STA2001 Assignment 8

1. (4.3-1). Let X and Y have the joint pmf

$$f(x,y) = \frac{x+y}{32}, \quad x = 1,2 \quad y = 1,2,3,4$$

(a) Display the joint pmf and the marginal pmfs on a graph like Figure 4.3-1(a).
(b) Find $g(x|y)$ and draw a figure like Figure 4.3-1(b), depicting the conditional pmfs for $y = 1,2,3,$ and 4.
(c) Find $h(y|x)$ and draw a figure like Figure 4.3-1(c), depicting the conditional pmfs for $x = 1$ and 2 .
(d) Find $P(1 \le Y \le 3|X=1)$, $P(Y \le 2|X=2)$, and $P(X=2|Y=3)$.
(e) Find $E(Y|X=1)$ and $\mathrm{Var}(Y|X=1)$.

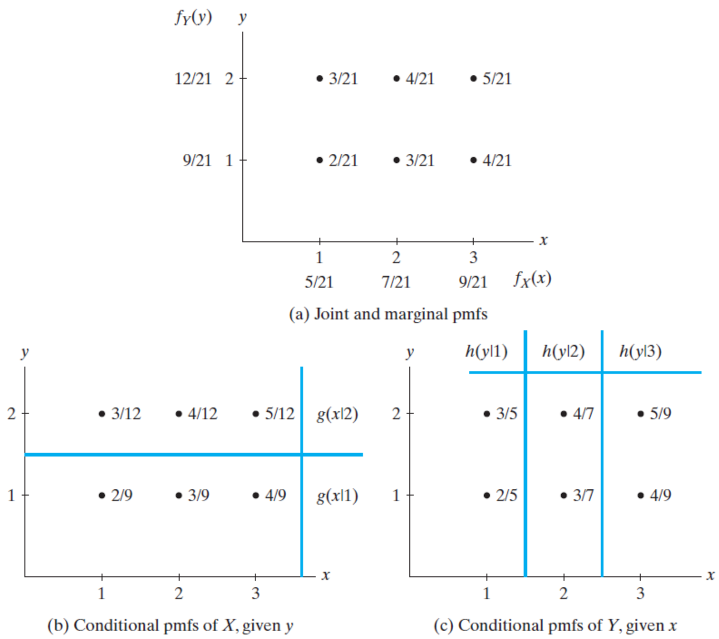
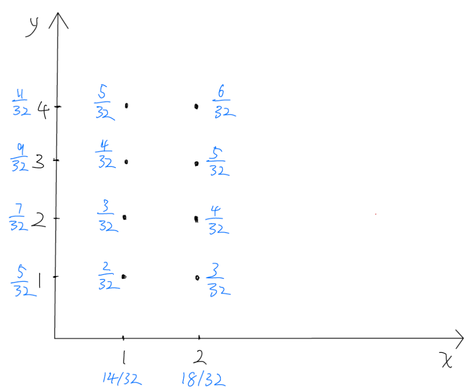


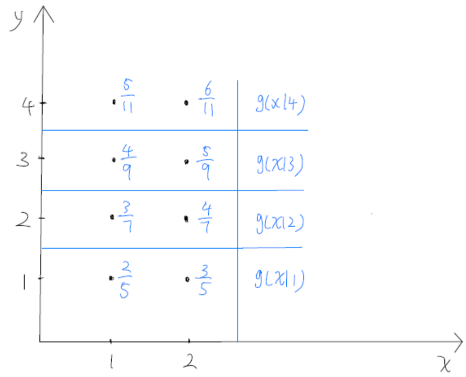
Figure 4.3-1 Joint, marginal, and conditional pmfs

Solution:

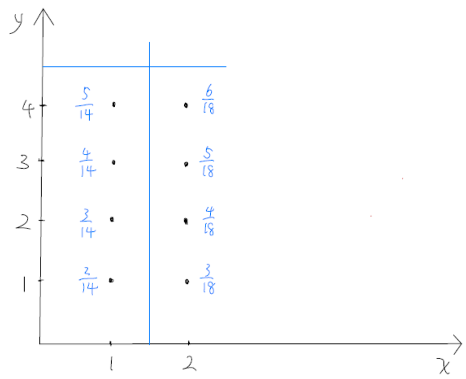
(a)



(b)



(c)



(d)

$$
\begin{aligned}
P(1 \le Y \le 3|X=1) &= 4/14 + 3/14 + 2/14 = 9/14 \\
P(Y \le 2|X=2) &= 4/18 + 3/18 = 7/18 \\
P(X=2|Y=3) &= 5/9
\end{aligned}
$$

(e)

$$
\begin{aligned}
E(Y|X=1) &= \sum_{y=1}^{4} y h(y|x=1) = 1 \times \frac{2}{14} + 2 \times \frac{3}{14} + 3 \times \frac{6}{14} + 4 \times \frac{5}{14} = \frac{20}{7} \\
\mathrm{Var}(Y|X=1) &= E\left(Y^2|X=1\right) - (E(Y|X=1))^2 = \frac{130}{14} - \frac{400}{49} = \frac{55}{49}
\end{aligned}
$$

2. (4.3-8). A fair six-sided die is rolled 30 independent times. Let X be the number of ones and Y the number of twos.

(a) What is the joint pmf of X and Y?
(b) Find the conditional pmf of X, given $Y = y$.
(c) Compute $E\left(X^2 - 4XY + 3Y^2\right)$.
(d) Prove that $Cov(X, Y) = -30 P_x P_y$, where P_x and P_y are the probabilities of getting number one and two for rolling one time, respectively.

Solution:

Refer to Example 4.3-3 on the textbook for how to obtain the relevant formulas.

(a) Given the information, (X, Y) follows a trinomial distribution whose pmf is given by

$$f(x,y) = \binom{30}{x, y, 30-x-y} \left(\frac{1}{6}\right)^x \left(\frac{1}{6}\right)^y \left(\frac{2}{3}\right)^{30-x-y}$$

Note that $n = 30$, $p_X = \frac{1}{6}$, $p_Y = \frac{1}{6}$, $1 - p_X - p_Y = \frac{2}{3}$.

(b) Given (X, Y) follows a trinomial distribution, the conditional distribution follows a binomial distribution,

$$f(x|y) = \binom{30-y}{x, (30-y-x)} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{30-x-y}$$

(c) Note that the marginal distributions also follow binomial distributions, and then we compute the following

$$\mu_X = 30 \times \frac{1}{6} = 5,\ \sigma_X^2 = 30 \times \frac{1}{6} \times \frac{5}{6} = 25/6$$

$$\mu_Y = 30 \times \frac{1}{6} = 5,\ \sigma_Y^2 = 30 \times \frac{1}{6} \times \frac{5}{6} = 25/6$$

$$\rho = -\sqrt{\frac{p_X p_Y}{(1-p_X)(1-p_Y)}} = -\frac{1}{5}$$

Hence,

$$\begin{aligned} E\left(X^2 - 4XY + 3Y^2\right) &= \left(\mu_X^2 + \sigma_X^2\right) - 4\left(\mu_X\mu_Y + \rho\sigma_X\sigma_Y\right) + 3\left(\mu_Y^2 + \sigma_Y^2\right) \\ &= 20 \end{aligned}$$

(d) Note that we have

$$X \sim b(n, p_X),\ Y \sim b(n, p_Y),\ Y|X \sim b(n-x, \frac{p_Y}{1-p_X})$$

According to the result in Lecture 16, example 2, the covariance of X and Y can be calculated as follows:

$$\begin{aligned} Cov(X,Y) &= \sum_{x \in S_x} (x - E[X]) f_X(x) \sum_{y \in S_Y(x)} (y - E[Y]) h(y|x) \\ &= \sum_{x=0}^{n} (x - np_X) f_X(x) \sum_{y=0}^{n-x} (y - np_Y) h(y|x) \\ &= \sum_{x=0}^{n} (x - np_X) f_X(x) \left((n-x)\frac{p_Y}{1-p_X} - np_Y\right) \\ &= \sum_{x=0}^{n} (x - np_X) f_X(x) \left(\frac{-xp_Y + np_Xp_Y}{1-p_X}\right) \\ &= \sum_{x=0}^{n} f_X(x) \left(\frac{-x^2p_Y + 2xnp_Xp_Y - n^2p_X^2p_Y}{1-p_X}\right) \\ &= \sum_{x=0}^{n} f_X(x) \left(\frac{-(x-np_X)^2)}{1-p_X}\right) p_Y \\ &= \left(\frac{-np_X(1-p_X)}{1-p_X}\right) p_Y \\ &= -np_Xp_Y \end{aligned}$$

where the second last equality uses $Var(X) = np_X(1-p_X)$.

Another solution: Note that $X+Y$ follows a binomial distribution, i.e., $X+Y \sim Binomial(30, P_x+P_y)$, then we have

$$\begin{aligned} Var(X+Y) &= Var(X) + Var(Y) - 2Cov(X,Y) \\ &= 30P_x(1-P_x) + 30P_y(1-P_y) - 2Cov(X,Y) \\ &= 30(P_x+P_y)(1-P_x-P_y), \end{aligned}$$

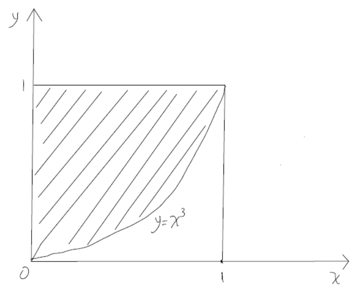
leading to $Cov(X,Y) = -30P_xP_y$.

3. (4.4-7). Let $f(x,y) = 4/3$, $0 < x < 1$, $x^3 < y < 1$, zero elsewhere.

   (a) Sketch the region where $f(x,y) > 0$.
   (b) Find $P(X > Y)$.

   Solution:

   (a)



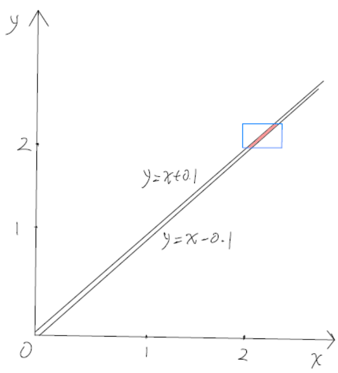
   (b)

$$\begin{aligned} P(X>Y) &= 1 - P(X \le Y) \\ &= 1 - \int_0^1 \int_x^1 \frac{4}{3}\, dy\, dx \\ &= 1 - \int_0^1 \frac{4}{3}(1-x)\, dx \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

4. (4.4-9). Two construction companies make bids of X and Y (in \$100,000 's) on a remodeling project. The joint pdf of X and Y is uniform on the space $2 < x < 2.5$, $2 < y < 2.3$. If X and Y are within 0.1 of each other, the companies will be asked to rebid; otherwise, the low bidder will be awarded the contract. What is the probability that they will be asked to rebid?

   Solution:

   We sketch the region as follows. The entire blue rectangular is the region where $f(x,y) > 0$, and the region in pink is the desired probability.



We are given the joint pdf,

$$f(x,y) = \frac{20}{3}, \quad 2 < x < 2.5,\ 2 < y < 2.3$$

As we are going to compute $P(|X-Y| < 0.1)$, we must be careful about the support of Y (i.e. partition the support of Y).

$$\begin{aligned}
P(|X-Y|<0.1) &= P(|X-Y|<0.1, 2<Y<2.1) + P(|X-Y|<0.1, 2.1<Y<2.3)\\
&= \int_2^{2.1}\int_2^{y+0.1} \frac{20}{3}\,dx\,dy + \int_{2.1}^{2.3}\int_{-0.1+y}^{0.1+y} \frac{20}{3}\,dx\,dy\\
&= \frac{1}{10} + \frac{8}{30}\\
&= \frac{11}{30}
\end{aligned}$$

5. (4.4-15). An automobile repair shop makes an initial estimate X (in thousands of dollars) of the amount of money needed to fix a car after an accident. Say X has the pdf

$$f(x) = 2e^{-2(x-0.2)}, \quad 0.2 < x < \infty$$

Given that $X = x$, the final payment Y has a uniform distribution between $x-0.1$ and $x+0.1$. What is the expected value of Y ?

Solution:

From the given information, the conditional density is given by

$$f(y|x) = \frac{1}{(x+0.1)-(x-0.1)} = 5, \quad x-0.1 < y < x+0.1$$

By Baye's theorem, the joint density is given by

$$f(x,y) = f(y|x)\cdot f(x) = 10e^{-2(x-0.2)}, \quad 0.2 < x < \infty,\ x-0.1 < y < x+0.1$$

Therefore, the expectation is computed as follows

$$\begin{aligned}
E(Y) &= \int_{0.2}^{+\infty}\int_{x-0.1}^{x+0.1} 10ye^{-2(x-0.2)}\,dy\,dx\\
&= \int_{0.2}^{+\infty} \left(5y^2e^{-2(x-0.2)}\right)\Big|_{x-0.1}^{x+0.1}\,dx\\
&= \int_{0.2}^{+\infty} 5(0.4x)e^{-2(x-0.2)}\,dx
\end{aligned}$$

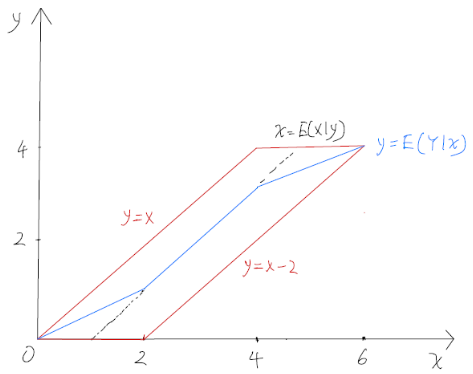
$$= \int_{0.2}^{+\infty} 2xe^{-2(x-0.2)}\,dx$$
$$= \left(-\frac{1}{2}(2x+1)e^{-2(x-0.2)}\right)\Big|_{0.2}^{+\infty}$$
$$= 0.7$$

6. (4.4-18). Let $f(x,y) = 1/8$, $0 \le y \le 4$, $y \le x \le y+2$, be the joint pdf of X and Y.
   (a) Sketch the region for which $f(x,y) > 0$.
   (b) Find $f_X(x)$, the marginal pdf of X.
   (c) Find $f_Y(y)$, the marginal pdf of Y.
   (d) Determine $h(y|x)$, the conditional pdf of Y, given that $X = x$.
   (e) Determine $g(x|y)$, the conditional pdf of X, given that $Y = y$.
   (f) Compute $E(Y|x)$, the conditional mean of Y, given that $X = x$.
   (g) Compute $E(X|y)$, the conditional mean of X, given that $Y = y$.
   (h) Graph $y = E(Y|x)$ on your sketch in part (a). Is $y = E(Y|x)$ linear?
   (i) Graph $x = E(X|y)$ on your sketch in part (a). Is $x = E(X|y)$ linear?

Solution:

(a) & (h) & (i)

The entire red parallelogram is the region where $f(x,y) > 0$, the blue line is the conditional mean required by (h), and the dot line in black and the blue straight line connected them is the conditional mean required by (i).



(b)

$$f_X(x) = \begin{cases} \int_0^x \frac{1}{8}\,dy = \frac{x}{8}, & 0 \le x \le 2 \\ \int_{x-2}^{x} \frac{1}{8}\,dy = \frac{1}{4}, & 2 < x < 4 \\ \int_{x-3}^{4} \frac{1}{8}\,dy = \frac{6-x}{8}, & 4 \le x \le 6 \end{cases}$$

(c)

$$f_Y(y) = \int_y^{y+2} \frac{1}{8}\,dx = \frac{1}{4}, \quad 0 \le y \le 4$$

(d)

$$h(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} 1/x, & 0 \le y \le x,\ 0 \le x \le 2 \\ 1/2, & x-2 \le y \le x,\ 2 < x < 4 \\ 1/(6-x), & x-2 \le y \le 4,\ 4 \le x \le 6 \end{cases}$$

(e)

$$g(x|y) = \frac{f(x,y)}{f_Y(y)} = 1/2, \quad y \le x \le y+2,\ 0 \le y \le 4$$

(f)

$$E(Y|x) = \begin{cases} \int_0^x y\frac{1}{x}\,dy = \frac{x}{2}, & 0 \le x \le 2 \\ \int_{x-2}^x y\frac{1}{2}\,dy = x-1, & 2 < x < 4 \\ \int_{x-2}^4 y\frac{1}{6-x}\,dy = \frac{x+2}{2}, & 4 <\le x \le 6 \end{cases}$$

(g)

$$E(X|y) = \int_y^{y+2} x\frac{1}{2}\,dx = y+1, \quad 0 < y < 4$$

(h) No.

(i) Yes.

7. (4.5-1). Let X and Y have a bivariate normal distribution with parameters $\mu_X = -3$, $\mu_Y = 10, \sigma_X^2 = 25$, $\sigma_Y^2 = 9$, and $\rho = 3/5$. Compute

(a) $P(-5 < X < 5)$
(b) $P(-5 < X < 5|Y = 13)$
(c) $P(7 < Y < 16)$
(d) $P(7 < Y < 16|X = 2)$

Solution:

Given a bivariate normal distribution, we can prove that the marginal distributions and conditional distributions ($X|y$ and $Y|x$) also follow normal distributions (check this on the textbook or lecture note). Based on this fact, we can answer the following questions.

(a) $P(-5 < X < 5) = \Phi\left(\frac{5-(-3)}{\sqrt{25}}\right) - \Phi\left(\frac{-5-(-3)}{\sqrt{25}}\right) = 0.6006$

(b)

$$\mu_{X|Y=13} = \mu_X + \rho\left(\frac{\sigma_X}{\sigma_Y}\right)(13-\mu_Y) = -3 + \frac{3}{5}\cdot\frac{5}{3}\cdot(13-10) = 0$$

$$\sigma^2_{X|Y=13} = \sigma_X^2\left(1-\rho^2\right) = 25 \times \left(1 - \left(\frac{3}{5}\right)^2\right) = 16$$

$$P(-5 < X < 5|Y = 13) = \Phi\left(\frac{5-0}{\sqrt{16}}\right) - \Phi\left(\frac{-5-0}{\sqrt{16}}\right) = 0.7888$$

(c) $P(7 < Y < 16) = \Phi\left(\frac{16-10}{\sqrt{9}}\right) - \Phi\left(\frac{7-10}{\sqrt{9}}\right) = 0.8185$

(d)

$$\mu_{Y|X=2} = \mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(2-\mu_X) = 10 + \frac{3}{5}\times\frac{3}{5}(2-1-3) = \frac{59}{5}$$

$$\sigma^2_{Y|X=2} = \sigma_Y^2\cdot(1-\rho^2) = 9\cdot\left(1-\left(\frac{3}{5}\right)^2\right) = \frac{144}{25}$$

$$\Phi(7 < X < 16|X = 2) = \Phi\left(\frac{16-59/5}{12/5}\right) - \Phi\left(\frac{7-59/5}{12/5}\right) = 0.9371$$

8. (4.5-6). For a freshman taking introductory statistics and majoring in psychology, let X equal the student's ACT mathematics score and Y the students ACT verbal score. Assume that X and Y have a bivariate normal distribution with $\mu_X = 22.7$, $\sigma_X^2 = 17.64$, $\mu_Y = 22.7, \sigma_Y^2 = 12.25$, and $\rho = 0.78$.

(a) Find $P(18.5 < Y < 25.5)$.
(b) Find $E(Y|x)$.
(c) Find $\text{Var}(Y|x)$.
(d) Find $P(18.5 < Y < 25.5|X = 23)$.
(e) Find $P(18.5 < Y < 25.5|X = 25)$.
(f) For $x = 21, 23$, and 25, draw a graph of $z = h(y|x)$ similar to Figure 4.5-1.

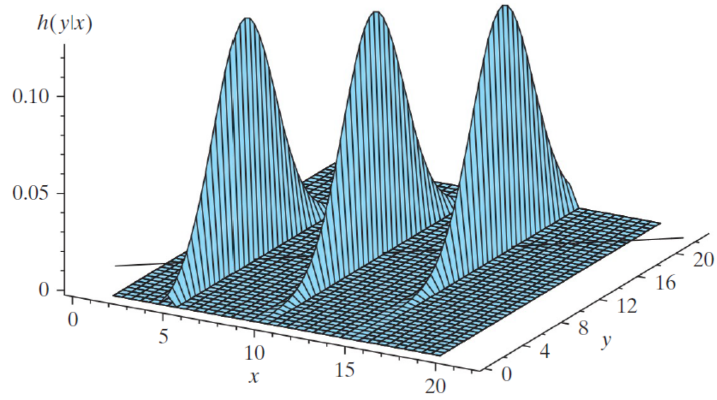


Figure 4.5-1 Conditional pdf of Y, given that $x = 5, 10, 15$

Solution:

(a) $P(18.5 < Y < 25.5) = \Phi(0.8) - \Phi(-1.2) = 0.6730$

(b) $E(Y|x) = 22.7 + 0.78\left(\sqrt{\frac{12.25}{17.64}}\right)(x - 22.7) = 0.65x + 7.945$

(c) $\text{Var}(Y|x) = 12.25\left(1 - 0.78^2\right) = 4.7971$

(d)

$$\begin{aligned} P(18.5 < Y < 25.5|X = 23) &= \Phi\left(\frac{25.5 - 0.65 \times 23 - 7.945}{\sqrt{4.7971}}\right) - \Phi\left(\frac{18.5 - 0.65 \times 23 - 7.945}{\sqrt{4.7971}}\right) \\ &= \Phi(1.189) - \Phi(-2.007) \\ &= 0.8604 \end{aligned}$$

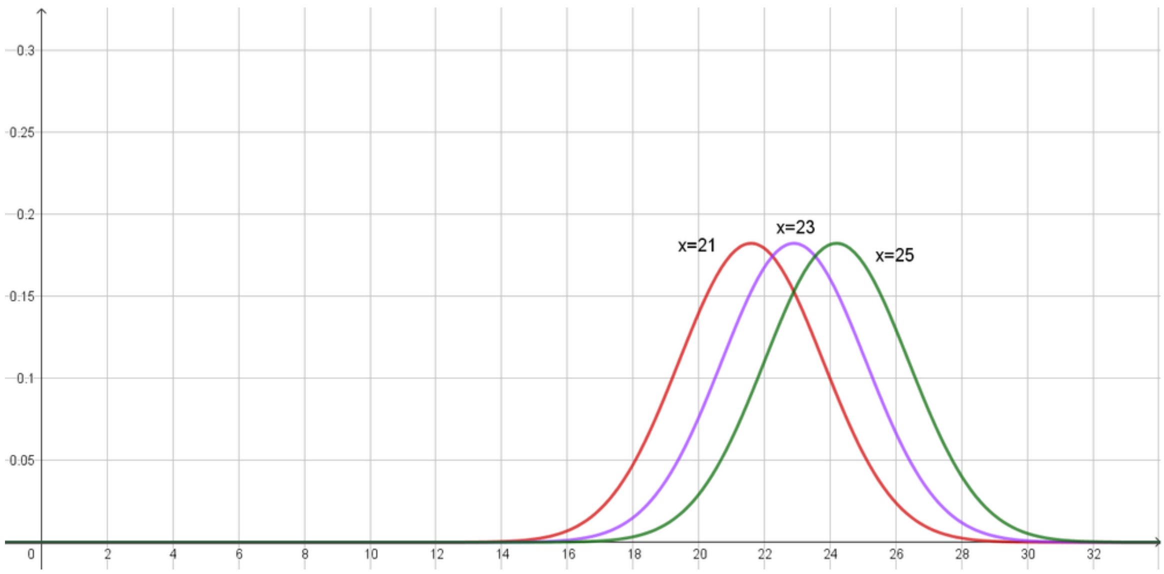
(e) $P(18.5 < Y < 25.5|X = 25) = \Phi(0.596) - \Phi(-2.60) = 0.7197$

(f)

For $x = 21$, $z = h(y|x)$ is a normal pdf with mean $E(Y|X = 21) = 7.945 + 0.65 \times (21) = 21.595$ and variance 4.7971.

For $x = 23$, $z = h(y|x)$ is a normal pdf with mean $E(Y|X = 23) = 7.945 + 0.65 \times (23) = 22.895$ and variance 4.7971.

For $x = 25$, $z = h(y|x)$ is a normal pdf with mean $E(Y|X = 25) = 7.945 + 0.65 \times (25) = 24.195$ and variance 4.7971.



9. Let

$$f(x,y) = \left(\frac{1}{2\pi}\right) e^{-\left(x^2+y^2\right)/2} \left[1 + xye^{-\left(x^2+y^2-2\right)/2}\right], \quad -\infty < x < \infty, -\infty < y < \infty$$

Show that $f(x,y)$ is a joint pdf and the two marginal pdfs are each normal. Note that X and Y can each be normal, but their joint pdf is not bivariate normal.

Solution:

First, we show that $f(x,y) \geq 0$ for $-\infty < x < \infty, -\infty < y < \infty$.

The case for $x > 0, y > 0$ and $x < 0, y < 0$ are obvious. WLOG, assume that $x < 0$ and $y > 0$. By some simple algebra, we can show that to prove $f(x,y) \geq 0$ is equivalent to prove

$$e^{-\left(x^2+y^2\right)/2} \cdot \left(1 + xye \cdot e^{-\left(x^2+y^2\right)/2}\right) > 0$$

Furthermore, it's equivalent to prove (again, just simple algebra)

$$\begin{aligned} -xy &< \exp\left(\frac{x^2+y^2}{2} - 1\right) \\ \Leftrightarrow \ln(-xy) &< \frac{x^2+y^2}{2} - 1 \end{aligned}$$

Note that this must be true because we have $\ln(-xy) < -xy - 1 < \frac{x^2+y^2}{2} - 1$, for $y > 0$ and $-x > 0$.

We will have the same result for $y < 0$ and $x > 0$, as the function $f(x,y)$ is symmetric. Therefore, we have proved $f(x,y) \geq 0$ for $-\infty < x < \infty, -\infty < y < \infty$.

Next, we need to show that $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dy\,dx = 1$.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right) e^{-\left(x^2+y^2\right)/2} \left[1 + xye^{-\left(x^2+y^2-2\right)/2}\right] dy \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right) e^{-\left(x^2+y^2\right)/2}\,dy + \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right) e^{-\left(x^2+y^2\right)/2} \left(xye^{-\left(x^2+y^2-2\right)/2}\right) dy \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy + 0 \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty \end{aligned}$$

At the second equality, the integrand of the second integral is an odd function on $\mathbb{R}$ so the integral equals to 0, and the integral at the 3rd equality is actually integrating a standard normal pdf so it's 1.

Similarly we can show that (as the given function $f(x, y)$ is symmetry),

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad -\infty < y < \infty$$

Furthermore,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, dy\, dx = \int_{-\infty}^{+\infty} f_X(x)\, dx = \int_{-\infty}^{+\infty} f_Y(y)\, dy = 1$$

So the marginals are normal distributed, and $f(x, y)$ is indeed a joint pdf.

10. Assume that X and Y are bivariate normal distributed with their joint probability density function described by

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}q(x, y)\right), x \in \mathbb{R}, y \in \mathbb{R},$$

$$q(x, y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]. \quad (1)$$

where $\mu_X \in \mathbb{R}, \mu_Y \in \mathbb{R}, \sigma_X > 0, \sigma_Y > 0$ and $|\rho| \leq 1$.

(a1) Prove that the marginal probability density function of X is the probability density function of $N(\mu_X, \sigma_X^2)$, and the condition probability density function of Y given $X = x$ is the probability density function of

$$N(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x - \mu_X), (1-\rho^2)\sigma_Y^2). \quad (2)$$

In other words, prove that $X \sim N(\mu_X, \sigma_X^2)$, and $Y|X = x \sim N(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x-\mu_X), (1-\rho^2)\sigma_Y^2)$.

(a2) Assume that $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$. Then prove that ρ in (1) is the correlation coefficient of X and Y, and moreover, X and Y are independent if and only if X and Y are uncorrelated.

Solution: We first prove (a1). By definition, the marginal probability density function of X, $f_X(x)$, can be derived from the joint probability density function $f(x, y)$ as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy. \quad (3)$$

Clearly, to prove (a1) is equivalent to prove that

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right). \quad (4)$$

To this goal, we arrange $f(x, y)$ as follows:

$$\begin{aligned} f(x, y) = &\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times \\ &\exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2 - \frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right) \\ = &\frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right) \times h(y|x) \end{aligned} \quad (5)$$

where

$$
\begin{aligned}
h(y|x) =& \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right)-\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right) \\
=& \frac{1}{\sqrt{2\pi}\sqrt{(1-\rho^2)\sigma_Y^2}} \exp\left(-\frac{1}{2}\left[\frac{y-[\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)]}{\sqrt{(1-\rho^2)\sigma_Y^2}}\right]^2\right)
\end{aligned} \tag{6}
$$

Clearly, to complete the proof, we only need to show that

$$
\int_{-\infty}^{\infty} h(y|x)dy = 1 \tag{7}
$$

which is true because $h(y|x)$ is the probability density function of $N(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x-\mu_X), (1-\rho^2)\sigma_Y^2)$. Actually, $h(y|x)$ is the conditional probability density function of Y given $X = x$ because

$$
h(y|x) = \frac{f(x,y)}{f_X(x)}. \tag{8}
$$

Now we prove (a2). By definition, the correlation coefficient of X and Y is defined as

$$
\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} \tag{9}
$$

where

$$
\mathrm{Cov}(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f(x,y)dxdy. \tag{10}
$$

Clearly, to prove (a2) is equivalent to prove $\rho(X,Y) = \rho$. To this goal, note that

$$
\begin{aligned}
\mathrm{Cov}(X,Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f_X(x)h(y|x)dxdy \\
&= \int_{-\infty}^{\infty}(x-\mu_X)f_X(x)\left(\int_{-\infty}^{\infty}(y-\mu_Y)h(y|x)\right)dydx \\
&= \int_{-\infty}^{\infty}(x-\mu_X)f_X(x)\left(\frac{\sigma_Y}{\sigma_X}\rho(x-\mu_X)\right)dx \\
&= \frac{\sigma_Y}{\sigma_X}\rho\int_{-\infty}^{\infty}(x-\mu_X)^2 f_X(x)dx \\
&= \frac{\sigma_Y}{\sigma_X}\rho\sigma_X^2 \\
&= \sigma_Y\sigma_X\rho
\end{aligned}
$$

which shows that $\rho(X,Y) = \rho$ and thus ρ is the correlation coefficient.

If X and Y are independent, then X and Y are uncorrelated because $\mathrm{Cov}(X,Y) = E(XY) - EXEY = 0$. So we only need to prove that the if X and Y are uncorrelated, then X and Y are independent. Since $\rho = 0$, it is easy to see that

$$
f(x,y) = f_X(x)f_Y(y) \tag{11}
$$

where $f_X(x)$ and $f_Y(y)$ are the probability density functions of X and Y, respectively, which implies that X and Y are independent.