Assignment 9 杜奕璇 122090095

14.8

⑥ Optimize $f(x,y)=x^2+y^2$ $g(x,y)=x^2y-2=0$

$\frac{\partial f}{\partial x}=2x$ $\frac{\partial f}{\partial y}=2y$ $\frac{\partial g}{\partial x}=2yx$ $\frac{\partial g}{\partial y}=x^2$

$\frac{2x}{2yx}=\frac{2y}{x^2}$ $\therefore 2y^2=x^2$ $\therefore y^3=1$ $\therefore y=1$ $x=\pm\sqrt{2}$

$\therefore$ point $(\pm\sqrt{2},1)$

⑪ $A=(2x)(2y)=4xy$ $g(x,y)=\frac{x^2}{16}+\frac{y^2}{9}-1=0$

$\nabla A=4y\vec{i}+4x\vec{j}$ $\nabla g=\frac{x}{8}\vec{i}+\frac{2y}{9}\vec{j}$ $\therefore \nabla A=\lambda\nabla g$ $\therefore \begin{cases}4y=\frac{x}{8}\lambda\\4x=\frac{2}{9}y\lambda\end{cases}$ $\lambda=\frac{32y}{x}$

$\therefore y=\pm\frac{3}{4}x \Rightarrow x^2=8$ $\therefore x=\pm2\sqrt{2}$ $x=2\sqrt{2}$ $y=\frac{3}{4}\times2\sqrt{2}=\frac{3\sqrt{2}}{2}$ $\therefore$ length $2x=4\sqrt{2}$ width $2y=3\sqrt{2}$

⑬ $\nabla f=2x\vec{i}+2y\vec{j}$ $\nabla g=(2x-2)\vec{i}+(2y-4)\vec{j}$ $\therefore \nabla f=\lambda\nabla g=2x\vec{i}+2y\vec{j}=\lambda[(2x-2)\vec{i}+(2y-4)\vec{j}]$

$\therefore \begin{cases}2x=\lambda(2x-2)\\2y=\lambda(2y-4)\end{cases}$ $\therefore \begin{cases}x=\frac{\lambda}{\lambda-1}\\y=\frac{2\lambda}{\lambda-1}\end{cases}$ $\therefore y=2x$ $\therefore \begin{cases}x=0\\y=0\end{cases}$ or $\begin{cases}x=2\\y=4\end{cases}$

$f(0,0)=0$ is the minimum value

$f(2,4)=20$ is the maximum value

㉙ $\nabla T=16x\vec{i}+4z\vec{j}+(4y-16)\vec{k}$ $\therefore \lambda=2$ or $x=0$

$\nabla g=8x\vec{i}+2y\vec{j}+8z\vec{k}$

i/ $\lambda=2$ $z=y=-\frac{4}{3}$

Then $4x^2+(-\frac{4}{3})^2+4(-\frac{4}{3})^2=16$ $\therefore x=\pm\frac{4}{3}$

ii/ $x=0$ $(y-4)(y+2)=0$ $\therefore y=4$ or $y=-2$

$y=4,\ z=0$

$y=-2,\ z=\pm\sqrt{3}$

The temperatures $T(\pm\frac{4}{3},-\frac{4}{3},-\frac{4}{3})=(642\frac{2}{3})^\circ$

$T(0,4,0)=600^\circ$

$T(0,-2,\sqrt{3})=(600-24\sqrt{3})^\circ$

$T(0,-2,-\sqrt{3})=(600+24\sqrt{3})^\circ$

So the $(\pm\frac{4}{3},-\frac{4}{3},-\frac{4}{3})$ are the hottest points

㉛ (a) P changes by a factor of $2^\alpha\cdot2^{1-\alpha}=2$

(b) $P(x,y)=120x^{\frac{3}{4}}y^{\frac{1}{4}}$ $G(x,y)=250x+400y-100000=0$ $\nabla P=\lambda\nabla G$

$5x=24y$

$\therefore x=300,\ y=62.5$ $P(300,62.5)\approx 24322$ units

㉝ $\nabla U=(y+2)\vec{i}+x\vec{j}$ $\nabla g=2\vec{i}+\vec{j}$ $\begin{cases}y+2=2\lambda\\x=\lambda\end{cases}$

$x=8$ and $y=14$

max: $\therefore U(8,14)=128$

(43) let $g_1(x,y,z) = y-x=0$, $g_2(x,y,z) = x^2+y^2+z^2-4=0$

$\nabla f = y\vec{i}+x\vec{j}+2z\vec{k}$ $\quad \nabla g_1 = -\vec{i}+\vec{j}$ $\quad \nabla g_2 = 2x\vec{i}+2y\vec{j}+2z\vec{k}$ $\quad \therefore \nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

$y\vec{i}+x\vec{j}+2z\vec{k} = \lambda(-\vec{i}+\vec{j}) + \mu(2x\vec{i}+2y\vec{j}+2z\vec{k})$

$\Rightarrow y = -\lambda+2x\mu$, $x = \lambda+2y\mu$, $2z = 2z\mu$ $\quad \Rightarrow z=0$ or $\mu=1$

i/ $z=0 \Rightarrow x=\pm\sqrt{2}$, $y=\pm\sqrt{2}$

$(\pm\sqrt{2},\pm\sqrt{2},0)$

ii/ $\mu=1 \Rightarrow x=0, y=0$, $z=\pm 2$

$(0,0,\pm 2)$

$\underbrace{f(0,0,\pm 2)=4}_{\text{maximum}}$ $\quad \underbrace{f(\pm\sqrt{2},\pm\sqrt{2},0)=2}_{\text{minimum}}$

(45) $\nabla f = \vec{i}+\vec{j}$ $\quad \nabla g = y\vec{i}+x\vec{j}$ $\quad y=x=\pm 4$

$\therefore (4,4)$ and $(-4,-4)$

no maximum or minimum value exists

(46) let $f(A,B,C) = \sum_{k=1}^{4} (Ax_k+By_k+C-z_k)^2 = C^2+(B+C-1)^2+(A+B+C-1)^2+(A+C+1)^2$

$f_A(A,B,C) = 4A+2B+4C$ $\quad f_A(A,B,C)=0$

$f_B(A,B,C) = 2A+4B+4C-4$ $\quad f_B(A,B,C)=0$

$f_C(A,B,C) = 4A+4B+8C-2$ $\quad f_C(A,B,C)=0$

$\therefore \begin{cases} A = -\frac{1}{2} \\ B = \frac{3}{2} \\ C = -\frac{1}{4} \end{cases}$ $\quad$ critical point $\left(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{4}\right)$

(47) (a) $f(a,b,c) = a^2b^2c^2$ $\quad a^2+b^2+c^2=r^2$

$\nabla f = 2ab^2c^2\vec{i} + 2a^2bc^2\vec{j} + 2a^2b^2c\vec{k}$ $\quad \nabla g = 2a\vec{i}+2b\vec{j}+2c\vec{k}$

$\therefore \nabla f = \lambda \nabla g$ $\quad \therefore \lambda=0$ or $a^2=b^2=c^2$

i/ $\lambda=0 \Rightarrow a^2b^2c^2=0$

ii/ $a^2=b^2=c^2 \Rightarrow f(a,b,c)=a^6$, $3a^2=r^2$ $\therefore f(a,b,c) = \left(\frac{r^2}{3}\right)^3$ is the maximum

(b) $(\sqrt{a},\sqrt{b},\sqrt{c})$ on the sphere if $a+b+c=r^2$

$abc = f(\sqrt{a},\sqrt{b},\sqrt{c}) \le \left(\frac{r^2}{3}\right)^3 \Rightarrow (abc)^{\frac{1}{3}} \le \frac{r^2}{3} = \frac{a+b+c}{3}$

14.9

② $f(x,y)=e^x\cos y \quad f_x=e^x\cos y \quad f_y=-e^x\sin y \quad f_{xx}=e^x\cos y \quad f_{xy}=-e^x\sin y$

$f_{yy}=-e^x\cos y \Rightarrow f(x,y)\approx f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{1}{2}[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)]$

$=1+x+\frac{1}{2}(x^2-y^2) \Rightarrow$ quadratic approximation

$f_{xxx}=e^x\cos y \quad f_{xxy}=-e^x\sin y \quad f_{xyy}=-e^x\cos y \quad f_{yyy}=e^x\sin y$

$\therefore f(x,y)=\text{quadratic}+\frac{1}{6}[x^3f_{xxx}(0,0)+3x^2yf_{xxy}(0,0)+3xy^2f_{xyy}(0,0)+y^3f_{yyy}(0,0)]=1+x+\frac{1}{2}(x^2-y^2)+\frac{1}{6}(x^3-3xy^2)$

cubic approximation

⑥ $f(x,y)=\ln(2x+y+1)$

$f_x=\frac{2}{2x+y+1} \quad f_y=\frac{1}{2x+y+1} \quad f_{xx}=\frac{-4}{(2x+y+1)^2} \quad f_{xy}=\frac{-2}{(2x+y+1)^2} \quad f_{yy}=\frac{-1}{(2x+y+1)^2}$

$f(x,y)\approx f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{1}{2}[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)]$

$=2x+y+\frac{1}{2}(-4x^2-4xy-y^2)=(2x+y)-\frac{1}{2}(2x+y)^2$ quadratic approximation

$f_{xxx}=\frac{16}{(2x+y+1)^3}, \quad f_{xxy}=\frac{8}{(2x+y+1)^3}, \quad f_{xyy}=\frac{4}{(2x+y+1)^3} \quad f_{yyy}=\frac{2}{(2x+y+1)^3}$

$f(x,y)\approx \text{quadratic}+\frac{1}{6}[x^3f_{xxx}(0,0)+3x^2yf_{xxy}(0,0)+3xy^2f_{xyy}(0,0)+y^3f_{yyy}(0,0)]$

$=(2x+y)-\frac{1}{2}(2x+y)^2+\frac{1}{3}(2x+y)^3$ cubic approximation

⑫ $f(x,y)=e^x\sin y$

$f_x=e^x\sin y \quad f_y=e^x\cos y \quad f_{xx}=e^x\sin y \quad f_{xy}=e^x\cos y \quad f_{yy}=-e^x\sin y$

$f(x,y)\approx f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{1}{2}[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)]$

$=y+xy$ quadratic approximation

$f_{xxx}=e^x\sin y \quad f_{xxy}=e^x\cos y \quad f_{xyy}=-e^x\sin y \quad f_{yyy}=-e^x\cos y$

$|x|\le 0.1 \quad |e^x\sin y|\le|e^{0.1}\sin 0.1|\approx 0.11 \quad |e^x\cos y|\le|e^{0.1}\cos 0.1|\approx 1.11$

Therefore, $E(x,y)\le\frac{1}{6}[0.11(0.1)^3+3(1.11)(0.1)^3+3(0.11)(0.1)^3+1.11(0.1)^3]\le 0.000814$

15.1

⑨ $\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}\,dy\,dx$

$=\int_0^{\ln 2}[e^{2x+y}]_1^{\ln 5}\,dx=\int_0^{\ln 2}(e^{2x+\ln 5}-e^{2x+1})\,dx=\frac{1}{2}(e^{2x+\ln 5}-e^{2x+1})\Big|_0^{\ln 2}=\frac{1}{2}(e^{2\ln 2+\ln 5}-e^{2\ln 2+1}-e^{\ln 5}+e)=\frac{3}{2}(5-e)$

⑩ $\int_0^1\int_1^2 xye^x\,dy\,dx$

$=\int_0^1\left[\frac{xe^xy^2}{2}\right]_1^2 dx=\int_0^1\frac{3xe^x}{2}dx=\frac{3}{2}\int_0^1 xe^x dx=\frac{3}{2}(x-1)e^x\Big|_0^1=\frac{3}{2}$

⑱ $\iint_R y\sin(x+y)\,dA \qquad R: -\pi\le x\le 0,\ 0\le y\le\pi$

$=\int_0^\pi[-\cos(x+y)\cdot y]_{-\pi}^0\,dy=\int_0^\pi[-y\cos y+y\cos(y-\pi)]\,dy=\int_0^\pi -2y\cos y\,dy=(-2y\sin y-2\cos y)\Big|_0^\pi=4$

㉑ $\iint_R \frac{xy^3}{x^2+1}dA=\int_0^1\int_0^2\frac{xy^3}{x^2+1}dy\,dx=\int_0^1\left[\frac{xy^4}{4(x^2+1)}\right]_0^2 dx=\int_0^1\frac{4x}{x^2+1}dx=2\ln 2$

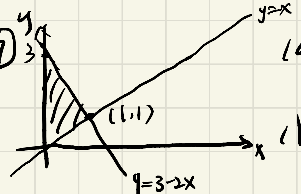
(26) $V=\iint_R f(x,y)\,dA=\int_0^2\int_0^2(16-x^2-y^2)\,dy\,dx=\int_0^2\left[16y-x^2y-\frac{1}{3}y^3\right]_0^2dx=\int_0^2\left(\frac{88}{3}-2x^2\right)dx=\left[\frac{88}{3}x-\frac{2}{3}x^3\right]_0^2=\frac{160}{3}$

(33) $\int_0^2\int_0^1\frac{x}{1+xy}\,dx\,dy=\int_0^1\int_0^2\frac{x}{1+xy}\,dy\,dx=\int_0^1\left|\ln(1+xy)\right|_0^2dx=\int_0^1\left|\ln(1+2x)\right|dx=\frac{1+2x}{2}[\ln(1+2x)-1]\Big]_0^1=\frac{3}{2}\ln3-1$

15.2

(12) (a) $\int_0^2\int_1^{e^x}dy\,dx$ (b) $\int_1^{e^2}\int_{\ln y}^2dx\,dy$

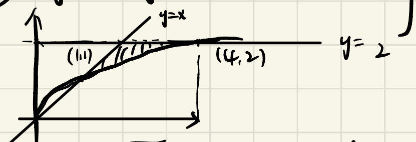
(17)



(a) $\int_0^1\int_x^{3-2x}dy\,dx$

(b) $\int_0^1\int_0^y dx\,dy+\int_1^3\int_0^{\frac{3-y}{2}}dx\,dy$

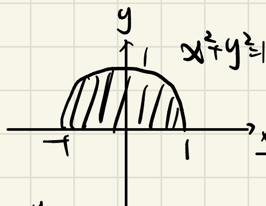
(22) $y=x$ $y=\sqrt{x}$

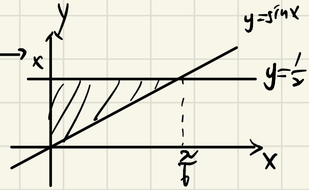


$\int_1^2\int_y^{y^2}dx\,dy=\int_1^2(y^2-y)\,dy=\left[\frac{y^3}{3}-\frac{y^2}{2}\right]_1^2=\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-\frac{1}{2}\right)=\frac{5}{6}$

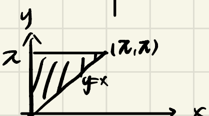
(41) $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}3y\,dx\,dy=\int_{-1}^1\int_0^{\sqrt{1-x^2}}3y\,dy\,dx$

$x=\pm\sqrt{1-y^2}\Rightarrow x^2=1-y^2\Rightarrow x^2+y^2=1$

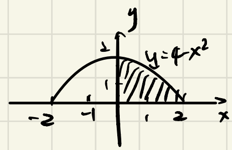




(44) $\int_0^{\frac{\pi}{6}}\int_{\sin x}^{\frac{1}{2}}xy^2\,dy\,dx=\int_0^{\frac{1}{2}}\int_0^{\sin^{-1}y}xy^2\,dx\,dy$



(47) $\int_0^\pi\int_x^\pi\frac{\sin y}{y}\,dy\,dx=\int_0^\pi\int_0^y\frac{\sin y}{y}\,dx\,dy=\int_0^\pi\sin y\,dy=2$



(50) $\int_0^2\int_0^{4-x^2}\frac{xe^{2y}}{4-y}\,dy\,dx=\int_0^4\int_0^{\sqrt{4-y}}\frac{xe^{2y}}{4-y}\,dx\,dy=\int_0^4\left[\frac{x^2e^{2y}}{2(4-y)}\right]_0^{\sqrt{4-y}}dy=\int_0^4\frac{e^{2y}}{2}\,dy=\frac{e^8-1}{4}$

(59) $V=\int_{-4}^1\int_{3x}^{4-x^2}(x+4)\,dy\,dx=\int_{-4}^1[xy+4y]_{3x}^{4-x^2}dx=\int_{-4}^1[x(4-x^2)+4(4-x^2)-3x^2-12x]\,dx=\left[-\frac{1}{4}x^4-\frac{7}{3}x^3-4x^2+16x\right]_{-4}^1$

$=\frac{157}{3}-\frac{1}{4}=\frac{625}{12}$

(72) $\int_0^\infty\int_0^\infty xe^{-(x+2y)}\,dx\,dy=\int_0^\infty e^{-2y}\lim_{b\to\infty}[-xe^{-x}-e^{-x}]_0^b\,dy=\int_0^\infty e^{-2y}\,dy=\frac{1}{2}\lim_{b\to\infty}(-e^{-2b}+1)=\frac{1}{2}$

(78) $\int_0^2(\tan^{-1}\pi x-\tan^{-1}x)\,dx=\int_0^2\int_x^{\pi x}\frac{1}{1+y^2}\,dy\,dx=\int_0^2\int_{\frac{y}{\pi}}^{y}\frac{1}{1+y^2}\,dx\,dy=\int_0^2\frac{(1-\frac{1}{\pi})y}{1+y^2}\,dy+\int_2^{2\pi}\frac{\left(2-\frac{y}{\pi}\right)}{1+y^2}\,dy$

$=\left(\frac{\pi-1}{2\pi}\right)\ln5+2\tan^{-1}2\pi-\frac{1}{2\pi}\ln(1+4\pi^2)-2\tan^{-1}2+\frac{1}{2\pi}\ln5=2\tan^{-1}2\pi-2\tan^{-1}2-\frac{1}{2\pi}\ln(1+4\pi^2)+\frac{\ln5}{2}$

(80) To minimize the integral, domain should include all points positive/negative

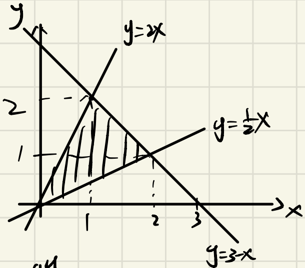
$x^2+y^2-9\le0$ or $x^2+y^2\le9$

15.3

(11) $\int_0^1\int_{\frac{x}{2}}^{2x} 1\,dy\,dx + \int_1^2\int_{\frac{x}{2}}^{3-x} 1\,dy\,dx$

$= \int_0^1 [y]_{\frac{x}{2}}^{2x}\,dx + \int_1^2 [y]_{\frac{x}{2}}^{3-x}\,dx$

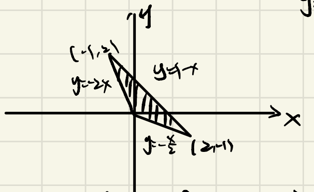
$= \int_0^1 (\frac{3}{2}x)\,dx + \int_1^2 (3-\frac{3}{2}x)\,dx = \frac{3}{2}$



(17) $\int_{-1}^0\int_{-2x}^{1-x} dy\,dx + \int_0^2\int_{-\frac{x}{2}}^{1-x} dy\,dx$

$= \int_{-1}^0 (1+x)\,dx + \int_0^2 (1-\frac{x}{2})\,dx$

$= \frac{3}{2}$



(21) Average height $= \frac{1}{4}\int_0^2\int_0^2 (x^2+y^2)\,dy\,dx = \frac{1}{4}\int_0^2 [x^2y+\frac{y^3}{3}]_0^2\,dx = \frac{1}{4}\int_0^2 (2x^2+\frac{8}{3})\,dx = \frac{8}{3}$

(26) $\int_0^1\int_{y^2}^{2y-y^2} 100(y+1)\,dx\,dy = \int_0^1 [100(y+1)x]_{y^2}^{2y-y^2}\,dy = \int_0^1 100(y+1)(2y-2y^2)\,dy = 200\int_0^1 (y-y^3)\,dy = 50$