MAT1002 Lecture 15, Thursday, Mar/16/2023

Outline

- Chain rule (14.4)
  - ↳ Implicit differentiation
- Directional derivatives and gradient vectors (14.5)

Chain Rule

Q: Suppose production $P = P(x,y) = kx^{\alpha}y^{1-\alpha}$, where labour $x = x(t)$ and capital $y = y(t)$ are both functions of time t. What is $\frac{dP}{dt}$ at a given time $t = t_0$?

The general concept here is the chain rule.

Special Case 1 $\quad f(x(t), y(t))$.

Suppose

1. $z = f(x,y)$ is diff'able at (x_0, y_0), and;
2. $x = x(t)$ and $y = y(t)$ are both diff'able at t_0 (and $x_0 = x(t_0)$, $y_0 = y(t_0)$.)

- Fix $t = t_0$, and consider a change of t-value by Δt. This creates Δx, Δy, and Δz.
- By differentiability of f at (x_0, y_0),

$$\Delta z = \left(f_x(x_0,y_0) + \varepsilon_1\right)\Delta x + \left(f_y(x_0,y_0) + \varepsilon_2\right)\Delta y, \qquad (1)$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

- By differentiability of x and y at t_0,

$$\Delta x = (x'(t_0) + \varepsilon_x)\Delta t, \quad \Delta y = (y'(t_0) + \varepsilon_y)\Delta t,$$

where $\varepsilon_x \to 0$ and $\varepsilon_y \to 0$ as $t \to 0$.

- Then ① becomes

$$\Delta z = \left(f_x(x_0,y_0) + \varepsilon_1\right)\left(x'(t_0) + \varepsilon_x\right)\Delta t + \left(f_y(x_0,y_0) + \varepsilon_2\right)\left(y'(t_0) + \varepsilon_y\right)\Delta t,$$

So $$\frac{\Delta z}{\Delta t} = \left(f_x(x_0,y_0) + \varepsilon_1\right)\left(x'(t_0) + \varepsilon_x\right) + \left(f_y(x_0,y_0) + \varepsilon_2\right)\left(y'(t_0) + \varepsilon_y\right) \quad ②$$

- As $\Delta t \to 0$, $(\Delta x, \Delta y) \to (0,0)$, so $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

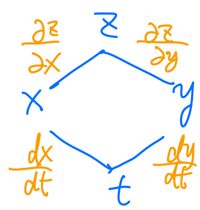
- Taking $\lim_{\Delta t \to 0}$ on both sides of ② yields

$$\left.\frac{dz}{dt}\right|_{t=t_0} = f_x(x_0,y_0)\,x'(t_0) + f_y(x_0,y_0)\,y'(t_0)$$

This chain rule is summarized in short as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

This can be remembered using the "branch diagram".



Example

Suppose that $z = x^2y + 3xy^4$, $x = \sin(2t)$ and $y = \cos t$. Find dz/dt when $t = 0$.

Ans : 6.

General Case 1 $\quad f(x_1(t), x_2(t), \dots, x_n(t))$.

Suppose $w = f(x_1, \dots, x_n)$, $x_1 = x_1(t), \dots, x_n = x_n(t)$, all differentiable. Then w is differentiable w.r.t. t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \dots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}.$$

Here x_i is the ith variable of f.

Q: How if $z = f(x,y)$, but x and y are both two-variable functions, say $x = x(s,t)$ and $y = y(s,t)$? E.g., z is altitude of mountain, by you want to know $\frac{\partial z}{\partial \theta}$, where $x = r\cos\theta$ and $y = r\sin\theta$?

Special Case 2

Suppose $z = f(x,y)$, $x = x(s,t)$, $y = y(s,t)$, all diff'able. Then z is diff'able in (s,t), and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Example

Suppose that $z = e^x \sin y$, $x = st^2$ and $y = s^2 t$. Find $\partial z/\partial s$ and $\partial z/\partial t$.

Sol: $\frac{\partial z}{\partial s} = \sin(s^2t)\, e^{st^2}\, t^2 + \cos(s^2t)\, e^{st^2} \cdot 2st$

$\frac{\partial z}{\partial t}$: exercise.

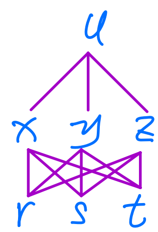
General Case

Theorem (Chain Rule)

Suppose that $u = f(x_1, x_2, \ldots, x_n)$ is a differentiable real-valued function with n variables, and suppose that for each i, $x_i = g_i(t_1, t_2, \ldots, t_m)$ is a differentiable real-valued function with m variables. Then u is differentiable with respect to $(t_1, t_2, \ldots, t_m)$, and

$$\frac{\partial u}{\partial t_j} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_j}.$$

$n=3$, $m=3$:



Implicit Differentiation

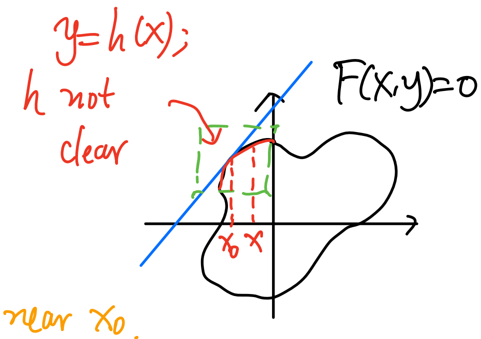
$y=h(x)$, h not clear.

A level curve of F

Suppose y is an implicit function of x given by $F(x,y)=0$ near a point (x_0,y_0), e.g., $F(x,y) = x^2+y^2-1=0$.

Then near the point, $F(x,h(x))=0$

$\forall$ points on "height" curve. Let $z = F(x,h(x)) = g(x)$.



Then $\left.\frac{dz}{dx}\right|_{(x,y)=(x_0,y_0)} = 0$. (1) On the other hand,

Height on curve

Since $z \equiv 0$ for x near x_0.

by the chain rule, if $F(x,y)$ is differentiable then

$$\frac{dz}{dx} = F_x \cdot \frac{dx}{dx} + F_y \cdot \frac{dy}{dx} = F_x + F_y \frac{dy}{dx}. \quad ②$$

By ① & ②,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \quad (\text{if } F_y \neq 0).$$

THEOREM ~~8~~ 14.4.8—A Formula for Implicit Differentiation Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \qquad (1)$$

Example

Find the slope of the tangent line to the curve $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

Sol: $F(x,y) = x^3 + y^3 - 6xy$; $F_x = 3x^2 - 6y$, $F_y = 3y^2 - 6x$

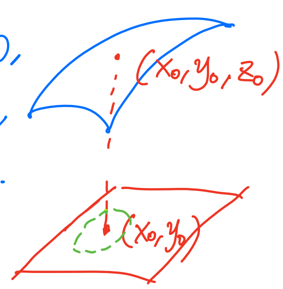
$$\left.\frac{dy}{dx}\right|_{(x,y)=(3,3)} = -\frac{F_x(3,3)}{F_y(3,3)} = -1. \qquad F_y(3,3) \neq 0$$

e.g. $x^2+y^2+z^2-1=0$

Similarly, a surface $F(x,y,z)=0$ may define z implicitly as a diff'able function of (x,y) near a point (x_0,y_0,z_0). Then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ can be found similarly: near (x_0,y_0,z_0), the surface is given by $F(x,y,h(x,y))=0$.

A level surface of F

$F(x,y,z)=0$,
$z=h(x,y)$,
h not clear.



"Temperature" Let $w = F(x,y,h(x,y)) = g(x,y)$. Then by considering w on $\underset{\text{Surface}}{z = h(x,y)}$ near (x_0,y_0), $\frac{\partial w}{\partial x} = 0 = \frac{\partial w}{\partial y}$. Also, by the chain rule,

Surface : $x = x$, $y = y$, $z = h(x,y)$

*: x & y are independent

$$\frac{\partial w}{\partial x} = F_x \cdot \frac{\partial x}{\partial x} + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial x} = F_x + F_z \frac{\partial z}{\partial x} \ (=0),$$

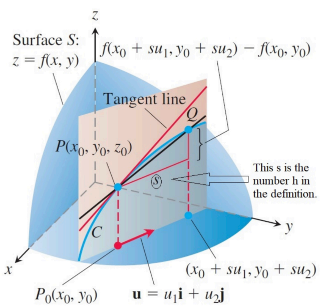
$$\frac{\partial w}{\partial y} = F_x \cdot \frac{\partial x}{\partial y} + F_y \frac{\partial y}{\partial y} + F_z \frac{\partial z}{\partial y} = F_y + F_z \frac{\partial z}{\partial y} \ (=0),$$

So

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}, \text{ whenever } F_z \neq 0.}$$

Exercise

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.



Directional Derivatives and Gradient Vectors

Q: Luigi is climbing a mountain and is now at horizontal position (x_0,y_0), how fast will the altitude increase or decrease if he move in the direction $\vec{u} = \langle u_1, u_2 \rangle$?

A: Directional derivative.

Definition

Let (x_0, y_0) be an interior point of the domain of a two-variable function f, and let $\vec{u}$ be a unit vector in $\mathbb{R}^2$. The (directional) derivative of f at (x_0, y_0) in the direction of $\vec{u}$, denoted by $D_{\vec{u}} f(x_0, y_0)$, is defined by

$$D_{\vec{u}} f(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}.$$

In vector notation (which may generalize to n-variable functions):

$$D_{\vec{u}} f(\vec{x_0}) = \lim_{h\to 0} \frac{f(\vec{x_0}+h\vec{u}) - f(\vec{x_0})}{h}.$$

Remark

Note that both partial derivatives are ~~specials~~ special cases of directional derivatives: $f_x = D_{\vec{u_1}} f$ and $f_y = D_{\vec{u_2}} f$, where $\vec{u_1} = \langle 1, 0\rangle$ and $\vec{u_2} = \langle 0, 1\rangle$.

Q: If f is differentiable, given (x_0, y_0) and $\vec{u}$ (unit), how to compute $D_{\vec{u}}(x_0, y_0)$?

Define $g(t) := f(\underbrace{x_0 + tu_1}_{x(t)}, \underbrace{y_0 + tu_2}_{y(t)})$. Then $g(0) = f(x_0, y_0)$, and

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0).$$

On the other hand, by the chain rule,

$$\begin{aligned} g'(0) &= f_x(x(0), y(0)) \cdot x'(0) + f_y(x(0), y(0)) \cdot y'(0) \\ &= f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2 = \nabla f(x_0, y_0) \cdot \vec{u}, \end{aligned}$$

where $\nabla f := \langle f_x, f_y \rangle$ is the gradient vector.

Theorem

If f is a differentiable two-variable function, then

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2.$$

In other words,

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u},$$

where $\nabla f := \langle f_x, f_y \rangle$.

~~Remark~~ Def

The vector ∇f defined above is called the gradient or the gradient vector of f.

Example

Find the directional derivative of $f(x,y) := xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\vec{v} = \langle 3, -4 \rangle$.

Solution

-1.

In general, for an n-variable function $f(x_1, \ldots, x_n)$, $\nabla f := \langle f_{x_1}, \ldots, f_{x_n} \rangle$. If f is diff'able at a point $\vec{x}_0 \in \mathbb{R}^n$, then it is still true that

$$D_{\vec{u}} f(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}.$$

e.g. Consider $f(x,y) := \begin{cases} xy^2/(x^2+y^4), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

- Let $\vec{u}$ be any unit vector.
- $$D_{\vec{u}} f(0,0) = \lim_{h \to 0} \frac{f(hu_1, hu_2) - f(0,0)}{h} = \lim_{h \to 0} \frac{h^3 u_1 u_2^2 / (h^2 u_1^2 + h^4 u_2^4)}{h}$$
$$= \lim_{h \to 0} \frac{u_1 u_2^2}{u_1^2 + h^2 u_2^4} = \begin{cases} u_2^2 / u_1, & \text{if } u_1 \neq 0; \\ 0, & \text{if } u_1 = 0. \end{cases}$$
- Hence, all directional derivatives exist for f at $(0,0)$.
- But f is not even cts at $(0,0)$ as we have seen in Lecture 14, let alone differentiable. "Existence of all directional derivatives $\not\Rightarrow$ differentiability"