STA2001 Assignment 3

Assignment 3
Deadline: 5:00 pm, Feb 24th

1. (2.1-3) For each of the following determine the constant c so that f(x) satisfies the conditions of being a pmf for a random variable X, and then depict each pmf as a line graph:

   (a) $f(x) = x/c, \quad x = 1, 2, 3, 4.$

   (b) $f(x) = c(1/4)^x, \quad x = 1, 2, 3, \ldots.$

   (c) $f(x) = \frac{c}{(x+1)(x+2)}, \quad x = 0, 1, 2, 3, \ldots$

   Hint: In part(c). write $f(x) = 1/(x+1) - 1/(x+2)$

   Solution:

   Using the condition that $\sum_{x \in S} f(s) = 1$, it is easy to determine the value of c.

   (a) c = 10

   (b) c = 3

   (c) c = 1

2. (2.1-12) let X be the number of accidents per week in a factory. Let the pmf of X be

$$f(x) = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}, \quad x = 0, 1, 2, \ldots$$

   Find the conditional probability of $X \geqslant 4$, given that $X \geqslant 1$.

   Solution:

$$\begin{aligned} P(X \geq 4 \mid X \geq 1) &= \frac{P(X \geq 4)}{P(X \geq 1)} = \frac{1 - P(X \leq 3)}{1 - P(X = 0)} \\ &= \frac{1 - [1 - 1/2 + 1/2 - 1/3 + 1/3 - 1/4 + 1/4 - 1/5]}{1 - [1 - 1/2]} = \frac{2}{5} \end{aligned}$$

3. (2.2-1) Find E(X) for each of the distributions given in Q1 (Exercise 2.1-3)

   Solution:

   (a) $E(X) = \sum_{x=1}^{4} xf(x) = \sum_{x=1}^{4} x * \frac{x}{10} = \frac{1}{10} * 1 + \frac{2}{10} * 2 + \frac{3}{10} * 3 + \frac{4}{10} * 4 = 3$

   (c) $E(X) = \sum_{x=1}^{\infty} xf(x) = \sum_{1}^{\infty} 3x * \frac{1}{4}x = 3 * 1 * \frac{1}{4} + 3 * 2 * \left(\frac{1}{4}\right)^2 + \cdots = \frac{4}{3}$
   Let $Sn = \sum_{x=1}^{\infty} 3x \left(\frac{1}{4}\right)^x = 3\left[(1/4) + 2(1/4)^2 + \cdots + n(1/4)^n\right]$
   $\frac{1}{4}Sn = 3\left[\left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{4}\right)^3 + \cdots + (n-1)\left(\frac{1}{4}\right)^n + n\left(\frac{1}{4}\right)^{n+1}\right]$

$\frac{3}{4}Sn = 3\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^{2}+\cdots+\left(\frac{1}{4}\right)^{n}-n\left(\frac{1}{4}\right)^{n+1}\right] = 3*\left[\frac{1}{4}*\frac{1-(1/4)^{n}}{1-1/4}\right)-n\left(\frac{1}{4}\right)^{n+1}\right]$
$Sn = \frac{4}{3}-\frac{n+4/3}{4^{n}}$
$E(x) = \sum_{x=1}^{\infty} Sn = \frac{4}{3}$

(f) $E(X) = \sum_{x=0}^{\infty} xf(x) = \sum_{x=0}^{\infty} x*\frac{1}{(x+1)(x+2)} = \sum_{x=0}^{\infty} \frac{x}{x+1}-\frac{x}{x+2} = 0+\frac{1}{2}-\frac{1}{3}+\frac{2}{3}-\frac{2}{4}+\cdots = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots \geq \frac{1}{2}+\frac{1}{2}+\cdots = \infty$
Therefore E(x) does not exist.

4. (2.2-5) In example of lecture 5 (Sec 2.2):"An enterprising man proposes a game: let the player throw a die$\cdots$". Let the reward becomes $Z = u(X) = X^{3}$.

(a) Find the pmf of Z, say h(z).

(b) Find E(Z).

(c) How much, on average, can the enterprising man expect to win on each play from players if he charges 10 dollars per play?

Solution:

(a) we know that $S_X = \{x : x = 1,2,3\}$, so $S_Z = \{z : z = u(x), x \in S_X\} = \{1,8,27\}$
$h(z) = P(Z=z) = P(u(X)=z) = P\left(X^{3}=z\right) = P\left(X = z^{\frac{1}{3}}\right) = \frac{4-z^{\frac{1}{3}}}{6}, z \in S_Z$

(b) $E(Z) = \sum_{z\in S_Z} zh(z) = 1\cdot h(1)+8\cdot h(8)+27\cdot h(27) = \frac{23}{3}$

(c) $10-E(Z) = \frac{7}{3}$ dollars.

5. (2.2-6) Let the pmf of X be defined by $f(x) = 6/\left(\pi^{2}x^{2}\right), x = 1,2,3,\ldots$ Show that $E(X) = +\infty$ and thus, does not exist.

Solution:

Note that $\sum_{x=1}^{\infty}\frac{6}{\pi^{2}x^{2}} = \frac{6}{\pi^{2}}\sum_{x=1}^{\infty}\frac{1}{x^{2}} = \frac{6}{\pi^{2}}\frac{\pi^{2}}{6} = 1$, so this is a pmf.
$E(X) = \sum_{x=1}^{\infty} x\frac{6}{\pi^{2}x^{2}} = \frac{6}{\pi^{2}}\sum_{x=1}^{\infty}\frac{1}{x} = +\infty$ and it is well known that the sum of this harmonic series is not finite.

6. (2.2-8) Let X be a random variable with support $\{1,2,3,5,15,25,50\}$, each point of which has the same probability $1/7$. Argue that c $=5$ is the value that minimizes $h(c) = E(|X-c|)$. Compare c with the value of b that minimizes $g(b) = E\left[(X-b)^{2}\right]$.

Solution:

$E(|X-c|) = \frac{1}{7}\sum_{x\in S}|x-c|$, where $S = \{1,2,3,5,15,25,30\}$.

When c $=5$, $E(|X-5|) = \frac{1}{7}[(5-1)+(5-2)+(5-3)+(5-5)+(15-5)+(25-5)+(50-5)]$

If c is either increased or decreased by 1, this expectation is increased by $1/7$.

Thus c $=5$, the median, minimizes this expectation while $b = E(X) = \mu$, the mean, minimizes $E\left[(X-b)^{2}\right]$

7. (2.3-2) For each of the following distributions, find

$$\mu = E(X),\ E[X(X-1)],\ \text{and } \sigma^2 = E[X(X-1)] + E(X) - \mu^2$$

(a) $f(x) = \frac{3!}{x!(3-x)!}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x}, x = 0, 1, 2, 3.$

(b) $f(x) = \frac{4!}{x!(4-x)!}\left(\frac{1}{2}\right)^4,\ x = 0, 1, 2, 3, 4.$

Solution:

(a)

$$\begin{aligned}
\mu &= E(X) \\
&= \sum_{x=1}^{3} x\frac{3!}{x!(3-x)!}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^{3-x} \\
&= 3\left(\frac{1}{4}\right)\sum_{k=0}^{2}\frac{2!}{k!(2-k)!}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{2-k} \\
&= 3\left(\frac{1}{4}\right)\left(\frac{1}{4}+\frac{3}{4}\right)^2 = \frac{3}{4} \\
E[X(X-1)] &= \sum_{x=2}^{3} x(x-1)\frac{3!}{x!(3-x)!}\left(\frac{1}{3}\right)^x\left(\frac{3}{4}\right)^{3-x} \\
&= 2(3)(1/4)\hat{}23/4 + 6(1/4)\hat{}3 = 6/16 \\
\sigma^2 &= E[X(X-1)] + E(X) - \mu^2 = \frac{9}{16}
\end{aligned}$$

Similarly, for (b) $\mu = 2, E[X(X-1)] = 3, \sigma^2 = 1$

8. (2.3-4) Let μ and σ^2 denote the mean and variance of the random variable X. Determine $E[(X-\mu)/\sigma]$ and $E\{[(X-\mu)/\sigma]^2\}$.

Solution:

$$\begin{aligned}
E[(X-\mu)/\sigma] &= (1/\sigma)[E(X)-\mu] = (1/\sigma)(\mu-\mu) = 0; \\
E[(X-\mu)/\sigma]^2 &= 1/(\sigma)^2 E\left[(X-\mu)^2\right] = \left(1\sigma^2\right) = \left(1/(\sigma)^2\right)(\sigma)^2 = 1
\end{aligned}$$

9. (2.3-6) Place eight chips in a bowl: Three have the number 1 on them, two have the number 2 , and three have the number 3 . Say each chip has a probability of 1/8 of being drawn at random, let the random variable X equal the number on the chip that is selected, so that the space of X is $S = \{1, 2, 3\}$.Make reasonable probability assignments to each of these three outcomes, and compute the mean μ and the variance σ^2 of this probability distribution.

Solution:

$$\begin{aligned}
&f(1)=\frac{3}{8},\\
&f(2)=\frac{2}{8},\\
&f(3)=\frac{3}{8},\\
&\mu=1*\frac{3}{8}+2*\frac{2}{8}+3*\frac{3}{8}=2\\
&\sigma^2=1^2*\frac{3}{8}+2^2*\frac{2}{8}+3^2*\frac{3}{8}-2^2=\frac{3}{4}
\end{aligned}$$

10. Let X follow a discrete uniform distribution on $\{a,\ldots,b\}$, where a and b are integers with $a\le b$. The pmf of X is

$$\mathrm{P}(X=x)=p(x)=\begin{cases}\frac{1}{b-a+1}, & \text{for } x\in\{a,\ldots,b\}\\ 0, & \text{otherwise}\end{cases}$$

(a) Find the mean $E(X)$ and variance $Var(X)$,

(b) Find moment generating function $M_X(t)$,

(c) Use moment generating function to calculate the mean $E(X)$,

(d) (optional) Use moment generating function to calculate the variance $\mathrm{Var}(X)$.

Solution:

(a) The expected value of X is $\mathrm{E}(X)=\sum_{x=a}^{b}\frac{x}{b-a+1}=\frac{a+(a+1)+\cdots+b}{b-a+1}=\frac{1}{b-a+1}\cdot\frac{(a+b)(b-a+1)}{2}=\frac{a+b}{2}$.

To find the variance of $X,\mathrm{Var}(X)$, we can make use of a newly defined r.v. $Y=X-a+1$. Write $n=b-a+1$. Then Y is discrete uniform on $\{1,\ldots,n\}$. Also, $\mathrm{Var}(X)=\mathrm{Var}(Y)$. Hence, we need only find $\mathrm{Var}(Y)$.

$$\mathrm{E}(Y)=\sum_{x=1}^{n}\frac{y}{n}=\frac{1}{n}\cdot(1+2+\cdots+n)=\frac{1}{n}\cdot\frac{n(n+1)}{2}=\frac{n+1}{2}$$

$$\mathrm{E}\left(Y^2\right)=\sum_{x=1}^{n}\frac{y^2}{n}=\frac{1}{n}\cdot\left(1^2+2^2+\cdots+n^2\right)=\frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6}=\frac{(n+1)(2n+1)}{6}$$

Therefore.

$$\begin{aligned}
\mathrm{Var}(X)=\mathrm{Var}(Y)=\mathrm{E}\left(Y^2\right)-[\mathrm{E}(Y)]^2&=\frac{(n+1)(2n+1)}{6}-\left(\frac{n+1}{2}\right)^2\\
&=\frac{2n^2+3n+1}{6}-\frac{n^2+2n+1}{4}\\
&=\frac{4n^2+6n+2-3n^2-6n-3}{12}\\
&=\frac{n^2-1}{12}\\
&=\frac{(b-a+1)^2-1}{12}.
\end{aligned}$$

(b) The mgf of X is

$$M_X(t) = \mathrm{E}\left(e^{tX}\right) = \sum_{x=a}^{b} \frac{e^{tx}}{b-a+1} = \frac{e^{ta}}{b-a+1}\left[1 + e^t + e^{2t} + \cdots + e^{(b-a)t}\right]$$

$$= \begin{cases} \frac{e^{at}\left(1-e^{(b-a+1)t}\right)}{(b-a+1)(1-e^t)}, & \text{for } t \neq 0 \\ 1, & \text{for } t = 0 \end{cases}$$

$$= \begin{cases} \frac{e^{at}-e^{(b+1)t}}{(b-a+1)(1-e^t)}, & \text{for } t \neq 0 \\ 1, & \text{for } t = 0 \end{cases}$$

(c) The derivative of $M_X(t)$ is

$$M_X'(t) = \frac{e^{at}\left(be^{t(-a+b+1)} + e^{t(-a+b+1)} - be^{t(-a+b+1)+t} + ae^t - a - e^t\right)}{\left(e^t-1\right)^2(a-b-1)}$$

$$M_X''(t) = \Big(a^2e^{at} - 2a^2e^{at+t} + a^2e^{at+2t} + 2ae^{at+t} - 2ae^{at+2t} + e^{at+t} + e^{at+2t} - b^2e^{bt+t}$$

$$+ 2b^2e^{bt+2t} - b^2e^{(b+2)t+t} - 2be^{bt+t} - e^{bt+t} + 2be^{bt+2t} - e^{bt+2t}\Big) \Big/ \left(\left(e^t-1\right)^3(a-b-1)\right)$$

Alternative Answer: (provided by Bingqian Wang et al.)
We can exchange the order of addition and derivative

$$M_X'(0) = \lim_{t\to 0} M_X'(t) = \lim_{t\to 0}\left(\frac{e^{ta}}{b-a+1}\left[1 + e^t + e^{2t} + \cdots + e^{(b-a)t}\right]\right)' \quad (1)$$

$$= \lim_{t\to 0}\left(\frac{1}{b-a+1}\left[ae^{at} + (a+1)e^{(a+1t} + \cdots + be^{bt}\right]\right) \quad (2)$$

$$= \frac{1}{b-a+1}[a + (a+1) + \cdots + b] \quad (3)$$

$$= \frac{a+b}{2} \quad (4)$$

$$M_X''(0) = \lim_{t\to 0} M_X''(t) = \lim_{t\to 0}\left(\frac{e^{ta}}{b-a+1}\left[1 + e^t + e^{2t} + \cdots + e^{(b-a)t}\right]\right)'' \quad (5)$$

$$= \lim_{t\to 0}\left(\frac{1}{b-a+1}\left[a^2e^{at} + (a+1)^2e^{(a+1t} + \cdots + b^2e^{bt}\right]\right) \quad (6)$$

$$= \frac{1}{b-a+1}[a^2 + (a+1)^2 + \cdots + b^2] \quad (7)$$

$$= \frac{2a^2 + 2ab - a + 2b^2 + b}{6} \quad (8)$$

Thus $\mathrm{M}_X''(0) - M_X'(0)^2 = \frac{(b-a+1)^2-1}{12}$