Determinants

A <u>$m \times n$ matrix</u> is an array of numbers arranged as follows:

$$m \text{ rows} \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & a_{2n} \\ a_{31} & & a_{ij} & a_{3n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \right.$$

$$\underbrace{\qquad\qquad\qquad}_{n \text{ columns}}$$

Each $n \times n$ matrix has a special number called the determinant. Determinants for 2×2 and 3×3 matrices can be computed as follows:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$:= \underset{(+)}{a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} - \underset{(-)}{a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}} + \underset{(+)}{a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$

More generally, for an $n \times n$ matrix M, let M_{ij} be the matrix obtained by deleting row i and column j from M.

e.g.

$$M = \begin{bmatrix} 1 & 2 & \cancel{3} \\ \cancel{4} & \cancel{5} & \cancel{6} \\ 7 & 8 & \cancel{11} \end{bmatrix} \qquad M_{2,3} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}.$$

Fix any $i \in \{1, 2, \ldots, n\}$. Then

$$\det M = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{i,j}) \quad \begin{bmatrix} \text{Expansion along} \\ \text{row } i \end{bmatrix}.$$

e.g. For a 3×3 matrix, if we expand along row 3 $(i=3)$, then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{3+1} a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + (-1)^{3+2} a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.$$

$$+ (-1)^{3+3} a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{22} & a_{22} \end{vmatrix}$$

e.g.

$$M = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$$

$$\det M = 2\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} \quad (i=1)$$

$$= 2(-2) - 5(8) + 4(7) = -16.$$

$$\det M = -3\begin{vmatrix} 5 & 4 \\ 4 & 6 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix} \quad (i=2)$$

$$= -3(14) + (-8) - 2(-17)$$

$$= -50 + 34 = -16.$$

More theory and computation techniques will be discussed in a linear algebra course.