MAT1002 Lecture 13, Thursday, Mar/09/2023

Outline

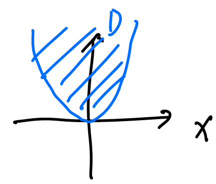
- Functions of several variables (14.1)
    - Point sets in $\mathbb{R}^n$
    - Graphs and level curves/surfaces
- Limits and continuity (14.2)

Functions of Several Variables

e.g. The area A of a rectangle depends on its length and width:

$$A = f(l, w) = lw, \quad l \geq 0, \quad w \geq 0.$$

e.g. $f(x,y) = kx^{1/3}y^{2/3}$, $D = \mathbb{R}^2$.

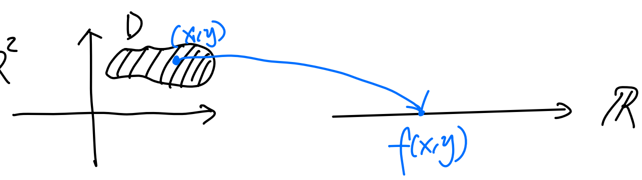


$f(x,y) = \sqrt{y - x^2}$, $D = \{(x,y) : y - x^2 \geq 0\} = \{(x,y) : y \geq x^2\}$.

- In general, $f(x_1, \dots, x_n)$ denotes a function (rule) with n real variables.
- Alternative notation: $f(\vec{x}) = f(x_1, \dots, x_n)$

  $\vec{x} = \langle x_1, \dots, x_n \rangle$ giving the point $(x_1, \dots, x_n)$

  vector notation is used to emphasize that there are multiple real variables.
- We focus on real-valued functions (scalar functions), which are $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$. In this course, $n = 2$ or 3 most of the time.
- Vizualization: $\mathbb{R}^2$ ($n=2$)



- For these functions, it is more natural to think of $\mathbb{R}^n$ as a set of points instead of a set of vectors (so the domain D is a point set).

Definition

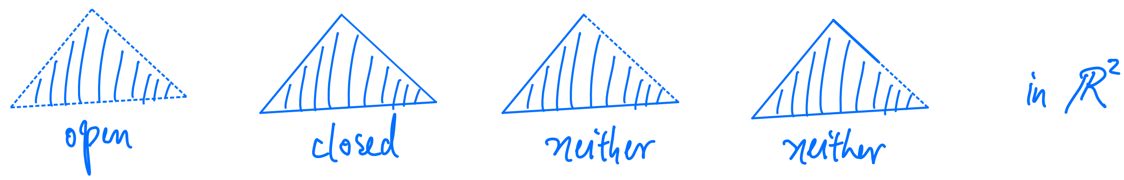
Let D be a subset of $\mathbb{R}^n$.

in $\mathbb{R}^2$, "open disk"

- A point P in $\mathbb{R}^n$ is called an interior point of D if some open ball centered at P completely lies in D.
- A point P in $\mathbb{R}^n$ is called a boundary point of D if every open ball centered at P intersects both D and $\mathbb{R}^n \setminus D$.
- The set D is said to be open if every point in D is an interior point of D; it is said to be closed if it contains all of its boundary points.

disk in $\mathbb{R}^2$

- The set D is said to be bounded if it lies in some ball with finite radius; it is said to be unbounded otherwise.



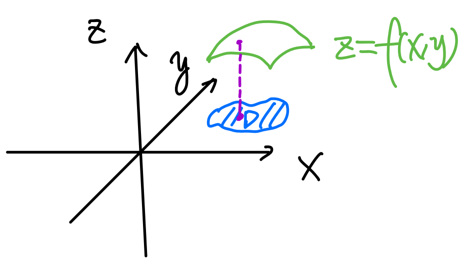
Graphs and Level Curves / Surfaces

Def: The graph of $f: D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, is the set

$$\{(x_1, \dots, x_n, f(x_1, \dots, x_n)) : (x_1, \dots, x_n) \in D\}.$$

Hence, the graph of an n-variable function lie in $\mathbb{R}^{n+1}$.

e.g. $n = 2$; $z = f(x,y)$



Think of climbing a mountain:

(x,y) : 2D position on ground level

$f(x,y)$: mountain altitude at (x,y)

e.g. $n=3$; $w=f(x,y,z)$, graph is

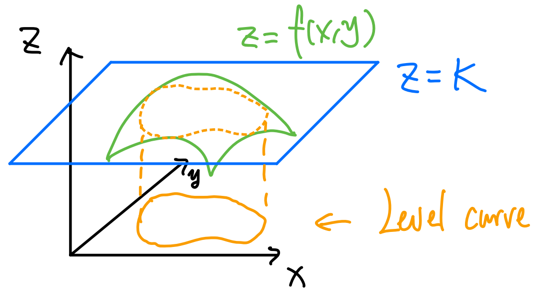
$$\{(x,y,z,w) : w=f(x,y,z),\ (x,y,z)\in D\}.$$

Can think of temperature: (x,y,z) : position in xyz-space

$f(x,y,z)$: temparature at (x,y,z).

Def: Given $f: D\to\mathbb{R}$, $D\subseteq\mathbb{R}^n$, and any constant $K\in\mathbb{R}$:

- For $n=2$, the set $\{(x,y)\in D : f(x,y)=K\}$ is called a level curve of f.
- For $n\geq 3$, the set $\{(x_1,\dots,x_n)\in D : f(x_1,\dots,x_n)=K\}$ is called a level surface of f.



Note that by definition, a level curve/surface of f lies in the DOMAIN of f.

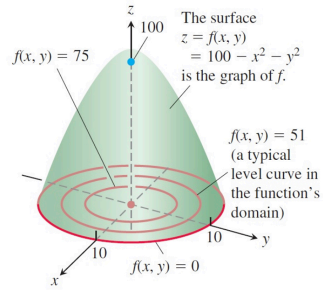
Remarks

For a function $f: D\to\mathbb{R}$ where $D\subseteq\mathbb{R}^n$:

- The graph of f lies in $\mathbb{R}^{n+1}$. ← Dimension is increased by 1
- Any level surface of f (or level curve if $n=2$) lies in D.

Same dimension.

e.g. The level curves of $f(x,y) = 100 - x^2 - y^2$.



e.g. What are the level surfaces of $f(x,y,z) = x^2+y^2-z^2$?

Explore using GeoGebra.

Limits

We look at the case where $n=2$ first.

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$\underbrace{|f(x,y) - L| < \epsilon}_{f(x,y) \text{ is } \varepsilon\text{-close to } L} \quad \text{whenever} \quad \underbrace{0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta}_{(x,y) \text{ is } \delta\text{-close to } (x_0,y_0) \text{ but not equal}}.$$

$f(x,y)$ is ε-close to L

(x,y) is δ-close to (x_0,y_0) but not equal

$\iff (x,y) \in B_\delta(x_0,y_0) \setminus \{(x_0,y_0)\}$.

For $n=3$ (three-variable), it is similar: we write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$$

Open ball
- Centre $= (x_0, y_0, z_0)$
- radius $= \delta$

if $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $\forall (x,y,z) \in D$,

$$|f(x,y,z) - L| < \varepsilon \quad \text{whenever} \quad (x,y,z) \in \underline{B_\delta(x_0,y_0,z_0)} \setminus \{(x_0,y_0,z_0)\}.$$

Limits can be defined similarly for other values of n.

The Basic Building Blocks

Using the definition of limits, it can be shown that

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0, \quad \lim_{(x,y)\to(x_0,y_0)} y = y_0, \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} k = k,$$

where k is a constant.

- Similarly, $\lim_{(x,y,z)\to(x_0,y_0,z_0)} z = z_0$, etc.
- Note that $\lim_{(x,y)\to(x_0,y_0)} x$ means $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$, where

  $f(x,y) = x$. $\leftarrow$ Graph is a cylindrical surface in $\mathbb{R}^3$ generated by a line.

Proof of $\lim_{(x,y)\to(x_0,y_0)} x = x_0$: (Optional)

- Let $\varepsilon > 0$. Want to show that $\exists\, \delta > 0$ s.t.

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \implies |x - x_0| < \varepsilon. \quad (*)$$

- Choose $\delta := \varepsilon$. Suppose $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$. Then

$$|x - x_0| = \sqrt{(x-x_0)^2} \leq \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta = \varepsilon.$$

- $(*)$ is proven.

$\square$

Properties of Limits

The following properties, although only stated for two-variable functions, hold for functions with finitely many variables.

THEOREM — Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$
5. *Quotient Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$

if n even, assume $f(x,y)\geq 0$ for all (x,y) in some $\left(B_\delta(x_0,y_0)\setminus\{(x_0,y_0)\}\right)\cap D$.

e.g. $$\lim_{(x,y)\to(4,8)} 5x^2y^{1/3} = \left(\lim_{(x,y)\to(4,8)} 5\right)\left(\lim_{(x,y)\to(4,8)} x^2\right)\left(\lim_{(x,y)\to(4,8)} y^{1/3}\right)$$

$$= 5\left(\lim_{(x,y)\to(4,8)} x\right)^2\left(\lim_{(x,y)\to(4,8)} y\right)^{1/3} = 5\cdot 4^2\cdot 8^{1/3} = 160.$$

e.g. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = 0$. Note that domain of function is

$D = \{(x,y) : x\geq 0, y\geq 0, x\neq y\}$, and:

- $(0,0)\notin D$, but D contains a curve approaching $(0,0)$.
- $\forall \delta>0$, $B_\delta(0,0)\cap D \neq \emptyset$.

Limits Along Paths (Curves)

For a one-variable function, we know that $\lim_{x \to x_0} f(x) = L$ if and only if

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L.$$

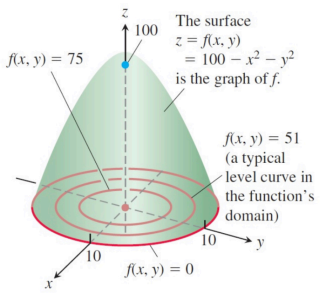
An analogous statement for functions with multiple variables is the following.

Theorem

Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^n$. Then $\lim_{\vec{x} \to \vec{x_0}} f(\vec{x}) = L$ if and only if

f converges to L as $\vec{x}$ approaches $\vec{x_0}$ along any ~~path~~ curve in D.

A curve in $\mathbb{R}^n$ is the range of a cts function $\vec{r} : I \to \mathbb{R}^n$.



Intuition: For the "mountain" $z = 100 - x^2 - y^2$, no matter which curve you travel along to approach $(0,0)$ with respect to the ground level, you would be approaching height 100.

For two-variable functions

Formally, if $\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$ and $\vec{r}(t)$ $\langle x(t), y(t) \rangle$ is any curve in $\mathbb{R}^2$ with $\lim_{t \to t_0} \vec{r}(t) = \langle x_0, y_0 \rangle$, then $\lim_{t \to t_0} f(\vec{r}(t)) = L$.

A consequence of the theorem is the following: if f does not converge to the same number along two different ~~paths~~ curves, then $\lim_{\vec{x} \to \vec{x_0}} f(x)$ does not exist.

e.g. For $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$:

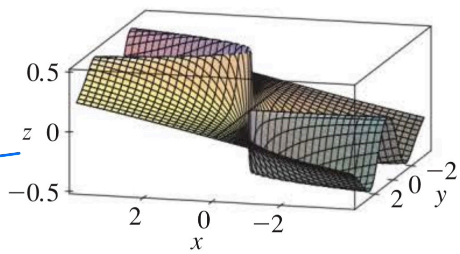
- On x-axis ($y=0$, $x\neq0$), $f(x,y)\equiv 1$.
- On y-axis ($x=0$, $y\neq 0$), $f(x,y)\equiv -1$.
- Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

Example

Show that the following limits do not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

(b) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$.



Sol: (a) Along the curve $y=kx$, $(x,y)\to(0,0)$ as $x\to 0$.

So $$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx}} \frac{xy}{x^2+y^2} = \lim_{x\to0}\frac{x(kx)}{x^2+(kx)^2} = \lim_{x\to 0}\frac{kx^2}{(1+k^2)x^2} = \frac{k}{1+k^2}$$

Along $y=x$, limit $=\frac{1}{2}$; along $y=-x$, limit $=-\frac{1}{2}$.

So limit D.N.E.

(b) Along the curve $x=ky^2$, $(x,y)\to(0,0)$ as $y\to 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=ky^2}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0}\frac{ky^4}{(k^2+1)y^4} = \frac{k}{k^2+1} = \begin{cases} \frac{1}{2}, & \text{if } k=1 \\ -\frac{1}{2}, & \text{if } k=-1. \end{cases}$$

Hence limit D.N.E.

Squeeze (Sandwich) Theorem

Theorem: If $g(x,y) \le f(x,y) \le h(x,y)$, $\forall (x,y) \in B_\delta(a,b)$ (for some $\delta > 0$ fixed) and

$$\lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L,$$

then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Proof: Similar to the version for one-variable functions.

e.g. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$. (Do this in at least two ways.)

Continuity

Definition

Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^n$, and let $\vec{x_0} \in D$. Then f is said to be continuous at $\vec{x_0}$ if

$$\lim_{\vec{x}\to\vec{x_0}} f(\vec{x}) = f(\vec{x_0}).$$

vector notation for the terminal point, to emphasize it has multiple variables.

We say that f is continuous if f is continuous at $\vec{x_0}$ for every $\vec{x_0} \in D$.

A consequence of properties of limits is that sums, differences, products, quotients and powers of continuous functions are continuous everywhere they are defined.

Hence polynomials, such as $f(x,y,z) = 48z^2y - 61xyz + 13z - 7$, are cts.

Example

The function defined by

$$f(x,y) := \begin{cases} \frac{3x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous on $\mathbb{R}^2$.

Compositions of Continuous Functions

Just like functions with one variable, the composition of two continuous functions is continuous. In particular:

If $f : D \to \mathbb{R}$ is a multiple-variable continuous function, and g is a single-variable continuous function whose domain contains the range of f, then $g \circ f$ is a continuous function on D.

e.g. $f(x,y) = \cos \frac{xy}{x^2+1}$ is continuous on $\mathbb{R}^2$, and

$g(x,y,z) = e^{x+y} \cos z$ is cts on $\mathbb{R}^3$.