Exercise 8.1 Define

$$f(x)=\begin{cases} e^{-1/x^2} & (x \neq 0), \\ 0 & (x=0). \end{cases}$$

Prove that f has derivatives of all orders at $x=0$ and that $f^{(n)}(0)=0$ for $n=1,2,3,\ldots$.

Solution. We have $\lim\limits_{x\to 0} x^k e^{-1/x^2}=0$ for all $k=0,\pm 1,\pm 2,\ldots$ by L'Hospital's rule. It is easily shown by induction that there is a polynomial p_n such that $f^{(n)}(x)=p_n\left(\frac{1}{x}\right)e^{-1/x^2}$ for $x\neq 0$. Assuming (by induction) that $f^{(n)}(0)=0$, we then have $f^{(n+1)}(0)=\lim\limits_{x\to 0} q_n\left(\dfrac{1}{x}\right)e^{-1/x^2}=0$, where $q_n(x)=xp_n(x)$.