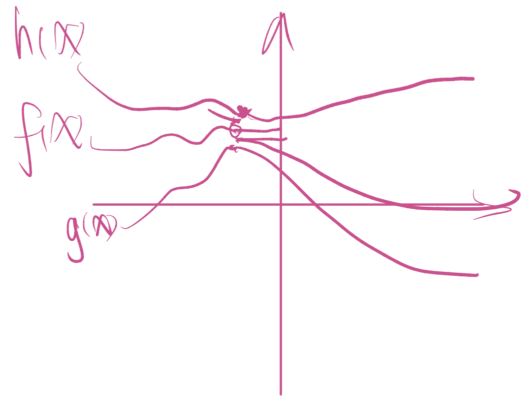


1. T F F T F

2. (i) C

(ii) B

(iii) $2x\sin^3(4x) + x^2 3\sin^2(4x)\cos(4x)\cdot 4$

(iv) $F'' = fg'' + 2f'g' + f''g$

(v) $$\frac{w(x)\left(u'(x)+2v(x)v'(x)\right) - \left(u(x)+v^2(x)\right)w'(x)}{w^2(x)}$$

(vi) velocity: $-4\ \text{cm/s}$

acceleration: $-4\sqrt{3}\ \text{cm/s}^2$

(vii) Toward negative x-axis / $-\infty$ / moving left.

(viii) $y = -\frac{1}{x} + \frac{1}{2}x^2 - \frac{1}{2}$ (or $\frac{x^3-x-2}{2x}$).

3. (i) <u>Proof</u>: Let $f(x) := x^3+3x+1$.

- Since $f(-1) = -3$, $f(0) = 1$, and f is continuous, there exists $c \in (-1, 0)$ such that $f(c) = 0$.

- Hence $x = c$ is a solution and $|c| < 1$. $\square$

(ii) $f'(x) = 3x^2+3$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{3} = \frac{-1}{3}.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -\frac{1}{3} - \frac{-\frac{1}{27}}{\frac{1}{3}+3} = -\frac{1}{3} + \frac{1}{90}$$

$$= \frac{-29}{90}$$

4. (i) $$\lim_{x\to 0}\frac{8x}{3\sin x - x} = \lim_{x\to 0} 8\,\frac{\frac{x}{\sin x}}{3-\frac{x}{\sin x}}$$

$$= 8\,\frac{\lim_{x\to 0}\frac{x}{\sin x}}{3-\lim_{x\to 0}\frac{x}{\sin x}} = \frac{8}{3-1} = 4.$$

(ii) $$\lim_{x\to -\infty}\sqrt{x^2+3x}-\sqrt{x^2-2x} = \lim_{x\to -\infty}\frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to -\infty}\frac{5x}{\sqrt{x^2(1+\frac{3}{x})}+\sqrt{x^2(1-\frac{2}{x})}} = \lim_{x\to -\infty}\frac{5x}{-x\left(\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}\right)}$$

$\sqrt{x^2} = -x$ for $x<0$

$$= \frac{-5}{\lim_{x\to -\infty}\sqrt{1+\frac{3}{x}}+\lim_{x\to -\infty}\sqrt{1-\frac{2}{x}}} = \frac{-5}{2}.$$

(iii) $$\lim_{x\to 9}\frac{\sin(\sqrt{x}-3)}{x-9} = \lim_{x\to 9}\frac{\sin(\sqrt{x}-3)}{\sqrt{x}-3}\,\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$$

$y=\sqrt{x}-3$

$$= \lim_{y\to 0}\frac{\sin y}{y}\,\lim_{x\to 9}\frac{1}{\sqrt{x}+3} = 1\cdot\frac{1}{6} = \frac{1}{6}.$$

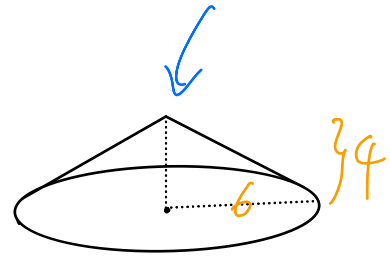
5. (i) $B(t) = M(t)N(t)$

$B'(t) = M(t)N'(t) + M'(t)N(t)$.

(ii) $B'(4) = M(4)N'(4) + M'(4)N(4)$

$= 1.2 \cdot 50 + 0.1 \cdot 820 = 60 + 82 = 142$ (g/week).

6. • Let $h(t)$ be the water level,
$r(t)$ be the radius of water surface
at time t, and
$V(t)$ be the volume of water.



• Then $r(t) = \frac{3}{2}(4-h(t))$, and at time t,

$$\begin{aligned} V &= \text{tank volume} - \text{cone above water} \\ &= \frac{1}{3}\pi \cdot 6^2 \cdot 4 - \frac{1}{3}\pi\left[\frac{3}{2}(4-h)\right]^2 \cdot (4-h) \\ &= \frac{1}{3}\pi\left(144 - \frac{9}{4}(4-h)^3\right) \end{aligned}$$

$$\left(= \frac{1}{3}\pi\left(-108h + 27h^2 - \frac{9}{4}h^3\right). \right)$$

• $\frac{dV}{dh} = \frac{9}{12} \cdot 3\pi(4-h)^2 = \frac{9}{4}\pi(4-h)^2.$

• Since $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$, at time t_0 when $h(t) = 2$,

$$\left.\frac{dh}{dt}\right|_{t=t_0} = \frac{\left.dV/dt\right|_{t=t_0}}{\left.dV/dh\right|_{h=2}} = \frac{0.5}{9\pi} = \frac{1}{18\pi}\ (m/min).$$

7. On the curve $\frac{x^2}{4}+\frac{y^2}{9}=1$, $\frac{dy}{dx}$ satisfies

$$\frac{1}{2}x+\frac{2}{9}y\cdot\frac{dy}{dx}=0,$$

So $\frac{dy}{dx}=-\frac{1}{2}x\cdot\frac{9}{2}\frac{1}{y}$.

When $x=1$ and $y>0$, we have $y=\frac{3\sqrt{3}}{2}$, and

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=-\frac{1}{2}\cdot\frac{9}{2}\frac{2}{3\sqrt{3}}\cdot 6=-\frac{9}{\sqrt{3}}=-3\sqrt{3}\ (m/s).$$

Alternatively, apply $\frac{d}{dt}$ on $\frac{x^2}{4}+\frac{y^2}{9}=1$. Then

$$\frac{1}{2}x\cdot x'(t)+\frac{2}{9}y\cdot y'(t)=0$$

$$\Rightarrow\ y'(t)=\frac{-1}{2}x\,x'(t)\cdot\frac{9}{2y}.$$

Substituting $x=1$ and $y=\frac{3\sqrt{3}}{2}$ in gives the same answer.

8. (i) Since f is differentiable everywhere, by the MVT,

$$f(0) = f(-6) + f'(c)6 = 4 + 6f'(c)$$

for some $c \in (-6, 0)$. Since $f'(c) \leq 3$, we have

$$f(0) \leq 4 + 6 \cdot 3 = 22.$$

largest possible.

(ii) Example appears when $f'(x) = 3$ for all $x \in (-6, 0)$.

One example could be

$$f(x) = 4 + 3(x+6) = 3x + 22.$$

9. (i) $f'(x) = 2\cos x(-\sin x) - 2\cos x$

$$= -2\cos x(\sin x + 1)$$

$f'(x) = 0 \iff x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

f'		$-$		$+$		$-$	
f	0	$\searrow$	$\frac{\pi}{2}$	$\nearrow$	$\frac{3\pi}{2}$	$\searrow$	2π

f is increasing on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

and decreasing on $\left[0, \frac{\pi}{2}\right]$ and on $\left[\frac{3\pi}{2}, 2\pi\right]$.

(ii) Critical points are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

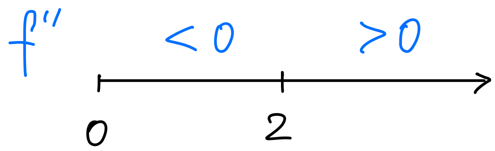
(iii) Local maxima occur at : $x=0$ & $x = \frac{3\pi}{2}$.

Local minima occur at : $x = \frac{\pi}{2}$ & $x = 2\pi$.

10. (i) $f'(x) = 3x^2 - 12x + 6$

$f''(x) = 6x - 12.$

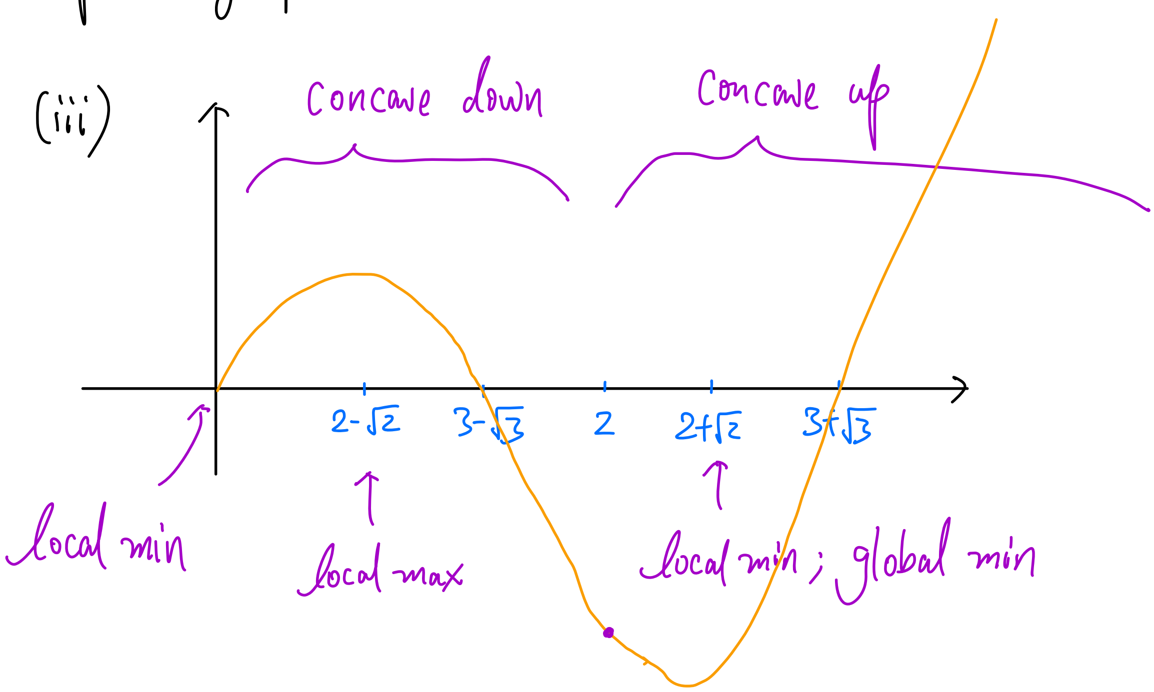
$f''(x) = 0 \iff x = 2.$



f is concave down on $[0,2]$ and concave up on $[2,\infty)$.

(ii) Since $y = f(x)$ has a tangent line at $x=2$ (because f is differentiable), $x = 2$ is an inflection point by part (i).

(iii)



11. Let $g(x) := (1+f(x)\sin(x))^{1/3}$.

Since $g(x)^3 - 1 = (g(x)-1)(g(x)^2+g(x)+1)$, we have

$$L := \lim_{x\to 0} \frac{g(x)-1}{x} = \lim_{x\to 0} \frac{g(x)^3-1}{x} \lim_{x\to 0} \frac{1}{g(x)^2+g(x)+1}$$

$$= \lim_{x\to 0} \frac{f(x)\sin x}{x} \lim_{x\to 0} \frac{1}{g(x)^2+g(x)+1}$$

$$= \lim_{x\to 0} f(x) \lim_{x\to 0} \frac{\sin x}{x} \left(\frac{1}{(\lim_{x\to 0} g(x))^2 + \lim_{x\to 0} g(x) + 1} \right).$$

Since $\lim_{x\to 0} g(x) = (1+f(0)\sin(0))^{1/3} = 1$

and $\lim_{x\to 0} f(x) = f(0) = 3$ by continuity of f, we have

$$L = 3 \cdot 1 \cdot \frac{1}{1^2+1+1} = 1.$$