MAT1002 Lecture 23, Thursday, Apr/13/2023

Outline

- Conservative fields (16.3)
  - ↳ Path independence
  - ↳ Fundamental theorem of line integrals
  - ↳ Loop property
  - ↳ Component test
  - ↳ Exact differential forms

Conservative Fields

A conservative force is a force with the property that the total work done in moving a particle between two points is independent of the path taken. Gravitational force is an example of a conservative force.

Definition

Let **F** be a vector field defined on an open region D. Then **F** is said to be conservative on D if the following condition holds: for any two points A and B in D, if C_1 and C_2 are piecewise smooth curves in D from A to B, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Any such line integral, which depends only on the initial and terminal points (but not the path itself), is said to be path independent.

Consider the vector field $\mathbf{F}(x, y) := \langle -y, x \rangle$. One can check that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

if C_1 is the line segment from $(0, 0)$ to $(1, 1)$ and C_2 is the curve given by

$$\langle t, t^2 \rangle, \quad t \in [0, 1].$$

Although both C_1 and C_2 are smooth curves from $(0, 0)$ to $(1, 1)$, the line integrals of **F** along them are different.

Hence, the field $\vec{F} := -y\vec{i} + x\vec{j}$ is not conservative.

Q: Which fields are conservative? What properties do they have?

Theorem (Fundamental Theorem of Line Integrals) (FTLI)

Let C be a piecewise smooth curve from A to B, parametrized by $\mathbf{r}(t)$, with $t \in [a,b]$. Let f be a real-valued function such that ∇f is continuous on a region D containing C. Then

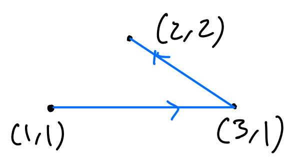
Piecewise smooth assumed

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

Proof: To be shown in class (also see the book). $\square$

As an immediate consequence, any continuous gradient field is conservative.

e.g. Find the work done by force $\vec{F} = \langle x, y \rangle$ along C, where C is



Sol: • $\vec{F} = \nabla f$, where $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$.

• Work $= \int_C \vec{F} \cdot d\vec{r} = \int_{(1,1)}^{(2,2)} \nabla f \cdot d\vec{r} \overset{\text{FTLI}}{=} f(2,2) - f(1,1)$

$= \frac{1}{2}(4+4) - \frac{1}{2}(1+1) = 4 - 1 = 3$.

Can use $\int_A^B$ to mean $\int_C$ if C is from A to B, given that the field is conservative.

Definition
If $\mathbf{F} = \nabla f$, then f is called a potential function for $\mathbf{F}$.

Q: Are there other conservative fields other than gradient fields?

Definition
A region D is said to be connected if any two points in D are connected by a curve lies entirely in D.

Theorem
Let $\mathbf{F}$ be a vector field whose components are continuous on an open connected region D. Then $\mathbf{F}$ is conservative on D if and only if $\mathbf{F}$ is a gradient field (that is, $\mathbf{F} = \nabla f$ for some f defined on D).

Will prove this for fields in $\mathbb{R}^2$. A proof for $\mathbb{R}^3$ is in the book (and is similar).

<u>Proof</u>: · "$\Leftarrow$" : This follows from the Fundamental Theorem of Line Integrals.

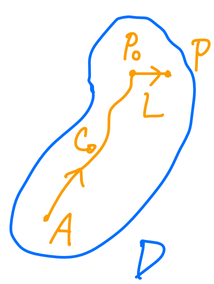
"$\Rightarrow$" : Assume that $\vec{F}$ is conservative.

- Fix $A := (a,b) \in D$. Define $f : D \to \mathbb{R}$ by $f(x,y) := \underbrace{\int_{(a,b)}^{(x,y)} \vec{F} \cdot d\vec{r}}_{\text{well-defined by path independence}}$

- Consider $P := (x,y)$. (What is f_x at (x,y)?)

- Fix $P_0 := (x_0, y)$ in D (say $x_0 < x$).

- Let C_0 be a piecewise smooth curve from A to P_0, and let L be the line segment from P_0 to P.



- Now $f(x,y) = \int_A^P \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}$

  curve from A to P

$$= \underbrace{\int_{C_0} \vec{F} \cdot d\vec{r}}_{K} + \int_L \vec{F} \cdot d\vec{r}$$

$L: \vec{r}(t) = \langle t, y \rangle$, $x_0 \le t \le x$

$\vec{r}'(t) = \langle 1, 0 \rangle$

$$= K + \int_{x_0}^{x} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = K + \int_{x_0}^{x} M(t, y)\, dt$$

- Consider moving the variable point P along the x-axis by a bit:

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{x_0}^{x} M(t, y)\, dt \overset{\text{By FTC}}{=\!=} M(x, y).$$

- Similarly, $\frac{\partial}{\partial y} f(x, y) = N(x, y)$.

- Hence, $\nabla f = \vec{F}$; $\vec{F}$ is a gradient field. $\square$

The Loop Property

- A given parametrization $\mathbf{r}(t)$, $a \le t \le b$, determines an orientation (or direction) of a curve C, with the positive direction corresponding to increasing values of t.

- In general, if C is a curve with a given orientation, we use $-C$ to denote the same curve with the opposite orientation. Then

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Theorem

Means: (a) $\Leftrightarrow$ (b); (a) if and only if (b).

The following statements are equivalent for any vector field $\mathbf{F}$.

(a) *The field* $\mathbf{F}$ *is conservative on* D.

(b) *For every closed curve* C *in* D, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

Proof: Shown in class (also in the book). $\square$

Example

The vector field defined by

$$\mathbf{F}(x, y) := \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

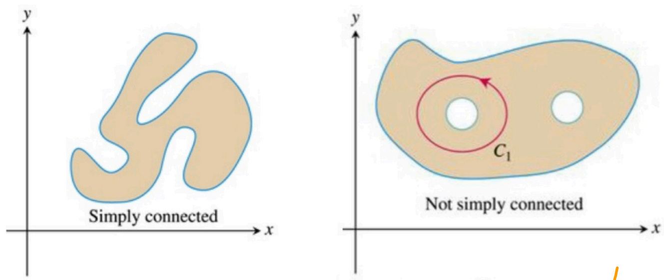
is not conservative on $\mathbb{R}^2 \setminus \{(0, 0)\}$, since its line integral along the unit circle is not equal to 0.

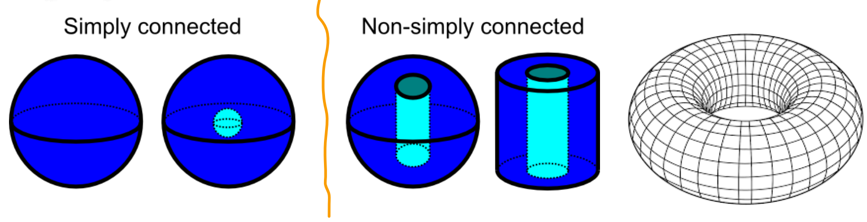
Component Test

“no hole” in 2D

A connected plane region D is said to be simply connected if every simple closed curve in D encloses only points that are in D.

Another way to think about it: a simple closed curve in D can be “shrank” to a point without leaving D. This can be generalized to 3D.





> **Theorem (Component Test for Conservative Fields)** (in $\mathbb{R}^2$)
>
> *Let $\mathbf{F}(x,y) := \langle M(x,y), N(x,y)\rangle$ be a vector field on an open simply connected domain D, such that M and N have continuous partial derivatives. Then $\mathbf{F}$ is conservative on D if and only if*
>
> $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
>
> *on D.*

Proof: Later.

Example

The vector field defined by $\mathbf{F}(x,y) := \langle -y, x\rangle$ is not conservative on $\mathbb{R}^2$, since it does not pass the component test.

Problem

Consider the vector field defined by

$$\mathbf{F}(x,y) := \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

We have shown that $\mathbf{F}$ is not conservative. On the other hand, one can check that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This seems to contradict the component test. What is wrong here?

Example

Consider the vector field defined by

$$\mathbf{F}(x, y) := \langle 3 + 2xy, x^2 - 3y^2 \rangle.$$

(a) Show that $\mathbf{F}$ is conservative on $\mathbb{R}^2$.

(b) Find a potential function for $\mathbf{F}$.

(c) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is parametrized by

$$\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t \rangle, \quad 0 \le t \le \pi.$$

Ans: (b) $f(x,y) = 3x + x^2y - y^3 + C$

(c) $e^{3\pi} + 1$.

Component Test in 3D

M, N, P cts on a simply connected region

There is also a component test for vector fields $\mathbf{F} := \mathbf{F}(x, y, z)$ in $\mathbb{R}^3$. Under assumptions similar to those ~~on Page 25~~ for the 2D component test, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

M N P

x y z

Exact Differential Forms

Consider $\vec{F} = \langle M, N, P \rangle$ (3D vector field). Then

- $\int_C \vec{F} \cdot d\vec{r} = \int_C \underbrace{M\,dx + N\,dy + P\,dz}_{\text{Differential form}}$.
- If $\vec{F} = \nabla f$ is a gradient field, then

$$\int_C M\,dx + N\,dy + P\,dz = \int_C \underbrace{f_x\,dx + f_y\,dy + f_z\,dz}_{\text{Total differential } df}$$

Def:
- An expression $M\,dx + N\,dy + P\,dz$ is a differential form.
- A differential form is exact on D if it is the total differential df for some real-valued (i.e. scalar) function f (on D).

Remark: $Mdx + Ndy + Pdz$ is exact if and only if $\vec{F}$ is conservative (with potential function f).

We summarize some of the key results ~~of this week~~ about conservative fields as a theorem.

Theorem

Let $\mathbf{F}$ *be a vector field whose components are continuous on an open connected region* D*. Then the following conditions are equivalent.*

(a) *The field* $\mathbf{F}$ *is conservative on* D.

(b) *There exists a function* f *such that* $\mathbf{F} = \nabla f$ *on* D.

(c) *For every closed curve* C *in* D, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

Under certain assumptions, (Simply connected D) whether (a) holds can be tested using the component test.