Review of Last Lecture

Key concepts and/or techniques:

1. Multivariate RV, $X_1, \cdots, X_n$
2. Independence of $X_1, \cdots, X_n$
3. Random sample of size n, i.e., i.i.d.
4. If $X_1, \cdots, X_n$ are independent, then

$$E[u_1(X_1)u_2(X_2)\cdots u_n(X_n)] = E[u_1(X_1)]E[u_2(X_2)]\cdots E[u_n(X_n)]$$

5. If $X_1, \cdots, X_n$ are independent, then

$$E[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i E[X_i]$$
$$Var[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i^2 Var[X_i]$$

Review of Last Lecture

- Discrete type: $X_1, X_2, \cdots, X_n$ are all discrete

  Joint pmf $f(x_1, \cdots, x_n) : \overline{S} \to (0, 1]$

  1. $f(x_1, \cdots, x_n) > 0, \quad (x_1, \cdots, x_n) \in \overline{S}$
  2. $\sum_{x_1, \cdots, x_n \in \overline{S}} f(x_1, \cdots, x_n) = 1$
  3. $P((X_1, \cdots, X_n) \in A) = \sum_{(x_1, \cdots, x_n) \in A} f(x_1, \cdots, x_n)$

- Continuous type: $X_1, X_2, \cdots, X_n$ are all continuous

  Joint pdf $f(x_1, \cdots, x_n) : \overline{S} \to (0, \infty)$

  1. $f(x_1, \cdots, x_n) > 0, \quad (x_1, \cdots, x_n) \in \overline{S}$
  2. $\int_{\overline{S}} f(x_1, \cdots, x_n) dx_1 \cdots dx_n = 1.$
  3. $P((X_1, \cdots, X_n) \in A) = \int_A f(x_1, \cdots, x_n) dx_1 \cdots dx_n.$

Review of Last Lecture

[N independent RVs]

The n RVs $X_1, \cdots, X_n$ are said to be (mutually) independent if

$$f(x_1, \cdots, x_n) = f_{X_1}(x_1) \cdot \cdots \cdot f_{X_n}(x_n),$$

where $f(x_1, \cdots, x_n)$ is the joint pmf or pdf of $X_1 \cdots X_n$, and $f_{X_i}(x_i)$ is the marginal pmf or pdf of X_i, $i = 1, \cdots, n$.

A necessary condition for the independence of the n RVs $X_1, \cdots, X_n$ is

$$\overline{S} = \overline{S_{X_1}} \times \cdots \times \overline{S_{X_n}}.$$

Remark: If $X_1, \cdots, X_n$ are independent, then any pair of them, any triple of them, $\cdots$, any $(n-1)$ of them are also independent.

Review of Last Lecture

[Theorem 5.3-1, page 191]

Assume that $X_1, X_2, \cdots, X_n$ are independent *RVs* and

$$Y = u_1(X_1)u_2(X_2)\cdots u_n(X_n)$$

If $E[u_i(X_i)], i = 1, \cdots, n$ exist. Then

$$E[Y] = E[u_1(X_1)u_2(X_2)\cdots u_n(X_n)]$$

$$= E[u_1(X_1)]E[u_2(X_2)]\cdots E[u_n(X_n)]$$

Remark: This is an extension of the result that when X and Y are independent, $E(XY) = E(X)E(Y)$.

Review of Last Lecture

[Theorem 5.3-2, page 192]

Assume that $X_1, X_2, \cdots, X_n$ are independent RVs with respective mean $\mu_1, \mu_2, \cdots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2$, respectively. Consider $Y = \sum_{i=1}^n a_i X_i$, where $a_1, a_2, \cdots, a_n$ are real constants. Then

$$E(Y) = \sum_{i=1}^n a_i \mu_i \quad \text{and} \quad Var(Y) = \sum_{i=1}^n a_i^2 \sigma_i^2.$$

STA2001 Probability and Statistics I

Lecture 21

Tianshi Chen

The Chinese University of Hong Kong, Shenzhen

Example 2, page 193

When $X_1, X_2, \cdots, X_n$ are independent and identically distributed RV with mean μ and variance σ^2. Consider

$$\overline{X} = \sum_{i=1}^{n} \frac{1}{n} X_i.$$

- $\overline{X}$ is a function of $X_1, X_2, \cdots, X_n$.
- $\overline{X}$ has the following mean and variance:

$$E(\overline{X}) = \sum_{i=1}^{n} \frac{1}{n}\mu = \mu,$$

$$Var(\overline{X}) = \sum_{i=1}^{n} (\frac{1}{n})^2 \sigma^2 = \frac{\sigma^2}{n}$$

Statistic

Definition

Any function of the random sample $X_1, X_2, \cdots, X_n$ that do not have any unknown parameters is called a statistic.

Definition

Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed with mean μ. Then the sample mean is defined as

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

and a <u>statistic</u> and also an <u>estimator</u> of mean μ.

In what follows, we will study the properties of the sample mean $\overline{X}$.

Section 5.4 Moment generating function technique

Motivation and Goal

Motivation

Mgf, if exists, uniquely determines the distribution of the RV. Therefore, the distribution of a RV can be equivalently found via its mgf.

Goal

To derive the distribution of functions of multivariate RVs $X_1, \cdots, X_n$ with mgf technique, where the function takes the form of

$$Y = \sum_{i=1}^{n} a_i X_i$$

Example 1, page 195

Example

Let X_1 and X_2 be independent RVs with uniform distribution on $\{1, 2, 3, 4\}$. Let $Y = X_1 + X_2$. What is the distribution of Y, i.e., pmf of Y?

Example 1, page 195

Example

Let X_1 and X_2 be independent RVs with uniform distribution on $\{1, 2, 3, 4\}$. Let $Y = X_1 + X_2$. What is the distribution of Y, i.e., pmf of Y?

$$
\begin{aligned}
M_Y(t) &= E(e^{tY}) = E[e^{t(X_1+X_2)}] \\
&= E(e^{tX_1}) \cdot E(e^{tX_2}) \\
&= M_{X_1}(t) \cdot M_{X_2}(t)
\end{aligned}
$$

Example 1, page 195

Example

Let X_1 and X_2 be independent RVs with uniform distribution on $\{1, 2, 3, 4\}$. Let $Y = X_1 + X_2$. What is the distribution of Y, i.e., pmf of Y?

$$\begin{aligned} M_Y(t) &= E(e^{tY}) = E[e^{t(X_1+X_2)}] \\ &= E(e^{tX_1}) \cdot E(e^{tX_2}) \\ &= M_{X_1}(t) \cdot M_{X_2}(t) \end{aligned}$$

$$f(x) = \frac{1}{4}, \quad x = 1, 2, 3, 4$$

$$\Rightarrow M_X(t) = E(e^{tX}) = \sum_{x=1}^{4} f(x)e^{tx} = \frac{1}{4}\sum_{x=1}^{4} e^{tx}$$

Example 1, page 195

$$
\begin{aligned}
M_Y(t) &= M_{X_1}(t) \cdot M_{X_2}(t) \\
&= \left(\frac{1}{4} \sum_{x_1=1}^{4} e^{tx_1} \right) \left(\frac{1}{4} \sum_{x_2=1}^{4} e^{tx_2} \right) \\
&= \frac{1}{16} e^{2t} + \frac{2}{16} e^{3t} + \frac{3}{16} e^{4t} + \frac{4}{16} e^{5t} + \frac{3}{16} e^{6t} + \frac{2}{16} e^{7t} + \frac{1}{16} e^{8t}
\end{aligned}
$$

Example 1, page 195

$$
\begin{aligned}
M_Y(t) &= M_{X_1}(t) \cdot M_{X_2}(t) \\
&= \left(\frac{1}{4} \sum_{x_1=1}^{4} e^{tx_1} \right) \left(\frac{1}{4} \sum_{x_2=1}^{4} e^{tx_2} \right) \\
&= \frac{1}{16} e^{2t} + \frac{2}{16} e^{3t} + \frac{3}{16} e^{4t} + \frac{4}{16} e^{5t} + \frac{3}{16} e^{6t} + \frac{2}{16} e^{7t} + \frac{1}{16} e^{8t}
\end{aligned}
$$

Then the pmf of Y can be derived as follows

$$
\begin{aligned}
&\overline{S_Y} = \{2, 3, \cdots, 8\} \\
&g(y) = P(Y = y) = \text{the coefficient of } e^{yt}, y \in \overline{S_Y}.
\end{aligned}
$$

Theorem 5.4-1, page 196

If $X_1, X_2, \cdots, X_n$ are independent RVs with respective mgfs $M_{X_i}(t)$

where $|t| < h_i$ for $h_i > 0, i = 1, 2, \cdots, n$. Then the

mgf of $Y = \sum_{i=1}^{n} a_i X_i$ is

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t),$$

where $|a_i t| < h_i, i = 1, \cdots, n$.

Proof of Theorem 5.4-1, page 196

$$M_Y(t) = E[e^{tY}] = E[e^{t\sum_{i=1}^{n} a_i X_i}]$$

$$= E[e^{ta_1X_1} e^{ta_2X_2} \cdots e^{ta_nX_n}]$$

$$\overset{\text{by Thm5.3-1 page 191}}{=\!=\!=\!=\!=} E[e^{a_1tX_1}] \cdots E[e^{a_ntX_n}]$$

$$\overset{M_X(t)=E[e^{tX}]}{=\!=\!=\!=} \prod_{i=1}^{n} M_{X_i}(a_i t)$$

$M_{X_i}(t)$ is defined for $|t| < h_i$,
$M_{X_i}(a_i t)$ is defined for $|a_i t| < h_i$.

Corollary 5.4-1, page 197

If $X_1, X_2, \cdots, X_n$ is a random sample of size n from a distribution with mgf $M(t)$, where $|t| < h$, then

(a) The mgf of $Y = \sum_{i=1}^{n} X_i$, is

$$M_Y(t) = \prod_{i=1}^{n} M(t) = (M(t))^n, \quad |t| < h$$

(b) The mgf of $\overline{X} = \sum_{i=1}^{n} \frac{1}{n} X_i$ is

$$M_{\overline{X}}(t) = \prod_{i=1}^{n} M\left(\frac{1}{n} t\right) = \left[M\left(\frac{t}{n}\right)\right]^n, \quad \left|\frac{t}{n}\right| < h$$

Example 2, page 197

Let $X_1, X_2, \cdots, X_n$ denote the outcome of n Bernoulli trials each with probability of success p. Let $Y = \sum_{i=1}^{n} X_i$, then what is the distribution of Y?

Example 2, page 197

Let $X_1, X_2, \cdots, X_n$ denote the outcome of n Bernoulli trials each with probability of success p. Let $Y = \sum_{i=1}^{n} X_i$, then what is the distribution of Y?

Recall that the mgf of $X_i, i = 1, 2, \cdots, n$ is

$$M(t) = 1 - p + pe^t, \quad -\infty < t < \infty$$

Then by Corollary 5.4-1,

$$M_Y(t) = \prod_{i=1}^{n}(1 - p + pe^t) = (1 - p + pe^t)^n \Longrightarrow Y \sim b(n, p)$$

Theorem 5.4-2, page 198

Let $X_1, X_2, \cdots, X_n$ be independent chi-square RVs with

$r_1, r_2, \cdots, r_n$ degrees of freedom, respectively, i.e.,

$X_i \sim \chi^2(r_i), i = 1, \cdots, n$. Then

$$Y = X_1 + X_2 + \cdots + X_n \quad \text{is} \quad \chi^2(r_1 + r_2 + \cdots + r_n)$$

Proof of Theorem 5.4-2, page 198

Recall from Thm 5.4-1, if $X_1, \cdots, X_n$ are independent RVs

with respective mgfs $M_{X_i}(t), |t| < h_i, i = 1, \cdots, n$. Then

$Y = \sum_{i=1}^{n} X_i$ has the mgf

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t), |t| < h_i, i = 1, \cdots, n.$$

Then recall that the mgf for chi-square distribution with degree of

freedom r_i is

$$M_{X_i}(t) = (1 - 2t)^{-\frac{r_i}{2}}$$

Proof of Theorem 5.4-2, page 198

Then we have

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$= (1-2t)^{-\frac{r_1}{2}}(1-2t)^{-\frac{r_2}{2}}\cdots(1-2t)^{-\frac{r_n}{2}}$$

$$= (1-2t)^{-\frac{1}{2}(r_1+\cdots+r_n)} \Rightarrow Y \sim \chi^2(r_1+\cdots+r_n)$$

Corollary 5.4-2, page 198

Let $Z_1, Z_2, \cdots, Z_n$ have standard normal distributions, $N(0,1)$. If these random variables are independent, then

$$W = Z_1^2 + Z_2^2 + \cdots + Z_n^2 \sim \chi^2(n)$$

Proof of Corollary 5.4-2, page 198

Recall from Theorem 3.3-2, if

$$X \sim N(\mu, \sigma^2) \quad \text{with} \quad \sigma^2 > 0,$$

then

$$\frac{(X-\mu)^2}{\sigma^2} \sim \chi^2(1)$$

For the current case, $Z_i^2 \sim \chi^2(1)$, $i = 1, \cdots, n$. Then by Theorem 5.4-2 and the independence of $Z_1, \cdots, Z_n$,

$$W = \sum_{i=1}^{n} Z_i^2 \sim \chi^2(n).$$

Corollary 5.4-3, page 198

If $X_1, X_2, \cdots, X_n$ are independent and have normal distributions $N(\mu_i, \sigma_i^2), i = 1, 2, ..., n$, respectively, then the distribution of

$$W = \sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i} \right)^2 \sim \chi^2(n)$$

Proof: Let $Z_i = \frac{X_i - \mu_i}{\sigma_i}$, obviously, $Z_i \sim N(0,1)$ and $Z_i^2 \sim \chi^2(n)$, $i = 1, \cdots, n$. Then by Theorem 5.4.2 and the independence of $X_1, \cdots, X_n$, we complete the proof.