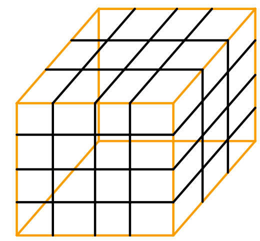
MAT1002 Lecture 20, Tuesday, Apr/04/2023

Outline

- Triple integrals (15.5)
- Cylindrical coordinates (15.7)
- Change of variable formula (15.8)

Triple integrals (Motivation: density and mass)

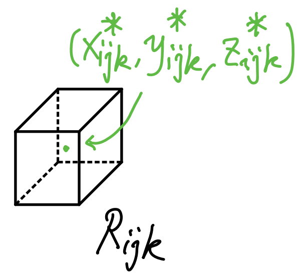


Let f be a three-variable function defined on the rectangular solid

$$R := [a,b] \times [c,d] \times [r,s].$$

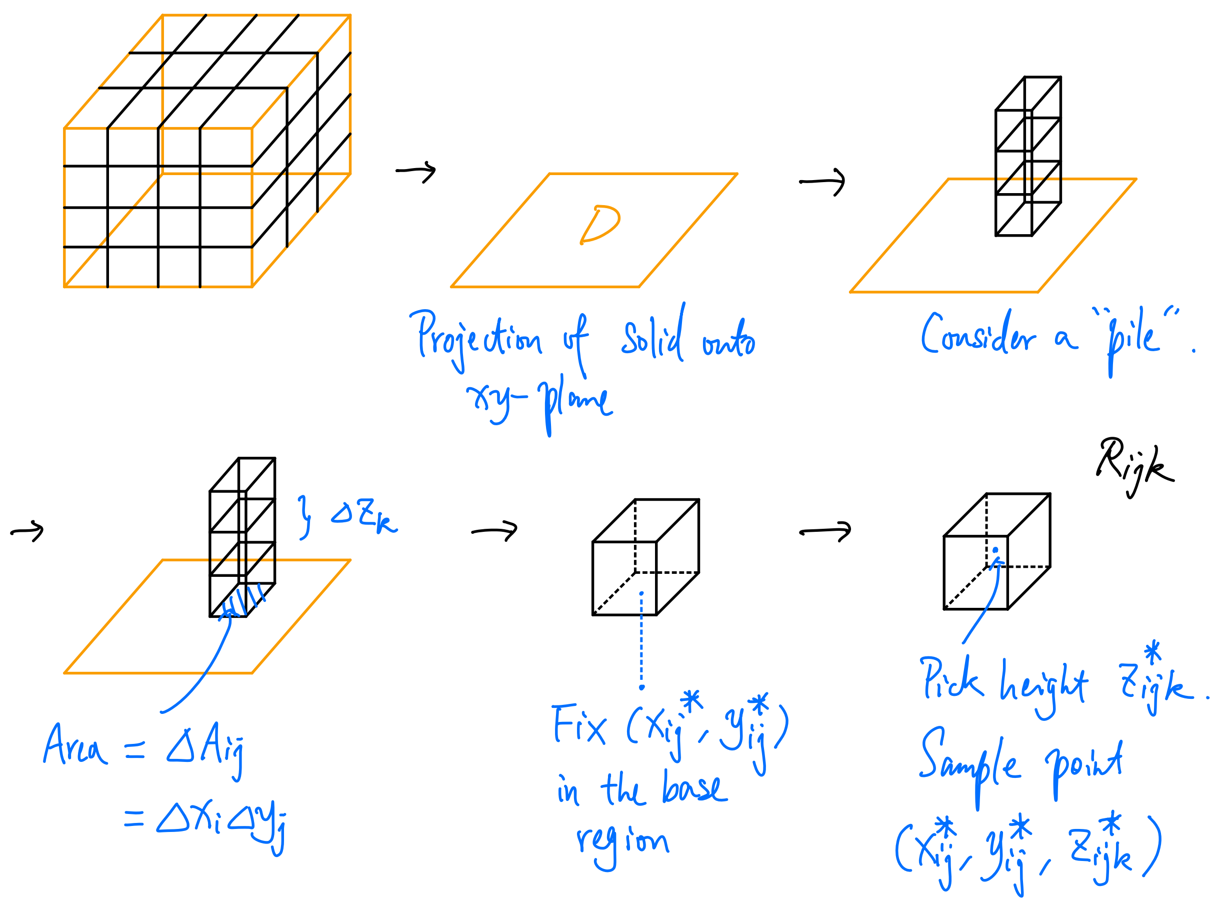
Then the triple integral of f over R is defined by

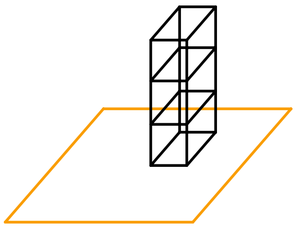
$$\iiint_R f(x,y,z)\, dV := \lim_{\|P\| \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk},$$



provided that the limit exists. The Riemann sum above is defined similarly to that defined ~~on Page 6~~. for double integrals.

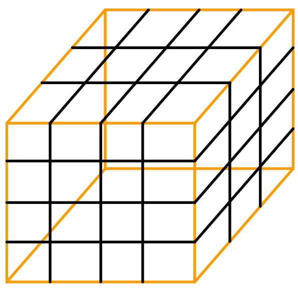
One way to think of triple integrals is that, if f is the density function, then $\iiint_R \underbrace{f(x,y,z)}_{\text{density at } (x,y,z),\ \text{say } kg/m^3}\, dV$ is the mass of the solid R.





Mass of this pile is

$$m_{ij} \approx \sum_{k=1}^{n} f(x_{ij}^*, y_{ij}^*, z_{ijk}^*)\,\Delta z_k\,\Delta A_{ij}$$



Mass of "Tofu" is

$$m = \sum_{i,j} m_{ij}$$

$$\approx \sum_{i,j}\left(\sum_{k} f(x_{ij}^*, y_{ij}^*, z_{ijk}^*)\,\Delta z_k\right)\Delta A_{ij}$$

- Taking limit as $\|P\| \to 0$ yields

$$m = \iint_D \left(\int_r^s f(x,y,z)\,dz\right) dA$$

$\iiint$ viewed as $\iint$

If f is cts

$$= \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx.$$

(two-variable Fubini's thm)

This is the three-variable Fubini's theorem.

Theorem (Fubini's Theorem) (Triple integrals)

If f is continuous on the rectangular solid

$$R = \underset{x}{[a,b]} \times \underset{y}{[c,d]} \times \underset{z}{[r,s]},$$

then

$$\iiint_R f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

And order does not matter.

Exercise

Evaluate $\iiint_R xyz^2\,dV$, where $R = [0,1] \times [-1,2] \times [0,3]$.

<u>Ans</u>: $= \int_0^1 \int_{-1}^2 \int_0^3 xyz^2\,dz\,dy\,dx = 27/4.$

<u>Remark</u> Note that

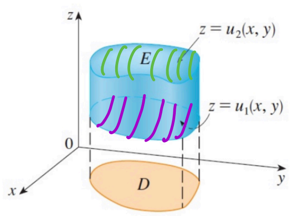
$$\int_a^b \int_c^d \int_r^s g_1(x) g_2(y) g_3(z)\,dz\,dy\,dx = \left(\int_a^b g_1(x)\,dx\right)\left(\int_c^d g_2(y)\,dy\right)\left(\int_r^s g_3(z)\,dz\right).$$

<u>Q</u>: How to compute $\iiint_E f(x,y,z)\,dV$ when E is not a rectangular solid?

Suppose that E has the form

$$\{(x, y, z) : (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\},$$

where D is the projection of E onto the xy-plane.



Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right) dA. \longleftarrow \quad \iiint \text{ viewed as } \iint$$

Note that in the inner integral on the right-hand side, x and y are held fixed. If D is a type-I region, then E can be written as

$$\{(x, y, z) : a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}.$$

By Fubini's theorem for double integrals, the equation above becomes

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

A similar strategy can be used when D is of type II or when E has other similar forms.

Example

Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Ans: $\frac{1}{24}$.

Definition
The volume $V(E)$ of a solid E in $\mathbb{R}^3$ is defined by

$$V(E) := \iiint_E dV := \iiint_E 1\, dV.$$

The average value of a function f over E is

$$\frac{1}{V(E)} \iiint_E f(x, y, z)\, dV.$$

Exercise
Use a triple integral to find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

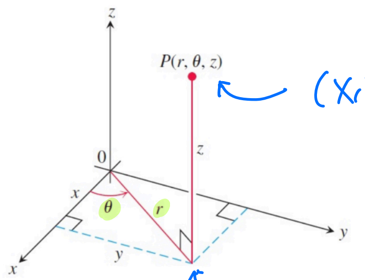
Ans: $\frac{1}{3}$.

Cylindrical Coordinates

Given any point P in the xyz-space, we can represent the point by (r, θ, z), where

- r and θ are polar coordinates of the projection of P onto the xy-plane, and;
- z is the z-coordinate of P in the Cartesian coordinates.

The coordinate system above in (r, θ, z) is called the cylindrical coordinate system.



$(x,y,z) = (r\cos\theta, r\sin\theta, z)$

$(x,y,0) = (r\cos\theta, r\sin\theta, 0)$

Cylindrical coordinates may be helpful in computing triple integrals when the solid E of integration has the form

$$E = \{(x,y,z) : (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\},$$

where D (the projection of E onto the xy-plane) is convenient to describe in polar coordinates. If D is given by

$$\alpha \le \theta \le \beta, \quad 0 \le h_1(\theta) \le r \le h_2(\theta),$$

$(\beta - \alpha \le 2\pi)$

then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

$$= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(x,y)}^{u_2(x,y)} f(r\cos\theta, r\sin\theta, z)\,dz\; r\,dr\,d\theta$$

e.g. A solid E lies inside the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at a point $P(x,y)$ is $K\sqrt{x^2+y^2}$ for some constant K. Find mass(E).

Ans $\frac{12\pi}{5}K$.

Exercise

Use cylindrical coordinates to evaluate the integral

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx.$$

Ans $16\pi/5$.

Moral of the story: $dV = r\,dz\,dr\,d\theta$ after changing from xyz- to cylindrical coordinates.

Change of Variable Formula (Double Integrals)

When discussing double integrals in polar coordinates last week, we saw that

$$\iint_D f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)r\,drd\theta,$$

where R is the polar region corresponding to the region D (which is in the xy-plane). Essentially, we made the substitution

$$x = r\cos\theta, \quad y = r\sin\theta,$$

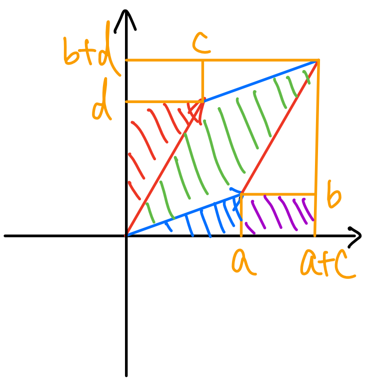
and replaced the differential dA with $r\,drd\theta$. More generally, if we make the substitution

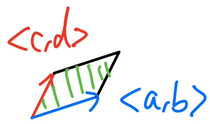
$$x = g(u,v), \quad y = h(u,v),$$

what would the formula become?

Geometric Properties of Determinants

Consider the following figure:



Area of 

$$= (a+c)(b+d) - 2\left(\tfrac{1}{2}ab\right) - 2\left(\tfrac{1}{2}cd\right) - 2(cb)$$

$$= ab + ad + bc + cd - ab - cd - 2cb$$

$$= ad - cb = \det\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Fact:

$\left|\det\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right|$ ← Abs. values

is the area of the parallelogram spanned by $\langle a,b\rangle$ and $\langle c,d\rangle$. ()

Another fact:

You can "take out" any constant from a row or from a column:

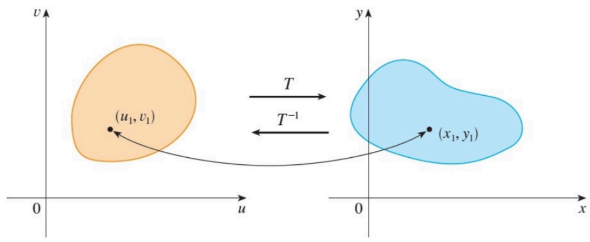
$$\det\begin{bmatrix} \lambda a & \mu c \\ \lambda b & \mu d \end{bmatrix} = \lambda \det\begin{bmatrix} a & \mu c \\ b & \mu d \end{bmatrix} = \lambda\mu \det\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Suppose that

$$x = g(u, v) \quad \text{and} \quad y = h(u, v),$$

where g and h have continuous partial derivatives. We may understand this as a transformation T that maps a point in the uv-plane to a point in the xy-plane:

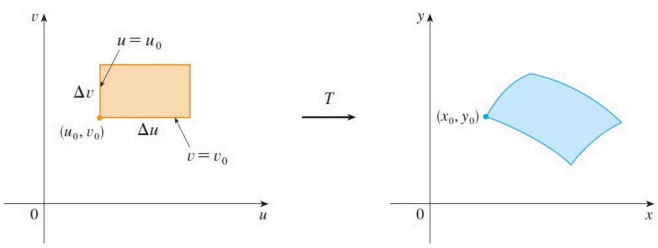
$$T(u, v) := (g(u, v), h(u, v)) = (x, y).$$



Let R be a region in the uv-plane, and let D be the image of R under the transformation T. That is,

$$D = T(R) := \{T(u, v) : (u, v) \in R\}.$$

Partition R into rectangles R_{ij}, and let D_{ij} be the image of R_{ij} under T.



Let $A(D_{ij})$ and $A(R_{ij})$ be the area of D_{ij} and R_{ij}, respectively.
When the norm of the partition is small,

$$A(D_{ij}) \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| A(R_{ij}) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u_i \Delta v_j,$$

absolute value of the Jacobian

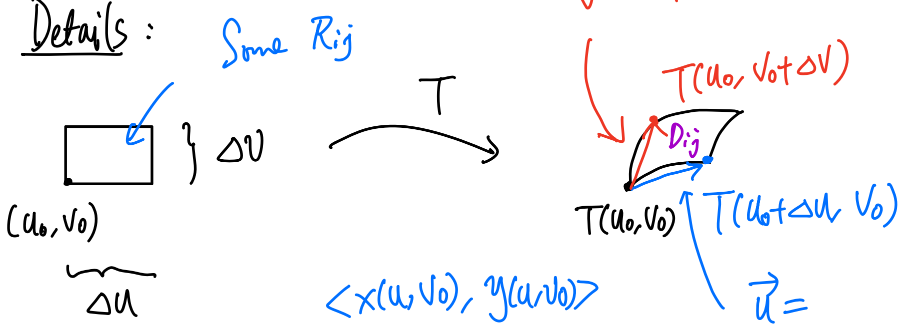
where

Jacobian, which is a determinant

$$\frac{\partial(x,y)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

The determinant $\frac{\partial(x,y)}{\partial(u,v)}$ above is called the Jacobian of the transformation.

Details :



$\vec{v} = T(u_0, v_0+\Delta v) - T(u_0, v_0) = \vec{r}_2(v_0+\Delta v) - \vec{r}_2(v_0)$
$\approx \vec{r}_2{}'(v_0)\Delta v$

$\vec{u} = T(u_0+\Delta u, v_0) - T(u_0, v_0)$
$= \vec{r}_1(u_0+\Delta u) - \vec{r}_1(u_0)$
$\approx \vec{r}_1{}'(u_0)\Delta u$

Fix v_0, let $\vec{r}_1(u) := \underbrace{T(u, v_0)}_{= \langle x(u,v_0),\, y(u,v_0)\rangle}$

Fix u_0, let $\vec{r}_2(v) := \underbrace{T(u_0, v)}_{\langle x(u_0,v),\, y(u_0,v)\rangle}$

Hence $A(D_{ij}) \approx \text{Area}(\vec{v}, \vec{u}) = \left|\det\begin{bmatrix}\vec{u} & \vec{v}\end{bmatrix}\right|$

$$= \left|\det\begin{bmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{bmatrix}\right| = \Delta u \Delta v \left| \underbrace{\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}}_{\frac{\partial(x,y)}{\partial(u,v)}} \right|$$

Evaluated at (u_0, v_0)

"Jacobian of T"

In terms of differentials, one can memorize the above as

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

Theorem (Change of Variables in Double Integrals)

Let f be continuous on a region D in the xy-plane. Suppose that

- *the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ maps a region R in the uv-plane onto D, and;*
- *the transformation is one-to-one in the interior of R, and;*
- *all partial derivatives of g and h are continuous.*

Then

$$\iint_D f(x,y)\,dA = \iint_R f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

e.g. (Polar transformation)

If a change of variables is given by the transformation

$$\underline{x = g(r,\theta) = r\cos\theta \quad \text{and} \quad y = h(r,\theta) = r\sin\theta,}$$

To satisfy the one-to-one condition, here we restrict to $r \geq 0$, i.e., domain of T is $\{(r,\theta) : r \geq 0, \theta \text{ is in an interval with length} = 2\pi\}$

then it follows that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \geqslant 0,$$

which gives

$$\iint_D f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

This is consistent with the formula we stated in ~~Week 9~~ last week.

Example

Evaluate the following integrals.

(a) $\iint_D e^{(x+y)/(x-y)}\,dA$, where D is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$.

(b) $\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$.

Ans

(a) $\frac{3}{4}(e-e^{-1})$

(b) $2e(e-2)$