STA2001 Tutorial 1

1. An insurance company looks at its auto insurance customers and finds that (a) all insure at least one car, (b) 90% insure more than one car (c) 25% insure a sports car, and (d) 15% insure more than one car, including a sports car. Find the probability that a customer selected at random insures exactly one car and it is not a sports car.

   Solution:

   We define the following events:

   S: the customers insure at least one car. So $P(S) = 100\%$.

   A: the customers insure more than one car. So $P(A) = 90\%$.

   B: the customers insure a sports car. So $P(B) = 25\%$.

   C: the customers insure more than one car including a sports car. So apparently we have $C = A \cap B$, and $P(C) = 15\%$.

   Then $A' \cap B'$ is exactly the event we are interested in, the customers insure one car and it's not a sports car, so the probability of this event is given by

$$\begin{aligned} P\left(A' \cap B'\right) &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - (90\% + 25\% - 15\%) \\ &= 0 \end{aligned}$$

   where the first equality is due to De Morgan's law.

2. 1.2-4. The "eating club" is hosting a make-your-own sundae at which the following are provided:

Ice Cream Flavors	Toppins
Chocolate	caramel
Cookies 'n' cream	Hot fudge
Strawberry	Marshmallow
Vanilla	M&Ms
	Nuts
	Strawberries

(a). How many sundaes are possible using one flavor of ice cream and three different toppings?

(b). How many sundaes are possible using one flavor of ice cream and from zero to six (different) toppings?

(c). How many different combinations of flavors of three scoops of ice cream are possible if it is permissible to make all three scoops the same flavor?

Solution:

(a). According to the multiplication principle, we need to choose 1 from 4 flavors (in 4 ways), then choose 3 from 6 toppings (by the unordered without replacement rule), which give us $4 \times \binom{6}{3} = 80$ kinds of sundae.

(b). Similarly, using one flavor and from 0 to 6 different toppings, we can make $4 \times \left[\binom{6}{0} + \binom{6}{1} + \cdots + \binom{6}{6}\right] = 4 \times 2^6 = 256$ kinds of sundaes.

(c). It is an combination problem with replacement.

We can find some 1-to-1 corresponding diagrams to describe all possible combinations. Specifically, we use three $'\square'$ to describe the scoops and three $'|'$ to separate them. Define the scoops in front of the first $'|'$ have the first flavor, chocolate; and the scoops behind the first but in front of the second $'|'$ have the second flavor, cookies 'n' cream, and so on. For example, $\square\square||\square|$ represents the combination of two scoops in chocolate and one in Strawberry. Actually, the number of distinguishable permutations of the $'\square'$ and $'|'$ is equal to the number of all possible combinations of flavors.

Since the scoops are indistinguishable and the flavors are distinguishable, $n = 4$ denotes the number of flavors and $r = 3$ denotes the number of scoops.

In total, we would have $\binom{n+r-1}{r} = \binom{4-1+3}{3} = 20$ combinations of flavors of three scoops of ice cream. Note that this is known as a combination of n objects taken r at a time with repetition.

Alternatively, we may consider the number of different kinds of flavors occur in these three scoops of ice cream, that is,

$$\binom{4}{1} + 2 \times \binom{4}{2} + \binom{4}{3} = 20$$

3. 1.2-7. In a state lottery, four digits are drawn at random one at a time with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select

   (a). 6, 7, 8, 9.

   (b). 6, 7, 8, 8.

   (c). 7, 7, 8, 8.

   (d). 7, 8, 8, 8.

   Solution:

   First we calculate the number of permutations that are ordered and with replacement (i.e. total number of the four digits sequence can be drawn), which is obviously given by $10 \times 10 \times 10 \times 10 = 10^4$.

   As we want to compute the probability that we win, then 10^4 will be the denominator and the numerator will be total number of the permutation of our selected four numbers in computing it.

   (a). The numerator will be $4!$ in this case, and thus the probability is given by $4!/10^4$.

   (b). Since there are two 8s in the selected digits, we need to eliminate their orders (because they are indistinguishable, says, 6788 and 6788 are the same one if we have exchanged the position of two 8s), so the numerator is $4!/2!$ and then the probability should be $4!/\left(10^4 \times 2!\right)$.

   (c). Similar with (b), since we have two 7s and two 8s, the numerator should be $4!/(\times 2! \times 2!)$, and then the probability is $4!/(10^4 \times 2! \times 2!)$.

   (d). Similar with (b), we have three 8s, and therefore the probability is given by $4!/(10^4 \times 3!)$.

4. Suppose that an experiment is repeated n times. The number of times that an event A actually occurred throughout these n performances is called the *frequency* of A, denoted by $\mathcal{N}(A)$. The ratio $f(A) := \mathcal{N}(A)/n$ is called the relative frequency of event A in these n repetitions of the experiment.

    1. For the sample space S, show $f(S) = 1$.
    2. For two events A and B, if A and B are *mutually exclusive* (i.e., $A \cap B = \emptyset$), prove $f(A \cup B) = f(A) + f(B)$.
    3. For any two events A and B, show that

    $$f(A \cup B) = f(A) + f(B) - f(A \cap B).$$

    Solution:

    1. As S includes all the outcomes, S occurs in every trial, and hence $\mathcal{N}(S) = n$ which implies that $f(S) = 1$.
    2. As A and B are mutually exclusive, if A occurs, then B cannot, and vice versa. Therefore, $\mathcal{N}(A \cup B) = \mathcal{N}(A) + \mathcal{N}(B)$, and hence $f(A \cup B) = f(A) + f(B)$.
    3. For each trial of the experiment, one of the four events occurs: $A \cap B'$, $A' \cap B$, $A \cap B$ and $(A \cap B)'$, which are mutually exclusive and exhaustive. By 2), we have

    $$\begin{align} f(A) &= f(A \cap B') + f(A \cap B) \tag{a1} \\ f(B) &= f(A' \cap B) + f(A \cap B) \tag{a2} \\ f(A \cup B) &= f(A \cap B') + f(A' \cap B) + f(A \cap B), \tag{a3} \end{align}$$

    where the last equality can be obtained by applying 2) iteratively.