STA2001 Assignment 8

1. (4.3-1). Let X and Y have the joint pmf

$$f(x,y)=\frac{x+y}{32},\quad x=1,2\quad y=1,2,3,4$$

(a) Display the joint pmf and the marginal pmfs on a graph like Figure 4.3-1(a).
(b) Find $g(x|y)$ and draw a figure like Figure 4.3-1(b), depicting the conditional pmfs for $y=1,2,3$, and 4.
(c) Find $h(y|x)$ and draw a figure like Figure 4.3-1(c), depicting the conditional pmfs for $x=1$ and 2 .
(d) Find $P(1\le Y\le 3|X=1)$, $P(Y\le 2|X=2)$, and $P(X=2|Y=3)$.
(e) Find $E(Y|X=1)$ and $\mathrm{Var}(Y|X=1)$.

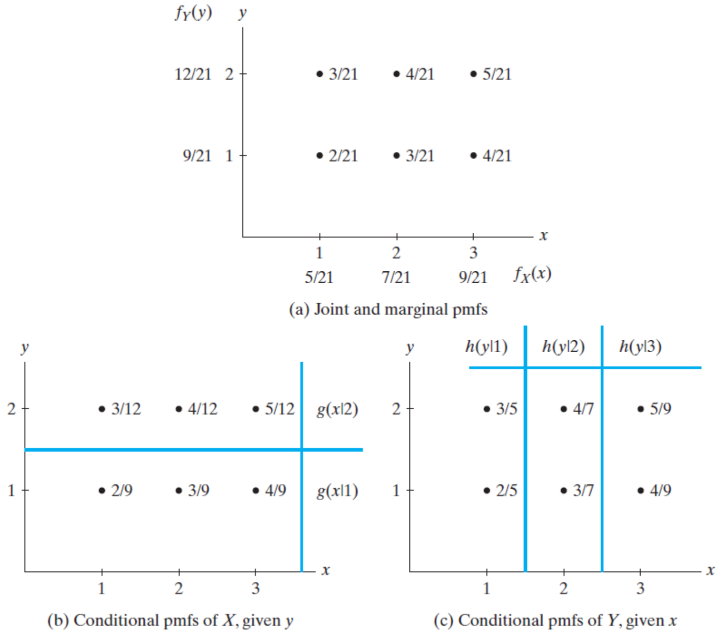

Figure 4.3-1 Joint, marginal, and conditional pmfs

2. (4.3-8). A fair six-sided die is rolled 30 independent times. Let X be the number of ones and Y the number of twos.

(a) What is the joint pmf of X and Y?
(b) Find the conditional pmf of X, given $Y=y$.
(c) Compute $E\left(X^2-4XY+3Y^2\right)$.
(d) Prove that $Cov(X,Y)=-30P_xP_y$, where P_x and P_y are the probabilities of getting number one and two for rolling one time, respectively.

3. (4.4-7). Let $f(x,y)=4/3$, $0<x<1$, $x^3<y<1$, zero elsewhere.

(a) Sketch the region where $f(x,y)>0$.
(b) Find $P(X>Y)$.

4. (4.4-9). Two construction companies make bids of X and Y (in \$100,000 's) on a remodeling project. The joint pdf of X and Y is uniform on the space $2<x<2.5$, $2<y<2.3$. If X and Y are within

0.1 of each other, the companies will be asked to rebid; otherwise, the low bidder will be awarded the contract. What is the probability that they will be asked to rebid?

5. (4.4-15). An automobile repair shop makes an initial estimate X (in thousands of dollars) of the amount of money needed to fix a car after an accident. Say X has the pdf

$$f(x) = 2e^{-2(x-0.2)}, \quad 0.2 < x < \infty$$

Given that $X = x$, the final payment Y has a uniform distribution between $x - 0.1$ and $x + 0.1$. What is the expected value of Y ?

6. (4.4-18). Let $f(x, y) = 1/8$, $0 \le y \le 4$, $y \le x \le y + 2$, be the joint pdf of X and Y.
(a) Sketch the region for which $f(x, y) > 0$.
(b) Find $f_X(x)$, the marginal pdf of X.
(c) Find $f_Y(y)$, the marginal pdf of Y.
(d) Determine $h(y|x)$, the conditional pdf of Y, given that $X = x$.
(e) Determine $g(x|y)$, the conditional pdf of X, given that $Y = y$.
(f) Compute $E(Y|x)$, the conditional mean of Y, given that $X = x$.
(g) Compute $E(X|y)$, the conditional mean of X, given that $Y = y$.
(h) Graph $y = E(Y|x)$ on your sketch in part (a). Is $y = E(Y|x)$ linear?
(i) Graph $x = E(X|y)$ on your sketch in part (a). Is $x = E(X|y)$ linear?

7. (4.5-1). Let X and Y have a bivariate normal distribution with parameters $\mu_X = -3$, $\mu_Y = 10, \sigma_X^2 = 25$, $\sigma_Y^2 = 9$, and $\rho = 3/5$. Compute
(a) $P(-5 < X < 5)$
(b) $P(-5 < X < 5|Y = 13)$
(c) $P(7 < Y < 16)$
(d) $P(7 < Y < 16|X = 2)$

8. (4.5-6). For a freshman taking introductory statistics and majoring in psychology, let X equal the student's ACT mathematics score and Y the students ACT verbal score. Assume that X and Y have a bivariate normal distribution with $\mu_X = 22.7$, $\sigma_X^2 = 17.64$, $\mu_Y = 22.7, \sigma_Y^2 = 12.25$, and $\rho = 0.78$.
(a) Find $P(18.5 < Y < 25.5)$.
(b) Find $E(Y|x)$.
(c) Find $\mathrm{Var}(Y|x)$.
(d) Find $P(18.5 < Y < 25.5|X = 23)$.
(e) Find $P(18.5 < Y < 25.5|X = 25)$.
(f) For $x = 21, 23$, and 25, draw a graph of $z = h(y|x)$ similar to Figure 4.5-1.

9. Let

$$f(x, y) = \left(\frac{1}{2\pi}\right) e^{-(x^2+y^2)/2} \left[1 + xye^{-(x^2+y^2-2)/2}\right], \quad -\infty < x < \infty, -\infty < y < \infty$$

Show that $f(x, y)$ is a joint pdf and the two marginal pdfs are each normal. Note that X and Y can each be normal, but their joint pdf is not bivariate normal.

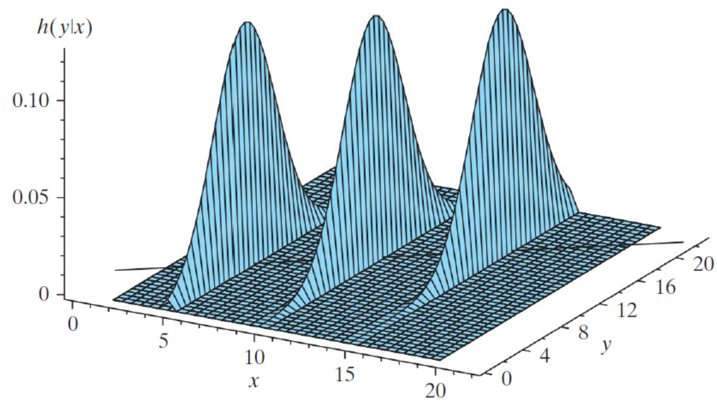

Figure 4.5-1 Conditional pdf of Y, given that $x = 5, 10, 15$

10. Assume that X and Y are bivariate normal distributed with their joint probability density function described by

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}q(x,y)\right), x \in \mathbb{R}, y \in \mathbb{R},$$
$$q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]. \tag{1}$$

where $\mu_X \in \mathbb{R}, \mu_Y \in \mathbb{R}, \sigma_X > 0, \sigma_Y > 0$ and $|\rho| \leq 1$.

(a1) Prove that the marginal probability density function of X is the probability density function of $N(\mu_X, \sigma_X^2)$, and the condition probability density function of Y given $X = x$ is the probability density function of

$$N(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x-\mu_X), (1-\rho^2)\sigma_Y^2). \tag{2}$$

In other words, prove that $X \sim N(\mu_X, \sigma_X^2)$, and $Y|X = x \sim N(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x-\mu_X), (1-\rho^2)\sigma_Y^2)$.

(a2) Assume that $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$. Then prove that ρ in (1) is the correlation coefficient of X and Y, and moreover, X and Y are independent if and only if X and Y are uncorrelated.