STA2001 Tutorial 10

1. 4.5-8. Let X and Y have a bivariate normal distribution with parameters $\mu_X = 10$, $\sigma_X^2 = 9$, $\mu_Y = 15$, $\sigma_Y^2 = 16$ and $\rho = 0$. Find

   (a) $P(13.6 < Y < 17.2)$
   (b) $E(Y|x)$
   (c) $\text{Var}(Y|x)$
   (d) $P(13.6 < Y < 17.2|X = 9.1)$

   Solution:

   (a) Note the marginal distribution for Y is a normal distribution, with parameters μ_Y and σ_Y^2.
   Therefore,

$$\begin{aligned} P(13.6 < Y < 17.2) &= P\left(\frac{13.6-15}{4} < \frac{Y-15}{4} < \frac{17.2-15}{4}\right) \\ &= P(-0.35 < Z < 0.55) \\ &= \Phi(0.55) - \Phi(-0.35) \\ &= \Phi(0.55) - (1 - \Phi(0.35)) \\ &= 0.3456 \end{aligned}$$

   where Z is the standard normal random variable.

   (b) As (X, Y) is a bivariate normal distribution, the conditional random variable $Y|X = x$ also follows a normal distribution, i.e. $Y|x \sim N(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_X), \sigma_X^2(1-\rho)^2)$ (You may the proof of this in the textbook).
   Hence,

$$E(Y|x) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 15 + 0 \times \frac{4}{3}(x - 10) = 15$$

   (c) Follows the answer to (b), we have

$$\text{Var}(Y|x) = \sigma_Y^2(1 - \rho^2) = 16 \times (1 - 0) = 16$$

   (d) Since (X, Y) follows a bivariate normal distribution and $\rho = 0$, X and Y are independent. Note that this case is a special case, and **in general not correlated does not imply independence.**
   Therefore we can drop the condition since X and Y are independent:

$$P(13.6 < Y < 17.2|X = 9.1) = P(13.6 < Y < 17.2) = 0.3456$$

2. 5.1-10. Let X has the uniform distribution $U(-1,3)$. Find the pdf of $Y = X^2$.

Solution:

Method 1:

Since X has the uniform distribution $U(-1,3)$, that is, $-1 < x < 3$, and thus the support of $Y = X^2$ is $0 \le y < 9$. Furthermore, the pdf of X is given by

$$f(x) = \frac{1}{4}, \qquad -1 < x < 3$$

When $0 \le y < 1$ and therefore $-1 < x < 1$, we obtain the two-to-one transformation represented by

$$x_1 = -\sqrt{y} \text{ when } -1 < x_1 < 0, \text{ so } \tfrac{dx_1}{dy} = \tfrac{-1}{2\sqrt{y}}$$
$$x_2 = \sqrt{y} \text{ when } 0 < x_2 < 1, \text{ so } \tfrac{dx_2}{dy} = \tfrac{1}{2\sqrt{y}}$$

When $1 < y < 9$ and therefore $1 < x < 3$, we obtain the one-to-one transformation represented by

$$x_3 = \sqrt{y} \text{ when } 1 < x_3 < 3, \text{ so } \tfrac{dx_3}{dy} = \tfrac{1}{2\sqrt{y}}$$

Thus,

$$g(y) = \begin{cases} \frac{1}{4} \cdot |\frac{-1}{2\sqrt{y}}| + \frac{1}{4} \cdot |\frac{1}{2\sqrt{y}}| = \frac{1}{4\sqrt{y}}, & 0 < y < 1 \\ \frac{1}{4} \cdot |\frac{1}{2\sqrt{y}}| = \frac{1}{8\sqrt{y}}, & 1 < y < 9 \end{cases}$$

Note: If you have questions regarding this method, please read the textbook section 5.1 carefully.

Method 2:

Let $f(x)$ and $F(x)$ be the pdf and cdf of the random variable X. Note that $-1 < x < 3$, so we must have the support for Y, $0 \le y < 9$.

Now we try to write the cdf of Y in terms of the cdf of X,

$$G(y) = P(Y \le y) = P(X^2 \le y)$$

Naturally we will write $P(-\sqrt{y} \le X \le \sqrt{y})$ in the next equality, but this is **NOT ALWAYS TRUE** in this case. For instance, let $y = 1.1$ and then $P(X^2 \le 1.1) = P(-\sqrt{1.1} \le X \le \sqrt{1.1})$, but this is obviously not true as X only takes values on the interval $(-1, 3)$. This tells us we need to consider different cases when y takes different values.

(1) When $0 < y < 1$ and therefore $-1 < x < 1$, we write

$$\begin{aligned} G(y) = P(X^2 \le y) &= P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= F(\sqrt{y}) - F(-\sqrt{y}) \end{aligned}$$

$$g(y) = \frac{d}{dy}G(y) = f(\sqrt{y})(\sqrt{y})' - f(-\sqrt{y})(-\sqrt{y})' = \frac{1}{4\sqrt{y}}, \quad 0 < y < 1$$

(2) When $1 \le y < 9$ and therefore $1 \le x < 3$, we write

$$\begin{aligned} G(y) = P(X^2 \le y) &= P(1 \le X \le \sqrt{y}) \\ &= F(\sqrt{y}) - F(1) \end{aligned}$$

$$g(y) = \frac{d}{dy}G(y) = f(\sqrt{y})(\sqrt{y})' = \frac{1}{8\sqrt{y}}, \quad 1 \le y < 9$$

The results are the same as we obtained using method 1.

3. 5.1-14. Let X be $N(0,1)$. Find the pdf of $Y = |X|$, a distribution that is often called the half-normal.

   Hint: Here $y \in S_y = \{y : 0 < y < \infty\}$. Consider the two transformations $x_1 = -y$, $-\infty < x_1 < 0$, and $x_2 = y$, $0 < x_2 < \infty$.

   Solution:

   Since the distribution of X is $N(0,1)$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad -\infty < x < \infty$$

   Note that $y = |x|$, so $0 \leq y < \infty$, and thus we have a two-to-one transformation.

   When $-\infty < x_1 < 0$, $x_1 = -y$. Therefore,

$$\frac{dx_1}{dy} = \frac{d-y}{dy} = -1$$

   When $0 < x_2 < \infty$, $x_2 = y$. Therefore,

$$\frac{dx_2}{dy} = \frac{dy}{dy} = 1$$

   Thus the pdf of Y is

$$\begin{aligned} g(y) &= f(-y)\left|\frac{dx_1}{dy}\right| + f(y)\left|\frac{dx_2}{dy}\right| \\ &= \frac{1}{\sqrt{2\pi}} e^{-(-y)^2/2} \cdot |-1| + \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \cdot |1| \\ &= \frac{2}{\sqrt{2\pi}} e^{-y^2/2} \end{aligned}$$

   where $0 < y < \infty$.