STA2001 Assignment 9

1. (5.3-11). Let X_1, X_2, X_3 be three independent random variables with binomial distributions $b(4, 1/2)$, $b(6, 1/3)$, and $b(12, 1/6)$, respectively. Find

   (a) $P(X_1 = 2, X_2 = 2, X_3 = 5)$
   (b) $E(X_1 X_2 X_3)$
   (c) The mean and the variance of $Y = X_1 + X_2 + X_3$.

2. (5.3-19). Two components operate in parallel in a device, so the device fails when and only when both components fail. The lifetimes, X_1 and X_2, of the respective components are independent and identically distributed with an exponential distribution with $\theta = 2$. The cost of operating the device is $Z = 2Y_1 + Y_2$, where $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$. Compute $E(Z)$.

3. (5.3-21). Flip $n = 8$ fair coins and remove all that came up heads. Flip the remaining coins (that came up tails) and remove the heads again. Continue flipping the remaining coins until each has come up heads. We shall find the pmf of Y, the number of trials needed. Let X_i equal the number of flips required to observe heads on coin i, $i = 1, 2, \ldots, 8$. Then $Y = \max(X_1, X_2, \ldots, X_8)$.

   (a) Show that $P(Y \leq y) = [1 - (1/2)^y]^8$.
   (b) Show that the pmf of Y is defined by $P(Y = y) = [1 - (1/2)^y]^8 - \left[1 - (1/2)^{y-1}\right]^8$, $y = 1, 2, \ldots$.

4. (5.4-3). Let X_1, X_2, X_3 be mutually independent random variables with Poisson distributions having means 2, 1, and 4, respectively.

   (a) Find the mgf of the sum $Y = X_1 + X_2 + X_3$.
   (b) How is Y distributed?
   (c) Compute $P(3 \leq Y \leq 9)$.

5. (5.4-8). Let $W = X_1 + X_2 + \cdots + X_h$, a sum of h mutually independent and identically distributed exponential random variables with mean θ. Show that W has a gamma distribution with parameters $\alpha = h$ and θ, respectively.

6. (5.4-9). Let X and Y, with respective pmfs $f(x)$ and $g(y)$, be independent discrete random variables, each of whose support is a subset of the nonnegative integers $0, 1, 2, \ldots$ Show that the pmf of $W = X + Y$ is given by the **convolution formula**

   $$h(w) = \sum_{x=0}^{w} f(x) g(w - x), \quad w = 0, 1, 2, \ldots$$

   Hint: Argue that $h(w) = P(W = w)$ is the probability of the $w + 1$ mutually exclusive events $(x, y = w - x), x = 0, 1, \ldots, w$.

7. (a) (5.4-6). Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a geometric distribution with $p = 1/3$. Find the mgf of $Y = X_1 + X_2 + X_3 + X_4 + X_5$. How is Y distributed?

   (b) (5.4-7). Let X_1, X_2, X_3 denote a random sample of size 3 from a gamma distribution with $\alpha = 7$ and $\theta = 5$. Find the mgf of $Y = X_1 + X_2 + X_3$. How is Y distributed?

8. (5.4-16). The number X of sick days taken during a year by an employee follows a Poisson distribution with mean 2. Let us observe four such employees. Assuming independence, compute the probability that their total number of sick days exceeds 2.

9. Consider a group of 4 people. Each person randomly sends a like to one of the rest 3 people (i.e. with probability $\frac{1}{3}$ each). We say a person is popular if the person received at least 2 likes. Let N denote the number of popular people among the four.

   (a) Find $E[N]$

   (b) Find $Var(N)$

   (c) Find $P(N = 0)$

   Hint: Let X_i denote the event that i is a popular person, $i = 1, 2, 3, 4$. Let $I_i \triangleq I(X_i)$ denote the indicator random variable that equals 1 if X_i is true and 0 otherwise. By definition, $N = I_1 + I_2 + I_3 + I_4$.

10. This question centers around a technique known as importance sampling. Suppose that X and Y are continuous random variables with probability density function (pdf) denoted by f_X and f_Y respectively on a common sample space $\overline{S}$. We wish to estimate

$$I = E(g(X)) = \int_{\overline{S}} g(x) f_X(x)\, dx,$$

where g is a function such that $\mathrm{Var}(g(X))$ is finite. At times it maybe difficult to sample from f_X. However it is perhaps easier to sample from f_Y. With importance sampling, we can sample from f_Y and estimate I. Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed (i.i.d) random variables with pdf f_Y, and define

$$w(y) = \begin{cases} \dfrac{f_X(y)}{f_Y(y)}, \text{ if } f_Y(y) > 0, \\ 0, \text{ otherwise.} \end{cases}$$

$$J_n = \frac{1}{n} \sum_{i=1}^{n} g(Y_i) w(Y_i).$$

Prove that

(a) $E(J_n) = E(g(Y)w(Y)) = I$.

(b) $\mathrm{Var}(J_n) = \dfrac{1}{n}\left(E\left(g^2(X)w(X)\right) - I^2\right)$.