

FIN2010 Financial Management

Time value of money I



Agenda

- What is the time value of money (TVM)?
- TVM of a single cash flow
 - APR and EAR
- TVM of cash flow streams



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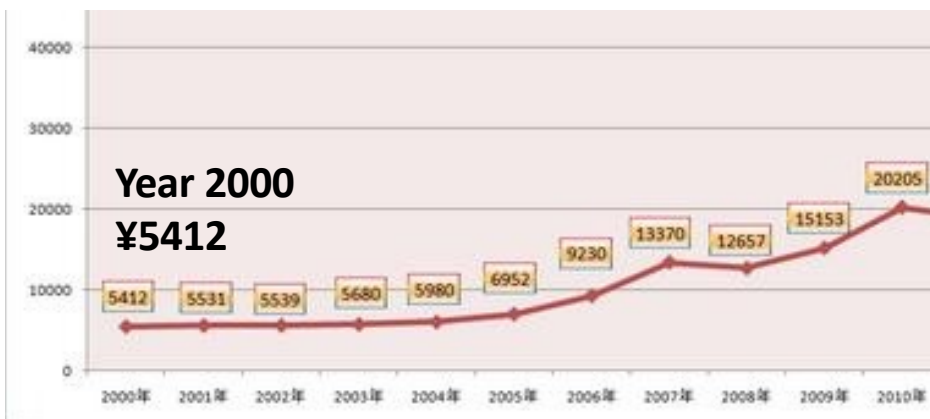


Time Value of Money

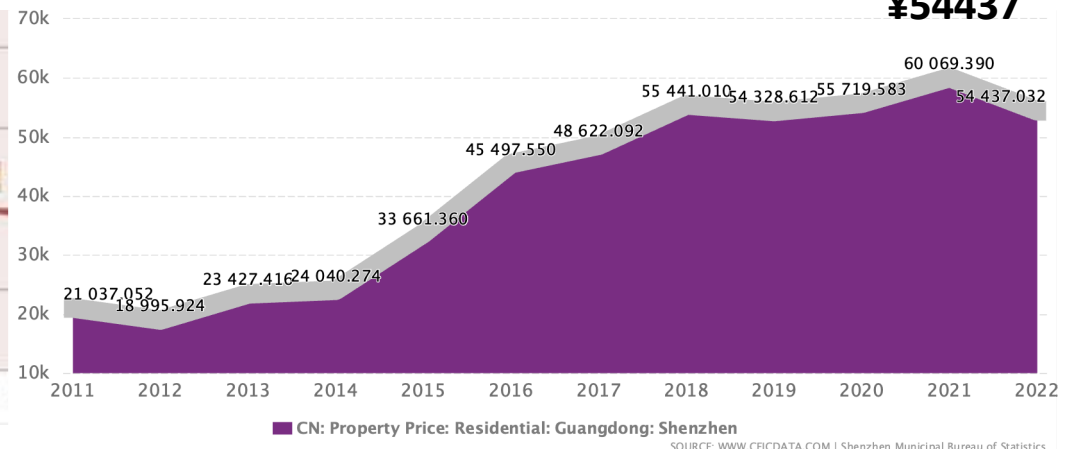
- Money has a time value
 - A dollar today is worth more than a dollar tomorrow

Property Price (/m²): Residential: Shenzhen 2000-2010 & 2011-2022

Year 2022
¥54437



[Sina](#)



[CEIC](#)

- If I had 1 million in 2000 and used it to buy an apartment in Shenzhen, I would have assets worth of ~10 million by 2022.



Key Concepts

- Present Value – what a cash flow is worth at the beginning of an investment period
- Future Value – what a cash flow is worth in the future
- Interest rate – “exchange rate” between earlier money and later money
 - Discount rate
 - Cost of capital
 - Opportunity cost of capital
 - Required return



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Future Values

- Suppose you invest 1,000 for one year at 5% per year.
What is the future value in one year?
 - $\text{Interest} = 1,000 \times 5\% = 50$
 - $\text{Value in one year} = \text{initial investment} + \text{interest} = 1,000 + 50 = 1,050$
 - $\text{Future value (FV)} = 1,000(1 + 5\%) = 1,050$



Future Values

- Suppose you leave the money in for another year. How much will you have two years from now?
 - **Compound interest:** interest is earned on both
 - the initial investment
 - the interest from previous period
 - $FV \text{ with compound interest} = 1050(1+0.05) = 1,102.50$
 - **Simple interest:** interest is earned only on
 - the initial investment
 - $FV \text{ with simple interest} = 1,000 + 50 + 50 = 1,100$
 - The extra 2.50 comes from the interest of $.05(50) = 2.50$ earned on the first interest payment



When do We Use Simple/Compound Interest?

In reality,

- Simple interest occurs if we do not reinvest the interest
 - Is rarely used.
 - E.g., Chinese banks pay simple interest on timed deposits
 - If you deposit 1000 into 工商银行 for 5 years, you will get
 $1000(1 + 2.65\% * 5) = 1132.5$
- Compound interest occurs if we do reinvest the interest
 - Is commonly used
 - E.g., if you owe money, your credit card balance compounds every day

(2023) 国有银行活期、定期存款利率表（整存整取）								
序号	银行名称	活期	三个月	半年	一年	二年	三年	五年
1	中国银行	0.25	1.25	1.45	1.65	2.15	2.60	2.65
2	农业银行	0.25	1.25	1.45	1.65	2.15	2.60	2.65
3	工商银行	0.25	1.25	1.45	1.65	2.15	2.60	2.65
4	建设银行	0.25	1.25	1.45	1.65	2.15	2.60	2.65
5	交通银行	0.25	1.25	1.45	1.65	2.15	2.60	2.65
6	邮储银行	0.25	1.25	1.46	1.68	2.15	2.60	2.65
		单位：% 数据来源：各大银行官网、APP、微信公众号等						

[21JINGJI](#)



Future Values: General Formula

- $FV = PV(1 + r)^t$
 - FV = future value
 - PV = present value
 - r = per period interest rate
 - t = number of periods
- Suppose you deposit 1,000 in a bank today for five years. The bank pays compound interest of 2.75% per year. How much will you have in five years?
 - $FV = 1,000(1+2.75\%)^5 = 1,145.27$
- What is the effect of compounding?
 - Simple interest = $1000 + 1000 \times 5 \times 2.75\% = 1137.5$
 - Compounding added 7.77 to the value of the investment

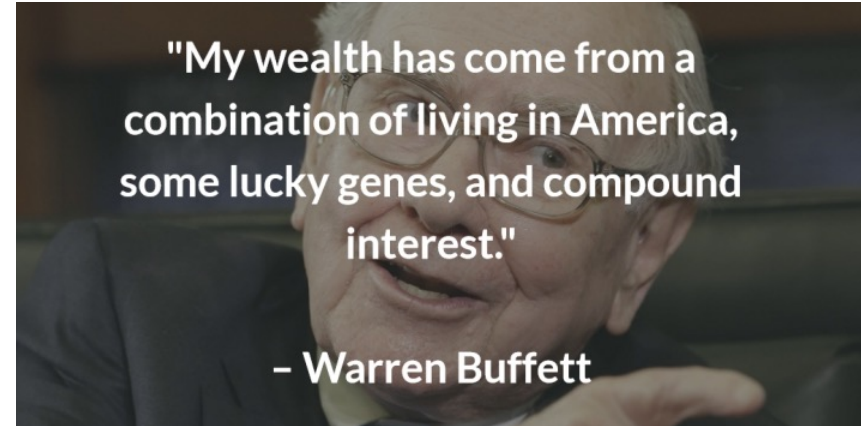
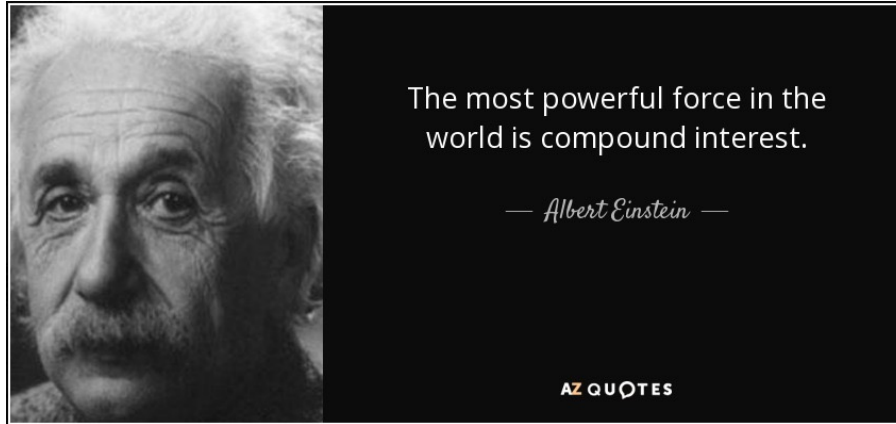


Interpretation of PV and FV

- PV and FV help us establish equivalence between cash now and cash in the future.
- Therefore, we can use PV and FV to compare cash flow at different points in time.
 - If the interest rate is 3%, will you prefer 1,000 today or 1,145.27 in five years?
 - $FV \text{ of } 1,000 = 1,000 * (1 + 3\%)^5 = 1,159.27.$
 - | Today | | 5 years later |
|-------|---|---------------|
| 1,000 | = | 1,159.27 |
| | | ↓ |
| | | 1,145.27 |
 - Therefore, 1,000 today is better than 1,145.27 in five years.



The Power of Compounding



- Between 1965 and 2022, Berkshire Hathaway (CEO: Warren Buffett) earned an average return of 19.9% per year.
- Suppose you invested \$1000 with Warren Buffet at the beginning of 1965, how much did you have at the end of 2022?
 - $\$1000 * (1 + 19.9\%)^{58} = \$37,283,151.9$
 - **When it comes to investment, stay cool and let time be your best friend!**



The Power of Compounding

搜狐新闻 > 最新要闻 > 世态万象

郑州21岁大学生因无力偿还60万元网贷跳楼自杀



大渝新闻

重庆一大学生疑因无力归还网贷自杀

- Not understanding the compounding interest has cost many people fortune and even lives!



香港中文大學(深圳)
The Chinese University of Hong Kong, Shenzhen

經管學院
School of Management and Economics

Solve for PV, t, r

- Start with the basic equation: $FV = PV(1 + r)^t$
- $PV = FV (1 + r)^{-t}$
- $R = \left(\frac{FV}{PV}\right)^{1/t} - 1$
- $t = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1+r)}$



Present Values-Example

- Suppose you need \$95,000 to pay your tuition next year. If you can earn 7% annually, how much do you need to invest today?

- $PV = 95,000(1.07)^{-1} = 88785.05$



Discount Rate-Example

- Li Ka-shing is famous for “hoarding” land for profit. His company bought 1.64 million m² of land (Yangjiashan) in Chongqing for 2.45B in 2007. His company offered to sell 1.03 million m² of the land (mostly undeveloped) for 20B in 2018. What is the annual appreciate rate of the land? Assume the per m² land price is the same for the entire area.



- Original cost of land for 1.03 million m² = $2.45\text{B} * \frac{1.03}{1.64} = 1.54\text{B}$
- $r = \left(\frac{20}{1.54}\right)^{1/11} - 1 = 26.3\%$



Number of Periods – Example

- Suppose you want to buy a new house. You currently have \$15,000, and you need \$20,000 for the down payment. You can earn 7.5% per year. How long does it take to save enough for the down payment?
 - $t = \ln(20,000/15,000) / \ln(1.075) = 3.98 \text{ years}$



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- What is the time value of money (TVM)?
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- TVM of cash flow streams



Annual Percentage Rate (APR)

- Interests can be charged in any frequency
 - Credit card: daily
 - Home mortgage: monthly
 - Bonds: semiannually
- Interest rate is required by law to be quoted in annual frequency, the so-called annual percentage rate (APR)
 - $APR = \text{per period rate} * \text{the number of periods per year (m)}$
 - So, $\text{per period rate} = APR / \text{number of periods per year (m)}$
 - $FV = PV * \left(1 + \frac{APR}{m}\right)^{m*n}$
where n is the number of years



APR vs. Period Rate

- What is the APR if the monthly rate is 0.5%?
 - $0.5\% \times 12 = 6\%$
- A 6% bond pays interests semiannually. What is the amount of each interest?
 - $6\% / 2 = 3\%$
- Currently the home mortgage rate is 9%, and interest is charged monthly. What is the interest rate per month?
 - $9\% / 12 = 0.75\%$



Effective Annual Rate (EAR) vs. APR

Annual Percentage Rate (APR) for Purchases	18.99% when you open your account. This APR will vary with the market based on the Prime Rate.
APR for Balance Transfers	18.99%. This APR will vary with the market based on the Prime Rate.
APR for Cash Advances	25.24%. This APR will vary with the market based on the Prime Rate.
Penalty APR and When it Applies	Up to 29.99%, based on your creditworthiness. This APR will vary with the market based on the Prime Rate. This APR may be applied to your account if you: <ol style="list-style-type: none"> 1. Make a late payment; 2. Go over your credit limit; or 3. Make a payment that is returned. <p>How Long Will the Penalty APR Apply? If your APRs are increased for any of these reasons, the Penalty APRs may apply indefinitely.</p>

If compounded annually, **annual** interest for a \$100 debt is \$18.99

If compounded semiannually, **annual** interest is $\$100 * [(1 + 0.1899/2)^2 - 1] = \19.89

If compounded monthly, **annual** interest is $\$100 * [(1 + 0.1899/12)^{12} - 1] = \20.73

If compounded daily, annual interest is $\$100 * [(1 + 0.1899/365)^{365} - 1] = \20.91

- **Effective annual rate:** the actual interest rate after accounting for compounding that occurs during the year

$$1 + EAR = \left(1 + \frac{APR}{m}\right)^m \longrightarrow EAR = \left(1 + \frac{APR}{m}\right)^m - 1$$

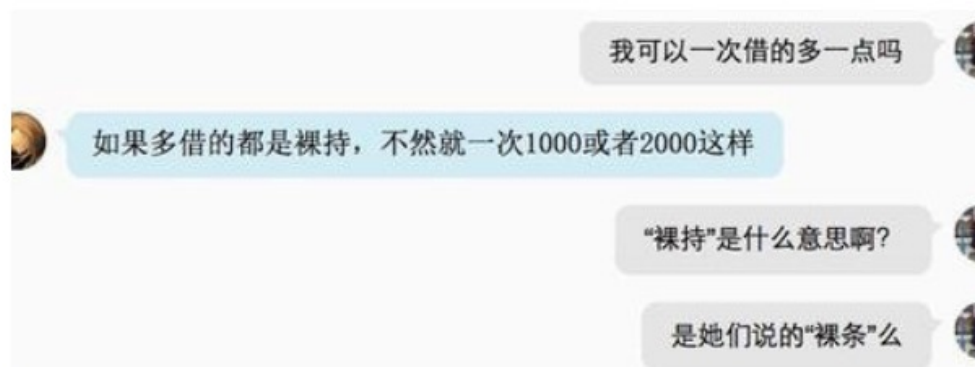
where m is the compounding frequency



EAR Example—P2P Lending

女大学生网贷被要求拍裸照抵押 借款周利息高达30%

2016年06月14日15:22 综合



专栏推荐



解决地方
不能完全
能没有约

樊纲

图解新闻

- $APR = 30\% * 52 = 1560\%$
- $EAR = (1 + 30\%)^{52} - 1 = 841499.4$



EAR-Continuous Compounding

- Sometimes investments or loans are figured based on continuous compounding. In other words, m goes to infinity
- $EAR = \lim_{m \rightarrow \infty} \left(1 + \frac{APR}{m}\right)^m - 1 = e^{APR} - 1$
- Example: What is the effective annual rate of 7% compounded continuously?
 - $EAR = e^{7\%} - 1 = 7.25\%$



When should I use EAR vs. APR?

- $FV = PV(1 + r)^t$
 - r should be the effective rate in each period

Assume you need to calculate FV in n years:

- If you are given an APR:
 - APR always comes in a pair: {APR, compounding frequency}
 - APR only tells you the interest rate in each compounding period
Compounding period interest = APR/m
 - For a period of any other lengths, do not use APR in this way. Instead, express time in terms of numbers of compounding periods:

$$FV = PV \left(1 + \frac{APR}{m} \right)^{nm}$$

- If you are given an EAR:
 - $FV = PV(1 + EAR)^n$, n expressed in terms of years



EAR vs. APR Example

12% APR, monthly compounding \leftrightarrow 12.68% EAR

- What is the relationship between the two numbers?
 - $1 + 12.68\% = (1 + 12\%/12)^{12}$
- What is the interest rate for a 2-month period?
 - Using APR: 1-month interest rate $r_{1m} = 1\%$,
so $r_{2m} = (1 + r_{1m})^2 - 1 = 2.01\%$
 - Using EAR: $r_{2m} = (1 + 12.68\%)^{2/12} - 1 = 2.01\%$
- What is the interest rate for a half-month period?
 - Using APR: so $r_{0.5m} = (1 + r_{1m})^{1/2} - 1 = 0.499\%$
 - Using EAR: $r_{0.5m} = (1 + 12.68\%)^{1/24} - 1 = 0.499\%$



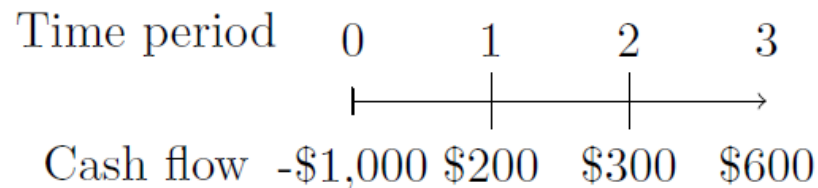
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- What is the time value of money (TVM)?
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 - APR and EAR
- **TVM of cash flow streams**



Cash Flows Streams

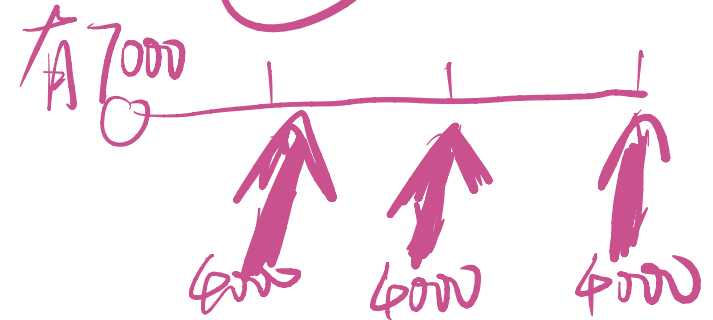
- Basic principle:
 - The PV (FV) of a stream of cash flows equals to the sum of PV (FV) of individual cash flows.
- A useful tool: timeline
 - A linear representation of the timing and amount of cash flows.
 - Lend out \$1,000 today and then receive \$200 in the first period, \$300 in the second period, \$600 in the third period.



Cash Flows Streams– FV Example

- You think you will be able to deposit \$4,000 at the end of each of the next three years in a bank account paying 8 percent interest.

- You currently have \$7,000 in the account.
- How much will you have in three years?
- How much will you have in four years?



- Find the value at year 3 of each cash flow and add them together

- Today (year 0): $FV = 7000(1.08)^3 = 8,817.98$
- Year 1: $FV = 4,000(1.08)^2 = 4,665.60$
- Year 2: $FV = 4,000(1.08) = 4,320$
- Year 3: value = 4,000
- Total value in 3 years = $8,817.98 + 4,665.60 + 4,320 + 4,000 = 21,803.58$
- Value at year 4 = $21,803.58(1.08) = 23,547.87$



Cash Flows Streams– PV Example

- You are offered an investment that will give you \$200 in one year, \$400 the next year, \$600 the following year, and \$800 at the end of the fourth year. You can earn 12% on your investment. What is the PV of this cash flow streams?
- Find the PV of each cash flows and add them
 - Year 1 CF: $200 / (1.12)^1 = 178.57$
 - Year 2 CF: $400 / (1.12)^2 = 318.88$
 - Year 3 CF: $600 / (1.12)^3 = 427.07$
 - Year 4 CF: $800 / (1.12)^4 = 508.41$
 - Total PV = $178.57 + 318.88 + 427.07 + 508.41 = 1,432.93$



Standardized Cash Flow Streams

- Annuity – finite series of equal payments that occur at regular intervals
 - If the first payment occurs at the **end** of the period, it is called an *ordinary annuity*
 - If the first payment occurs at the **beginning** of the period, it is called an *annuity due*
 - Examples: rent payment, tuition payment
- Perpetuity – infinite series of payments that occur at regular intervals
 - **Equal** payments: level perpetuity
 - Payments **grow at a constant rate**: growth perpetuity
 - Example: stock dividend



Review – Geometric Series

Sum of geometric series $a + ax + ax^2 + \cdots + ax^{n-1}$:

$$sum = a \frac{1 - x^n}{1 - x}$$

Notes:

- a : the first term of the series
- n : the number of terms in total
- x : common ratio

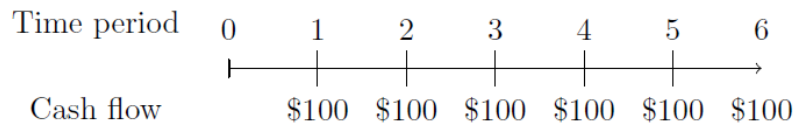
Infinite geometric series $a + ax + ax^2 + \cdots$:

$$sum = \frac{a}{1 - x}, |x| < 1$$

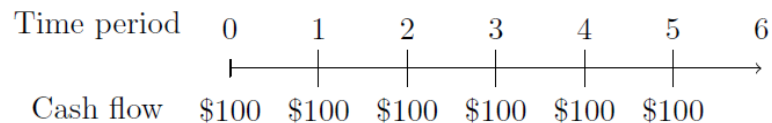


Standardized Cash Flow Streams-Examples

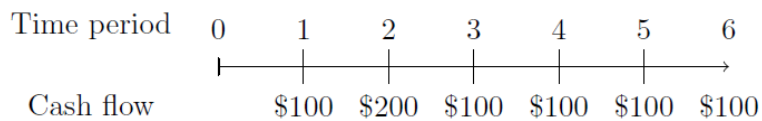
a.



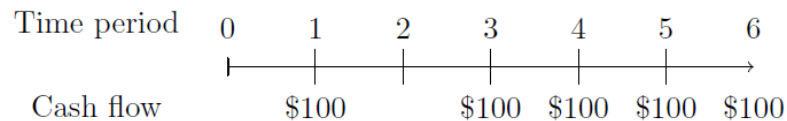
b.



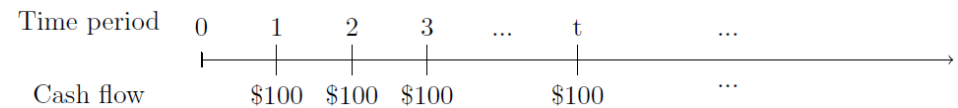
c.



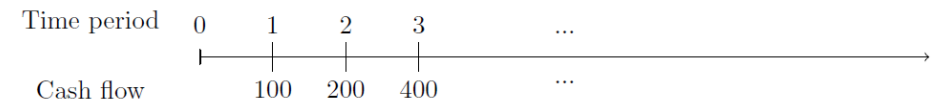
d.



e.



f.

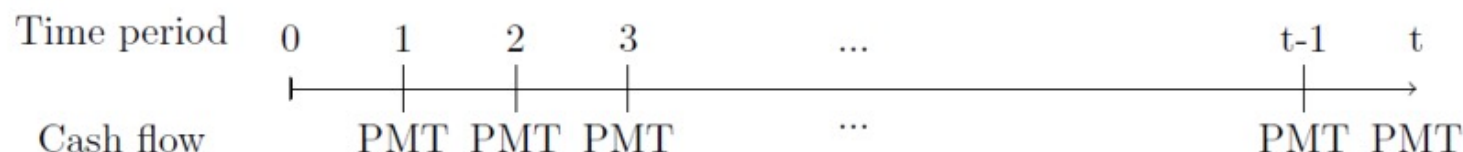


- a. An ordinary annuity from 0 to 6. An annuity due from 1-7
- b. An annuity due from 0-6
- c. Complex cash flows from 0-6. An annuity due from 3-7
- d. Complex cash flows from 0-6. An ordinary annuity from 2-6
- e. Level perpetuity.
- f. Growth perpetuity



Annuity

t periods, the amount of each cash flow(PMT), periodic interest rate(r), number of periods per year (m).



$$\begin{aligned}
 FV &= PMT(1+r)^{t-1} + PMT(1+r)^{t-2} + \dots + PMT(1+r)^0 \\
 &= PMT[(1+r)^{t-1} + (1+r)^{t-2} + \dots + (1+r)^0] \quad \text{Inside the bracket: a geometric sequence} \\
 &= PMT\left[\frac{(1+r)^t - 1}{r}\right]
 \end{aligned}$$

$$\begin{aligned}
 PV &= PMT * \frac{1}{(1+r)^1} + PMT * \frac{1}{(1+r)^2} + PMT * \frac{1}{(1+r)^3} + \dots + PMT * \frac{1}{(1+r)^t} \\
 &= PMT\left[\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \dots + \frac{1}{(1+r)^t}\right] : \text{Geometric sequence} \\
 &= PMT * \frac{1 - \frac{1}{(1+r)^t}}{r}
 \end{aligned}$$



Annuities Due



$$\begin{aligned}
 FV &= PMT * (1 + r)^t + PMT * (1 + r)^{t-1} + PMT * (1 + r)^{t-2} + \dots + PMT * (1 + r)^1 \\
 &= PMT[(1 + r)^t + (1 + r)^{t-1} + (1 + r)^{t-2} + \dots + (1 + r)^1] \\
 &= PMT * \frac{(1 + r)^t - 1}{r} (1 + r) \\
 &= FV_{OA}(1 + r)
 \end{aligned}$$

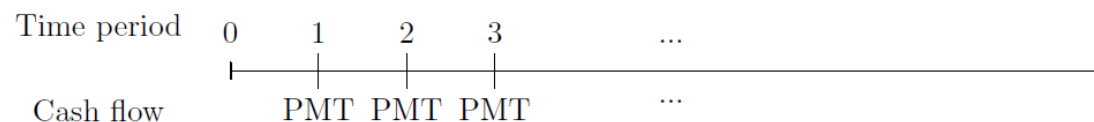
$$\begin{aligned}
 PV &= PMT * \frac{1}{(1 + r)^0} + PMT * \frac{1}{(1 + r)^1} + PMT * \frac{1}{(1 + r)^2} + \dots + PMT * \frac{1}{(1 + r)^{t-1}} \\
 &= PMT[\frac{1}{(1 + r)^0} + \frac{1}{(1 + r)^1} + \frac{1}{(1 + r)^2} + \frac{1}{(1 + r)^{t-1}}] \\
 &= PMT * \frac{1 - \frac{1}{(1+r)^t}}{r} (1 + r) \\
 &= PV_{OA}(1 + r)
 \end{aligned}$$

OA stands for ordinary annuity that are of the same PMT, r and t



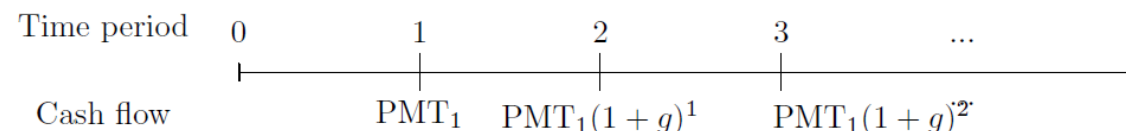
Perpetuity

- **Level perpetuity**



$$\begin{aligned}
 PV &= PMT * \frac{1}{(1+r)^1} + PMT * \frac{1}{(1+r)^2} + PMT * \frac{1}{(1+r)^3} + \dots \\
 &= PMT \left[\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \dots \right] \\
 &= PMT * \frac{1}{r}
 \end{aligned}$$

- **Growth perpetuity**



$$\begin{aligned}
 PV &= PMT * \frac{1}{(1+r)^1} + PMT(1+g) * \frac{1}{(1+r)^2} + PMT(1+g)^2 * \frac{1}{(1+r)^3} + \dots \\
 &= PMT \left[\frac{(1+g)^0}{(1+r)^1} + \frac{(1+g)^1}{(1+r)^2} + \frac{(1+g)^2}{(1+r)^3} + \dots \right] \\
 &= PMT * \frac{1}{r-g} \quad (\text{Note that it has to be } g < r)
 \end{aligned}$$

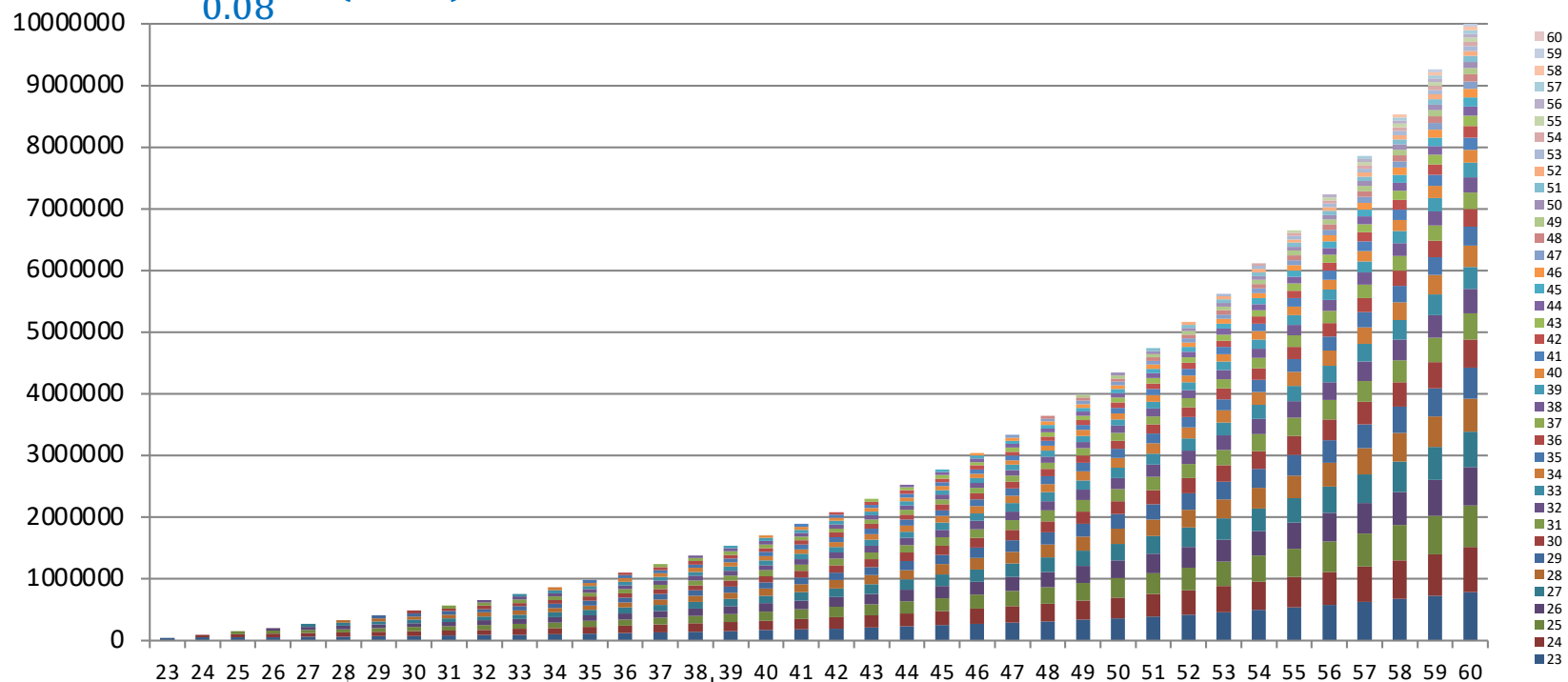


Example—Retirement Planning

- Suppose you want to have 10 million when you retire at the age 60. You plan to save x on every birthday from 23 to 59. Annual interest rate = 8%.

$$- x \cdot 1.08^{37} + x \cdot 1.08^{36} + \dots + x \cdot 1.08^1 = 10,000,000$$

$$- x \frac{1.08^{37}-1}{0.08} (1.08) = 10,000,000 \Rightarrow x = 45,596$$



Some Takeaways

- Wealth grows exponentially
- Savings of 1.23M in total over the course of 37 years will grow to 10M
- Savings are much more valuable when you are young
 - The first saving will grow 17.25 times after 37 years
 - The first 8 (out of 37) savings accounts for 48% of terminal wealth
- You only reach the half-way mark when you are 52, 29 years into the savings plan



Summary

- For a single cash flow: $FV = PV * \left(1 + \frac{APR}{m}\right)^{m*n}$
 - APR: how interest rate is quoted. Equals to per period interest times m (#periods per year)
 - EAR: actual interest accrued in a year
- For cash flow streams: FV (or PV) is the sum of FV (or PV) of individual cash flows
- Standardized cash flow streams
 - Ordinary annuity: $PV = PMT \frac{1-(1+r)^{-n}}{r}$
 - Annuities due: $PV = PMT \left(1 + \frac{1-(1+r)^{-n+1}}{r}\right) = PMT(1+r) \frac{1-(1+r)^{-n}}{r}$
 - Level perpetuity: $PV = PMT/r$
 - Growth perpetuity: $PV = \frac{PMT}{r-g}$

