STA2001 Probability and statistical Inference I

Assignment 2
Deadline: 23:59 pm, Jan 26th

1. Prove the properties of the conditional probability.

   (a) $P(A \mid B) \geq 0$ and $P(A \mid B) \leq 1$.

   (b) $P(B \mid B) = 1$.

   (c) for mutually exclusive events $A_1, A_2, A_3, \cdots, A_k$, then $P\left(A_1 \cup A_2 \cup \cdots \cup A_k \mid B\right) = P\left(A_1 \mid B\right) + P\left(A_2 \mid B\right) + \cdots + P\left(A_k \mid B\right)$.

   (d) $P\left(A' \mid B\right) = 1 - P(A \mid B)$.

   (e) $P(A \cup C \mid B) = P(A \mid B) + P(C \mid B) - P(A \cap C \mid B)$.

   Solution:

   (a) $P(A \mid B) = P(A \cap B)/P(B)$ Since $P(A \cap B) \geq 0, P(A \cap B) \leq P(B)$ thus $0 \leq P(A \cap B)/P(B) \leq 1$ thus $P(A \mid B) \geq 0$ and $P(A \mid B) \leq 1$.

   (b) $P(B \mid B) = P(B \cap B)/P(B) = P(B)/P(B) = 1$.

   (c)
$$\begin{aligned} P\left(A_1 \cup A_2 \cup \cdots \cup A_k \mid B\right) &= P\left[\left(A_1 \cup A_2 \cup \cdots \cup A_K\right) \cap B\right]/P(B) \\ &= P\left[\left(A_1 \cap B\right) \cup \left(A_2 \cap B\right) \cup \cdots \cup \left(A_K \cap B\right)\right]/P(B) \end{aligned}$$

   Since $A_1, A_2, \cdots A_k$ are mutually exclusive, $\left(A_1 \cap B\right), \left(A_2 \cap B\right), \cdots \left(A_K \cap B\right)$ are also mutually exclusive, thus $P\left(A_1 \cup A_2 \cup \cdots \cup A_K \mid B\right) = \left[P\left(A_1 \cap B\right) + P\left(A_2 \cap B\right) + \cdots\right.$ $\left. p\left(A_K \mid B\right)\right]/P(B) = P\left(A_1 \mid B\right) + P\left(A_2 \mid B\right) + \cdots + P\left(A_K \mid B\right)$.

   (d)
$$\begin{aligned} P\left(A' \mid B\right) &= P((S-A) \mid B) \mid P(B) \\ &= P((S-A) \cap B)/P(B) \\ &= [P(B) - P(A \cap B)]/P(B) \\ &= 1 - P(A \cap B)/P(B) \end{aligned}$$

   (e)
$$\begin{aligned} P[(A \cup C) \cap B]/P(B) &= P[(A \cap B) \cup (C \cap B)]/P(B) \\ &= [P(A \cap B) + P(C \cap B) - P(A \cap B \cap C)]/P(B) \\ &= P(A \mid B) + P(C \mid B) - P(A \cap C \mid B). \end{aligned}$$

2. (1.3-8) An urn contains 17 balls marked LOSE and 3 balls marked WIN. You and an opponent take turns selecting a single ball at random from the urn without replacement.The person who selects the third WIN ball wins the game. It does not matter who selected the first two WIN balls.

   (a) If you draw first, find the probability that you win the game on your second draw.
   (b) If you draw first, find the probability that your opponent wins the game on his second draw.
   (c) If you draw first, what is the probability that you win? Hint: You could win on your second, third, fourth, $\cdots$, or tenth draw, but not on your first.
   (d) Would you prefer to draw first or second? Why?

   Solution:

   (a)
   $$\frac{3}{20} * \frac{2}{19} * \frac{1}{18} = \frac{1}{1140}.$$

   (b)
   $$\boxdot\ \boxdot\ \boxdot | \boxdot$$

   we should ensure there are 2 win bolls and one lose in the first three draw and then we need to multiply by the probability of getting win ball in fourth draw.

   $$\frac{\binom{3}{2} * \binom{17}{1}}{\binom{20}{3}} * \frac{1}{17} = \frac{1}{380}$$

   (c) the probability that you win in the second draw:

   $$\boxdot\ \boxdot\ |\boxdot$$

   $$\frac{\binom{3}{2}\binom{17}{0}}{\binom{2}{2}} * \frac{1}{18}$$

   the probability that you win in the third draw:

   $$\boxdot\ \boxdot\ \boxdot\ \boxdot\ |\boxdot$$

   $$\frac{\binom{3}{2} * \binom{17}{2}}{\binom{20}{4}} * \frac{1}{16}$$

the probability that you win in the fourth draw:

$$\boxdot\,\boxdot\,\boxdot\,\boxdot\,\boxdot|\boxdot$$

$$\frac{\binom{3}{2} * \binom{17}{4}}{\binom{20}{6}} * \frac{1}{14}$$

From the examples you could infer the answer should be :

$$\sum_{k=1}^{9} \frac{\binom{3}{2} * \binom{17}{2k-2}}{\binom{20}{2k}} * \frac{1}{20-2k} = \frac{35}{76} = 0.4605$$

$*$ Here, k represents the times that you win. because of the rule that the one who draws the third win ball will win the game, so you could only have 9 opportunities to win. So k could be 1 to 9.

(d) Draw second. The probability of winning is $1 - 0.4605 = 0.8395$

3. (1.3-9) An urn contains four balls numbered 1 through 4. The balls are selected one at a time without replacement. A match occurs if the ball numbered m is the mth ball selected. Let the event A_i denote a match on the ith draw, i $= 1, 2, 3, 4$.

(a) Show that $P(A_i) = \frac{3!}{4!}$ for each i.

(b) Show that $P(A_i \cap A_j) = \frac{2!}{4!}$, $i \neq j$.

(c) Show that $P(A_i \cap A_j \cap A_k) = \frac{1!}{4!}, i \neq j, i \neq k, j \neq k$.

(d) Show that the probability of at least one match is $P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}$.

(e) Extend this exercise so that there are n balls in the urn. What is the limit of the probability of at least one match as n increases without bound?

Solution:

(a) For any $i \in 1, 2, 3, 4$,the position of ith ball is fixed and other three balls are flexible. so $P(A_i) = \frac{3P3}{4P4} = \frac{3!}{4!}$.

(b) For any $i \in 1, 2, 3, 4$,the position of ith and jth ball are fixed and other two balls are flexible. so $P(A_i \cap A_j) = \frac{2P2}{4P4} = \frac{2!}{4!}, i \neq j$.

(c) Since the position of three balls are fixed, the position of left one must be fixed. so $P(A_i \cap A_j \cap A_k) = \frac{1}{4P4} = \frac{1!}{4!}, i \neq j, i \neq k, j \neq k$.

(d)

$$
\begin{aligned}
&P\left(A_1 \cup A_2 \cup A_3 \cup A_4\right)=P\left[\left(A_1 \cup A_2 \cup A_3\right) \cup A_4\right] \\
&=P\left(A_1 \cup A_2 \cup A_3\right)+P\left(A_4\right)-P\left[\left(A_1 \cup A_2 \cup A_3\right) \cap A_4\right] \\
&=P\left(A_1\right)+P\left(A_2\right)+P\left(A_3\right)-P\left(A_1 \cap A_2\right)-P\left(A_1 \cap A_3\right)-P\left(A_2 \cap A_3\right)+P\left(A_1 \cap A_2 \cap\right. \\
&\left.A_3\right)+P\left(A_4\right)-P\left[\left(A_1 \cap A_4\right) \cup\left(A_2 \cap A_4\right) \cup\left(A_3 \cap A_4\right)\right] \\
&=P\left(A_1\right)+P\left(A_2\right)+P\left(A_3\right)+P\left(A_4\right)-P\left(A_1 \cap A_2\right)-P\left(A_1 \cap A_3\right)-P\left(A_2 \cap A_3\right)+P\left(A_1 \cap\right. \\
&\left.A_2 \cap A_3\right)-\left[P\left(A_1 \cap A_4\right)+P\left(A_2 \cap A_4\right)+P\left(A_3 \cap A_4\right)-P\left(A_1 \cap A_2 \cap A_4\right)-P\left(A_1 \cap A_3 \cap\right.\right. \\
&\left.\left.A_4\right)-P\left(A_2 \cap A_3 \cap A_4\right)+P\left(A_1 \cap A_2 \cap A_3 \cap A_4\right)\right] \\
&=P\left(A_1\right)+P\left(A_2\right)+P\left(A_3\right)+P\left(A_4\right)-P\left(A_1 \cap A_2\right)-P\left(A_1 \cap A_3\right)-P\left(A_2 \cap A_3\right)+P\left(A_1 \cap\right. \\
&\left.A_2 \cap A_3\right)-P\left(A_1 \cap A_4\right)-P\left(A_2 \cap A_4\right)-P\left(A_3 \cap A_4\right)+P\left(A_1 \cap A_2 \cap A_4\right)+P\left(A_1 \cap A_3 \cap\right. \\
&\left.A_4\right)+P\left(A_2 \cap A_3 \cap A_4\right)-P\left(A_1 \cap A_2 \cap A_3 \cap A_4\right) \\
&=4 * P\left(A_i\right)-6 * P\left(A_i \cap A_j\right)+4 * P\left(A_i \cap A_j \cap A_k\right)-P\left(A_i \cap A_j \cap A_k \cap A_h\right) \\
&=4 * \frac{3!}{4!}-6 * \frac{2!}{4!}+4 * \frac{1!}{4!}-\frac{1!}{4!}=1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}.
\end{aligned}
$$

(e) as $e^x \approx 1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}$ if x $=-1$ then we could get $e^x \approx 1-1+\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^n}{n!}$ We know that

$$
\lim_{n \rightarrow \infty} P\left(A_1 \cup A_2 \cup \ldots \cup A_n\right)=\lim_{n \rightarrow \infty}\left[1-\left(1-1+\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^{n+1}}{n!}\right)\right]=1-e^{-1}.
$$

4. (1.3-13) In the gambling game craps, a pair of dice is rolled and the outcome of the experiment is the sum of the points on the up sides of the six-sided dice. The bettor wins on the first roll if the sum is 7 or 11 . The bettor loses on the first roll if the sum is 2,3 , or 12 . If the sum is $4,5,6,8,9$, or 10 , that number is called the bettors point. Once the point is established, the rule is as follows: If the bettor rolls a 7 before the point, the bettor loses; but if the point is rolled before a 7 , the bettor wins.

(a) List the 36 outcomes in the sample space for the roll of a pair of dice. Assume that each of them has a probability of 1/36.

(b) Find the probability that the bettor wins on the first roll. That is, find the probability of rolling a 7 or 11, $P(7$ or $11)$.

(c) Given that 8 is the outcome on the first roll, find the probability that the bettor now rolls the point 8 before rolling a 7 and thus wins. Note that at this stage in the game the only outcomes of interest are 7 and 8 . Thus find $P(8 \mid 7$ or $8)$.

(d) The probability that a bettor rolls an 8 on the first roll and then wins is given by $P(8) P(8 \mid 7$ or $8)$. Show that this probability is $(5/36)(5/11)$.

(e) Show that the total probability that a bettor wins in the game of craps is 0.49293. Hint: Note that the bettor can win in one of several mutually exclusive ways: by rolling a 7 or an 11 on the first roll or by establishing one of the points $4,5,6,8,9$, or 10 on the first roll and then obtaining that point on successive rolls before a 7 comes up.

Solution:

(a)

$$\begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array}$$

(b)

$$\begin{array}{cccccc} 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$

$$P(7 \text{ or } 11) = \frac{8}{36} = \frac{2}{9}.$$

(c)

$$P(8|7 \text{ or } 8) = \frac{p(8 \cap 7 \text{ or } 8)}{p(7 \text{ or } 8)} = \frac{5}{11}.$$

(d) Because $P(8)$ and $P(8/7 \text{ or } 8)$ are independent, so

$$P(8) * P(8/7 \text{ or } 8) = \frac{5}{36} * \frac{5}{11}.$$

(e) There are two ways to get a bettor wins:
(1) rolling a 7 or an 11 on the first roll
(2) rolling a point on the first roll and then rolling a 7 on successive rolls.
For instance: if we roll a point 4 then we should keep rolling 4 before 7 on successive rolls. then we could win the game: $P(4) * P(4/7 \text{ or } 4)$. other points is the same like point 4

$$P(A \text{ bettor wins }) = \frac{2}{9} + 2 * \left(\frac{5}{36} * \frac{5}{11} + \frac{4}{36} * \frac{4}{10} + \frac{3}{36} * \frac{3}{9} \right) = 0.49293.$$

5. Consider two independent rolls of a fair six-sided die, and the following events:

$$\begin{aligned} A &= \{\text{1st roll is 1, 2, or 3}\}, \\ B &= \{\text{1st roll is 3, 4, or 5 }\}, \\ C &= \{\text{the sum of the two rolls is 9}\}. \end{aligned}$$

Prove or disprove the following statements.
(i) The events A, B, and C satisfy the equation

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

(ii) The events A, B and C are pairwise independent.
(iii) The events A, B and C are mutually independent.
(iv) The events A, B are dependent.

Solution:
We have

$$P(A \cap B) = \frac{1}{6} \neq \frac{1}{2} \times \frac{1}{2} = P(A)P(B)$$
$$P(A \cap C) = \frac{1}{36} \neq \frac{1}{2} \times \frac{4}{36} = P(A)P(C)$$
$$P(B \cap C) = \frac{1}{12} \neq \frac{1}{2} \times \frac{4}{36} = P(B)P(C)$$

On the other hand, we have

$$P(A \cap B \cap C) = \frac{1}{36} = \frac{1}{2} \times \frac{1}{2} \times \frac{4}{36} = P(A)P(B)P(C).$$

Hence, the events A,B and C are not mutually independent and they are not pairwise independent.

6. Consider 3 urns. Urn A contains 2 white and 4 red balls; urn B contains 8 white and 4 red balls; and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white, given that exactly 2 white balls were selected?

   Solution:

   The probability in the question should be

$$\begin{aligned} &P(\text{Ball from A white}|\text{2 white balls selected}) \\ &= \frac{P(\text{Ball from A white} \cap \text{2 white balls selected})}{P(\text{2 white balls selected})} \end{aligned}$$

   First, consider $P(\text{Ball from } A \text{ white} \cap \text{2 white balls selected})$. This means that the ball chosen from A must be white and either the ball from B or C is white and the other one is not. The probability of drawing a white ball from A is $\frac{2}{6}$. Likewise, the probability of drawing a white ball from B, C is $\frac{8}{12}$ and $\frac{1}{4}$ respectively. The probability of not drawing the white ball from B, C is $\frac{4}{12}$ and $\frac{3}{4}$ respectively. So,

$$P(\text{Ball from A white} \cap \text{2 white balls selected}) = \frac{2}{6}(\frac{8}{12} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{4}{12})$$

   Now consider $P(\text{2 white balls selected})$. There are 3 ways we can choose the two white balls: choose a white ball from A and B and a red ball from C, choose a white ball from B and C and a red ball from A and choose a white ball from A and C and a red ball from B. So the

$$\frac{2}{6}\cdot\frac{8}{12}\cdot\frac{3}{4}+\frac{4}{6}\cdot\frac{8}{12}\cdot\frac{1}{4}+\frac{2}{6}\cdot\frac{4}{12}\cdot\frac{1}{4}$$

Therefore the probability is $\frac{7}{11}$.

7. (1.4-18) An eight-team single-elimination tournament is set up as follows: For example, eight students (called $A - H$) set up a tournament among themselves. The top-listed student in each bracket calls heads or tails when his or her opponent flips a coin. If the call is correct, the student moves on to the next bracket.

   (a) How many coin flips are required to determine the tournament winner?

   (b) What is the probability that you can predict all of the winners?

   (c) In NCAA Division I basketball, after the "play-in" games, 64 teams participate in a single-elimination tournament to determine the national champion. Considering only the remaining 64 teams, how many games are required to determine the national champion?

   (d) Assume that for any given game, either team has an equal chance of winning. (That is probably not true.) On page 43 of the March 22, 1999, issue, Time claimed that the "mathematical odds of predicting all 63 NCAA games correctly is 1 in 75 million." Do you agree with this statement? If not, why not?

   Solution:

   (a) Assume that we have n competitors then $n-1$ coin flips are required to determine the tournament winner,so the answer is $(8-1=7)$ times.

   (b) Because n-1 coin flips are required to determine the tournament winner, each flips we have probability of 1/2 to predict correctly. so the probability that you can predict all of the winners should be $\left(\frac{1}{2}\right)$

   $$\left(\frac{1}{2}\right)^7$$

   (c) There are 64 teams participate in tournament , so $(64-1=63)$ flips should be arranged 63 times.

   (d) There are 63 games which we should arrange, and each of them, the probability we predict correctly should be 1/2, then the probability of predicting all 63 games correctly should be $\left(\frac{1}{2}\right)^{63} = 1/9223372036854775800$ which is less than 1/75000000.

8. (1.5-6) A life insurance company issues standard, preferred, and ultrapreferred policies. Of the companys policyholders of a certain age, 60% have standard policies and

a probability of 0.01 of dying in the next year, 30% have preferred policies and a probability of 0.008 of dying in the next year, and 10% have ultrapreferred policies and a probability of 0.007 of dying in the next year. A policyholder of that age dies in the next year.What are the conditional probabilities of the deceased having had a standard, a preferred, and an ultrapreferred policy?

Solution:

Let B be the event that the policyholder dies. Let A_1, A_2, A_3 be the events that the deceased is standard, preferred and ultra-preferred, respectively and they are mutually exclusive and exhuastive. Then

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(A_1) * P(B \mid A_1)}{P(A_1) * P(B \mid A_1) + P(A_2) * P(B \mid A_2) + P(A_3) * P(B \mid A_3)}$$

$$= \frac{0.6 * 0.01}{0.06 * 0.01 + 0.3 * 0.008 + 0.1 * 0.007} = \frac{60}{91} = 0.659$$

$$P(A_2 \mid B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{P(A_2) * P(B \mid A_2)}{P(A_1) * P(B \mid A_1) + P(A_2) * P(B \mid A_2) + P(A_3) * P(B \mid A_3)}$$

$$= \frac{24}{91} = 0.264$$

$$P(A_3 \mid B) = \frac{P(A_3 \cap B)}{P(B)} = \frac{P(A_3) * P(B \mid A_3)}{P(A_1) * P(B \mid A_1) + P(A_2) * P(B \mid A_2) + P(A_3) * P(B \mid A_3)}$$

$$= \frac{7}{91} = 0.077$$

9. (1.5-10) Suppose we want to investigate the percentage of abused children in a certain population. To do this, doctors examine some of these children taken at random from that population. However, doctors are not perfect: They sometimes classify an abused child (A^+)as one not abused (D^-)or they classify a nonabused child (A^-)as one that is abused (D^+). Suppose these error rates are $P(D^- \mid A^+) = 0.08$ and $P(D^+ \mid A^-) = 0.05$, respectively; thus, $P(D^+ \mid A^+) = 0.92$ and $P(D^- \mid A^-) = 0.95$ are the probabilities of the correct decisions. Let us pretend that only 2% of all children are abused; that is, $P(A^+) = 0.02$ and $P(A^-) = 0.98$.

   (a) Select a child at random. What is the probability that the doctor classifies this child as abused? That is, compute
   $$P\left(D^+\right) = P\left(A^+\right) P\left(D^+ \mid A^+\right) + P\left(A^-\right) P\left(D^+ \mid A^-\right).$$

   (b) Compute $P(A^- \mid D^+)$and $P(A^+ \mid D^+)$.

   (c) Compute $P(A^- \mid D^-)$and $P(A^+ \mid D^-)$.

   (d) Are the probabilities in (b) and (c) alarming? This happens because the error rates of 0.08 and 0.05 are high relative to the fraction 0.02 of abused children in the population.

Solution:

For following question solution : A^+indicates those children who are real abused A^- indicates those children who are not abused actually.D^+indicates those who are diagnose as abused D^-indicates those who are diagnose as nonabused.And they are mutually exclusive and exhaustive.

	D^+	D^-	Total
A^+	0.92	0.08	0.02
A^-	0.05	0.95	0.98
Total	0.0674	0.9326	1

(a)
$$P\left(D^+\right) = 0.02 * 0.92 + 0.98 * 0.05 = 0.0184 + 0.0490 = 0.0674$$

(b)
$$P\left(A^- \mid D^+\right) = \frac{P\left(A^-\right) * P\left(D^+ \mid A^-\right)}{P\left(D^+\right)} = \frac{0.0490}{0.0674} = 0.727$$
$$P\left(A^+ \mid D^+\right) = \frac{P\left(A^+\right) * P\left(D^+ \mid A^+\right)}{P\left(D^+\right)} = \frac{0.0184}{0.0674} = 0.273$$

(c)
$$P\left(A^- \mid D^-\right) = \frac{P\left(A^-\right) * P\left(D^- \mid A^-\right)}{P\left(D^-\right)} = \frac{0.98 * 0.95}{0.02 * 0.08 + 0.98 * 0.95} = \frac{9310}{9326} = 0.998$$
$$P\left(A^+ \mid D^-\right) = 1 - P\left(A^- \mid D^-\right) = 1 - 0.998 = 0.002$$

(d) Yes. Although the correct rate are high enough here, however the percent of real abused child is too small comparing with the error rate 0.08 and 0.05. Even if $P\left(D^+ |A^+|\right)$and $P\left(D^- \mid A^-\right)$is high $P\left(A^+ \mid D^+\right)$and $P\left(A^+ \mid D^-\right)$are still very low.

10. Die A has 4 red and 2 white faces, whereas die B has 1 red and 5 white faces. A fair coin is flipped only once at the beginning of the game. If it lands on heads, the game continues with die A; if it lands on tails, then die B is to be used.

    (a) Find the probability that the first throw of the die results in red.

    (b) If the first two throws result in red, what is the probability that die A is used?

    (c) If the first two throws result in red, what is the probability of red at the third throw?

Solution:

Let A denote the event that die A is used. Let R_i denote the event that the i-th roll results in red.

(a) We know that $P\left(R_1 \mid A\right)=4/6, P\left(R_1 \mid A^{'}\right)=1/6, P(A)=1/2$, hence

$$\begin{aligned} P\left(R_1\right) &= P\left(R_1 \mid A\right) P(A)+P\left(R_1 \mid A^{'}\right) P\left(A^{'}\right) \\ &= \frac{4}{6} \cdot \frac{1}{2}+\frac{1}{6} \cdot \frac{1}{2}=\frac{5}{12} \end{aligned}$$

(b) By Baye's Law we have

$$\begin{aligned} P\left(A \mid R_1 \cap R_2\right) &= \frac{P\left(R_1 \cap R_2 \mid A\right) P(A)}{P\left(R_1 \cap R_2\right)} \\ &= \frac{P\left(R_1 \cap R_2 \mid A\right) P(A)}{P\left(R_1 \cap R_2 \mid A\right) P(A)+P\left(R_1 \cap R_2 \mid A'\right) P\left(A'\right)} \end{aligned}$$

R_1 and R_2 are conditionally independent, conditioning on A, hence

$$P\left(R_1 \cap R_2 \mid A\right)=P\left(R_1 \mid A\right) P\left(R_2 \mid A\right)=\left(\frac{4}{6}\right)^2=\frac{4}{9}$$

and

$$P\left(R_1 \cap R_2 \mid A^{'}\right)=P\left(R_1 \mid A^{'}\right) P\left(R_2 \mid A^{'}\right)=\left(\frac{1}{6}\right)^2=\frac{1}{36}$$

Hence

$$P\left(A \mid R_1 \cap R_2\right)=\frac{(4/9) \cdot(1/2)}{(4/9) \cdot(1/2)+(1/36) \cdot(1/2)}=\frac{16}{17}$$

(c) We have

$$\begin{aligned} P\left(R_3 \mid R_1 R_2\right) &= \frac{P\left(R_1 R_2 R_3\right)}{P\left(R_1 R_2\right)} \\ &= \frac{P\left(R_1 R_2 R_3 \mid A\right) P(A)+P\left(R_1 R_2 R_3 \mid A'\right) P\left(A'\right)}{P\left(R_1 R_2 \mid A\right) P(A)+P\left(R_1 R_2 \mid A'\right) P\left(A'\right)} \end{aligned}$$

R_1, R_2, R_3 are conditionally mutually independent, conditioning on A, hence

$$\begin{aligned} & P\left(R_3 \mid R_1 R_2\right) \\ & =\frac{P\left(R_1 \mid A\right) P\left(R_2 \mid A\right) P\left(R_3 \mid A\right) P(A)+P\left(R_1 \mid A'\right) P\left(R_2 \mid A'\right) P\left(R_3 \mid A'\right) P\left(A'\right)}{P\left(R_1 \mid A\right) P\left(R_2 \mid A\right) P(A)+P\left(R_1 \mid A'\right) P\left(R_2 \mid A'\right) P\left(A'\right)} \\ & =\frac{(4/6)^3 \cdot(1/2)+(1/6)^3 \cdot(1/2)}{(4/6)^2 \cdot(1/2)+(1/6)^2 \cdot(1/2)} \\ & =\frac{65}{102} \end{aligned}$$

Alternative Answer:

By Law of total probability (conditional version) we have

$$P\left(R_3 \mid R_1R_2\right) = P\left(R_3 \mid R_1R_2A\right)P\left(A \mid R_1R_2\right) + P\left(R_3 \mid R_1R_2A^{'}\right)P\left(A^{'} \mid R_1R_2\right)$$

Condition on A, R_1, R_2, R_3 are independent, hence $P\left(R_3 \mid R_1R_2A\right) = P\left(R_3 \mid A\right)$. Similarly, $P\left(R_3 \mid R_1R_2A^{'}\right) = P\left(R_3 \mid A^{'}\right)$
It is computed in the next question that

$$P\left(A \mid R_1R_2\right) = \frac{16}{17}$$

Thus

$$\begin{aligned} P\left(R_3 \mid R_1R_2\right) &= P\left(R_3 \mid A\right)P\left(A \mid R_1R_2\right) + P\left(R_3 \mid A^{'}\right)P\left(A^{'} \mid R_1R_2\right) \\ &= \frac{4}{6} \cdot \frac{16}{17} + \frac{1}{6} \cdot \left(1 - \frac{16}{17}\right) = \frac{65}{102} \end{aligned}$$