

CHAPTER 16 INTEGRALS AND VECTOR FIELDS

16.1 LINE INTEGRALS

1. $\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j} \Rightarrow x = t \text{ and } y = 1-t \Rightarrow y = 1-x \Rightarrow (c)$
2. $\mathbf{r} = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1, \text{ and } z = t \Rightarrow (e)$
3. $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} \Rightarrow x = 2 \cos t \text{ and } y = 2 \sin t \Rightarrow x^2 + y^2 = 4 \Rightarrow (g)$
4. $\mathbf{r} = t\mathbf{i} \Rightarrow x = t, y = 0, \text{ and } z = 0 \Rightarrow (a)$
5. $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t, y = t, \text{ and } z = t \Rightarrow (d)$
6. $\mathbf{r} = t\mathbf{j} + (2-2t)\mathbf{k} \Rightarrow y = t \text{ and } z = 2-2t \Rightarrow z = 2-2y \Rightarrow (b)$
7. $\mathbf{r} = (t^2 - 1)\mathbf{j} + 2t\mathbf{k} \Rightarrow y = t^2 - 1 \text{ and } z = 2t \Rightarrow y = \frac{z^2}{4} - 1 \Rightarrow (f)$
8. $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{k} \Rightarrow x = 2 \cos t \text{ and } z = 2 \sin t \Rightarrow x^2 + z^2 = 4 \Rightarrow (h)$
9. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{2}; x = t \text{ and } y = 1-t \Rightarrow x + y = t + (1-t) = 1$
 $\Rightarrow \int_C f(x, y, z) ds = \int_0^1 f(t, 1-t, 0) \left| \frac{d\mathbf{r}}{dt} \right| dt = \int_0^1 (1)(\sqrt{2}) dt = \left[\sqrt{2}t \right]_0^1 = \sqrt{2}$
10. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j} + \mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{2}; x = t, y = 1-t, \text{ and } z = 1 \Rightarrow x - y + z - 2$
 $= t - (1-t) + 1 - 2 = 2t - 2 \Rightarrow \int_C f(x, y, z) ds = \int_0^1 (2t-2)\sqrt{2} dt = \sqrt{2} \left[t^2 - 2t \right]_0^1 = -\sqrt{2}$
11. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2-2t)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{4+1+4} = 3; xy + y + z = (2t)t + t + (2-2t)$
 $\Rightarrow \int_C f(x, y, z) ds = \int_0^1 (2t^2 - t + 2) 3 dt = 3 \left[\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t \right]_0^1 = 3 \left(\frac{2}{3} - \frac{1}{2} + 2 \right) = \frac{13}{2}$
12. $\mathbf{r}(t) = (4 \cos t)\mathbf{i} + (4 \sin t)\mathbf{j} + 3t\mathbf{k}, -2\pi \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-4 \sin t)\mathbf{i} + (4 \cos t)\mathbf{j} + 3\mathbf{k}$
 $\Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{16 \sin^2 t + 16 \cos^2 t + 9} = 5; \sqrt{x^2 + y^2} = \sqrt{16 \cos^2 t + 16 \sin^2 t} = 4 \Rightarrow \int_C f(x, y, z) ds = \int_{-2\pi}^{2\pi} (4)(5) dt$
 $= \left[20t \right]_{-2\pi}^{2\pi} = 80\pi$
13. $\mathbf{r}(t) = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(-\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = (1-t)\mathbf{i} + (2-3t)\mathbf{j} + (3-2t)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$
 $\Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1+9+4} = \sqrt{14}; x + y + z = (1-t) + (2-3t) + (3-2t) = 6-6t \Rightarrow \int_C f(x, y, z) ds$
 $= \int_0^1 (6-6t) \sqrt{14} dt = 6\sqrt{14} \left[t - \frac{t^2}{2} \right]_0^1 = (6\sqrt{14}) \left(\frac{1}{2} \right) = 3\sqrt{14}$

14. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 1 \leq t \leq \infty \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{3}; \frac{\sqrt{3}}{x^2+y^2+z^2} = \frac{\sqrt{3}}{t^2+t^2+t^2} = \frac{\sqrt{3}}{3t^2}$
 $\Rightarrow \int_C f(x, y, z) ds = \int_1^\infty \left(\frac{\sqrt{3}}{3t^2} \right) \sqrt{3} dt = \left[-\frac{1}{t} \right]_1^\infty = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) = 1$
15. $C_1: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1+4t^2}; x + \sqrt{y} - z^2 = t + \sqrt{t^2} - 0 = t + |t| = 2t$ since $t \geq 0 \Rightarrow \int_{C_1} f(x, y, z) ds = \int_0^1 2t\sqrt{1+4t^2} dt = \left[\frac{1}{6}(1+4t^2)^{3/2} \right]_0^1 = \frac{1}{6}(5)^{3/2} - \frac{1}{6} = \frac{1}{6}(5\sqrt{5} - 1);$
 $C_2: \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y} - z^2 = 1 + \sqrt{1} - t^2 = 2 - t^2$
 $\Rightarrow \int_{C_2} f(x, y, z) ds = \int_0^1 (2 - t^2)(1) dt = \left[2t - \frac{1}{3}t^3 \right]_0^1 = 2 - \frac{1}{3} = \frac{5}{3};$
therefore $\int_C f(x, y, z) ds = \int_{C_1} f(x, y, z) ds + \int_{C_2} f(x, y, z) ds = \frac{5}{6}\sqrt{5} + \frac{5}{2}$
16. $C_1: \mathbf{r}(t) = t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y} - z^2 = 0 + \sqrt{0} - t^2 = -t^2$
 $\Rightarrow \int_{C_1} f(x, y, z) ds = \int_0^1 (-t^2)(1) dt = \left[-\frac{t^3}{3} \right]_0^1 = -\frac{1}{3};$
 $C_2: \mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y} - z^2 = 0 + \sqrt{t} - 1 = \sqrt{t} - 1$
 $\Rightarrow \int_{C_2} f(x, y, z) ds = \int_0^1 (\sqrt{t} - 1)(1) dt = \left[\frac{2}{3}t^{3/2} - t \right]_0^1 = \frac{2}{3} - 1 = -\frac{1}{3};$
 $C_3: \mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y} - z^2 = t + \sqrt{1} - 1 = t$
 $\Rightarrow \int_{C_3} f(x, y, z) ds = \int_0^1 (t)(1) dt = \left[\frac{t^2}{2} \right]_0^1 = \frac{1}{2}$
 $\Rightarrow \int_C f(x, y, z) ds = \int_{C_1} f ds + \int_{C_2} f ds + \int_{C_3} f ds = -\frac{1}{3} + \left(-\frac{1}{3} \right) + \frac{1}{2} = -\frac{1}{6}$
17. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 0 < a \leq t \leq b \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{3}; \frac{x+y+z}{x^2+y^2+z^2} = \frac{t+t+t}{t^2+t^2+t^2} = \frac{1}{t}$
 $\Rightarrow \int_C f(x, y, z) ds = \int_a^b \left(\frac{1}{t} \right) \sqrt{3} dt = \left[\sqrt{3} \ln |t| \right]_a^b = \sqrt{3} \ln \left(\frac{b}{a} \right),$ since $0 < a \leq b$
18. $\mathbf{r}(t) = (a \cos t)\mathbf{j} + (a \sin t)\mathbf{k}, 0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{j} + (a \cos t)\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = |a|;$
 $-\sqrt{x^2 + z^2} = -\sqrt{0 + a^2 \sin^2 t} = \begin{cases} -|a| \sin t, & 0 \leq t \leq \pi \\ |a| \sin t, & \pi \leq t \leq 2\pi \end{cases} \Rightarrow \int_C f(x, y, z) ds = \int_0^\pi -|a|^2 \sin t dt + \int_\pi^{2\pi} |a|^2 \sin t dt$
 $= \left[a^2 \cos t \right]_0^\pi - \left[a^2 \cos t \right]_\pi^{2\pi} = \left[a^2(-1) - a^2 \right] - \left[a^2 - a^2(-1) \right] = -4a^2$
19. (a) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t\mathbf{j}, 0 \leq t \leq 4 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \frac{1}{2}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \frac{\sqrt{5}}{2} \Rightarrow \int_C x ds = \int_0^4 t \frac{\sqrt{5}}{2} dt = \frac{\sqrt{5}}{2} \int_0^4 t dt = \left[\frac{\sqrt{5}}{4} t^2 \right]_0^4 = 4\sqrt{5}$
(b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1+4t^2} \Rightarrow \int_C x ds = \int_0^2 t\sqrt{1+4t^2} dt$
 $= \left[\frac{1}{12} (1+4t^2)^{3/2} \right]_0^2 = \frac{17\sqrt{17}-1}{12}$

20. (a) $\mathbf{r}(t) = t\mathbf{i} + 4t\mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 4\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{17} \Rightarrow \int_C \sqrt{x+2y} \, ds = \int_0^1 \sqrt{t+2(4t)} \sqrt{17} \, dt$
 $= \sqrt{17} \int_0^1 \sqrt{9t} \, dt = 3\sqrt{17} \int_0^1 \sqrt{t} \, dt = \left[2\sqrt{17} t^{2/3} \right]_0^1 = 2\sqrt{17}$
- (b) $C_1 : \mathbf{r}(t) = t\mathbf{i}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1$; $C_2 : \mathbf{r}(t) = \mathbf{i} + t\mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1$
 $\int_C \sqrt{x+2y} \, ds = \int_{C_1} \sqrt{x+2y} \, ds + \int_{C_2} \sqrt{x+2y} \, ds = \int_0^1 \sqrt{t+2(0)} \, dt + \int_0^1 \sqrt{1+2(t)} \, dt$
 $= \int_0^1 \sqrt{t} \, dt + \int_0^1 \sqrt{1+2t} \, dt = \left[\frac{2}{3} t^{2/3} \right]_0^1 + \left[\frac{1}{3} (1+2t)^{3/2} \right]_0^1 = \frac{2}{3} + \left(\frac{5\sqrt{5}}{3} - \frac{1}{3} \right) = \frac{5\sqrt{5}+1}{3}$
21. $\mathbf{r}(t) = 4t\mathbf{i} - 3t\mathbf{j}$, $-1 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = 4\mathbf{i} - 3\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 5 \Rightarrow \int_C y e^{x^2} \, ds = \int_{-1}^2 (-3t) e^{(4t)^2} \cdot 5 \, dt$
 $= -15 \int_{-1}^2 t e^{16t^2} \, dt = \left[-\frac{15}{32} e^{16t^2} \right]_{-1}^2 = -\frac{15}{32} e^{64} + \frac{15}{32} e^{16} = \frac{15}{32} (e^{16} - e^{64})$
22. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t} = 1$
 $\Rightarrow \int_C (x - y + 3) \, ds = \int_0^{2\pi} (\cos t - \sin t + 3) \cdot 1 \, dt = [\sin t + \cos t + 3t]_0^{2\pi} = 6\pi$
23. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$, $1 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + 3t^2\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{(2t)^2 + (3t^2)^2} = t\sqrt{4+9t^2}$
 $\Rightarrow \int_C \frac{x^2}{y^{4/3}} \, ds = \int_1^2 \frac{(t^2)^2}{(t^3)^{4/3}} \cdot t\sqrt{4+9t^2} \, dt = \int_1^2 t\sqrt{4+9t^2} \, dt = \left[\frac{1}{27} (4+9t^2)^{3/2} \right]_1^2 = \frac{80\sqrt{10}-13\sqrt{13}}{27}$
24. $\mathbf{r}(t) = t^3\mathbf{i} + t^4\mathbf{j}$, $\frac{1}{2} \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 3t^2\mathbf{i} + 4t^3\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{(3t^2)^2 + (4t^3)^2} = t^2\sqrt{9+16t^2}$
 $\Rightarrow \int_C \frac{\sqrt{y}}{x} \, ds = \int_{1/2}^1 \frac{\sqrt{t^4}}{t^3} \cdot t^2\sqrt{9+16t^2} \, dt = \int_{1/2}^1 t\sqrt{9+16t^2} \, dt = \left[\frac{1}{48} (9+16t^2)^{3/2} \right]_{1/2}^1 = \frac{125-13\sqrt{13}}{48}$
25. $C_1 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1+4t^2}$; $C_2 : \mathbf{r}(t) = (1-t)\mathbf{i} + (1-t)\mathbf{j}$, $0 \leq t \leq 1$
 $\Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{2} \Rightarrow \int_C (x + \sqrt{y}) \, ds = \int_{C_1} (x + \sqrt{y}) \, ds + \int_{C_2} (x + \sqrt{y}) \, ds$
 $= \int_0^1 (t + \sqrt{t^2}) \sqrt{1+4t^2} \, dt + \int_0^1 ((1-t) + \sqrt{1-t}) \sqrt{2} \, dt = \int_0^1 2t\sqrt{1+4t^2} \, dt + \int_0^1 (1-t + \sqrt{1-t}) \sqrt{2} \, dt$
 $= \left[\frac{1}{6} (1+4t^2)^{3/2} \right]_0^1 + \sqrt{2} \left[t - \frac{1}{2} t^2 - \frac{2}{3} (1-t)^{3/2} \right]_0^1 = \frac{5\sqrt{5}-1}{6} + \frac{7\sqrt{2}}{6} = \frac{5\sqrt{5}+7\sqrt{2}-1}{6}$
26. $C_1 : \mathbf{r}(t) = t\mathbf{i}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1$; $C_2 : \mathbf{r}(t) = \mathbf{i} + t\mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1$;
 $C_3 : \mathbf{r}(t) = (1-t)\mathbf{i} + \mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1$; $C_4 : \mathbf{r}(t) = (1-t)\mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1$;
 $\Rightarrow \int_C \frac{1}{x^2+y^2+1} \, ds = \int_{C_1} \frac{1}{x^2+y^2+1} \, ds + \int_{C_2} \frac{1}{x^2+y^2+1} \, ds + \int_{C_3} \frac{1}{x^2+y^2+1} \, ds + \int_{C_4} \frac{1}{x^2+y^2+1} \, ds$

$$\begin{aligned}
&= \int_0^1 \frac{dt}{t^2+1} + \int_0^1 \frac{dt}{t^2+2} + \int_0^1 \frac{dt}{(1-t)^2+2} + \int_0^1 \frac{dt}{(1-t)^2+1} \\
&= \left[\tan^{-1} t \right]_0^1 + \frac{1}{\sqrt{2}} \left[\tan^{-1} \left(\frac{t}{\sqrt{2}} \right) \right]_0^1 + \frac{1}{\sqrt{2}} \left[\tan^{-1} \left(\frac{t-1}{\sqrt{2}} \right) \right]_0^1 + \left[-\tan^{-1}(1-t) \right]_0^1 = \frac{\pi}{2} + \frac{2}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \right)
\end{aligned}$$

$$27. \quad \mathbf{r}(x) = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + \frac{x^2}{2}\mathbf{j}, 0 \leq x \leq 2 \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + x\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dx} \right| = \sqrt{1+x^2}; f(x, y) = f\left(x, \frac{x^2}{2}\right) = \frac{x^3}{\left(\frac{x^2}{2}\right)} = 2x$$

$$\Rightarrow \int_C f \, ds = \int_0^2 (2x)\sqrt{1+x^2} \, dx = \left[\frac{2}{3}(1+x^2)^{3/2} \right]_0^2 = \frac{2}{3}(5^{3/2}-1) = \frac{10\sqrt{5}-2}{3}$$

$$28. \quad \mathbf{r}(t) = (1-t)\mathbf{i} + \frac{1}{2}(1-t)^2\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1+(1-t)^2}; f(x, y) = f\left((1-t), \frac{1}{2}(1-t)^2\right) = \frac{(1-t)+\frac{1}{4}(1-t)^4}{\sqrt{1+(1-t)^2}}$$

$$\begin{aligned}
\Rightarrow \int_C f \, ds &= \int_0^1 \frac{(1-t)+\frac{1}{4}(1-t)^4}{\sqrt{1+(1-t)^2}} \sqrt{1+(1-t)^2} \, dt = \int_0^1 \left((1-t) + \frac{1}{4}(1-t)^4 \right) dt = \left[-\frac{1}{2}(1-t)^2 - \frac{1}{20}(1-t)^5 \right]_0^1 \\
&= 0 - \left(-\frac{1}{2} - \frac{1}{20} \right) = \frac{11}{20}
\end{aligned}$$

$$29. \quad \mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j}, 0 \leq t \leq \frac{\pi}{2} \Rightarrow \frac{d\mathbf{r}}{dt} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2; f(x, y) = f(2 \cos t, 2 \sin t) = 2 \cos t + 2 \sin t \Rightarrow \int_C f \, ds = \int_0^{\pi/2} (2 \cos t + 2 \sin t)(2) \, dt = [4 \sin t - 4 \cos t]_0^{\pi/2} = 4 - (-4) = 8$$

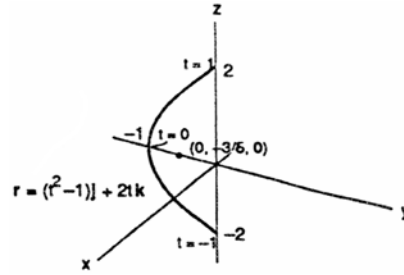
$$30. \quad \mathbf{r}(t) = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j}, 0 \leq t \leq \frac{\pi}{4} \Rightarrow \frac{d\mathbf{r}}{dt} = (2 \cos t)\mathbf{i} + (-2 \sin t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2; f(x, y) = f(2 \sin t, 2 \cos t) = 4 \sin^2 t - 2 \cos t \Rightarrow \int_C f \, ds = \int_0^{\pi/4} (4 \sin^2 t - 2 \cos t)(2) \, dt = [4t - 2 \sin 2t - 4 \sin t]_0^{\pi/4} = \pi - 2(1 + \sqrt{2})$$

$$31. \quad y = x^2, 0 \leq x \leq 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1+4t^2} \Rightarrow A = \int_C f(x, y) \, ds = \int_C (x + \sqrt{y}) \, ds = \int_0^2 \left(t + \sqrt{t^2} \right) \sqrt{1+4t^2} \, dt = \int_0^2 2t\sqrt{1+4t^2} \, dt = \left[\frac{1}{6}(1+4t^2)^{3/2} \right]_0^2 = \frac{17\sqrt{17}-1}{6}$$

$$32. \quad 2x+3y=6, 0 \leq x \leq 6 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + \left(2 - \frac{2}{3}t\right)\mathbf{j}, 0 \leq t \leq 6 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \frac{2}{3}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \frac{\sqrt{13}}{3} \Rightarrow A = \int_C f(x, y) \, ds = \int_C (4+3x+2y) \, ds = \int_0^6 \left(4+3t+2\left(2 - \frac{2}{3}t\right) \right) \frac{\sqrt{13}}{3} \, dt = \frac{\sqrt{13}}{3} \int_0^6 \left(8 + \frac{5}{3}t \right) \, dt = \frac{\sqrt{13}}{3} \left[8t + \frac{5}{6}t^2 \right]_0^6 = 26\sqrt{13}$$

$$33. \quad \mathbf{r}(t) = (t^2-1)\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2\sqrt{t^2+1}; M = \int_C \delta(x, y, z) \, ds = \int_0^1 \delta(t) \left(2\sqrt{t^2+1} \right) \, dt = \int_0^1 \left(\frac{3}{2}t \right) \left(2\sqrt{t^2+1} \right) \, dt = \left[(t^2+1)^{3/2} \right]_0^1 = 2^{3/2} - 1 = 2\sqrt{2} - 1$$

34. $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, -1 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k}$
 $\Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2\sqrt{t^2 + 1}; M = \int_C \delta(x, y, z) ds$
 $= \int_{-1}^1 \left(15\sqrt{t^2 - 1} + 2 \right) \left(2\sqrt{t^2 + 1} \right) dt$
 $= \int_{-1}^1 30(t^2 + 1) dt = \left[30\left(\frac{t^3}{3} + t\right) \right]_{-1}^1 = 60\left(\frac{1}{3} + 1\right) = 80;$
 $M_{xz} = \int_C y\delta(x, y, z) ds = \int_{-1}^1 (t^2 - 1) \left[30(t^2 + 1) \right] dt$
 $= \int_{-1}^1 30(t^4 - 1) dt = \left[30\left(\frac{t^5}{5} - t\right) \right]_{-1}^1 = 60\left(\frac{1}{5} - 1\right) = -48 \Rightarrow \bar{y} = \frac{M_{xz}}{M} = -\frac{48}{80} = -\frac{3}{5};$
 $M_{yz} = \int_C x\delta(x, y, z) ds = \int_C 0\delta ds = 0 \Rightarrow \bar{x} = 0; \bar{z} = 0$ by symmetry (since δ is independent of z) $\Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(0, -\frac{3}{5}, 0\right)$



35. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{2 + 2 + 4t^2} = 2\sqrt{1 + t^2};$
 (a) $M = \int_C \delta ds = \int_0^1 (3t) \left(2\sqrt{1 + t^2} \right) dt = \left[2(1 + t^2)^{3/2} \right]_0^1 = 2(2^{3/2} - 1) = 4\sqrt{2} - 2$
 (b) $M = \int_C \delta ds = \int_0^1 (1) \left(2\sqrt{1 + t^2} \right) dt = \left[t\sqrt{1 + t^2} + \ln(t + \sqrt{1 + t^2}) \right]_0^1 = \left[\sqrt{2} + \ln(1 + \sqrt{2}) \right] - (0 + \ln 1)$
 $= \sqrt{2} + \ln(1 + \sqrt{2})$

36. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} + t^{1/2}\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1 + 4 + t} = \sqrt{5 + t};$
 $M = \int_C \delta ds = \int_0^2 (3\sqrt{5 + t}) (\sqrt{5 + t}) dt = \int_0^2 3(5 + t) dt = \left[\frac{3}{2}(5 + t)^2 \right]_0^2 = \frac{3}{2}(7^2 - 5^2) = \frac{3}{2}(24) = 36;$
 $M_{yz} = \int_C x\delta ds = \int_0^2 t[3(5 + t)] dt = \int_0^2 (15t + 3t^2) dt = \left[\frac{15}{2}t^2 + t^3 \right]_0^2 = 30 + 8 = 38;$
 $M_{xz} = \int_C y\delta ds = \int_0^2 2t[3(5 + t)] dt = 2 \int_0^2 (15t + 3t^2) dt = 76; M_{xy} = \int_C z\delta ds = \int_0^2 \frac{2}{3}t^{3/2}[3(5 + t)] dt$
 $= \int_0^2 (10t^{3/2} + 2t^{5/2}) dt = \left[4t^{5/2} + \frac{4}{7}t^{7/2} \right]_0^2 = 4(2)^{5/2} + \frac{4}{7}(2)^{7/2} = 16\sqrt{2} + \frac{32}{7}\sqrt{2} = \frac{144}{7}\sqrt{2}$
 $\Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{38}{36} = \frac{19}{18}, \bar{y} = \frac{M_{xz}}{M} = \frac{76}{36} = \frac{19}{9}, \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{144\sqrt{2}}{7 \cdot 36} = \frac{4}{7}\sqrt{2}$

37. Let $x = a \cos t$ and $y = a \sin t, 0 \leq t \leq 2\pi$. Then $\frac{dx}{dt} = -a \sin t, \frac{dy}{dt} = a \cos t, \frac{dz}{dt} = 0$
 $\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = a dt; I_z = \int_C (x^2 + y^2) \delta ds = \int_0^{2\pi} (a^2 \sin^2 t + a^2 \cos^2 t) a \delta dt$
 $= \int_0^{2\pi} a^3 \delta dt = 2\pi a^3.$

38. $\mathbf{r}(t) = t\mathbf{j} + (2 - 2t)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} - 2\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{5}; M = \int_C \delta ds = \int_0^1 \delta \sqrt{5} dt = \delta \sqrt{5};$
 $I_x = \int_C (y^2 + z^2) \delta ds = \int_0^1 [t^2 + (2 - 2t)^2] \delta \sqrt{5} dt = \int_0^1 (5t^2 - 8t + 4) \delta \sqrt{5} dt = \delta \sqrt{5} \left[\frac{5}{3}t^3 - 4t^2 + 4t \right]_0^1 = \frac{5}{3}\delta \sqrt{5};$

$$I_y = \int_C (x^2 + z^2) \delta \, ds = \int_0^1 [0^2 + (2-2t)^2] \delta \sqrt{5} \, dt = \int_0^1 (4t^2 - 8t + 4) \delta \sqrt{5} \, dt = \delta \sqrt{5} \left[\frac{4}{3} t^3 - 4t^2 + 4t \right]_0^1 = \frac{4}{3} \delta \sqrt{5};$$

$$I_z = \int_C (x^2 + y^2) \delta \, ds = \int_0^1 (0^2 + t^2) \delta \sqrt{5} \, dt = \delta \sqrt{5} \left[\frac{t^3}{3} \right]_0^1 = \frac{1}{3} \delta \sqrt{5}$$

39. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2};$

(a) $I_z = \int_C (x^2 + y^2) \delta \, ds = \int_0^{2\pi} (\cos^2 t + \sin^2 t) \delta \sqrt{2} \, dt = 2\pi \delta \sqrt{2}$

(b) $I_z = \int_C (x^2 + y^2) \delta \, ds = \int_0^{4\pi} \delta \sqrt{2} \, dt = 4\pi \delta \sqrt{2}$

40. $\mathbf{r}(t) = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + \frac{2\sqrt{2}}{3} t^{3/2} \mathbf{k}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j} + \sqrt{2}t \mathbf{k}$

$$\Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{(t+1)^2} = t+1 \text{ for } 0 \leq t \leq 1; M = \int_C \delta \, ds = \int_0^1 (t+1) \, dt = \left[\frac{1}{2}(t+1)^2 \right]_0^1 = \frac{1}{2}(2^2 - 1^2) = \frac{3}{2};$$

$$M_{xy} = \int_C z \delta \, ds = \int_0^1 \left(\frac{2\sqrt{2}}{3} t^{3/2} \right) (t+1) \, dt = \frac{2\sqrt{2}}{3} \int_0^1 (t^{5/2} + t^{3/2}) \, dt = \frac{2\sqrt{2}}{3} \left[\frac{2}{7} t^{7/2} + \frac{2}{5} t^{5/2} \right]_0^1$$

$$= \frac{2\sqrt{2}}{3} \left(\frac{2}{7} + \frac{2}{5} \right) = \frac{2\sqrt{2}}{3} \left(\frac{24}{35} \right) = \frac{16\sqrt{2}}{35} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{16\sqrt{2}}{35} \right) \left(\frac{2}{3} \right) = \frac{32\sqrt{2}}{105};$$

$$I_z = \int_C (x^2 + y^2) \delta \, ds = \int_0^1 (t^2 \cos^2 t + t^2 \sin^2 t) (t+1) \, dt = \int_0^1 (t^3 + t^2) \, dt = \left[\frac{t^4}{4} + \frac{t^3}{3} \right]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

41. $\delta(x, y, z) = 2 - z$ and $\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k}$, $0 \leq t \leq \pi \Rightarrow M = 2\pi - 2$ as found in Example 3 of the text; also

$$\left| \frac{d\mathbf{r}}{dt} \right| = 1; I_x = \int_C (y^2 + z^2) \delta \, ds = \int_0^\pi (\cos^2 t + \sin^2 t) (2 - \sin t) \, dt = \int_0^\pi (2 - \sin t) \, dt = 2\pi - 2$$

42. $\mathbf{r}(t) = t\mathbf{i} + \frac{2\sqrt{2}}{3} t^{3/2} \mathbf{j} + \frac{t^2}{2} \mathbf{k}$, $0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \sqrt{2} t^{1/2} \mathbf{j} + t\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1 + 2t + t^2} = \sqrt{(1+t)^2} = 1+t$ for

$$0 \leq t \leq 2; M = \int_C \delta \, ds = \int_0^2 \left(\frac{1}{t+1} \right) (1+t) \, dt = \int_0^2 dt = 2; M_{yz} = \int_C x \delta \, ds = \int_0^2 t \left(\frac{1}{t+1} \right) (1+t) \, dt = \left[\frac{t^2}{2} \right]_0^2 = 2;$$

$$M_{xz} = \int_C y \delta \, ds = \int_0^2 \frac{2\sqrt{2}}{3} t^{3/2} \, dt = \left[\frac{4\sqrt{2}}{15} t^{5/2} \right]_0^2 = \frac{32}{15}; M_{xy} = \int_C z \delta \, ds = \int_0^2 \frac{t^2}{2} \, dt = \left[\frac{t^3}{6} \right]_0^2 = \frac{4}{3} \Rightarrow \bar{x} = \frac{M_{yz}}{M} = 1,$$

$$\bar{y} = \frac{M_{xz}}{M} = \frac{16}{15}, \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{2}{3}; I_x = \int_C (y^2 + z^2) \delta \, ds = \int_0^2 \left(\frac{8}{9} t^3 + \frac{1}{4} t^4 \right) \, dt = \left[\frac{2}{9} t^4 + \frac{t^5}{20} \right]_0^2 = \frac{32}{9} + \frac{32}{20} = \frac{232}{45};$$

$$I_y = \int_C (x^2 + z^2) \delta \, ds = \int_0^2 \left(t^2 + \frac{1}{4} t^4 \right) \, dt = \left[\frac{t^3}{3} + \frac{t^5}{20} \right]_0^2 = \frac{8}{3} + \frac{32}{20} = \frac{64}{15};$$

$$I_z = \int_C (x^2 + y^2) \delta \, ds = \int_0^2 \left(t^2 + \frac{8}{9} t^3 \right) \, dt = \left[\frac{t^3}{3} + \frac{2}{9} t^4 \right]_0^2 = \frac{8}{3} + \frac{32}{9} = \frac{56}{9}$$

43-46. Example CAS commands:

Maple:

f:=(x,y,z) -> sqrt(1+30*x^2+10*y);

g:=t -> t;

h:=t -> t^2;

k:=t -> 3*t^2;

a,b:=0.2;

$$ds := (D(g)^2 + D(h)^2 + D(k)^2)^{1/2}; \quad \#(a)$$

$$'ds' = ds(t) * 'dt';$$

$$F := f(g, h, k); \quad \#(b)$$

$$F(t)' = F(t);$$

$$\text{Int}(f, s = C..NULL) = \text{Int}(\text{simplify}(F(t) * ds(t)), t = a..b); \quad \#(c)$$

$$`` = \text{value}(\text{rhs}(\%));$$

Mathematica: (functions and domains may vary)

Clear[x, y, z, r, t, f]

f[x_, y_, z_] := Sqrt(1 + 30x^2 + 10y)

{a, b} = {0, 2};

x[t_] := t

y[t_] := t^2

z[t_] := 3t^2

r[t_] := {x[t], y[t], z[t]}

v[t_] := D[r[t], t]

mag[vector_] := Sqrt[vector.vector]

Integrate[f[x(t), y(t), z[t]] mag[v[t]], {t, a, b}]

N[%]

16.2 VECTOR FIELDS AND LINE INTEGRALS: WORK, CIRCULATION, AND FLUX

$$1. \quad f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x) = -x(x^2 + y^2 + z^2)^{-3/2}; \text{ similarly,}$$

$$\frac{\partial f}{\partial y} = -y(x^2 + y^2 + z^2)^{-3/2} \text{ and } \frac{\partial f}{\partial z} = -z(x^2 + y^2 + z^2)^{-3/2} \Rightarrow \nabla f = \frac{-xi - yj - zk}{(x^2 + y^2 + z^2)^{3/2}}$$

$$2. \quad f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} \ln(x^2 + y^2 + z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{2} \left(\frac{1}{x^2 + y^2 + z^2} \right) (2x) = \frac{x}{x^2 + y^2 + z^2}; \text{ similarly,}$$

$$\frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2 + z^2} \text{ and } \frac{\partial f}{\partial z} = \frac{z}{x^2 + y^2 + z^2} \Rightarrow \nabla f = \frac{xi + yj + zk}{x^2 + y^2 + z^2}$$

$$3. \quad g(x, y, z) = e^z - \ln(x^2 + y^2) \Rightarrow \frac{\partial g}{\partial x} = -\frac{2x}{x^2 + y^2}, \frac{\partial g}{\partial y} = -\frac{2y}{x^2 + y^2} \text{ and } \frac{\partial g}{\partial z} = e^z \Rightarrow \nabla g = \left(\frac{-2x}{x^2 + y^2} \right) \mathbf{i} - \left(\frac{2y}{x^2 + y^2} \right) \mathbf{j} + e^z \mathbf{k}$$

$$4. \quad g(x, y, z) = xy + yz + xz \Rightarrow \frac{\partial g}{\partial x} = y + z, \frac{\partial g}{\partial y} = x + z, \text{ and } \frac{\partial g}{\partial z} = y + x \Rightarrow \nabla g = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$$

$$5. \quad |\mathbf{F}| \text{ inversely proportional to the square of the distance from } (x, y) \text{ to the origin} \Rightarrow \sqrt{(M(x, y))^2 + (N(x, y))^2}$$

$$= \frac{k}{x^2 + y^2}, k > 0; \mathbf{F} \text{ points toward the origin} \Rightarrow \mathbf{F} \text{ is in the direction of } \mathbf{n} = \frac{-x}{\sqrt{x^2 + y^2}} \mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}} \mathbf{j} \Rightarrow \mathbf{F} = a \mathbf{n}, \text{ for}$$

$$\text{some constant } a > 0. \text{ Then } M(x, y) = \frac{-ax}{\sqrt{x^2 + y^2}} \text{ and } N(x, y) = \frac{-ay}{\sqrt{x^2 + y^2}} \Rightarrow \sqrt{(M(x, y))^2 + (N(x, y))^2} = a$$

$$\Rightarrow a = \frac{k}{x^2 + y^2} \Rightarrow \mathbf{F} = \frac{-kx}{(x^2 + y^2)^{3/2}} \mathbf{i} - \frac{ky}{(x^2 + y^2)^{3/2}} \mathbf{j}, \text{ for any constant } k > 0$$

6. Given $x^2 + y^2 = a^2 + b^2$, let $x = \sqrt{a^2 + b^2} \cos t$ and $y = -\sqrt{a^2 + b^2} \sin t$. Then
 $\mathbf{r} = \left(\sqrt{a^2 + b^2} \cos t \right) \mathbf{i} - \left(\sqrt{a^2 + b^2} \sin t \right) \mathbf{j}$ traces the circle in a clockwise direction as t goes from 0 to 2π
 $\Rightarrow \mathbf{v} = \left(-\sqrt{a^2 + b^2} \sin t \right) \mathbf{i} - \left(\sqrt{a^2 + b^2} \cos t \right) \mathbf{j}$ is tangent to the circle in a clockwise direction. Thus, let
 $\mathbf{F} = \mathbf{v} \Rightarrow \mathbf{F} = y\mathbf{i} - x\mathbf{j}$ and $\mathbf{F}(0, 0) = \mathbf{0}$.
7. Substitute the parametric representations for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$.
- (a) $\mathbf{F} = 3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 9t \Rightarrow \int_0^1 9t \, dt = \frac{9}{2}$
- (b) $\mathbf{F} = 3t^2\mathbf{i} + 2t\mathbf{j} + 4t^4\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 7t^2 + 16t^7 \Rightarrow \int_0^1 (7t^2 + 16t^7) \, dt = \left[\frac{7}{3}t^3 + 2t^8 \right]_0^1 = \frac{7}{3} + 2 = \frac{13}{3}$
- (c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = 3t\mathbf{i} + 2t\mathbf{j}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 5t \Rightarrow \int_0^1 5t \, dt = \frac{5}{2}$;
 $\mathbf{F}_2 = 3\mathbf{i} + 2\mathbf{j} + 4t\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 4t \Rightarrow \int_0^1 4t \, dt = 2 \Rightarrow \frac{5}{2} + 2 = \frac{9}{2}$
8. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$.
- (a) $\mathbf{F} = \left(\frac{1}{t^2 + 1} \right) \mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{t^2 + 1} \Rightarrow \int_0^1 \frac{1}{t^2 + 1} \, dt = \left[\tan^{-1} t \right]_0^1 = \frac{\pi}{4}$
- (b) $\mathbf{F} = \left(\frac{1}{t^2 + 1} \right) \mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{2t}{t^2 + 1} \Rightarrow \int_0^1 \frac{2t}{t^2 + 1} \, dt = \left[\ln(t^2 + 1) \right]_0^1 = \ln 2$
- (c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = \left(\frac{1}{t^2 + 1} \right) \mathbf{j}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = \frac{1}{t^2 + 1}$; $\mathbf{F}_2 = \frac{1}{2}\mathbf{j}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0 \Rightarrow \int_0^1 \frac{1}{t^2 + 1} \, dt = \frac{\pi}{4}$
9. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$.
- (a) $\mathbf{F} = \sqrt{t}\mathbf{i} - 2t\mathbf{j} + \sqrt{t}\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2\sqrt{t} - 2t \Rightarrow \int_0^1 (2\sqrt{t} - 2t) \, dt = \left[\frac{4}{3}t^{3/2} - t^2 \right]_0^1 = \frac{1}{3}$
- (b) $\mathbf{F} = t^2\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 4t^4 - 3t^2 \Rightarrow \int_0^1 (4t^4 - 3t^2) \, dt = \left[\frac{4}{5}t^5 - t^3 \right]_0^1 = -\frac{1}{5}$
- (c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = -2t\mathbf{j} + \sqrt{t}\mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -2t \Rightarrow \int_0^1 -2t \, dt = -1$;
 $\mathbf{F}_2 = \sqrt{t}\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow \int_0^1 1 \, dt = 1 \Rightarrow -1 + 1 = 0$
10. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$.
- (a) $\mathbf{F} = t^2\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 \Rightarrow \int_0^1 3t^2 \, dt = 1$

$$(b) \quad \mathbf{F} = t^3\mathbf{i} - t^6\mathbf{j} + t^5\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^7 + 4t^8 \Rightarrow \int_0^1 (t^3 + 2t^7 + 4t^8) dt$$

$$= \left[\frac{t^4}{4} + \frac{t^8}{4} + \frac{4}{9}t^9 \right]_0^1 = \frac{17}{18}$$

$$(c) \quad \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = t^2\mathbf{i} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = t^2 \Rightarrow \int_0^1 t^2 dt = \frac{1}{3};$$

$$\mathbf{F}_2 = \mathbf{i} + t\mathbf{j} + t\mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow \int_0^1 t dt = \frac{1}{2} \Rightarrow \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

11. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$.

$$(a) \quad \mathbf{F} = (3t^2 - 3t)\mathbf{i} + 3t\mathbf{j} + \mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 + 1 \Rightarrow \int_0^1 (3t^2 + 1) dt = \left[t^3 + t \right]_0^1 = 2$$

$$(b) \quad \mathbf{F} = (3t^2 - 3t)\mathbf{i} + 3t^4\mathbf{j} + \mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 4t^3 + 3t^2 - 3t$$

$$\Rightarrow \int_0^1 (6t^5 + 4t^3 + 3t^2 - 3t) dt = \left[t^6 + t^4 + t^3 - \frac{3}{2}t^2 \right]_0^1 = \frac{3}{2}$$

$$(c) \quad \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = (3t^2 - 3t)\mathbf{i} + \mathbf{k} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 3t^2 - 3t$$

$$\Rightarrow \int_0^1 (3t^2 - 3t) dt = \left[t^3 - \frac{3}{2}t^2 \right]_0^1 = -\frac{1}{2}; \mathbf{F}_2 = 3t\mathbf{j} + \mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow \int_0^1 1 dt = 1 \Rightarrow -\frac{1}{2} + 1 = \frac{1}{2}$$

12. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$.

$$(a) \quad \mathbf{F} = 2t\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t \Rightarrow \int_0^1 6t dt = \left[3t^2 \right]_0^1 = 3$$

$$(b) \quad \mathbf{F} = (t^2 + t^4)\mathbf{i} + (t^4 + t)\mathbf{j} + (t + t^2)\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 5t^4 + 3t^2$$

$$\Rightarrow \int_0^1 (6t^5 + 5t^4 + 3t^2) dt = \left[t^6 + t^5 + t^3 \right]_0^1 = 3$$

$$(c) \quad \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 2t \Rightarrow \int_0^1 2t dt = 1;$$

$$\mathbf{F}_2 = (1+t)\mathbf{i} + (t+1)\mathbf{j} + 2\mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 2 \Rightarrow \int_0^1 2 dt = 2 \Rightarrow 1 + 2 = 3$$

$$13. \quad x = t, y = 2t + 1, 0 \leq t \leq 3 \Rightarrow dx = dt \Rightarrow \int_C (x - y) dx = \int_0^3 (t - (2t + 1)) dt = \int_0^3 (-t - 1) dt = \left[-\frac{1}{2}t^2 - t \right]_0^3 = -\frac{15}{2}$$

$$14. \quad x = t, y = t^2, 1 \leq t \leq 2 \Rightarrow dy = 2t dt \Rightarrow \int_C \frac{x}{y} dy = \int_1^2 \frac{t}{t^2} (2t) dt = \int_1^2 2 dt = [2t]_1^2 = 2$$

$$15. \quad C_1 : x = t, y = 0, 0 \leq t \leq 3 \Rightarrow dy = 0; C_2 : x = 3, y = t, 0 \leq t \leq 3 \Rightarrow dy = dt \Rightarrow \int_C (x^2 + y^2) dy$$

$$= \int_{C_1} (x^2 + y^2) dx + \int_{C_2} (x^2 + y^2) dx = \int_0^3 (t^2 + 0^2) \cdot 0 + \int_0^3 (3^2 + t^2) dt = \int_0^3 (9 + t^2) dt = \left[9t + \frac{1}{3}t^3 \right]_0^3 = 36$$

$$16. \quad C_1 : x = t, y = 3t, 0 \leq t \leq 1 \Rightarrow dx = dt; C_2 : x = 1 - t, y = 3, 0 \leq t \leq 1 \Rightarrow dx = -dt; C_3 : x = 0, y = 3 - t, 0 \leq t \leq 3$$

$$\begin{aligned} \Rightarrow dx = 0 &\Rightarrow \int_C \sqrt{x+y} \, dx = \int_{C_1} \sqrt{x+y} \, dx + \int_{C_2} \sqrt{x+y} \, dx + \int_{C_3} \sqrt{x+y} \, dx \\ &= \int_0^1 \sqrt{t+3t} \, dt + \int_0^1 \sqrt{(1-t)+3}(-1) \, dt + \int_0^3 \sqrt{0+(3-t)} \cdot 0 \, dt = \int_0^1 2\sqrt{t} \, dt - \int_0^1 \sqrt{4-t} \, dt \\ &= \left[\frac{4}{3} t^{3/2} \right]_0^1 - \left[\frac{2}{3} (4-t)^{3/2} \right]_0^1 = \frac{4}{3} + \left(2\sqrt{3} - \frac{16}{3} \right) = 2\sqrt{3} - 4 \end{aligned}$$

$$17. \quad \mathbf{r}(t) = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}, 0 \leq t \leq 1 \Rightarrow dx = dt, dy = 0, dz = 2t \, dt$$

$$(a) \quad \int_C (x+y-z) \, dx = \int_0^1 (t-1-t^2) \, dt = \left[\frac{1}{2} t^2 - t - \frac{1}{3} t^3 \right]_0^1 = -\frac{5}{6}$$

$$(b) \quad \int_C (x+y-z) \, dy = \int_0^1 (t-1-t^2) \cdot 0 \, dt = 0$$

$$(c) \quad \int_C (x+y-z) \, dz = \int_0^1 (t-1-t^2) 2t \, dt = \int_0^1 (2t^2 - 2t - 2t^3) \, dt = \left[\frac{2}{3} t^3 - t^2 - \frac{1}{2} t^4 \right]_0^1 = -\frac{5}{6}$$

$$18. \quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos t)\mathbf{k}, 0 \leq t \leq \pi \Rightarrow dx = -\sin t \, dt, dy = \cos t \, dt, dz = \sin t \, dt$$

$$(a) \quad \int_C xz \, dx = \int_0^\pi (\cos t)(-\cos t)(-\sin t) \, dt = \int_0^\pi \cos^2 t \sin t \, dt = \left[-\frac{1}{3} (\cos t)^3 \right]_0^\pi = \frac{2}{3}$$

$$(b) \quad \int_C xz \, dy = \int_0^\pi (\cos t)(-\cos t)(\cos t) \, dt = -\int_0^\pi \cos^3 t \, dt = -\int_0^\pi (1 - \sin^2 t) \cos t \, dt = \left[\frac{1}{3} (\sin t)^3 - \sin t \right]_0^\pi = 0$$

$$\begin{aligned} (c) \quad \int_C xy \, dz &= \int_0^\pi (\cos t)(\sin t)(-\cos t)(\sin t) \, dt = -\int_0^\pi \cos^2 t \sin^2 t \, dt = -\frac{1}{4} \int_0^\pi \sin^2 2t \, dt = -\frac{1}{4} \int_0^\pi \frac{1 - \cos 4t}{2} \, dt \\ &= -\frac{1}{8} \int_0^\pi (1 - \cos 4t) \, dt = \left[-\frac{1}{8} t + \frac{1}{32} \sin 4t \right]_0^\pi = -\frac{\pi}{8} \end{aligned}$$

$$19. \quad \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1, \text{ and } \mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} \Rightarrow \mathbf{F} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$$

$$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t^3 \Rightarrow \text{work} = \int_0^1 2t^3 \, dt = \frac{1}{2}$$

$$20. \quad \mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \frac{t}{6}\mathbf{k}, 0 \leq t \leq 2\pi, \text{ and } \mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x+y)\mathbf{k}$$

$$\Rightarrow \mathbf{F} = (2 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + (\cos t + \sin t)\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{1}{6}\mathbf{k}$$

$$\begin{aligned} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= 3 \cos^2 t - 2 \sin^2 t + \frac{1}{6} \cos t + \frac{1}{6} \sin t \Rightarrow \text{work} = \int_0^{2\pi} \left(3 \cos^2 t - 2 \sin^2 t + \frac{1}{6} \cos t + \frac{1}{6} \sin t \right) dt \\ &= \left[\frac{3}{2} t + \frac{3}{4} \sin 2t - t + \frac{\sin 2t}{2} + \frac{1}{6} \sin t - \frac{1}{6} \cos t \right]_0^{2\pi} = \pi \end{aligned}$$

$$21. \quad \mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 2\pi, \text{ and } \mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + (\sin t)\mathbf{j} + (\cos t)\mathbf{k} \text{ and}$$

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \cos t - \sin^2 t + \cos t \Rightarrow \text{work} = \int_0^{2\pi} (t \cos t - \sin^2 t + \cos t) \, dt \\ &= \left[\cos t + t \sin t - \frac{t}{2} + \frac{\sin 2t}{4} + \sin t \right]_0^{2\pi} = -\pi \end{aligned}$$

22. $\mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{t}{6}\mathbf{k}$, $0 \leq t \leq 2\pi$, and $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + (\cos^2 t)\mathbf{j} + (12 \sin t)\mathbf{k}$ and
 $\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \cos t - \sin t \cos^2 t + 2 \sin t$
 $\Rightarrow \text{work} = \int_0^{2\pi} \left(t \cos t - \sin t \cos^2 t + 2 \sin t \right) dt = \left[\cos t + t \sin t + \frac{1}{3} \cos^3 t - 2 \cos t \right]_0^{2\pi} = 0$
23. $x = t$ and $y = x^2 = t^2 \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j}$, $-1 \leq t \leq 2$, and $\mathbf{F} = xy\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t^3\mathbf{i} + (t+t^2)\mathbf{j}$ and
 $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + (2t^2 + 2t^3) = 3t^3 + 2t^2 \Rightarrow \int_C xy \, dx + (x+y) \, dy = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \, dt = \int_{-1}^2 (3t^3 + 2t^2) \, dt$
 $= \left[\frac{3}{4}t^4 + \frac{2}{3}t^3 \right]_{-1}^2 = \left(12 + \frac{16}{3} \right) - \left(\frac{3}{4} - \frac{2}{3} \right) = \frac{45}{4} + \frac{18}{3} = \frac{69}{4}$
24. Along $(0,0)$ to $(1,0)$: $\mathbf{r} = t\mathbf{i}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t\mathbf{i} + t\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t$;
 Along $(1,0)$ to $(0,1)$: $\mathbf{r} = (1-t)\mathbf{i} + t\mathbf{j}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (1-2t)\mathbf{i} + \mathbf{j}$ and
 $\frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t$;
 Along $(0,1)$ to $(0,0)$: $\mathbf{r} = (1-t)\mathbf{j}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (t-1)\mathbf{i} + (1-t)\mathbf{j}$ and
 $\frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t-1 \Rightarrow \int_C (x-y) \, dx + (x+y) \, dy = \int_0^1 t \, dt + \int_0^1 2t \, dt + \int_0^1 (t-1) \, dt = \int_0^1 (4t-1) \, dt$
 $= \left[2t^2 - t \right]_0^1 = 2 - 1 = 1$
25. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = y^2\mathbf{i} + y\mathbf{j}$, $2 \geq y \geq -1$, and $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} = y^4\mathbf{i} - y\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dy} = 2y\mathbf{i} + \mathbf{j}$ and $\mathbf{F} \cdot \frac{d\mathbf{r}}{dy} = 2y^5 - y$
 $\Rightarrow \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_2^{-1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dy} \, dy = \int_2^{-1} (2y^5 - y) \, dy = \left[\frac{1}{3}y^6 - \frac{1}{2}y^2 \right]_2^{-1} = \left(\frac{1}{3} - \frac{1}{2} \right) - \left(\frac{64}{3} - \frac{4}{2} \right) = \frac{3}{2} - \frac{63}{3} = -\frac{39}{2}$
26. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq \frac{\pi}{2}$, and $\mathbf{F} = y\mathbf{i} - x\mathbf{j} \Rightarrow \mathbf{F} = (\sin t)\mathbf{i} - (\cos t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$
 $\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin^2 t - \cos^2 t = -1 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (-1) \, dt = -\frac{\pi}{2}$
27. $\mathbf{r} = (\mathbf{i} + \mathbf{j}) + t(\mathbf{i} + 2\mathbf{j}) = (1+t)\mathbf{i} + (1+2t)\mathbf{j}$, $0 \leq t \leq 1$, and $\mathbf{F} = xy\mathbf{i} + (y-x)\mathbf{j} \Rightarrow \mathbf{F} = (1+3t+2t^2)\mathbf{i} + t\mathbf{j}$ and
 $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 1+5t+2t^2 \Rightarrow \text{work} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \, dt = \int_0^1 (1+5t+2t^2) \, dt = \left[t + \frac{5}{2}t^2 + \frac{2}{3}t^3 \right]_0^1 = \frac{25}{6}$
28. $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$, and $\mathbf{F} = \nabla f = 2(x+y)\mathbf{i} + 2(x+y)\mathbf{j}$
 $\Rightarrow \mathbf{F} = 4(\cos t + \sin t)\mathbf{i} + 4(\cos t + \sin t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j}$
 $\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -8(\sin t \cos t + \sin^2 t) + 8(\cos^2 t + \cos t \sin t) = 8(\cos^2 t - \sin^2 t) = 8 \cos 2t$
 $\Rightarrow \text{work} = \int_C \nabla f \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \, dt = \int_0^{2\pi} 8 \cos 2t \, dt = [4 \sin 2t]_0^{2\pi} = 0$
29. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$, $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$,
 $\mathbf{F}_1 = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, and $\mathbf{F}_2 = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 0$ and $\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1$

$$\Rightarrow \text{Circ}_1 = \int_0^{2\pi} 0 \, dt = 0 \text{ and } \text{Circ}_2 = \int_0^{2\pi} dt = 2\pi; \mathbf{n} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n} = \cos^2 t + \sin^2 t = 1 \text{ and}$$

$$\mathbf{F}_2 \cdot \mathbf{n} = 0 \Rightarrow \text{Flux}_1 = \int_0^{2\pi} dt = 2\pi \text{ and } \text{Flux}_2 = \int_0^{2\pi} 0 \, dt = 0$$

$$(b) \quad \mathbf{r} = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}, 0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (4 \cos t)\mathbf{j}, \mathbf{F}_1 = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}, \text{ and}$$

$$\mathbf{F}_2 = (-4 \sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 15 \sin t \cos t \text{ and } \mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = 4 \Rightarrow \text{Circ}_1 = \int_0^{2\pi} 15 \sin t \cos t \, dt$$

$$= \left[\frac{15}{2} \sin^2 t \right]_0^{2\pi} = 0 \text{ and } \text{Circ}_2 = \int_0^{2\pi} 4 \, dt = 8\pi; \mathbf{n} = \left(\frac{4}{\sqrt{17}} \cos t \right)\mathbf{i} + \left(\frac{1}{\sqrt{17}} \sin t \right)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n}$$

$$= \frac{4}{\sqrt{17}} \cos^2 t + \frac{4}{\sqrt{17}} \sin^2 t \text{ and } \mathbf{F}_2 \cdot \mathbf{n} = -\frac{15}{\sqrt{17}} \sin t \cos t \Rightarrow \text{Flux}_1 = \int_0^{2\pi} (\mathbf{F}_1 \cdot \mathbf{n}) |\mathbf{v}| \, dt = \int_0^{2\pi} \left(\frac{4}{\sqrt{17}} \right) \sqrt{17} \, dt$$

$$= 8\pi \text{ and } \text{Flux}_2 = \int_0^{2\pi} (\mathbf{F}_2 \cdot \mathbf{n}) |\mathbf{v}| \, dt = \int_0^{2\pi} \left(-\frac{15}{\sqrt{17}} \sin t \cos t \right) \sqrt{17} \, dt = \left[-\frac{15}{2} \sin^2 t \right]_0^{2\pi} = 0$$

$$30. \quad \mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, 0 \leq t \leq 2\pi, \mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j}, \text{ and } \mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j},$$

$$\mathbf{F}_1 = (2a \cos t)\mathbf{i} - (3a \sin t)\mathbf{j}, \text{ and } \mathbf{F}_2 = (2a \cos t)\mathbf{i} + (a \cos t - a \sin t)\mathbf{j} \Rightarrow \mathbf{n} |\mathbf{v}| = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j},$$

$$\mathbf{F}_1 \cdot \mathbf{n} |\mathbf{v}| = 2a^2 \cos^2 t - 3a^2 \sin^2 t, \text{ and } \mathbf{F}_2 \cdot \mathbf{n} |\mathbf{v}| = 2a^2 \cos^2 t + a^2 \sin t \cos t - a^2 \sin^2 t$$

$$\Rightarrow \text{Flux}_1 = \int_0^{2\pi} (2a^2 \cos^2 t - 3a^2 \sin^2 t) \, dt = 2a^2 \left[\frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{2\pi} - 3a^2 \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = -\pi a^2, \text{ and}$$

$$\text{Flux}_2 = \int_0^{2\pi} (2a^2 \cos^2 t - a^2 \sin t \cos t - a^2 \sin^2 t) \, dt$$

$$= 2a^2 \left[\frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{2\pi} + \frac{a^2}{2} \left[\sin^2 t \right]_0^{2\pi} - a^2 \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = \pi a^2$$

$$31. \quad \mathbf{F}_1 = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, \frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 0 \Rightarrow \text{Circ}_1 = 0; M_1 = a \cos t,$$

$$N_1 = a \sin t, dx = -a \sin t \, dt, dy = a \cos t \, dt \Rightarrow \text{Flux}_1 = \int_C M_1 \, dy - N_1 \, dx = \int_0^\pi (a^2 \cos^2 t + a^2 \sin^2 t) \, dt$$

$$= \int_0^\pi a^2 \, dt = a^2 \pi;$$

$$\mathbf{F}_2 = t\mathbf{i}, \frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow \text{Circ}_2 = \int_{-a}^a t \, dt = 0; M_2 = t, N_2 = 0, dx = dt, dy = 0$$

$$\Rightarrow \text{Flux}_2 = \int_C M_2 \, dy - N_2 \, dx = \int_{-a}^a 0 \, dt = 0;$$

$$\text{therefore, } \text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0 \text{ and } \text{Flux} = \text{Flux}_1 + \text{Flux}_2 = a^2 \pi$$

$$32. \quad \mathbf{F}_1 = (a^2 \cos^2 t)\mathbf{i} + (a^2 \sin^2 t)\mathbf{j}, \frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t$$

$$\Rightarrow \text{Circ}_1 = \int_0^\pi (-a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t) \, dt = -\frac{2a^3}{3}; M_1 = a^2 \cos^2 t, N_1 = a^2 \sin^2 t, dy = a \cos t \, dt,$$

$$dx = -a \sin t \, dt \Rightarrow \text{Flux}_1 = \int_C M_1 \, dy - N_1 \, dx = \int_0^\pi (a^3 \cos^3 t + a^3 \sin^3 t) \, dt = \frac{4}{3} a^3;$$

$$\mathbf{F}_2 = t^2 \mathbf{i}, \frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t^2 \Rightarrow \text{Circ}_2 = \int_{-a}^a t^2 \, dt = \frac{2a^3}{3}; M_2 = t^2, N_2 = 0, dy = 0, dx = dt$$

$$\Rightarrow \text{Flux}_2 = \int_C M_2 \, dy - N_2 \, dx = 0; \text{therefore, } \text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0 \text{ and } \text{Flux} = \text{Flux}_1 + \text{Flux}_2 = \frac{4}{3} a^3$$

33. $\mathbf{F}_1 = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^2 \sin^2 t + a^2 \cos^2 t = a^2$
 $\Rightarrow \text{Circ}_1 = \int_0^\pi a^2 dt = a^2 \pi$; $M_1 = -a \sin t$, $N_1 = a \cos t$, $dx = -a \sin t dt$, $dy = a \cos t dt$
 $\Rightarrow \text{Flux}_1 = \int_C M_1 dy - N_1 dx = \int_0^\pi (-a^2 \sin t \cos t + a^2 \sin t \cos t) dt = 0$; $\mathbf{F}_2 = t\mathbf{j}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$
 $\Rightarrow \text{Circ}_2 = 0$; $M_2 = 0$, $N_2 = t$, $dx = dt$, $dy = 0 \Rightarrow \text{Flux}_2 = \int_C M_2 dy - N_2 dx = \int_{-a}^a -t dt = 0$; therefore,
 $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = a^2 \pi$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$
34. $\mathbf{F}_1 = (-a^2 \sin^2 t)\mathbf{i} + (a^2 \cos^2 t)\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^3 \sin^3 t + a^3 \cos^3 t$
 $\Rightarrow \text{Circ}_1 = \int_0^\pi (a^3 \sin^3 t + a^3 \cos^3 t) dt = \frac{4}{3} a^3$; $M_1 = -a^2 \sin^2 t$, $N_1 = a^2 \cos^2 t$, $dy = a \cos t dt$, $dx = -a \sin t dt$
 $\Rightarrow \text{Flux}_1 = \int_C M_1 dy - N_1 dx = \int_0^\pi (-a^3 \cos t \sin^2 t + a^3 \sin t \cos^2 t) dt = \frac{2}{3} a^3$; $\mathbf{F}_2 = t^2 \mathbf{j}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$
 $\Rightarrow \text{Circ}_2 = 0$; $M_2 = 0$, $N_2 = t^2$, $dy = 0$, $dx = dt \Rightarrow \text{Flux}_2 = \int_C M_2 dy - N_2 dx = \int_{-a}^a -t^2 dt = -\frac{2}{3} a^3$; therefore,
 $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = \frac{4}{3} a^3$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$
35. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq \pi$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2 + y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ and
 $\mathbf{F} = (\cos t + \sin t)\mathbf{i} - (\cos^2 t + \sin^2 t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t - \sin^2 t - \cos t \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T} ds$
 $= \int_0^\pi (-\sin t \cos t - \sin^2 t - \cos t) dt = \left[-\frac{1}{2} \sin^2 t - \frac{t}{2} + \frac{\sin 2t}{4} - \sin t \right]_0^\pi = -\frac{\pi}{2}$
- (b) $\mathbf{r} = (1-2t)\mathbf{i}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2 + y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = -2\mathbf{i}$ and $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t)^2 \mathbf{j} \Rightarrow$
 $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 4t - 2 \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^1 (4t - 2) dt = \left[2t^2 - 2t \right]_0^1 = 0$
- (c) $\mathbf{r}_1 = (1-t)\mathbf{i} - t\mathbf{j}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2 + y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j}$
 $\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = (2t-1) + (1-2t+2t^2) = 2t^2 \Rightarrow \text{Flow}_1 = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = \int_0^1 2t^2 dt = \frac{2}{3}$; $\mathbf{r}_2 = -t\mathbf{i} + (t-1)\mathbf{j}$,
 $0 \leq t \leq 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2 + y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} + \mathbf{j}$ and $\mathbf{F} = -\mathbf{i} - (t^2 + t^2 - 2t + 1)\mathbf{j}$
 $= -\mathbf{i} - (2t^2 - 2t + 1)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = 1 - (2t^2 - 2t + 1) = 2t - 2t^2 \Rightarrow \text{Flow}_2 = \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = \int_0^1 (2t - 2t^2) dt$
 $= \left[t^2 - \frac{2}{3} t^3 \right]_0^1 = \frac{1}{3} \Rightarrow \text{Flow} = \text{Flow}_1 + \text{Flow}_2 = \frac{2}{3} + \frac{1}{3} = 1$
36. From $(1, 0)$ to $(0, 1)$: $\mathbf{r}_1 = (1-t)\mathbf{i} + t\mathbf{j}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2 + y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} + \mathbf{j}$,
 $\mathbf{F} = \mathbf{i} - (1-2t+2t^2)\mathbf{j}$, and $\mathbf{n}_1 |\mathbf{v}_1| = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_1 |\mathbf{v}_1| = 2t - 2t^2 \Rightarrow \text{Flux}_1 = \int_0^1 (2t - 2t^2) dt = \left[t^2 - \frac{2}{3} t^3 \right]_0^1 = \frac{1}{3}$;
From $(0, 1)$ to $(-1, 0)$: $\mathbf{r}_2 = -t\mathbf{i} + (1-t)\mathbf{j}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2 + y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} - \mathbf{j}$,
 $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j}$, and $\mathbf{n}_2 |\mathbf{v}_2| = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_2 |\mathbf{v}_2| = (2t-1) + (-1+2t-2t^2) = -2+4t-2t^2$
 $\Rightarrow \text{Flux}_2 = \int_0^1 (-2+4t-2t^2) dt = \left[-2t+2t^2-\frac{2}{3}t^3 \right]_0^1 = -\frac{2}{3}$;

From $(-1, 0)$ to $(1, 0)$: $\mathbf{r}_3 = (-1+2t)\mathbf{i}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_3}{dt} = 2\mathbf{i}$,

$$\mathbf{F} = (-1+2t)\mathbf{i} - (1-4t+4t^2)\mathbf{j}, \text{ and } \mathbf{n}_3 \cdot |\mathbf{v}_3| = -2\mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_3 \cdot |\mathbf{v}_3| = 2(1-4t+4t^2)$$

$$\Rightarrow \text{Flux}_3 = 2 \int_0^1 (1-4t+4t^2) dt = 2 \left[t - 2t^2 + \frac{4}{3}t^3 \right]_0^1 = \frac{2}{3} \Rightarrow \text{Flux} = \text{Flux}_1 + \text{Flux}_2 + \text{Flux}_3 = \frac{1}{3} - \frac{2}{3} + \frac{2}{3} = \frac{1}{3}$$

$$\begin{aligned} 37. \quad (a) \quad y = 2x, 0 \leq x \leq 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= ((2t)^2\mathbf{i} + 2(t)(2t)\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j}) \\ &= 4t^2 + 8t^2 = 12t^2 \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^2 12t^2 dt = \left[4t^3 \right]_0^2 = 32 \end{aligned}$$

$$\begin{aligned} (b) \quad y = x^2, 0 \leq x \leq 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= \left((t^2)^2\mathbf{i} + 2(t)(t^2)\mathbf{j} \right) \cdot (\mathbf{i} + 2t\mathbf{j}) \\ &= t^4 + 4t^4 = 5t^4 \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^2 5t^4 dt = \left[t^5 \right]_0^2 = 32 \end{aligned}$$

$$\begin{aligned} (c) \quad \text{answers will vary, one possible path is } y = \frac{1}{2}x^3, 0 \leq x \leq 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^3\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} &= \mathbf{i} + 3t^2\mathbf{j} \\ \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= \left(\left(\frac{1}{2}t^3 \right)^2\mathbf{i} + 2(t)\left(\frac{1}{2}t^3 \right)\mathbf{j} \right) \cdot (\mathbf{i} + 3t^2\mathbf{j}) = \frac{1}{4}t^6 + \frac{3}{2}t^6 = \frac{7}{4}t^6 \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^2 \frac{7}{4}t^6 dt \\ &= \left[\frac{1}{4}t^7 \right]_0^2 = 32 \end{aligned}$$

$$38. \quad (a) \quad C_1 : \mathbf{r}(t) = (1-t)\mathbf{i} + \mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((1)\mathbf{i} + ((1-t) + 2(1))\mathbf{j}) \cdot (-\mathbf{i}) = -1;$$

$$C_2 : \mathbf{r}(t) = -\mathbf{i} + (1-t)\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((1-t)\mathbf{i} + ((-1) + 2(1-t))\mathbf{j}) \cdot (-\mathbf{j}) = 2t-1;$$

$$C_3 : \mathbf{r}(t) = (t-1)\mathbf{i} - \mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((-1)\mathbf{i} + ((t-1) + 2(-1))\mathbf{j}) \cdot (\mathbf{i}) = -1;$$

$$C_4 : \mathbf{r}(t) = \mathbf{i} + (t-1)\mathbf{j}, 0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((t-1)\mathbf{i} + ((1) + 2(t-1))\mathbf{j}) \cdot (\mathbf{j}) = 2t-1;$$

$$\begin{aligned} \Rightarrow \text{Flow} &= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt + \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt + \int_{C_3} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt + \int_{C_4} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \\ &= \int_0^2 (-1) dt + \int_0^2 (2t-1) dt + \int_0^2 (-1) dt + \int_0^2 (2t-1) dt = [-t]_0^2 + [t^2 - t]_0^2 + [-t]_0^2 + [t^2 - t]_0^2 \\ &= -2 + 2 - 2 + 2 = 0 \end{aligned}$$

$$\begin{aligned} (b) \quad x^2 + y^2 = 4 \Rightarrow \mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j}, 0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} &= (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} \\ \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= ((2 \sin t)\mathbf{i} + (2 \cos t + 2(2 \sin t))\mathbf{j}) \cdot ((-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j}) = -4 \sin^2 t + 4 \cos^2 t + 8 \sin t \cos t \\ &= 4 \cos 2t + 4 \sin 2t \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^{2\pi} (4 \cos 2t + 4 \sin 2t) dt = [2 \sin 2t - 2 \cos 2t]_0^{2\pi} = 0 \end{aligned}$$

(c) answers will vary, one possible path is:

$$C_1 : \mathbf{r}(t) = t\mathbf{i}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((0)\mathbf{i} + (t + 2(1))\mathbf{j}) \cdot (\mathbf{i}) = 0;$$

$$C_2 : \mathbf{r}(t) = (1-t)\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = (t\mathbf{i} + ((1-t) + 2t)\mathbf{j}) \cdot (-\mathbf{i} + \mathbf{j}) = 1;$$

$$C_3 : \mathbf{r}(t) = (1-t)\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((1-t)\mathbf{i} + (0 + 2(1-t))\mathbf{j}) \cdot (-\mathbf{j}) = 2t-1;$$

$$\begin{aligned} \Rightarrow \text{Flow} &= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt + \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt + \int_{C_3} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^1 (0) dt + \int_0^1 (1) dt + \int_0^1 (2t-1) dt \\ &= 0 + [t]_0^1 + [t^2 - t]_0^1 = 1 + (-1) = 0 \end{aligned}$$

39. $\mathbf{F} = -\frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$ on $x^2 + y^2 = 4$;

at $(2, 0)$, $\mathbf{F} = \mathbf{j}$; at $(0, 2)$, $\mathbf{F} = -\mathbf{i}$;

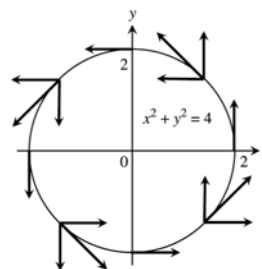
at $(-2, 0)$, $\mathbf{F} = -\mathbf{j}$; at $(0, -2)$, $\mathbf{F} = \mathbf{i}$;

at $(\sqrt{2}, \sqrt{2})$, $\mathbf{F} = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$;

at $(\sqrt{2}, -\sqrt{2})$, $\mathbf{F} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$;

at $(-\sqrt{2}, \sqrt{2})$, $\mathbf{F} = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$;

at $(-\sqrt{2}, -\sqrt{2})$, $\mathbf{F} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$



40. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ on $x^2 + y^2 = 1$; at $(1, 0)$, $\mathbf{F} = \mathbf{i}$;

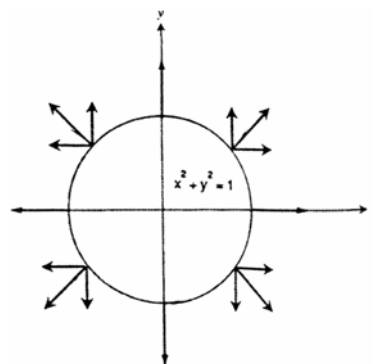
at $(-1, 0)$, $\mathbf{F} = -\mathbf{i}$; at $(0, 1)$, $\mathbf{F} = \mathbf{j}$;

at $(0, -1)$, $\mathbf{F} = -\mathbf{j}$; at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\mathbf{F} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$;

at $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\mathbf{F} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$;

at $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $\mathbf{F} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$;

at $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $\mathbf{F} = -\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$.



41. (a) $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is to have a magnitude $\sqrt{a^2 + b^2}$ and to be tangent to $x^2 + y^2 = a^2 + b^2$ in a counterclockwise direction. Thus $x^2 + y^2 = a^2 + b^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$ is the slope of the tangent line at any point on the circle $\Rightarrow y' = -\frac{a}{b}$ at (a, b) . Let $\mathbf{v} = -b\mathbf{i} + a\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{a^2 + b^2}$, with \mathbf{v} in a counterclockwise direction and tangent to the circle. Then let $P(x, y) = -y$ and $Q(x, y) = x$
 $\Rightarrow \mathbf{G} = -y\mathbf{i} + x\mathbf{j} \Rightarrow$ for (a, b) on $x^2 + y^2 = a^2 + b^2$ we have $\mathbf{G} = -b\mathbf{i} + a\mathbf{j}$ and $|\mathbf{G}| = \sqrt{a^2 + b^2}$.

(b) $\mathbf{G} = \left(\sqrt{x^2 + y^2}\right)\mathbf{F} = \left(\sqrt{a^2 + b^2}\right)\mathbf{F}$.

42. (a) From Exercise 41, part a, $-y\mathbf{i} + x\mathbf{j}$ is a vector tangent to the circle and pointing in a counterclockwise direction $\Rightarrow y\mathbf{i} - x\mathbf{j}$ is a vector tangent to the circle pointing in a clockwise direction $\Rightarrow \mathbf{G} = \frac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2 + y^2}}$ is a unit vector tangent to the circle and pointing in a clockwise direction.
- (b) $\mathbf{G} = -\mathbf{F}$

43. The slope of the line through (x, y) and the origin is $\frac{y}{x} \Rightarrow \mathbf{v} = x\mathbf{i} + y\mathbf{j}$ is a vector parallel to that line and pointing away from the origin $\Rightarrow \mathbf{F} = -\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$ is the unit vector pointing toward the origin.

44. (a) From Exercise 43, $-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ is a unit vector through (x, y) pointing toward the origin and we want $|\mathbf{F}|$ to

$$\text{have magnitude } \sqrt{x^2+y^2} \Rightarrow \mathbf{F} = \sqrt{x^2+y^2} \left(-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}} \right) = -x\mathbf{i} - y\mathbf{j}.$$

$$(b) \text{ We want } |\mathbf{F}| = \frac{C}{\sqrt{x^2+y^2}} \text{ where } C \neq 0 \text{ is a constant } \Rightarrow \mathbf{F} = \frac{C}{\sqrt{x^2+y^2}} \left(-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}} \right) = -C \left(\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}} \right).$$

45. Yes. The work and area have the same numerical value because $\text{work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y\mathbf{i} \cdot d\mathbf{r}$

$$= \int_b^a [f(t)\mathbf{i}] \cdot \left[\mathbf{i} + \frac{df}{dt} \mathbf{j} \right] dt \quad [\text{On the path, } y \text{ equals } f(t)]$$

$$= \int_a^b f(t) dt = \text{Area under the curve} \quad [\text{because } f(t) > 0]$$

46. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + f(x)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + f'(x)\mathbf{j}$; $\mathbf{F} = \frac{k}{\sqrt{x^2+y^2}}(x\mathbf{i} + y\mathbf{j})$ has constant magnitude k and points away from

$$\text{the origin } \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dx} = \frac{kx}{\sqrt{x^2+y^2}} + \frac{k \cdot y \cdot f'(x)}{\sqrt{x^2+y^2}} = \frac{kx + k \cdot f(x) \cdot f'(x)}{\sqrt{x^2 + [f(x)]^2}} = k \frac{d}{dx} \sqrt{x^2 + [f(x)]^2}, \text{ by the chain rule}$$

$$\begin{aligned} \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T} ds &= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dx} dx = \int_a^b k \frac{d}{dx} \sqrt{x^2 + [f(x)]^2} dx = k \left[\sqrt{x^2 + [f(x)]^2} \right]_a^b \\ &= k \left(\sqrt{b^2 + [f(b)]^2} - \sqrt{a^2 + [f(a)]^2} \right), \text{ as claimed.} \end{aligned}$$

47. $\mathbf{F} = -4t^3\mathbf{i} + 8t^2\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 12t^3 \Rightarrow \text{Flow} = \int_0^2 12t^3 dt = \left[3t^4 \right]_0^2 = 48$

48. $\mathbf{F} = 12t^2\mathbf{j} + 9t^2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = 3\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 72t^2 \Rightarrow \text{Flow} = \int_0^1 72t^2 dt = \left[24t^3 \right]_0^1 = 24$

49. $\mathbf{F} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + 1$

$$\Rightarrow \text{Flow} = \int_0^\pi (-\sin t \cos t + 1) dt = \left[\frac{1}{2} \cos^2 t + t \right]_0^\pi = \left(\frac{1}{2} + \pi \right) - \left(\frac{1}{2} + 0 \right) = \pi$$

50. $\mathbf{F} = (-2 \sin t)\mathbf{i} - (2 \cos t)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -4 \sin^2 t - 4 \cos^2 t + 4 = 0$
 $\Rightarrow \text{Flow} = 0$

51. $C_1: \mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, 0 \leq t \leq \frac{\pi}{2} \Rightarrow \mathbf{F} = (2 \cos t)\mathbf{i} + 2t\mathbf{j} + (2 \sin t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$

$$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -2 \cos t \sin t + 2t \cos t + 2 \sin t = -\sin 2t + 2t \cos t + 2 \sin t$$

$$\Rightarrow \text{Flow}_1 = \int_0^{\pi/2} (-\sin 2t + 2t \cos t + 2 \sin t) dt = \left[\frac{1}{2} \cos 2t + 2t \sin t + 2 \cos t - 2 \cos t \right]_0^{\pi/2} = -1 + \pi;$$

$$C_2: \mathbf{r} = \mathbf{j} + \frac{\pi}{2}(1-t)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \mathbf{F} = \pi(1-t)\mathbf{j} + 2\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = -\frac{\pi}{2}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\pi$$

$$\Rightarrow \text{Flow}_2 = \int_0^1 -\pi dt = \left[-\pi t \right]_0^1 = -\pi;$$

$$C_3: \mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \mathbf{F} = 2t\mathbf{i} + 2(1-t)\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t$$

$$\Rightarrow \text{Flow}_3 = \int_0^1 2t \, dt = \left[t^2 \right]_0^1 = 1 \Rightarrow \text{Circulation} = (-1 + \pi) - \pi + 1 = 0$$

52. $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$, where $f(x, y, z) = \frac{1}{2}(x^2, y^2 + x^2) \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(f(\mathbf{r}(t)))$ by the chain rule $\Rightarrow \text{Circulation} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_a^b \frac{d}{dt}(f(\mathbf{r}(t))) dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. Since C is an entire ellipse, $\mathbf{r}(b) = \mathbf{r}(a)$, thus the Circulation = 0.

53. Let $x = t$ be the parameter $\Rightarrow y = x^2 = t^2$ and $z = x = t \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1$ from $(0, 0, 0)$ to $(1, 1, 1)$
 $\Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$ and $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^3 - t^3 = 2t^3 \Rightarrow \text{Flow} = \int_0^1 2t^3 \, dt = \frac{1}{2}$

54. (a) $\mathbf{F} = \nabla(xy^2z^3) \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} = \frac{df}{dt}$, where $f(x, y, z) = xy^2z^3 \Rightarrow \oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_a^b \frac{d}{dt}(f(\mathbf{r}(t))) dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0$ since C is an entire ellipse.

$$(b) \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_{(1,1,1)}^{(2,1,-1)} \frac{d}{dt}(xy^2z^3) dt = \left[xy^2z^3 \right]_{(1,1,1)}^{(2,1,-1)} = (2)(1)^2(-1)^3 - (1)(1)^2(1)^3 = -2 - 1 = -3$$

55-60. Example CAS commands:

Maple:

```
with( LinearAlgebra );#55
F:=r->< r[1]*r[2]^6|3*r[1]*(r[1]*r[2]^5+2)>;
r:=t->< 2*cos(t)| sin(t) >;
a,b:= 0.2*Pi;
dr:=map(diff,r(t),t); # (a)
F(r(t)); # (b)
q1:=simplify( F(r(t)) . dr ) assuming t::real; # (c)
q2:=Int( q1, t = a..b );
value( q2 );
```

Mathematica: (functions and bounds will vary):

Exercises 55 and 56 use vectors in 2 dimensions

```
Clear[x, y, t, f, r, v]
f[x_,y_]:= {x y^6, 3x(x y^5 + 2)}
{a,b} = {0, 2 Pi};
x[t_]:= 2 Cos[t]
y[t_]:= Sin[t]
r[t_]:= {x[t], y[t]}
v[t_]:= r'[t]
integrand=f[x[t], y[t]]. v[t]/Simplify
Integrate[integrand, {t, a, b}]
N[%]
```

If the integration takes too long or cannot be done, use NIntegrate to integrate numerically. This is suggested for Exercises 57 - 60 that use vectors in 3 dimensions. Be certain to leave spaces between variables to be multiplied.

```

Clear[x, y, z, t, f, r, v]
f[x_, y_, z_] := [y + y z Cos[x, y, z], x^2 + x z Cos[x, y, z], z + x y Cos[x y z]]
[a, b] = {0, 2π};
x[t_] := 2 Cos[t]
y[t_] := 3 Sin[t]
z[t_] := 1
r[t_] := {x[t], y[t], z[t]}
v[t_] := r'[t]
integrand = f[x[t], y[t], z[t]] · v[t] / Simplify
NIntegrate[integrand (t, a, b)]

```

16.3 PATH INDEPENDENCE, POTENTIAL FUNCTIONS, AND CONSERVATIVE FIELDS

1. $\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative
2. $\frac{\partial P}{\partial y} = x \cos z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y \cos z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \sin z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative
3. $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial N}{\partial z} \Rightarrow$ Not Conservative
4. $\frac{\partial N}{\partial x} = 1 \neq -1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative
5. $\frac{\partial N}{\partial x} = 0 \neq 1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative
6. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y} \Rightarrow$ Conservative
7. $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 3y \Rightarrow g(y, z) = \frac{3y^2}{2} + h(z) \Rightarrow f(x, y, z) = x^2 + \frac{3y^2}{2} + h(z)$
 $\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 4z \Rightarrow h(z) = 2z^2 + C \Rightarrow f(x, y, z) = x^2 + \frac{3y^2}{2} + 2z^2 + C$
8. $\frac{\partial f}{\partial x} = y + z \Rightarrow f(x, y, z) = (y + z)x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x + z \Rightarrow \frac{\partial g}{\partial y} = z \Rightarrow g(y, z) = zy + h(z)$
 $\Rightarrow f(x, y, z) = (y + z)x + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = x + y + h'(z) = x + y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = (y + z)x + zy + C$
9. $\frac{\partial f}{\partial x} = e^{y+2z} \Rightarrow f(x, y, z) = xe^{y+2z} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xe^{y+2z} + \frac{\partial g}{\partial y} = xe^{y+2z} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$
 $= xe^{y+2z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2xe^{y+2z} + h'(z) = 2xe^{y+2z} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xe^{y+2z} + C$
10. $\frac{\partial f}{\partial x} = y \sin z \Rightarrow f(x, y, z) = xy \sin z + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y} = x \sin z \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$
 $\Rightarrow f(x, y, z) = xy \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy \cos z + h'(z) = xy \cos z \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xy \sin z + C$

11. $\frac{\partial f}{\partial z} = \frac{z}{y^2+z^2} \Rightarrow f(x, y, z) = \frac{1}{2} \ln(y^2+z^2) + g(x, y) \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} = \ln x + \sec^2(x+y) \Rightarrow g(x, y)$
 $= (x \ln x - x) + \tan(x+y) + h(y) \Rightarrow f(x, y, z) = \frac{1}{2} \ln(y^2+z^2) + (x \ln x - x) + \tan(x+y) + h(y)$
 $\Rightarrow \frac{\partial f}{\partial y} = \frac{y}{y^2+z^2} + \sec^2(x+y) + h'(y) = \sec^2(x+y) + \frac{y}{y^2+z^2} \Rightarrow h'(y) = 0 \Rightarrow h(y) = C \Rightarrow f(x, y, z)$
 $= \frac{1}{2} \ln(y^2+z^2) + (x \ln x - x) + \tan(x+y) + C$
12. $\frac{\partial f}{\partial x} = \frac{y}{1+x^2y^2} \Rightarrow f(x, y, z) = \tan^{-1}(xy) + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x}{1+x^2y^2} + \frac{\partial g}{\partial y} = \frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}}$
 $\Rightarrow \frac{\partial g}{\partial y} = \frac{z}{\sqrt{1-y^2z^2}} \Rightarrow g(y, z) = \sin^{-1}(yz) + h(z) \Rightarrow f(x, y, z) = \tan^{-1}(xy) + \sin^{-1}(yz) + h(z)$
 $\Rightarrow \frac{\partial f}{\partial z} = \frac{y}{\sqrt{1-y^2z^2}} + h'(z) = \frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z} \Rightarrow h'(z) = \frac{1}{z} \Rightarrow h(z) = \ln|z| + C$
 $\Rightarrow f(x, y, z) = \tan^{-1}(xy) + \sin^{-1}(yz) + \ln|z| + C$
13. Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz$ is exact;
 $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y \Rightarrow g(y, z) = y^2 + h(z) \Rightarrow f(x, y, z) = x^2 + y^2 + h(z)$
 $\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 2z \Rightarrow h(z) = z^2 + C \Rightarrow f(x, y, z) = x^2 + y^2 + z^2 + C \Rightarrow \int_{(0,0,0)}^{(2,3,-6)} 2x dx + 2y dy + 2z dz$
 $= f(2, 3, -6) - f(0, 0, 0) = 2^2 + 3^2 + (-6)^2 = 49$
14. Let $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz$ is exact;
 $\frac{\partial f}{\partial x} = yz \Rightarrow f(x, y, z) = xyz + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$
 $\Rightarrow f(x, y, z) = xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy + h'(z) = xy \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xyz + C$
 $\Rightarrow \int_{(1,1,2)}^{(3,5,0)} yz dx + xz dy + xy dz = f(3, 5, 0) - f(1, 1, 2) = 0 - 2 = -2$
15. Let $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 - z^2)\mathbf{j} - 2yz\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -2z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 2x = \frac{\partial M}{\partial y}$
 $\Rightarrow M dx + N dy + P dz$ is exact; $\frac{\partial f}{\partial x} = 2xy \Rightarrow f(x, y, z) = x^2y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 - z^2 \Rightarrow \frac{\partial g}{\partial y} = -z^2$
 $\Rightarrow g(y, z) = -yz^2 + h(z) \Rightarrow f(x, y, z) = x^2y - yz^2 + h(z) \Rightarrow \frac{\partial f}{\partial z} = -2yz + h'(z) = -2yz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$
 $\Rightarrow f(x, y, z) = x^2y - yz^2 + C \Rightarrow \int_{(0,0,0)}^{(1,2,3)} 2xy dx + (x^2 - z^2) dy - 2yz dz = f(1, 2, 3) - f(0, 0, 0) = 2 - 2(3)^2$
 $= -16$
16. Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} - y^2\mathbf{j} - \left(\frac{4}{1+z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$
 $\Rightarrow M dx + N dy + P dz$ is exact; $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -y^2 \Rightarrow g(y, z) = -\frac{y^3}{3} + h(z)$
 $\Rightarrow f(x, y, z) = x^2 - \frac{y^3}{3} + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = -\frac{4}{1+z^2} \Rightarrow h(z) = -4 \tan^{-1} z + C$

$$\begin{aligned}\Rightarrow f(x, y, z) &= x^2 - \frac{y^3}{3} - 4 \tan^{-1} z + C \Rightarrow \int_{(0,0,0)}^{(3,3,1)} 2x \, dx - y^2 \, dy - \frac{4}{1+z^2} \, dz = f(3, 3, 1) - f(0, 0, 0) \\ &= \left(9 - \frac{27}{3} - 4 \cdot \frac{\pi}{4}\right) - (0 - 0 - 0) = -\pi\end{aligned}$$

$$\begin{aligned}17. \text{ Let } \mathbf{F}(x, y, z) &= (\sin y \cos x)\mathbf{i} + (\cos y \sin x)\mathbf{j} + \mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \cos y \cos x = \frac{\partial M}{\partial y} \\ \Rightarrow M \, dx + N \, dy + P \, dz &\text{ is exact; } \frac{\partial f}{\partial x} = \sin y \cos x \Rightarrow f(x, y, z) = \sin y \sin x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \cos y \sin x + \frac{\partial g}{\partial y} \\ &= \cos y \sin x \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = \sin y \sin x + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 1 \Rightarrow h(z) = z + C \\ \Rightarrow f(x, y, z) &= \sin y \sin x + z + C \Rightarrow \int_{(1,0,0)}^{(0,1,1)} \sin y \cos x \, dx + \cos y \sin x \, dy + dz = f(0, 1, 1) - f(1, 0, 0) \\ &= (0 + 1) - (0 + 0) = 1\end{aligned}$$

$$\begin{aligned}18. \text{ Let } \mathbf{F}(x, y, z) &= (2 \cos y)\mathbf{i} + \left(\frac{1}{y} - 2x \sin y\right)\mathbf{j} + \left(\frac{1}{z}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2 \sin y = \frac{\partial M}{\partial y} \\ \Rightarrow M \, dx + N \, dy + P \, dz &\text{ is exact; } \frac{\partial f}{\partial x} = 2 \cos y \Rightarrow f(x, y, z) = 2x \cos y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -2x \sin y + \frac{\partial g}{\partial y} \\ &= \frac{1}{y} - 2x \sin y \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y} \Rightarrow g(y, z) = \ln |y| + h(z) \Rightarrow f(x, y, z) = 2x \cos y + \ln |y| + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = \frac{1}{z} \\ \Rightarrow h(z) &= \ln |z| + C \Rightarrow f(x, y, z) = 2x \cos y + \ln |y| + \ln |z| + C \\ \Rightarrow \int_{(0,2,1)}^{(1,\pi/2,2)} 2 \cos y \, dx + \left(\frac{1}{y} - 2x \sin y\right) dy + \frac{1}{z} \, dz &= f\left(1, \frac{\pi}{2}, 2\right) - f(0, 2, 1) \\ &= \left(2 \cdot 0 + \ln \frac{\pi}{2} + \ln 2\right) - (0 \cdot \cos 2 + \ln 2 + \ln 1) = \ln \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}19. \text{ Let } \mathbf{F}(x, y, z) &= 3x^2\mathbf{i} + \left(\frac{z^2}{y}\right)\mathbf{j} + (2z \ln y)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = \frac{2z}{y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \\ \Rightarrow M \, dx + N \, dy + P \, dz &\text{ is exact; } \frac{\partial f}{\partial x} = 3x^2 \Rightarrow f(x, y, z) = x^3 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = \frac{z^2}{y} \Rightarrow g(y, z) = z^2 \ln y + h(z) \\ \Rightarrow f(x, y, z) &= x^3 + z^2 \ln y + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2z \ln y + h'(z) = 2z \ln y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \\ \Rightarrow f(x, y, z) &= x^3 + z^2 \ln y + C \Rightarrow \int_{(1,1,1)}^{(1,2,3)} 3x^2 \, dx + \frac{z^2}{y} \, dy + 2z \ln y \, dz = f(1, 2, 3) - f(1, 1, 1) \\ &= (1 + 9 \ln 2 + C) - (1 + 0 + C) = 9 \ln 2\end{aligned}$$

$$\begin{aligned}20. \text{ Let } \mathbf{F}(x, y, z) &= (2x \ln y - yz)\mathbf{i} + \left(\frac{x^2}{y} - xz\right)\mathbf{j} - (xy)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{2x}{y} - z = \frac{\partial M}{\partial y} \\ \Rightarrow M \, dx + N \, dy + P \, dz &\text{ is exact; } \frac{\partial f}{\partial x} = 2x \ln y - yz \Rightarrow f(x, y, z) = x^2 \ln y - xyz + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial g}{\partial y} \\ &= \frac{x^2}{y} - xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = x^2 \ln y - xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = -xy + h'(z) = -xy \Rightarrow h'(z) = 0 \\ \Rightarrow h(z) &= C \Rightarrow f(x, y, z) = x^2 \ln y - xyz + C \Rightarrow \int_{(1,2,1)}^{(2,1,1)} (2x \ln y - yz) \, dx + \left(\frac{x^2}{y} - xz\right) dy - xy \, dz \\ &= f(2, 1, 1) - f(1, 2, 1) = (4 \ln 1 - 2 + C) - (\ln 2 - 2 + C) = -\ln 2\end{aligned}$$

$$\begin{aligned}21. \text{ Let } \mathbf{F}(x, y, z) &= \left(\frac{1}{y}\right)\mathbf{i} + \left(\frac{1}{z} - \frac{x}{y^2}\right)\mathbf{j} - \left(\frac{y}{z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -\frac{1}{z^2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{y^2} = \frac{\partial M}{\partial y} \\ \Rightarrow M \, dx + N \, dy + P \, dz &\text{ is exact; } \frac{\partial f}{\partial x} = \frac{1}{y} \Rightarrow f(x, y, z) = \frac{x}{y} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{\partial g}{\partial y} = \frac{1}{z} - \frac{x}{y^2}\end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{z} \Rightarrow g(y, z) = \frac{y}{z} + h(z) \Rightarrow f(x, y, z) = \frac{x}{y} + \frac{y}{z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = -\frac{y}{z^2} + h'(z) = -\frac{y}{z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \\ \Rightarrow f(x, y, z) = \frac{x}{y} + \frac{y}{z} + C \Rightarrow \int_{(1,1,1)}^{(2,2,2)} \frac{1}{y} dx + \left(\frac{1}{z} - \frac{x}{y^2}\right) dy - \frac{y}{z^2} dz = f(2, 2, 2) - f(1, 1, 1) = \left(\frac{2}{2} + \frac{2}{2} + C\right) - \left(\frac{1}{1} + \frac{1}{1} + C\right) \\ = 0 \end{aligned}$$

22. Let $\mathbf{F}(x, y, z) = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{x^2 + y^2 + z^2}$ (and let $\rho^2 = x^2 + y^2 + z^2 \Rightarrow \frac{\partial \rho}{\partial x} = \frac{x}{\rho}, \frac{\partial \rho}{\partial y} = \frac{y}{\rho}, \frac{\partial \rho}{\partial z} = \frac{z}{\rho}$)
 $\Rightarrow \frac{\partial P}{\partial y} = -\frac{4yz}{\rho^4} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{4xz}{\rho^4} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{4xy}{\rho^4} = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz$ is exact;
 $\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2 + z^2} \Rightarrow f(x, y, z) = \ln(x^2 + y^2 + z^2) + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2 + z^2} + \frac{\partial g}{\partial y} = \frac{2y}{x^2 + y^2 + z^2}$
 $\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = \ln(x^2 + y^2 + z^2) + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{2z}{x^2 + y^2 + z^2} + h'(z)$
 $= \frac{2z}{x^2 + y^2 + z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = \ln(x^2 + y^2 + z^2) + C$
 $\Rightarrow \int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x dx + 2y dy + 2z dz}{x^2 + y^2 + z^2} = f(2, 2, 2) - f(-1, -1, -1) = \ln 12 - \ln 3 = \ln 4$
23. $\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = (1+t)\mathbf{i} + (1+2t)\mathbf{j} + (1-2t)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow dx = dt, dy = 2 dt, dz = -2 dt$
 $\Rightarrow \int_{(1,1,1)}^{(2,3,-1)} y dx + x dy + 4 dz = \int_0^1 (2t+1) dt + (t+1)(2 dt) + 4(-2) dt = \int_0^1 (4t-5) dt = \left[2t^2 - 5t\right]_0^1 = -3$
24. $\mathbf{r} = t(3\mathbf{j} + 4\mathbf{k}), 0 \leq t \leq 1 \Rightarrow dx = 0, dy = 3 dt, dz = 4 dt \Rightarrow \int_{(0,0,0)}^{(0,3,4)} x^2 dx + yz dy + \left(\frac{y^2}{2}\right) dz$
 $= \int_0^1 (12t^2) (3 dt) + \left(\frac{9t^2}{2}\right) (4 dt) = \int_0^1 54t^2 dt = \left[18t^3\right]_0^1 = 18$
25. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz$ is exact $\Rightarrow \mathbf{F}$ is conservative
 \Rightarrow path independence
26. $\frac{\partial P}{\partial y} = -\frac{yz}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{xz}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{xy}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{\partial M}{\partial y}$
 $\Rightarrow M dx + N dy + P dz$ is exact $\Rightarrow \mathbf{F}$ is conservative \Rightarrow path independence
27. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{2x}{y^2} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative \Rightarrow there exists an f so that $\mathbf{F} = \nabla f$;
 $\frac{\partial f}{\partial x} = \frac{2x}{y} \Rightarrow f(x, y) = \frac{x^2}{y} + g(y) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2} \Rightarrow g'(y) = \frac{1}{y^2} \Rightarrow g(y) = -\frac{1}{y} + C$
 $\Rightarrow f(x, y) = \frac{x^2}{y} - \frac{1}{y} + C \Rightarrow \mathbf{F} = \nabla\left(\frac{x^2-1}{y}\right)$
28. $\frac{\partial P}{\partial y} = \cos z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{e^x}{y} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative \Rightarrow there exists an f so that $\mathbf{F} = \nabla f$;
 $\frac{\partial f}{\partial x} = e^x \ln y \Rightarrow f(x, y, z) = e^x \ln y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{e^x}{y} + \frac{\partial g}{\partial y} = \frac{e^x}{y} + \sin z \Rightarrow \frac{\partial g}{\partial y} = \sin z \Rightarrow g(y, z)$

$$= y \sin z + h(z) \Rightarrow f(x, y, z) = e^x \ln y + y \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = y \cos z + h'(z) = y \cos z \Rightarrow h'(z) = 0 \\ \Rightarrow h(z) = C \Rightarrow f(x, y, z) = e^x \ln y + y \sin z + C \Rightarrow \mathbf{F} = \nabla(e^x \ln y + y \sin z)$$

29. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial y} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative \Rightarrow there exists an f so that $\mathbf{F} = \nabla f$;
 $\frac{\partial f}{\partial x} = x^2 + y \Rightarrow f(x, y, z) = \frac{1}{3}x^3 + xy + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = y^2 + x \Rightarrow \frac{\partial g}{\partial y} = y^2 \Rightarrow g(y, z) = \frac{1}{3}y^3 + h(z)$
 $\Rightarrow f(x, y, z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = ze^z \Rightarrow h(z) = ze^z - e^z + C$
 $\Rightarrow f(x, y, z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z + C \Rightarrow \mathbf{F} = \nabla\left(\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right)$
- (a) work $= \int_A^B \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = \left(\frac{1}{3} + 0 + 0 + e - e\right) - \left(\frac{1}{3} + 0 + 0 - 1\right) = 1$
- (b) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$
- (c) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

Note: Since \mathbf{F} is conservative, $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from $(1, 0, 0)$ to $(1, 0, 1)$.

30. $\frac{\partial P}{\partial y} = xe^{yz} + xye^{yz} + \cos y = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = ye^{yz} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = ze^{yz} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative \Rightarrow there exists an f so
that $\mathbf{F} = \nabla f$; $\frac{\partial f}{\partial x} = e^{yz} \Rightarrow f(x, y, z) = xe^{yz} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xze^{yz} + \frac{\partial g}{\partial y} = xze^{yz} + z \cos y \Rightarrow \frac{\partial g}{\partial y} = z \cos y$
 $\Rightarrow g(y, z) = z \sin y + h(z) \Rightarrow f(x, y, z) = xe^{yz} + z \sin y + h(z) \Rightarrow \frac{\partial f}{\partial z} = xye^{yz} + \sin y + h'(z) = xye^{yz} + \sin y$
 $\Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xe^{yz} + z \sin y + C \Rightarrow \mathbf{F} = \nabla(xe^{yz} + z \sin y)$
- (a) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[xe^{yz} + z \sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = (1+0) - (1+0) = 0$
- (b) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[xe^{yz} + z \sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = 0$
- (c) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[xe^{yz} + z \sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = 0$

Note: Since \mathbf{F} is conservative, $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from $(1, 0, 1)$ to $(1, \frac{\pi}{2}, 0)$.

31. (a) $\mathbf{F} = \nabla(x^3 y^2) \Rightarrow \mathbf{F} = 3x^2 y^2 \mathbf{i} + 2x^3 y \mathbf{j}$; let C_1 be the path from $(-1, 1)$ to $(0, 0) \Rightarrow x = t - 1$ and $y = -t + 1, 0 \leq t \leq 1 \Rightarrow \mathbf{F} = 3(t-1)^2(-t+1)^2 \mathbf{i} + 2(t-1)^3(-t+1) \mathbf{j} = 3(t-1)^4 \mathbf{i} - 2(t-1)^4 \mathbf{j}$ and $\mathbf{r}_1 = (t-1)\mathbf{i} + (-t+1)\mathbf{j} \Rightarrow d\mathbf{r}_1 = dt \mathbf{i} - dt \mathbf{j} \Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_0^1 [3(t-1)^4 + 2(t-1)^4] dt$
 $= \int_0^1 5(t-1)^4 dt = \left[(t-1)^5\right]_0^1 = 1$; let C_2 be the path from $(0, 0)$ to $(1, 1) \Rightarrow x = t$ and $y = t,$
 $0 \leq t \leq 1 \Rightarrow \mathbf{F} = 3t^4 \mathbf{i} + 2t^4 \mathbf{j}$ and $\mathbf{r}_2 = t\mathbf{i} + t\mathbf{j} \Rightarrow d\mathbf{r}_2 = dt \mathbf{i} + dt \mathbf{j} \Rightarrow \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = \int_0^1 (3t^4 + 2t^4) dt$
 $= \int_0^1 5t^4 dt = 1 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = 2$

- (b) Since $f(x, y) = x^3 y^2$ is a potential function for \mathbf{F} , $\int_{(-1,1)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(-1, 1) = 2$
32. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = -2x \sin y = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative \Rightarrow there exists an f so that $\mathbf{F} = \nabla f$;
 $\frac{\partial f}{\partial x} = 2x \cos y \Rightarrow f(x, y, z) = x^2 \cos y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -x^2 \sin y + \frac{\partial g}{\partial y} = -x^2 \sin y \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$
 $\Rightarrow f(x, y, z) = x^2 \cos y + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = x^2 \cos y + C \Rightarrow \mathbf{F} = \nabla(x^2 \cos y)$
- (a) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = \left[x^2 \cos y \right]_{(1,0)}^{(0,1)} = 0 - 1 = -1$
- (b) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = \left[x^2 \cos y \right]_{(-1,\pi)}^{(1,0)} = 1 - (-1) = 2$
- (c) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = \left[x^2 \cos y \right]_{(-1,0)}^{(1,0)} = 1 - 1 = 0$
- (d) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = \left[x^2 \cos y \right]_{(1,0)}^{(1,0)} = 1 - 1 = 0$
33. (a) If the differential form is exact, then $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \Rightarrow 2ay = cy$ for all $y \Rightarrow 2a = c$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow 2cx = 2cx$ for all x , and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow by = 2ay$ for all $y \Rightarrow b = 2a$ and $c = 2a$
- (b) $\mathbf{F} = \nabla f \Rightarrow$ the differential form with $a = 1$ in part (a) is exact $\Rightarrow b = 2$ and $c = 2$
34. $\mathbf{F} = \nabla f \Rightarrow g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(x,y,z)} \nabla f \cdot d\mathbf{r} = f(x, y, z) - f(0, 0, 0) \Rightarrow \frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} - 0$, $\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} - 0$, and $\frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} - 0 \Rightarrow \nabla g = \nabla f = \mathbf{F}$, as claimed
35. The path will not matter; the work along any path will be the same because the field is conservative.
36. The field is not conservative, for otherwise the work would be the same along C_1 and C_2 .
37. Let the coordinates of points A and B be (x_A, y_A, z_A) and (x_B, y_B, z_B) , respectively. The force $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is conservative because all the partial derivatives of M , N , and P are zero. Therefore, the potential function is $f(x, y, z) = ax + by + cz + C$, and the work done by the force in moving a particle along any path from A to B is $f(B) - f(A) = f(x_B, y_B, z_B) - f(x_A, y_A, z_A)$
 $= (ax_B + by_B + cz_B + C) - (ax_A + by_A + cz_A + C) = a(x_B - x_A) + b(y_B - y_A) + c(z_B - z_A) = \mathbf{F} \cdot \overrightarrow{BA}$
38. (a) Let $-GmM = C \Rightarrow \mathbf{F} = C \left[\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{y}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k} \right]$
 $\Rightarrow \frac{\partial P}{\partial y} = \frac{-3yzC}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{-3xzC}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \frac{-3xyC}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} = \nabla f$ for some f ;
 $\frac{\partial f}{\partial x} = \frac{xC}{(x^2 + y^2 + z^2)^{3/2}} \Rightarrow f(x, y, z) = -\frac{C}{(x^2 + y^2 + z^2)^{1/2}} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{yC}{(x^2 + y^2 + z^2)^{3/2}} + \frac{\partial g}{\partial y}$

$$= \frac{yC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{zC}{(x^2+y^2+z^2)^{3/2}} + h'(z) = \frac{zC}{(x^2+y^2+z^2)^{3/2}} \\ \Rightarrow h(z) = C_1 \Rightarrow f(x, y, z) = -\frac{C}{(x^2+y^2+z^2)^{1/2}} + C_1. \text{ Let } C_1 = 0 \Rightarrow f(x, y, z) = \frac{GmM}{(x^2+y^2+z^2)^{1/2}} \text{ is a potential} \\ \text{function for } \mathbf{F}.$$

- (b) If s is the distance of (x, y, z) from the origin, then $s = \sqrt{x^2 + y^2 + z^2}$. The work done by the gravitational field \mathbf{F} is work $= \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \left[\frac{GmM}{\sqrt{x^2+y^2+z^2}} \right]_{P_1}^{P_2} = \frac{GmM}{s_2} - \frac{GmM}{s_1} = GmM \left(\frac{1}{s_2} - \frac{1}{s_1} \right)$, as claimed.

16.4 GREEN'S THEOREM IN THE PLANE

1. $M = -y = -a \sin t$, $N = x = a \cos t$, $dx = -a \sin t \, dt$, $dy = a \cos t \, dt \Rightarrow \frac{\partial M}{\partial y} = 0$, $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$, and $\frac{\partial N}{\partial y} = 0$;

$$\text{Equation (3): } \oint_C M \, dy - N \, dx = \int_0^{2\pi} [(-a \sin t)(a \cos t) - (a \cos t)(-a \sin t)] \, dt = \int_0^{2\pi} 0 \, dt = 0;$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy = \iint_R 0 \, dx \, dy = 0, \text{ Flux}$$

$$\text{Equation (4): } \oint_C M \, dx + N \, dy = \int_0^{2\pi} [(-a \sin t)(-a \sin t) - (a \cos t)(a \cos t)] \, dt = \int_0^{2\pi} a^2 \, dt = 2\pi a^2;$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy = \int_{-a}^a \int_{-c}^{\sqrt{a^2-x^2}} 2 \, dy \, dx = \int_{-a}^a 4\sqrt{a^2-x^2} \, dx = 4 \left[\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_{-a}^a \\ = 2a^2 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 2a^2 \pi, \text{ Circulation}$$

2. $M = y = a \sin t$, $N = 0$, $dx = -a \sin t \, dt$, $dy = a \cos t \, dt \Rightarrow \frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = 0$;

$$\text{Equation (3): } \oint_C M \, dy - N \, dx = \int_0^{2\pi} a^2 \sin t \cos t \, dt = a^2 \left[\frac{1}{2} \sin^2 t \right]_0^{2\pi} = 0; \iint_R 0 \, dx \, dy = 0, \text{ Flux}$$

$$\text{Equation (4): } \oint_C M \, dx + N \, dy = \int_0^{2\pi} (-a^2 \sin^2 t) \, dt = -a^2 \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = -\pi a^2;$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy = \iint_R -1 \, dx \, dy = \int_0^{2\pi} \int_0^a -r \, dr \, d\theta = \int_0^{2\pi} -\frac{a^2}{2} \, d\theta = -\pi a^2, \text{ Circulation}$$

3. $M = 2x = 2a \cos t$, $N = -3y = -3a \sin t$, $dx = -a \sin t \, dt$, $dy = a \cos t \, dt \Rightarrow \frac{\partial M}{\partial x} = 2$, $\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = -3$;

$$\text{Equation (3): } \oint_C M \, dy - N \, dx = \int_0^{2\pi} [(2a \cos t)(a \cos t) + (3a \sin t)(-a \sin t)] \, dt \\ = \int_0^{2\pi} (2a^2 \cos^2 t - 3a^2 \sin^2 t) \, dt = 2a^2 \left[\frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{2\pi} - 3a^2 \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = 2\pi a^2 - 3\pi a^2 = -\pi a^2;$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy = \iint_R -1 \, dx \, dy = \int_0^{2\pi} \int_0^a -r \, dr \, d\theta = \int_0^{2\pi} -\frac{a^2}{2} \, d\theta = -\pi a^2, \text{ Flux}$$

$$\text{Equation (4): } \oint_C M \, dx + N \, dy = \int_0^{2\pi} [(2a \cos t)(-a \sin t) + (-3a \sin t)(a \cos t)] \, dt \\ = \int_0^{2\pi} (-2a^2 \sin t \cos t - 3a^2 \sin t \cos t) \, dt = -5a^2 \left[\frac{1}{2} \sin^2 t \right]_0^{2\pi} = 0; \iint_R 0 \, dx \, dy = 0, \text{ Circulation}$$

$$4. \quad M = -x^2y = -a^3 \cos^2 t, \quad N = xy^2 = a^3 \cos t \sin^2 t, \quad dx = -a \sin t \, dt, \quad dy = a \cos t \, dt \\ \Rightarrow \frac{\partial M}{\partial x} = -2xy, \quad \frac{\partial M}{\partial y} = -x^2, \quad \frac{\partial N}{\partial x} = y^2, \quad \text{and} \quad \frac{\partial N}{\partial y} = 2xy;$$

$$\text{Equation (3):} \quad \oint_C M \, dy - N \, dx = \int_0^{2\pi} \left(-a^4 \cos^3 t \sin t + a^4 \cos t \sin^3 t \right) dt = \left[\frac{a^4}{4} \cos^4 t + \frac{a^4}{4} \sin^4 t \right]_0^{2\pi} = 0;$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy = \iint_R (-2xy + 2xy) \, dx \, dy = 0, \text{ Flux}$$

$$\text{Equation (4):} \quad \oint_C M \, dx + N \, dy = \int_0^{2\pi} \left(a^4 \cos^2 t \sin^2 t + a^4 \cos^2 t \sin^2 t \right) dt = \int_0^{2\pi} \left(2a^4 \cos^2 t \sin^2 t \right) dt \\ = \int_0^{2\pi} \frac{1}{2} a^4 \sin^2 2t \, dt = \frac{a^4}{4} \int_0^{4\pi} \sin^2 u \, du = \frac{a^4}{4} \left[\frac{u}{2} - \frac{\sin 2u}{4} \right]_0^{4\pi} = \frac{\pi a^4}{2}; \quad \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy = \iint_R (y^2 + x^2) \, dx \, dy \\ = \int_0^{2\pi} \int_0^a r^2 \cdot r \, dr \, d\theta = \int_0^{2\pi} \frac{a^4}{4} d\theta = \frac{\pi a^4}{2}, \text{ Circulation}$$

$$5. \quad M = x - y, \quad N = y - x \Rightarrow \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = -1, \quad \frac{\partial N}{\partial y} = 1 \Rightarrow \text{Flux} = \iint_R 2 \, dx \, dy = \int_0^1 \int_0^1 2 \, dx \, dy = 2;$$

$$\text{Circ} = \iint_R [-1 - (-1)] \, dx \, dy = 0$$

$$6. \quad M = x^2 + 4y, \quad N = x + y^2 \Rightarrow \frac{\partial M}{\partial x} = 2x, \quad \frac{\partial M}{\partial y} = 4, \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (2x + 2y) \, dx \, dy \\ = \int_0^1 \int_0^1 (2x + 2y) \, dx \, dy = \int_0^1 \left[x^2 + 2xy \right]_0^1 dy = \int_0^1 (1 + 2y) \, dy = \left[y + y^2 \right]_0^1 = 2; \text{ Circ} = \iint_R (1 - 4) \, dx \, dy \\ = \int_0^1 \int_0^1 -3 \, dx \, dy = -3$$

$$7. \quad M = y^2 - x^2, \quad N = x^2 + y^2 \Rightarrow \frac{\partial M}{\partial x} = -2x, \quad \frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2x, \quad \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (-2x + 2y) \, dx \, dy \\ = \int_0^3 \int_0^x (-2x + 2y) \, dy \, dx = \int_0^3 \left(-2x^2 + x^2 \right) dx = \left[-\frac{1}{3} x^3 \right]_0^3 = -9; \text{ Circ} = \iint_R (2x - 2y) \, dx \, dy \\ = \int_0^3 \int_0^x (2x - 2y) \, dy \, dx = \int_0^3 x^2 \, dx = 9$$

$$8. \quad M = x + y, \quad N = -(x^2 + y^2) \Rightarrow \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -2x, \quad \frac{\partial N}{\partial y} = -2y \Rightarrow \text{Flux} = \iint_R (1 - 2y) \, dx \, dy \\ = \int_0^1 \int_0^x (1 - 2y) \, dy \, dx = \int_0^1 \left(x - x^2 \right) dx = \frac{1}{6}; \text{ Circ} = \iint_R (-2x - 1) \, dx \, dy = \int_0^1 \int_0^x (-2x - 1) \, dy \, dx \\ = \int_0^1 (-2x^2 - x) \, dx = -\frac{7}{6}$$

$$9. \quad M = xy + y^2, \quad N = x - y \Rightarrow \frac{\partial M}{\partial x} = y, \quad \frac{\partial M}{\partial y} = x + 2y, \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = -1 \Rightarrow \text{Flux} = \iint_R (y + (-1)) \, dy \, dx \\ = \int_0^1 \int_{x^2}^{\sqrt{x}} (y - 1) \, dy \, dx = \int_0^1 \left(\frac{1}{2} x - \sqrt{x} - \frac{1}{2} x^4 + x^2 \right) dx = -\frac{11}{60}; \text{ Circ} = \iint_R (1 - (x + 2y)) \, dy \, dx \\ = \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - 2y) \, dy \, dx = \int_0^1 \left(\sqrt{x} - x^{3/2} - x - x^2 + x^3 + x^4 \right) dx = -\frac{7}{60}$$

$$10. \quad M = x + 3y, N = 2x - y \Rightarrow \frac{\partial M}{\partial x} = 1, \frac{\partial M}{\partial y} = 3, \frac{\partial N}{\partial x} = 2, \frac{\partial N}{\partial y} = -1 \Rightarrow \text{Flux} = \iint_R (1 + (-1)) \, dy \, dx = 0$$

$$\text{Circ} = \iint_R (2 - 3) \, dy \, dx = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{(2-x^2)/2}}^{\sqrt{(2-x^2)/2}} (-1) \, dy \, dx = -\frac{2}{\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2} \, dx = -\pi\sqrt{2}$$

$$11. \quad M = x^3 y^2, N = \frac{1}{2} x^4 y \Rightarrow \frac{\partial M}{\partial x} = 3x^2 y^2, \frac{\partial M}{\partial y} = 2x^3 y, \frac{\partial N}{\partial x} = 2x^3 y, \frac{\partial N}{\partial y} = \frac{1}{2} x^4 \Rightarrow \text{Flux} = \iint_R \left(3x^2 y^2 + \frac{1}{2} x^4 \right) \, dy \, dx \\ = \int_0^2 \int_{x^2-x}^x \left(3x^2 y^2 + \frac{1}{2} x^4 \right) \, dy \, dx = \int_0^2 \left(3x^5 - \frac{7}{2} x^6 + 3x^7 - x^8 \right) \, dx = \frac{64}{9}; \text{Circ} = \iint_R (2x^3 y - 2x^3 y) \, dy \, dx = 0$$

$$12. \quad M = \frac{x}{1+y^2}, N = \tan^{-1} y \Rightarrow \frac{\partial M}{\partial x} = \frac{1}{1+y^2}, \frac{\partial M}{\partial y} = \frac{-2xy}{(1+y^2)^2}, \frac{\partial N}{\partial x} = 0, \frac{\partial N}{\partial y} = \frac{1}{1+y^2} \Rightarrow \text{Flux} = \iint_R \left(\frac{1}{1+y^2} + \frac{1}{1+y^2} \right) \, dx \, dy \\ = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{1+y^2} \, dx \, dy = \int_{-1}^1 \frac{4\sqrt{1-y^2}}{1+y^2} \, dy = 4\pi\sqrt{2} - 4\pi; \text{Circ} = \iint_R \left(0 - \left(\frac{-2xy}{(1+y^2)^2} \right) \right) \, dy \, dx \\ = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\frac{2xy}{(1+y^2)^2} \right) \, dx \, dy = \int_{-1}^1 (0) \, dy = 0$$

$$13. \quad M = x + e^x \sin y, N = x + e^x \cos y \Rightarrow \frac{\partial M}{\partial x} = 1 + e^x \sin y, \frac{\partial M}{\partial y} = e^x \cos y, \frac{\partial N}{\partial x} = 1 + e^x \cos y, \frac{\partial N}{\partial y} = -e^x \sin y \\ \Rightarrow \text{Flux} = \iint_R dx \, dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2} \cos 2\theta \right) \, d\theta = \left[\frac{1}{4} \sin 2\theta \right]_{-\pi/4}^{\pi/4} = \frac{1}{2}; \\ \text{Circ} = \iint_R (1 + e^x \cos y - e^x \cos y) \, dx \, dy = \iint_R dx \, dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2} \cos 2\theta \right) \, d\theta = \frac{1}{2}$$

$$14. \quad M = \tan^{-1} \frac{y}{x}, N = \ln(x^2 + y^2) \Rightarrow \frac{\partial M}{\partial x} = \frac{-y}{x^2 + y^2}, \frac{\partial M}{\partial y} = \frac{x}{x^2 + y^2}, \frac{\partial N}{\partial x} = \frac{2x}{x^2 + y^2}, \frac{\partial N}{\partial y} = \frac{2y}{x^2 + y^2} \\ \Rightarrow \text{Flux} = \iint_R \left(\frac{-y}{x^2 + y^2} + \frac{2y}{x^2 + y^2} \right) \, dx \, dy = \int_0^\pi \int_1^2 \left(\frac{r \sin \theta}{r^2} \right) r \, dr \, d\theta = \int_0^\pi \sin \theta \, d\theta = 2; \\ \text{Circ} = \iint_R \left(\frac{2x}{x^2 + y^2} - \frac{x}{x^2 + y^2} \right) \, dx \, dy = \int_0^\pi \int_1^2 \left(\frac{r \cos \theta}{r^2} \right) r \, dr \, d\theta = \int_0^\pi \cos \theta \, d\theta = 0$$

$$15. \quad M = xy, N = y^2 \Rightarrow \frac{\partial M}{\partial x} = y, \frac{\partial M}{\partial y} = x, \frac{\partial N}{\partial x} = 0, \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (y + 2y) \, dy \, dx = \int_0^1 \int_{x^2}^x 3y \, dy \, dx \\ = \int_0^1 \left(\frac{3x^2}{2} - \frac{3x^4}{2} \right) \, dx = \frac{1}{5}; \text{Circ} = \iint_R -x \, dy \, dx = \int_0^1 \int_{x^2}^x -x \, dy \, dx = \int_0^1 (-x^2 + x^3) \, dx = -\frac{1}{12}$$

$$16. \quad M = -\sin y, N = x \cos y \Rightarrow \frac{\partial M}{\partial x} = 0, \frac{\partial M}{\partial y} = -\cos y, \frac{\partial N}{\partial x} = \cos y, \frac{\partial N}{\partial y} = -x \sin y \\ \Rightarrow \text{Flux} = \iint_R (-x \sin y) \, dx \, dy = \int_0^{\pi/2} \int_0^{\pi/2} (-x \sin y) \, dx \, dy = \int_0^{\pi/2} \left(-\frac{\pi^2}{8} \sin y \right) \, dy = -\frac{\pi^2}{8}; \\ \text{Circ} = \iint_R [\cos y - (-\cos y)] \, dx \, dy = \int_0^{\pi/2} \int_0^{\pi/2} 2 \cos y \, dx \, dy = \int_0^{\pi/2} \pi \cos y \, dy = [\pi \sin y]_0^{\pi/2} = \pi$$

17. $M = 3xy - \frac{x}{1+y^2}$, $N = e^x + \tan^{-1} y \Rightarrow \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2}$, $\frac{\partial N}{\partial y} = \frac{1}{1+y^2}$
 $\Rightarrow \text{Flux} = \iint_R \left(3y - \frac{1}{1+y^2} + \frac{1}{1+y^2} \right) dx dy = \iint_R 3y dx dy = \int_0^{2\pi} \int_0^{a(1+\cos \theta)} (3r \sin \theta) r dr d\theta$
 $= \int_0^{2\pi} a^3 (1+\cos \theta)^3 (\sin \theta) d\theta = \left[-\frac{a^3}{4} (1+\cos \theta)^4 \right]_0^{2\pi} = -4a^3 - (-4a^3) = 0$
18. $M = y + e^x \ln y$, $N = \frac{e^x}{y} \Rightarrow \frac{\partial M}{\partial y} = 1 + \frac{e^x}{y}$, $\frac{\partial N}{\partial x} = \frac{e^x}{y} \Rightarrow \text{Circ} = \iint_R \left[\frac{e^x}{y} - \left(1 + \frac{e^x}{y} \right) \right] dx dy = \iint_R (-1) dx dy$
 $= \int_{-1}^1 \int_{x^4+1}^{3-x^2} -dy dx = -\int_{-1}^1 \left[(3-x^2) - (x^4+1) \right] dx = \int_{-1}^1 (x^4 + x^2 - 2) dx = -\frac{44}{15}$
19. $M = 2xy^3$, $N = 4x^2y^2 \Rightarrow \frac{\partial M}{\partial y} = 6xy^2$, $\frac{\partial N}{\partial x} = 8xy^2 \Rightarrow \text{work} = \oint_C 2xy^3 dx + 4x^2y^2 dy = \iint_R (8xy^2 - 6xy^2) dx dy$
 $= \int_0^1 \int_0^{x^3} 2xy^2 dy dx = \int_0^1 \frac{2}{3} x^{10} dx = \frac{2}{33}$
20. $M = 4x - 2y$, $N = 2x - 4y \Rightarrow \frac{\partial M}{\partial y} = -2$, $\frac{\partial N}{\partial x} = 2 \Rightarrow \text{work} = \oint_C (4x - 2y) dx + (2x - 4y) dy$
 $= \iint_R [2 - (-2)] dx dy = 4 \iint_R dx dy = 4(\text{Area of the circle}) = 4(\pi \cdot 4) = 16\pi$
21. $M = y^2$, $N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 2x \Rightarrow \oint_C y^2 dx + x^2 dy = \iint_R (2x - 2y) dy dx$
 $= \int_0^1 \int_0^{1-x} (2x - 2y) dy dx = \int_0^1 (-3x^2 + 4x - 1) dx = \left[-x^3 + 2x^2 - x \right]_0^1 = -1 + 2 - 1 = 0$
22. $M = 3y$, $N = 2x \Rightarrow \frac{\partial M}{\partial y} = 3$, $\frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C 3y dx + 2x dy = \iint_R (2 - 3) dx dy = \int_0^\pi \int_0^{\sin x} (-1) dy dx$
 $= -\int_0^\pi \sin x dx = -2$
23. $M = 6y + x$, $N = y + 2x \Rightarrow \frac{\partial M}{\partial y} = 6$, $\frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C (6y + x) dx + (y + 2x) dy = \iint_R (2 - 6) dy dx$
 $= -4(\text{Area of the circle}) = -16\pi$
24. $M = 2x + y^2$, $N = 2xy + 3y \Rightarrow \frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 2y \Rightarrow \oint_C (2x + y^2) dx + (2xy + 3y) dy = \iint_R (2y - 2y) dx dy = 0$
25. $M = x = a \cos t$, $N = y = a \sin t \Rightarrow dx = -a \sin t dt$, $dy = a \cos t dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx$
 $= \frac{1}{2} \int_0^{2\pi} (a^2 \cos^2 t + a^2 \sin^2 t) dt = \frac{1}{2} \int_0^{2\pi} a^2 dt = \pi a^2$
26. $M = x = a \cos t$, $N = y = b \sin t \Rightarrow dx = -a \sin t dt$, $dy = b \cos t dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx$
 $= \frac{1}{2} \int_0^{2\pi} (ab \cos^2 t + ab \sin^2 t) dt = \frac{1}{2} \int_0^{2\pi} ab dt = \pi ab$

27. $M = x = \cos^3 t$, $N = y = \sin^3 t \Rightarrow dx = -3 \cos^2 t \sin t dt$, $dy = 3 \sin^2 t \cos t dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx$
 $= \frac{1}{2} \int_0^{2\pi} (3 \sin^2 t \cos^2 t) (\cos^2 t + \sin^2 t) dt = \frac{1}{2} \int_0^{2\pi} (3 \sin^2 t \cos^2 t) dt = \frac{3}{8} \int_0^{2\pi} \sin^2 2t dt = \frac{3}{16} \int_0^{4\pi} \sin^2 u du$
 $= \frac{3}{16} \left[\frac{u}{2} - \frac{\sin 2u}{4} \right]_0^{4\pi} = \frac{3}{8} \pi$
28. $C_1: M = x = t$, $N = y = 0 \Rightarrow dx = dt$, $dy = 0$; $C_2: M = x = (2\pi - t) - \sin(2\pi - t) = 2\pi - t + \sin t$,
 $N = y = 1 - \cos(2\pi - t) = 1 - \cos t \Rightarrow dx = (\cos t - 1) dt$, $dy = \sin t dt$
 $\Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx = \frac{1}{2} \oint_{C_1} x dy - y dx + \frac{1}{2} \oint_{C_2} x dy - y dx$
 $= \frac{1}{2} \int_0^{2\pi} (0) dt + \frac{1}{2} \int_0^{2\pi} [(2\pi - t + \sin t)(\sin t) - (1 - \cos t)(\cos t - 1)] dt = -\frac{1}{2} \int_0^{2\pi} (2 \cos t + t \sin t - 2 - 2\pi \sin t) dt$
 $= -\frac{1}{2} [3 \sin t - t \cos t - 2t - 2\pi \cos t]_0^{2\pi} = 3\pi$
29. (a) $M = f(x)$, $N = g(y) \Rightarrow \frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = 0 \Rightarrow \oint_C f(x) dx + g(y) dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R 0 dx dy = 0$
(b) $M = ky$, $N = hx \Rightarrow \frac{\partial M}{\partial y} = k$, $\frac{\partial N}{\partial x} = h \Rightarrow \oint_C ky dx + hx dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$
 $= \iint_R (h - k) dx dy = (h - k)(\text{Area of the region})$
30. $M = xy^2$, $N = x^2 y + 2x \Rightarrow \frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = 2xy + 2 \Rightarrow \oint_C xy^2 dx + (x^2 y + 2x) dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$
 $= \iint_R (2xy + 2 - 2xy) dx dy = 2 \iint_R dx dy = 2 \text{ times the area of the square}$
31. The integral is 0 for any simple closed plane curve C . The reasoning: By the tangential form of Green's Theorem, with $M = 4x^3 y$ and $N = x^4$, $\oint_C 4x^3 y dx + x^4 dy = \iint_R \left[\frac{\partial}{\partial x}(x^4) - \frac{\partial}{\partial y}(4x^3 y) \right] dx dy$
 $= \iint_R \underbrace{(4x^3 - 4x^3)}_0 dx dy = 0.$
32. The integral is 0 for any simple closed curve C . The reasoning: By the normal form of Green's theorem, with
 $M = x^3$ and $N = -y^3$, $\oint_C -y^3 dy + x^3 dx = \iint_R \left[\underbrace{\frac{\partial}{\partial x}(-y^3)}_0 - \underbrace{\frac{\partial}{\partial y}(x^3)}_0 \right] dx dy = 0.$
33. Let $M = x$ and $N = 0 \Rightarrow \frac{\partial M}{\partial x} = 1$ and $\frac{\partial N}{\partial y} = 0 \Rightarrow \oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy \Rightarrow \oint_C x dy$
 $\iint_R (1 + 0) dx dy \Rightarrow \text{Area of } R = \iint_R dx dy = \oint_C x dy$; similarly, $M = y$ and $N = 0 \Rightarrow \frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 0$
 $\Rightarrow \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y} \right) dy dx \Rightarrow \oint_C y dx = \iint_R (0 + 1) dy dx \Rightarrow -\oint_C y dx = \iint_R dx dy = \text{Area of } R$

34. $\int_a^b f(x) dx = \text{Area of } R = -\oint_C y dx$, from Exercise 33
35. Let $\delta(x, y) = 1 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{\iint_R x \delta(x, y) dA}{\iint_R \delta(x, y) dA} = \frac{\iint_R x dA}{\iint_R dA} = \frac{\iint_R x dA}{A} \Rightarrow A\bar{x} = \iint_R x dA = \iint_R (x+0) dx dy$
 $= \oint_C \frac{x^2}{2} dy$, $A\bar{x} = \iint_R x dA = \iint_R (0+x) dx dy = -\oint_C xy dx$, and $A\bar{x} = \iint_R x dA = \iint_R \left(\frac{2}{3}x + \frac{1}{3}x\right) dx dy$
 $= \oint_C \frac{1}{3}x^2 dy - \frac{1}{3}xy dx \Rightarrow \frac{1}{2}\oint_C x^2 dy = -\oint_C xy dx = \frac{1}{3}\oint_C x^2 dy - xy dx = A\bar{x}$
36. If $\delta(x, y) = 1$ then $I_y = \iint_R x^2 \delta(x, y) dA = \iint_R x^2 dA = \iint_R (x^2 + 0) dy dx = \frac{1}{3}\oint_C x^3 dy$,
 $\iint_R x^2 dA = \iint_R (0+x^2) dy dx = -\oint_C x^2 y dx$, and $\iint_R x^2 dA = \iint_R \left(\frac{3}{4}x^2 + \frac{1}{4}x^2\right) dy dx$
 $= \oint_C \frac{1}{4}x^3 dy - \frac{1}{4}x^2 y dx = \frac{1}{4}\oint_C x^3 dy - x^2 y dx \Rightarrow \frac{1}{3}\oint_C x^3 dy = -\oint_C x^2 y dx = \frac{1}{4}\oint_C x^3 dy - x^2 y dx = I_y$
37. $M = \frac{\partial f}{\partial y}$, $N = -\frac{\partial f}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y^2}$, $\frac{\partial N}{\partial x} = -\frac{\partial^2 f}{\partial x^2} \Rightarrow \oint_C \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = \iint_R \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) dx dy = 0$ for such curves C
38. $M = \frac{1}{4}x^2 y + \frac{1}{3}y^3$, $N = x \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{4}x^2 + y^2$, $\frac{\partial N}{\partial x} = 1 \Rightarrow \text{Curl} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - \left(\frac{1}{4}x^2 + y^2\right) > 0$ in the interior of the ellipse $\frac{1}{4}x^2 + y^2 = 1 \Rightarrow \text{work} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(1 - \frac{1}{4}x^2 - y^2\right) dx dy$ will be maximized on the region $R = \{(x, y) | \text{curl } \mathbf{F} \geq 0\}$ or over the region enclosed by $1 = \frac{1}{4}x^2 + y^2$
39. (a) $\nabla f = \left(\frac{2x}{x^2+y^2}\right)\mathbf{i} + \left(\frac{2y}{x^2+y^2}\right)\mathbf{j} \Rightarrow M = \frac{2x}{x^2+y^2}$, $N = \frac{2y}{x^2+y^2}$; since M, N are discontinuous at $(0, 0)$, we compute $\int_C \nabla f \cdot \mathbf{n} ds$ directly since Green's Theorem does not apply. Let $x = a \cos t$, $y = a \sin t$
 $\Rightarrow dx = -a \sin t dt$, $dy = a \cos t dt$, $M = \frac{2}{a} \cos t$, $N = \frac{2}{a} \sin t$, $0 \leq t \leq 2\pi$, so $\int_C \nabla f \cdot \mathbf{n} ds = \int_C M dy - N dx$
 $= \int_0^{2\pi} \left[\left(\frac{2}{a} \cos t\right)(a \cos t) - \left(\frac{2}{a} \sin t\right)(-a \sin t)\right] dt = \int_0^{2\pi} 2(\cos^2 t + \sin^2 t) dt = 4\pi$. Note that this holds for any $a > 0$, so $\int_C \nabla f \cdot \mathbf{n} ds = 4\pi$ for any circle C centered at $(0, 0)$ traversed counterclockwise and $\int_C \nabla f \cdot \mathbf{n} ds = -4\pi$ if C is traversed clockwise.
- (b) If K does not enclose the point $(0, 0)$ we may apply Green's Theorem: $\int_C \nabla f \cdot \mathbf{n} ds = \int_C M dy - N dx$
 $= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy = \iint_R \left(\frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2}\right) dx dy = \iint_R 0 dx dy = 0$. If K does enclose the point $(0, 0)$ we proceed as follows:
 Choose a small enough so that the circle C centered at $(0, 0)$ of radius a lies entirely within K . Green's Theorem applies to the region R that lies between K and C . Thus, as before, $0 = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy$
 $= \int_K M dy - N dx + \int_C M dy - N dx$ where K is traversed counterclockwise and C is traversed clockwise.

Hence by part (a) $0 = \left[\int_K M \, dy - N \, dx \right] - 4\pi \Rightarrow 4\pi = \int_K M \, dy - N \, dx = \int_K \nabla f \cdot \mathbf{n} \, ds$. We have shown:

$$\int_K \nabla f \cdot \mathbf{n} \, ds = \begin{cases} 0 & \text{if } (0, 0) \text{ lies inside } K \\ 4\pi & \text{if } (0, 0) \text{ lies outside } K \end{cases}$$

40. Assume a particle has a closed trajectory in R and let C_1 be the path $\Rightarrow C_1$ encloses a simply connected region $R_1 \Rightarrow C_1$ is a simple closed curve. Then the flux over R_1 is $\oint_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds = 0$, since the velocity vectors \mathbf{F} are tangent to C_1 . But $0 = \oint_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds = \oint_{C_1} M \, dy - N \, dx = \iint_{R_1} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy \Rightarrow M_x + N_y = 0$, which is a contradiction. Therefore, C_1 cannot be a closed trajectory.

$$\begin{aligned} 41. \quad \int_{g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x} dx \, dy &= N(g_2(y), y) - N(g_1(y), y) \Rightarrow \int_c^d \int_{g_1(y)}^{g_2(y)} \left(\frac{\partial N}{\partial x} \right) dx \, dy = \int_c^d [N(g_2(y), y) - N(g_1(y), y)] dy \\ &= \int_c^d N(g_2(y), y) dy - \int_c^d N(g_1(y), y) dy = \int_c^d N(g_2(y), y) dy + \int_d^c N(g_1(y), y) dy = \int_{C_2} N \, dy + \int_{C_1} N \, dy \\ &= \oint_C N \, dy \Rightarrow \oint_C N \, dy = \iint_R \frac{\partial N}{\partial x} dx \, dy \end{aligned}$$

42. The curl of a conservative two-dimensional field is zero. The reasoning. A two-dimensional field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ can be considered to be the restriction to the xy -plane of a three-dimensional field whose k component, P , is zero, and whose \mathbf{i} and \mathbf{j} components are independent of z . For such a field to be conservative, we must have $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ by the component test in Section 16.3 $\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial x}$, and $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$.

43-46. Example CAS commands:

Maple:

```
with( plots );#43
M:=(x,y) -> 2*x-y;
N:=(x,y) -> x+3*y;
C:= x^2+4*y^2=4;
implicitplot( C, x=-2..2,y= 2..2, scaling=constrained, title="#43(a)(Section 16.4)");
curlF_k:=D[1](N) - D[2](M):                               #(b)
'curlF_k'=curlF_k(x,y);
top,bot:=solve( C, y);                                     #(c)
left,right:=-2,2;
q1:=Int( Int( curlF_k(x,y),y=bot..top ), x=left..right );
value( q1 );
```

Mathematica: (functions and bounds will vary)

The **ImplicitPlot** command will be useful for 43 and 44, but is not needed for 45 and 46. In 46, the equation of the line from (0, 4) to (2, 0) must be determined first.

```
Clear[x, y, f]
<<Graphics`ImplicitPlot`
```

```

f[x_, y_] := {2x - y, x + 3y}
curve = x^2 + 4y^2 == 4
ImplicitPlot[curve, {x, -3, 3}, {y, -2, 2}, AspectRatio -> Automatic, AxesLabel -> {x, y}];
ybonds = Solve[curve, y]
{y1, y2} = y /. bounds;
integrand := D[f[x, y][[2]], x] - D[f[x, y][[1]], y] / Simplify
Integrate[integrand, {x, -2, 2}, {y, y1, y2}]
N[%]

```

Bounds for y are determined differently in 45 and 46. In 46, note equation of the line from $(0, 4)$ to $(2, 0)$.

```

Clear[x, y, f]
f[x_, y_] := {x Exp[y], 4x^2 Log[y]}
ybound = 4 - 2x
Plot[{0, ybound}, {x, 0, 2.1}, AspectRatio -> Automatic, AxesLabel -> {x, y}];
integrand := D[f[x, y][[2]], x] - D[f[x, y][[1]], y] // Simplify
Integrate[integrand, {x, 0, 2}, {y, 0, ybound}]
N[%]

```

16.5 SURFACES AND AREA

- In cylindrical coordinates, let $x = r \cos \theta$, $y = r \sin \theta$, $z = \left(\sqrt{x^2 + y^2}\right)^2 = r^2$.
Then $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r^2\mathbf{k}$, $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.
- In cylindrical coordinates, let $x = r \cos \theta$, $y = r \sin \theta$, $z = 9 - x^2 - y^2 = 9 - r^2$.
Then $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (9 - r^2)\mathbf{k}$; $z \geq 0 \Rightarrow 9 - r^2 \geq 0 \Rightarrow r^2 \leq 9 \Rightarrow -3 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$.
But $-3 \leq r \leq 0$ gives the same points as $0 \leq r \leq 3$, so let $0 \leq r \leq 3$.
- In cylindrical coordinates, let $x = r \cos \theta$, $y = r \sin \theta$, $z = \frac{\sqrt{x^2 + y^2}}{2} \Rightarrow z = \frac{r}{2}$.
Then $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \left(\frac{r}{2}\right)\mathbf{k}$. For $0 \leq z \leq 3$, $0 \leq \frac{r}{2} \leq 3 \Rightarrow 0 \leq r \leq 6$; to get only the first octant, let $0 \leq \theta \leq \frac{\pi}{2}$.
- In cylindrical coordinates, let $x = r \cos \theta$, $y = r \sin \theta$, $z = 2\sqrt{x^2 + y^2} \Rightarrow z = 2r$.
Then $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + 2r\mathbf{k}$. For $2 \leq z \leq 4$, $2 \leq 2r \leq 4 \Rightarrow 1 \leq r \leq 2$, and let $0 \leq \theta \leq 2\pi$.
- In cylindrical coordinates, let $x = r \cos \theta$, $y = r \sin \theta$; since $x^2 + y^2 = r^2 \Rightarrow z^2 = 9 - (x^2 + y^2) = 9 - r^2 \Rightarrow z = \sqrt{9 - r^2}$, $z \geq 0$. Then $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \sqrt{9 - r^2}\mathbf{k}$. Let $0 \leq \theta \leq 2\pi$. For the domain of r :

$$z = \sqrt{x^2 + y^2} \text{ and } x^2 + y^2 + z^2 = 9 \Rightarrow x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2 = 9 \Rightarrow 2(x^2 + y^2) = 9 \Rightarrow 2r^2 = 9 \\ \Rightarrow r = \frac{3}{\sqrt{2}} \Rightarrow 0 \leq r \leq \frac{3}{\sqrt{2}}.$$

6. In cylindrical coordinates, $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \sqrt{4 - r^2}\mathbf{k}$ (see Exercise 5 above with $x^2 + y^2 + z^2 = 4$, instead of $x^2 + y^2 + z^2 = 9$). For the first octant, let $0 \leq \theta \leq \frac{\pi}{2}$. For the domain of r :

$$z = \sqrt{x^2 + y^2} \text{ and } x^2 + y^2 + z^2 = 4 \Rightarrow x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2 = 4 \Rightarrow 2(x^2 + y^2) = 4 \Rightarrow 2r^2 = 4 \Rightarrow r = \sqrt{2}.$$

Thus, let $\sqrt{2} \leq r \leq 2$ (to get the portion of the sphere between the cone and the xy -plane).

7. In spherical coordinates, $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $\rho = \sqrt{x^2 + y^2 + z^2} \Rightarrow \rho^2 = 3 \Rightarrow \rho = \sqrt{3}$

$$\Rightarrow z = \sqrt{3} \cos \phi \text{ for the sphere; } z = \frac{\sqrt{3}}{2} = \sqrt{3} \cos \phi \Rightarrow \cos \phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}; z = -\frac{\sqrt{3}}{2} \Rightarrow -\frac{\sqrt{3}}{2} = \sqrt{3} \cos \phi$$

$$\Rightarrow \cos \phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}. \text{ Then } \mathbf{r}(\phi, \theta) = (\sqrt{3} \sin \phi \cos \theta)\mathbf{i} + (\sqrt{3} \sin \phi \sin \theta)\mathbf{j} + (\sqrt{3} \cos \phi)\mathbf{k},$$

$$\frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3} \text{ and } 0 \leq \theta \leq 2\pi.$$

8. In spherical coordinates, $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $\rho = \sqrt{x^2 + y^2 + z^2} \Rightarrow \rho^2 = 8 \Rightarrow \rho = \sqrt{8} = 2\sqrt{2}$

$$\Rightarrow x = 2\sqrt{2} \sin \phi \cos \theta, y = 2\sqrt{2} \sin \phi \sin \theta, \text{ and } z = 2\sqrt{2} \cos \phi. \text{ Thus let}$$

$$\mathbf{r}(\phi, \theta) = (2\sqrt{2} \sin \phi \cos \theta)\mathbf{i} + (2\sqrt{2} \sin \phi \sin \theta)\mathbf{j} + (2\sqrt{2} \cos \phi)\mathbf{k}; z = -2 \Rightarrow -2 = 2\sqrt{2} \cos \phi$$

$$\Rightarrow \cos \phi = -\frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{3\pi}{4}; z = 2\sqrt{2} \Rightarrow 2\sqrt{2} = 2\sqrt{2} \cos \phi \Rightarrow \cos \phi = 1 \Rightarrow \phi = 0. \text{ Thus } 0 \leq \phi \leq \frac{3\pi}{4} \text{ and}$$

$$0 \leq \theta \leq 2\pi.$$

9. Since $z = 4 - y^2$, we can let \mathbf{r} be a function of x and $y \Rightarrow \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - y^2)\mathbf{k}$. Then $z = 0$

$$\Rightarrow 0 = 4 - y^2 \Rightarrow y = \pm 2. \text{ Thus, let } -2 \leq y \leq 2 \text{ and } 0 \leq x \leq 2.$$

10. Since $y = x^2$, we can let \mathbf{r} be a function of x and $z \Rightarrow \mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$. Then $y = 2$

$$\Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}. \text{ Thus, let } -\sqrt{2} \leq x \leq \sqrt{2} \text{ and } 0 \leq z \leq 3.$$

11. When $x = 0$, let $y^2 + z^2 = 9$ be the circular section in the yz -plane. Use polar coordinates in the yz -plane

$$\Rightarrow y = 3 \cos \theta \text{ and } z = 3 \sin \theta. \text{ Thus let } x = u \text{ and } \theta = v \Rightarrow \mathbf{r}(u, v) = u\mathbf{i} + (3 \cos v)\mathbf{j} + (3 \sin v)\mathbf{k} \text{ where}$$

$$0 \leq u \leq 3, \text{ and } 0 \leq v \leq 2\pi.$$

12. When $y = 0$, let $x^2 + z^2 = 4$ be the circular section in the xz -plane. Use polar coordinates in the xz -plane

$$\Rightarrow x = 2 \cos \theta \text{ and } z = 2 \sin \theta. \text{ Thus let } y = u \text{ and } \theta = v \Rightarrow \mathbf{r}(u, v) = (2 \cos v)\mathbf{i} + u\mathbf{j} + (2 \sin v)\mathbf{k} \text{ where}$$

$$-2 \leq u \leq 2, \text{ and } 0 \leq v \leq \pi \text{ (since we want the portion above the } xy\text{-plane)}.$$

13. (a) $x + y + z = 1 \Rightarrow z = 1 - x - y$. In cylindrical coordinates, let $x = r \cos \theta$ and $y = r \sin \theta$

$$\Rightarrow z = 1 - r \cos \theta - r \sin \theta \Rightarrow \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (1 - r \cos \theta - r \sin \theta)\mathbf{k}, 0 \leq \theta \leq 2\pi$$

$$\text{and } 0 \leq r \leq 3.$$

- (b) In a fashion similar to cylindrical coordinates, but working in the yz -plane instead of the xy -plane, let $y = u \cos v$, $z = u \sin v$ where $u = \sqrt{y^2 + z^2}$ and v is the angle formed by (x, y, z) , $(x, 0, 0)$, and $(x, y, 0)$ with $(x, 0, 0)$ as vertex. Since $x + y + z = 1 \Rightarrow x = 1 - y - z \Rightarrow x = 1 - u \cos v - u \sin v$, then \mathbf{r} is a function of u and $v \Rightarrow \mathbf{r}(u, v) = (1 - u \cos v - u \sin v)\mathbf{i} + (u \cos v)\mathbf{j} + (u \sin v)\mathbf{k}$, $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$.
14. (a) In a fashion similar to cylindrical coordinates, but working in the xz -plane instead of the xy -plane, let $x = u \cos v$, $z = u \sin v$ where $u = \sqrt{x^2 + z^2}$ and v is the angle formed by (x, y, z) , $(y, 0, 0)$, and $(x, y, 0)$ with vertex $(y, 0, 0)$. Since $x - y + 2z = 2 \Rightarrow y = x + 2z - 2$, then $\mathbf{r}(u, v) = (u \cos v)\mathbf{i} + (u \cos v + 2u \sin v - 2)\mathbf{j} + (u \sin v)\mathbf{k}$, $0 \leq u \leq \sqrt{3}$ and $0 \leq v \leq 2\pi$.
- (b) In a fashion similar to cylindrical coordinates, but working in the yz -plane instead of the xy -plane, let $y = u \cos v$, $z = u \sin v$ where $u = \sqrt{y^2 + z^2}$ and v is the angle formed by (x, y, z) , $(x, 0, 0)$, and $(x, y, 0)$ with vertex $(x, 0, 0)$. Since $x - y + 2z = 2 \Rightarrow x = y - 2z + 2$, then $\mathbf{r}(u, v) = (u \cos v - 2u \sin v + 2)\mathbf{i} + (u \cos v)\mathbf{j} + (u \sin v)\mathbf{k}$, $0 \leq u \leq \sqrt{2}$ and $0 \leq v \leq 2\pi$.
15. Let $x = w \cos v$ and $z = w \sin v$. Then $(x - 2)^2 + z^2 = 4 \Rightarrow x^2 - 4x + z^2 = 0$
 $\Rightarrow w^2 \cos^2 v - 4w \cos v + w^2 \sin^2 v = 0 \Rightarrow w^2 - 4w \cos v = 0 \Rightarrow w = 0$ or $w - 4 \cos v = 0 \Rightarrow w = 0$ or $w = 4 \cos v$. Now $w = 0 \Rightarrow x = 0$ and $y = 0$, which is a line not a cylinder. Therefore, let $w = 4 \cos v \Rightarrow x = (4 \cos v)(\cos v) = 4 \cos^2 v$ and $z = 4 \cos v \sin v$. Finally, let $y = u$. Then $\mathbf{r}(u, v) = (4 \cos^2 v)\mathbf{i} + u\mathbf{j} + (4 \cos v \sin v)\mathbf{k}$, $-\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$ and $0 \leq u \leq 3$.
16. Let $y = w \cos v$ and $z = w \sin v$. Then $y^2 + (z - 5)^2 = 25 \Rightarrow y^2 + z^2 - 10z = 0$
 $\Rightarrow w^2 \cos^2 v + w^2 \sin^2 v - 10w \sin v = 0 \Rightarrow w^2 - 10w \sin v = 0 \Rightarrow w(w - 10 \sin v) = 0 \Rightarrow w = 0$ or $w = 10 \sin v$. Now $w = 0 \Rightarrow y = 0$ and $z = 0$, which is a line not a cylinder. Therefore, let $w = 10 \sin v \Rightarrow y = 10 \sin v \cos v$ and $z = 10 \sin^2 v$. Finally, let $x = u$. Then $\mathbf{r}(u, v) = u\mathbf{i} + (10 \sin v \cos v)\mathbf{j} + (10 \sin^2 v)\mathbf{k}$, $0 \leq u \leq 10$ and $0 \leq v \leq \pi$.
17. Let $x = r \cos \theta$ and $y = r \sin \theta$. Then $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \left(\frac{2 - r \sin \theta}{2}\right)\mathbf{k}$, $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$
 $\Rightarrow \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} - \left(\frac{\sin \theta}{2}\right)\mathbf{k}$ and $\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} - \left(\frac{r \cos \theta}{2}\right)\mathbf{k}$
 $\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -\frac{\sin \theta}{2} \\ -r \sin \theta & r \cos \theta & -\frac{r \cos \theta}{2} \end{vmatrix}$
 $= \left(\frac{-r \sin \theta \cos \theta}{2} + \frac{(\sin \theta)(r \cos \theta)}{2}\right)\mathbf{i} + \left(\frac{r \sin^2 \theta}{2} + \frac{r \cos^2 \theta}{2}\right)\mathbf{j} + (r \cos^2 \theta + r \sin^2 \theta)\mathbf{k} = \frac{r}{2}\mathbf{j} + r\mathbf{k}$
 $\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{r^2}{4} + r^2} = \frac{\sqrt{5}r}{2} \Rightarrow A = \int_0^{2\pi} \int_0^1 \frac{\sqrt{5}r}{2} dr d\theta = \int_0^{2\pi} \left[\frac{\sqrt{5}r^2}{4}\right]_0^1 d\theta = \int_0^{2\pi} d\theta = \frac{\pi\sqrt{5}}{2}$
18. Let $x = r \cos \theta$ and $y = r \sin \theta \Rightarrow z = -x = -r \cos \theta$, $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. Then $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} - (r \cos \theta)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} - (\cos \theta)\mathbf{k}$ and $\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} + (r \sin \theta)\mathbf{k}$

$$\begin{aligned}\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -\cos \theta \\ -r \sin \theta & r \cos \theta & r \sin \theta \end{vmatrix} \\ &= (r \sin^2 \theta + r \cos^2 \theta) \mathbf{i} + (r \sin \theta \cos \theta - r \sin \theta \cos \theta) \mathbf{j} + (r \cos^2 \theta + r \sin^2 \theta) \mathbf{k} = r \mathbf{i} + r \mathbf{k} \\ \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| &= \sqrt{r^2 + r^2} = r\sqrt{2} \Rightarrow A = \int_0^{2\pi} \int_0^2 r\sqrt{2} dr d\theta = \int_0^{2\pi} \left[\frac{r^2\sqrt{2}}{2} \right]_0^2 d\theta = \int_0^{2\pi} 2\sqrt{2} d\theta = 4\pi\sqrt{2}\end{aligned}$$

19. Let $x = r \cos \theta$ and $y = r \sin \theta \Rightarrow z = 2\sqrt{x^2 + y^2} = 2r, 1 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$. Then
 $\mathbf{r}(r, \theta) = (r \cos \theta) \mathbf{i} + (r \sin \theta) \mathbf{j} + 2r \mathbf{k} \Rightarrow \mathbf{r}_r = (\cos \theta) \mathbf{i} + (\sin \theta) \mathbf{j} + 2 \mathbf{k}$ and $\mathbf{r}_\theta = (-r \sin \theta) \mathbf{i} + (r \cos \theta) \mathbf{j}$

$$\begin{aligned}\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 2 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = (-2r \cos \theta) \mathbf{i} - (2r \sin \theta) \mathbf{j} + (r \cos^2 \theta + r \sin^2 \theta) \mathbf{k} \\ \Rightarrow (-2r \cos \theta) \mathbf{i} - (2r \sin \theta) \mathbf{j} + r \mathbf{k} &\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^2 \cos^2 \theta + 4r^2 \sin^2 \theta + r^2} = \sqrt{5r^2} = r\sqrt{5} \\ \Rightarrow A &= \int_0^{2\pi} \int_1^3 r\sqrt{5} dr d\theta = \int_0^{2\pi} \left[\frac{r^2\sqrt{5}}{2} \right]_1^3 d\theta = \int_0^{2\pi} 4\sqrt{5} d\theta = 8\pi\sqrt{5}\end{aligned}$$

20. Let $x = r \cos \theta$ and $y = r \sin \theta \Rightarrow z = \frac{\sqrt{x^2 + y^2}}{3} = \frac{r}{3}, 3 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$. Then

$$\begin{aligned}\mathbf{r}(r, \theta) &= (r \cos \theta) \mathbf{i} + (r \sin \theta) \mathbf{j} + \left(\frac{r}{3}\right) \mathbf{k} \Rightarrow \mathbf{r}_r = (\cos \theta) \mathbf{i} + (\sin \theta) \mathbf{j} + \left(\frac{1}{3}\right) \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta) \mathbf{i} + (r \cos \theta) \mathbf{j} \\ \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & \frac{1}{3} \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = \left(-\frac{1}{3}r \cos \theta\right) \mathbf{i} - \left(\frac{1}{3}r \sin \theta\right) \mathbf{j} + \left(r \cos^2 \theta + r \sin^2 \theta\right) \mathbf{k} \\ &= \left(-\frac{1}{3}r \cos \theta\right) \mathbf{i} - \left(\frac{1}{3}r \sin \theta\right) \mathbf{j} + r \mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{1}{9}r^2 \cos^2 \theta + \frac{1}{9}r^2 \sin^2 \theta + r^2} = \sqrt{\frac{10r^2}{9}} = \frac{r\sqrt{10}}{3} \\ \Rightarrow A &= \int_0^{2\pi} \int_3^4 \frac{r\sqrt{10}}{3} dr d\theta = \int_0^{2\pi} \left[\frac{r^2\sqrt{10}}{6} \right]_3^4 d\theta = \int_0^{2\pi} \frac{7\sqrt{10}}{6} d\theta = \frac{7\pi\sqrt{10}}{3}\end{aligned}$$

21. Let $x = r \cos \theta$ and $y = r \sin \theta \Rightarrow r^2 = x^2 + y^2 = 1, 1 \leq z \leq 4$ and $0 \leq \theta \leq 2\pi$. Then

$$\begin{aligned}\mathbf{r}(z, \theta) &= (\cos \theta) \mathbf{i} + (\sin \theta) \mathbf{j} + z \mathbf{k} \Rightarrow \mathbf{r}_z = \mathbf{k} \text{ and } \mathbf{r}_\theta = (-\sin \theta) \mathbf{i} + (\cos \theta) \mathbf{j} \\ \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos \theta) \mathbf{i} + (\sin \theta) \mathbf{j} \Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1 \\ \Rightarrow A &= \int_0^{2\pi} \int_1^4 1 dr d\theta = \int_0^{2\pi} 3 d\theta = 6\pi\end{aligned}$$

22. Let $x = u \cos v$ and $z = u \sin v \Rightarrow u^2 = x^2 + z^2 = 10, -1 \leq y \leq 1, 0 \leq v \leq 2\pi$. Then

$$\mathbf{r}(y, v) = (u \cos v) \mathbf{i} + y \mathbf{j} + (u \sin v) \mathbf{k} = (\sqrt{10} \cos v) \mathbf{i} + y \mathbf{j} + (\sqrt{10} \sin v) \mathbf{k} \Rightarrow \mathbf{r}_v = (-\sqrt{10} \sin v) \mathbf{i} + (\sqrt{10} \cos v) \mathbf{k}$$

$$\text{and } \mathbf{r}_y = \mathbf{j} \Rightarrow \mathbf{r}_v \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{10} \sin v & 0 & \sqrt{10} \cos v \\ 0 & 1 & 0 \end{vmatrix} = (-\sqrt{10} \cos v) \mathbf{i} - (\sqrt{10} \sin v) \mathbf{k} \Rightarrow |\mathbf{r}_v \times \mathbf{r}_y| = \sqrt{10}$$

$$\Rightarrow A = \int_0^{2\pi} \int_{-1}^1 \sqrt{10} \, du \, dv = \int_0^{2\pi} \left[\sqrt{10} u \right]_{-1}^1 dv = \int_0^{2\pi} 2\sqrt{10} \, dv = 4\pi\sqrt{10}$$

23. $z = 2 - x^2 - y^2$ and $z = \sqrt{x^2 + y^2} \Rightarrow z = 2 - z^2 \Rightarrow z^2 + z - 2 = 0 \Rightarrow z = -2$ or $z = 1$. Since $z = \sqrt{x^2 + y^2} \geq 0$, we get $z = 1$ where the cone intersects the paraboloid. When $x = 0$ and $y = 0$, $z = 2 \Rightarrow$ the vertex of the paraboloid is $(0, 0, 2)$. Therefore, z ranges from 1 to 2 on the “cap” $\Rightarrow r$ ranges from 1 (when $x^2 + y^2 = 1$) to 0 (when $x = 0$ and $y = 0$ at the vertex). Let $x = r \cos \theta$, $y = r \sin \theta$, and $z = 2 - r^2$. Then

$$\mathbf{r}(r, \theta) = (r \cos \theta) \mathbf{i} + (r \sin \theta) \mathbf{j} + (2 - r^2) \mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi \Rightarrow \mathbf{r}_r = (\cos \theta) \mathbf{i} + (\sin \theta) \mathbf{j} - 2r \mathbf{k} \text{ and}$$

$$\mathbf{r}_\theta = (-r \sin \theta) \mathbf{i} + (r \cos \theta) \mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix}$$

$$= (2r^2 \cos \theta) \mathbf{i} + (2r^2 \sin \theta) \mathbf{j} + r \mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^4 \cos^2 \theta + 4r^4 \sin^2 \theta + r^2} = r\sqrt{4r^2 + 1}$$

$$\Rightarrow A = \int_0^{2\pi} \int_0^1 r\sqrt{4r^2 + 1} \, dr \, d\theta = \int_0^{2\pi} \left[\frac{1}{12} (4r^2 + 1)^{3/2} \right]_0^1 d\theta = \int_0^{2\pi} \left(\frac{5\sqrt{5}-1}{12} \right) d\theta = \frac{\pi}{6} (5\sqrt{5} - 1)$$

24. Let $x = r \cos \theta$, $y = r \sin \theta$ and $z = x^2 + y^2 = r^2$. Then $\mathbf{r}(r, \theta) = (r \cos \theta) \mathbf{i} + (r \sin \theta) \mathbf{j} + r^2 \mathbf{k}$, $1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi \Rightarrow \mathbf{r}_r = (\cos \theta) \mathbf{i} + (\sin \theta) \mathbf{j} + 2r \mathbf{k}$ and $\mathbf{r}_\theta = (-r \sin \theta) \mathbf{i} + (r \cos \theta) \mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = (-2r^2 \cos \theta) \mathbf{i} - (2r^2 \sin \theta) \mathbf{j} + r \mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta|$$

$$= \sqrt{4r^4 \cos^2 \theta + 4r^4 \sin^2 \theta + r^2} = r\sqrt{4r^2 + 1} \Rightarrow A = \int_0^{2\pi} \int_1^2 r\sqrt{4r^2 + 1} \, dr \, d\theta = \int_0^{2\pi} \left[\frac{1}{12} (4r^2 + 1)^{3/2} \right]_1^2 d\theta$$

$$= \int_0^{2\pi} \left(\frac{17\sqrt{17}-5\sqrt{5}}{12} \right) d\theta = \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5})$$

25. Let $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi \Rightarrow \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2}$ on the sphere. Next, $x^2 + y^2 + z^2 = 2$ and $z = \sqrt{x^2 + y^2} \Rightarrow z^2 + z^2 = 2 \Rightarrow z^2 = 1 \Rightarrow z = 1$ since $z \geq 0 \Rightarrow \phi = \frac{\pi}{4}$. For the lower portion of the sphere cut by the cone, we get $\phi = \pi$. Then

$$\mathbf{r}(\phi, \theta) = (\sqrt{2} \sin \phi \cos \theta) \mathbf{i} + (\sqrt{2} \sin \phi \sin \theta) \mathbf{j} + (\sqrt{2} \cos \phi) \mathbf{k}, \quad \frac{\pi}{4} \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

$$\Rightarrow \mathbf{r}_\phi = (\sqrt{2} \sin \phi \cos \theta) \mathbf{i} + (\sqrt{2} \cos \phi \sin \theta) \mathbf{j} - (\sqrt{2} \sin \phi) \mathbf{k} \text{ and } \mathbf{r}_\theta = (-\sqrt{2} \sin \phi \sin \theta) \mathbf{i} + (\sqrt{2} \sin \phi \cos \theta) \mathbf{j}$$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2} \sin \phi \cos \theta & \sqrt{2} \cos \phi \sin \theta & -\sqrt{2} \sin \phi \\ -\sqrt{2} \sin \phi \sin \theta & \sqrt{2} \sin \phi \cos \theta & 0 \end{vmatrix} = (2 \sin^2 \phi \cos \theta) \mathbf{i} + (2 \sin^2 \phi \sin \theta) \mathbf{j} + (2 \sin \phi \cos \phi) \mathbf{k}$$

$$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{4\sin^4\phi \cos^2\theta + 4\sin^4\phi \sin^2\theta + 4\sin^2\phi \cos^2\phi} = \sqrt{4\sin^2\phi} = 2|\sin\phi| = 2\sin\phi$$

$$\Rightarrow A = \int_0^{2\pi} \int_{\pi/4}^{\pi} 2\sin\phi \, d\phi \, d\theta = \int_0^{2\pi} (2 + \sqrt{2}) \, d\theta = (4 + 2\sqrt{2})\pi$$

26. Let $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi \Rightarrow \rho = \sqrt{x^2 + y^2 + z^2} = 2$ on the sphere. Next,

$$z = -1 \Rightarrow -1 = 2\cos\phi \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}; \quad z = \sqrt{3} \Rightarrow \sqrt{3} = 2\cos\phi \Rightarrow \cos\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}.$$

$$\text{Then } \mathbf{r}(\phi, \theta) = (2\sin\phi \cos\theta)\mathbf{i} + (2\sin\phi \sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}, \quad \frac{\pi}{6} \leq \phi \leq \frac{2\pi}{3}, \quad 0 \leq \theta \leq 2\pi$$

$$\Rightarrow \mathbf{r}_\phi = (2\cos\phi \cos\theta)\mathbf{i} + (2\cos\phi \sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k} \text{ and } \mathbf{r}_\theta = (-2\sin\phi \sin\theta)\mathbf{i} + (2\sin\phi \cos\theta)\mathbf{j}$$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi \cos\theta & 2\cos\phi \sin\theta & -2\sin\phi \\ -2\sin\phi \sin\theta & 2\sin\phi \cos\theta & 0 \end{vmatrix} = (4\sin^2\phi \cos\theta)\mathbf{i} + (4\sin^2\phi \sin\theta)\mathbf{j} + (4\sin\phi \cos\phi)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi \cos^2\theta + 16\sin^4\phi \sin^2\theta + 16\sin^2\phi \cos^2\phi}$$

$$\Rightarrow A = \int_0^{2\pi} \int_{\pi/6}^{2\pi/3} 4\sin\phi \, d\phi \, d\theta = \int_0^{2\pi} (2 + 2\sqrt{3}) \, d\theta = (4 + 4\sqrt{3})\pi$$

27. The parametrization $\mathbf{r}(r, \theta) = (r \cos\theta)\mathbf{i} + (r \sin\theta)\mathbf{j} + r\mathbf{k}$

$$\text{at } P_0 = (\sqrt{2}, \sqrt{2}, 2) \Rightarrow \theta = \frac{\pi}{4}, r = 2,$$

$$\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \mathbf{k} \text{ and}$$

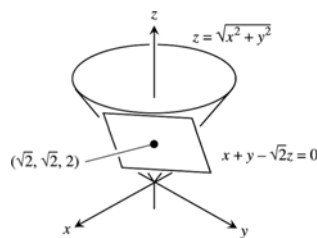
$$\mathbf{r}_\theta = (-r \sin\theta)\mathbf{i} + (r \cos\theta)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix} = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k}$$

$$\Rightarrow \text{the tangent plane is } 0 = (-\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k}) \cdot [(x - \sqrt{2})\mathbf{i} + (y - \sqrt{2})\mathbf{j} + (z - 2)\mathbf{k}] \Rightarrow \sqrt{2}x + \sqrt{2}y - 2z = 0, \text{ or}$$

$$x + y - \sqrt{2}z = 0. \text{ The parametrization } \mathbf{r}(r, \theta) \Rightarrow x = r \cos\theta, y = r \sin\theta \text{ and } z = r \Rightarrow x^2 + y^2 = r^2 = z^2$$

$$\Rightarrow \text{the surface is } z = \sqrt{x^2 + y^2}.$$



28. The parametrization

$$\mathbf{r}(\phi, \theta) = (4\sin\phi \cos\theta)\mathbf{i} + (4\sin\phi \sin\theta)\mathbf{j} + (4\cos\phi)\mathbf{k} \text{ at}$$

$$P_0 = (\sqrt{2}, \sqrt{2}, 2\sqrt{3}) \Rightarrow \rho = 4 \text{ and}$$

$$z = 2\sqrt{3} = 4\cos\phi \Rightarrow \phi = \frac{\pi}{6}; \text{ also } x = \sqrt{2} \text{ and } y = \sqrt{2}$$

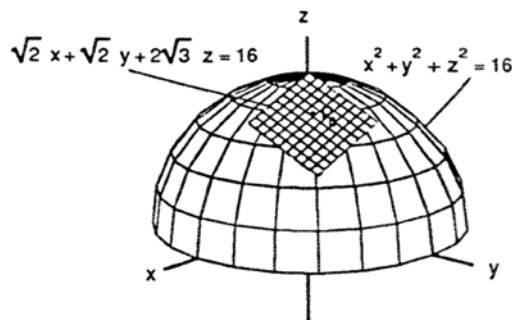
$$\Rightarrow \theta = \frac{\pi}{4}. \text{ Then}$$

$$\mathbf{r}_\phi = (4\cos\phi \cos\theta)\mathbf{i} + (4\cos\phi \sin\theta)\mathbf{j} - (4\sin\phi)\mathbf{k}$$

$$= \sqrt{6}\mathbf{i} + \sqrt{6}\mathbf{j} - 2\mathbf{k} \text{ and}$$

$$\mathbf{r}_\theta = (-4\sin\phi \sin\theta)\mathbf{i} + (4\sin\phi \cos\theta)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} \text{ at } P_0 \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{6} & \sqrt{6} & -2 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix} = 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k}$$

$$\Rightarrow \text{the tangent plane is } (2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k}) \cdot [(x - \sqrt{2})\mathbf{i} + (y - \sqrt{2})\mathbf{j} + (z - 2\sqrt{3})\mathbf{k}] = 0$$



$$\Rightarrow \sqrt{2}x + \sqrt{2}y + 2\sqrt{3}z = 16, \text{ or } x + y + \sqrt{6}z = 8\sqrt{2}. \text{ The parametrization}$$

$$\Rightarrow x = 4\sin\phi\cos\theta, y = 4\sin\phi\sin\theta, z = 4\cos\phi \Rightarrow \text{the surface is } x^2 + y^2 + z^2 = 16, z \geq 0.$$

29. The parametrization

$$\mathbf{r}(\theta, z) = (3\sin 2\theta)\mathbf{i} + (6\sin^2\theta)\mathbf{j} + z\mathbf{k} \text{ at}$$

$$P_0 = \left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right) \Rightarrow \theta = \frac{\pi}{3} \text{ and } z = 0. \text{ Then}$$

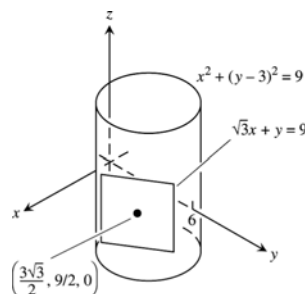
$$\mathbf{r}_\theta = (6\cos 2\theta)\mathbf{i} + (12\sin\theta\cos\theta)\mathbf{j} = -3\mathbf{i} + 3\sqrt{3}\mathbf{j} \text{ and}$$

$$\mathbf{r}_z = \mathbf{k} \text{ at } P_0 \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3\sqrt{3} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\sqrt{3}\mathbf{i} + 3\mathbf{j}$$

$$\Rightarrow \text{the tangent plane is } (3\sqrt{3}\mathbf{i} + 3\mathbf{j}) \cdot \left[\left(x - \frac{3\sqrt{3}}{2}\right)\mathbf{i} + \left(y - \frac{9}{2}\right)\mathbf{j} + (z - 0)\mathbf{k}\right] = 0 \Rightarrow \sqrt{3}x + y = 9. \text{ The parametrization}$$

$$\Rightarrow x = 3\sin 2\theta \text{ and } y = 6\sin^2\theta \Rightarrow x^2 + y^2 = 9\sin^2 2\theta + (6\sin^2\theta)^2$$

$$= 9(4\sin^2\theta\cos^2\theta) + 36\sin^4\theta = 6(6\sin^2\theta) = 6y \Rightarrow x^2 + y^2 - 6y + 9 = 9 \Rightarrow x^2 + (y - 3)^2 = 9$$



30. The parametrization $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$ at

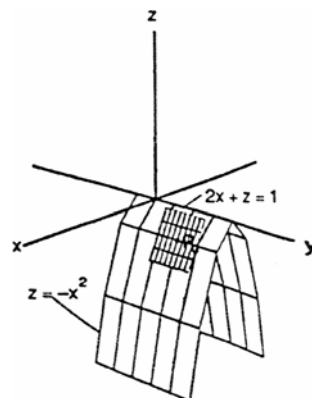
$$P_0 = (1, 2, -1) \Rightarrow \mathbf{r}_x = \mathbf{i} - 2x\mathbf{k} = \mathbf{i} - 2\mathbf{k} \text{ and } \mathbf{r}_y = \mathbf{j}$$

$$\text{at } P_0 \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 2\mathbf{i} + \mathbf{k} \Rightarrow \text{the tangent}$$

$$\text{plane is } (2\mathbf{i} + \mathbf{k}) \cdot [(x-1)\mathbf{i} + (y-2)\mathbf{j} + (z+1)\mathbf{k}] = 0$$

$$\Rightarrow 2x + z = 1. \text{ The parametrization } \Rightarrow x = x, y = y$$

$$\text{and } z = -x^2 \Rightarrow \text{the surface is } z = -x^2$$



31. (a) An arbitrary point on the circle C is $(x, z) = (R + r\cos u, r\sin u) \Rightarrow (x, y, z)$ is on the torus with $x = (R + r\cos u)\cos v$, $y = (R + r\cos u)\sin v$, and $z = r\sin u$, $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$

(b) $\mathbf{r}_u = (-r\sin u\cos v)\mathbf{i} - (r\sin u\sin v)\mathbf{j} + (r\cos u)\mathbf{k}$ and $\mathbf{r}_v = (-(R + r\cos u)\sin v)\mathbf{i} + ((R + r\cos u)\cos v)\mathbf{j}$

$$\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin u\cos v & -r\sin u\sin v & r\cos u \\ -(R + r\cos u)\sin v & (R + r\cos u)\cos v & 0 \end{vmatrix}$$

$$= -(R + r\cos u)(r\cos v\cos u)\mathbf{i} - (R + r\cos u)(r\sin v\cos u)\mathbf{j} + (-r\sin u)(R + r\cos u)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v|^2 = (R + r\cos u)^2 (r^2 \cos^2 v \cos^2 u + r^2 \sin^2 v \cos^2 u + r^2 \sin^2 u) \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = r(R + r\cos u)$$

$$\Rightarrow A = \int_0^{2\pi} \int_0^{2\pi} (rR + r^2 \cos u) du dv = \int_0^{2\pi} 2\pi rR dv = 4\pi^2 rR$$

32. (a) The point (x, y, z) is on the surface for fixed $x = f(u)$ when $y = g(u) \sin\left(\frac{\pi}{2} - v\right)$ and $z = g(u) \cos\left(\frac{\pi}{2} - v\right) \Rightarrow x = f(u), y = g(u) \cos v$, and $z = g(u) \sin v \Rightarrow \mathbf{r}(u, v) = f(u)\mathbf{i} + (g(u) \cos v)\mathbf{j} + (g(u) \sin v)\mathbf{k}$, $0 \leq v \leq 2\pi, a \leq u \leq b$
- (b) Let $u = y$ and $x = u^2 \Rightarrow f(u) = u^2$ and $g(u) = u \Rightarrow \mathbf{r}(u, v) = u^2\mathbf{i} + (u \cos v)\mathbf{j} + (u \sin v)\mathbf{k}$, $0 \leq v \leq 2\pi, 0 \leq u$
33. (a) Let $w^2 + \frac{z^2}{c^2} = 1$ where $w = \cos \phi$ and $\frac{z}{c} = \sin \phi \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \phi \Rightarrow \frac{x}{a} = \cos \phi \cos \theta$ and $\frac{y}{b} = \cos \phi \sin \theta$
 $\Rightarrow x = a \cos \theta \cos \phi, y = b \sin \theta \cos \phi$, and $z = c \sin \phi$
 $\Rightarrow \mathbf{r}(\theta, \phi) = (a \cos \theta \cos \phi)\mathbf{i} + (b \sin \theta \cos \phi)\mathbf{j} + (c \sin \phi)\mathbf{k}$
- (b) $\mathbf{r}_\theta = (-a \sin \theta \cos \phi)\mathbf{i} + (b \cos \theta \cos \phi)\mathbf{j}$ and $\mathbf{r}_\phi = (-a \cos \theta \sin \phi)\mathbf{i} - (b \sin \theta \sin \phi)\mathbf{j} + (c \cos \phi)\mathbf{k}$
- $$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin \theta \cos \phi & b \cos \theta \cos \phi & 0 \\ -a \cos \theta \sin \phi & -b \sin \theta \sin \phi & c \cos \phi \end{vmatrix}$$
- $$= (bc \cos \theta \cos^2 \phi)\mathbf{i} + (ac \sin \theta \cos^2 \phi)\mathbf{j} + (ab \sin \phi \cos \phi)\mathbf{k}$$
- $$\Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_\phi|^2 = b^2 c^2 \cos^2 \theta \cos^4 \phi + a^2 c^2 \sin^2 \theta \cos^4 \phi + a^2 b^2 \sin^2 \phi \cos^2 \phi$$
- , and the result follows.
- $$A = \int_0^{2\pi} \int_0^{2\pi} |\mathbf{r}_\theta \times \mathbf{r}_\phi| d\phi d\theta = \int_0^{2\pi} \int_0^{2\pi} [a^2 b^2 \sin^2 \phi \cos^2 \phi + b^2 c^2 \cos^2 \theta \cos^4 \phi + a^2 c^2 \sin^2 \theta \cos^4 \phi]^{1/2} d\phi d\theta$$
34. (a) $\mathbf{r}(\theta, u) = (\cosh u \cos \theta)\mathbf{i} + (\cosh u \sin \theta)\mathbf{j} + (\sinh u)\mathbf{k}$
- (b) $\mathbf{r}(\theta, u) = (a \cosh u \cos \theta)\mathbf{i} + (b \cosh u \sin \theta)\mathbf{j} + (c \sinh u)\mathbf{k}$
35. $\mathbf{r}(\theta, u) = (5 \cosh u \cos \theta)\mathbf{i} + (5 \cosh u \sin \theta)\mathbf{j} + (5 \sinh u)\mathbf{k} \Rightarrow \mathbf{r}_\theta = (-5 \cosh u \sin \theta)\mathbf{i} + (5 \cosh u \cos \theta)\mathbf{j}$ and $\mathbf{r}_u = (5 \sinh u \cos \theta)\mathbf{i} + (5 \sinh u \sin \theta)\mathbf{j} + (5 \cosh u)\mathbf{k} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_u = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 \cosh u \sin \theta & 5 \cosh u \cos \theta & 0 \\ 5 \sinh u \cos \theta & 5 \sinh u \sin \theta & 5 \cosh u \end{vmatrix}$
- $$= (25 \cosh^2 u \cos \theta)\mathbf{i} + (25 \cosh^2 u \sin \theta)\mathbf{j} - (25 \cosh u \sinh u)\mathbf{k}$$
- . At the point
- $(x_0, y_0, 0)$
- , where
- $x_0^2 + y_0^2 = 25$
- we have
- $5 \sinh u = 0 \Rightarrow u = 0$
- and
- $x_0 = 25 \cos \theta, y_0 = 25 \sin \theta \Rightarrow$
- the tangent plane is
- $5(x_0 \mathbf{i} + y_0 \mathbf{j}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + z\mathbf{k}] = 0 \Rightarrow x_0 x - x_0^2 + y_0 y - y_0^2 = 0 \Rightarrow x_0 x + y_0 y = 25$
36. Let $\frac{z^2}{c^2} - w^2 = 1$ where $\frac{z}{c} = \cosh u$ and $w = \sinh u \Rightarrow w^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow \frac{x}{a} = w \cos \theta$ and $\frac{y}{b} = w \sin \theta$
 $\Rightarrow x = a \sinh u \cos \theta, y = b \sinh u \sin \theta$, and $z = c \cosh u$
 $\Rightarrow \mathbf{r}(\theta, u) = (a \sinh u \cos \theta)\mathbf{i} + (b \sinh u \sin \theta)\mathbf{j} + (c \cosh u)\mathbf{k}$, $0 \leq \theta \leq 2\pi, -\infty < u < \infty$
37. $\mathbf{p} = \mathbf{k}, \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (2y)^2 + (-1)^2} = \sqrt{4x^2 + 4y^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1$;
 $z = 2 \Rightarrow x^2 + y^2 = 2$; thus $S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{4x^2 + 4y^2 + 1} dx dy = \iint_R \sqrt{4r^2 \cos^2 \theta + 4r^2 \sin^2 \theta + 1} r dr d\theta$
 $= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \left[\frac{1}{12} (4r^2 + 1)^{3/2} \right]_0^{\sqrt{2}} d\theta = \int_0^{2\pi} \frac{13}{6} d\theta = \frac{13}{3} \pi$

38. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $2 \leq x^2 + y^2 \leq 6 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA$
- $$= \iint_R \sqrt{4x^2 + 4y^2 + 1} dx dy = \iint_R \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \int_{\sqrt{2}}^{\sqrt{6}} \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \left[\frac{1}{12} (4r^2 + 1)^{3/2} \right]_{\sqrt{2}}^{\sqrt{6}} d\theta$$
- $$= \int_0^{2\pi} \frac{49}{6} d\theta = \frac{49}{3} \pi$$
39. $\mathbf{p} = \mathbf{k}$, $\nabla f = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow |\nabla f| = 3$ and $|\nabla f \cdot \mathbf{p}| = 2$; $x = y^2$ and $x = 2 - y^2$ intersect at $(1, 1)$ and $(1, -1)$
- $$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \frac{3}{2} dx dy = \int_{-1}^1 \int_{y^2}^{2-y^2} \frac{3}{2} dx dy = \int_{-1}^1 (3 - 3y^2) dy = 4$$
40. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} - 2\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4} = 2\sqrt{x^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 2 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA$
- $$= \iint_R \frac{2\sqrt{x^2 + 1}}{2} dx dy = \int_0^{\sqrt{3}} \int_0^x \sqrt{x^2 + 1} dy dx = \int_0^{\sqrt{3}} x\sqrt{x^2 + 1} dx = \left[\frac{1}{3} (x^2 + 1)^{3/2} \right]_0^{\sqrt{3}} = \frac{1}{3} (4)^{3/2} - \frac{1}{3} = \frac{7}{3}$$
41. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (-2)^2 + (-2)^2} = \sqrt{4x^2 + 8} = 2\sqrt{x^2 + 2}$ and $|\nabla f \cdot \mathbf{p}| = 2$
- $$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \frac{2\sqrt{x^2 + 2}}{2} dx dy = \int_0^2 \int_0^{3x} \sqrt{x^2 + 2} dy dx = \int_0^2 3x\sqrt{x^2 + 2} dx = \left[(x^2 + 2)^{3/2} \right]_0^2 = 6\sqrt{6} - 2\sqrt{2}$$
42. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{8} = 2\sqrt{2}$ and $|\nabla f \cdot \mathbf{p}| = 2z$; $x^2 + y^2 + z^2 = 2$ and $z = \sqrt{x^2 + y^2} \Rightarrow x^2 + y^2 = 1$; thus, $S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \frac{2\sqrt{2}}{2z} dA = \sqrt{2} \iint_R \frac{1}{z} dA$
- $$= \sqrt{2} \iint_R \frac{1}{\sqrt{2 - (x^2 + y^2)}} dA = \sqrt{2} \int_0^{2\pi} \int_0^1 \frac{r dr d\theta}{\sqrt{2 - r^2}} = \sqrt{2} \int_0^{2\pi} (-1 + \sqrt{2}) d\theta = 2\pi(2 - \sqrt{2})$$
43. $\mathbf{p} = \mathbf{k}$, $\nabla f = c\mathbf{i} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{c^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{c^2 + 1} dx dy$
- $$= \int_0^{2\pi} \int_0^1 \sqrt{c^2 + 1} r dr d\theta = \int_0^{2\pi} \frac{\sqrt{c^2 + 1}}{2} d\theta = \pi\sqrt{c^2 + 1}$$
44. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} + 2z\mathbf{j} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (2z)^2} = 2$ and $|\nabla f \cdot \mathbf{p}| = 2z$ for the upper surface, $z \geq 0$
- $$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \frac{2}{2z} dA = \iint_R \frac{1}{\sqrt{1 - x^2}} dx dy = 2 \int_{-1/2}^{1/2} \int_0^{1/2} \frac{1}{\sqrt{1 - x^2}} dy dx = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1 - x^2}} dx$$
- $$= \left[\sin^{-1} x \right]_{-1/2}^{1/2} = \frac{\pi}{6} - \left(-\frac{\pi}{6} \right) = \frac{\pi}{3}$$

45. $\mathbf{p} = \mathbf{i}$, $\nabla f = \mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{1^2 + (2y)^2 + (2z)^2} = \sqrt{1 + 4y^2 + 4z^2}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $1 \leq y^2 + z^2 \leq 4$
 $\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{1 + 4y^2 + 4z^2} dy dz = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2 \cos^2 \theta + 4r^2 \sin^2 \theta} r dr d\theta$
 $= \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} r dr d\theta = \int_0^{2\pi} \left[\frac{1}{12} (1 + 4r^2)^{3/2} \right]_1^2 d\theta = \int_0^{2\pi} \frac{1}{12} (17\sqrt{17} - 5\sqrt{5}) d\theta = \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5})$
46. $\mathbf{p} = \mathbf{j}$, $\nabla f = 2x\mathbf{i} + \mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4z^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $y = 0$ and $x^2 + y + z^2 = 2 \Rightarrow x^2 + z^2 = 2$;
 thus $S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{4x^2 + 4z^2 + 1} dx dz = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \frac{13}{6} d\theta = \frac{13}{3} \pi$
47. $\mathbf{p} = \mathbf{k}$, $\nabla f = \left(2x - \frac{2}{x}\right)\mathbf{i} + \sqrt{15}\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{\left(2x - \frac{2}{x}\right)^2 + (\sqrt{15})^2 + (-1)^2} = \sqrt{4x^2 + 8 + \frac{4}{x^2}} = \sqrt{\left(2x + \frac{2}{x}\right)^2}$
 $= 2x + \frac{2}{x}$, on $1 \leq x \leq 2$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \left(2x + 2x^{-1}\right) dx dy = \int_0^1 \int_1^2 \left(2x + 2x^{-1}\right) dx dy$
 $= \int_0^1 \left[x^2 + 2 \ln x \right]_1^2 dy = \int_0^1 (3 + 2 \ln 2) dy = 3 + 2 \ln 2$
48. $\mathbf{p} = \mathbf{k}$, $\nabla f = 3\sqrt{x}\mathbf{i} + 3\sqrt{y}\mathbf{j} - 3\mathbf{k} \Rightarrow |\nabla f| = \sqrt{9x + 9y + 9} = 3\sqrt{x + y + 1}$ and $|\nabla f \cdot \mathbf{p}| = 3$
 $\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{x + y + 1} dx dy = \int_0^1 \int_0^1 \sqrt{x + y + 1} dx dy = \int_0^1 \left[\frac{2}{3} (x + y + 1)^{3/2} \right]_0^1 dy$
 $= \int_0^1 \left[\frac{2}{3} (y + 2)^{3/2} - \frac{2}{3} (y + 1)^{3/2} \right] dy = \left[\frac{4}{15} (y + 2)^{5/2} - \frac{4}{15} (y + 1)^{5/2} \right]_0^1 = \frac{4}{15} \left[(3)^{5/2} - (2)^{5/2} - (2)^{5/2} + 1 \right]$
 $= \frac{4}{15} (9\sqrt{3} - 8\sqrt{2} + 1)$
49. $f_x(x, y) = 2x$, $f_y(x, y) = 2y \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1} \Rightarrow \text{Area} = \iint_R \sqrt{4x^2 + 4y^2 + 1} dx dy$
 $= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{4r^2 + 1} r dr d\theta = \frac{\pi}{6} (13\sqrt{13} - 1)$
50. $f_x(y, z) = -2y$, $f_z(y, z) = -2z \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{4y^2 + 4z^2 + 1} \Rightarrow \text{Area} = \iint_R \sqrt{4y^2 + 4z^2 + 1} dy dz$
 $= \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1} r dr d\theta = \frac{\pi}{6} (5\sqrt{5} - 1)$
51. $f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$, $f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}} \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 1} = \sqrt{2} \Rightarrow \text{Area} = \iint_{R_{xy}} \sqrt{2} dx dy$
 $= \sqrt{2} \text{ (Area between the ellipse and the circle)} = \sqrt{2} (6\pi - \pi) = 5\pi\sqrt{2}$
52. Over R_{xy} : $z = 2 - \frac{2}{3}x - 2y \Rightarrow f_x(x, y) = -\frac{2}{3}$, $f_y(x, y) = -2 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{4}{9} + 4 + 1} = \frac{7}{3}$
 $\Rightarrow \text{Area} = \iint_{R_{xy}} \frac{7}{3} dA = \frac{7}{3} \text{ (Area of the shadow triangle in the } xy\text{-plane)} = \left(\frac{7}{3}\right)\left(\frac{3}{2}\right) = \frac{7}{2}.$

Over R_{xz} : $y = 1 - \frac{1}{3}x - \frac{1}{2}z \Rightarrow f_x(x, z) = -\frac{1}{3}, f_z(x, z) = -\frac{1}{2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{\frac{1}{9} + \frac{1}{4} + 1} = \frac{7}{6}$
 $\Rightarrow \text{Area} = \iint_{R_{xz}} \frac{7}{6} dA = \frac{7}{6}$ (Area of the shadow triangle in the xz -plane) $= \left(\frac{7}{6}\right)(3) = \frac{7}{2}$.

Over R_{yz} : $x = 3 - 3y - \frac{3}{2}z \Rightarrow f_y(y, z) = -3, f_z(y, z) = -\frac{3}{2} \Rightarrow \sqrt{f_y^2 + f_z^2 + 1} = \sqrt{9 + \frac{9}{4} + 1} = \frac{7}{2}$
 $\Rightarrow \text{Area} = \iint_{R_{yz}} \frac{7}{2} dA = \frac{7}{2}$ (Area of the shadow triangle in the yz -plane) $= \left(\frac{7}{2}\right)(1) = \frac{7}{2}$.

53. $y = \frac{2}{3}z^{3/2} \Rightarrow f_x(x, z) = 0, f_z(x, z) = z^{1/2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{z+1}; y = \frac{16}{3} \Rightarrow \frac{2}{3}z^{3/2} \Rightarrow z = 4$
 $\Rightarrow \text{Area} = \int_0^4 \int_0^1 \sqrt{z+1} dx dz = \int_0^4 \sqrt{z+1} dz = \frac{2}{3}(5\sqrt{5} - 1)$

54. $y = 4 - z \Rightarrow f_x(x, z) = 0, f_z(x, z) = -1 \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{2} \Rightarrow \text{Area} = \iint_{R_{xz}} \sqrt{2} dA = \int_0^2 \int_0^{4-z^2} \sqrt{2} dx dz$
 $= \sqrt{2} \int_0^2 (4 - z^2) dz = \frac{16\sqrt{2}}{3}$

55. $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k} \Rightarrow \mathbf{r}_x(x, y) = \mathbf{i} + f_x(x, y)\mathbf{k}, \mathbf{r}_y(x, y) = \mathbf{j} + f_y(x, y)\mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x, y) \\ 0 & 1 & f_y(x, y) \end{vmatrix} = -f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}$$

$$\Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{(-f_x(x, y))^2 + (-f_y(x, y))^2 + 1^2} = \sqrt{f_x(x, y)^2 + f_y(x, y)^2 + 1}$$

$$\Rightarrow d\sigma = \sqrt{f_x(x, y)^2 + f_y(x, y)^2 + 1} dA$$

56. S is obtained by rotating $y = f(x), a \leq x \leq b$ about the x -axis where $f(x) \geq 0$

(a) Let (x, y, z) be a point on S . Consider the cross section when $x = x^*$, the cross section is a circle with radius $r = f(x^*)$. The set of parametric equations for this circle are given by $y(\theta) = r \cos \theta = f(x^*) \cos \theta$ and $z(\theta) = r \sin \theta = f(x^*) \sin \theta$ where $0 \leq \theta \leq 2\pi$. Since x can take on any value between a and b we have $x(x, \theta) = x, y(x, \theta) = f(x) \cos \theta, z(x, \theta) = f(x) \sin \theta$ where $a \leq x \leq b$ and $0 \leq \theta \leq 2\pi$. Thus $\mathbf{r}(x, \theta) = x\mathbf{i} + f(x) \cos \theta \mathbf{j} + f(x) \sin \theta \mathbf{k}$

(b) $\mathbf{r}_x(x, \theta) = \mathbf{i} + f'(x) \cos \theta \mathbf{j} + f'(x) \sin \theta \mathbf{k}$ and $\mathbf{r}_\theta(x, \theta) = -f(x) \sin \theta \mathbf{j} + f(x) \cos \theta \mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x) \cos \theta & f'(x) \sin \theta \\ 0 & -f(x) \sin \theta & f(x) \cos \theta \end{vmatrix} = f(x) \cdot f'(x) \mathbf{i} - f(x) \cos \theta \mathbf{j} - f(x) \sin \theta \mathbf{k}$$

$$\Rightarrow |\mathbf{r}_x \times \mathbf{r}_\theta| = \sqrt{(f(x) \cdot f'(x))^2 + (-f(x) \cos \theta)^2 + (-f(x) \sin \theta)^2} = f(x) \sqrt{1 + (f'(x))^2}$$

$$A = \int_a^b \int_0^{2\pi} f(x) \sqrt{1 + (f'(x))^2} d\theta dx = \int_a^b \left[f(x) \sqrt{1 + (f'(x))^2} \theta \right]_0^{2\pi} dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

16.6 SURFACE INTEGRALS

$$\begin{aligned}
 1. \text{ Let the parametrization be } \mathbf{r}(x, z) &= x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j} \text{ and } \mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
 &= 2x\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2 + 1} \Rightarrow \iint_S G(x, y, z) d\sigma = \int_0^3 \int_0^2 x\sqrt{4x^2 + 1} dx dz = \int_0^3 \left[\frac{1}{12} (4x^2 + 1)^{3/2} \right]_0^2 dz \\
 &= \int_0^3 \frac{1}{12} (17\sqrt{17} - 1) dz = \frac{17\sqrt{17} - 1}{4}
 \end{aligned}$$

$$2. \text{ Let the parametrization be } \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \sqrt{4 - y^2}\mathbf{k}, -2 \leq y \leq 2 \Rightarrow \mathbf{r}_x = \mathbf{i} \text{ and } \mathbf{r}_y = \mathbf{j} - \frac{y}{\sqrt{4 - y^2}}\mathbf{k}$$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{y}{\sqrt{4 - y^2}} \end{vmatrix} = \frac{y}{\sqrt{4 - y^2}}\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\frac{y^2}{4 - y^2} + 1} = \frac{2}{\sqrt{4 - y^2}}$$

$$\Rightarrow \iint_S G(x, y, z) d\sigma = \int_1^4 \int_{-2}^2 \sqrt{4 - y^2} \left(\frac{2}{\sqrt{4 - y^2}} \right) dy dx = 24$$

$$3. \text{ Let the parametrization be } \mathbf{r}(\phi, \theta) = (\sin \phi \cos \theta)\mathbf{i} + (\sin \phi \sin \theta)\mathbf{j} + (\cos \phi)\mathbf{k} \text{ (spherical coordinates with } \rho = 1 \text{ on the sphere), } 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi \Rightarrow \mathbf{r}_\phi = (\cos \phi \cos \theta)\mathbf{i} + (\cos \phi \sin \theta)\mathbf{j} - (\sin \phi)\mathbf{k} \text{ and }$$

$$\begin{aligned}
 \mathbf{r}_\theta &= (-\sin \phi \sin \theta)\mathbf{i} + (\sin \phi \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \phi \sin \theta & \sin \phi \cos \theta & 0 \end{vmatrix} \\
 &= (\sin^2 \phi \cos \theta)\mathbf{i} + (\sin^2 \phi \sin \theta)\mathbf{j} + (\sin \phi \cos \theta)\mathbf{k} \Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{\sin^4 \phi \cos^2 \theta + \sin^4 \phi \sin^2 \theta + \sin^2 \phi \cos^2 \phi} \\
 &= \sin \phi; x = \sin \phi \cos \theta \Rightarrow G(x, y, z) = \cos^2 \theta \sin^2 \phi \Rightarrow \iint_S G(x, y, z) d\sigma = \int_0^{2\pi} \int_0^\pi (\cos^2 \theta \sin^2 \phi) (\sin \phi) d\phi d\theta \\
 &= \int_0^{2\pi} \int_0^\pi (\cos^2 \theta) (1 - \cos^2 \phi) (\sin \phi) d\phi d\theta; \begin{bmatrix} u = \cos \phi \\ du = -\sin \phi d\phi \end{bmatrix} \rightarrow \int_0^{2\pi} \int_1^{-1} (\cos^2 \theta) (u^2 - 1) du d\theta \\
 &= \int_0^{2\pi} (\cos^2 \theta) \left[\frac{u^3}{3} - u \right]_1^{-1} d\theta = \frac{4}{3} \int_0^{2\pi} \cos^2 \theta d\theta = \frac{4}{3} \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi} = \frac{4\pi}{3}
 \end{aligned}$$

$$4. \text{ Let the parametrization be } \mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k} \text{ (spherical coordinates with } \rho = a, a \geq 0, \text{ on the sphere), } 0 \leq \phi \leq \frac{\pi}{2} \text{ (since } z \geq 0), 0 \leq \theta \leq 2\pi$$

$$\Rightarrow \mathbf{r}_\phi = (a \cos \phi \cos \theta)\mathbf{i} + (a \cos \phi \sin \theta)\mathbf{j} - (a \sin \phi)\mathbf{k} \text{ and } \mathbf{r}_\theta = (-a \sin \phi \sin \theta)\mathbf{i} + (a \sin \phi \cos \theta)\mathbf{j}$$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} = (a^2 \sin^2 \phi \cos \theta)\mathbf{i} + (a^2 \sin^2 \phi \sin \theta)\mathbf{j} + (a^2 \sin \phi \cos \phi)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4 \sin^4 \phi \cos^2 \theta + a^4 \sin^4 \phi \sin^2 \theta + a^4 \sin^2 \phi \cos^2 \phi} = a^2 \sin \phi; z = a \cos \phi$$

$$\Rightarrow G(x, y, z) = a^2 \cos^2 \phi \Rightarrow \iint_S G(x, y, z) d\sigma = \int_0^{2\pi} \int_0^{\pi/2} (a^2 \cos^2 \phi) (a^2 \sin \phi) d\phi d\theta = \frac{2}{3} \pi a^4$$

5. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - x - y)\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$

$$= \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{3} \Rightarrow \iint_S F(x, y, z) d\sigma = \int_0^1 \int_0^1 (4 - x - y) \sqrt{3} dy dx = \int_0^1 \sqrt{3} \left[4y - xy - \frac{y^2}{2} \right]_0^1 dx$$

$$= \int_0^1 \sqrt{3} \left(\frac{7}{2} - x \right) dx = \sqrt{3} \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^1 = 3\sqrt{3}$$

6. Let the parametrization be $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}$, $0 \leq r \leq 1$ (since $0 \leq z \leq 1$) and $0 \leq \theta \leq 2\pi$

$$\Rightarrow \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix}$$

$$= (-r \cos \theta)\mathbf{i} - (r \sin \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(-r \cos \theta)^2 + (-r \sin \theta)^2 + r^2} = r\sqrt{2}; z = r \text{ and } x = r \cos \theta$$

$$\Rightarrow F(x, y, z) = r - r \cos \theta \Rightarrow \iint_S F(x, y, z) d\sigma = \int_0^{2\pi} \int_0^1 (r - r \cos \theta) (r\sqrt{2}) dr d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 (1 - \cos \theta) r^2 dr d\theta = \frac{2\pi\sqrt{2}}{3}$$

7. Let the parametrization be $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (1 - r^2)\mathbf{k}$, $0 \leq r \leq 1$ (since $0 \leq z \leq 1$) and $0 \leq \theta \leq 2\pi \Rightarrow \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} - 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = (2r^2 \cos \theta)\mathbf{i} + (2r^2 \sin \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta|$$

$$= \sqrt{(2r^2 \cos \theta)^2 + (2r^2 \sin \theta)^2 + r^2} = r\sqrt{1 + 4r^2}; z = 1 - r^2 \text{ and } x = r \cos \theta \Rightarrow H(x, y, z) = (r^2 \cos^2 \theta) \sqrt{1 + 4r^2}$$

$$\Rightarrow \iint_S H(x, y, z) d\sigma = \int_0^{2\pi} \int_0^1 (r^2 \cos^2 \theta) (\sqrt{1 + 4r^2}) (r\sqrt{1 + 4r^2}) dr d\theta = \int_0^{2\pi} \int_0^1 r^3 (1 + 4r^2) \cos^2 \theta dr d\theta = \frac{11\pi}{12}$$

8. Let the parametrization be $\mathbf{r}(\phi, \theta) = (2 \sin \phi \cos \theta)\mathbf{i} + (2 \sin \phi \sin \theta)\mathbf{j} + (2 \cos \phi)\mathbf{k}$ (spherical coordinates with $\rho = 2$ on the sphere), $0 \leq \phi \leq \frac{\pi}{4}$; $x^2 + y^2 + z^2 = 4$ and $z = \sqrt{x^2 + y^2} = z^2 + z^2 = 4 \Rightarrow z^2 = 2 \Rightarrow z = \sqrt{2}$ (since $z \geq 0$) $\Rightarrow 2 \cos \phi = \sqrt{2} \Rightarrow \cos \phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$, $0 \leq \theta \leq 2\pi$; $\mathbf{r}_\phi = (2 \cos \phi \cos \theta)\mathbf{i} + (2 \cos \phi \sin \theta)\mathbf{j} - (2 \sin \phi)\mathbf{k}$

$$\text{and } \mathbf{r}_\theta = (-2 \sin \phi \sin \theta)\mathbf{i} + (2 \sin \phi \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 \cos \phi \cos \theta & 2 \cos \phi \sin \theta & -2 \sin \phi \\ -2 \sin \phi \sin \theta & 2 \sin \phi \cos \theta & 0 \end{vmatrix}$$

$$= (4 \sin^2 \phi \cos \theta)\mathbf{i} + (4 \sin^2 \phi \sin \theta)\mathbf{j} + (4 \sin \phi \cos \phi)\mathbf{k}$$

$$\begin{aligned} \Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{16 \sin^4 \phi \cos^2 \theta + 16 \sin^4 \phi \sin^2 \theta + 16 \sin^2 \phi \cos^2 \phi} = 4 \sin \phi; y = 2 \sin \phi \sin \theta \text{ and} \\ z = 2 \cos \phi &\Rightarrow H(x, y, z) = 4 \cos \phi \sin \phi \sin \theta \Rightarrow \iint_S H(x, y, z) d\sigma = \int_0^{2\pi} \int_0^{\pi/4} (4 \cos \phi \sin \phi \sin \theta)(4 \sin \phi) d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} 16 \sin^2 \phi \cos \phi \sin \theta d\phi d\theta = 0 \end{aligned}$$

9. The bottom face S of the cube is in the xy -plane $\Rightarrow z = 0 \Rightarrow G(x, y, 0) = x + y$ and $f(x, y, z) = z = 0 \Rightarrow \mathbf{p} = \mathbf{k}$

$$\text{and } \nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx dy \Rightarrow \iint_S G d\sigma = \iint_R (x + y) dx dy = \int_0^a \int_0^a (x + y) dx dy$$

$$= \int_0^a \left(\frac{a^2}{2} + ay \right) dy = a^3. \text{ Because of symmetry, we also get } a^3 \text{ over the face of the cube in the } xz\text{-plane and}$$

a^3 over the face of the cube in the yz -plane. Next, on the top of the cube, $G(x, y, z) = G(x, y, a) = x + y + a$ and $f(x, y, z) = z = a \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx dy$

$$\Rightarrow \iint_S G d\sigma = \iint_R (x + y + a) dx dy = \int_0^a \int_0^a (x + y + a) dx dy = \int_0^a \int_0^a (x + y) dx dy + \int_0^a \int_0^a a dx dy = 2a^3. \text{ Because of}$$

symmetry, the integral is also $2a^3$ over each of the other two faces. Therefore,

$$\iint_{\text{cube}} (x + y + z) d\sigma = 3(a^3 + 2a^3) = 9a^3.$$

10. On the face S in the xz -plane, we have $y = 0 \Rightarrow f(x, y, z) = y = 0$ and $G(x, y, z) = G(x, 0, z) = z \Rightarrow \mathbf{p} = \mathbf{j}$ and

$$\nabla f = \mathbf{j} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx dz \Rightarrow \iint_S G d\sigma = \iint_S (y + z) d\sigma = \int_0^1 \int_0^2 z dx dz = \int_0^1 2z dz = 1.$$

On the face in the xy -plane, we have $z = 0 \Rightarrow f(x, y, z) = z = 0$ and $G(x, y, z) = G(x, y, 0) = y \Rightarrow \mathbf{p} = \mathbf{k}$ and

$$\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx dy \Rightarrow \iint_S G d\sigma = \iint_S y d\sigma = \int_0^1 \int_0^2 y dx dy = 1.$$

On the triangular face in the plane $x = 2$ we have $f(x, y, z) = x = 2$ and $G(x, y, z) = G(2, y, z) = y + z$

$$\Rightarrow \mathbf{p} = \mathbf{i} \text{ and } \nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz dy \Rightarrow \iint_S G d\sigma = \iint_S (y + z) d\sigma$$

$$= \int_0^1 \int_0^{1-y} (y + z) dz dy = \int_0^1 \frac{1}{2} (1 - y^2) dy = \frac{1}{3}.$$

On the triangular face in the yz -plane we have $x = 0 \Rightarrow f(x, y, z) = x = 0$ and $G(x, y, z) = G(0, y, z) = y + z$

$$\Rightarrow \mathbf{p} = \mathbf{i} \text{ and } \nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz dy \Rightarrow \iint_S G d\sigma = \iint_S (y + z) d\sigma$$

$$= \int_0^1 \int_0^{1-y} (y + z) dz dy = \frac{1}{3}.$$

Finally, on the sloped face, we have $y + z = 1 \Rightarrow f(x, y, z) = y + z = 1$ and $G(x, y, z) = y + z = 1 \Rightarrow \mathbf{p} = \mathbf{k}$ and

$$\nabla f = \mathbf{j} + \mathbf{k} \Rightarrow |\nabla f| = \sqrt{2} \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{2} dx dy \Rightarrow \iint_S G d\sigma = \iint_S (y + z) d\sigma = \int_0^1 \int_0^2 \sqrt{2} dx dy = 2\sqrt{2}.$$

Therefore, $\iint_{\text{wedge}} G(x, y, z) d\sigma = 1 + 1 + \frac{1}{3} + \frac{1}{3} + 2\sqrt{2} = \frac{8}{3} + 2\sqrt{2}$

11. On the faces in the coordinate planes, $G(x, y, z) = 0 \Rightarrow$ the integral over these faces is 0.

On the face $x = a$, we have $f(x, y, z) = x = a$ and $G(x, y, z) = G(a, y, z) = ayz \Rightarrow \mathbf{p} = \mathbf{i}$ and

$$\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dy dz \Rightarrow \iint_S G d\sigma = \iint_S ayz d\sigma = \int_0^c \int_0^b ayz dy dz = \frac{ab^2c^2}{4}.$$

On the face $y = b$, we have $f(x, y, z) = y = b$ and $G(x, y, z) = G(x, b, z) = bxz \Rightarrow \mathbf{p} = \mathbf{j}$ and

$$\nabla f = \mathbf{j} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx dz \Rightarrow \iint_S G d\sigma = \iint_S bxz d\sigma = \int_0^c \int_0^a bxz dx dz = \frac{a^2bc^2}{4}.$$

On the face $z = c$, we have $f(x, y, z) = z = c$ and $G(x, y, z) = G(x, y, c) = cxy \Rightarrow \mathbf{p} = \mathbf{k}$ and

$$\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dy dx \Rightarrow \iint_S G d\sigma = \iint_S cxy d\sigma = \int_0^b \int_0^a cxy dx dy = \frac{a^2b^2c}{4}.$$

$$\text{Therefore, } \iint_S G(x, y, z) d\sigma = \frac{abc(ab+ac+bc)}{4}.$$

12. On the face $x = a$, we have $f(x, y, z) = x = a$ and $G(x, y, z) = G(a, y, z) = ayz \Rightarrow \mathbf{p} = \mathbf{i}$ and

$$\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz dy \Rightarrow \iint_S G d\sigma = \iint_S ayz d\sigma = \int_{-b}^b \int_{-c}^c ayz dz dy = 0. \text{ Because of the symmetry of } G \text{ on all the other faces, all the integrals are 0, and } \iint_S G(x, y, z) d\sigma = 0.$$

13. $f(x, y, z) = 2x + 2y + z = 2 \Rightarrow \nabla f = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $G(x, y, z) = x + y + (2 - 2x - 2y) = 2 - x - y \Rightarrow \mathbf{p} = \mathbf{k}$,

$$|\nabla f| = 3 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = 3 dy dx; \quad z = 0 \Rightarrow 2x + 2y = 2 \Rightarrow y = 1 - x \Rightarrow \iint_S G d\sigma = \iint_S (2 - x - y) d\sigma \\ = 3 \int_0^1 \int_0^{1-x} (2 - x - y) dy dx = 3 \int_0^1 \left[(2 - x)(1 - x) - \frac{1}{2}(1 - x)^2 \right] dx = 3 \int_0^1 \left(\frac{3}{2} - 2x + \frac{x^2}{2} \right) dx = 2$$

14. $f(x, y, z) = y^2 + 4z = 16 \Rightarrow \nabla f = 2y\mathbf{j} + 4\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 16} = 2\sqrt{y^2 + 4}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 4$

$$\Rightarrow d\sigma = \frac{2\sqrt{y^2+4}}{4} dx dy \Rightarrow \iint_S G d\sigma = \int_{-4}^4 \int_0^1 \left(x\sqrt{y^2+4} \right) \left(\frac{\sqrt{y^2+4}}{2} \right) dx dy = \int_{-4}^4 \int_0^1 \frac{x(y^2+4)}{2} dx dy \\ = \int_{-4}^4 \frac{1}{4} (y^2 + 4) dy = \frac{1}{2} \left[\frac{y^3}{3} + 4y \right]_0^4 = \frac{1}{2} \left(\frac{64}{3} + 16 \right) = \frac{56}{3}$$

15. $f(x, y, z) = x + y^2 - z = 0 \Rightarrow \nabla f = \mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 2} = \sqrt{2}\sqrt{2y^2 + 1}$ and

$$\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{2}\sqrt{2y^2+1}}{1} dx dy \Rightarrow \iint_S G d\sigma = \int_0^1 \int_0^y (x + y^2 - x) \sqrt{2}\sqrt{2y^2+1} dx dy \\ = \sqrt{2} \int_0^1 \int_0^y y^2 \sqrt{2y^2+1} dx dy = \sqrt{2} \int_0^1 y^3 \sqrt{2y^2+1} dy = \frac{6\sqrt{6}+\sqrt{2}}{30}$$

16. $f(x, y, z) = x^2 + y - z = 0 \Rightarrow \nabla f = 2x\mathbf{i} + \mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 2} = \sqrt{2}\sqrt{2x^2 + 1}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1$

$$\Rightarrow d\sigma = \frac{\sqrt{2}\sqrt{2x^2+1}}{1} dx dy \Rightarrow \iint_S G d\sigma = \int_{-1}^1 \int_0^1 x\sqrt{2}\sqrt{2x^2+1} dx dy = \sqrt{2} \int_{-1}^1 \int_0^1 x\sqrt{2x^2+1} dx dy \\ = \frac{3\sqrt{6}-\sqrt{2}}{6} \int_0^1 dy = \frac{3\sqrt{6}-\sqrt{2}}{3}$$

$$\begin{aligned}
 17. \quad f(x, y, z) = 2x + y + z = 2 &\Rightarrow \nabla f = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\nabla f| = \sqrt{6} \text{ and } \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{6}}{1} dy \, dx \\
 &\Rightarrow \iint_S G \, d\sigma = \int_0^1 \int_{1-x}^{2-2x} xy(2-2x-y)\sqrt{6} \, dy \, dx = \sqrt{6} \int_0^1 \int_{1-x}^{2-2x} (2xy - 2x^2y - xy^2) \, dy \, dx \\
 &= \sqrt{6} \int_0^1 \left(\frac{2}{3}x - 2x^2 + 2x^3 - \frac{2}{3}x^4 \right) dx = \frac{\sqrt{6}}{30}
 \end{aligned}$$

$$\begin{aligned}
 18. \quad f(x, y, z) = x + y = 1 &\Rightarrow \nabla f = \mathbf{i} + \mathbf{j} \Rightarrow |\nabla f| = \sqrt{2} \text{ and } \mathbf{p} = \mathbf{j} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{2}}{1} dz \, dx \\
 &\Rightarrow \iint_S G \, d\sigma = \int_0^1 \int_0^1 (x - (1-x) - z) \sqrt{2} \, dz \, dx = \sqrt{2} \int_0^1 \int_0^1 (2x - z - 1) \, dz \, dx = \sqrt{2} \int_0^1 \left(2x - \frac{3}{2} \right) dx = -\frac{\sqrt{2}}{2}
 \end{aligned}$$

$$\begin{aligned}
 19. \quad \text{Let the parametrization be } \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - y^2)\mathbf{k}, \quad 0 \leq x \leq 1, -2 \leq y \leq 2; z = 0 \Rightarrow 0 = 4 - y^2 \\
 \Rightarrow y = \pm 2; \mathbf{r}_x = \mathbf{i} \text{ and } \mathbf{r}_y = \mathbf{j} - 2y\mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} |\mathbf{r}_x \times \mathbf{r}_y| \, dy \, dx \\
 = (2xy - 3z) \, dy \, dx = [2xy - 3(4 - y^2)] \, dy \, dx \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^1 \int_{-2}^2 (2xy + 3y^2 - 12) \, dy \, dx \\
 = \int_0^1 [xy^2 + y^3 - 12y]_{-2}^2 \, dx = \int_0^1 (-32) \, dx = -32
 \end{aligned}$$

$$\begin{aligned}
 20. \quad \text{Let the parametrization be } \mathbf{r}(x, y) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}, \quad -1 \leq x \leq 1, 0 \leq z \leq 2 \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j} \text{ and } \mathbf{r}_z = \mathbf{k} \\
 \Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} |\mathbf{r}_x \times \mathbf{r}_z| \, dz \, dx = -x^2 dz \, dx \\
 \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_{-1}^1 \int_0^2 (-x^2) \, dz \, dx = -\frac{4}{3}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \text{Let the parametrization be } \mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k} \text{ (spherical coordinates with } \\
 \rho = a, a \geq 0, \text{ on the sphere), } 0 \leq \phi \leq \frac{\pi}{2} \text{ (for the first octant), } 0 \leq \theta \leq \frac{\pi}{2} \text{ (for the first octant)} \\
 \Rightarrow \mathbf{r}_\phi = (a \cos \phi \cos \theta)\mathbf{i} + (a \cos \phi \sin \theta)\mathbf{j} - (a \sin \phi)\mathbf{k} \text{ and } \mathbf{r}_\theta = (-a \sin \phi \sin \theta)\mathbf{i} + (a \sin \phi \cos \theta)\mathbf{j} \\
 \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} = (a^2 \sin^2 \phi \cos \theta)\mathbf{i} + (a^2 \sin^2 \phi \sin \theta)\mathbf{j} + (a^2 \sin \phi \cos \phi)\mathbf{k} \\
 \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, d\theta \, d\phi = a^3 \cos^2 \phi \sin \phi \, d\theta \, d\phi \text{ since } \mathbf{F} = z\mathbf{k} = (a \cos \phi)\mathbf{k} \\
 \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} a^3 \cos^2 \phi \sin \phi \, d\phi \, d\theta = \frac{\pi a^3}{6}
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \text{Let the parametrization be } \mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k} \text{ (spherical coordinates with } \\
 \rho = a, a \geq 0, \text{ on the sphere), } 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi \\
 \Rightarrow \mathbf{r}_\phi = (a \cos \phi \cos \theta)\mathbf{i} + (a \cos \phi \sin \theta)\mathbf{j} - (a \sin \phi)\mathbf{k} \text{ and } \mathbf{r}_\theta = (-a \sin \phi \sin \theta)\mathbf{i} + (a \sin \phi \cos \theta)\mathbf{j}
 \end{aligned}$$

$$\begin{aligned}
\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} = (a^2 \sin^2 \phi \cos \theta) \mathbf{i} + (a^2 \sin^2 \phi \sin \theta) \mathbf{j} + (a^2 \sin \phi \cos \phi) \mathbf{k} \\
\Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \mathbf{F} \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, d\theta \, d\phi = (a^3 \sin^3 \phi \cos^2 \theta + a^3 \sin^3 \phi \sin^2 \theta + a^3 \sin \phi \cos^2 \phi) \, d\theta \, d\phi \\
&= a^3 \sin \phi \, d\theta \, d\phi \text{ since } \mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k} \\
\Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \int_0^{2\pi} \int_0^\pi a^3 \sin \phi \, d\phi \, d\theta = 4\pi a^3
\end{aligned}$$

23. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (2a - x - y)\mathbf{k}$, $0 \leq x \leq a$, $0 \leq y \leq a \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k}$

$$\begin{aligned}
\Rightarrow \mathbf{r}_x \times \mathbf{r}_y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} |\mathbf{r}_x \times \mathbf{r}_y| \, dy \, dx \\
&= [2xy + 2y(2a - x - y) + 2x(2a - x - y)] \, dy \, dx \text{ since } \mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k} \\
&= 2xy\mathbf{i} + 2y(2a - x - y)\mathbf{j} + 2x(2a - x - y)\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma \\
&= \int_0^a \int_0^a [2xy + 2y(2a - x - y) + 2x(2a - x - y)] \, dy \, dx = \int_0^a \int_0^a (4ay - 2y^2 + 4ax - 2x^2 - 2xy) \, dy \, dx \\
&= \int_0^a \left(\frac{4}{3}a^3 + 3a^2x - 2ax^2 \right) dx = \left(\frac{4}{3} + \frac{3}{2} - \frac{2}{3} \right) a^4 = \frac{13a^4}{6}
\end{aligned}$$

24. Let the parametrization be $\mathbf{r}(\theta, z) = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + z\mathbf{k}$, $0 \leq z \leq a$, $0 \leq \theta \leq 2\pi$ (where $r = \sqrt{x^2 + y^2} = 1$ on the cylinder) $\Rightarrow \mathbf{r}_\theta = (-\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$

$$\begin{aligned}
\Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_z}{|\mathbf{r}_\theta \times \mathbf{r}_z|} |\mathbf{r}_\theta \times \mathbf{r}_z| \, dz \, d\theta = (\cos^2 \theta + \sin^2 \theta) \, dz \, d\theta = dz \, d\theta, \text{ since } \mathbf{F} = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + z\mathbf{k} \\
\Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \int_0^{2\pi} \int_0^a 1 \, dz \, d\theta = 2\pi a
\end{aligned}$$

25. Let the parametrization be $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}$, $0 \leq r \leq 1$ (since $0 \leq z \leq 1$) and $0 \leq \theta \leq 2\pi$

$$\begin{aligned}
\Rightarrow \mathbf{r}_r &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & 1 \end{vmatrix} \\
&= (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|} |\mathbf{r}_\theta \times \mathbf{r}_r| \, d\theta \, dr = (r^3 \sin \theta \cos^2 \theta + r^2) \, d\theta \, dr \text{ since} \\
\mathbf{F} &= (r^2 \sin \theta \cos \theta)\mathbf{i} - r\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_0^1 (r^3 \sin \theta \cos^2 \theta + r^2) \, dr \, d\theta = \int_0^{2\pi} \left(\frac{1}{4} \sin \theta \cos^2 \theta + \frac{1}{3} \right) \, d\theta \\
&= \left[-\frac{1}{12} \cos^3 \theta + \frac{\theta}{3} \right]_0^{2\pi} = \frac{2\pi}{3}
\end{aligned}$$

26. Let the parametrization be $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + 2r\mathbf{k}$, $0 \leq r \leq 1$ (since $0 \leq z \leq 2$) and $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \Rightarrow \mathbf{r}_r &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & 2 \end{vmatrix} \\ &= (2r \cos \theta)\mathbf{i} + (2r \sin \theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|} |\mathbf{r}_\theta \times \mathbf{r}_r| \, d\theta \, dr \\ &= (2r^3 \sin^2 \theta \cos \theta + 4r^3 \cos \theta \sin \theta + r) \, d\theta \, dr \text{ since } \mathbf{F} = (r^2 \sin^2 \theta)\mathbf{i} + (2r^2 \cos \theta)\mathbf{j} - \mathbf{k} \\ \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \int_0^{2\pi} \int_0^1 (2r^3 \sin^2 \theta \cos \theta + 4r^3 \cos \theta \sin \theta + r) \, dr \, d\theta = \int_0^{2\pi} \left(\frac{1}{2} \sin^2 \theta \cos \theta + \cos \theta \sin \theta + \frac{1}{2} \right) d\theta \\ &= \left[\frac{1}{6} \sin^3 \theta + \frac{1}{2} \sin^2 \theta + \frac{1}{2} \theta \right]_0^{2\pi} = \pi \end{aligned}$$

27. Let the parametrization be $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}$, $1 \leq r \leq 2$ (since $1 \leq z \leq 2$) and $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \Rightarrow \mathbf{r}_r &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & 1 \end{vmatrix} \\ &= (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|} |\mathbf{r}_\theta \times \mathbf{r}_r| \, d\theta \, dr = (-r^2 \cos^2 \theta - r^2 \sin^2 \theta - r^3) \, d\theta \, dr \\ &= (-r^2 - r^3) \, d\theta \, dr \text{ since } \mathbf{F} = (-r \cos \theta)\mathbf{i} - (r \sin \theta)\mathbf{j} + r^2\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_1^2 (-r^2 - r^3) \, dr \, d\theta = -\frac{73\pi}{6} \end{aligned}$$

28. Let the parametrization be $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r^2\mathbf{k}$, $0 \leq r \leq 1$ (since $0 \leq z \leq 1$) and $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \Rightarrow \mathbf{r}_r &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + 2r\mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & 2r \end{vmatrix} \\ &= (2r^2 \cos \theta)\mathbf{i} + (2r^2 \sin \theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|} |\mathbf{r}_\theta \times \mathbf{r}_r| \, d\theta \, dr = (8r^3 \cos^2 \theta + 8r^3 \sin^2 \theta - 2r) \, d\theta \, dr \\ &= (8r^3 - 2r) \, d\theta \, dr \text{ since } \mathbf{F} = (4r \cos \theta)\mathbf{i} + (4r \sin \theta)\mathbf{j} + 2\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_0^1 (8r^3 - 2r) \, dr \, d\theta = 2\pi \end{aligned}$$

29. $g(x, y, z) = z$, $\mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R (\mathbf{F} \cdot \mathbf{k}) \, dA$

$$= \int_0^2 \int_0^3 3 \, dy \, dx = 18$$

30. $g(x, y, z) = y$, $\mathbf{p} = -\mathbf{j} \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R (\mathbf{F} \cdot -\mathbf{j}) \, dA$

$$= \int_{-1}^2 \int_2^7 2 \, dz \, dx = \int_{-1}^2 2(7-2) \, dx = 10(2+1) = 30$$

31. $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a; \mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{z^2}{a};$
 $|\nabla g \cdot \mathbf{k}| = 2z \Rightarrow d\sigma = \frac{2a}{2z} dA \Rightarrow \text{Flux} = \iint_S \left(\frac{z^2}{a}\right) \left(\frac{a}{z}\right) dA = \iint_S z dA = \iint_S \sqrt{a^2 - (x^2 + y^2)} dx dy$
 $= \int_0^{\pi/2} \int_0^a \sqrt{a^2 - r^2} r dr d\theta = \frac{\pi a^3}{6}$
32. $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a; \mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{-xy}{a} + \frac{xy}{a} = 0;$
 $|\nabla g \cdot \mathbf{k}| = 2z \Rightarrow d\sigma = \frac{2a}{2z} dA \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S 0 d\sigma = 0$
33. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z} dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{xy}{a} - \frac{xy}{a} + \frac{z}{a} = \frac{z}{a} \Rightarrow \text{Flux} = \iint_R \left(\frac{z}{a}\right) \left(\frac{a}{z}\right) dA = \iint_R 1 dA = \frac{\pi a^2}{4}$
34. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z} dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{zx^2}{a} + \frac{zy^2}{a} + \frac{z^3}{a} = z \left(\frac{x^2 + y^2 + z^2}{a} \right) = az$
 $\Rightarrow \text{Flux} = \iint_R (za) \left(\frac{a}{z}\right) = \iint_R a^2 dx dy = a^2 (\text{Area of } R) = \frac{1}{4} \pi a^4 dx dy$
35. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z} dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{a} = a \Rightarrow \text{Flux} = \iint_R a \left(\frac{a}{z}\right) dA = \iint_R \frac{a^2}{z} dA$
 $= \iint_R \frac{a^2}{\sqrt{a^2 - (x^2 + y^2)}} dA = \int_0^{\pi/2} \int_0^a \frac{a^2}{\sqrt{a^2 - r^2}} r dr d\theta = \int_0^{\pi/2} a^2 \left[-\sqrt{a^2 - r^2} \right]_0^a d\theta = \frac{\pi a^3}{2}$
36. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z} dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{\left(\frac{x^2}{a}\right) + \left(\frac{y^2}{a}\right) + \left(\frac{z^3}{a}\right)}{\sqrt{x^2 + y^2 + z^2}} = \frac{\left(\frac{a^2}{a}\right)}{a} = 1$
 $\Rightarrow \text{Flux} = \iint_R \frac{a}{z} dx dy = \iint_R \frac{a}{\sqrt{a^2 - (x^2 + y^2)}} dx dy = \int_0^{\pi/2} \int_0^a \frac{a}{\sqrt{a^2 - r^2}} r dr d\theta = \frac{\pi a^2}{2}$
37. $g(x, y, z) = y^2 + z = 4 \Rightarrow \nabla g = 2y\mathbf{j} + \mathbf{k} \Rightarrow |\nabla g| = \sqrt{4y^2 + 1} \Rightarrow \mathbf{n} = \frac{2y\mathbf{j} + \mathbf{k}}{\sqrt{4y^2 + 1}} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{2xy - 3z}{\sqrt{4y^2 + 1}};$
 $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2 + 1} dA \Rightarrow \text{Flux} = \iint_R \left(\frac{2xy - 3z}{\sqrt{4y^2 + 1}}\right) \sqrt{4y^2 + 1} dA = \iint_R (2xy - 3z) dA;$
 $z = 0 \text{ and } z = 4 - y^2 \Rightarrow y^2 = 4 \Rightarrow \text{Flux} = \iint_R \left[2xy - 3(4 - y^2) \right] dA = \int_0^1 \int_{-2}^2 (2xy - 12 + 3y^2) dy dx$
 $= \int_0^1 \left[xy^2 - 12y + y^3 \right]_{-2}^2 dx = \int_0^1 (-32) dx = -32$
38. $g(x, y, z) = x^2 + y^2 - z = 0 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4(x^2 + y^2) + 1}$
 $\Rightarrow \mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}}{\sqrt{4(x^2 + y^2) + 1}} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{8x^2 + 8y^2 - 2}{\sqrt{4(x^2 + y^2) + 1}}; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4(x^2 + y^2) + 1} dA$

$$\Rightarrow \text{Flux} = \iint_R \left(\frac{8x^2 + 8y^2 - 2}{\sqrt{4(x^2 + y^2) + 1}} \right) \sqrt{4(x^2 + y^2) + 1} dA = \iint_R (8x^2 + 8y^2 - 2) dA; z = 1 \text{ and } x^2 + y^2 = z$$

$$\Rightarrow x^2 + y^2 = 1 \Rightarrow \text{Flux} = \int_0^{2\pi} \int_0^1 (8r^2 - 2) r dr d\theta = 2\pi$$

$$39. g(x, y, z) = y - e^x = 0 \Rightarrow \nabla g = -e^x \mathbf{i} + \mathbf{j} \Rightarrow |\nabla g| = \sqrt{e^{2x} + 1} \Rightarrow \mathbf{n} = \frac{e^x \mathbf{i} - \mathbf{j}}{\sqrt{e^{2x} + 1}} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{-2e^x - 2y}{\sqrt{e^{2x} + 1}};$$

$$\mathbf{p} = \mathbf{i} \Rightarrow |\nabla g \cdot \mathbf{p}| = e^x \Rightarrow d\sigma = \frac{\sqrt{e^{2x} + 1}}{e^x} dA \Rightarrow \text{Flux} = \iint_R \left(\frac{-2e^x - 2y}{\sqrt{e^{2x} + 1}} \right) \left(\frac{\sqrt{e^{2x} + 1}}{e^x} \right) dA = \iint_R \frac{-2e^x - 2e^x}{e^x} dA$$

$$= \iint_R (-4) dA = \int_0^1 \int_1^2 (-4) dy dz = -4$$

$$40. g(x, y, z) = y - \ln x = 0 \Rightarrow \nabla g = -\frac{1}{x} \mathbf{i} + \mathbf{j} \Rightarrow |\nabla g| = \sqrt{\frac{1}{x^2} + 1} = \frac{\sqrt{1+x^2}}{x} \text{ since } 1 \leq x \leq e \Rightarrow \mathbf{n} = \frac{\left(\frac{-1}{x} \mathbf{i} + \mathbf{j}\right)}{\left(\frac{\sqrt{1+x^2}}{x}\right)} = \frac{-\mathbf{i} + x\mathbf{j}}{\sqrt{1+x^2}}$$

$$\Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{2xy}{\sqrt{1+x^2}}; \mathbf{p} = \mathbf{j} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{1+x^2}}{x} dA \Rightarrow \text{Flux} = \iint_R \left(\frac{2xy}{\sqrt{1+x^2}} \right) \left(\frac{\sqrt{1+x^2}}{x} \right) dA$$

$$= \int_0^1 \int_1^e 2y dx dz = \int_1^e \int_0^1 2 \ln x dz dx = \int_1^e 2 \ln x dx = 2[x \ln x - x]_1^e = 2(e - e) - 2(0 - 1) = 2$$

$$41. \text{ On the face } z = a: g(x, y, z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1; \mathbf{n} = \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} = 2xz = 2ax \text{ since } z = a; d\sigma = dx dy$$

$$\Rightarrow \text{Flux} = \iint_R 2ax dx dy = \int_0^a \int_0^a 2ax dx dy = a^4.$$

$$\text{On the face } z = 0: g(x, y, z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1; \mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -2xz = 0 \text{ since } z = 0;$$

$$d\sigma = dx dy \Rightarrow \text{Flux} = \iint_R 0 dx dy = 0.$$

$$\text{On the face } x = a: g(x, y, z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1; \mathbf{n} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \mathbf{n} = 2xy = 2ay \text{ since } x = a;$$

$$d\sigma = dy dz \Rightarrow \text{Flux} = \int_0^a \int_0^a 2ay dy dz = a^4.$$

$$\text{On the face } x = 0: g(x, y, z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1; \mathbf{n} = -\mathbf{i} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -2xy = 0 \text{ since } x = 0 \Rightarrow \text{Flux} = 0.$$

$$\text{On the face } y = a: g(x, y, z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1; \mathbf{n} = \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n} = 2yz = 2az \text{ since } y = a;$$

$$d\sigma = dz dx \Rightarrow \text{Flux} = \int_0^a \int_0^a 2az dz dx = a^4.$$

$$\text{On the face } y = 0: g(x, y, z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1; \mathbf{n} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -2yz = 0 \text{ since } y = 0 \Rightarrow \text{Flux} = 0.$$

$$\text{Therefore, Total Flux} = 3a^4.$$

$$42. \text{ Across the cap: } g(x, y, z) = x^2 + y^2 + z^2 = 25 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 10$$

$$\Rightarrow \mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{5} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{x^2 z}{5} + \frac{y^2 z}{5} + \frac{z}{5}; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 2z \text{ since } z \geq 0 \Rightarrow d\sigma = \frac{10}{2z} dA$$

$$\Rightarrow \text{Flux}_{\text{cap}} = \iint_{\text{cap}} \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_R \left(\frac{x^2 z}{5} + \frac{y^2 z}{5} + \frac{z}{5} \right) \left(\frac{5}{z} \right) dA = \iint_R (x^2 + y^2 + 1) dx dy = \int_0^{2\pi} \int_0^4 (r^2 + 1) r dr d\theta$$

$$= \int_0^{2\pi} 72 d\theta = 144\pi.$$

Across the bottom: $g(x, y, z) = z = 3 \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1 \Rightarrow \mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -1; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1$
 $\Rightarrow d\sigma = dA \Rightarrow \text{Flux}_{\text{bottom}} = \iint_{\text{bottom}} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R -1 \, dA = -1(\text{Area of the circular region}) = -16\pi$. Therefore,
 $\text{Flux} = \text{Flux}_{\text{cap}} + \text{Flux}_{\text{bottom}} = 128\pi$

$$43. \quad \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \text{ since } z \geq 0 \Rightarrow d\sigma = \frac{2a}{2z} dA \\ = \frac{a}{z} dA; M = \iint_S \delta \, d\sigma = \frac{\delta}{8} (\text{surface area of sphere}) = \frac{\delta\pi a^2}{2}; M_{xy} = \iint_S z\delta \, d\sigma = \delta \iint_R z\left(\frac{a}{z}\right) dA = a\delta \iint_R dA \\ = a\delta \int_0^{\pi/2} \int_0^a r \, dr \, d\theta = \frac{\delta\pi a^3}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\delta\pi a^3}{4}\right)\left(\frac{2}{\delta\pi a^2}\right) = \frac{a}{2}. \text{ Because of symmetry, } \bar{x} = \bar{y} = \frac{a}{2} \\ \Rightarrow \text{the centroid is } \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right).$$

$$44. \quad \nabla f = 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 4z^2} = \sqrt{4(y^2 + z^2)} = 6; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{k}| = 2z \text{ since } z \geq 0 \Rightarrow d\sigma = \frac{6}{2z} dA \\ = \frac{3}{z} dA; M = \iint_S 1 \, d\sigma = \int_{-3}^3 \int_0^3 \frac{3}{z} \, dx \, dy = \int_{-3}^3 \int_0^3 \frac{3}{\sqrt{9-y^2}} \, dx \, dy = 9\pi; M_{xy} = \iint_S z \, d\sigma = \int_{-3}^3 \int_0^3 z\left(\frac{3}{z}\right) \, dx \, dy = 54; \\ M_{xz} = \iint_S y \, d\sigma = \int_{-3}^3 \int_0^3 y\left(\frac{3}{z}\right) \, dx \, dy = \int_{-3}^3 \int_0^3 \frac{3y}{\sqrt{9-y^2}} \, dx \, dy = 0; M_{yz} = \iint_S x \, d\sigma = \int_{-3}^3 \int_0^3 \frac{3x}{\sqrt{9-y^2}} \, dx \, dy = \frac{27}{2}\pi. \\ \text{Therefore, } \bar{x} = \frac{\left(\frac{27}{2}\pi\right)}{9\pi} = \frac{3}{2}, \bar{y} = 0, \text{ and } \bar{z} = \frac{54}{9\pi} = \frac{6}{\pi}$$

$$45. \quad \text{Because of symmetry, } \bar{x} = \bar{y} = 0; M = \iint_S \delta \, d\sigma = \delta \iint_S d\sigma = (\text{Area of } S)\delta = 3\pi\sqrt{2}\delta; \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} \\ \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2}; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{x^2 + y^2 + z^2}}{2z} dA \\ = \frac{\sqrt{x^2 + y^2 + (x^2 + y^2)}}{z} dA = \frac{\sqrt{2}\sqrt{x^2 + y^2}}{z} dA \Rightarrow M_{xy} = \delta \iint_R z\left(\frac{\sqrt{2}\sqrt{x^2 + y^2}}{z}\right) dA = \delta \iint_R \sqrt{2}\sqrt{x^2 + y^2} \, dA \\ = \delta \int_0^{2\pi} \int_1^2 \sqrt{2}r^2 \, dr \, d\theta = \frac{14\pi\sqrt{2}}{3}\delta \Rightarrow \bar{z} = \frac{\left(\frac{14\pi\sqrt{2}}{3}\delta\right)}{3\pi\sqrt{2}\delta} = \frac{14}{9} \Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{14}{9}\right). \text{ Next, } I_z = \iint_S (x^2 + y^2) \delta \, d\sigma \\ = \iint_R (x^2 + y^2) \left(\frac{\sqrt{2}\sqrt{x^2 + y^2}}{z}\right) \delta \, dA = \delta\sqrt{2} \iint_R (x^2 + y^2) \, dA = \delta\sqrt{2} \int_0^{2\pi} \int_1^2 r^3 \, dr \, d\theta = \frac{15\pi\sqrt{2}}{2}\delta \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{\sqrt{10}}{2}$$

$$46. \quad f(x, y, z) = 4x^2 + 4y^2 - z^2 = 0 \Rightarrow \nabla f = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{64x^2 + 64y^2 + 4z^2} \\ = 2\sqrt{16x^2 + 16y^2 + z^2} = 2\sqrt{4z^2 + z^2} = 2\sqrt{5} \, z \text{ since } z \geq 0; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{5}z}{2z} dA = \sqrt{5} \, dA \\ \Rightarrow I_z = \iint_S (x^2 + y^2) \delta \, d\sigma = \delta\sqrt{5} \iint_R (x^2 + y^2) \, dx \, dy = \delta\sqrt{5} \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^3 \, dr \, d\theta = \frac{3\sqrt{5}\pi\delta}{2}$$

$$47. \quad (\text{a}) \text{ Let the diameter lie on the } z\text{-axis and let } f(x, y, z) = x^2 + y^2 + z^2 = a^2, z \geq 0 \text{ be the upper hemisphere} \\ \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a, a > 0; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \text{ since } z \geq 0 \\ \Rightarrow d\sigma = \frac{a}{z} dA \Rightarrow I_z = \iint_S (x^2 + y^2) \left(\frac{a}{z}\right) d\sigma = a\delta \iint_R \frac{x^2 + y^2}{\sqrt{a^2 - (x^2 + y^2)}} dA = a\delta \int_0^{2\pi} \int_0^a \frac{r^2}{\sqrt{a^2 - r^2}} r \, dr \, d\theta$$

$$= a\delta \int_0^{2\pi} \left[-r^2 \sqrt{a^2 - r^2} - \frac{2}{3} (a^2 - r^2)^{3/2} \right]_0^a d\theta = a\delta \int_0^{2\pi} \frac{2}{3} a^3 d\theta = \frac{4\pi}{3} a^4 \delta \Rightarrow \text{the moment of inertia is } \frac{8\pi}{3} a^4 \delta \text{ for the whole sphere}$$

(b) $I_L = I_{\text{c.m.}} + mh^2$, where m is the mass of the body and h is the distance between the parallel lines; now,
 $I_{\text{c.m.}} = \frac{8\pi}{3} a^4 \delta$ (from part a) and $\frac{m}{2} = \iint_S \delta \, d\sigma = \delta \iint_R \left(\frac{a}{z} \right) dA = a\delta \iint_R \frac{1}{\sqrt{a^2 - (x^2 + y^2)}} dy \, dx$
 $= a\delta \int_0^{2\pi} \int_0^a \frac{1}{\sqrt{a^2 - r^2}} r \, dr \, d\theta = a\delta \int_0^{2\pi} \left[-\sqrt{a^2 - r^2} \right]_0^a d\theta = a\delta \int_0^{2\pi} a \, d\theta = 2\pi a^2 \delta$ and $h = a$
 $\Rightarrow I_L = \frac{8\pi}{3} a^4 \delta + 4\pi a^2 \delta a^2 = \frac{20\pi}{3} a^4 \delta$

48. Let $z = \frac{h}{a} \sqrt{x^2 + y^2}$ be the cone from $z = 0$ to $z = h$, $h > 0$. Because of symmetry, $\bar{x} = 0$ and $\bar{y} = 0$;

$$\begin{aligned} z = \frac{h}{a} \sqrt{x^2 + y^2} &\Rightarrow f(x, y, z) = \frac{h^2}{a^2} (x^2 + y^2) - z^2 = 0 \Rightarrow \nabla f = \frac{2xh^2}{a^2} \mathbf{i} + \frac{2yh^2}{a^2} \mathbf{j} - 2z\mathbf{k} \\ \Rightarrow |\nabla f| &= \sqrt{\frac{4x^2h^4}{a^4} + \frac{4y^2h^4}{a^4} + 4z^2} = 2\sqrt{\frac{h^4}{a^4} (x^2 + y^2) + \frac{h^2}{a^2} (x^2 + y^2)} = 2\sqrt{\left(\frac{h^2}{a^2}\right) (x^2 + y^2) \left(\frac{h^2}{a^2} + 1\right)} = 2\sqrt{z^2 \left(\frac{h^2 + a^2}{a^2}\right)} \\ &= \left(\frac{2z}{a}\right) \sqrt{h^2 + a^2} \text{ since } z \geq 0; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{\left(\frac{2z}{a}\right) \sqrt{h^2 + a^2}}{2z} dA = \frac{\sqrt{h^2 + a^2}}{a} dA; \\ M &= \iint_S d\sigma = \iint_R \frac{\sqrt{h^2 + a^2}}{a} dA = \frac{\sqrt{h^2 + a^2}}{a} (\pi a^2) = \pi a \sqrt{h^2 + a^2}; M_{xy} = \iint_S z \, d\sigma = \iint_R z \left(\frac{\sqrt{h^2 + a^2}}{a} \right) dA \\ &= \frac{\sqrt{h^2 + a^2}}{a} \iint_R \frac{h}{a} \sqrt{x^2 + y^2} \, dx \, dy = \frac{h\sqrt{h^2 + a^2}}{a^2} \int_0^{2\pi} \int_0^a r^2 \, dr \, d\theta = \frac{2\pi ah\sqrt{h^2 + a^2}}{3} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac{2h}{3} \Rightarrow \text{the centroid is} \\ &\left(0, 0, \frac{2h}{3} \right) \end{aligned}$$

16.7 STOKES' THEOREM

$$\begin{aligned} 1. \quad \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (2 - 0)\mathbf{k} = 2\mathbf{k} \text{ and } \mathbf{n} = \mathbf{k} \Rightarrow \text{curl } \mathbf{F} \cdot \mathbf{n} = 2 \Rightarrow d\sigma = dx \, dy \\ \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R 2 \, dA = 2(\text{Area of the ellipse}) = 4\pi \end{aligned}$$

$$\begin{aligned} 2. \quad \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (3 - 2)\mathbf{k} = \mathbf{k} \text{ and } \mathbf{n} = \mathbf{k} \Rightarrow \text{curl } \mathbf{F} \cdot \mathbf{n} = 1 \Rightarrow d\sigma = dx \, dy \\ \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R dx \, dy = \text{Area of circle} = 9\pi \end{aligned}$$

$$3. \quad \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix} = -x\mathbf{i} - 2x\mathbf{j} + (z-1)\mathbf{k} \text{ and } \mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} \Rightarrow \text{curl } \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{3}}(-x-2x+z-1)$$

$$\Rightarrow d\sigma = \frac{\sqrt{3}}{1} dA \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \frac{1}{\sqrt{3}}(-3x+z-1)\sqrt{3} dA = \int_0^1 \int_0^{1-x} [-3x+(1-x-y)-1] dy dx$$

$$= \int_0^1 \int_0^{1-x} (-4x-y) dy dx = \int_0^1 -\left[4x(1-x) + \frac{1}{2}(1-x)^2\right] dx = -\int_0^1 \left(\frac{1}{2} + 3x - \frac{7}{2}x^2\right) dx = -\frac{5}{6}$$

$$4. \quad \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} = (2y-2z)\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k} \text{ and } \mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$$

$$\Rightarrow \text{curl } \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{3}}(2y-2z+2z-2x+2x-2y) = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S 0 d\sigma = 0$$

$$5. \quad \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+y^2 & x^2+y^2 \end{vmatrix} = 2y\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k} \text{ and } \mathbf{n} = \mathbf{k} \Rightarrow \text{curl } \mathbf{F} \cdot \mathbf{n} = 2x-2y$$

$$\Rightarrow d\sigma = dx dy \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^1 \int_{-1}^1 (2x-2y) dx dy = \int_{-1}^1 [x^2 - 2xy]_{-1}^1 dy = \int_{-1}^1 -4y dy = 0$$

$$6. \quad \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 3x^2y^2\mathbf{k} \text{ and } \mathbf{n} = \frac{2xi+2yj+2zk}{2\sqrt{x^2+y^2+z^2}} = \frac{xi+yj+zk}{4} \Rightarrow \text{curl } \mathbf{F} \cdot \mathbf{n} = -\frac{3}{4}x^2y^2z;$$

$$d\sigma = \frac{4}{z} dA \quad (\text{Section 16.6, Example 6, with } a=4) \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(-\frac{3}{4}x^2y^2z\right)\left(\frac{4}{z}\right) dA$$

$$= -3 \int_0^{2\pi} \int_0^2 (r^2 \cos^2 \theta)(r^2 \sin^2 \theta) r dr d\theta = -3 \int_0^{2\pi} \left[\frac{r^6}{6}\right]_0^2 (\cos \theta \sin \theta)^2 d\theta = -32 \int_0^{2\pi} \frac{1}{4} \sin^2 2\theta d\theta$$

$$= -4 \int_0^{4\pi} \sin^2 u du = -4 \left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = -8\pi$$

$$7. \quad x = 3 \cos t \text{ and } y = 2 \sin t \Rightarrow \mathbf{F} = (2 \sin t)\mathbf{i} + (9 \cos^2 t)\mathbf{j} + (9 \cos^2 t + 16 \sin^4 t) \sin e^{\sqrt{(6 \sin t \cos t)(0)}} \mathbf{k} \text{ at the base of}$$

the shell; $\mathbf{r} = (3 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} \Rightarrow d\mathbf{r} = (-3 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -6 \sin^2 t + 18 \cos^3 t$

$$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \int_0^{2\pi} \left(-6 \sin^2 t + 18 \cos^3 t\right) dt = \left[-3t + \frac{3}{2} \sin 2t + 6(\sin t)(\cos^2 t + 2)\right]_0^{2\pi} = -6\pi$$

$$\begin{aligned}
 8. \quad \text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z + \frac{1}{2+x} & \tan^{-1} y & x + \frac{1}{4+z} \end{vmatrix} = -2\mathbf{j}; f(x, y, z) = 4x^2 + y + z^2 \Rightarrow \nabla f = 8x\mathbf{i} + \mathbf{j} + 2z\mathbf{k} \\
 \Rightarrow \mathbf{n} &= \frac{\nabla f}{|\nabla f|} \text{ and } \mathbf{p} = \mathbf{j} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = |\nabla f| dA; \nabla \times \mathbf{F} \cdot \mathbf{n} = \frac{1}{|\nabla f|} (-2\mathbf{j} \cdot \nabla f) = \frac{-2}{|\nabla f|} \\
 \Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma &= -2 dA \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_R -2 dA = -2 (\text{Area of } R) = -2(\pi \cdot 1 \cdot 2) = -4\pi, \text{ where } R \text{ is the} \\
 &\text{elliptic region in the } xz\text{-plane enclosed by } 4x^2 + z^2 = 4.
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \text{Flux of } \nabla \times \mathbf{F} &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}, \text{ so let } C \text{ be parametrized by } \mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, 0 \leq t \leq 2\pi \\
 \Rightarrow \frac{d\mathbf{r}}{dt} &= (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ay \sin t + ax \cos t = a^2 \sin^2 t + a^2 \cos^2 t = a^2 \\
 \Rightarrow \text{Flux of } \nabla \times \mathbf{F} &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} a^2 dt = 2\pi a^2
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \nabla \times (y\mathbf{i}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\mathbf{k}; \mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2xi + 2yj + 2zk}{2\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow \nabla \times (y\mathbf{i}) \cdot \mathbf{n} = -z; d\sigma = \frac{1}{z} dA \\
 (\text{Section 16.6, Example 6, with } a=1) &\Rightarrow \iint_S \nabla \times (y\mathbf{i}) \cdot \mathbf{n} d\sigma = \iint_R (-z) \left(\frac{1}{2} dA \right) = -\iint_R dA = -\pi, \text{ where } R \text{ is the disk} \\
 &x^2 + y^2 \leq 1 \text{ in the } xy\text{-plane.}
 \end{aligned}$$

11. For the upper hemisphere with $z \geq 0$, the boundary C is the unit circle of radius 1 centered at the origin in the xy -plane. An outward normal on the upper hemisphere corresponds to counterclockwise circulation around the boundary, so the boundary can be parametrized as $\mathbf{r}(\theta) = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + 0\mathbf{k}$, with $0 \leq \theta \leq 2\pi$. Thus $d\mathbf{r} = (-\sin \theta d\theta)\mathbf{i} + (\cos \theta d\theta)\mathbf{j}$. For the field $\mathbf{A} = (y + \sqrt{z})\mathbf{i} + e^{xyz}\mathbf{j} + (\cos xz)\mathbf{k}$, the flux of $\mathbf{F} = \nabla \times \mathbf{A}$ across the upper hemisphere is, by Stokes' Theorem, equal to the circulation of \mathbf{A} on the boundary. Since $z = 0$ and $y = \sin \theta$ on the boundary, the field \mathbf{A} on the boundary is $(\sin \theta)\mathbf{i} + \mathbf{j} + \mathbf{k}$. The circulation of \mathbf{A} on C is

$$\begin{aligned}
 \oint_C \mathbf{A} \cdot d\mathbf{r} &= \oint_C ((\sin \theta)\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot ((-\sin \theta d\theta)\mathbf{i} + (\cos \theta d\theta)\mathbf{j}) = \int_0^{2\pi} (\cos \theta - \sin^2 \theta) d\theta \\
 &= \int_0^{2\pi} \left(\cos \theta + \frac{1}{2}(\cos 2\theta - 1) \right) d\theta = -\pi
 \end{aligned}$$

12. Since the outward normal on the bottom hemisphere corresponds to clockwise circulation on the boundary, the flux of \mathbf{F} through the bottom hemisphere will be π and the total flux through the sphere will be 0.

$$\begin{aligned}
 13. \quad \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}; \mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} - 2r\mathbf{k} \text{ and } \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} \\
 \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = (2r^2 \cos \theta)\mathbf{i} + (2r^2 \sin \theta)\mathbf{j} + r\mathbf{k}; \mathbf{n} = \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|} \text{ and } d\sigma = |\mathbf{r}_r \times \mathbf{r}_\theta| dr d\theta
 \end{aligned}$$

$$\begin{aligned}\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma &= (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) \, dr \, d\theta = (10r^2 \cos \theta + 4r^2 \sin \theta + 3r) \, dr \, d\theta \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma \\ &= \int_0^{2\pi} \int_0^2 (10r^2 \cos \theta + 4r^2 \sin \theta + 3r) \, dr \, d\theta = \int_0^{2\pi} \left[\frac{10}{3} r^3 \cos \theta + \frac{4}{3} r^3 \sin \theta + \frac{3}{2} r^2 \right]_0^2 d\theta \\ &= \int_0^{2\pi} \left(\frac{80}{3} \cos \theta + \frac{32}{3} \sin \theta + 6 \right) d\theta = 6(2\pi) = 12\pi\end{aligned}$$

$$14. \quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x+z \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}; \mathbf{r}_r \times \mathbf{r}_\theta = (2r^2 \cos \theta)\mathbf{i} + (2r^2 \sin \theta)\mathbf{j} + r\mathbf{k} \text{ and}$$

$$\begin{aligned}\nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma &= (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) \, dr \, d\theta \text{ (see Exercise 13 above)} \\ \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \int_0^{2\pi} \int_0^3 (-2r^2 \cos \theta - 4r^2 \sin \theta - 2r) \, dr \, d\theta = \int_0^{2\pi} \left[-\frac{2}{3} r^3 \cos \theta - \frac{4}{3} r^3 \sin \theta - r^2 \right]_0^3 d\theta \\ &= \int_0^{2\pi} (-18 \cos \theta - 36 \sin \theta - 9) \, d\theta = -9(2\pi) = -18\pi\end{aligned}$$

$$15. \quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & 2y^3 z & 3z \end{vmatrix} = -2y^3 \mathbf{i} + 0\mathbf{j} - x^2 \mathbf{k}; \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = (-r \cos \theta)\mathbf{i} - (r \sin \theta)\mathbf{j} + r\mathbf{k} \text{ and}$$

$$\begin{aligned}\nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma &= (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) \, dr \, d\theta \text{ (see Exercise 13 above)} \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R (2ry^3 \cos \theta - rx^2) \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (2r^4 \sin^3 \theta \cos \theta - r^3 \cos^2 \theta) \, dr \, d\theta = \int_0^{2\pi} \left(\frac{2}{5} \sin^3 \theta \cos \theta - \frac{1}{4} \cos^2 \theta \right) d\theta = \left[\frac{1}{10} \sin^4 \theta - \frac{1}{4} \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \right]_0^{2\pi} \\ &= -\frac{\pi}{4}\end{aligned}$$

$$16. \quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}; \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k} \text{ and}$$

$$\begin{aligned}\nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma &= (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) \, dr \, d\theta \text{ (see Exercise 13 above)} \\ \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \int_0^{2\pi} \int_0^5 (r \cos \theta + r \sin \theta + r) \, dr \, d\theta = \int_0^{2\pi} \left[(\cos \theta + \sin \theta + 1) \frac{r^2}{2} \right]_0^5 d\theta = \left(\frac{25}{2} \right) (2\pi) = 25\pi\end{aligned}$$

$$17. \quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 5-2x & z^2-2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 5\mathbf{k}; \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{3} \cos \phi \cos \theta & \sqrt{3} \cos \phi \sin \theta & -\sqrt{3} \sin \phi \\ -\sqrt{3} \sin \phi \sin \theta & \sqrt{3} \sin \phi \cos \theta & 0 \end{vmatrix}$$

$$= (3 \sin^2 \phi \cos \theta)\mathbf{i} + (3 \sin^2 \phi \sin \theta)\mathbf{j} + (3 \sin \phi \cos \phi)\mathbf{k}; \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) \, d\phi \, d\theta \text{ (see Exercise 13}$$

$$\text{above)} \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_0^{\pi/2} (-15 \cos \phi \sin \phi) \, d\phi \, d\theta = \int_0^{2\pi} \left[\frac{15}{2} \cos^2 \phi \right]_0^{\pi/2} d\theta = \int_0^{2\pi} -\frac{15}{2} \, d\theta = -15\pi$$

$$\begin{aligned}
18. \quad \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x \end{vmatrix} = -2z\mathbf{i} - \mathbf{j} - 2y\mathbf{k}; \quad \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix} \\
&= (4\sin^2\phi\cos\theta)\mathbf{i} + (4\sin^2\phi\sin\theta)\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}; \quad \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) \, d\phi \, d\theta \quad (\text{see Exercise 13 above}) \\
&\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R (-8z\sin^2\phi\cos\theta - 4\sin^2\phi\sin\theta - 8y\sin\phi\cos\theta) \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/2} (-16\sin^2\phi\cos\phi\cos\theta - 4\sin^2\phi\sin\theta - 16\sin^2\phi\sin\theta\cos\theta) \, d\phi \, d\theta \\
&= \int_0^{2\pi} \left[-\frac{16}{3}\sin^3\phi\cos\theta - 4\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)(\sin\theta) - 16\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)(\sin\theta\cos\theta) \right]_0^{\pi/2} d\theta \\
&= \int_0^{2\pi} \left(-\frac{16}{3}\cos\theta - \pi\sin\theta - 4\pi\sin\theta\cos\theta \right) d\theta = \left[-\frac{16}{3}\sin\theta + \pi\cos\theta - 2\pi\sin^2\theta \right]_0^{2\pi} = 0
\end{aligned}$$

19. We first compute the circulation of $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$ on the curve C given by

$\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + (3 - 2\cos^3 t)\mathbf{k}$ for $0 \leq t \leq 2\pi$. On C $\mathbf{F} = (2\sin t)\mathbf{i} - (2\cos t)\mathbf{j} + (4\cos^2 t)\mathbf{k}$, and $d\mathbf{r} = (-2\sin t \, dt)\mathbf{i} + (2\cos t \, dt)\mathbf{j} - (6\sin t \cos^2 t \, dt)\mathbf{k}$.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C ((2\sin t)\mathbf{i} - (2\cos t)\mathbf{j} + (4\cos^2 t)\mathbf{k}) \cdot ((-2\sin t \, dt)\mathbf{i} + (2\cos t \, dt)\mathbf{j} - (6\sin t \cos^2 t \, dt)\mathbf{k}) \\
&= -4 \int_0^{2\pi} (\sin^2 t + \cos^2 t + 6\sin t \cos^4 t) \, dt = -4 \int_0^{2\pi} (1 + 6\sin t \cos^4 t) \, dt = -8\pi.
\end{aligned}$$

Now we find the flux of $\nabla \times \mathbf{F}$ across the surface S . Note that counterclockwise circulation on C corresponds to inward normals on the cylindrical portion of S and upward normals on the base disk.

For the field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$, $\nabla \times \mathbf{F} = (-2x)\mathbf{j} - 2\mathbf{k}$.

On the base disk, the unit upward normal is \mathbf{k} so $\nabla \times \mathbf{F} \cdot \mathbf{n} = ((-2x)\mathbf{j} - 2\mathbf{k}) \cdot \mathbf{k} = -2$. The integral of the constant -2 over a disk of area 4π is -8π , so to verify Stokes' Theorem in this case it remains to show that the flux across the cylindrical portion of S is 0.

We'll reuse the parameter t and parametrize the cylinder by $\mathbf{s}(t, z) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + z\mathbf{k}$ with

$0 \leq z \leq 3 - 2\cos^3 t$. An inward unit normal is $(-\cos t)\mathbf{i} + (-\sin t)\mathbf{j}$ and the area element is $d\sigma = 2 \, dz \, dt$. On the cylinder the field $\nabla \times \mathbf{F} = (-4\cos t)\mathbf{j} - 2\mathbf{k}$. Thus

$$\begin{aligned}
\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \iint_S ((-4\cos t)\mathbf{j} - 2\mathbf{k}) \cdot ((-\cos t)\mathbf{i} + (-\sin t)\mathbf{j}) \, d\sigma \\
&= \int_0^{2\pi} \int_0^{3-2\cos^3 t} (4\sin t \cos t) \, 2 \, dz \, dt = 8 \int_0^{2\pi} (\sin t \cos t)(3 - 2\cos^3 t) \, dt \\
&= 8 \int_0^{2\pi} (3\sin t \cos t - 2\sin t \cos^4 t) \, dt = 8 \left(\frac{3}{2}\sin^2 t + \frac{2}{5}\cos^5 t \right) \Big|_0^{2\pi} = 0
\end{aligned}$$

20. The boundary C of the paraboloid S given by $z = 4 - x^2 - y^2$ is the circle of radius 2 centered at the origin in the xy -plane. An upward normal on the paraboloid corresponds to counterclockwise circulation around C , so we can parametrize C by $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}$ for $0 \leq t \leq 2\pi$, with $d\mathbf{r} = (-2\sin t \, dt)\mathbf{i} + (2\cos t \, dt)\mathbf{j}$. On C the field $\mathbf{F} = 2xy\mathbf{i} + x\mathbf{j} + (y+z)\mathbf{k}$ is equal to $(8\cos t \sin t)\mathbf{i} + (2\cos t)\mathbf{j} + (2\sin t)\mathbf{k}$.

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C ((8 \cos t \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + (2 \sin t)\mathbf{k}) \cdot ((-2 \sin t) dt)\mathbf{i} + (2 \cos t) dt)\mathbf{j}) \\ &= 4 \int_0^{2\pi} (-4 \cos t \sin^2 t + \cos^2 t) dt = 4 \left[-\frac{4}{3} \sin^3 t + \frac{1}{4} \sin 2t + \frac{1}{2} t \right]_0^{2\pi} = 4\pi\end{aligned}$$

Now for comparison we integrate $\nabla \times \mathbf{F} = \mathbf{i} + (1 - 2x)\mathbf{k}$ over the paraboloid S . We can parametrize the paraboloid as $\mathbf{s}(u, t) = (u \cos t)\mathbf{i} + (u \sin t)\mathbf{j} + (4 - u^2)\mathbf{k}$ with $0 \leq u \leq 2$ and $0 \leq t \leq 2\pi$. Thus on S the field $\nabla \times \mathbf{F}$ is equal to $\mathbf{i} + (1 - 2u \cos t)\mathbf{k}$.

First we find the vector area element:

$$\begin{aligned}\mathbf{s}_u \times \mathbf{s}_t &= ((\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (-2u)\mathbf{k}) \times ((-u \sin t)\mathbf{i} + (u \cos t)\mathbf{j} + 0\mathbf{k}) \\ &= (2u^2 \cos t)\mathbf{i} + (2u^2 \sin t)\mathbf{j} + u\mathbf{k}\end{aligned}$$

which is upward, as we require. The integral of the outward component of the field is then

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot (\mathbf{s}_u \times \mathbf{s}_t) du dt &= \int_0^{2\pi} \int_0^2 (\mathbf{i} + (1 - 2u \cos t)\mathbf{k}) \cdot ((2u^2 \cos t)\mathbf{i} + (2u^2 \sin t)\mathbf{j} + u\mathbf{k}) du dt \\ &= \int_0^{2\pi} \int_0^2 u du dt = \int_0^{2\pi} \left[\frac{u^2}{2} \right]_0^2 dt = 4\pi\end{aligned}$$

Thus the circulation of \mathbf{F} around the boundary of the paraboloid is equal to the flux of $\nabla \times \mathbf{F}$ through the paraboloid.

21. (a) $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \text{curl } \mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S 0 d\sigma = 0$
- (b) Let $f(x, y, z) = x^2 y^2 z^3 \Rightarrow \nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0} \Rightarrow \text{curl } \mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S 0 d\sigma = 0$
- (c) $\mathbf{F} = \nabla \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{0} \Rightarrow \nabla \times \mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S 0 d\sigma = 0$
- (d) $\mathbf{F} = \nabla f \Rightarrow \nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S 0 d\sigma = 0$
22. $\mathbf{F} = \nabla f = \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} (2x)\mathbf{i} - \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} (2y)\mathbf{j} - \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} (2z)\mathbf{k}$
 $= -x(x^2 + y^2 + z^2)^{-3/2} \mathbf{i} - y(x^2 + y^2 + z^2)^{-3/2} \mathbf{j} - z(x^2 + y^2 + z^2)^{-3/2} \mathbf{k}$
- (a) $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, 0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$
 $\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -x(x^2 + y^2 + z^2)^{-3/2} (-a \sin t) - y(x^2 + y^2 + z^2)^{-3/2} (a \cos t)$
 $= \left(-\frac{a \cos t}{a^3}\right)(-a \sin t) - \left(\frac{a \sin t}{a^3}\right)(a \cos t) = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$
- (b) $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S \nabla \times \nabla f \cdot \mathbf{n} d\sigma = \iint_S \mathbf{0} \cdot \mathbf{n} d\sigma = \iint_S 0 d\sigma = 0$

23. Let $\mathbf{F} = 2y\mathbf{i} + 3z\mathbf{j} - x\mathbf{k} \Rightarrow \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3z & -x \end{vmatrix} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}; \mathbf{n} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3} \Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} = -2$

$\Rightarrow \oint_C 2y \, dx + 3z \, dy - x \, dz = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S -2 \, d\sigma = -2 \iint_S d\sigma$, where $\iint_S d\sigma$ is the area of the region enclosed by C on the plane S : $2x + 2y + z = 2$

24. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$

25. Suppose $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ exists such that $\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Then $\frac{\partial}{\partial x} \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) = \frac{\partial}{\partial x} (x) \Rightarrow \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} = 1$.

Likewise, $\frac{\partial}{\partial y} \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) = \frac{\partial}{\partial y} (y) \Rightarrow \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} = 1$ and $\frac{\partial}{\partial z} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{\partial}{\partial z} (z) \Rightarrow \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 1$.

Summing the calculated equations $\Rightarrow \left(\frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 P}{\partial y \partial x} \right) + \left(\frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 N}{\partial x \partial z} \right) + \left(\frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 M}{\partial z \partial y} \right) = 3$ or $0 = 3$

(assuming the second mixed partials are equal). This result is a contradiction, so there is no field \mathbf{F} such that $\text{curl } \mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

26. Yes: If $\nabla \times \mathbf{F} = \mathbf{0}$, then the circulation of \mathbf{F} around the boundary C of any oriented surface S in the domain of \mathbf{F} is zero. The reason is this: By Stokes' theorem, $\text{circulation} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S \mathbf{0} \cdot \mathbf{n} \, d\sigma = 0$.

27. $r = \sqrt{x^2 + y^2} \Rightarrow r^4 = (x^2 + y^2)^2 \Rightarrow \mathbf{F} = \nabla(r^4) = 4x(x^2 + y^2)\mathbf{i} + 4y(x^2 + y^2)\mathbf{j} = M\mathbf{i} + N\mathbf{j}$

$\Rightarrow \oint_C \nabla(r^4) \cdot \mathbf{n} \, ds = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C M \, dy - N \, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy$

$= \iint_R \left[4(x^2 + y^2) + 8x^2 + 4(x^2 + y^2) + 8y^2 \right] dA = \iint_R 16(x^2 + y^2) dA = 16 \iint_R x^2 dA + 16 \iint_R y^2 dA$

$= 16I_y + 16I_x$.

28. $\frac{\partial P}{\partial y} = 0, \frac{\partial N}{\partial z} = 0, \frac{\partial M}{\partial z} = 0, \frac{\partial P}{\partial x} = 0, \frac{\partial N}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \Rightarrow \text{curl } \mathbf{F} = \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right] \mathbf{k} = \mathbf{0}$.

However, $x^2 + y^2 = 1 \Rightarrow \mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$

$\Rightarrow \mathbf{F} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_0^{2\pi} 1 \, dt = 2\pi$ which is not zero.

16.8 THE DIVERGENCE THEOREM AND A UNIFIED THEORY

1. $\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}} \Rightarrow \operatorname{div} \mathbf{F} = \frac{xy - xy}{(x^2 + y^2)^{3/2}} = 0$
2. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} \Rightarrow \operatorname{div} \mathbf{F} = 1 + 1 = 2$
3. $\mathbf{F} = -\frac{GM(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}}$

$$\Rightarrow \operatorname{div} \mathbf{F} = -GM \left[\frac{(x^2 + y^2 + z^2)^{3/2} - 3x^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \right] - GM \left[\frac{(x^2 + y^2 + z^2)^{3/2} - 3y^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \right]$$

$$- GM \left[\frac{(x^2 + y^2 + z^2)^{3/2} - 3z^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \right] = -GM \left[\frac{3(x^2 + y^2 + z^2)^2 - 3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{7/2}} \right] = 0$$
4. $z = a^2 - r^2$ in cylindrical coordinates $\Rightarrow z = a^2 - (x^2 + y^2) \Rightarrow \mathbf{v} = (a^2 - x^2 - y^2)\mathbf{k} \Rightarrow \operatorname{div} \mathbf{v} = 0$
5. $\frac{\partial}{\partial x}(y - x) = -1, \frac{\partial}{\partial y}(z - y) = -1, \frac{\partial}{\partial z}(y - x) = 0 \Rightarrow \nabla \cdot \mathbf{F} = -2 \Rightarrow \text{Flux} = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 -2 \, dx \, dy \, dz = -2(2^3) = -16$
6. $\frac{\partial}{\partial x}(x^2) = 2x, \frac{\partial}{\partial y}(y^2) = 2y, \frac{\partial}{\partial z}(z^2) = 2z \Rightarrow \nabla \cdot \mathbf{F} = 2x + 2y + 2z$
 - (a) $\text{Flux} = \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) \, dx \, dy \, dz = \int_0^1 \int_0^1 \left[x^2 + 2x(y + z) \right]_0^1 \, dy \, dz = \int_0^1 \int_0^1 (1 + 2y + 2z) \, dy \, dz$
 $= \int_0^1 \left[y(1 + 2z) + y^2 \right]_0^1 \, dz = \int_0^1 (2 + 2z) \, dz = \left[2z + z^2 \right]_0^1 = 3$
 - (b) $\text{Flux} = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x + 2y + 2z) \, dx \, dy \, dz = \int_{-1}^1 \int_{-1}^1 \left[x^2 + 2x(y + z) \right]_{-1}^1 \, dy \, dz = \int_{-1}^1 \int_{-1}^1 (4y + 4z) \, dy \, dz$
 $= \int_{-1}^1 \left[2y^2 + 4yz \right]_{-1}^1 \, dz = \int_{-1}^1 8z \, dz = \left[4z^2 \right]_{-1}^1 = 0$
 - (c) In cylindrical coordinates, $\text{Flux} = \iiint_D (2x + 2y + 2z) \, dx \, dy \, dz$
 $= \int_0^1 \int_0^{2\pi} \int_0^2 (2r \cos \theta + 2r \sin \theta + 2z) r \, dr \, d\theta \, dz = \int_0^1 \int_0^{2\pi} \left[\frac{2}{3} r^3 \cos \theta + \frac{2}{3} r^3 \sin \theta + zr^2 \right]_0^2 \, d\theta \, dz$
 $= \int_0^1 \int_0^{2\pi} \left(\frac{16}{3} \cos \theta + \frac{16}{3} \sin \theta + 4z \right) d\theta \, dz = \int_0^1 \left[\frac{16}{3} \sin \theta - \frac{16}{3} \cos \theta + 4z\theta \right]_0^{2\pi} \, dz = \int_0^1 8\pi z \, dz = \left[4\pi z^2 \right]_0^1 = 4\pi$
7. $\frac{\partial}{\partial x}(y) = 0, \frac{\partial}{\partial y}(xy) = x, \frac{\partial}{\partial z}(-z) = -1 \Rightarrow \nabla \cdot \mathbf{F} = x - 1; z = x^2 + y^2 \Rightarrow z = r^2$ in cylindrical coordinates

$$\Rightarrow \text{Flux} = \iiint_D (x - 1) \, dz \, dy \, dx = \int_0^{2\pi} \int_0^2 \int_0^{r^2} (r \cos \theta - 1) \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (r^3 \cos \theta - r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^5}{5} \cos \theta - \frac{r^4}{4} \right]_0^2 \, d\theta = \int_0^{2\pi} \left(\frac{32}{5} \cos \theta - 4 \right) d\theta = \left[\frac{32}{5} \sin \theta - 4\theta \right]_0^{2\pi} = -8\pi$$

8. $\frac{\partial}{\partial x}(x^2) = 2x, \frac{\partial}{\partial y}(xz) = 0, \frac{\partial}{\partial z}(3z) = 3 \Rightarrow \nabla \cdot \mathbf{F} = 2x + 3 \Rightarrow \text{Flux} = \iiint_D (2x+3) dV$
 $= \int_0^{2\pi} \int_0^\pi \int_0^2 (2\rho \sin \phi \cos \theta + 3) (\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{2\pi} \int_0^\pi \left[\frac{\rho^4}{2} \sin \phi \cos \theta + \rho^3 \right]_0^2 \sin \phi d\phi d\theta$
 $= \int_0^{2\pi} \int_0^\pi (8 \sin \phi \cos \theta + 8) \sin \phi d\phi d\theta = \int_0^{2\pi} \left[8 \left(\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right) \cos \theta - 8 \cos \phi \right]_0^\pi d\theta = \int_0^{2\pi} (4\pi \cos \theta + 16) d\theta = 32\pi$
9. $\frac{\partial}{\partial x}(x^2) = 2x, \frac{\partial}{\partial y}(-2xy) = -2x, \frac{\partial}{\partial z}(3xz) = 3x \Rightarrow \text{Flux} = \iiint_D 3x dx dy dz$
 $= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 (3\rho \sin \phi \cos \theta) (\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{\pi/2} \int_0^{\pi/2} 12 \sin^2 \phi \cos \theta d\phi d\theta = \int_0^{\pi/2} 3\pi \cos \theta d\theta = 3\pi$
10. $\frac{\partial}{\partial x}(6x^2 + 2xy) = 12x + 2y, \frac{\partial}{\partial y}(2y + x^2z) = 2, \frac{\partial}{\partial z}(4x^2y^3) = 0 \Rightarrow \nabla \cdot \mathbf{F} = 12x + 2y + 2$
 $\Rightarrow \text{Flux} = \iiint_D (12x + 2y + 2) dV = \int_0^3 \int_0^{\pi/2} \int_0^2 (12r \cos \theta + 2r \sin \theta + 2) r dr d\theta dz$
 $= \int_0^3 \int_0^{\pi/2} \left(32 \cos \theta + \frac{16}{3} \sin \theta + 4 \right) d\theta dz = \int_0^3 \left(32 + 2\pi + \frac{16}{3} \right) dz = 112 + 6\pi$
11. $\frac{\partial}{\partial x}(2xz) = 2z, \frac{\partial}{\partial y}(-xy) = -x, \frac{\partial}{\partial z}(-z^2) = -2z \Rightarrow \nabla \cdot \mathbf{F} = -x \Rightarrow \text{Flux} = \iiint_D -x dV$
 $= \int_0^2 \int_0^{\sqrt{16-4x^2}} \int_0^{4-y} (-x) dz dy dx = \int_0^2 \int_0^{\sqrt{16-4x^2}} (xy - 4x) dy dx = \int_0^2 \left[\frac{1}{2} x (16 - 4x^2) - 4x \sqrt{16 - 4x^2} \right] dx$
 $= \left[4x^2 - \frac{1}{2} x^4 + \frac{1}{3} (16 - 4x^2)^{3/2} \right]_0^2 = -\frac{40}{3}$
12. $\frac{\partial}{\partial x}(x^3) = 3x^2, \frac{\partial}{\partial y}(y^3) = 3y^2, \frac{\partial}{\partial z}(z^3) = 3z^2 \Rightarrow \nabla \cdot \mathbf{F} = 3x^2 + 3y^2 + 3z^2 \Rightarrow \text{Flux} = \iiint_D 3(x^2 + y^2 + z^2) dV$
 $= 3 \int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 (\rho^2 \sin \phi) d\rho d\phi d\theta = 3 \int_0^{2\pi} \int_0^\pi \frac{a^5}{5} \sin \phi d\phi d\theta = 3 \int_0^{2\pi} \frac{2a^5}{5} d\theta = \frac{12\pi a^5}{5}$
13. Let $\rho = \sqrt{x^2 + y^2 + z^2}$. Then $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}, \frac{\partial \rho}{\partial y} = \frac{y}{\rho}, \frac{\partial \rho}{\partial z} = \frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}(\rho x) = \left(\frac{\partial \rho}{\partial x} \right) x + \rho = \frac{x^2}{\rho} + \rho,$
 $\frac{\partial}{\partial y}(\rho y) = \left(\frac{\partial \rho}{\partial y} \right) y + \rho = \frac{y^2}{\rho} + \rho, \frac{\partial}{\partial z}(\rho z) = \left(\frac{\partial \rho}{\partial z} \right) z + \rho = \frac{z^2}{\rho} + \rho \Rightarrow \nabla \cdot \mathbf{F} = \frac{x^2 + y^2 + z^2}{\rho} + 3\rho = 4\rho,$ since
 $\rho = \sqrt{x^2 + y^2 + z^2} \Rightarrow \text{Flux} = \iiint_D 4\rho dV = \int_0^{2\pi} \int_0^\pi \int_1^{\sqrt{2}} (4\rho) (\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{2\pi} \int_0^\pi 3 \sin \phi d\phi d\theta$
 $= \int_0^{2\pi} 6 d\theta = 12\pi$

14. Let $\rho = \sqrt{x^2 + y^2 + z^2}$. Then $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}$, $\frac{\partial \rho}{\partial y} = \frac{y}{\rho}$, $\frac{\partial \rho}{\partial z} = \frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x} \left(\frac{x}{\rho} \right) = \frac{1}{\rho} - \left(\frac{x}{\rho^2} \right) \frac{\partial \rho}{\partial x} = \frac{1}{\rho} - \frac{x^2}{\rho^3}$. Similarly,
 $\frac{\partial}{\partial y} \left(\frac{y}{\rho} \right) = \frac{1}{\rho} - \frac{y^2}{\rho^3}$ and $\frac{\partial}{\partial z} \left(\frac{z}{\rho} \right) = \frac{1}{\rho} - \frac{z^2}{\rho^3} \Rightarrow \nabla \cdot \mathbf{F} = \frac{3}{\rho} - \frac{x^2 + y^2 + z^2}{\rho^3} = \frac{2}{\rho}$
 $\Rightarrow \text{Flux} = \iiint_D \frac{2}{\rho} dV = \int_0^{2\pi} \int_0^\pi \int_1^2 \left(\frac{2}{\rho} \right) (\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{2\pi} \int_0^\pi 3 \sin \phi d\phi d\theta = \int_0^{2\pi} 6 d\theta = 12\pi$
15. $\frac{\partial}{\partial x} (5x^3 + 12xy^2) = 15x^2 + 12y^2$, $\frac{\partial}{\partial y} (y^3 + e^y \sin z) = 3y^2 + e^y \sin z$, $\frac{\partial}{\partial z} (5z^3 + e^y \cos z) = 15z^2 - e^y \sin z$
 $\Rightarrow \nabla \cdot \mathbf{F} = 15x^2 + 15y^2 + 15z^2 = 15\rho^2 \Rightarrow \text{Flux} = \iiint_D 15\rho^2 dV = \int_0^{2\pi} \int_0^\pi \int_1^{\sqrt{2}} (15\rho^2) (\rho^2 \sin \phi) d\rho d\phi d\theta$
 $= \int_0^{2\pi} \int_0^\pi (12\sqrt{2} - 3) \sin \phi d\phi d\theta = \int_0^{2\pi} (24\sqrt{2} - 6) d\theta = (48\sqrt{2} - 12)\pi$
16. $\frac{\partial}{\partial x} \left[\ln(x^2 + y^2) \right] = \frac{2x}{x^2 + y^2}$, $\frac{\partial}{\partial y} \left(-\frac{2z}{x} \tan^{-1} \frac{y}{x} \right) = \left(-\frac{2z}{x} \right) \left[\frac{\left(\frac{1}{x} \right)}{1 + \left(\frac{y}{x} \right)^2} \right] = -\frac{2z}{x^2 + y^2}$, $\frac{\partial}{\partial z} \left(z\sqrt{x^2 + y^2} \right) = \sqrt{x^2 + y^2}$
 $\Rightarrow \nabla \cdot \mathbf{F} = \frac{2x}{x^2 + y^2} - \frac{2z}{x^2 + y^2} + \sqrt{x^2 + y^2} \Rightarrow \text{Flux} = \iiint_D \left(\frac{2x}{x^2 + y^2} - \frac{2z}{x^2 + y^2} + \sqrt{x^2 + y^2} \right) dz dy dx$
 $= \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{-1}^2 \left(\frac{2r \cos \theta}{r^2} - \frac{2z}{r^2} + r \right) dz r dr d\theta = \int_0^{2\pi} \int_1^{\sqrt{2}} \left(6 \cos \theta - \frac{3}{r} + 3r^2 \right) dr d\theta$
 $= \int_0^{2\pi} \left[6(\sqrt{2} - 1) \cos \theta - 3 \ln \sqrt{2} + 2\sqrt{2} - 1 \right] d\theta = 2\pi \left(-\frac{3}{2} \ln 2 + 2\sqrt{2} - 1 \right)$
17. (a) $\frac{\partial}{\partial x}(x) = 1$, $\frac{\partial}{\partial y}(y) = 1$, $\frac{\partial}{\partial z}(z) = 1 \Rightarrow \nabla \cdot \mathbf{F} = 3 \Rightarrow \text{Flux} = \iiint_D 3 dV = 3 \iiint_D dV = 3$ (Volume of the solid)
 (b) If \mathbf{F} is orthogonal to \mathbf{n} at every point of S , then $\mathbf{F} \cdot \mathbf{n} = 0$ everywhere $\Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = 0$. But the flux is 3 (Volume of the solid) $\neq 0$, so \mathbf{F} is not orthogonal to \mathbf{n} at every point.
18. Yes, the outward flux through the top is 5. The reason is this: Since $\nabla \cdot \mathbf{F} = \nabla \cdot (x\mathbf{i} - 2y\mathbf{j} + (z+3)\mathbf{k}) = 1 - 2 + 1 = 0$, the outward flux across the closed cubelike surface is 0 by the Divergence Theorem. The flux across the top is therefore the negative of the flux across the sides and base. Routine calculations show that the sum of these latter fluxes is -5 . (The flux across the sides that lie in the xz -plane and the yz -plane are 0, while the flux across the xy -plane is -3 .) Therefore the flux across the top is 5.
19. For the field $\mathbf{F} = (y \cos 2x)\mathbf{i} + (y^2 \sin 2x)\mathbf{j} + (x^2 y + x)\mathbf{k}$, $\nabla \cdot \mathbf{F} = -2y \sin 2x + 2y \sin 2x + 1 = 1$. If \mathbf{F} were the curl of a field \mathbf{A} whose component functions have continuous second partial derivatives, then we would have $\text{div } \mathbf{F} = \text{div}(\text{curl } \mathbf{A}) = \nabla \cdot (\nabla \times \mathbf{A}) = 0$. Since $\text{div } \mathbf{F} = 1$, \mathbf{F} is not the curl of such a field.

20. From the Divergence Theorem, $\iint_S \nabla f \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \nabla f dV = \iiint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dV$. Now,
 $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} \ln(x^2 + y^2 + z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2 + z^2}$, $\frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2 + z^2}$, $\frac{\partial f}{\partial z} = \frac{z}{x^2 + y^2 + z^2}$
 $\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{-x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2}$, $\frac{\partial^2 f}{\partial y^2} = \frac{x^2 - y^2 + z^2}{(x^2 + y^2 + z^2)^2}$, $\frac{\partial^2 f}{\partial z^2} = \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{x^2 + y^2 + z^2}$

$$\begin{aligned}\Rightarrow \iint_S \nabla f \cdot \mathbf{n} \, d\sigma &= \iiint_D \frac{dV}{x^2+y^2+z^2} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \frac{\rho^2 \sin \phi}{\rho^2} \, d\rho \, d\phi \, d\theta = \int_0^{\pi/2} \int_0^{\pi/2} a \sin \phi \, d\phi \, d\theta \\ &= \int_0^{\pi/2} [-a \cos \phi]_0^{\pi/2} d\theta = \int_0^{\pi/2} a \, d\theta = \frac{\pi a}{2}\end{aligned}$$

21. The integral's value never exceeds the surface area of S . Since $|\mathbf{F}| \leq 1$, we have $|\mathbf{F} \cdot \mathbf{n}| = |\mathbf{F}| |\mathbf{n}| \leq (1)(1) = 1$ and

$$\begin{aligned}\iint_D \iint_S \nabla \cdot \mathbf{F} \, d\sigma &= \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma && \text{[Divergence Theorem]} \\ &\leq \iint_S |\mathbf{F} \cdot \mathbf{n}| \, d\sigma && \text{[A property of integrals]} \\ &\leq \iint_S (1) \, d\sigma && [|\mathbf{F} \cdot \mathbf{n}| \leq 1] \\ &= \text{Area of } S.\end{aligned}$$

22. $\nabla \cdot \mathbf{F} = -2x - 4y - 6z + 12 \Rightarrow \text{Flux} = \int_0^a \int_0^b \int_0^1 (-2x - 4y - 6z + 12) \, dz \, dy \, dx = \int_0^a \int_0^b (-2x - 4y + 9) \, dy \, dx =$
 $\int_0^a (-2xb - 2b^2 + 9b) \, dx = -a^2b - 2ab^2 + 9ab = ab(-a - 2b + 9) = f(a, b); \frac{\partial f}{\partial a} = -2ab - 2b^2 + 9b$ and
 $\frac{\partial f}{\partial b} = -a^2 - 4ab + 9a$ so that $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0 \Rightarrow b(-2a - 2b + 9) = 0$ and $a(-a - 4b + 9) = 0 \Rightarrow b = 0$ or
 $-2a - 2b + 9 = 0$, and $a = 0$ or $-a - 4b + 9 = 0$. Now $b = 0$ or $a = 0 \Rightarrow \text{Flux} = 0; -2a - 2b + 9 = 0$ and
 $-a - 4b + 9 = 0 \Rightarrow 3a - 9 = 0 \Rightarrow a = 3 \Rightarrow b = \frac{3}{2}$ so that $f(3, \frac{3}{2}) = \frac{27}{2}$ is the maximum flux.

23. By the Divergence Theorem, the net outward flux of the field $\mathbf{F} = xy\mathbf{i} + (\sin xz + y^2)\mathbf{j} + (e^{xy^2} + x)\mathbf{k}$ over the surface S will be equal to the integral of $\nabla \cdot \mathbf{F} = y + 2y = 3y$ over the region D bounded by S . We will integrate using the area in the zx -plane bounded by $z = 0$ and $z = 1 - x^2$ as the base. The y height at any point (x, z) will be $2 - z$. Thus the integral of $\text{div } \mathbf{F}$ over D is

$$\begin{aligned}\int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} 3y \, dy \, dz \, dx &= \int_{-1}^1 \int_0^{1-x^2} \left[\frac{3}{2} y^2 \right]_0^{2-z} dz \, dx = \int_{-1}^1 \int_0^{1-x^2} \frac{3}{2} (2-z)^2 dz \, dx \\ &= \int_{-1}^1 \left[-\frac{1}{2} (2-z)^3 \right]_0^{1-x^2} dx = \int_{-1}^1 4 - \frac{1}{2} (x^2 + 1)^3 dx = \left(\frac{7}{2} x - \frac{1}{3} x^3 - \frac{3}{10} x^5 - \frac{1}{14} x^7 \right) \Big|_{-1}^1 = \frac{184}{35}\end{aligned}$$

24. The field $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) / (x^2 + y^2 + z^2)^{3/2}$ is discussed in Example 5 in Section 16.8, where we show that the flux of \mathbf{F} across any closed surface enclosing the origin is 4π . Note that the divergence of \mathbf{F} is not defined at the origin, so we need an argument like that shown in Example 5.

25. (a) $\text{div}(\mathbf{gF}) = \nabla \cdot \mathbf{gF} = \frac{\partial}{\partial x}(gM) + \frac{\partial}{\partial y}(gN) + \frac{\partial}{\partial z}(gP) = \left(g \frac{\partial M}{\partial x} + M \frac{\partial g}{\partial x} \right) + \left(g \frac{\partial N}{\partial y} + N \frac{\partial g}{\partial y} \right) + \left(g \frac{\partial P}{\partial z} + P \frac{\partial g}{\partial z} \right)$
 $= \left(M \frac{\partial g}{\partial x} + N \frac{\partial g}{\partial y} + P \frac{\partial g}{\partial z} \right) + g \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) = g \nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$

$$\begin{aligned}
 \text{(b)} \quad \nabla \times (g\mathbf{F}) &= \left[\frac{\partial}{\partial y}(gP) - \frac{\partial}{\partial z}(gN) \right] \mathbf{i} + \left[\frac{\partial}{\partial z}(gM) - \frac{\partial}{\partial x}(gP) \right] \mathbf{j} + \left[\frac{\partial}{\partial x}(gN) - \frac{\partial}{\partial y}(gM) \right] \mathbf{k} \\
 &= \left(P \frac{\partial g}{\partial y} + g \frac{\partial P}{\partial y} - N \frac{\partial g}{\partial z} - g \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(M \frac{\partial g}{\partial z} + g \frac{\partial M}{\partial z} - P \frac{\partial g}{\partial x} - g \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(N \frac{\partial g}{\partial x} + g \frac{\partial N}{\partial x} - M \frac{\partial g}{\partial y} - g \frac{\partial M}{\partial y} \right) \mathbf{k} \\
 &= \left(P \frac{\partial g}{\partial y} - N \frac{\partial g}{\partial z} \right) \mathbf{i} + \left(g \frac{\partial P}{\partial y} - g \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(M \frac{\partial g}{\partial z} - P \frac{\partial g}{\partial x} \right) \mathbf{j} + \left(g \frac{\partial M}{\partial z} - g \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(N \frac{\partial g}{\partial x} - M \frac{\partial g}{\partial y} \right) \mathbf{k} + \left(g \frac{\partial N}{\partial x} - g \frac{\partial M}{\partial y} \right) \mathbf{k} \\
 &= g \nabla \times \mathbf{F} + \nabla g \times \mathbf{F}
 \end{aligned}$$

26. (a) Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k}$

$$\Rightarrow a\mathbf{F}_1 + b\mathbf{F}_2 = (aM_1 + bM_2)\mathbf{i} + (aN_1 + bN_2)\mathbf{j} + (aP_1 + bP_2)\mathbf{k}$$

$$\Rightarrow \nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = \left(a \frac{\partial M_1}{\partial x} + b \frac{\partial M_2}{\partial x} \right) + \left(a \frac{\partial N_1}{\partial y} + b \frac{\partial N_2}{\partial y} \right) + \left(a \frac{\partial P_1}{\partial z} + b \frac{\partial P_2}{\partial z} \right)$$

$$= a \left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z} \right) + b \left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z} \right) = a(\nabla \cdot \mathbf{F}_1) + b(\nabla \cdot \mathbf{F}_2)$$

(b) Define \mathbf{F}_1 and \mathbf{F}_2 as in part a

$$\begin{aligned}
 \Rightarrow \nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2) &= \left[\left(a \frac{\partial P_1}{\partial y} + b \frac{\partial P_2}{\partial y} \right) - \left(a \frac{\partial N_1}{\partial z} + b \frac{\partial N_2}{\partial z} \right) \right] \mathbf{i} + \left[\left(a \frac{\partial M_1}{\partial z} + b \frac{\partial M_2}{\partial z} \right) - \left(a \frac{\partial P_1}{\partial x} + b \frac{\partial P_2}{\partial x} \right) \right] \mathbf{j} \\
 &\quad + \left[\left(a \frac{\partial N_1}{\partial x} + b \frac{\partial N_2}{\partial x} \right) - \left(a \frac{\partial M_1}{\partial y} + b \frac{\partial M_2}{\partial y} \right) \right] \mathbf{k} \\
 &= a \left[\left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y} \right) \mathbf{k} \right] + b \left[\left(\frac{\partial P_2}{\partial y} - \frac{\partial N_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y} \right) \mathbf{k} \right] \\
 &= a \nabla \times \mathbf{F}_1 + b \nabla \times \mathbf{F}_2
 \end{aligned}$$

$$\text{(c)} \quad \mathbf{F}_1 \times \mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M_1 & N_1 & P_1 \\ M_2 & N_2 & P_2 \end{vmatrix} = (N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k}$$

$$\Rightarrow \nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \nabla \cdot [(N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k}]$$

$$= \frac{\partial}{\partial x}(N_1P_2 - P_1N_2) - \frac{\partial}{\partial y}(M_1P_2 - P_1M_2) + \frac{\partial}{\partial z}(M_1N_2 - N_1M_2)$$

$$\begin{aligned}
 &= \left(P_2 \frac{\partial N_1}{\partial x} + N_1 \frac{\partial P_2}{\partial x} - N_2 \frac{\partial P_1}{\partial x} - P_1 \frac{\partial N_2}{\partial x} \right) - \left(M_1 \frac{\partial P_2}{\partial y} + P_2 \frac{\partial M_1}{\partial y} - P_1 \frac{\partial M_2}{\partial y} - M_2 \frac{\partial P_1}{\partial y} \right) \\
 &\quad + \left(M_1 \frac{\partial N_2}{\partial z} + N_2 \frac{\partial M_1}{\partial z} - N_1 \frac{\partial M_2}{\partial z} - M_2 \frac{\partial N_1}{\partial z} \right)
 \end{aligned}$$

$$= M_2 \left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z} \right) + N_2 \left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x} \right) + P_2 \left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y} \right) + M_1 \left(\frac{\partial N_2}{\partial z} - \frac{\partial P_2}{\partial x} \right) + N_1 \left(\frac{\partial P_2}{\partial x} - \frac{\partial M_2}{\partial y} \right) + P_1 \left(\frac{\partial M_2}{\partial y} - \frac{\partial N_2}{\partial z} \right)$$

$$= \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$$

27. Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k}$.

$$\text{(a)} \quad \mathbf{F}_1 \times \mathbf{F}_2 = (N_1P_2 - P_1N_2)\mathbf{i} + (P_1M_2 - M_1P_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k}$$

$$\begin{aligned}
 \Rightarrow \nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) &= \left[\frac{\partial}{\partial y}(M_1N_2 - N_1M_2) - \frac{\partial}{\partial z}(P_1M_2 - M_1P_2) \right] \mathbf{i} + \left[\frac{\partial}{\partial z}(N_1P_2 - P_1N_2) - \frac{\partial}{\partial x}(M_1N_2 - N_1M_2) \right] \mathbf{j} \\
 &\quad + \left[\frac{\partial}{\partial x}(P_1M_2 - M_1P_2) - \frac{\partial}{\partial y}(N_1P_2 - P_1N_2) \right] \mathbf{k}
 \end{aligned}$$

$$\text{consider the } \mathbf{i}\text{-component only: } \frac{\partial}{\partial y}(M_1N_2 - N_1M_2) - \frac{\partial}{\partial z}(P_1M_2 - M_1P_2)$$

$$= N_2 \frac{\partial M_1}{\partial y} + M_1 \frac{\partial N_2}{\partial y} - M_2 \frac{\partial N_1}{\partial y} - N_1 \frac{\partial M_2}{\partial y} - M_2 \frac{\partial P_1}{\partial z} - P_1 \frac{\partial M_2}{\partial z} + P_2 \frac{\partial M_1}{\partial z} + M_1 \frac{\partial P_2}{\partial z}$$

$$= \left(N_2 \frac{\partial M_1}{\partial y} + P_2 \frac{\partial M_1}{\partial z} \right) - \left(N_1 \frac{\partial M_2}{\partial y} + P_1 \frac{\partial M_2}{\partial z} \right) + \left(\frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z} \right) M_1 - \left(\frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z} \right) M_2$$

$$= \left(M_2 \frac{\partial M_1}{\partial x} + N_2 \frac{\partial M_1}{\partial y} + P_2 \frac{\partial M_1}{\partial z} \right) - \left(M_1 \frac{\partial M_2}{\partial x} + N_1 \frac{\partial M_2}{\partial y} + P_1 \frac{\partial M_2}{\partial z} \right) + \left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z} \right) M_1 - \left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z} \right) M_2$$

$$\text{Now, i-comp of } (\mathbf{F}_2 \cdot \nabla) \mathbf{F}_1 = \left(M_2 \frac{\partial}{\partial x} + N_2 \frac{\partial}{\partial y} + P_2 \frac{\partial}{\partial z} \right) M_1 = \left(M_2 \frac{\partial M_1}{\partial x} + N_2 \frac{\partial M_1}{\partial y} + P_2 \frac{\partial M_1}{\partial z} \right);$$

$$\text{likewise, i-comp of } (\mathbf{F}_1 \cdot \nabla) \mathbf{F}_2 = \left(M_1 \frac{\partial M_2}{\partial x} + N_1 \frac{\partial M_2}{\partial y} + P_1 \frac{\partial M_2}{\partial z} \right);$$

$$\text{i comp of } (\nabla \cdot \mathbf{F}_2) \mathbf{F}_1 = \left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z} \right) M_1 \text{ and i-comp of } (\nabla \cdot \mathbf{F}_1) \mathbf{F}_2 = \left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z} \right) M_2.$$

Similar results hold for the **j** and **k** components of $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2)$. In summary, since the corresponding components are equal, we have the result $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla) \mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla) \mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2) \mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1) \mathbf{F}_2$

(b) Here again we consider only the **i**-component of each expression. Thus, the **i**-comp of $\nabla (\mathbf{F}_1 \cdot \mathbf{F}_2)$

$$= \frac{\partial}{\partial x} (M_1 M_2 + N_1 N_2 + P_1 P_2) = \left(M_1 \frac{\partial M_2}{\partial x} + M_2 \frac{\partial M_1}{\partial x} + N_1 \frac{\partial N_2}{\partial x} + N_2 \frac{\partial N_1}{\partial x} + P_1 \frac{\partial P_2}{\partial x} + P_2 \frac{\partial P_1}{\partial x} \right)$$

$$\text{i-comp of } (\mathbf{F}_1 \cdot \nabla) \mathbf{F}_2 = \left(M_1 \frac{\partial M_2}{\partial x} + N_1 \frac{\partial M_2}{\partial y} + P_1 \frac{\partial M_2}{\partial z} \right),$$

$$\text{i-comp of } (\mathbf{F}_2 \cdot \nabla) \mathbf{F}_1 = \left(M_2 \frac{\partial M_1}{\partial x} + N_2 \frac{\partial M_1}{\partial y} + P_2 \frac{\partial M_1}{\partial z} \right),$$

$$\text{i-comp of } \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) = N_1 \left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y} \right) - P_1 \left(\frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x} \right), \text{ and}$$

$$\text{i-comp of } \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1) = N_2 \left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y} \right) - P_2 \left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x} \right).$$

Since corresponding components are equal, we see that

$$\nabla (\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla) \mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla) \mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1), \text{ as claimed.}$$

$$28. \quad (a) \quad \text{From the Divergence Theorem, } \iint_S \nabla f \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \nabla f dV = \iiint_D (\nabla^2 f) dV = \iiint_D 0 dV = 0$$

$$(b) \quad \text{From the Divergence Theorem, } \iint_S f \nabla f \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot f \nabla f dV. \text{ Now,}$$

$$\begin{aligned} f \nabla f &= \left(f \frac{\partial f}{\partial x} \right) \mathbf{i} + \left(f \frac{\partial f}{\partial y} \right) \mathbf{j} + \left(f \frac{\partial f}{\partial z} \right) \mathbf{k} \Rightarrow \nabla \cdot f \nabla f = \left[f \frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial x} \right)^2 \right] + \left[f \frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial y} \right)^2 \right] + \left[f \frac{\partial^2 f}{\partial z^2} + \left(\frac{\partial f}{\partial z} \right)^2 \right] \\ &= f \nabla^2 f + |\nabla f|^2 = 0 + |\nabla f|^2 \text{ since } f \text{ is harmonic} \Rightarrow \iint_S f \nabla f \cdot \mathbf{n} d\sigma = \iiint_D |\nabla f|^2 dV, \text{ as claimed.} \end{aligned}$$

$$\begin{aligned} 29. \quad \iint_S f \nabla g \cdot \mathbf{n} d\sigma &= \iiint_D \nabla \cdot f \nabla g dV = \iiint_D \nabla \cdot \left(f \frac{\partial g}{\partial x} \mathbf{i} + f \frac{\partial g}{\partial y} \mathbf{j} + f \frac{\partial g}{\partial z} \mathbf{k} \right) dV \\ &= \iiint_D \left(f \frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z} \frac{\partial g}{\partial z} \right) dV \\ &= \iiint_D \left[f \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \right) + \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial g}{\partial z} \right) \right] dV = \iiint_D (f \nabla^2 g + \nabla f \cdot \nabla g) dV \end{aligned}$$

30. By Exercise 29, $\iint_S f \nabla g \cdot \mathbf{n} \, d\sigma = \iiint_D \left(f \nabla^2 g + \nabla f \cdot \nabla g \right) dV$ and by interchanging the roles of f and g ,
- $$\iint_S g \nabla f \cdot \mathbf{n} \, d\sigma = \iiint_D \left(g \nabla^2 f + \nabla g \cdot \nabla f \right) dV.$$
- Subtracting the second equation from the first yields:
- $$\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} \, d\sigma = \iiint_D \left(f \nabla^2 g - g \nabla^2 f \right) dV \text{ since } \nabla f \cdot \nabla g = \nabla g \cdot \nabla f.$$
31. (a) The integral $\iiint_D p(t, x, y, z) \, dV$ represents the mass of the fluid at any time t . The equation says that the instantaneous rate of change of mass is flux of the fluid through the surface S enclosing the region D : the mass decreases if the flux is outward (so the fluid flows out of D), and increases if the flow is inward (interpreting \mathbf{n} as the outward pointing unit normal to the surface).
- (b) $\iiint_D \frac{\partial p}{\partial t} \, dV = \frac{d}{dt} \iiint_D p \, dV = - \iint_S p \mathbf{v} \cdot \mathbf{n} \, d\sigma = - \iiint_D \nabla \cdot p \mathbf{v} \, dV \Rightarrow \frac{\partial p}{\partial t} = - \nabla \cdot p \mathbf{v}$
- Since the law is to hold for all regions D , $\nabla \cdot p \mathbf{v} + \frac{\partial p}{\partial t} = 0$, as claimed
32. (a) ∇T points in the direction of maximum change of the temperature, so if the solid is heating up at the point the temperature is greater in a region surrounding the point $\Rightarrow \nabla T$ points away from the point $\Rightarrow -\nabla T$ points toward the point $\Rightarrow -\nabla T$ points in the direction the heat flows.
- (b) Assuming the Law of Conservation of Mass (Exercise 31) with $-k \nabla T = p \mathbf{v}$ and $c \rho T = p$, we have
- $$\frac{d}{dt} \iiint_D c \rho T \, dV = - \iint_S -k \nabla T \cdot \mathbf{n} \, d\sigma \Rightarrow \text{the continuity equation, } \nabla \cdot (-k \nabla T) + \frac{\partial}{\partial t} (c \rho T) = 0$$
- $$\Rightarrow c \rho \frac{\partial T}{\partial t} = - \nabla \cdot (-k \nabla T) = k \nabla^2 T \Rightarrow \frac{\partial T}{\partial t} = \frac{k}{c \rho} \nabla^2 T = K \nabla^2 T, \text{ as claimed}$$

CHAPTER 16 PRACTICE EXERCISES

1. Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t, y = t, z = t, 0 \leq t \leq 1 \Rightarrow f(g(t), h(t), k(t)) = 3 - 3t^2$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1, \frac{dz}{dt} = 1 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{3} dt \Rightarrow \int_C f(x, y, z) \, ds = \int_0^1 \sqrt{3} (3 - 3t^2) \, dt = 2\sqrt{3}$
- Path 2: $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1 \Rightarrow x = t, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = 2t - 3t^2 + 3$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{2} dt \Rightarrow \int_{C_1} f(x, y, z) \, ds = \int_0^1 \sqrt{2} (2t - 3t^2 + 3) \, dt = 3\sqrt{2};$
- $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1, z = t \Rightarrow f(g(t), h(t), k(t)) = 2 - 2t$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_2} f(x, y, z) \, ds = \int_0^1 (2 - 2t) \, dt = 1$
- $\Rightarrow \int_C f(x, y, z) \, ds = \int_{C_1} f(x, y, z) \, ds + \int_{C_2} f(x, y, z) \, ds = 3\sqrt{2} + 1$
2. Path 1: $\mathbf{r}_1 = t\mathbf{i} \Rightarrow x = t, y = 0, z = 0 \Rightarrow f(g(t), h(t), k(t)) = t^2$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_1} f(x, y, z) \, ds = \int_0^1 t^2 \, dt = \frac{1}{3};$
- $\mathbf{r}_2 = \mathbf{i} + t\mathbf{j} \Rightarrow x = 1, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = 1 + t$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_2} f(x, y, z) ds = \int_0^1 (1+t) dt = \frac{3}{2};$$

$$\mathbf{r}_3 = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1, z = t \Rightarrow f(g(t), h(t), k(t)) = 2 - t \text{ and } \frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1$$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_3} f(x, y, z) ds = \int_0^1 (2-t) dt = \frac{3}{2}$$

$$\Rightarrow \int_{\text{Path 1}} f(x, y, z) ds = \int_{C_1} f(x, y, z) ds + \int_{C_2} f(x, y, z) ds + \int_{C_3} f(x, y, z) ds = \frac{10}{3}$$

$$\text{Path 2 : } \mathbf{r}_4 = t\mathbf{i} + t\mathbf{j} \Rightarrow x = t, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = t^2 + t \text{ and } \frac{dx}{dt} = 1, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{2} dt \Rightarrow \int_{C_4} f(x, y, z) ds = \int_0^1 \sqrt{2} (t^2 + t) dt = \frac{5}{6}\sqrt{2};$$

$$\mathbf{r}_3 = \mathbf{i} + \mathbf{j} + t\mathbf{k} \text{ (see above)} \Rightarrow \int_{C_3} f(x, y, z) ds = \frac{3}{2}$$

$$\Rightarrow \int_{\text{Path 2}} f(x, y, z) ds = \int_{C_3} f(x, y, z) ds + \int_{C_4} f(x, y, z) ds = \frac{5}{6}\sqrt{2} + \frac{3}{2} = \frac{5\sqrt{2}+9}{6}$$

$$\text{Path 3 : } \mathbf{r}_5 = t\mathbf{k} \Rightarrow x = 0, y = 0, z = t, 0 \leq t \leq 1 \Rightarrow f(g(t), h(t), k(t)) = -t \text{ and } \frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1$$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_5} f(x, y, z) ds = \int_0^1 -t dt = -\frac{1}{2};$$

$$\mathbf{r}_6 = t\mathbf{j} + \mathbf{k} \Rightarrow x = 0, y = t, z = 1, 0 \leq t \leq 1 \Rightarrow f(g(t), h(t), k(t)) = t - 1 \text{ and } \frac{dx}{dt} = 0, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_6} f(x, y, z) ds = \int_0^1 (t-1) dt = -\frac{1}{2};$$

$$\mathbf{r}_7 = t\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow x = t, y = 1, z = 1, 0 \leq t \leq 1 \Rightarrow f(g(t), h(t), k(t)) = t^2 \text{ and } \frac{dx}{dt} = 1, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = dt \Rightarrow \int_{C_7} f(x, y, z) ds = \int_0^1 t^2 dt = \frac{1}{3}$$

$$\Rightarrow \int_{\text{Path 3}} f(x, y, z) ds = \int_{C_5} f(x, y, z) ds + \int_{C_6} f(x, y, z) ds + \int_{C_7} f(x, y, z) ds = -\frac{1}{2} - \frac{1}{2} + \frac{1}{3} = -\frac{2}{3}$$

3. $\mathbf{r} = (a \cos t)\mathbf{j} + (a \sin t)\mathbf{k} \Rightarrow x = 0, y = a \cos t, z = a \sin t \Rightarrow f(g(t), h(t), k(t)) = \sqrt{a^2 \sin^2 t} = a |\sin t|$ and

$$\frac{dx}{dt} = 0, \frac{dy}{dt} = -a \sin t, \frac{dz}{dt} = a \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = a dt$$

$$\Rightarrow \int_c f(x, y, z) ds = \int_0^{2\pi} a^2 |\sin t| dt = \int_0^\pi a^2 \sin t dt + \int_\pi^{2\pi} (-a^2 \sin t) dt = 4a^2$$

4. $\mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} \Rightarrow x = \cos t + t \sin t, y = \sin t - t \cos t, z = 0$

$$\Rightarrow f(g(t), h(t), k(t)) = \sqrt{(\cos t + t \sin t)^2 + (\sin t - t \cos t)^2} = \sqrt{1+t^2} \text{ and}$$

$$\frac{dx}{dt} = -\sin t + \sin t + t \cos t = t \cos t, \frac{dy}{dt} = \cos t - \cos t + t \sin t = t \sin t, \frac{dz}{dt} = 0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

$$= \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} dt = |t| dt = t dt \text{ since } 0 \leq t \leq \sqrt{3} \Rightarrow \int_c f(x, y, z) ds = \int_0^{\sqrt{3}} t \sqrt{1+t^2} dt = \frac{7}{3}$$

5. $\frac{\partial P}{\partial y} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial M}{\partial y}$
 $\Rightarrow M dx + N dy + P dz$ is exact; $\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x+y+z}} \Rightarrow f(x, y, z) = 2\sqrt{x+y+z} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{1}{\sqrt{x+y+z}} + \frac{\partial g}{\partial y}$
 $= \frac{1}{\sqrt{x+y+z}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = 2\sqrt{x+y+z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{1}{\sqrt{x+y+z}} + h'(z)$
 $= \frac{1}{\sqrt{x+y+z}} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = 2\sqrt{x+y+z} + C \Rightarrow \int_{(-1, 1, 1)}^{(4, -3, 0)} \frac{dx+dy+dz}{\sqrt{x+y+z}}$
 $= f(4, -3, 0) - f(-1, 1, 1) = 2\sqrt{1} - 2\sqrt{1} = 0$
6. $\frac{\partial P}{\partial y} = -\frac{1}{2\sqrt{yz}} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz$ is exact; $\frac{\partial f}{\partial x} = 1$
 $\Rightarrow f(x, y, z) = x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -\sqrt{\frac{z}{y}} \Rightarrow g(y, z) = -2\sqrt{yz} + h(z) \Rightarrow f(x, y, z) = x - 2\sqrt{yz} + h(z)$
 $\Rightarrow \frac{\partial f}{\partial z} = -\sqrt{\frac{y}{z}} + h'(z) = -\sqrt{\frac{y}{z}} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = x - 2\sqrt{yz} + C$
 $\Rightarrow \int_{(1, 1, 1)}^{(10, 3, 3)} dx - \sqrt{\frac{z}{y}} dy - \sqrt{\frac{y}{z}} dz = f(10, 3, 3) - f(1, 1, 1) = (10 - 2 \cdot 3) - (1 - 2 \cdot 1) = 4 + 1 = 5$
7. $\frac{\partial M}{\partial z} = -y \cos z \neq y \cos z = \frac{\partial P}{\partial x} \Rightarrow \mathbf{F}$ is not conservative; $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} - \mathbf{k}, 0 \leq t \leq 2\pi$
 $\Rightarrow d\mathbf{r} = (-2 \sin t)\mathbf{i} - (2 \cos t)\mathbf{j} \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} [(-2 \sin t)(\sin(-1))(-2 \sin t) + (2 \cos t)(\sin(-1))(-2 \cos t)] dt$
 $= 4 \sin(1) \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 8\pi \sin(1)$
8. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 3x^2 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = 0$
9. Let $M = 8x \sin y$ and $N = -8y \cos x \Rightarrow \frac{\partial M}{\partial y} = 8x \cos y$ and $\frac{\partial N}{\partial x} = 8y \sin x \Rightarrow \int_C 8x \sin y dx - 8y \cos x dy$
 $= \iint_R (8y \sin x - 8x \cos y) dy dx = \int_0^{\pi/2} \int_0^{\pi/2} (8y \sin x - 8x \cos y) dy dx = \int_0^{\pi/2} (\pi^2 \sin x - 8x) dx = -\pi^2 + \pi^2 = 0$
10. Let $M = y^2$ and $N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 2x \Rightarrow \int_C y^2 dx + x^2 dy = \iint_R (2x - 2y) dx dy$
 $= \int_0^{2\pi} \int_0^2 (2r \cos \theta - 2r \sin \theta) r dr d\theta = \int_0^{2\pi} \frac{16}{3} (\cos \theta - \sin \theta) d\theta = 0$
11. Let $z = 1 - x - y \Rightarrow f_x(x, y) = -1$ and $f_y(x, y) = -1 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{3} \Rightarrow \text{Surface Area} = \iint_R \sqrt{3} dx dy$
 $= \sqrt{3} \text{ (Area of the circular region in the } xy\text{-plane)} = \pi\sqrt{3}$
12. $\nabla f = -3\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}, \mathbf{p} = \mathbf{i} \Rightarrow |\nabla f| = \sqrt{9 + 4y^2 + 4z^2}$ and $|\nabla f \cdot \mathbf{p}| = 3$
 $\Rightarrow \text{Surface Area} = \iint_R \frac{\sqrt{9 + 4y^2 + 4z^2}}{3} dy dz = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{\sqrt{9 + 4r^2}}{3} r dr d\theta = \frac{1}{3} \int_0^{2\pi} \left(\frac{7}{4} \sqrt{21} - \frac{9}{4} \right) d\theta = \frac{\pi}{6} (7\sqrt{21} - 9)$

$$13. \quad \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}, \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2 \text{ and } |\nabla f \cdot \mathbf{p}| = |2z| = 2z \text{ since}$$

$$z \geq 0 \Rightarrow \text{Surface Area} = \iint_R \frac{2}{2z} dA = \iint_R \frac{1}{z} dA = \iint_R \frac{1}{\sqrt{1-x^2-y^2}} dx dy = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-r^2}} r dr d\theta$$

$$= \int_0^{2\pi} \left[-\sqrt{1-r^2} \right]_0^{1/\sqrt{2}} d\theta = \int_0^{2\pi} \left(1 - \frac{1}{\sqrt{2}} \right) d\theta = 2\pi \left(1 - \frac{1}{\sqrt{2}} \right)$$

$$14. \quad (a) \quad \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}, \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 4 \text{ and } |\nabla f \cdot \mathbf{p}| = 2z \text{ since}$$

$$z \geq 0 \Rightarrow \text{Surface Area} = \iint_R \frac{4}{2z} dA = \iint_R \frac{2}{z} dA = 2 \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{2}{\sqrt{4-r^2}} r dr d\theta = 4\pi - 8$$

$$(b) \quad \mathbf{r} = 2 \cos \theta \Rightarrow d\mathbf{r} = -2 \sin \theta d\theta; ds^2 = r^2 d\theta^2 + dr^2 \text{ (Arc length in polar coordinates)}$$

$$\Rightarrow ds^2 = (2 \cos \theta)^2 d\theta^2 + dr^2 = 4 \cos^2 \theta d\theta^2 + 4 \sin^2 \theta d\theta^2 = 4 d\theta^2 \Rightarrow ds = 2 d\theta; \text{ the height of the cylinder is } z = \sqrt{4-r^2} = \sqrt{4-4\cos^2 \theta} = 2|\sin \theta| = 2 \sin \theta \text{ if } 0 \leq \theta \leq \frac{\pi}{2}$$

$$\Rightarrow \text{Surface Area} = \int_{-\pi/2}^{\pi/2} h ds = 2 \int_0^{\pi/2} (2 \sin \theta)(2 d\theta) = 8$$

$$15. \quad f(x, y, z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow \nabla f = \left(\frac{1}{a}\right)\mathbf{i} + \left(\frac{1}{b}\right)\mathbf{j} + \left(\frac{1}{c}\right)\mathbf{k} \Rightarrow |\nabla f| = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \text{ and } \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = \frac{1}{c} \text{ since}$$

$$c > 0 \Rightarrow \text{Surface Area} = \iint_R \frac{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}{\left(\frac{1}{c}\right)} dA = c \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \iint_R dA = \frac{1}{2} abc \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}, \text{ since the area of the triangular region } R \text{ is } \frac{1}{2} ab. \text{ To check this result, let } \mathbf{v} = a\mathbf{i} + c\mathbf{k} \text{ and } \mathbf{w} = -a\mathbf{i} + b\mathbf{j}; \text{ the area can be found by computing } \frac{1}{2} |\mathbf{v} \times \mathbf{w}|.$$

$$16. \quad (a) \quad \nabla f = 2y\mathbf{j} - \mathbf{k}, \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 1} \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2 + 1} dx dy$$

$$\Rightarrow \iint_S g(x, y, z) d\sigma = \iint_R \frac{yz}{\sqrt{4y^2 + 1}} \sqrt{4y^2 + 1} dx dy = \iint_R y(y^2 - 1) dx dy = \int_{-1}^1 \int_0^3 (y^3 - y) dx dy$$

$$= \int_{-1}^1 3(y^3 - y) dy = 3 \left[\frac{y^4}{4} - \frac{y^2}{2} \right]_{-1}^1 = 0$$

$$(b) \quad \iint_S g(x, y, z) d\sigma = \iint_R \frac{z}{\sqrt{4y^2 + 1}} \sqrt{4y^2 + 1} dx dy = \int_{-1}^1 \int_0^3 (y^2 - 1) dx dy = \int_{-1}^1 3(y^2 - 1) dy = 3 \left[\frac{y^3}{3} - y \right]_{-1}^1 = -4$$

$$17. \quad \nabla f = 2y\mathbf{j} + 2z\mathbf{k}, \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 4z^2} = 2\sqrt{y^2 + z^2} = 10 \text{ and } |\nabla f \cdot \mathbf{p}| = 2z \text{ since } z \geq 0$$

$$\Rightarrow d\sigma = \frac{10}{2z} dx dy = \frac{5}{z} dx dy = \iint_S g(x, y, z) d\sigma = \iint_R (x^4 y) (y^2 + z^2) \left(\frac{5}{z} \right) dx dy$$

$$= \iint_R (x^4 y) (25) \left(\frac{5}{\sqrt{25-y^2}} \right) dx dy = \int_0^4 \int_0^1 \frac{125y}{\sqrt{25-y^2}} x^4 dx dy = \int_0^4 \frac{25y}{\sqrt{25-y^2}} dy = 50$$

18. Define the coordinate system so that the origin is at the center of the earth, the z -axis is the earth's axis (north is the positive z direction), and the xz -plane contains the earth's prime meridian. Let S denote the surface

which is Wyoming so then S is part of the surface $z = (R^2 - x^2 - y^2)^{1/2}$. Let R_{xy} be the projection of S onto

the xy -plane. The surface area of Wyoming is $\iint_S 1 \, d\sigma = \iint_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$

$$= \iint_{R_{xy}} \sqrt{\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + 1} \, dA = \iint_{R_{xy}} \frac{R}{(R^2 - x^2 - y^2)^{1/2}} \, dA = \int_{\theta_1}^{\theta_2} \int_{R \sin 45^\circ}^{R \sin 49^\circ} R (R^2 - r^2)^{-1/2} r \, dr \, d\theta \quad (\text{where } \theta_1 \text{ and}$$

$$\theta_2 \text{ are the radian equivalent to } 104^\circ 3' \text{ and } 111^\circ 3', \text{ respectively}) = \int_{\theta_1}^{\theta_2} \left[-R (R^2 - r^2)^{1/2} \right]_{R \sin 45^\circ}^{R \sin 49^\circ} d\theta$$

$$= \int_{\theta_1}^{\theta_2} \left[R (R^2 - R^2 \sin^2 45^\circ)^{1/2} - R (R^2 - R^2 \sin^2 49^\circ)^{1/2} \right] d\theta = (\theta_2 - \theta_1) R^2 (\cos 45^\circ - \cos 49^\circ)$$

$$= \frac{7\pi}{180} R^2 (\cos 45^\circ - \cos 49^\circ) = \frac{7\pi}{180} (6371)^2 (\cos 45^\circ - \cos 49^\circ) \approx 253,143 \text{ km}^2.$$

19. A possible parametrization is $\mathbf{r}(\phi, \theta) = (6 \sin \phi \cos \theta)\mathbf{i} + (6 \sin \phi \sin \theta)\mathbf{j} + (6 \cos \phi)\mathbf{k}$ (spherical coordinates); now

$$\rho = 6 \text{ and } z = -3 \Rightarrow -3 = 6 \cos \phi \Rightarrow \cos \phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3} \text{ and } z = 3\sqrt{3} \Rightarrow 3\sqrt{3} = 6 \cos \phi$$

$$\Rightarrow \cos \phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6} \Rightarrow \frac{\pi}{6} \leq \phi \leq \frac{2\pi}{3}; \text{ also } 0 \leq \theta \leq 2\pi$$

20. A possible parametrization is $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} - \left(\frac{r^2}{2}\right)\mathbf{k}$ (cylindrical coordinates); now

$$r = \sqrt{x^2 + y^2} \Rightarrow z = -\frac{r^2}{2} \text{ and } -2 \leq z \leq 0 \Rightarrow -2 \leq -\frac{r^2}{2} \leq 0 \Rightarrow 4 \geq r^2 \geq 0 \Rightarrow 0 \leq r \leq 2 \text{ since } r \geq 0; \text{ also } 0 \leq \theta \leq 2\pi$$

21. A possible parametrization is $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (1 + r)\mathbf{k}$ (cylindrical coordinates);

$$\text{now } r = \sqrt{x^2 + y^2} \Rightarrow z = 1 + r \text{ and } 1 \leq z \leq 3 \Rightarrow 1 \leq 1 + r \leq 3 \Rightarrow 0 \leq r \leq 2; \text{ also } 0 \leq \theta \leq 2\pi$$

22. A possible parametrization is $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \left(3 - x - \frac{y}{2}\right)\mathbf{k}$ for $0 \leq x \leq 2$ and $0 \leq y \leq 2$

23. Let $x = u \cos v$ and $z = u \sin v$, where $u = \sqrt{x^2 + z^2}$ and v is the angle in the xz -plane with the x -axis

$$\Rightarrow \mathbf{r}(u, v) = (u \cos v)\mathbf{i} + 2u^2\mathbf{j} + (u \sin v)\mathbf{k} \text{ is a possible parametrization; } 0 \leq y \leq 2 \Rightarrow 2u^2 \leq 2 \Rightarrow u^2 \leq 1$$

$$\Rightarrow 0 \leq u \leq 1 \text{ since } u \geq 0; \text{ also, for just the upper half of the paraboloid, } 0 \leq v \leq \pi$$

24. A possible parametrization is $(\sqrt{10} \sin \phi \cos \theta)\mathbf{i} + (\sqrt{10} \sin \phi \sin \theta)\mathbf{j} + (\sqrt{10} \cos \phi)\mathbf{k}$, $0 \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \theta \leq \frac{\pi}{2}$

$$25. \quad \mathbf{r}_u = \mathbf{i} + \mathbf{j}, \mathbf{r}_v = \mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{6}$$

$$\Rightarrow \text{Surface Area} = \int \int_{R_{uv}} |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv = \int_0^1 \int_0^1 \sqrt{6} \, du \, dv = \sqrt{6}$$

26. $\iint_S (xy - z^2) d\sigma = \int_0^1 \int_0^1 [(u+v)(u-v) - v^2] \sqrt{6} du dv = \sqrt{6} \int_0^1 \int_0^1 (u^2 - 2v^2) du dv$
 $= \sqrt{6} \int_0^1 \left[\frac{u^2}{3} - 2uv^2 \right]_0^1 dv = \sqrt{6} \int_0^1 \left(\frac{1}{3} - 2v^2 \right) dv = \sqrt{6} \left[\frac{1}{3}v - \frac{2}{3}v^3 \right]_0^1 = -\frac{\sqrt{6}}{3} = -\sqrt{\frac{2}{3}}$
27. $\mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$, $\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 1 \end{vmatrix}$
 $= (\sin \theta)\mathbf{i} - (\cos \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\sin^2 \theta + \cos^2 \theta + r^2} = \sqrt{1+r^2} \Rightarrow \text{Surface Area} = \iint_{R_{r\theta}} |\mathbf{r}_r \times \mathbf{r}_\theta| dr d\theta$
 $= \int_0^{2\pi} \int_0^1 \sqrt{1+r^2} dr d\theta = \int_0^{2\pi} \left[\frac{r}{2} \sqrt{1+r^2} + \frac{1}{2} \ln(r + \sqrt{1+r^2}) \right]_0^1 d\theta = \int_0^{2\pi} \left[\frac{1}{2} \sqrt{2} + \frac{1}{2} \ln(1 + \sqrt{2}) \right] d\theta$
 $= \pi \left[\sqrt{2} + \ln(1 + \sqrt{2}) \right]$
28. $\iint_S \sqrt{x^2 + y^2 + 1} d\sigma = \int_0^{2\pi} \int_0^1 \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + 1} \sqrt{1+r^2} dr d\theta = \int_0^{2\pi} \int_0^1 (1+r^2) dr d\theta$
 $= \int_0^{2\pi} \left[r + \frac{r^3}{3} \right]_0^1 d\theta = \int_0^{2\pi} \frac{4}{3} d\theta = \frac{8}{3} \pi$
29. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow$ Conservative
30. $\frac{\partial P}{\partial y} = \frac{-3zy}{(x^2 + y^2 + z^2)^{-5/2}} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{-3xz}{(x^2 + y^2 + z^2)^{-5/2}} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \frac{-3xy}{(x^2 + y^2 + z^2)^{-5/2}} = \frac{\partial M}{\partial y} \Rightarrow$ Conservative
31. $\frac{\partial P}{\partial y} = 0 \neq ye^z = \frac{\partial N}{\partial z} \Rightarrow$ Not Conservative
32. $\frac{\partial P}{\partial y} = \frac{x}{(x+yz)^2} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{-y}{(x+yz)^2} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \frac{-z}{(x+yz)^2} = \frac{\partial M}{\partial y} \Rightarrow$ Conservative
33. $\frac{\partial f}{\partial x} = 2 \Rightarrow f(x, y, z) = 2x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y + z \Rightarrow g(y, z) = y^2 + zy + h(z) \Rightarrow f(x, y, z) = 2x + y^2 + zy + h(z)$
 $= 2x + y^2 + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = y + h'(z) = y + 1 \Rightarrow h'(z) = 1 \Rightarrow h(z) = z + C \Rightarrow f(x, y, z) = 2x + y^2 + zy + z + C$
34. $\frac{\partial f}{\partial x} = z \cos xz \Rightarrow f(x, y, z) = \sin xz + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = e^y \Rightarrow g(y, z) = e^y + h(z) \Rightarrow f(x, y, z) = \sin xz + e^y + h(z)$
 $= \sin xz + e^y + h(z) \Rightarrow \frac{\partial f}{\partial z} = x \cos xz + h'(z) = x \cos xz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = \sin xz + e^y + C$
35. Over Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 1 \Rightarrow x = t, y = t, z = t$ and $d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) dt \Rightarrow \mathbf{F} = 2t^2\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$
 $\Rightarrow \mathbf{F} \cdot d\mathbf{r} = (3t^2 + 1) dt \Rightarrow \text{Work} = \int_0^1 (3t^2 + 1) dt = 2;$
 Over Path 2: $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$, $0 \leq t \leq 1 \Rightarrow x = t, y = t, z = 0$ and $d\mathbf{r}_1 = (\mathbf{i} + \mathbf{j}) dt \Rightarrow \mathbf{F}_1 = 2t^2\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$

$$\Rightarrow \mathbf{F}_1 \cdot d\mathbf{r}_1 = (2t^2 + 1) dt \Rightarrow \text{Work}_1 = \int_0^1 (2t^2 + 1) dt = \frac{5}{3}; \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow x = 1, y = 1, z = t \text{ and}$$

$$d\mathbf{r}_2 = \mathbf{k} dt \Rightarrow \mathbf{F}_2 = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot d\mathbf{r}_2 = dt \Rightarrow \text{Work}_2 = \int_0^1 dt = 1 \Rightarrow \text{Work} = \text{Work}_1 + \text{Work}_2 = \frac{5}{3} + 1 = \frac{8}{3}$$

36. Over Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow x = t, y = t, z = t$ and $d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) dt \Rightarrow \mathbf{F} = 2t^2\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$

$$\Rightarrow \mathbf{F} \cdot d\mathbf{r} = (3t^2 + 1) dt \Rightarrow \text{Work} = \int_0^1 (3t^2 + 1) dt = 2;$$

Over Path 2: Since f is conservative, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around any simply closed curve C . Thus consider

$$\int_{\text{curve}} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}, \text{ where } C_1 \text{ is the path from } (0, 0, 0) \text{ to } (1, 1, 0) \text{ to } (1, 1, 1) \text{ and } C_2 \text{ is the}$$

$$\text{path from } (1, 1, 1) \text{ to } (0, 0, 0). \text{ Now, from Path 1 above, } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -2 \Rightarrow 0 = \int_{\text{curve}} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + (-2)$$

$$\Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2$$

37. (a) $\mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} \Rightarrow x = e^t \cos t, y = e^t \sin t$ from $(1, 0)$ to $(e^{2\pi}, 0) \Rightarrow 0 \leq t \leq 2\pi$

$$\Rightarrow \frac{d\mathbf{r}}{dt} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} \text{ and } \mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}} = \frac{(e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}}{(e^{2t} \cos^2 t + e^{2t} \sin^2 t)^{3/2}}$$

$$= \left(\frac{\cos t}{e^{2t}}\right)\mathbf{i} + \left(\frac{\sin t}{e^{2t}}\right)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \left(\frac{\cos^2 t}{e^t} - \frac{\sin t \cos t}{e^t} + \frac{\sin^2 t}{e^t} + \frac{\sin t \cos t}{e^t}\right) = e^{-t} \Rightarrow \text{Work} = \int_0^{2\pi} e^{-t} dt = 1 - e^{-2\pi}$$

(b) $\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}} \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{(x^2 + y^2)^{3/2}} \Rightarrow f(x, y, z) = -(x^2 + y^2)^{-1/2} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{y}{(x^2 + y^2)^{3/2}} + \frac{\partial g}{\partial y}$

$$= \frac{y}{(x^2 + y^2)^{3/2}} \Rightarrow g(y, z) = C \Rightarrow f(x, y, z) = -(x^2 + y^2)^{-1/2} \text{ is a potential function for } \mathbf{F}$$

$$\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = f(e^{2\pi}, 0) - f(1, 0) = 1 - e^{-2\pi}$$

38. (a) $\mathbf{F} = \nabla(x^2 z e^y) \Rightarrow \mathbf{F}$ is conservative $\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path C

(b) $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(1, 0, 0)}^{(1, 0, 2\pi)} \nabla(x^2 z e^y) \cdot d\mathbf{r} = (x^2 z e^y) \Big|_{(1, 0, 0)}^{(1, 0, 2\pi)} - (x^2 z e^y) \Big|_{(1, 0, 0)} = 2\pi - 0 = 2\pi$

39. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -y & 3z^2 \end{vmatrix} = -2y\mathbf{k};$ unit normal to the plane is $\mathbf{n} = \frac{2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}}{\sqrt{4 + 36 + 9}} = \frac{2}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}$

$$\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} = \frac{6}{7}y; \mathbf{p} = \mathbf{k} \text{ and } f(x, y, z) = 2x + 6y - 3z \Rightarrow |\nabla f \cdot \mathbf{p}| = 3 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \frac{7}{3} dA$$

$$\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \frac{6}{7} y d\sigma = \iint_R \left(\frac{6}{7} y\right) \left(\frac{7}{3} dA\right) = \iint_R 2y dA = \int_0^{2\pi} \int_0^1 2r \sin \theta r dr d\theta = \int_0^{2\pi} \frac{2}{3} \sin \theta d\theta = 0$$

40. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y & x + y & 4y^2 - z \end{vmatrix} = 8y\mathbf{i}$; the circle lies in the plane $f(x, y, z) = y + z = 0$ with unit normal $\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R 0 \, d\sigma = 0$
41. (a) $\mathbf{r} = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow x = \sqrt{2}t, y = \sqrt{2}t, z = 4 - t^2 \Rightarrow \frac{dx}{dt} = \sqrt{2}, \frac{dy}{dt} = \sqrt{2}, \frac{dz}{dt} = -2t$
 $\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{4 + 4t^2} dt \Rightarrow M = \int_C \delta(x, y, z) \, ds = \int_0^1 3t\sqrt{4 + 4t^2} \, dt = \left[\frac{1}{4}(4 + 4t)^{3/2}\right]_0^1$
 $= 4\sqrt{2} - 2$
- (b) $M = \int_C \delta(x, y, z) \, ds = \int_0^1 \sqrt{4 + 4t^2} \, dt = \left[t\sqrt{1 + t^2} + \ln\left(t + \sqrt{1 + t^2}\right)\right]_0^1 = \sqrt{2} + \ln(1 + \sqrt{2})$
42. $\mathbf{r} = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k}, 0 \leq t \leq 2 \Rightarrow x = t, y = 2t, z = \frac{2}{3}t^{3/2} \Rightarrow \frac{dx}{dt} = 1, \frac{dy}{dt} = 2, \frac{dz}{dt} = t^{1/2}$
 $\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{t + 5} \, dt \Rightarrow M = \int_C \delta(x, y, z) \, ds = \int_0^2 3\sqrt{5 + t}\sqrt{t + 5} \, dt = \int_0^2 3(t + 5) \, dt = 36$;
 $M_{yz} = \int_C x\delta \, ds = \int_0^2 3t(t + 5) \, dt = 38$; $M_{xz} = \int_C y\delta \, ds = \int_0^2 6t(t + 5) \, dt = 76$; $M_{xy} = \int_C z\delta \, ds$
 $= \int_0^2 2t^{3/2}(t + 5) \, dt = \frac{144}{7}\sqrt{2} \Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{38}{36} = \frac{19}{18}, \bar{y} = \frac{M_{xz}}{M} = \frac{76}{36} = \frac{19}{9}, \bar{z} = \frac{M_{xy}}{M} = \frac{\left(\frac{144}{7}\sqrt{2}\right)}{36} = \frac{4}{7}\sqrt{2}$
43. $\mathbf{r} = t\mathbf{i} + \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\mathbf{j} + \left(\frac{t^2}{2}\right)\mathbf{k}, 0 \leq t \leq 2 \Rightarrow x = t, y = \frac{2\sqrt{2}}{3}t^{3/2}, z = \frac{t^2}{2} \Rightarrow \frac{dx}{dt} = 1, \frac{dy}{dt} = \sqrt{2}t^{1/2}, \frac{dz}{dt} = t$
 $\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{1 + 2t + t^2} \, dt = \sqrt{(t + 1)^2} \, dt = |t + 1| \, dt = (t + 1) \, dt$ on the domain given.
Then $M = \int_C \delta \, ds = \int_0^2 \left(\frac{1}{t + 1}\right)(t + 1) \, dt = \int_0^2 dt = 2$; $M_{yz} = \int_C x\delta \, ds = \int_0^2 t\left(\frac{1}{t + 1}\right)(t + 1) \, dt = \int_0^2 t \, dt = 2$;
 $M_{xz} = \int_C y\delta \, ds = \int_0^2 \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\left(\frac{1}{t + 1}\right)(t + 1) \, dt = \int_0^2 \frac{2\sqrt{2}}{3}t^{3/2} \, dt = \frac{32}{15}$; $M_{xy} = \int_C z\delta \, ds$
 $= \int_0^2 \left(\frac{t^2}{2}\right)\left(\frac{1}{t + 1}\right)(t + 1) \, dt = \int_0^2 \frac{t^2}{2} \, dt = \frac{4}{3} \Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{2}{2} = 1$; $\bar{y} = \frac{M_{xz}}{M} = \frac{\left(\frac{32}{15}\right)}{2} = \frac{16}{15}$; $\bar{z} = \frac{M_{xy}}{M} = \frac{\left(\frac{4}{3}\right)}{2} = \frac{2}{3}$;
 $I_x = \int_C (y^2 + z^2) \delta \, ds = \int_0^2 \left(\frac{8}{9}t^3 + \frac{t^4}{4}\right) dt = \frac{232}{45}$; $I_y = \int_C (x^2 + z^2) \delta \, ds = \int_0^2 \left(t^2 + \frac{t^4}{4}\right) dt = \frac{64}{15}$;
 $I_z = \int_C (y^2 + x^2) \delta \, ds = \int_0^2 \left(t^2 + \frac{8}{9}t^3\right) dt = \frac{56}{9}$
44. $\bar{z} = 0$ because the arch is in the xy -plane, and $\bar{x} = 0$ because the mass is distributed symmetrically with respect to the y -axis; $\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, 0 \leq t \leq \pi \Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
 $= \sqrt{(-a \sin t)^2 + (a \cos t)^2} dt = a \, dt$, since $a \geq 0$; $M = \int_C \delta \, ds = \int_C (2a - y) \, ds = \int_0^\pi (2a - a \sin t) \, a \, dt$

$$\begin{aligned}
&= 2a^2\pi - 2a^2; M_{xz} = \int_C y \delta \, dt = \int_C y(2a - y) \, ds = \int_0^\pi (a \sin t)(2a - a \sin t) \, dt = \int_0^\pi (2a^2 \sin t - a^2 \sin^2 t) \, dt \\
&= \left[-2a^2 \cos t - a^2 \left(\frac{t}{2} - \frac{\sin 2t}{4} \right) \right]_0^\pi = 4a^2 - \frac{a^2\pi}{2} \Rightarrow \bar{y} = \frac{\left(4a^2 - \frac{a^2\pi}{2} \right)}{2a^2\pi - 2a^2} = \frac{8-\pi}{4\pi-4} \Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(0, \frac{8-\pi}{4\pi-4}, 0 \right)
\end{aligned}$$

45. $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}, 0 \leq t \leq \ln 2 \Rightarrow x = e^t \cos t, y = e^t \sin t, z = e^t \Rightarrow \frac{dx}{dt} = (e^t \cos t - e^t \sin t),$

$$\begin{aligned}
\frac{dy}{dt} &= (e^t \sin t + e^t \cos t), \frac{dz}{dt} = e^t \Rightarrow \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2} dt \\
&= \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2} dt = \sqrt{3e^{2t}} dt = \sqrt{3}e^t dt; M = \int_C \delta \, ds = \int_0^{\ln 2} \sqrt{3} e^t dt \\
&= \sqrt{3}; M_{xy} = \int_C z \delta \, ds = \int_0^{\ln 2} (\sqrt{3} e^t)(e^t) dt = \int_0^{\ln 2} \sqrt{3} e^{2t} dt = \frac{3\sqrt{3}}{2} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac{\left(\frac{3\sqrt{3}}{2} \right)}{\sqrt{3}} = \frac{3}{2}; \\
I_z &= \int_C (x^2 + y^2) \delta \, ds = \int_0^{\ln 2} (e^{2t} \cos^2 t + e^{2t} \sin^2 t)(\sqrt{3} e^t) dt = \int_0^{\ln 2} \sqrt{3} e^{3t} dt = \frac{7\sqrt{3}}{3}
\end{aligned}$$

46. $\mathbf{r}(t) = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 3t\mathbf{k}, 0 \leq t \leq 2\pi \Rightarrow x = 2 \sin t, y = 2 \cos t, z = 3t \Rightarrow \frac{dx}{dt} = 2 \cos t, \frac{dy}{dt} = -2 \sin t,$

$$\begin{aligned}
\frac{dz}{dt} &= 3 \Rightarrow \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2} dt = \sqrt{4+9} dt = \sqrt{13} dt; M = \int_C \delta \, ds = \int_0^{2\pi} \delta \sqrt{13} dt = 2\pi\delta\sqrt{13}; \\
M_{xy} &= \int_C z \delta \, ds = \int_0^{2\pi} (3t)(\delta\sqrt{13}) dt = 6\delta\pi^2\sqrt{13}; M_{yz} = \int_C x \delta \, ds = \int_0^{2\pi} (2 \sin t)(\delta\sqrt{13}) dt = 0; \\
M_{xz} &= \int_C y \delta \, ds = \int_0^{2\pi} (2 \cos t)(\delta\sqrt{13}) dt = 0 \Rightarrow \bar{x} = \bar{y} = 0 \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{6\delta\pi^2\sqrt{13}}{2\delta\pi\sqrt{13}} = 3\pi \Rightarrow (0, 0, 3\pi) \text{ is the} \\
&\text{center of mass}
\end{aligned}$$

47. Because of symmetry $\bar{x} = \bar{y} = 0$. Let $f(x, y, z) = x^2 + y^2 + z^2 = 25 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$$\begin{aligned}
&\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 10 \text{ and } \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z, \text{ since } z \geq 0 \Rightarrow M = \iint_R \delta(x, y, z) \, d\delta \\
&= \iint_R z \left(\frac{10}{2z} \right) dA = \iint_R 5 \, dA = 5 \text{ (Area of the circular region)} = 80\pi; M_{xy} = \iint_R z \delta \, d\delta = \iint_R 5z \, dA \\
&= \iint_R 5\sqrt{25 - x^2 - y^2} \, dx \, dy = \int_0^{2\pi} \int_0^4 \left(5\sqrt{25 - r^2} \right) r \, dr \, d\theta = \int_0^{2\pi} \frac{490}{3} d\theta = \frac{980}{3}\pi \Rightarrow \bar{z} = \frac{\left(\frac{980}{3}\pi \right)}{80\pi} = \frac{49}{12} \\
&\Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{49}{12} \right); I_z = \iint_R (x^2 + y^2) \delta \, d\sigma = \iint_R 5(x^2 + y^2) \, dx \, dy = \int_0^{2\pi} \int_0^4 5r^3 \, dr \, d\theta = \int_0^{2\pi} 320 \, d\theta \\
&= 640\pi
\end{aligned}$$

48. On the face $z = 1: g(x, y, z) = z = 1$ and $\mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$

$$\Rightarrow I = \iint_R (x^2 + y^2) \, dA = 2 \int_0^{\pi/4} \int_0^{\sec \theta} r^3 \, dr \, d\theta = \frac{2}{3}; \text{ On the face } z = 0: g(x, y, z) = z = 0 \Rightarrow \nabla g = \mathbf{k} \text{ and } \mathbf{p} = \mathbf{k}$$

$$\Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (x^2 + y^2) \, dA = \frac{2}{3}; \text{ On the face } y = 0: g(x, y, z) = y = 0$$

$$\Rightarrow \nabla g = \mathbf{j} \text{ and } \mathbf{p} = \mathbf{j} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (x^2 + 0) \, dA = \int_0^1 \int_0^1 x^2 \, dx \, dz = \frac{1}{3}; \text{ On the face}$$

$$\begin{aligned}
 y=1: g(x, y, z) = y=1 &\Rightarrow \nabla g = \mathbf{j} \text{ and } \mathbf{p} = \mathbf{j} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (x^2 + 1^2) dA \\
 &= \int_0^1 \int_0^1 (x^2 + 1) dx dz = \frac{4}{3}; \text{ On the face } x=1: g(x, y, z) = x=1 \Rightarrow \nabla g = \mathbf{i} \text{ and } \mathbf{p} = \mathbf{i} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \\
 &\Rightarrow d\sigma = dA \Rightarrow I = \iint_R (1^2 + y^2) dA = \int_0^1 \int_0^1 (1 + y^2) dy dz = \frac{4}{3}; \text{ On the face } x=0: g(x, y, z) = x=0 \Rightarrow \nabla g = -\mathbf{i} \\
 &\text{and } \mathbf{p} = -\mathbf{i} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (0^2 + y^2) dA = \int_0^1 \int_0^1 y^2 dy dz = \frac{1}{3} \\
 &\Rightarrow I_z = \frac{2}{3} + \frac{2}{3} + \frac{1}{3} + \frac{4}{3} + \frac{4}{3} + \frac{1}{3} = \frac{14}{3}
 \end{aligned}$$

$$\begin{aligned}
 49. \quad M = 2xy + x \text{ and } N = xy - y &\Rightarrow \frac{\partial M}{\partial x} = 2y + 1, \frac{\partial M}{\partial y} = 2x, \frac{\partial N}{\partial x} = y, \frac{\partial N}{\partial y} = x - 1 \\
 \Rightarrow \text{Flux} &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy = \iint_R (2y + 1 + x - 1) dy dx = \int_0^1 \int_0^1 (2y + x) dy dx = \frac{3}{2}; \\
 \text{Circ} &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R (y - 2x) dy dx = \int_0^1 \int_0^1 (y - 2x) dy dx = -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 50. \quad M = y - 6x^2 \text{ and } N = x + y^2 &\Rightarrow \frac{\partial M}{\partial x} = -12x, \frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = 1, \frac{\partial N}{\partial y} = 2y \\
 \Rightarrow \text{Flux} &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy = \iint_R (-12x + 2y) dx dy = \int_0^1 \int_0^1 (-12x + 2y) dx dy = \int_0^1 (4y^2 + 2y - 6) dy = -\frac{11}{3}; \\
 \text{Circ} &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R (1 - 1) dx dy = 0
 \end{aligned}$$

$$\begin{aligned}
 51. \quad M = -\frac{\cos y}{x} \text{ and } N = \ln x \sin y &\Rightarrow \frac{\partial M}{\partial y} = \frac{\sin y}{x} \text{ and } \frac{\partial N}{\partial x} = \frac{\sin y}{x} \Rightarrow \oint_C \ln x \sin y dy - \frac{\cos y}{x} dx \\
 &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R \left(\frac{\sin y}{x} - \frac{\sin y}{x} \right) dx dy = 0
 \end{aligned}$$

$$52. \quad (a) \quad \text{Let } M = x \text{ and } N = y \Rightarrow \frac{\partial M}{\partial x} = 1, \frac{\partial M}{\partial y} = 0, \frac{\partial N}{\partial x} = 0, \frac{\partial N}{\partial y} = 1$$

$$\Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy = \iint_R (1 + 1) dx dy = 2 \iint_R dx dy = 2(\text{Area of the region})$$

(b) Let C be a closed curve to which Green's Theorem applies and let \mathbf{n} be the unit normal vector to C . Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ and assume \mathbf{F} is orthogonal to \mathbf{n} at every point of C . Then the flux density of \mathbf{F} at every point of C is 0 since $\mathbf{F} \cdot \mathbf{n} = 0$ at every point of $C \Rightarrow \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$ at every point of C

$$\Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy = \iint_R 0 dx dy = 0. \text{ But part (a) above states that the flux is}$$

$2(\text{Area of the region}) \Rightarrow$ the area of the region would be 0 \Rightarrow contradiction. Therefore, \mathbf{F} cannot be orthogonal to \mathbf{n} at every point of C .

$$\begin{aligned}
 53. \quad \frac{\partial}{\partial x}(2xy) = 2y, \frac{\partial}{\partial y}(2yz) = 2z, \frac{\partial}{\partial z}(2xz) = 2x &\Rightarrow \nabla \cdot \mathbf{F} = 2y + 2z + 2x \Rightarrow \text{Flux} = \iiint_D (2x + 2y + 2z) dV \\
 &= \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) dx dy dz = \int_0^1 \int_0^1 (1 + 2y + 2z) dy dz = \int_0^1 (2 + 2z) dz = 3
 \end{aligned}$$

54. $\frac{\partial}{\partial x}(xz) = z, \frac{\partial}{\partial z}(yz) = z, \frac{\partial}{\partial z}(1) = 0 \Rightarrow \nabla \cdot \mathbf{F} = 2z \Rightarrow \text{Flux} = \iiint_D 2z \, r \, dr \, d\theta \, dz$

$$\int_0^{2\pi} \int_0^4 \int_3^{\sqrt{25-r^2}} 2z \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^4 r(16-r^2) \, dr \, d\theta = \int_0^{2\pi} 64 \, d\theta = 128\pi$$
55. $\frac{\partial}{\partial x}(-2x) = -2, \frac{\partial}{\partial y}(-3y) = -3, \frac{\partial}{\partial z}(z) = 1 \Rightarrow \nabla \cdot \mathbf{F} = -4; x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 = z \Rightarrow z = 1$

$$\Rightarrow x^2 + y^2 = 1 \Rightarrow \text{Flux} = \iiint_D -4 \, dV = -4 \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = -4 \int_0^{2\pi} \int_0^1 \left(r\sqrt{2-r^2} - r^3 \right) dr \, d\theta$$

$$= -4 \int_0^{2\pi} \left(-\frac{7}{12} + \frac{2}{3}\sqrt{2} \right) d\theta = \frac{2}{3}\pi(7-8\sqrt{2})$$
56. $\frac{\partial}{\partial x}(6x+y) = 6, \frac{\partial}{\partial y}(-x-z) = 0, \frac{\partial}{\partial z}(4yz) = 4y \Rightarrow \nabla \cdot \mathbf{F} = 6+4y; z = \sqrt{x^2+y^2} = r$

$$\Rightarrow \text{Flux} = \iiint_D (6+4y) \, dV = \int_0^{\pi/2} \int_0^1 \int_0^r (6+4r \sin \theta) \, dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 (6r^2 + 4r^3 \sin \theta) \, dr \, d\theta$$

$$= \int_0^{\pi/2} (2 + \sin \theta) \, d\theta = \pi + 1$$
57. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 0 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV = 0$
58. $\mathbf{F} = 3xz^2\mathbf{i} + y\mathbf{j} - z^3\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 3z^2 + 1 - 3z^2 = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV$

$$= \int_0^4 \int_0^{\sqrt{16-x^2}/2} \int_0^{y/2} 1 \, dz \, dy \, dx = \int_0^4 \left(\frac{16-x^2}{16} \right) dx = \left[x - \frac{x^3}{48} \right]_0^4 = \frac{8}{3}$$
59. $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = y^2 + x^2 + 0 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV$

$$= \iiint_D (x^2 + y^2) \, dV = \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2 \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 2r^3 \, dr \, d\theta = \int_0^{2\pi} \frac{1}{2} \, d\theta = \pi$$
60. (a) $\mathbf{F} = (3z+1)\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 3 \Rightarrow \text{Flux across the hemisphere} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV = \iiint_D 3 \, dV$

$$= 3\left(\frac{1}{2}\right)\left(\frac{4}{3}\pi a^3\right) = 2\pi a^3$$
- (b) $f(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{4a^2} = 2a$ since
 $a \geq 0 \Rightarrow \mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2a} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = (3z+1)\left(\frac{z}{a}\right); \mathbf{p} = \mathbf{k} \Rightarrow \nabla f \cdot \mathbf{p} = \nabla f \cdot \mathbf{k} = 2z \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$
 since $z \geq 0 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} = \frac{2a}{2z} dA = \frac{a}{z} dA \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_{R_{xy}} (3z+1)\left(\frac{z}{a}\right)\left(\frac{a}{z}\right) dA = \iint_{R_{xy}} (3z+1) \, dx \, dy$

$$= \iint_{R_{xy}} \left(3\sqrt{a^2 - x^2 - y^2} + 1 \right) dx \, dy = \int_0^{2\pi} \int_0^a \left(3\sqrt{a^2 - r^2} + 1 \right) r \, dr \, d\theta = \int_0^{2\pi} \left(\frac{a^2}{2} + a^3 \right) d\theta = \pi a^2 + 2\pi a^3,$$

 which is the flux across the hemisphere. Across the base we find $\mathbf{F} = [3(0)+1]\mathbf{k} = \mathbf{k}$ since $z = 0$ in the

$$\begin{aligned}
 xy\text{-plane} &\Rightarrow \mathbf{n} = -\mathbf{k} \text{ (outward normal)} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -1 \Rightarrow \text{Flux across the base} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma \\
 &= \iint_{R_{xy}} (-1) \, dx \, dy = -\pi a^2. \text{ Therefore, the total flux across the closed surface is} \\
 &\left(\pi a^2 + 2\pi a^3 \right) - \pi a^2 = 2\pi a^3.
 \end{aligned}$$

CHAPTER 16 ADDITIONAL AND ADVANCED EXERCISES

- $dx = (-2 \sin t + 2 \sin 2t) \, dt$ and $dy = (2 \cos t - 2 \cos 2t) \, dt$; Area $= \frac{1}{2} \oint_C x \, dy - y \, dx$
 $= \frac{1}{2} \int_0^{2\pi} [(2 \cos t - \cos 2t)(2 \cos t - 2 \cos 2t) - (2 \sin t - \sin 2t)(-2 \sin t + 2 \sin 2t)] \, dt$
 $= \frac{1}{2} \int_0^{2\pi} [6 - (6 \cos t \cos 2t + 6 \sin t \sin 2t)] \, dt = \frac{1}{2} \int_0^{2\pi} (6 - 6 \cos t) \, dt = 6\pi$
- $dx = (-2 \sin t - 2 \sin 2t) \, dt$ and $dy = (2 \cos t - 2 \cos 2t) \, dt$; Area $= \frac{1}{2} \oint_C x \, dy - y \, dx$
 $= \frac{1}{2} \int_0^{2\pi} [(2 \cos t + \cos 2t)(2 \cos t - 2 \cos 2t) - (2 \sin t - \sin 2t)(-2 \sin t - 2 \sin 2t)] \, dt$
 $= \frac{1}{2} \int_0^{2\pi} [2 - 2(\cos t \cos 2t - \sin t \sin 2t)] \, dt = \frac{1}{2} \int_0^{2\pi} (2 - 2 \cos 3t) \, dt = \frac{1}{2} \left[2t - \frac{2}{3} \sin 3t \right]_0^{2\pi} = 2\pi$
- $dx = \cos 2t \, dt$ and $dy = \cos t \, dt$; Area $= \frac{1}{2} \oint_C x \, dy - y \, dx = \frac{1}{2} \int_0^\pi \left(\frac{1}{2} \sin 2t \cos t - \sin t \cos 2t \right) \, dt$
 $= \frac{1}{2} \int_0^\pi [\sin t \cos^2 t - (\sin t)(2 \cos^2 t - 1)] \, dt = \frac{1}{2} \int_0^\pi (-\sin t \cos^2 t + \sin t) \, dt = \frac{1}{2} \left[\frac{1}{3} \cos^3 t - \cos t \right]_0^\pi = -\frac{1}{3} + 1 = \frac{2}{3}$
- $dx = (-2a \sin t - 2a \cos 2t) \, dt$ and $dy = (b \cos t) \, dt$; Area $= \frac{1}{2} \oint_C x \, dy - y \, dx$
 $= \frac{1}{2} \int_0^{2\pi} \left[(2ab \cos^2 t - ab \cos t \sin 2t) - (-2ab \sin^2 t - 2ab \sin t \cos 2t) \right] \, dt$
 $= \frac{1}{2} \int_0^{2\pi} [2ab - 2ab \cos^2 t \sin t + 2ab(\sin t)(2 \cos^2 t - 1)] \, dt = \frac{1}{2} \int_0^{2\pi} (2ab + 2ab \cos^2 t \sin t - 2ab \sin t) \, dt$
 $= \frac{1}{2} \left[2abt - \frac{2}{3} ab \cos^3 t + 2ab \cos t \right]_0^{2\pi} = 2\pi ab$
- $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ is $\mathbf{0}$ only at the point $(0, 0, 0)$, and $\text{curl } \mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is never $\mathbf{0}$.
 - $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{k}$ is $\mathbf{0}$ only on the line $x = t, y = 0, z = 0$ and $\text{curl } \mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j}$ is never $\mathbf{0}$.
 - $\mathbf{F}(x, y, z) = z\mathbf{i}$ is $\mathbf{0}$ only when $z = 0$ (the xy -plane) and $\text{curl } \mathbf{F}(x, y, z) = \mathbf{j}$ is never $\mathbf{0}$.
- $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$ and $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{R}$, so \mathbf{F} is parallel to \mathbf{n} when $yz^2 = \frac{cx}{R}$, $xz^2 = \frac{cy}{R}$,
and $2xyz = \frac{cz}{R} \Rightarrow \frac{yz^2}{x} = \frac{xz^2}{y} = 2xy \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$ and $z^2 = \pm \frac{c}{R} = 2x^2 \Rightarrow z = \pm \sqrt{2}x$. Also,
 $x^2 + y^2 + z^2 = R^2 \Rightarrow x^2 + x^2 + 2x^2 = R^2 \Rightarrow 4x^2 = R^2 \Rightarrow x = \pm \frac{R}{2}$. Thus the points are: $\left(\frac{R}{2}, \frac{R}{2}, \frac{\sqrt{2}R}{2} \right)$,

$$\left(\frac{R}{2}, \frac{R}{2}, -\frac{\sqrt{2}R}{2}\right), \left(-\frac{R}{2}, -\frac{R}{2}, \frac{\sqrt{2}R}{2}\right), \left(-\frac{R}{2}, -\frac{R}{2}, -\frac{\sqrt{2}R}{2}\right), \left(\frac{R}{2}, -\frac{R}{2}, \frac{\sqrt{2}R}{2}\right), \left(\frac{R}{2}, -\frac{R}{2}, -\frac{\sqrt{2}R}{2}\right), \\ \left(-\frac{R}{2}, \frac{R}{2}, \frac{\sqrt{2}R}{2}\right), \left(-\frac{R}{2}, \frac{R}{2}, -\frac{\sqrt{2}R}{2}\right)$$

7. Set up the coordinate system so that $(a, b, c) = (0, R, 0) \Rightarrow \delta(x, y, z) = \sqrt{x^2 + (y-R)^2 + z^2}$
 $= \sqrt{x^2 + y^2 + z^2 - 2Ry + R^2} = \sqrt{2R^2 - 2Ry}$; let $f(x, y, z) = x^2 + y^2 + z^2 - R^2$ and $\mathbf{p} = \mathbf{i}$
 $\Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2R \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{i}|} dz dy = \frac{2R}{2x} dz dy$
 $\Rightarrow \text{Mass} = \iint_S \delta(x, y, z) d\sigma = \iint_{R_{yz}} \sqrt{2R^2 - 2Ry} \left(\frac{R}{x}\right) dz dy = R \iint_{R_{yz}} \frac{\sqrt{2R^2 - 2Ry}}{\sqrt{R^2 - y^2 - z^2}} dz dy$
 $= 4R \int_{-R}^R \int_0^{\sqrt{R^2 - y^2}} \frac{\sqrt{2R^2 - 2Ry}}{\sqrt{R^2 - y^2 - z^2}} dz dy = 4R \int_{-R}^R \sqrt{2R^2 - 2Ry} \sin^{-1} \left(\frac{z}{\sqrt{R^2 - y^2}} \right) \bigg|_0^{\sqrt{R^2 - y^2}} dy$
 $= 2\pi R \int_{-R}^R \sqrt{2R^2 - 2Ry} dy = 2\pi R \left(\frac{-1}{3R} \right) (2R^2 - 2Ry)^{3/2} \bigg|_{-R}^R = \frac{16\pi R^3}{3}$

8. $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \theta\mathbf{k}$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 1 \end{vmatrix}$
 $= (\sin \theta)\mathbf{i} - (\cos \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{1 + r^2}$; $\delta = 2\sqrt{x^2 + y^2} = 2\sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = 2r$
 $\Rightarrow \text{Mass} = \iint_S \delta(x, y, z) d\sigma = \int_0^{2\pi} \int_0^1 2r\sqrt{1 + r^2} dr d\theta = \int_0^{2\pi} \left[\frac{2}{3} (1 + r^2)^{3/2} \right]_0^1 d\theta = \int_0^{2\pi} \frac{2}{3} (2\sqrt{2} - 1) d\theta$
 $= \frac{4\pi}{3} (2\sqrt{2} - 1)$

9. $M = x^2 + 4xy$ and $N = -6y \Rightarrow \frac{\partial M}{\partial x} = 2x + 4y$ and $\frac{\partial N}{\partial y} = -6 \Rightarrow \text{Flux} = \int_0^b \int_0^a (2x + 4y - 6) dx dy$
 $= \int_0^b (a^2 + 4ay - 6a) dy = a^2b + 2ab^2 - 6ab$. We want to minimize
 $f(a, b) = a^2b + 2ab^2 - 6ab = ab(a + 2b - 6)$. Thus, $f_a(a, b) = 2ab + 2b^2 - 6b = 0$ and
 $f_b(a, b) = a^2 + 4ab - 6a = 0 \Rightarrow b(2a + 2b - 6) = 0 \Rightarrow b = 0$ or $b = -a + 3$. Now $b = 0 \Rightarrow a^2 - 6a = 0 \Rightarrow a = 0$
or $a = 6 \Rightarrow (0, 0)$ and $(6, 0)$ are critical points. On the other hand, $b = -a + 3 \Rightarrow a^2 + 4a(-a + 3) - 6a = 0$
 $\Rightarrow -3a^2 + 6a = 0 \Rightarrow a = 0$ or $a = 2 \Rightarrow (0, 3)$ and $(2, 1)$ are also critical points. The flux at $(0, 0) = 0$, the flux at
 $(6, 0) = 0$, the flux at $(0, 3) = 0$ and the flux at $(2, 1) = -4$. Therefore, the flux is minimized at $(2, 1)$ with
value -4 .

10. A plane through the origin has equation $ax + by + cz = 0$. Consider first the case when $c \neq 0$. Assume the plane
is given by $z = ax + by$ and let $f(x, y, z) = x^2 + y^2 + z^2 = 4$. Let C denote the circle of intersection of the plane
with the sphere. By Stokes' Theorem, $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$, where \mathbf{n} is a unit normal to the plane. Let

$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (ax + by)\mathbf{k}$ be a parametrization of the surface. Then $\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & a \\ 0 & 1 & b \end{vmatrix} = -a\mathbf{i} - b\mathbf{j} + \mathbf{k}$

$$\Rightarrow d\sigma = |\mathbf{r}_x \times \mathbf{r}_y| dx dy = \sqrt{a^2 + b^2 + 1} dx dy. \text{ Also, } \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \text{ and } \mathbf{n} = \frac{a\mathbf{i} + b\mathbf{j} - \mathbf{k}}{\sqrt{a^2 + b^2 + 1}}$$

$$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_{R_{xy}} \frac{a+b-1}{\sqrt{a^2+b^2+1}} \sqrt{a^2+b^2+1} dx dy = \iint_{R_{xy}} (a+b-1) dx dy = (a+b-1) \iint_{R_{xy}} dx dy. \text{ Now}$$

$$x^2 + y^2 + (ax + by)^2 = 4 \Rightarrow \left(\frac{a^2+1}{4}\right)x^2 + \left(\frac{b^2+1}{4}\right)y^2 + \left(\frac{ab}{2}\right)xy = 1 \Rightarrow \text{the region } R_{xy} \text{ is the interior of the ellipse}$$

$Ax^2 + Bxy + Cy^2 = 1$ in the xy -plane, where $A = \frac{a^2+1}{4}$, $B = \frac{ab}{2}$, and $C = \frac{b^2+1}{4}$. The area of the ellipse is

$$\frac{2\pi}{\sqrt{4AC-B^2}} = \frac{4\pi}{\sqrt{a^2+b^2+1}} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = h(a, b) = \frac{4\pi(a+b-1)}{\sqrt{a^2+b^2+1}}. \text{ Thus we optimize } H(a, b) = \frac{(a+b-1)^2}{a^2+b^2+1}:$$

$$\frac{\partial H}{\partial a} = \frac{2(a+b-1)(b^2+1+a-ab)}{(a^2+b^2+1)^2} = 0 \text{ and } \frac{\partial H}{\partial b} = \frac{2(a+b-1)(a^2+1+b-ab)}{(a^2+b^2+1)^2} = 0 \Rightarrow a+b-1=0, \text{ or } b^2+1+a-ab=0 \text{ and}$$

$$a^2+1+b-ab=0 \Rightarrow a+b-1=0, \text{ or } a^2-b^2+(b-a)=0 \Rightarrow a+b-1=0, \text{ or } (a-b)(a+b-1)=0$$

$$\Rightarrow a+b-1=0 \text{ or } a=b. \text{ The critical values } a+b-1=0 \text{ give a saddle. If } a=b, \text{ then } 0=b^2+1+a-ab$$

$$\Rightarrow a^2+1+a-a^2=0 \Rightarrow a=-1 \Rightarrow b=-1. \text{ Thus, the point } (a, b) = (-1, -1) \text{ gives a local extremum for}$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \Rightarrow z = -x - y \Rightarrow x + y + z = 0 \text{ is the desired plane, if } c \neq 0.$$

Note: Since $h(-1, -1)$ is negative, the circulation about \mathbf{n} is clockwise, so $-\mathbf{n}$ is the correct pointing normal for the counterclockwise circulation. Thus $\iint_S \nabla \times \mathbf{F} \cdot (-\mathbf{n}) d\sigma$ actually gives the maximum circulation.

If $c = 0$, one can see that the corresponding problem is equivalent to the calculation above when $b = 0$, which does not lead to a local extreme.

11. (a) Partition the string into small pieces. Let $\Delta_i s$ be the length of the i^{th} piece. Let (x_i, y_i) be a point in the i^{th} piece. The work done by gravity in moving the i^{th} piece to the x -axis is approximately

$$W_i = (gx_i y_i \Delta_i s) y_i \text{ where } x_i y_i \Delta_i s \text{ is approximately the mass of the } i^{\text{th}} \text{ piece. The total work done by}$$

$$\text{gravity in moving the string to the } x\text{-axis is } \sum_i W_i = \sum_i gx_i y_i^2 \Delta_i s \Rightarrow \text{Work} = \int_C gxy^2 ds$$

$$\begin{aligned} \text{(b) } \text{Work} &= \int_C gxy^2 ds = \int_0^{\pi/2} g(2\cos t)(4\sin^2 t)\sqrt{4\sin^2 t + 4\cos^2 t} dt = 16g \int_0^{\pi/2} \cos t \sin^2 t dt \\ &= \left[16g \left(\frac{\sin^3 t}{3} \right) \right]_0^{\pi/2} = \frac{16}{3} g \end{aligned}$$

$$\text{(c) } \bar{x} = \frac{\int_C x(xy) ds}{\int_C xy ds} \text{ and } \bar{y} = \frac{\int_C y(xy) ds}{\int_C xy ds}; \text{ the mass of the string is } \int_C xy ds \text{ and the weight of the string is}$$

$$g \int_C xy ds. \text{ Therefore, the work done in moving the point mass at } (\bar{x}, \bar{y}) \text{ to the } x\text{-axis is}$$

$$W = \left(g \int_C xy ds \right) \bar{y} = g \int_C xy^2 ds = \frac{16}{3} g.$$

12. (a) Partition the sheet into small pieces. Let $\Delta_i\sigma$ be the area of the i^{th} piece and select a point (x_i, y_i, z_i) in the i^{th} piece. The mass of the i^{th} piece is approximately $x_i y_i \Delta_i\sigma$. The work done by gravity in moving the i^{th} piece to the xy -plane is approximately $(gx_i y_i \Delta_i\sigma)z_i = gx_i y_i z_i \Delta_i\sigma \Rightarrow \text{Work} = \iint_S gxyz \, d\sigma$.

$$\begin{aligned} \text{(b)} \quad \iint_S gxyz \, d\sigma &= g \iint_{R_{xy}} xy(1-x-y)\sqrt{1+(-1)^2+(-1)^2} \, dA = \sqrt{3}g \int_0^1 \int_0^{1-x} (xy - x^2y - xy^2) \, dy \, dx \\ &= \sqrt{3}g \int_0^1 \left[\frac{1}{2}xy^2 - \frac{1}{2}x^2y - \frac{1}{3}xy^3 \right]_0^{1-x} dx = \sqrt{3}g \int_0^1 \left[\frac{1}{6}x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 \right] dx \\ &= \sqrt{3}g \left[\frac{1}{12}x^2 - \frac{1}{6}x^3 + \frac{1}{6}x^4 - \frac{1}{30}x^5 \right]_0^1 = \sqrt{3}g \left(\frac{1}{12} - \frac{1}{30} \right) = \frac{\sqrt{3}g}{20} \end{aligned}$$

- (c) The center of mass of the sheet is the point $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{z} = \frac{M_{xy}}{M}$ with $M_{xy} = \iint_S xyz \, d\sigma$ and

$M = \iint_S xy \, d\sigma$. The work done by gravity in moving the point mass at $(\bar{x}, \bar{y}, \bar{z})$ to the xy -plane is

$$gM\bar{z} = gM \left(\frac{M_{xy}}{M} \right) = gM_{xy} = \iint_S gxyz \, d\sigma = \frac{\sqrt{3}g}{20}.$$

13. (a) Partition the sphere $x^2 + y^2 + (z-2)^2 = 1$ into small pieces. Let $\Delta_i\sigma$ be the surface area of the i^{th} piece and let (x_i, y_i, z_i) be a point on the i^{th} piece. The force due to pressure on the i^{th} piece is approximately $w(4-z_i)\Delta_i\sigma$. The total force on S is approximately $\sum_i w(4-z_i)\Delta_i\sigma$. This gives the actual force to be

$$\iint_S w(4-z) \, d\sigma.$$

- (b) The upward buoyant force is a result of the \mathbf{k} -component of the force on the ball due to liquid pressure. The force on the ball at (x, y, z) is $w(4-z)(-\mathbf{n}) = w(z-4)\mathbf{n}$, where \mathbf{n} is the outer unit normal at (x, y, z) . Hence the \mathbf{k} -component of this force is $w(z-4)\mathbf{n} \cdot \mathbf{k} = w(z-4)\mathbf{k} \cdot \mathbf{n}$. The (magnitude of the) buoyant force on the ball is obtained by adding up all these \mathbf{k} -components to obtain $\iint_S w(z-4)\mathbf{k} \cdot \mathbf{n} \, d\sigma$.

- (c) The Divergence Theorem says $\iint_S w(z-4)\mathbf{k} \cdot \mathbf{n} \, d\sigma = \iiint_D \text{div}(w(z-4)\mathbf{k}) \, dV = \iiint_D w \, dV$, where D is

$$x^2 + y^2 + (z-2)^2 \leq 1 \Rightarrow \iint_S w(z-4)\mathbf{k} \cdot \mathbf{n} \, d\sigma = w \iiint_D 1 \, dV = \frac{4}{3}\pi w, \text{ the weight of the fluid if it were to occupy the region } D.$$

14. The surface S is $z = \sqrt{x^2 + y^2}$ from $z = 1$ to $z = 2$. Partition S into small pieces and let $\Delta_i\sigma$ be the area of the i^{th} piece. Let (x_i, y_i, z_i) be a point on the i^{th} piece. Then the magnitude of the force on the i^{th} piece due to liquid pressure is approximately $F_i = w(2-z_i)\Delta_i\sigma \Rightarrow$ the total force on S is approximately

$$\begin{aligned} \sum_i F_i &= \sum w(2-z_i)\Delta_i\sigma \Rightarrow \text{the actual force is } \iint_S w(2-z) \, d\sigma = \iint_{R_{xy}} w \left(2 - \sqrt{x^2 + y^2} \right) \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} \, dA \\ &= \iint_{R_{xy}} \sqrt{2}w \left(2 - \sqrt{x^2 + y^2} \right) dA = \int_0^{2\pi} \int_1^2 \sqrt{2}w(2-r)r \, dr \, d\theta = \int_0^{2\pi} \sqrt{2}w \left[r^2 - \frac{1}{3}r^3 \right]_1^2 d\theta = \int_0^{2\pi} \frac{2\sqrt{2}w}{3} d\theta = \frac{4\sqrt{2}\pi w}{3} \end{aligned}$$

15. Assume that S is a surface to which Stokes' Theorem applies. Then $\oint_C \mathbf{E} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{E}) \cdot \mathbf{n} \, d\sigma$
 $= \iint_S \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \cdot \mathbf{n} \, d\sigma = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathbf{n} \, d\sigma$. Thus the voltage around a loop equals the negative of the rate of change of magnetic flux through the loop.
16. According to Gauss's Law, $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 4\pi GmM$ for any surface enclosing the origin. But if $\mathbf{F} = \nabla \times \mathbf{H}$ then the integral over such a closed surface would have to be 0 by the Divergence Theorem since $\operatorname{div} \mathbf{F} = 0$.
17. $\oint_C f \nabla g \cdot d\mathbf{r} = \iint_S \nabla \times (f \nabla g) \cdot \mathbf{n} \, d\sigma$ (Stokes' Theorem)
 $= \iint_S (f \nabla \times \nabla g + \nabla f \times \nabla g) \cdot \mathbf{n} \, d\sigma$ (Section 16.8, Exercise 19b)
 $= \iint_S [(f)(\mathbf{0}) + \nabla f \times \nabla g] \cdot \mathbf{n} \, d\sigma$ (Section 16.7, Equation 8)
 $= \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n} \, d\sigma$
18. $\nabla \times \mathbf{F}_1 = \nabla \times \mathbf{F}_2 \Rightarrow \nabla \times (\mathbf{F}_2 - \mathbf{F}_1) = \mathbf{0} \Rightarrow \mathbf{F}_2 - \mathbf{F}_1$ is conservative $\Rightarrow \mathbf{F}_2 - \mathbf{F}_1 = \nabla f$; also, $\nabla \cdot \mathbf{F}_1 = \nabla \cdot \mathbf{F}_2$
 $\Rightarrow \nabla \cdot (\mathbf{F}_2 - \mathbf{F}_1) = 0 \Rightarrow \nabla^2 f = 0$ (so f is harmonic). Finally, on the surface S , $\nabla f \cdot \mathbf{n} = (\mathbf{F}_2 - \mathbf{F}_1) \cdot \mathbf{n}$
 $= \mathbf{F}_2 \cdot \mathbf{n} - \mathbf{F}_1 \cdot \mathbf{n} = 0$. Now, $\nabla \cdot (f \nabla f) = \nabla f \cdot \nabla f + f \nabla^2 f$ so the Divergence Theorem gives
 $\iiint_D |\nabla f|^2 \, dV + \iiint_D f \nabla^2 f \, dV = \iiint_D \nabla \cdot (f \nabla f) \, dV = \iint_S f \nabla f \cdot \mathbf{n} \, d\sigma = 0$, and since $\nabla^2 f = 0$ we have
 $\iiint_D |\nabla f|^2 \, dV + 0 = 0 \Rightarrow \iiint_D |\mathbf{F}_2 - \mathbf{F}_1|^2 \, dV = 0 = \iint_S f \nabla f \cdot \mathbf{n} \, d\sigma = 0$, as claimed.
19. False; let $\mathbf{F} = y\mathbf{i} + x\mathbf{j} \neq \mathbf{0} \Rightarrow \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) = 0$ and $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$
20. $|\mathbf{r}_u \times \mathbf{r}_v|^2 = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 \sin^2 \theta = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 (1 - \cos^2 \theta) = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 - |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 \cos^2 \theta = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 - (\mathbf{r}_u \cdot \mathbf{r}_v)^2$
 $\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v|^2 = EG - F^2 \Rightarrow d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv = \sqrt{EG - F^2} \, du \, dv$
21. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow \nabla \cdot \mathbf{r} = 1 + 1 + 1 = 3 \Rightarrow \iiint_D \nabla \cdot \mathbf{r} \, dV = 3 \iiint_D dV = 3V \Rightarrow V = \frac{1}{3} \iiint_D \nabla \cdot \mathbf{r} \, dV = \frac{1}{3} \iint_S \mathbf{r} \cdot \mathbf{n} \, d\sigma$, by the Divergence Theorem