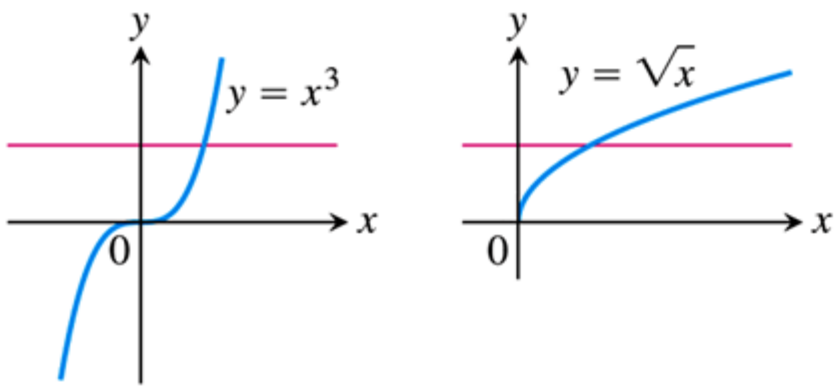
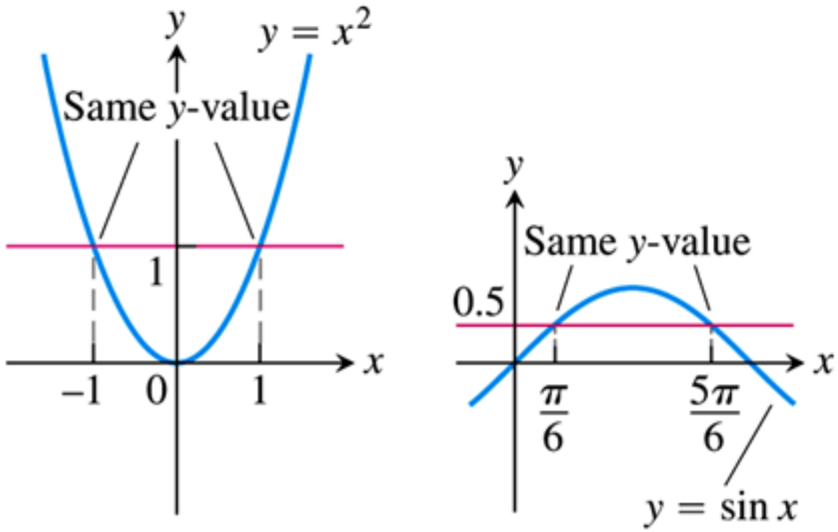
7.1

Inverse Functions and Their Derivatives

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

FIGURE 7.1 (a) $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$. (b) $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

The Horizontal Line Test for One-to-One Functions
A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Definition

Let $f : D \to Y$ be a function.

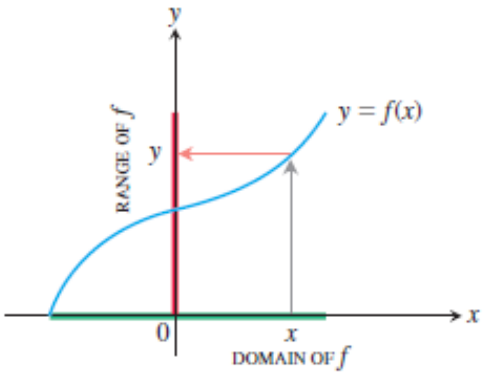
- We say that f is one-to-one (or injective) if $f(x_1) \neq f(x_2)$ for all distinct x_1 and x_2 in D (that is, $x_1 \neq x_2$).
- We say that f is onto (or surjective) if, for every $y \in Y$, there exists $x \in D$ such that $f(x) = y$.
- We say that f is bijective if it is both one-to-one and onto. A bijective function is called a bijection.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

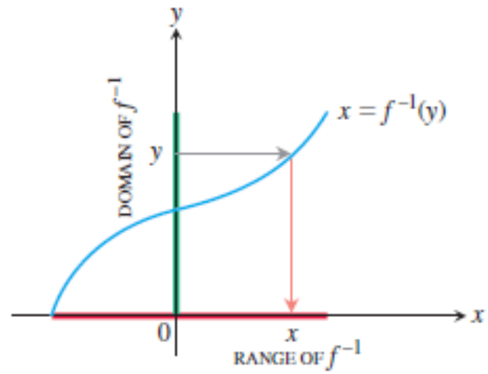
$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

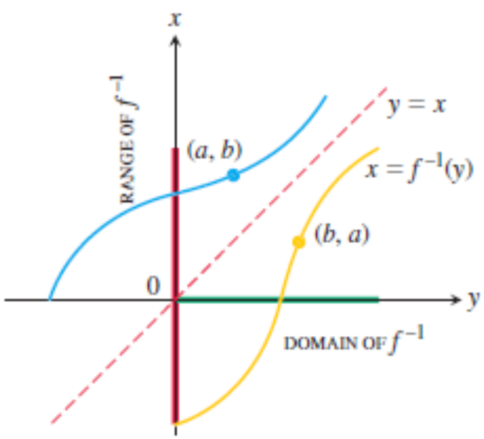
If f is continuous and f^{-1} exists, then f^{-1} is continuous.



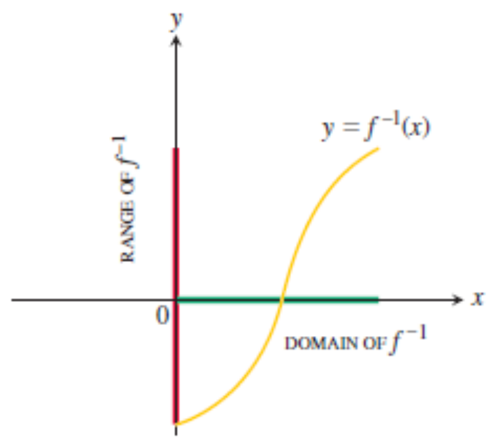
(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



(b) The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.



(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 7.2 The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

EXAMPLE 4 Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution For $x \geq 0$, the graph satisfies the horizontal line test, so the function is one-to-one and has an inverse. To find the inverse, we first solve for x in terms of y:

$$y = x^2$$
$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

We then interchange x and y, obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$ (Figure 7.4).

Notice that the function $y = x^2, x \geq 0$, with domain *restricted* to the nonnegative real numbers, *is* one-to-one (Figure 7.4) and has an inverse. On the other hand, the function $y = x^2$, with no domain restrictions, *is not* one-to-one (Figure 7.1b) and therefore has no inverse. ■

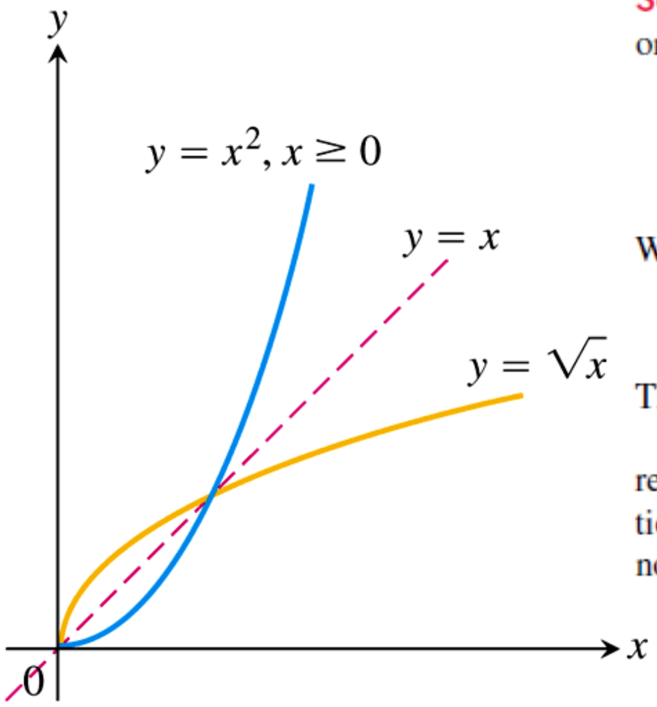
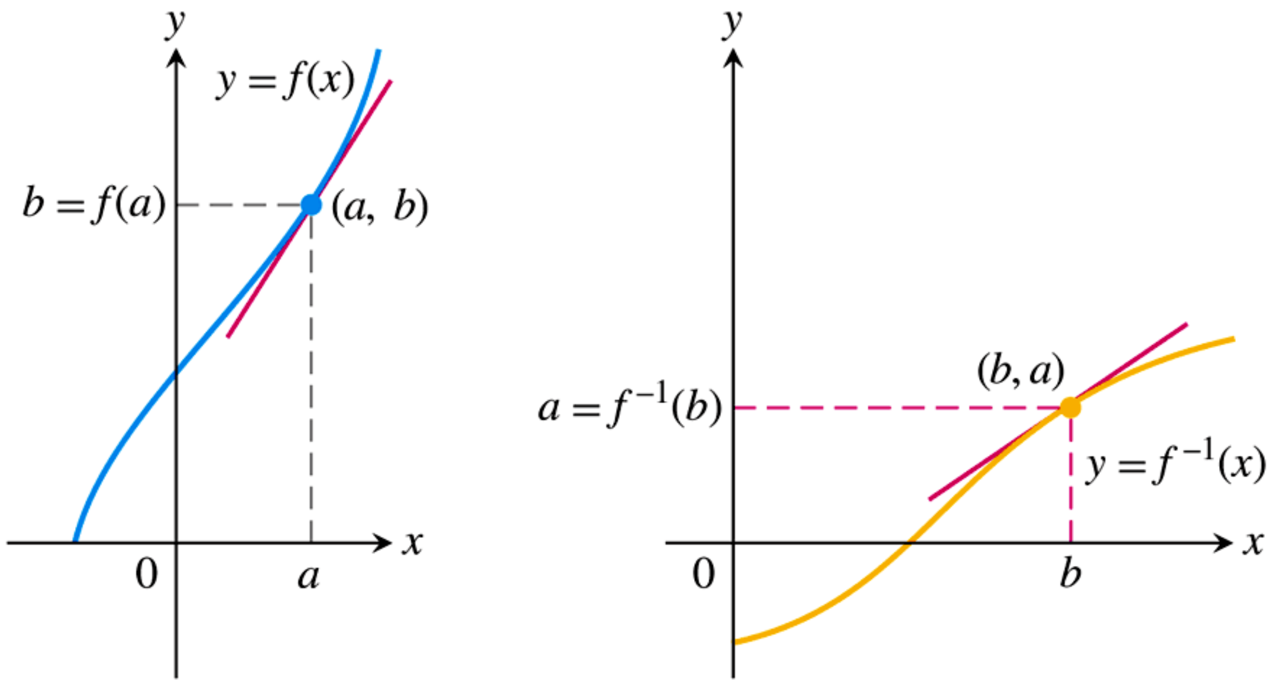


FIGURE 7.4 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 4).

Derivatives of Inverses of Differentiable Functions

Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$, the reflected line has slope $1/m$.



The slopes are reciprocal: $(f^{-1})'(b) = \dfrac{1}{f'(a)}$ or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

FIGURE 7.5 The graphs of inverse functions have reciprocal slopes at corresponding points.

THEOREM 1—The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Theorem 1 makes two assertions. The first of these has to do with the conditions under which f^{-1} is differentiable; the second assertion is a formula for the derivative of f^{-1} when it exists. While we omit the proof of the first assertion, the second one is proved in the following way:

$$f(f^{-1}(x)) = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}f(f^{-1}(x)) = 1 \qquad \text{Differentiating both sides}$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx}f^{-1}(x) = 1 \qquad \text{Chain Rule}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}. \qquad \text{Solving for the derivative}$$

EXAMPLE 5 The function $f(x) = x^2, x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

Let's verify that Theorem 1 gives the same formula for the derivative of $f^{-1}(x)$:

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{2(f^{-1}(x))} && f'(x) = 2x \text{ with } x \text{ replaced by } f^{-1}(x) \\ &= \frac{1}{2(\sqrt{x})}.\end{aligned}$$

Theorem 1 gives a derivative that agrees with the known derivative of the square root function.

Let's examine Theorem 1 at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem 1 says that the derivative of f at 2, which is $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, which is $(f^{-1})'(4)$, are reciprocals. It states that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{2x}\bigg|_{x=2} = \frac{1}{4}.$$

See Figure 7.6. ■

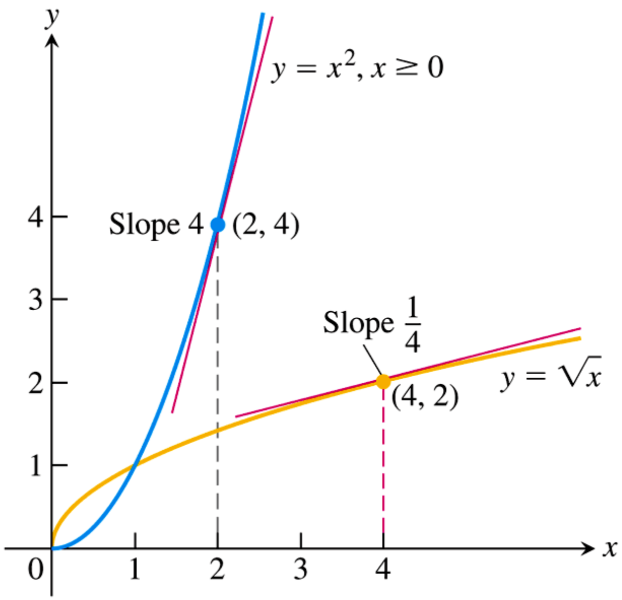


FIGURE 7.6 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 5).

EXAMPLE 6 Let $f(x) = x^3 - 2, x > 0$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution We apply Theorem 1 to obtain the value of the derivative of f^{-1} at $x = 6$:

$$\left.\frac{df}{dx}\right|_{x=2} = \left.3x^2\right|_{x=2} = 12$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{12}. \qquad \text{Eq. (1)}$$

See Figure 7.7. ■

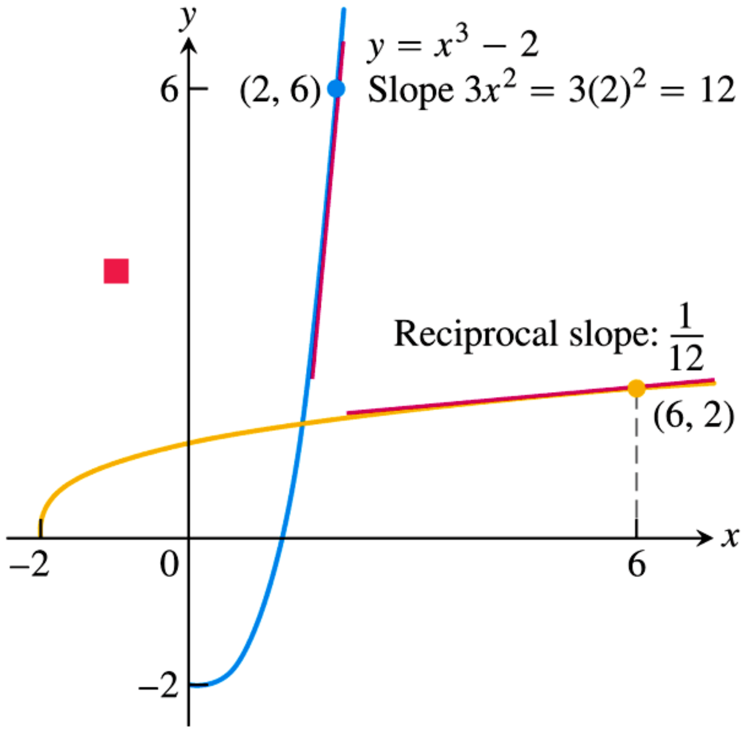


FIGURE 7.7 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 6).

7.2

Natural Logarithms

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\, dt, \qquad x > 0. \tag{1}$$

From the Fundamental Theorem of Calculus, $\ln x$ is a continuous function. Geometrically, if $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Figure 7.8). For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x to 1, and the function is not defined for $x \leq 0$. From the Zero Width Interval Rule for definite integrals, we also have

$$\ln 1 = \int_1^1 \frac{1}{t} dt = 0.$$

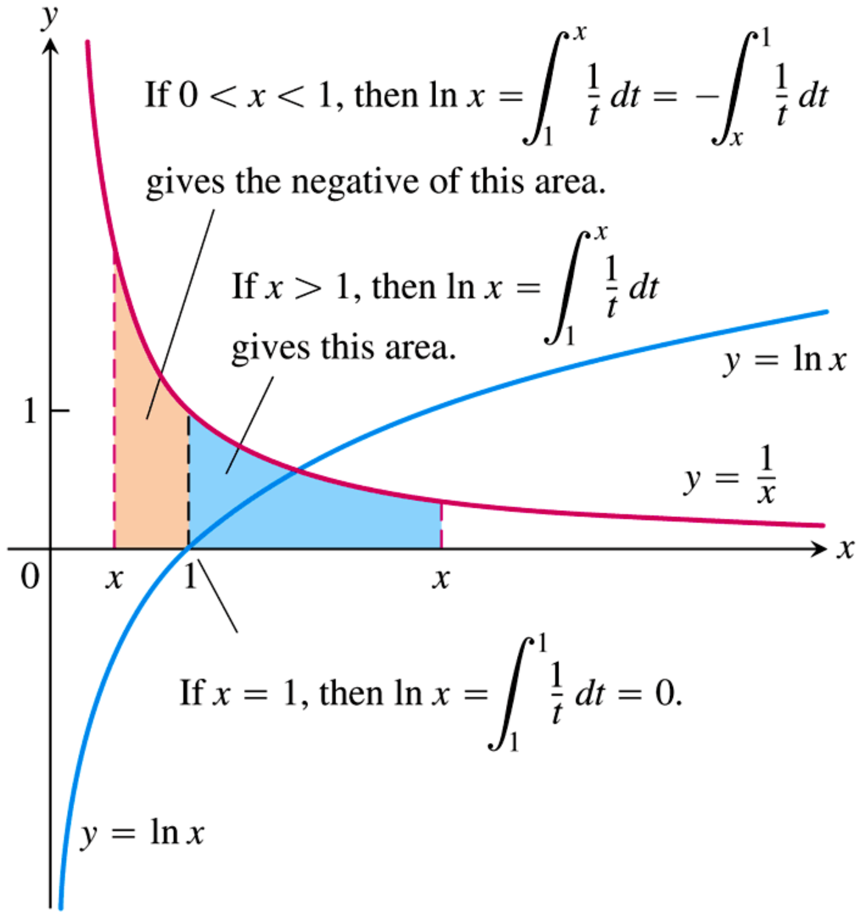


FIGURE 7.8 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the x-axis as x moves from 1 to the left.

TABLE 7.1 Typical 2-place values of $\ln x$

x	$\ln x$
0	undefined
0.05	-3.00
0.5	-0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

There is an important number between $x = 2$ and $x = 3$ whose natural logarithm equals 1. This number, which we now define, exists because $\ln x$ is a continuous function and therefore satisfies the Intermediate Value Theorem on $[2, 3]$.

DEFINITION The **number e** is that number in the domain of the natural logarithm satisfying

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

From FTC1, we have

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

Note that $y = \ln x$ is a solution to the initial value problem $dy/dx = 1/x$, $x > 0$, with $y(1) = 0$.

From the Chain Rule, we have

$$\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0. \qquad (2)}$$

EXAMPLE 1 We use Equation (2) to find derivatives.

(a) $\dfrac{d}{dx}\ln 2x = \dfrac{1}{2x}\dfrac{d}{dx}(2x) = \dfrac{1}{2x}(2) = \dfrac{1}{x}, \quad x > 0$

(b) Equation (2) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$$

(c) Equation (2) with $u = |x|$ gives an important derivative:

$$\begin{aligned}
\frac{d}{dx}\ln|x| &= \frac{d}{du}\ln u\cdot\frac{du}{dx} && u = |x|, x \neq 0\\
&= \frac{1}{u}\cdot\frac{x}{|x|} && \frac{d}{dx}(|x|) = \frac{x}{|x|}\\
&= \frac{1}{|x|}\cdot\frac{x}{|x|} && \text{Substitute for } u.\\
&= \frac{x}{x^2}\\
&= \frac{1}{x}.
\end{aligned}$$

So $1/x$ is the derivative of $\ln x$ on the domain $x > 0$, and the derivative of $\ln(-x)$ on the domain $x < 0$. ■

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad x \neq 0 \tag{4}$$

THEOREM 2—Algebraic Properties of the Natural Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$
2. *Quotient Rule*: $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ — Rule 2 with $b = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ — For r rational

Proof that $\ln bx = \ln b + \ln x$ The argument starts by observing that $\ln bx$ and $\ln x$ have the same derivative:

$$\frac{d}{dx}\ln(bx) = \frac{b}{bx} = \frac{1}{x} = \frac{d}{dx}\ln x.$$

According to Corollary 2 of the Mean Value Theorem, the functions must differ by a constant, which means that

$$\ln bx = \ln x + C$$

for some constant C.

Since this last equation holds for all positive values of x, it must hold for $x = 1$. Hence,

$$\begin{aligned} \ln(b \cdot 1) &= \ln 1 + C \\ \ln b &= 0 + C \qquad \ln 1 = 0 \\ C &= \ln b. \end{aligned}$$

By substituting we conclude that

$$\ln bx = \ln b + \ln x. \qquad \blacksquare$$

Proof that $\ln x^r = r \ln x$

$$\begin{aligned}\frac{d}{dx}\ln x^r &= \frac{1}{x^r}\frac{d}{dx}(x^r)\\ &= \frac{1}{x^r}rx^{r-1}\\ &= r\cdot\frac{1}{x} = \frac{d}{dx}(r\ln x).\end{aligned}$$

Since $\ln x^r$ and $r \ln x$ have the same derivative,

$$\ln x^r = r\ln x + C$$

for some constant C. Taking x to be 1 identifies C as zero

The Graph and Range of ln x

The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so $\ln x$ is an increasing function of x. The second derivative, $-1/x^2$, is negative, so the graph of $\ln x$ is concave down. (See Figure 7.9a.)

We can estimate the value of ln 2 by considering the area under the graph of $y = 1/x$ and above the interval $[1, 2]$. In Figure 7.9(b) a rectangle of height $1/2$ over the interval $[1, 2]$ fits under the graph. Therefore the area under the graph, which is ln 2, is greater than the area, $1/2$, of the rectangle. So $\ln 2 > 1/2$. Knowing this we have

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}.$$

This result shows that $\ln(2^n) \to \infty$ as $n \to \infty$. Since $\ln x$ is an increasing function, we get that

$$\lim_{x\to\infty} \ln x = \infty.$$

We also have

$$\lim_{x\to 0^+} \ln x = \lim_{t\to\infty} \ln t^{-1} = \lim_{t\to\infty} (-\ln t) = -\infty. \qquad x = 1/t = t^{-1}$$

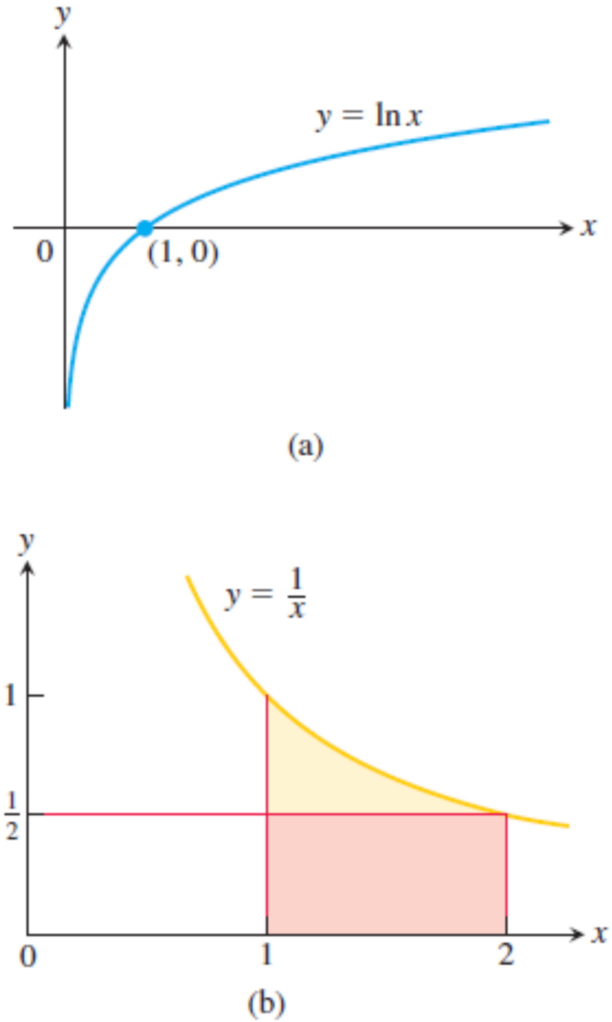


FIGURE 7.9 (a) The graph of the natural logarithm. (b) The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \le x \le 2$.

From Example 1, we have the following.

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}du = \ln|u| + C. \tag{3}$$

Equation (3) applies anywhere on the domain of $1/u$, the points where $u \neq 0$. It says that integrals of a certain *form* lead to logarithms. If $u = f(x)$, then $du = f'(x)\,dx$ and

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

whenever $f(x)$ is a differentiable function that is never zero.

EXAMPLE 3 Here we recognize an integral of the form $\int \frac{du}{u}$.

$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3 + 2\sin\theta}\,d\theta = \int_1^5 \frac{2}{u}\,du \qquad \begin{aligned} &u = 3 + 2\sin\theta, \quad du = 2\cos\theta\,d\theta, \\ &u(-\pi/2) = 1, \quad u(\pi/2) = 5 \end{aligned}$$

$$= 2\ln|u| \Big]_1^5$$

$$= 2\ln|5| - 2\ln|1| = 2\ln 5$$

Note that $u = 3 + 2\sin\theta$ is always positive on $[-\pi/2, \pi/2]$, so Equation (3) applies. ■

The Integrals of tan *x*, cot *x*, sec *x*, and csc *x*

Equation (3) tells us how to integrate these trigonometric functions.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u} \qquad \begin{aligned} &u = \cos x > 0 \text{ on } (-\pi/2, \pi/2), \\ &du = -\sin x\,dx \end{aligned}$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C. \qquad \text{Reciprocal Rule}$$

For the cotangent,

$$\int \cot x\, dx = \int \frac{\cos x\, dx}{\sin x} = \int \frac{du}{u} \qquad \begin{aligned} u &= \sin x, \\ du &= \cos x\, dx \end{aligned}$$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

To integrate $\sec x$, we multiply and divide by $(\sec x + \tan x)$ as an algebraic form of 1.

$$\int \sec x\, dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C \qquad \begin{aligned} u &= \sec x + \tan x, \\ du &= (\sec x \tan x + \sec^2 x) dx \end{aligned}$$

For $\csc x$, we multiply and divide by $(\csc x + \cot x)$ as an algebraic form of 1.

$$\int \csc x\, dx = \int \csc x \frac{(\csc x + \cot x)}{(\csc x + \cot x)} dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx$$

$$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C \qquad \begin{aligned} u &= \csc x + \cot x, \\ du &= (-\csc x \cot x - \csc^2 x)\, dx \end{aligned}$$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u\,du = \ln|\sec u| + C \qquad \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 5 Find dy/dx if

$$y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\begin{aligned}
\ln y &= \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1} \\
&= \ln((x^2 + 1)(x + 3)^{1/2}) - \ln(x - 1) && \text{Quotient Rule} \\
&= \ln(x^2 + 1) + \ln(x + 3)^{1/2} - \ln(x - 1) && \text{Product Rule} \\
&= \ln(x^2 + 1) + \frac{1}{2}\ln(x + 3) - \ln(x - 1). && \text{Power Rule}
\end{aligned}$$

We then take derivatives of both sides with respect to x, using Equation (2) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

Finally, we substitute for y from the original equation:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

7.6

Inverse Trigonometric Functions

Defining the Inverses

The six basic trigonometric functions are not one-to-one (their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one. The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse $\sin^{-1} x$ (Figure 7.21).

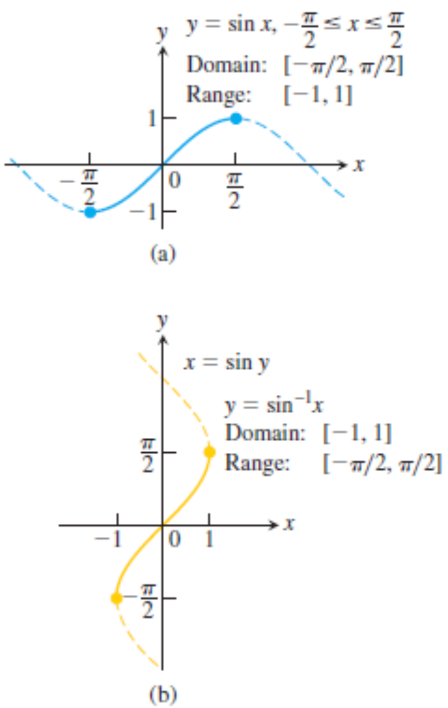


FIGURE 7.22 The graphs of (a) $y = \sin x, -\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

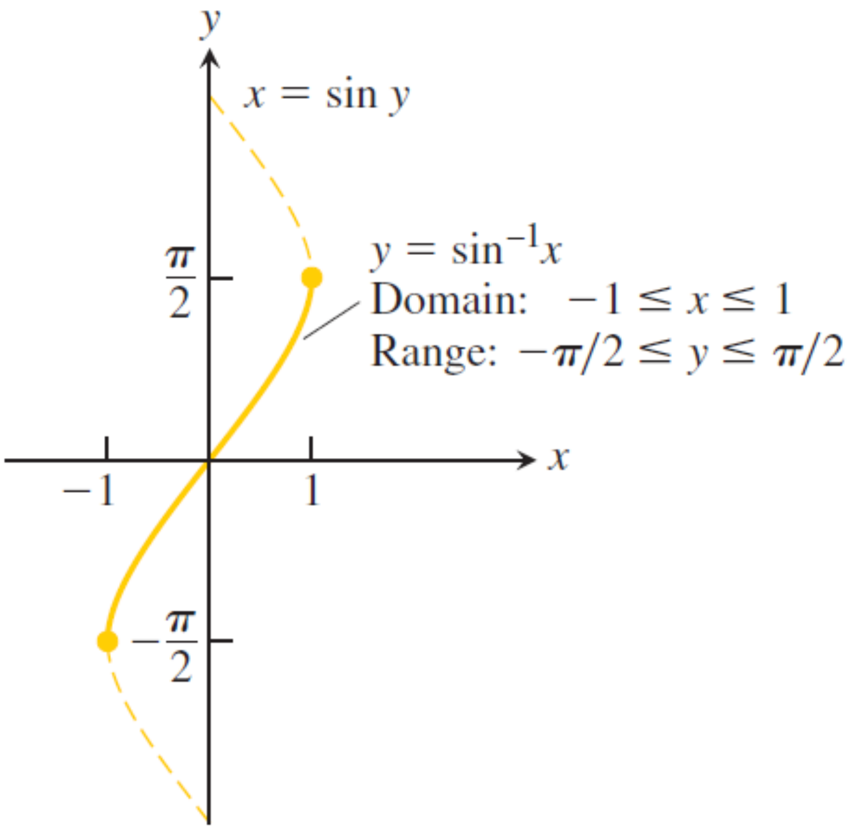
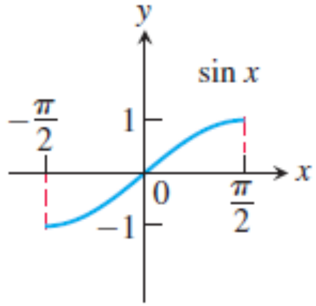
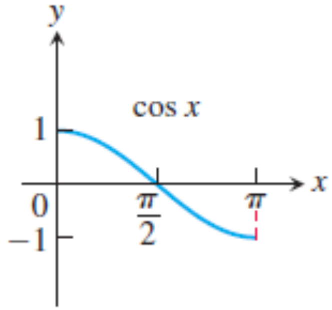


FIGURE 7.21 The graph of $y = \sin^{-1} x$.

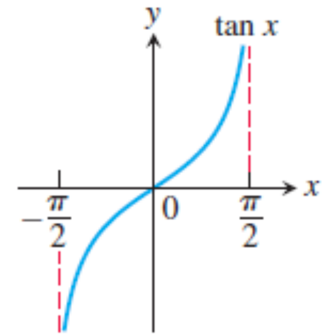
Domain Restrictions



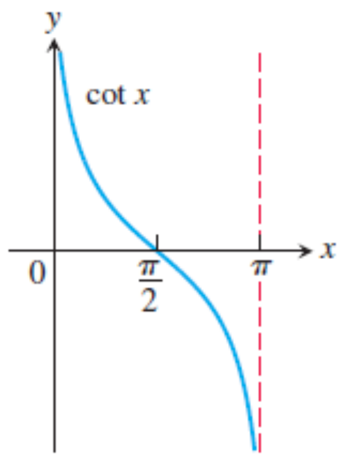
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



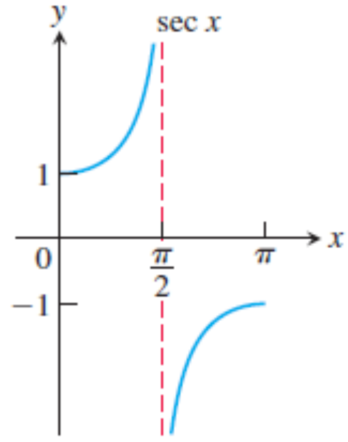
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



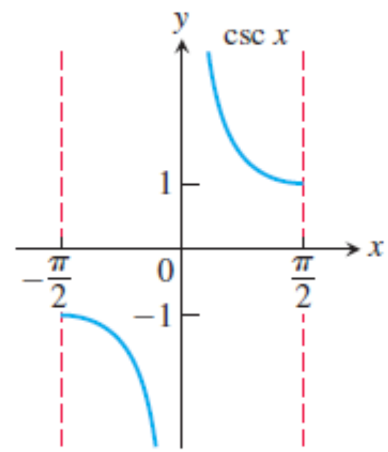
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

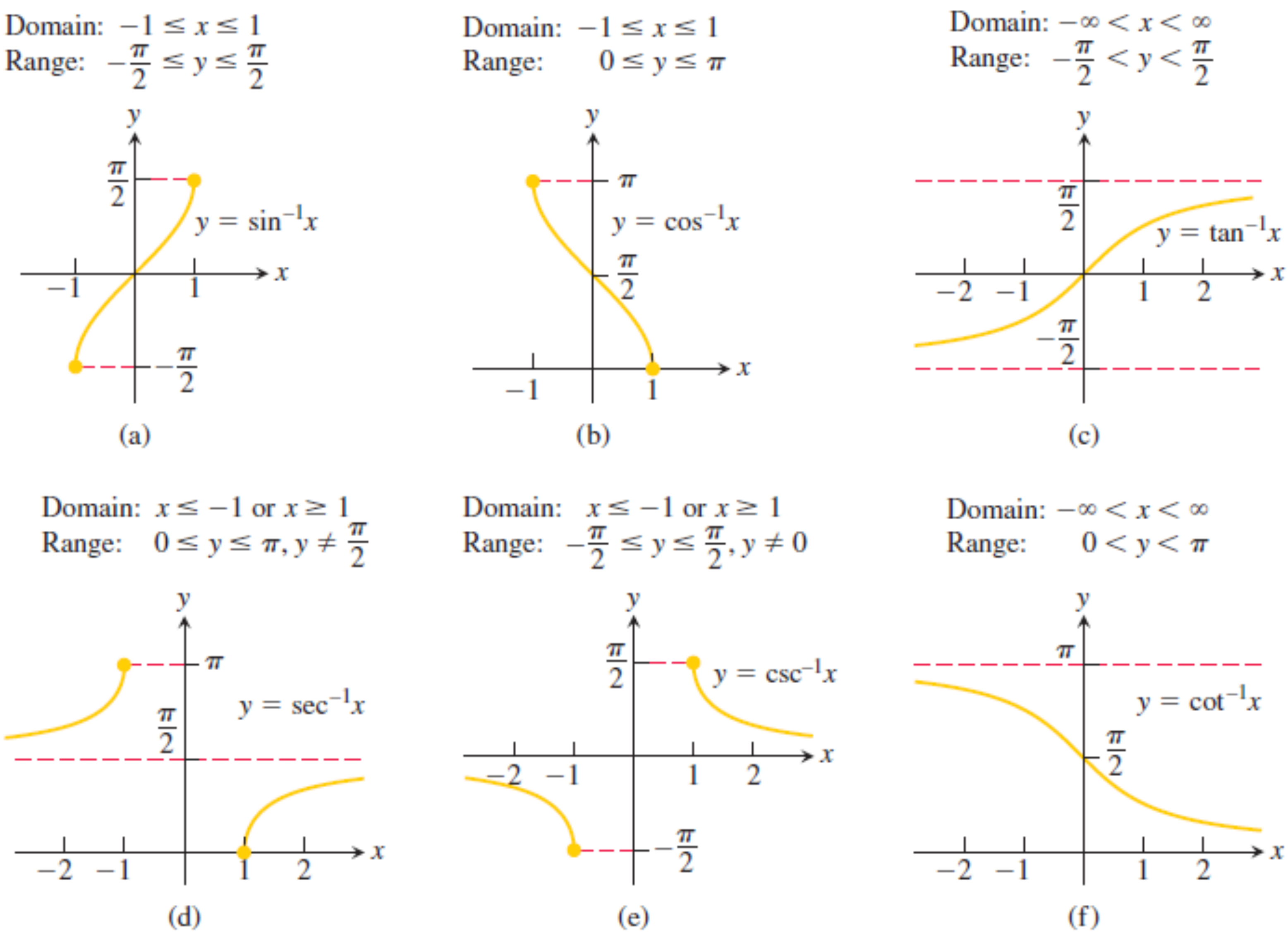


FIGURE 7.23 Graphs of the six basic inverse trigonometric functions.

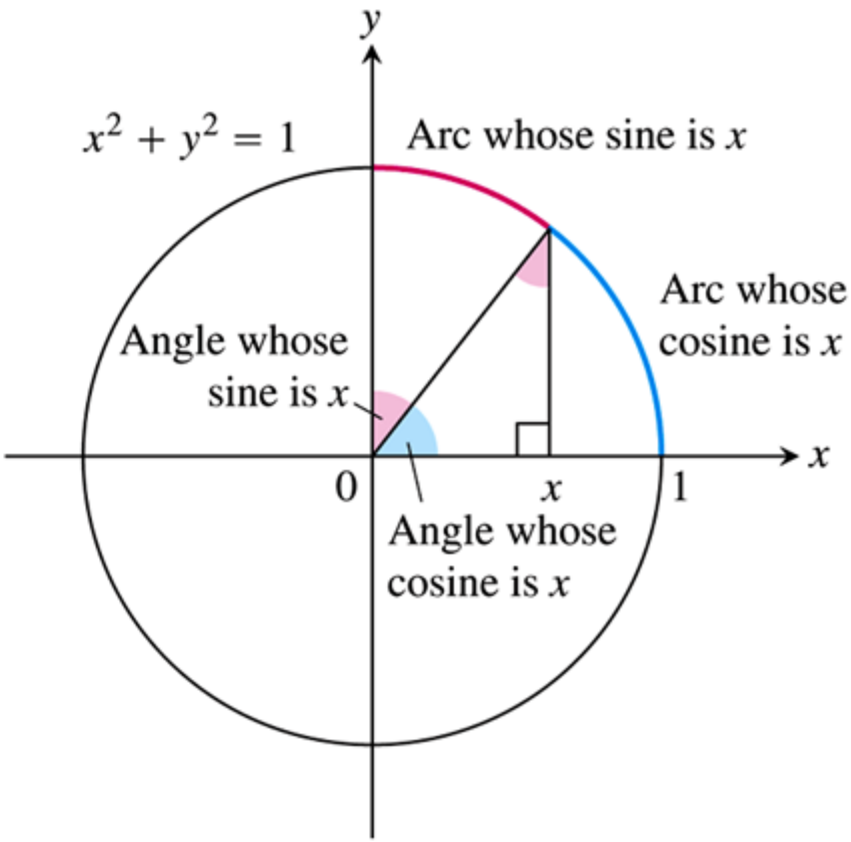
DEFINITION

$\boldsymbol{y = \sin^{-1} x}$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$\boldsymbol{y = \cos^{-1} x}$ is the number in $[0, \pi]$ for which $\cos y = x$.

The "Arc" in Arcsine and Arccosine

For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."



Some Identities

$$\sin^{-1} x + \cos^{-1} x = \pi/2$$

$$\tan^{-1} x + \cot^{-1} x = \pi/2$$

$$\sec^{-1} x + \csc^{-1} x = \pi/2$$

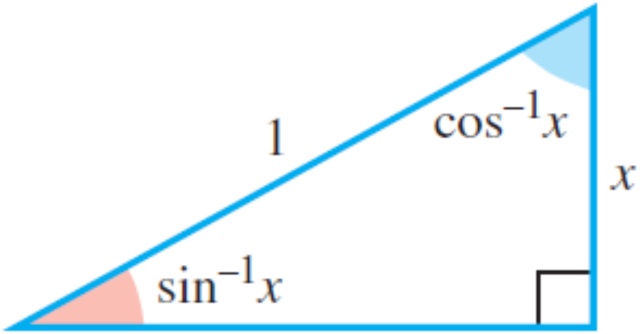


FIGURE 7.28 $\sin^{-1} x$ and $\cos^{-1} x$ are complementary angles (so their sum is $\pi/2$).

The Derivative of $y = \sin^{-1} u$

We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 1 in Section 7.1 therefore assures us that the inverse function $y = \sin^{-1} x$ is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangents to the graph are vertical at these points (see Figure 7.30).

We find the derivative of $y = \sin^{-1} x$ by applying Theorem 1 with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$:

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\cos(\sin^{-1} x)} && f'(u) = \cos u \\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \cos u = \sqrt{1 - \sin^2 u} \\
&= \frac{1}{\sqrt{1 - x^2}}. && \sin(\sin^{-1} x) = x
\end{aligned}
$$

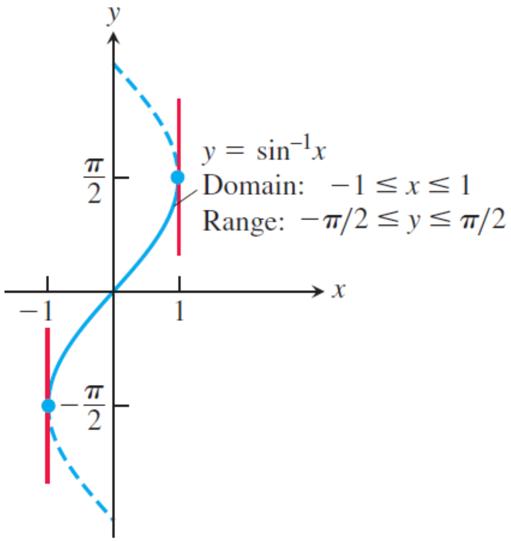


FIGURE 7.30 The graph of $y = \sin^{-1} x$ has vertical tangents at $x = -1$ and $x = 1$.

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get the general formula

$$
\boxed{\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \qquad |u| < 1.}
$$

EXAMPLE 4 Using the Chain Rule, we calculate the derivative

$$\frac{d}{dx}(\sin^{-1}x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}.$$

The Derivative of $y = \tan^{-1} u$

We find the derivative of $y = \tan^{-1} x$ by applying Theorem 1 with $f(x) = \tan x$ and $f^{-1}(x) = \tan^{-1} x$. Theorem 1 can be applied because the derivative of $\tan x$ is positive for $-\pi/2 < x < \pi/2$:

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\sec^2(\tan^{-1} x)} && f'(u) = \sec^2 u \\
&= \frac{1}{1 + \tan^2(\tan^{-1} x)} && \sec^2 u = 1 + \tan^2 u \\
&= \frac{1}{1+x^2}. && \tan(\tan^{-1} x) = x
\end{aligned}$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\boxed{\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}.}$$

The Derivative of $y = \sec^{-1} u$

Since the derivative of $\sec x$ is positive for $0 < x < \pi/2$ and $\pi/2 < x < \pi$, Theorem 1 says that the inverse function $y = \sec^{-1} x$ is differentiable. Instead of applying the formula in Theorem 1 directly, we find the derivative of $y = \sec^{-1} x$, $|x| > 1$, using implicit differentiation and the Chain Rule as follows:

$$y = \sec^{-1} x$$

$$\sec y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}x \qquad \text{Differentiate both sides.}$$

$$\sec y \tan y \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}. \qquad \text{Since } |x| > 1, y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \text{ and } \sec y \tan y \neq 0.$$

To express the result in terms of x, we use the relationships

$$\sec y = x \qquad \text{and} \qquad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2 - 1}}.$$

Can we do anything about the $\pm$ sign? A glance at Figure 7.31 shows that the slope of the graph $y = \sec^{-1} x$ is always positive. Thus,

$$\frac{d}{dx}\sec^{-1} x = \begin{cases} +\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x < -1. \end{cases}$$

With the absolute value symbol, we can write a single expression that eliminates the "$\pm$" ambiguity:

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\boxed{\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \qquad |u| > 1.}$$

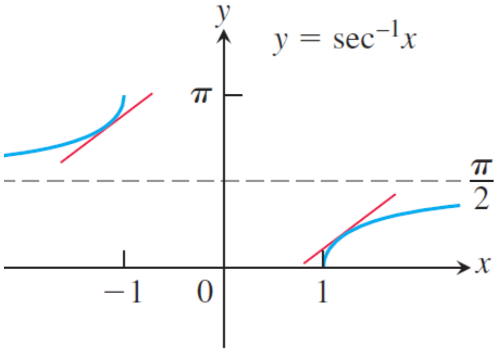


FIGURE 7.31 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

EXAMPLE 5 Using the Chain Rule and derivative of the arcsecant function, we find

$$\begin{aligned}
\frac{d}{dx}\sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}}\frac{d}{dx}(5x^4) \\
&= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) \qquad 5x^4 > 1 > 0 \\
&= \frac{4}{x\sqrt{25x^8 - 1}}.
\end{aligned}$$

■

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

These identities were seen earlier and they can be exploited to find other derivatives. For example

$$\frac{d}{dx}(\cos^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right) \qquad \text{Identity}$$

$$= -\frac{d}{dx}(\sin^{-1} x)$$

$$= -\frac{1}{\sqrt{1 - x^2}}. \qquad \text{Derivative of arcsine}$$

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1 + u^2} \frac{du}{dx}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$$

Integration Formulas

The derivative formulas in Table 7.3 yield three useful integration formulas in Table 7.4. The formulas are readily verified by differentiating the functions on the right-hand sides.

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

EXAMPLE 6 These examples illustrate how we use Table 7.4.

(a) $$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big]_{\sqrt{2}/2}^{\sqrt{3}/2}$$ $a = 1, u = x$ in Table 7.4, Formula 1

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

(b) $$\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}}$$ $a = \sqrt{3}, u = 2x$, and $du/2 = dx$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$ Table 7.4, Formula 1

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

EXAMPLE 7 Evaluate

(a) $\displaystyle\int \frac{dx}{\sqrt{4x - x^2}}$ **(b)** $\displaystyle\int \frac{dx}{4x^2 + 4x + 2}$

Solution

(a) The expression $\sqrt{4x - x^2}$ does not match any of the formulas in Table 7.4, so we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x - 2)^2.$$

Then we substitute $a = 2$, $u = x - 2$, and $du = dx$ to get

$$\begin{aligned}
\int \frac{dx}{\sqrt{4x - x^2}} &= \int \frac{dx}{\sqrt{4 - (x - 2)^2}} \\
&= \int \frac{du}{\sqrt{a^2 - u^2}} && a = 2,\ u = x - 2, \text{ and } du = dx \\
&= \sin^{-1}\left(\frac{u}{a}\right) + C && \text{Table 7.4, Formula 1} \\
&= \sin^{-1}\left(\frac{x - 2}{2}\right) + C
\end{aligned}$$

(b) We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$$

$$= 4\left(x + \frac{1}{2}\right)^2 + 1 = (2x + 1)^2 + 1.$$

Then,

$$\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x + 1)^2 + 1} = \frac{1}{2}\int \frac{du}{u^2 + a^2} \qquad a = 1, u = 2x + 1, \text{ and } du/2 = dx$$

$$= \frac{1}{2} \cdot \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 7.4, Formula 2}$$

$$= \frac{1}{2}\tan^{-1}(2x + 1) + C \qquad a = 1, u = 2x + 1$$

Week 9

Assignment 9

7.1: #9,10,32,34,41,42,44,56,57,58
7.6: #14,32,47,52,65,69,73,80,83,88,89,107,114
7.2: #2,24,36,46,53,54,55,65,68,77,82

These questions will need to be submitted.
Deadline: 10 PM, Friday, Nov 17

Required Reading (Textbook)

- Sections 7.1, 7.2, 7.6

Quiz 3 next week (Week 10, Nov 13 – 17)

Scope = 5.4, 5.5, 5.6, 6.1, 6.3; 30 minutes.