MAT1002 Lecture 27, Thursday, Apr/27/2023

Outline

- Divergence and the divergence theorem (16.8)
- Summary of "differentiation operators" and "big theorems"

Divergence

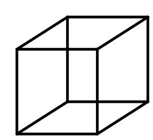
> **Def**: Given a vector field $\vec{F}$, the **divergence of $\vec{F}$** is
> $$\mathrm{div}(\vec{F}) := \nabla \cdot \vec{F}.$$

- If $\vec{F} = \langle M, N, P \rangle$, then $\mathrm{div}\,\vec{F} = \frac{\partial}{\partial x}M + \frac{\partial}{\partial y}N + \frac{\partial}{\partial z}P$.
- If $\vec{F} = \langle M, N \rangle$, then $\mathrm{div}\,\vec{F} = \frac{\partial}{\partial x}M + \frac{\partial}{\partial y}N$.

What would be a "physical" meaning of $\mathrm{div}(\vec{F})$?

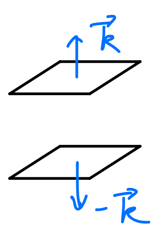
Fix a point $P(x_0, y_0, z_0)$, and consider a (very tiny) cubic solid E containing P. Let S be the surface of E (the "shell").

Consider the (outward) flux of $\vec{F}$ across S.



$E = [a,b] \times [c,d] \times [s,t]$

(Assume M, N, P have continuous partials)



- Flux I

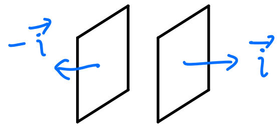
$$= \iint_{[a,b]\times[c,d]} \vec{F}(x,y,t)\,\vec{k}\,d\sigma$$

$$- \iint_{[a,b]\times[c,d]} \vec{F}(x,y,s)\,\vec{k}\,d\sigma$$

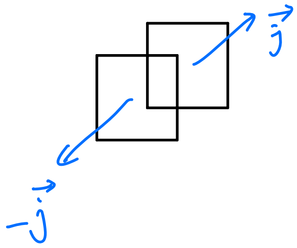
$$= \int_a^b \int_c^d \Big(P(x,y,t) - P(x,y,s)\Big)\,dy\,dx = \int_a^b \int_c^d \int_s^t \frac{\partial}{\partial z}P(x,y,z)\,dz\,dy\,dx$$

$$= \iiint_E \frac{\partial P}{\partial z}\,dV$$

- Similarly,



$$\text{Flux } 2 = \iiint_E \frac{\partial M}{\partial x}\, dV$$



$$\text{Flux } 3 = \iiint_E \frac{\partial N}{\partial y}\, dV$$

- $\text{Flux} = \int_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_E \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}\right) dV = \iiint_E \operatorname{div}\vec{F}\, dV.$

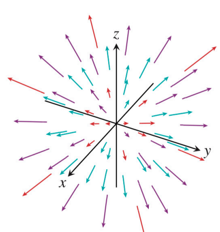
Since E is arbitrarily tiny, $(\operatorname{div}\vec{F})(P)$ is the flux density:

$$\lim_{V(E)\to 0} \frac{\text{flux across shell } S \text{ surrounding } P}{\text{volume of solid } E \text{ enclosed by } S}.$$

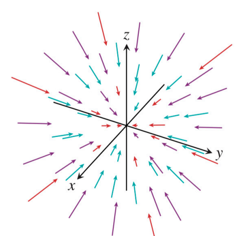
$$(\operatorname{div}\vec{F})(P) \begin{cases} >0 \Rightarrow \text{expanding (diverging) at } P \\ <0 \Rightarrow \text{Compressing (shrinking) at } P \\ =0 \Rightarrow \text{neither.} \end{cases}$$

$\langle x,y,z\rangle$

$\operatorname{div}\vec{F} = 3$



(a)



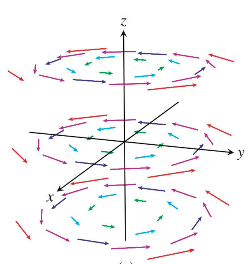
(b)

$\langle -x,-y,-z\rangle$

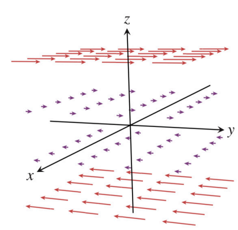
$\operatorname{div}\vec{F} = -3$

$\langle -y,x,0\rangle$

$\operatorname{div}\vec{F}=0$



(c)



(d)

$\langle 0,z,0\rangle$

$\operatorname{div}\vec{F} = 0$

We saw that any curl field is "surface independent". Another property of a curl field is the following.

Theorem

If $\mathbf{F}$ *is a vector field in* $\mathbb{R}^3$ *whose components have continuous second partial derivatives, then*

$$\mathit{div}\,(\mathit{curl}\,\mathbf{F}) \equiv 0.$$

This follows from direct computation with the mixed derivative theorem.

Proof:

$$\begin{aligned}\text{div}\,(\text{curl}\,\mathbf{F}) &= \nabla \cdot (\nabla \times \mathbf{F}) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} \\ &= 0.\end{aligned}$$

Example

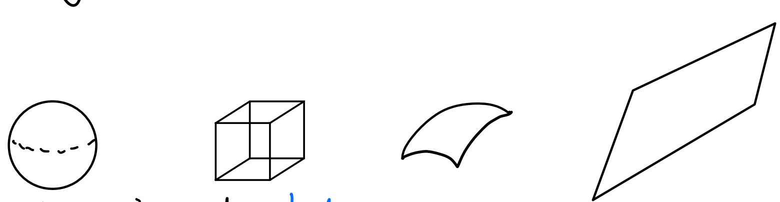
Show that $\mathbf{F}(x, y, z) := \langle xy, xyz, -y^2\rangle$ is not the curl field of any field.

Divergence Theorem

Roughly speaking, a closed surface is a surface that divides the space into two regions, "inside" (bounded) and "outside" (unbounded).

Equivalently, a closed surface is a surface with no boundary curve.

e.g.



$x^2+y^2+z^2=a^2$ cubic shell

closed

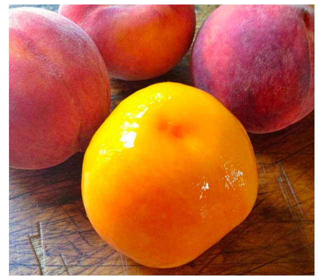
not closed

e.g. $x^2+y^2+z^2 \le a^2$: closed bdd solid (closed ball $\overline{B_a}$).

$x^2+y^2+z^2 = a^2$: closed surface that is the boundary of $\overline{B_a}$.

e.g. Whole peach : closed bdd solid.

peach skin : closed surface that is the boundary of the peach.



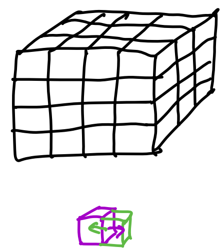
Theorem (Divergence Theorem) A.K.A. Gauss' divergence thm

Let $\mathbf{F} = \langle M, N, P \rangle$ *be a vector fields whose components have continuous partial derivatives. If* E *is a bounded solid having* S *as its boundary, where* S *is a closed, piecewise smooth surface, oriented* outward *(i.e. its unit normals point out from* E*), then*

$$\underbrace{\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma}_{\text{Total flux across } S} = \iiint_E \underbrace{\operatorname{div} \mathbf{F}}_{\text{"Microscopic flux" at a point}} \, dV.$$

Total flux across S

"Microscopic flux" at a point, $\approx$ flux across a tiny closed surface enclosing the point



When summing microscopic flux, adjacent face get cancelled.

e.g.1 Find the flux of $\vec{F} = \langle z, y, x \rangle$ across the sphere $x^2+y^2+z^2=1$.

if no direction is specified, flux is outward

- If we compute the flux directly as a surface integral, it will be very complicated.
- If we use divergence thm:

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_{\underset{x^2+y^2+z^2\le 1}{B}} \operatorname{div}\vec{F}\cdot dV = \iiint_B 1\, dV = \mathrm{Vol}(B) = \frac{4}{3}\pi 1^3 = \frac{4}{3}\pi.$$

e.g.2 Find the flux of $\vec{F} = \langle x^2, 4xyz, ze^x \rangle$ aross the boundary of the box $0 \le x \le 3$, $0 \le y \le 2$, $0 \le z \le 1$.

- Computing using surface integrals is tedious (six faces).
- If we use divergence thm:

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_{\underset{\text{box}}{E}} \operatorname{div}\vec{F}\, dV = \iiint_E (2x+4xz+e^x)\, dV$$

$$= \int_0^3\int_0^2\int_0^1 \underbrace{(2x+4xz+e^x)}_{\text{no } y}\, dz\, dy\, dx$$

$$= \left(\int_0^2 dy\right)\left(\int_0^3\int_0^1 (2x+4xz+e^x)\, dz\, dx\right)$$

$$= 2\int_0^3 \left[(2x+e^x)z + 2xz^2\right]_{z=0}^{1} dx = 2\int_0^3 (2x+e^x+2x)\, dx$$

$$= 2\left(2x^2+e^x\right)\Big|_0^3 = 2(18+e^3-1) = 2(17+e^3).$$

More General Solids

The divergence thm can be used on solids that can be decomposed into finitely many bounded solids, each of which having a closed surface as its boundary.

e.g. $E: \ b^2 \leq x^2+y^2+z^2 \leq a^2$ $\quad$ $(0<b<a)$ $\quad$ ($\overline{B_a}$, with B_b removed)

Purple surfaces are in common but have different orientations, so flux across them cancel each other.

Flux across boundary of E

$= $ flux across " " E_1 + flux across " " E_2

$$= \iiint_{E_1} \operatorname{div}\vec{F}\, dV + \iiint_{E_2} \operatorname{div}\vec{F}\, dV \quad \text{(div thm, base form)}$$

$$= \iiint_{E} \operatorname{div}\vec{F}\, dV$$

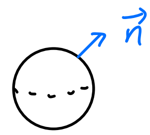
e.g. Consider $\vec{F} = \dfrac{1}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z\rangle$.

(a) Show that the outward flux across any closed surface S enclosing the origin is the same.

(b) Find the value of such a flux.

Sol: (a) · Fix S. Pick a small enough so that the ball $\overline{B_a}$ lies in the interior of the solid enclosed by S. Let E be the solid inside S but outside of S_a.

· Suppose S_a is oriented outward.



· By the divergence theorem (previous example),

$$\iint_{S \cup -S_a} \vec{F}\cdot\vec{n}\, d\sigma = \iiint_E \operatorname{div}(\vec{F})\, dV. \qquad ①$$

· Let $\rho := \rho(x,y,z) = \sqrt{x^2+y^2+z^2}$. Then

$$\frac{\partial M}{\partial x} = \frac{\rho^3 - x\cdot\frac{3}{2}\rho\cdot 2x}{\rho^6} = \frac{\rho^2-3x^2}{\rho^5}.$$

Similarly, $\dfrac{\partial N}{\partial y} = \dfrac{\rho^2-3y^2}{\rho^5}$, $\dfrac{\partial P}{\partial z} = \dfrac{\rho^2-3z^2}{\rho^5}$.

· $\operatorname{div}(\vec{F}) = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} + \dfrac{\partial P}{\partial z} = \dfrac{3\rho^2-3\rho^2}{\rho^5} = 0$, $\forall (x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$.

· By ①, $\iint_{S\cup -S_a} \vec{F}\cdot\vec{n}\, d\sigma = 0$.

· But $\iint_{S\cup -S_a} \vec{F}\cdot\vec{n}\, d\sigma = \iint_S \vec{F}\cdot\vec{n}\, d\sigma - \iint_{S_a} \vec{F}\cdot\vec{n}\, d\sigma$,

So $\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S_a} \vec{F}\cdot\vec{n}\, d\sigma$. We finish part (a) in part (b) by showing that $\iint_{S_a} \vec{F}\cdot\vec{n}\, d\sigma$ is independent of the choice of a.

(b) Fix $a>0$. The outward unit normal of S_a is

$$\vec{n} = \frac{\langle 2x, 2y, 2z\rangle}{2\sqrt{x^2+y^2+z^2}}.$$

Then
$$\iint_{S_a} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S_a} \frac{\langle x,y,z\rangle\cdot\langle 2x,2y,2z\rangle}{2(x^2+y^2+z^2)^2}\, d\sigma$$

$$= \iint_{S_a} \frac{1}{a^2}\, d\sigma = \frac{1}{a^2} \overbrace{\text{Area}(S_a)}^{4\pi a^2} = 4\pi$$

This is the value of any flux required in (a).

$\square$

Summary of "Differentiation Operators"

$\boxed{\text{Purple} = \text{boundary of Green}}$

① $f(x) \xrightarrow{\frac{d}{dx}} f'(x)$

scalar (real-valued)   scalar

Fundamental Thm of Calculus:

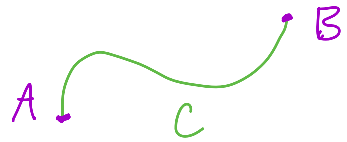
$$\int_{[a,b]} f'(x)\,dx := \int_a^b f'(x)\,dx = f(b) - f(a)$$

② $f(x,y,z) \xrightarrow{\nabla} \nabla f(x,y,z)$

Scalar → vector

Fundamental Thm of Line Integrals:

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

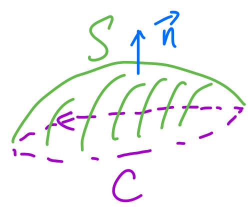


③ $\vec{F}(x,y,z) \xrightarrow{\text{curl}} \text{curl}\,\vec{F} = \nabla \times \vec{F}$

Vector → Vector

Stokes' Thm (3D)

$$\iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$



Green's Thm (Circulation) (xy-plane)

$$\iint_R \text{curl}\,\vec{F} \cdot \vec{k}\, dA = \oint_C \vec{F} \cdot d\vec{r}$$

④ $\vec{F} \xrightarrow{\text{div}} \text{div}\,\vec{F} = \nabla \cdot \vec{F}$

vector → Scalar

Divergence Thm (3D)

$$\iiint_E \text{div}\,\vec{F}\, dV = \iint_S \vec{F} \cdot \vec{n}\, d\sigma$$

Green's Thm (Flux) (2D)

$$\iint_R \text{div}\,\vec{F}\, dA = \oint_C \vec{F} \cdot \vec{n}\, ds$$