5.4

The Fundamental Theorem of Calculus

Mean Value Theorem for Definite Integrals

In the previous section we defined the average value of a continuous function over a closed interval $[a, b]$ as the definite integral $\int_a^b f(x)\,dx$ divided by the length or width $b - a$ of the interval. The Mean Value Theorem for Definite Integrals asserts that this average value is *always* taken on at least once by the function f in the interval.

The graph in Figure 5.16 shows a *positive* continuous function $y = f(x)$ defined over the interval $[a, b]$. Geometrically, the Mean Value Theorem says that there is a number c in $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function and base width $b - a$ has exactly the same area as the region beneath the graph of f from a to b.

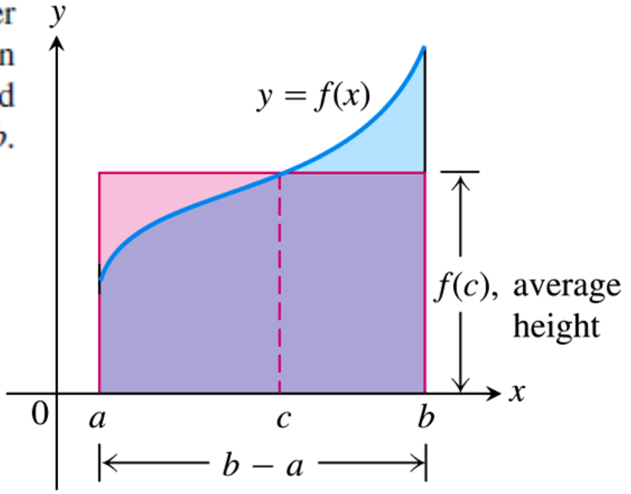


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b - a) = \int_a^b f(x)\,dx.$$

THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

Proof If we divide both sides of the Max-Min Inequality (Table 5.6, Rule 6) by $(b - a)$, we obtain

$$\min f \leq \frac{1}{b - a} \int_a^b f(x)\, dx \leq \max f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions (Section 2.5) says that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $(1/(b - a)) \int_a^b f(x)\, dx$ at some point c in $[a, b]$. ■

The continuity of f is important here. It is possible that a discontinuous function never equals its average value (Figure 5.17).

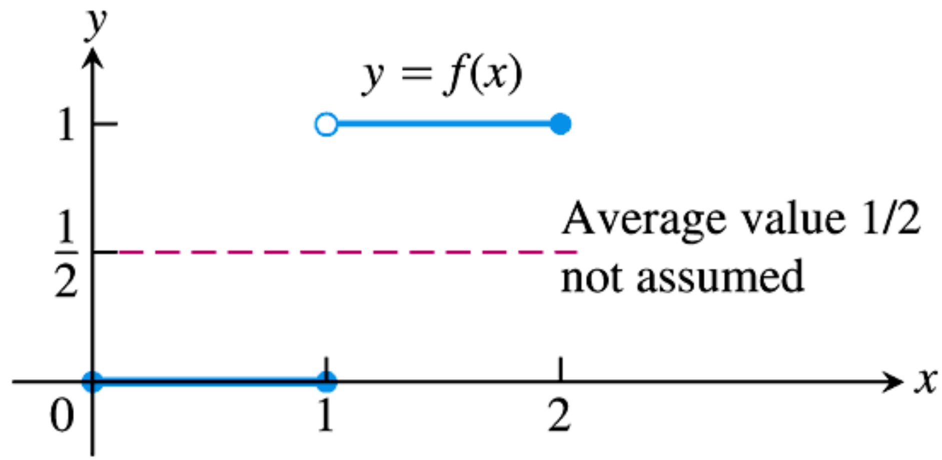


FIGURE 5.17 A discontinuous function need not assume its average value.

EXAMPLE 1 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution The average value of f on $[a, b]$ is

$$\text{av}(f) = \frac{1}{b - a}\int_a^b f(x)\,dx = \frac{1}{b - a} \cdot 0 = 0.$$

By the Mean Value Theorem, f assumes this value at some point $c \in [a, b]$. ■

The Fundamental Theorem of Calculus

If $f(t)$ is an integrable function over a finite interval I, then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t)\,dt. \qquad (1)$$

For example, if f is nonnegative and x lies to the right of a, then $F(x)$ is the area under the graph from a to x (Figure 5.18). The variable x is the upper limit of integration of an integral, but F is just like any other real-valued function of a real variable. For each value of the input x, there is a well-defined numerical output, in this case the definite integral of f from a to x.

If $f \geq 0$ on $[a, b]$, then the computation of $F'(x)$ from the definition of the derivative means taking the limit as $h \to 0$ of the difference quotient

$$\frac{F(x+h) - F(x)}{h}.$$

For $h > 0$, the numerator is obtained by subtracting two areas, so it is the area under the graph of f from x to $x + h$ (Figure 5.19). If h is small, this area is approximately equal to the area of the rectangle of height $f(x)$ and width h, which can be seen from Figure 5.19. That is,

$$F(x+h) - F(x) \approx hf(x).$$

Dividing both sides of this approximation by h and letting $h \to 0$, it is reasonable to expect that

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

This result is true even if the function f is not positive, and it forms the first part of the Fundamental Theorem of Calculus.

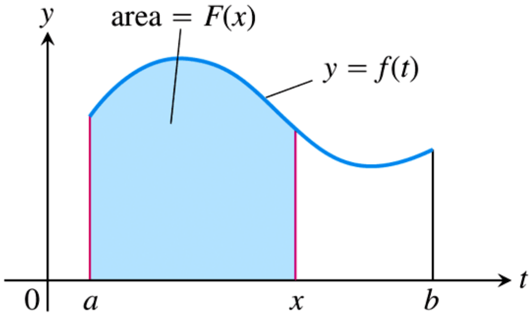


FIGURE 5.18 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

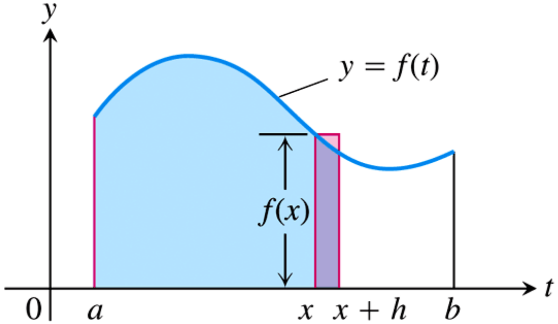


FIGURE 5.19 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

THEOREM 4—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

The above is often abbreviated as FTC1.

Differentiation and integration may be viewed as inverse operations.

Earlier, we saw that the indefinite integral is the antiderivative; i.e., we differentiate a function (resulting in its derivative), and then integrate, we get back the original function. We now see that after integration, differentiating the integral gets us back the original function.

Proof of Theorem 4 We prove the Fundamental Theorem, Part 1, by applying the definition of the derivative directly to the function $F(x)$, when x and $x + h$ are in (a, b). This means writing out the difference quotient

$$\frac{F(x + h) - F(x)}{h} \tag{3}$$

and showing that its limit as $h \to 0$ is the number $f(x)$ for each x in (a, b). Doing so, we find

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt \right] \\ &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt. && \text{Table 5.6, Rule 5} \end{aligned}$$

According to the Mean Value Theorem for Definite Integrals, the value before taking the limit in the last expression is one of the values taken on by f in the interval between x and $x + h$. That is, for some number c in this interval,

$$\frac{1}{h} \int_x^{x+h} f(t)\, dt = f(c). \tag{4}$$

As $h \to 0$, $x + h$ approaches x, forcing c to approach x also (because c is trapped between x and $x + h$). Since f is continuous at x, $f(c)$ approaches $f(x)$:

$$\lim_{h \to 0} f(c) = f(x). \tag{5}$$

In conclusion, we have

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt \\ &= \lim_{h \to 0} f(c) && \text{Eq. (4)} \\ &= f(x). && \text{Eq. (5)} \end{aligned}$$

If $x = a$ or b, then the limit of Equation (3) is interpreted as a one-sided limit with $h \to 0^+$ or $h \to 0^-$, respectively. Then Theorem 1 in Section 3.2 shows that F is continuous over $[a, b]$. This concludes the proof. ■

EXAMPLE 2 Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\,dt$ **(b)** $y = \int_x^5 3t \sin t\,dt$

(c) $y = \int_1^{x^2} \cos t\,dt$ **(d)** $y = \int_{1+3x^2}^{4} \frac{1}{2+t}\,dt$

Solution We calculate the derivatives with respect to the independent variable x.

(a) $$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\,dt = x^3 + 1$$ Eq. (2) with $f(t) = t^3 + 1$

(b) $$\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t \sin t\,dt = \frac{d}{dx}\left(-\int_5^x 3t \sin t\,dt\right)$$ Table 5.6, Rule 1

$$= -\frac{d}{dx}\int_5^x 3t \sin t\,dt$$

$$= -3x \sin x$$ Eq. (2) with $f(t) = 3t \sin t$

(c) The upper limit of integration is not x but x^2. This makes y a composite of the two functions,

$$y = \int_1^u \cos t \, dt \qquad \text{and} \qquad u = x^2.$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \left(\frac{d}{du} \int_1^u \cos t \, dt \right) \cdot \frac{du}{dx} \\
&= \cos u \cdot \frac{du}{dx} \\
&= \cos(x^2) \cdot 2x \\
&= 2x \cos x^2
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{d}{dx} \int_{1+3x^2}^4 \frac{1}{2+t} \, dt &= \frac{d}{dx} \left(-\int_4^{1+3x^2} \frac{1}{2+t} \, dt \right) && \text{Rule 1} \\
&= -\frac{d}{dx} \int_4^{1+3x^2} \frac{1}{2+t} \, dt \\
&= -\frac{1}{2+(1+3x^2)} \cdot \frac{d}{dx}(1+3x^2) && \text{Eq. (2) and the Chain Rule} \\
&= -\frac{2x}{1+x^2}
\end{aligned}$$

■

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2

If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t)\,dt.$$

Thus, if F is *any* antiderivative of f, then $F(x) = G(x) + C$ for some constant C for $a < x < b$ (by Corollary 2 of the Mean Value Theorem for Derivatives, Section 4.2).

Since both F and G are continuous on $[a, b]$, we see that $F(x) = G(x) + C$ also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \to a^+$ and $x \to b^-$).

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 \\ &= \int_a^b f(t)\,dt. \qquad \blacksquare \end{aligned}$$

We often use the notation

$$F(x)\Big]_a^b = F(b) - F(a)$$

So the equation of FTC2 can be written as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \qquad \text{where} \qquad F' = f$$

Other common notations are $F(x)\big|_a^b$ and $[F(x)]_a^b$.

EXAMPLE 3 We calculate several definite integrals using the Evaluation Theorem, rather than by taking limits of Riemann sums.

(a) $$\int_0^{\pi} \cos x\,dx = \sin x\Big]_0^{\pi}$$

$$= \sin\pi - \sin 0 = 0 - 0 = 0$$

$$\frac{d}{dx}\sin x = \cos x$$

(b) $$\int_{-\pi/4}^{0} \sec x \tan x\,dx = \sec x\Big]_{-\pi/4}^{0}$$

$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

(c) $$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$$

$$= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right]$$

$$= [8 + 1] - [5] = 4$$

$$\frac{d}{dx}\left(x^{3/2} + \frac{4}{x}\right) = \frac{3}{2}x^{1/2} - \frac{4}{x^2}$$

The Integral of a Rate

We can interpret Part 2 of the Fundamental Theorem in another way. If F is any antiderivative of f, then $F' = f$. The equation in the theorem can then be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Now $F'(x)$ represents the rate of change of the function $F(x)$ with respect to x, so the last equation asserts that the integral of F' is just the *net change* in F as x changes from a to b. Formally, we have the following result.

THEOREM 5—The Net Change Theorem The net change in a differentiable function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\,dx. \tag{6}$$

EXAMPLE 4 Here are several interpretations of the Net Change Theorem.

(a) If $c(x)$ is the cost of producing x units of a certain commodity, then $c'(x)$ is the marginal cost (Section 3.4). From Theorem 5,

$$\int_{x_1}^{x_2} c'(x)\,dx = c(x_2) - c(x_1),$$

which is the cost of increasing production from x_1 units to x_2 units.

(b) If an object with position function $s(t)$ moves along a coordinate line, its velocity is $v(t) = s'(t)$. Theorem 5 says that

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1),$$

so the integral of velocity is the **displacement** over the time interval $t_1 \leq t \leq t_2$. On the other hand, the integral of the speed $|v(t)|$ is the **total distance traveled** over the time interval. This is consistent with our discussion in Section 5.1. ■

If we rearrange Equation (6) as

$$F(b) = F(a) + \int_a^b F'(x)\,dx,$$

we see that the Net Change Theorem also says that the final value of a function $F(x)$ over an interval $[a, b]$ equals its initial value $F(a)$ plus its net change over the interval. So if $v(t)$ represents the velocity function of an object moving along a coordinate line, this means that the object's final position $s(t_2)$ over a time interval $t_1 \leq t \leq t_2$ is its initial position $s(t_1)$ plus its net change in position along the line (see Example 4b).

Total Area

Area is always a nonnegative quantity. The Riemann sum contains terms such as $f(c_k)\,\Delta x_k$ that give the area of a rectangle when $f(c_k)$ is positive. When $f(c_k)$ is negative, then the product $f(c_k)\,\Delta x_k$ is the negative of the rectangle's area. When we add up such terms for a negative function, we get the negative of the area between the curve and the x-axis. If we then take the absolute value, we obtain the correct positive area.

EXAMPLE 7 Figure 5.21 shows the graph of the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.

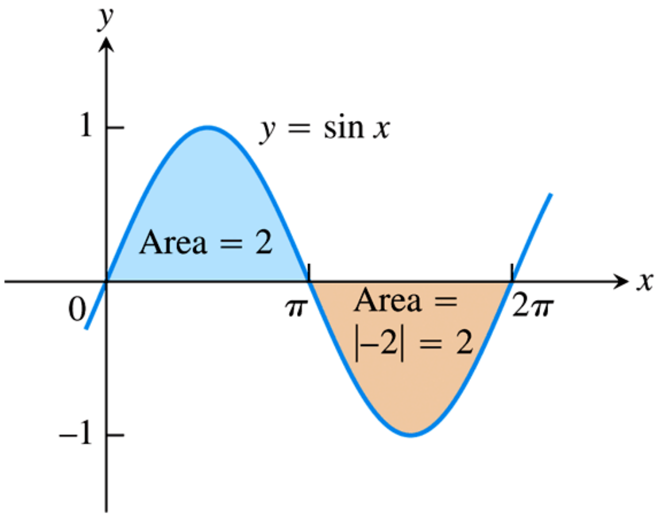


FIGURE 5.21 The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals (Example 7).

Solution

(a) The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x\,dx = -\cos x\Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

The definite integral is zero because the portions of the graph above and below the x-axis make canceling contributions.

(b) The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$

$$\int_{\pi}^{2\pi} \sin x\,dx = -\cos x\Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values,

$$\text{Area} = |2| + |-2| = 4.$$ ■

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

5.5

Indefinite Integrals and the Substitution Method

As with differentials, when computing integrals we have

$$du = \frac{du}{dx}\,dx.$$

EXAMPLE 1 Find the integral $\int (x^3 + x)^5(3x^2 + 1)\,dx$.

Solution We set $u = x^3 + x$. Then

$$du = \frac{du}{dx}\,dx = (3x^2 + 1)\,dx,$$

so that by substitution we have

$$\int (x^3 + x)^5(3x^2 + 1)\,dx = \int u^5\,du \qquad \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\,dx.$$

$$= \frac{u^6}{6} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{(x^3 + x)^6}{6} + C \qquad \text{Substitute } x^3 + x \text{ for } u.$$

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f:

$$\begin{aligned} \frac{d}{dx}F(g(x)) &= F'(g(x)) \cdot g'(x) && \text{Chain Rule} \\ &= f(g(x)) \cdot g'(x). && F' = f \end{aligned}$$

If we make the substitution $u = g(x)$, then

$$\begin{aligned} \int f(g(x))g'(x)\,dx &= \int \frac{d}{dx}F(g(x))\,dx \\ &= F(g(x)) + C && \text{Theorem 8 in Chapter 4} \\ &= F(u) + C && u = g(x) \\ &= \int F'(u)\,du && \text{Theorem 8 in Chapter 4} \\ &= \int f(u)\,du. && F' = f \end{aligned}$$

■

The Substitution Method to evaluate $\int f(g(x))g'(x)\,dx$

1. Substitute $u = g(x)$ and $du = (du/dx)\,dx = g'(x)\,dx$ to obtain $\int f(u)\,du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

EXAMPLE 4 Find $\int \cos(7\theta + 3)\, d\theta$.

Solution We let $u = 7\theta + 3$ so that $du = 7\, d\theta$. The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7, using the same procedure as in Example 2. Then,

$$\int \cos(7\theta + 3)\, d\theta = \frac{1}{7}\int \cos(7\theta + 3)\cdot 7\, d\theta \qquad \text{Place factor } 1/7 \text{ in front of integral.}$$

$$= \frac{1}{7}\int \cos u\, du \qquad \text{Let } u = 7\theta + 3,\ du = 7\, d\theta.$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate.}$$

$$= \frac{1}{7}\sin(7\theta + 3) + C. \qquad \text{Substitute } 7\theta + 3 \text{ for } u.$$

There is another approach to this problem. With $u = 7\theta + 3$ and $du = 7\, d\theta$ as before, we solve for $d\theta$ to obtain $d\theta = (1/7)\, du$. Then the integral becomes

$$\int \cos(7\theta + 3)\, d\theta = \int \cos u \cdot \frac{1}{7}\, du \qquad \text{Let } u = 7\theta + 3,\ du = 7\, d\theta, \text{ and } d\theta = (1/7)\, du.$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate.}$$

$$= \frac{1}{7}\sin(7\theta + 3) + C. \qquad \text{Substitute } 7\theta + 3 \text{ for } u.$$

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 3)$. ■

EXAMPLE 6 Evaluate $\int x\sqrt{2x+1}\,dx$.

Solution Our previous integration in Example 2 suggests the substitution $u = 2x + 1$ with $du = 2\,dx$. Then,

$$\sqrt{2x+1}\,dx = \frac{1}{2}\sqrt{u}\,du.$$

However, in this case the integrand contains an extra factor of x multiplying the term $\sqrt{2x+1}$. To adjust for this, we solve the substitution equation $u = 2x + 1$ to obtain $x = (u-1)/2$, and find that

$$x\sqrt{2x+1}\,dx = \frac{1}{2}(u-1)\cdot\frac{1}{2}\sqrt{u}\,du.$$

The integration now becomes

$$\begin{aligned}
\int x\sqrt{2x+1}\,dx &= \frac{1}{4}\int (u-1)\sqrt{u}\,du = \frac{1}{4}\int (u-1)u^{1/2}\,du && \text{Substitute.} \\
&= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right) du && \text{Multiply terms.} \\
&= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C && \text{Integrate.} \\
&= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C. && \text{Replace } u \text{ by } 2x+1. \blacksquare
\end{aligned}$$

EXAMPLE 8 Evaluate $\displaystyle\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$.

Solution We can use the substitution method of integration as an exploratory tool: Substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$ or we might even press our luck and take u to be the entire cube root. Here is what happens in each case, and both substitutions are successful.

Method 1: Substitute $u = z^2 + 1$.

$$\begin{aligned}
\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} &= \int \frac{du}{u^{1/3}} && \text{Let } u = z^2+1,\ du = 2z\,dz. \\
&= \int u^{-1/3}\,du && \text{In the form } \textstyle\int u^n\,du \\
&= \frac{u^{2/3}}{2/3} + C && \text{Integrate.} \\
&= \frac{3}{2}u^{2/3} + C \\
&= \frac{3}{2}(z^2+1)^{2/3} + C && \text{Replace } u \text{ by } z^2+1.
\end{aligned}$$

Method 2: Substitute $u = \sqrt[3]{z^2 + 1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2 + 1}} = \int \frac{3u^2\,du}{u}$$

Let $u = \sqrt[3]{z^2 + 1}$,
$u^3 = z^2 + 1,\ 3u^2\,du = 2z\,dz$.

$$= 3\int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C$$

Integrate.

$$= \frac{3}{2}(z^2 + 1)^{2/3} + C$$

Replace u by $(z^2 + 1)^{1/3}$. ■

5.6

Definite Integral Substitutions and the Area Between Curves

THEOREM 7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Proof Let F denote any antiderivative of f. Then,

$$\begin{aligned}
\int_a^b f(g(x)) \cdot g'(x)\, dx &= F(g(x))\Big]_{x=a}^{x=b} && \begin{aligned}&\tfrac{d}{dx}F(g(x))\\ &= F'(g(x))g'(x)\\ &= f(g(x))g'(x)\end{aligned}\\
&= F(g(b)) - F(g(a))\\
&= F(u)\Big]_{u=g(a)}^{u=g(b)}\\
&= \int_{g(a)}^{g(b)} f(u)\, du. && \text{Fundamental Theorem, Part 2} \quad \blacksquare
\end{aligned}$$

EXAMPLE 1 Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

$y = x^3+1 \quad \frac{dy}{dx} = 3x^2 \quad \frac{2}{3}y^{\frac{3}{2}}$

$\int_{-1}^{1} y^{\frac{1}{2}}\,dy = \frac{2}{3}(x^3+1)^{\frac{3}{2}}$

Solution We have two choices.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 7.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_0^2 \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du \qquad \text{Let } u = x^3+1,\ du = 3x^2\,dx.$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{2}{3}(x^3+1)^{3/2} + C \qquad \text{Replace } u \text{ by } x^3+1.$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1} \qquad \text{Use the integral just found, with limits of integration for } x.$$

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3} \qquad \blacksquare$$

THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

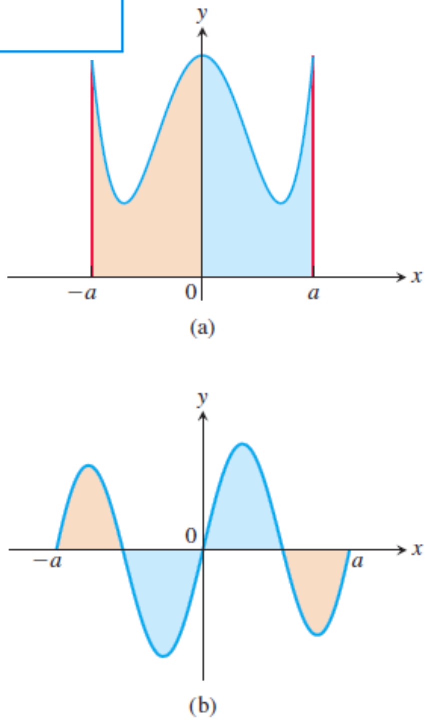


FIGURE 5.23 (a) For f an even function, the integral from $-a$ to a is twice the integral from 0 to a. (b) For f an odd function, the integral from $-a$ to a equals 0.

Recall from Week 1

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

Proof of Part (a)

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx && \text{Additivity Rule for Definite Integrals} \\
&= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx && \text{Order of Integration Rule} \\
&= -\int_{0}^{a} f(-u)(-du) + \int_{0}^{a} f(x)\,dx && \text{Let } u=-x,\ du=-dx. \text{ When } x=0,\ u=0. \text{ When } x=-a,\ u=a. \\
&= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx \\
&= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx && f \text{ is even, so } f(-u)=f(u). \\
&= 2\int_{0}^{a} f(x)\,dx
\end{aligned}$$

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

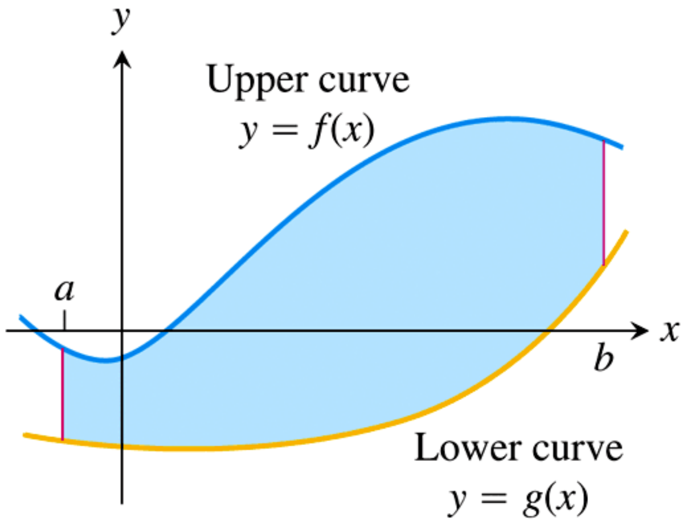
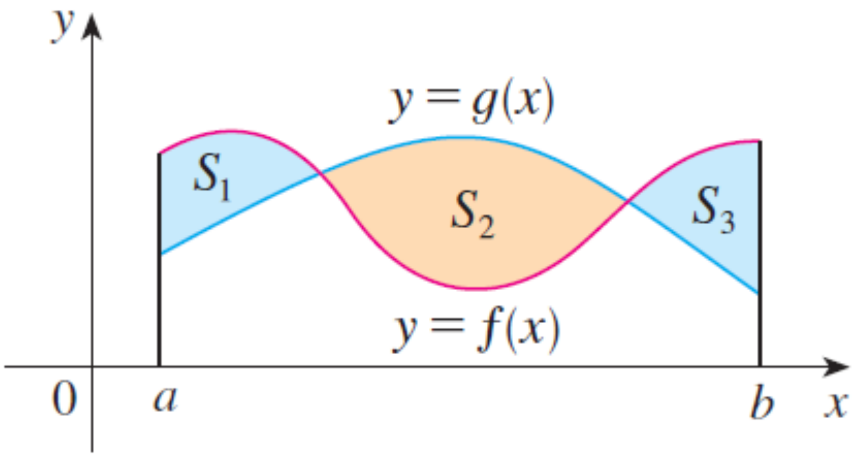


FIGURE 5.25 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

More Generally

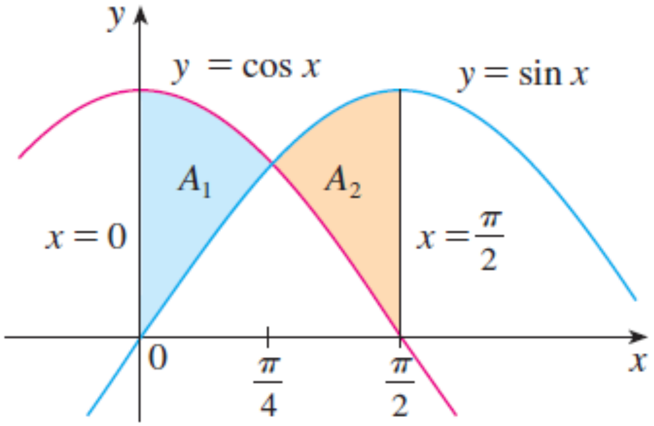
The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\, dx$$



EXAMPLE Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Observe that $\cos x \geqslant \sin x$ when $0 \leqslant x \leqslant \pi/4$ but $\sin x \geqslant \cos x$ when $\pi/4 \leqslant x \leqslant \pi/2$. Therefore the required area is



$$
\begin{aligned}
A &= \int_0^{\pi/2} |\cos x - \sin x|\, dx = A_1 + A_2 \\
&= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx \\
&= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\
&= 2\sqrt{2} - 2
\end{aligned}
$$

In this particular example we could have saved some work by noticing that the region is symmetric about $x = \pi/4$ and so

$$A = 2A_1 = 2\int_0^{\pi/4} (\cos x - \sin x)\, dx$$ ■

EXAMPLE 5 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution The sketch (Figure 5.28) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (both formulas agree at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure 5.28.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\begin{aligned}
\sqrt{x} &= x - 2 && \text{Equate } f(x) \text{ and } g(x). \\
x &= (x-2)^2 = x^2 - 4x + 4 && \text{Square both sides.} \\
x^2 - 5x + 4 &= 0 && \text{Rewrite.} \\
(x-1)(x-4) &= 0 && \text{Factor.} \\
x = 1, \qquad x &= 4. && \text{Solve.}
\end{aligned}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$

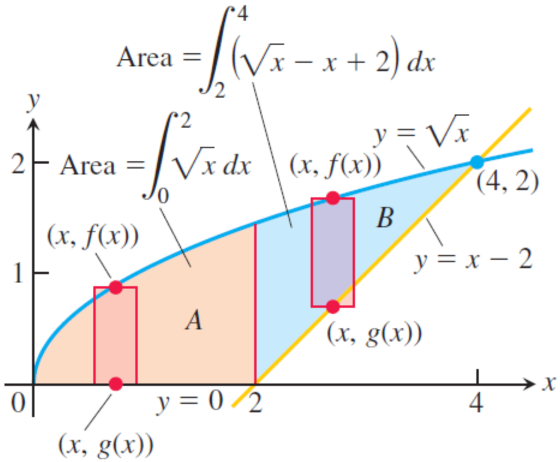


FIGURE 5.28 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 6.

We add the areas of subregions A and B to find the total area:

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 \left(\sqrt{x} - x + 2\right) dx}_{\text{area of } B}$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

■

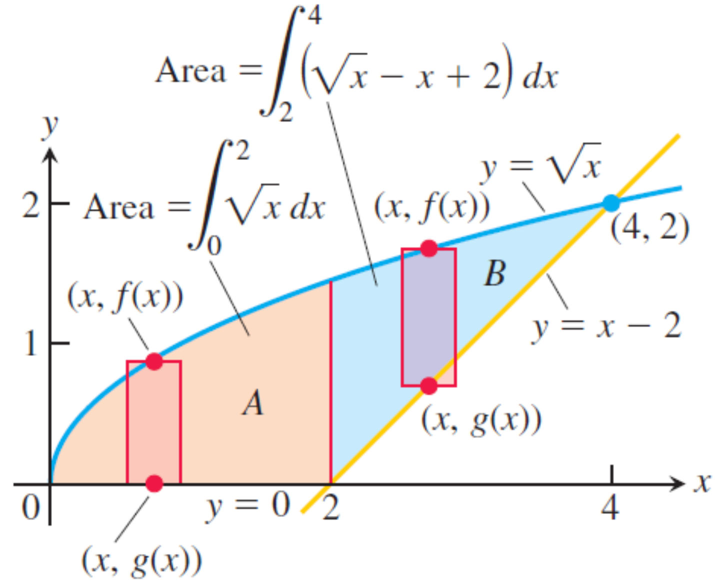
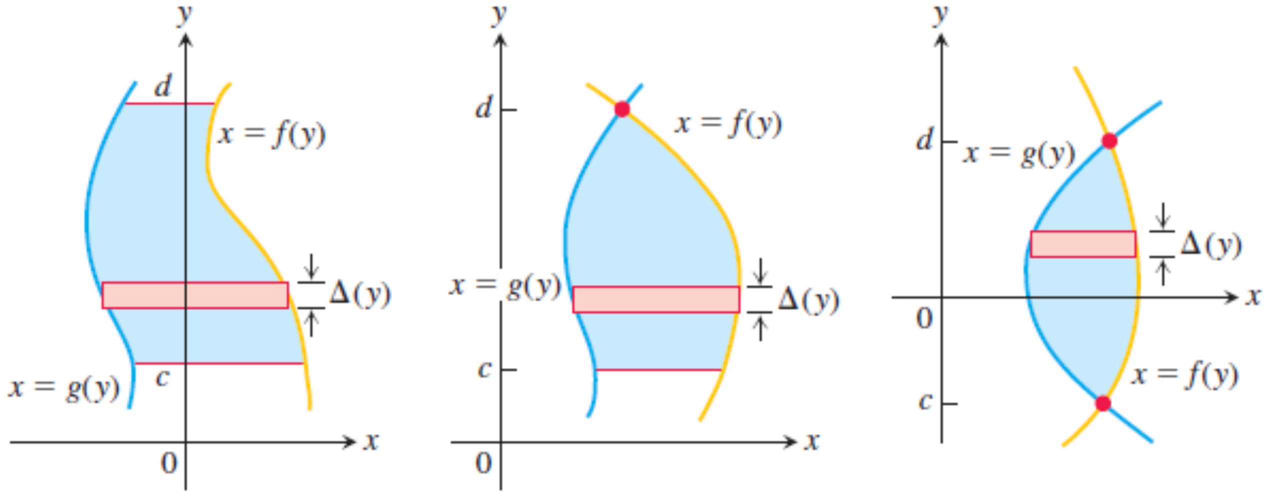


FIGURE 5.28 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 6.

Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these:



use the formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

EXAMPLE 6 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution We first sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y-values (Figure 5.29). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y:

$$
\begin{aligned}
y + 2 &= y^2 && \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2. \\
y^2 - y - 2 &= 0 && \text{Rewrite.} \\
(y + 1)(y - 2) &= 0 && \text{Factor.} \\
y = -1, \qquad y &= 2 && \text{Solve.}
\end{aligned}
$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

The area of the region is

$$
\begin{aligned}
A = \int_c^d [f(y) - g(y)]\,dy &= \int_0^2 [y + 2 - y^2]\,dy \\
&= \int_0^2 [2 + y - y^2]\,dy \\
&= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 \\
&= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}.
\end{aligned}
$$

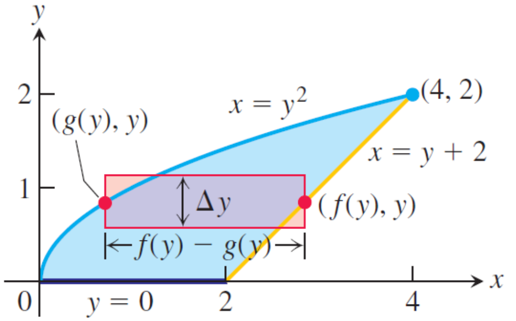


FIGURE 5.29 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 6).

Week 7

Assignment 7*

5.4:#1,3,4,8,9,12,17,27,28,40,42,43,46,64,68,71,73

5.5: #18,19,21,22,26,32,42,43,46,54,55,62

5.6: #9,14,25,61,70,72,78,79,83,84,85,87

*The questions above are practice only – no submission is needed (due to the coming midterm exam).

Required Reading (Textbook)

- Sections 5.4 to 5.6

Midterm to be held on Saturday, Oct 28, 2023.

Scope = Chapters 2, 3, and 4 (except 2.3).