MAT1002 Final Exam Reference Solution

1. T, F, F $\quad$ (e.g. $u_n = (-1)^{n+1}\frac{1}{n}$) $\quad$ 2 pt each; no partial marks.

6

2. (i) 2.

27

(ii) They are spheres of the form $x^2+y^2+z^2 = 4-\frac{1}{k^2}$, 2 pts

$k \geq \frac{1}{2}$. $\quad$ 1 pt

(iii) Along C, $\frac{dH}{dt}\Big|_{t=0} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} + f_z\frac{dz}{dt} + f_t\Big|_{t=0}$.

or at $(x(0), y(0), z(0), 0)$

(iv) 1.

(v) $\left< -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right>$ $\quad$ has to be normalized

(vi) $\frac{1}{\sqrt{2}}x^2 - \frac{1}{4\sqrt{2}}x^3 + \frac{3}{32\sqrt{2}}x^4$

(vii) $-\left<38, 6, 12\right>$ (which is $-\nabla H(3,4,5)$).

(Any vector of the form $\alpha\left<19,3,6\right>$ with $\alpha < 0$ is O.K.)

(viii) $$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{\cos\phi}^{2\cos\phi} \rho^3\cos\phi\sin\phi \, d\rho \, d\phi \, d\theta$$

can swap

No partial marks except (ii).

(ix) $\frac{2}{3}\pi$. (By symmetry, $\int_C x^2 ds = \int_C y^2 ds = \int_C z^2 ds$;

then $\int_C x^2 ds = \frac{1}{3}\int_C (x^2+y^2+z^2) ds = \frac{1}{3}\int_C ds = \frac{1}{3}2\pi(1)$.)

3. (i) Let $g(x,y,z) = xz^2 - yz + \cos(xy)$. Then

$\nabla g(x,y,z) = \langle z^2 - y\sin(xy), -z - x\sin(xy), 2xz - y \rangle$ 1

$\Rightarrow \nabla g(0,0,1) = \langle 1, -1, 0 \rangle$ 1

M is given by $1(x-0) - 1(y-0) + 0(z-1) = 0$ or

$$x - y = 0.$$ 2

l is given by

$$\left.\begin{aligned} &\langle x,y,z \rangle = \langle 0,0,1 \rangle + t\langle 1,-1,0 \rangle = \langle t, -t, 1 \rangle \\ &(\text{or } x = t,\ y = -t,\ z = 1,\ t \in \mathbb{R}). \end{aligned}\right\}$$ 2

(ii) Since $\langle 0,0,1 \rangle = \vec{r}(1)$, it suffices to show that $\vec{r}'(1) \perp \nabla g(0,0,1)$. reasoning 2

Since $\vec{r}'(t) = \langle \frac{1}{t}, 1 + \ln t, 1 \rangle$, $\vec{r}'(1) = \langle 1, 1, 1 \rangle$. Then

$\vec{r}'(1) \cdot \nabla g(0,0,1) = \langle 1,1,1 \rangle \cdot \langle 1,-1,0 \rangle = 0$, and we are done. 2

$\square$

4. No. We show that $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$ by showing that the limit does not exist. For each fixed $k \in \mathbb{R}$, let

$$L_k := \lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=kx}} f(x,y) = \lim_{x\to 0} \frac{kx^4}{kx^4+(k-1)^2x^2}.$$

· Consider two path test: ②

If $k \neq 1$, then

$$L_k = \lim_{x\to 0} \frac{kx^2}{kx^2+(k-1)^2} = 0.$$

· Get two different limits by correct computation: ③

If $k=1$, then

$$L_k = \lim_{x\to 0} \frac{x^4}{x^4} = 1.$$

By the two-path test, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

· Since $\lim_{(x,y)\to(0,0)}$ D.N.E, it is $\neq f(0,0)$,

so not continuous. ①

5. Let $g_1(x,y,z) := x^2+z^2-2$, $g_2(x,y,z) := x+y-1$.

Then $\nabla g_1 = \langle 2x, 0, 2z \rangle$ and $\nabla g_2 = \langle 1, 1, 0 \rangle$,

which are never parallel and never zero with the constraints.

Solve
$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 . \end{cases} \iff \begin{cases} 1 = \lambda_1 2x + \lambda_2 & ① \\ 1 = \lambda_2 & ② \\ 1 = \lambda_1 2z & ③ \\ x^2+z^2 = 2 & ④ \\ x+y = 1 & ⑤ \end{cases}$$

Have this idea : 2

$$\left.\begin{matrix} ② \Rightarrow \lambda_2 = 1 \\ ① \end{matrix}\right\} \Rightarrow \lambda_1 2x = 0.$$ Since $\lambda_1 \neq 0$ by ③, we

have $x=0$. Then

$$⑤ \Rightarrow y = 1 \quad \& \quad ④ \Rightarrow z = \pm\sqrt{2},$$

So $(x,y,z) = (0,1,\pm\sqrt{2})$. Correct sol : 2

$f(0,1,\sqrt{2}) = 1+\sqrt{2}$ is global maximum 1

and $f(0,1,-\sqrt{2}) = 1-\sqrt{2}$ is global minimum. 1

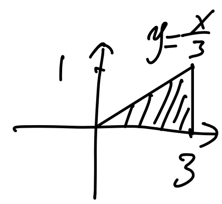
6. (i) $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$

This idea: 2

$$= \iint_D e^{x^2}\,dA \qquad D = \{(x,y) : 0 \le y \le 1,\ 3y \le x \le 3\}$$

$$= \{(x,y) : 0 \le x \le 3,\ 0 \le y \le \tfrac{x}{3}\}.$$

$$= \int_0^3 \int_0^{\frac{x}{3}} e^{x^2}\,dy\,dx \quad 2$$



$$= \frac{1}{3}\int_0^3 x e^{x^2}\,dx$$

$$= \frac{1}{3}\int_0^9 e^u \frac{1}{2}\,du$$

$$= \frac{1}{6}(e^9 - 1). \quad 1$$

(ii) Let $x = u + v^2$ and $y = v$. Then

Tried change-of-variable: 1

$$u = x - y^2 \quad \text{and} \quad v = y.$$

Since $u=0$ if and only if $x = y^2$, $u=1$ if and only if $x = y^2+1$, and

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -2y \\ 0 & 1 \end{vmatrix} = 1,$$

Remember Jacobian: 1

$$I = \int_0^1\int_0^1 v(u+v^2)^4\,du\,dv = \int_0^1 \int_{y^2}^{y^2+1} y x^4\,dx\,dy \quad 2$$

$$= \frac{1}{5}\int_0^1 y\left[(y^2+1)^5 - y^{10}\right]dy \overset{w=y^2}{=\!=} \frac{1}{10}\int_0^1 \left((w+1)^5 - w^5\right)dw$$

$$= \frac{1}{10}\int_0^1 (5w^4 + 10w^3 + 10w^2 + 5w + 1)\,dw$$

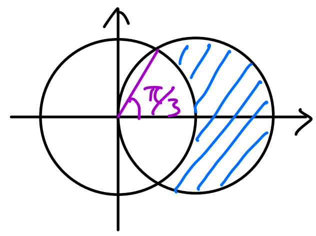
$$= \frac{1}{10}\left(1 + \frac{10}{4} + \frac{10}{3} + \frac{5}{2} + 1\right) = \frac{1}{10}\left(\frac{12+30+40+30+12}{12}\right) = \frac{31}{30}. \quad 1$$

7. Let D be the region. In polar coordinates.

$$x^2+y^2=1 \iff r=1,$$

$$(x-1)^2+y^2=1 \iff x^2+y^2-2x=0$$
$$\iff r^2-2r\cos\theta=0$$
$$\iff r=2\cos\theta \quad \text{or} \quad r=0 \ \text{(origin)}.$$

$$2\cos\theta=1 \iff \cos\theta=\frac{1}{2}$$
$$\iff \theta=\pm\frac{\pi}{3}.$$



Explained (see above): ① pt each

Now

$$A(D)=\iint_D dA=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_1^{2\cos\theta} r\,dr\,d\theta$$

①

$$=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(4\cos^2\theta-1)\,d\theta$$

$$=\int_0^{\frac{\pi}{3}}(4\cos^2\theta-1)\,d\theta$$

$$=\int_0^{\frac{\pi}{3}}\left[2(1+\cos 2\theta)-1\right]d\theta$$

$$=\frac{\pi}{3}+\int_0^{\frac{\pi}{3}}2\cos(2\theta)\,d\theta$$

$$=\frac{\pi}{3}+\left(\sin(2\theta)\right)\Big|_{\theta=0}^{\frac{\pi}{3}}$$

$$=\frac{\pi}{3}+\frac{\sqrt{3}}{2}.$$

②

8. Consider $z = \pm\sqrt{1-(x^2+y^2)^2}$, with domain

$$D = \{(x,y): x^2+y^2 \le 1\}.$$

1

$$V(E) = \iiint_E dV = \iint_D \int_{-\sqrt{1-(x^2+y^2)^2}}^{\sqrt{1-(x^2+y^2)^2}} dz\, dA$$

Cylindrical

$$= \int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^4}}^{\sqrt{1-r^4}} r\, dz\, dr\, d\theta \quad 2$$

$$= 2\pi \int_0^1 2r\sqrt{1-r^4}\, dr$$

$u = r^2$

$$= 2\pi \int_0^1 \sqrt{1-u^2}\, du$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \pi \int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$$

$$= \pi \left(\frac{\pi}{2} + \frac{1}{2}\sin(2\theta)\Big|_{\theta=\frac{\pi}{2}}^{0}\right)$$

$$= \frac{\pi^2}{2}. \quad 2$$

9. (i) Since $\vec{F}$ is conservative on D, by component test,

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow \frac{k}{y}e^{kx} = \frac{e^{kx}}{y} \Rightarrow k=1$$

· Mention the test: ①

· Remember the form of the test: ②

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \Rightarrow m\cos z = \cos z \Rightarrow m=1 \quad ②$$

(ii) By (i), $\vec{F} = \langle e^x \ln y, \frac{e^x}{y} + \sin z, y\cos z \rangle$.

Since $\vec{F}$ is conservative, $\vec{F} = \nabla f$ for some f.

· $f_x = e^x \ln y \Rightarrow f = \int f_x \, dx = e^x \ln y + K(y,z)$. ①

· By ①, $f_y = \frac{e^x}{y} + \frac{\partial K}{\partial y}(y,z)$. ①

· But $f_y = N = \frac{e^x}{y} + \sin z$, so $\frac{\partial K}{\partial y}(y,z) = \sin z$ and

$$K(y,z) = y\sin z + B(z).$$

· Now ① $\Rightarrow f = e^x \ln y + y\sin z + B(z)$

$$\Rightarrow f_z = y\cos z + B'(z).$$

Some intermediate steps: ①

· But $f_z = P = y\cos z$, So $B'(z) = 0 \Rightarrow B(z) \equiv C$.

One possible potential: $f(x,y,z) = e^x \ln y + y\sin z$. ③

Could have $+C$.

10. (i) By Green's theorem, 1

$$\oint_C x\,dy \underset{1}{=} \oint_C 0\,dx + x\,dy \underset{1}{=} \iint_R \left(\frac{\partial x}{\partial x} + \frac{\partial 0}{\partial y}\right) dA \underset{1}{=} \iint_R dA \underset{1}{=} A(R).$$

(ii) $x = \sqrt{\sin t}\,\cos t$, $y = \sin^{\frac{3}{2}} t$, 2 $0 \le t \le \pi$. 1

(iii)

$$A(R) = \oint_C x\,dy = \int_0^{\pi} \sin^{\frac{1}{2}} t \cos t \,\tfrac{3}{2} \sin^{\frac{1}{2}} t \cos t \, dt$$ 2

$$= \frac{3}{2}\int_0^{\pi} \sin t \cos^2 t \, dt$$

$$= \frac{3}{2}\int_{-1}^{1} (u)^2 \, du$$

$$= \frac{3}{2}\int_{-1}^{1} u^2 \, du$$

$$= 3\int_0^1 u^2 \, du$$

$$= 1.$$ 3

11. The surface S has boundary being the circle

$$x=2,\quad y=2\cos t,\quad z=2\sin t,\quad 0\le t\le 2\pi,$$ ②

which is counterclockwise with respect to the given direction of $\vec{n}$.

By Stokes' theorem, ②

$(dx=0)$

$$\iint_S \operatorname{curl}(\vec{F})\cdot\vec{n}\,d\sigma = \oint_C \vec{F}\cdot d\vec{r} = \oint_C N\,dy + P\,dz$$

$$= \int_0^{2\pi} \left[8\cos t\,(-2)\sin t\,dt + 16\sin^2 t\cdot 2\cos t\right]dt$$

$$= \left(-8\sin^2 t + \frac{32}{3}\sin^3 t\right)\Big|_{t=0}^{2\pi} = 0.$$ ②

(Alternatively, at plane $x=2$, $N=4y$, $P=4z^2$,

$$\oint_C N\,dy + P\,dz \overset{\text{Green's}}{=} \iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)dA = \iint_R 0\,dA = 0.)$$

12. By the divergence theorem (or Gauss' theorem):

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E z^3\,dV$$

①

get div($\vec{F}$): ①

In spherical coordinates:

$4\rho\neq 0$

$$z=\sqrt{\frac{x^2+y^2}{3}} \iff \rho\cos\phi = \frac{\rho\sin\phi}{\sqrt{3}} \iff \tan\phi=\sqrt{3}$$

$$\iff \phi=\frac{\pi}{3}$$

$$I=\iiint_E z^3\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^3\cos^3\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=2\pi\left(\int_0^{\frac{\pi}{3}}\cos^3\phi\sin\phi\,d\phi\right)\left(\int_0^1\rho^5\,d\rho\right)$$

$$=2\pi\left(\int_{\frac{1}{2}}^1 u^3\,du\right)\frac{1}{6}$$

$$=\frac{1}{3}\pi\,\frac{1}{4}\left(1-\frac{1}{16}\right)=\frac{5\pi}{64}$$

②

Get the right form of the iterated integral: ②