MAT1002 Lecture 25, Thursday, Apr/20/2023

Outline

- Surfaces in $\mathbb{R}^3$ and areas (16.5)
  - ↳ Tangent planes
  - ↳ Surface areas
- Surface integrals (16.6)
  - ↳ Scalar functions: mass of a surface
  - ↳ Vector field: Flux

Tangent Planes to Parametric Surfaces

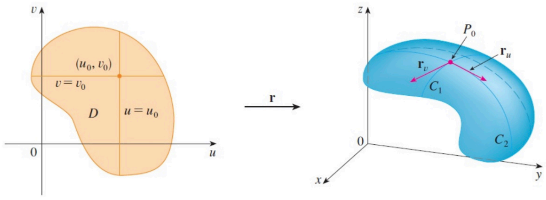
More precisely,

Let S be a parametric surface given by

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle,$$

~~where x, y and z have continuous partial derivatives.~~ Let $P_0 = \mathbf{r}(u_0, v_0)$ be a point in S. If we hold u constant by setting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a parametrization (with parameter v) of a curve C_1 lying on S, as shown below.

$\overrightarrow{OP_0} = \vec{r}(u_0,v_0)$



A tangent vector to C_1 at P_0 is given by

$$\mathbf{r}_v := \left\langle \frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0)\right\rangle.$$

Similarly, by holding v constant, $\mathbf{r}(u, v_0)$ gives a curve C_2 lying on S, and its tangent vector at P_0 is

$$\mathbf{r}_u := \left\langle \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0)\right\rangle.$$

Def: • A parametric surface S is called smooth if $\vec{r}_u$ and $\vec{r}_v$ are cts, and $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ for every point in the parameter domain. The tangent plane to a smooth parametric surface at a point is the plane through the point with normal vector $\vec{r}_u \times \vec{r}_v$.

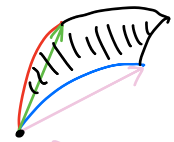
Surface Areas: Parametric Forms

Suppose $\vec{r}$ is one-to-one in the interior of the parameter domain D.

Let S be a smooth parametric surface. The surface area of S is equal to the sum of the areas of many small subregions, as shown below. The area $\Delta\sigma$ of each subregion is approximately the area of a parallelogram, which is

$$|\Delta u \mathbf{r}_u \times \Delta v \mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v.$$

$\vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0, v_0) \approx \vec{r}_v \Delta v$

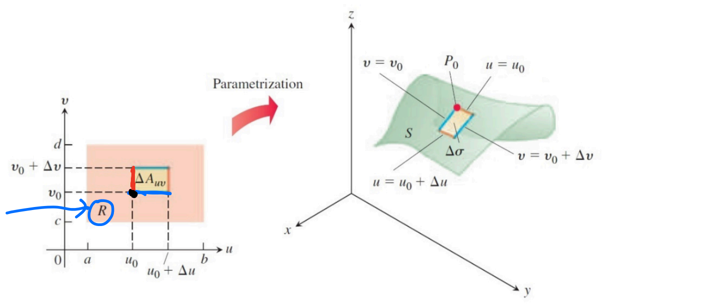


$\vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0, v_0)$

$\approx \frac{\partial \vec{r}}{\partial u}(u_0, v_0)\Delta u$

$= \vec{r}_u \Delta u$

A "small" piece



Parameter domain; may not be a rectangle in general

By adding terms of the form $|\mathbf{r}_u \times \mathbf{r}_v|\Delta u \Delta v$ and taking limit as $\|P\| \to 0$, we define the surface area of S to be $\iint_S d\sigma$, where

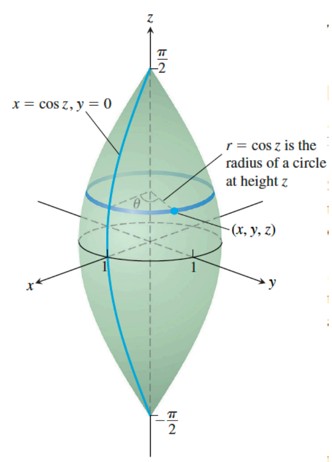
Area of a small region of S

$$\iint_S d\sigma := \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv,$$

and R is the parameter domain.

(16.5.6)

e.g. Parametrize the following surface "American football", and find its area.



Ans: $2\pi[\sqrt{2} + \ln(1+\sqrt{2})]$.

Explicit form : $z=f(x,y)$

For a smooth surface S given by $z=f(x,y)$, $(x,y)\in R$, we have

$$x=x,\quad y=y,\quad z=f(x,y),\quad (x,y)\in R.$$

One can show that surface area of S is

*: One can easily check that

$$A=\iint_R \sqrt{f_x^2+f_y^2+1}\,dx\,dy\,,\qquad \vec{r}_x\times\vec{r}_y=\langle -f_x,-f_y,1\rangle$$

where R is the domain of f (i.e., projection of S onto xy-plane).

Example

Let S be the surface given by $z=\sqrt{x^2+y^2}$, $z\le 2$.

(a) Find the tangent plane to S at $(\sqrt{2}/2,\sqrt{2}/2,1)$.

(b) Find the surface area of S.

Ans :

(a) $-\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y+z=0$

(b) $4\sqrt{2}\pi$.

Sol (a) $\vec{r}_x=\langle 1,0,\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\cdot 2x\rangle$ $\qquad \vec{r}_y=\langle 0,1,\frac{y}{\sqrt{x^2+y^2}}\rangle$

$$=\langle 1,0,\frac{x}{\sqrt{x^2+y^2}}\rangle$$

At P_0, $\vec{r}_x=\langle 1,0,\frac{\sqrt{2}}{2}\rangle$, $\vec{r}_y=\langle 0,1,\frac{\sqrt{2}}{2}\rangle$

$$\vec{r}_x\times\vec{r}_y=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ 1&0&\sqrt{2}/2\\ 0&1&\sqrt{2}/2\end{vmatrix}=\langle -\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},1\rangle$$

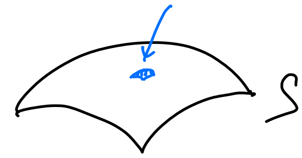
$(0,0,0)\in S \Rightarrow$ Ans.

(b) $$A=\iint_R |\vec{r}_x\times\vec{r}_y|\,dx\,dy=\iint_R \sqrt{2}\,dx\,dy=\sqrt{2}\,A(R)=\sqrt{2}\,\pi(2)^2$$

$$=4\sqrt{2}\pi.$$

Surface Integrals

Mass $\approx g(x_{ij}^*, y_{ij}^*, z_{ij}^*)\Delta\sigma_{ij}$



Let S be a smooth surface. Suppose that the planar density of of S at (x, y, z) is given by $g(x, y, z)$, say kg/m^2. Then the mass of S is given by the surface integral of g,

$$\iint_S g(x, y, z)d\sigma,$$

where $\iint_S g(x,y,z)\,d\sigma$ is the limit of $\sum_k g(x_k^*, y_k^*, z_k^*)\Delta\sigma_k$ as the areas of the small regions $(\Delta\sigma_k)$ goes to 0.

If S is in parametric form given by $(x, y, z) = \mathbf{r}(u, v)$, then

$$\iint_S g(x, y, z)d\sigma = \iint_R g(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv,$$

where R is the parameter domain.

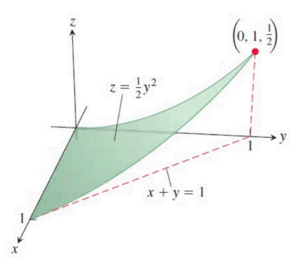
If S is in explicit form $z = f(x, y)$, where f is defined on R, then

$$\iint_S g(x, y, z)d\sigma = \iint_R g(x, y, f(x, y))\sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy.$$

Example

Evaluate $\iint_S \sqrt{x(1 + 2z)}\, d\sigma$ on the portion of the surface $z = y^2/2$ over the triangular region R: $x \geq 0$, $y \geq 0$, $x + y \leq 1$ in the xy-plane.

This is example 16.6.4.



Sol:

$$I = \iint_R \sqrt{x(1+y^2)}\,\sqrt{0+y^2+1}\, dx\, dy$$

$$= \iint_R \sqrt{x}\,(1+y^2)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-x} \sqrt{x}\,(1+y^2)\, dy\, dx$$

$$= \ldots \text{ (see the book).}$$

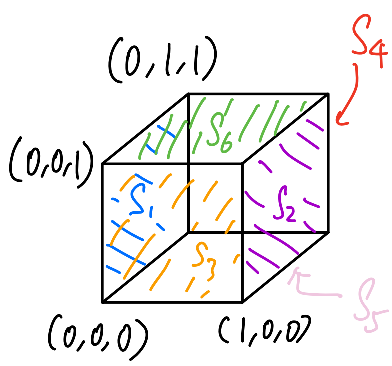
Piecewise-Smooth Surfaces

Suppose $S = S_1 \cup S_2 \cup \dots \cup S_n$, where

- each S_i is smooth, and ;
- $S_i \cap S_j$ is ϕ, a point, or a curve.

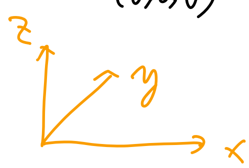
Then $\iint_S f(x,y,z)\,d\sigma = \sum_{i=1}^{n} \iint_{S_i} f(x,y,z)\,d\sigma$.

e.g. (16.6.2) Let $f(x,y,z) = xyz$ and let S be the cube surface on the right. Find $I := \iint_S f(x,y,z)\,d\sigma$.



Sol. • There are six faces, corresponding to

$x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.
S_1 $\quad S_2$ $\quad S_3$ $\quad S_4$ $\quad S_5$ $\quad S_6$



• $\iint_{S_1} f(x,y,z)\,d\sigma = \iint_{S_3} f(x,y,z)\,d\sigma = \iint_{S_5} f(x,y,z)\,d\sigma = 0$ since $f(x,y,z) \equiv 0$ on these faces.

S_6 : $z=1$, $D = [0,1] \times [0,1]$.

• For S_6, by explicit form, we have

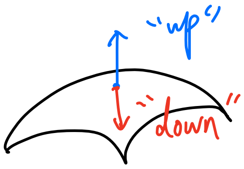
$$\iint_{S_6} f(x,y,z)\,d\sigma = \int_0^1 \int_0^1 xy\sqrt{0^2+0^2+1}\,dx\,dy = \left(\int_0^1 x\,dx\right)\left(\int_0^1 y\,dy\right)$$
$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

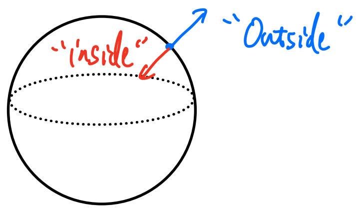
• By symmetry, $\iint_{S_2} = \iint_{S_4} = \frac{1}{4}$. Hence $I = 0 \cdot 3 + \frac{1}{4} \cdot 3 = \frac{3}{4}$.

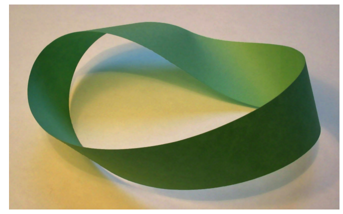
Orientable Surfaces

Roughly speaking, these are surfaces with two sides.

e.g.

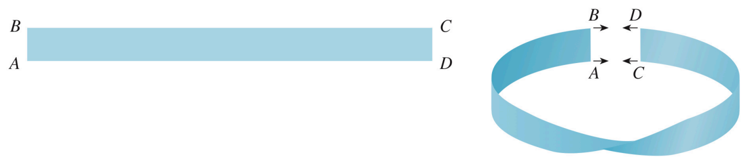






Not all surfaces are orientable. e.g., the Möbius strip.

In this course, we will only talk about orientable surfaces.



Flux : 3D

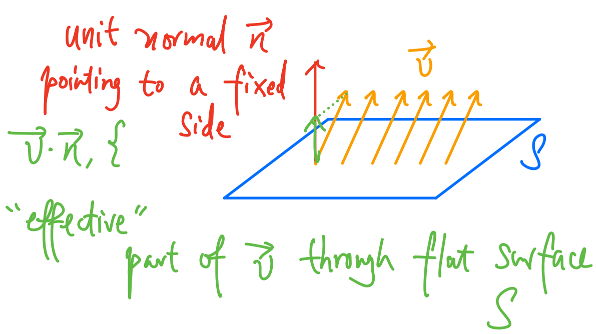
Suppose some flowing fluid in the 3D-space has constant velocity $\vec{v}$. (m/sec)

How much fluid (in terms of volume) flows through a surface S per unit time (with respect to some given direction, say "up")?

It is

$$(\vec{v}\cdot\vec{n})\ \text{Area}(S)$$

$$\text{m/sec} \cdot \text{m}^2 = \text{m}^3/\text{sec}$$



This is the concept of 3D-flux. Adding all the microscopic flux in a Riemann sum motivates the following definition.

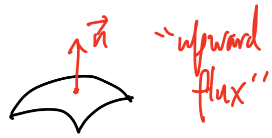
Definition

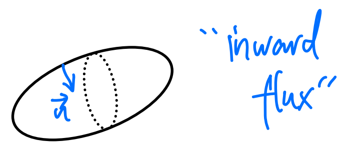
Let $\mathbf{F}$ be a continuous vector field in $\mathbb{R}^3$, defined on an orientable surface S, with a unit normal vector $\mathbf{n}$ specifying ~~the orientation~~ a "side" of S. The surface integral of $\mathbf{F}$ over S, or the flux of $\mathbf{F}$ across S, is defined to be

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

- $\vec{n} = \vec{n}(x, y, z)$
- $\vec{n}$ specifies a side of the surface and the measuring direction of the flux.

e.g.





Flux : Computation

If S is a surface with parametrization $\mathbf{r}(u, v)$, then the unit normal vectors to S at a point are

$$\mathbf{n} = \pm \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

Here, the sign is determined by the context or question, which you need to check.

Suppose that the flux direction is given by the one with "$+$". Then

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \iint_R \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv \\ &= \iint_R \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv. \end{aligned}$$

1

If S is given by $z = f(x,y)$, $(x,y) \in R$, then by using x and y themselves as parameters, we have

$$\vec{r}_x = \langle 1, 0, f_x \rangle, \quad \vec{r}_y = \langle 0, 1, f_y \rangle \quad \text{and} \quad \vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1 \rangle,$$

and the formula ① above becomes

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_R \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) \, dx\,dy$$
$$= \iint_R (-M f_x - N f_y + P) \, dx\,dy$$

Example

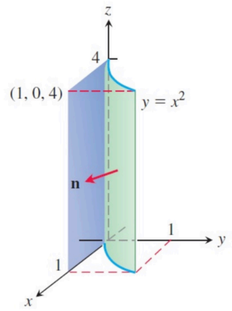
Find the flux of $\mathbf{F} := \langle yz, x, -z^2 \rangle$ across the surface

$$S := \{(x, y, z) : \; y = x^2, \, 0 \le x \le 1, \, 0 \le z \le 4\},$$

in the direction of $\mathbf{n}$ indicated in the figure.

Example 16.6.5,

Ans = 2.



e.g. Find the inward flux of $\vec{F} = \langle z, y, x \rangle$ across the sphere $x^2 + y^2 + z^2 = 1$.

Ans: $-4\pi/3$.

Remark The computation above is relatively messy by direct parametrizations, but will become one-line after studying Chap 16.8.