

STA2001 Probability and Statistics (I)

Lecture 14

Tianshi Chen

The Chinese University of Hong Kong, Shenzhen

Chapter 4. Bivariate Distribution

Section 4.1 Bivariate Distribution of Discrete Type

Motivation

Very often, we are interested to study two random experiments jointly, each of whose outcome is a scalar, or a random experiment whose outcome is a pair of two scalars.

1. observe college students to obtain information such as height x and weight y .
2. observe high school students to obtain information such as rank x and score of college entrance examination y .

Motivation

Very often, we are interested to study two random experiments jointly, each of whose outcome is a scalar, or a random experiment whose outcome is a pair of two scalars.

1. observe college students to obtain information such as height x and weight y .
 2. observe high school students to obtain information such as rank x and score of college entrance examination y .
- ▶ a random experiment whose outcome is a scalar,
→ univariate RV
 - ▶ two random experiments jointly each of whose outcome is a scalar, or a random experiment whose outcome is a pair of two scalars, → bivariate RV

Bivariate RV

Definition

Let (X, Y) be a pair of RVs with their range denoted by $\overline{S} \subseteq R^2$. Then (X, Y) or X and Y is said to be a bivariate RV. If \overline{S} is finite or countably infinite, then (X, Y) is said to be a discrete bivariate RV.

Moreover, let $\overline{S}_X \subseteq R$ and $\overline{S}_Y \subseteq R$ denote the range of X and Y , respectively.

$$\overline{S} = \{\text{all possible values of } (X, Y)\}$$

$$\overline{S}_X = \{\text{all possible values of } X\} = \{x | (x, y) \in \overline{S}\}$$

$$\overline{S}_Y = \{\text{all possible values of } Y\} = \{y | (x, y) \in \overline{S}\}$$

Then, it holds that

$$\overline{S} \subseteq \overline{S}_X \times \overline{S}_Y = \{(x, y) | x \in \overline{S}_X, y \in \overline{S}_Y\}$$

Example 1, Page 134

Roll a pair of 4-sided fair dice. Then the original sample space

$$S = \left\{ \begin{array}{cccc} (1, 1), & (1, 2), & (1, 3), & (1, 4), \\ (2, 1), & (2, 2), & (2, 3), & (2, 4), \\ (3, 1), & (3, 2), & (3, 3), & (3, 4), \\ (4, 1), & (4, 2), & (4, 3), & (4, 4) \end{array} \right\},$$

where the two numbers in each pair represent the outcome of the first die and the second die, respectively.

Example 1, Page 134

Roll a pair of 4-sided fair dice. Then the original sample space

$$S = \left\{ \begin{array}{cccc} (1, 1), & (1, 2), & (1, 3), & (1, 4), \\ (2, 1), & (2, 2), & (2, 3), & (2, 4), \\ (3, 1), & (3, 2), & (3, 3), & (3, 4), \\ (4, 1), & (4, 2), & (4, 3), & (4, 4) \end{array} \right\},$$

where the two numbers in each pair represent the outcome of the first die and the second die, respectively.

Now let X denote the smaller and Y the larger outcome of the pair of dice, e.g., if the outcome is $(3, 2)$ or $(2, 3)$, then $X = 2$, $Y = 3$.

Example 1, Page 134

Roll a pair of 4-sided fair dice. Then the original sample space

$$S = \left\{ \begin{array}{cccc} (1, 1), & (1, 2), & (1, 3), & (1, 4), \\ (2, 1), & (2, 2), & (2, 3), & (2, 4), \\ (3, 1), & (3, 2), & (3, 3), & (3, 4), \\ (4, 1), & (4, 2), & (4, 3), & (4, 4) \end{array} \right\},$$

where the two numbers in each pair represent the outcome of the first die and the second die, respectively.

Now let X denote the smaller and Y the larger outcome of the pair of dice, e.g., if the outcome is $(3, 2)$ or $(2, 3)$, then $X = 2$, $Y = 3$.

Sample space $\bar{S} \subseteq \bar{S}_X \times \bar{S}_Y$:

$$\bar{S}_X = \bar{S}_Y = \{1, 2, 3, 4\}, \bar{S} = \left\{ \begin{array}{cccc} (1, 1), & (1, 2), & (1, 3), & (1, 4), \\ & (2, 2), & (2, 3), & (2, 4), \\ & & (3, 3), & (3, 4), \\ & & & (4, 4) \end{array} \right\},$$

where the two numbers in each pair represent the possible values of X and Y , respectively.

Joint pmf

Definition

The function $f(x, y) : \bar{S} \rightarrow (0, 1]$ is called the joint probability mass function (joint pmf) of X and Y or (X, Y) , if

1. $f(x, y) > 0$ for $(x, y) \in \bar{S}$,
2. $\sum_{(x,y) \in \bar{S}} f(x, y) = 1$,
3. For $A \subseteq \bar{S}$,

$$P[(X, Y) \in A] \triangleq P(\{(X, Y) \in A\}) = \sum_{(x,y) \in A} f(x, y)$$

which defines the probability function for a set A . In particular, taking $A = \{(x, y)\}$ yields the probability of $X = x$ and $Y = y$, i.e.,

$$P(X = x, Y = y) = f(x, y)$$

Example 1 [Continued]

Question:

$$P(X = 2, Y = 3) = ?, \quad P(X = 2, Y = 2) = ?$$

Example 1 [Continued]

Question:

$$P(X = 2, Y = 3) = ?, \quad P(X = 2, Y = 2) = ?$$

$$P(X = 2, Y = 3) = \frac{1}{16} + \frac{1}{16} = \frac{2}{16}$$

$$P(X = 2, Y = 2) = \frac{1}{16}$$

Question: What is the joint pmf $f(x, y)$?

Example 1 [Continued]

Question:

$$P(X = 2, Y = 3) = ?, \quad P(X = 2, Y = 2) = ?$$

$$P(X = 2, Y = 3) = \frac{1}{16} + \frac{1}{16} = \frac{2}{16}$$

$$P(X = 2, Y = 2) = \frac{1}{16}$$

Question: What is the joint pmf $f(x, y)$?

$$\bar{S} = \left\{ \begin{array}{cccc} (1, 1) & (1, 2) & (1, 3) & (1, 4) \\ & (2, 2) & (2, 3) & (2, 4) \\ & & (3, 3) & (3, 4) \\ & & & (4, 4) \end{array} \right\}$$

$$f(x, y) = \begin{cases} \frac{2}{16}, & 1 \leq x < y \leq 4 \\ \frac{1}{16}, & 1 \leq x = y \leq 4. \end{cases}$$

A Remark on The Computation of The Probability

$$\text{For } A \subseteq \overline{S}, P[(X, Y) \in A] \triangleq P(\{(X, Y) \in A\}) = \sum_{(x,y) \in A} f(x, y),$$

where the double summation can be split into 2 single summation.
Let

$$A_X = \{x | (x, y) \in A\}, A_Y(x) = \{y | (x, y) \in A\}, \text{ for } x \in A_X$$

Then

$$P((X, Y) \in A) = \sum_{x \in A_X} \sum_{y \in A_Y(x)} f(x, y)$$

Let

$$A_Y = \{y | (x, y) \in A\}, A_X(y) = \{x | (x, y) \in A\}, \text{ for } y \in A_Y$$

Then

$$P((X, Y) \in A) = \sum_{y \in A_Y} \sum_{x \in A_X(y)} f(x, y)$$

Marginal pmf

Definition

Let (X, Y) be a bivariate RV or X and Y be two RVs and have the joint pmf $f(x, y) : \bar{S} \rightarrow (0, 1]$. Sometimes, we are interested in the pmf of X or Y alone, which is called the marginal pmf of X or Y and described by

For $x \in \bar{S}_X$,

$$\begin{aligned} f_X(x) &= P_X(X = x) \triangleq P(\{X = x, Y \in \bar{S}_Y(x)\}) \\ &= \sum_{y \in \bar{S}_Y(x)} f(x, y) \end{aligned}$$

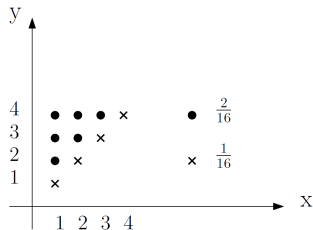
where

$$\bar{S}_Y(x) = \{y | (x, y) \in \bar{S}\} \text{ for the given } x \in \bar{S}_X.$$

Example 1 [Continued]

$$f(x, y) = \begin{cases} \frac{2}{16}, & 1 \leq x < y \leq 4 \\ \frac{1}{16}, & 1 \leq x = y \leq 4. \end{cases}$$

What is $f_X(x)$, $f_Y(y)$?



Example 1 [Continued]

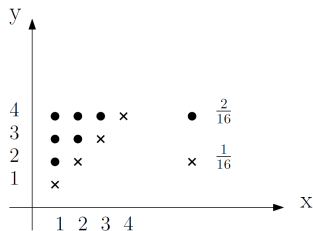
$$f(x, y) = \begin{cases} \frac{2}{16}, & 1 \leq x < y \leq 4 \\ \frac{1}{16}, & 1 \leq x = y \leq 4. \end{cases}$$

What is $f_X(x)$, $f_Y(y)$?

First, $\overline{S_X} = \overline{S_Y} = \{1, 2, 3, 4\}$.

$$f_X(x) = \sum_{y \in \overline{S_Y}(x)} f(x, y), x \in \overline{S_X} = \{1, 2, 3, 4\}$$

$$\implies f_X(1) = \frac{7}{16}, \quad f_X(2) = \frac{5}{16}, \quad f_X(3) = \frac{3}{16}, \quad f_X(4) = \frac{1}{16}$$



Marginal pmf

Definition

Let (X, Y) be a bivariate RV or X and Y be two RVs and have the joint pmf $f(x, y) : \bar{S} \rightarrow (0, 1]$. Sometimes, we are interested in the pmf of X or Y alone, which is called the marginal pmf of X or Y and described by

For $y \in \bar{S}_Y$,

$$\begin{aligned} f_Y(y) &= P_Y(Y = y) \triangleq P(\{X \in \bar{S}_X(y), Y = y\}) \\ &= \sum_{x \in \bar{S}_X(y)} f(x, y) \end{aligned}$$

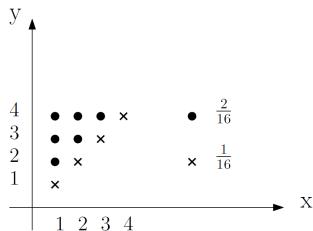
where

$$\bar{S}_X(y) = \{x | (x, y) \in \bar{S}\} \text{ for the given } y \in \bar{S}_Y.$$

Example 1 [Continued]

$$f(x, y) = \begin{cases} \frac{2}{16}, & 1 \leq x < y \leq 4 \\ \frac{1}{16}, & 1 \leq x = y \leq 4. \end{cases}$$

What is $f_X(x)$, $f_Y(y)$?



First, $\overline{S_X} = \overline{S_Y} = \{1, 2, 3, 4\}$.

$$f_X(x) = \sum_{y \in \overline{S_Y}(x)} f(x, y), x \in \overline{S_X} = \{1, 2, 3, 4\}$$

$$\Rightarrow f_X(1) = \frac{7}{16}, \quad f_X(2) = \frac{5}{16}, \quad f_X(3) = \frac{3}{16}, \quad f_X(4) = \frac{1}{16}$$

$$f_Y(y) = \sum_{x \in \overline{S_X}(y)} f(x, y), y \in \overline{S_Y} = \{1, 2, 3, 4\}$$

$$\Rightarrow f_Y(1) = \frac{1}{16}, \quad f_Y(2) = \frac{3}{16}, \quad f_Y(3) = \frac{5}{16}, \quad f_Y(4) = \frac{7}{16}$$

Remarks on Marginal pmf

It is crucial to understand the following definitions

$$\overline{S}, \overline{S_X}, \overline{S_Y}, \overline{S_X}(y), \overline{S_Y}(x)$$

$$\overline{S} = \{\text{all possible values of } (X, Y)\}$$

$$\overline{S_X} = \{\text{all possible values of } X\} = \{x | (x, y) \in \overline{S}\}$$

$$\overline{S_Y} = \{\text{all possible values of } Y\} = \{y | (x, y) \in \overline{S}\}$$

$$\overline{S_X}(y) = \{x | (x, y) \in \overline{S}\} \text{ for a given } y \in \overline{S_Y}$$

$$\overline{S_Y}(x) = \{y | (x, y) \in \overline{S}\} \text{ for a given } x \in \overline{S_X}$$

Trinomial Distribution

Description: The random experiment has three mutually exclusive and exhaustive outcomes:

- ▶ “perfect”,
- ▶ “second”
- ▶ “defective”

We repeat the experiment n independent times, and moreover, the probabilities

- ▶ p_X : the probability of “perfect”,
- ▶ p_Y : the probability of “second”
- ▶ p_Z : the probability of “defective”

remain the same for each repetition. Such n repetitions can be called a trinomial experiment.

For the trinomial experiment, we are interested in the number of perfects, the number of seconds and the number of defectives.

Trinomial Distribution

For the n trinomial trials, we let

- ▶ X be number of perfects,
- ▶ Y be number of seconds,
- ▶ $Z = n - X - Y$ be the number of defectives

We are interested in the joint pmf of (X, Y) , $f(x, y) : \bar{S} \rightarrow \mathbb{R}^2$

- ▶ $\bar{S} = \{(x, y) | x + y \leq n, x = 0, 1, \dots, n, y = 0, 1, \dots, n\}$
- ▶ $f(x, y) = P(X = x, Y = y)$ which is the probability of having x perfects, y seconds, and $n - x - y$ defectives

Trinomial Distribution

Joint pmf: to calculate $f(x, y) = P(X = x, Y = y)$,

- ▶ the probability for each way of having x perfects, y seconds, and $n - x - y$ defectives is

$$p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}$$

- ▶ the total number of ways of having x perfects, y seconds, and $n - x - y$ defectives is

$$\binom{n}{x, y, n-x-y} = \frac{n!}{x!y!(n-x-y)!}$$

Therefore, the joint pmf for trinomial distribution is

$$f(x, y) = \frac{n!}{x!y!(n-x-y)!} p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}, (x, y) \in \bar{S}$$

It's called trinomial distribution because of the trinomial expansion.

Trinomial Distribution

$$\begin{aligned}(a + b + c)^n &= \sum_{x=0}^n \binom{n}{x} a^x (b + c)^{n-x} \\&= \sum_{x=0}^n \binom{n}{x} a^x \sum_{y=0}^{n-x} \binom{n-x}{y} b^y c^{n-x-y} \\&= \sum_{x=0}^n \sum_{y=0}^{n-x} \frac{n!}{x!y!(n-x-y)!} a^x b^y c^{n-x-y}\end{aligned}$$

Marginal pmf: to calculate $f_X(x)$ or $f_Y(y)$

Trinomial Distribution

$$\begin{aligned}(a + b + c)^n &= \sum_{x=0}^n \binom{n}{x} a^x (b + c)^{n-x} \\&= \sum_{x=0}^n \binom{n}{x} a^x \sum_{y=0}^{n-x} \binom{n-x}{y} b^y c^{n-x-y} \\&= \sum_{x=0}^n \sum_{y=0}^{n-x} \frac{n!}{x!y!(n-x-y)!} a^x b^y c^{n-x-y}\end{aligned}$$

Marginal pmf: to calculate $f_X(x)$ or $f_Y(y)$

$$\begin{aligned}f_X(x) &= \sum_{y \in \overline{S_Y}(x)} f(x, y) = \sum_{y=0}^{n-x} \binom{n}{x} \binom{n-x}{y} p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y} \\&= \binom{n}{x} p_X^x (1 - p_X)^{n-x}\end{aligned}$$

Without summing, we know $X \sim b(n, p_X)$ and $Y \sim b(n, p_Y)$

Independent Random Variables

Definition

The random variables X and Y are said to be independent if for every $x \in \overline{S_X}$ and $y \in \overline{S_Y}$

$$f(x, y) = f_X(x)f_Y(y)$$

or equivalently,

$$P(X = x, Y = y) = P_X(X = x)P_Y(Y = y).$$

X and Y are said to be dependent if otherwise.

When X and Y are independent,

$$\overline{S} = \overline{S_X} \times \overline{S_Y}, \quad \overline{S} \text{ is said to be rectangular}$$

which is a necessary condition for independence of X and Y .

Independent Random Variables

Definition

The random variables X and Y are said to be independent if for every $x \in \overline{S_X}$ and $y \in \overline{S_Y}$

$$P(X = x, Y = y) = P_X(X = x)P_Y(Y = y)$$

or equivalently,

$$f(x, y) = f_X(x)f_Y(y).$$

The definition of independent RVs has root in the definition of independent events.

$$A = \{X = x, Y \in \overline{S_Y}(x)\}, B = \{X \in \overline{S_X}(y), Y = y\}$$

X and Y are independent if and only if A and B are independent.

Example 2, Page 135

Let the joint pmf of X and Y be defined by

$$f(x, y) = \frac{x+y}{21}, \quad x = 1, 2, 3, \quad y = 1, 2.$$

$$\bar{S} = \{(x, y) | x = 1, 2, 3, \quad y = 1, 2.\}$$

$$f : \bar{S} \longrightarrow (0, 1] \text{ with } \overline{S_X} = \{1, 2, 3\}, \quad \overline{S_Y} = \{1, 2\}.$$

Question

Are X and Y independent or dependent?

Example 2, Page 135

$$f_X(x) = \sum_{y \in \overline{S_Y}(x)} f(x, y) = \sum_{y=1}^2 \frac{x+y}{21} = \frac{2x+3}{21}, \quad x = 1, 2, 3.$$

$$f_Y(y) = \sum_{x \in \overline{S_X}(y)} f(x, y) = \sum_{x=1}^3 \frac{x+y}{21} = \frac{3y+6}{21}, \quad y = 1, 2$$

$$f(x, y) = \frac{x+y}{21} \neq \frac{2x+3}{21} \cdot \frac{3y+6}{21} = f_X(x)f_Y(y)$$

$\Rightarrow X$ and Y are dependent

What is the implication of independent RVs?