STA2001 Assignment 7

1. (4.1-3). Let the joint pmf of X and Y be defined by

$$f(x,y) = \frac{x+y}{32}$$

$x = 1, 2, y = 1, 2, 3, 4.$

(a) Find $f_X(x)$, the marginal pmf of X
(b) Find $f_Y(y)$, the marginal pmf of Y
(c) Find $P(X > Y)$
(d) Find $P(Y = 2X)$
(e) Find $P(X + Y = 3)$
(f) Find $P(X \leq 3 - Y)$
(g) Are X and Y independent or dependent? Why or why not?
(h) Find the means and the variances of X and Y

2. (4.1-4). Select an (even) integer randomly from the set $\{12, 14, 16, 18, 20, 22\}$. Then select an integer randomly from the set $\{12, 13, 14, 15, 16, 17\}$. Let X equal the integer that is selected from the first set and let Y equal the sum of the two integers.

(a) Show the joint pmf of X and Y on the space of X and Y.
(b) Compute the marginal pmfs.
(c) Are X and Y independent? Why or why not?

3. (4.1-5). Roll a pair of four-sided dice, one red and one black. Let X equal the outcome on the red die and let Y equal the sum of the two dice.

(a) On graph paper, describe the space of X and Y.
(b) Define the joint pmf on the space (similar to Figure 4.1-1).
(c) Give the marginal pmf of X in the margin.
(d) Give the marginal pmf of Y in the margin.
(e) Are X and Y dependent or independent? Why or why not?

4. (4.1-8). In a smoking survey among men between the ages of 25 and 30. 63% prefer to date nonsmokers, 13% prefer to date smokers, and 24% dont care. Suppose nine such men are selected randomly. Let X equal the number who prefer to date nonsmokers and Y equal the number who prefer to date smokers.

(a) Determine the joint pmf of X and Y. Be sure to include the support of the pmf.
(b) Find the marginal pmf of X. Again include the support.

5. (4.1-9). A manufactured item is classified as good, a second, or defective with probabilities 6/10, 3/10, and 1/10, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.

(a) Give the joint pmf of X and Y, $f(x, y)$.
(b) Sketch the set of integers (x, y) for which $f(x, y) > 0$. From the shape of this region, can X and Y be independent? Why or why not?
(c) Find $P(X = 10, Y = 4)$.
(d) Give the marginal pmf of X.
(e) Find $P(X \leq 11)$.

6. (4.2-2). Let X and Y have the joint pmf defined by $f(0,0) = f(1,2) = 0.2$, $f(0,1) = f(1,1) = 0.3$.
   (a) Depict the points and corresponding probabilities on a graph.
   (b) Give the marginal pmfs in the 'margins.'
   (c) Compute $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \mathrm{Cov}(X,Y)$, and ρ.

7. (4.2-3). Roll a fair four-sided die twice. Let X equal the outcome on the first roll, and let Y equal the sum of the two rolls. Determine $\mu_X, \mu_Y, \sigma_x^2, \sigma_Y^2, \mathrm{Cov}(X,Y)$, and ρ.

8. (4.2-9). A car dealer sells X cars each day and always tries to sell an extended warranty on each of these cars. (In our opinion, most of these warranties are not good deals.) Let Y be the number of extended warranties sold; then $Y \leq X$. The joint pmf of X and Y is given by

$$f(x,y) = c(x+1)(4-x)(y+1)(3-y)$$

   $x = 0,1,2,3$, $y = 0,1,2$, with $y \leq x$.
   (a) Find the value of c.
   (b) Sketch the support of X and Y.
   (c) Record the marginal pmfs $f_X(x)$ and $f_Y(y)$ in the margins.
   (d) Are X and Y independent?
   (e) Compute μ_X and σ_X^2.
   (f) Compute μ_Y and σ_Y^2.
   (g) Compute $\mathrm{Cov}(X,Y)$
   (h) Determine ρ, the correlation coefficient.

9. Let Y_1, Y_2, Y_3 be independent random variables which have the Bernoulli distribution with the probability of success p.
   (a) Define a new random variable $Z = Y_1 + Y_2 + Y_3$. For independent Y_1, Y_2, Y_3, we have

$$\mathrm{E}(e^{tZ}) = \mathrm{E}(e^{t(Y_1+Y_2+Y_3)}) = \mathrm{E}(e^{tY_1})\mathrm{E}(e^{tY_2})\mathrm{E}(e^{tY_3}) = (1-p+pe^t)^3$$

   What's the distribution of Z?
   (b) For $k = 1,2$, let

$$X_k = \begin{cases} 1, & Y_1+Y_2+Y_3 = k, \\ -1 & Y_1+Y_2+Y_3 \neq k. \end{cases}$$

      i. Find the joint pmf of X_1, X_2.
      ii. Find the marginal pmfs of X_1 and X_2, respectively.
      iii. Find the value of the success probability p that minimizes $\mathrm{E}(X_1X_2)$.
      iv. Compute $\mathrm{Cov}(X_1 - X_2, X_2)$.

10. (2023 Final Q10) Suppose the joint distribution of two discrete random variables X, Y is given as

$$P_{XY}(n,m) = \frac{\lambda^n e^{-\lambda}}{n!} \binom{n}{m} p^m (1-p)^{n-m}$$

   where $0 \leq m \leq n$ and $0 \leq p \leq 1$.
   (a) Find the marginal pmf $P_Y(m)$.

(b) Find the marginal pmf $P_X(n)$.

(c) Find the conditional pmf $P_{X|Y}(n|m)$.

(d) (optional) Find the expectation $E[XY]$.

Hint: $\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$