36x - 4\sqrt{6}x^{3/2} + x^2\right)dx = \left[36x - 24\sqrt{6}\cdot\frac{2}{3}x^{3/2} + 18x^2 - 4\sqrt{6}\cdot\frac{2}{5}x^{5/2} + \frac{x^3}{3}\right]_0^6$
$= 216 - 16\cdot\sqrt{6}\sqrt{6}\cdot 6 + 18\cdot 6^2 - \frac{8}{5}\sqrt{6}\ \sqrt{6}\cdot 6^2 + \frac{6^3}{3} = 216 - 576 + 648 - \frac{1728}{5} + 72 = 360 - \frac{1728}{5} = \frac{1800-1728}{5} = \frac{72}{5}$

5. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(2\sqrt{x} - \frac{x^2}{4}\right)^2 = \frac{\pi}{4}\left(4x - x^{5/2} + \frac{x^4}{16}\right);\ \ a = 0, b = 4 \ \Rightarrow V = \int_a^b A(x)\,dx$
$= \frac{\pi}{4}\int_0^4\left(4x - x^{5/2} + \frac{x^4}{16}\right)dx = \frac{\pi}{4}\left[2x^2 - \frac{2}{7}x^{7/2} + \frac{x^5}{5\cdot 16}\right]_0^4 = \frac{\pi}{4}\left(32 - 32\cdot\frac{8}{7} + \frac{2}{5}\cdot 32\right) = \frac{32\pi}{4}\left(1 - \frac{8}{7} + \frac{2}{5}\right)$
$= \frac{8\pi}{35}(35 - 40 + 14) = \frac{72\pi}{35}$

6. $A(x) = \frac{1}{2}(\text{edge})^2\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}\left[2\sqrt{x} - \left(-2\sqrt{x}\right)\right]^2$
$= \frac{\sqrt{3}}{4}\left(4\sqrt{x}\right)^2 = 4\sqrt{3}x;\ \ a = 0, b = 1$
$\Rightarrow V = \int_a^b A(x)\,dx = \int_0^1 4\sqrt{3}x\,dx = \left[2\sqrt{3}x^2\right]_0^1 = 2\sqrt{3}$



7. (a) *disk method*:
$V = \int_a^b \pi[R(x)]^2\,dx = \int_{-1}^1 \pi\left(3x^4\right)^2 dx$
$= \pi\int_{-1}^1 9x^8 dx = \pi\left[x^9\right]_{-1}^1 = 2\pi$



(b) *shell method*:
$V = \int_a^b 2\pi\left(\substack{\text{shell}\\\text{radius}}\right)\left(\substack{\text{shell}\\\text{height}}\right)dx = \int_0^1 2\pi x\left(3x^4\right)dx = 2\pi\cdot 3\int_0^1 x^5 dx = 2\pi\cdot 3\left[\frac{x^6}{6}\right]_0^1 = \pi$
Note: The lower limit of integration is 0 rather than -1.

(c) *shell method*:
$V = \int_a^b 2\pi\left(\substack{\text{shell}\\\text{radius}}\right)\left(\substack{\text{shell}\\\text{height}}\right)dx = 2\pi\int_{-1}^1(1-x)\left(3x^4\right)dx = 2\pi\left[\frac{3x^3}{5} - \frac{x^6}{2}\right]_{-1}^1 = 2\pi\left[\left(\frac{3}{5} - \frac{1}{2}\right) - \left(-\frac{3}{5} - \frac{1}{2}\right)\right] = \frac{12\pi}{5}$

(d) *washer method*:

$R(x)=3, r(x)=3-3x^4=3\left(1-x^4\right) \Rightarrow V=\int_a^b \pi\left([R(x)]^2-[r(x)]^2\right) dx=\int_{-1}^1 \pi\left[9-9\left(1-x^4\right)^2\right] dx$

$=9\pi \int_{-1}^1\left[1-\left(1-2x^4+x^8\right)\right] dx=9\pi \int_{-1}^1\left(2x^4-x^8\right) dx=9\pi\left[\frac{2x^5}{5}-\frac{x^9}{9}\right]_{-1}^1=18\pi\left[\frac{2}{5}-\frac{1}{9}\right]=\frac{2\pi \cdot 13}{5}=\frac{26\pi}{5}$

8. (a) *washer method*:

$R(x)=\frac{4}{x^3}, r(x)=\frac{1}{2} \Rightarrow V=\int_a^b \pi\left([R(x)]^2-[r(x)]^2\right) dx=\int_1^2 \pi\left[\left(\frac{4}{x^3}\right)^2-\left(\frac{1}{2}\right)^2\right] dx=\pi\left[-\frac{16}{5} x^{-5}-\frac{x}{4}\right]_1^2$

$=\pi\left[\left(\frac{-16}{5 \cdot 32}-\frac{1}{2}\right)-\left(-\frac{16}{5}-\frac{1}{4}\right)\right]=\pi\left(-\frac{1}{10}-\frac{1}{2}+\frac{16}{5}+\frac{1}{4}\right)=\frac{\pi}{20}(-2-10+64+5)=\frac{57\pi}{20}$

(b) *shell method*:

$V=2\pi \int_1^2 x\left(\frac{4}{x^3}-\frac{1}{2}\right) dx=2\pi\left[-4x^{-1}-\frac{x^2}{4}\right]_1^2=2\pi\left[\left(-\frac{4}{2}-1\right)-\left(-4-\frac{1}{4}\right)\right]=2\pi\left(\frac{5}{4}\right)=\frac{5\pi}{2}$

(c) *shell method*:

$V=2\pi \int_a^b\left(\begin{array}{c}\text{shell} \\ \text{radius}\end{array}\right)\left(\begin{array}{c}\text{shell} \\ \text{height}\end{array}\right) dx=2\pi \int_1^2(2-x)\left(\frac{4}{x^3}-\frac{1}{2}\right) dx=2\pi \int_1^2\left(\frac{8}{x^3}-\frac{4}{x^2}-1+\frac{x}{2}\right) dx$

$=2\pi\left[-\frac{4}{x^2}+\frac{4}{x}-x+\frac{x^2}{4}\right]_1^2=2\pi\left[(-1+2-2+1)-\left(-4+4-1+\frac{1}{4}\right)\right]=\frac{3\pi}{2}$

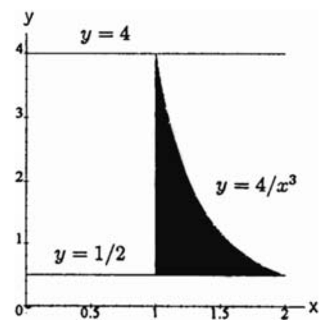
(d) *washer method*:

$V=\int_a^b \pi\left([R(x)]^2-[r(x)]^2\right) dx$

$=\pi \int_1^2\left[\left(\frac{7}{2}\right)^2-\left(4-\frac{4}{x^3}\right)^2\right] dx$

$=\frac{49\pi}{4}-16\pi \int_1^2\left(1-2x^{-3}+x^{-6}\right) dx$

$=\frac{49\pi}{4}-16\pi\left[x+x^{-2}-\frac{x^{-5}}{5}\right]_1^2$



$=\frac{49\pi}{4}-16\pi\left[\left(2+\frac{1}{4}-\frac{1}{5 \cdot 32}\right)-\left(1+1-\frac{1}{5}\right)\right]=\frac{49\pi}{4}-16\pi\left(\frac{1}{4}-\frac{1}{160}+\frac{1}{5}\right)=\frac{49\pi}{4}-\frac{16\pi}{160}(40-1+32)=\frac{49\pi}{4}-\frac{71\pi}{10}$

$=\frac{103\pi}{20}$

9. (a) *disk method*:

$V=\pi \int_1^5\left(\sqrt{x-1}\right)^2 dx=\pi \int_1^5(x-1) dx=\pi\left[\frac{x^2}{2}-x\right]_1^5=\pi\left[\left(\frac{25}{2}-5\right)-\left(\frac{1}{2}-1\right)\right]=\pi\left(\frac{24}{2}-4\right)=8\pi$

(b) *washer method*:

$R(y)=5, r(y)=y^2+1 \Rightarrow V=\int_c^d \pi\left([R(y)]^2-[r(y)]^2\right) dy=\pi \int_{-2}^2\left[25-\left(y^2+1\right)^2\right] dy$

$=\pi \int_{-2}^2\left(25-y^4-2y^2-1\right) dy=\pi \int_{-2}^2\left(24-y^4-2y^2\right) dy=\pi\left[24y-\frac{y^5}{5}-\frac{2}{3} y^3\right]_{-2}^2$

$=2\pi\left(24 \cdot 2-\frac{32}{5}-\frac{2}{3} \cdot 8\right)=32\pi\left(3-\frac{2}{5}-\frac{1}{3}\right)=\frac{32\pi}{15}(45-6-5)=\frac{1088\pi}{15}$

(c) *disk method*:

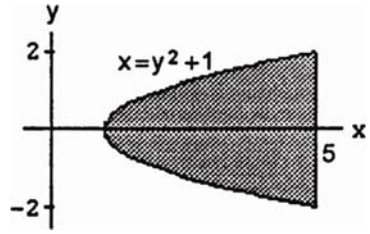
$R(y)=5-\left(y^2+1\right)=4-y^2$

$\Rightarrow V=\int_c^d \pi\left[R(y)\right]^2 dy=\int_{-2}^{2}\pi\left(4-y^2\right)^2 dy$

$=\pi\int_{-2}^{2}\left(16-8y^2+y^4\right)dy=\pi\left[16y-\frac{8y^3}{3}+\frac{y^5}{5}\right]_{-2}^{2}$

$=2\pi\left(32-\frac{64}{3}+\frac{32}{5}\right)=64\pi\left(1-\frac{2}{3}+\frac{1}{5}\right)$

$=\frac{64\pi}{15}(15-10+3)=\frac{512\pi}{15}$

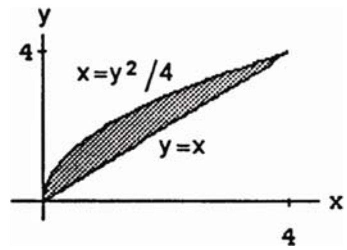


10. (a) *shell method*:

$V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy$

$=\int_0^4 2\pi y\left(y-\frac{y^2}{4}\right)dy=2\pi\int_0^4\left(y^2-\frac{y^3}{4}\right)dy$

$=2\pi\left[\frac{y^3}{3}-\frac{y^4}{16}\right]_0^4=2\pi\left(\frac{64}{3}-\frac{64}{4}\right)=\frac{2\pi}{12}\cdot 64=\frac{32\pi}{3}$



(b) *shell method*:

$V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^4 2\pi x\left(2\sqrt{x}-x\right)dx=2\pi\int_0^4\left(2x^{3/2}-x^2\right)dx=2\pi\left[\frac{4}{5}x^{5/2}-\frac{x^3}{3}\right]_0^4$

$=2\pi\left(\frac{4}{5}\cdot 32-\frac{64}{3}\right)=\frac{128\pi}{15}$

(c) *shell method*:

$V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^4 2\pi(4-x)\left(2\sqrt{x}-x\right)dx=2\pi\int_0^4\left(8x^{1/2}-4x-2x^{3/2}+x^2\right)dx$

$=2\pi\left[\frac{16}{3}x^{3/2}-2x^2-\frac{4}{5}x^{5/2}+\frac{x^3}{3}\right]_0^4=2\pi\left(\frac{16}{3}\cdot 8-32-\frac{4}{5}\cdot 32+\frac{64}{3}\right)=64\pi\left(\frac{4}{3}-1-\frac{4}{5}+\frac{2}{3}\right)=64\pi\left(1-\frac{4}{5}\right)=\frac{64\pi}{5}$

(d) *shell method*:

$V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^4 2\pi(4-y)\left(y-\frac{y^2}{4}\right)dy=2\pi\int_0^4\left(4y-y^2-y^2+\frac{y^3}{4}\right)dy$

$=2\pi\int_0^4\left(4y-2y^2+\frac{y^3}{4}\right)dy=2\pi\left[2y^2-\frac{2}{3}y^3+\frac{y^4}{16}\right]_0^4=2\pi\left(32-\frac{2}{3}\cdot 64+16\right)=32\pi\left(2-\frac{8}{3}+1\right)=\frac{32\pi}{3}$

11. *disk method*:

$R(x)=\tan x, a=0, b=\frac{\pi}{3}\Rightarrow V=\pi\int_0^{\pi/3}\tan^2 x\,dx=\pi\int_0^{\pi/3}\left(\sec^2 x-1\right)dx=\pi\left[\tan x-x\right]_0^{\pi/3}=\frac{\pi\left(3\sqrt{3}-\pi\right)}{3}$

12. *disk method*:

$V=\pi\int_0^{\pi}(2-\sin x)^2 dx=\pi\int_0^{\pi}\left(4-4\sin x+\sin^2 x\right)dx=\pi\int_0^{\pi}\left(4-4\sin x+\frac{1-\cos 2x}{2}\right)dx$

$=\pi\left[4x+4\cos x+\frac{x}{2}-\frac{\sin 2x}{4}\right]_0^{\pi}=\pi\left[\left(4\pi-4+\frac{\pi}{2}-0\right)-(0+4+0-0)\right]=\pi\left(\frac{9\pi}{2}-8\right)=\frac{\pi}{2}(9\pi-16)$

13. (a) *disk method*:

$V=\pi\int_0^2\left(x^2-2x\right)^2 dx=\pi\int_0^2\left(x^4-4x^3+4x^2\right)dx=\pi\left[\frac{x^5}{5}-x^4+\frac{4}{3}x^3\right]_0^2=\pi\left(\frac{32}{5}-16+\frac{32}{3}\right)$

$=\frac{16\pi}{15}(6-15+10)=\frac{16\pi}{15}$

(b) *washer method*:

$$V=\int_0^2 \pi\left[1^2-\left(x^2-2x+1\right)^2\right]dx=\int_0^2 \pi\,dx-\int_0^2 \pi(x-1)^4\,dx=2\pi-\left[\pi\frac{(x-1)^5}{5}\right]_0^2=2\pi-\pi\cdot\frac{2}{5}=\frac{8\pi}{5}$$

(c) *shell method*:

$$V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=2\pi\int_0^2(2-x)\left[-\left(x^2-2x\right)\right]dx=2\pi\int_0^2(2-x)\left(2x-x^2\right)dx$$

$$=2\pi\int_0^2\left(4x-2x^2-2x^2+x^3\right)dx=2\pi\int_0^2\left(x^3-4x^2+4x\right)dx=2\pi\left[\frac{x^4}{4}-\frac{4}{3}x^3+2x^2\right]_0^2=2\pi\left(4-\frac{32}{3}+8\right)$$

$$=\frac{2\pi}{3}(36-32)=\frac{8\pi}{3}$$

(d) *washer method*:

$$V=\pi\int_0^2\left[2-\left(x^2-2x\right)\right]^2dx-\pi\int_0^2 2^2dx=\pi\int_0^2\left[4-4\left(x^2-2x\right)+\left(x^2-2x\right)^2\right]dx-8\pi$$

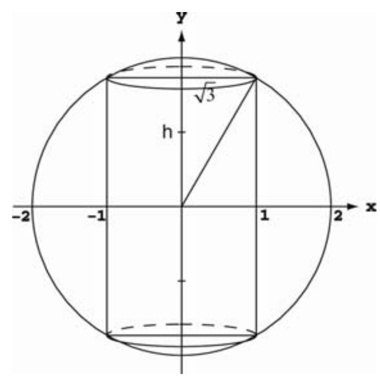
$$=\pi\int_0^2\left(4-4x^2+8x+x^4-4x^3+4x^2\right)dx-8\pi=\pi\int_0^2\left(x^4-4x^3+8x+4\right)dx-8\pi$$

$$=\pi\left[\frac{x^5}{5}-x^4+4x^2+4x\right]_0^2-8\pi=\pi\left(\frac{32}{5}-16+16+8\right)-8\pi=\frac{\pi}{5}(32+40)-8\pi=\frac{72\pi}{5}-\frac{40\pi}{5}=\frac{32\pi}{5}$$

14. *disk method*:

$$V=2\pi\int_0^{\pi/4}4\tan^2x\,dx=8\pi\int_0^{\pi/4}\left(\sec^2x-1\right)dx=8\pi\left[\tan x-x\right]_0^{\pi/4}=2\pi(4-\pi)$$

15. The material removed from the sphere consists of a cylinder and two "caps." From the diagram, the height of the cylinder is 2h, where $h^2+\left(\sqrt{3}\right)^2=2^2$, *i.e.* $h=1$. Thus $V_{cyl}=(2h)\pi\left(\sqrt{3}\right)^2=6\pi\text{ m}^3$. To get the volume of a cap, use the disk method and $x^2+y^2=2^2$: $V_{cap}=\int_1^2\pi x^2dy$ $=\int_1^2\pi\left(4-y^2\right)dy=\pi\left[4y-\frac{y^3}{3}\right]_1^2=\pi\left[\left(8-\frac{8}{3}\right)-\left(4-\frac{1}{3}\right)\right]$ $=\frac{5\pi}{3}\text{m}^3$. Therefore, $V_{removed}=V_{cyl}+2V_{cap}=6\pi+\frac{10\pi}{3}$ $=\frac{28\pi}{3}\text{m}^3$.



16. We rotate the region enclosed by the curve $y=\sqrt{12\left(1-\frac{4x^2}{121}\right)}$ and the x-axis around the x-axis. To find the volume we use the *disk* method: $V=\int_a^b\pi\left[R(x)\right]^2dx=\int_{-11/2}^{11/2}\pi\left(\sqrt{12\left(1-\frac{4x^2}{121}\right)}\right)^2dx=\pi\int_{-11/2}^{11/2}12\left(1-\frac{4x^2}{121}\right)dx$

$$=12\pi\int_{-11/2}^{11/2}\left(1-\frac{4x^2}{121}\right)dx=12\pi\left[x-\frac{4x^3}{363}\right]_{-11/2}^{11/2}=24\pi\left[\frac{11}{2}-\left(\frac{4}{363}\right)\left(\frac{11}{2}\right)^3\right]=132\pi\left[1-\left(\frac{4}{363}\right)\left(\frac{11^2}{4}\right)\right]=132\pi\left(1-\frac{1}{3}\right)$$

$$=\frac{264\pi}{3}=88\pi\approx 276\text{ cm}^3$$

17. $y = x^{1/2} - \frac{x^{3/2}}{3} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(\frac{1}{x} - 2 + x\right) \Rightarrow L = \int_1^4 \sqrt{1 + \frac{1}{4}\left(\frac{1}{x} - 2 + x\right)}\, dx$

$\Rightarrow L = \int_1^4 \sqrt{\frac{1}{4}\left(\frac{1}{x} + 2 + x\right)}\, dx = \int_1^4 \sqrt{\frac{1}{4}\left(x^{-1/2} + x^{1/2}\right)^2}\, dx = \int_1^4 \frac{1}{2}\left(x^{-1/2} + x^{1/2}\right) dx = \frac{1}{2}\left[2x^{1/2} + \frac{2}{3}x^{3/2}\right]_1^4$

$= \frac{1}{2}\left[\left(4 + \frac{2}{3}\cdot 8\right) - \left(2 + \frac{2}{3}\right)\right] = \frac{1}{2}\left(2 + \frac{14}{3}\right) = \frac{10}{3}$

18. $x = y^{2/3} \Rightarrow \frac{dx}{dy} = \frac{2}{3}y^{-1/3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4y^{-2/3}}{9} \Rightarrow L = \int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^8 \sqrt{1 + \frac{4}{9y^{2/3}}}\, dy = \int_1^8 \frac{\sqrt{9y^{2/3}+4}}{3y^{1/3}}\, dy$

$= \frac{1}{3}\int_1^8 \sqrt{9y^{2/3} + 4}\left(y^{-1/3}\right) dy;\ [u = 9y^{2/3} + 4 \Rightarrow du = 6y^{-1/3}dy;\ y = 1 \Rightarrow u = 13, y = 8 \Rightarrow u = 40]$

$\to L = \frac{1}{18}\int_{13}^{40} u^{1/2} du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \approx 7.634$

19. $\frac{dy}{dx} = \frac{1}{2}x^{1/5} - \frac{1}{2}x^{-1/5} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}x^{2/5} + \frac{1}{2} + \frac{1}{4}x^{-2/5} = \left(\frac{1}{2}x^{1/5} + \frac{1}{2}x^{-1/5}\right)^2$

$\int_1^{32} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^{32}\left(\frac{1}{2}x^{1/5} + \frac{1}{2}x^{-1/5}\right) dx = \left(\frac{5}{12}x^{6/5} + \frac{5}{8}x^{4/5}\right)\Big|_1^{32} = \frac{285}{8}$

20. $x = \frac{1}{12}y^3 + \frac{1}{y} \Rightarrow \frac{dx}{dy} = \frac{1}{4}y^2 - \frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4} \Rightarrow L = \int_1^2 \sqrt{1 + \left(\frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}\right)}\, dy$

$= \int_1^2 \sqrt{\frac{1}{16}y^4 + \frac{1}{2} + \frac{1}{y^4}}\, dy = \int_1^2 \sqrt{\left(\frac{1}{4}y^2 + \frac{1}{y^2}\right)^2}\, dy = \int_1^2 \left(\frac{1}{4}y^2 + \frac{1}{y^2}\right) dy = \left[\frac{1}{12}y^3 - \frac{1}{y}\right]_1^2$

$= \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right) = \frac{7}{12} + \frac{1}{2} = \frac{13}{12}$

21. $S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx;\ \frac{dy}{dx} = \frac{1}{\sqrt{2x+1}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{2x+1} \Rightarrow S = \int_0^3 2\pi\sqrt{2x+1}\sqrt{1 + \frac{1}{2x+1}}\, dx$

$= 2\pi\int_0^3 \sqrt{2x+1}\sqrt{\frac{2x+2}{2x+1}}\, dx = 2\sqrt{2}\pi\int_0^3 \sqrt{x+1}\, dx = 2\sqrt{2}\pi\left[\frac{2}{3}(x+1)^{3/2}\right]_0^3 = 2\sqrt{2}\pi\cdot\frac{2}{3}(8-1) = \frac{28\pi\sqrt{2}}{3}$

22. $S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx;\ \frac{dy}{dx} = x^2 \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^4 \Rightarrow S = \int_0^1 2\pi\cdot\frac{x^3}{3}\sqrt{1+x^4}\, dx = \frac{\pi}{6}\int_0^1 \sqrt{1+x^4}\left(4x^3\right) dx$

$= \frac{\pi}{6}\left[\frac{2}{3}\left(1+x^4\right)^{3/2}\right]_0^1 = \frac{\pi}{9}\left(2\sqrt{2} - 1\right)$

23. $S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy;\ \frac{dx}{dy} = \frac{\left(\frac{1}{2}\right)(4-2y)}{\sqrt{4y-y^2}} = \frac{2-y}{\sqrt{4y-y^2}} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = \frac{4y-y^2+4-4y+y^2}{4y-y^2} = \frac{4}{4y-y^2}$

$\Rightarrow S = \int_1^2 2\pi\sqrt{4y-y^2}\sqrt{\frac{4}{4y-y^2}}\, dy = 4\pi\int_1^2 dx = 4\pi$

24. $S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy;\ \frac{dx}{dy} = \frac{1}{2\sqrt{y}} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4y} = \frac{4y+1}{4y} \Rightarrow S = \int_2^6 2\pi\sqrt{y}\cdot\frac{\sqrt{4y+1}}{\sqrt{4y}}\, dy = \pi\int_2^6 \sqrt{4y+1}\, dy$

$= \frac{\pi}{4}\left[\frac{2}{3}(4y+1)^{3/2}\right]_2^6 = \frac{\pi}{6}(125-27) = \frac{\pi}{6}(98) = \frac{49\pi}{3}$

25. The equipment alone: the force required to lift the equipment is equal to its weight $\Rightarrow F_1(x) = 100\ N$. The work done is $W_1 = \int_a^b F_1(x)\,dx = \int_0^{40} 100\,dx = [100x]_0^{40} = 4000$ J; the rope alone: the force required to lift the rope is equal to the weight of the rope paid out at elevation $x \Rightarrow F_2(x) = 0.8(40 - x)$. The work done is $W_2 = \int_a^b F_2(x)\,dx = \int_0^{40} 0.8(40 - x)\,dx = 0.8\left[40x - \frac{x^2}{2}\right]_0^{40} = 0.8\left(40^2 - \frac{40^2}{2}\right) = \frac{(0.8)(1600)}{2} = 640$ J; the total work is $W = W_1 + W_2 = 4000 + 640 = 4640$ J

26. The force required to lift the water is equal to the water's weight, which varies steadily from $9.8 \cdot 4000$ N to $9.8 \cdot 2000$ N over the 1500 m elevation. When the truck is x m off the base of Mt. Washington, the water weight is $F(x) = 9.8 \cdot 4000 \cdot \left(\frac{2 \cdot 1500 - x}{2 \cdot 1500}\right) = (39{,}200)\left(1 - \frac{x}{3000}\right)$ N. The work done is

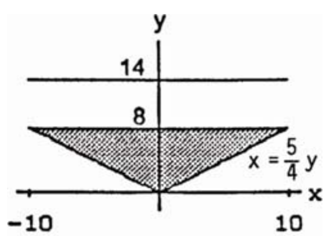
$$W = \int_a^b F(x)\,dx = \int_0^{1500} 39{,}200\left(1 - \frac{x}{3000}\right)dx$$

$$= 39{,}200\left[x - \frac{x^2}{2 \cdot 3000}\right]_0^{1500} = 39{,}200\left(1500 - \frac{1500^2}{4 \cdot 1500}\right) = \left(\frac{3}{4}\right)(39{,}200)(1500) = 44{,}100{,}000 \text{ J}$$

27. Using a proportionality constant of 1, the work in lifting the weight of w N from $r - a$ to a is $\int_{r-a}^{r} wt\,dt = w\left[\frac{t^2}{2}\right]_{r-a}^{r} = \frac{w}{2}\left(r^2 - (r-a)^2\right) = \frac{w}{2}(2ar - a^2)$.

28. Force constant: $F = kx \Rightarrow 200 = k(0.8) \Rightarrow k = 250$ N/m; the 300 N force stretches the spring $x = \frac{F}{k} = \frac{300}{250} = 1.2$ m; the work required to stretch the spring that far is then $W = \int_0^{1.2} F(x)\,dx = \int_0^{1.2} 250x\,dx$ $= \int_0^{1.2} 250x\,dx = \left[125x^2\right]_0^{1.2} = 125(1.2)^2 = 180$ J

29. We imagine the water divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0, 8]$. The typical slab between the planes at y and $y + \Delta y$ has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{5}{4}y\right)^2 \Delta y$ $= \frac{25\pi}{16} y^2 \Delta y$ m^3. The force $F(y)$ required to lift this slab is equal to its weight: $F(y) = 9800\Delta V$ $= \frac{(9800)(25)}{16}\pi y^2 \Delta y$ N. The distance through which $F(y)$ must act to lift this slab to the level 6 m above the top is about $(6 + 8 - y)$ m, so the work done lifting the slab is about $\Delta W = \frac{(9800)(25)}{16}\pi y^2(14 - y)\Delta y$ J. The work done lifting all the slabs from $y = 0$ to $y = 8$ to the level 6 m above the top is approximately



Reservoir's Cross Section

$W \approx \sum_0^8 \frac{(9800)(25)}{16}\pi y^2(14 - y)\Delta y$ J so the work to pump the water is the limit of these Riemann sums as the norm of the partition goes to zero: $W = \int_0^8 \frac{(9800)(25)}{(16)}\pi y^2(14 - y)\,dy = \frac{(9800)(25)\pi}{16}\int_0^8 \left(14y^2 - y^3\right)dy$

$$= (9800)\left(\frac{25\pi}{16}\right)\left[\frac{14}{3}y^3 - \frac{y^4}{4}\right]_0^8 = (9800)\left(\frac{25\pi}{16}\right)\left(\frac{14}{3} \cdot 8^3 - \frac{8^4}{4}\right) \approx 65{,}680{,}230 \text{ J}$$

30. The same as in Exercise 29, but change the distance through which $F(y)$ must act to $(8-y)$ rather than $(6+8-y)$. Also change the upper limit of integration from 8 to 5. The integral is:

$$W=\int_0^5 \frac{(9800)(25)\pi}{16} y^2(8-y)\,dy=(9800)\left(\frac{25\pi}{16}\right)\int_0^5\left(8y^2-y^3\right)dy=(9800)\left(\frac{25\pi}{16}\right)\left[\frac{8}{3}y^3-\frac{y^4}{4}\right]_0^5$$

$$=(9800)\left(\frac{25\pi}{16}\right)\left(\frac{8}{3}\cdot 5^3-\frac{5^4}{4}\right)\approx 8{,}518{,}707\text{ J}$$

31. The tank's cross section looks like the figure in Exercise 29 with right edge given by $x=\frac{5}{10}y=\frac{y}{2}$. A typical horizontal slab has volume $\Delta V=\pi(\text{radius})^2(\text{thickness})=\pi\left(\frac{y}{2}\right)^2\Delta y=\frac{\pi}{4}y^2\Delta y$. The force required to lift this slab is its weight: $F(y)=9000\cdot\frac{\pi}{4}y^2\Delta y$. The distance through which $F(y)$ must act is $(2+10-y)$ m, so the work to pump the liquid is $W=9000\int_0^{10}\pi(12-y)\left(\frac{y^2}{4}\right)dy=2250\pi\left[\frac{12y^3}{3}-\frac{y^4}{4}\right]_0^{10}=3{,}375{,}000\pi$ J; the time needed to empty the tank is $\frac{3{,}375{,}000\pi\text{ J}}{41{,}250\text{ J}}\approx 257\,\text{s}$.

32. A typical horizontal slab has volume about $\Delta V=(6)(2x)\Delta y=(6)\left(2\sqrt{1.5625-y^2}\right)\Delta y$ and the force required to lift this slab is its weight $F(y)=(8950)(6)\left(2\sqrt{1.5625-y^2}\right)\Delta y$. The distance through which $F(y)$ must act is $(2+1.25-y)$ m, so the work to pump the olive oil from the half-full tank is

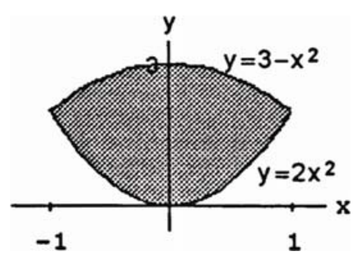
$$W=8950\int_{-1.25}^{0}(3.25-y)(6)\left(2\sqrt{1.5625-y^2}\right)dy$$

$$=107{,}400\int_{-1.25}^{0}3.25\sqrt{1.5625-y^2}\,dy+53{,}700\int_{-1.25}^{0}\left(1.5625-y^2\right)^{1/2}(-2y)\,dy=349{,}050\cdot$$

(area of a quarter circle having radius 1.25) $+\frac{2}{3}(53{,}700)\left[\left(1.5625-y^2\right)^{3/2}\right]_{-1.25}^{0}$

$$=(349{,}050)(0.390625\pi)+69{,}922=498{,}270\text{ J}$$

33. Intersection points: $3-x^2=2x^2\Rightarrow 3x^2-3=0$ $\Rightarrow 3(x-1)(x+1)=0\Rightarrow x=-1$ or $x=1$. Symmetry suggests that $\bar{x}=0$. The typical *vertical* strip has center of mass: $(\tilde{x},\tilde{y})=\left(x,\frac{2x^2+\left(3-x^2\right)}{2}\right)=\left(x,\frac{x^2+3}{2}\right)$, length: $\left(3-x^2\right)-2x^2=3\left(1-x^2\right)$, width: dx,

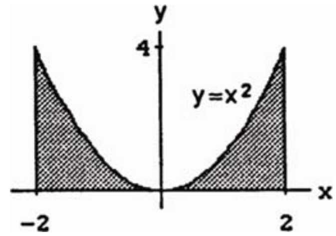


area: $dA=3\left(1-x^2\right)dx$, and mass: $dm=\delta\cdot dA=3\delta\left(1-x^2\right)dx\Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm=\frac{3}{2}\delta\left(x^2+3\right)\left(1-x^2\right)dx=\frac{3}{2}\delta\left(-x^4-2x^2+3\right)dx\Rightarrow M_x=\int\tilde{y}\,dm=\frac{3}{2}\delta\int_{-1}^{1}\left(-x^4-2x^2+3\right)dx$

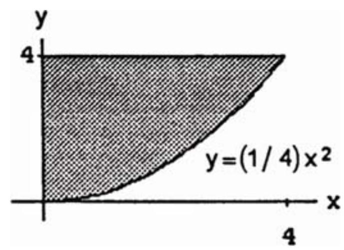
$=\frac{3}{2}\delta\left[-\frac{x^5}{5}-\frac{2x^3}{3}+3x\right]_{-1}^{1}=3\delta\left(-\frac{1}{5}-\frac{2}{3}+3\right)=\frac{3\delta}{15}(-3-10+45)=\frac{32\delta}{5}$; $M=\int dm=3\delta\int_{-1}^{1}\left(1-x^2\right)dx$

$=3\delta\left[x-\frac{x^3}{3}\right]_{-1}^{1}=6\delta\left(1-\frac{1}{3}\right)=4\delta\Rightarrow\bar{y}=\frac{M_x}{M}=\frac{32\delta}{5\cdot 4\delta}=\frac{8}{5}$. Therefore, the centroid is $(\bar{x},\bar{y})=\left(0,\frac{8}{5}\right)$.

34. Symmetry suggests that $\overline{x}=0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y})=\left(x, \frac{x^2}{2}\right)$, length: x^2, width: dx, area: $dA=x^2\ dx$, mass: $dm=\delta\cdot dA=\delta x^2\ dx \Rightarrow$ the moment about the x-axis is $\tilde{y}\ dm=\frac{\delta}{2}x^2\cdot x^2\ dx$ $=\frac{\delta}{2}x^4\ dx \Rightarrow M_x=\int \tilde{y}\ dm=\frac{\delta}{2}\int_{-2}^{2}x^4 dx=\frac{\delta}{10}\left[x^5\right]_{-2}^{2}$



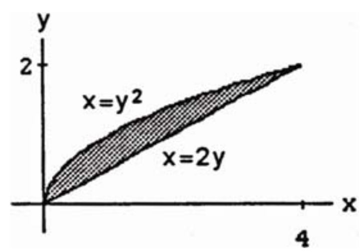
35. The typical *vertical* strip has: center of mass: $(\tilde{x}, \tilde{y})$ $=\left(x, \frac{4+\frac{x^2}{4}}{2}\right)$, length: $4-\frac{x^2}{4}$, width: dx, area: $dA=\left(4-\frac{x^2}{4}\right)dx$, mass: $dm=\delta\cdot dA$ $=\delta\left(4-\frac{x^2}{4}\right)dx \Rightarrow$ the moment about the x-axis is



$\tilde{y}\ dm=\delta\cdot\frac{\left(4+\frac{x^2}{4}\right)}{2}\left(4-\frac{x^2}{4}\right)dx=\frac{\delta}{2}\left(16-\frac{x^4}{16}\right)dx$; moment about: $\tilde{x}\ dm=\delta\left(4-\frac{x^2}{4}\right)\cdot x\ dx=\ \delta\left(4x-\frac{x^3}{4}\right)dx$.

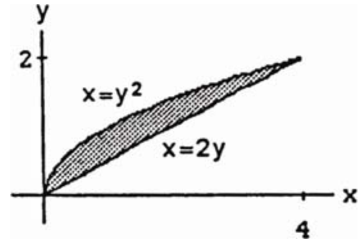
Thus, $M_x=\int \tilde{y}\ dm=\frac{\delta}{2}\int_0^4\left(16-\frac{x^4}{16}\right)dx=\frac{\delta}{2}\left[16x-\frac{x^5}{5\cdot 16}\right]_0^4=\frac{\delta}{2}\left[64-\frac{64}{5}\right]=\frac{128\delta}{5}$; $M_y=\int \tilde{x}\ dm$ $=\delta\int_0^4\left(4x-\frac{x^3}{4}\right)dx=\delta\left[2x^2-\frac{x^4}{16}\right]_0^4=\delta(32-16)=16\delta$; $M=\int dm=\delta\int_0^4\left(4-\frac{x^2}{4}\right)dx=\delta\left[4x-\frac{x^3}{12}\right]_0^4$ $=\delta\left(16-\frac{64}{12}\right)=\frac{32\delta}{3}\Rightarrow \overline{x}=\frac{M_y}{M}=\frac{16\cdot\delta\cdot 3}{32\cdot\delta}=\frac{3}{2}$ and $\overline{y}=\frac{M_x}{M}=\frac{128\cdot\delta\cdot 3}{5\cdot 32\cdot\delta}=\frac{12}{5}$. Centroid is $(\overline{x},\overline{y})=\left(\frac{3}{2},\frac{12}{5}\right)$.

36. A typical *horizontal* strip has: center of mass: $(\tilde{x}, \tilde{y})=\left(\frac{y^2+2y}{2}, y\right)$, length: $2y-y^2$, width: dy, area: $dA=\left(2y-y^2\right)dy$, mass: $dm=\delta\cdot dA$ $=\delta\left(2y-y^2\right)dy$; the moment about the x-axis is $\tilde{y}\ dm=\delta\cdot y\cdot\left(2y-y^2\right)dy=\delta\left(2y^2-y^3\right)dy$;



the moment about the y-axis is $\tilde{x}\ dm=\delta\cdot\frac{\left(y^2+2y\right)}{2}\cdot\left(2y-y^2\right)dy=\frac{\delta}{2}\left(4y^2-y^4\right)dy \Rightarrow M_x=\int \tilde{y}\ dm$ $=\delta\int_0^2\left(2y^2-y^3\right)dy=\delta\left[\frac{2}{3}y^3-\frac{y^4}{4}\right]_0^2=\delta\left(\frac{2}{3}\cdot 8-\frac{16}{4}\right)=\delta\left(\frac{16}{3}-\frac{16}{4}\right)=\frac{\delta\cdot 16}{12}=\frac{4\delta}{3}$; $M_y=\int \tilde{x}\ dm$ $=\frac{\delta}{2}\int_0^2\left(4y^2-y^4\right)dy=\frac{\delta}{2}\left[\frac{4}{3}y^3-\frac{y^5}{5}\right]_0^2=\frac{\delta}{2}\left(\frac{4\cdot 8}{3}-\frac{32}{5}\right)=\frac{32\delta}{15}$; $M=\int dm=\delta\int_0^2\left(2y-y^2\right)dy=\delta\left[y^2-\frac{y^3}{3}\right]_0^2$ $=\delta\left(4-\frac{8}{3}\right)=\frac{4\delta}{3}\Rightarrow \overline{x}=\frac{M_y}{M}=\frac{\delta\cdot 32\cdot 3}{15\cdot\delta\cdot 4}=\frac{8}{5}$ and $\tilde{y}=\frac{M_x}{M}=\frac{4\cdot\delta\cdot 3}{3\cdot 4\cdot\delta}=1$. Therefore, the centroid is $(\overline{x},\overline{y})=\left(\frac{8}{5},1\right)$.

37. A typical horizontal strip has: center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{y^2+2y}{2}, y\right)$, length: $2y - y^2$, width: dy, area: $dA = \left(2y - y^2\right) dy$, mass: $dm = \delta \cdot dA$ $= (1+y)\left(2y - y^2\right) dy \Rightarrow$ the moment about the x-axis is $\tilde{y} = dm = y(1+y)\left(2y - y^2\right) dy$ $= \left(2y^2 + 2y^3 - y^3 - y^4\right) dy = \left(2y^2 + y^3 - y^4\right) dy$; the moment about the y-axis is $\tilde{x}\, dm = \left(\frac{y^2+2y}{2}\right)(1+y)\left(2y - y^2\right) dy = \frac{1}{2}\left(4y^2 - y^4\right)(1+y)\, dy = \frac{1}{2}\left(4y^2 + 4y^3 - y^4 - y^5\right) dy \Rightarrow M_x = \int \tilde{y}\, dm$ $= \int_0^2 \left(2y^2 + y^3 - y^4\right) dy = \left[\frac{2}{3}y^3 + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^2 = \left(\frac{16}{3} + \frac{16}{4} - \frac{32}{5}\right) = 16\left(\frac{1}{3} + \frac{1}{4} - \frac{2}{5}\right) = \frac{16}{60}(20 + 15 - 24) = \frac{4}{15}(11) = \frac{44}{15}$; $M_y = \int \tilde{x}\, dm = \int_0^2 \frac{1}{2}\left(4y^2 + 4y^3 - y^4 - y^5\right) dy = \frac{1}{2}\left[\frac{4}{3}y^3 + y^4 - \frac{y^5}{5} - \frac{y^6}{6}\right]_0^2 = \frac{1}{2}\left(\frac{4 \cdot 2^3}{3} + 2^4 - \frac{2^5}{5} - \frac{2^6}{6}\right)$ $= 4\left(\frac{4}{3} + 2 - \frac{4}{5} - \frac{8}{6}\right) = 4\left(2 - \frac{4}{5}\right) = \frac{24}{5}$; $M = \int dm = \int_0^2 (1+y)\left(2y - y^2\right) dy = \int_0^2 \left(2y + y^2 - y^3\right) dy$ $= \left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2 = \left(4 + \frac{8}{3} - \frac{16}{4}\right) = \frac{8}{3} \Rightarrow \bar{x} = \frac{M_y}{M} = \left(\frac{24}{5}\right)\left(\frac{3}{8}\right) = \frac{9}{5}$ and $\bar{y} = \frac{M_x}{M} = \left(\frac{44}{15}\right)\left(\frac{3}{8}\right) = \frac{44}{40} = \frac{11}{10}$. Therefore, the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{9}{5}, \frac{11}{10}\right)$.



38. A typical vertical strip has: center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{3}{2x^{3/2}}\right)$, length: $\frac{3}{x^{3/2}}$, width: dx, area: $dA = \frac{3}{x^{3/2}}\, dx$, mass: $dm = \delta \cdot dA = \delta \cdot \frac{3}{x^{3/2}}\, dx \Rightarrow$ the moment about the x-axis is $\tilde{y}\, dm = \frac{3}{2x^{3/2}} \cdot \delta \frac{3}{x^{3/2}}\, dx = \frac{9\delta}{2x^3}\, dx$; the moment about the y-axis is $\tilde{x}\, dm = x \cdot \delta \frac{3}{x^{3/2}}\, dx = \frac{3\delta}{x^{1/2}}\, dx$.

(a) $M_x = \delta \int_1^9 \frac{1}{2}\left(\frac{9}{x^3}\right) dx = \frac{9\delta}{2}\left[-\frac{x^{-2}}{2}\right]_1^9 = \frac{20\delta}{9}$; $M_y = \delta \int_1^9 x\left(\frac{3}{x^{3/2}}\right) dx = 3\delta\left[2x^{1/2}\right]_1^9 = 12\delta$;

$M = \delta \int_1^9 \frac{3}{x^{3/2}}\, dx = -6\delta\left[x^{-1/2}\right]_1^9 = 4\delta \Rightarrow \bar{x} = \frac{My}{M} = \frac{12\delta}{4\delta} = 3$ and $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{20\delta}{9}\right)}{4\delta} = \frac{5}{9}$

(b) $M_x = \int_1^9 \frac{x}{2}\left(\frac{9}{x^3}\right) dx = \frac{9}{2}\left[-\frac{1}{x}\right]_1^9 = 4$; $M_y = \int_1^9 x^2\left(\frac{3}{x^{3/2}}\right) dx = \left[2x^{3/2}\right]_1^9 = 52$; $M = \int_1^9 x\left(\frac{3}{x^{3/2}}\right) dx = 6\left[x^{1/2}\right]_1^9$ $= 12 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{13}{3}$ and $\bar{y} = \frac{M_x}{M} = \frac{1}{3}$

39. $F = \int_a^b W \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right) \cdot L(y)\, dy \Rightarrow F = 2\int_0^2 (9800)(2-y)(2y)\, dy = 39{,}200 \int_0^2 \left(2y - y^2\right) dy = 39{,}200\left[y^2 - \frac{y^3}{3}\right]_0^2$ $= (39{,}200)\left(4 - \frac{8}{3}\right) = (39{,}200)\left(\frac{4}{3}\right) = 52{,}267$ J

40. $F = \int_a^b W \cdot \left(\begin{smallmatrix}\text{strip}\\\text{depth}\end{smallmatrix}\right) \cdot L(y)\, dy \Rightarrow F = \int_0^{0.5} 11{,}800(0.5-y)(2y+4)\, dy = 11{,}800\int_0^{0.5}\left(y+2-2y^2-4y\right) dy$

$= 11{,}800\int_0^{0.5}\left(2-3y-2y^2\right) dy = 11{,}800\left[\frac{10}{3}y - \frac{3}{2}y^2 - \frac{2}{3}y^3\right]_0^{0.5} = (11{,}800)\left[1-\left(\frac{3}{2}\right)(0.25)-\left(\frac{2}{3}\right)(0.125)\right]$

$= (11{,}800)\left(1-\frac{75}{200}-\frac{250}{3\cdot 1000}\right) = \left(\frac{11{,}800}{3000}\right)(3000-75\cdot 15-250) = \frac{(11{,}800)(1625)}{3000} \approx 6392 \text{ N}.$

41. $F = \int_a^b W \cdot \left(\begin{smallmatrix}\text{strip}\\\text{depth}\end{smallmatrix}\right) \cdot L(y)\, dy \Rightarrow F = 9800\int_0^4 (9-y)\left(2\cdot\frac{\sqrt{y}}{2}\right) dy = 9800\int_0^4\left(9y^{1/2}-3y^{3/2}\right) dy$

$= 9800\left[6y^{3/2}-\frac{2}{5}y^{5/2}\right]_0^4 = (9800)\left(6\cdot 8-\frac{2}{5}\cdot 32\right) = \left(\frac{9800}{5}\right)(48\cdot 5-64) = \frac{(9800)(176)}{5} = 344{,}960 \text{ N}$

42. Place the origin at the bottom of the tank. Then $F = \int_0^h W \cdot \left(\begin{smallmatrix}\text{strip}\\\text{depth}\end{smallmatrix}\right) \cdot L(y)\, dy$, $h =$ the height of the mercury column, strip depth

$= h-y,\ L(y) = 0.09 \Rightarrow F = \int_0^h 133{,}350(h-y)\cdot(0.09)\, dy = 12{,}001.5\int_0^h (h-y)\, dy = 12{,}001.5\left[hy-\frac{y^2}{2}\right]_0^h$

$= 12{,}001.5\left(h^2-\frac{h^2}{2}\right) = \frac{12{,}001.5}{2}h^2$. Now solve $\frac{12{,}001.5}{2}h^2 = 150{,}000$ to get $h \approx 5$ m. The volume of the mercury is $s^2h = 0.3^2\cdot 5 = 0.45 \text{ m}^3$.

CHAPTER 6 ADDITIONAL AND ADVANCED EXERCISES

1. $V = \pi\int_a^b [f(x)]^2\, dx = b^2 - ab \Rightarrow \pi\int_a^x [f(t)]^2\, dt = x^2 - ax$ for all $x > a \Rightarrow \pi[f(x)]^2 = 2x - a$

$\Rightarrow f(x) = \sqrt{\frac{2x-a}{\pi}}$

2. $V = \pi\int_0^a [f(x)]^2\, dx = a^2 + a \Rightarrow \pi\int_a^x [f(t)]^2\, dt = x^2 + x$ for all $x > a \Rightarrow \pi[f(x)]^2 = 2x+1 \Rightarrow f(x) = \frac{\sqrt{2x+1}}{\pi}$

3. $s(x) = Cx \Rightarrow \int_0^x \sqrt{1+[f'(t)]^2}\, dt = Cx \Rightarrow \sqrt{1+[f'(x)]^2} = C \Rightarrow f'(x) = \sqrt{C^2-1}$ for $C \ge 1$

$\Rightarrow f(x) = \int_0^x \sqrt{C^2-1}\, dt + k$. Then $f(0) = a \Rightarrow a = 0 + k \Rightarrow f(x) = \int_0^x \sqrt{C^2-1}\, dt + a \Rightarrow f(x) = x\sqrt{C^2-1} + a$, where $C \ge 1$.

4. (a) The graph of $f(x) = \sin x$ traces out a path from $(0, 0)$ to $(\alpha, \sin\alpha)$ whose length is $L = \int_0^\alpha \sqrt{1+\cos^2\theta}\, d\theta$. The line segment from $(0, 0)$ to $(\alpha, \sin\alpha)$ has length $\sqrt{(\alpha-0)^2+(\sin\alpha-0)^2} = \sqrt{\alpha^2+\sin^2\alpha}$. Since the shortest distance between two points is the length of the straight line segment joining them, we have immediately that $\int_0^\alpha \sqrt{1+\cos^2\theta}\, d\theta > \sqrt{\alpha^2+\sin^2\alpha}$ if $0 < \alpha \le \frac{\pi}{2}$.

(b) In general, if $y = f(x)$ is continuously differentiable and $f(0) = 0$, then $\int_0^\alpha \sqrt{1+[f'(t)]^2}\, dt > \sqrt{\alpha^2+f^2(\alpha)}$ for $\alpha > 0$.

5. We can find the centroid and then use Pappus' Theorem to calculate the volume. $f(x)=x,\ g(x)=x^2,$

$f(x)=g(x)\Rightarrow x=x^2\Rightarrow x^2-x=0\Rightarrow x=0, x=1;\ \ \delta=1;\ \ M=\int_0^1\left(x-x^2\right)dx=\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1$

$=\left(\frac{1}{2}-\frac{1}{3}\right)-0=\frac{1}{6};\ \ \bar{x}=\frac{1}{1/6}\int_0^1 x\left(x-x^2\right)dx=6\int_0^1\left(x^2-x^3\right)dx=6\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1=6\left(\frac{1}{3}-\frac{1}{4}\right)-0=\frac{1}{2};$

$\bar{y}=\frac{1}{1/6}\int_0^1\frac{1}{2}\left[x^2-\left(x^2\right)^2\right]dx=3\int_0^1\left(x^2-x^4\right)dx=3\left[\frac{1}{3}x^3-\frac{1}{5}x^5\right]_0^1=3\left(\frac{1}{3}-\frac{1}{5}\right)-0=\frac{2}{3}\Rightarrow$ The centroid is $\left(\frac{1}{2},\frac{2}{5}\right).$

ρ is the distance from $\left(\frac{1}{2},\frac{2}{5}\right)$ to the axis of rotation, $y=x$. To calculate this distance we must find the point on $y=x$ that also lies on the line perpendicular to $y=x$ that passes through $\left(\frac{1}{2},\frac{2}{5}\right)$. The equation of this line is $y-\frac{2}{5}=-1\left(x-\frac{1}{2}\right)\Rightarrow x+y=\frac{9}{10}$. The point of intersection of the lines $x+y=\frac{9}{10}$ and $y=x$ is $\left(\frac{9}{20},\frac{9}{20}\right)$. Thus, $\rho=\sqrt{\left(\frac{9}{10}-\frac{1}{2}\right)^2+\left(\frac{9}{20}-\frac{2}{5}\right)^2}=\frac{1}{10\sqrt{2}}$. Thus $V=2\pi\left(\frac{1}{10\sqrt{2}}\right)\left(\frac{1}{6}\right)=\frac{\pi}{30\sqrt{2}}$.

6. Since the slice is made at an angle of $45°$, the volume of the wedge is half the volume of the cylinder of radius $\frac{1}{2}$ and height 1. Thus, $V=\frac{1}{2}\left[\pi\left(\frac{1}{2}\right)^2(1)\right]=\frac{\pi}{8}$.

7. $y=2\sqrt{x}\Rightarrow ds=\sqrt{\frac{1}{x}+1}\,dx\Rightarrow A=\int_0^3 2\sqrt{x}\sqrt{\frac{1}{x}+1}\,dx=\frac{4}{3}\left[(1+x)^{3/2}\right]_0^3=\frac{28}{3}$

8. This surface is a triangle having a base of $2\pi a$ and a height of $2\pi ak$. Therefore the surface area is $\frac{1}{2}(2\pi a)(2\pi ak)=2\pi^2a^2k$.

9. $F=ma=t^2\Rightarrow\frac{d^2}{dt^2}=a=\frac{t^2}{m}\Rightarrow v=\frac{dx}{dt}=\frac{t^3}{3m}+C;\ \ v=0$ when $t=0\Rightarrow C=0\Rightarrow\frac{dx}{dt}=\frac{t^2}{3m}\Rightarrow x=\frac{t^4}{12m}+C_1;$ $x=0$ when $t=0\Rightarrow C_1=0\Rightarrow x=\frac{t^4}{12m}$. Then $x=h\Rightarrow t=(12\,mh)^{1/4}$. The work done is

$W=\int F\,dx=\int_0^{(12mh)^{1/4}}F(t)\cdot\frac{dx}{dt}\,dt=\int_0^{(12\,mh)^{1/4}}t^2\cdot\frac{t^3}{3m}\,dt=\frac{1}{3m}\left[\frac{t^6}{6}\right]_0^{(12mh)^{1/4}}=\left(\frac{1}{18m}\right)(12mh)^{6/4}=\frac{(12mh)^{3/2}}{18m}$

$=\frac{12mh\cdot\sqrt{12mh}}{18m}=\frac{2h}{3}\cdot 2\sqrt{3mh}=\frac{4h}{3}\sqrt{3mh}$

10. Converting to grams and centimeters, 3.6 N/cm = 360 N/m. Thus, $F=360x\Rightarrow W=\int_0^{0.15}360x\,dx$

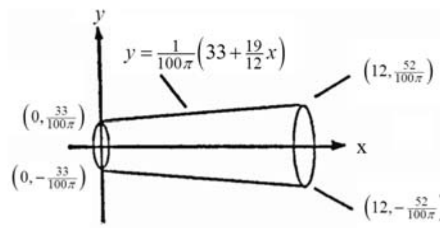
$=\left[180x^2\right]_0^{0.15}=4.05$ J. Since $W=\frac{1}{2}mv_0^2-\frac{1}{2}mv_1^2$, where $W=4.05$ J, m = 0.05 kg and $v_1=0$ m/s, we have $4.05=\left(\frac{1}{2}\right)\left(\frac{1}{20}v_0^2\right)\Rightarrow v_0^2=4.05\cdot 40$. For the projectile height, $s=-9.8t^2+v_0t$ (since $s=0$ at $t=0$) $\Rightarrow\frac{ds}{dt}=v=-9.8t+v_0$. At the top of the ball's path, $v=0\Rightarrow t=\frac{v_0}{9.8}$ and the height is

$s=-4.9\left(\frac{v_0}{9.8}\right)^2+v_0\left(\frac{v_0}{9.8}\right)=\frac{v_0^2}{19.6}=\frac{4.05\cdot 40}{19.6}=8.26$ m.

11. From the symmetry of $y=1-x^n$, n even, about the y-axis for $-1\le x\le 1$, we have $\bar{x}=0$. To find $\bar{y}=\frac{M_x}{M}$, we use the vertical strips technique. The typical strip has center of mass: $(\tilde{x},\tilde{y})=\left(x,\frac{1-x^n}{2}\right)$, length: $1-x^n$,

width: dx, area: $dA = \left(1-x^n\right)dx$, mass: $dm = 1\cdot dA = \left(1-x^n\right)dx$. The moment of the strip about the x-axis is $\tilde{y}\,dm = \frac{\left(1-x^n\right)^2}{2}\,dx \Rightarrow M_x = \int_{-1}^{1}\frac{\left(1-x^n\right)^2}{2}\,dx = 2\int_0^1 \frac{1}{2}\left(1-2x^n+x^{2n}\right)dx = \left[x-\frac{2x^{n+1}}{n+1}+\frac{x^{2n+1}}{2n+1}\right]_0^1 = 1-\frac{2}{n+1}+\frac{1}{2n+1}$ $= \frac{(n+1)(2n+1)-2(2n+1)+(n+1)}{(n+1)(2n+1)} = \frac{2n^2+3n+1-4n-2+n+1}{(n+1)(2n+1)} = \frac{2n^2}{(n+1)(2n+1)}$. Also, $M = \int_{-1}^{1} dA = \int_{-1}^{1}\left(1-x^n\right)dx$ $= 2\int_0^1\left(1-x^n\right)dx = 2\left[x-\frac{x^{n+1}}{n+1}\right]_0^1 = 2\left(1-\frac{1}{n+1}\right) = \frac{2n}{n+1}$. Therefore, $\overline{y} = \frac{M_x}{M} = \frac{2n^2}{(n+1)(2n+1)}\cdot\frac{(n+1)}{2n} = \frac{n}{2n+1} \Rightarrow \left(0, \frac{n}{2n+1}\right)$ is the location of the centroid. As $n\to\infty$, $\overline{y}\to\frac{1}{2}$ so the limiting position of the centroid is $\left(0, \frac{1}{2}\right)$.

12. Align the telephone pole along the x-axis as shown in the accompanying figure. The slope of the top length of pole is $\frac{\left(\frac{52}{100\pi}-\frac{33}{100\pi}\right)}{12} = \frac{1}{100\pi}\cdot\frac{1}{12}\cdot(52-33)$ $= \frac{19}{100\pi\cdot 12}$. Thus, $y = \frac{33}{100\pi}+\frac{19}{100\pi\cdot 12}x$ $= \frac{1}{100\pi}\left(33+\frac{19}{12}x\right)$ is an equation of the line representing the top of the pole. Then



$M_y = \int_a^b x\cdot\pi y^2\,dx = \pi\int_0^{12} x\left[\frac{1}{100\pi}\left(33+\frac{19}{12}x\right)\right]^2 dx = \frac{1}{10{,}000\pi}\int_0^{12} x\left(33+\frac{19}{12}x\right)^2 dx$;
$M = \int_a^b \pi y^2\,dx = \pi\int_0^{12}\left[\frac{1}{100\pi}\left(33+\frac{19}{12}x\right)\right]^2 dx = \frac{1}{10{,}000\pi}\int_0^{12}\left(33+\frac{19}{12}x\right)^2 dx$. Thus, $\overline{x} = \frac{M_y}{M} \approx \frac{151{,}596}{22{,}036} \approx 6.88$ (using a calculator to compute the integrals). By symmetry about the x-axis, $\overline{y} = 0$ so the center of mass is about 6.9 meters from the top of the pole.

13. (a) Consider a single vertical strip with center of mass $(\tilde{x}, \tilde{y})$. If the plate lies to the right of the line, then the moment of this strip about the line $x = b$ is $(\tilde{x}-b)\,dm = (\tilde{x}-b)\,\delta\,dA \Rightarrow$ the plate's first moment about $x = b$ is the integral $\int(x-b)\delta\,dA = \int\delta x\,dA - \int\delta b\,dA = M_y - b\delta A$.

(b) If the plate lies to the left of the line, the moment of a vertical strip about the line $x = b$ is $(b-\tilde{x})\,dm$ $= (b-\tilde{x})\,\delta\,dA \Rightarrow$ the plate's first moment about $x = b$ is $\int(b-x)\delta\,dA = \int b\delta\,dA - \int\delta x\,dA = b\delta A - M_y$.

14. (a) By symmetry of the plate about the x-axis, $\overline{y} = 0$. A typical vertical strip has center of mass: $(\tilde{x}, \tilde{y})$ $= (x, 0)$, length: $4\sqrt{ax}$, width: dx, area: $4\sqrt{ax}\,dx$, mass: $dm = \delta\,dA = kx\cdot 4\sqrt{ax}\,dx$, for some proportionality constant k. The moment of the strip about the y-axis is $M_y = \int\tilde{x}\,dm = \int_0^a 4k\,x^2\sqrt{ax}\,dx$ $= 4k\sqrt{a}\int_0^a x^{5/2}dx = 4k\sqrt{a}\left[\frac{2}{7}x^{7/2}\right]_0^a = 4k\,a^{1/2}\cdot\frac{2}{7}a^{7/2} = \frac{8k\,a^4}{7}$. Also, $M = \int dm = \int_0^a 4k\,x\sqrt{ax}\,dx$ $= 4k\sqrt{a}\int_0^a x^{3/2}dx = 4k\sqrt{a}\left[\frac{2}{5}x^{5/2}\right]_0^a = 4k\,a^{1/2}\cdot\frac{2}{5}a^{5/2} = \frac{8k\,a^3}{5}$. Thus, $\overline{x} = \frac{M_y}{M} = \frac{8k\,a^4}{7}\cdot\frac{5}{8k\,a^3} = \frac{5}{7}a$ $\Rightarrow (\overline{x}, \overline{y}) = \left(\frac{5a}{7}, 0\right)$ is the center of mass.

(b) A typical horizontal strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{\frac{y^2}{4a}+a}{2}, y\right) = \left(\frac{y^2+4a^2}{8a}, y\right)$, length: $a-\frac{y^2}{4a}$, width: dy, area: $\left(a-\frac{y^2}{4a}\right)dy$, mass: $dm = \delta\,dA = |y|\left(a-\frac{y^2}{4a}\right)dy$. Thus,

$M_x = \int \tilde{y}\, dm = \int_{-2a}^{2a} y|y|\left(a - \frac{y^2}{4a}\right) dy = \int_{-2a}^{0} -y^2\left(a - \frac{y^2}{4a}\right) dy + \int_{0}^{2a} y^2\left(a - \frac{y^2}{4a}\right) dy$

$= \int_{-2a}^{0}\left(-ay^2 + \frac{y^4}{4a}\right) dy + \int_{0}^{2a}\left(ay^2 - \frac{y^4}{4a}\right) dy = \left[-\frac{a}{3}y^3 + \frac{y^5}{20a}\right]_{-2a}^{0} + \left[\frac{a}{3}y^3 - \frac{y^5}{20a}\right]_{0}^{2a} = -\frac{8a^4}{3} + \frac{32a^5}{20a} + \frac{8a^4}{3} - \frac{32a^5}{20a} = 0;$

$M_y = \int \tilde{x}\, dm = \int_{-2a}^{2a}\left(\frac{y^2+4a^2}{8a}\right)|y|\left(a - \frac{y^2}{4a}\right) dy = \frac{1}{8a}\int_{-2a}^{2a}|y|\left(y^2 + 4a^2\right)\left(\frac{4a^2-y^2}{4a}\right) dy = \frac{1}{32a^2}\int_{-2a}^{2a}|y|\left(16a^4 - y^4\right) dy$

$= \frac{1}{32a^2}\int_{-2a}^{0}\left(-16a^4y + y^5\right) dy + \frac{1}{32a^2}\int_{0}^{2a}\left(16a^4y - y^5\right) dy = \frac{1}{32a^2}\left[-8a^4y^2 + \frac{y^6}{6}\right]_{-2a}^{0} + \frac{1}{32a^2}\left[8a^4y^2 - \frac{y^6}{6}\right]_{0}^{2a}$

$= \frac{1}{32a^2}\left[8a^4 \cdot 4a^2 - \frac{64a^6}{6}\right] + \frac{1}{32a^2}\left[8a^4 \cdot 4a^2 - \frac{64a^6}{6}\right] = \frac{1}{16a^2}\left(32a^6 - \frac{32a^6}{3}\right) = \frac{1}{16a^2} \cdot \frac{2}{3}\left(32a^6\right) = \frac{4}{3}a^4;$

$M = \int dm = \int_{-2a}^{2a}|y|\left(\frac{4a^2-y^2}{4a}\right) dy = \frac{1}{4a}\int_{-2a}^{2a}|y|\left(4a^2 - y^2\right) dy = \frac{1}{4\text{a}}\int_{-2a}^{0}\left(-4a^2y + y^3\right) dy + \frac{1}{4a}\int_{0}^{2a}\left(4a^2y - y^3\right) dy$

$= \frac{1}{4a}\left[-2a^2y^2 + \frac{y^4}{4}\right]_{-2a}^{0} + \frac{1}{4a}\left[2a^2y^2 - \frac{y^4}{4}\right]_{0}^{2a} = 2 \cdot \frac{1}{4a}\left(2a^2 \cdot 4a^2 - \frac{16a^4}{4}\right) = \frac{1}{2a}\left(8a^4 - 4a^4\right) = 2a^3$. Therefore,

$\bar{x} = \frac{M_y}{M} = \left(\frac{4}{3}a^4\right)\left(\frac{1}{2a^3}\right) = \frac{2a}{3}$ and $\bar{y} = \frac{M_x}{M} = 0$ is the center of mass.

15. (a) On $[0, a]$ a typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{b^2-x^2}+\sqrt{a^2-x^2}}{2}\right)$,

length: $\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}$, width: dx, area: $dA = \left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx$, mass: $dm = \delta\, dA$

$= \delta\left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx$. On $[a, b]$ a typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{b^2-x^2}}{2}\right)$,

length: $\sqrt{b^2 - x^2}$, width: dx, area: $dA = \sqrt{b^2 - x^2}\, dx$, mass: $dm = \delta\, dA = \delta\sqrt{b^2 - x^2}\, dx$. Thus,

$M_x = \int \tilde{y}\, dm = \int_0^a \frac{1}{2}\left(\sqrt{b^2 - x^2} + \sqrt{a^2 - x^2}\right)\delta\left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx + \int_a^b \frac{1}{2}\sqrt{b^2 - x^2}\,\delta\sqrt{b^2 - x^2}\, dx$

$= \frac{\delta}{2}\int_0^a\left[\left(b^2 - x^2\right) - \left(a^2 - x^2\right)\right] dx + \frac{\delta}{2}\int_a^b\left(b^2 - x^2\right) dx = \frac{\delta}{2}\int_0^a\left(b^2 - a^2\right) dx + \frac{\delta}{2}\int_a^b\left(b^2 - x^2\right) dx$

$= \frac{\delta}{2}\left[\left(b^2 - a^2\right)x\right]_0^a + \frac{\delta}{2}\left[b^2x - \frac{x^3}{3}\right]_a^b = \frac{\delta}{2}\left[\left(b^2 - a^2\right)a\right] + \frac{\delta}{2}\left[\left(b^3 - \frac{b^3}{3}\right) - \left(b^2a - \frac{a^3}{3}\right)\right]$

$= \frac{\delta}{2}\left(ab^2 - a^3\right) + \frac{\delta}{2}\left(\frac{2}{3}b^3 - ab^2 + \frac{a^3}{3}\right) = \frac{\delta b^3}{3} - \frac{\delta a^3}{3} = \delta\left(\frac{b^3-a^3}{3}\right);$

$M_y = \int \tilde{x}\, dm = \int_0^a x\,\delta\left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx + \int_a^b x\,\delta\sqrt{b^2 - x^2}\, dx$

$= \delta\int_0^a x\left(b^2 - x^2\right)^{1/2} dx - \delta\int_0^a x\left(a^2 - x^2\right)^{1/2} dx + \delta\int_a^b x\left(b^2 - x^2\right)^{1/2} dx$

$= \frac{-\delta}{2}\left[\frac{2\left(b^2-x^2\right)^{3/2}}{3}\right]_0^a + \frac{\delta}{2}\left[\frac{2\left(a^2-x^2\right)^{3/2}}{3}\right]_0^a - \frac{\delta}{2}\left[\frac{2\left(b^2-x^2\right)^{3/2}}{3}\right]_a^b$

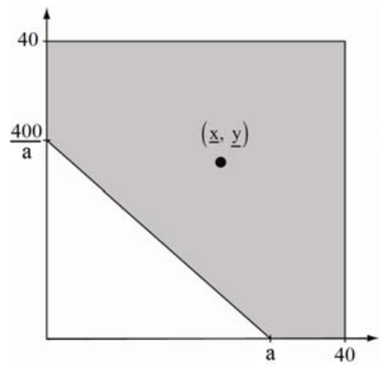
$= -\frac{\delta}{3}\left[\left(b^2 - a^2\right)^{3/2} - \left(b^2\right)^{3/2}\right] + \frac{\delta}{3}\left[0 - \left(a^2\right)^{3/2}\right] - \frac{\delta}{3}\left[0 - \left(b^2 - a^2\right)^{3/2}\right] = \frac{\delta b^3}{3} - \frac{\delta a^3}{3} = \frac{\delta\left(b^3-a^3\right)}{3} = M_x;$

We calculate the mass geometrically: $M = \delta A = \delta\left(\frac{\pi b^2}{4}\right) - \delta\left(\frac{\pi a^2}{4}\right) = \frac{\delta\pi}{4}\left(b^2 - a^2\right)$. Thus, $\overline{x} = \frac{M_y}{M}$

$= \frac{\delta\left(b^3-a^3\right)}{3} \cdot \frac{4}{\delta\pi\left(b^2-a^2\right)} = \frac{4}{3\pi}\left(\frac{b^3-a^3}{b^2-a^2}\right) = \frac{4}{3\pi}\frac{(b-a)\left(a^2+ab+b^2\right)}{(b-a)(b+a)} = \frac{4\left(a^2+ab+b^2\right)}{3\pi(a+b)}$; likewise $\overline{y} = \frac{M_x}{M} = \frac{4\left(a^2+ab+b^2\right)}{3\pi(a+b)}$.

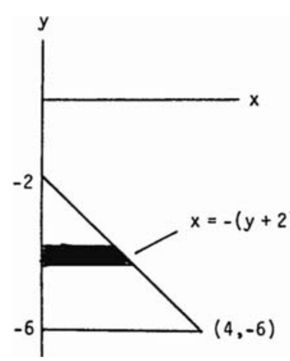
(b) $\lim\limits_{b\to a} \frac{4}{3\pi}\left(\frac{a^2+ab+b^2}{a+b}\right) = \left(\frac{4}{3\pi}\right)\left(\frac{a^2+a^2+a^2}{a+a}\right) = \left(\frac{4}{3\pi}\right)\left(\frac{3a^2}{2a}\right) = \frac{2a}{\pi} \Rightarrow (\overline{x},\overline{y}) = \left(\frac{2a}{\pi}, \frac{2a}{\pi}\right)$ is the limiting position of the centroid as $b \to a$. This is the centroid of a circle of radius a (and the two circles coincide when $b = a$).

16. Since the area of the triangle is 400, the diagram may be labeled as shown at the right. The centroid of the triangle is $\left(\frac{a}{3}, \frac{400}{3a}\right)$. The shaded portion is $1600 - 400 = 1200$. Write $(\underline{x}\ \underline{y})$ for the centroid of the remaining region. The centroid of the whole square is obviously $(20, 20)$. Think of the square as a sheet of uniform density, so that the centroid of the square is the average of the centroids of the two regions, weighted by area: $20 = \frac{400\left(\frac{a}{3}\right)+1200(\underline{x})}{1600}$ and



$20 = \frac{400\left(\frac{400}{3a}\right)+1200(\underline{y})}{1600}$ which we solve to get $\underline{x} = \frac{80}{3} - \frac{a}{9}$ and $\underline{y} = \frac{80/3(a-5/3)}{a}$. Set $\underline{x} = 22$ cm (Given). It follows that $a = 42$, whence $\underline{y} = \frac{4840}{189} \approx 25.6$ cm. The distances of the centroid $(\underline{x}, \underline{y})$ from the other sides are easily computed. (If we set $\underline{y} = 22$ cm above, we will find $\underline{x} = 25.6$ cm.)

17. The submerged triangular plate is depicted in the figure at the right. The hypotenuse of the triangle has slope $-1 \Rightarrow y-(-2) = -(x-0) \Rightarrow x = -(y+2)$ is an equation of the hypotenuse. Using a typical horizontal strip, the fluid pressure is



$F = \int (9800)\cdot\left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}\text{strip}\\ \text{length}\end{smallmatrix}\right) dy$

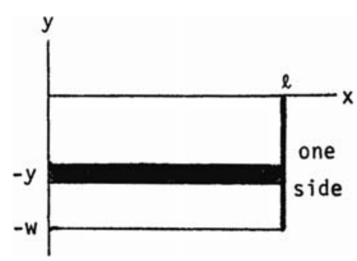
$= \int_{-6}^{-2} (9800)(-y)\left[-(y+2)\right] dy$

$= 9800\int_{-6}^{-2}\left(y^2+2y\right) dy = 62.4\left[\frac{y^3}{3}+y^2\right]_{-6}^{-2}$

$= (9800)\left[\left(-\frac{8}{3}+4\right)-\left(-\frac{216}{3}+36\right)\right]$

$= (9800)\left(\frac{208}{3}-32\right) = \frac{(9800)(112)}{3} \approx 365{,}867$ N

18. Consider a rectangular plate of length ℓ and width w. The length is parallel with the surface of the fluid of weight density ω. The force on one side of the plate is $F = \omega\int_{-w}^{0}(-y)(\ell)\, dy = -\omega\ell\left[\frac{y^2}{2}\right]_{-w}^{0} = \frac{\omega\ell w^2}{2}$.
The average force on one side of the plate is



$F_{av} = \frac{\omega}{w}\int_{-w}^{0}(-y)\, dy = \frac{\omega}{w}\left[-\frac{y^2}{2}\right]_{-w}^{0} = \frac{\omega w}{2}$. Therefore the force $\frac{\omega\ell w^2}{2} = \left(\frac{\omega w}{2}\right)(\ell w)$
= (the average pressure up and down) · (the area of the plate).