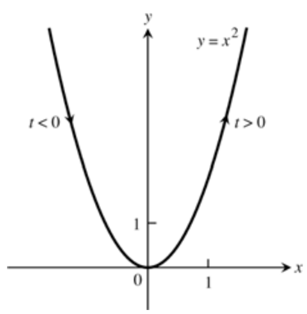
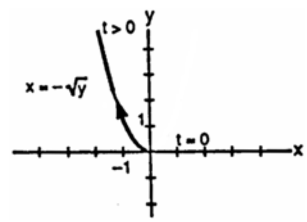
CHAPTER 11 PARAMETRIC EQUATIONS AND POLAR COORDINATES

11.1 PARAMETRIZATIONS OF PLANE CURVES

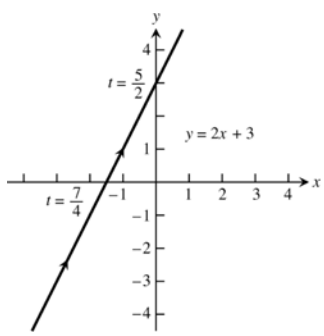
1. $x = 3t,\ y = 9t^2,\ -\infty < t < \infty \Rightarrow y = x^2$



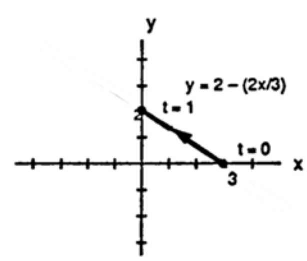
2. $x = -\sqrt{t},\ y = t,\ t \ge 0 \Rightarrow x = -\sqrt{y}$
or $y = x^2,\ x \le 0$



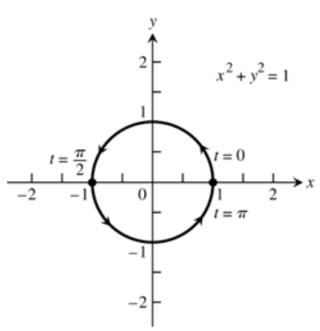
3. $x = 2t - 5,\ y = 4t - 7,\ -\infty < t < \infty$
$\Rightarrow x + 5 = 2t \Rightarrow 2(x+5) = 4t$
$\Rightarrow y = 2(x+5) - 7 \Rightarrow y = 2x + 3$



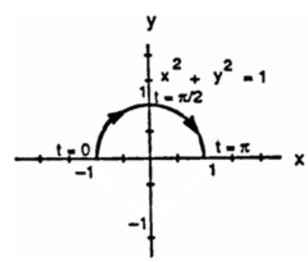
4. $x = 3 - 3t,\ y = 2t,\ 0 \le t \le 1 \Rightarrow \frac{y}{2} = t$
$\Rightarrow x = 3 - 3\left(\frac{y}{2}\right) \Rightarrow 2x = 6 - 3y$
$\Rightarrow y = 2 - \frac{2}{3}x,\ 0 \le x \le 3$



5. $x = \cos 2t,\ y = \sin 2t,\ 0 \le t \le \pi$
$\Rightarrow \cos^2 2t + \sin^2 2t = 1 \Rightarrow x^2 + y^2 = 1$

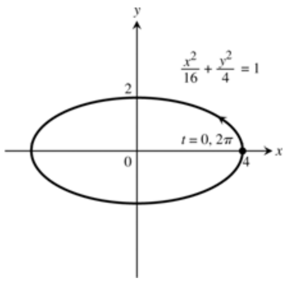


6. $x = \cos(\pi - t),\ y = \sin(\pi - t),\ 0 \le t \le \pi$
$\Rightarrow \cos^2(\pi - t) + \sin^2(\pi - t) = 1$
$\Rightarrow x^2 + y^2 = 1,\ y \ge 0$



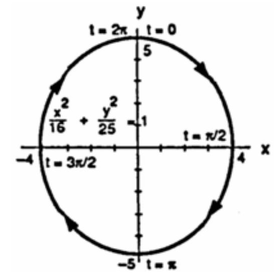
7. $x = 4\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$

$\Rightarrow \frac{16\cos^2 t}{16} + \frac{4\sin^2 t}{4} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$



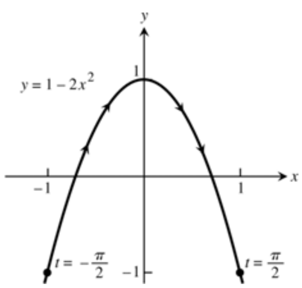
8. $x = 4\sin t,\ y = 5\cos t, 0 \le t \le 2\pi$

$\Rightarrow \frac{16\sin^2 t}{16} + \frac{25\cos^2 t}{25} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{25} = 1$



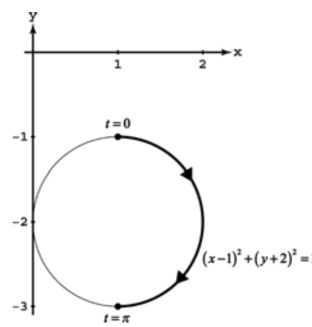
9. $x = \sin t,\ y = \cos 2t, -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$\Rightarrow y = \cos 2t = 1 - 2\sin^2 t \Rightarrow y = 1 - 2x^2$



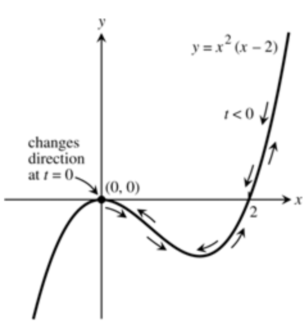
10. $x = 1 + \sin t,\ y = \cos t - 2, 0 \le t \le \pi$

$\Rightarrow \sin^2 t + \cos^2 t = 1 \Rightarrow (x-1)^2 + (y+2)^2 = 1$



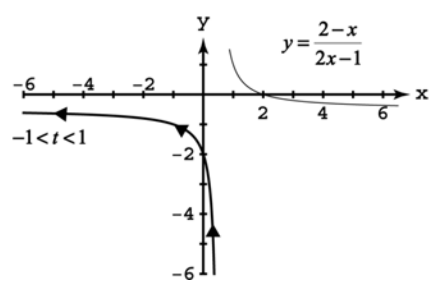
11. $x = t^2,\ y = t^6 - 2t^4, -\infty < t < \infty$

$\Rightarrow y = \left(t^2\right)^3 - 2\left(t^2\right)^2 \Rightarrow y = x^3 - 2x^2$



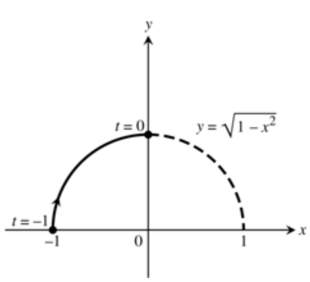
12. $x = \frac{t}{t-1},\ y = \frac{t-2}{t+1}, -1 < t < 1$

$\Rightarrow t = \frac{x}{x-1}, \Rightarrow y = \frac{2-x}{2x-1}$



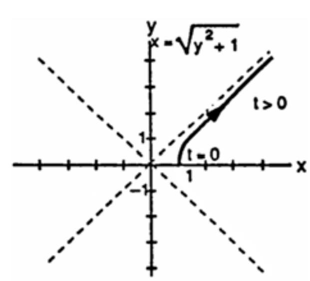
13. $x = t,\ y = \sqrt{1-t^2}, -1 \le t \le 0$

$\Rightarrow y = \sqrt{1-x^2}$

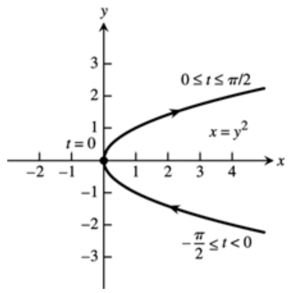


14. $x = \sqrt{t+1},\ y = \sqrt{t},\ t \ge 0$

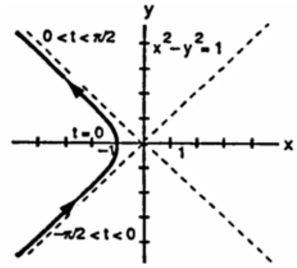
$\Rightarrow y^2 = t \Rightarrow x = \sqrt{y^2+1},\ y \ge 0$



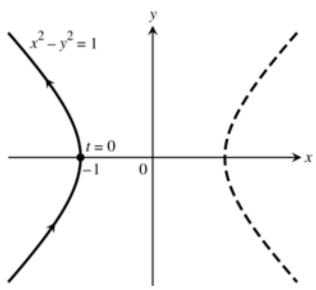
15. $x = \sec^2 t - 1,\ y = \tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}$
$\Rightarrow \sec^2 t - 1 = \tan^2 t \Rightarrow x = y^2$



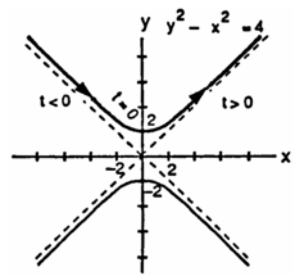
16. $x = -\sec t,\ y = \tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}$
$\Rightarrow \sec^2 t - \tan^2 t = 1 \Rightarrow x^2 - y^2 = 1$



17. $x = -\cosh t,\ y = \sinh t, -\infty < 1 < \infty$
$\Rightarrow \cosh^2 t - \sinh^2 t = 1 \Rightarrow x^2 - y^2 = 1$



18. $x = 2 \sinh t,\ y = 2 \cosh t, -\infty < t < \infty$
$\Rightarrow 4 \cosh^2 t - 4 \sinh^2 t = 4 \Rightarrow y^2 - x^2 = 4$



19. (a) $x = a \cos t, y = -a \sin t, 0 \le t \le 2\pi$
(b) $x = a \cos t, y = a \sin t, 0 \le t \le 2\pi$
(c) $x = a \cos t, y = -a \sin t, 0 \le t \le 4\pi$
(d) $x = a \cos t, y = a \sin t, 0 \le t \le 4\pi$

20. (a) $x = a \sin t, y = b \cos t, \frac{\pi}{2} \le t \le \frac{5\pi}{2}$
(b) $x = a \cos t, y = b \sin t, 0 \le t \le 2\pi$
(c) $x = a \sin t, y = b \cos t, \frac{\pi}{2} \le t \le \frac{9\pi}{2}$
(d) $x = a \cos t, y = b \sin t, 0 \le t \le 4\pi$

21. Using $(-1, -3)$ we create the parametric equations $x = -1 + at$ and $y = -3 + bt$, representing a line which goes through $(-1, -3)$ at $t = 0$. We determine a and b so that the line goes through $(4, 1)$ when $t = 1$. Since $4 = -1 + a \Rightarrow a = 5$. Since $1 = -3 + b \Rightarrow b = 4$. Therefore, one possible parameterization is $x = -1 + 5t$, $y = -3 + 4t, 0 \le t \le 1$.

22. Using $(-1, 3)$ we create the parametric equations $x = -1 + at$ and $y = 3 + bt$, representing a line which goes through $(-1, 3)$ at $t = 0$. We determine a and b so that the line goes through $(3, -2)$ when $t = 1$. Since $3 = -1 + a \Rightarrow a = 4$. Since $-2 = 3 + b \Rightarrow b = -5$. Therefore, one possible parameterization is $x = -1 + 4t$, $y = 3 - 5t, 0 \le t \le 1$.

23. The lower half of the parabola is given by $x = y^2 + 1$ for $y \le 0$. Substituting t for y, we obtain one possible parameterization $x = t^2 + 1, y = t, t \le 0$.

24. The vertex of the parabola is at $(-1, -1)$, so the left half of the parabola is given by $y = x^2 + 2x$ for $x \le -1$. Substituting t for x, we obtain one possible parameterization: $x = t, y = t^2 + 2t, t \le -1$.

25. For simplicity, we assume that x and y are linear functions of t and that the point (x, y) starts at $(2, 3)$ for $t = 0$ and passes through $(-1, -1)$ at $t = 1$. Then $x = f(t)$, where $f(0) = 2$ and $f(1) = -1$. Since slope $= \frac{\Delta x}{\Delta t} = \frac{-1-2}{1-0} = -3$, $x = f(t) = -3t + 2 = 2 - 3t$. Also, $y = g(t)$, where $g(0) = 3$ and $g(1) = -1$. Since slope $= \frac{\Delta y}{\Delta t} = \frac{-1-3}{1-0} = -4$. $y = g(t) = -4t + 3 = 3 - 4t$. One possible parameterization is: $x = 2 - 3t$, $y = 3 - 4t, t \ge 0$.

26. For simplicity, we assume that x and y are linear functions of t and that the point (x, y) starts at $(-1, 2)$ for $t = 0$ and passes through $(0, 0)$ at $t = 1$. Then $x = f(t)$, where $f(0) = -1$ and $f(1) = 0$. Since slope $= \frac{\Delta x}{\Delta t} = \frac{0-(-1)}{1-0} = 1$, $x = f(t) = 1t + (-1) = -1 + t$. Also, $y = g(t)$, where $g(0) = 2$ and $g(1) = 0$. Since slope $= \frac{\Delta y}{\Delta t} = \frac{0-2}{1-0} = -2$. $y = g(t) = -2t + 2 = 2 - 2t$. One possible parameterization is: $x = -1 + t$, $y = 2 - 2t, t \ge 0$.

27. Since we only want the top half of a circle, $y \ge 0$, so let $x = 2\cos t, y = 2|\sin t|, 0 \le t \le 4\pi$

28. Since we want x to stay between -3 and 3, let $x = 3\sin t$, then $y = (3\sin t)^2 = 9\sin^2 t$, thus $x = 3\sin t$, $y = 9\sin^2 t, 0 \le t < \infty$

29. $x^2 + y^2 = a^2 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$; let $t = \frac{dy}{dx} \Rightarrow -\frac{x}{y} = t \Rightarrow x = -yt$. Substitution yields $y^2t^2 + y^2 = a^2 \Rightarrow y = \frac{a}{\sqrt{1+t^2}}$ and $x = \frac{-at}{\sqrt{1+t}}, -\infty < t < \infty$

30. In terms of θ, parametric equations for the circle are $x = a\cos\theta, y = a\sin\theta, 0 \le \theta < 2\pi$. Since $\theta = \frac{s}{a}$, the arc length parametrizations are: $x = a\cos\frac{s}{a}$, $y = a\sin\frac{s}{a}$, and $0 \le \frac{s}{a} < 2\pi \Rightarrow 0 \le s \le 2\pi a$ is the interval for s.

31. Drop a vertical line from the point (x, y) to the x-axis, then θ is an angle in a right triangle, and from trigonometry we know that $\tan\theta = \frac{y}{x} \Rightarrow y = x\tan\theta$. The equation of the line through $(0, 2)$ and $(4, 0)$ is given by $y = -\frac{1}{2}x + 2$. Thus $x\tan\theta = -\frac{1}{2}x + 2 \Rightarrow x = \frac{4}{2\tan\theta+1}$ and $y = \frac{4\tan\theta}{2\tan\theta+1}$ where $0 \le \theta \le \frac{\pi}{2}$.

32. Drop a vertical line from the point (x, y) to the x-axis, then θ is an angle in a right triangle, and from trigonometry we know that $\tan\theta = \frac{y}{x} \Rightarrow y = x\tan\theta$. Since $y = \sqrt{x} \Rightarrow y^2 = x \Rightarrow (x\tan\theta)^2 = x$ $\Rightarrow x = \cot^2\theta \Rightarrow y = \cot\theta$ where $0 < \theta \le \frac{\pi}{2}$.

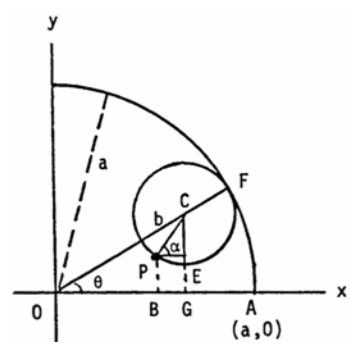
33. The equation of the circle is given by $(x-2)^2 + y^2 = 1$. Drop a vertical line from the point (x, y) on the circle to the x-axis, then θ is an angle in a right triangle. So that we can start at $(1, 0)$ and rotate in a clockwise direction, let $x = 2 - \cos\theta$, $y = \sin\theta$, $0 \le \theta \le 2\pi$.

34. Drop a vertical line from the point (x, y) to the x-axis, then θ is an angle in a right triangle, whose height is y and whose base is $x + 2$. By trigonometry we have $\tan\theta = \frac{y}{x+2} \Rightarrow y = (x+2)\tan\theta$. The equation of the circle

is given by $x^2+y^2=1 \Rightarrow x^2+\left((x+2)\tan\theta\right)^2=1 \Rightarrow x^2\sec^2\theta+4x\tan^2\theta+4\tan^2\theta-1=0.$ Solving for x we obtain $x=\frac{-4\tan^2\theta\pm\sqrt{\left(4\tan^2\theta\right)^2-4\sec^2\theta\left(4\tan^2\theta-1\right)}}{2\sec^2\theta}=\frac{-4\tan^2\theta\pm2\sqrt{1-3\tan^2\theta}}{2\sec^2\theta}=-2\sin^2\theta\pm\cos\theta\sqrt{\cos^2\theta-3\sin^2\theta}$ $=-2+2\cos^2\theta\pm\cos\theta\sqrt{4\cos^2\theta-3}$ and $y=\left(-2+2\cos^2\theta\pm\cos\theta\sqrt{4\cos^2\theta-3}+2\right)\tan\theta$ $=2\sin\theta\cos\theta\pm\sin\theta\sqrt{4\cos^2\theta-3}.$ Since we only need to go from $(1,0)$ to $(0,1)$, let $x=-2+2\cos^2\theta+\cos\theta\sqrt{4\cos^2\theta-3},\ \ y=2\sin\theta\cos\theta+\sin\theta\sqrt{4\cos^2\theta-3}, 0\le\theta\le\tan^{-1}\left(\frac{1}{2}\right).$ To obtain the upper limit for θ, note that $x=0$ and $y=1$, using $y=(x+2)\tan\theta\Rightarrow 1=2\tan\theta\Rightarrow\theta=\tan^{-1}\left(\frac{1}{2}\right).$

35. Extend the vertical line through A to the x-axis and let C be the point of intersection. Then $OC=AQ=x$ and $\tan t=\frac{2}{OC}=\frac{2}{x}\Rightarrow x=\frac{2}{\tan t}=2\cot t;$ $\sin t=\frac{2}{OA}\Rightarrow OA=\frac{2}{\sin t};$ and $(AB)(OA)=(AQ)^2\Rightarrow AB\left(\frac{2}{\sin t}\right)=x^2$ $\Rightarrow AB\left(\frac{2}{\sin t}\right)=\left(\frac{2}{\tan t}\right)^2\Rightarrow AB=\frac{2\sin t}{\tan^2 t}.$ Next $y=2-AB\sin t\Rightarrow y=2-\left(\frac{2\sin t}{\tan^2 t}\right)\sin t=2-\frac{2\sin^2 t}{\tan^2 t}$ $=2-2\cos^2 t=2\sin^2 t.$ Therefore let $x=2\cot t$ and $y=2\sin^2 t,\ \ 0<t<\pi.$

36. Arc PF = Arc AF since each is the distance rolled and



$\frac{\text{Arc }PF}{b}=\angle FCP\Rightarrow \text{Arc }PF=b(\angle FCP);$

$\frac{\text{Arc }AF}{a}=\theta\Rightarrow \text{Arc }AF=a\theta\Rightarrow a\theta=b(\angle FCP)\Rightarrow\angle FCP=\frac{a}{b}\theta;$

$\angle OCG=\frac{\pi}{2}-\theta;\ \ \angle OCG=\angle OCP+\angle PCE=\angle OCP+\left(\frac{\pi}{2}-\alpha\right).$

Now $\angle OCP=\pi-\angle FCP=\pi-\frac{a}{b}\theta.$ Thus $\angle OCG=\pi-\frac{a}{b}\theta+\frac{\pi}{2}-\alpha$

$\Rightarrow\frac{\pi}{2}-\theta=\pi-\frac{a}{b}\theta+\frac{\pi}{2}-\alpha\Rightarrow\alpha=\pi-\frac{a}{b}\theta+\theta=\pi-\left(\frac{a-b}{b}\theta\right).$

Then $x=OG-BG=OG-PE=(a-b)\cos\theta-b\cos\alpha$

$=(a-b)\cos\theta-b\cos\left(\pi-\frac{a-b}{b}\theta\right)=(a-b)\cos\theta+b\cos\left(\frac{a-b}{b}\theta\right).$

Also $y=EG=CG-CE\ =(a-b)\sin\theta-b\sin\alpha=(a-b)\sin\theta-b\sin\left(\pi-\frac{a-b}{b}\theta\right)$

$=(a-b)\sin\theta-b\sin\left(\frac{a-b}{b}\theta\right).$ Therefore $x=(a-b)\cos\theta+b\cos\left(\frac{a-b}{b}\theta\right)$ and $y=(a-b)\sin\theta-b\sin\left(\frac{a-b}{b}\theta\right).$

If $b=\frac{a}{4}$, then $x=\left(a-\frac{a}{4}\right)\cos\theta+\frac{a}{4}\cos\left(\frac{a-\left(\frac{a}{4}\right)}{\left(\frac{a}{4}\right)}\theta\right)=\frac{3a}{4}\cos\theta+\frac{a}{4}\cos 3\theta$

$=\frac{3a}{4}\cos\theta+\frac{a}{4}(\cos\theta\cos2\theta-\sin\theta\sin2\theta)=\frac{3a}{4}\cos\theta+\frac{a}{4}\left((\cos\theta)\left(\cos^2\theta-\sin^2\theta\right)-(\sin\theta)(2\sin\theta\cos\theta)\right)$

$=\frac{3a}{4}\cos\theta+\frac{a}{4}\cos^3\theta-\frac{a}{4}\cos\theta\sin^2\theta-\frac{2a}{4}\sin^2\theta\cos\theta=\frac{3a}{4}\cos\theta+\frac{a}{4}\cos^3\theta-\frac{3a}{4}(\cos\theta)\left(1-\cos^2\theta\right)$

$=a\cos^3\theta;\ \ y=\left(a-\frac{a}{4}\right)\sin\theta-\frac{a}{4}\sin\left(\frac{a-\left(\frac{a}{4}\right)}{\left(\frac{a}{4}\right)}\theta\right)=\frac{3a}{4}\sin\theta-\frac{a}{4}\sin3\theta$

$=\frac{3a}{4}\sin\theta-\frac{a}{4}(\sin\theta\cos2\theta+\cos\theta\sin2\theta)=\frac{3a}{4}\sin\theta-\frac{a}{4}\left((\sin\theta)\left(\cos^2\theta-\sin^2\theta\right)+(\cos\theta)(2\sin\theta\cos\theta)\right)$

$=\frac{3a}{4}\sin\theta-\frac{a}{4}\sin\theta\cos^2\theta+\frac{a}{4}\sin^3\theta-\frac{2a}{4}\cos^2\theta\sin\theta=\frac{3a}{4}\sin\theta-\frac{3a}{4}\sin\theta\cos^2\theta+\frac{a}{4}\sin^3\theta$

$=\frac{3a}{4}\sin\theta-\frac{3a}{4}(\sin\theta)\left(1-\sin^2\theta\right)+\frac{a}{4}\sin^3\theta=a\sin^3\theta.$

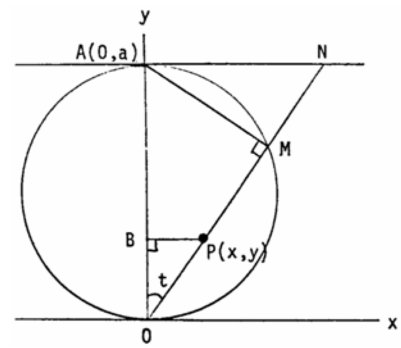
37. Draw line AM in the figure and note that $\angle AMO$ is a right angle since it is an inscribed angle which spans the diameter of a circle. Then $AN^2 = MN^2 + AM^2$. Now, $OA = a$, $\frac{AN}{a} = \tan t$, and $\frac{AM}{a} = \sin t$. Next $MN = OP$

$\Rightarrow OP^2 = AN^2 - AM^2 = a^2 \tan^2 t - a^2 \sin^2 t$

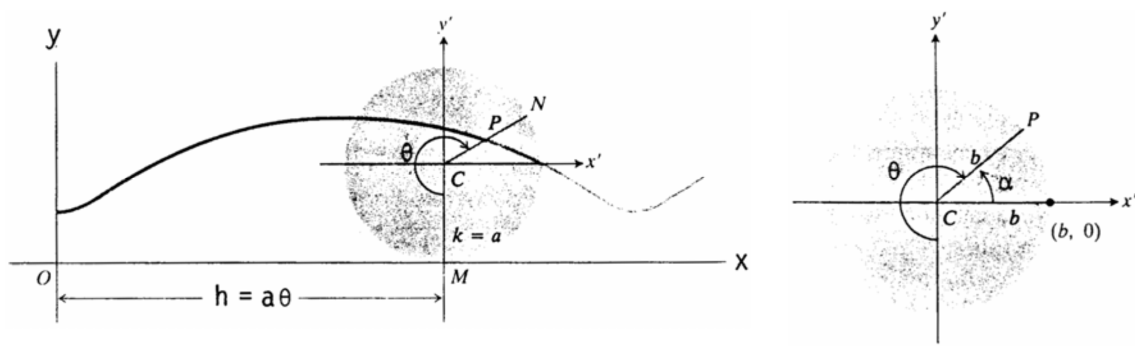
$\Rightarrow OP = \sqrt{a^2 \tan^2 t - a^2 \sin^2 t} = (a \sin t)\sqrt{\sec^2 t - 1} = \frac{a \sin^2 t}{\cos t}$.

In triangle BPO, $x = OP \sin t = \frac{a \sin^3 t}{\cos t} = a \sin^2 t \tan t$ and

$y = OP \cos t = a \sin^2 t \Rightarrow x = a \sin^2 t \tan t$ and $y = a \sin^2 t$.



38. Let the x-axis be the line the wheel rolls along with the y-axis through a low point of the trochoid (see the accompanying figure).



Let θ denote the angle through which the wheel turns. Then $h = a\theta$ and $k = a$. Next introduce $x'y'$-axes parallel to the xy-axes and having their origin at the center C of the wheel. Then $x' = b \cos \alpha$ and $y' = b \sin \alpha$, where $\alpha = \frac{3\pi}{2} - \theta$. It follows that $x' = b \cos\left(\frac{3\pi}{2} - \theta\right) = -b \sin \theta$ and $y' = b \sin\left(\frac{3\pi}{2} - \theta\right) = -b \cos \theta \Rightarrow x = h + x' = a\theta - b \sin \theta$ and $y = k + y' = a - b \cos \theta$ are parametric equations of the trochoid.

39. $D = \sqrt{(x-2)^2 + \left(y - \frac{1}{2}\right)^2} \Rightarrow D^2 = (x-2)^2 + \left(y - \frac{1}{2}\right)^2 = (t-2)^2 + \left(t^2 - \frac{1}{2}\right)^2 \Rightarrow D^2 = t^4 - 4t + \frac{17}{4}$

$\Rightarrow \frac{d\left(D^2\right)}{dt} = 4t^3 - 4 = 0 \Rightarrow t = 1$. The second derivative is always positive for $t \neq 0 \Rightarrow t = 1$ gives a local minimum for D^2 (and hence D) which is an absolute minimum since it is the only extremum $\Rightarrow$ the closest point on the parabola is $(1, 1)$.

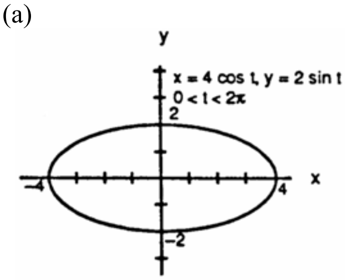
40. $D = \sqrt{\left(2 \cos t - \frac{3}{4}\right)^2 + (\sin t - 0)^2} \Rightarrow D^2 = \left(2 \cos t - \frac{3}{4}\right)^2 + \sin^2 t$

$\Rightarrow \frac{d\left(D^2\right)}{dt} = 2\left(2 \cos t - \frac{3}{4}\right)(-2 \sin t) + 2 \sin t \cos t = (-2 \sin t)\left(3 \cos t - \frac{3}{2}\right) = 0 \Rightarrow -2 \sin t = 0$ or $3 \cos t - \frac{3}{2} = 0$

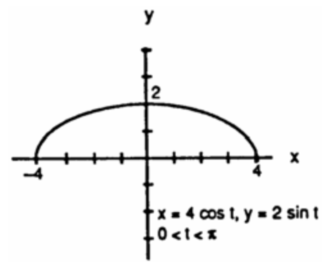
$\Rightarrow t = 0, \pi$ or $t = \frac{\pi}{3}, \frac{5\pi}{3}$. Now $\frac{d^2\left(D^2\right)}{dt^2} = -6 \cos^2 t + 3 \cos t + 6 \sin^2 t$ so that $\frac{d^2\left(D^2\right)}{dt^2}(0) = -3 \Rightarrow$ relative maximum, $\frac{d^2\left(D^2\right)}{dt^2}(\pi) = -9 \Rightarrow$ relative maximum, $\frac{d^2\left(D^2\right)}{dt^2}\left(\frac{\pi}{3}\right) = \frac{9}{2} \Rightarrow$ relative minimum, and

$\frac{d^2(D^2)}{dt^2}\left(\frac{5\pi}{3}\right) = \frac{9}{2} \Rightarrow$ relative minimum. Therefore both $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ give points on the ellipse closest to the point $\left(\frac{3}{4}, 0\right) \Rightarrow \left(1, \frac{\sqrt{3}}{2}\right)$ and $\left(1, -\frac{\sqrt{3}}{2}\right)$ are the desired points.

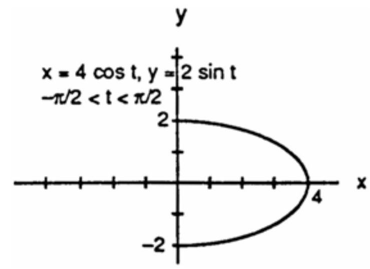
41. (a)



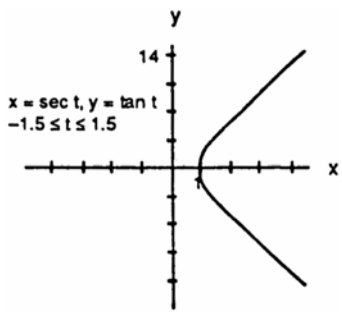
(b)



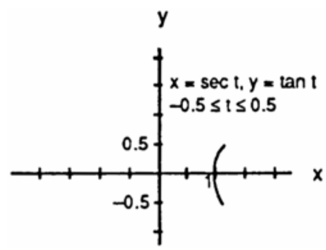
(c)



42. (a)



(b)



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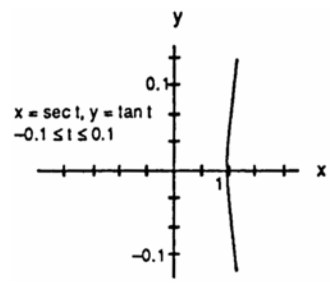
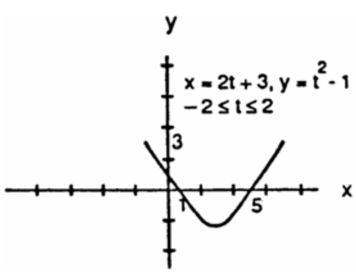
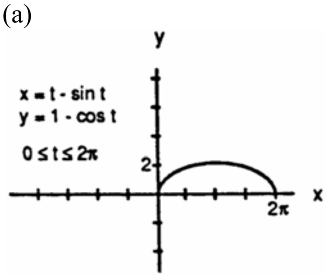


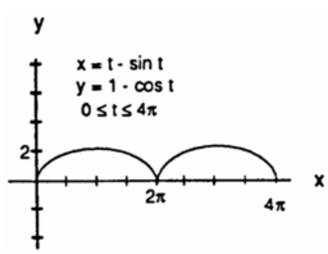
43.



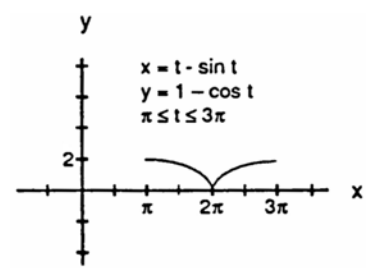
44. (a)



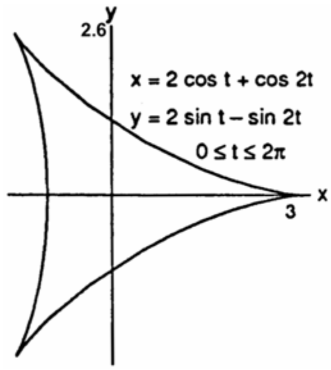
(b)



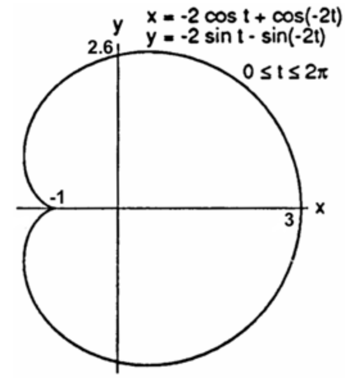
(c)



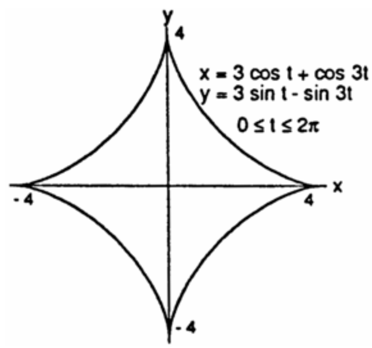
45. (a)



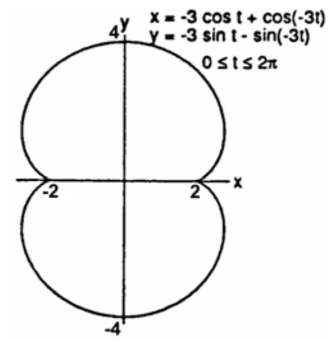
(b)



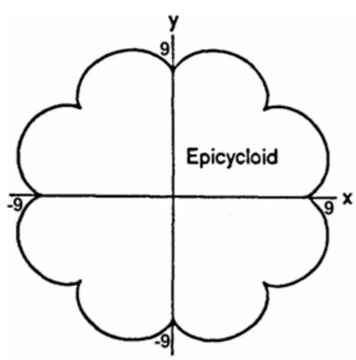
46. (a)



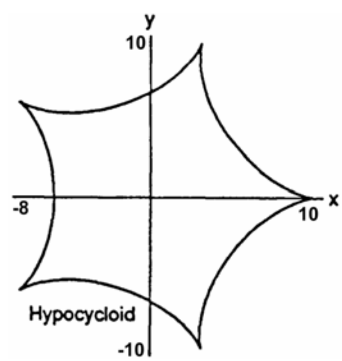
(b)



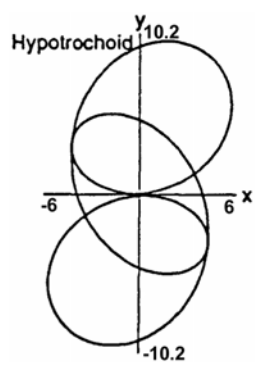
47. (a)



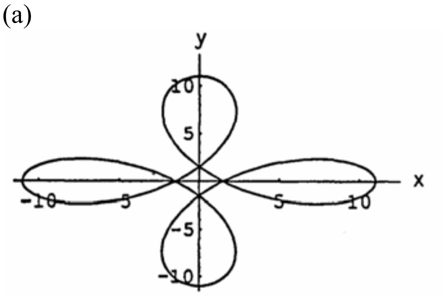
(b)



(c)

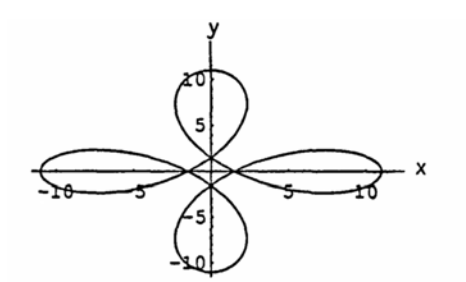


48. (a)



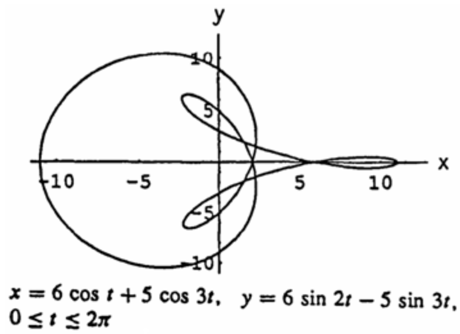
$x = 6\cos t + 5\cos 3t$, $y = 6\sin t - 5\sin 3t$,
$0 \le t \le 2\pi$

(b)

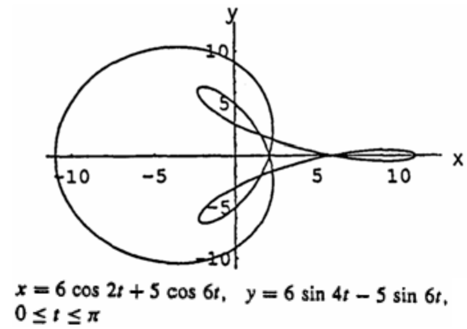


$x = 6\cos 2t + 5\cos 6t$, $y = 6\sin 2t - 5\sin 6t$,
$0 \le t \le \pi$

(c)



(d)



11.2 CALCULUS WITH PARAMETRIC CURVES

1. $t=\frac{\pi}{4} \Rightarrow x=2\cos\frac{\pi}{4}=\sqrt{2},\ \ y=2\sin\frac{\pi}{4}=\sqrt{2};\ \ \frac{dx}{dt}=-2\sin t,\ \ \frac{dy}{dt}=2\cos t \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2\cos t}{-2\sin t}=-\cot t$
$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}=-\cot\frac{\pi}{4}=-1;$ tangent line is $y-\sqrt{2}=-1\left(x-\sqrt{2}\right)$ or $y=-x+2\sqrt{2};\ \ \frac{dy'}{dt}=\csc^2 t$
$\Rightarrow \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=\frac{\csc^2 t}{-2\sin t}=-\frac{1}{2\sin^3 t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}}=-\sqrt{2}$

2. $t=-\frac{1}{6} \Rightarrow x=\sin\left(2\pi\left(-\frac{1}{6}\right)\right)=\sin\left(-\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2},\ \ y=\cos\left(2\pi\left(-\frac{1}{6}\right)\right)=\cos\left(-\frac{\pi}{3}\right)=\frac{1}{2};\ \ \frac{dx}{dt}=2\pi\cos 2\pi t,$
$\frac{dy}{dt}=-2\pi\sin 2\pi t \Rightarrow \frac{dy}{dx}=\frac{-2\pi\sin 2\pi t}{2\pi\cos 2\pi t}=-\tan 2\pi t \Rightarrow \left.\frac{dy}{dx}\right|_{t=-\frac{1}{6}}=-\tan\left(2\pi\left(-\frac{1}{6}\right)\right)=-\tan\left(-\frac{\pi}{3}\right)=\sqrt{3};$
tangent line is $y-\frac{1}{2}=\sqrt{3}\left[x-\left(-\frac{\sqrt{3}}{2}\right)\right]$ or $y=\sqrt{3}x+2;\ \ \frac{dy'}{dt}=-2\pi\sec^2 2\pi t \Rightarrow \frac{d^2y}{dx^2}=\frac{-2\pi\sec^2 2\pi t}{2\pi\cos 2\pi t}$
$=-\frac{1}{\cos^3 2\pi t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-\frac{1}{6}}=-8$

3. $t=\frac{\pi}{4} \Rightarrow x=4\sin\frac{\pi}{4}=2\sqrt{2},\ \ y=2\cos\frac{\pi}{4}=\sqrt{2};\ \ \frac{dx}{dt}=4\cos t,\ \ \frac{dy}{dt}=-2\sin t \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-2\sin t}{4\cos t}=-\frac{1}{2}\tan t$
$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}=-\frac{1}{2}\tan\frac{\pi}{4}=-\frac{1}{2};$ tangent line is $y-\sqrt{2}=-\frac{1}{2}\left(x-2\sqrt{2}\right)$ or $y=-\frac{1}{2}x+2\sqrt{2};\ \ \frac{dy'}{dt}=-\frac{1}{2}\sec^2 t$
$\Rightarrow \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=\frac{-\frac{1}{2}\sec^2 t}{4\cos t}=-\frac{1}{8\cos^3 t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}}=-\frac{\sqrt{2}}{4}$

4. $t=\frac{2\pi}{3} \Rightarrow x=\cos\frac{2\pi}{3}=-\frac{1}{2},\ \ y=\sqrt{3}\cos\frac{2\pi}{3}=-\frac{\sqrt{3}}{2};\ \ \frac{dx}{dt}=-\sin t,\ \ \frac{dy}{dt}=-\sqrt{3}\sin t \Rightarrow \frac{dy}{dx}=\frac{-\sqrt{3}\sin t}{-\sin t}=\sqrt{3}$
$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}}=\sqrt{3};$ tangent line is $y-\left(-\frac{\sqrt{3}}{2}\right)=\sqrt{3}\left[x-\left(-\frac{1}{2}\right)\right]$ or $y=\sqrt{3}\,x;\ \ \frac{dy'}{dt}=0 \Rightarrow \frac{d^2y}{dx^2}=\frac{0}{-\sin t}=0$
$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{2\pi}{3}}=0$

5. $t=\frac{1}{4} \Rightarrow x=\frac{1}{4},\ \ y=\frac{1}{2};\ \ \frac{dx}{dt}=1,\ \ \frac{dy}{dt}=\frac{1}{2\sqrt{t}} \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{1}{2\sqrt{t}} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{1}{4}}=\frac{1}{2\sqrt{\frac{1}{4}}}=1;$ tangent line is $y-\frac{1}{2}=1\cdot\left(x-\frac{1}{4}\right)$ or $y=x+\frac{1}{4};\ \ \frac{dy'}{dt}=-\frac{1}{4}t^{-3/2} \Rightarrow \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=-\frac{1}{4}t^{-3/2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{4}}=-2$

6. $t=-\frac{\pi}{4} \Rightarrow x=\sec^2\left(-\frac{\pi}{4}\right)-1=1,\ \ y=\tan\left(-\frac{\pi}{4}\right)=-1;\ \ \frac{dx}{dt}=2\sec^2 t\tan t,\ \ \frac{dy}{dt}=\sec^2 t$ $\Rightarrow \frac{dy}{dx}=\frac{\sec^2 t}{2\sec^2 t\tan t}=\frac{1}{2\tan t}=\frac{1}{2}\cot t \Rightarrow \left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{4}}=\frac{1}{2}\cot\left(-\frac{\pi}{4}\right)=-\frac{1}{2};$ tangent line is $y-(-1)=-\frac{1}{2}(x-1)$ or $y=-\frac{1}{2}x-\frac{1}{2};\ \ \frac{dy'}{dt}=-\frac{1}{2}\csc^2 t \Rightarrow \frac{d^2y}{dx^2}=\frac{-\frac{1}{2}\csc^2 t}{2\sec^2 t\tan t}=-\frac{1}{4}\cot^3 t \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-\frac{\pi}{4}}=\frac{1}{4}$

7. $t=\frac{\pi}{6} \Rightarrow x=\sec\frac{\pi}{6}=\frac{2}{\sqrt{3}},\ \ y=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}};\ \ \frac{dx}{dt}=\sec t\tan t,\ \ \frac{dy}{dt}=\sec^2 t \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\sec^2 t}{\sec t\tan t}=\csc t$ $\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}}=\csc\frac{\pi}{6}=2;$ tangent line is $y-\frac{1}{\sqrt{3}}=2\left(x-\frac{2}{\sqrt{3}}\right)$ or $y=2x-\sqrt{3};\ \ \frac{dy'}{dt}=-\csc t\cot t$ $\Rightarrow \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=\frac{-\csc t\cot t}{\sec t\tan t}=-\cot^3 t \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{6}}=-3\sqrt{3}$

8. $t=3 \Rightarrow x=-\sqrt{3+1}=-2,\ \ y=\sqrt{3(3)}=3;\ \ \frac{dx}{dt}=-\frac{1}{2}(t+1)^{-1/2},\ \ \frac{dy}{dt}=\frac{3}{2}(3t)^{-1/2}\ \frac{dy}{dx}=\frac{\left(\frac{3}{2}\right)(3t)^{-1/2}}{\left(-\frac{1}{2}\right)(t+1)^{-1/2}}=-\frac{3\sqrt{t+1}}{\sqrt{3t}}$ $\Rightarrow \left.\frac{dy}{dx}\right|_{t=3}=\frac{-3\sqrt{3+1}}{\sqrt{3(3)}}=-2;$ tangent line is $y-3=-2\left(x-(-2)\right)$ or $y=-2x-1;$ $\frac{dy'}{dt}=\frac{\sqrt{3t}\left[-\frac{3}{2}(t+1)^{-1/2}\right]+3\sqrt{t+1}\left[\frac{3}{2}(3t)^{-1/2}\right]}{3t}=\frac{3}{2t\sqrt{3t}\sqrt{t+1}} \Rightarrow \frac{d^2y}{dx^2}=\frac{\left(\frac{3}{2t\sqrt{3t}\sqrt{t+1}}\right)}{\left(\frac{-1}{2\sqrt{t+1}}\right)}=-\frac{3}{t\sqrt{3t}} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=3}=-\frac{1}{3}$

9. $t=-1 \Rightarrow x=5,\ \ y=1;\ \ \frac{dx}{dt}=4t,\ \ \frac{dy}{dt}=4t^3 \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{4t^3}{4t}=t^2 \Rightarrow \left.\frac{dy}{dx}\right|_{t=-1}=(-1)^2=1;$ tangent line is $y-1=1\cdot(x-5)$ or $y=x-4;\ \ \frac{dy'}{dt}=2t \Rightarrow \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=\frac{2t}{4t}=\frac{1}{2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-1}=\frac{1}{2}$

10. $t=1 \Rightarrow x=1,\ \ y=-2;\ \ \frac{dx}{dt}=-\frac{1}{t^2},\ \ \frac{dy}{dt}=\frac{1}{t} \Rightarrow \frac{dy}{dx}=\frac{\left(\frac{1}{t}\right)}{\left(-\frac{1}{t^2}\right)}=-t \Rightarrow \left.\frac{dy}{dx}\right|_{t=1}=-1;$ tangent line is $y-(-2)=-1(x-1)$ or $y=-x-1;\ \ \frac{dy'}{dt}=-1 \Rightarrow \frac{d^2y}{dx^2}=\frac{-1}{\left(-\frac{1}{t^2}\right)}=t^2 \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=1}=1$

11. $t=\frac{\pi}{3} \Rightarrow x=\frac{\pi}{3}-\sin\frac{\pi}{3}=\frac{\pi}{3}-\frac{\sqrt{3}}{2}$, $y=1-\cos\frac{\pi}{3}=1-\frac{1}{2}=\frac{1}{2}$; $\frac{dx}{dt}=1-\cos t$, $\frac{dy}{dt}=\sin t \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\sin t}{1-\cos t}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}}=\frac{\sin\left(\frac{\pi}{3}\right)}{1-\cos\left(\frac{\pi}{3}\right)}=\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}=\sqrt{3}$; tangent line is $y-\frac{1}{2}=\sqrt{3}\left(x-\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right) \Rightarrow y=\sqrt{3x}-\frac{\pi\sqrt{3}}{3}+2$;

$\frac{dy'}{dt}=\frac{(1-\cos t)(\cos t)-(\sin t)(\sin t)}{(1-\cos t)^2}=\frac{-1}{1-\cos t} \Rightarrow \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=\frac{\left(\frac{-1}{1-\cos t}\right)}{1-\cos t}=\frac{-1}{(1-\cos t)^2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}}=-4$

12. $t=\frac{\pi}{2} \Rightarrow x=\cos\frac{\pi}{2}=0$, $y=1+\sin\frac{\pi}{2}=2$; $\frac{dx}{dt}=-\sin t$, $\frac{dy}{dt}=\cos t \Rightarrow \frac{dy}{dx}=\frac{\cos t}{-\sin t}=-\cot t$

$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}}=-\cot\frac{\pi}{2}=0$; tangent line is $y=2$; $\frac{dy'}{dt}=\csc^2 t \Rightarrow \frac{d^2y}{dx^2}=\frac{\csc^2 t}{-\sin t}=-\csc^3 t \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}}=-1$

13. $t=2 \Rightarrow x=\frac{1}{2+1}=\frac{1}{3}$, $y=\frac{2}{2-1}=2$; $\frac{dx}{dt}=\frac{-1}{(t+1)^2}$, $\frac{dy}{dt}=\frac{-1}{(t-1)^2} \Rightarrow \frac{dy}{dx}=\frac{(t+1)^2}{(t-1)^2} \Rightarrow \left.\frac{dy}{dx}\right|_{t=2}=\frac{(2+1)^2}{(2-1)^2}=9$; tangent line is

$y=9x-1$; $\frac{dy'}{dt}=-\frac{4(t+1)}{(t-1)^3} \Rightarrow \frac{d^2y}{dx^2}=\frac{4(t+1)^3}{(t-1)^3} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=2}=\frac{4(2+1)^3}{(2-1)^3}=108$

14. $t=0 \Rightarrow x=0+e^0=1$, $y=1-e^0=0$; $\frac{dx}{dt}=1+e^t$, $\frac{dy}{dt}=-e^t \Rightarrow \frac{dy}{dx}=\frac{-e^t}{1+e^t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=0}=\frac{-e^0}{1+e^0}=-\frac{1}{2}$; tangent line is

$y=-\frac{1}{2}x+\frac{1}{2}$; $\frac{dy'}{dt}=\frac{-e^t}{\left(1+e^t\right)^2} \Rightarrow \frac{d^2y}{dx^2}=\frac{-e^t}{\left(1+e^t\right)^3} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=0}=\frac{-e^0}{\left(1+e^0\right)^3}=-\frac{1}{8}$

15. $x^3+2t^2=9 \Rightarrow 3x^2\frac{dx}{dt}+4t=0 \Rightarrow 3x^2\frac{dx}{dt}=-4t \Rightarrow \frac{dx}{dt}=\frac{-4t}{3x^2}$; $2y^3-3t^2=4 \Rightarrow 6y^2\frac{dy}{dt}-6t=0$

$\Rightarrow \frac{dy}{dx}=\frac{6t}{6y^2}=\frac{t}{y^2}$; thus $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\left(\frac{t}{y^2}\right)}{\left(\frac{-4t}{3x^2}\right)}=\frac{t\left(3x^2\right)}{y^2(-4t)}=\frac{3x^2}{-4y^2}$; $t=2 \Rightarrow x^3+2(2)^2=9 \Rightarrow x^3+8=9 \Rightarrow x^3=1$

$\Rightarrow x=1$; $t=2 \Rightarrow 2y^3-3(2)^2=4 \Rightarrow 2y^3=16 \Rightarrow y^3=8 \Rightarrow y=2$; therefore $\left.\frac{dy}{dx}\right|_{t=2}=\frac{3(1)^2}{-4(2)^2}=-\frac{3}{16}$

16. $x=\sqrt{5-\sqrt{t}} \Rightarrow \frac{dx}{dt}=\frac{1}{2}\left(5-\sqrt{t}\right)^{-1/2}\left(-\frac{1}{2}t^{-1/2}\right)=-\frac{1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}$; $y(t-1)=\sqrt{t} \Rightarrow y+(t-1)\frac{dy}{dt}=\frac{1}{2}t^{-1/2}$

$\Rightarrow (t-1)\frac{dy}{dt}=\frac{1}{2\sqrt{t}}-y \Rightarrow \frac{dy}{dt}=\frac{\frac{1}{2\sqrt{t}}-y}{(t-1)}=\frac{1-2y\sqrt{t}}{2t\sqrt{t}-2\sqrt{t}}$; thus $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{1-2y\sqrt{t}}{2t\sqrt{t}-2\sqrt{t}}}{\frac{-1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}}=\frac{1-2y\sqrt{t}}{2\sqrt{t}(t-1)}\cdot\frac{4\sqrt{t}\sqrt{5-\sqrt{t}}}{-1}$

$=\frac{2\left(1-2y\sqrt{t}\right)\sqrt{5-\sqrt{t}}}{1-t}$; $t=4 \Rightarrow x=\sqrt{5-\sqrt{4}}=\sqrt{3}$; $t=4 \Rightarrow y\cdot 3=\sqrt{4} \Rightarrow y=\frac{2}{3}$ therefore,

$\left.\frac{dy}{dx}\right|_{t=4}=\frac{2\left(1-2\left(\frac{2}{3}\right)\sqrt{4}\right)\sqrt{5-\sqrt{4}}}{1-4}=\frac{10\sqrt{3}}{9}$

17. $x+2x^{3/2}=t^2+t \Rightarrow \frac{dx}{dt}+3x^{1/2}\frac{dx}{dt}=2t+1 \Rightarrow \left(1+3x^{1/2}\right)\frac{dx}{dt}=2t+1 \Rightarrow \frac{dx}{dt}=\frac{2t+1}{1+3x^{1/2}}$; $y\sqrt{t+1}+2t\sqrt{y}=4$

$\Rightarrow \frac{dy}{dt}\sqrt{t+1}+y\left(\frac{1}{2}\right)(t+1)^{-1/2}+2\sqrt{y}+2t\left(\frac{1}{2}y^{-1/2}\right)\frac{dy}{dt}=0 \Rightarrow \frac{dy}{dt}\sqrt{t+1}+\frac{y}{2\sqrt{t+1}}+2\sqrt{y}+\left(\frac{t}{\sqrt{y}}\right)\frac{dy}{dt}=0$

$\Rightarrow\left(\sqrt{t+1}+\frac{t}{\sqrt{y}}\right)\frac{dy}{dt}=\frac{-y}{2\sqrt{t+1}}-2\sqrt{y}\Rightarrow\frac{dy}{dt}=\frac{\left(\frac{-y}{2\sqrt{t+1}}-2\sqrt{y}\right)}{\left(\sqrt{t+1}+\frac{t}{\sqrt{y}}\right)}=\frac{-y\sqrt{y}-4y\sqrt{t+1}}{2\sqrt{y}(t+1)+2t\sqrt{t+1}}$; thus $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\left(\frac{-y\sqrt{y}-4y\sqrt{t+1}}{2\sqrt{y}(t+1)+2t\sqrt{t+1}}\right)}{\left(\frac{2t+1}{1+3x^{1/2}}\right)}$;

$t=0\Rightarrow x+2x^{3/2}=0\Rightarrow x\left(1+2x^{1/2}\right)=0\Rightarrow x=0$; $t=0\Rightarrow y\sqrt{0+1}+2(0)\sqrt{y}=4\Rightarrow y=4$;

therefore $\left.\frac{dy}{dx}\right|_{t=0}=\frac{\left(\frac{-4\sqrt{4}-4(4)\sqrt{0+1}}{2\sqrt{4}(0+1)+2(0)\sqrt{0+1}}\right)}{\left(\frac{2(0)+1}{1+3(0)^{1/2}}\right)}=-6$

18. $x\sin t+2x=t\Rightarrow\frac{dx}{dt}\sin t+x\cos t+2\frac{dx}{dt}=1\Rightarrow(\sin t+2)\frac{dx}{dt}=1-x\cos t\Rightarrow\frac{dx}{dt}=\frac{1-x\cos t}{\sin t+2}$;

$t\sin t-2t=y\Rightarrow\sin t+t\cos t-2=\frac{dy}{dt}$; thus $\frac{dy}{dx}=\frac{\sin t+t\cos t-2}{\left(\frac{1-x\cos t}{\sin t+2}\right)}$; $t=\pi\Rightarrow x\sin\pi+2x=\pi\Rightarrow x=\frac{\pi}{2}$;

therefore $\left.\frac{dy}{dx}\right|_{t=\pi}=\frac{\sin\pi+\pi\cos\pi-2}{\left(\frac{1-\left(\frac{\pi}{2}\right)\cos\pi}{\sin\pi+2}\right)}=\frac{-4\pi-8}{2+\pi}=-4$

19. $x=t^3+t,\ \ y+2t^3=2x+t^2\Rightarrow\frac{dx}{dt}=3t^2+1,\ \ \frac{dy}{dt}+6t^2=2\frac{dx}{dt}+2t\Rightarrow\frac{dy}{dt}=2\left(3t^2+1\right)+2t-6t^2=2t+2$

$\Rightarrow\frac{dy}{dx}=\frac{2t+2}{3t^2+1}\Rightarrow\left.\frac{dy}{dx}\right|_{t=1}=\frac{2(1)+2}{3(1)^2+1}=1$

20. $t=\ln(x-t),\ \ y=te^t\Rightarrow1=\frac{1}{x-t}\left(\frac{dx}{dt}-1\right)\Rightarrow x-t=\frac{dx}{dt}-1\Rightarrow\frac{dx}{dt}=x-t+1,\ \ \frac{dy}{dt}=te^t+e^t;\ \Rightarrow\frac{dy}{dx}=\frac{te^t+e^t}{x-t+1}$;

$t=0\Rightarrow0=\ln(x-0)\Rightarrow x=1\Rightarrow\left.\frac{dy}{dx}\right|_{t=0}=\frac{(0)e^0+e^0}{1-0+1}=\frac{1}{2}$

21. $A=\int_0^{2\pi}y\,dx=\int_0^{2\pi}a(1-\cos t)a(1-\cos t)dt=a^2\int_0^{2\pi}(1-\cos t)^2dt=a^2\int_0^{2\pi}\left(1-2\cos t+\cos^2 t\right)dt$

$=a^2\int_0^{2\pi}\left(1-2\cos t+\frac{1+\cos 2t}{2}\right)dt=a^2\int_0^{2\pi}\left(\frac{3}{2}-2\cos t+\frac{1}{2}\cos 2t\right)dt=a^2\left[\frac{3}{2}t-2\sin t+\frac{1}{4}\sin 2t\right]_0^{2\pi}$

$=a^2(3\pi-0+0)-0=3\pi a^2$

22. $A=\int_0^1x\,dy=\int_0^1\left(t-t^2\right)\left(-e^{-t}\right)dt\ \ \left[u=t-t^2\Rightarrow du=(1-2t)\,dt;\ dv=\left(-e^{-t}\right)dt\Rightarrow v=e^{-t}\right]$

$=\left[e^{-t}\left(t-t^2\right)\right]_0^1-\int_0^1e^{-t}(1-2t)dt\ \ \left[u=1-2t\Rightarrow du=-2dt;\ dv=e^{-t}dt\Rightarrow v=-e^{-t}\right]$

$=\left[e^{-t}\left(t-t^2\right)\right]_0^1-\left(\left[-e^{-t}(1-2t)\right]_0^1-\int_0^12e^{-t}dt\right)=\left[e^{-t}\left(t-t^2\right)+e^{-t}(1-2t)-2e^{-t}\right]_0^1$

$=\left(e^{-1}(0)+e^{-1}(-1)-2e^{-1}\right)-\left(e^0(0)+e^0(1)-2e^0\right)=1-3e^{-1}=1-\frac{3}{e}$

23. $A=2\int_\pi^0y\,dx=2\int_\pi^0(b\sin t)(-a\sin t)\,dt=2ab\int_0^\pi\sin^2t\,dt=2ab\int_0^\pi\frac{1-\cos 2t}{2}\,dt=ab\int_0^\pi(1-\cos 2t)\,dt$

$=ab\left[t-\frac{1}{2}\sin 2t\right]_0^\pi=ab\left((\pi-0)-0\right)=\pi ab$

24. (a) $x = t^2,\ y = t^6,\ 0 \le t \le 1 \Rightarrow A = \int_0^1 y\,dx = \int_0^1 \left(t^6\right) 2t\,dt = \int_0^1 2t^7\,dt = \left[\frac{1}{4}t^8\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$

(b) $x = t^3,\ y = t^9,\ 0 \le t \le 1 \Rightarrow A = \int_0^1 y\,dx = \int_0^1 \left(t^9\right) 3t^2\,dt = \int_0^1 3t^{11}\,dt = \left[\frac{1}{4}t^{12}\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$

25. $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = 1 + \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (1+\cos t)^2} = \sqrt{2 + 2\cos t}$

$\Rightarrow$ Length $= \int_0^\pi \sqrt{2 + 2\cos t}\,dt = \sqrt{2}\int_0^\pi \sqrt{\left(\frac{1-\cos t}{1-\cos t}\right)(1+\cos t)}\,dt = \sqrt{2}\int_0^\pi \sqrt{\frac{\sin^2 t}{1-\cos t}}\,dt = \sqrt{2}\int_0^\pi \frac{\sin t}{\sqrt{1-\cos t}}\,dt$

(since $\sin t \ge 0$ on $[0, \pi]$); $\left[u = 1 - \cos t \Rightarrow du = \sin t\,dt;\ t = 0 \Rightarrow u = 0,\ t = \pi \Rightarrow u = 2\right]$

$\to \sqrt{2}\int_0^2 u^{-1/2}\,du = \sqrt{2}\left[2u^{1/2}\right]_0^2 = 4$

26. $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 3t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(3t^2\right)^2 + (3t)^2} = \sqrt{9t^4 + 9t^2} = 3t\sqrt{t^2+1}$ $\left(\text{since } t \ge 0 \text{ on } \left[0, \sqrt{3}\right]\right)$

$\Rightarrow$ Length $= \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\,dt$; $\left[u = t^2 + 1 \Rightarrow \frac{3}{2}\,du = 3t\,dt;\ t = 0 \Rightarrow u = 1,\ t = \sqrt{3} \Rightarrow u = 4\right]$

$\to \int_1^4 \frac{3}{2}u^{1/2}\,du = \left[u^{3/2}\right]_1^4 = (8 - 1) = 7$

27. $\frac{dx}{dt} = t$ and $\frac{dy}{dt} = (2t+1)^{1/2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^2 + (2t+1)} = \sqrt{(t+1)^2} = |t+1| = t+1$ since $0 \le t \le 4$

$\Rightarrow$ Length $= \int_0^4 (t+1)\,dt = \left[\frac{t^2}{2} + t\right]_0^4 = (8+4) = 12$

28. $\frac{dx}{dt} = (2t+3)^{1/2}$ and $\frac{dy}{dt} = 1 + t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t+3) + (1+t)^2} = \sqrt{t^2 + 4t + 4} = |t+2| = t+2$

since $0 \le t \le 3 \Rightarrow$ Length $= \int_0^3 (t+2)\,dt = \left[\frac{t^2}{2} + 2t\right]_0^3 = \frac{21}{2}$

29. $\frac{dx}{dt} = 8t\cos t$ and $\frac{dy}{dt} = 8t\sin t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(8t\cos t)^2 + (8t\sin t)^2} = \sqrt{64t^2\cos^2 t + 64t^2\sin^2 t}$

$= |8t| = 8t$ since $0 \le t \le \frac{\pi}{2} \Rightarrow$ Length $= \int_0^{\pi/2} 8t\,dt = \left[4t^2\right]_0^{\pi/2} = \pi^2$

30. $\frac{dx}{dt} = \left(\frac{1}{\sec t + \tan t}\right)\left(\sec t\tan t + \sec^2 t\right) - \cos t = \sec t - \cos t$ and $\frac{dy}{dt} = -\sin t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{(\sec t - \cos t)^2 + (-\sin t)^2} = \sqrt{\sec^2 t - 1} = \sqrt{\tan^2 t} = |\tan t| = \tan t$ since $0 \le t \le \frac{\pi}{3}$

$\Rightarrow$ Length $= \int_0^{\pi/3} \tan t\,dt = \int_0^{\pi/3} \frac{\sin t}{\cos t}\,dt = \left[-\ln|\cos t|\right]_0^{\pi/3} = -\ln\frac{1}{2} + \ln 1 = \ln 2$

31. $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

$\Rightarrow$ Area $= \int 2\pi y\, ds = \int_0^{2\pi} 2\pi(2+\sin t)(1)dt = 2\pi\left[2t - \cos t\right]_0^{2\pi} = 2\pi\left[(4\pi - 1) - (0-1)\right] = 8\pi^2$

32. $\frac{dx}{dt} = t^{1/2}$ and $\frac{dy}{dt} = t^{-1/2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t + t^{-1}} = \sqrt{\frac{t^2+1}{t}} \Rightarrow$ Area $= \int 2\pi x\, ds = \int_0^{\sqrt{3}} 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{\frac{t^2+1}{t}}\, dt$

$= \frac{4\pi}{3}\int_0^{\sqrt{3}} t\sqrt{t^2+1}\, dt;\ \left[u = t^2 + 1 \Rightarrow du = 2t\, dt;\ t = 0 \Rightarrow u = 1,\ t = \sqrt{3} \Rightarrow u = 4\right]$

$\rightarrow \int_1^4 \frac{2\pi}{3}\sqrt{u}\, du = \left[\frac{4\pi}{9}u^{3/2}\right]_1^4 = \frac{28\pi}{9}$

Note: $\int_0^{\sqrt{3}} 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{\frac{t^2+1}{t}}\, dt$ is an improper integral but $\lim_{t\to 0^+} f(t)$ exists and is equal to 0, where $f(t) = 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{\frac{t^2+1}{t}}$. Thus the discontinuity is removable: define $F(t) = f(t)$ for $t > 0$ and $F(0) = 0 \Rightarrow \int_0^{\sqrt{3}} F(t)\, dt = \frac{28\pi}{9}$.

33. $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = t + \sqrt{2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1^2 + \left(t+\sqrt{2}\right)^2} = \sqrt{t^2 + 2\sqrt{2}t + 3}$

$\Rightarrow$ Area $= \int 2\pi x\, ds = \int_{-\sqrt{2}}^{\sqrt{2}} 2\pi\left(t + \sqrt{2}\right)\sqrt{t^2 + 2\sqrt{2}t + 3}\, dt;$

$\left[u = t^2 + 2\sqrt{2}t + 3 \Rightarrow du = \left(2t + 2\sqrt{2}\right)dt;\ t = -\sqrt{2} \Rightarrow u = 1, t = \sqrt{2} \Rightarrow u = 9\right]$

$\rightarrow \int_1^9 \pi\sqrt{u}\, du = \left[\frac{2}{3}\pi u^{3/2}\right]_1^9 = \frac{2\pi}{3}(27 - 1) = \frac{52\pi}{3}$

34. From Exercise 30, $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \tan t \Rightarrow$ Area $= \int 2\pi y\, ds = \int_0^{\pi/3} 2\pi\cos t\tan t\, dt = 2\pi\int_0^{\pi/3}\sin t\, dt$

$= 2\pi\left[-\cos t\right]_0^{\pi/3} = 2\pi\left[-\frac{1}{2} - (-1)\right] = \pi$

35. $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 1 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2^2 + 1^2} = \sqrt{5} \Rightarrow$ Area $= \int 2\pi y\, ds = \int_0^1 2\pi(t+1)\sqrt{5}\, dt$

$= 2\pi\sqrt{5}\left[\frac{t^2}{2} + t\right]_0^1 = 3\pi\sqrt{5}$. Check: slant height is $\sqrt{5} \Rightarrow$ Area is $\pi(1+2)\sqrt{5} = 3\pi\sqrt{5}$.

36. $\frac{dx}{dt} = h$ and $\frac{dy}{dt} = r \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{h^2 + r^2} \Rightarrow$ Area $= \int 2\pi y\, ds = \int_0^1 2\pi rt\sqrt{h^2 + r^2}\, dt$

$= 2\pi r\sqrt{h^2 + r^2}\int_0^1 t\, dt = 2\pi r\sqrt{h^2 + r^2}\left[\frac{t^2}{2}\right]_0^1 = \pi r\sqrt{h^2 + r^2}$.

Check: slant height is $\sqrt{h^2 + r^2} \Rightarrow$ Area is $\pi r\sqrt{h^2 + r^2}$.

37. Let the density be $\delta = 1$. Then $x = \cos t + t\sin t \Rightarrow \frac{dx}{dt} = t\cos t$, and $y = \sin t - t\cos t \Rightarrow \frac{dy}{dt} = t\sin t$

$\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(t\cos t)^2 + (t\sin t)^2} = |t|\, dt = t\, dt$ since $0 \le t \le \frac{\pi}{2}$. The curve's mass is

$M = \int dm = \int_0^{\pi/2} t\, dt = \frac{\pi^2}{8}$. Also $M_x = \int \tilde{y}\, dm = \int_0^{\pi/2} (\sin t - t\cos t)\, t\, dt = \int_0^{\pi/2} t\sin t\, dt - \int_0^{\pi/2} t^2 \cos t\, dt$

$= [\sin t - t\cos t]_0^{\pi/2} - \left[t^2 \sin t - 2\sin t + 2t\cos t\right]_0^{\pi/2} = 3 - \frac{\pi^2}{4}$, where we integrated by parts. Therefore,

$\overline{y} = \frac{M_x}{M} = \frac{\left(3 - \frac{\pi^2}{4}\right)}{\left(\frac{\pi^2}{8}\right)} = \frac{24}{\pi^2} - 2$. Next, $M_y = \int \tilde{x}\, dm = \int_0^{\pi/2} (\cos t + t\sin t)\, t\, dt = \int_0^{\pi/2} t\cos t\, dt + \int_0^{\pi/2} t^2 \sin t\, dt$

$= [\cos t + t\sin t]_0^{\pi/2} + \left[-t^2\cos t + 2\cos t + 2t\sin t\right]_0^{\pi/2} = \frac{3\pi}{2} - 3$, again integrating by parts. Hence,

$\overline{x} = \frac{M_y}{M} = \frac{\left(\frac{3\pi}{2} - 3\right)}{\left(\frac{\pi^2}{8}\right)} = \frac{12}{\pi} - \frac{24}{\pi^2}$. Therefore $(\overline{x}, \overline{y}) = \left(\frac{12}{\pi} - \frac{24}{\pi^2}, \frac{24}{\pi^2} - 2\right)$.

38. Let the density be $\delta = 1$. Then $x = e^t \cos t \Rightarrow \frac{dx}{dt} = e^t \cos t - e^t \sin t$, and $y = e^t \sin t \Rightarrow \frac{dy}{dx} = e^t \sin t + e^t \cos t$

$\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{\left(e^t\cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2}\, dt = \sqrt{2e^{2t}}\, dt = \sqrt{2}\, e^t\, dt.$

The curve's mass is $M = \int dm = \int_0^{\pi} \sqrt{2}\, e^t\, dt = \sqrt{2}\, e^{\pi} - \sqrt{2}$. Also $M_x = \int \tilde{y}\, dm = \int_0^{\pi} \left(e^t \sin t\right)\left(\sqrt{2}\, e^t\right) dt$

$= \int_0^{\pi} \sqrt{2}\, e^{2t} \sin t\, dt = \sqrt{2}\left[\frac{e^{2t}}{5}(2\sin t - \cos t)\right]_0^{\pi} = \sqrt{2}\left(\frac{e^{2\pi}}{5} + \frac{1}{5}\right) \Rightarrow \overline{y} = \frac{M_x}{M} = \frac{\sqrt{2}\left(\frac{e^{2\pi}}{5} + \frac{1}{5}\right)}{\sqrt{2}e^{\pi} - \sqrt{2}} = \frac{e^{2\pi} + 1}{5\left(e^{\pi} - 1\right)}.$

Next $M_y = \int \tilde{x}\, dm = \int_0^{\pi} \left(e^t \cos t\right)\left(\sqrt{2}e^t\right) dt = \int_0^{\pi} \sqrt{2}\, e^{2t} \cos t\, dt = \sqrt{2}\left[\frac{e^{2t}}{5}(2\cos t + \sin t)\right]_0^{\pi} = -\sqrt{2}\left(\frac{2e^{2\pi}}{5} + \frac{2}{5}\right)$

$\Rightarrow \overline{x} = \frac{M_y}{M} = \frac{-\sqrt{2}\left(\frac{2e^{2\pi}}{5} + \frac{2}{5}\right)}{\sqrt{2}\, e^{\pi} - \sqrt{2}} = -\frac{2e^{2\pi} + 2}{5\left(e^{\pi} - 1\right)}$. Therefore $(\overline{x}, \overline{y}) = \left(-\frac{2e^{2\pi} + 2}{5\left(e^{\pi} - 1\right)}, \frac{e^{2\pi} + 1}{5\left(e^{\pi} - 1\right)}\right)$.

39. Let the density be $\delta = 1$. Then $x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t$, and $y = t + \sin t \Rightarrow \frac{dy}{dt} = 1 + \cos t$

$\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(-\sin t)^2 + (1 + \cos t)^2}\, dt = \sqrt{2 + 2\cos t}\, dt$. The curve's mass is

$M = \int dm = \int_0^{\pi} \sqrt{2 + 2\cos t}\, dt = \sqrt{2}\int_0^{\pi} \sqrt{1 + \cos t}\, dt = \sqrt{2}\int_0^{\pi} \sqrt{2\cos^2\left(\frac{t}{2}\right)}\, dt = 2\int_0^{\pi} \left|\cos\left(\frac{t}{2}\right)\right| dt = 2\int_0^{\pi} \cos\left(\frac{t}{2}\right) dt$

$\left(\text{since } 0 \le t \le \pi \Rightarrow 0 \le \frac{t}{2} \le \frac{\pi}{2}\right) = 2\left[2\sin\left(\frac{t}{2}\right)\right]_0^{\pi} = 4$. Also $M_x = \int \tilde{y}\, dm = \int_0^{\pi} (t + \sin t)\left(2\cos\frac{t}{2}\right) dt$

$= \int_0^{\pi} 2t\cos\left(\frac{t}{2}\right) dt + \int_0^{\pi} 2\sin t\cos\left(\frac{t}{2}\right) dt = 2\left[4\cos\left(\frac{t}{2}\right) + 2t\sin\left(\frac{t}{2}\right)\right]_0^{\pi} + 2\left[-\frac{1}{3}\cos\left(\frac{3}{2}t\right) - \cos\left(\frac{1}{2}t\right)\right]_0^{\pi} = 4\pi - \frac{16}{3}$

$\Rightarrow \overline{y} = \frac{M_x}{M} = \frac{\left(4\pi - \frac{16}{3}\right)}{4} = \pi - \frac{4}{3}$. Next $M_y = \int \tilde{x}\, dm = \int_0^{\pi} (\cos t)\left(2\cos\frac{t}{2}\right) dt = 2\int_0^{\pi} \cos t\cos\left(\frac{t}{2}\right) dt$

$= 2\left[\sin\left(\frac{t}{2}\right) + \frac{\sin\left(\frac{3}{2}t\right)}{3}\right]_0^{\pi} = 2 - \frac{2}{3} = \frac{4}{3} \Rightarrow \overline{x} = \frac{M_y}{M} = \frac{\left(\frac{4}{3}\right)}{4} = \frac{1}{3}$. Therefore $(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \pi - \frac{4}{3}\right)$.

40. Let the density be $\delta = 1$. Then $x = t^3 \Rightarrow \frac{dx}{dt} = 3t^2$, and $y = \frac{3t^2}{2} \Rightarrow \frac{dy}{dt} = 3t \Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $= \sqrt{\left(3t^2\right)^2 + \left(3t\right)^2}\, dt = 3\,|t|\,\sqrt{t^2+1}\, dt = 3t\sqrt{t^2+1}\, dt$ since $0 \le t \le \sqrt{3}$. The curve's mass is $M = \int dm = \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\, dt = \left[\left(t^2+1\right)^{3/2}\right]_0^{\sqrt{3}} = 7$. Also $M_x = \int \tilde{y}\, dm = \int_0^{\sqrt{3}} \frac{3t^2}{2}\left(3t\sqrt{t^2+1}\right) dt$ $= \frac{9}{2}\int_0^{\sqrt{3}} t^3\sqrt{t^2+1}\, dt = \frac{87}{5} = 17.4$ (by computer) $\Rightarrow \overline{y} = \frac{M_x}{M} = \frac{17.4}{7} \approx 2.49$. Next $M_y = \int \tilde{x}\, dm$ $= \int_0^{\sqrt{3}} t^3 \cdot 3t\sqrt{t^2+1}\, dt = 3\int_0^{\sqrt{3}} t^4\sqrt{t^2+1}\, dt \approx 16.4849$ (by computer) $\Rightarrow \overline{x} = \frac{M_y}{M} = \frac{16.4849}{7} \approx 2.35$. Therefore, $(\overline{x}, \overline{y}) \approx (2.35, 2.49)$.

41. (a) $\frac{dx}{dt} = -2\sin 2t$ and $\frac{dy}{dt} = 2\cos 2t \Rightarrow \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(-2\sin 2t\right)^2 + \left(2\cos 2t\right)^2} = 2$

$\Rightarrow$ Length $= \int_0^{\pi/2} 2\, dt = \left[2t\right]_0^{\pi/2} = \pi$

(b) $\frac{dx}{dt} = \pi\cos \pi t$ and $\frac{dy}{dt} = -\pi\sin \pi t \Rightarrow \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\pi\cos \pi t\right)^2 + \left(-\pi\sin \pi t\right)^2} = \pi$

$\Rightarrow$ Length $= \int_{-1/2}^{1/2} \pi\, dt = \left[\pi t\right]_{-1/2}^{1/2} = \pi$

42. (a) $x = g(y)$ has the parametrization $x = g(y)$ and $y = y$ for $c \le y \le d \Rightarrow \frac{dx}{dy} = g'(y)$ and $\frac{dy}{dy} = 1$; then

Length $= \int_c^d \sqrt{\left(\frac{dy}{dy}\right)^2 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\, dy$

(b) $x = y^{3/2}, 0 \le y \le \frac{4}{3} \Rightarrow \frac{dx}{dy} = \frac{3}{2}y^{1/2} \Rightarrow L = \int_0^{4/3} \sqrt{1 + \left(\frac{3}{2}y^{1/2}\right)^2}\, dy = \int_0^{4/3} \sqrt{1 + \frac{9}{4}y\, dy} = \left[\frac{4}{9} \cdot \frac{2}{3}\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^{4/3}$

$= \frac{8}{27}(4)^{3/2} - \frac{8}{27}(1)^{3/2} = \frac{56}{27}$

(c) $x = \frac{3}{2}y^{2/3}, 0 \le y \le 1 \Rightarrow \frac{dx}{dy} = y^{-1/3} \Rightarrow L = \int_0^1 \sqrt{1 + \left(y^{-1/3}\right)^2}\, dy = \int_0^1 \sqrt{1 + \frac{1}{y^{2/3}}}\, dy = \lim_{a \to 0^+} \int_a^1 \sqrt{\frac{y^{2/3}+1}{y^{2/3}}}\, dy$

$= \lim_{a \to 0^+} \frac{3}{2}\int_a^1 \left(y^{2/3}+1\right)^{1/2}\left(\frac{2}{3}y^{-1/3}\right) dy = \lim_{a \to 0^+} \left[\frac{3}{2} \cdot \frac{2}{3}\left(y^{2/3}+1\right)^{3/2}\right]_a^1 = \lim_{a \to 0^+} \left((2)^{3/2} - \left(a^{2/3}+1\right)^{3/2}\right)$

$= 2\sqrt{2} - 1$

43. $x = (1 + 2\sin\theta)\cos\theta, \quad y = (1 + 2\sin\theta)\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos^2\theta - \sin\theta(1 + 2\sin\theta)$,

$\frac{dy}{d\theta} = 2\cos\theta\sin\theta + \cos\theta(1 + 2\sin\theta) \Rightarrow \frac{dy}{dx} = \frac{2\cos\theta\sin\theta + \cos\theta(1+2\sin\theta)}{2\cos^2\theta - \sin\theta(1+2\sin\theta)} = \frac{4\cos\theta\sin\theta + \cos\theta}{2\cos^2\theta - 2\sin^2\theta - \sin\theta} = \frac{2\sin 2\theta + \cos\theta}{2\cos 2\theta - \sin\theta}$

(a) $x = \left(1 + 2\sin(0)\right)\cos(0) = 1, \quad y = \left(1 + 2\sin(0)\right)\sin(0) = 0;\ \left.\frac{dy}{dx}\right|_{\theta=0} = \frac{2\sin(2(0)) + \cos(0)}{2\cos(2(0)) - \sin(0)} = \frac{0+1}{2-0} = \frac{1}{2}$

(b) $x = \left(1 + 2\sin\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{\pi}{2}\right) = 0, \quad y = \left(1 + 2\sin\left(\frac{\pi}{2}\right)\right)\sin\left(\frac{\pi}{2}\right) = 3;\ \left.\frac{\mathrm{dy}}{\mathrm{dx}}\right|_{\theta=\pi/2} = \frac{2\sin\left(2\left(\frac{\pi}{2}\right)\right) + \cos\left(\frac{\pi}{2}\right)}{2\cos\left(2\left(\frac{\pi}{2}\right)\right) - \sin\left(\frac{\pi}{2}\right)} = \frac{0+0}{-2-1} = 0$

(c) $x = \left(1 + 2\sin\left(\frac{4\pi}{3}\right)\right)\cos\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}-1}{2}$, $y = \left(1 + 2\sin\left(\frac{4\pi}{3}\right)\right)\sin\left(\frac{4\pi}{3}\right) = \frac{3-\sqrt{3}}{2}$;

$$\left.\frac{dy}{dx}\right|_{\theta=4\pi/3} = \frac{2\sin\left(2\left(\frac{4\pi}{3}\right)\right) + \cos\left(\frac{4\pi}{3}\right)}{2\cos\left(2\left(\frac{4\pi}{3}\right)\right) - \sin\left(\frac{4\pi}{3}\right)} = \frac{\sqrt{3} - \frac{1}{2}}{-1 + \frac{\sqrt{3}}{2}} = \frac{2\sqrt{3}-1}{\sqrt{3}-2} = -\left(4 + 3\sqrt{3}\right)$$

44. $x = t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi \Rightarrow \frac{dx}{dt} = 1,\ \frac{dy}{dt} = \sin t \Rightarrow \frac{dy}{dx} = \frac{\sin t}{1} = \sin t \Rightarrow \frac{d}{dt}\left(\frac{dy}{dx}\right) = \cos t \Rightarrow \frac{d^2y}{dx^2} = \frac{\cos t}{1} = \cos t.$

The maximum and minimum slope will occur at points that maximize/minimize $\frac{dy}{dx}$, in other words, points where $\frac{d^2y}{dx^2} = 0 \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2} \Rightarrow \frac{d^2y}{dx^2} = \underset{\pi/2}{+++|}\underset{3\pi/2}{---|}+++$

(a) the maximum slope is $\left.\frac{dy}{dx}\right|_{t=\pi/2} = \sin\left(\frac{\pi}{2}\right) = 1$, which occurs at $x = \frac{\pi}{2},\ y = 1 - \cos\left(\frac{\pi}{2}\right) = 1$

(b) the minimum slope is $\left.\frac{dy}{dx}\right|_{t=3\pi/2} = \sin\left(\frac{3\pi}{2}\right) = -1$, which occurs at $x = \frac{3\pi}{2},\ y = 1 - \cos\left(\frac{3\pi}{2}\right) = 1$

45. $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = 2\cos 2t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos 2t}{\cos t} = \frac{2\left(2\cos^2 t - 1\right)}{\cos t}$; then $\frac{dy}{dx} = 0 \Rightarrow \frac{2\left(2\cos^2 t - 1\right)}{\cos t} = 0$ $\Rightarrow 2\cos^2 t - 1 = 0 \Rightarrow \cos t = \pm\frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. In the 1st quadrant: $t = \frac{\pi}{4} \Rightarrow x = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $y = \sin 2\left(\frac{\pi}{4}\right) = 1 \Rightarrow \left(\frac{\sqrt{2}}{2}, 1\right)$ is the point where the tangent line is horizontal. At the origin: $x = 0$ and $y = 0$ $\Rightarrow \sin t = 0 \Rightarrow t = 0$ or $t = \pi$ and $\sin 2t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$; thus $t = 0$ and $t = \pi$ give the tangent lines at the origin. Tangents at origin: $\left.\frac{dy}{dx}\right|_{t=0} = 2 \Rightarrow y = 2x$ and $\left.\frac{dy}{dx}\right|_{t=\pi} = -2 \Rightarrow y = -2x$

46. $\frac{dx}{dt} = 2\cos 2t$ and $\frac{dy}{dt} = 3\cos 3t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos 3t}{2\cos 2t} = \frac{3(\cos 2t\cos t - \sin 2t\sin t)}{2\left(2\cos^2 t - 1\right)}$

$$= \frac{3\left[\left(2\cos^2 t - 1\right)(\cos t) - 2\sin t\cos t\sin t\right]}{2\left(2\cos^2 t - 1\right)} = \frac{(3\cos t)\left(2\cos^2 t - 1 - 2\sin^2 t\right)}{2\left(2\cos^2 t - 1\right)} = \frac{(3\cos t)\left(4\cos^2 t - 3\right)}{2\left(2\cos^2 t - 1\right)};$$

then $\frac{dy}{dx} = 0 \Rightarrow \frac{(3\cos t)\left(4\cos^2 t - 3\right)}{2\left(2\cos^2 t - 1\right)} = 0 \Rightarrow 3\cos t = 0$ or $4\cos^2 t - 3 = 0 : 3\cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$ and $4\cos^2 t - 3 = 0 \Rightarrow \cos t = \pm\frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. In the 1st quadrant: $t = \frac{\pi}{6} \Rightarrow x = \sin 2\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $y = \sin 3\left(\frac{\pi}{6}\right) = 1 \Rightarrow \left(\frac{\sqrt{3}}{2}, 1\right)$ is the point where the graph has a horizontal tangent. At the origin: $x = 0$ and $y = 0 \Rightarrow \sin 2t = 0$ and $\sin 3t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3} \Rightarrow t = 0$ and $t = \pi$ give the tangent lines at the origin. Tangents at the origin: $\left.\frac{dy}{dx}\right|_{t=0} = \frac{3\cos 0}{2\cos 0} = \frac{3}{2} \Rightarrow y = \frac{3}{2}x$, and $\left.\frac{dy}{dx}\right|_{t=\pi} = \frac{3\cos(3\pi)}{2\cos(2\pi)} = -\frac{3}{2} \Rightarrow y = -\frac{3}{2}x$

47. (a) $x = a(t - \sin t),\ y = a(1 - \cos t),\ 0 \le t \le 2\pi \Rightarrow \frac{dx}{dt} = a(1 - \cos t),\ \frac{dy}{dt} = a\sin t$

$\Rightarrow$ Length $= \int_0^{2\pi} \sqrt{\left(a(1 - \cos t)\right)^2 + (a\sin t)^2}\, dt = \int_0^{2\pi} \sqrt{a^2 - 2a^2\cos t + a^2\cos^2 t + a^2\sin^2 t}\, dt$

$= a\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos t}\ dt = a\sqrt{2}\int_0^{2\pi}\sqrt{2\sin^2\left(\frac{t}{2}\right)}\ dt = 2a\int_0^{2\pi}\sin\left(\frac{t}{2}\right)dt = \left[-4a\cos\left(\frac{t}{2}\right)\right]_0^{2\pi}$

$= -4a\cos\pi + 4a\cos(0) = 8a$

(b) $a = 1 \Rightarrow x = t - \sin t,\ y = 1-\cos t,\ 0 \le t \le 2\pi \Rightarrow \frac{dx}{dt} = 1-\cos t,\ \frac{dy}{dt} = \sin t$

$\Rightarrow$ Surface area $= \int_0^{2\pi} 2\pi(1-\cos t)\sqrt{(1-\cos t)^2 + (\sin t)^2}\ dt$

$= \int_0^{2\pi} 2\pi(1-\cos t)\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\ dt = 2\pi\int_0^{2\pi}(1-\cos t)\sqrt{2-2\cos t}\ dt$

$= 2\sqrt{2}\pi\int_0^{2\pi}(1-\cos t)^{3/2}dt = 2\sqrt{2\pi}\int_0^{2\pi}\left(1-\cos\left(2\cdot\frac{t}{2}\right)\right)^{3/2}dt = 2\sqrt{2}\pi\int_0^{2\pi}\left(2\sin^2\left(\frac{t}{2}\right)\right)^{3/2}dt$

$= 8\pi\int_0^{2\pi}\sin^3\left(\frac{t}{2}\right)dt\ \left[u = \frac{t}{2} \Rightarrow du = \frac{1}{2}dt \Rightarrow dt = 2\ du; t = 0 \Rightarrow u = 0, t = 2\pi \Rightarrow u = \pi\right]$

$= 16\pi\int_0^{\pi}\sin^3 u\ du = 16\pi\int_0^{\pi}\sin^2 u\sin u\ du = 16\pi\int_0^{\pi}\left(1-\cos^2 u\right)\sin u\,du$

$= 16\pi\int_0^{\pi}\sin u\ du - 16\pi\int_0^{\pi}\cos^2 u\sin u\,du = \left[-16\pi\cos u + \frac{16\pi}{3}\cos^3 u\right]_0^{\pi}$

$= \left(16\pi - \frac{16\pi}{3}\right) - \left(-16\pi + \frac{16\pi}{3}\right) = \frac{64\pi}{3}$

48. $x = t-\sin t,\ y = 1-\cos t,\ 0 \le t \le 2\pi;$ Volume $= \int_0^{2\pi}\pi y^2 dx = \int_0^{2\pi}\pi(1-\cos t)^2(1-\cos t)\,dt$

$= \pi\int_0^{2\pi}\left(1-3\cos t+3\cos^2 t-\cos^3 t\right)dt = \pi\int_0^{2\pi}\left(1-3\cos t+3\left(\frac{1+\cos 2t}{2}\right)-\cos^2 t\cos t\right)dt$

$= \pi\int_0^{2\pi}\left(\frac{5}{2}-3\cos t+\frac{3}{2}\cos 2t-\left(1-\sin^2 t\right)\cos t\right)dt = \pi\int_0^{2\pi}\left(\frac{5}{2}-4\cos t+\frac{3}{2}\cos 2t+\sin^2 t\cos t\right)dt$

$= \pi\left[\frac{5}{2}t-4\sin t+\frac{3}{4}\sin 2t+\frac{1}{3}\sin^3 t\right]_0^{2\pi} = \pi(5\pi-0+0+0)-0 = 5\pi^2$

49-52. Example CAS commands:

Maple:

```
with( plots );
with( student );
x := t -> t^3/3;
y := t -> t^2/2;
a := 0;
b := 1;
N := [2, 4, 8 ];
for n in N do
tt := [seq( a+i*(b-a)/n, i=0..n )];
pts := [seq([x(t),y(t)],t=tt)];
L := simplify(add( student[distance](pts[i+1],pts[i], i=1..n ));          # (b)
T := sprintf("#49(a) (Section 11.2)\nn=%3d L=%8.5f\n", n, L );
P[n] := plot( [[x(t),y(t),t=a..b],pts], title=T ):                         # (a)
end do:
```

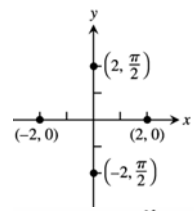
```
display( [seq(P[n],n=N)], insequence=true );
ds := t ->sqrt( simplify(D(x)(t)^2 + D(y)(t)^2) );                    # (c)
L := Int( ds(t), t=a..b ):
L = evalf(L);
```

11.3 POLAR COORDINATES

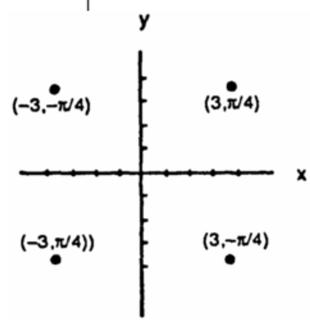
1. $a, e; b, g; c, h; d, f$

2. $a, f; b, h; c, g; d, e$

3. (a) $\left(2, \frac{\pi}{2}+2n\pi\right)$ and $\left(-2, \frac{\pi}{2}+(2n+1)\pi\right)$, n an integer
   (b) $(2, 2n\pi)$ and $(-2, (2n+1)\pi)$, n an integer
   (c) $\left(2, \frac{3\pi}{2}+2n\pi\right)$ and $\left(-2, \frac{3\pi}{2}+(2n+1)\pi\right)$, n an integer
   (d) $(2, (2n+1)\pi)$ and $(-2, 2n\pi)$, n an integer



4. (a) $\left(3, \frac{\pi}{4}+2n\pi\right)$ and $\left(-3, \frac{5\pi}{4}+2n\pi\right)$, n an integer
   (b) $\left(-3, \frac{\pi}{4}+2n\pi\right)$ and $\left(3, \frac{5\pi}{4}+2n\pi\right)$, n an integer
   (c) $\left(3, -\frac{\pi}{4}+2n\pi\right)$ and $\left(-3, \frac{3\pi}{4}+2n\pi\right)$, n an integer
   (d) $\left(-3, -\frac{\pi}{4}+2n\pi\right)$ and $\left(3, \frac{3\pi}{4}+2n\pi\right)$, n an integer



5. (a) $x = r\cos\theta = 3\cos 0 = 3$, $y = r\sin\theta = 3\sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(3, 0)$
   (b) $x = r\cos\theta = -3\cos 0 = -3$, $y = r\sin\theta = -3\sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(-3, 0)$
   (c) $x = r\cos\theta = 2\cos\frac{2\pi}{3} = -1$, $y = r\sin\theta = 2\sin\frac{2\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(-1, \sqrt{3}\right)$
   (d) $x = r\cos\theta = 2\cos\frac{7\pi}{3} = 1$, $y = r\sin\theta = 2\sin\frac{7\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(1, \sqrt{3}\right)$
   (e) $x = r\cos\theta = -3\cos\pi = 3$, $y = r\sin\theta = -3\sin\pi = 0 \Rightarrow$ Cartesian coordinates are $(3, 0)$
   (f) $x = r\cos\theta = 2\cos\frac{\pi}{3} = 1$, $y = r\sin\theta = 2\sin\frac{\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(1, \sqrt{3}\right)$
   (g) $x = r\cos\theta = -3\cos 2\pi = -3$, $y = r\sin\theta = -3\sin 2\pi = 0 \Rightarrow$ Cartesian coordinates are $(-3, 0)$
   (h) $x = r\cos\theta = -2\cos\left(-\frac{\pi}{3}\right) = -1$, $y = r\sin\theta = -2\sin\left(-\frac{\pi}{3}\right) = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(-1, \sqrt{3}\right)$

6. (a) $x = \sqrt{2}\cos\frac{\pi}{4} = 1$, $y = \sqrt{2}\sin\frac{\pi}{4} = 1 \Rightarrow$ Cartesian coordinates are $(1, 1)$
   (b) $x = 1\cos 0 = 1$, $y = 1\sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(1, 0)$
   (c) $x = 0\cos\frac{\pi}{2} = 0$, $y = 0\sin\frac{\pi}{2} = 0 \Rightarrow$ Cartesian coordinates are $(0, 0)$
   (d) $x = -\sqrt{2}\cos\left(\frac{\pi}{4}\right) = -1$, $y = -\sqrt{2}\sin\left(\frac{\pi}{4}\right) = -1 \Rightarrow$ Cartesian coordinates are $(-1, -1)$
   (e) $x = -3\cos\frac{5\pi}{6} = \frac{3\sqrt{3}}{2}$, $y = -3\sin\frac{5\pi}{6} = -\frac{3}{2} \Rightarrow$ Cartesian coordinates are $\left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$
   (f) $x = 5\cos\left(\tan^{-1}\frac{4}{3}\right) = 3$, $y = 5\sin\left(\tan^{-1}\frac{4}{3}\right) = 4 \Rightarrow$ Cartesian coordinates are $(3, 4)$
   (g) $x = -1\cos 7\pi = 1$, $y = -1\sin 7\pi = 0 \Rightarrow$ Cartesian coordinates are $(1, 0)$

(h) $x = 2\sqrt{3}\cos\frac{2\pi}{3} = -\sqrt{3},\ y = 2\sqrt{3}\sin\frac{2\pi}{3} = 3 \Rightarrow$ Cartesian coordinates are $\left(-\sqrt{3}, 3\right)$

7. (a) $(1,1) \Rightarrow r = \sqrt{1^2+1^2} = \sqrt{2},\ \sin\theta = \frac{1}{\sqrt{2}}$ and $\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \Rightarrow$ Polar coordinates are $\left(\sqrt{2}, \frac{\pi}{4}\right)$

(b) $(-3,0) \Rightarrow r = \sqrt{(-3)^2+0^2} = 3,\ \sin\theta = 0$ and $\cos\theta = -1 \Rightarrow \theta = \pi \Rightarrow$ Polar coordinates are $(3, \pi)$

(c) $\left(\sqrt{3}, -1\right) \Rightarrow r = \sqrt{\left(\sqrt{3}\right)^2+(-1)^2} = 2,\ \sin\theta = -\frac{1}{2}$ and $\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{11\pi}{6} \Rightarrow$ Polar coordinates are $\left(2, \frac{11\pi}{6}\right)$

(d) $(-3,4) \Rightarrow r = \sqrt{(-3)^2+4^2} = 5,\ \sin\theta = \frac{4}{5}$ and $\cos\theta = -\frac{3}{5} \Rightarrow \theta = \pi - \arctan\left(\frac{4}{3}\right) \Rightarrow$ Polar coordinates are $\left(5, \pi - \arctan\left(\frac{4}{3}\right)\right)$

8. (a) $(-2,-2) \Rightarrow r = \sqrt{(-2)^2+(-2)^2} = 2\sqrt{2},\ \sin\theta = -\frac{1}{\sqrt{2}}$ and $\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = -\frac{3\pi}{4} \Rightarrow$ Polar coordinates are $\left(2\sqrt{2}, -\frac{3\pi}{4}\right)$

(b) $(0,3) \Rightarrow r = \sqrt{0^2+3^2} = 3,\ \sin\theta = 1$ and $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow$ Polar coordinates are $\left(3, \frac{\pi}{2}\right)$

(c) $\left(-\sqrt{3}, 1\right) \Rightarrow r = \sqrt{\left(-\sqrt{3}\right)^2+1^2} = 2,\ \sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{5\pi}{6} \Rightarrow$ Polar coordinates are $\left(2, \frac{5\pi}{6}\right)$

(d) $(5,-12) \Rightarrow r = \sqrt{5^2+(-12)^2} = 13,\ \sin\theta = -\frac{12}{13}$ and $\cos\theta = \frac{5}{12} \Rightarrow \theta = -\arctan\left(\frac{12}{5}\right) \Rightarrow$ Polar coordinates are $\left(13, -\arctan\left(\frac{12}{5}\right)\right)$

9. (a) $(3,3) \Rightarrow r = -\sqrt{3^2+3^2} = -3\sqrt{2},\ \sin\theta = -\frac{1}{\sqrt{2}}$ and $\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{5\pi}{4} \Rightarrow$ Polar coordinates are $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$

(b) $(-1,0) \Rightarrow r = -\sqrt{(-1)^2+0^2} = -1,\ \sin\theta = 0$ and $\cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow$ Polar coordinates are $(-1, 0)$

(c) $\left(-1, \sqrt{3}\right) \Rightarrow r = -\sqrt{(-1)^2+\left(\sqrt{3}\right)^2} = -2,\ \sin\theta = -\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{5\pi}{3} \Rightarrow$ Polar coordinates are $\left(-2, \frac{5\pi}{3}\right)$

(d) $(4,-3) \Rightarrow r = -\sqrt{4^2+(-3)^2} = -5,\ \sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5} \Rightarrow \theta = \pi - \arctan\left(\frac{3}{4}\right) \Rightarrow$ Polar coordinates are $\left(-5, \pi - \arctan\left(\frac{4}{3}\right)\right)$

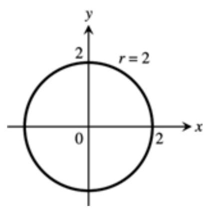
10. (a) $(-2,0) \Rightarrow r = -\sqrt{(-2)^2+0^2} = -2,\ \sin\theta = 0$ and $\cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow$ Polar coordinates are $(-2, 0)$

(b) $(1,0) \Rightarrow r = -\sqrt{1^2+0^2} = -1,\ \sin\theta = 0$ and $\cos\theta = -1 \Rightarrow \theta = \pi$ or $\theta = -\pi \Rightarrow$ Polar coordinates are $(-1, \pi)$ or $(-1, -\pi)$

(c) $(0,-3) \Rightarrow r = -\sqrt{0^2+(-3)^2} = -3,\ \sin\theta = 1$ and $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow$ Polar coordinates are $\left(-3, \frac{\pi}{2}\right)$

(d) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \Rightarrow r = -\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = -1, \sin\theta = -\frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{7\pi}{6}$ or $\theta = -\frac{5\pi}{6} \Rightarrow$ Polar coordinates are $\left(-1, \frac{7\pi}{6}\right)$ or $\left(-1, -\frac{5\pi}{6}\right)$

11.



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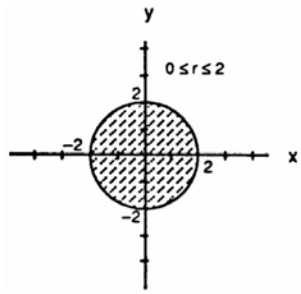

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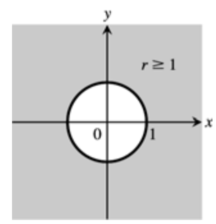

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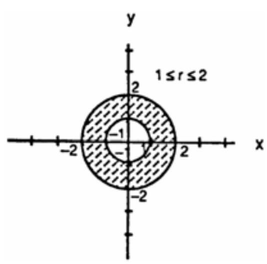

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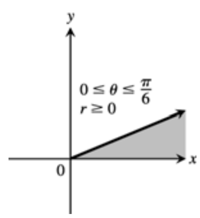

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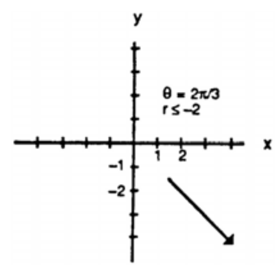

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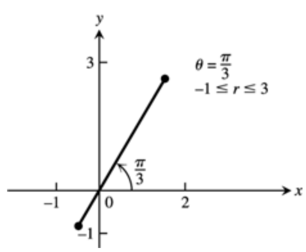

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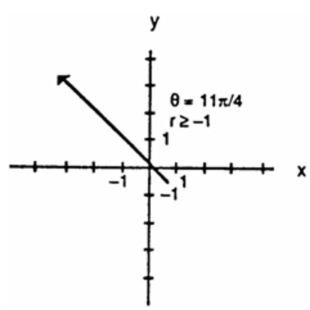

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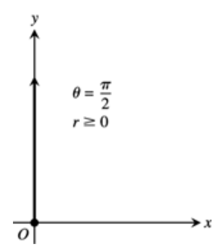

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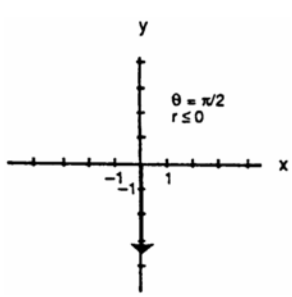
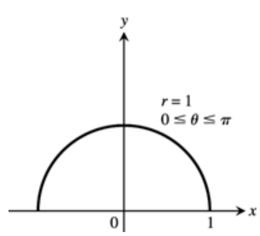


21.



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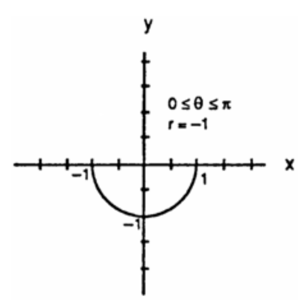

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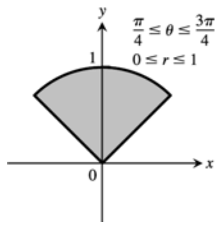

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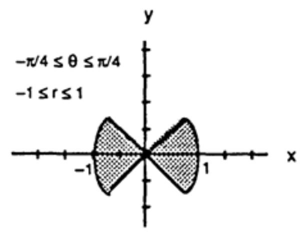

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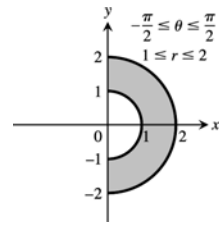

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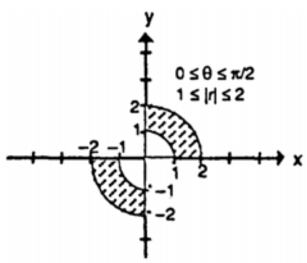


27. $r \cos \theta = 2 \Rightarrow x = 2,$ vertical line through $(2, 0)$

28. $r \sin \theta = -1 \Rightarrow y = -1,$ horizontal line through $(0, -1)$

29. $r \sin \theta = 0 \Rightarrow y = 0,$ the x-axis

30. $r \cos \theta = 0 \Rightarrow x = 0,$ the y-axis

31. $r = 4 \csc \theta \Rightarrow r = \frac{4}{\sin \theta} \Rightarrow r \sin \theta = 4 \Rightarrow y = 4,$ a horizontal line through $(0, 4)$

32. $r = -3 \sec \theta \Rightarrow r = \frac{-3}{\cos \theta} \Rightarrow r \cos \theta = -3 \Rightarrow x = -3,$ a vertical line through $(-3, 0)$

33. $r \cos \theta + r \sin \theta = 1 \Rightarrow x + y = 1,$ line with slope $m = -1$ and intercept $b = 1$

34. $r \sin \theta = r \cos \theta \Rightarrow y = x,$ line with slope $m = 1$ and intercept $b = 0$

35. $r^2 = 1 \Rightarrow x^2 + y^2 = 1,$ circle with center $C = (0, 0)$ and radius 1

36. $r^2 = 4r \sin \theta \Rightarrow x^2 + y^2 = 4y \Rightarrow x^2 + y^2 - 4y + 4 = 4 \Rightarrow x^2 + (y-2)^2 = 4,$ circle with center $C = (0, 2)$ and radius 2

37. $r = \frac{5}{\sin \theta - 2 \cos \theta} \Rightarrow r \sin \theta - 2r \cos \theta = 5 \Rightarrow y - 2x = 5,$ line with slope $m = 2$ and intercept $b = 5$

38. $r^2 \sin 2\theta = 2 \Rightarrow 2r^2 \sin \theta \cos \theta = 2 \Rightarrow (r \sin \theta)(r \cos \theta) = 1 \Rightarrow xy = 1,$ hyperbola with focal axis $y = x$

39. $r = \cot \theta \csc \theta = \left(\frac{\cos \theta}{\sin \theta}\right)\left(\frac{1}{\sin \theta}\right) \Rightarrow r \sin^2 \theta = \cos \theta \Rightarrow r^2 \sin^2 \theta = r \cos \theta \Rightarrow y^2 = x,$ parabola with vertex $(0, 0)$ which opens to the right

40. $r=4\tan\theta\sec\theta \Rightarrow r=4\left(\frac{\sin\theta}{\cos^2\theta}\right) \Rightarrow r\cos^2\theta=4\sin\theta \Rightarrow r^2\cos^2\theta=4r\sin\theta \Rightarrow x^2=4y$, parabola with vertex $=(0, 0)$ which opens upward

41. $r=(\csc\theta)e^{r\cos\theta} \Rightarrow r\sin\theta=e^{r\cos\theta} \Rightarrow y=e^x$, graph of the natural exponential function

42. $r\sin\theta=\ln r+\ln\cos\theta=\ln(r\cos\theta) \Rightarrow y=\ln x$, graph of the natural exponential function

43. $r^2+2r^2\cos\theta\sin\theta=1 \Rightarrow x^2+y^2+2xy=1 \Rightarrow x^2+2xy+y^2=1 \Rightarrow (x+y)^2=1 \Rightarrow x+y=\pm 1$, two parallel straight lines of slope -1 and y-intercepts $b=\pm 1$

44. $\cos^2\theta=\sin^2\theta \Rightarrow r^2\cos^2\theta=r^2\sin^2\theta \Rightarrow x^2=y^2 \Rightarrow |x|=|y| \Rightarrow \pm x=y$, two perpendicular lines through the origin with slopes 1 and -1, respectively.

45. $r^2=-4r\cos\theta \Rightarrow x^2+y^2=-4x \Rightarrow x^2+4x+y^2=0 \Rightarrow x^2+4x+4+y^2=4 \Rightarrow (x+2)^2+y^2=4$, a circle with center $C(-2, 0)$ and radius 2

46. $r^2=-6r\sin\theta \Rightarrow x^2+y^2=-6y \Rightarrow x^2+y^2+6y=0 \Rightarrow x^2+y^2+6y+9=9 \Rightarrow x^2+(y+3)^2=9$, a circle with center $C(0, -3)$ and radius 3

47. $r=8\sin\theta \Rightarrow r^2=8r\sin\theta \Rightarrow x^2+y^2=8y \Rightarrow x^2+y^2-8y=0 \Rightarrow x^2+y^2-8y+16=16 \Rightarrow x^2+(y-4)^2=16$, a circle with center $C(0, 4)$ and radius 4

48. $r=3\cos\theta \Rightarrow r^2=3r\cos\theta \Rightarrow x^2+y^2=3x \Rightarrow x^2+y^2-3x=0 \Rightarrow x^2-3x+\frac{9}{4}+y^2=\frac{9}{4} \Rightarrow \left(x-\frac{3}{2}\right)^2+y^2=\frac{9}{4}$, a circle with center $C\left(\frac{3}{2}, 0\right)$ and radius $\frac{3}{2}$

49. $r=2\cos\theta+2\sin\theta \Rightarrow r^2=2r\cos\theta+2r\sin\theta \Rightarrow x^2+y^2=2x+2y \Rightarrow x^2-2x+y^2-2y=0$
$\Rightarrow (x-1)^2+(y-1)^2=2$, a circle with center $C(1,1)$ and radius $\sqrt{2}$

50. $r=2\cos\theta-\sin\theta \Rightarrow r^2=2r\cos\theta-r\sin\theta \Rightarrow x^2+y^2=2x-y \Rightarrow x^2-2x+y^2+y=0$
$\Rightarrow (x-1)^2+\left(y+\frac{1}{2}\right)^2=\frac{5}{4}$, a circle with center $C\left(1, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{2}$

51. $r\sin\left(\theta+\frac{\pi}{6}\right)=2 \Rightarrow r\left(\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\right)=2 \Rightarrow \frac{\sqrt{3}}{2}r\sin\theta+\frac{1}{2}r\cos\theta=2 \Rightarrow \frac{\sqrt{3}}{2}y+\frac{1}{2}x=2$
$\Rightarrow \sqrt{3}\,y+x=4$, line with slope $m=-\frac{1}{\sqrt{3}}$ and intercept $b=\frac{4}{\sqrt{3}}$

52. $r\sin\left(\frac{2\pi}{3}-\theta\right)=5 \Rightarrow r\left(\sin\frac{2\pi}{3}\cos\theta-\cos\frac{2\pi}{3}\sin\theta\right)=5 \Rightarrow \frac{\sqrt{3}}{2}r\cos\theta+\frac{1}{2}r\sin\theta=5 \Rightarrow \frac{\sqrt{3}}{2}x+\frac{1}{2}y=5$
$\Rightarrow \sqrt{3}x+y=10$, line with slope $m=-\sqrt{3}$ and intercept $b=10$

53. $x=7 \Rightarrow r\cos\theta=7$

54. $y=1 \Rightarrow r\sin\theta=1$

55. $x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \theta = \frac{\pi}{4}$

56. $x - y = 3 \Rightarrow r\cos\theta - r\sin\theta = 3$

57. $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$ or $r = -2$

58. $x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1 \Rightarrow r^2\left(\cos^2\theta - \sin^2\theta\right) = 1 \Rightarrow r^2\cos 2\theta = 1$

59. $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow 4x^2 + 9y^2 = 36 \Rightarrow 4r^2\cos^2\theta + 9r^2\sin^2\theta = 36$

60. $xy = 2 \Rightarrow (r\cos\theta)(r\sin\theta) = 2 \Rightarrow r^2\cos\theta\sin\theta = 2 \Rightarrow 2r^2\cos\theta\sin\theta = 4 \Rightarrow r^2\sin 2\theta = 4$

61. $y^2 = 4x \Rightarrow r^2\sin^2\theta = 4r\cos\theta \Rightarrow r\sin^2\theta = 4\cos\theta$

62. $x^2 + xy + y^2 = 1 \Rightarrow x^2 + y^2 + xy = 1 \Rightarrow r^2 + r^2\sin\theta\cos\theta = 1 \Rightarrow r^2\left(1 + \sin\theta\cos\theta\right) = 1$

63. $x^2 + (y-2)^2 = 4 \Rightarrow x^2 + y^2 - 4y + 4 = 4 \Rightarrow x^2 + y^2 = 4y \Rightarrow r^2 = 4r\sin\theta \Rightarrow r = 4\sin\theta$

64. $(x-5)^2 + y^2 = 25 \Rightarrow x^2 - 10x + 25 + y^2 = 25 \Rightarrow x^2 + y^2 = 10x \Rightarrow r^2 = 10r\cos\theta \Rightarrow r = 10\cos\theta$

65. $(x-3)^2 + (y+1)^2 = 4 \Rightarrow x^2 - 6x + 9 + y^2 + 2y + 1 = 4 \Rightarrow x^2 + y^2 = 6x - 2y - 6 \Rightarrow r^2 = 6r\cos\theta - 2r\sin\theta - 6$

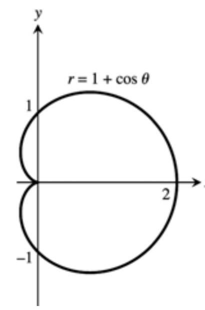
66. $(x+2)^2 + (y-5)^2 = 16 \Rightarrow x^2 + 4x + 4 + y^2 - 10y + 25 = 16 \Rightarrow x^2 + y^2 = -4x + 10y - 13$
$\Rightarrow r^2 = -4r\cos\theta + 10r\sin\theta - 13$

67. $(0, \theta)$ where θ is any angle

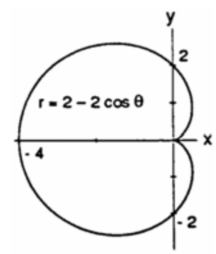
68. (a) $x = a \Rightarrow r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta} \Rightarrow r = a\sec\theta$
(b) $y = b \Rightarrow r\sin\theta = b \Rightarrow r = \frac{b}{\sin\theta} \Rightarrow r = b\csc\theta$

11.4 GRAPHING POLAR COORDINATE EQUATIONS

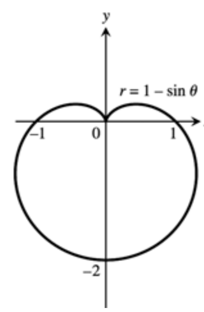
1. $1+\cos(-\theta)=1+\cos\theta=r \Rightarrow$ symmetric about the x-axis; $1+\cos(-\theta)\neq -r$ and $1+\cos(\pi-\theta)=1-\cos\theta\neq r \Rightarrow$ not symmetric about the y-axis; therefore not symmetric about the origin



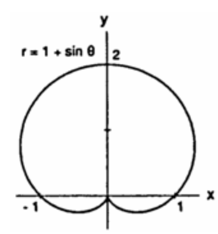
2. $2-2\cos(-\theta)=2-2\cos\theta=r \Rightarrow$ symmetric about the x-axis; $2-2\cos(-\theta)\neq -r$ and $2-2\cos(\pi-\theta)=2+2\cos\theta\neq r$ $\Rightarrow$ not symmetric about the y-axis; therefore not symmetric about the origin



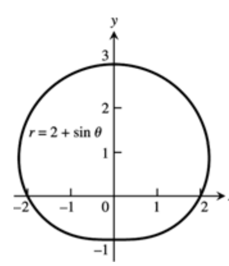
3. $1-\sin(-\theta)=1+\sin\theta\neq r$ and $1-\sin(\pi-\theta)=1-\sin\theta\neq -r$ $\Rightarrow$ not symmetric about the x-axis; $1-\sin(\pi-\theta)=1-\sin\theta=r$ $\Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



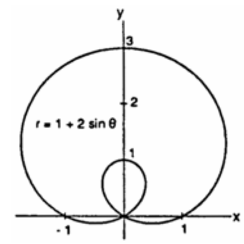
4. $1+\sin(-\theta)=1-\sin\theta\neq r$ and $1+\sin(\pi-\theta)=1+\sin\theta\neq -r$ $\Rightarrow$ not symmetric about the x-axis; $1+\sin(\pi-\theta)=1+\sin\theta=r$ $\Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



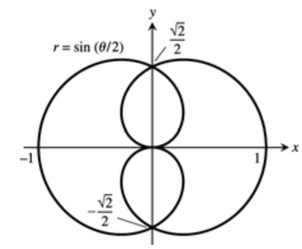
5. $2+\sin(-\theta)=2-\sin\theta\neq r$ and $2+\sin(\pi-\theta)=2+\sin\theta\neq -r \Rightarrow$ not symmetric about the x-axis; $2+\sin(\pi-\theta)=2+\sin\theta=r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



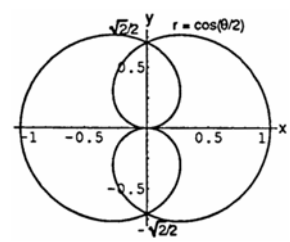
6. $1+2\sin(-\theta)=1-2\sin\theta\neq r$ and $1+2\sin(\pi-\theta)=1+2\sin\theta\neq -r \Rightarrow$ not symmetric about the x-axis; $1+2\sin(\pi-\theta)=1+2\sin\theta=r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



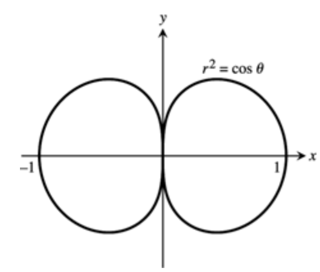
7. $\sin\left(-\frac{\theta}{2}\right)=-\sin\left(\frac{\theta}{2}\right)=-r \Rightarrow$ symmetric about the y-axis; $\sin\left(\frac{2\pi-\theta}{2}\right)=\sin\left(\frac{\theta}{2}\right)$, so the graph is symmetric about the x-axis, and hence the origin.



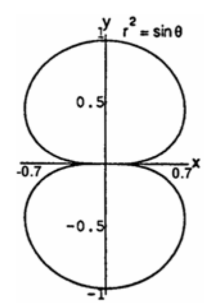
8. $\cos\left(-\frac{\theta}{2}\right)=\cos\left(\frac{\theta}{2}\right)=r \Rightarrow$ symmetric about the x-axis; $\cos\left(\frac{2\pi-\theta}{2}\right)=\cos\left(\frac{\theta}{2}\right)$, so the graph is symmetric about the y-axis, and hence the origin.



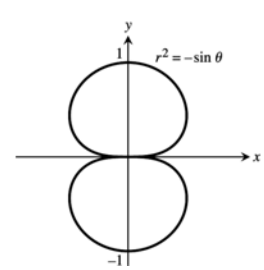
9. $\cos(-\theta)=\cos\theta=r^2 \Rightarrow (r,-\theta)$ and $(-r,-\theta)$ are on the graph when (r,θ) is on the graph $\Rightarrow$ symmetric about the x-axis and y-axis; therefore symmetric about the origin



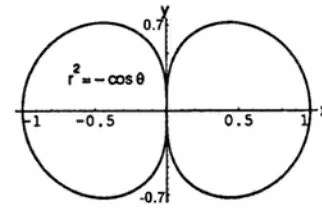
10. $\sin(\pi-\theta)=\sin\theta=r^2 \Rightarrow (r,\pi-\theta)$ and $(-r,\pi-\theta)$ are on the graph when (r,θ) is on the graph $\Rightarrow$ symmetric about the y-axis and the x-axis; therefore symmetric about the origin



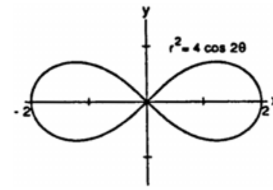
11. $-\sin(\pi-\theta)=-\sin\theta=r^2 \Rightarrow (r,\pi-\theta)$ and $(-r,\pi-\theta)$ are on the graph when (r,θ) is on the graph $\Rightarrow$ symmetric about the y-axis and the x-axis; therefore symmetric about the origin



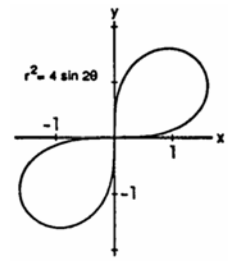
12. $-\cos(-\theta) = -\cos\theta = r^2 \Rightarrow (r, -\theta)$ and $(-r, -\theta)$ are on the graph when (r, θ) is on the graph $\Rightarrow$ symmetric about the x-axis and the y-axis; therefore symmetric about the origin



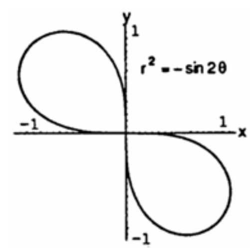
13. Since $(\pm r, -\theta)$ are on the graph when (r, θ) is on the graph, $(\pm r)^2 = 4\cos 2(-\theta) \Rightarrow r^2 = 4\cos 2\theta$, the graph is symmetric about the x-axis and the y-axis $\Rightarrow$ the graph is symmetric about the origin



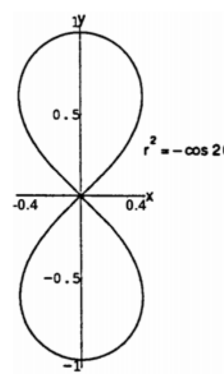
14. Since (r, θ) on the graph $\Rightarrow (-r, \theta)$ is on the graph, $(\pm r)^2 = 4\sin 2\theta \Rightarrow r^2 = 4\sin 2\theta$, the graph is symmetric about the origin. But $4\sin 2(-\theta) = -4\sin 2\theta \neq r^2$ and $4\sin 2(\pi - \theta) = 4\sin(2\pi - 2\theta) = 4\sin(-2\theta) = -4\sin 2\theta \neq r^2 \Rightarrow$ the graph is not symmetric about the x-axis; therefore the graph is not symmetric about the y-axis



15. Since (r, θ) on the graph $\Rightarrow (-r, \theta)$ is on the graph, $(\pm r)^2 = -\sin 2\theta \Rightarrow r^2 = -\sin 2\theta$, the graph is symmetric about the origin. But $-\sin 2(-\theta) = -(-\sin 2\theta) = \sin 2\theta \neq r^2$ and $-\sin 2(\pi - \theta) = -\sin(2\pi - 2\theta) = -\sin(-2\theta) = -(-\sin 2\theta) = \sin 2\theta \neq r^2$ $\Rightarrow$ the graph is not symmetric about the x-axis; therefore the graph is not symmetric about the y-axis



16. Since $(\pm r, -\theta)$ are on the graph when (r, θ) is on the graph, $(\pm r)^2 = -\cos 2(-\theta) \Rightarrow r^2 = -\cos 2\theta$, the graph is symmetric about the x-axis and the y-axis $\Rightarrow$ the graph is symmetric about the origin.

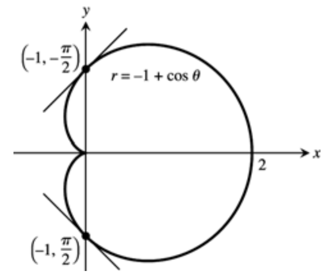


17. $\theta = \frac{\pi}{2} \Rightarrow r = -1 \Rightarrow \left(-1, \frac{\pi}{2}\right)$, and $\theta = -\frac{\pi}{2} \Rightarrow r = -1 \Rightarrow \left(-1, -\frac{\pi}{2}\right)$;

$r' = \frac{dr}{d\theta} = -\sin\theta$; Slope $= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-\sin^2\theta + r\cos\theta}{-\sin\theta\cos\theta - r\sin\theta}$

$\Rightarrow$ Slope at $\left(-1, \frac{\pi}{2}\right)$ is $\frac{-\sin^2\left(\frac{\pi}{2}\right) + (-1)\cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}\cos\frac{\pi}{2} - (-1)\sin\frac{\pi}{2}} = -1$;

Slope at $\left(-1, -\frac{\pi}{2}\right)$ is $\frac{-\sin^2\left(-\frac{\pi}{2}\right) + (-1)\cos\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}\right) - (-1)\sin\left(-\frac{\pi}{2}\right)} = 1$

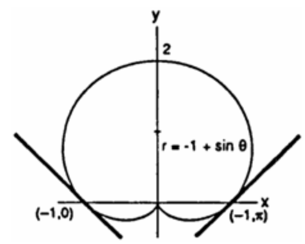


18. $\theta = 0 \Rightarrow r = -1 \Rightarrow (-1, 0)$, and $\theta = \pi \Rightarrow r = -1 \Rightarrow (-1, \pi)$;

$r' = \frac{dr}{d\theta} = \cos\theta$; Slope $= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + r\cos\theta}{\cos\theta\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + r\cos\theta}{\cos^2\theta - r\sin\theta}$

$\Rightarrow$ Slope at $(-1, 0)$ is $\frac{\cos 0\sin 0 + (-1)\cos 0}{\cos^2 0 - (-1)\sin 0} = -1$;

Slope at $(-1, \pi)$ is $\frac{\cos\pi\sin\pi + (-1)\cos\pi}{\cos^2\pi - (-1)\sin\pi} = 1$



19. $\theta = \frac{\pi}{4} \Rightarrow r = 1 \Rightarrow \left(1, \frac{\pi}{4}\right)$; $\theta = -\frac{\pi}{4} \Rightarrow r = -1 \Rightarrow \left(-1, -\frac{\pi}{4}\right)$;

$\theta = \frac{3\pi}{4} \Rightarrow r = -1 \Rightarrow \left(-1, \frac{3\pi}{4}\right)$; $\theta = -\frac{3\pi}{4} \Rightarrow r = 1 \Rightarrow \left(1, -\frac{3\pi}{4}\right)$;

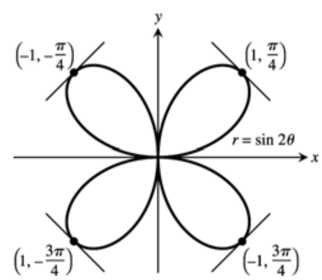
$r' = \frac{dr}{d\theta} = 2\cos 2\theta$; Slope $= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{2\cos 2\theta\sin\theta + r\cos\theta}{2\cos 2\theta\cos\theta - r\sin\theta}$

$\Rightarrow$ Slope at $\left(1, \frac{\pi}{4}\right)$ is $\frac{2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right) + (1)\cos\left(\frac{\pi}{4}\right)}{2\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - (1)\sin\left(\frac{\pi}{4}\right)} = -1$;

Slope at $\left(-1, -\frac{\pi}{4}\right)$ is $\frac{2\cos\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{4}\right) + (-1)\cos\left(-\frac{\pi}{4}\right)}{2\cos\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{4}\right) - (-1)\sin\left(-\frac{\pi}{4}\right)} = 1$;

Slope at $\left(-1, \frac{3\pi}{4}\right)$ is $\frac{2\cos\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right) + (-1)\cos\left(\frac{3\pi}{4}\right)}{2\cos\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{4}\right) - (-1)\sin\left(\frac{3\pi}{4}\right)} = 1$;

Slope at $\left(1, -\frac{3\pi}{4}\right)$ is $\frac{2\cos\left(-\frac{3\pi}{2}\right)\sin\left(-\frac{3\pi}{4}\right) + (1)\cos\left(-\frac{3\pi}{4}\right)}{2\cos\left(-\frac{3\pi}{2}\right)\cos\left(-\frac{3\pi}{4}\right) - (1)\sin\left(-\frac{3\pi}{4}\right)} = -1$;



20. $\theta = 0 \Rightarrow r = 1 \Rightarrow (1, 0)$; $\theta = \frac{\pi}{2} \Rightarrow r = -1 \Rightarrow \left(-1, \frac{\pi}{2}\right)$;

$\theta = -\frac{\pi}{2} \Rightarrow r = -1 \Rightarrow \left(-1, -\frac{\pi}{2}\right)$; $\theta = \pi \Rightarrow r = 1 \Rightarrow (1, \pi)$;

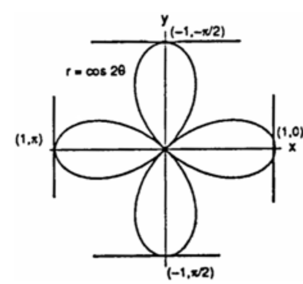
$r' = \frac{dr}{d\theta} = -2\sin 2\theta$; Slope $= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-2\sin 2\theta\sin\theta + r\cos\theta}{-2\sin 2\theta\cos\theta - r\sin\theta}$

$\Rightarrow$ Slope at $(1, 0)$ is $\frac{-2\sin 0\sin 0 + \cos 0}{-2\sin 0\cos 0 - \sin 0}$, which is undefined;

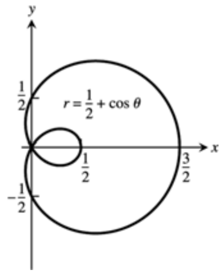
Slope at $\left(-1, \frac{\pi}{2}\right)$ is $\frac{-2\sin 2\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) + (-1)\cos\left(\frac{\pi}{2}\right)}{-2\sin 2\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - (-1)\sin\left(\frac{\pi}{2}\right)} = 0$;

Slope at $\left(-1, -\frac{\pi}{2}\right)$ is $\frac{-2\sin 2\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + (-1)\cos\left(-\frac{\pi}{2}\right)}{-2\sin 2\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}\right) - (-1)\sin\left(-\frac{\pi}{2}\right)} = 0$;

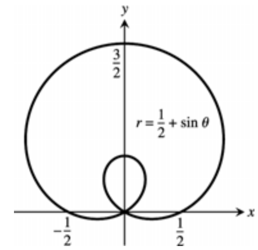
Slope at $(1, \pi)$ is $\frac{-2\sin 2\pi\sin\pi + \cos\pi}{-2\sin 2\pi\cos\pi - \sin\pi}$, which is undefined



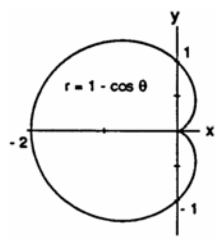
21. (a)



(b)



22. (a)



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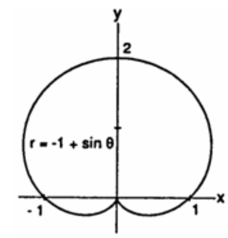
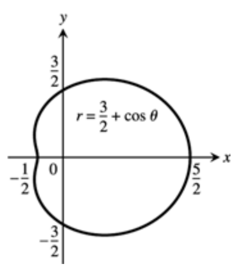


23. (a)



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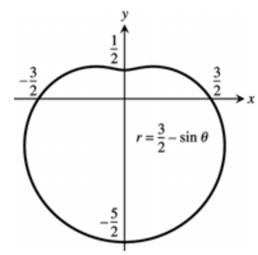
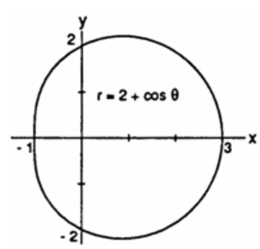
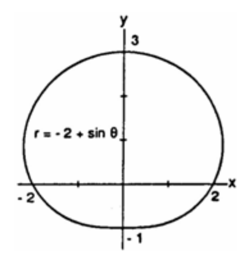


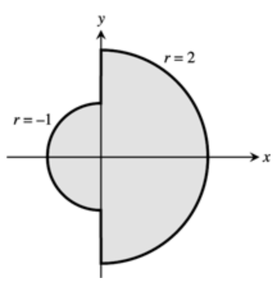
24. (a)



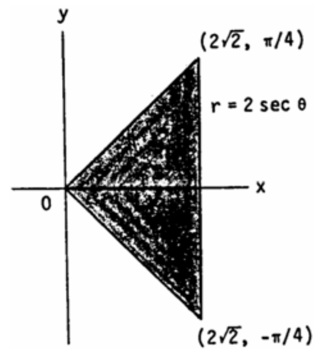
(b)



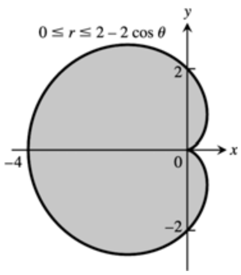
25.



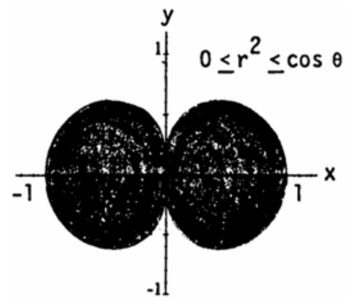
26. $r = 2\sec\theta \Rightarrow r = \frac{2}{\cos\theta} \Rightarrow r\cos\theta = 2 \Rightarrow x = 2$



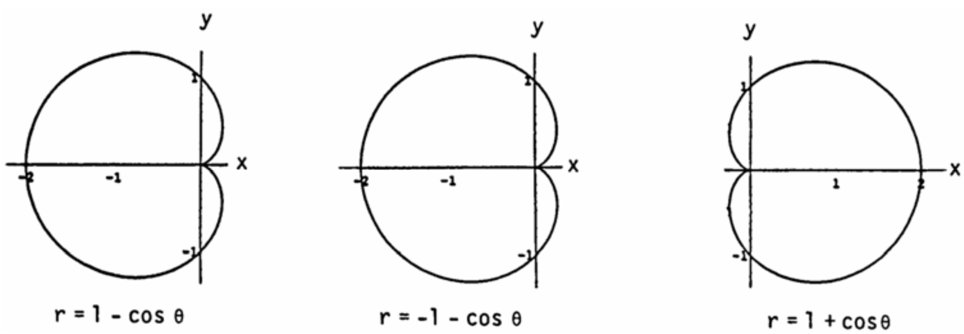
27.



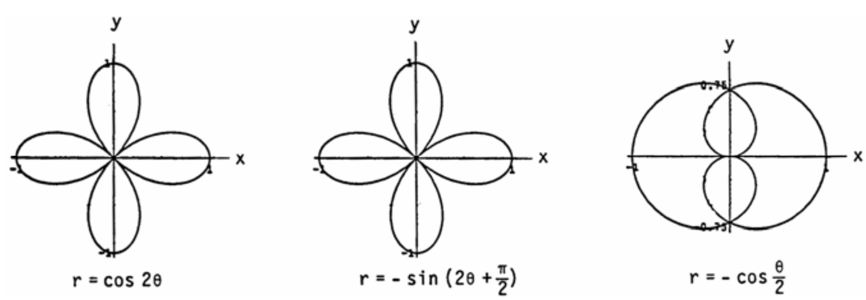
28.



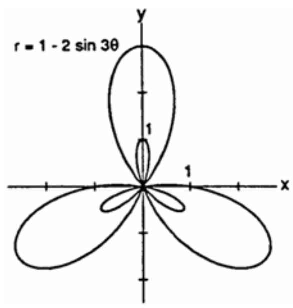
29. Note that (r, θ) and $(-r, \theta + \pi)$ describe the same point in the plane. Then $r = 1 - \cos\theta \Leftrightarrow -1 - \cos(\theta + \pi)$ $= -1 - (\cos\theta\cos\pi - \sin\theta\sin\pi) = -1 + \cos\theta = -(1 - \cos\theta) = -r$; therefore (r, θ) is on the graph of $r = 1 - \cos\theta \Leftrightarrow (-r, \theta + \pi)$ is on the graph of $r = -1 - \cos\theta \Rightarrow$ the answer is (a).



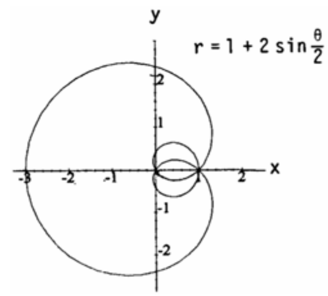
30. Note that (r, θ) and $(-r, \theta + \pi)$ describe the same point in the plane. Then $r = \cos 2\theta \Leftrightarrow -\sin\left(2(\theta + \pi) + \frac{\pi}{2}\right)$ $= -\sin\left(2\theta + \frac{5\pi}{2}\right) = -\sin(2\theta)\cos\left(\frac{5\pi}{2}\right) - \cos(2\theta)\sin\left(\frac{5\pi}{2}\right) = -\cos 2\theta = -r$; therefore (r, θ) is on the graph of $r = -\sin\left(2\theta + \frac{\pi}{2}\right) \Rightarrow$ the answer is (a).



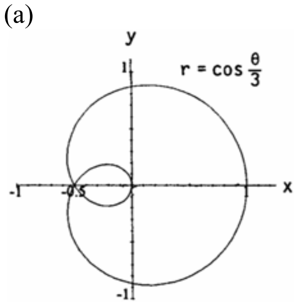
31.



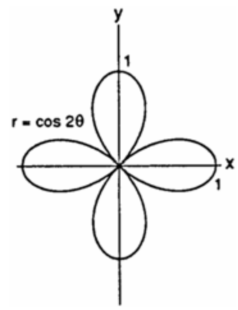
32.



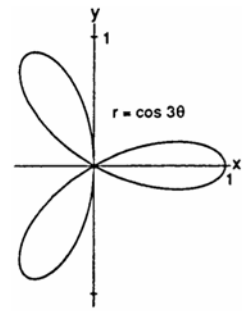
33. (a)



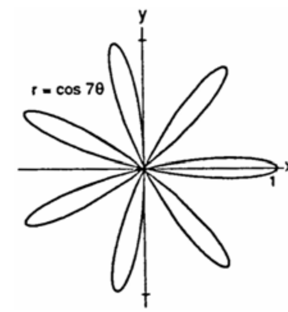
(b)



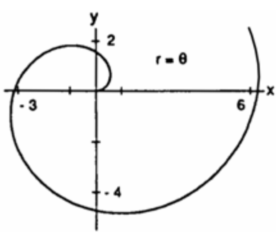
(c)



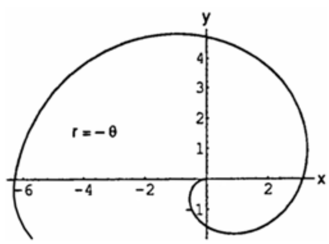
(d)



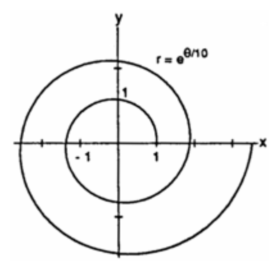
34. (a)



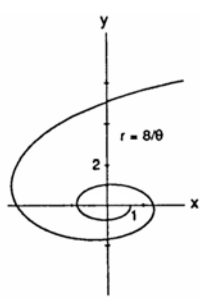
(b)



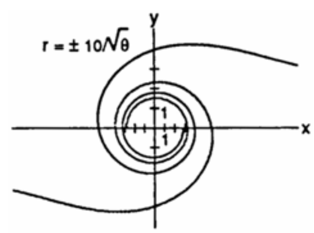
(c)



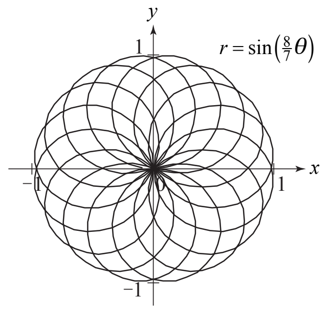
(d)



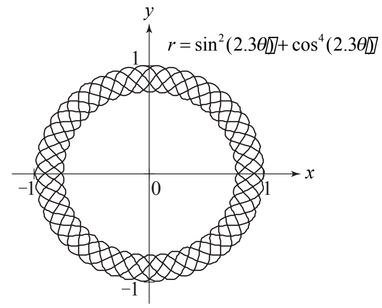
(e)



35.



36.



11.5 AREAS AND LENGTHS IN POLAR COORDINATES

1. $A = \int_0^{\pi} \frac{1}{2}\theta^2\, d\theta = \left[\frac{1}{6}\theta^3\right]_0^{\pi} = \frac{\pi^3}{6}$

2. $A = \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\sin\theta)^2\, d\theta = 2\int_{\pi/4}^{\pi/2} \sin^2\theta\, d\theta = 2\int_{\pi/4}^{\pi/2} \frac{1-\cos 2\theta}{2}\, d\theta = \int_{\pi/4}^{\pi/2} (1-\cos 2\theta)d\theta = \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2}$
$= \left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{4} + \frac{1}{2}$

3. $A = \int_0^{2\pi} \frac{1}{2}(4 + 2\cos\theta)^2\, d\theta = \int_0^{2\pi} \frac{1}{2}\left(16 + 16\cos\theta + 4\cos^2\theta\right) d\theta = \int_0^{2\pi} \left[8 + 8\cos\theta + 2\left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta$
$= \int_0^{2\pi} (9 + 8\cos\theta + \cos 2\theta)\, d\theta = \left[9\theta + 8\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 18\pi$

4. $A = \int_0^{2\pi} \frac{1}{2}\left[a(1+\cos\theta)\right]^2\, d\theta = \int_0^{2\pi} \frac{1}{2}a^2\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta = \frac{1}{2}a^2 \int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$
$= \frac{1}{2}a^2 \int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta = \frac{1}{2}a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3}{2}\pi a^2$

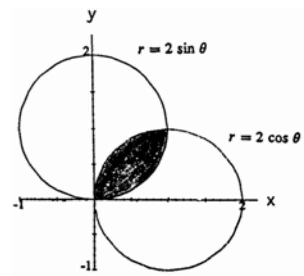
5. $A = 2\int_0^{\pi/4} \frac{1}{2}\cos^2 2\theta\, d\theta = \int_0^{\pi/4} \frac{1+\cos 4\theta}{2}\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4} = \frac{\pi}{8}$

6. $A = \int_{-\pi/6}^{\pi/6} \frac{1}{2}(\cos 3\theta)^2\, d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta\, d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{1+\cos 6\theta}{2}\, d\theta = \frac{1}{4}\int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta)\, d\theta$
$= \frac{1}{4}\left[\theta + \frac{1}{6}\sin 6\theta\right]_{-\pi/6}^{\pi/6} = \frac{1}{4}\left(\frac{\pi}{6} + 0\right) - \frac{1}{4}\left(-\frac{\pi}{6} + 0\right) = \frac{\pi}{12}$

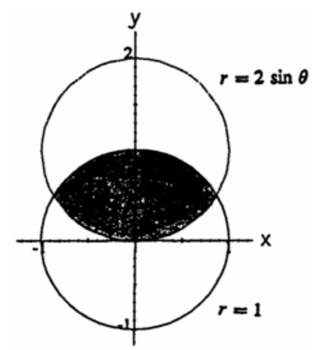
7. $A = \int_0^{\pi/2} \frac{1}{2}(4\sin 2\theta)\, d\theta = \int_0^{\pi/2} 2\sin 2\theta\, d\theta = \left[-\cos 2\theta\right]_0^{\pi/2} = 2$

8. $A = (6)(2)\int_0^{\pi/6} \frac{1}{2}(2\sin 3\theta)\, d\theta = 12\int_0^{\pi/6} \sin 3\theta\, d\theta = 12\left[-\frac{\cos 3\theta}{3}\right]_0^{\pi/6} = 4$

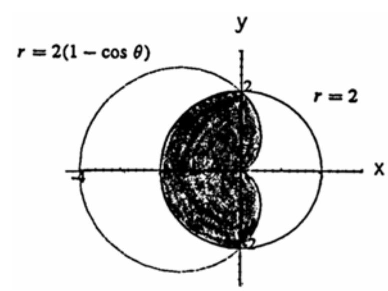
9. $r = 2\cos\theta$ and $r = 2\sin\theta \Rightarrow 2\cos\theta = 2\sin\theta$
$\Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$; therefore
$A = 2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2 d\theta = \int_0^{\pi/4} 4\sin^2\theta\, d\theta$
$= \int_0^{\pi/4} 4\left(\frac{1-\cos 2\theta}{2}\right) d\theta = \int_0^{\pi/4} (2 - 2\cos 2\theta)\, d\theta$
$= \left[2\theta - \sin 2\theta\right]_0^{\pi/4} = \frac{\pi}{2} - 1$



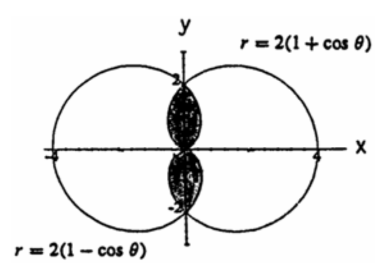
10. $r = 1$ and $r = 2\sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2}$
$\Rightarrow \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$; therefore
$A = \pi(1)^2 - \int_{\pi/6}^{5\pi/6} \frac{1}{2}[(2\sin\theta)^2 - 1^2]\, d\theta$
$= \pi - \int_{\pi/6}^{5\pi/6} \left(2\sin^2\theta - \frac{1}{2}\right) d\theta = \pi - \int_{\pi/6}^{5\pi/6} \left(1 - \cos 2\theta - \frac{1}{2}\right) d\theta$
$= \pi - \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2} - \cos 2\theta\right) d\theta = \pi - \left[\frac{1}{2}\theta - \frac{\sin 2\theta}{2}\right]_{\pi/6}^{5\pi/6}$
$= \pi - \left(\frac{5\pi}{12} - \frac{1}{2}\sin\frac{5\pi}{3}\right) + \left(\frac{\pi}{12} - \frac{1}{2}\sin\frac{\pi}{3}\right) = \frac{4\pi - 3\sqrt{3}}{6}$



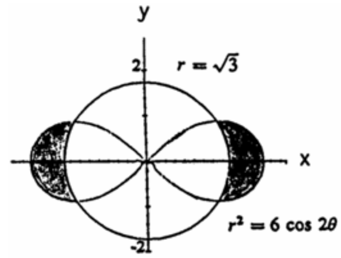
11. $r = 2$ and $r = 2(1 - \cos\theta) \Rightarrow 2 = 2(1 - \cos\theta) \Rightarrow \cos\theta = 0$
$\Rightarrow \theta = \pm\frac{\pi}{2}$; therefore
$A = 2\int_0^{\pi/2} \frac{1}{2}\left[2(1 - \cos\theta)\right]^2 d\theta + \frac{1}{2}$ area of the circle
$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \cos^2\theta\right) d\theta + \left(\frac{1}{2}\pi\right)(2)^2$
$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta + 2\pi$
$= \int_0^{\pi/2} \left(4 - 8\cos\theta + 2 + 2\cos 2\theta\right) d\theta + 2\pi$
$= \left[6\theta - 8\sin\theta + \sin 2\theta\right]_0^{\pi/2} + 2\pi = 5\pi - 8$



12. $r = 2(1 - \cos\theta)$ and $r = 2(1 + \cos\theta)$
$\Rightarrow 1 - \cos\theta = 1 + \cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$; the graph also gives the point of intersection $(0, 0)$; therefore
$A = 2\int_0^{\pi/2} \frac{1}{2}\left[2(1 - \cos\theta)\right]^2 d\theta + 2\int_{\pi/2}^{\pi} \frac{1}{2}\left[2(1 + \cos\theta)\right]^2 d\theta$
$= \int_0^{\pi/2} 4(1 - 2\cos\theta + \cos^2\theta)\, d\theta + \int_{\pi/2}^{\pi} 4(1 + 2\cos\theta + \cos^2\theta)\, d\theta$
$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta + \int_{\pi/2}^{\pi} 4\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$
$= \int_0^{\pi/2} (6 - 8\cos\theta + 2\cos 2\theta)\, d\theta + \int_{\pi/2}^{\pi} (6 + 8\cos\theta + 2\cos 2\theta)\, d\theta$
$= \left[6\theta - 8\sin\theta + \sin 2\theta\right]_0^{\pi/2} + \left[6\theta + 8\sin\theta + \sin 2\theta\right]_{\pi/2}^{\pi} = 6\pi - 16$

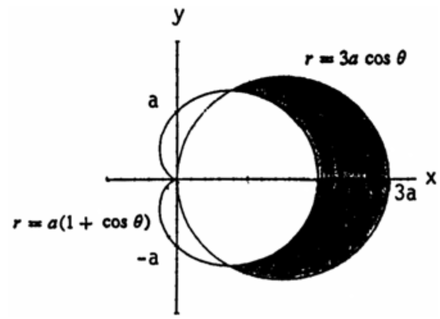


13. $r=\sqrt{3}$ and $r^2=6\cos 2\theta \Rightarrow 3=6\cos 2\theta \Rightarrow \cos 2\theta=\frac{1}{2}$
$\Rightarrow \theta=\frac{\pi}{6}$ (in the 1st quadrant); we use symmetry of the graph to find the area, so $A=4\int_0^{\pi/6}\left[\frac{1}{2}(6\cos 2\theta)-\frac{1}{2}\left(\sqrt{3}\right)^2\right]d\theta$
$=2\int_0^{\pi/6}(6\cos 2\theta-3)\,d\theta=2\left[3\sin 2\theta-3\theta\right]_0^{\pi/6}=3\sqrt{3}-\pi$



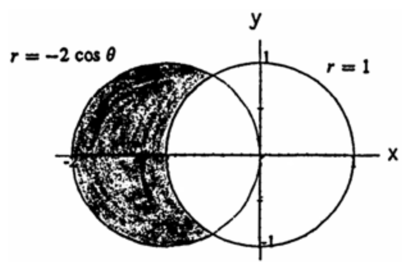
14. $r=3a\cos\theta$ and $r=a(1+\cos\theta) \Rightarrow 3a\cos\theta=a(1+\cos\theta)$
$\Rightarrow 3\cos\theta=1+\cos\theta \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$ or $-\frac{\pi}{3}$; the graph also gives the point of intersection $(0,0)$; therefore

$$A=2\int_0^{\pi/3}\frac{1}{2}\left[(3a\cos\theta)^2-a^2(1+\cos\theta)^2\right]d\theta$$
$$=\int_0^{\pi/3}\left(9a^2\cos^2\theta-a^2-2a^2\cos\theta-a^2\cos^2\theta\right)d\theta$$
$$=\int_0^{\pi/3}\left(8a^2\cos^2\theta-2a^2\cos\theta-a^2\right)d\theta$$
$$=\int_0^{\pi/3}\left[4a^2(1+\cos 2\theta)-2a^2\cos\theta-a^2\right]d\theta$$
$$=\int_0^{\pi/3}\left(3a^2+4a^2\cos 2\theta-2a^2\cos\theta\right)d\theta$$
$$=\left[3a^2\theta+2a^2\sin 2\theta-2a^2\sin\theta\right]_0^{\pi/3}$$
$$=\pi a^2+2a^2\left(\tfrac{1}{2}\right)-2a^2\left(\tfrac{\sqrt{3}}{2}\right)=a^2\left(\pi+1-\sqrt{3}\right)$$



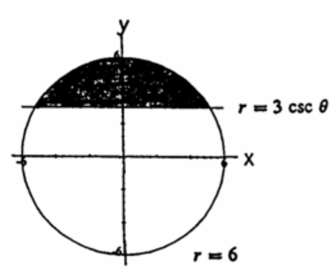
15. $r=1$ and $r=-2\cos\theta \Rightarrow 1=-2\cos\theta \Rightarrow \cos\theta=-\frac{1}{2} \Rightarrow \theta=\frac{2\pi}{3}$ in quadrant II; therefore

$$A=2\int_{2\pi/3}^{\pi}\frac{1}{2}\left[(-2\cos\theta)^2-1^2\right]d\theta=\int_{2\pi/3}^{\pi}\left(4\cos^2\theta-1\right)d\theta$$
$$=\int_{2\pi/3}^{\pi}\left[2(1+\cos 2\theta)-1\right]d\theta=\int_{2\pi/3}^{\pi}(1+2\cos 2\theta)\,d\theta$$
$$=\left[\theta+\sin 2\theta\right]_{2\pi/3}^{\pi}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$$

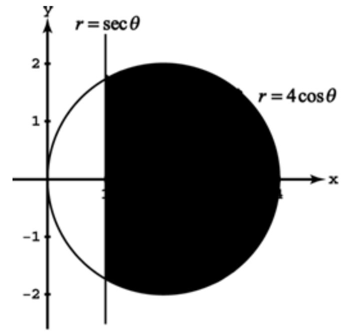


16. $r=6$ and $r=3\csc\theta \Rightarrow 6\sin\theta=3 \Rightarrow \sin\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6}$ or $\frac{5\pi}{6}$; therefore

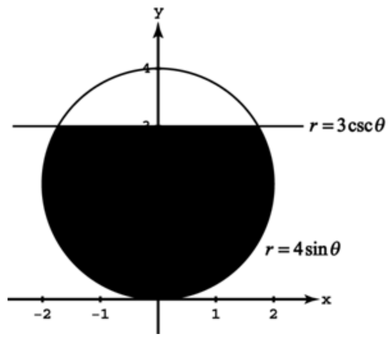
$$A=\int_{\pi/6}^{5\pi/6}\frac{1}{2}\left(6^2-9\csc^2\theta\right)d\theta=\int_{\pi/6}^{5\pi/6}\left(18-\frac{9}{2}\csc^2\theta\right)d\theta$$
$$=\left[18\theta+\frac{9}{2}\cot\theta\right]_{\pi/6}^{5\pi/6}=\left(15\pi-\frac{9}{2}\sqrt{3}\right)-\left(3\pi+\frac{9}{2}\sqrt{3}\right)=12\pi-9\sqrt{3}$$



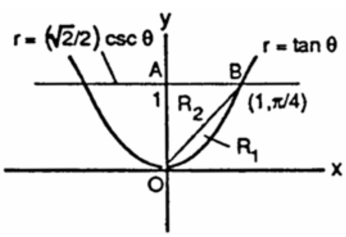
17. $r = \sec\theta$ and $r = 4\cos\theta \Rightarrow 4\cos\theta = \sec\theta \Rightarrow \cos^2\theta = \frac{1}{4}$
$\Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$; therefore
$A = 2\int_0^{\pi/3} \frac{1}{2}\left(16\cos^2\theta - \sec^2\theta\right) d\theta$
$= \int_0^{\pi/3}\left(8 + 8\cos 2\theta - \sec^2\theta\right) d\theta = \left[8\theta + 4\sin 2\theta - \tan\theta\right]_0^{\pi/3}$
$= \left(\frac{8\pi}{3} + 2\sqrt{3} - \sqrt{3}\right) - (0 + 0 - 0) = \frac{8\pi}{3} + \sqrt{3}$



18. $r = 3\csc\theta$ and $r = 4\sin\theta \Rightarrow 4\sin\theta = 3\csc\theta \Rightarrow \sin^2\theta = \frac{3}{4}$
$\Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$; therefore
$A = 4\pi - 2\int_{\pi/3}^{\pi/2} \frac{1}{2}\left(16\sin^2\theta - 9\csc^2\theta\right) d\theta$
$= 4\pi - \int_{\pi/3}^{\pi/2}\left(8 - 8\cos 2\theta - 9\csc^2\theta\right) d\theta$
$= 4\pi - \left[8\theta - 4\sin 2\theta + 9\cot\theta\right]_{\pi/3}^{\pi/2}$
$= 4\pi - \left[(4\pi - 0 + 0) - \left(\frac{8\pi}{3} - 2\sqrt{3} + 3\sqrt{3}\right)\right] = \frac{8\pi}{3} + \sqrt{3}$



19. (a) $r = \tan\theta$ and $r = \left(\frac{\sqrt{2}}{2}\right)\csc\theta \Rightarrow \tan\theta = \left(\frac{\sqrt{2}}{2}\right)\csc\theta$
$\Rightarrow \sin^2\theta = \left(\frac{\sqrt{2}}{2}\right)\cos\theta \Rightarrow 1 - \cos^2\theta = \left(\frac{\sqrt{2}}{2}\right)\cos\theta$
$\Rightarrow \cos^2\theta + \left(\frac{\sqrt{2}}{2}\right)\cos\theta - 1 = 0 \Rightarrow \cos\theta = -\sqrt{2}$ or $\frac{\sqrt{2}}{2}$
(use the quadratic formula) $\Rightarrow \theta = \frac{\pi}{4}$ (the solution in the first quadrant); therefore the area of R_1 is



$A_1 = \int_0^{\pi/4} \frac{1}{2}\tan^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/4}\left(\sec^2\theta - 1\right) d\theta = \frac{1}{2}\left[\tan\theta - \theta\right]_0^{\pi/4} = \frac{1}{2}\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{1}{2} - \frac{\pi}{8}$;
$AO = \left(\frac{\sqrt{2}}{2}\right)\csc\frac{\pi}{2} = \frac{\sqrt{2}}{2}$ and $OB = \left(\frac{\sqrt{2}}{2}\right)\csc\frac{\pi}{4} = 1 \Rightarrow AB = \sqrt{1^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{2}}{2} \Rightarrow$ the area of R_2 is
$A_2 = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{4}$; therefore the area of the region shaded in the text is $2\left(\frac{1}{2} - \frac{\pi}{8} + \frac{1}{4}\right) = \frac{3}{2} - \frac{\pi}{4}$. Note: The area must be found this way since no common interval generates the region. For example, the interval $0 \le \theta \le \frac{\pi}{4}$ generates the arc OB of $r = \tan\theta$ but does not generate the segment AB of the liner $= \frac{\sqrt{2}}{2}\csc\theta$. Instead the interval generates the half-line from B to $+\infty$ on the line $r = \frac{\sqrt{2}}{2}\csc\theta$.

(b) $\lim_{\theta \to \frac{\pi}{2}^-} \tan\theta = \infty$ and the line $x = 1$ is $r = \sec\theta$ in polar coordinates; then
$\lim_{\theta \to \frac{\pi}{2}^-}(\tan\theta - \sec\theta) = \lim_{\theta \to \frac{\pi}{2}^-}\left(\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}\right) = \lim_{\theta \to \frac{\pi}{2}^-}\left(\frac{\sin\theta - 1}{\cos\theta}\right) = \lim_{\theta \to \frac{\pi}{2}^-}\left(\frac{\cos\theta}{-\sin\theta}\right) = 0 \Rightarrow r = \tan\theta$ approaches
$r = \sec\theta$ as $\theta \to \left(\frac{\pi}{2}\right)^- \Rightarrow r = \sec\theta$(or $x = 1$) is a vertical asymptote of $r = \tan\theta$. Similarly, $r = -\sec\theta$(or $x = -1$) is a vertical asymptote of $r = \tan\theta$.

20. It is not because the circle is generated twice from $\theta = 0$ to 2π. The area of the cardioid is
$A = 2\int_0^{\pi} \frac{1}{2}(\cos\theta + 1)^2\, d\theta = \int_0^{\pi}\left(\cos^2\theta + 2\cos\theta + 1\right) d\theta = \int_0^{\pi}\left(\frac{1+\cos 2\theta}{2} + 2\cos\theta + 1\right) d\theta$
$= \left[\frac{3\theta}{2} + \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi} = \frac{3\pi}{2}$. The area of the circle is $A = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4} \Rightarrow$ the area requested is actually $\frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$

21. $r = \theta^2, 0 \le \theta \le \sqrt{5} \Rightarrow \frac{dr}{d\theta} = 2\theta$; therefore Length $= \int_0^{\sqrt{5}} \sqrt{\left(\theta^2\right)^2 + (2\theta)^2}\, d\theta = \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta$
$= \int_0^{\sqrt{5}} |\,\theta\,| \sqrt{\theta^2 + 4}\, d\theta = \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\, d\theta$; (since $\theta \ge 0$)
$\left[u = \theta^2 + 4 \Rightarrow \frac{1}{2}\, du = \theta\, d\theta; \theta = 0 \Rightarrow u = 4, \theta = \sqrt{5} \Rightarrow u = 9\right] \to \int_4^9 \frac{1}{2}\sqrt{u}\, du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_4^9 = \frac{19}{3}$

22. $r = \frac{e^{\theta}}{\sqrt{2}}, 0 \le \theta \le \pi \Rightarrow \frac{dr}{d\theta} = \frac{e^{\theta}}{\sqrt{2}}$; therefore Length $= \int_0^{\pi} \sqrt{\left(\frac{e^{\theta}}{\sqrt{2}}\right)^2 + \left(\frac{e^{\theta}}{\sqrt{2}}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{2\left(\frac{e^{2\theta}}{2}\right)}\, d\theta$
$= \int_0^{\pi} e^{\theta}\, d\theta = \left[e^{\theta}\right]_0^{\pi} = e^{\pi} - 1$

23. $r = 1 + \cos\theta \Rightarrow \frac{dr}{d\theta} = -\sin\theta$; therefore Length $= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$
$= 2\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta = 2\int_0^{\pi} \sqrt{\frac{4(1+\cos\theta)}{2}}\, d\theta = 4\int_0^{\pi} \sqrt{\frac{1+\cos\theta}{2}} d\theta = 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta = 4\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8$

24. $r = a\sin^2\frac{\theta}{2}, 0 \le \theta \le \pi, a > 0 \Rightarrow \frac{dr}{d\theta} = a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$; therefore Length $= \int_0^{\pi} \sqrt{\left(a\sin^2\frac{\theta}{2}\right)^2 + \left(a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}\, d\theta$
$= \int_0^{\pi} \sqrt{a^2\sin^4\frac{\theta}{2} + a^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\, d\theta = \int_0^{\pi} a\left|\sin\frac{\theta}{2}\right| \sqrt{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}\, d\theta = a\int_0^{\pi} \sin\left(\frac{\theta}{2}\right) d\theta$ (since $0 \le \theta \le \pi$)
$= \left[-2a\cos\frac{\theta}{2}\right]_0^{\pi} = 2a$

25. $r = \frac{6}{1+\cos\theta}, 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} = \frac{6\sin\theta}{(1+\cos\theta)^2}$; therefore Length $= \int_0^{\pi/2} \sqrt{\left(\frac{6}{1+\cos\theta}\right)^2 + \left(\frac{6\sin\theta}{(1+\cos\theta)^2}\right)^2}\, d\theta$
$= \int_0^{\pi/2} \sqrt{\frac{36}{(1+\cos\theta)^2} + \frac{36\sin^2\theta}{(1+\cos\theta)^4}}\, d\theta = 6\int_0^{\pi/2} \left|\frac{1}{1+\cos\theta}\right| \sqrt{1 + \frac{\sin^2\theta}{(1+\cos\theta)^2}}\, d\theta$
$= 6\int_0^{\pi/2} \left(\frac{1}{1+\cos\theta}\right) \sqrt{\frac{1+2\cos\theta+\cos^2\theta+\sin^2\theta}{(1+\cos\theta)^2}}\, d\theta$ $\left(\text{since } \frac{1}{1+\cos\theta} > 0 \text{ on } 0 \le \theta \le \frac{\pi}{2}\right)$
$= 6\int_0^{\pi/2} \left(\frac{1}{1+\cos\theta}\right) \sqrt{\frac{2+2\cos\theta}{(1+\cos\theta)^2}}\, d\theta = 6\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{(1+\cos\theta)^{3/2}} = 6\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\left(2\cos^2\frac{\theta}{2}\right)^{3/2}} = 3\int_0^{\pi/2} \left|\sec^3\frac{\theta}{2}\right| d\theta$
$= 3\int_0^{\pi/2} \sec^3\frac{\theta}{2}\, d\theta = 6\int_0^{\pi/4} \sec^3 u\, du = 6\left(\left[\frac{\sec u \tan u}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec u\, du\right)$ (use tables)
$= 6\left(\frac{1}{\sqrt{2}} + \left[\frac{1}{2}\ln|\sec u + \tan u|\right]_0^{\pi/4}\right) = 3\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$

26. $r=\frac{2}{1-\cos\theta}, \frac{\pi}{2}\le\theta\le\pi \Rightarrow \frac{dr}{d\theta}=\frac{-2\sin\theta}{(1-\cos\theta)^2}$; therefore Length $=\int_{\pi/2}^{\pi}\sqrt{\left(\frac{2}{1-\cos\theta}\right)^2+\left(\frac{-2\sin\theta}{(1-\cos\theta)^2}\right)^2}\,d\theta$

$=\int_{\pi/2}^{\pi}\sqrt{\frac{4}{(1-\cos\theta)^2}\left(1+\frac{\sin^2\theta}{(1-\cos\theta)^2}\right)}\,d\theta=\int_{\pi/2}^{\pi}\left|\frac{2}{1-\cos\theta}\right|\sqrt{\frac{(1-\cos\theta)^2+\sin^2\theta}{(1-\cos\theta)^2}}\,d\theta$

$=2\int_{\pi/2}^{\pi}\left(\frac{1}{1-\cos\theta}\right)\sqrt{\frac{1-2\cos\theta+\cos^2\theta+\sin^2\theta}{(1-\cos\theta)^2}}\,d\theta$ (since $1-\cos\theta\ge 0$ on $\frac{\pi}{2}\le\theta\le\pi$)

$=2\int_{\pi/2}^{\pi}\left(\frac{1}{1-\cos\theta}\right)\sqrt{\frac{2-2\cos\theta}{(1-\cos\theta)^2}}\,d\theta=2\sqrt{2}\int_{\pi/2}^{\pi}\frac{d\theta}{(1-\cos\theta)^{3/2}}=2\sqrt{2}\int_{\pi/2}^{\pi}\frac{d\theta}{\left(2\sin^2\frac{\theta}{2}\right)^{3/2}}=\int_{\pi/2}^{\pi}\left|\csc^3\frac{\theta}{2}\right|d\theta=\int_{\pi/2}^{\pi}\csc^3\left(\frac{\theta}{2}\right)d\theta$

(since $\csc\frac{\theta}{2}\ge 0$ on $\frac{\pi}{2}\le\theta\le\pi$) $=\int_{\pi/4}^{\pi/2}\csc^3 u\,du=2\left(\left[-\frac{\csc u\cot u}{2}\right]_{\pi/4}^{\pi/2}+\frac{1}{2}\int_{\pi/4}^{\pi/2}\csc u\,du\right)$ (use tables)

$=2\left(\frac{1}{\sqrt{2}}-\left[\frac{1}{2}\ln|\csc u+\cot u|\right]_{\pi/4}^{\pi/2}\right)=2\left[\frac{1}{\sqrt{2}}+\frac{1}{2}\ln\left(\sqrt{2}+1\right)\right]=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$

27. $r=\cos^3\frac{\theta}{3}\Rightarrow\frac{dr}{d\theta}=-\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}$; therefore Length $=\int_0^{\pi/4}\sqrt{\left(\cos^3\frac{\theta}{3}\right)^2+\left(-\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}\right)^2}\,d\theta$

$=\int_0^{\pi/4}\sqrt{\cos^6\left(\frac{\theta}{3}\right)+\sin^2\left(\frac{\theta}{3}\right)\cos^4\left(\frac{\theta}{3}\right)}\,d\theta=\int_0^{\pi/4}\left(\cos^2\frac{\theta}{3}\right)\sqrt{\cos^2\left(\frac{\theta}{3}\right)+\sin^2\left(\frac{\theta}{3}\right)}\,d\theta=\int_0^{\pi/4}\cos^2\left(\frac{\theta}{3}\right)d\theta$

$=\int_0^{\pi/4}\frac{1+\cos\left(\frac{2\theta}{3}\right)}{2}\,d\theta=\frac{1}{2}\left[\theta+\frac{3}{2}\sin\frac{2\theta}{3}\right]_0^{\pi/4}=\frac{\pi}{8}+\frac{3}{8}$

28 $r=\sqrt{1+\sin 2\theta}, 0\le\theta\le\pi\sqrt{2}\Rightarrow\frac{dr}{d\theta}=\frac{1}{2}(1+\sin 2\theta)^{-1/2}(2\cos 2\theta)=(\cos 2\theta)(1+\sin 2\theta)^{-1/2}$; therefore

Length $=\int_0^{\pi\sqrt{2}}\sqrt{(1+\sin 2\theta)+\frac{\cos^2 2\theta}{(1+\sin 2\theta)}}\,d\theta=\int_0^{\pi\sqrt{2}}\sqrt{\frac{1+2\sin 2\theta+\sin^2 2\theta+\cos^2 2\theta}{1+\sin 2\theta}}\,d\theta=\int_0^{\pi\sqrt{2}}\sqrt{\frac{2+2\sin 2\theta}{1+\sin 2\theta}}\,d\theta$

$=\int_0^{\pi\sqrt{2}}\sqrt{2}\,d\theta=\left[\sqrt{2}\theta\right]_0^{\pi\sqrt{2}}=2\pi$

29. Let $r=f(\theta)$. Then $x=f(\theta)\cos\theta\Rightarrow\frac{dx}{d\theta}=f'(\theta)\cos\theta-f(\theta)\sin\theta\Rightarrow\left(\frac{dx}{d\theta}\right)^2=\left[f'(\theta)\cos\theta-f(\theta)\sin\theta\right]^2$

$=\left[f'(\theta)\right]^2\cos^2\theta-2f'(\theta)f(\theta)\sin\theta\cos\theta+\left[f(\theta)\right]^2\sin^2\theta$; $y=f(\theta)\sin\theta\Rightarrow\frac{dy}{d\theta}=f'(\theta)\sin\theta+f(\theta)\cos\theta$

$\Rightarrow\left(\frac{dy}{d\theta}\right)^2=\left[f'(\theta)\sin\theta+f(\theta)\cos\theta\right]^2=\left[f'(\theta)\right]^2\sin^2\theta+2f'(\theta)f(\theta)\sin\theta\cos\theta+\left[f(\theta)\right]^2\cos^2\theta$. Therefore

$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\left[f'(\theta)\right]^2\left(\cos^2\theta+\sin^2\theta\right)+\left[f(\theta)\right]^2\left(\cos^2\theta+\sin^2\theta\right)=\left[f'(\theta)\right]^2+\left[f(\theta)\right]^2=r^2+\left(\frac{dr}{d\theta}\right)^2$.

Thus, $L=\int_\alpha^\beta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$.

30. (a) $r=a\Rightarrow\frac{dr}{d\theta}=0$; Length $=\int_0^{2\pi}\sqrt{a^2+0^2}\,d\theta=\int_0^{2\pi}|a|\,d\theta=\left[a\theta\right]_0^{2\pi}=2\pi a$

(b) $r=a\cos\theta\Rightarrow\frac{dr}{d\theta}=-a\sin\theta$; Length $=\int_0^{\pi}\sqrt{(a\cos\theta)^2+(-a\sin\theta)^2}\,d\theta=\int_0^{\pi}\sqrt{a^2\left(\cos^2\theta+\sin^2\theta\right)}\,d\theta$

$=\int_0^{\pi}|a|\,d\theta=\left[a\theta\right]_0^{\pi}=\pi a$

(c) $r=a\sin\theta\Rightarrow\frac{dr}{d\theta}=a\cos\theta$; Length $=\int_0^{\pi}\sqrt{(a\cos\theta)^2+(a\sin\theta)^2}\,d\theta=\int_0^{\pi}\sqrt{a^2\left(\cos^2\theta+\sin^2\theta\right)}\,d\theta$

$=\int_0^{\pi}|a|\,d\theta=\left[a\theta\right]_0^{\pi}=\pi a$

31. (a) $r_{av} = \frac{1}{2\pi - 0}\int_0^{2\pi} a(1-\cos\theta)\,d\theta = \frac{a}{2\pi}\left[\theta - \sin\theta\right]_0^{2\pi} = a$

(b) $r_{av} = \frac{1}{2\pi - 0}\int_0^{2\pi} a\,d\theta = \frac{1}{2\pi}\left[a\theta\right]_0^{2\pi} = a$

(c) $r_{av} = \frac{1}{\left(\frac{\pi}{2}\right)-\left(-\frac{\pi}{2}\right)}\int_{-\pi/2}^{\pi/2} a\cos\theta\,d\theta = \frac{1}{\pi}\left[a\sin\theta\right]_{-\pi/2}^{\pi/2} = \frac{2a}{\pi}$

32. $r = 2f(\theta), \alpha \le \theta \le \beta \Rightarrow \frac{dr}{d\theta} = 2f'(\theta) \Rightarrow r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left[2f(\theta)\right]^2 + \left[2f'(\theta)\right]^2$

$\Rightarrow$ Length $= \int_\alpha^\beta \sqrt{4\left[f(\theta)\right]^2 + 4\left[f'(\theta)\right]^2}\,d\theta = 2\int_\alpha^\beta \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2}\,d\theta$

which is twice the length of the curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$.

11.6 CONIC SECTIONS

1. $x = \frac{y^2}{8} \Rightarrow 4p = 8 \Rightarrow p = 2$; focus is $(2,0)$, directrix is $x = -2$

2. $x = -\frac{y^2}{4} \Rightarrow 4p = 4 \Rightarrow p = 1$; focus is $(-1,0)$, directrix is $x = 1$

3. $y = -\frac{x^2}{6} \Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2}$; focus is $\left(0, -\frac{3}{2}\right)$, directrix is $y = \frac{3}{2}$

4. $y = \frac{x^2}{2} \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}$; focus is $\left(0, \frac{1}{2}\right)$, directrix is $y = -\frac{1}{2}$

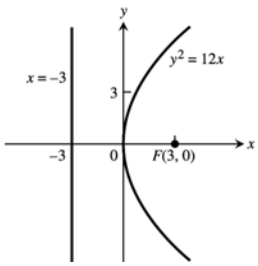
5. $\frac{x^2}{4} - \frac{y^2}{9} = 1 \Rightarrow c = \sqrt{4+9} = \sqrt{13} \Rightarrow$ foci are $\left(\pm\sqrt{13}, 0\right)$; vertices are $(\pm 2, 0)$; asymptotes are $y = \pm\frac{3}{2}x$

6. $\frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow c = \sqrt{9-4} = \sqrt{5} \Rightarrow$ foci are $\left(0, \pm\sqrt{5}\right)$; vertices are $(0, \pm 3)$

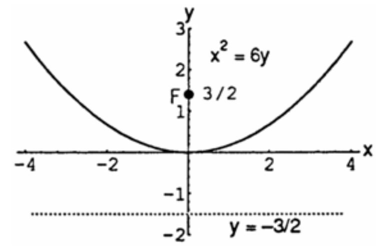
7. $\frac{x^2}{2} + y^2 = 1 \Rightarrow c = \sqrt{2-1} = 1 \Rightarrow$ foci are $(\pm 1, 0)$; vertices are $\left(\pm\sqrt{2}, 0\right)$

8. $\frac{y^2}{4} - x^2 = 1 \Rightarrow c = \sqrt{4+1} = \sqrt{5} \Rightarrow$ foci are $\left(0, \pm\sqrt{5}\right)$; vertices are $(0, \pm 2)$; asymptotes are $y = \pm 2x$

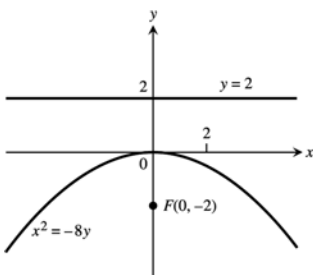
9. $y^2 = 12x \Rightarrow x = \frac{y^2}{12} \Rightarrow 4p = 12 \Rightarrow p = 3;$
focus is $(3, 0)$, directrix is $x = -3$



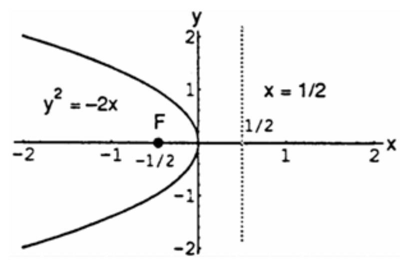
10. $x^2 = 6y \Rightarrow y = \frac{x^2}{6} \Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2};$
focus is $\left(0, \frac{3}{2}\right)$, directrix is $y = -\frac{3}{2}$



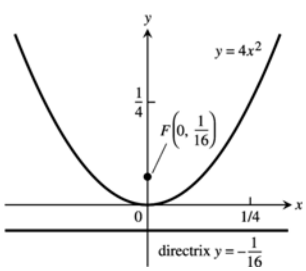
11. $x^2 = -8y \Rightarrow y = \frac{x^2}{-8} \Rightarrow 4p = 8 \Rightarrow p = 2;$
focus is $(0, -2)$, directrix is $y = 2$



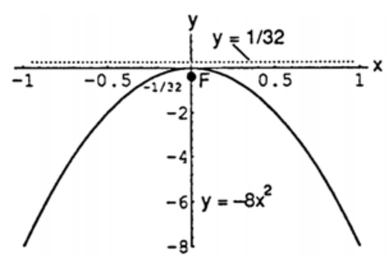
12. $y^2 = -2x \Rightarrow x = \frac{y^2}{-2} \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2};$
focus is $\left(-\frac{1}{2}, 0\right)$, directrix is $x = \frac{1}{2}$



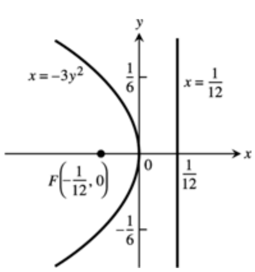
13. $y = 4x^2 \Rightarrow y = \frac{x^2}{\left(\frac{1}{4}\right)} \Rightarrow 4p = \frac{1}{4} \Rightarrow p = \frac{1}{16};$
focus is $\left(0, \frac{1}{16}\right)$, directrix is $y = -\frac{1}{16}$



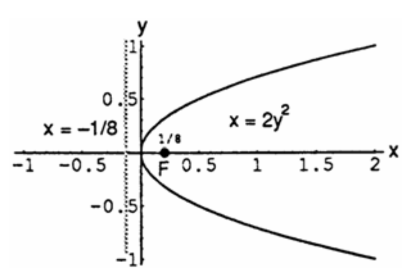
14. $y = -8x^2 \Rightarrow y = -\frac{x^2}{\left(\frac{1}{8}\right)} \Rightarrow 4p = \frac{1}{8} \Rightarrow p = \frac{1}{32};$
focus is $\left(0, -\frac{1}{32}\right)$, directrix is $y = \frac{1}{32}$



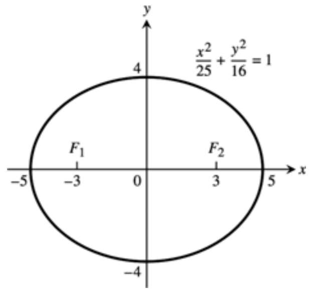
15. $x = -3y^2 \Rightarrow x = -\frac{y^2}{\left(\frac{1}{3}\right)} \Rightarrow 4p = \frac{1}{3} \Rightarrow p = \frac{1}{12};$
focus is $\left(-\frac{1}{12}, 0\right)$, directrix is $x = \frac{1}{12}$



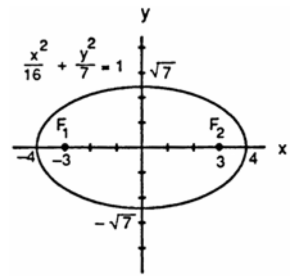
16. $x = 2y^2 \Rightarrow x = \frac{y^2}{\left(\frac{1}{2}\right)} \Rightarrow 4p = \frac{1}{2} \Rightarrow p = \frac{1}{8};$
focus is $\left(\frac{1}{8}, 0\right)$, directrix is $x = -\frac{1}{8}$



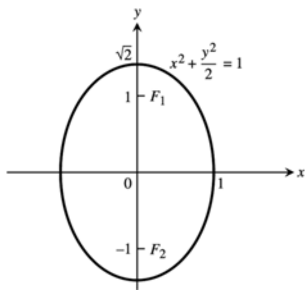
17. $16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = 3$



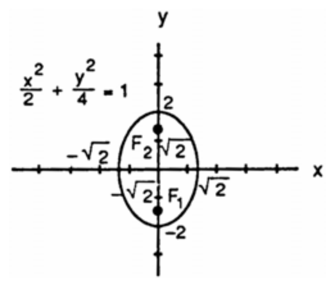
18. $7x^2 + 16y^2 = 112 \Rightarrow \frac{x^2}{16} + \frac{y^2}{7} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{16 - 7} = 3$



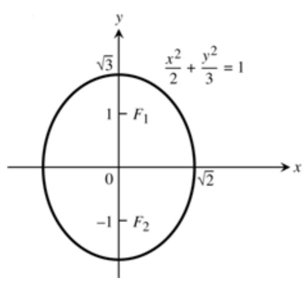
19. $2x^2 + y^2 = 2 \Rightarrow x^2 + \frac{y^2}{2} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{2 - 1} = 1$



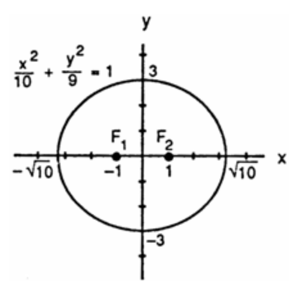
20. $2x^2 + y^2 = 4 \Rightarrow \frac{x^2}{2} + \frac{y^2}{4} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{4 - 2} = \sqrt{2}$



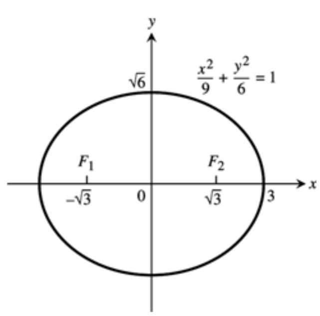
21. $3x^2 + 2y^2 = 6 \Rightarrow \frac{x^2}{2} + \frac{y^2}{3} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{3 - 2} = 1$



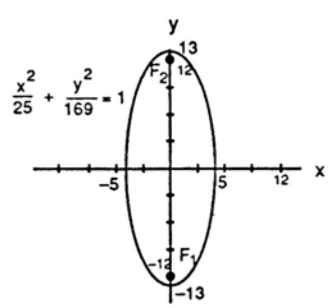
22. $9x^2 + 10y^2 = 90 \Rightarrow \frac{x^2}{10} + \frac{y^2}{9} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{10 - 9} = 1$



23. $6x^2 + 9y^2 = 54 \Rightarrow \frac{x^2}{9} + \frac{y^2}{6} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{9 - 6} = \sqrt{3}$



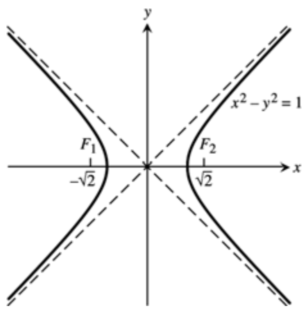
24. $169x^2 + 25y^2 = 4225 \Rightarrow \frac{x^2}{25} + \frac{y^2}{169} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{169 - 25} = 12$



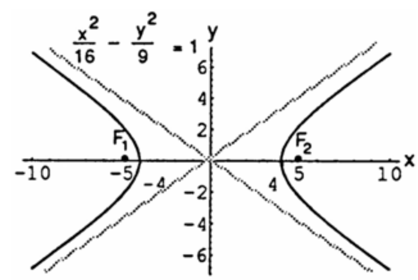
25. Foci: $\left(\pm\sqrt{2}, 0\right)$, Vertices: $(\pm 2, 0) \Rightarrow a = 2,\ c = \sqrt{2} \Rightarrow b^2 = a^2 - c^2 = 4 - \left(\sqrt{2}\right)^2 = 2 \Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1$

26. Foci: $(0, \pm 4)$, Vertices: $(0, \pm 5) \Rightarrow a = 5,\ c = 4 \Rightarrow b^2 = 25 - 16 = 9 \Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1$

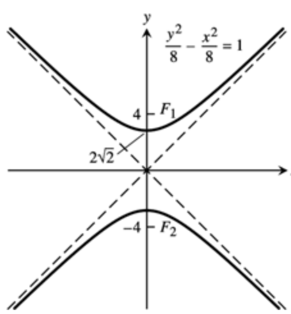
27. $x^2 - y^2 = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2}$; asymptotes are $y = \pm x$



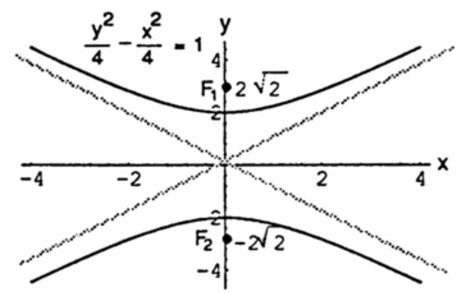
28. $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$ $= \sqrt{16+9} = 5$; asymptotes are $y = \pm \frac{3}{4} x$



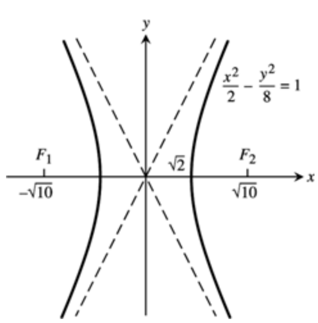
29. $y^2 - x^2 = 8 \Rightarrow \frac{y^2}{8} - \frac{x^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$ $= \sqrt{8+8} = 4$; asymptotes are $y = \pm x$



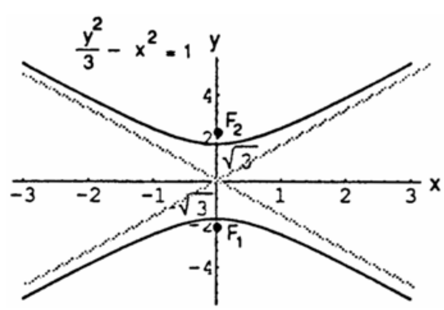
30. $y^2 - x^2 = 4 \Rightarrow \frac{y^2}{4} - \frac{x^2}{4} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$ $= \sqrt{4+4} = 2\sqrt{2}$; asymptotes are $y = \pm x$



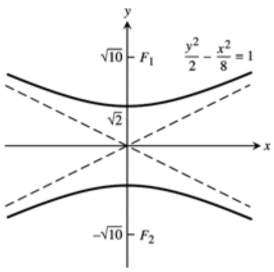
31. $8x^2 - 2y^2 = 16 \Rightarrow \frac{x^2}{2} - \frac{y^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$ $= \sqrt{2+8} = \sqrt{10}$; asymptotes are $y = \pm 2x$



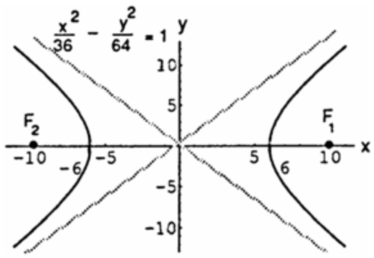
32. $y^2 - 3x^2 = 3 \Rightarrow \frac{y^2}{3} - x^2 = 1 \Rightarrow c = \sqrt{a^2 + b^2}$ $= \sqrt{3+1} = 2$; asymptotes are $y = \pm\sqrt{3}x$



33. $8y^2 - 2x^2 = 16 \Rightarrow \frac{y^2}{2} - \frac{x^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$ $= \sqrt{2+8} = \sqrt{10}$; asymptotes are $y = \pm\frac{x}{2}$



34. $64x^2 - 36y^2 = 2304 \Rightarrow \frac{x^2}{36} - \frac{y^2}{64} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$ $= \sqrt{36+64} = 10$; asymptotes are $y = \pm\frac{4}{3}$



35. Foci: $\left(0, \pm\sqrt{2}\right)$, Asymptotes: $y = \pm x \Rightarrow c = \sqrt{2}$ and $\frac{a}{b} = 1 \Rightarrow a = b \Rightarrow c^2 = a^2 + b^2 = 2a^2 \Rightarrow 2 = 2a^2 \Rightarrow a = 1$ $\Rightarrow b = 1 \Rightarrow y^2 - x^2 = 1$

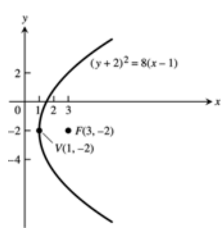
36. Foci: $(\pm 2, 0)$, Asymptotes: $y = \pm\frac{1}{\sqrt{3}}x \Rightarrow c = 2$ and $\frac{b}{a} = \frac{1}{\sqrt{3}} \Rightarrow b = \frac{a}{\sqrt{3}} \Rightarrow c^2 = a^2 + b^2 = a^2 + \frac{a^2}{3} = \frac{4a^2}{3}$ $\Rightarrow 4 = \frac{4a^2}{3} \Rightarrow a^2 = 3 \Rightarrow a = \sqrt{3} \Rightarrow b = 1 \Rightarrow \frac{x^2}{3} - y^2 = 1$

37. Vertices: $(\pm 3, 0)$, Asymptotes: $y = \pm\frac{4}{3}x \Rightarrow a = 3$ and $\frac{b}{a} = \frac{4}{3} \Rightarrow b = \frac{4}{3}(3) = 4 \Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$

38. Vertices: $(0, \pm 2)$, Asymptotes: $y = \pm\frac{1}{2}x \Rightarrow a = 2$ and $\frac{a}{b} = \frac{1}{2} \Rightarrow b = 2(2) = 4 \Rightarrow \frac{y^2}{4} - \frac{x^2}{16} = 1$

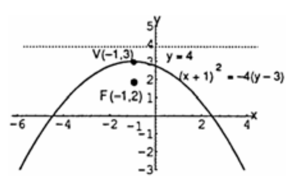
39. (a) $y^2 = 8x \Rightarrow 4p = 8 \Rightarrow p = 2 \Rightarrow$ directrix is $x = -2$, focus is $(2, 0)$, and vertex is $(0, 0)$; therefore the new directrix is $x = -1$, the new focus is $(3, -2)$, and the new vertex is $(1, -2)$

(b)



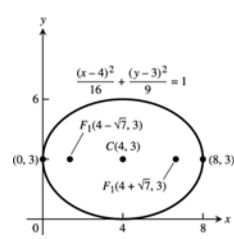
40. (a) $x^2 = -4y \Rightarrow 4p = 4 \Rightarrow p = 1 \Rightarrow$ directrix is $y = 1$, focus is $(0, -1)$ and vertex is $(0, 0)$; therefore the new directrix is $y = 4$, the new focus is $(-1, 2)$, and the new vertex is $(-1, 3)$

(b)



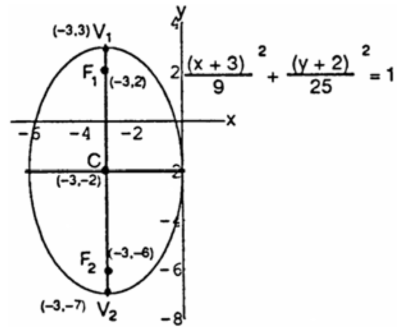
41. (a) $\frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(-4, 0)$ and $(4, 0)$; $c = \sqrt{a^2 - b^2} = \sqrt{7} \Rightarrow$ foci are $\left(\sqrt{7}, 0\right)$ and $\left(-\sqrt{7}, 0\right)$; therefore the new center is $(4, 3)$, the new vertices are $(0, 3)$ and $(8, 3)$, and the new foci are $\left(4 \pm \sqrt{7}, 3\right)$

(b)



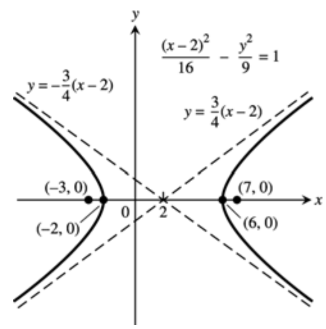
42. (a) $\frac{x^2}{9}+\frac{y^2}{25}=1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, 5)$ and $(0, -5)$; $c=\sqrt{a^2-b^2}=\sqrt{16}=4$ $\Rightarrow$ foci are $(0, 4)$ and $(0, -4)$; therefore the new center is $(-3, -2)$, the new vertices are $(-3, 3)$ and $(-3, -7)$, and the new foci are $(-3, 2)$ and $(-3, -6)$

(b)



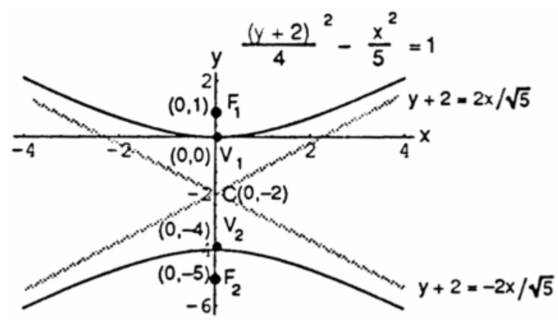
43. (a) $\frac{x^2}{16}-\frac{y^2}{9}=1 \Rightarrow$ center is $(0, 0)$, vertices are $(-4, 0)$ and $(4, 0)$, and the asymptotes are $\frac{x}{4}=\pm\frac{y}{3}$ or $y=\pm\frac{3x}{4}$; $c=\sqrt{a^2+b^2}=\sqrt{25}=5$ $\Rightarrow$ foci are $(-5, 0)$ and $(5, 0)$; therefore the new center is $(2, 0)$, the new vertices are $(-2, 0)$ and $(6, 0)$, the new foci are $(-3, 0)$ and $(7, 0)$, and the new asymptotes are $y=\pm\frac{3(x-2)}{4}$

(b)



44. (a) $\frac{y^2}{4}-\frac{x^2}{5}=1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, -2)$ and $(0, 2)$, and the asymptotes are $\frac{y}{2}=\pm\frac{x}{\sqrt{5}}$ or $y=\pm\frac{2x}{\sqrt{5}}$; $c=\sqrt{a^2+b^2}=\sqrt{9}=3$ $\Rightarrow$ foci are $(0, 3)$ and $(0, -3)$; therefore the new center is $(0, -2)$, the new vertices are $(0, -4)$ and $(0, 0)$, the new foci are $(0, 1)$ and $(0, -5)$, and the new asymptotes are $y+2=\pm\frac{2x}{\sqrt{5}}$

(b)



45. $y^2=4x \Rightarrow 4p=4 \Rightarrow p=1 \Rightarrow$ focus is $(1, 0)$, directrix is $x=-1$, and vertex is $(0, 0)$; therefore the new vertex is $(-2, -3)$, the new focus is $(-1, -3)$, and the new directrix is $x=-3$; the new equation is $(y+3)^2=4(x+2)$

46. $y^2=-12x \Rightarrow 4p=12 \Rightarrow p=3 \Rightarrow$ focus is $(-3, 0)$, directrix is $x=3$, and vertex is $(0, 0)$; therefore the new vertex is $(4, 3)$, the new focus is $(1, 3)$, and the new directrix is $x=7$; the new equation is $(y-3)^2=-12(x-4)$

47. $x^2=8y \Rightarrow 4p=8 \Rightarrow p=2 \Rightarrow$ focus is $(0, 2)$, directrix is $y=-2$ and vertex is $(0, 0)$; therefore the new vertex is $(1, -7)$, the new focus is $(1, -5)$, and the new directrix is $y=-9$; the new equation is $(x-1)^2=8(y+7)$

48. $x^2 = 6y \Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2} \Rightarrow$ focus is $\left(0, \frac{3}{2}\right)$, directrix is $y = -\frac{3}{2}$, and vertex is $(0, 0)$; therefore the new vertex is $(-3,-2)$, the new focus is $\left(-3,-\frac{1}{2}\right)$, and the new directrix is $y = -\frac{7}{2}$; the new equation is $(x+3)^2 = 6(y+2)$

49. $\frac{x^2}{6} + \frac{y^2}{9} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, 3)$ and $(0,-3)$; $c = \sqrt{a^2 - b^2} = \sqrt{9-6} = \sqrt{3} \Rightarrow$ foci are $\left(0, \sqrt{3}\right)$ and $\left(0, -\sqrt{3}\right)$; therefore the new center is $(-2,-1)$, the new vertices are $(-2, 2)$ and $(-2,-4)$, and the new foci are $\left(-2, -1 \pm \sqrt{3}\right)$; the new equation is $\frac{(x+2)^2}{6} + \frac{(y+1)^2}{9} = 1$

50. $\frac{x^2}{2} + y^2 = 1 \Rightarrow$ center is $(0, 0)$, vertices are $\left(\sqrt{2}, 0\right)$ and $\left(-\sqrt{2}, 0\right)$; $c = \sqrt{a^2 - b^2} = \sqrt{2-1} = 1 \Rightarrow$ foci are $(-1, 0)$ and $(1, 0)$; therefore the new center is $(3, 4)$, the new vertices are $\left(3 \pm \sqrt{2}, 4\right)$, and the new foci are $(2, 4)$ and $(4, 4)$; the new equation is $\frac{(x-3)^2}{2} + (y-4)^2 = 1$

51. $\frac{x^2}{3} + \frac{y^2}{2} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $\left(\sqrt{3}, 0\right)$ and $\left(-\sqrt{3}, 0\right)$; $c = \sqrt{a^2 - b^2} = \sqrt{3-2} = 1 \Rightarrow$ foci are $(-1, 0)$ and $(1, 0)$; therefore the new center is $(2, 3)$, the new vertices are $\left(2 \pm \sqrt{3}, 3\right)$, and the new foci are $(1, 3)$ and $(3, 3)$; the new equation is $\frac{(x-2)^2}{3} + \frac{(y-3)^2}{2} = 1$

52. $\frac{x^2}{16} + \frac{y^2}{25} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, 5)$ and $(0,-5)$; $c = \sqrt{a^2 - b^2} = \sqrt{25-16} = 3 \Rightarrow$ foci are $(0, 3)$ and $(0, -3)$; therefore the new center is $(-4, -5)$, the new vertices are $(-4, 0)$ and $(-4, -10)$, and the new foci are $(-4,-2)$ and $(-4,-8)$; the new equation is $\frac{(x+4)^2}{16} + \frac{(y+5)^2}{25} = 1$

53. $\frac{x^2}{4} - \frac{y^2}{5} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(2, 0)$ and $(-2, 0)$; $c = \sqrt{a^2 + b^2} = \sqrt{4+5} = 3 \Rightarrow$ foci are $(3, 0)$ and $(-3, 0)$; the asymptotes are $\pm\frac{x}{2} = \frac{y}{\sqrt{5}} \Rightarrow y = \pm\frac{\sqrt{5}x}{2}$; therefore the new center is $(2, 2)$, the new vertices are $(4, 2)$ and $(0, 2)$, and the new foci are $(5, 2)$ and $(-1, 2)$; the new asymptotes are $y - 2 = \pm\frac{\sqrt{5}(x-2)}{2}$; the new equation is $\frac{(x-2)^2}{4} - \frac{(y-2)^2}{5} = 1$

54. $\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(4, 0)$ and $\left(-4, 0\right)$; $c = \sqrt{a^2 + b^2} = \sqrt{16+9} = 5 \Rightarrow$ foci are $(-5, 0)$ and $(5, 0)$; the asymptotes are $\pm\frac{x}{4} = \frac{y}{3} \Rightarrow y = \pm\frac{3x}{4}$; therefore the new center is $(-5,-1)$, the new vertices are $(-1,-1)$ and $(-9,-1)$, and the new foci are $(-10,-1)$ and $(0,-1)$; the new asymptotes are $y+1 = \pm\frac{3(x+5)}{4}$; the new equation is $\frac{(x+5)^2}{16} - \frac{(y+1)^2}{9} = 1$

55. $y^2 - x^2 = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, 1)$ and $(0, -1)$; $c = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2} \Rightarrow$ foci are $\left(0, \pm\sqrt{2}\right)$; the asymptotes are $y = \pm x$; therefore the new center is $(-1,-1)$, the new vertices are $(-1, 0)$ and $(-1,-2)$, and the new foci are $\left(-1, -1 \pm \sqrt{2}\right)$; the new asymptotes are $y+1 = \pm(x+1)$; the new equation is $(y+1)^2 - (x+1)^2 = 1$

56. $\frac{y^2}{3} - x^2 = 1 \Rightarrow$ center is $(0, 0)$, vertices are $\left(0, \sqrt{3}\right)$ and $\left(0, -\sqrt{3}\right)$; $c = \sqrt{a^2 + b^2} = \sqrt{3+1} = 2 \Rightarrow$ foci are $(0, 2)$ and $(0, -2)$; the asymptotes are $\pm x = \frac{y}{\sqrt{3}} \Rightarrow y = \pm\sqrt{3}x$; therefore the new center is $(1, 3)$, the new vertices are $\left(1, 3 \pm \sqrt{3}\right)$, and the new foci are $(1, 5)$ and $(1, 1)$; the new asymptotes are $y - 3 = \pm\sqrt{3}(x-1)$; the new equation is $\frac{(y-3)^2}{3} - (x-1)^2 = 1$

57. $x^2 + 4x + y^2 = 12 \Rightarrow x^2 + 4x + 4 + y^2 = 12 + 4 \Rightarrow (x+2)^2 + y^2 = 16$; this is a circle: center at $C(-2, 0)$, $a = 4$

58. $2x^2 + 2y^2 - 28x + 12y + 114 = 0 \Rightarrow x^2 - 14x + 49 + y^2 + 6y + 9 = -57 + 49 + 9 \Rightarrow (x-7)^2 + (y+3)^2 = 1$; this is a circle: center at $C(7, -3)$, $a = 1$

59. $x^2 + 2x + 4y - 3 = 0 \Rightarrow x^2 + 2x + 1 = -4y + 3 + 1 \Rightarrow (x+1)^2 = -4(y-1)$; this is a parabola: $V(-1, 1)$, $F(-1, 0)$

60. $y^2 - 4y - 8x - 12 = 0 \Rightarrow y^2 - 4y + 4 = 8x + 12 + 4 \Rightarrow (y-2)^2 = 8(x+2)$; this is a parabola: $V(-2, 2)$, $F(0, 2)$

61. $x^2 + 5y^2 + 4x = 1 \Rightarrow x^2 + 4x + 4 + 5y^2 = 5 \Rightarrow (x+2)^2 + 5y^2 = 5 \Rightarrow \frac{(x+2)^2}{5} + y^2 = 1$; this is an ellipse: the center is $(-2, 0)$, the vertices are $\left(-2 \pm \sqrt{5}, 0\right)$; $c = \sqrt{a^2 - b^2} = \sqrt{5-1} = 2 \Rightarrow$ the foci are $(-4, 0)$ and $(0, 0)$

62. $9x^2 + 6y^2 + 36y = 0 \Rightarrow 9x^2 + 6\left(y^2 + 6y + 9\right) = 54 \Rightarrow 9x^2 + 6(y+3)^2 = 54 \Rightarrow \frac{x^2}{6} + \frac{(y+3)^2}{9} = 1$; this is an ellipse: the center is $(0, -3)$ the vertices are $(0, 0)$ and $(0, -6)$; $c = \sqrt{a^2 - b^2} = \sqrt{9-6} = \sqrt{3} \Rightarrow$ the foci are $\left(0, -3 \pm \sqrt{3}\right)$

63. $x^2 + 2y^2 - 2x - 4y = -1 \Rightarrow x^2 - 2x + 1 + 2\left(y^2 - 2y + 1\right) = 2 \Rightarrow (x-1)^2 + 2(y-1)^2 = 2 \Rightarrow \frac{(x-1)^2}{2} + (y-1)^2 = 1$; this is an ellipse: the center is $(1, 1)$, the vertices are $\left(1 \pm \sqrt{2}, 1\right)$; $c = \sqrt{a^2 - b^2} = \sqrt{2-1} = 1 \Rightarrow$ the foci are $(2, 1)$ and $(0, 1)$

64. $4x^2 + y^2 + 8x - 2y = -1 \Rightarrow 4\left(x^2 + 2x + 1\right) + y^2 - 2y + 1 = 4 \Rightarrow 4(x+1)^2 + (y-1)^2 = 4 \Rightarrow (x+1)^2 + \frac{(y-1)^2}{4} = 1$; this is an ellipse: the center is $(-1, 1)$ the vertices are $(-1, 3)$ and $(-1, -1)$; $c = \sqrt{a^2 - b^2} = \sqrt{4-1} = \sqrt{3} \Rightarrow$ the foci are $\left(-1, 1 \pm \sqrt{3}\right)$

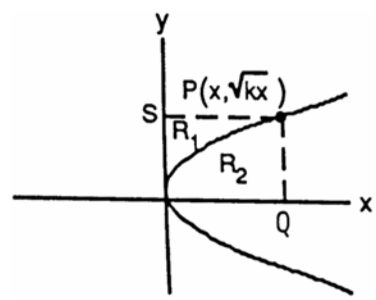
65. $x^2 - y^2 - 2x + 4y = 4 \Rightarrow x^2 - 2x + 1 - \left(y^2 - 4y + 4\right) = 1 \Rightarrow (x-1)^2 - (y-2)^2 = 1$; this is a hyperbola: the center is $(1, 2)$, the vertices are $(2, 2)$ and $(0, 2)$; $c = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2} \Rightarrow$ the foci are $\left(1 \pm \sqrt{2}, 2\right)$; the asymptotes are $y - 2 = \pm(x-1)$

66. $x^2 - y^2 + 4x - 6y = 6 \Rightarrow x^2 + 4x + 4 - \left(y^2 + 6y + 9\right) = 1 \Rightarrow (x+2)^2 - (y+3)^2 = 1$; this is a hyperbola: the center is $(-2, -3)$, the vertices are $(-1, -3)$ and $(-3, -3)$; $c = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2} \Rightarrow$ the foci are $\left(-2 \pm \sqrt{2}, -3\right)$; the asymptotes are $y + 3 = \pm(x+2)$

67. $2x^2 - y^2 + 6y = 3 \Rightarrow 2x^2 - \left(y^2 - 6y + 9\right) = -6 \Rightarrow \frac{(y-3)^2}{6} - \frac{x^2}{3} = 1$; this is a hyperbola: the center is $(0, 3)$, the vertices are $\left(0, 3 \pm \sqrt{6}\right)$; $c = \sqrt{a^2 + b^2} = \sqrt{6+3} = 3 \Rightarrow$ the foci are $(0, 6)$ and $(0, 0)$; the asymptotes are $\frac{y-3}{\sqrt{6}} = \pm \frac{x}{\sqrt{3}} \Rightarrow y = \pm\sqrt{2}x + 3$

68. $y^2 - 4x^2 + 16x = 24 \Rightarrow y^2 - 4\left(x^2 - 4x + 4\right) = 8 \Rightarrow \frac{y^2}{8} - \frac{(x-2)^2}{2} = 1$; this is a hyperbola: the center is $(2, 0)$, the vertices are $\left(2, \pm\sqrt{8}\right)$; $c = \sqrt{a^2 + b^2} = \sqrt{8+2} = \sqrt{10} \Rightarrow$ the foci are $\left(2, \pm\sqrt{10}\right)$; the asymptotes are $\frac{y}{\sqrt{8}} = \pm \frac{x-2}{\sqrt{2}} \Rightarrow y = \pm 2(x-2)$

69. (a) $y^2 = kx \Rightarrow x = \frac{y^2}{k}$; the volume of the solid formed by revolving R_1 about the y-axis is $V_1 = \int_0^{\sqrt{kx}} \pi \left(\frac{y^2}{k}\right)^2 dy = \frac{\pi}{k^2} \int_0^{\sqrt{kx}} y^4 dy = \frac{\pi x^2 \sqrt{kx}}{5}$; the volume of the right circular cylinder formed by revolving PQ about the y-axis is $V_2 = \pi x^2 \sqrt{kx} \Rightarrow$ the volume of the solid formed by revolving R_2 about the y-axis is $V_3 = V_2 - V_1 = \frac{4\pi x^2 \sqrt{kx}}{5}$. Therefore we can see the ratio of V_3 to V_1 is 4:1.



(b) The volume of the solid formed by revolving R_2 about the x-axis is $V_1 = \int_0^x \pi\left(\sqrt{kt}\right)^2 dt = \pi k \int_0^x t\, dt = \frac{\pi k x^2}{2}$. The volume of the right circular cylinder formed by revolving PS about the x-axis is $V_2 = \pi\left(\sqrt{kx}\right)^2 x = \pi k x^2 \Rightarrow$ the volume of the solid formed by revolving R_1 about the x-axis is $V_3 = V_2 - V_1 = \pi k x^2 - \frac{\pi k x^2}{2} = \frac{\pi k x^2}{2}$ Therefore the ratio of V_3 to V_1 is 1:1.

70. $y = \int \frac{w}{H} x\, dx = \frac{w}{H}\left(\frac{x^2}{2}\right) + C = \frac{wx^2}{2H} + C$; $y = 0$ when $x = 0 \Rightarrow 0 = \frac{w(0)^2}{2H} + C \Rightarrow C = 0$; therefore $y = \frac{wx^2}{2H}$ is the equation of the cable's curve

71. $x^2 = 4py$ and $y = p \Rightarrow x^2 = 4p^2 \Rightarrow x = \pm 2p$. Therefore the line $y = p$ cuts the parabola at points $(-2p, p)$ and $(2p, p)$, and these points are $\sqrt{[2p - (-2p)]^2 + (p-p)^2} = 4p$ units apart.

72. $\lim\limits_{x\to\infty}\left(\frac{b}{a}x-\frac{b}{a}\sqrt{x^2-a^2}\right)=\frac{b}{a}\lim\limits_{x\to\infty}\left(x-\sqrt{x^2-a^2}\right)=\frac{b}{a}\lim\limits_{x\to\infty}\left[\frac{\left(x-\sqrt{x^2-a^2}\right)\left(x+\sqrt{x^2-a^2}\right)}{x+\sqrt{x^2-a^2}}\right]=\frac{b}{a}\lim\limits_{x\to\infty}\left[\frac{x^2-\left(x^2-a^2\right)}{x+\sqrt{x^2-a^2}}\right]$

$=\frac{b}{a}\lim\limits_{x\to\infty}\left[\frac{a^2}{x+\sqrt{x^2-a^2}}\right]=0$

73. Let $y=\sqrt{1-\frac{x^2}{4}}$ on the interval $0\le x\le 2$. The area of the inscribed rectangle is given by $A(x)=2x\left(2\sqrt{1-\frac{x^2}{4}}\right)=4x\sqrt{1-\frac{x^2}{4}}$ (since the length is $2x$ and the height is $2y$) $\Rightarrow A'(x)=4\sqrt{1-\frac{x^2}{4}}-\frac{x^2}{\sqrt{1-\frac{x^2}{4}}}$. Thus $A'(x)=0\Rightarrow 4\sqrt{1-\frac{x^2}{4}}-\frac{x^2}{\sqrt{1-\frac{x^2}{4}}}=0\Rightarrow 4\left(1-\frac{x^2}{4}\right)-x^2=0\Rightarrow x^2=2\Rightarrow x=\sqrt{2}$ (only the positive square root lies in the interval). Since $A(0)=A(2)=0$ we have that $A\left(\sqrt{2}\right)=4$ is the maximum area when the length is $2\sqrt{2}$ and the height is $\sqrt{2}$.

74. (a) Around the x-axis: $9x^2+4y^2=36\Rightarrow y^2=9-\frac{9}{4}x^2\Rightarrow y=\pm\sqrt{9-\frac{9}{4}x^2}$ and we use the positive root

$\Rightarrow V=2\int_0^2\pi\left(\sqrt{9-\frac{9}{4}x^2}\right)^2dx=2\int_0^2\pi\left(9-\frac{9}{4}x^2\right)dx=2\pi\left[9x-\frac{3}{4}x^3\right]_0^2=24\pi$

(b) Around the y-axis: $9x^2+4y^2=36\Rightarrow x^2=4-\frac{4}{9}y^2\Rightarrow x=\pm\sqrt{4-\frac{4}{9}y^2}$ and we use the positive root

$\Rightarrow V=2\int_0^3\pi\left(\sqrt{4-\frac{4}{9}y^2}\right)^2dy=2\int_0^3\pi\left(4-\frac{4}{9}y^2\right)dy=2\pi\left[4y-\frac{4}{27}y^3\right]_0^3=16\pi$

75. $9x^2-4y^2=36\Rightarrow y^2=\frac{9x^2-36}{4}\Rightarrow y=\pm\frac{3}{2}\sqrt{x^2-4}$ on the interval $2\le x\le 4\Rightarrow V=\int_2^4\pi\left(\frac{3}{2}\sqrt{x^2-4}\right)^2dx$

$=\frac{9\pi}{4}\int_2^4\left(x^2-4\right)dx=\frac{9\pi}{4}\left[\frac{x^3}{3}-4x\right]_2^4=\frac{9\pi}{4}\left[\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)\right]=\frac{9\pi}{4}\left(\frac{56}{3}-8\right)=\frac{3\pi}{4}(56-24)=24\pi$

76. Let $P_1(-p,y_1)$ be any point on $x=-p$, and let $P(x,y)$ be a point where a tangent intersects $y^2=4px$. Now $y^2=4px\Rightarrow 2y\frac{dy}{dx}=4p\Rightarrow\frac{dy}{dx}=\frac{2p}{y}$; then the slope of a tangent line from P_1 is $\frac{y-y_1}{x-(-p)}=\frac{dy}{dx}=\frac{2p}{y}$ $\Rightarrow y^2-yy_1=2px+2p^2$. Since $x=\frac{y^2}{4p}$, we have $y^2-yy_1=2p\left(\frac{y^2}{4p}\right)+2p^2\Rightarrow y^2-yy_1=\frac{1}{2}y^2+2p^2$ $\Rightarrow\frac{1}{2}y^2-yy_1-2p^2=0\Rightarrow y=\frac{2y_1\pm\sqrt{4y_1^2+16p^2}}{2}=y_1\pm\sqrt{y_1^2+4p^2}$. Therefore the slopes of the two tangents from P_1 are $m_1=\frac{2p}{y_1+\sqrt{y_1^2+4p^2}}$ and $m_2=\frac{2p}{y_1-\sqrt{y_1^2+4p^2}}\Rightarrow m_1m_2=\frac{4p^2}{y_1^2-\left(y_1^2+4p^2\right)}=-1\Rightarrow$ the lines are perpendicular

77. $(x-2)^2+(y-1)^2=5\Rightarrow 2(x-2)+2(y-1)\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{x-2}{y-1}$; $y=0\Rightarrow(x-2)^2+(0-1)^2=5\Rightarrow(x-2)^2=4$ $\Rightarrow x=4$ or $x=0\Rightarrow$ the circle crosses the x-axis at $(4,0)$ and $(0,0)$; $x=0\Rightarrow(0-2)^2+(y-1)^2=5$ $\Rightarrow(y-1)^2=1\Rightarrow y=2$ or $y=0\Rightarrow$ the circle crosses the y-axis at $(0,2)$ and $(0,0)$.
At $(4,0)$: $\frac{dy}{dx}=-\frac{4-2}{0-1}=2\Rightarrow$ the tangent line is $y=2(x-4)$ or $y=2x-8$

At $(0,0)$: $\frac{dy}{dx} = -\frac{0-2}{0-1} = -2 \Rightarrow$ the tangent line is $y = -2x$

At $(0,2)$: $\frac{dy}{dx} = -\frac{0-2}{2-1} = 2 \Rightarrow$ the tangent line is $y - 2 = 2x$ or $y = 2x + 2$

78. $x^2 - y^2 = 1 \Rightarrow x = \pm\sqrt{1+y^2}$ on the interval $-3 \le y \le 3 \Rightarrow V = \int_{-3}^{3} \pi\left(\sqrt{1+y^2}\right)^2 dy = 2\int_0^3 \pi\left(\sqrt{1+y^2}\right)^2 dy$

$= 2\pi\int_0^3 \left(1+y^2\right) dy = 2\pi\left[y + \frac{y^3}{3}\right]_0^3 = 24\pi$

79. Let $y = \sqrt{16 - \frac{16}{9}x^2}$ on the interval $-3 \le x \le 3$. Since the plate is symmetric about the y-axis, $\overline{x} = 0$.

For a vertical strip: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{16-\frac{16}{9}x^2}}{2}\right)$, length $= \sqrt{16 - \frac{16}{9}x^2}$, width $= dx \Rightarrow$ area $= dA = \sqrt{16 - \frac{16}{9}x^2}\, dx$

$\Rightarrow$ mass $= dm = \delta\, dA = \delta\sqrt{16 - \frac{16}{9}x^2}\, dx.$

Moment of the strip about the x-axis: $\tilde{y}\; dm = \frac{\sqrt{16-\frac{16}{9}x^2}}{2}\left(\delta\sqrt{16 - \frac{16}{9}x^2}\right) dx = \delta\left(8 - \frac{8}{9}x^2\right) dx$ so the moment of the plate about the x-axis is $M_x = \int \tilde{y}\; dm = \int_{-3}^{3} \delta\left(8 - \frac{8}{9}x^2\right) dx = \delta\left[8x - \frac{8}{27}x^3\right]_{-3}^{3} = 32\delta$; also the mass of the plate is $M = \int_{-3}^{3} \delta\sqrt{16 - \frac{16}{9}x^2}\, dx = \int_{-3}^{3} 4\delta\sqrt{1 - \left(\frac{1}{3}x\right)^2}\, dx$; $\left[u = \frac{x}{3} \Rightarrow 3\; du = dx;\; x = -3 \Rightarrow u = -1, x = 3 \Rightarrow u = 1\right]$

$\rightarrow 4\delta\int_{-1}^{1} 3\sqrt{1-u^2}\, du = 12\delta\int_{-1}^{1} \sqrt{1-u^2}\, du = 12\delta\left[\frac{1}{2}\left(u\sqrt{1-u^2} + \sin^{-1} u\right)\right]_{-1}^{1} = 6\pi\delta \Rightarrow \overline{y} = \frac{M_x}{M} = \frac{32\delta}{6\pi\delta} = \frac{16}{3\pi}.$

Therefore the center of mass is $\left(0, \frac{16}{3\pi}\right)$.

80. $y = \sqrt{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{x^2+1} \Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{x^2+1}} = \sqrt{\frac{2x^2+1}{x^2+1}}$

$\Rightarrow S = \int_0^{\sqrt{2}} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\sqrt{2}} 2\pi\sqrt{x^2+1}\sqrt{\frac{2x^2+1}{x^2+1}}\, dx = \int_0^{\sqrt{2}} 2\pi\sqrt{2x^2+1}\, dx$; $\begin{bmatrix} u = \sqrt{2}x \\ du = \sqrt{2}dx \end{bmatrix}$

$\rightarrow \frac{2\pi}{\sqrt{2}}\int_0^2 \sqrt{u^2+1}\; du = \frac{2\pi}{\sqrt{2}}\left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left(u + \sqrt{u^2+1}\right)\right)\right]_0^2 = \frac{\pi}{\sqrt{2}}\left[2\sqrt{5} + \ln\left(2+\sqrt{5}\right)\right]$

81. (a) $\tan\beta = m_L \Rightarrow \tan\beta = f'(x_0)$ where $f(x) = \sqrt{4px}$;

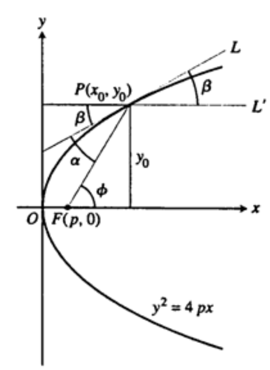
$f'(x) = \frac{1}{2}(4px)^{-1/2}(4p) = \frac{2p}{\sqrt{4px}}$

$\Rightarrow f'(x_0) = \frac{2p}{\sqrt{4px_0}} = \frac{2p}{y_0} \Rightarrow \tan\beta = \frac{2p}{y_0}.$

(b) $\tan\phi = m_{FP} = \frac{y_0 - 0}{x_0 - p} = \frac{y_0}{x_0 - p}$

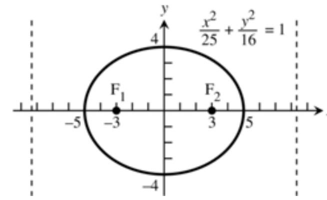
(c) $\tan\alpha = \frac{\tan\phi - \tan\beta}{1 + \tan\phi\tan\beta} = \frac{\left(\frac{y_0}{x_0-p} - \frac{2p}{y_0}\right)}{1 + \left(\frac{y_0}{x_0-p}\right)\left(\frac{2p}{y_0}\right)}$

$= \frac{y_0^2 - 2p(x_0-p)}{y_0(x_0-p+2p)} = \frac{4px_0 - 2px_0 + 2p^2}{y_0(x_0+p)} = \frac{2p(x_0+p)}{y_0(x_0+p)} = \frac{2p}{y_0}$

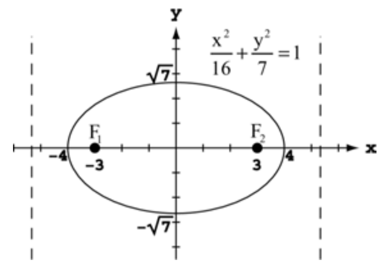


11.7 CONICS IN POLAR COORDINATES

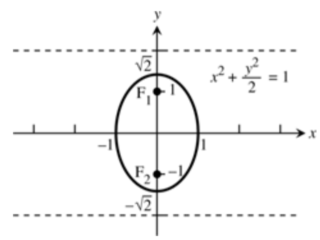
1. $16x^2+25y^2=400 \Rightarrow \frac{x^2}{25}+\frac{y^2}{16}=1 \Rightarrow c=\sqrt{a^2-b^2}$
$=\sqrt{25-16}=3 \Rightarrow e=\frac{c}{a}=\frac{3}{5}$; $F(\pm 3,0)$; directrices are
$x=0\pm\frac{a}{e}=\pm\frac{5}{\left(\frac{3}{5}\right)}=\pm\frac{25}{3}$



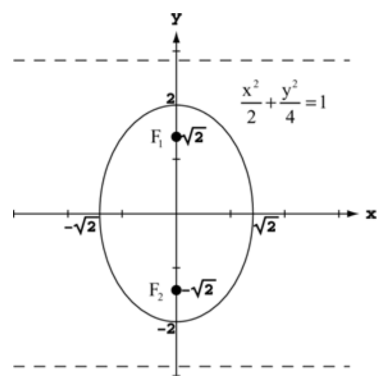
2. $7x^2+16y^2=112 \Rightarrow \frac{x^2}{16}+\frac{y^2}{7}=1 \Rightarrow c=\sqrt{a^2-b^2}$
$=\sqrt{16-7}=3 \Rightarrow e=\frac{c}{a}=\frac{3}{4}$; $F(\pm 3,0)$;
directrices are $x=0\pm\frac{a}{e}=\pm\frac{4}{\left(\frac{3}{4}\right)}=\pm\frac{16}{3}$



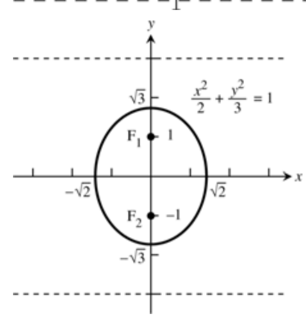
3. $2x^2+y^2=2 \Rightarrow x^2+\frac{y^2}{2}=1 \Rightarrow c=\sqrt{a^2-b^2}$
$=\sqrt{2-1}=1 \Rightarrow e=\frac{c}{a}=\frac{1}{\sqrt{2}}$; $F(0,\pm 1)$;
directrices are $y=0\pm\frac{a}{e}=\pm\frac{\sqrt{2}}{\left(\frac{1}{\sqrt{2}}\right)}=\pm 2$



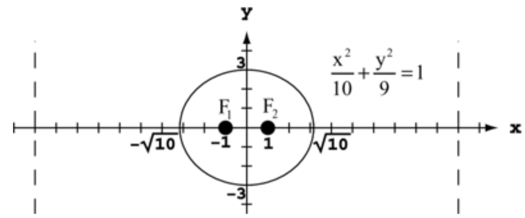
4. $2x^2+y^2=4 \Rightarrow \frac{x^2}{2}+\frac{y^2}{4}=1 \Rightarrow c=\sqrt{a^2-b^2}$
$=\sqrt{4-2}=\sqrt{2} \Rightarrow e=\frac{c}{a}=\frac{\sqrt{2}}{2}$; $F\left(0,\pm\sqrt{2}\right)$;
directrices are $y=0\pm\frac{a}{e}=\pm\frac{2}{\left(\frac{\sqrt{2}}{2}\right)}=\pm 2\sqrt{2}$



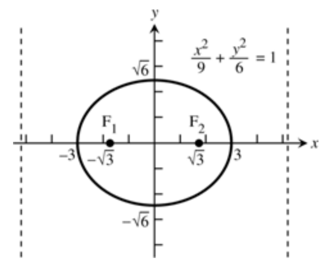
5. $3x^2+2y^2=6 \Rightarrow \frac{x^2}{2}+\frac{y^2}{3}=1 \Rightarrow c=\sqrt{a^2-b^2}$
$=\sqrt{3-2}=1 \Rightarrow e=\frac{c}{a}=\frac{1}{\sqrt{3}}$; $F(0,\pm 1)$;
directrices are $y=0\pm\frac{a}{e}=\pm\frac{\sqrt{3}}{\left(\frac{1}{\sqrt{3}}\right)}=\pm 3$



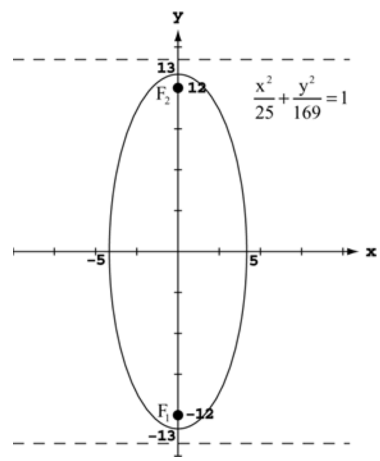
6. $9x^2+10y^2=90 \Rightarrow \frac{x^2}{10}+\frac{y^2}{9}=1 \Rightarrow c=\sqrt{a^2-b^2}$
$=\sqrt{10-9}=1 \Rightarrow e=\frac{c}{a}=\frac{1}{\sqrt{10}}$; $F(\pm 1,0)$;
directrices are $x=0\pm\frac{a}{e}=\pm\frac{\sqrt{10}}{\left(\frac{1}{\sqrt{10}}\right)}=\pm 10$



7. $6x^2+9y^2=54 \Rightarrow \frac{x^2}{9}+\frac{y^2}{6}=1 \Rightarrow c=\sqrt{a^2-b^2}$
$=\sqrt{9-6}=\sqrt{3} \Rightarrow e=\frac{c}{a}=\frac{\sqrt{3}}{3}$; $F\left(\pm\sqrt{3},0\right)$;
directrices are $x=0\pm\frac{a}{e}=\pm\frac{3}{\left(\frac{\sqrt{3}}{3}\right)}=\pm 3\sqrt{3}$



8. $169x^2+25y^2=4225 \Rightarrow \frac{x^2}{25}+\frac{y^2}{169}=1 \Rightarrow c=\sqrt{a^2-b^2}$
$=\sqrt{169-25}=12 \Rightarrow e=\frac{c}{a}=\frac{12}{13}$;
directrices are $y=0\pm\frac{a}{e}=\pm\frac{13}{\left(\frac{12}{13}\right)}=\pm\frac{169}{12}$



9. Foci: $(0,\pm 3)$, $e=0.5 \Rightarrow c=3$ and $a=\frac{c}{e}=\frac{3}{0.5}=6 \Rightarrow b^2=36-9=27 \Rightarrow \frac{x^2}{27}+\frac{y^2}{36}=1$

10. Foci: $(\pm 8,0)$, $e=0.2 \Rightarrow c=8$ and $a=\frac{c}{e}=\frac{8}{0.2}=40 \Rightarrow b^2=1600-64=1536 \Rightarrow \frac{x^2}{1600}+\frac{y^2}{1536}=1$

11. Vertices: $(0,\pm 70)$, $e=0.1 \Rightarrow a=70$ and $c=ae=70(0.1)=7 \Rightarrow b^2=4900-49=4851 \Rightarrow \frac{x^2}{4851}+\frac{y^2}{4900}=1$

12. Vertices: $(\pm 10,0)$, $e=0.24 \Rightarrow a=10$ and $c=ae=10(0.24)=2.4 \Rightarrow b^2=100-5.76=94.24 \Rightarrow \frac{x^2}{100}+\frac{y^2}{94.24}=1$

13. Focus: $\left(\sqrt{5},0\right)$, Directrix: $x=\frac{9}{\sqrt{5}} \Rightarrow c=ae=\sqrt{5}$ and $\frac{a}{e}=\frac{9}{\sqrt{5}} \Rightarrow \frac{ae}{e^2}=\frac{9}{\sqrt{5}} \Rightarrow \frac{\sqrt{5}}{e^2}=\frac{9}{\sqrt{5}} \Rightarrow e^2=\frac{5}{9} \Rightarrow e=\frac{\sqrt{5}}{3}$.
Then $PF=\frac{\sqrt{5}}{3}PD \Rightarrow \sqrt{\left(x-\sqrt{5}\right)^2+\left(y-0\right)^2}=\frac{\sqrt{5}}{3}\left|x-\frac{9}{\sqrt{5}}\right| \Rightarrow \left(x-\sqrt{5}\right)^2+y^2=\frac{5}{9}\left(x-\frac{9}{\sqrt{5}}\right)^2$
$\Rightarrow x^2-2\sqrt{5}x+5+y^2=\frac{5}{9}\left(x^2-\frac{18}{\sqrt{5}}x+\frac{81}{5}\right) \Rightarrow \frac{4}{9}x^2+y^2=4 \Rightarrow \frac{x^2}{9}+\frac{y^2}{4}=1$

14. Focus: (4, 0), Directrix: $x=\frac{16}{3} \Rightarrow c=ae=4$ and $\frac{a}{e}=\frac{16}{3} \Rightarrow \frac{ae}{e^2}=\frac{16}{3} \Rightarrow \frac{4}{e^2}=\frac{16}{3} \Rightarrow e^2=\frac{3}{4} \Rightarrow e=\frac{\sqrt{3}}{2}$.

Then $PF=\frac{\sqrt{3}}{2}PD \Rightarrow \sqrt{(x-4)^2+(y-0)^2}=\frac{\sqrt{3}}{2}\left|x-\frac{16}{3}\right| \Rightarrow (x-4)^2+y^2=\frac{3}{4}\left(x-\frac{16}{3}\right)^2$

$\Rightarrow x^2-8x+16+y^2=\frac{3}{4}\left(x^2-\frac{32}{3}x+\frac{256}{9}\right) \Rightarrow \frac{1}{4}x^2+y^2=\frac{16}{3} \Rightarrow \frac{x^2}{\left(\frac{64}{3}\right)}+\frac{y^2}{\left(\frac{16}{3}\right)}=1$

15. Focus: (–4, 0), Directrix: $x=-16 \Rightarrow c=ae=4$ and $\frac{a}{e}=16 \Rightarrow \frac{ae}{e^2}=16 \Rightarrow \frac{4}{e^2}=16 \Rightarrow e^2=\frac{1}{4} \Rightarrow e=\frac{1}{2}$.

Then $PF=\frac{1}{2}PD \Rightarrow \sqrt{(x+4)^2+(y-0)^2}=\frac{1}{2}|x+16| \Rightarrow (x+4)^2+y^2=\frac{1}{4}(x+16)^2$

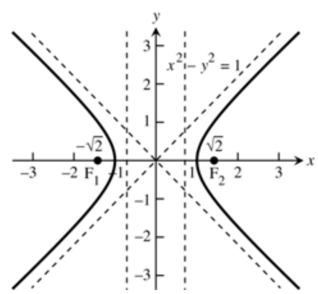
$\Rightarrow x^2+8x+16+y^2=\frac{1}{4}\left(x^2+32x+256\right) \Rightarrow \frac{3}{4}x^2+y^2=48 \Rightarrow \frac{x^2}{64}+\frac{y^2}{48}=1$

16. Focus: $\left(-\sqrt{2},0\right)$, Directrix: $x=-2\sqrt{2} \Rightarrow c=ae=\sqrt{2}$ and $\frac{a}{e}=2\sqrt{2} \Rightarrow \frac{ae}{e^2}=2\sqrt{2} \Rightarrow \frac{\sqrt{2}}{e^2}=2\sqrt{2} \Rightarrow e^2=\frac{1}{2}$

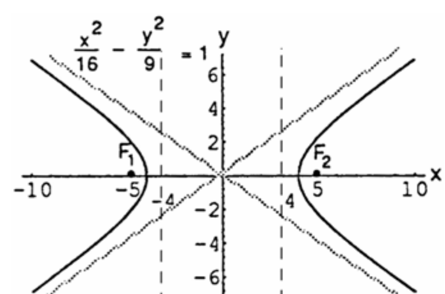
$\Rightarrow e=\frac{1}{\sqrt{2}}$. Then $PF=\frac{1}{\sqrt{2}}PD \Rightarrow \sqrt{\left(x+\sqrt{2}\right)^2+(y-0)^2}=\frac{1}{\sqrt{2}}\left|x+2\sqrt{2}\right| \Rightarrow \left(x+\sqrt{2}\right)^2+y^2=\frac{1}{2}\left(x+2\sqrt{2}\right)^2$

$\Rightarrow x^2+2\sqrt{2}x+2+y^2=\frac{1}{2}\left(x^2+4\sqrt{2}x+8\right) \Rightarrow \frac{1}{2}x^2+y^2=2 \Rightarrow \frac{x^2}{4}+\frac{y^2}{2}=1$

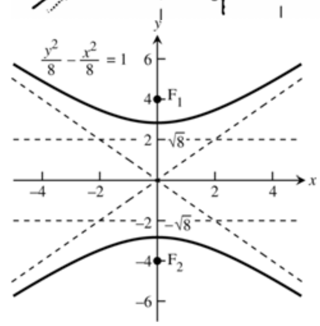
17. $x^2-y^2=1 \Rightarrow c=\sqrt{a^2+b^2}=\sqrt{1+1}=\sqrt{2} \Rightarrow e=\frac{c}{a}$
$=\frac{\sqrt{2}}{1}=\sqrt{2}$; asymptotes are $y=\pm x$; $F\left(\pm\sqrt{2},0\right)$;
directrices are $x=0\pm\frac{a}{e}=\pm\frac{1}{\sqrt{2}}$



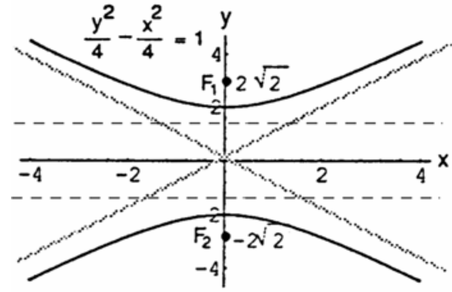
18. $9x^2-16y^2=144 \Rightarrow \frac{x^2}{16}-\frac{y^2}{9}=1 \Rightarrow c=\sqrt{a^2+b^2}$
$=\sqrt{16+9}=5 \Rightarrow e=\frac{c}{a}=\frac{5}{4}$; asymptotes are $y=\pm\frac{3}{4}x$;
$F(\pm 5,0)$; directrices are $x=0\pm\frac{a}{e}=\pm\frac{16}{5}$



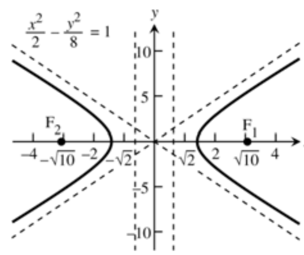
19. $y^2-x^2=8 \Rightarrow \frac{y^2}{8}-\frac{x^2}{8}=1 \Rightarrow c=\sqrt{a^2+b^2}=\sqrt{8+8}=4$
$\Rightarrow e=\frac{c}{a}=\frac{4}{\sqrt{8}}=\sqrt{2}$; asymptotes are $y=\pm x$; $F(0,\pm 4)$;
directrices are $y=0\pm\frac{a}{e}=\pm\frac{\sqrt{8}}{\sqrt{2}}=\pm 2$



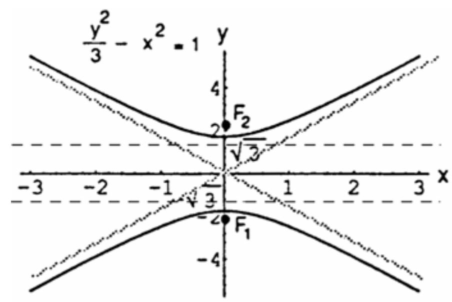
20. $y^2 - x^2 = 4 \Rightarrow \frac{y^2}{4} - \frac{x^2}{4} = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{4+4} = 2\sqrt{2}$
$\Rightarrow e = \frac{c}{a} = \frac{2\sqrt{2}}{2} = \sqrt{2}$; asymptotes are $y = \pm x$; $F\left(0, \pm 2\sqrt{2}\right)$;
directrices are $y = 0 \pm \frac{a}{e} = \pm \frac{2}{\sqrt{2}} = \pm\sqrt{2}$



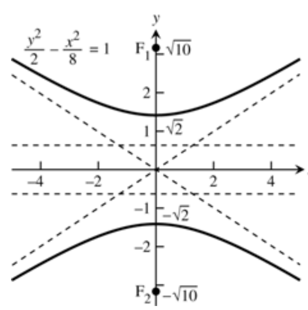
21. $8x^2 - 2y^2 = 16 \Rightarrow \frac{x^2}{2} - \frac{y^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$
$= \sqrt{2+8} = \sqrt{10} \Rightarrow e = \frac{c}{a} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$;
asymptotes are $y = \pm 2x$; $F\left(\pm\sqrt{10}, 0\right)$;
directrices are $x = 0 \pm \frac{a}{e} = \pm \frac{\sqrt{2}}{\sqrt{5}} = \pm \frac{2}{\sqrt{10}}$



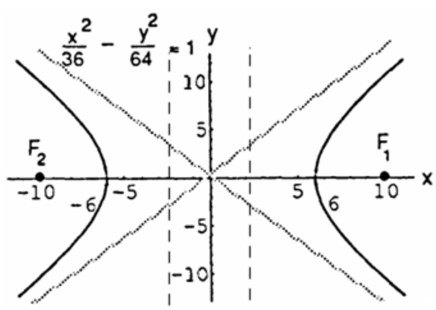
22. $y^2 - 3x^2 = 3 \Rightarrow \frac{y^2}{3} - x^2 = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{3+1} = 2$
$\Rightarrow e = \frac{c}{a} = \frac{2}{\sqrt{3}}$; asymptotes are $y = \pm\sqrt{3}\,x$; $F(0, \pm 2)$;
directrices are $y = 0 \pm \frac{a}{e} = \pm \frac{\sqrt{3}}{\left(\frac{2}{\sqrt{3}}\right)} = \pm \frac{3}{2}$



23. $8y^2 - 2x^2 = 16 \Rightarrow \frac{y^2}{2} - \frac{x^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$
$= \sqrt{2+8} = \sqrt{10} \Rightarrow e = \frac{c}{a} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$; asymptotes are $y = \pm \frac{x}{2}$;
$F\left(0, \pm\sqrt{10}\right)$; directrices are $y = 0 \pm \frac{a}{e} = \pm \frac{\sqrt{2}}{\sqrt{5}} = \pm \frac{2}{\sqrt{10}}$



24. $64x^2 - 36y^2 = 2304 \Rightarrow \frac{x^2}{36} - \frac{y^2}{64} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$
$= \sqrt{36+64} = 10 \Rightarrow e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}$;
asymptotes are $y = \pm \frac{4}{3}x$; $F(\pm 10, 0)$;
directrices are $x = 0 \pm \frac{a}{e} = \pm \frac{6}{\left(\frac{5}{3}\right)} = \pm \frac{18}{5}$



25. Vertices $(0, \pm 1)$ and $e = 3 \Rightarrow a = 1$ and $e = \frac{c}{a} = 3 \Rightarrow c = 3a = 3 \Rightarrow b^2 = c^2 - a^2 = 9 - 1 = 8 \Rightarrow y^2 - \frac{x^2}{8} = 1$

26. Vertices $(\pm 2, 0)$ and $e = 2 \Rightarrow a = 2$ and $e = \frac{c}{a} = 2 \Rightarrow c = 2a = 4 \Rightarrow b^2 = c^2 - a^2 = 16 - 4 = 12 \Rightarrow \frac{x^2}{4} - \frac{y^2}{12} = 1$

27. Foci $(\pm 3, 0)$ and $e = 3 \Rightarrow c = 3$ and $e = \frac{c}{a} = 3 \Rightarrow c = 3a \Rightarrow a = 1 \Rightarrow b^2 = c^2 - a^2 = 9 - 1 = 8 \Rightarrow x^2 - \frac{y^2}{8} = 1$

28. Foci $(0, \pm 5)$ and $e = 1.25 \Rightarrow c = 5$ and $e = \frac{c}{a} = 1.25 = \frac{5}{4} \Rightarrow c = \frac{5}{4}a \Rightarrow 5 = \frac{5}{4}a \Rightarrow a = 4$
$\Rightarrow b^2 = c^2 - a^2 = 25 - 16 = 9 \Rightarrow \frac{y^2}{16} - \frac{x^2}{9} = 1$

29. $e = 1, x = 2 \Rightarrow k = 2 \Rightarrow r = \frac{2(1)}{1 + (1)\cos\theta} = \frac{2}{1 + \cos\theta}$

30. $e = 1, y = 2 \Rightarrow k = 2 \Rightarrow r = \frac{2(1)}{1 + (1)\sin\theta} = \frac{2}{1 + \sin\theta}$

31. $e = 5, y = -6 \Rightarrow k = 6 \Rightarrow r = \frac{6(5)}{1 - 5\sin\theta} = \frac{30}{1 - 5\sin\theta}$

32. $e = 2, x = 4 \Rightarrow k = 4 \Rightarrow r = \frac{4(2)}{1 + 2\cos\theta} = \frac{8}{1 + 2\cos\theta}$

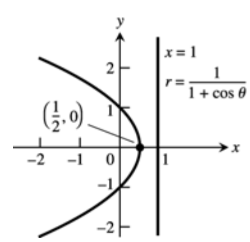
33. $e = \frac{1}{2}, x = 1 \Rightarrow k = 1 \Rightarrow r = \frac{\left(\frac{1}{2}\right)(1)}{1 + \left(\frac{1}{2}\right)\cos\theta} = \frac{1}{2 + \cos\theta}$

34. $e = \frac{1}{4}, x = -2 \Rightarrow k = 2 \Rightarrow r = \frac{\left(\frac{1}{4}\right)(2)}{1 - \left(\frac{1}{4}\right)\cos\theta} = \frac{2}{4 - \cos\theta}$

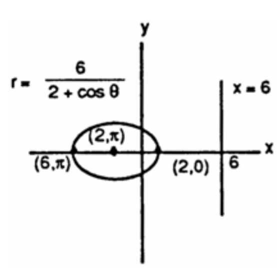
35. $e = \frac{1}{5}, y = -10 \Rightarrow k = 10 \Rightarrow r = \frac{\left(\frac{1}{5}\right)(10)}{1 - \left(\frac{1}{5}\right)\sin\theta} = \frac{10}{5 - \sin\theta}$

36. $e = \frac{1}{3}, y = 6 \Rightarrow k = 6 \Rightarrow r = \frac{\left(\frac{1}{3}\right)(6)}{1 + \left(\frac{1}{3}\right)\sin\theta} = \frac{6}{3 + \sin\theta}$

37. $r = \frac{1}{1 + \cos\theta} \Rightarrow e = 1, k = 1 \Rightarrow x = 1$



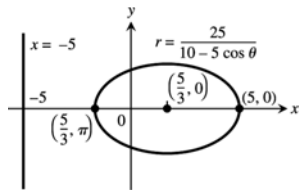
38. $r = \frac{6}{2 + \cos\theta} = \frac{3}{1 + \left(\frac{1}{2}\right)\cos\theta} \Rightarrow e = \frac{1}{2}, k = 6 \Rightarrow x = 6;$
$a\left(1 - e^2\right) = ke \Rightarrow a\left[1 - \left(\frac{1}{2}\right)^2\right] = 3 \Rightarrow \frac{3}{4}a = 3$
$\Rightarrow a = 4 \Rightarrow ea = 2$



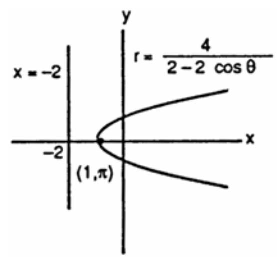
39. $r=\frac{25}{10-5\cos\theta}\Rightarrow r=\frac{\left(\frac{25}{10}\right)}{1-\left(\frac{5}{10}\right)\cos\theta}=\frac{\left(\frac{5}{2}\right)}{1-\left(\frac{1}{2}\right)\cos\theta}$

$\Rightarrow e=\frac{1}{2}, k=5\Rightarrow x=-5;\ \ a\left(1-e^2\right)=ke$

$\Rightarrow a\left[1-\left(\frac{1}{2}\right)^2\right]=\frac{5}{2}\Rightarrow \frac{3}{4}a=\frac{5}{2}\Rightarrow a=\frac{10}{3}\Rightarrow ea=\frac{5}{3}$



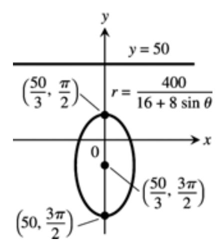
40. $r=\frac{4}{2-2\cos\theta}\Rightarrow r=\frac{2}{1-\cos\theta}\Rightarrow e=1, k=2\Rightarrow x=-2$



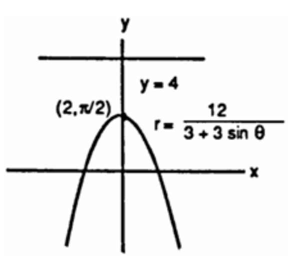
41. $r=\frac{400}{16+8\sin\theta}\Rightarrow r=\frac{\left(\frac{400}{16}\right)}{1+\left(\frac{8}{16}\right)\sin\theta}\Rightarrow r=\frac{25}{1+\left(\frac{1}{2}\right)\sin\theta}$

$e=\frac{1}{2}, k=50\Rightarrow y=50;\ \ a\left(1-e^2\right)=ke$

$\Rightarrow a\left[1-\left(\frac{1}{2}\right)^2\right]=25\Rightarrow \frac{3}{4}a=25\Rightarrow a=\frac{100}{3}\Rightarrow ea=\frac{50}{3}$

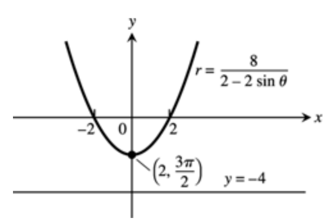


42. $r=\frac{12}{3+3\sin\theta}\Rightarrow r=\frac{4}{1+\sin\theta}\Rightarrow\ e=1,\ k=4\Rightarrow y=4$



43. $r=\frac{8}{2-2\sin\theta}\Rightarrow r=\frac{4}{1-\sin\theta}$

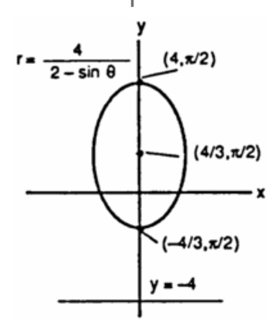
$\Rightarrow\ e=1, k=4\Rightarrow y=-4$



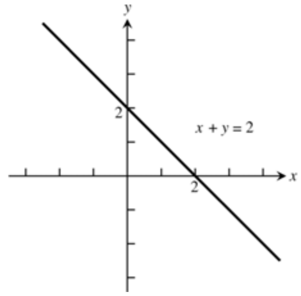
44. $r=\frac{4}{2-\sin\theta}\Rightarrow r=\frac{2}{1-\left(\frac{1}{2}\right)\sin\theta}\Rightarrow\ e=\frac{1}{2}, k=4\Rightarrow y=-4;$

$a\left(1-e^2\right)=ke\Rightarrow a\left[1-\left(\frac{1}{2}\right)^2\right]=2$

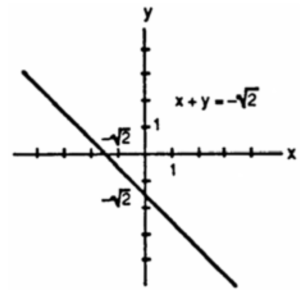
$\Rightarrow \frac{3}{4}a=2\Rightarrow a=\frac{8}{3}\Rightarrow ea=\frac{4}{3}$



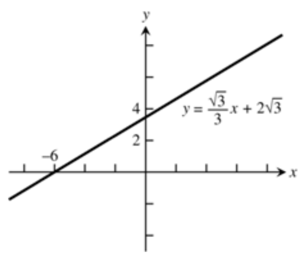
45. $r\cos\left(\theta-\frac{\pi}{4}\right)=\sqrt{2} \Rightarrow r\left(\cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}\right)=\sqrt{2}$
$\Rightarrow \frac{1}{\sqrt{2}}r\cos\theta+\frac{1}{\sqrt{2}}r\sin\theta=\sqrt{2} \Rightarrow \frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y=\sqrt{2}$
$\Rightarrow x+y=2 \Rightarrow y=2-x$



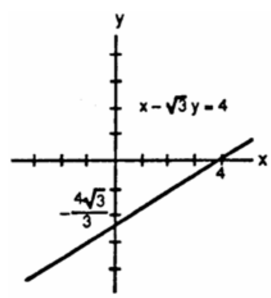
46. $r\cos\left(\theta+\frac{3\pi}{4}\right)=1 \Rightarrow r\left(\cos\theta\cos\frac{3\pi}{4}-\sin\theta\sin\frac{3\pi}{4}\right)=1$
$\Rightarrow -\frac{2}{\sqrt{2}}r\cos\theta-\frac{\sqrt{2}}{2}r\sin\theta=1 \Rightarrow x+y=-\sqrt{2}$
$\Rightarrow y=-x-\sqrt{2}$



47. $r\cos\left(\theta-\frac{2\pi}{3}\right)=3 \Rightarrow r\left(\cos\theta\cos\frac{2\pi}{3}+\sin\theta\sin\frac{2\pi}{3}\right)=3$
$\Rightarrow -\frac{1}{2}r\cos\theta+\frac{\sqrt{3}}{2}r\sin\theta=3 \Rightarrow -\frac{1}{2}x+\frac{\sqrt{3}}{2}y=3$
$\Rightarrow -x+\sqrt{3}y=6 \Rightarrow y=\frac{\sqrt{3}}{3}x+2\sqrt{3}$



48. $r\cos\left(\theta+\frac{\pi}{3}\right)=2 \Rightarrow r\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)=2$
$\Rightarrow \frac{1}{2}r\cos\theta-\frac{\sqrt{3}}{2}r\sin\theta=2 \Rightarrow \frac{1}{2}x-\frac{\sqrt{3}}{2}y=2$
$\Rightarrow x-\sqrt{3}y=4 \Rightarrow y=\frac{\sqrt{3}}{3}x-\frac{4\sqrt{3}}{3}$



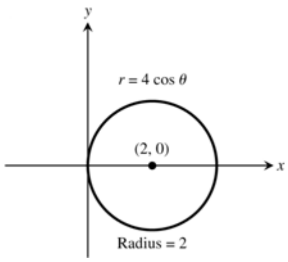
49. $\sqrt{2}x+\sqrt{2}y=6 \Rightarrow \sqrt{2}\,r\cos\theta+\sqrt{2}\,r\sin\theta=6 \Rightarrow r\left(\frac{\sqrt{2}}{2}\cos\theta+\frac{\sqrt{2}}{2}\sin\theta\right)=3 \Rightarrow r\left(\cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta\right)=3$
$\Rightarrow r\cos\left(\theta-\frac{\pi}{4}\right)=3$

50. $\sqrt{3}x-y=1 \Rightarrow \sqrt{3}r\cos\theta-r\sin\theta=1 \Rightarrow r\left(\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta\right)=\frac{1}{2} \Rightarrow r\left(\cos\frac{\pi}{6}\cos\theta-\sin\frac{\pi}{6}\sin\theta\right)=\frac{1}{2}$
$\Rightarrow r\cos\left(\theta+\frac{\pi}{6}\right)=\frac{1}{2}$

51. $y=-5 \Rightarrow r\sin\theta=-5 \Rightarrow -r\sin\theta=5 \Rightarrow r\sin(-\theta)=5 \Rightarrow r\cos\left(\frac{\pi}{2}-(-\theta)\right)=5 \Rightarrow r\cos\left(\theta+\frac{\pi}{2}\right)=5$

52. $x = -4 \Rightarrow r\cos\theta = -4 \Rightarrow -r\cos\theta = 4 \Rightarrow r\cos(\theta - \pi) = 4$

53.



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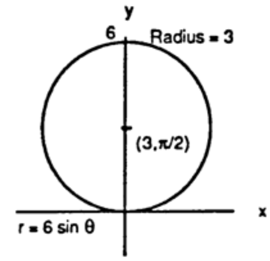

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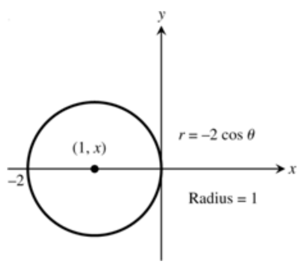

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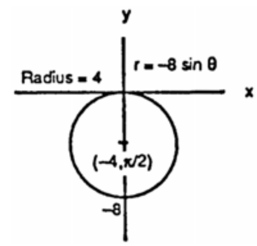
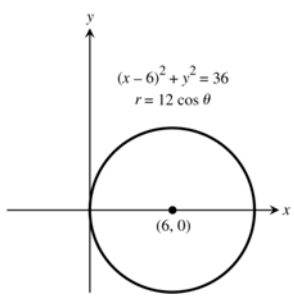
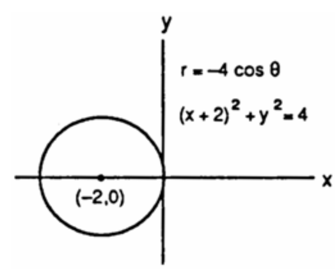


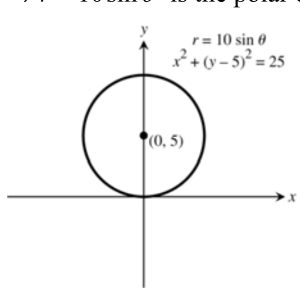
57. $(x-6)^2 + y^2 = 36 \Rightarrow C = (6,0), a = 6$
$\Rightarrow r = 12\cos\theta$ is the polar equation



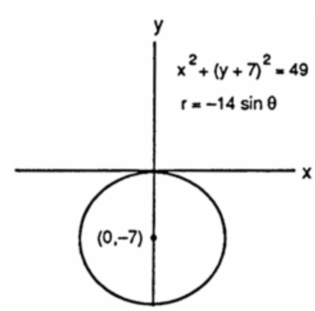
58. $(x+2)^2 + y^2 = 4 \Rightarrow C = (-2,0), a = 2$
$\Rightarrow r = -4\cos\theta$ is the polar equation



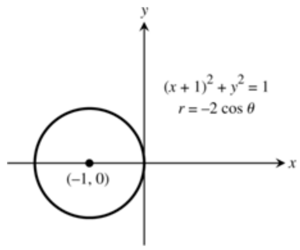
59. $x^2 + (y-5)^2 = 25 \Rightarrow C = (0,5), a = 5$
$\Rightarrow r = 10\sin\theta$ is the polar equation



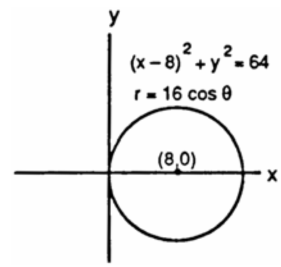
60. $x^2 + (y+7)^2 = 49 \Rightarrow C = (0,-7), a = 7$
$\Rightarrow r = -14\sin\theta$ is the polar equation



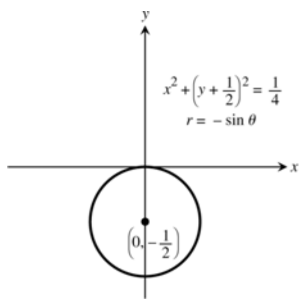
61. $x^2 + 2x + y^2 = 0 \Rightarrow (x+1)^2 + y^2 = 1$
$\Rightarrow C = (-1, 0),\quad a = 1 \Rightarrow r = -2\cos\theta$
is the polar equation



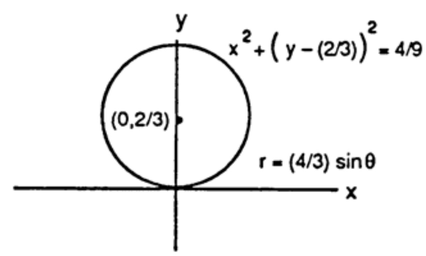
62. $x^2 - 16x + y^2 = 0 \Rightarrow (x-8)^2 + y^2 = 64$
$\Rightarrow C = (8, 0),\quad a = 8 \Rightarrow r = 16\cos\theta$
is the polar equation



63. $x^2 + y^2 + y = 0 \Rightarrow x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}$
$\Rightarrow C = \left(0, -\frac{1}{2}\right),\quad a = \frac{1}{2} \Rightarrow r = -\sin\theta$
is the polar equation



64. $x^2 + y^2 - \frac{4}{3}y = 0 \Rightarrow x^2 + \left(y - \frac{2}{3}\right)^2 = \frac{4}{9}$
$\Rightarrow C = \left(0, \frac{2}{3}\right),\quad a = \frac{2}{3} \Rightarrow r = \frac{3}{4}\sin\theta$
is the polar equation



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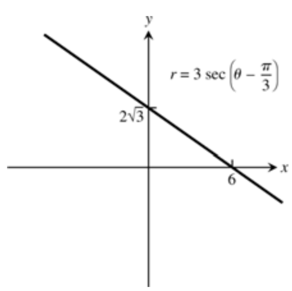

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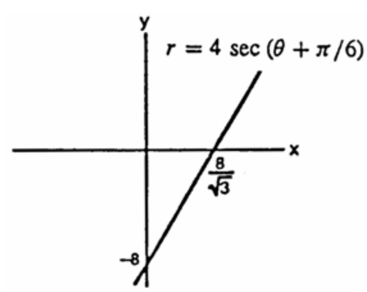

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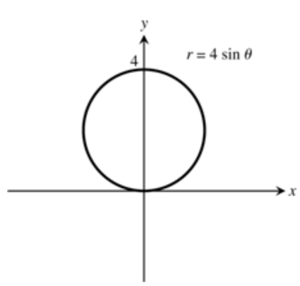

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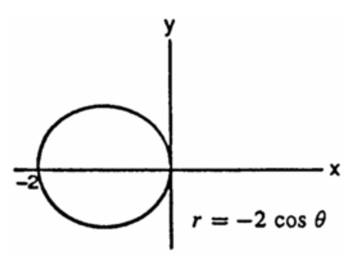

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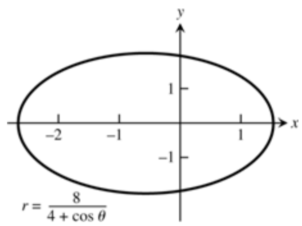

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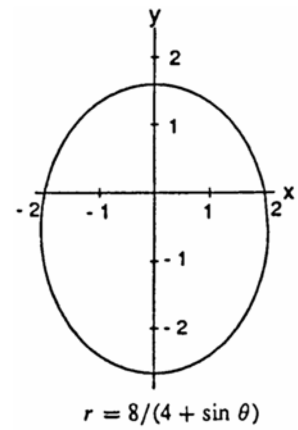

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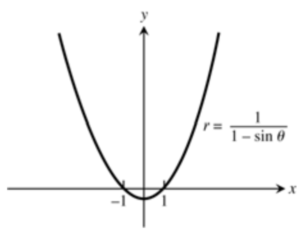

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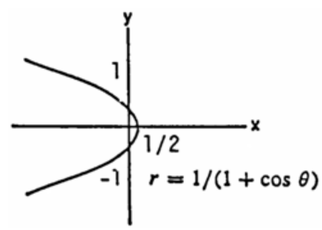
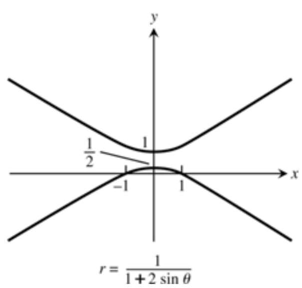
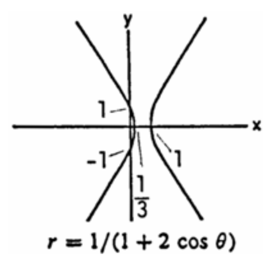


73.



74.



75. (a) Perihelion $= a - ae = a(1-e)$, Aphelion $= ea + a = a(1+e)$

(b)

Planet	Perihelion	Aphelion
Mercury	0.3075 AU	0.4667 AU
Venus	0.7184 AU	0.7282 AU
Earth	0.9833 AU	1.0167 AU
Mars	1.3817 AU	1.6663 AU
Jupiter	4.9512 AU	5.4548 AU
Saturn	9.0210 AU	10.0570 AU
Uranus	18.2977 AU	20.0623 AU
Neptune	29.8135 AU	30.3065 AU

76. Mercury: $r = \frac{(0.3871)\left(1-0.2056^2\right)}{1+0.2056\cos\theta} = \frac{0.3707}{1+0.2056\cos\theta}$

Venus: $r = \frac{(0.7233)\left(1-0.0068^2\right)}{1+0.0068\cos\theta} = \frac{0.7233}{1+0.0068\cos\theta}$

Earth: $r = \frac{1\left(1-0.0167^2\right)}{1+0.0167\cos\theta} = \frac{0.9997}{1+0.0617\cos\theta}$

Mars: $r = \frac{(1.524)\left(1-0.0934^2\right)}{1+0.0934\cos\theta} = \frac{1.511}{1+0.0934\cos\theta}$

Jupiter: $r = \frac{(5.203)\left(1-0.0484^2\right)}{1+0.0484\cos\theta} = \frac{5.191}{1+0.0484\cos\theta}$

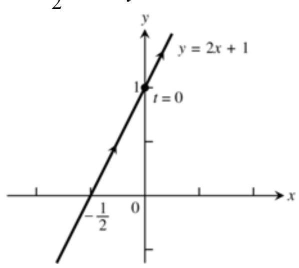
Saturn: $r = \frac{(9.539)\left(1-0.0543^2\right)}{1+0.0543\cos\theta} = \frac{9.511}{1+0.0543\cos\theta}$

Uranus : $r = \frac{(19.18)\left(1-0.0460^2\right)}{1+0.0460\cos\theta} = \frac{19.14}{1+0.0460\cos\theta}$

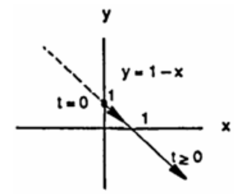
Neptune: $r = \frac{(30.06)\left(1-0.0082^2\right)}{1+0.0082\cos\theta} = \frac{30.06}{1+0.0082\cos\theta}$

CHAPTER 11 PRACTICE EXERCISES

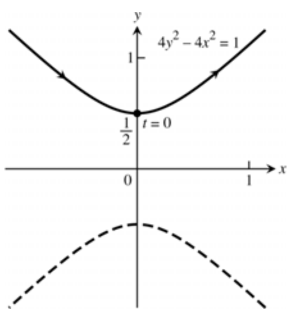
1. $x=\frac{t}{2}$ and $y=t+1 \Rightarrow 2x=t \Rightarrow y=2x+1$



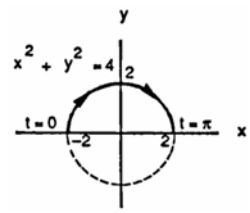
2. $x=\sqrt{t}$ and $y=1-\sqrt{t} \Rightarrow y=1-x$



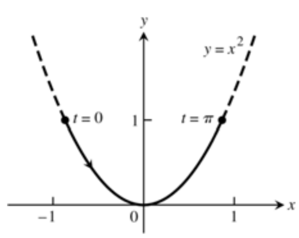
3. $x=\frac{1}{2}\tan t$ and $y=\frac{1}{2}\sec t \Rightarrow x^2=\frac{1}{4}\tan^2 t$ and $y^2=\frac{1}{4}\sec^2 t \Rightarrow 4x^2=\tan^2 t$ and $4y^2=\sec^2 t$ $\Rightarrow 4x^2+1=4y^2 \Rightarrow 4y^2-4x^2=1$



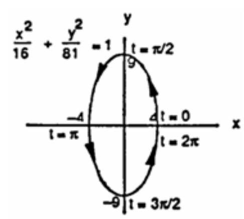
4. $x=-2\cos t$ and $y=2\sin t \Rightarrow x^2=4\cos^2 t$ and $y^2=4\sin^2 t \Rightarrow x^2+y^2=4$



5. $x=-\cos t$ and $y=\cos^2 t \Rightarrow y=(-x)^2=x^2$



6. $x=4\cos t$ and $y=9\sin t \Rightarrow x^2=16\cos^2 t$ and $y^2=81\sin^2 t \Rightarrow \frac{x^2}{16}+\frac{y^2}{81}=1$



7. $16x^2+9y^2=144 \Rightarrow \frac{x^2}{9}+\frac{y^2}{16}=1 \Rightarrow a=3$ and $b=4 \Rightarrow x=3\cos t$ and $y=4\sin t,\ 0\le t\le 2\pi$

8. $x^2+y^2=4 \Rightarrow x=-2\cos t$ and $y=2\sin t,\ 0\le t\le 6\pi$

9. $x=\frac{1}{2}\tan t,\ y=\frac{1}{2}\sec t \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\frac{1}{2}\sec t\tan t}{\frac{1}{2}\sec^2 t}=\frac{\tan t}{\sec t}=\sin t \Rightarrow \left.\frac{dy}{dx}\right|_{t=\pi/3}=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$;
$t=\frac{\pi}{3} \Rightarrow x=\frac{1}{2}\tan\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ and $y=\frac{1}{2}\sec\frac{\pi}{3}=1 \Rightarrow y=\frac{\sqrt{3}}{2}x+\frac{1}{4}$; $\frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=\frac{\cos t}{\frac{1}{2}\sec^2 t}=2\cos^3 t$
$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\pi/3}=2\cos^3\left(\frac{\pi}{3}\right)=\frac{1}{4}$

10. $x = 1 + \frac{1}{t^2},\ y = 1 - \frac{3}{t} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{3}{t^2}\right)}{\left(-\frac{2}{t^3}\right)} = -\frac{3}{2}t \Rightarrow \left.\frac{dy}{dx}\right|_{t=2} = -\frac{3}{2}(2) = -3;\ t = 2 \Rightarrow x = 1 + \frac{1}{2^2} = \frac{5}{4}$ and

$y = 1 - \frac{3}{2} = -\frac{1}{2} \Rightarrow y = -3x + \frac{13}{4};\ \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left(-\frac{3}{2}\right)}{\left(-\frac{2}{t^3}\right)} = \frac{3}{4}t^3 \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{3}{4}(2)^3 = 6$

11. (a) $x = 4t^2,\ y = t^3 - 1 \Rightarrow t = \pm\frac{\sqrt{x}}{2} \Rightarrow y = \left(\pm\frac{\sqrt{x}}{2}\right)^3 - 1 = \pm\frac{x^{3/2}}{8} - 1$

(b) $x = \cos t,\ y = \tan t \Rightarrow \sec t = \frac{1}{x} \Rightarrow \tan^2 t + 1 = \sec^2 t \Rightarrow y^2 = \frac{1}{x^2} - 1 = \frac{1-x^2}{x^2} \Rightarrow y = \pm\frac{\sqrt{1-x^2}}{x}$

12. (a) The line through $(1, -2)$ with slope 3 is $y = 3x - 5 \Rightarrow x = t,\ y = 3t - 5,\ -\infty < t < \infty$

(b) $(x-1)^2 + (y+2)^2 = 9 \Rightarrow x - 1 = 3\cos t,\ y + 2 = 3\sin t \Rightarrow x = 1 + 3\cos t,\ y = -2 + 3\sin t,\ 0 \le t \le 2\pi$

(c) $y = 4x^2 - x \Rightarrow x = t,\ y = 4t^2 - t,\ -\infty < t < \infty$

(d) $9x^2 + 4y^2 = 36 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow x = 2\cos t,\ y = 3\sin t,\ 0 \le t \le 2\pi$

13. $y = x^{1/2} - \frac{x^{3/2}}{3} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(\frac{1}{x} - 2 + x\right) \Rightarrow L = \int_1^4 \sqrt{1 + \frac{1}{4}\left(\frac{1}{x} - 2 + x\right)}\, dx$

$\Rightarrow L = \int_1^4 \sqrt{\frac{1}{4}\left(\frac{1}{x} + 2 + x\right)}\, dx = \int_1^4 \sqrt{\frac{1}{4}\left(x^{-1/2} + x^{1/2}\right)^2}\, dx = \int_1^4 \frac{1}{2}\left(x^{-1/2} + x^{1/2}\right) dx = \frac{1}{2}\left[2x^{1/2} + \frac{2}{3}x^{3/2}\right]_1^4$

$= \frac{1}{2}\left[\left(4 + \frac{2}{3}\cdot 8\right) - \left(2 + \frac{2}{3}\right)\right] = \frac{1}{2}\left(2 + \frac{14}{3}\right) = \frac{10}{3}$

14. $x = y^{2/3} \Rightarrow \frac{dx}{dy} = \frac{2}{3}y^{-1/3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4y^{-2/3}}{9} \Rightarrow L = \int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^8 \sqrt{1 + \frac{4}{9y^{2/3}}}\, dy = \int_1^8 \sqrt{\frac{9y^{2/3} + 4}{3y^{1/3}}}\, dy$

$= \frac{1}{3}\int_1^8 \sqrt{9y^{2/3} + 4}\left(y^{-1/3}\right) dy;\ \left[u = 9y^{2/3} + 4 \Rightarrow du = 6y^{-1/3}dy;\ y = 1 \Rightarrow u = 13,\ y = 8 \Rightarrow u = 40\right]$

$\rightarrow L = \frac{1}{18}\int_{13}^{40} u^{1/2} du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \approx 7.634$

15. $y = \frac{5}{12}x^{6/5} - \frac{5}{8}x^{4/5} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{1/5} - \frac{1}{2}x^{-1/5} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)$

$\Rightarrow L = \int_1^{32} \sqrt{1 + \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)}\, dx \Rightarrow L = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{2/5} + 2 + x^{-2/5}\right)}\, dx = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{1/5} + x^{-1/5}\right)^2}\, dx$

$= \int_1^{32} \frac{1}{2}\left(x^{1/5} + x^{-1/5}\right) dx = \frac{1}{2}\left[\frac{5}{6}x^{6/5} + \frac{5}{4}x^{4/5}\right]_1^{32} = \frac{1}{2}\left[\left(\frac{5}{6}\cdot 2^6 + \frac{5}{4}\cdot 2^4\right) - \left(\frac{5}{6} + \frac{5}{4}\right)\right] = \frac{1}{2}\left(\frac{315}{6} + \frac{75}{4}\right)$

$= \frac{1}{48}(1260 + 450) = \frac{1710}{48} = \frac{285}{8}$

16. $x = \frac{1}{12}y^3 + \frac{1}{y} \Rightarrow \frac{dx}{dy} = \frac{1}{4}y^2 - \frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4} \Rightarrow L = \int_1^2 \sqrt{1 + \left(\frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}\right)}\, dy$

$= \int_1^2 \sqrt{\frac{1}{16}y^4 + \frac{1}{2} + \frac{1}{y^4}}\, dy = \int_1^2 \sqrt{\left(\frac{1}{4}y^2 + \frac{1}{y^2}\right)^2}\, dy = \int_1^2 \left(\frac{1}{4}y^2 + \frac{1}{y^2}\right) dy = \left[\frac{1}{12}y^3 - \frac{1}{y}\right]_1^2$

$= \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right) = \frac{7}{12} + \frac{1}{2} = \frac{13}{12}$

17. $\frac{dx}{dt} = -5\sin t + 5\sin 5t$ and $\frac{dy}{dt} = 5\cos t - 5\cos 5t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{(-5\sin t + 5\sin 5t)^2 + (5\cos t - 5\cos 5t)^2} = 5\sqrt{\sin^2 5t - 2\sin t\sin 5t + \sin^2 t + \cos^2 t - 2\cos t\cos 5t + \cos^2 5t}$

$= 5\sqrt{2 - 2(\sin t\sin 5t + \cos t\cos 5t)} = 5\sqrt{2(1-\cos 4t)} = 5\sqrt{4\left(\frac{1}{2}\right)(1-\cos 4t)} = 10\sqrt{\sin^2 2t} = 10|\sin 2t|$

$= 10\sin 2t\left(\text{since } 0 \le t \le \frac{\pi}{2}\right) \Rightarrow \text{Length} = \int_0^{\pi/2} 10\sin 2t\, dt = [-5\cos 2t]_0^{\pi/2} = (-5)(-1) - (-5)(1) = 10$

18. $\frac{dx}{dt} = 3t^2 - 12t$ and $\frac{dy}{dt} = 3t^2 + 12t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(3t^2-12t\right)^2 + \left(3t^2+12t\right)^2} = \sqrt{288t^2 + 18t^4}$

$= 3\sqrt{2}\,|t|\,\sqrt{16+t^2} \Rightarrow \text{Length} = \int_0^1 3\sqrt{2}|t|\sqrt{16+t^2}\,dt = 3\sqrt{2}\int_0^1 t\sqrt{16+t^2}\,dt;$

$\left[u = 16 + t^2 \Rightarrow du = 2t\,dt \Rightarrow \frac{1}{2}du = t\,dt;\ t = 0 \Rightarrow u = 16;\ t = 1 \Rightarrow u = 17\right];$

$\to \frac{3\sqrt{2}}{2}\int_{16}^{17}\sqrt{u}\,du = \frac{3\sqrt{2}}{2}\left[\frac{2}{3}u^{3/2}\right]_{16}^{17} = \frac{3\sqrt{2}}{2}\left(\frac{2}{3}(17)^{3/2} - \frac{2}{3}(16)^{3/2}\right) = \frac{3\sqrt{2}}{2}\cdot\frac{2}{3}\left((17)^{3/2} - 64\right)$

$= \sqrt{2}\left((17)^{3/2} - 64\right) \approx 8.617.$

19. $\frac{dx}{d\theta} = -3\sin\theta$ and $\frac{dy}{d\theta} = 3\cos\theta \Rightarrow \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{(-3\sin\theta)^2 + (3\cos\theta)^2} = \sqrt{3\left(\sin^2\theta + \cos^2\theta\right)} = 3$

$\Rightarrow \text{Length} = \int_0^{3\pi/2} 3\,d\theta = 3\int_0^{3\pi/2} d\theta = 3\left(\frac{3\pi}{2} - 0\right) = \frac{9\pi}{2}$

20. $x = t^2$ and $y = \frac{t^3}{3} - t, -\sqrt{3} \le t \le \sqrt{3} \Rightarrow \frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = t^2 - 1 \Rightarrow \text{Length} = \int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{(2t)^2 + \left(t^2-1\right)^2}\,dt$

$= \int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{t^4 + 2t^2 + 1}\,dt = \int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{\left(t^2+1\right)^2}\,dt = \int_{-\sqrt{3}}^{\sqrt{3}}\left(t^2+1\right)dt = \left[\frac{t^3}{3} + t\right]_{-\sqrt{3}}^{\sqrt{3}} = 4\sqrt{3}$

21. $x = \frac{t^2}{2}$ and $y = 2t, 0 \le t \le \sqrt{5} \Rightarrow \frac{dx}{dt} = t$ and $\frac{dy}{dt} = 2 \Rightarrow \text{Surface Area} = \int_0^{\sqrt{5}} 2\pi(2t)\sqrt{t^2+4}\,dt$

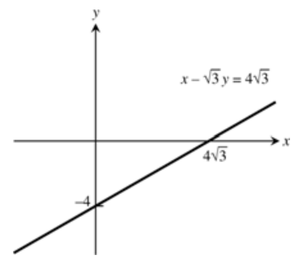
$\left[u = t^2 + 4 \Rightarrow du = 2t\,dt;\ t = 0 \Rightarrow u = 4, t = \sqrt{5} \Rightarrow u = 9\right] \to \int_4^9 2\pi u^{1/2}du = 2\pi\left[\frac{2}{3}u^{3/2}\right]_4^9 = \frac{76\pi}{3}$

22. $x = t^2 + \frac{1}{2t}$ and $y = 4\sqrt{t}, \frac{1}{\sqrt{2}} \le t \le 1 \Rightarrow \frac{dx}{dt} = 2t - \frac{1}{2t^2}$ and $\frac{dy}{dt} = \frac{2}{\sqrt{t}}$

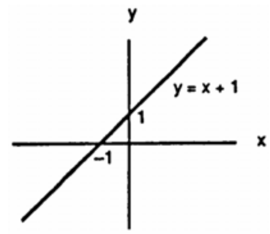
$\Rightarrow \text{Surface Area} = \int_{1/\sqrt{2}}^{1} 2\pi\left(t^2 + \frac{1}{2t}\right)\sqrt{\left(2t - \frac{1}{2t^2}\right)^2 + \left(\frac{2}{\sqrt{t}}\right)^2}\,dt = 2\pi\int_{1/\sqrt{2}}^{1}\left(t^2 + \frac{1}{2t}\right)\sqrt{\left(2t + \frac{1}{2t^2}\right)^2}\,dt$

$= 2\pi\int_{1\sqrt{2}}^{1}\left(t^2 + \frac{1}{2t}\right)\left(2t + \frac{1}{2t^2}\right)dt = 2\pi\int_{1\sqrt{2}}^{1}\left(2t^3 + \frac{3}{2} + \frac{1}{4}t^{-3}\right)dt = 2\pi\left[\frac{1}{2}t^4 + \frac{3}{2}t - \frac{1}{8}t^{-2}\right]_{1/\sqrt{2}}^{1} = 2\pi\left(2 - \frac{3\sqrt{2}}{4}\right)$

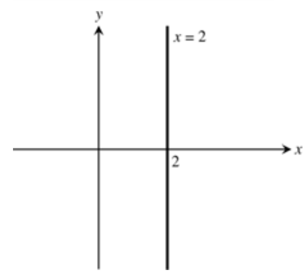
23. $r\cos\left(\theta+\frac{\pi}{3}\right)=2\sqrt{3}\Rightarrow r\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)=2\sqrt{3}$
$\Rightarrow \frac{1}{2}r\cos\theta-\frac{\sqrt{3}}{2}r\sin\theta=2\sqrt{3}$
$\Rightarrow r\cos\theta-\sqrt{3}\,r\sin\theta=4\sqrt{3}\Rightarrow x-\sqrt{3}y=4\sqrt{3}$
$\Rightarrow y=\frac{\sqrt{3}}{3}x-4$



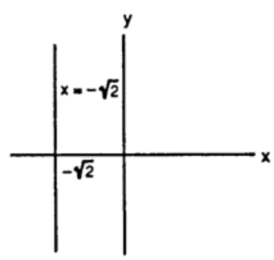
24. $r\cos\left(\theta-\frac{3\pi}{4}\right)=\frac{\sqrt{2}}{2}\Rightarrow r\left(\cos\theta\cos\frac{3\pi}{4}+\sin\theta\sin\frac{3\pi}{4}\right)=\frac{\sqrt{2}}{2}$
$\Rightarrow -\frac{\sqrt{2}}{2}r\cos\theta+\frac{\sqrt{2}}{2}r\sin\theta=\frac{\sqrt{2}}{2}\Rightarrow -x+y=1$
$\Rightarrow y=x+1$



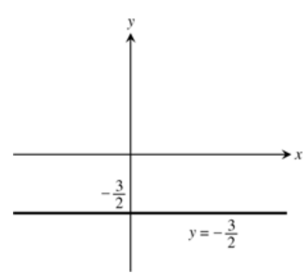
25. $r=2\sec\theta\Rightarrow r=\frac{2}{\cos\theta}\Rightarrow r\cos\theta=2\Rightarrow x=2$



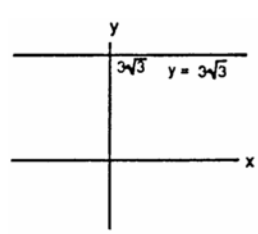
26. $r=-\sqrt{2}\sec\theta\Rightarrow r\cos\theta=-\sqrt{2}\Rightarrow x=-\sqrt{2}$



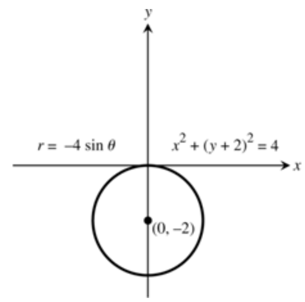
27. $r=-\frac{3}{2}\csc\theta\Rightarrow r\sin\theta=-\frac{3}{2}\Rightarrow y=-\frac{3}{2}$



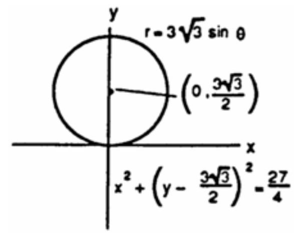
28. $r=3\sqrt{3}\csc\theta\Rightarrow r\sin\theta=3\sqrt{3}\Rightarrow y=3\sqrt{3}$



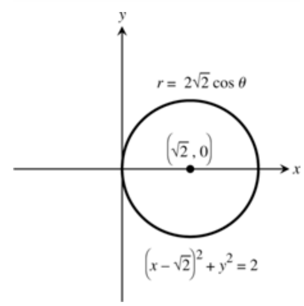
29. $r = -4\sin\theta \Rightarrow r^2 = -4r\sin\theta \Rightarrow x^2 + y^2 + 4y = 0$
$\Rightarrow x^2 + (y+2)^2 = 4;$
circle with center $(0, -2)$ and radius 2.



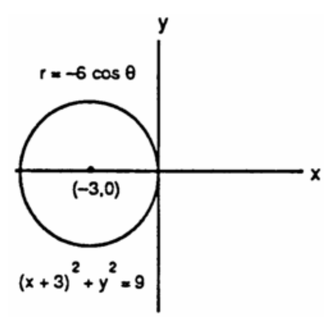
30. $r = 3\sqrt{3}\sin\theta \Rightarrow r^2 = 3\sqrt{3}\, r\sin\theta$
$\Rightarrow x^2 + y^2 - 3\sqrt{3}\, y = 0 \Rightarrow x^2 + \left(y - \frac{3\sqrt{3}}{2}\right)^2 = \frac{27}{4};$
circle with center $\left(0, \frac{3\sqrt{3}}{2}\right)$ and radius $\frac{3\sqrt{3}}{2}$



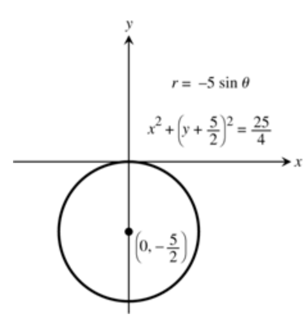
31. $r = 2\sqrt{2}\cos\theta \Rightarrow r^2 = 2\sqrt{2}\, r\cos\theta$
$\Rightarrow x^2 + y^2 - 2\sqrt{2}x = 0 \Rightarrow \left(x - \sqrt{2}\right)^2 + y^2 = 2;$
circle with center $\left(\sqrt{2}, 0\right)$ and radius $\sqrt{2}$



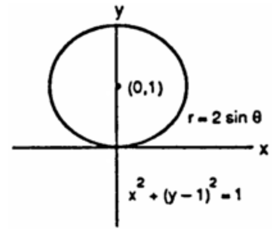
32. $r = -6\cos\theta \Rightarrow r^2 = -6r\cos\theta \Rightarrow x^2 + y^2 + 6x = 0$
$\Rightarrow (x+3)^3 + y^2 = 9;$
circle with center $(-3, 0)$ and radius 3



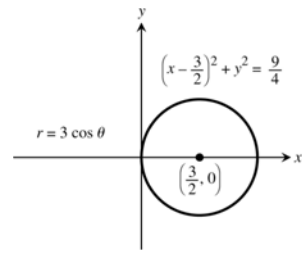
33. $x^2 + y^2 + 5y = 0 \Rightarrow x^2 + \left(y + \frac{5}{2}\right)^2 = \frac{25}{4}$
$\Rightarrow C = \left(0, -\frac{5}{2}\right)$ and $a = \frac{5}{2};$
$r^2 + 5r\sin\theta = 0 \Rightarrow r = -5\sin\theta$



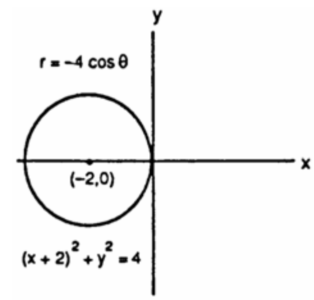
34. $x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y-1)^2 = 1$
$\Rightarrow C = (0,1)$ and $a = 1;$
$r^2 - 2r\sin\theta = 0 \Rightarrow r = 2\sin\theta$



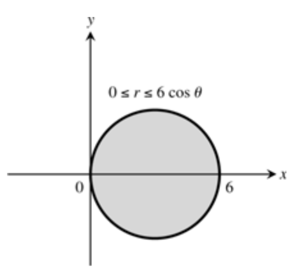
35. $x^2 + y^2 - 3x = 0 \Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$
$\Rightarrow C = \left(\frac{3}{2}, 0\right)$ and $a = \frac{3}{2};$
$r^2 - 3r\cos\theta = 0 \Rightarrow r = 3\cos\theta$



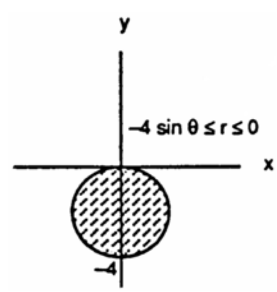
36. $x^2 + y^2 + 4x = 0 \Rightarrow (x+2)^2 + y^2 = 4$
$\Rightarrow C = (-2, 0)$ and $a = 2;$
$r^2 + 4r\cos\theta = 0 \Rightarrow r = -4\cos\theta$



37.



38.



39. d 40. e 41. l 42. f

43. k 44. h 45. i 46. j

47. $A = 2\int_0^\pi \frac{1}{2} r^2 d\theta = \int_0^\pi (2-\cos\theta)^2\, d\theta = \int_0^\pi \left(4 - 4\cos\theta + \cos^2\theta\right) d\theta = \int_0^\pi \left(4 - 4\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$
$= \int_0^\pi \left(\frac{9}{2} - 4\cos\theta + \frac{\cos 2\theta}{2}\right) d\theta = \left[\frac{9}{2}\theta - 4\sin\theta + \frac{\sin 2\theta}{4}\right]_0^\pi = \frac{9}{2}\pi$

48. $A = \int_0^{\pi/3} \frac{1}{2}\left(\sin^2 3\theta\right) d\theta = \int_0^{\pi/3} \left(\frac{1-\cos 6\theta}{2}\right) d\theta = \frac{1}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{\pi}{12}$

49. $r=1+\cos 2\theta$ and $r=1 \Rightarrow 1=1+\cos 2\theta \Rightarrow 0=\cos 2\theta \Rightarrow 2\theta=\frac{\pi}{2} \Rightarrow \theta=\frac{\pi}{4}$; therefore

$$A=4\int_0^{\pi/4}\frac{1}{2}\left[(1+\cos 2\theta)^2-1^2\right]d\theta=2\int_0^{\pi/4}\left(1+2\cos 2\theta+\cos^2 2\theta-1\right)d\theta$$

$$=2\int_0^{\pi/4}\left(2\cos 2\theta+\frac{1}{2}+\frac{\cos 4\theta}{2}\right)d\theta=2\left[\sin 2\theta+\frac{1}{2}\theta+\frac{\sin 4\theta}{8}\right]_0^{\pi/4}=2\left(1+\frac{\pi}{8}+0\right)=2+\frac{\pi}{4}$$

50. The circle lies interior to the cardioid. Thus,

$$A=2\int_{-\pi/2}^{\pi/2}\frac{1}{2}\left[2(1+\sin\theta)\right]^2 d\theta-\pi \text{ (the integral is the area of the cardioid minus the area of the circle)}$$

$$=\int_{-\pi/2}^{\pi/2}4\left(1+2\sin\theta+\sin^2\theta\right)d\theta-\pi=\int_{-\pi/2}^{\pi/2}(6+8\sin\theta-2\cos 2\theta)\,d\theta-\pi=\left[6\theta-8\cos\theta-\sin 2\theta\right]_{-\pi/2}^{\pi/2}-\pi$$

$$=\left[3\pi-(-3\pi)\right]-\pi=5\pi$$

51. $r=-1+\cos\theta \Rightarrow \frac{dr}{d\theta}=-\sin\theta$; Length $=\int_0^{2\pi}\sqrt{(-1+\cos\theta)^2+(-\sin\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta$

$$=\int_0^{2\pi}\sqrt{\frac{4(1-\cos\theta)}{2}}\,d\theta=\int_0^{2\pi}2\sin\frac{\theta}{2}\,d\theta=\left[-4\cos\frac{\theta}{2}\right]_0^{2\pi}=(-4)(-1)-(-4)(1)=8$$

52. $r=2\sin\theta+2\cos\theta,\ 0\le\theta\le\frac{\pi}{2} \Rightarrow \frac{dr}{d\theta}=2\cos\theta-2\sin\theta;\ r^2+\left(\frac{dr}{d\theta}\right)^2=(2\sin\theta+2\cos\theta)^2+(2\cos\theta-2\sin\theta)^2$

$$=8\left(\sin^2\theta+\cos^2\theta\right)=8 \Rightarrow L=\int_0^{\pi/2}\sqrt{8}\,d\theta=\left[2\sqrt{2}\theta\right]_0^{\pi/2}=2\sqrt{2}\left(\frac{\pi}{2}\right)=\pi\sqrt{2}$$

53. $r=8\sin^3\left(\frac{\theta}{3}\right),\ 0\le\theta\le\frac{\pi}{4} \Rightarrow \frac{dr}{d\theta}=8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right);\ r^2+\left(\frac{dr}{d\theta}\right)^2=\left[8\sin^3\left(\frac{\theta}{3}\right)\right]^2+\left[8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)\right]^2$

$$=64\sin^4\left(\frac{\theta}{3}\right) \Rightarrow L=\int_0^{\pi/4}\sqrt{64\sin^4\left(\frac{\theta}{3}\right)}\,d\theta=\int_0^{\pi/4}8\sin^2\left(\frac{\theta}{3}\right)d\theta=\int_0^{\pi/4}8\left[\frac{1-\cos\left(\frac{2\theta}{3}\right)}{2}\right]d\theta$$

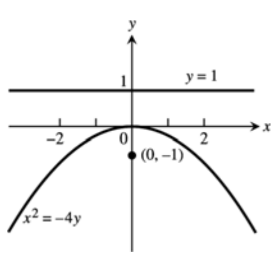
$$=\int_0^{\pi/4}\left[4-4\cos\left(\frac{2\theta}{3}\right)\right]d\theta=\left[4\theta-6\sin\left(\frac{2\theta}{3}\right)\right]_0^{\pi/4}=4\left(\frac{\pi}{4}\right)-6\sin\left(\frac{\pi}{6}\right)-0=\pi-3$$

54. $r=\sqrt{1+\cos 2\theta} \Rightarrow \frac{dr}{d\theta}=\frac{1}{2}(1+\cos 2\theta)^{-1/2}(-2\sin 2\theta)=\frac{-\sin 2\theta}{\sqrt{1+\cos 2\theta}} \Rightarrow \left(\frac{dr}{d\theta}\right)^2=\frac{\sin^2 2\theta}{1+\cos 2\theta}$

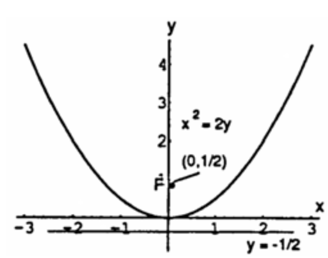
$$\Rightarrow r^2+\left(\frac{dr}{d\theta}\right)^2=1+\cos 2\theta+\frac{\sin^2 2\theta}{1+\cos 2\theta}=\frac{(1+\cos 2\theta)^2+\sin^2 2\theta}{1+\cos 2\theta}=\frac{1+2\cos 2\theta+\cos^2 2\theta+\sin^2 2\theta}{1+\cos 2\theta}$$

$$=\frac{2+2\cos 2\theta}{1+\cos 2\theta}=2 \Rightarrow L=\int_{-\pi/2}^{\pi/2}\sqrt{2}\,d\theta=\sqrt{2}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]=\sqrt{2}\pi$$

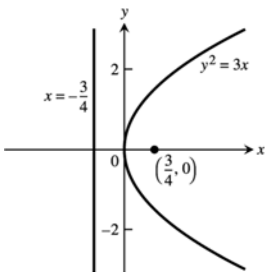
55. $x^2=-4y \Rightarrow y=-\frac{x^2}{4} \Rightarrow 4p=4 \Rightarrow p=1$;
therefore Focus is $(0,-1)$. Directrix is $y=1$



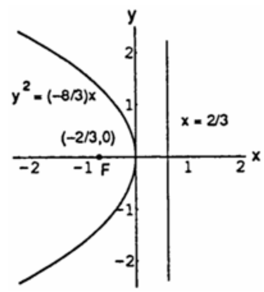
56. $x^2=2y \Rightarrow \frac{x^2}{2}=y \Rightarrow 4p=2 \Rightarrow p=\frac{1}{2}$;
therefore Focus is $\left(0,\frac{1}{2}\right)$; Directrix is $y=-\frac{1}{2}$



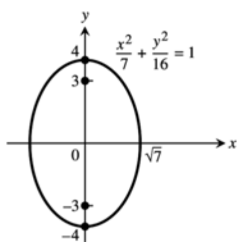
57. $y^2 = 3x \Rightarrow x = \frac{y^2}{3} \Rightarrow 4p = 3 \Rightarrow p = \frac{3}{4}$;
therefore Focus is $\left(\frac{3}{4}, 0\right)$, Directrix is $x = -\frac{3}{4}$



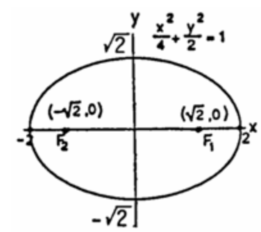
58. $y^2 = -\frac{8}{3}x \Rightarrow x = -\frac{y^2}{\left(\frac{8}{3}\right)} \Rightarrow 4p = \frac{8}{3} \Rightarrow p = \frac{2}{3}$;
therefore Focus is $\left(-\frac{2}{3}, 0\right)$, Directrix is $x = \frac{2}{3}$



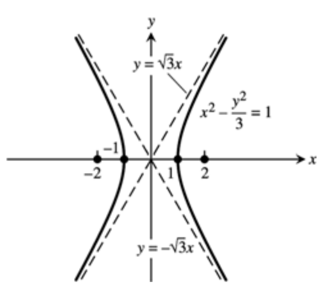
59. $16x^2 + 7y^2 = 112 \Rightarrow \frac{x^2}{7} + \frac{y^2}{16} = 1$
$\Rightarrow c^2 = 16 - 7 = 9 \Rightarrow c = 3$; $e = \frac{c}{a} = \frac{3}{4}$



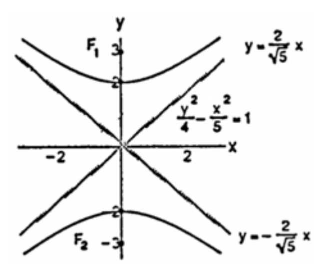
60. $x^2 + 2y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1 \Rightarrow c^2 = 4 - 2 = 2$
$\Rightarrow c = \sqrt{2}$; $e = \frac{c}{a} = \frac{\sqrt{2}}{2}$



61. $3x^2 - y^2 = 3 \Rightarrow x^2 - \frac{y^2}{3} = 1 \Rightarrow c^2 = 1 + 3 = 4$
$\Rightarrow c = 2$; $e = \frac{c}{a} = \frac{2}{1} = 2$;
the asymptotes are $y = \pm\sqrt{3}x$



62. $5y^2 - 4x^2 = 20 \Rightarrow \frac{y^2}{4} - \frac{x^2}{5} = 1 \Rightarrow c^2 = 4 + 5 = 9$
$\Rightarrow c = 3$, $e = \frac{c}{a} = \frac{3}{2}$; the asymptotes are $y = \pm\frac{2}{\sqrt{5}}x$



63. $x^2 = -12y \Rightarrow -\frac{x^2}{12} = y \Rightarrow 4p = 12 \Rightarrow p = 3 \Rightarrow$ focus is $(0, -3)$, directrix is $y = 3$, vertex is $(0, 0)$; therefore new vertex is $(2, 3)$, new focus is $(2, 0)$, new directrix is $y = 6$, and the new equation is $(x-2)^2 = -12(y-3)$

64. $y^2 = 10x \Rightarrow \frac{y^2}{10} = x \Rightarrow 4p = 10 \Rightarrow p = \frac{5}{2} \Rightarrow$ focus is $\left(\frac{5}{2}, 0\right)$, directrix is $x = -\frac{5}{2}$, vertex is $(0, 0)$; therefore new vertex is $\left(-\frac{1}{2}, -1\right)$, new focus is $(2, -1)$, new directrix is $x = -3$, and the new equation is
$(y+1)^2 = 10\left(x + \frac{1}{2}\right)$

65. $\frac{x^2}{9}+\frac{y^2}{25}=1 \Rightarrow a=5$ and $b=3 \Rightarrow c=\sqrt{25-9}=4 \Rightarrow$ foci are $(0, \pm 4)$, vertices are $(0, \pm 5)$, center is $(0, 0)$; therefore the new center is $(-3, -5)$, new foci are $(-3, -1)$ and $(-3, -9)$, new vertices are $(-3, -10)$ and $(-3, 0)$, and the new equation is $\frac{(x+3)^2}{9}+\frac{(y+5)^2}{25}=1$

66. $\frac{x^2}{169}+\frac{y^2}{144}=1 \Rightarrow a=13$ and $b=12 \Rightarrow c=\sqrt{169-144}=5 \Rightarrow$ foci are $(\pm 5, 0)$, vertices are $(\pm 13, 0)$, center is $(0, 0)$; therefore the new center is $(5, 12)$, new foci are $(10, 12)$ and $(0, 12)$, new vertices are $(18, 12)$ and $(-8, 12)$, and the new equation is $\frac{(x-5)^2}{169}+\frac{(y-12)^2}{144}=1$

67. $\frac{y^2}{8}-\frac{x^2}{2}=1 \Rightarrow a=2\sqrt{2}$ and $b=\sqrt{2} \Rightarrow c=\sqrt{8+2}=\sqrt{10} \Rightarrow$ foci are $\left(0, \pm\sqrt{10}\right)$, vertices are $\left(0, \pm 2\sqrt{2}\right)$, center is $(0, 0)$, and the asymptotes are $y=\pm 2x$; therefore the new center is $\left(2, 2\sqrt{2}\right)$, new foci are $\left(2, 2\sqrt{2}\pm\sqrt{10}\right)$, new vertices are $\left(2, 4\sqrt{2}\right)$ and $(2, 0)$, the new asymptotes are $y=2x-4+2\sqrt{2}$ and $y=-2x+4+2\sqrt{2}$; the new equation is $\frac{\left(y-2\sqrt{2}\right)^2}{8}-\frac{(x-2)^2}{2}=1$

68. $\frac{x^2}{36}-\frac{y^2}{64}=1 \Rightarrow a=6$ and $b=8 \Rightarrow c=\sqrt{36+64}=10 \Rightarrow$ foci are $(\pm 10, 0)$, vertices are $(\pm 6, 0)$, the center is $(0, 0)$ and the asymptotes are $\frac{y}{8}=\pm\frac{x}{6}$ or $y=\pm\frac{4}{3}x$; therefore the new center is $(-10, -3)$, the new foci are $(-20, -3)$ and $(0, -3)$, the new vertices are $(-16, -3)$ and $(-4, -3)$, the new asymptotes are $y=\frac{4}{3}x+\frac{31}{3}$ and $y=-\frac{4}{3}x-\frac{49}{3}$; the new equation is $\frac{(x+10)^2}{36}-\frac{(y+3)^2}{64}=1$

69. $x^2-4x-4y^2=0 \Rightarrow x^2-4x+4-4y^2=4 \Rightarrow (x-2)^2-4y^2=4 \Rightarrow \frac{(x-2)^2}{4}-y^2=1$, a hyperbola; $a=2$ and $b=1 \Rightarrow c=\sqrt{1+4}=\sqrt{5}$; the center is $(2,0)$, the vertices are $(0,0)$ and $(4,0)$; the foci are $\left(2\pm\sqrt{5}, 0\right)$ and the asymptotes are $y=\pm\frac{x-2}{2}$

70. $4x^2-y^2+4y=8 \Rightarrow 4x^2-y^2+4y-4=4 \Rightarrow 4x^2-(y-2)^2=4 \Rightarrow x^2-\frac{(y-2)^2}{4}=1$, a hyperbola; $a=1$ and $b=2 \Rightarrow c=\sqrt{1+4}=\sqrt{5}$; the center is $(0, 2)$, the vertices are $(1, 2)$ and $(-1, 2)$, the foci are $\left(\pm\sqrt{5}, 2\right)$ and the asymptotes are $y=\pm 2x+2$

71. $y^2-2y+16x=-49 \Rightarrow y^2-2y+1=-16x-48 \Rightarrow (y-1)^2=-16(x+3)$, a parabola; the vertex is $(-3, 1)$; $4p=16 \Rightarrow p=4 \Rightarrow$ the focus is $(-7, 1)$ and the directrix is $x=1$

72. $x^2-2x+8y=-17 \Rightarrow x^2-2x+1=-8y-16 \Rightarrow (x-1)^2=-8(y+2)$, a parabola; the vertex is $(1, -2)$; $4p=8 \Rightarrow p=2 \Rightarrow$ the focus is $(1, -4)$ and the directrix is $y=0$

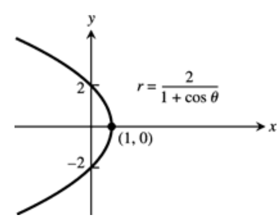
73. $9x^2+16y^2+54x-64y=-1 \Rightarrow 9\left(x^2+6x\right)+16\left(y^2-4y\right)=-1 \Rightarrow 9\left(x^2+6x+9\right)+16\left(y^2-4y+4\right)=144$
$\Rightarrow 9(x+3)^2+16(y-2)^2=144 \Rightarrow \frac{(x+3)^2}{16}+\frac{(y-2)^2}{9}=1,$ an ellipse; the center is $(-3, 2)$; $a=4$ and $b=3$
$\Rightarrow c=\sqrt{16-9}=\sqrt{7}$; the foci are $\left(-3\pm\sqrt{7}, 2\right)$; the vertices are $(1, 2)$ and $(-7, 2)$

74. $25x^2+9y^2-100x+54y=44 \Rightarrow 25\left(x^2-4x\right)+9\left(y^2+6y\right)=44 \Rightarrow 25\left(x^2-4x+4\right)+9\left(y^2+6y+9\right)=225$
$\Rightarrow \frac{(x-2)^2}{9}+\frac{(y+3)^2}{25}=1,$ an ellipse; the center is $(2, -3)$; $a=5$ and $b=3 \Rightarrow c=\sqrt{25-9}=4$; the foci are $(2, 1)$ and $(2, -7)$; the vertices are $(2, 2)$ and $(2, -8)$

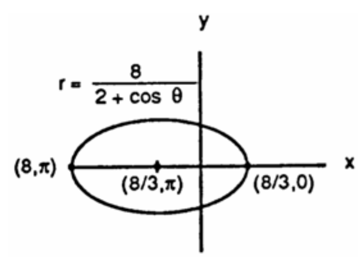
75. $x^2+y^2-2x-2y=0 \Rightarrow x^2-2x+1+y^2-2y+1=2 \Rightarrow (x-1)^2+(y-1)^2=2,$ a circle with center $(1, 1)$ and radius $=\sqrt{2}$

76. $x^2+y^2+4x+2y=1 \Rightarrow x^2+4x+4+y^2+2y+1=6 \Rightarrow (x+2)^2+(y+1)^2=6,$ a circle with center $(-2, -1)$ and radius $=\sqrt{6}$

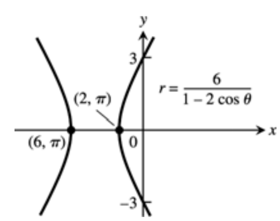
77. $r=\frac{2}{1+\cos\theta} \Rightarrow e=1 \Rightarrow$ parabola with vertex at $(1, 0)$



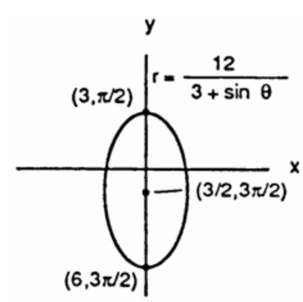
78. $r=\frac{8}{2+\cos\theta} \Rightarrow r=\frac{4}{1+\left(\frac{1}{2}\right)\cos\theta} \Rightarrow e=\frac{1}{2} \Rightarrow$ ellipse;
$ke=4 \Rightarrow \frac{1}{2}k=4 \Rightarrow k=8$; $k=\frac{a}{e}-ea \Rightarrow 8=\frac{a}{\left(\frac{1}{2}\right)}-\frac{1}{2}a$
$\Rightarrow a=\frac{16}{3} \Rightarrow ea=\left(\frac{1}{2}\right)\left(\frac{16}{3}\right)=\frac{8}{3}$; therefore the center is $\left(\frac{8}{3}, \pi\right)$; vertices are $(8, \pi)$ and $\left(\frac{8}{3}, 0\right)$



79. $r=\frac{6}{1-2\cos\theta} \Rightarrow e=2 \Rightarrow$ hyperbola; $ke=6 \Rightarrow 2k=6$
$\Rightarrow k=3 \Rightarrow$ vertices are $(2, \pi)$ and $(6, \pi)$



80. $r=\frac{12}{3+\sin\theta} \Rightarrow r=\frac{4}{1+\left(\frac{1}{3}\right)\sin\theta} \Rightarrow e=\frac{1}{3}; ke=4$
$\Rightarrow \frac{1}{3}k=4 \Rightarrow k=12$; $a\left(1-e^2\right))4 \Rightarrow a\left[1-\left(\frac{1}{3}\right)^2\right]=4$
$\Rightarrow a=\frac{9}{2} \Rightarrow ea=\left(\frac{1}{3}\right)\left(\frac{9}{2}\right)=\frac{3}{2}$; therefore the center is $\left(\frac{3}{2}, \frac{3\pi}{2}\right)$; vertices are $\left(3, \frac{\pi}{2}\right)$ and $\left(6, \frac{3\pi}{2}\right)$



81. $e=2$ and $r\cos\theta=2 \Rightarrow x=2$ is directrix $\Rightarrow k=2$; the conic is a hyperbola; $r=\frac{ke}{1+e\cos\theta} \Rightarrow r=\frac{(2)(2)}{1+2\cos\theta}$ $\Rightarrow r=\frac{4}{1+2\cos\theta}$

82. $e=1$ and $r\cos\theta=-4 \Rightarrow x=-4$ is directrix $\Rightarrow k=4$; the conic is a parabola; $r=\frac{ke}{1-e\cos\theta} \Rightarrow r=\frac{(4)(1)}{1-\cos\theta}$ $\Rightarrow r=\frac{4}{1-\cos\theta}$

83. $e=\frac{1}{2}$ and $r\sin\theta=2 \Rightarrow y=2$ is directrix $\Rightarrow k=2$; the conic is an ellipse; $r=\frac{ke}{1+e\sin\theta} \Rightarrow r=\frac{(2)\left(\frac{1}{2}\right)}{1+\left(\frac{1}{2}\right)\sin\theta}$ $\Rightarrow r=\frac{2}{2+\sin\theta}$

84. $e=\frac{1}{3}$ and $r\sin\theta=-6 \Rightarrow y=-6$ is directrix $\Rightarrow k=6$; the conic is an ellipse; $r=\frac{ke}{1-e\sin\theta} \Rightarrow r=\frac{(6)\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)\sin\theta}$ $\Rightarrow r=\frac{6}{3-\sin\theta}$

85. (a) Around the x-axis: $9x^2+4y^2=36 \Rightarrow y^2=9-\frac{9}{4}x^2 \Rightarrow y=\pm\sqrt{9-\frac{9}{4}x^2}$ and we use the positive root:

$$V=2\int_0^2 \pi\left(\sqrt{9-\frac{9}{4}x^2}\right)^2 dx=2\int_0^2 \pi\left(9-\frac{9}{4}x^2\right)dx=2\pi\left[9x-\frac{3}{4}x^3\right]_0^2=24\pi$$

(b) Around the y-axis: $9x^2+4y^2=36 \Rightarrow x^2=4-\frac{4}{9}y^2 \Rightarrow x=\pm\sqrt{4-\frac{4}{9}y^2}$ and we use the positive root:

$$V=2\int_0^3 \pi\left(\sqrt{4-\frac{4}{9}y^2}\right)^2 dy=2\int_0^3 \pi\left(4-\frac{4}{9}y^2\right)dy=2\pi\left[4y-\frac{4}{27}y^3\right]_0^3=16\pi$$

86. $9x^2-4y^2=36, x=4 \Rightarrow y^2=\frac{9x^2-36}{4} \Rightarrow y=\frac{3}{2}\sqrt{x^2-4}$; $V=\int_2^4 \pi\left(\frac{3}{2}\sqrt{x^2-4}\right)^2 dx=\frac{9\pi}{4}\int_2^4\left(x^2-4\right)dx$

$$=\frac{9\pi}{4}\left[\frac{x^3}{3}-4x\right]_2^4=\frac{9\pi}{4}\left[\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)\right]=\frac{9\pi}{4}\left(\frac{56}{3}-\frac{24}{3}\right)=\frac{3\pi}{4}(32)=24\pi$$

87. (a) $r=\frac{k}{1+e\cos\theta} \Rightarrow r+er\cos\theta=k \Rightarrow \sqrt{x^2+y^2}+ex=k \Rightarrow \sqrt{x^2+y^2}=k-ex \Rightarrow x^2+y^2=k^2-2kex+e^2x^2$

$\Rightarrow x^2-e^2x^2+y^2+2kex-k^2=0 \Rightarrow \left(1-e^2\right)x^2+y^2+2kex-k^2=0$

(b) $e=0 \Rightarrow x^2+y^2-k^2=0 \Rightarrow x^2+y^2=k^2 \Rightarrow$ circle;

$0<e<1 \Rightarrow e^2<1 \Rightarrow e^2-1<0 \Rightarrow B^2-4AC=0^2-4\left(1-e^2\right)(1)=4\left(e^2-1\right)<0 \Rightarrow$ ellipse;

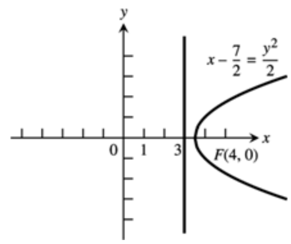
$e=1 \Rightarrow B^2-4AC=0^2-4(0)(1)=0 \Rightarrow$ parabola;

$e>1 \Rightarrow e^2>1 \Rightarrow B^2-4AC=0^2-4\left(1-e^2\right)(1)=4e^2-4>0 \Rightarrow$ hyperbola

88. Let $\left(r_1,\theta_1\right)$ be a point on the graph where $r_1=a\theta_1$. Let $\left(r_2,\theta_2\right)$ be on the graph where $r_2=a\theta_2$ and $\theta_2=\theta_1+2\pi$. Then r_1 and r_2 lie on the same ray on consecutive turns of the spiral and the distance between the two points is $r_2-r_1=a\theta_2-a\theta_1=a\left(\theta_2-\theta_1\right)=2\pi a$, which is constant.

CHAPTER 11 ADDITIONAL AND ADVANCED EXERCISES

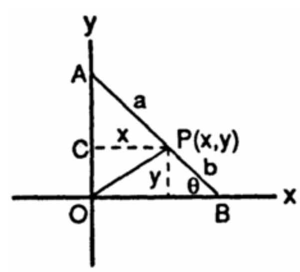
1. Directrix $x=3$ and focus $(4, 0) \Rightarrow$ vertex is $\left(\frac{7}{2}, 0\right)$
$\Rightarrow p=\frac{1}{2} \Rightarrow$ the equation is $x-\frac{7}{2}=\frac{y^2}{2}$



2. $x^2-6x-12y+9=0 \Rightarrow x^2-6x+9=12y \Rightarrow \frac{(x-3)^2}{12}=y \Rightarrow$ vertex is $(3, 0)$ and $p=3 \Rightarrow$ focus is $(3, 3)$ and the directrix is $y=-3$

3. $x^2=4y \Rightarrow$ vertex is $(0, 0)$ and $p=1 \Rightarrow$ focus is $(0, 1)$; thus the distance from $P(x, y)$ to the vertex is $\sqrt{x^2+y^2}$ and the distance from P to the focus is $\sqrt{x^2+(y-1)^2} \Rightarrow \sqrt{x^2+y^2}=2\sqrt{x^2+(y-1)^2}$
$\Rightarrow x^2+y^2=4\left[x^2+(y-1)^2\right] \Rightarrow x^2+y^2=4x^2+4y^2-8y+4 \Rightarrow 3x^2+3y^2-8y+4=0$, which is a circle

4. Let the segment $a+b$ intersect the y-axis in point A and intersect the x-axis in point B so that $PB=b$ and $PA=a$ (see figure). Draw the horizontal line through P and let it intersect the y-axis in point C. Let $\angle PBO=\theta$
$\Rightarrow \angle APC=\theta$. Then $\sin\theta=\frac{y}{b}$ and $\cos\theta=\frac{x}{a}$
$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=\cos^2\theta+\sin^2\theta=1.$



5. Vertices are $(0, \pm 2) \Rightarrow a=2$; $e=\frac{c}{a} \Rightarrow 0.5=\frac{c}{2} \Rightarrow c=1 \Rightarrow$ foci are $(0, \pm 1)$

6. Let the center of the ellipse be $(x, 0)$; directrix $x=2$, focus $(4, 0)$, and $e=\frac{2}{3} \Rightarrow \frac{a}{e}-c=2 \Rightarrow \frac{a}{e}=2+c$
$\Rightarrow a=\frac{2}{3}(2+c)$. Also $c=ae=\frac{2}{3}a \Rightarrow a=\frac{2}{3}\left(2+\frac{2}{3}a\right) \Rightarrow a=\frac{4}{3}+\frac{4}{9}a \Rightarrow \frac{5}{9}a=\frac{4}{3} \Rightarrow a=\frac{12}{5}$;
$x-2=\frac{a}{e} \Rightarrow x-2=\left(\frac{12}{5}\right)\left(\frac{3}{2}\right)=\frac{18}{5} \Rightarrow x=\frac{28}{5} \Rightarrow$ the center is $\left(\frac{28}{5}, 0\right)$; $x-4=c \Rightarrow c=\frac{28}{5}-4=\frac{8}{5}$ so that
$c^2=a^2-b^2=\left(\frac{12}{5}\right)^2-\left(\frac{8}{5}\right)^2=\frac{80}{25}$; therefore the equation is $\frac{\left(x-\frac{28}{5}\right)^2}{\left(\frac{144}{25}\right)}+\frac{y^2}{\left(\frac{80}{25}\right)}=1$ or $\frac{25\left(x-\frac{28}{5}\right)^2}{144}+\frac{5y^2}{16}=1$

7. Let the center of the hyperbola be $(0, y)$.
(a) Directrix $y=-1$, focus $(0, -7)$ and $e=2 \Rightarrow c-\frac{a}{e}=6 \Rightarrow \frac{a}{e}=c-6 \Rightarrow a=2c-12$. Also $c=ae=2a$
$\Rightarrow a=2(2a)-12 \Rightarrow a=4 \Rightarrow c=8$; $y-(-1)=\frac{a}{e}=\frac{4}{2}=2 \Rightarrow y=1 \Rightarrow$ the center is $(0,1)$;
$c^2=a^2+b^2 \Rightarrow b^2=c^2-a^2=64-16=48$; therefore the equation is $\frac{(y-1)^2}{16}-\frac{x^2}{48}=1$

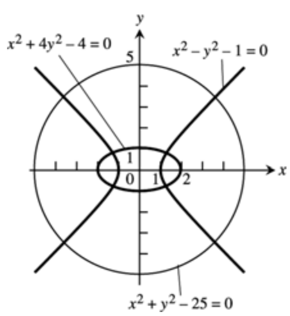
(b) $e = 5 \Rightarrow c - \frac{a}{e} = 6 \Rightarrow \frac{a}{e} = c - 6 \Rightarrow a = 5c - 30$. Also, $c = ae = 5a \Rightarrow a = 5(5a) - 30 \Rightarrow 24a = 30 \Rightarrow a = \frac{5}{4}$ $\Rightarrow c = \frac{25}{4}$; $y - (-1) = \frac{a}{e} = \frac{\left(\frac{5}{4}\right)}{5} = \frac{1}{4} \Rightarrow y = -\frac{3}{4} \Rightarrow$ the center is $\left(0, -\frac{3}{4}\right)$; $c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$ $= \frac{625}{16} - \frac{25}{16} = \frac{75}{2}$; therefore the equation is $\frac{\left(y+\frac{3}{4}\right)^2}{\left(\frac{25}{16}\right)} - \frac{x^2}{\left(\frac{75}{2}\right)} = 1$ or $\frac{16\left(y+\frac{3}{4}\right)^2}{25} - \frac{2x^2}{75} = 1$

8. The center is $(0, 0)$ and $c = 2 \Rightarrow 4 = a^2 + b^2 \Rightarrow b^2 = 4 - a^2$. The equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \Rightarrow \frac{49}{a^2} - \frac{144}{b^2} = 1$ $\Rightarrow \frac{49}{a^2} - \frac{144}{\left(4-a^2\right)} = 1 \Rightarrow 49\left(4 - a^2\right) - 144a^2 = a^2\left(4 - a^2\right) \Rightarrow 196 - 49a^2 - 144a^2 = 4a^2 - a^4$ $\Rightarrow a^4 - 197a^2 + 196 = 0 \Rightarrow \left(a^2 - 196\right)\left(a^2 - 1\right) = 0 \Rightarrow a = 14$ or $a = 1$; $a = 14 \Rightarrow b^2 = 4 - (14)^2 < 0$ which is impossible; $a = 1 \Rightarrow b^2 = 4 - 1 = 3$; therefore the equation is $y^2 - \frac{x^2}{3} = 1$

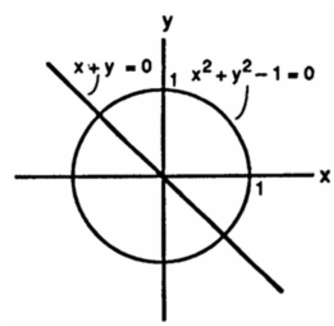
9. $b^2x^2 + a^2y^2 = a^2b^2 \Rightarrow \frac{dy}{dx} = -\frac{b^2x}{a^2y}$; at $\left(x_1, y_1\right)$ the tangent line is $y - y_1 = \left(-\frac{b^2x_1}{a^2y_1}\right)\left(x - x_1\right)$ $\Rightarrow a^2yy_1 + b^2xx_1 = b^2x_1^2 + a^2y_1^2 = a^2b^2 \Rightarrow b^2xx_1 + a^2yy_1 - a^2b^2 = 0$

10. $b^2x^2 - a^2y^2 = a^2b^2 \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$; at $\left(x_1, y_1\right)$ the tangent line is $y - y_1 = \left(\frac{b^2x_1}{a^2y_1}\right)\left(x - x_1\right)$ $\Rightarrow b^2xx_1 - a^2yy_1 = b^2x_1^2 - a^2y_1^2 = a^2b^2 \Rightarrow b^2xx_1 - a^2yy_1 - a^2b^2 = 0$

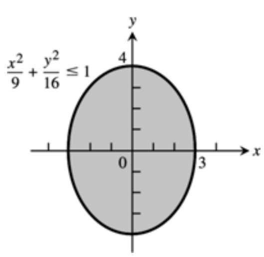
11.



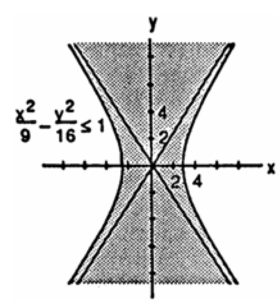
12.



13.



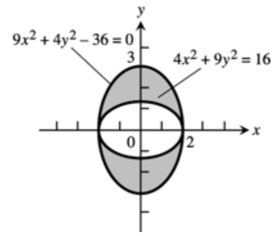
14.



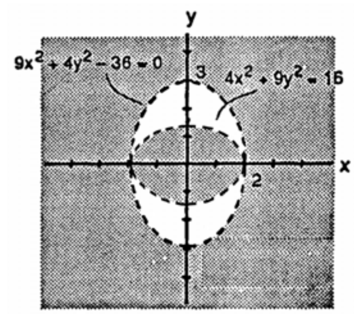
15. $\left(9x^2+4y^2-36\right)\left(4x^2+9y^2-16\right)\le 0$

$\Rightarrow 9x^2+4y^2-36\le 0$ and $4x^2+9y^2-16\ge 0$ or

$9x^2+4y^2-36\ge 0$ and $4x^2+9y^2-16\le 0$



16. $\left(9x^2+4y^2-36\right)\left(4x^2+9y^2-16\right)>0,$ which is the complement of the set in Exercise 15



17. (a) $x=e^{2t}\cos t$ and $y=e^{2t}\sin t \Rightarrow x^2+y^2=e^{4t}\cos^2 t+e^{4t}\sin^2 t=e^{4t}$. Also $\frac{y}{x}=\frac{e^{2t}\sin t}{e^{2t}\cos t}=\tan t$

$\Rightarrow t=\tan^{-1}\left(\frac{y}{x}\right)\Rightarrow x^2+y^2=e^{4\tan^{-1}(y/x)}$ is the Cartesian equation. Since $r^2=x^2+y^2$ and

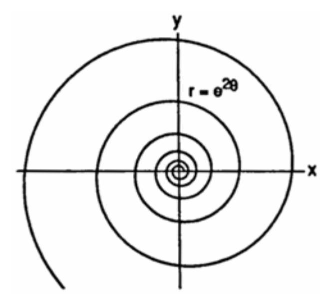
$\theta=\tan^{-1}\left(\frac{y}{x}\right)$, the polar equation is $r^2=e^{4\theta}$ or $r=e^{2\theta}$ for $r>0$

(b) $ds^2=r^2d\theta^2+dr^2; r=e^{2\theta}\Rightarrow dr=2e^{2\theta}d\theta$

$\Rightarrow ds^2=r^2d\theta^2+\left(2e^{2\theta}d\theta\right)^2=\left(e^{2\theta}\right)^2d\theta^2+4e^{4\theta}d\theta^2$

$=5e^{4\theta}d\theta^2\Rightarrow ds=\sqrt{5}e^{2\theta}d\theta$

$\Rightarrow L=\int_0^{2\pi}\sqrt{5}e^{2\theta}d\theta=\left[\frac{\sqrt{5}e^{2\theta}}{2}\right]_0^{2\pi}=\frac{\sqrt{5}}{2}\left(e^{4\pi}-1\right)$



18. $r=2\sin^3\left(\frac{\theta}{3}\right)\Rightarrow dr=2\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)d\theta\Rightarrow ds^2=r^2d\theta^2+dr^2$

$=\left[2\sin^3\left(\frac{\theta}{3}\right)\right]^2d\theta^2+\left[2\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)d\theta\right]^2=4\sin^6\left(\frac{\theta}{3}\right)d\theta^2+4\sin^4\left(\frac{\theta}{3}\right)\cos^2\left(\frac{\theta}{3}\right)d\theta^2$

$=\left[4\sin^4\left(\frac{\theta}{3}\right)\right]\left[\sin^2\left(\frac{\theta}{3}\right)+\cos^2\left(\frac{\theta}{3}\right)\right]d\theta^2=4\sin^4\left(\frac{\theta}{3}\right)d\theta^2\Rightarrow ds=2\sin^2\left(\frac{\theta}{3}\right)d\theta.$

Then $L=\int_0^{3\pi}2\sin^2\left(\frac{\theta}{3}\right)d\theta=\int_0^{3\pi}\left[1-\cos\left(\frac{2\theta}{3}\right)\right]d\theta=\left[\theta-\frac{3}{2}\sin\left(\frac{2\theta}{3}\right)\right]_0^{3\pi}=3\pi$

19. $e=2$ and $r\cos\theta=2\Rightarrow x=2$ is the directrix $\Rightarrow k=2;$ the conic is a hyperbola with $r=\frac{ke}{1+e\cos\theta}$

$\Rightarrow r=\frac{(2)(2)}{1+2\cos\theta}=\frac{4}{1+2\cos\theta}$

20. $e = 1$ and $r\cos\theta = -4 \Rightarrow x = -4$ is the directrix $\Rightarrow k = 4$; the conic is a parabola with $r = \frac{ke}{1 - e\cos\theta}$

$\Rightarrow r = \frac{(4)(1)}{1 - \cos\theta} = \frac{4}{1 - \cos\theta}$

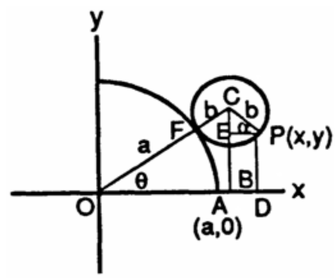
21. $e = \frac{1}{2}$ and $r\sin\theta = 2 \Rightarrow y = 2$ is the directrix $\Rightarrow k = 2$; the conic is an ellipse with $r = \frac{ke}{1 + e\sin\theta}$

$\Rightarrow r = \frac{2\left(\frac{1}{2}\right)}{1 + \left(\frac{1}{2}\right)\sin\theta} = \frac{2}{2 + \sin\theta}$

22. $e = \frac{1}{3}$ and $r\sin\theta = -6 \Rightarrow y = -6$ is the directrix $\Rightarrow k = 6$; the conic is an ellipse with $r = \frac{ke}{1 - e\sin\theta}$

$\Rightarrow r = \frac{6\left(\frac{1}{3}\right)}{1 - \left(\frac{1}{3}\right)\sin\theta} = \frac{6}{3 - \sin\theta}$

23. Arc PF = Arc AF since each is the distance rolled;



$\angle PCF = \frac{\text{Arc } PF}{b} \Rightarrow \text{Arc } PF = b(\angle PCF)$;

$\theta = \frac{\text{Arc } AF}{a} \Rightarrow \text{Arc } AF = a\theta \Rightarrow a\theta = b(\angle PCF)$

$\Rightarrow \angle PCF = \left(\frac{a}{b}\right)\theta$; $\angle OCB = \frac{\pi}{2} - \theta$ and

$\angle OCB = \angle PCF - \angle PCE = \angle PCF - \left(\frac{\pi}{2} - \alpha\right)$

$= \left(\frac{a}{b}\right)\theta - \left(\frac{\pi}{2} - \alpha\right) \Rightarrow \frac{\pi}{2} - \theta = \left(\frac{a}{b}\right)\theta - \left(\frac{\pi}{2} - \alpha\right)$

$\Rightarrow \frac{\pi}{2} - \theta = \left(\frac{a}{b}\right)\theta - \frac{\pi}{2} + \alpha \Rightarrow \alpha = \pi - \theta - \left(\frac{a}{b}\right)\theta$

$\Rightarrow \alpha = \pi - \left(\frac{a+b}{b}\right)\theta.$

Now $x = OB + BD = OB + EP = (a+b)\cos\theta + b\cos\alpha = (a+b)\cos\theta + b\cos\left(\pi - \left(\frac{a+b}{b}\right)\theta\right)$

$= (a+b)\cos\theta + b\cos\pi\cos\left(\left(\frac{a+b}{b}\right)\theta\right) + b\sin\pi\sin\left(\left(\frac{a+b}{b}\right)\theta\right) = (a+b)\cos\theta - b\cos\left(\left(\frac{a+b}{b}\right)\theta\right)$ and

$y = PD = CB - CE = (a+b)\sin\theta - b\sin\alpha = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right)$

$= (a+b)\sin\theta - b\sin\pi\cos\left(\left(\frac{a+b}{b}\right)\theta\right) + b\cos\pi\sin\left(\left(\frac{a+b}{b}\right)\theta\right) = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right)$;

therefore $x = (a+b)\cos\theta - b\cos\left(\left(\frac{a+b}{b}\right)\theta\right)$ and $y = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right)$

24. $x = a(t - \sin t) \Rightarrow \frac{dx}{dt} = a(1 - \cos t)$ and let $\delta = 1 \Rightarrow dm = dA = y\,dx = y\left(\frac{dx}{dt}\right)dt = a(1-\cos t)a(1-\cos t)\,dt$

$= a^2(1-\cos t)^2\,dt$; then $A = \int_0^{2\pi} a^2(1-\cos t)^2\,dt = a^2\int_0^{2\pi}\left(1 - 2\cos t + \cos^2 t\right)dt$

$= a^2\int_0^{2\pi}\left(1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos 2t\right)dt = a^2\left[\frac{3}{2}t - 2\sin t + \frac{\sin 2t}{4}\right]_0^{2\pi} = 3\pi a^2$; $\tilde{x} = x = a(t - \sin t)$ and

$\tilde{y} = \frac{1}{2}y = \frac{1}{2}a(1-\cos t) \Rightarrow M_x = \int \tilde{y}\,dm = \int \tilde{y}\,\delta\,dA = \int_0^{2\pi}\frac{1}{2}a(1-\cos t)\,a^2(1-\cos t)^2\,dt = \frac{1}{2}a^3\int_0^{2\pi}(1-\cos t)^3\,dt$

$= \frac{a^3}{2}\int_0^{2\pi}\left(1 - 3\cos t + 3\cos^2 t - \cos^3 t\right)dt = \frac{a^3}{2}\int_0^{2\pi}\left[1 - 3\cos t + \frac{3}{2} + \frac{3\cos 2t}{2} - \left(1 - \sin^2 t\right)(\cos t)\right]dt$

$= \frac{a^3}{2}\left[\frac{5}{2}t - 3\sin t + \frac{3\sin 2t}{4} - \sin t + \frac{\sin^3 t}{3}\right]_0^{2\pi} = \frac{5\pi a^3}{2}$. Therefore $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{5\pi a^3}{2}\right)}{3\pi a^2} = \frac{5}{6}a.$

Also, $M_y = \int \tilde{x}\,dm = \int \tilde{x}\delta\,dA = \int_0^{2\pi} a(t-\sin t)a^2(1-\cos t)^2\,dt$

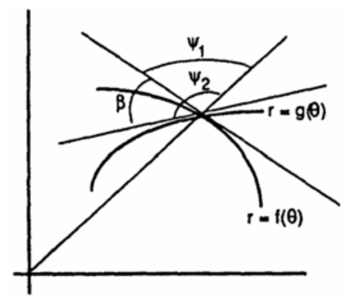
$= a^3\int_o^{2\pi}\left(t - 2t\cos t + t\cos^2 t - \sin t + 2\sin t\cos t - \sin t\cos^2 t\right)dt$

$= a^3\left[\frac{t^2}{2} - 2\cos t - 2t\sin t + \frac{1}{4}t^2 + \frac{1}{8}\cos 2t + \frac{t}{4}\sin 2t + \cos t + \sin^2 t + \frac{\cos^3 t}{3}\right]_0^{2\pi} = 3\pi^2 a^3.$

Thus $\bar{x} = \frac{M_y}{M} = \frac{3\pi^2 a^3}{3\pi a^2} = \pi a \Rightarrow \left(\pi a, \frac{5}{6}a\right)$ is the center of mass.

25. $\beta = \psi_2 - \psi_1 \Rightarrow \tan\beta = \tan(\psi_2 - \psi_1) = \frac{\tan\psi_2 - \tan\psi_1}{1+\tan\psi_2\tan\psi_1}$; the curves will be orthogonal when $\tan\beta$ is undefined, or when $\tan\psi_2 = \frac{-1}{\tan\psi_1} \Rightarrow \frac{r}{g'(\theta)} = \frac{-1}{\left[\frac{r}{f'(\theta)}\right]}$

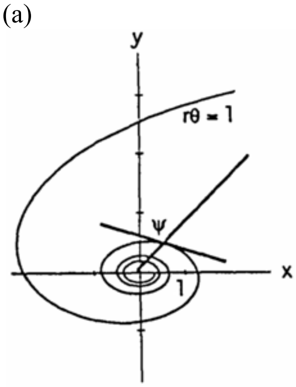
$\Rightarrow r^2 = -f'(\theta)g'(\theta)$



26. $r = \sin^4\left(\frac{\theta}{4}\right) \Rightarrow \frac{dr}{d\theta} = \sin^3\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right) \Rightarrow \tan\psi = \frac{\sin^4\left(\frac{\theta}{4}\right)}{\sin^3\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right)} = \tan\left(\frac{\theta}{4}\right)$

27. $r = 2a\sin 3\theta \Rightarrow \frac{dr}{d\theta} = 6a\cos 3\theta \Rightarrow \tan\psi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{2a\sin 3\theta}{6a\cos 3\theta} = \frac{1}{3}\tan 3\theta$; when $\theta = \frac{\pi}{6}$, $\tan\psi = \frac{1}{3}\tan\frac{\pi}{2} \Rightarrow \psi = \frac{\pi}{2}$

28. (a)



(b) $r\theta = 1 \Rightarrow r = \theta^{-1} \Rightarrow \frac{dr}{d\theta} = -\theta^{-2}$

$\Rightarrow \tan\psi\,|_{\theta=1} = \frac{\theta^{-1}}{-\theta^{-2}} = -\theta \Rightarrow \lim_{\theta\to\infty}\tan\psi = -\infty$

$\Rightarrow \psi \to \frac{\pi}{2}$ from the right as the spiral winds in around the origin.

29. $\tan\psi_1 = \frac{\sqrt{3}\cos\theta}{-\sqrt{3}\sin\theta} = -\cot\theta$ is $-\frac{1}{\sqrt{3}}$ at $\theta = \frac{\pi}{3}$; $\tan\psi_2 = \frac{\sin\theta}{\cos\theta} = \tan\theta$ is $\sqrt{3}$ at $\theta = \frac{\pi}{3}$; since the product of these slopes is -1, the tangents are perpendicular

30. $\tan\psi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{a(1-\cos\theta)}{a\sin\theta}$ is 1 at $\theta = \frac{\pi}{2} \Rightarrow \psi = \frac{\pi}{4}$