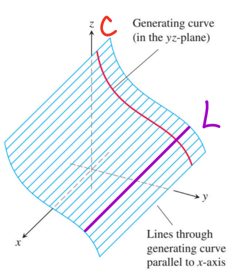
MAT1002 Lecture 11, Thursday, Mar/02/2023

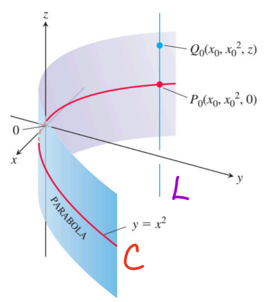
Outline

- Cylinders and quadric surfaces (12.6)
- Vector-valued functions (13.1)
- Limits and continuity (13.1)
- Derivatives (13.1)

Cylinders and Quadric Surfaces

Def: Let C be a plane curve lying in $\mathbb{R}^3$. If L is a line that passes through C and is not on the plane in which C lies, and S is the union of all lines which are parallel to L and which pass through C, then S is called a cylinder or a cylindrical surface.





e.g. $\{(x,y,z): x^2+y^2=1\}$ is a cylindrical surface, which is a (usual) circular cylinder with radius 1.

Def: A quadric surface is a the set of all points (x,y,z) in $\mathbb{R}^3$ that satisfy an equation $f(x,y,z)=0$, where $f(x,y,z)$ is a degree-two polynomial; that is, f has the form

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J.$$

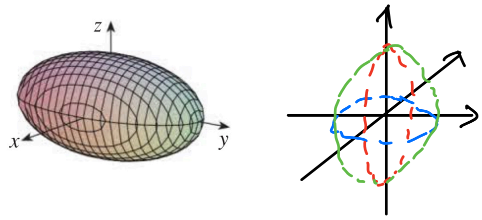
We will only look at some basic forms in this course.

An xy-trace of a quadric surface S is the intersection of S with a fixed plane $z=z_0$. The terms xz-trace and yz-trace are defined similarly (intersection with $y=y_0$ and $x=x_0$, respectively).

There are six basic types of quadric surfaces.

1. The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ describes an ellipsoid.

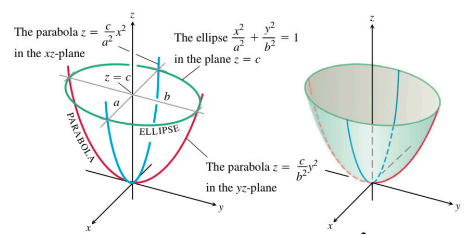
- All (nonempty) traces are ellipses.
- If $a=b=c$, it is a sphere.



From now on, by traces we mean nonempty traces.

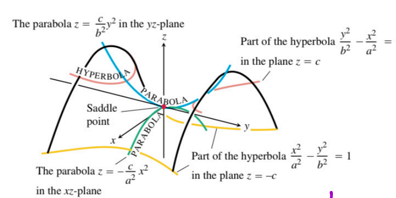
2. The equation $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ describes an elliptic paraboloid.

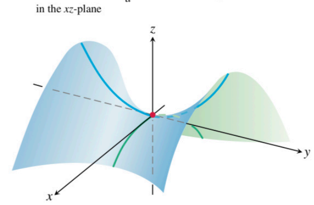
- All xy-traces are ellipses.
- All xz- and yz-traces are parabolas.



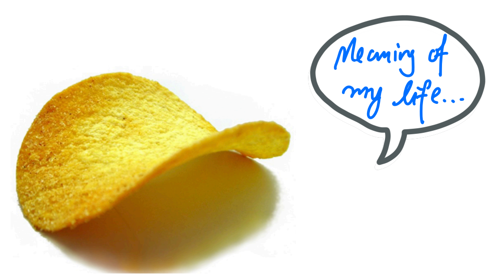
3. The equation $\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ describes an hyperbolic paraboloid.

- All xy-traces are hyperbolas
(except at plane $z=0$, the xy-trace is two lines $y=\pm\frac{b}{a}x$).
- All xz- and yz-traces are parabolas.





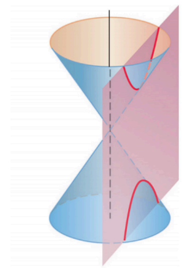
here $c>0$.

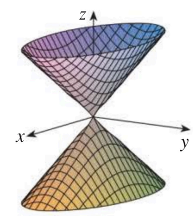


4. The equation $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ describes an elliptic cone (or cone).

or $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

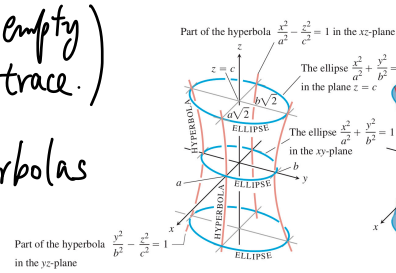
- All xy-traces are ellipses.
- All xz- and yz-traces are hyperbolas (except at $x=0$ and $y=0$, at which the traces are pairs of lines).

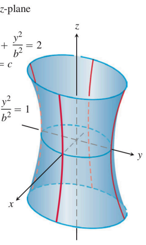




5. The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ describes an hyperboloid of one sheet.

- All xy-traces are ellipses. (No empty xy-trace.)
- All xz- and yz-traces are hyperbolas (except at $x=a$ or $y=b$).



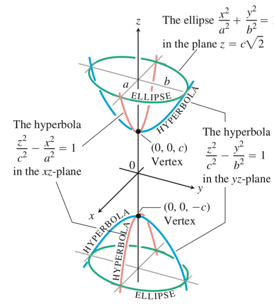


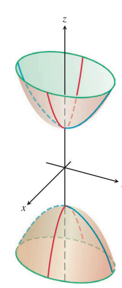
6. The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ describes an hyperboloid of two sheets.

- All xy-traces are ellipses.

(There may be empty xy-traces, e.g., $z = \frac{c}{2}$.)

- All xz- and yz-traces are hyperbolas.



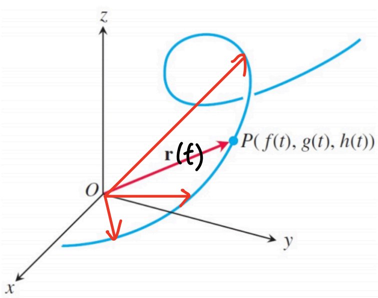


Vector-Valued Functions (on Intervals)

Definition
A vector-valued function or a vector function (on I) is a function $\vec{r}: I \to \mathbb{R}^n$, where I is an interval and $\mathbb{R}^n$ is viewed as a set of vectors.

In this course, n is usually 2 or 3.

Remark
For a vector function $\vec{r}: I \to \mathbb{R}^n$, we have $\vec{r}(t) = \langle r_1(t), \ldots, r_n(t) \rangle$, where each r_i is a function from I to $\mathbb{R}$.



A parametric curve given by $x = f_1(t)$, $y = f_2(t)$, $t \in I$ can be viewed as the range of the function $\vec{r}: I \to \mathbb{R}^2$, $\vec{r}(t) := \langle f_1(t), f_2(t) \rangle$ (the points given by the position vector $\vec{r}(t)$).

Example
Sketch the curve parametrized by the function

$$\vec{r}(t) := \langle \cos(t), \sin(t), t \rangle, \quad t \in \mathbb{R}.$$

This curve is called a helix.

Limits and Continuity

Definition
Let $\vec{r}: I \to \mathbb{R}^n$ be a vector function. We say that $\vec{r}$ has a limit $\vec{L}$ as t approaches t_0, and write

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L},$$

if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $t \in I$, $0 < |t - t_0| < \delta$ implies $|\vec{r}(t) - \vec{L}| < \epsilon$.

The terminal point of $\vec{r}(t)$ can be arbitrarily close to that of $\vec{L}$ if t is sufficiently close to t_0.

The following is a convenient equivalent condition.

Theorem ("Component-Wise Limits")
A vector function $\vec{r} = \langle r_1, r_2, \dots r_n \rangle$ satisfies

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L} = \langle L_1, L_2, \dots, L_n \rangle$$

if and only if $\lim_{t \to t_0} r_i(t) = L_i$ for all i.

Proof: Omitted.

Example

If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\lim_{t \to \pi/4} \mathbf{r}(t) = \left(\lim_{t \to \pi/4} \cos t\right)\mathbf{i} + \left(\lim_{t \to \pi/4} \sin t\right)\mathbf{j} + \left(\lim_{t \to \pi/4} t\right)\mathbf{k}$$
$$= \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}.$$

Definition
A vector function $\vec{r} : I \to \mathbb{R}^n$ is said to be continuous at $t_0 \in I$ if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0),$$

and it is called a continuous function if it is continuous at all points in I.

(one-sided) continuous at endpoints

Remark By componentwise limits, it follows that $\vec{r} = \langle r_1, \dots, r_n \rangle$ is continuous at $t_0 \iff r_i$ is continuous at t_0, $\forall i$.

Example
The function $\vec{r}(t) := \langle \cos(t), \sin(t), \lfloor t \rfloor \rangle$ is discontinuous at every $t \in \mathbb{Z}$, since $\lfloor t \rfloor$ is discontinuous at all integers.

Derivatives

Definition Let $\vec r : I \to \mathbb{R}^n$ be a function, and let t be an interior point of I.

- ~~A vector~~ The function $\vec r$ is said to be differentiable at t if the limit

$$\vec r\,'(t) := \frac{d\vec r}{dt} := \lim_{h\to 0} \frac{\overbrace{\vec r(t+h) - \vec r(t)}^{\Delta \vec r}}{\underbrace{h}_{\Delta t}}$$

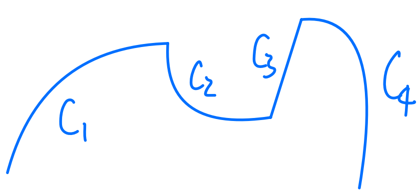
  exists.
- The vector $\vec r\,'(t)$ is called the derivative of $\vec r$ at t.
- If $\vec r\,'(t) \neq \vec 0$, $\vec r\,'(t)$ is also called the tangent vector to the curve (given by $\vec r$) at the point $P = P(t)$, where $\overrightarrow{OP} = \vec r(t)$.
- The corresponding unit tangent vector is

$$\vec T(t) := \frac{\vec r\,'(t)}{|\vec r\,'(t)|}$$

- The tangent line to the curve at a point $P(t)$ is the line through $P(t)$ in the direction of $\vec r\,'(t)$.

- $\vec r$ is said to be smooth if $\vec r\,'$ is continuous and never $\vec 0$ on I.
- A curve in $\mathbb{R}^n$ is said to be smooth if it has at least one smooth parametrization.

e.g.



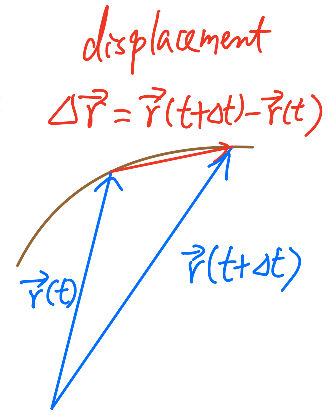
$\leftarrow$ Not smooth, but piecewise smooth

$\leftarrow$ Each C_i is smooth

Physical Meanings

If $\vec{r}(t)$ is the position vector of a particle moving along a curve at time t, then:

- $\vec{r}\,'(t)$ is the particle's velocity vector at time t, and;
- $|\vec{r}\,'(t)|$ is the particle's speed at time t.



Note that

$$\text{Velocity} = \vec{r}\,'(t) = |\vec{r}\,'(t)| \left(\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \right) = (\text{Speed})(\text{Direction}).$$

- $\dfrac{d\vec{r}\,'}{dt} = \dfrac{d(d\vec{r}/dt)}{dt} = \dfrac{d^2\vec{r}}{dt^2}$ is the particle's acceleration vector.
- $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ is the direction of motion at time t.

The derivative of a vector function can be computed by taking derivatives component-wise.

Theorem
If $\vec{r}(t) := \langle r_1(t), r_2(t), \ldots, r_n(t) \rangle$ is a vector function, where each r_i is differentiable, then

$$\vec{r}\,'(t) = \langle r_1'(t), r_2'(t), \ldots, r_n'(t) \rangle.$$

This follows from componentwise limits. (Details shown in lecture.)

e.g. A curve is traced out by $\vec{r}(t) := \langle 1+t^3, te^{-t}, \sin(2t) \rangle$.
Find direction of motion and acceleration of the particle at the point $(1,0,0)$.

Sol: $\vec{r}'(t) = \langle 3t^2, -te^{-t}+e^{-t}, 2\cos(2t)\rangle$;

$\vec{r}(t) = \langle 1,0,0\rangle \iff t=0$

Direction

$\Rightarrow \vec{r}'(0) = \langle 0,1,2\rangle \Rightarrow \vec{T}(0) = \frac{\vec{r}'(0)}{|\vec{r}'(0)|} = \frac{1}{\sqrt{5}}\langle 0,1,2\rangle.$

$\frac{d\vec{r}'(t)}{dt} = \langle 6t, te^{-t}-e^{-t}-e^{-t}, -4\sin(2t)\rangle$

$= \langle 6t, (t-2)e^{-t}, -4\sin(2t)\rangle$

Acceleration

$\Rightarrow \left.\frac{d\vec{r}'(t)}{dt}\right|_{t=0} = \langle 0,-2,0\rangle.$

Differentiation Rules

Differentiation Rules for Vector Functions

Let **u** and **v** be differentiable vector functions of t, **C** a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \mathbf{u}'(t)+\mathbf{v}'(t)$
4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t)-\mathbf{v}(t)] = \mathbf{u}'(t)-\mathbf{v}'(t)$
5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Scalar function/ real-valued function

So "product rule" also works for dot/cross products.

Properties 1, 2, 3, 4 and 7 can be proven componentwise.

Property 5 can be verified directly (see the book).

To prove property 6, we use the following fact:

$$\lim_{t\to t_0}\left(\vec{a}(t)\times\vec{b}(t)\right) = \left(\lim_{t\to t_0}\vec{a}(t)\right)\times\left(\lim_{t\to t_0}\vec{b}(t)\right).$$

This fact can be proven by componentwise limits and direct verification (details: exercise).

<u>Proof of Property 6</u>:

$$\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \lim_{h\to 0}\frac{\vec{u}(t+h)\times\vec{v}(t+h) - \vec{u}(t)\times\vec{v}(t)}{h}$$

$$= \lim_{h\to 0}\frac{\vec{u}(t+h)\times\vec{v}(t+h) - \vec{u}(t+h)\times\vec{v}(t) + \vec{u}(t+h)\times\vec{v}(t) - \vec{u}(t)\times\vec{v}(t)}{h}$$

$$= \lim_{h\to 0}\frac{\vec{u}(t+h)\times\left(\vec{v}(t+h)-\vec{v}(t)\right)}{h} + \lim_{h\to 0}\frac{\left(\vec{u}(t+h)-\vec{u}(t)\right)\times\vec{v}(t)}{h}$$

$$= \lim_{h\to 0}\underbrace{\vec{u}(t+h)}_{\vec{u}\text{ cts since differentiable}}\times\lim_{h\to 0}\frac{\left(\vec{v}(t+h)-\vec{v}(t)\right)}{h} + \lim_{h\to 0}\frac{\left(\vec{u}(t+h)-\vec{u}(t)\right)}{h}\times\lim_{h\to 0}\vec{v}(t)$$

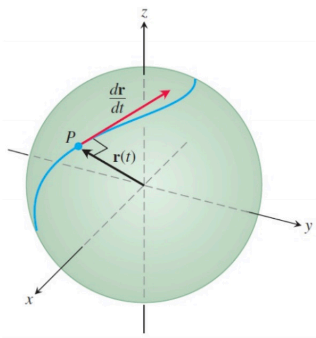
$$= \vec{u}(t)\times\vec{v}'(t) + \vec{u}'(t)\times\vec{v}(t).$$

$\square$

Vector Functions of Constant Length

Suppose that a curve given by a differentiable function $\vec{r}$ lies on a sphere centered at the origin. Then at any t, the tangent vector $\vec{r}'(t)$ is perpendicular to the position vector $\vec{r}(t)$. This can be shown by algebraically proving the following statement: if $\vec{r}$ is a differentiable vector function of constant length, then for all t,

$$\vec{r}(t)\cdot\vec{r}'(t) = 0.$$



Proof: $|\vec{r}(t)| \equiv a$ Constantly equal to

$\Rightarrow \vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 \equiv a^2$

$\Rightarrow \vec{r}(t)\cdot\vec{r}\,'(t) + \vec{r}\,'(t)\vec{r}(t) \equiv 0$ Dot product rule

$\Rightarrow 2\vec{r}(t)\cdot\vec{r}\,'(t) \equiv 0$

$\Rightarrow \vec{r}(t)\cdot\vec{r}\,'(t) \equiv 0$ $\square$

The conclusion above holds for all differentiable vector functions $\vec{r}: I \to \mathbb{R}^n$ that have a constant length $(|\vec{r}(t)| = K,\ \forall t \in I)$.