STA2001 Assignment 7

1. (4.1-3). Let the joint pmf of X and Y be defined by

$$f(x,y) = \frac{x+y}{32}$$

$x = 1, 2, y = 1, 2, 3, 4.$

(a) Find $f_X(x)$, the marginal pmf of X
(b) Find $f_Y(y)$, the marginal pmf of Y
(c) Find $P(X > Y)$
(d) Find $P(Y = 2X)$
(e) Find $P(X + Y = 3)$
(f) Find $P(X \leq 3 - Y)$
(g) Are X and Y independent or dependent? Why or why not?
(h) Find the means and the variances of X and Y

Solution:

(a) $f_X(x) = \sum_{y=1}^{4} \frac{x+y}{32} = \frac{x+1+x+2+x+3+x+4}{32} = \frac{4x+10}{32} = \frac{2x+5}{16},\ x = 1, 2$

(b) $f_Y(y) = \sum_{x=1}^{2} \frac{x+y}{32} = \frac{1+y+2+y}{32} = \frac{2y+3}{32},\ y = 1, 2, 3, 4.$

(c) $P(X > Y) = P(X = 2, Y = 1) = \frac{2+1}{32} = \frac{3}{32}$

(d) $P(Y = 2X) = P(X = 1, Y = 2) + P(X = 2, Y = 4) = \frac{1+2}{32} + \frac{2+4}{32} = \frac{9}{32}$

(e) $P(X + Y = 3) = P(X = 1, Y = 2) + P(X = 2, Y = 1) = \frac{1+2}{32} + \frac{2+1}{32} = \frac{3}{16}$

(f) $P(X \leq 3 - Y) = P(X + Y \leq 3) = P(X = 1, Y = 1) + P(X = 1, Y = 2) + P(X = 2, Y = 1) = \frac{2+3+3}{32} = \frac{1}{4}$

(g) $f(x,y) = \frac{x+y}{32} \neq f_X(x) \cdot f_Y(y) = \frac{2x+5}{16} \cdot \frac{2y+3}{32}$, so they are not independent.

(h) $E(X) = \sum_{x=1}^{2} x f_X(x) = 1 \cdot f_X(1) + 2 \cdot f_X(2) = 1 \times \frac{7}{16} + 2 \times \frac{9}{16} = \frac{25}{16}$

$E(X^2) = \sum_{x=1}^{2} x^2 f_X(x) = 1^2 \cdot f_X(1) + 2^2 \cdot f_X(2) = 1 \times \frac{7}{16} + 4 \times \frac{9}{16} = \frac{43}{16}$

$\mathrm{Var}(X) = E\left(X^2\right) - (EX)^2 = \frac{43}{16} - \left(\frac{25}{16}\right)^2 = \frac{63}{256}$

$E(Y) = \sum_{y=1}^{4} y f_Y(y) = \frac{5}{22} \times 1 + \frac{7}{32} \times 2 + \frac{9}{32} \times 3 + \frac{11}{32} \times 4 = \frac{90}{32} = \frac{45}{16}$

$E(Y^2) = \sum_{y=1}^{4} y^2 f_Y(y) = 1^2 \times \frac{5}{32} + 2^2 \times \frac{7}{32} + 3^2 \times \frac{9}{32} + 4^2 \times \frac{11}{32} = \frac{145}{16}$

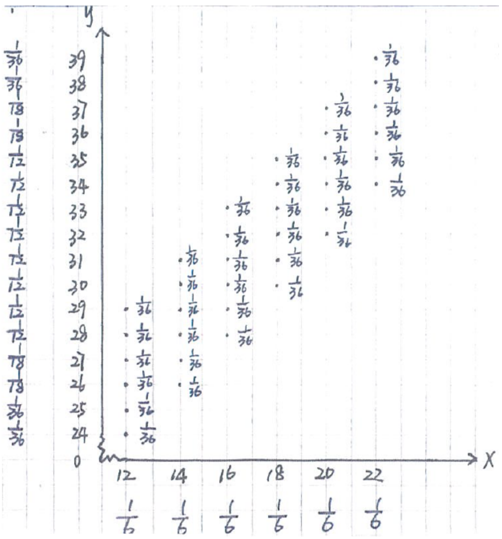
$\mathrm{Var}(X) = E\left(X^2\right) - (EX)^2 = \frac{145}{16} - \left(\frac{45}{16}\right)^2 = \frac{295}{256}$

2. (4.1-4). Select an (even) integer randomly from the set $\{12, 14, 16, 18, 20, 22\}$. Then select an integer randomly from the set $\{12, 13, 14, 15, 16, 17\}$. Let X equal the integer that is selected from the first set and let Y equal the sum of the two integers.

(a) Show the joint pmf of X and Y on the space of X and Y.
(b) Compute the marginal pmfs.
(c) Are X and Y independent? Why or why not?

Solution:

(a) $f(x,y) = \frac{1}{36}$, $x = 12, 14, 16, 18, 20, 22$, $y = 24, 26, 28, 30, 32, 34, 25, 27, 29, 31, 33, 35, 36, 37, 38, 39.$



(b) $f_X(x) = \frac{1}{6}$, $x = 12, 14, 16, 18, 20, 22$.

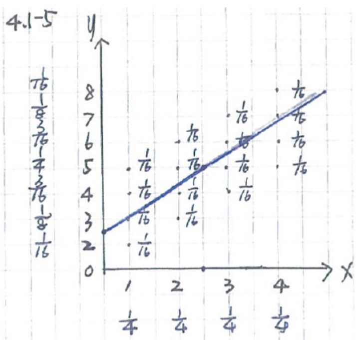
$$f_Y(y) = \begin{cases} \frac{1}{36}, & y = 24, 25, 38, 39 \\ \frac{1}{18}, & y = 26, 27, 36, 37 \\ \frac{1}{12}, & y = 28, 29, 30, 31, 32, 33, 34, 35. \end{cases}$$

(c) Since $f(x, y) \neq f_X(x) \cdot f_Y(y)$, they are not independent.

3. (4.1-5). Roll a pair of four-sided dice, one red and one black. Let X equal the outcome on the red die and let Y equal the sum of the two dice.
   (a) On graph paper, describe the space of X and Y.
   (b) Define the joint pmf on the space (similar to Figure 4.1-1).
   (c) Give the marginal pmf of X in the margin.
   (d) Give the marginal pmf of Y in the margin.
   (e) Are X and Y dependent or independent? Why or why not?

Solution:

(a)



(b) $f(x, y) = \frac{1}{16}$, $x = 1, 2, 3, 4$, $y = x + 1, x + 2, x + 3, x + 4$.

(c) $f_X(x) = \frac{1}{4}$, $x = 1, 2, 3, 4$.

(d)

$$f_Y(y) = \begin{cases} \frac{1}{16}, & y = 2, 8 \\ \frac{1}{8}, & y = 3, 7 \\ \frac{3}{16}, & y = 4, 6 \\ \frac{1}{4}, & y = 5 \end{cases}$$

(e) From the graph, we know the space is not a rectangular, so X and Y are not independent.

4. (4.1-8). In a smoking survey among men between the ages of 25 and 30. 63% prefer to date nonsmokers, 13% prefer to date smokers, and 24% dont care. Suppose nine such men are selected randomly. Let X equal the number who prefer to date nonsmokers and Y equal the number who prefer to date smokers.

(a) Determine the joint pmf of X and Y. Be sure to include the support of the pmf.
(b) Find the marginal pmf of X. Again include the support.

Solution:

(a) From the given information, we know the X and Y have a trinomial pmf with parameters $n = 9$, $p_X = 0.63$ and $p_Y = 0.13$,

$$\begin{aligned} f(x, y) &= \frac{n!}{x!y!(n-x-y)!} p_X^x p_Y^y \left(1 - p_X - p_Y\right)^{n-x-y} \\ &= \frac{9!}{x!y!(9-x-y)!} (0.63)^x (0.13)^y (0.24)^{9-x-y} \end{aligned}$$

where the support is given by $S = \{(x, y) | x \in \mathbb{Z}^+, y \in \mathbb{Z}^+, 0 \leq x + y \leq 9\}$.

(b) We first prove that if X and Y have the trinomial pmf $f(x, y)$ with parameters n, p_X and p_Y, then X and Y have marginal binomial distributions $b(n, p_X)$ and $b(n, p_Y)$, respectively. We here prove that $X \sim b(n, p_X)$. The proof that $Y \sim b(n, p_Y)$ is likewise.

$$\begin{aligned} f_X(x) &= \sum_{y=0}^{n-x} f(x, y) \\ &= \sum_{y=0}^{n-x} \frac{n!}{x!y!(n-x-y)!} p_X^x p_Y^y \left(1 - p_X - p_Y\right)^{n-x-y} \\ &= \frac{n!}{x!(n-x)!} p_X^x \sum_{y=0}^{n-x} \frac{(n-x)!}{y!(n-x-y)!} p_Y^y (1 - p_X - p_Y)^{n-x-y} \\ &= \frac{n!}{x!(n-x)!} p_X^x (1 - p_X - p_Y + p_Y)^{n-x} \\ &= \frac{n!}{x!(n-x)!} p_X^x (1 - p_X)^{n-x}. \end{aligned}$$

Therefore $X \sim b(9, 0.63)$ and we have

$$f_X(x) = \frac{9!}{x!(9-x)!} 0.63^x (1 - 0.63)^{9-x}, \quad x = 0, 1, \ldots, 9.$$

5. (4.1-9). A manufactured item is classified as good, a second, or defective with probabilities 6/10,

3/10, and 1/10, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.

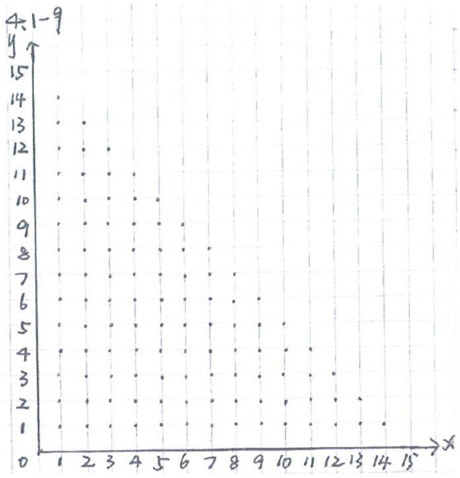
(a) Give the joint pmf of X and Y, $f(x, y)$.
(b) Sketch the set of integers (x, y) for which $f(x, y) > 0$. From the shape of this region, can X and Y be independent? Why or why not?
(c) Find $P(X = 10, Y = 4)$.
(d) Give the marginal pmf of X.
(e) Find $P(X \le 11)$.

Solution:

(a) From the given information, we know that it's a trinomial distribution,

$$f(x, y) = \frac{15!}{x!y!(15 - x - y)!} \left(\frac{6}{10}\right)^x \left(\frac{3}{10}\right)^y \left(\frac{1}{10}\right)^{15-x-y}, \quad 0 \le x + y \le 15$$

(b) From the shape of this region, the space is not rectangular. Therefore, X and Y are not independent.



(c) $P(X = 10, Y = 4) = \frac{15!}{10!4!1!} 0.6^{10} 0.3^4 0.1^1 = 0.0735$

(d) As it's the marginal pmf of a trinomial distribution, so it's a binomial distribution (prove it by yourself),

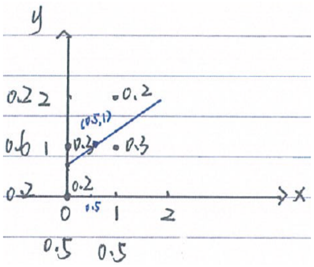
$$f_X(x) = \frac{15!}{x!(15 - x)!} (0.6)^x (0.4)^{15-x}, \quad 0 \le x \le 15$$

(e) $P(X \le 11) = \sum_{k=0}^{11} f_X(k) = 1 - \sum_{k=12}^{15} f_X(k) = 0.9095.$

6. (4.2-2). Let X and Y have the joint pmf defined by $f(0, 0) = f(1, 2) = 0.2$, $f(0, 1) = f(1, 1) = 0.3$.
(a) Depict the points and corresponding probabilities on a graph.
(b) Give the marginal pmfs in the 'margins.'
(c) Compute $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \mathrm{Cov}(X, Y)$, and ρ.

Solution:

(a) & (b)



We can also easily compute the marginal pmfs given the joint pmf,

$$f_X(x) = \begin{cases} 0.5, & x = 0 \\ 0.5, & x = 1 \end{cases}$$

$$f_Y(y) = \begin{cases} 0.2, & y = 0 \\ 0.6, & y = 1 \\ 0.2, & y = 2 \end{cases}$$

(c)

$$\begin{aligned}
\mu_X &= 0.5 \times 0 + 0.5 \times 1 = 0.5 \\
\mu_Y &= 0.2 \times 0 + 0.6 \times 1 + 0.2 \times 2 = 1 \\
\sigma_X^2 &= \sum_x (x - \mu_X)^2 f_X(x) = (0 - 0.5)^2 \times 0.5 + (1 - 0.5)^2 \times 0.5 = 0.25 \\
\sigma_Y^2 &= \sum_y (y - \mu_Y)^2 f_Y(y) = (0 - 1)^2 \times 0.2 + (1 - 1)^2 \times 0.6 + (2 - 1)^2 \times 0.2 = 0.4 \\
\mathrm{Cov}(X, Y) = E(XY) - \mu_X \mu_Y &= \sum_{(x,y) \in S} xyf(x, y) - \mu_X \mu_Y = 0.7 - 0.5 \times 1 \times 1 = 0.2 \\
\rho = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} &= \frac{1/5}{\sqrt{1/4}\sqrt{2/5}} = \frac{\sqrt{10}}{5} \approx 0.6325
\end{aligned}$$

7. (4.2-3). Roll a fair four-sided die twice. Let X equal the outcome on the first roll, and let Y equal the sum of the two rolls. Determine $\mu_X, \mu_Y, \sigma_x^2, \sigma_Y^2, \mathrm{Cov}(X, Y)$, and ρ.

Solution:

According to given information, we can easily write down the joint pmf and the marginal pmfs,

$$\begin{aligned}
f(x, y) &= \frac{1}{16}, \quad x = 1, 2, 3, 4, \; y = x + 1, x + 2, x + 3, x + 4. \\
f_X(x) &= \frac{1}{4}, \quad x = 1, 2, 3, 4. \\
f_Y(y) &= \begin{cases} \frac{1}{16}, & y = 2, 8 \\ \frac{1}{8}, & y = 3, 7 \\ \frac{3}{16}, & y = 4, 6 \\ \frac{1}{4}, & y = 5 \end{cases}
\end{aligned}$$

Then we can compute the following,

$$\mu_X = \sum_{x=1}^{4} x f_X(x) = 1 \times \frac{1}{4} + 2 \times \frac{1}{4} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{10}{4} = \frac{5}{2}$$

$$\mu_Y = \sum_{y=2}^{8} y f_Y(y) = 2 \cdot \frac{1}{16} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{3}{16} + 5 \cdot \frac{1}{4} + \frac{1}{16} + 7 \cdot \frac{1}{8} + 6 \times \frac{3}{4} = 5$$

$$\mu_X^2 = \sum_{x=1}^{4} x^2 f_X(x) = 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{4} + 3^2 \cdot \frac{1}{4} + 4^2 \cdot \frac{1}{4} = 7.5$$

$$\mu_Y^2 = \sum_{y=2}^{8} y^2 f_Y(y) = 2^2 \cdot \frac{1}{16} + 3^2 \cdot \frac{1}{8} + 4^2 \cdot \frac{3}{16} + 5^2 \cdot \frac{1}{4} + +6^2 \cdot \frac{3}{16} + 7^2 \frac{1}{8} + 8^2 \cdot \frac{1}{16} = 27.5$$

$$\mathrm{Var}(X) = E(X^2) - (EX)^2 = 7.5 - 2.5^2 = 1.25$$

$$\mathrm{Var}(Y) = E(Y^2) - (EY)^2 = 27.5 - 5^2 = 2.5$$

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = \sum_{(x,y)\in S} xyf(x, y) - \mu_X \mu_Y = 1.25, \text{ where } S \text{ is the support of the joint pmf.}$$

$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\sqrt{2}}{2}$$

8. (4.2-9). A car dealer sells X cars each day and always tries to sell an extended warranty on each of these cars. (In our opinion, most of these warranties are not good deals.) Let Y be the number of extended warranties sold; then $Y \leq X$. The joint pmf of X and Y is given by

$$f(x, y) = c(x + 1)(4 - x)(y + 1)(3 - y)$$

$x = 0, 1, 2, 3,\ y = 0, 1, 2$, with $y \leq x$.

(a) Find the value of c.
(b) Sketch the support of X and Y.
(c) Record the marginal pmfs $f_X(x)$ and $f_Y(y)$ in the margins.
(d) Are X and Y independent?
(e) Compute μ_X and σ_X^2.
(f) Compute μ_Y and σ_Y^2.
(g) Compute $\mathrm{Cov}(X, Y)$
(h) Determine ρ, the correlation coefficient.

Solution:

(a) Let S be the support of the joint distribution (i.e. the space that the joint pmf takes positive values), so we must have

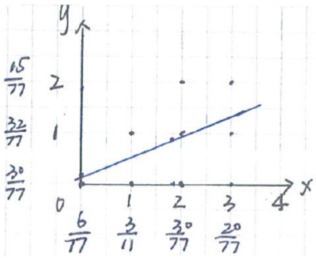
$$\sum_{(x,y)\in S} f(x, y) = 1$$

Therefore, given this condition and the joint pmf $f(x, y)$ we could write an equation with respect to c,

$$f(0, 0) + f(1, 1) + f(1, 0) + f(2, 0) + f(2, 1) + f(2, 2) + f(3, 0) + f(3, 1) + f(3, 2) = 1$$

Solve this equation, we have $c = \frac{1}{154}$.

(b)



(c)

$$f_X(x) = \begin{cases} \frac{6}{77}, & x = 0 \\ \frac{21}{77}, & x = 1 \\ \frac{30}{77}, & x = 2 \\ \frac{20}{77}, & x = 3 \end{cases} \qquad f_Y(y) = \begin{cases} \frac{30}{77}, & y = 0 \\ \frac{32}{77}, & y = 1 \\ \frac{15}{77}, & y = 2 \end{cases}$$

(d) Since the space is not a rectangular, X and Y are not independent.

(e) $\mu_x = \sum_{x=0}^{3} x f_X(x) = \frac{141}{77}$

$\sigma_X^2 = E\left(X^2\right) - (EX)^2 = \frac{4836}{5929}$

(f) $\mu_Y = \sum_{y=0}^{2} y f_Y(y) = \frac{62}{77}$

$\sigma_Y^2 = E\left(Y^2\right) - (EY)^2 = \frac{3240}{5929}$

(g) $\operatorname{Cov}(X,Y) = E(XY) - \mu_X \mu_Y = \frac{1422}{5929}$

(h) $\rho = \frac{\operatorname{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{79}{24180}\sqrt{12090} \approx 0.3592$

9. Let Y_1, Y_2, Y_3 be independent random variables which have the Bernoulli distribution with the probability of success p.

(a) Define a new random variable $Z = Y_1 + Y_2 + Y_3$. For independent Y_1, Y_2, Y_3, we have

$$\mathrm{E}(e^{tZ}) = \mathrm{E}(e^{t(Y_1+Y_2+Y_3)}) = \mathrm{E}(e^{tY_1})\mathrm{E}(e^{tY_2})\mathrm{E}(e^{tY_3}) = (1 - p + pe^t)^3$$

What's the distribution of Z?

(b) For $k = 1, 2$, let

$$X_k = \begin{cases} 1, & Y_1 + Y_2 + Y_3 = k, \\ -1 & Y_1 + Y_2 + Y_3 \neq k. \end{cases}$$

i. Find the joint pmf of X_1, X_2.

ii. Find the marginal pmfs of X_1 and X_2, respectively.

iii. Find the value of the success probability p that minimizes $\mathrm{E}(X_1 X_2)$.

iv. Compute $\operatorname{Cov}(X_1 - X_2, X_2)$.

Solution:

(a) Z has a binomial distribution $Z = Y_1 + Y_2 + Y_3 \sim b(3, p)$

(b) i. Let $f(X_1, X_2)$ be the joint pmf of X_1 and X_2,then we have

$$\begin{aligned} f(-1,-1) &= P\{X_1 = -1, X_2 = -1\} \\ &= P\{Y_1 + Y_2 + Y_3 \neq 1, Y_1 + Y_2 + Y_3 \neq 2\} \\ &= P\{Y_1 + Y_2 + Y_3 = 0\} + P\{Y_1 + Y_2 + Y_3 = 3\} \\ &= (1-p)^3 + p^3 \end{aligned}$$

$$
\begin{aligned}
f(-1,1) &= P\{X_1 = -1, X_2 = 1\} \\
&= P\{Y_1 + Y_2 + Y_3 \neq 1, Y_1 + Y_2 + Y_3 = 2\} \\
&= \binom{3}{2} p^2(1-p) \\
&= 3p^2(1-p)
\end{aligned}
$$

$$
\begin{aligned}
f(1,-1) &= P\{X_1 = 1, X_2 = -1\} \\
&= P\{Y_1 + Y_2 + Y_3 = 1, Y_1 + Y_2 + Y_3 \neq 2\} \\
&= P\{Y_1 + Y_2 + Y_3 = 1\} \\
&= \binom{3}{1} p(1-p)^2 = 3p(1-p)^2
\end{aligned}
$$

$$
\begin{aligned}
f(1,1) &= P\{X_1 = 1, X_2 = 1\} \\
&= P\{Y_1 + Y_2 + Y_3 = 1, Y_1 + Y_2 + Y_3 = 2\} = 0
\end{aligned}
$$

ii. Then we could get the marginal pmf of X_k as follows:

$$
f_{X1}(x) = \begin{cases} 1 - 3p + 6p^2 - 3p^3, & X_1 = -1, \\ 3p(1-p)^2, & X_1 = 1. \end{cases}
$$

$$
f_{X2}(x) = \begin{cases} 1 - 3p^2 + 3p^3, & X_2 = -1, \\ 3p^2(1-p), & X_2 = 1. \end{cases}
$$

iii.

$$
\begin{aligned}
&\mathrm{E}(X_1X_2) = 1 \times P\{X_1 = -1, X_2 = -1\} + (-1) \times P\{X_1 = -1, X_2 = 1\} \\
&+ (-1) \times P\{X_1 = 1, X_2 = -1\} + 1 \times P\{X_1 = 1, X_2 = 1\} = 1 - 6p + 6p^2
\end{aligned}
$$

When $p = \frac{1}{2}$, $\mathrm{E}(X_1X_2)$ gets the minimum value $-\frac{1}{2}$.

iv.

$$
\begin{aligned}
\mathrm{Cov}(X_1 - X_2, X_2) &= \mathrm{Cov}(X_1, X_2) - \mathrm{Cov}(X_2, X_2) \\
&= \mathrm{E}(X_1X_2) - \mathrm{E}(X_1)\mathrm{E}(X_2) - \mathrm{Var}(X_2) \\
&= -12p^2 - 24p^3 + 144p^4 - 180p^5 + 72p^6
\end{aligned}
$$

10. (2023 Final Q10) Suppose the joint distribution of two discrete random variables X, Y is given as

$$
P_{XY}(n,m) = \frac{\lambda^n e^{-\lambda}}{n!} \begin{pmatrix} n \\ m \end{pmatrix} p^m (1-p)^{n-m}
$$

where $0 \leq m \leq n$ and $0 \leq p \leq 1$.

(a) Find the marginal pmf $P_Y(m)$.

(b) Find the marginal pmf $P_X(n)$.

(c) Find the conditional pmf $P_{X|Y}(n|m)$.

(d) (optional) Find the expectation $E[XY]$.

Hint: $\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$

Solution:

(a) Since we have

$$P_{XY}(n,m) = \frac{(p\lambda)^m e^{-\lambda}}{m!}\left(\frac{\lambda^{n-m}(1-p)^{n-m}}{(n-m)!}\right),$$

then the marginal pmf is

$$\begin{aligned} P_Y(m) &= \frac{(p\lambda)^m e^{-\lambda}}{m!}\sum_{n=m}^{\infty}\left(\frac{\lambda^{n-m}(1-p)^{n-m}}{(n-m)!}\right) \\ &= \frac{(p\lambda)^m e^{-\lambda}}{m!}\sum_{n=0}^{\infty}\left(\frac{\lambda^{n}(1-p)^{n}}{(n)!}\right) \\ &= \frac{(p\lambda)^m e^{-\lambda}}{m!} e^{\lambda(1-p)} \\ &= \frac{(p\lambda)^m e^{-\lambda p}}{m!}, \quad m = 0, 1, \cdots \end{aligned}$$

(b) Since we have

$$P_{XY}(n,m) = \frac{\lambda^n e^{-\lambda}}{n!}\binom{n}{m} p^m (1-p)^{n-m}$$

then the marginal pmf is

$$\begin{aligned} P_X(n) &= \frac{\lambda^n e^{-\lambda}}{n!}\sum_{m=0}^{n}\binom{n}{m} p^m (1-p)^{n-m} \\ &= \frac{\lambda^n e^{-\lambda}}{n!}, \quad n = 0, 1, \cdots \end{aligned}$$

where the summation equals 1 as the binominal distribution.

(c) The conditional distribution of X given Y is

$$\begin{aligned} P_{X|Y}(n \mid m) &= \frac{P_{XY}(n,m)}{P_Y(m)} = \frac{1}{P_Y(m)}\frac{\lambda^n e^{-\lambda}}{n!}\binom{n}{m} p^m (1-p)^{n-m} \\ &= \underbrace{\frac{\lambda^m e^{-\lambda}}{P_Y(m) m!}}_{C} \times \frac{(\lambda(1-p))^{n-m}}{(n-m)!}, \quad n = m, m+1, \cdots \end{aligned}$$

Note that C is the normalization factor which does not depend on n. Based on the form of $P_{X|Y}(n \mid m)$, and the fact that $\sum_{n=m}^{\infty} P_{X|Y}(n \mid m) = 1$, we have

$$C = e^{-\lambda(1-p)}$$

that is $P_{X|Y}$ is a shifted Poisson distribution.

(d) If Z is a Poisson random variable with parameter $\lambda(1-p)$, then conditioned on $Y = m$ we have $X = Z + m$. Hence, $E[Z + m \mid Y = m] = \lambda(1-p) + m$.

Then the expectation is

$$\begin{aligned}
E[XY] &= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} mnp_{X|Y}(n|m)p_Y(m) \\
&= \sum_{m=0}^{\infty} mp_Y(m) \sum_{n=m}^{\infty} np_{X|Y}(n|m) \\
&= \sum_{m=0}^{\infty} mp_Y(m)E[X|Y=m] \\
&= \sum_{m=0}^{\infty} m\frac{(p\lambda)^m e^{-\lambda p}}{m!}(m+\lambda-\lambda p) \\
&= \sum_{m=0}^{\infty} m^2 \frac{(p\lambda)^m e^{-\lambda p}}{m!} + (\lambda-\lambda p)\sum_{m=0}^{\infty} m\frac{(p\lambda)^m e^{-\lambda p}}{m!} \\
&= E[Y^2] + (\lambda-\lambda p)E[Y] \\
&= \lambda p + (\lambda p)^2 + (\lambda-\lambda p)\lambda p \\
&= \lambda + \lambda^2 p
\end{aligned}$$

where Y is a Poisson distribution.