MAT 1002 Final Exam, 4:00-6:30 pm, May 18, 2021

Your Name and Student ID:

Your Lecture Class(e.g, L1) and **your tutorial class** (e.g, T01):

Instruction: (i) This is a closed-book and closed-notes exam; no calculators, no dictionaries and no cell phones; (ii) Show your work unless otherwise instructed–a correct answer without showing your work when required shall be given **no credits**; (iii) Write down ALL your work and your answers(including the answers for short questions) in the Answer Book.

1. (30 points) **Short Questions** (for these questions, NO need to show your work, just write down your answers in the Exam Book; NO partial credits for each question)

   (i). If $a_n \leq b_n$ for all $n \geq N$ (for some fixed integer N), and the series $\sum b_n$ converges, then $\sum a_n$ must also converge.

   True False

   (ii). Find curvature of the curve given by

   $$\overrightarrow{r}(t) =< \cos t + t \sin t, \sin t - t \cos t, 3 >, \quad 0 < t < \infty.$$

   ____.

   (iii). If $f = f(x, y)$ has all directional derivatives at (a, b), then f must be differentiable at (a, b).

   True False

   (iv). Let

   $$f(x, y) = xy\frac{x^2 - 2y^2}{x^2 + y^2}.$$

   Find $f_y(x, 0)$, where $x \neq 0$.

   ____.

   (v). Suppose today's temperature function is given by $T(x, y) = 43 - y^2 - 2y + xy - x$, and you are taking this exam at CUHKSZ which is located at $(0, 0)$. To escape from the sweltering (hot) weather the fastest, in which direction should you head to?

   ____.

(vi). If $M = M(x, y)$ and $N = N(x, y)$ both have continuous partial directives on an open region D, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ on D, then the vector field $\overrightarrow{F} = \langle M, N \rangle$ must be conservative on D.

True False

(vii). If $f = f(x, y)$ is continuous on a closed and bounded region D, then f must attain its absolute maximum and absolute minimum values in D.

True False

(viii). For the critical points of the function $f(x, y) = 2x^4 + y^4 - 2x^2 - 2y^2$, which one of the following statements is correct?

(a) $(0, 0)$ is a local minimum point.
(b) $(0, 1)$ is a local maximum point.
(c) $(0, -1)$ is a saddle point.
(d) There are no local maximum points among all the critical points.

True False

(ix). Let $\overrightarrow{V}(x, y, z) =< x^2 - y, 4z, x^2 >$ be the velocity vector field of a gas flowing in space. At point $(1, 1, 1)$ which of the following is true?

(a) The gas is expanding.
(b) The gas is contracting.
(c) Neither of the above.

(x). For the gas mentioned above and at point $(1, 1, 1)$, find a vector around which the gas rotates most rapidly:

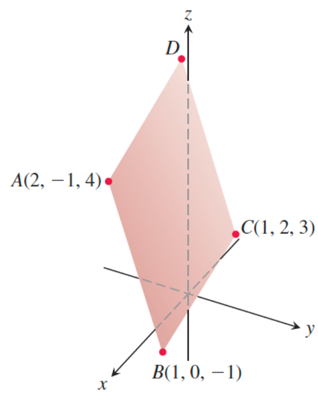
____.

2. (8 points) Find all values of x for which the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ converges; and indicate if the convergence is absolute or conditional.

3. (6 points) Find the following limit

$$\lim_{x \to 0} \frac{2x^2(1 - \cos(x^2)) - x^6}{\sin(x^{10})}.$$

4. (12 points) The parallelogram shown below has vertices $A(2, -1, 4)$, $B(1, 0, -1)$, $C(1, 2, 3)$ and D. Find



(a) The cosine of the interior angle at B.

(b) The vector projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$.

(c) The area of the parallelogram.

(d) An equation for the plane containing the parallelogram.

5. (15 points) Consider the surface $S : \cos(\pi yz) + 4xz^2 = 1$.

   (a) Find an equation of the tangent plane at $(1/2, 1, -1)$.

   (b) Let $z = f(x, y)$ be the function implicitly defined by $\cos(\pi yz) + 4xz^2 = 1$. Find the derivative of $f(x, y)$ at the point $(1/2, 1)$ in the direction of $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$.

   (c) Find parametric equations of the tangent line of the contour curve $f(x, y) = -1$ in the plane $z = -1$, with the point of tangency being $(1/2, 1, -1)$.

6. (9 points) Let $f(x, y)$ be such that f and its partial derivatives up to order 2 are continuous in the rectangle

$$R = \{(a, b) \mid -1 < a, b < 1\}$$

Use Taylor's theorem for functions of a single variable to prove that for any point $(x, y) \in R$ there exists $c \in (0, 1)$ such that

$$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2}\left(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right)\Big|_{(cx,cy)} .$$

7. (8 points) Find the maximum value of $f(x, y, z) = x+2y+5z$ on the sphere $x^2+y^2+z^2 = 1$ by the method of Lagrange multipliers.

8. (8 points) Consider the integral

$$\int_0^9 \int_0^1 \int_{2y}^2 \frac{4 \sin x^2}{\sqrt{z}} dxdydz.$$

(a) Sketch the solid on which the triple integral of the integrand is equal to the above iterated integral.

(b) Find a way to evaluate the integral.

9. (6 points) Consider the solid ball B of radius 2 in the xyz-space with equation $x^2 + y^2 + z^2 \leq 4$. If we take out from B the portion inscribed by the cylinder $x^2 + y^2 = 1$, what is the volume of the remaining solid?

10. (6 points) Compute $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$, where $\overrightarrow{F} =< \arctan e^x + 4y, \ln(1+y^2) + x >$, and C is the circle $x^2 + y^2 = 1$, oriented counter-clockwise.

11. (6 points) Let S be the unit upper hemisphere

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0,$$

oriented by the unit outer normal vector field $\overrightarrow{n}$; let $\overrightarrow{F} =< y, x, (x^2+y^4)^{3/2} \sin(e^{\sqrt{xyz}}) >$. Compute

$$\int\int_S (\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\, d\sigma.$$

12. (6 points) Let Ω be the part of the unit ball $x^2 + y^2 + z^2 \leq 1$ inside the first octant; let S be the boundary of Ω, oriented by the unit outer normal vector field $\overrightarrow{n}$. Compute

$$\int\int_S \overrightarrow{F} \cdot \overrightarrow{n}\, d\sigma,$$

where $\overrightarrow{F} =< x^2, -2xy, xz >$.