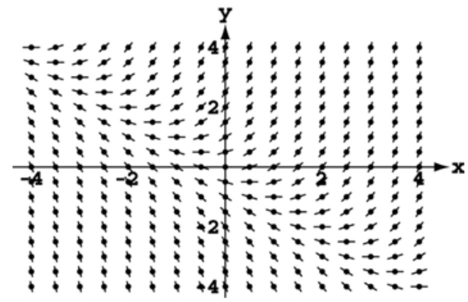
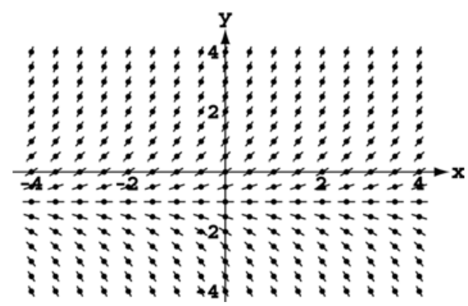
CHAPTER 9 FIRST-ORDER DIFFERENTIAL EQUATIONS

9.1 SOLUTIONS, SLOPE FIELDS AND EULER'S METHOD

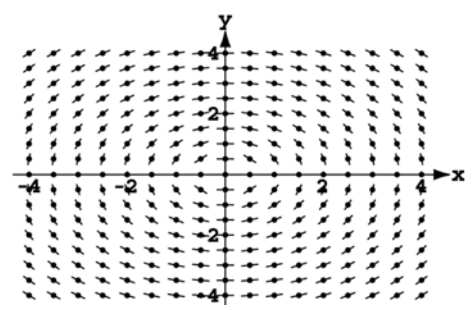
1. $y' = x + y \Rightarrow$ slope of 0 for the line $y = -x$.
For $x, y > 0$, $y' = x + y \Rightarrow$ slope > 0 in Quadrant I.
For $x, y < 0$, $y' = x + y \Rightarrow$ slope < 0 in Quadrant III.
For $|y| > |x|$, $y > 0$, $x < 0$, $y' = x + y \Rightarrow$ slope > 0 in Quadrant II above $y = -x$.
For $|y| < |x|$, $y > 0$, $x < 0$, $y' = x + y \Rightarrow$ slope < 0 in Quadrant II below $y = -x$.
For $|y| < |x|$, $x > 0$, $y < 0$, $y' = x + y \Rightarrow$ slope > 0 in Quadrant IV above $y = -x$.
For $|y| > |x|$, $x > 0$, $y < 0$, $y' = x + y \Rightarrow$ slope < 0 in Quadrant IV below $y = -x$.
All of the conditions are seen in slope field (d).



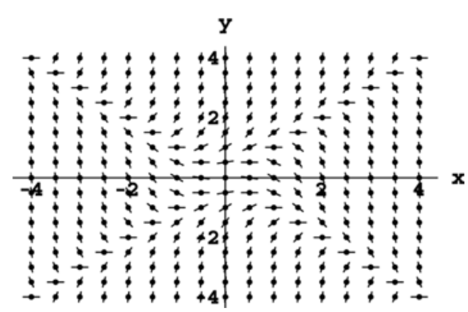
2. $y' = y + 1 \Rightarrow$ slope is constant for a given value of y, slope is 0 for $y = -1$, slope is positive for $y > -1$ and negative for $y < -1$. These characteristics are evident in slope field (c)



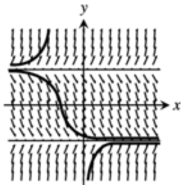
3. $y' = -\frac{x}{y} \Rightarrow$ slope $= 1$ on $y = -x$ and -1 on $y = x$. $y' = -\frac{x}{y}$ $\Rightarrow$ slope $= 0$ on the y-axis, excluding $(0, 0)$, and is undefined on the x-axis. Slopes are positive for $x > 0$, $y < 0$ and $x < 0$, $y > 0$ (Quadrants II and IV), otherwise negative, Field (a) is consistent with these conditions.



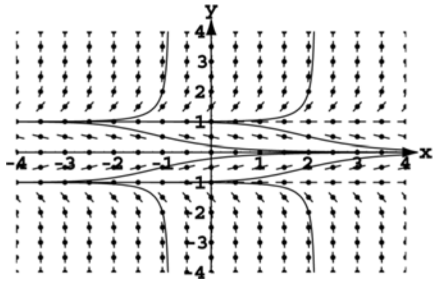
4. $y' = y^2 - x^2 \Rightarrow$ slope is 0 for $y = x$ and for $y = -x$. For $|y| > |x|$ slope is positive and for $|y| < |x|$ slope is negative. Field (b) has these characteristics.



5.



6.



7. $y = -1 + \int_1^x (t - y(t))\,dt \Rightarrow \frac{dy}{dx} = x - y(x);\ \ y(1) = -1 + \int_1^1 (t - y(t))\,dt = -1;\ \ \frac{dy}{dx} = x - y,\ y(1) = -1$

8. $y = \int_1^x \frac{1}{t}\,dt \Rightarrow \frac{dy}{dx} = \frac{1}{x};\ \ y(1) = \int_1^1 \frac{1}{t}\,dt = 0;\ \ \frac{dy}{dx} = \frac{1}{x},\ y(1) = 0$

9. $y = 2 - \int_0^x (1 + y(t))\sin t\,dt \Rightarrow \frac{dy}{dx} = -(1 + y(x))\sin x;\ \ y(0) = 2 - \int_0^0 (1 + y(t))\sin t\,dt = 2;\ \ \frac{dy}{dx} = -(1+y)\sin x,$
$y(0) = 2$

10. $y = 1 + \int_0^x y(t)\,dt \Rightarrow \frac{dy}{dx} = y(x);\ \ y(0) = 1 + \int_0^0 y(t)dt = 1;\ \ \frac{dy}{dx} = y,\ y(0) = 1$

11. $y_1 = y_0 + \left(1 - \frac{y_0}{x_0}\right)dx = -1 + \left(1 - \frac{-1}{2}\right)(.5) = -0.25,$
$y_2 = y_1 + \left(1 - \frac{y_1}{x_1}\right)dx = -0.25 + \left(1 - \frac{-0.25}{2.5}\right)(.5) = 0.3,$
$y_3 = y_2 + \left(1 - \frac{y_2}{x_2}\right)dx = 0.3 + \left(1 - \frac{0.3}{3}\right)(.5) = 0.75;$
$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = 1 \Rightarrow P(x) = \frac{1}{x},\ Q(x) = 1 \Rightarrow \int P(x)\,dx = \int \frac{1}{x}\,dx = \ln|x| = \ln x,\ x > 0 \Rightarrow v(x) = e^{\ln x} = x$
$\Rightarrow y = \frac{1}{x}\int x \cdot 1\,dx = \frac{1}{x}\left(\frac{x^2}{2} + C\right);\ \ x = 2,\ y = -1 \Rightarrow -1 = 1 + \frac{C}{2} \Rightarrow C = -4 \Rightarrow y = \frac{x}{2} - \frac{4}{x}$
$\Rightarrow y(3.5) = \frac{3.5}{2} - \frac{4}{3.5} = \frac{4.25}{7} \approx 0.6071$

12. $y_1 = y_0 + x_0(1 - y_0)\,dx = 0 + 1(1-0)(.2) = .2,$
$y_2 = y_1 + x_1(1 - y_1)\,dx = .2 + 1.2(1 - .2)(.2) = .392,$
$y_3 = y_2 + x_2(1 - y_2)\,dx = .392 + 1.4(1 - .392)(.2) = .5622;$
$\frac{dy}{1-y} = x\,dx \Rightarrow -\ln|1-y| = \frac{x^2}{2} + C;\ \ x = 1,\ y = 0 \Rightarrow -\ln 1 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2} \Rightarrow \ln|1-y| = -\frac{x^2}{2} + \frac{1}{2}$
$\Rightarrow y = 1 - e^{(1-x^2)/2} \Rightarrow y(1.6) \approx .5416$

13. $y_1 = y_0 + (2x_0 y_0 + 2y_0)\,dx = 3 + [2(0)(3) + 2(3)](.2) = 4.2,$
$y_2 = y_1 + (2x_1 y_1 + 2y_1)\,dx = 4.2 + [2(.2)(4.2) + 2(4.2)](.2) = 6.216,$
$y_3 = y_2 + (2x_2 y_2 + 2y_2)\,dx = 6.216 + [2(.4)(6.216) + 2(6.216)](.2) = 9.6969;$
$\frac{dy}{dx} = 2y(x+1) \Rightarrow \frac{dy}{y} = 2(x+1)dx \Rightarrow \ln|y| = (x+1)^2 + C;\ \ x = 0,\ y = 3 \Rightarrow \ln 3 = 1 + C \Rightarrow C = \ln 3 - 1$
$\Rightarrow \ln y = (x+1)^2 + \ln 3 - 1 \Rightarrow y = e^{(x+1)^2 + \ln 3 - 1} = e^{\ln 3} e^{x^2 + 2x} = 3e^{x(x+2)} \Rightarrow y(.6) \approx 14.2765$

14. $y_1 = y_0 + y_0^2\left(1+2x_0\right)dx = 1 + 1^2\left[1+2(-1)\right](.5) = .5,$
$y_2 = y_1 + y_1^2\left(1+2x_1\right)dx = .5 + (.5)^2\left[1+2(-.5)\right](.5) = .5,$
$y_3 = y_2 + y_2^2\left(1+2x_2\right)dx = .5 + (.5)^2\left[1+2(0)\right](.5) = .625;$
$\frac{dy}{y^2} = \left(1+2x\right)dx \Rightarrow -\frac{1}{y} = x + x^2 + C;\;\; x=-1,\, y=1 \Rightarrow -1 = -1 + (-1)^2 + C \Rightarrow C = -1 \Rightarrow \frac{1}{y} = 1 - x - x^2$
$\Rightarrow y = \frac{1}{1-x-x^2} \Rightarrow y(.5) = \frac{1}{1-.5-(.5)^2} = 4$

15. $y_1 = y_0 + 2x_0e^{x_0^2}\,dx = 2 + 2(0)(.1) = 2,$
$y_2 = y_1 + 2x_1e^{x_1^2}\,dx = 2 + 2(.1)e^{(.1)^2}(.1) = 2.0202,$
$y_3 = y_2 + 2x_2e^{x_2^2}\,dx = 2.0202 + 2(.2)e^{(.2)^2}(.1) = 2.0618,$
$dy = 2xe^{x^2}dx \Rightarrow y = e^{x^2} + C;\;\; y(0) = 2 \Rightarrow 2 = 1 + C \Rightarrow C = 1 \Rightarrow y = e^{x^2} + 1 \Rightarrow y(.3) = e^{(.3)^2} + 1 \approx 2.0942$

16. $y_1 = y_0 + \left(y_0e^{x_0}\right)dx = 2 + \left(2\cdot e^0\right)(.5) = 3,$
$y_2 = y_1 + \left(y_1e^{x_1}\right)dx = 3 + \left(3\cdot e^{0.5}\right)(.5) = 5.47308,$
$y_3 = y_2 + \left(y_2e^{x_2}\right)dx = 5.47308 + \left(5.47308\cdot e^{1.0}\right)(.5) = 12.9118,$
$\frac{dy}{dx} = ye^x \Rightarrow \frac{dy}{y} = e^x\,dx \Rightarrow \ln|y| = e^x + C;\;\; x=0,\, y=2 \Rightarrow \ln 2 = 1 + C \Rightarrow C = \ln 2 - 1 \Rightarrow \ln|y| = e^x + \ln 2 - 1$
$\Rightarrow y = 2e^{e^x-1} \Rightarrow y(1.5) = 2e^{e^{1.5}-1} \approx 65.0292$

17. $y_1 = 1 + 1(.2) = 1.2,$
$y_2 = 1.2 + (1.2)(.2) = 1.44,$
$y_3 = 1.44 + (1.44)(.2) = 1.728,$
$y_4 = 1.728 + (1.728)(.2) = 2.0736,$
$y_5 = 2.0736 + (2.0736)(.2) = 2.48832;$
$\frac{dy}{y} = dx \Rightarrow \ln y = x + C_1 \Rightarrow y = Ce^x;\;\; y(0) = 1 \Rightarrow 1 = Ce^0 \Rightarrow C = 1 \Rightarrow y = e^x \Rightarrow y(1) = e \approx 2.7183$

18. $y_1 = 2 + \left(\frac{2}{1}\right)(.2) = 2.4,$
$y_2 = 2.4 + \left(\frac{2.4}{1.2}\right)(.2) = 2.8,$
$y_3 = 2.8 + \left(\frac{2.8}{1.4}\right)(.2) = 3.2,$
$y_4 = 3.2 + \left(\frac{3.2}{1.6}\right)(.2) = 3.6,$
$y_5 = 3.6 + \left(\frac{3.6}{1.8}\right)(.2) = 4;$
$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln y = \ln x + C \Rightarrow y = kx;\;\; y(1) = 2 \Rightarrow 2 = k \Rightarrow y = 2x \Rightarrow y(2) = 4$

19. $y_1 = -1 + \left[\frac{(-1)^2}{\sqrt{1}}\right](.5) = -.5,$
$y_2 = -.5 + \left[\frac{(-.5)^2}{\sqrt{1.5}}\right](.5) = -.39794,$

$y_3 = -.39794 + \left[\frac{(-.39794)^2}{\sqrt{2}}\right](.5) = -.34195,$

$y_4 = -.34195 + \left[\frac{(-.34195)^2}{\sqrt{2.5}}\right](.5) = -.30497,$

$y_5 = -.27812,\ y_6 = -.25745,\ y_7 = -.24088,\ y_8 = -.2272;$

$\frac{dy}{y^2} = \frac{dx}{\sqrt{x}} \Rightarrow -\frac{1}{y} = 2\sqrt{x} + C;\quad y(1) = -1 \Rightarrow 1 = 2 + C \Rightarrow C = -1 \Rightarrow y = \frac{1}{1-2\sqrt{x}} \Rightarrow y(5) = \frac{1}{1-2\sqrt{5}} \approx -.2880$

20. $y_1 = 1 + (0 \cdot \sin 1)\left(\frac{1}{3}\right) = 1,$

$y_2 = 1 + \left(\frac{1}{3} \cdot \sin 1\right)\left(\frac{1}{3}\right) = 1.09350,$

$y_3 = 1.09350 + \left(\frac{2}{3} \cdot \sin 1.09350\right)\left(\frac{1}{3}\right) = 1.29089,$

$y_4 = 1.29089 + \left(\frac{3}{3} \cdot \sin 1.29089\right)\left(\frac{1}{3}\right) = 1.61125,$

$y_5 = 1.61125 + \left(\frac{4}{3} \cdot \sin 1.61125\right)\left(\frac{1}{3}\right) = 2.05533,$

$y_6 = 2.05533 + \left(\frac{5}{3} \cdot \sin 2.05533\right)\left(\frac{1}{3}\right) = 2.54694;$

$y' = x \sin y \Rightarrow \csc y\, dy = x\, dx \Rightarrow -\ln|\csc y + \cot y| = \frac{1}{2}x^2 + C \Rightarrow \csc y + \cot y = e^{-\frac{1}{2}x^2 + C} = Ce^{-\frac{1}{2}x^2}$

$\Rightarrow \frac{1+\cos y}{\sin y} = Ce^{-\frac{1}{2}x^2} \Rightarrow \cot\left(\frac{y}{2}\right) = Ce^{-\frac{1}{2}x^2};\ \ y(0) = 1 \Rightarrow \cot\left(\frac{1}{2}\right) = Ce^0 = C \Rightarrow \cot\left(\frac{y}{2}\right) = \cot\left(\frac{1}{2}\right)e^{-\frac{1}{2}x^2}$

$\Rightarrow y = 2\cot^{-1}\left(\cot\left(\frac{1}{2}\right)e^{-\frac{1}{2}x^2}\right), y(2) = 2\cot^{-1}\left(\cot\left(\frac{1}{2}\right)e^{-2}\right) = 2.65591$

21. $y = -1 - x + (1 + x_0 + y_0)e^{x - x_0} \Rightarrow y(x_0) = -1 - x_0 + (1 + x_0 + y_0)e^{x_0 - x_0} = -1 - x_0 + (1 + x_0 + y_0)(1) = y_0$

$\frac{dy}{dx} = -1 + (1 + x_0 + y_0)e^{x - x_0} \Rightarrow y = -1 - x + (1 + x_0 + y_0)e^{x - x_0} = \frac{dy}{dx} - x \Rightarrow \frac{dy}{dx} = x + y$

22. $y' = f(x),\ y(x_0) = y_0 \Rightarrow y = \int_{x_0}^{x} f(t)\, dt + C,\quad y(x_0) = \int_{x_0}^{x_0} f(t)\, dt + C = C \Rightarrow C = y_0 \Rightarrow y = \int_{x_0}^{x} f(t)\, dt + y_0$

23-34. Example CAS commands:

Maple:

```
ode := diff( y(x), x ) = y(x);
icA := [0,1];
icB := [0, 2];
icC := [0,-1];
DEplot( ode, y(x), x=0..2, [icA,icB,icC], arrows=slim, linecolor=blue, title="#23 (Section 9.1)" );
```

Mathematica:

To plot vector fields, you must begin by loading a graphics package.

```
<<Graphics`PlotField`
```

To control lengths and appearance of vectors, select the Help browser, type PlotVectorField and select Go.

```
Clear[x, y, f]
yprime = y (2 – y);
```

```
pv = PlotVectorField[{1, yprime}, {x, −5, 5}, {y, −4, 6}, Axes → True, AxesLabel → {x, y}];
```

To draw solution curves with Mathematica, you must first solve the differential equation. This will be done with the DSolve command. The y[x] and x at the end of the command specify the dependent and independent variables. The command will not work unless the y in the differential equation is referenced as y[x].

```
equation = y'[x]==y[x] (2 − y[x]) ;
initcond = y[a] == b;
sols = DSolve[{equation, initcond}, y[x], x]
vals = {{0, 1/2}, {0, 3/2}, {0, 2}, {0, 3}}
f[{a_, b_}] = sols[[1, 1, 2]];
solnset = Map[f, vals]
ps = Plot[Evaluate[solnset, {x, −5, 5}];
Show[pv, ps, PlotRange → {−4, 6}];
```

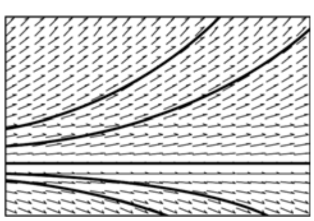
The code for problems such as 31 & 32 is similar for the direction field, but the analytical solutions involve complicated inverse functions, so the numerical solver NDSolve is used. Note that a domain interval is specified.

```
equation = y'[x] == Cos[2x − y[x]] ;
initcond = y[0] == 2;
sol = NDSolve[{equation, initcond}, y[x], {x, 0, 5}]
ps = Plot[Evaluate[y[x]/.sol, {x, 0, 5}];
N[y[x] /. sol/.x → 2]
Show[pv, ps, PlotRange → {0, 5}];
```

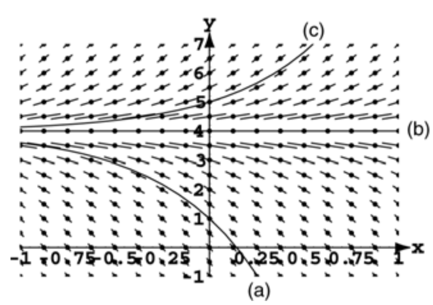
Solutions for 33 can be found one at a time and plots named and shown together. No direction fields here. For 34, the direction field code is similar, but the solution is found implicitly using integrations. The plot requires loading another special graphics package.

```
<<Graphics`ImplicitPlot`
Clear[x,y]
solution[c_] = Integrate[2 (y − 1), y] == Integrate[3x^2 + 4x + 2, x] + c
values = {−6, −4, −2, 0, 2, 4, 6};
solns = Map[solution, values];
ps = ImplicitPlot[solns, {x, −3, 3}, {y, −3, 3}]
Show[pv, ps]
```

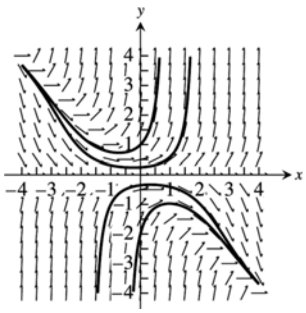
23.



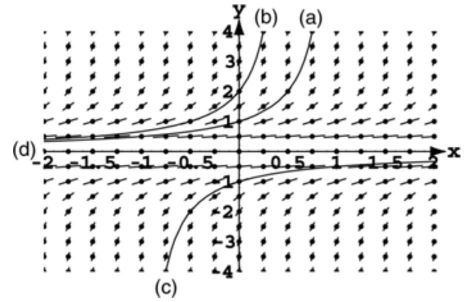
24.



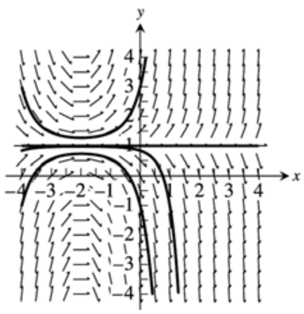
25.



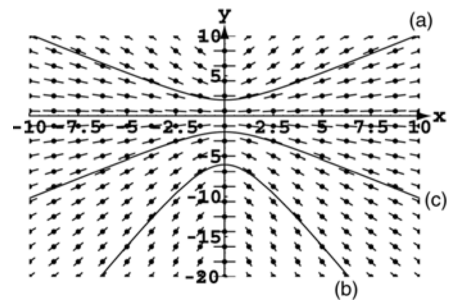
26.



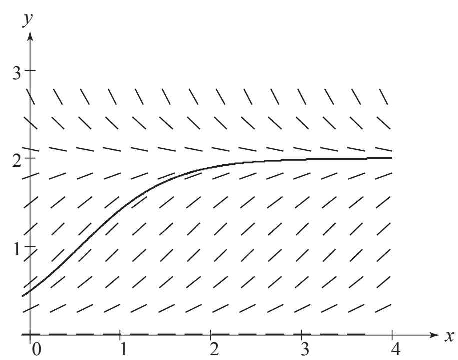
27.



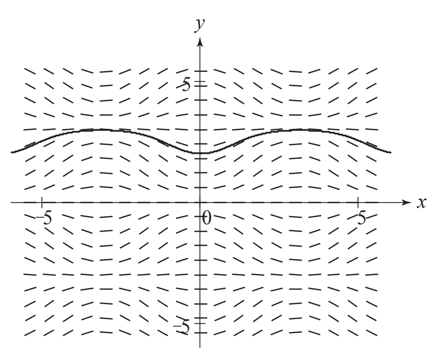
28.



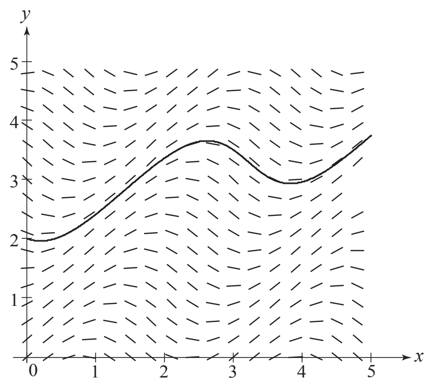
29. The general solution is $y = \dfrac{2}{1+ce^{-2t}}$. The particular solution $y = \dfrac{2}{1+3e^{-2t}}$ is shown below together with the slope field.



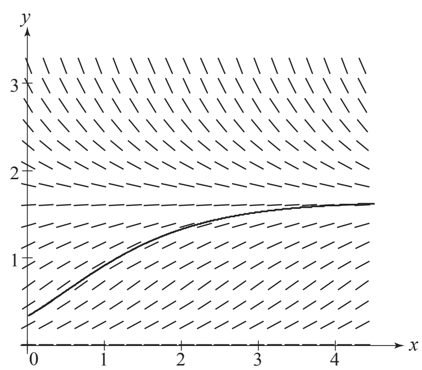
30. The general solution is $y = \pi + \tan^{-1}\left(\dfrac{2e^{c-\cos x}}{1-e^{2(c-\cos x)}}\right)$. The required particular solution with $c = 1+\ln(\csc 2 - \cot 2) \approx 1.443$ is shown below together with the slope field.



31. The particular solution with $y(0)=2$ is shown together with the slope field.



32. The particular solution with $y(0)=1/3$ is shown together with the slope field.



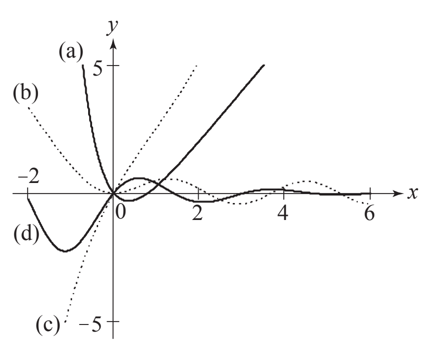
33. The particular solutions with $y(0)=0$ are:

(a) $y=2e^{-2x}+2x-2$

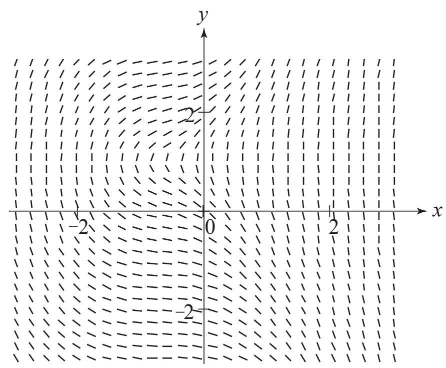
(b) $y=\frac{1}{5}\sin 2x-\frac{2}{5}\cos 2x+\frac{2}{5}e^{-x}$

(c) $y=2e^{x/2}-2e^{-x}$

(d) $y=e^{-x/2}\left(\frac{4}{17}\cos 2x+\frac{16}{17}\sin 2x\right)-\frac{4}{17}e^{-x}$



34. (a)



(b) The solution is given implicitly by $y^2 - 2y = x^3 + 2x^2 + 2x + c.$

35. $\frac{dy}{dx} = 2xe^{x^2}$, $y(0) = 2 \Rightarrow y_{n+1} = y_n + 2x_n e^{x_n^2} dx = y_n + 2x_n e^{x_n^2}(0.1) = y_n + 0.2x_n e^{x_n^2}$
On a TI-84 calculator home screen, type the following commands:
2 STO > y: 0 STO > x: y (enter)
y + 0.2*x*e^(x^2) STO > y: x + 0.1 STO > x: y (enter, 10 times)
The last value displayed gives $y_{\text{Euler}}(1) \approx 3.45835$
The exact solution: $dy = 2xe^{x^2} dx \Rightarrow y = e^{x^2} + C$; $y(0) = 2 = e^0 + C \Rightarrow C = 1 \Rightarrow y = 1 + e^{x^2}$
$\Rightarrow y_{\text{exact}}(1) = 1 + e \approx 3.71828$

36. $\frac{dy}{dx} = 2y^2(x-1)$, $y(2) = -\frac{1}{2} \Rightarrow y_{n+1} = y_n + 2y_n^2(x_n - 1)dx = y_n + 0.2y_n^2(x_n - 1)$
On a TI-84 calculator home screen, type the following commands:
–0.5 STO > y : 2 STO > x : y (enter)
y + 0.2*y^2 (x – 1) STO > y: x + 0.1 STO > x : y (enter, 10 times)
The last value displayed gives $y_{\text{Euler}}(2) \approx -0.19285$
The exact solution: $\frac{dy}{dx} = 2y^2(x-1) \Rightarrow \frac{dy}{y^2} = (2x-2)\,dx \Rightarrow -\frac{1}{y} = x^2 - 2x + C \Rightarrow \frac{1}{y} = -x^2 + 2x + C$
$y(2) = -\frac{1}{2} \Rightarrow \frac{1}{-1/2} = -(2)^2 + 2(2) + C = C \Rightarrow C = -2 \Rightarrow \frac{1}{y} = -x^2 + 2x - 2 \Rightarrow y = \frac{1}{-x^2+2x-2}$
$y(3) = \frac{1}{-(3)^2+2(3)-2} = -0.2$

37. $\frac{dy}{dx} = \frac{\sqrt{x}}{y}$, $y > 0$, $y(0) = 1 \Rightarrow y_{n+1} = y_n + \frac{\sqrt{x_n}}{y_n} dx = y_n + \frac{\sqrt{x_n}}{y_n}(0.1) = y_n + 0.1\frac{\sqrt{x_n}}{y_n}$
On a TI-84 calculator home screen, type the following commands:
1 STO > y: 0 STO > x: y (enter)
y + 0.1*($\sqrt{\text{x}}$/y) STO > y: x + 0.1 STO > x: y (enter, 10 times)
The last value displayed gives $y_{\text{Euler}}(1) \approx 1.5000$
The exact solution: $dy = \frac{\sqrt{x}}{y} dx \Rightarrow y\,dy = \sqrt{x}\,dx \Rightarrow \frac{y^2}{2} = \frac{2}{3}x^{3/2} + C$; $\frac{(y(0))^2}{2} = \frac{1^2}{2} = \frac{1}{2} = \frac{2}{3}(0)^{3/2} + C \Rightarrow C = \frac{1}{2}$
$\Rightarrow \frac{y^2}{2} = \frac{2}{3}x^{3/2} + \frac{1}{2} \Rightarrow y = \sqrt{\frac{4}{3}x^{3/2} + 1} \Rightarrow y_{\text{exact}}(1) = \sqrt{\frac{4}{3}(1)^{3/2} + 1} \approx 1.5275$

38. $\frac{dy}{dx} = 1 + y^2,\ \ y(0) = 0 \Rightarrow y_{n+1} = y_n + \left(1 + y_n^2\right)dx = y_n + \left(1 + y_n^2\right)(0.1) = y_n + 0.1\left(1 + y_n^2\right)$

On a TI-84 calculator home screen, type the following commands:

0 STO > y: 0 STO > x: y (enter)

$\mathrm{y} + 0.1*(1 + \mathrm{y}^2)$ STO > y: x + 0.1 STO > x: y (enter, 10 times)

The last value displayed gives $\mathrm{y}_{\mathrm{Euler}}(1) \approx 1.3964$

The exact solution: $dy = \left(1 + y^2\right)dx \Rightarrow \frac{dy}{1+y^2} = dx \Rightarrow \tan^{-1} y = x + C;\ \ \tan^{-1} y(0) = \tan^{-1} 0 = 0 = 0 + C$

$\Rightarrow C = 0 \Rightarrow \tan^{-1} y = x \Rightarrow y = \tan x \Rightarrow y_{\text{exact}}(1) = \tan 1 \approx 1.5574$

39. Example CAS commands:

Maple:

```
ode := diff( y(x), x ) = x + y(x);ic := y(0)=-7/10;
x0 := -4;x1 := 4;y0 := -4;y1 := 4;
b := 1;
P1 := DEplot( ode, y(x), x=x0..x1, y=y0..y1, arrows=thin, title="#39(a) (Section 9.1)" ):
P1;
Ygen := unapply( rhs(dsolve( ode, y(x) )), x,_C1 );                                   # (b)
P2 := seq( plot( Ygen(x,c), x=x0..x1, y=y0..y1, color=blue ), c=-2..2 ):              # (c)
display( [P1,P2], title="#39(c) (Section 9.1)" );
CC := solve( Ygen(0,C)=rhs(ic), C );                                                  # (d)
Ypart := Ygen(x,CC);
P3 := plot( Ypart, x=0..b, title="#39(d) (Section 9.1)" ):
P3;
euler4 := dsolve( {ode,ic}, numeric, method=classical[foreuler], stepsize = (x1-x0)/4 ):  # (e)
P4 := odeplot( euler4, [x,y(x)], x=0..b, numpoints=4, color=blue ):
display( [P3,P4], title="#39(e) (Section 9.1)" );
euler8 := dsolve( {ode,ic}, numeric, method=classical[foreuler], stepsize=(x1-x0)/8 ):    # (f )
P5 := odeplot( euler8, [x,y(x)], x=0..b, numpoints=8, color=green ):
euler16 := dsolve( {ode,ic}, numeric, method=classical[foreuler], stepsize=(x1-x0)/16 ):
P6 := odeplot( euler16, [x,y(x)], x=0..b, numpoints=16, color=pink ):
euler32 := dsolve( {ode,ic}, numeric, method=classical[foreuler], stepsize=(x1-x0)/32 ):
P7 := odeplot( euler32, [x,y(x)], x=0..b, numpoints=32, color=cyan ):
display( [P3,P4,P5,P6,P7], title="#39(f ) (Section 9.1)" );
<< N | h    |`percent error`> ,                                                      # (g)
< 4 | (x1-x0)/ 4 | evalf[5]( abs(1-eval(y(x),euler4(b))/eval(Ypart,x=b))*100 ) >,
< 8 | (x1-x0)/ 8 | evalf[5]( abs(1-eval(y(x),euler8(b))/eval(Ypart,x=b))*100 ) >,
< 16 | (x1-x0)/16 | evalf[5]( abs(1-eval(y(x),euler16(b))/eval(Ypart,x=b))*100 ) >,
< 32 | (x1-x0)/ 32 | evalf[5]( abs(1-eval(y(x),euler32(b))/eval(Ypart,x=b))*100 ) >>;
```

39-42. Example CAS commands:

Mathematica: (assigned functions, step sizes, and values for initial conditions may vary)
Problems 39 - 42 involve use of code from Problems 23 - 34 together with the above code for Euler's method.

9.2 FIRST-ORDER LINEAR EQUATIONS

1. $x\frac{dy}{dx}+y=e^x \Rightarrow \frac{dy}{dx}+\left(\frac{1}{x}\right)y=\frac{e^x}{x},\quad P(x)=\frac{1}{x}, Q(x)=\frac{e^x}{x}$

   $\int P(x)\,dx=\int\frac{1}{x}\,dx=\ln|x|=\ln x,\ x>0 \Rightarrow v(x)=e^{\int P(x)\,dx}=e^{\ln x}=x$

   $y=\frac{1}{v(x)}\int v(x)Q(x)\,dx=\frac{1}{x}\int x\left(\frac{e^x}{x}\right)dx=\frac{1}{x}\left(e^x+C\right)=\frac{e^x+C}{x}, x>0$

2. $e^x\frac{dy}{dx}+2e^xy=1 \Rightarrow \frac{dy}{dx}+2y=e^{-x},\quad P(x)=2, Q(x)=e^{-x}$

   $\int P(x)dx=\int 2dx=2x \Rightarrow v(x)=e^{\int P(x)\,dx}=e^{2x}$

   $y=\frac{1}{e^{2x}}\int e^{2x}\cdot e^{-x}dx=\frac{1}{e^{2x}}\int e^x dx=\frac{1}{e^{2x}}\left(e^x+C\right)=e^{-x}+Ce^{-2x}$

3. $xy'+3y=\frac{\sin x}{x^2},\ x>0 \Rightarrow \frac{dy}{dx}+\left(\frac{3}{x}\right)y=\frac{\sin x}{x^3},\quad P(x)=\frac{3}{x}, Q(x)=\frac{\sin x}{x^3}$

   $\int\frac{3}{x}\,dx=3\ln|x|=\ln x^3,\ x>0 \Rightarrow v(x)=e^{\ln x^3}=x^3$

   $y=\frac{1}{x^3}\int x^3\left(\frac{\sin x}{x^3}\right)dx=\frac{1}{x^3}\int\sin x\,dx=\frac{1}{x^3}\left(-\cos x+C\right)=\frac{C-\cos x}{x^3},\ x>0$

4. $y'+\left(\tan x\right)y=\cos^2 x, -\frac{\pi}{2}<x<\frac{\pi}{2} \Rightarrow \frac{dy}{dx}+\left(\tan x\right)y=\cos^2 x,\quad P(x)=\tan x, Q(x)=\cos^2 x$

   $\int\tan x\,dx=\int\frac{\sin x}{\cos x}\,dx=-\ln|\cos x|=\ln(\cos x)^{-1}, -\frac{\pi}{2}<x<\frac{\pi}{2} \Rightarrow v(x)=e^{\ln(\cos x)^{-1}}=(\cos x)^{-1}=\sec x$

   $y=\frac{1}{\sec x}\int\sec x\cdot\cos^2 x\,dx=(\cos x)\int\cos x\,dx=(\cos x)\left(\sin x+C\right)=\sin x\cos x+C\cos x$

5. $x\frac{dy}{dx}+2y=1-\frac{1}{x}, x>0 \Rightarrow \frac{dy}{dx}+\left(\frac{2}{x}\right)y=\frac{1}{x}-\frac{1}{x^2},\quad P(x)=\frac{2}{x}, Q(x)=\frac{1}{x}-\frac{1}{x^2}$

   $\int\frac{2}{x}\,dx=2\ln|x|=\ln x^2,\ x>0 \Rightarrow v(x)=e^{\ln x^2}=x^2$

   $y=\frac{1}{x^2}\int x^2\left(\frac{1}{x}-\frac{1}{x^2}\right)dx=\frac{1}{x^2}\int(x-1)\,dx=\frac{1}{x^2}\left(\frac{x^2}{2}-x+C\right)=\frac{1}{2}-\frac{1}{x}+\frac{C}{x^2},\ x>0$

6. $(1+x)y'+y=\sqrt{x} \Rightarrow \frac{dy}{dx}+\left(\frac{1}{1+x}\right)y=\frac{\sqrt{x}}{1+x},\quad P(x)=\frac{1}{1+x}, Q(x)=\frac{\sqrt{x}}{1+x}$

   $\int\frac{1}{1+x}dx=\ln(1+x)$, since $x>0 \Rightarrow v(x)=e^{\ln(1+x)}=1$

   $y=\frac{1}{1+x}\int(1+x)\left(\frac{\sqrt{x}}{1+x}\right)dx=\frac{1}{1+x}\int\sqrt{x}\,dx=\left(\frac{1}{1+x}\right)\left(\frac{2}{3}x^{3/2}+C\right)=\frac{2x^{3/2}}{3(1+x)}+\frac{C}{1+x}$

7. $\frac{dy}{dx}-\frac{1}{2}y=\frac{1}{2}e^{x/2} \Rightarrow P(x)=-\frac{1}{2},\ Q(x)=\frac{1}{2}e^{x/2} \Rightarrow \int P(x)\,dx=-\frac{1}{2}x \Rightarrow v(x)=e^{-x/2}$

   $\Rightarrow y=\frac{1}{e^{-x/2}}\int e^{-x/2}\left(\frac{1}{2}e^{x/2}\right)dx=e^{x/2}\int\frac{1}{2}dx=e^{x/2}\left(\frac{1}{2}x+C\right)=\frac{1}{2}xe^{x/2}+Ce^{x/2}$

8. $\frac{dy}{dx}+2y=2xe^{-2x}\Rightarrow P(x)=2,\ \ Q(x)=2xe^{-2x}\Rightarrow\int P(x)dx=\int 2\,dx=2x\Rightarrow v(x)=e^{2x}$
$\Rightarrow y=\frac{1}{e^{2x}}\int e^{2x}\left(2xe^{-2x}\right)dx=\frac{1}{e^{2x}}\int 2x\,dx=e^{-2x}\left(x^2+C\right)=x^2e^{-2x}+Ce^{-2x}$

9. $\frac{dy}{dx}-\left(\frac{1}{x}\right)y=2\ln x\Rightarrow P(x)=-\frac{1}{x},\ \ Q(x)=2\ln x\Rightarrow\int P(x)\,dx=-\int\frac{1}{x}\,dx=-\ln x, x>0$
$\Rightarrow v(x)=e^{-\ln x}=\frac{1}{x}\Rightarrow y=x\int\left(\frac{1}{x}\right)(2\ln x)\,dx=x\left[(\ln x)^2+C\right]=x(\ln x)^2+Cx$

10. $\frac{dy}{dx}+\left(\frac{2}{x}\right)y=\frac{\cos x}{x^2},\ x>0\Rightarrow P(x)=\frac{2}{x},\ \ Q(x)=\frac{\cos x}{x^2}\Rightarrow\int P(x)\,dx=\int\frac{2}{x}\,dx=2\ln|x|=\ln x^2,\ x>0$
$\Rightarrow v(x)=e^{\ln x^2}=x^2\Rightarrow y=\frac{1}{x^2}\int x^2\left(\frac{\cos x}{x^2}\right)dx=\frac{1}{x^2}\int\cos x\,dx=\frac{1}{x^2}(\sin x+C)=\frac{\sin x+C}{x^2}$

11. $\frac{ds}{dt}+\left(\frac{4}{t-1}\right)s=\frac{t+1}{(t-1)^3}\Rightarrow P(t)=\frac{4}{t-1},\ \ Q(t)=\frac{t+1}{(t-1)^3}\Rightarrow\int P(t)\,dt=\int\frac{4}{t-1}\,dt=4\ln|t-1|=\ln\,(t-1)^4$
$\Rightarrow v(t)=e^{\ln(t-1)^4}=(t-1)^4\Rightarrow s=\frac{1}{(t-1)^4}\int(t-1)^4\left[\frac{t+1}{(t-1)^3}\right]dt=\frac{1}{(t-1)^4}\int\left(t^2-1\right)dt=\frac{1}{(t-1)^4}\left(\frac{t^3}{3}-t+C\right)$
$=\frac{t^3}{3(t-1)^4}-\frac{t}{(t-1)^4}+\frac{C}{(t-1)^4}$

12. $(t+1)\frac{ds}{dt}+2s=3(t+1)+\frac{1}{(t+1)^2}\Rightarrow\frac{ds}{dt}+\left(\frac{2}{t+1}\right)s=3+\frac{1}{(t+1)^3}\Rightarrow P(t)=\frac{2}{t+1},\ \ Q(t)=3+(t+1)^{-3}$
$\Rightarrow\int P(t)\,dt=\int\frac{2}{t+1}\,dt=2\ln|t+1|=\ln\,(t+1)^2\Rightarrow v(t)=e^{\ln\,(t+1)^2}=(t+1)^2$
$\Rightarrow s=\frac{1}{(t+1)^2}\int(t+1)^2\left[3+(t+1)^{-3}\right]dt=\frac{1}{(t+1)^2}\int\left[3(t+1)^2+(t+1)^{-1}\right]dt$
$=\frac{1}{(t+1)^2}\left[(t+1)^3+\ln|t+1|+C\right]=(t+1)+(t+1)^{-2}\ln\,(t+1)+\frac{C}{(t+1)^2},\ t>-1$

13. $\frac{dr}{d\theta}+(\cot\theta)\,r=\sec\theta\Rightarrow P(\theta)=\cot\theta,\ \ Q(\theta)=\sec\theta\Rightarrow\int P(\theta)\,d\theta=\int\cot\theta\,d\theta=\ln|\sin\theta|\Rightarrow v(\theta)=e^{\ln|\sin\theta|}$
$=\sin\theta$ because $0<\theta<\frac{\pi}{2}\Rightarrow r=\frac{1}{\sin\theta}\int(\sin\theta)(\sec\theta)\,d\theta=\frac{1}{\sin\theta}\int\tan\theta\,d\theta=\frac{1}{\sin\theta}(\ln|\sec\theta|+C)$
$=(\csc\theta)\,(\ln|\sec\theta|+C)$

14. $\tan\theta\frac{dr}{d\theta}+r=\sin^2\theta\Rightarrow\frac{dr}{d\theta}+\frac{r}{\tan\theta}=\frac{\sin^2\theta}{\tan\theta}\Rightarrow\frac{dr}{d\theta}+(\cot\theta)\,r=\sin\theta\cos\theta\Rightarrow P(\theta)=\cot\theta,\ \ Q(\theta)=\sin\theta\cos\theta$
$\Rightarrow\int P(\theta)\,d\theta=\int\cot\theta\,d\theta=\ln|\sin\theta|=\ln(\sin\theta)$ since $0<\theta<\frac{\pi}{2}\Rightarrow v(\theta)=e^{\ln(\sin\theta)}=\sin\theta$
$\Rightarrow r=\frac{1}{\sin\theta}\int(\sin\theta)(\sin\theta\cos\theta)\,d\theta=\frac{1}{\sin\theta}\int\sin^2\theta\cos\theta\,d\theta=\left(\frac{1}{\sin\theta}\right)\left(\frac{\sin^3\theta}{3}+C\right)=\frac{\sin^2\theta}{3}+\frac{C}{\sin\theta}$

15. $\frac{dy}{dt}+2y=3\Rightarrow P(t)=2,\ \ Q(t)=3\Rightarrow\int P(t)\,dt=\int 2\,dt=2t\Rightarrow v(t)=e^{2t}\Rightarrow y=\frac{1}{e^{2t}}\int 3e^{2t}\,dt=\frac{1}{e^{2t}}\left(\frac{3}{2}e^{2t}+C\right);$
$y(0)=1\Rightarrow\frac{3}{2}+C=1\Rightarrow C=-\frac{1}{2}\Rightarrow y=\frac{3}{2}-\frac{1}{2}e^{-2t}$

16. $\frac{dy}{dt}+\frac{2y}{t}=t^2\Rightarrow P(t)=\frac{2}{t},\ \ Q(t)=t^2\Rightarrow\int P(t)\,dt=2\ln|t|\Rightarrow v(t)=e^{\ln t^2}=t^2\Rightarrow y=\frac{1}{t^2}\int\left(t^2\right)\left(t^2\right)dt$
$=\frac{1}{t^2}\int t^4\,dt=\frac{1}{t^2}\left(\frac{t^5}{5}+C\right)=\frac{t^3}{5}+\frac{C}{t^2};\ \ y(2)=1\Rightarrow\frac{8}{5}+\frac{C}{4}=1\Rightarrow C=-\frac{12}{5}\Rightarrow y=\frac{t^3}{5}-\frac{12}{5t^2}$

17. $\frac{dy}{d\theta}+\left(\frac{1}{\theta}\right)y=\frac{\sin\theta}{\theta}\Rightarrow P(\theta)=\frac{1}{\theta},\ \ Q(\theta)=\frac{\sin\theta}{\theta}\Rightarrow\int P(\theta)d\theta=\ln|\theta|\Rightarrow v(\theta)=e^{\ln|\theta|}=|\theta|\Rightarrow y=\frac{1}{|\theta|}\int|\theta|\left(\frac{\sin\theta}{\theta}\right)d\theta$
$=\frac{1}{\theta}\int\theta\left(\frac{\sin\theta}{\theta}\right)d\theta$ for $\theta\neq 0\Rightarrow y=\frac{1}{\theta}\int\sin\theta\,d\theta=\frac{1}{\theta}(-\cos\theta+C)=-\frac{1}{\theta}\cos\theta+\frac{C}{\theta};\ \ y\left(\frac{\pi}{2}\right)=1\Rightarrow C=\frac{\pi}{2}$
$\Rightarrow y=-\frac{1}{\theta}\cos\theta+\frac{\pi}{2\theta}$

18. $\frac{dy}{d\theta}-\left(\frac{2}{\theta}\right)y=\theta^2\sec\theta\tan\theta\Rightarrow P(\theta)=-\frac{2}{\theta},\ \ Q(\theta)=\theta^2\sec\theta\tan\theta\Rightarrow\int P(\theta)\,d\theta=-2\ln|\theta|\Rightarrow v(\theta)=e^{-2\ln|\theta|}$
$=\theta^{-2}\Rightarrow y=\frac{1}{\theta^{-2}}\int\left(\theta^{-2}\right)\left(\theta^2\sec\theta\tan\theta\right)d\theta=\theta^2\int\sec\theta\tan\theta\,d\theta=\theta^2\left(\sec\theta+C\right)=\theta^2\sec\theta+C\theta^2;$
$y\left(\frac{\pi}{3}\right)=2\Rightarrow 2=\left(\frac{\pi^2}{9}\right)(2)+C\left(\frac{\pi^2}{9}\right)\Rightarrow C=\frac{18}{\pi^2}-2\Rightarrow y=\theta^2\sec\theta+\left(\frac{18}{\pi^2}-2\right)\theta^2$

19. $(x+1)\frac{dy}{dx}-2\left(x^2+x\right)y=\frac{e^{x^2}}{x+1}\Rightarrow\frac{dy}{dx}-2\left[\frac{x(x+1)}{x+1}\right]y=\frac{e^{x^2}}{(x+1)^2}\Rightarrow\frac{dy}{dx}-2xy=\frac{e^{x^2}}{(x+1)^2}\Rightarrow P(x)=-2x,\ \ Q(x)=\frac{e^{x^2}}{(x+1)^2}$
$\Rightarrow\int P(x)\,dx=\int -2x\,dx=-x^2\Rightarrow v(x)=e^{-x^2}\Rightarrow y=\frac{1}{e^{-x^2}}\int e^{-x^2}\left[\frac{e^{x^2}}{(x+1)^2}\right]dx=e^{x^2}\int\frac{1}{(x+1)^2}\,dx=e^{x^2}\left[\frac{(x+1)^{-1}}{-1}+C\right]$
$=-\frac{e^{x^2}}{x+1}+Ce^{x^2};\ \ y(0)=5\Rightarrow -\frac{1}{0+1}+C=5\Rightarrow -1+C=5\Rightarrow C=6\Rightarrow y=6e^{x^2}-\frac{e^{x^2}}{x+1}$

20. $\frac{dy}{dx}+xy=x\Rightarrow P(x)=x,\ \ Q(x)=x\Rightarrow\int P(x)\,dx=\int x\,dx=\frac{x^2}{2}\Rightarrow v(x)=e^{x^2/2}\Rightarrow y=\frac{1}{e^{x^2/2}}\int e^{x^2/2}\cdot x\,dx$
$=\frac{1}{e^{x^2/2}}\left(e^{x^2/2}+C\right)=1+\frac{C}{e^{x^2/2}};\ \ y(0)=-6\Rightarrow 1+C=-6\Rightarrow C=-7\Rightarrow y=1-\frac{7}{e^{x^2/2}}$

21. $\frac{dy}{dt}-ky=0\Rightarrow P(t)=-k,\ \ Q(t)=0\Rightarrow\int P(t)\,dt=\int -k\,dt=-kt\Rightarrow v(t)=e^{-kt}\Rightarrow y=\frac{1}{e^{-kt}}\int\left(e^{-kt}\right)(0)\,dt$
$=e^{kt}(0+C)=Ce^{kt};\ \ y(0)=y_0\Rightarrow C=y_0\Rightarrow y=y_0e^{kt}$

22. (a) $\frac{du}{dt}+\frac{k}{m}u=0\Rightarrow P(t)=\frac{k}{m},\ \ Q(t)=0\Rightarrow\int P(t)\,dt=\int\frac{k}{m}\,dt=\frac{k}{m}t=\frac{kt}{m}\Rightarrow u(t)=e^{kt/m}$
$\Rightarrow y=\frac{1}{e^{kt/m}}\int e^{kt/m}\cdot 0\,dt=\frac{C}{e^{kt/m}};\ \ u(0)=u_0\Rightarrow\frac{C}{e^{k(0)/m}}=u_0\Rightarrow C=u_0\Rightarrow u=u_0\,e^{-(k/m)t}$

(b) $\frac{du}{dt}=-\frac{k}{m}u\Rightarrow\frac{du}{u}=-\frac{k}{m}\,dt\Rightarrow\ln u=-\frac{k}{m}t+C\Rightarrow u=e^{-(k/m)t+C}\Rightarrow u=e^{-(k/m)t}\cdot e^C.$ Let $e^C=C_1$.
Then $u=\frac{1}{e^{(k/m)t}}\cdot C_1$ and $u(0)=u_0=\frac{1}{e^{(k/m)(0)}}\cdot C_1=C_1$. So $u=u_0\,e^{-(k/m)t}$

23. $x\int\frac{1}{x}\,dx=x\left(\ln|x|+C\right)=x\ln|x|+Cx\Rightarrow$ (b) is correct

24. $\frac{1}{\cos x}\int\cos x\,dx=\frac{1}{\cos x}\left(\sin x+C\right)=\tan x+\frac{C}{\cos x}\Rightarrow$ (b) is correct

25. Steady State $=\frac{V}{R}$ and we want $i=\frac{1}{2}\left(\frac{V}{R}\right)\Rightarrow\frac{1}{2}\left(\frac{V}{R}\right)=\frac{V}{R}\left(1-e^{-Rt/L}\right)\Rightarrow\frac{1}{2}=1-e^{-Rt/L}\Rightarrow -\frac{1}{2}=-e^{-Rt/L}$
$\Rightarrow\ln\frac{1}{2}=-\frac{Rt}{L}\Rightarrow -\frac{L}{R}\ln\frac{1}{2}=t\Rightarrow t=\frac{L}{R}\ln 2$ s

26. (a) $\frac{di}{dt}+\frac{R}{L}i=0\Rightarrow\frac{1}{i}di=-\frac{R}{L}dt\Rightarrow\ln i=-\frac{Rt}{L}+C_1\Rightarrow i=e^{C_1}e^{-Rt/L}=Ce^{-Rt/L};\ \ i(0)=I\Rightarrow I=C$
$\Rightarrow i=Ie^{-Rt/L}$ ampere

(b) $\frac{1}{2}I = I\,e^{-Rt/L} \Rightarrow e^{-Rt/L} = \frac{1}{2} \Rightarrow -\frac{Rt}{L} = \ln\frac{1}{2} = -\ln 2 \Rightarrow t = \frac{L}{R}\ln 2$ .seconds

(c) $t = \frac{L}{R} \Rightarrow i = I\,e^{(-Rt/L)(L/R)} = I\,e^{-t}$ ampere

27. (a) $t = \frac{3L}{R} \Rightarrow i = \frac{V}{R}\left(1 - e^{(-R/L)(3L/R)}\right) = \frac{V}{R}\left(1 - e^{-3}\right) \approx 0.9502\frac{V}{R}$ amp, or about 95% of the steady state value

(b) $t = \frac{2L}{R} \Rightarrow i = \frac{V}{R}\left(1 - e^{(-R/L)(2L/R)}\right) = \frac{V}{R}\left(1 - e^{-2}\right) \approx 0.8647\frac{V}{R}$ amp or about 86% of the steady state value

28. (a) $\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L} \Rightarrow P(t) = \frac{R}{L},\ Q(t) = \frac{V}{L} \Rightarrow \int P(t)\,dt = \int \frac{R}{L}\,dt = \frac{Rt}{L} \Rightarrow v(t) = e^{Rt/L} \Rightarrow i = \frac{1}{e^{Rt/L}}\int e^{Rt/L}\left(\frac{V}{L}\right)dt$

$= \frac{1}{e^{Rt/L}}\left[\frac{L}{R}e^{Rt/L}\left(\frac{V}{L}\right) + C\right] = \frac{V}{R} + Ce^{-(R/L)t}$

(b) $i(0) = 0 \Rightarrow \frac{V}{R} + C = 0 \Rightarrow C = -\frac{V}{R} \Rightarrow i = \frac{V}{R} - \frac{V}{R}e^{-Rt/L}$

(c) $i = \frac{V}{R} \Rightarrow \frac{di}{dt} = 0 \Rightarrow \frac{di}{dt} + \frac{R}{L}i = 0 + \left(\frac{R}{L}\right)\left(\frac{V}{R}\right) = \frac{V}{L} \Rightarrow i = \frac{V}{R}$ is a solution of Eq. (6); $i = Ce^{-(R/L)t}$

29. $y' - y = -y^2$; we have $n = 2$, so let $u = y^{1-2} = y^{-1}$. Then $y = u^{-1}$ and $\frac{du}{dx} = -1y^{-2}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -y^2\frac{du}{dx}$

$\Rightarrow -u^{-2}\frac{du}{dx} - u^{-1} = -u^{-2} \Rightarrow \frac{du}{dx} + u = 1$. With $e^{\int dx} = e^x$ as the integrating factor, we have

$e^x\left(\frac{du}{dx} + u\right) = \frac{d}{dx}\left(e^x u\right) = e^x$. Integrating, we get $e^x u = e^x + C \Rightarrow u = 1 + \frac{C}{e^x} = \frac{1}{y} \Rightarrow y = \frac{1}{1+\frac{C}{e^x}} = \frac{e^x}{e^x + C}$

30. $y' - y = xy^2$; we have $n = 2$, so let $u = y^{-1}$. Then $y = u^{-1}$ and $\frac{du}{dx} = -y^{-2}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -y^2\frac{du}{dx} = -u^{-2}\frac{du}{dx}$.

Substituting: $-u^{-2}\frac{du}{dx} - u^{-1} = xu^{-2} \Rightarrow \frac{du}{dx} + u = -x$. Using $e^{\int dx} = e^x$ as an integrating factor:

$e^x\left(\frac{du}{dx} + u\right) = \frac{d}{dx}\left(e^x u\right) = -x\,e^x \Rightarrow e^x u = e^x(1 - x) + C \Rightarrow u = \frac{e^x(1-x)+C}{e^x} \Rightarrow y = u^{-1} = \frac{e^x}{e^x - xe^x + C}$

31. $xy' + y = y^{-2} \Rightarrow y' + \left(\frac{1}{x}\right)y = \left(\frac{1}{x}\right)y^{-2}$. Let $u = y^{1-(-2)} = y^3 \Rightarrow y = u^{1/3}$ and $y^{-2} = u^{-2/3}$.

$\frac{du}{dx} = 3y^2\frac{dy}{dx} \Rightarrow y' = \frac{dy}{dx} = \left(\frac{1}{3}\right)\left(\frac{du}{dx}\right)\left(y^{-2}\right) = \left(\frac{1}{3}\right)\left(\frac{du}{dx}\right)\left(u^{-2/3}\right)$. Thus we have

$\left(\frac{1}{3}\right)\left(\frac{du}{dx}\right)\left(u^{-2/3}\right) + \left(\frac{1}{x}\right)u^{1/3} = \left(\frac{1}{x}\right)u^{-2/3} \Rightarrow \frac{du}{dx} + \left(\frac{3}{x}\right)u = \left(\frac{3}{x}\right)$. The integrating factor is $v(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x}$

$= e^{\ln x^3} = x^3$. Thus $\frac{d}{dx}\left(x^3 u\right) = \left(\frac{3}{x}\right)x^3 = 3x^2 \Rightarrow x^3 u = x^3 + C \Rightarrow u = 1 + \frac{C}{x^3} = y^3 \Rightarrow y = \left(1 + \frac{C}{x^3}\right)^{1/3}$

32. $x^2 y' + 2xy = y^3 \Rightarrow y' + \left(\frac{2}{x}\right)y = \left(\frac{1}{x^2}\right)y^3$. $P(x) = \left(\frac{2}{x}\right), Q(x) = \left(\frac{1}{x^2}\right), n = 3$. Let $u = y^{1-3} = y^{-2}$. Substituting

gives $\frac{du}{dx} + (-2)\left(\frac{2}{x}\right)u = -2\left(\frac{1}{x^2}\right) \Rightarrow \frac{du}{dx} + \left(\frac{-4}{x}\right)u = \frac{-2}{x^2}$. Let the integrating factor, $v(x)$, be

$e^{\int\left(\frac{-4}{x}\right)dx} = e^{\ln x^{-4}} = x^{-4}$. Thus $\frac{d}{dx}\left(x^{-4}u\right) = -2x^{-6} \Rightarrow x^{-4}u = \frac{2}{5}x^{-5} + C \Rightarrow u = \frac{2}{5x} + Cx^4 = y^{-2}$

$\Rightarrow y = \left(\frac{2}{5x} + Cx^4\right)^{-1/2}$

9.3 APPLICATIONS

1. Note that the total mass is $66+7=73$ kg, therefore, $v=v_0e^{-(k/m)t} \Rightarrow v=9e^{-3.9t/73}$

   (a) $s(t)=\int 9e^{-3.9t/73}dt=-\frac{2190}{13}e^{-3.9t/73}+C$

   Since $s(0)=0$ we have $C=\frac{2190}{13}$ and $\lim_{t\to\infty} s(t)=\lim_{t\to\infty}\frac{2190}{13}\left(1-e^{-3.9t/73}\right)=\frac{2190}{13}\approx 168.5$

   The cyclist will coast about 168.5 meters.

   (b) $1=9e^{-3.9/73} \Rightarrow \frac{3.9t}{73}=\ln 9 \Rightarrow t=\frac{73\ln 9}{3.9}\approx 41.13$ s

   It will take about 41.13 seconds.

2. $v=v_0e^{-(k/m)t} \Rightarrow v=9e^{-(59{,}000/51{,}000{,}000)t} \Rightarrow v=9e^{-59t/51{,}000}$

   (a) $s(t)=\int 9e^{-59t/51{,}000}\,dt=-\frac{459{,}000}{59}e^{-59t/51{,}000}+C$

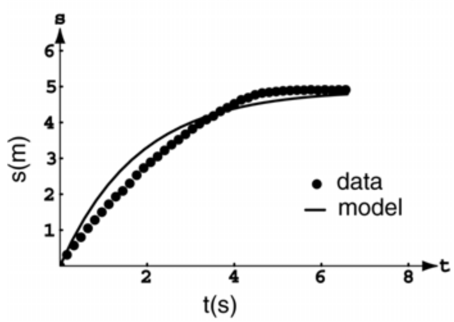
   Since $s(0)=0$ we have $C=\frac{459{,}000}{59}$ and $\lim_{t\to\infty} s(t)=\lim_{t\to\infty}\frac{459{,}000}{59}\left(1-e^{-59t/51{,}000}\right)=\frac{459{,}000}{59}\approx 7780$ m

   The ship sill coast about 7780 m, or 7.78 km.

   (b) $1=9e^{-59t/51{,}000} \Rightarrow \frac{59t}{51{,}000}=\ln 9 \Rightarrow t=\frac{51{,}000\ln 9}{59}\approx 1899.3$ s

   It will take about 31.65 minutes.

3. The total distance traveled $=\frac{v_0m}{k} \Rightarrow \frac{(2.75)(39.92)}{k}=4.91 \Rightarrow k=22.36.$ Therefore, the distance traveled is given by the function $s(t)=4.91\left(1-e^{-(22.36/39.92)t}\right)$. The graph shows $s(t)$ and the data points.

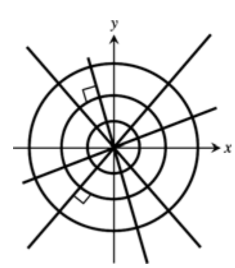


4. $\frac{v_0m}{k}=$ coasting distance $\Rightarrow \frac{(0.80)(49.90)}{k}=1.32 \Rightarrow k=\frac{998}{33}$

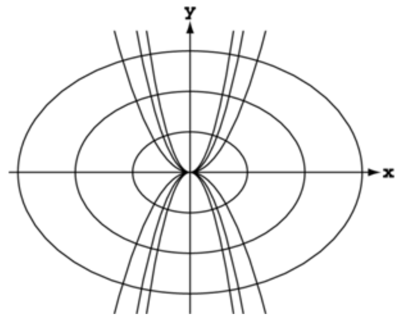
   We know that $\frac{v_0m}{k}=1.32$ and $\frac{k}{m}=\frac{998}{33(49.9)}=\frac{20}{33}$.

   Using Equation 2, we have: $s(t)=\frac{v_0m}{k}\left(1-e^{-(k/m)t}\right)=1.32\left(1-e^{-20t/33}\right)\approx 1.32\left(1-e^{-0.606t}\right)$

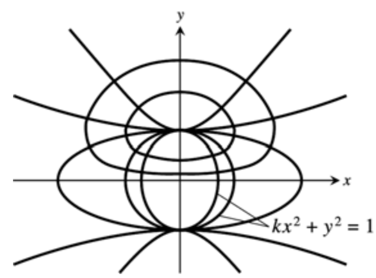
5. $y=mx \Rightarrow \frac{y}{x}=m \Rightarrow \frac{xy'-y}{x^2}=0 \Rightarrow y'=\frac{y}{x}$. So for orthogonals: $\frac{dy}{dx}=-\frac{x}{y} \Rightarrow y\,dy=-x\,dx \Rightarrow \frac{y^2}{2}+\frac{x^2}{2}=C$ $\Rightarrow x^2+y^2=C_1$



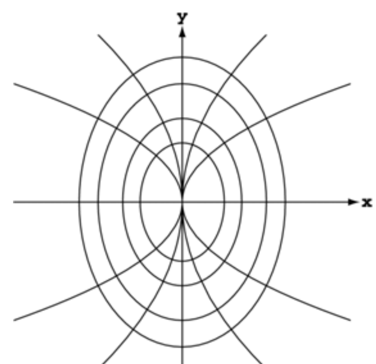
6. $y = cx^2 \Rightarrow \frac{y}{x^2} = c \Rightarrow \frac{x^2 y' - 2xy}{x^4} = 0 \Rightarrow x^2 y' = 2xy$
$\Rightarrow y' = \frac{2y}{x}$. So for the orthogonals: $\frac{dy}{dx} = -\frac{x}{2y}$
$\Rightarrow 2y\,dy = -x\,dx \Rightarrow y^2 = -\frac{x^2}{2} + C \Rightarrow y = \pm\sqrt{-\frac{x^2}{2} + C},$
$C > 0$



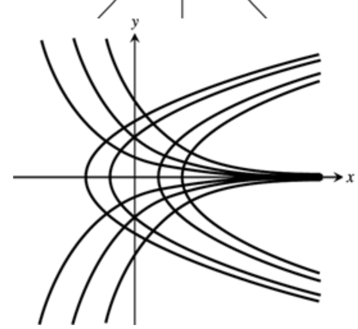
7. $kx^2 + y^2 = 1 \Rightarrow 1 - y^2 = kx^2 \Rightarrow \frac{1-y^2}{x^2} = k$
$\Rightarrow \frac{x^2(2y)y' - \left(1-y^2\right)2x}{x^4} = 0 \Rightarrow -2yx^2 y' = \left(1-y^2\right)(2x)$
$\Rightarrow y' = \frac{\left(1-y^2\right)(2x)}{-2xy^2} = \frac{\left(1-y^2\right)}{-xy}$. So for the orthogonals:
$\frac{dy}{dx} = \frac{xy}{1-y^2} \Rightarrow \frac{\left(1-y^2\right)}{y}dy = x\,dx \Rightarrow \ln y - \frac{y^2}{2} = \frac{x^2}{2} + C$



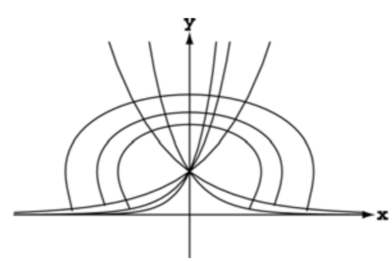
8. $2x^2 + y^2 = c^2 \Rightarrow 4x + 2yy' = 0 \Rightarrow y' = -\frac{4x}{2y} = -\frac{2x}{y}$. For orthogonals: $\frac{dy}{dx} = \frac{y}{2x} \Rightarrow \frac{dy}{y} = \frac{dx}{2x} \Rightarrow \ln y = \frac{1}{2}\ln x + C$
$\Rightarrow \ln y = \ln x^{1/2} + \ln C_1 \Rightarrow y = C_1\,|\,x\,|^{1/2}$



9. $y = ce^{-x} \Rightarrow \frac{y}{e^{-x}} = c \Rightarrow \frac{e^{-x}y' - y\left(e^{-x}\right)(-1)}{\left(e^{-x}\right)^2} = 0$
$\Rightarrow e^{-x}y' = -ye^{-x} \Rightarrow y' = -y$. So for the orthogonals:
$\frac{dy}{dx} = \frac{1}{y} \Rightarrow y\,dy = dx \Rightarrow \frac{y^2}{2} = x + C \Rightarrow y^2 = 2x + C_1$
$\Rightarrow y = \pm\sqrt{2x + C_1}$

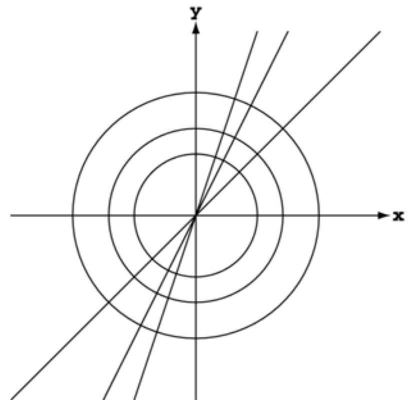


10. $y = e^{kx} \Rightarrow \ln y = kx \Rightarrow \frac{\ln y}{x} = k \Rightarrow \frac{x\left(\frac{1}{y}\right)y' - \ln y}{x^2} = 0$
$\Rightarrow \left(\frac{x}{y}\right)y' - \ln y = 0 \Rightarrow y' = \frac{y \ln y}{x}$. So for the orthogonals: $\frac{dy}{dx} = \frac{-x}{y \ln y} \Rightarrow y \ln y\,dy = -x\,dx$
$\Rightarrow \frac{1}{2}y^2 \ln y - \frac{1}{4}\left(y^2\right) = \left(-\frac{1}{2}x^2\right) + C$
$\Rightarrow y^2 \ln y - \frac{y^2}{2} = -x^2 + C_1$

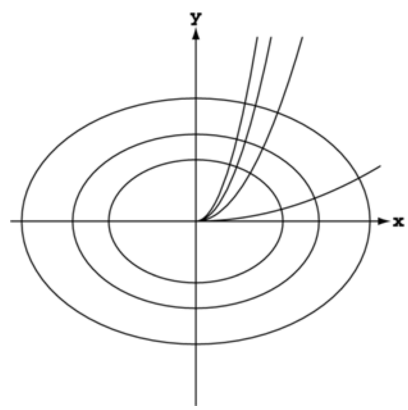


11. $2x^2+3y^2=5$ and $y^2=x^3$ intersect at $(1,1)$. Also, $2x^2+3y^2=5 \Rightarrow 4x+6y\,y'=0 \Rightarrow y'=-\frac{4x}{6y}$ $\Rightarrow y'(1,1)=-\frac{2}{3}$ and $y_1^2=x^3 \Rightarrow 2y_1y_1'=3x^2 \Rightarrow y_1'=\frac{3x^2}{2y_1} \Rightarrow y_1'(1,1)=\frac{3}{2}$. Since $y'\cdot y_1'=\left(-\frac{2}{3}\right)\left(\frac{3}{2}\right)=-1$, the curves are orthogonal.

12. (a) $x\,dx+y\,dy=0 \Rightarrow \frac{x^2}{2}+\frac{y^2}{2}=C$ the general equation of the family with slope $y'=-\frac{x}{y}$.
For the orthogonals: $y'=\frac{y}{x} \Rightarrow \frac{dy}{y}=\frac{dx}{x}$
$\Rightarrow \ln y=\ln x+C$ or $y=C_1x$ (where $C_1=e^C$) is the general equation of the orthogonals.



(b) $x\,dy-2y\,dx=0 \Rightarrow 2y\,dx=x\,dy \Rightarrow \frac{dy}{2y}=\frac{dx}{x}$
$\Rightarrow \frac{1}{2}\left(\frac{dy}{y}\right)=\frac{dx}{x} \Rightarrow \frac{1}{2}\ln y=\ln x+C \Rightarrow y=C_1x^2$ is the equation for the solution family.
$\frac{1}{2}\ln y-\ln x=C \Rightarrow \frac{1}{2}\frac{y'}{y}-\frac{1}{x}=0 \Rightarrow y'=\frac{2y}{x}$
$\Rightarrow$ slope of orthogonals is $\frac{dy}{dx}=-\frac{x}{2y}$
$\Rightarrow 2y\,dy=-x\,dx \Rightarrow y^2=-\frac{x^2}{2}+C$ is the general equation of the orthogonals.



13. Let $y(t)=$ the amount of salt in the container and $V(t)=$ the total volume of liquid in the tank at time t. Then, the departure rate is $\frac{y(t)}{V(t)}$ (the outflow rate).

(a) Rate entering $=0.2\frac{\text{kg}}{\text{L}}\cdot 20\frac{\text{L}}{\text{min}}=4\frac{\text{kg}}{\text{min}}$

(b) Volume $=V(t)=400\text{ L}+(20t\text{ L}-16t\text{ L})=(400+4t)\text{L}$

(c) The volume at time t is $(400+4t)$ L. The amount of salt in the tank at time t is y kgs. So the concentration at any time t is $\frac{y}{400+4t}\frac{\text{kg}}{\text{L}}$. Then, the rate leaving $=\frac{y}{400+4t}\left(\frac{\text{kg}}{\text{L}}\right)\cdot 16\left(\frac{\text{L}}{\text{min}}\right)=\frac{16y}{400+4t}\left(\frac{\text{kg}}{\text{min}}\right)$

(d) $\frac{dy}{dt}=4-\frac{4y}{100+t} \Rightarrow \frac{dy}{dt}+\left(\frac{4}{100+t}\right)y=4 \Rightarrow P(t)=\frac{4}{100+t},\ Q(t)=4 \Rightarrow \int P(t)\,dt=\int\frac{4}{100+t}\,dt=4\ln(100+t)$
$\Rightarrow v(t)=e^{4\ln(100+t)}=(100+t)^4 \Rightarrow y=\frac{1}{(100+t)^4}\int(100+t)^4(4\,dt)=\frac{4}{(100+t)^4}\left(\frac{(100+t)^5}{5}+C\right)$
$=0.8(100+t)+\frac{C}{(100+t)^4};\ \ y(0)=20 \Rightarrow 0.8(100+0)+\frac{C}{(100+0)^4}=20 \Rightarrow C=-(60)(100)^4$
$\Rightarrow y=0.8(100+t)-\frac{(60)(100)^4}{(100+t)^4} \Rightarrow y=0.8(100+t)-\frac{60}{\left(1+\frac{t}{100}\right)^4}$

(e) $y(25)=0.8(100+25)-\frac{(60)(100)^4}{(100+25)^4}\approx 75.424\text{ kg} \Rightarrow \text{concentration}=\frac{y(25)}{\text{volume}}\approx\frac{75.424}{500}\approx 0.151\frac{\text{kg}}{\text{L}}$

14. (a) $\frac{dV}{dt} = (20-12) = 8 \Rightarrow V = 400 + 8t$

The tank is full when $V = 800 = 400 + 8t \Rightarrow t = 50\,\text{min}$

(b) Let $y(t)$ be the amount of concentrate in the tank at time t.

$\frac{dy}{dt} = \left(\frac{1}{20}\,\frac{\text{kg}}{\text{L}}\right)\left(20\,\frac{\text{L}}{\text{min}}\right) - \left(\frac{y}{400+8t}\,\frac{\text{kg}}{\text{L}}\right)\left(12\,\frac{\text{L}}{\text{min}}\right) \Rightarrow \frac{dy}{dt} = 1 - \frac{3}{2}\left(\frac{y}{50+t}\right) \Rightarrow \frac{dy}{dt} + \frac{3}{2(t+50)}\,y = 1 \quad Q(t) = 1;$

$P(t) = \frac{3}{2}\left(\frac{1}{t+50}\right) \Rightarrow \int P(t)\,dt = \frac{3}{2}\int \frac{1}{t+50}\,dt = \frac{3}{2}\ln(t+50)$ since $t + 50 > 0 \Rightarrow v(t) = e^{\int P(t)\,dt} = e^{\frac{3}{2}\ln(t+50)}$

$= (t+50)^{3/2} \Rightarrow y(t) = \frac{1}{(t+50)^{3/2}}\int 1(t+50)^{3/2}\,dt = \frac{2}{5}(t+50)^{-3/2}\left[(t+50)^{5/2} + C\right] \Rightarrow y(t) = \frac{2}{5}t + 20 + \frac{\frac{2}{5}C}{(t+50)^{3/2}}$

Apply the initial condition (i.e., distilled water in the tank at $t = 0$):

$y(0) = 0 = 20 + \frac{\frac{2}{5}C}{50^{3/2}} \Rightarrow C = -50^{5/2} \Rightarrow y(t) = \frac{2}{5}t + 20 - \frac{\frac{2}{5}50^{5/2}}{(t+50)^{3/2}}$. When the tank is full at $t = 50$,

$y(50) = 40 - \frac{\frac{2}{5}50^{5/2}}{100^{3/2}} \approx 32.93$ kgs of concentrate.

15. Let y be the amount of fertilizer in the tank at time t. Then rate entering $= 0.1\,\frac{\text{kg}}{\text{L}} \cdot 4\,\frac{\text{L}}{\text{min}} = 0.4\,\frac{\text{kg}}{\text{min}}$ and the volume

in the tank at time t is $V(t) = 400\ (\text{L}) + \left[4\left(\frac{\text{L}}{\text{min}}\right) - 12\left(\frac{\text{L}}{\text{min}}\right)\right] t\ \text{min} = (400 - 8t)$ L. Hence rate out

$= \left(\frac{y}{100-2t}\right)3 = \frac{3y}{100-2t}\,\frac{\text{kg}}{\text{min}} \Rightarrow \frac{dy}{dt} = \left(0.4 - \frac{3y}{100-2t}\right)\frac{\text{kg}}{\text{min}} \Rightarrow \frac{dy}{dt} + \left(\frac{3}{100-2t}\right)y = 0.4 \Rightarrow P(t) = \frac{3}{100-2t},\ \ Q(t) = 0.4$

$\Rightarrow \int P(t)\,dt = \int \frac{3}{100-2t}\,dt = \frac{3\ln(100-2t)}{-2} \Rightarrow v(t) = e^{(-3\ln(100-2t))/2} = (100-2t)^{-3/2}$

$\Rightarrow y = \frac{0.4}{(100-2t)^{-3/2}}\int (100-2t)^{-3/2}\,dt = 0.4(100-2t)^{-3/2}\left[\frac{-2(100-2t)^{-1/2}}{-2} + C\right] = 0.4(100-2t) + 0.4C(100-2t)^{3/2};$

$y(0) = 0 \Rightarrow [100 - 2(0)] + C[100-2(0)]^{3/2} \Rightarrow C(100)^{3/2} = -100 \Rightarrow C = -(100)^{-1/2} = -\frac{1}{10}$

$\Rightarrow y = 0.4(100-2t) - \frac{(100-2t)^{3/2}}{25}$. Let $\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -0.8 - \frac{\left(\frac{3}{2}\right)(100-2t)^{1/2}(-2)}{25} = -0.8 + \frac{3\sqrt{100-2t}}{25} = 0$

$\Rightarrow 20 = 3\sqrt{100-2t} \Rightarrow 400 = 9(100-2t) \Rightarrow 400 = 900 - 18t \Rightarrow -500 = -18t \Rightarrow t \approx 27.8\,\text{min}$, the time to reach the maximum. The maximum amount is then $y(27.8) = [40 - 0.8(27.8)] - \frac{[100-2(27.8)]^{3/2}}{25} \approx 5.93$ kg.

16. Let $y = y(t)$ be the amount of carbon monoxide (CO) in the room at time t. The amount of CO entering the room is $\left(\frac{4}{100} \times \frac{8}{1000}\right) = \frac{32}{100{,}000}\,\frac{\text{m}^3}{\text{min}}$, and the amount of CO leaving the room is $\left(\frac{y}{120}\right)\left(\frac{8}{1000}\right) = \frac{y}{15{,}000}\,\frac{\text{m}^3}{\text{min}}$.

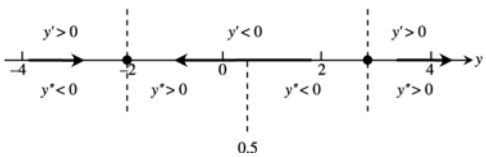
Thus, $\frac{dy}{dt} = \frac{32}{100{,}000} - \frac{y}{15{,}000} \Rightarrow \frac{dy}{dt} + \frac{1}{15{,}000}\,y = \frac{32}{100{,}000} \Rightarrow P(t) = \frac{1}{15{,}000},\ \ Q(t) = \frac{32}{100{,}000} \Rightarrow v(t) = e^{t/15{,}000}$

$\Rightarrow y = \frac{1}{e^{t/15{,}000}}\int \frac{32}{100{,}000}\,e^{t/15{,}000}\,dt \Rightarrow y = e^{-t/15{,}000}\left(\frac{32 \cdot 15{,}000}{100{,}000}\,e^{t/15{,}000} + C\right) = e^{-t/15{,}000}\left(4.8e^{t/15{,}000} + C\right);$

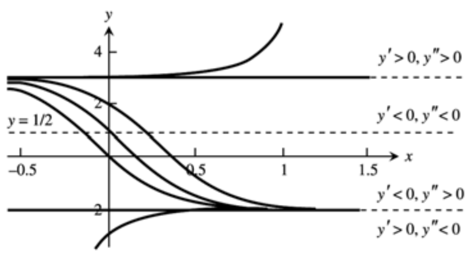
$y(0) = 0 \Rightarrow 0 = 1(4.8 + C) \Rightarrow C = -4.8 \Rightarrow y = 4.8 - 4.8e^{-t/15{,}000}$. When the concentration of CO is 0.01% in the room, the amount of CO satisfies $\frac{y}{120} = \frac{.01}{100} \Rightarrow y = 0.012\ \text{m}^3$. When the room contains this amount we have $0.012 = 4.8 - 4.8e^{-t/15{,}000} \Rightarrow \frac{4.788}{4.8} = e^{-t/15{,}000} \Rightarrow t = -15{,}000\ln\left(\frac{4.788}{4.8}\right) \approx 37.55\,\text{min}$.

9.4 GRAPHICAL SOLUTIONS OF AUTONOMOUS EQUATIONS

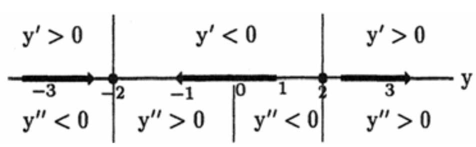
1. $y' = (y+2)(y-3)$
   (a) $y = -2$ is a stable equilibrium value and $y = 3$ is an unstable equilibrium.
   (b) $y'' = (2y-1)y' = 2(y+2)\left(y-\frac{1}{2}\right)(y-3)$



   (c)



2. $y' = (y+2)(y-2)$
   (a) $y = -2$ is a stable equilibrium value and $y = 2$ is an unstable equilibrium.
   (b) $y'' = 2yy' = 2(y+2)y(y-2)$



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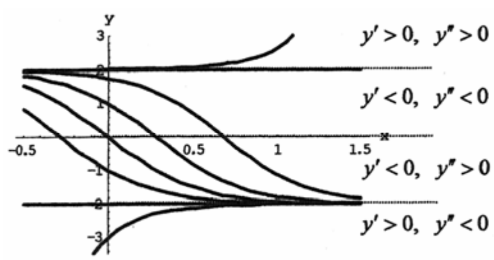
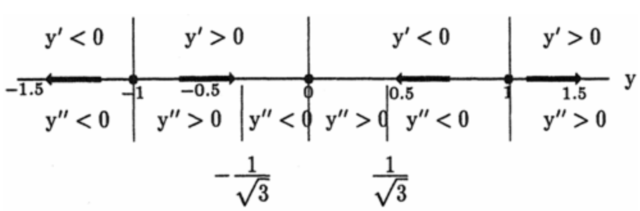


3. $y' = y^3 - y = (y+1)y(y-1)$
   (a) $y = -1$ and $y = 1$ are unstable equilibria and $y = 0$ is a stable equilibrium value.
   (b) $y'' = \left(3y^2 - 1\right)y' = 3(y+1)\left(y+\frac{1}{\sqrt{3}}\right)y\left(y-\frac{1}{\sqrt{3}}\right)(y-1)$



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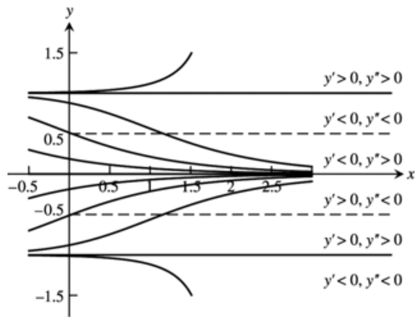
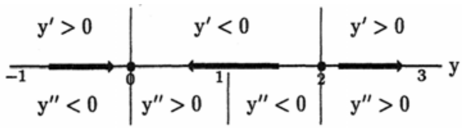


4. $y' = y(y-2)$

(a) $y = 0$ is a stable equilibrium value and $y = 2$ is an unstable equilibrium.

(b) $y'' = (2y-2)y' = 2y(y-1)(y-2)$



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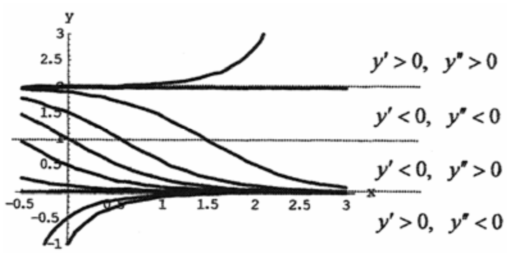
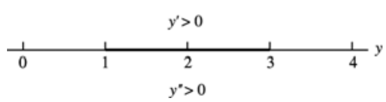


5. $y' = \sqrt{y},\ y > 0$

(a) There are no equilibrium values.

(b) $y'' = \frac{1}{2\sqrt{y}}\, y' = \frac{1}{2\sqrt{y}}\sqrt{y} = \frac{1}{2}$



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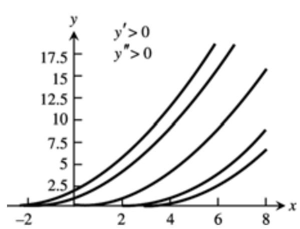
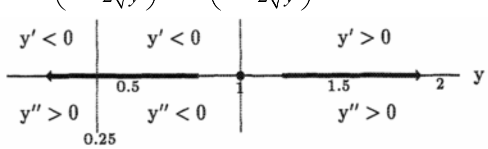


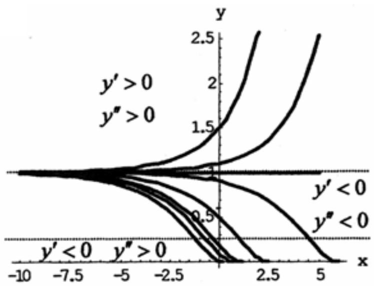
6. $y' = y - \sqrt{y},\ y > 0$

(a) $y = 1$ is an unstable equilibrium.

(b) $y'' = \left(1 - \frac{1}{2\sqrt{y}}\right)y' = \left(1 - \frac{1}{2\sqrt{y}}\right)\left(y - \sqrt{y}\right) = \left(\sqrt{y} - \frac{1}{2}\right)\left(\sqrt{y} - 1\right)$



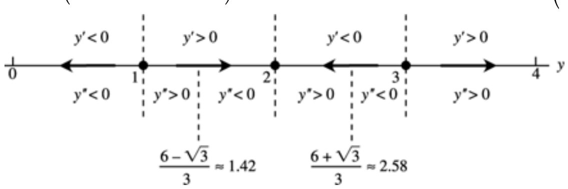
(c)



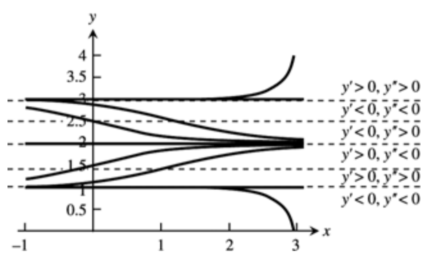
7. $y' = (y-1)(y-2)(y-3)$

(a) $y = 1$ and $y = 3$ are unstable equilibria and $y = 2$ is a stable equilibrium value.

(b) $y'' = \left(3y^2 - 12y + 11\right)(y-1)(y-2)(y-3) = 3(y-1)\left(y - \frac{6-\sqrt{3}}{3}\right)(y-2)\left(y - \frac{6+\sqrt{3}}{3}\right)(y-3)$



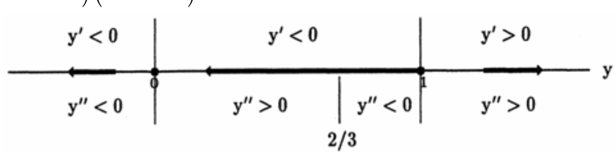
(c)



8. $y' = y^3 - y^2 = y^2(y-1)$

(a) $y = 0$ and $y = 1$ are unstable equilibria.

(b) $y'' = \left(3y^2 - 2y\right)\left(y^3 - y^2\right) = y^3(3y-2)(y-1)$



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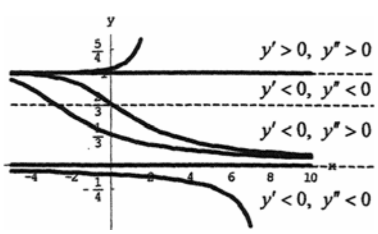
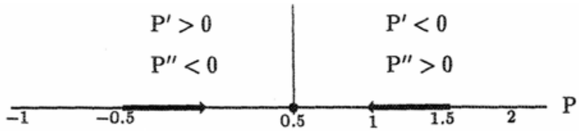
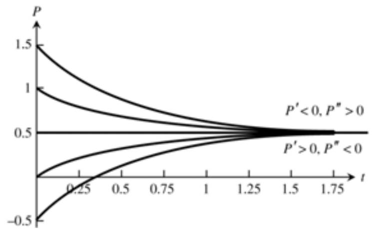
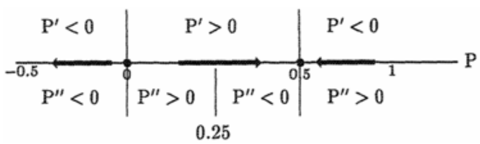


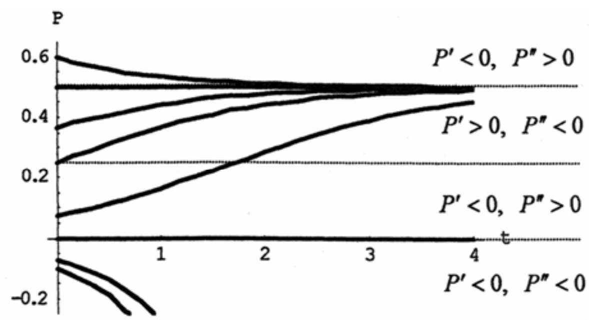
9. $\frac{dp}{dt} = 1 - 2P$ has a stable equilibrium at $P = \frac{1}{2}$. $\frac{d^2P}{dt^2} = -2\frac{dp}{dt} = -2(1 - 2P)$



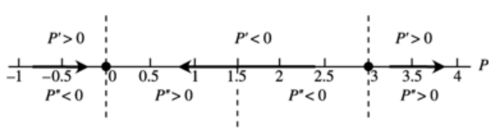


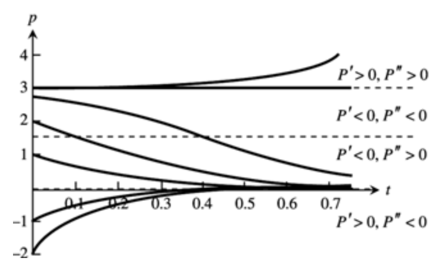
10. $\frac{dP}{dt} = P(1 - 2P)$ has an unstable equilibrium at $P = 0$ and a stable equilibrium at $P = \frac{1}{2}$.
$\frac{d^2P}{dt^2} = (1 - 4P)\frac{dP}{dt} = P(1 - 4P)(1 - 2P)$





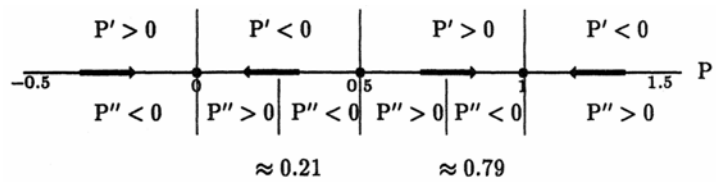
11. $\frac{dP}{dt} = 2P(P - 3)$ has a stable equilibrium at $P = 0$ and an unstable equilibrium at $P = 3$.
$\frac{d^2P}{dt^2} = 2(2P - 3)\frac{dP}{dt} = 4P(2P - 3)(P - 3)$

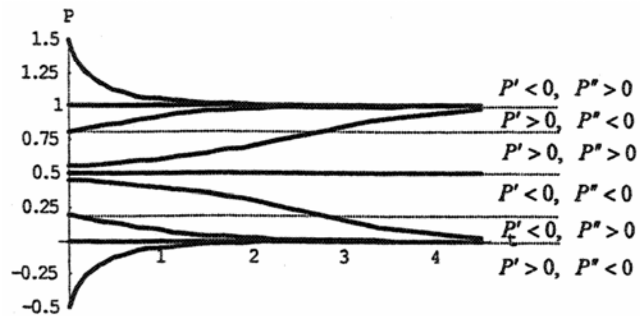




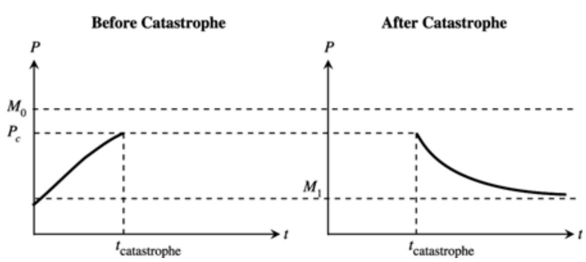
12. $\frac{dP}{dt} = 3P(1-P)\left(P-\frac{1}{2}\right)$ has a stable equilibria at $P=0$ and $P=1$ and an unstable equilibrium at $P=\frac{1}{2}$.

$\frac{d^2P}{dt^2} = -\frac{3}{2}\left(6P^2-6P+1\right)\frac{dP}{dt} = \frac{3}{2}P\left(P-\frac{3-\sqrt{3}}{6}\right)\left(P-\frac{1}{2}\right)\left(P-\frac{3+\sqrt{3}}{6}\right)(P-1)$



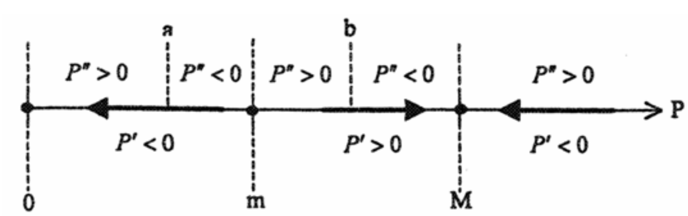


13.



Before the catastrophe, the population exhibits logistic growth and $P(t) \to M_0$, the stable equilibrium. After the catastrophe, the population declines logistically and $P(t) \to M_1$, the new stable equilibrium.

14. $\frac{dP}{dt} = rP(M-P)(P-m), r, M, m > 0$



The model has 3 equilibrium points. The rest point $P=0$, $P=M$ are asymptotically stable while $P=m$ is unstable. For initial populations greater than m, the model predicts P approaches M for large t. For initial populations less than m, the model predicts extinction. Points of inflection occur at $P=a$ and $P=b$ where $a = \frac{1}{3}\left[M+m-\sqrt{M^2-mM+m^2}\right]$ and $b = \frac{1}{3}\left[M+m+\sqrt{M^2-mM+m^2}\right]$.

(a) The model is reasonable in the sense that if $P<m$, then $P \to 0$ as $t \to \infty$; if $m<P<M$, then $P \to M$ as $t \to \infty$; if $P>M$, then $P \to M$ as $t \to \infty$.

(b) It is different if the population falls below m, for then $P \to 0$ as $t \to \infty$ (extinction). It is probably a more realistic model for that reason because we know some populations have become extinct after the population level became too low.

(c) For $P > M$ we see that $\frac{dP}{dt} = rP(M-P)(P-m)$ is negative. Thus the curve is everywhere decreasing. Moreover, $P \equiv M$ is a solution to the differential equation. Since the equation satisfies the existence and uniqueness conditions, solution trajectories cannot cross. Thus, $P \to M$ as $t \to \infty$.

(d) See the initial discussion above.

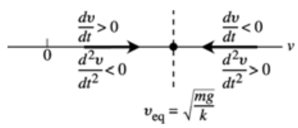
(e) See the initial discussion above.

15. $\frac{dv}{dt} = g - \frac{k}{m}v^2, g, k, m > 0$ and $v(t) \geq 0$

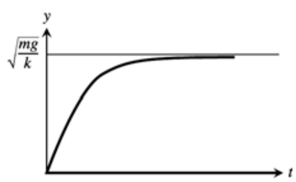
Equilibrium: $\frac{dv}{dt} = g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{mg}{k}}$

Concavity: $\frac{d^2y}{dt^2} = -2\left(\frac{k}{m}v\right)\frac{dv}{dt} = -2\left(\frac{k}{m}v\right)\left(g - \frac{k}{m}v^2\right)$

(a)



(b)



(c) $V_{\text{terminal}} = \sqrt{\frac{441}{0.2}} = 46.96\,\frac{\text{m}}{\text{s}} = 169$ km/h for the 45-kg skydiver

$V_{\text{terminal}} = \sqrt{\frac{784}{0.2}} = 62.6\,\frac{\text{m}}{\text{s}} \approx 225$ km/h for the 80-kg skydiver

16. $F = F_p - F_r$

$ma = mg - k\sqrt{v}$

$\frac{dv}{dt} = g - \frac{k}{m}\sqrt{v}, v(0) = v_0$

Thus, $\frac{dv}{dt} = 0$ implies $v = \left(\frac{mg}{k}\right)^2$, the terminal velocity. If $v_0 < \left(\frac{mg}{k}\right)^2$, the object will fall faster and faster, approaching the terminal velocity; if $v_0 > \left(\frac{mg}{k}\right)^2$, the object will slow down to the terminal velocity.

17. $F = F_p - F_r$

$ma = 200 - 50|v|$

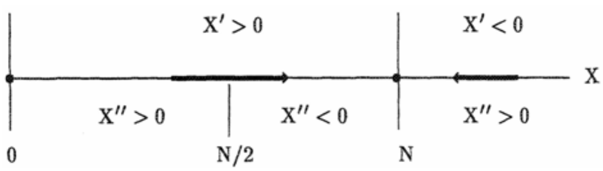
$\frac{dv}{dt} = \frac{1}{m}(200 - 50|v|)$

The maximum velocity occurs when $\frac{dv}{dt} = 0$ or $v = 4\,\frac{\text{m}}{\text{s}}$.

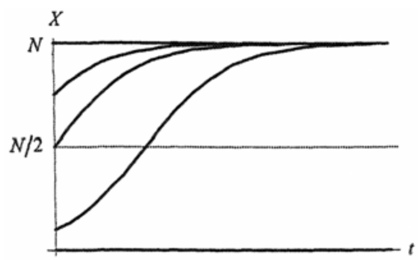
18. (a) The model seems reasonable because the rate of spread of a piece of information, an innovation, or a cultural fad is proportional to the product of the number of individuals who have it (X) and those who do not $(N-X)$. When X is small, there are only a few individuals to spread the item so the rate of spread is slow. On the other hand, when $(N-X)$ is small the rate of spread will be slow because there are only a few individuals who can receive it during the interval of time. The rate of spread will be fastest when

both X and $(N-X)$ are large because then there are a lot of individuals to spread the item and a lot of individuals to receive it.

(b) There is a stable equilibrium at $X = N$ and an unstable equilibrium at $X = 0$.
$\frac{d^2x}{dt^2} = k\frac{dx}{dt}(N-X) - kX\frac{dx}{dt} = k^2X(N-X)(N-2X) \Rightarrow$ inflection point at $X = 0$, $X = \frac{N}{2}$, and $X = N$.



(c)



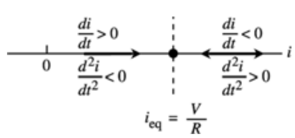
(d) The spread rate is most rapid when $x = \frac{N}{2}$. Eventually all of the people will receive the item.

19. $L\frac{di}{dt} + Ri = V \Rightarrow \frac{di}{dt} = \frac{V}{L} - \frac{R}{L}i = \frac{R}{L}\left(\frac{V}{R} - i\right), V, L, R > 0$

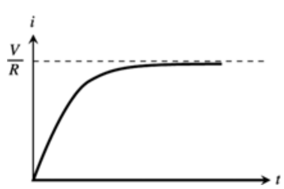
Equilibrium: $\frac{di}{dt} = \frac{R}{L}\left(\frac{V}{R} - i\right) = 0 \Rightarrow i = \frac{V}{R}$

Concavity: $\frac{d^2i}{dt^2} = -\left(\frac{R}{L}\right)\frac{di}{dt} = -\left(\frac{R}{L}\right)^2\left(\frac{V}{R} - i\right)$

Phase Line:

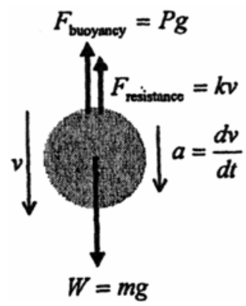


If the switch is closed at $t = 0$, then $i(0) = 0$, and the graph of the solution looks like this:



As $t \to \infty$, it $\to i_{\text{steady state}} = \frac{V}{R}$. (In the steady state condition, the self-inductance acts like a simple wire connector and, as a result, the current through the resistor can be calculated using the familiar version of Ohm's Law.)

20. (a) Free body diagram of the pearl:

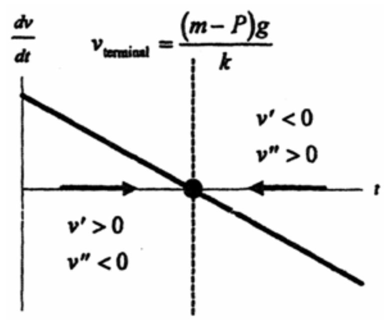


(b) Use Newton's Second Law, summing forces in the direction of the acceleration:
$mg - Pg - kv = ma \Rightarrow \frac{dv}{dt} = \left(\frac{m-P}{m}\right)g - \frac{k}{m}v.$

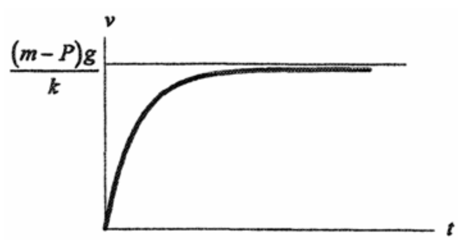
(c) Equilibrium: $\frac{dv}{dt} = \frac{k}{m}\left(\frac{(m-P)g}{k} - v\right) = 0$

$\Rightarrow v_{\text{terminal}} = \frac{(m-P)g}{k}$

Concavity: $\frac{d^2v}{dt^2} = -\frac{k}{m}\frac{dv}{dt} = -\left(\frac{k}{m}\right)^2\left(\frac{(m-P)g}{k} - v\right)$



(d)



(e) The terminal velocity of the pearl is $\frac{(m-P)g}{k}$.

9.5 SYSTEMS OF EQUATIONS AND PHASE PLANES

1. Seasonal variations, nonconformity of the environments, effects of other interactions, unexpected disasters, etc.

2. $x = r\cos\theta \Rightarrow \frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dr}{dt} = y + x - x\left(x^2+y^2\right) = r\sin\theta + r\cos\theta - r^3\cos\theta$

   $y = r\sin\theta \Rightarrow \frac{dy}{dt} = r\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{dr}{dt} = -x + y - y\left(x^2+y^2\right) = -r\cos\theta + r\sin\theta - r^3\sin\theta$

   Solve for $\frac{dr}{dt}$ by adding $\cos\theta \times \text{eq}(1)$ to to $\sin\theta\times\text{eq}(2)$:

   $\cos^2\theta\frac{dr}{dt} + \sin^2\theta\frac{dr}{dt} = \cos\theta\left(r\sin\theta + r\cos\theta - r^3\cos\theta\right) + \sin\theta\left(-r\cos\theta + r\sin\theta - r^3\sin\theta\right)$

   $\Rightarrow \frac{dr}{dt} = r\sin\theta\cos\theta + r\cos^2\theta - r^3\cos^2\theta - r\sin\theta\cos\theta + r\sin^2\theta - r^3\sin^2\theta = r - r^3 = r\left(1-r^2\right)$

   Solve for $\frac{d\theta}{dt}$ by adding $(-\sin\theta)\times\text{eq}(1)$ to $\cos\theta\times\text{eq}(2)$: eq(1) to $\cos\theta\times\text{eq}(2)$:

   $r\sin^2\theta\frac{d\theta}{dt} + r\cos^2\theta\frac{d\theta}{dt} = -\sin\theta\left(r\sin\theta + r\cos\theta - r^3\cos\theta\right) + \cos\theta\left(-r\cos\theta + r\sin\theta - r^3\sin\theta\right)$

   $\Rightarrow r\frac{d\theta}{dt} = -r\sin^2\theta - r\sin\theta\cos\theta + r^3\sin\theta\cos\theta - r\cos^2\theta + r\sin\theta\cos\theta - r^3\sin\theta\cos\theta = -r \Rightarrow \frac{d\theta}{dt} = -1$

   If $r = 1$ (that is, the trajectory starts on the circle $x^2 + y^2 = 1$), then $\left.\frac{dr}{dt}\right|_{r=1} = (1)\left(1-(1)^2\right) = 0,$ thus the trajectory remains on the circle, and rotates around the circle in a clockwise direction, since $\frac{d\theta}{dt} = -1.$ The solution is periodic since at any point (x, y) on the trajectory, $(x, y) = \left(r\cos\theta, r\sin\theta\right) = \left(1\cos\theta, 1\sin\theta\right)$ $= \left(\cos\theta, \sin\theta\right) \Rightarrow$ both x and y are periodic.

3. This model assumes that the number of interactions is proportional to the product of x and y:

   $\frac{dx}{dt} = (a - b\,y)\,x,\ a < 0,$

   $\frac{dy}{dt} = m\left(1-\frac{y}{M}\right)y - n\,x\,y = y\left(m - \frac{m}{M}y - nx\right).$

   To find the equilibrium points:

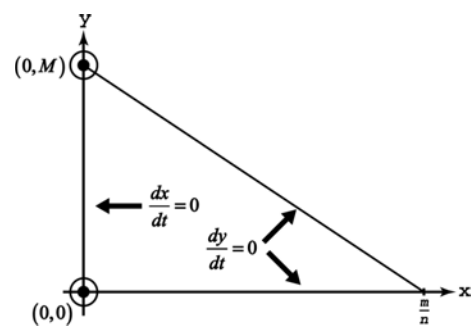
   $\frac{dx}{dt} = 0 \Rightarrow (a - b\,y)x = 0 \Rightarrow x = 0$ or $y = \frac{a}{b}$

   (remember $\frac{a}{b} < 0$);

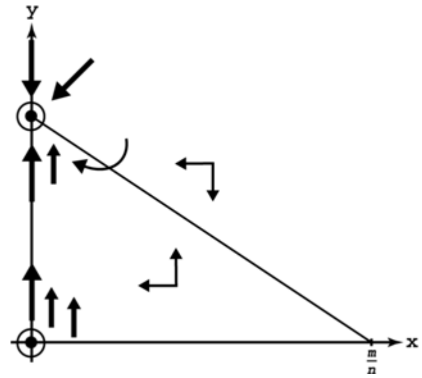
   $\frac{dy}{dt} = 0 \Rightarrow y\left(m - \frac{m}{M}y - n\,x\right) \Rightarrow y = 0$ .or

   $y = -\frac{Mn}{m}x + M;$

   Thus there are two equlibrium points, both occur when $x = 0, (0, 0)$ and $(0, M)$.



Implies coexistence is not possible because eventually trout die out and bass reach their population limit.



4. The coefficients a, b, m, and n need to be determined by sampling or by analyzing historical data. Then, more specific graphical predictions can be made. These predictions would then have to be compared to actual population growth patterns. If the predictions match actual results, we have partially validated our model. If necessary, more tests could be run. However, it should be remembered that the primary purpose of a graphical analysis is to analyze the behavior qualitatively. With reference to Figure 9.29, attempt to maintain the fish populations in Region B through stocking and regulation (open and closed seasons). For example, should Regions A or D be entered, restocking the appropriate species can cause a return to Region B.

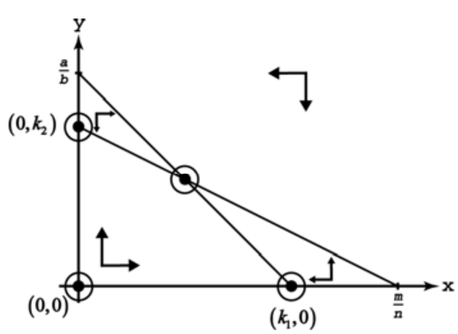
5. (a) Logistic growth occurs in the absence of the competitor, and simple interaction of the species: growth dominates the competition when either population is small so it is difficult to drive either species to extinction.

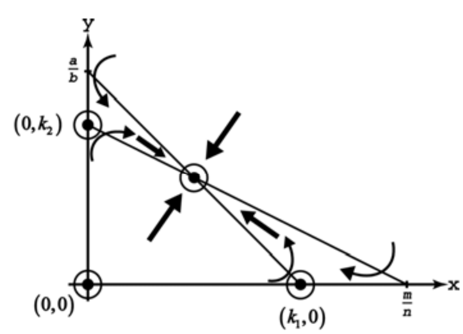
   (b) $a =$ per capita growth rate for trout
   $m =$ per capita growth rate for bass
   $b =$ intensity of competition to the trout
   $n =$ intensity of competition to the bass
   $k_1 =$ environmental carrying capacity for the trout
   $k_2 =$ environmental carrying capacity for the bass

   (c) $\frac{dx}{dt} = 0 \Rightarrow a\left(1-\frac{x}{k_1}\right)x - bxy = \left[a\left(1-\frac{x}{k_1}\right) - by\right]x = 0 \Rightarrow x = 0$ or $a\left(1-\frac{x}{k_1}\right) - by = 0 \Rightarrow x = 0$ or $y = \frac{a}{b} - \frac{a}{bk_1}x$; $\frac{dy}{dt} = 0 \Rightarrow m\left(1-\frac{y}{k_2}\right)y - nxy = \left[m\left(1-\frac{y}{k_2}\right) - nx\right]y = 0 \Rightarrow y = 0$ or $m\left(1-\frac{y}{k_2}\right) - nx = 0$ $\Rightarrow y = 0$ or $y = k_2 - \frac{nk_2}{m}x$. There are five cases to consider.

   Case I: $\frac{a}{b} > k_2$ and $\frac{m}{n} > k_1$.

   By picking $\frac{a}{b} > k_2$ and $\frac{m}{n} > k_1$ we ensure an equilibrium point exists inside the first quadrant.

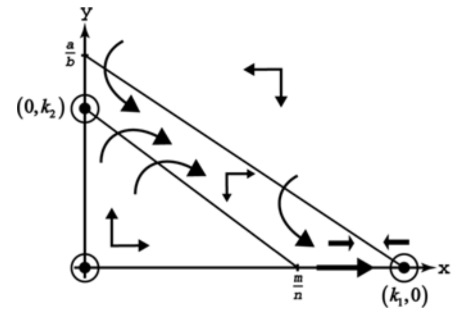




Graphical analysis implies four equilibrium points exist: $(0, 0), (k_1, 0), (0, k_2)$, and $\left(\frac{a\,m\,k_1 - b\,m\,k_1 k_2}{a\,m - b\,n\,k_1 k_2}, \frac{a\,m\,k_2 - a\,n\,k_1 k_2}{a\,m - b\,n\,k_1 k_2}\right)$ (the point of intersection of the two boundaries in the first quadrant). All of these equilibrium points are unstable except for the point of intersection. The possibility of coexistence is predicted by this model.

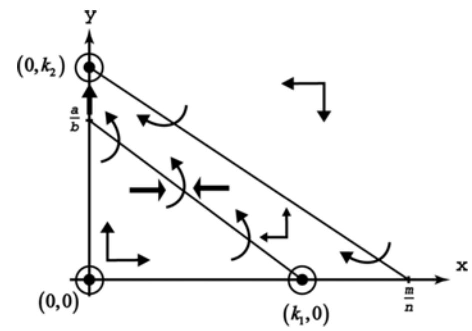
Case II: $\frac{a}{b} > k_2$ and $\frac{m}{n} < k_1$.

$(0, k_2)$: unstable
$(k_1, 0)$: stable
$(0, 0)$: unstable
Trout wins: $(k_1, 0)$
Not sensitive
No coexistence



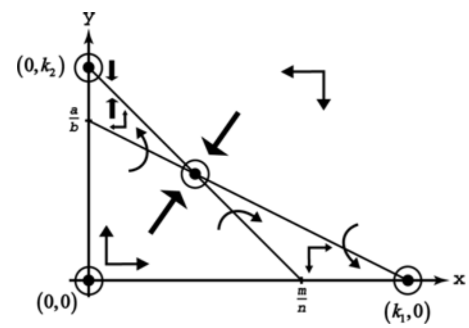
Case III: $\frac{a}{b} < k_2$ and $\frac{m}{n} > k_1$.

$(0, k_2)$: stable
$(k_1, 0)$: unstable
$(0, 0)$: unstable
Bass wins: $(0, k_2)$
Not sensitive
No coexistence



Case IV: $\frac{a}{b} < k_2$ and $\frac{m}{n} < k_1$.

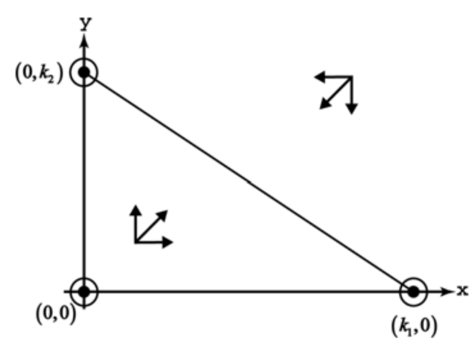
$(0, k_2)$: stable
$(k_1\ 0)$: stable
$(0, 0)$: unstable
$\left(\frac{a\,m\,k_1 - b\,m\,k_1 k_2}{a\,m - b\,n\,k_1 k_2}, \frac{a\,m\,k_2 - a\,n\,k_1 k_2}{a\,m - b\,n\,k_1 k_2}\right)$: unstable
Bass or trout: $(0, k_2)$ or $(k_1, 0)$
Very sensitive
Coexistence is <u>possible</u> but not predicted



If we assume $\frac{a}{b} < k_2$ and $\frac{m}{n} < k_1$ then graphical analysis implies four equilibrium points exist: $(0, k_2), (k_1, 0), (0, 0)$, and $\left(\frac{a\,m\,k_1 - b\,m\,k_1 k_2}{a\,m - b\,n\,k_1 k_2}, \frac{a\,m\,k_2 - a\,n\,k_1 k_2}{a\,m - b\,n\,k_1 k_2}\right)$ (the point of intersection of the two boundaries in the first quadrant).

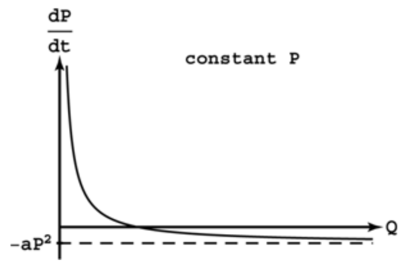
Case V: $\frac{a}{b} = k_2$ and $\frac{a}{b\,k_1} = \frac{n\,k_2}{m}$ (lines coincide).

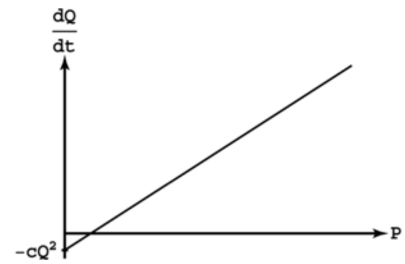
$(0, k_2)$: stable
$(k_1, 0)$: stable
$(0, 0)$: unstable
Line segment joining $(0, k_2)$ and $(k_1, 0)$: stable
Bass wins: $(0, k_2)$
Not sensitive
Coexistence is likely outcome



Note that all points on the line segment joining $(0, k_2)$ and $(k_1, 0)$ are rest points.

6. For a fixed price, as Q increases, $\frac{dp}{dt}$ gets smaller and, possibly, becomes negative. This observation implies that as the quantity supplied increases, the price will not rise as fast. If Q gets high enough, then the price will decrease. Next, consider $\frac{dQ}{dt}$: For a fixed quantity, as P increases, $\frac{dQ}{dt}$ gets larger. Thus, as the market price increases, the quantity supplied will increase at a faster rate. If P is too small, $\frac{dQ}{dt}$ will be negative and the quantity supplied will decrease. This observation is the traditional explanation of the effect of market price levels on the quantity supplied.

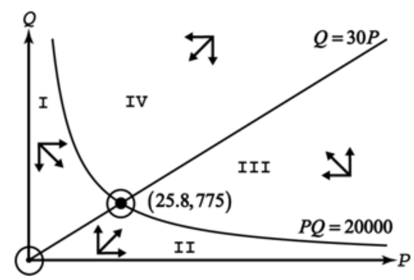




(a) $\frac{dP}{dt} = 0$ and $\frac{dQ}{dt} = 0$ gives the equilibrium points (P, Q): $(0, 0)$ and $(25.8, 775)$. Now $\frac{dP}{dt} > 0$ when $PQ < 20{,}000$ and $P > 0$; $\frac{dP}{dt} < 0$ otherwise. $\frac{dQ}{dt} > 0$ when $P > \frac{Q}{30}$ and $Q > 0$; $\frac{dQ}{dt} < 0$ otherwise.

(b) These considerations give the following graphical analysis:

Region	**dP/dt**	**dQ/dt**
I	> 0	< 0
II	> 0	> 0
III	< 0	< 0
IV	< 0	> 0



The equilibrium point $(0, 0)$ is unstable. The graphical analysis for the point $(25.8, 775)$ is inconclusive: trajectories near the point may be periodic, or may spiral toward or away from the point.

(c) The curve $\frac{dP}{dt} = 0$ or $PQ = 20000$ can be thought of as the demand curve; $\frac{dQ}{dt} = 0$ or $Q = 30P$ can be viewed as the supply curve.

7. (a) $\frac{dx}{dt} = ax - bxy = (a - by)x$ and $\frac{dy}{dt} = my - nxy = (m - nx)y \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(m-nx)y}{(a-by)x}$

(b) $\frac{dy}{dx} = \frac{(m-nx)y}{(a-by)x} \Rightarrow \left(\frac{a}{y} - b\right) dy = \left(\frac{m}{x} - n\right) dx \Rightarrow \int \left(\frac{a}{y} - b\right) dy = \int \left(\frac{m}{x} - n\right) dx \Rightarrow a \ln|y| - by = m \ln|x| - nx + C$

$\Rightarrow \ln\left|y^a\right| + \ln e^{-by} = \ln\left|x^m\right| + \ln e^{-nx} + \ln e^C \Rightarrow \ln\left|y^a e^{-by}\right| = \ln\left|x^m e^{-nx} e^C\right| \Rightarrow y^a e^{-by} = x^m e^{-nx} e^C$,

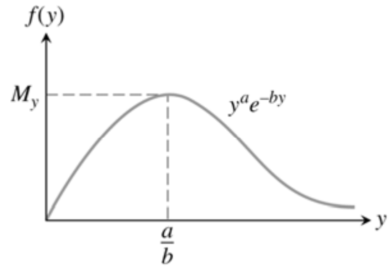
let $K = e^C \Rightarrow y^a e^{-by} = Kx^m e^{-nx}$

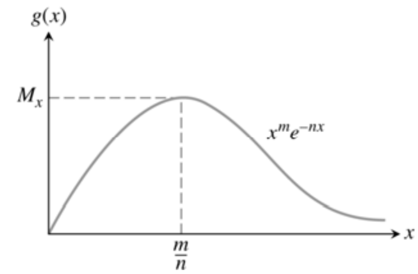
(c) $f(y) = y^a e^{-by} \Rightarrow f'(y) = ay^{a-1}e^{-by} - by^a e^{-by} = y^{a-1}e^{-by}(a - by)$ and $f'(y) = 0 \Rightarrow y = 0$ or $y = \frac{a}{b}$;

$f''\left(\frac{a}{b}\right) = -b\left(\frac{a}{b}\right)^{a-1} e^{-a} < 0 \Rightarrow f(y)$ has a unique max of $M_y = \left(\frac{a}{eb}\right)^a$ when $y = \frac{a}{b}$.

$g(x) = x^m e^{-nx} \Rightarrow g'(x) = mx^{m-1}e^{-nx} - nx^m e^{-nx} = x^{m-1}e^{-nx}(m - nx)$ and $g'(x) = 0 \Rightarrow x = 0$ or $x = \frac{m}{n}$;

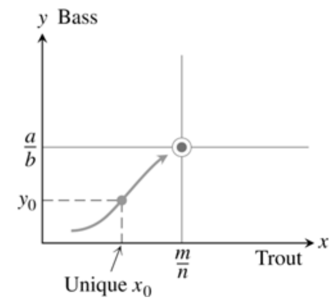
$g''\left(\frac{m}{n}\right) = -n\left(\frac{m}{n}\right)^{m-1} e^{-m} < 0 \Rightarrow g(x)$ has a unique max of $M_x = \left(\frac{m}{en}\right)^m$ when $x = \frac{m}{n}$.





(d) Consider trajectory $(x, y) \to \left(\frac{m}{n}, \frac{a}{b}\right)$. For $y^a e^{-by} = Kx^m e^{-nx} \Rightarrow \frac{y^a}{e^{by}} \cdot \frac{e^{nx}}{x^m} = K$, taking the limit of both sides $\Rightarrow \lim\limits_{\substack{x \to m/n \\ y \to a/b}} \left(\frac{y^a}{e^{by}} \cdot \frac{e^{nx}}{x^m} \right) = \lim\limits_{\substack{x \to m/n \\ y \to a/b}} K \Rightarrow \frac{M_y}{M_x} = K$. Thus, $\frac{y^a}{e^{by}} = \frac{M_y}{M_x} \frac{x^m}{e^{nx}}$ represents the equation any solution trajectory must satisfy if the trajectory approaches the rest point asymptotically.

(e) Pick initial condition $y_0 < \frac{a}{b}$. Then, from the figure at right, $f(y_0) < M_y$ implies $\frac{M_y}{M_x} \frac{x^m}{e^{nx}} = \frac{y_0^a}{e^b y_0} < M_y$ and thus $\frac{x^m}{e^{nx}} < M_x$. From the figure for $g(x)$, there exists a unique $x_0 < \frac{m}{n}$ satisfying $\frac{x^m}{e^{nx}} < M_x$. That is, for each $y < \frac{a}{b}$ there is a unique x satisfying $\frac{y^a}{e^{by}} = \frac{M_y}{M_x} \frac{x^m}{e^{nx}}$. Thus, there can exist only one trajectory solution approaching $\left(\frac{m}{n}, \frac{a}{b}\right)$. (You can think of the point (x_0, y_0) as the initial condition for that trajectory.)



(f) Likewise there exists a unique trajectory when $y_0 > \frac{a}{b}$. Again, $f(y_0) < M_y$ implies $\frac{M_y}{M_x} \frac{x^m}{e^{nx}} = \frac{y_0^a}{e^{by_0}} < M_y$ and thus $\frac{x^m}{e^{nx}} < M_x$. From the figure for $g(x)$, there exists a unique $x_0 > \frac{m}{n}$ satisfying $\frac{x^m}{e^{nx}} < M_x$. That is, for each $y > \frac{a}{b}$ there is a unique x satisfying $\frac{y^a}{e^{by}} = \frac{M_y}{M_x} \frac{x^m}{e^{nx}}$. Thus, there can exist only one trajectory solution approaching $\left(\frac{m}{n}, \frac{a}{b}\right)$.

8. Let $z = y' = \frac{dy}{dx} \Rightarrow \frac{dz}{dx} = z' = y''$, then given the differential equation $y'' = F(x, y, y')$, we can write it as the following system of first order differential equations:

$$\frac{dy}{dx} = z$$
$$\frac{dz}{dx} = F(x, y, z)$$

In general, for the nth order differential equation given by $y^{(n)} = F\left(x, y, y', y'', \ldots, y^{(n-1)}\right)$, let $z_1 = y' = \frac{dy}{dx}$ $\Rightarrow \frac{dz_1}{dx} = z_1' = y''$, let $z_2 = z_1' = y'' \Rightarrow \frac{dz_2}{dx} = z_2' = y''', \ldots$, let $z_{n-1} = z_{n-2}' = y^{(n-1)} \Rightarrow z_{n-1}' = y^{(n)}$. This gives us the following system of first order differential equations:

$$\frac{dy}{dx} = z_1$$
$$\frac{dz_1}{dx} = z_2$$
$$\frac{dz_2}{dx} = z_3$$
$$\vdots$$

$$\frac{dz_{n-2}}{dx} = z_{n-1}$$
$$\frac{dz_{n-1}}{dx} = F\left(x, y, z_1, z_2, \ldots, z_{n-1}\right)$$

9. In the absence of foxes $\Rightarrow b = 0 \Rightarrow \frac{dx}{dt} = a\,x$ and the population of rabbits grows at a rate proportional to the number of rabbits.

10. In the absence of rabbits $\Rightarrow d = 0 \Rightarrow \frac{dy}{dt} = -c\,y$ and the population of foxes decays (since the foxes have no food source) at a rate proportional to the number of foxes.

11. $\frac{dx}{dt} = \left(a - b\,y\right)x = 0 \Rightarrow y = \frac{a}{b}$ or $x = 0$; $\frac{dy}{dt} = \left(-c + d\,x\right)y = 0 \Rightarrow x = \frac{c}{d}$ or $y = 0 \Rightarrow$ equilibrium points at $(0, 0)$ or $\left(\frac{c}{d}, \frac{a}{b}\right)$. For the point $(0, 0)$, there are no rabbits and no foxes. It is an unstable equilibrium point, if there are no foxes, but a few rabbits are introduced, then $\frac{dx}{dt} = a \Rightarrow$ the rabbit population will grow exponentially away from $(0, 0)$.

12. Let $x(t)$ and $y(t)$ both be positive and suppose that they satisfy the differential equations $\frac{dx}{dt} = (a - b\,y)x$ and $\frac{dy}{dt} = (-c + d\,x)y$. Let $C(t) = a\ln y(t) - b\cdot y(t) - d\cdot x(t) + c\ln x(t) \Rightarrow C'(t) = a\frac{y'(t)}{y(t)} - b\cdot y'(t) - d\cdot x'(t) + c\frac{x'(t)}{x(t)}$
$= \left(\frac{a}{y(t)} - b\right)y'(t) + \left(\frac{c}{x(t)} - d\right)x'(t) = \left(\frac{a}{y(t)} - b\right)\left(-c + d\cdot x(t)\right)x(t) + \left(\frac{c}{x(t)} - d\right)\left(a - b\cdot y(t)\right)y(t) = 0$
Since $C'(t) = 0 \Rightarrow C(t) =$ constant.

13. Consider a particular trajectory and suppose that (x_0, y_0) is such that $x_0 < \frac{c}{d}$ and $y_0 < \frac{a}{b}$, then $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} < 0 \Rightarrow$ the rabbit population is increasing while the fox population is decreasing, points on the trajectory are moving down and to the right; if $x_0 > \frac{c}{d}$ and $y_0 < \frac{a}{b}$, then $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} > 0 \Rightarrow$ both the rabbit and fox populations are increasing, points on the trajectory are moving up and to the right; if $x_0 > \frac{c}{d}$ and $y_0 > \frac{a}{b}$, then $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} > 0 \Rightarrow$ the rabbit population is decreasing while the fox population is increasing, points on the trajectory are moving up and to the left; and finally if $x_0 < \frac{c}{d}$ and $y_0 > \frac{a}{b}$, then $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} < 0 \Rightarrow$ both the rabbit and fox populations are decreasing, points on the trajectory are moving down and to the left. Thus, points travel around the trajectory in a counterclockwise direction. Note that we will follow the same trajectory if (x_0, y_0) starts at a different point on the trajectory.

14. There are three possible cases: If the rabbit population begins (before the wolf) and ends (after the wolf) at a value larger than the equilibrium level of $x = \frac{c}{d}$, then the trajectory moves closer to the equilibrium and the maximum value of the foxes is smaller. If the rabbit population begins (before the wolf) and ends (after the wolf) at a value smaller than the equilibrium level of $x = \frac{c}{d}$, but greater than 0, then the trajectory moves further from the equilibrium and the maximum value of the foxes is greater. If the rabbit population begins and ends very near the equilibrium value, then the trajectory will stay near the equilibrium value, since it is a stable equilibrium, and the fox population will remain roughly the same.

CHAPTER 9 PRACTICE EXERCISES

1. $y' = xe^y\sqrt{x-2} \Rightarrow e^{-y}dy = x\sqrt{x-2}dx \Rightarrow -e^{-y} = \frac{2(x-2)^{3/2}(3x+4)}{15} + C \Rightarrow e^{-y} = \frac{-2(x-2)^{3/2}(3x+4)}{15} - C$
$\Rightarrow -y = \ln\left[\frac{-2(x-2)^{3/2}(3x+4)}{15} - C\right] \Rightarrow y = -\ln\left[\frac{-2(x-2)^{3/2}(3x+4)}{15} - C\right]$

2. $y' = xye^{x^2} \Rightarrow \frac{dy}{y} = e^{x^2}x\,dx \Rightarrow \ln y = \frac{1}{2}e^{x^2} + C$

3. $\sec x\,dy + x\cos^2 y\,dx = 0 \Rightarrow \frac{dy}{\cos^2 y} = -\frac{x\,dx}{\sec x} \Rightarrow \tan y = -\cos x - x\sin x + C$

4. $2x^2dx - 3\sqrt{y}\csc x\,dy = 0 \Rightarrow 3\sqrt{y}\,dy = \frac{2x^2}{\csc x}dx \Rightarrow 2y^{3/2} = 2\left(2-x^2\right)\cos x + 4x\sin x + C$
$\Rightarrow y^{3/2} = \left(2-x^2\right)\cos x + 2x\sin x + C_1$

5. $y' = \frac{e^y}{xy} \Rightarrow ye^{-y}dy = \frac{dx}{x} \Rightarrow (y+1)e^{-y} = -\ln|x| + C$

6. $y' = xe^{x-y}\csc y \Rightarrow y' = \frac{xe^x}{e^y}\csc y \Rightarrow \frac{e^y}{\csc y}dy = xe^x dx \Rightarrow \frac{e^y}{2}\left(\sin y - \cos y\right) = (x-1)e^x + C$

7. $x(x-1)dy - y\,dx = 0 \Rightarrow x(x-1)dy = y\,dx \Rightarrow \frac{dy}{y} = \frac{dx}{x(x-1)} \Rightarrow \ln y = \ln(x-1) - \ln(x) + C$
$\Rightarrow \ln y = \ln(x-1) - \ln(x) + \ln C_1 \Rightarrow \ln y = \ln\left(\frac{C_1(x-1)}{x}\right) \Rightarrow y = \frac{C_1(x-1)}{x}$

8. $y' = \left(y^2-1\right)\left(x^{-1}\right) \Rightarrow \frac{dy}{y^2-1} = \frac{dx}{x} \Rightarrow \frac{\ln\left(\frac{y-1}{y+1}\right)}{2} = \ln x + C \Rightarrow \ln\left(\frac{y-1}{y+1}\right) = 2\ln x + \ln C_1 \Rightarrow \frac{y-1}{y+1} = C_1x^2$

9. $2y' - y = xe^{x/2} \Rightarrow y' - \frac{1}{2}y = \frac{x}{2}e^{x/2}$.
$P(x) = -\frac{1}{2}, v(x) = e^{\int\left(-\frac{1}{2}\right)dx} = e^{-x/2}$.
$e^{-x/2}y' - \frac{1}{2}e^{-x/2}y = \left(e^{-x/2}\right)\left(\frac{x}{2}\right)\left(e^{x/2}\right) = \frac{x}{2} \Rightarrow \frac{d}{dx}\left(e^{-x/2}y\right) = \frac{x}{2} \Rightarrow e^{-x/2}y = \frac{x^2}{4} + C \Rightarrow y = e^{x/2}\left(\frac{x^2}{4} + C\right)$

10. $\frac{y'}{2} + y = e^{-x}\sin x \Rightarrow y' + 2y = 2e^{-x}\sin x$.
$P(x) = 2, v(x) = e^{\int 2dx} = e^{2x}$.
$e^{2x}y' + 2e^{2x}y = 2e^{2x}e^{-x}\sin x = 2e^x\sin x \Rightarrow \frac{d}{dx}\left(e^{2x}y\right) = 2e^x\sin x \Rightarrow e^{2x}y = e^x\left(\sin x - \cos x\right) + C$
$\Rightarrow y = e^{-x}\left(\sin x - \cos x\right) + Ce^{-2x}$

11. $xy' + 2y = 1 - x^{-1} \Rightarrow y' + \left(\frac{2}{x}\right)y = \frac{1}{x} - \frac{1}{x^2}$.

$v(x) = e^{2\int \frac{dx}{x}} = e^{2\ln x} = e^{\ln x^2} = x^2$.

$x^2 y' + 2xy = x - 1 \Rightarrow \frac{d}{dx}\left(x^2 y\right) = x - 1 \Rightarrow x^2 y = \frac{x^2}{2} - x + C \Rightarrow y = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}$

12. $xy' - y = 2x \ln x \Rightarrow y' - \left(\frac{1}{x}\right)y = 2\ln x$.

$v(x) = e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$.

$\left(\frac{1}{x}\right)y' - \left(\frac{1}{x}\right)^2 y = \frac{2}{x}\ln x \Rightarrow \frac{d}{dx}\left(\frac{1}{x}\cdot y\right) = \frac{2}{x}\ln x \Rightarrow \frac{1}{x}\cdot y = \left[\ln x\right]^2 + C \Rightarrow y = x\left[\ln x\right]^2 + Cx$

13. $\left(1 + e^x\right)dy + \left(ye^x + e^{-x}\right)dx = 0 \Rightarrow \left(1 + e^x\right)y' + e^x y = -e^{-x} \Rightarrow y' = \frac{e^x}{1+e^x}y = \frac{-e^{-x}}{1+e^x}$.

$v(x) = e^{\int \frac{e^x dx}{1+e^x}} = e^{\ln\left(e^x+1\right)} = e^x + 1$.

$\left(e^x + 1\right)y' + \left(e^x + 1\right)\left(\frac{e^x}{1+e^x}\right)y = \frac{-e^{-x}}{1+e^x}\left(e^x + 1\right) \Rightarrow \frac{d}{dx}\left[\left(e^x + 1\right)y\right] = -e^{-x} \Rightarrow \left(e^x + 1\right)y = e^{-x} + C$

$\Rightarrow y = \frac{e^{-x}+C}{e^x+1} = \frac{e^{-x}+C}{1+e^x}$

14. $e^{-x}dy + \left(e^{-x}y - 4x\right)dx = 0 \Rightarrow \frac{dy}{dx} + y = 4xe^x \Rightarrow p(x) = 1,\ \ v(x) = e^{\int 1dx} = e^x \Rightarrow e^x\frac{dy}{dx} + ye^x = 4x\,e^{2x}$

$\Rightarrow \frac{d}{dx}\left(y\,e^x\right) = 4x\,e^{2x} \Rightarrow y\,e^x = \int 4x\,e^{2x}dx \Rightarrow y\,e^x = 2xe^{2x} - e^{2x} + C \Rightarrow y = 2xe^x - e^x + Ce^{-x}$

15. $\left(x + 3y^2\right)dy + y\,dx = 0 \Rightarrow x\,dy + y\,dx = -3y^2dy \Rightarrow \frac{d}{dx}(xy) = -3y^2dy \Rightarrow xy = -y^3 + C$

16. $x\,dy + \left(3y - x^{-2}\cos x\right)dx = 0 \Rightarrow y' + \left(\frac{3}{x}\right)y = x^{-3}\cos x$. Let $v(y) = e^{\int \frac{3dx}{x}} = e^{3\ln x} = e^{\ln x^3} = x^3$.

Then $x^3 y' + 3x^2 y = \cos x$ and $x^3 y = \int \cos x\,dx = \sin x + C$. So $y = x^{-3}\left(\sin x + C\right)$

17. $(x+1)\frac{dy}{dx} + 2y = x \Rightarrow y' + \left(\frac{2}{x+1}\right)y = \frac{x}{x+1}$. Let $v(x) = e^{\int \frac{2}{x+1}dx} = e^{2\ln(x+1)} = e^{\ln(x+1)^2} = (x+1)^2$.

So $y'(x+1)^2 + \frac{2}{(x+1)}(x+1)^2 y = \frac{x}{(x+1)}(x+1)^2 \Rightarrow \frac{d}{dx}\left[y(x+1)^2\right] = x(x+1) \Rightarrow y(x+1)^2 = \int x(x+1)dx$

$\Rightarrow y(x+1)^2 = \frac{x^3}{3} + \frac{x^2}{2} + C \Rightarrow y = (x+1)^{-2}\left(\frac{x^3}{3} + \frac{x^2}{2} + C\right)$. We have $y(0) = 1 \Rightarrow 1 = C$. So

$y = (x+1)^{-2}\left(\frac{x^3}{3} + \frac{x^2}{2} + 1\right)$

18. $x\frac{dy}{dx} + 2y = x^2 + 1 \Rightarrow y' + \left(\frac{2}{x}\right)y = x + \frac{1}{x}$. Let $v(x) = e^{\int\left(\frac{2}{x}\right)dx} = e^{\ln x^2} = x^2$. So $x^2 y' + 2xy = x^3 + x$

$\Rightarrow \frac{d}{dx}\left(x^2 y\right) = x^3 + x \Rightarrow x^2 y = \frac{x^4}{4} + \frac{x^2}{2} + C \Rightarrow y = \frac{x^2}{4} + \frac{C}{x^2} + \frac{1}{2}$. We have $y(1) = 1 \Rightarrow 1 = \frac{1}{4} + C + \frac{1}{2} \Rightarrow C = \frac{1}{4}$.

So $y = \frac{x^2}{4} + \frac{1}{4x^2} + \frac{1}{2} = \frac{x^4+2x^2+1}{4x^2}$

19. $\frac{dy}{dx}+3x^2y=x^2$. Let $v(x)=e^{\int 3x^2dx}=e^{x^3}$. So $e^{x^3}y'+3x^2e^{x^3}y=x^2e^{x^3}\Rightarrow\frac{d}{dx}\left(e^{x^3}y\right)=x^2e^{x^3}$
$\Rightarrow e^{x^3}y=\frac{1}{3}e^{x^3}+C$. We have $y(0)=-1\Rightarrow e^{0^3}(-1)=\frac{1}{3}e^{0^3}+C\Rightarrow -1=\frac{1}{3}+C\Rightarrow C=-\frac{4}{3}$ and $e^{x^3}y=\frac{1}{3}e^{x^3}-\frac{4}{3}$
$\Rightarrow y=\frac{1}{3}-\frac{4}{3}e^{-x^3}$

20. $xdy+(y-\cos x)dx=0\Rightarrow xy'+y-\cos x=0\Rightarrow y'+\left(\frac{1}{x}\right)y=\frac{\cos x}{x}$. Let $v(x)=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$.
So $xy'+x\left(\frac{1}{x}\right)y=\cos x\Rightarrow\frac{d}{dx}(xy)=\cos x\Rightarrow xy=\int\cos x\,dx\Rightarrow xy=\sin x+C$.
We have $y\left(\frac{\pi}{2}\right)=0\Rightarrow\left(\frac{\pi}{2}\right)0=1+C\Rightarrow C=-1$. So $xy=-1+\sin x\Rightarrow y=\frac{-1+\sin x}{x}$

21. $xy'+(x-2)y=3x^3e^{-x}\Rightarrow y'+\left(\frac{x-2}{x}\right)y=3x^2e^{-x}$. Let $v(x)=e^{\int\left(\frac{x-2}{x}\right)dx}=e^{x-2\ln x}=\frac{e^x}{x^2}$.
So $\frac{e^x}{x^2}y'+\frac{e^x}{x^2}\left(\frac{x-2}{x}\right)y=3\Rightarrow\frac{d}{dx}\left(y\cdot\frac{e^x}{x^2}\right)=3\Rightarrow y\cdot\frac{e^x}{x^2}=3x+C$. We have $y(1)=0\Rightarrow 0=3(1)+C\Rightarrow C=-3$
$\Rightarrow y\cdot\frac{e^x}{x^2}=3x-3\Rightarrow y=x^2e^{-x}(3x-3)$

22. $y\,dx+(3x-xy+2)dy=0\Rightarrow\frac{dx}{dy}+\frac{3x-xy+2}{y}=0\Rightarrow\frac{dx}{dy}+\frac{3x}{y}-x=-\frac{2}{y}\Rightarrow\frac{dx}{dy}+\left(\frac{3}{y}-1\right)x=-\frac{2}{y}$.
$P(y)=\frac{3}{y}-1\Rightarrow\int P(y)\,dy=3\ln y-y\Rightarrow v(y)=e^{3\ln y-y}=y^3e^{-y}$
$y^3e^{-y}x'+y^3e^{-y}\left(\frac{3}{y}-1\right)x=-2y^2e^{-y}\Rightarrow y^3e^{-y}x=\int -2y^2e^{-y}dy=2e^{-y}\left(y^2+2y+2\right)+C$
$\Rightarrow y^3=\frac{2\left(y^2+2y+2\right)+Ce^y}{x}$. We have $y(2)=-1\Rightarrow -1=\frac{2(1-2+2)+Ce^{-1}}{2}\Rightarrow C=-4e$ and $\Rightarrow y^3=\frac{2\left(y^2+2y+2\right)-4e^{y+1}}{x}$

23. To find the approximate values let $y_n=y_{n-1}+\left(y_{n-1}+\cos x_{n-1}\right)(0.1)$ with $x_0=0$, $y_0=0$, and 20 steps. Use a spreadsheet, graphing calculator, or CAS to obtain the values in the following table.

x	y
0	0
0.1	0.1000
0.2	0.2095
0.3	0.3285
0.4	0.4568
0.5	0.5946
0.6	0.7418
0.7	0.8986
0.8	1.0649
0.9	1.2411
1.0	1.4273

x	y
1.1	1.6241
1.2	1.8319
1.3	2.0513
1.4	2.2832
1.5	2.5285
1.6	2.7884
1.7	3.0643
1.8	3.3579
1.9	3.6709
2.0	4.0057

24. To find the approximate values let $y_n = y_{n-1} + (2 - y_{n-1})(2x_{n-1} + 3)(0.1)$ with $x_0 = -3, y_0 = 1,$ and 20 steps. Use a spreadsheet, graphing calculator, or CAS to obtain the values in the following table.

x	y
–3.0	1.0000
–2.9	0.7000
–2.8	0.3360
–2.7	–0.0966
–2.6	–0.5998
–2.5	–1.1718
–2.4	–1.8062
–2.3	–2.4913
–2.2	–3.2099
–2.1	–3.9393
–2.0	–4.6520

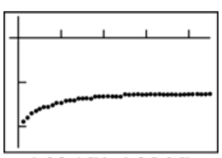
x	y
–1.9	–5.3172
–1.8	–5.9026
–1.7	–6.3768
–1.6	–6.7119
–1.5	–6.8861
–1.4	–6.8861
–1.3	–6.7084
–1.2	–6.3601
–1.1	–5.8585
–1.0	–5.2298

25. To estimate $y(3)$, let $y = y_{n-1} + \left(\frac{x_{n-1} - 2y_{n-1}}{x_{n-1} + 1}\right)(0.05)$ with initial values $x_0 = 0, y_0 = 1,$ and 60 steps. Use a spreadsheet, graphing calculator, or CAS to obtain $y(3) \approx 0.8981.$

26. To estimate $y(4)$, let $z_n = y_{n-1} + \left(\frac{x_{n-1}^2 - 2y_{n-1} + 1}{x_{n-1}}\right)(0.05)$ with initial values $x_0 = 1, y_0 = 1,$ and 60 steps. Use a spreadsheet, graphing calculator, or CAS to obtain $y(4) \approx 4.4974.$

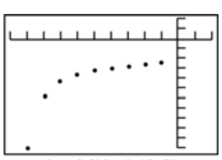
27. Let $y_n = y_{n-1} + \left(\frac{1}{e^{x_{n-1} + y_{n-1} + 2}}\right)(dx)$ with starting values $x_0 = 0$ and $y_0 = 2,$ and steps of 0.1 and $-0.1.$ Use a spreadsheet, programmable calculator, or CAS to generate the following graphs.

(a)



[–0.2, 4.5] by [–2.5, 0.5]

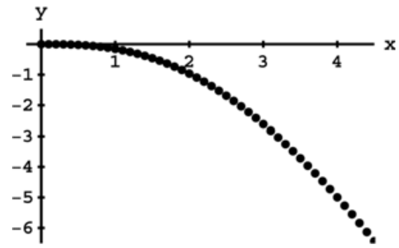
(b) Note that we choose a small interval of x-values because the y-values decrease very rapidly and our calculator cannot handle the calculations for $x \le -1$. (This occurs because the analytic solution is $y = -2 + \ln\left(2 - e^{-x}\right)$, which has an asymptote at $x = -\ln 2 \approx 0.69$. Obviously, the Euler approximations are misleading for $x \le -0.7$.)



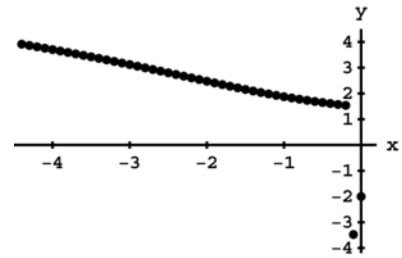
[–1, 0.2] by [–10, 2]

28. Let $y_n = y_{n-1} - \left(\frac{x_{n-1}^2 + y_{n-1}}{e^{y_{n-1}} + x_{n-1}}\right)(dx)$ with starting values $x_0 = 0$ and $y_0 = 0$, and steps of 0.1 and -0.1. Use a spreadsheet, programmable calculator, or CAS to generate the following graphs.

(a)



(b)



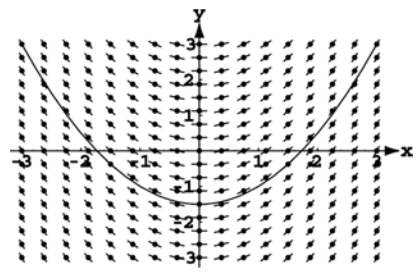
29.

x	1	1.2	1.4	1.6	1.8	2.0
y	-1	-0.8	-0.56	-0.28	0.04	0.4

$\frac{dy}{dx} = x \Rightarrow dy = x\,dx \Rightarrow y = \frac{x^2}{2} + C;\quad x = 1$ and $y = -1$

$\Rightarrow -1 = \frac{1}{2} + C \Rightarrow C = -\frac{3}{2} \Rightarrow y(\text{exact}) = \frac{x^2}{2} - \frac{3}{2}$

$\Rightarrow y(2) = \frac{2^2}{2} - \frac{3}{2} = \frac{1}{2}$ is the exact value.



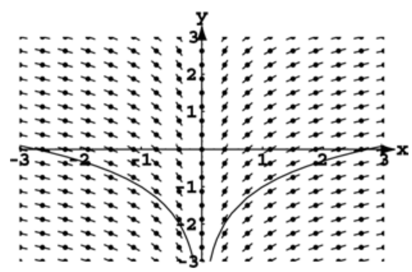
30.

x	1	1.2	1.4	1.6	1.8	2.0
y	-1	-0.8	-0.6333	-0.4904	-0.3654	-0.2544

$\frac{dy}{dx} = \frac{1}{x} \Rightarrow dy = \frac{1}{x}dx \Rightarrow y = \ln|x| + C;\quad x = 1$ and $y = -1$

$\Rightarrow -1 = \ln 1 + C \Rightarrow C = -1 \Rightarrow y(\text{exact}) = \ln|x| - 1$

$\Rightarrow y(2) = \ln 2 - 1 \approx -0.3069$ is the exact value.



31.

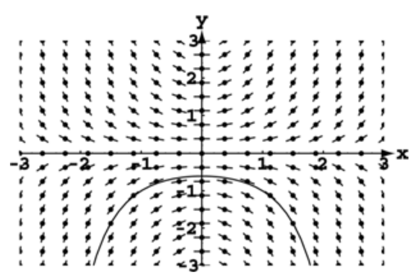
x	1	1.2	1.4	1.6	1.8	2.0
y	-1	-1.2	-0.488	-1.9046	-2.5141	-3.4192

$\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x\,dx \Rightarrow \ln|y| = \frac{x^2}{2} + C$

$\Rightarrow y = e^{\frac{x^2}{2} + C} = e^{\frac{x^2}{2}} \cdot e^C = C_1 e^{\frac{x^2}{2}};\quad x = 1$ and $y = -1$

$\Rightarrow -1 = C_1 e^{1/2} \Rightarrow C_1 = -e^{1/2} \Rightarrow y(\text{exact}) = -e^{1/2} \cdot e^{\frac{x^2}{2}}$

$= -e^{\left(x^2 - 1\right)/2} \Rightarrow y(2) = -e^{3/2} \approx -4.4817$ is the exact value.



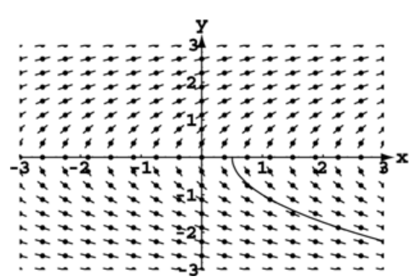
32.

x	1	1.2	1.4	1.6	1.8	2.0
y	-1	-1.2	-1.3667	-1.5130	-1.6452	-1.7688

$\frac{dy}{dx} = \frac{1}{y} \Rightarrow y\,dy = dx \Rightarrow \frac{y^2}{y} = x + C;\quad x = 1$ and $y = -1$

$\frac{1}{2} = 1 + C \Rightarrow C = -\frac{1}{2} \Rightarrow y^2 = 2x - 1$

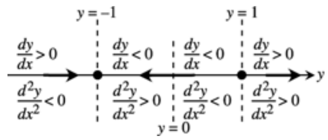
$\Rightarrow y(\text{exact}) = \sqrt{2x-1} \Rightarrow y(2) = -\sqrt{3} \approx -1.7321$ is the exact value.



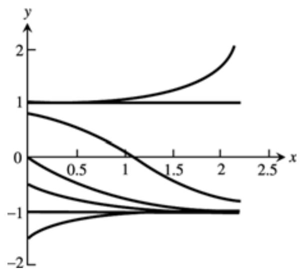
33. $\frac{dy}{dx} = y^2 - 1 \Rightarrow y' = (y+1)(y-1)$. We have $y' = 0 \Rightarrow (y+1) = 0, (y-1) = 0 \Rightarrow y = -1, 1$.

(a) Equilibrium points are -1 (stable) and 1 (unstable)

(b) $y' = y^2 - 1 \Rightarrow y'' = 2yy' \Rightarrow y'' = 2y\left(y^2 - 1\right) = 2y(y+1)(y-1)$. So $y'' = 0 \Rightarrow y = 0, y = -1, y = 1$.



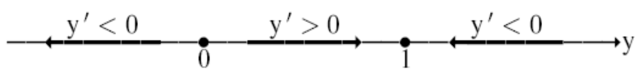
(c)



34. $\frac{dy}{dx} = y - y^2 \Rightarrow y' = y(1-y)$. We have $y' = 0 \Rightarrow y(1-y) = 0 \Rightarrow y = 0, 1 - y = 0 \Rightarrow y = 0, 1$.

(a) The equilibrium points are 0 and 1. So, 0 is unstable and 1 is stable.

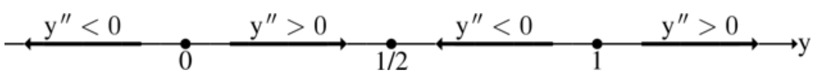
(b) Let $\rightarrow =$ increasing. $\leftarrow =$ decreasing.



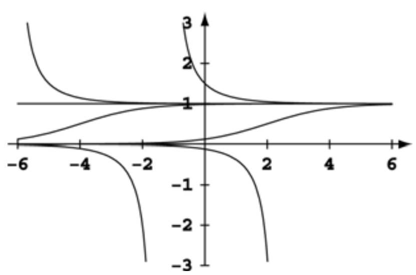
$y' = y - y^2 \Rightarrow y'' = y' - 2yy' \Rightarrow y'' = \left(y - y^2\right) - 2y\left(y - y^2\right) = y - y^2 - 2y^2 + 2y^3 \Rightarrow y'' = 2y^3 - 3y^2 + y$

$= y\left(2y^2 - 3y + 1\right) \Rightarrow y'' = y(2y-1)(y-1)$. So, $y'' = 0 \Rightarrow y = 0, 2y - 1 = 0, y - 1 = 0 \Rightarrow y = 0, y = \frac{1}{2}, y = 1$.

Let $\rightarrow =$ concave up, $\leftarrow =$ concave down.



(c)



35. (a) Force $=$ Mass times Acceleration (Newton's Second Law) or $F = ma$. Let $a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds}$. Then

$ma = -mgR^2 s^{-2} \Rightarrow a = -gR^2 s^{-2} \Rightarrow v\,\frac{dv}{ds} = -gR^2 s^{-2} \Rightarrow v\,dv = -gR^2 s^{-2} ds \Rightarrow \int v\,dv = \int -gR^2 s^{-2} ds$

$\Rightarrow \frac{v^2}{2} = \frac{gR^2}{s} + C_1 \Rightarrow v^2 = \frac{2gR^2}{s} + 2C_1 = \frac{2gR^2}{s} + C$. When $t = 0, v = v_0$ and $s = R \Rightarrow v_0^2 = \frac{2gR^2}{R} + C$

$\Rightarrow C = v_0^2 - 2gR \Rightarrow v^2 = \frac{2gR^2}{s} + v_0^2 - 2gR$

(b) If $v_0=\sqrt{2gR}$, the $v^2=\frac{2gR^2}{s}\Rightarrow v=\sqrt{\frac{2gR^2}{s}}$, since $v\ge 0$ if $v_0\ge\sqrt{2gR}$. Then $\frac{ds}{dt}=\frac{\sqrt{2gR^2}}{\sqrt{s}}$

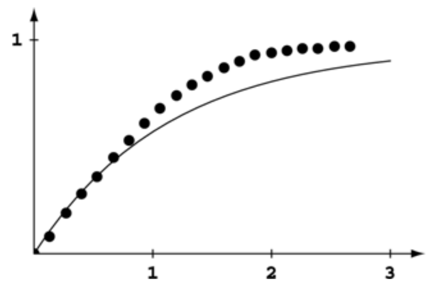
$\Rightarrow \sqrt{s}\,ds=\sqrt{2gR^2}\,dt\Rightarrow\int s^{1/2}ds=\int\sqrt{2gR^2}\,dt\Rightarrow\frac{2}{3}s^{3/2}=\sqrt{2gR^2}\,t+C_1\Rightarrow s^{3/2}=\left(\frac{3}{2}\sqrt{2gR^2}\right)t+C;\ \ t=0$

and $s=R\Rightarrow R^{3/2}=\left(\frac{3}{2}\sqrt{2gR^2}\right)(0)+C\Rightarrow C=R^{3/2}\Rightarrow s^{3/2}=\left(\frac{3}{2}\sqrt{2gR^2}\right)t+R^{3/2}=\left(\frac{3}{2}R\sqrt{2g}\right)t+R^{3/2}$

$=R^{3/2}\left[\left(\frac{3}{2}R^{-1/2}\sqrt{2g}\right)t+1\right]=R^{3/2}\left[\left(\frac{3\sqrt{2gR}}{2R}\right)t+1\right]=R^{3/2}\left[\left(\frac{3v_0}{2R}\right)t+1\right]\Rightarrow s=R\left[1+\left(\frac{3v_0}{2R}\right)t\right]^{2/3}$

36. $\frac{v_0m}{k}=$ coasting distance $\Rightarrow\frac{(0.86)(30.84)}{k}=0.97\Rightarrow k\approx 27.343.$ $s(t)=\frac{v_0m}{k}\left(1-e^{-(k/m)t}\right)$

$\Rightarrow s(t)=0.97\left(1-e^{-(27.343/30.84)t}\right)\Rightarrow s(t)=0.97\left(1-e^{-0.8866t}\right)$. A graph of the model is shown superimposed on a graph of the data.



CHAPTER 9 ADDITIONAL AND ADVANCED EXERCISES

1. (a) $\frac{dy}{dt}=k\frac{A}{V}(c-y)\Rightarrow dy=-k\frac{A}{V}(y-c)\,dt\Rightarrow\frac{dy}{y-c}=-k\frac{A}{V}dt\Rightarrow\int\frac{dy}{y-c}=-\int k\frac{A}{V}dt\Rightarrow\ln|y-c|=-k\frac{A}{V}t+C_1$

$\Rightarrow y-c=\pm e^{C_1}e^{-k\frac{A}{V}t}$. Apply the initial condition, $y(0)=y_0\Rightarrow y_0=c+C\Rightarrow C=y_0-c$

$\Rightarrow y=c+(y_0-c)e^{-k\frac{A}{V}t}$.

(b) Steady state solution: $y_\infty=\lim_{t\to\infty}y(t)=\lim_{t\to\infty}\left[c+(y_0-c)e^{-k\frac{A}{V}t}\right]=c+(y_0-c)(0)=c$

2. $\frac{d(mv)}{dt}=F+(v+u)\frac{dm}{dt}\Rightarrow F=\frac{d(mv)}{dt}-(v+u)\frac{dm}{dt}\Rightarrow F=m\frac{dv}{dt}+v\frac{dm}{dt}-v\frac{dm}{dt}-u\frac{dm}{dt}\Rightarrow F=m\frac{dv}{dt}-u\frac{dm}{dt}$.

$\frac{dm}{dt}=-b\Rightarrow m=-|b|t+C$. At $t=0, m=m_0$, so $C=m_0$ and $m=m_0-|b|t$.

Thus $F=(m_0-|b|t)\frac{dv}{dt}-u|b|=-(m_0-|b|t)|g|\Rightarrow\frac{dv}{dt}=-g+\frac{u|b|}{m_0-|b|t}\Rightarrow v=-gt-u\ln\left(\frac{m_0-|b|t}{m_0}\right)+C_1$

$v=0$ at $t=0\Rightarrow C_1=0$. So $v=-gt-u\ln\left(\frac{m_0-|b|t}{m_0}\right)=\frac{dy}{dt}\Rightarrow y=\int\left[-gt-u\ln\left(\frac{m_0-|b|t}{m_0}\right)\right]dt$ and $u=c, y=0$ at

$t=0\Rightarrow y=-\frac{1}{2}gt^2+c\left[t+\left(\frac{m_0-|b|t}{|b|}\right)\ln\left(\frac{m_0-|b|t}{m_0}\right)\right]$

3. (a) Let y be any function such that $v(x)y = \int v(x)Q(x)dx + C,\ \ v(x) = e^{\int P(x)dx}$. Then $\frac{d}{dx}\left(v(x)\cdot y\right) = v(x)\cdot y' + y\cdot v'(x) = v(x)Q(x)$. We have $v(x) = e^{\int P(x)dx}$ $\Rightarrow v'(x) = e^{\int P(x)dx}P(x) = v(x)P(x)$. Thus $v(x)\cdot y' + y\cdot v(x)P(x) = v(x)Q(x) \Rightarrow y' + yP(x) = Q(x) \Rightarrow$ the given y is a solution.

(b) If v and Q are continuous on $[a, b]$ and $x \in (a, b)$, then $\frac{d}{dx}\left[\int_{x_0}^{x} v(t)Q(t)dt\right] = v(x)Q(x)$ $\Rightarrow \int_{x_0}^{x} v(t)Q(t)dt = \int v(x)Q(x)dx$. So $C = y_0 v(x_0) - \int v(x)Q(x)dx$. From part (a), $v(x)y = \int v(x)Q(x)dx + C$. Substituting for C: $v(x)y = \int v(x)Q(x)dx + y_0 v(x_0) - \int v(x)Q(x)dx$ $\Rightarrow v(x)y = y_0 v(x_0)$ when $x = x_0$.

4. (a) $y' + P(x)y = 0, y(x_0) = 0$. Use $v(x) = e^{\int P(x)dx}$ as an integrating factor. Then $\frac{d}{dx}\left(v(x)y\right) = 0$ $\Rightarrow v(x)y = C \Rightarrow y = Ce^{-\int P(x)dx}$ and $y_1 = C_1 e^{-\int P(x)dx},\ \ y_2 = C_2 e^{-\int P(x)dx},\ \ y_1(x_0) = y_2(x_0) = 0,$ $y_1 - y_2 = (C_1 - C_2)e^{-\int P(x)dx} = C_3 e^{-\int P(x)dx}$ and $y_1 - y_2 = 0 - 0 = 0$. So $y_1 - y_2$ is a solution to $y' + P(x)y = 0$ with $y(x_0) = 0$.

(b) $\frac{d}{dx}\left(v(x)[y_1(x) - y_2(x)]\right) = \frac{d}{dx}\left(e^{\int P(x)dx}\left[e^{-\int P(x)dx}(C_1 - C_2)\right]\right) = \frac{d}{dx}(C_1 - C_2) = \frac{d}{dx}(C_3) = 0.$

$\int \frac{d}{dx}\left(v(x)[y_1(x) - y_2(x)]\right)dx = \left(v(x)[y_1(x) - y_2(x)]\right) = \int 0\ dx = C$

(c) $y_1 = C_1 e^{-\int P(x)\,dx},\ \ y_2 = C_2 e^{-\int P(x)\,dx},\ \ y = y_1 - y_2$. So $y(x_0) = 0 \Rightarrow C_1 e^{-\int P(x)dx} - C_2 e^{-\int P(x)dx} = 0$ $\Rightarrow C_1 - C_2 = 0 \Rightarrow C_1 = C_2 \Rightarrow y_1(x) = y_2(x)$ for $a < x < b$.

5. $\left(x^2 + y^2\right)dx + xy\,dy = 0 \Rightarrow \frac{dy}{dx} = \frac{-\left(x^2+y^2\right)}{xy} = -\frac{x}{y} - \frac{y}{x} = -\frac{1}{y/x} - \frac{y}{x} = F\left(\frac{y}{x}\right) \Rightarrow F(v) = \frac{1}{v} - v \Rightarrow \frac{dx}{x} + \frac{dv}{v - F(v)} = 0$

$\Rightarrow \frac{dx}{x} + \frac{dv}{v - \left(-\frac{1}{v} - v\right)} = 0 \Rightarrow \int \frac{dx}{x} + \int \frac{v\,dv}{2v^2+1} = C \Rightarrow \ln|x| + \frac{1}{4}\ln\left|2v^2 + 1\right| = C \Rightarrow 4\ln|x| + \ln\left|2\left(\frac{y}{x}\right)^2 + 1\right| = C$

$\Rightarrow \ln\left|x^4\right| + \ln\left|\frac{2y^2+x^2}{x^2}\right| = C \Rightarrow \ln\left|x^2\left(2y^2 + x^2\right)\right| = C \Rightarrow x^2\left(2y^2 + x^2\right) = e^C \Rightarrow x^2\left(2y^2 + x^2\right) = C$

6. $x^2 dy + \left(y^2 - xy\right)dx = 0 \Rightarrow \frac{dy}{dx} = \frac{-\left(y^2 - xy\right)}{x^2} \Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^2 + \frac{y}{x} = F\left(\frac{y}{x}\right) \Rightarrow F(v) = -v^2 + v \Rightarrow \frac{dx}{x} + \frac{dv}{v - \left(-v^2 + v\right)} = 0$

$\Rightarrow \int \frac{dx}{x} + \int \frac{dv}{v^2} = C \Rightarrow \ln|x| - \frac{1}{v} = C \Rightarrow \ln|x| - \frac{1}{y/x} = C \Rightarrow \ln|x| - \frac{x}{y} = C$

7. $\left(xe^{y/x} + y\right)dx - x\,dy = 0 \Rightarrow \frac{dy}{dx} = \frac{xe^{y/x} + y}{x} = e^{y/x} + \frac{y}{x} = F\left(\frac{y}{x}\right) \Rightarrow F(v) = e^v + v \Rightarrow \frac{dx}{x} + \frac{dv}{v - \left(e^v + v\right)} = 0$

$\Rightarrow \int \frac{dx}{x} - \int \frac{dv}{e^v} = C \Rightarrow \ln|x| + e^{-v} = C \Rightarrow \ln|x| + e^{-y/x} = C$

8. $(x+y)\,dy+(x-y)\,dx=0 \Rightarrow \frac{dy}{dx}=\frac{-(x-y)}{x+y}=\frac{\frac{y}{x}-1}{1+\frac{y}{x}}=F\left(\frac{y}{x}\right) \Rightarrow F(v)=\frac{v-1}{1+v} \Rightarrow \frac{dx}{x}+\frac{dv}{v-\left(\frac{v-1}{1+v}\right)}=0 \Rightarrow \int \frac{dx}{x}+\int \frac{(1+v)\,dv}{v^2+1}=0$

$\Rightarrow \int \frac{dx}{x}+\int \frac{dv}{v^2+1}+\int \frac{v\,dv}{v^2+1}=0 \Rightarrow \ln|x|+\tan^{-1} v+\frac{1}{2}\ln\left|v^2+1\right|=C \Rightarrow 2\ln|x|+2\tan^{-1} v+\ln\left|\left(\frac{y}{x}\right)^2+1\right|=C$

$\Rightarrow \ln\left|x^2\right|+2\tan^{-1}\left(\frac{y}{x}\right)+\ln\left|\frac{y^2+x^2}{x^2}\right|=C \Rightarrow 2\tan^{-1}\left(\frac{y}{x}\right)+\ln\left|y^2+x^2\right|=C$

9. $y'=\frac{y}{x}+\cos\left(\frac{y-x}{x}\right)=\frac{y}{x}+\cos\left(\frac{y}{x}-1\right)=F\left(\frac{y}{x}\right) \Rightarrow F(v)=v+\cos(v-1) \Rightarrow \frac{dx}{x}+\frac{dv}{v-(v+\cos(v-1))}=0$

$\Rightarrow \int \frac{dx}{x}-\int \sec(v-1)\,dv=0 \Rightarrow \ln|x|-\ln\left|\sec(v-1)+\tan(v-1)\right|=C \Rightarrow \ln|x|-\ln\left|\sec\left(\frac{y}{x}-1\right)+\tan\left(\frac{y}{x}-1\right)\right|=C$

10. $\left(x\sin\frac{y}{x}-y\cos\frac{y}{x}\right)dx+x\cos\frac{y}{x}\,dy=0 \Rightarrow \frac{dy}{dx}=\frac{-\left(x\sin\frac{y}{x}-y\cos\frac{y}{x}\right)}{x\cos\frac{y}{x}}=\frac{y}{x}-\tan\frac{y}{x}=F\left(\frac{y}{x}\right) \Rightarrow F(v)=v-\tan v$

$\Rightarrow \frac{dx}{x}+\frac{dv}{v-(v-\tan v)}=0 \Rightarrow \int \frac{dx}{x}+\int \cot v\,dv=0 \Rightarrow \ln|x|+\ln|\sin v|=C \Rightarrow \ln|x|+\ln\left|\sin\frac{y}{x}\right|=C$