STA2001 Probability and Statistics (I)

Lecture 3

Tianshi Chen

The Chinese University of Hong Kong, Shenzhen

Review

- For random experiments that satisfy

  Assumption 1: S contains m possible outcomes

  $$e_k, \quad k = 1, 2, \cdots, m, \quad i.e., \quad S = \{e_1, e_2, \cdots, e_m\}.$$

  Assumption 2: The m outcomes are "equally likely"

  $$P(\{e_k\}) = \frac{1}{m}, \quad k = 1, \cdots, m.$$

  $$P(A) = \frac{N(A)}{N(S)},$$

  where $N(X)$ is the number of outcomes in $X \subseteq S$.

Review

- Revisit the method of enumeration by multiplication principle
  - permutation
  - combination
  - distinguishable permutation

Section 1.3 Conditional Probability

Motivation Example

Consider a number of tulip bulbs

	Early(E)	Late(L)	Totals
Red(R)	5	8	13
Yellow(Y)	3	4	7
Totals	8	12	20

Experiment 1: Select one bulb randomly.

- Sample space $S = \{\text{all bulbs}\}$.
- Assumption: all bulbs are "equally likely".

Consider the event $R = \{\text{the selected bulb is red}\}$, what is $P(R)$?

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$$P(R) = \frac{N(R)}{N(S)} = \frac{13}{20}$$

Motivation Example

Consider a number of tulip bulbs

	Early(E)	Late(L)	Totals
Red(R)	5	8	13
Yellow(Y)	3	4	7
Totals	8	12	20

Experiment 2: Select one bulb from the ones that bloom early.

- Sample space reduces to $E = \{\text{all bulbs that bloom early}\}$.
- Assumption: all bulbs are "equally likely".

Consider the event $R = \{\text{the selected bulb is red}\}$, what is the probability of the event R, denoted by $P(R|E)$?

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Consider the event $R = \{\text{the selected bulb is red}\}$, what is the probability of the event R, denoted by $P(R|E)$?

$$P(R|E) = \frac{N(R \cap E)}{N(E)} = \frac{5}{8}$$

We have defined a new probability function associated with the reduced sample space E.

Motivation Example

We study the problem of how to define a new probability function associated with a reduced sample space $E \subseteq S$, where S is the original sample space.

1. We have defined the probability function associated with the reduced sample space E directly.

2. We can also define it by linking to the probability function associated with the original sample space S.

Motivation Example

Under the assumptions that

1. S is finite
2. All outcomes are "equally likely"

the above example give us the idea

$$P(R|E) = \frac{N(R \cap E)}{N(E)} = \frac{N(R \cap E)/N(S)}{N(E)/N(S)} = \frac{P(R \cap E)}{P(E)}$$

leading to the next definition

Conditional Probability

Definition

The conditional probability of an event A, given that the event B has occurred, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) > 0$.

- B is the sample space for $P(A|B)$
- Independent of Assumptions 1 & 2 on the previous slide.

Conditional Probability

$\Rightarrow$ proof

Conditional probability satisfies the probability axioms

1. $P(A|B) \geq 0$.

2. $P(B|B) = 1$.

3. If $A_1, A_2, A_3, \cdots$ are countable and mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \cdots | B) = P(A_1|B) + P(A_2|B) + \cdots$$

$$\frac{P((A_1 \cup A_2 \cup \cdots) \cap B)}{P(B)} = \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} + \cdots$$

Example 2

$$P(A) = 0.4, \quad P(B) = 0.5, \quad P(A \cap B) = 0.3,$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.5} = 0.6$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.4} = 0.75$$

Can $P(A|B) > 1$ or $P(A|B) < 0$?

Example 2

$$P(A) = 0.4, \quad P(B) = 0.5, \quad P(A \cap B) = 0.3,$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.5} = 0.6$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.4} = 0.75$$

Can $P(A|B) > 1$ or $P(A|B) < 0$?

No, $P(A|B)$ is a probability function.

Example 3 (Shooting Game)

Question

25 balloons of which, 10 are yellow, 8 red, 7 green.

$$A = \{\text{the first balloon shot is yellow}\} \qquad \frac{10}{25} = \frac{2}{5}$$

$$B = \{\text{the second balloon shot is yellow}\}$$

What is the probability that the first two balloons shot are all yellow?

$$\frac{\cancel{2}}{5} \times \frac{\cancel{9}\,3}{\cancel{24}\,\cancel{8}\,4} = \frac{3}{20}$$

Example 3 (Shooting Game)

Question

25 balloons of which, 10 are yellow, 8 red, 7 green.

$$A = \{\text{the first balloon shot is yellow}\}$$

$$B = \{\text{the second balloon shot is yellow}\}$$

What is the probability that the first two balloons shot are all yellow?

$$P(A) = \frac{10}{25}, \quad P(B|A) = \frac{9}{24}$$

$$\Rightarrow P(A \cap B) = P(A)P(B|A) = \frac{10}{25} \cdot \frac{9}{24}$$

Multiplication Rule

Definition

The probability that two events, A and B both occur is given by the multiplication rule

$$P(A \cap B) = P(A)P(B|A), \quad \text{provided} \quad P(A) > 0$$

or by

$$P(A \cap B) = P(B)P(A|B), \quad \text{provided} \quad P(B) > 0$$

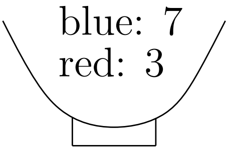
Example 4

$$\frac{3}{10}\cdot\frac{7}{9} = \frac{7}{30}$$

Question

A bowl contains 10 chips in total, 7 blue and 3 red. Drawn 2 chips successively at random and without replacement. What is the probability that the 1st draw is red and the 2nd draw is blue?

Example 4



$$A = \{\text{1st draw is red}\}$$

$$B = \{\text{2nd draw is blue}\}$$

$$P(A) = \frac{3}{10}, \quad P(B|A) = \frac{7}{9}$$

$$P(A \cap B) = P(B|A) \cdot P(A) = \frac{3}{10} \cdot \frac{7}{9} = \frac{7}{30}$$

Multiplication Rule for Three Events

Definition

The probability that three events, A, B and C all occur is given by the multiplication rule

$$P(A \cap B \cap C) = P((A \cap B) \cap C) = P(A \cap B)P(C|A \cap B)$$

$$\text{where} \quad P(A \cap B) = P(A)P(B|A)$$

$$\Rightarrow P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Induction principle can be used to derive the cases for more than three events.

Example 5

1 , 1   2 , 1   3 , 1   4 , 1
1 , 2   2 , 2   3 , 2   4 , 2
1 , 3   2 , 3   3 , 3   4 , 3
1 , 4   2 , 4   3 , 4   4 , 4

Question

Roll a pair of 4-sided dice and observe the sum of the dice

$A = \{\text{a sum of 3 is rolled}\}$

$B = \{\text{a sum of 3 or a sum of 5 is rolled}\}$

$C = \{\text{a sum of 3 is rolled before a sum of 5 is rolled}\}$

What are $P(A), P(B), P(C)$?

Example 5

Consider $P(A)$ and $P(B)$:

the sample space $S = \{(1,1),(1,2),\cdots,(4,4)\}$

$$P(A) = \frac{N(A)}{N(S)} = \frac{2}{16}, \quad P(B) = \frac{N(B)}{N(S)} = \frac{6}{16}$$

Consider P(C):

- Method 1 [by definition]:
  A. Figure out the simplified random experiment
  B. Figure out the corresponding sample space and the event

Example 5

For A, repeat the experiment of rolling a pair of 4-sided dice and record the sum of dice. For each repetition, we keep rolling the dice till we see either a sum of 3 or a sum of 5. Then we stop because we have an answer to the problem whether a sum of 3 is rolled before a sum of 5 is rolled.

Example 5

For instance

$$\text{Repetition 1} : 2, 4, 6, 3.$$
$$\text{Repetition 2} : 8, 6, 7, 4, 5$$
$$\text{Repetition 3} : 6, 5.$$

The sums other than 3 and 5 do not matter and we can remove them.

Repetition 1: a sum of 3 first
Repetition 2: a sum of 5 first
Repetition 3: a sum of 5 first

The problem reduces to roll the pair of dice (that gives the sum either 3 or 5) once and compute the probability that the sum is a 3.

Example 5

For B, the reduced sample space

$$S_r = \left\{\begin{matrix}(1,2),(2,1)\\(2,3),(3,2)\\(1,4),(4,1)\end{matrix}\right\} \text{give a sum of 3 or 5}$$

$$P(C) = P(\{\text{roll the pair of dice once and the sum is a 3}\})$$

$$= \frac{N(\{\text{roll the pair of dice once and the sum is 3}\})}{N(S_r)}$$

$$= \frac{2}{6}$$

Example 5

- Method 2 [by conditional probability]:

$$P(C) = P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/16}{6/16} = \frac{2}{6}$$

Note that the event "$A|B$" is the same as event "C".

This is because

A. Event C is concerned with the cases where the sum is either a 3 or a 5. 'B happened" means that the sum is either a 3 or a 5.

B. If B happened then $A|B$ is nothing but the event "roll the pair of dice (that gives the sum either 3 or 5) once, and the sum is 3".

Section 1.4 Independent Events

Motivation

Motivation

For certain pair of events, the occurrence of one of them does not change the probability of the occurrence of the other.

Example 1

Experiment: flip a coin twice and observe the sequence of heads and tails.

Sample space: $S = \{HH, HT, TH, TT\}$

Assumption: the four outcomes are "equally likely"

Events:

$$\begin{aligned} A &= \{\text{heads on the first flip}\} = \{HH, HT\} \\ B &= \{\text{tails on the second flip}\} = \{HT, TT\} \\ C &= \{\text{tails on both flips}\} = \{TT\} \end{aligned}$$

Example 1

$$P(A) = \frac{2}{4}, \quad P(B) = \frac{2}{4}, \quad P(C) = \frac{1}{4}$$

Example 1

$$P(A) = \frac{2}{4}, \quad P(B) = \frac{2}{4}, \quad P(C) = \frac{1}{4}$$

Given that C has occurred, then

$$P(B|C) = 1 \text{ because } C \subset B \text{ or } \frac{P(B \cap C)}{P(C)} = \frac{P(C)}{P(C)} = 1$$

Given that A has occurred, then

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{1/4}{2/4} = \frac{1}{2} = P(B)$$

Given that B has occurred, then

$$P(A|B) = \frac{1}{2} = P(A)$$

$$P(A|B) = \frac{P(AB)}{P_B} = \frac{P_A \cdot P_B}{P_B} = P_A$$

Example 1

So we have

$$P(B|A) = P(B), \text{and } P(A|B) = P(A)$$

the occurrence of one of them does not affect the probability of the occurrence of the other. Leading to the definition of independent events.

Independent Events

Definition

Events A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

Otherwise, events A and B are called dependent events

- When $P(A) \neq 0$ and $P(B) \neq 0$, we have

$$P(A|B) = P(A), \qquad P(B|A) = P(B)$$

Example 2, page 38

Question

A red die and a white die are rolled.

$$S = \{(1,1),(1,2),\cdots\}, \quad N(S) = 36$$

$$A = \{\text{4 on the red die}\}, \quad B = \{\text{sum of dice is odd}\}$$

Assuming the two dice are fair. Are A and B independent?

Example 2, page 38

A red die and a white die are rolled.

$$S = \{(1,1),(1,2),\cdots\}, \quad N(S) = 36$$

$$A = \{\text{4 on the red die}\}, \quad B = \{\text{sum of dice is odd}\}$$

Assuming the two dice are fair. Are A and B independent?

$$P(A) = \frac{6}{36}, \quad P(B) = \frac{18}{36}, \quad P(A \cap B) = \frac{3}{36}$$

$$P(A \cap B) = \frac{3}{36} = P(A)P(B) = \frac{6}{36} \cdot \frac{18}{36}$$

$\Rightarrow$ A and B are independent.