1. (i) E

(ii) $5x + 5x^2 + \left(\frac{5}{2!} - \frac{125}{3!}\right)x^3$

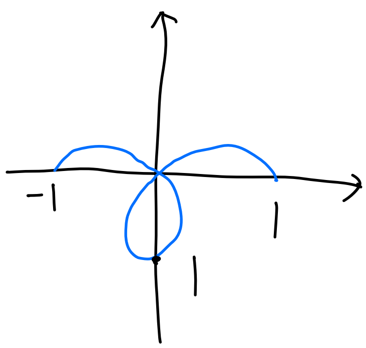
(or $5x + 5x^2 - \frac{55}{3}x^3$; or $x\left(5 + 5x - \frac{55}{3}x^2\right)$)

(iii) F

(iv) F

(v) $\sqrt{65}$

(vi)



$x = r\cos\theta$

$y = r\sin\theta$

$r = 2\cos\theta\sin\theta$

$r = 2\frac{x}{r}\frac{y}{r}$

$xy = 2r^3$

2. (i) No.

- $a_n := 0 \ \forall n$; Σa_n cvgs.
- $\sum_{n=1}^{\infty} \cos(a_n) = \sum_{n=1}^{\infty} 1 = \infty$, dvgs.

(ii) Yes.

- Since Σa_n cvgs, $a_n \to 0$ as $n \to \infty$.

  so $\exists N$ s.t. $0 \leq a_n < 1$ for all $n \geq N$.
- Then $0 \leq a_n^2 < a_n$ for all $n \geq N$.
- By direct comparison, Σa_n^2 cvgs.

(iii) No.

- $a_n := \frac{1}{n^2}$. Then $\Sigma a_n = \Sigma \frac{1}{n^2}$ cvgs.
- But $\Sigma \sqrt{a_n} = \Sigma \frac{1}{n}$ dvgs.

3. (i) · $\sin(\frac{1}{n}) > 0$, $\forall n = 1, 2, \ldots$

· $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to\infty} \frac{\sin 1/x}{1/x} \underset{y:=\frac{1}{x}}{=} \lim_{y\to 0^+} \frac{\sin y}{y} = 1.$

· Since $\Sigma \frac{1}{n}$ dvgs, $\Sigma \sin(\frac{1}{n})$ also dvgs

by limit comparison test.

(ii) · Since $\sqrt[n]{n} < \sqrt[n]{n+1} \leq \sqrt[n]{2n} = \sqrt[n]{2}\sqrt[n]{n}$

$\sqrt[n]{n} \to 1$, $\sqrt[n]{2}\sqrt[n]{n} \to 1 \cdot 1 = 1$

$\lim_{n\to\infty} \sqrt[n]{n+1} = 1$

· $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} \frac{1}{\sqrt[n]{n+1}} = 1 \cdot 1 = 1 \neq 0$

· Series dvgs (n^{th} term test)

(iii) · $\int_{48}^{\infty} \frac{1}{x \ln x} dx \overset{u=\ln x}{=\!=} \int_{\ln 48}^{\infty} \frac{1}{u} du$

$= \lim_{b\to\infty} \left(\ln(\ln b) - \ln(\ln 48) \right) = \infty$

$\Rightarrow \Sigma \frac{1}{n \ln n}$ dvgs

· Since $\frac{1}{n \ln n}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$,

$\Sigma \frac{(-1)^n}{n \ln n}$ cvgs (alternating series test)

$\Rightarrow \sum \frac{(-1)^n}{n \ln n}$ cvgs conditionally.

(iv) $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1\cdot 3\cdot \ldots \cdot (2n+1)}{4^{n+1} 2^{n+1} (n+1)!} \cdot \frac{4^n 2^n n!}{1\cdot 3\cdot \ldots \cdot (2n-1)}$

$= \lim_{n\to\infty} \frac{(2n+1)}{4\cdot 2\cdot (n+1)} = \frac{1}{4} < 1.$

By ratio test, series cvgs absolutely.

4.

Solution. Note

$$e^{2x^9} = 1 + 2x^9 + \frac{4x^{18}}{2!} + o(x^{18})$$

$$\sin x = x - \frac{x^3}{3!} + o(x^3)$$

$$\sqrt{1+x^{11}} = 1 + \frac{1}{2}x^{11} + o(x^{11})$$

Using big-Oh notation correctly is O.K.

we have

$$\lim_{x\to 0^+} \frac{e^{2x^9} - 1 - 2x^8 \sin x}{\sqrt{1+x^{11}} - 1} = \lim_{x\to 0^+} \frac{1 + 2x^9 - 1 - 2x^9 + \dfrac{x^{11}}{3} + o(x^{11})}{1 + \frac{1}{2}x^{11} - 1 + o(x^{11})}$$

$$= \cancel{\frac{1}{3}}\ \frac{2}{3}.$$

5. (i) $\binom{-\frac{1}{2}}{0} = 1, \ \binom{-\frac{1}{2}}{1} = -\frac{1}{2}, \ \binom{-\frac{1}{2}}{2} = \frac{-\frac{1}{2}(-\frac{3}{2})}{2!}, \ \dots,$

$$\binom{-\frac{1}{2}}{n} = \frac{(-1)^n \, 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n \cdot n!}$$

$$\frac{1}{\sqrt{1-t}} = (1-t)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n \, 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n \cdot n!} (-t)^n$$

$$= \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n \cdot n!} t^n .$$

$$\Rightarrow \frac{3x^2}{\sqrt{4-x^2}} = \frac{3}{2} \frac{x^2}{\sqrt{1-\frac{x^2}{4}}} = \frac{3}{2} x^2 \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n \cdot n!} \frac{x^{2n}}{4^n}$$

$$= \frac{3}{2} \sum_{n=0}^{\infty} \underbrace{\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^{3n} \cdot n!} x^{2n+2}}_{a_n} .$$

(ii)

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{2n+1}{8(n+1)} x^2 = \frac{x^2}{4} .$$

- By ratio test, series cvgs for $\frac{x^2}{4} < 1$.
- $\frac{x^2}{4} < 1 \iff |x| < 2$.
- Radius of convergence : $R = 2$.

6.

- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$;
- As an alternating series $\sum (-1)^k u_k$,

$$u_k = \frac{x^{2k}}{(2k)!}$$

- On $[-1, 1]$. $u_k \leq \frac{1}{(2k)!}$.
- $k = 3 \Rightarrow \frac{1}{(2k)!} = \frac{1}{720} < 0.006$,

  $k = 2 \Rightarrow \frac{1}{(2k)!} = \frac{1}{24} > 0.006$
- Take $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ $(k=2)$

deg $n=4$

7. (i)

~~*Proof.*~~ l_1 is in the direction of $\boldsymbol{v}_1 = \langle 0, 1, 1\rangle$ and l_2 is in the direction of $\boldsymbol{v}_2 = \langle 1, 2, 1\rangle$. Thus, the normal vector of the plane should be perpendicular to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, which is

$$\langle 0, 1, 1\rangle \times \langle 1, 2, 1\rangle = \langle -1, 1, -1\rangle.$$

It also needs to pass the origin, so the equation if $-x + y - z = 0$.

(ii) Since the plane are parallel to these two lines, the distance can be measured by using any point on the line. Choose $(1, -1, 2) \in l_1$ and $(-1, -2, 1) \in l_2$, we have

$$d_1 = \frac{|-1-1-2|}{\sqrt{3}} = \frac{4}{\sqrt{3}} \qquad d_2 = \frac{|1-2-1|}{\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

8.

Proof. Use the rule of cosine to compute the angles, we have

$$\begin{aligned}
\cos\theta_1 &= \frac{\langle 3,1,2,0,5\rangle \cdot \langle 1,0,0,6,5\rangle}{\sqrt{39}\sqrt{62}} = \frac{28}{\sqrt{39\times 62}} \\
\cos\theta_2 &= \frac{\langle 3,1,2,0,5\rangle \cdot \langle 2,2,6,0,3\rangle}{\sqrt{39}\sqrt{53}} = \frac{35}{\sqrt{39\times 53}} \\
\cos\theta_3 &= \frac{\langle 3,1,2,0,5\rangle \cdot \langle 2,1,1,0,6\rangle}{\sqrt{39}\sqrt{42}} = \frac{39}{\sqrt{39\times 42}}.
\end{aligned}$$

Easy to see, the numerator of $\cos\theta_3$ is the largest while it's denominator is the smallest, thus $\cos\theta_3$ is the largest value among the three cosines, thus the angle it the smallest. It is more likely that the app will recommend you the 3rd music $(2,1,1,0,6)$. $\square$

9.

Sol: (a)

$$L = \int_0^{2\pi} \sqrt{r^2 + |\frac{dr}{d\theta}|^2} d\theta = 2\int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta} d\theta = 4\int_0^{2\pi} |\cos\frac{\theta}{2}| d\theta = 8\int_0^{\pi} \cos\frac{\theta}{2} d\theta = 16.$$

(b)

$$S = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \cancel{6\pi}. \quad \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (1+2\cos\theta+\cos^2\theta) \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 6\pi$$

(c)The polar coordinate of the point $P(\frac{3}{2}, \frac{3\sqrt{3}}{2})$ is $(3, \frac{\pi}{3})$. The position vector is

$$\vec{r} = 2\langle \cos t + \cos^2 t, \sin t + \cos t \sin t \rangle,$$

The velocity vector is

$$\frac{d\vec{r}}{dt} = 2\langle -\sin t - \sin(2t), \cos t + \cos(2t) \rangle, \quad |\frac{d\vec{r}}{dt}| = 4|\cos(\frac{t}{2})|$$

At the point P,

$$\frac{d\vec{r}}{dt} = \langle -2\sqrt{3}, 0 \rangle, \quad \vec{r} = \langle \frac{3}{2}, \frac{3\sqrt{3}}{2} \rangle$$

then

$$\cos(\theta) = -1/2 \quad \text{and the angle} \quad \theta = 2\pi/3.$$

$x = r\cos\theta$

$\frac{x}{\cos\theta} = 2 + 2\cos\theta$

$x = 2\cos\theta + 2\cos^2\theta$

$y = r\sin\theta$

$\frac{y}{\sin\theta} = 2+$

10. Sol:

(a) $x = a + a\cos(t), y = at + a\sin(t), t \in \mathbb{R}$.

(b) P_2 corresponds to $t = \frac{\pi}{2}$.

$$\vec{r}'(t) = a < -\sin t, 1 + \cos t >$$

$$|\vec{r}'(t)| = a\sqrt{\sin^2 t + 1 + \cos^2 t + 2\cos t} = a\sqrt{2}\sqrt{1+\cos t}$$

$$= 2a\cos\frac{t}{2} \quad \left(t \in (0, \pi)\right)$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{< -2\sin\frac{t}{2}\cos\frac{t}{2}, 2\cos^2\frac{t}{2} >}{2\cos\frac{t}{2}}$$

$$= < -\sin\frac{t}{2}, \cos\frac{t}{2} >$$

$$\vec{T}'(t) = \frac{1}{2} < -\cos\frac{t}{2}, \sin\frac{t}{2} >$$

$$k = \frac{|\vec{T}'(\frac{\pi}{2})|}{|\vec{r}'(\frac{\pi}{2})|} = \frac{\sqrt{2}}{4a}.$$

11. (i) $\vec{a}(t) = \langle 0, -g \rangle$.

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du$$

$$= \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle + \langle 0, -gt \rangle$$

$$= \langle v_0 \cos\alpha, v_0 \sin\alpha - gt \rangle.$$

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\,du$$

$$= \langle x_0, y_0 \rangle + \langle v_0(\cos\alpha)t, v_0(\sin\alpha)t - \frac{1}{2}gt^2 \rangle$$

$\therefore$ $x(t) = x_0 + v_0(\cos\alpha)t$

$y(t) = y_0 + v_0(\sin\alpha)t - \frac{1}{2}gt^2$

(ii). Want $y(t_0) = 0$, $t_0 > 0$.

- $y(t) = 0 \iff t = \dfrac{-v_0(\sin\alpha) \pm \sqrt{v_0^2 \sin^2\alpha + 2gy_0}}{-g}$

- Since $t_0 > 0$, take "$-$" above.