Review of the last lecture

Key concepts and/or techniques:

- Bivariate RV: (X, Y) or X and Y with range $\overline{S} \subseteq \overline{S_X} \times \overline{S_Y} \subseteq \mathbb{R}^2$
- Joint pmf $f(x, y) : \overline{S} \to (0, 1]$
- How to derive Marginal pmf from the joint pmf

$$f_X(x) = P_X(X = x) \triangleq P\left(\{X = x, Y \in \overline{S_Y}(x)\}\right) = \sum_{y \in \overline{S_Y}(x)} f(x, y)$$

- Trinomial distribution: $(X, Y) \sim \text{Trinomial}(n, p_X, p_Y)$

$$f(x, y) = \frac{n!}{x!y!(n - x - y)!} p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}, (x, y) \in \overline{S}, \overline{S}$$

- X and Y are independent if $f(x, y) = f_X(x) f_Y(y)$

Review of the last lecture

Definition[Joint pmf]

The function $f(x,y) : \overline{S} \to (0,1]$ is called the joint probability mass function (joint pmf) of X and Y or (X,Y), if

1. $f(x,y) > 0$ for $(x,y) \in \overline{S}$,
2. $\sum_{(x,y)\in\overline{S}} f(x,y) = 1$,
3. For $A \subseteq \overline{S}$,

$$P[(X,Y) \in A] \stackrel{\Delta}{=} P(\{(X,Y) \in A\}) = \sum_{(x,y)\in A} f(x,y)$$

which defines the probability function for a set A. In particular, taking $A = \{(x,y)\}$ yields the probability of $X = x$ and $Y = y$, i.e.,

$$P(X = x, Y = y) = f(x,y)$$

Review of the last lecture

Definition[Marginal pmf]

Let (X, Y) be a bivariate RV, or X and Y be two RVs, and have the joint pmf $f(x, y) : \overline{S} \to (0, 1]$. Sometimes, we are interested in the pmf of X or Y alone, which is called the marginal pmf of X or Y and described by

For $x \in \overline{S_X}$,

$$\begin{aligned} f_X(x) = P_X(X = x) &\overset{\Delta}{=} P\left(\left\{X = x, Y \in \overline{S_Y}(x)\right\}\right) \\ &= \sum_{y \in \overline{S_Y}(x)} f(x, y) \end{aligned}$$

where

$$\overline{S_Y}(x) = \left\{y \middle| (x, y) \in \overline{S}\right\} \text{ for the given } x \in \overline{S_X}.$$

Review of the last lecture

It is crucial to understand the following definitions

$\overline{S}, \overline{S_X}, \overline{S_Y}, \overline{S_X}(y), \overline{S_Y}(x)$

$$
\begin{aligned}
\overline{S} &= \{\text{all possible values of } (X, Y)\} \\
\overline{S_X} &= \{\text{all possible values of } X\} = \left\{x | (x, y) \in \overline{S}\right\} \\
\overline{S_Y} &= \{\text{all possible values of } Y\} = \left\{y | (x, y) \in \overline{S}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\overline{S_X}(y) &= \left\{x | (x, y) \in \overline{S}\right\} \text{ for a given y} \in \overline{S_Y} \\
\overline{S_Y}(x) &= \left\{y | (x, y) \in \overline{S}\right\} \text{ for a given x} \in \overline{S_X}
\end{aligned}
$$

Review of the last lecture

Definition

The random variables X and Y are said to be independent if for every $x \in \overline{S_X}$ and $y \in \overline{S_Y}$

$$f(x, y) = f_X(x) f_Y(y)$$

or equivalently,

$$P(X = x, Y = y) = P_X(X = x) P_Y(Y = y).$$

X and Y are said to be dependent if otherwise.

When X and Y are independent,

$$\overline{S} = \overline{S_X} \times \overline{S_Y}, \quad \overline{S} \text{ is said to be rectangular}$$

which is a necessary condition for independence of X and Y.

STA2001 Probability and Statistics (I)

Lecture 15

Tianshi Chen

The Chinese University of Hong Kong, Shenzhen

Mathematical Expectation

Let X and Y be discrete RVs with their joint pmf

$$f(x, y) : \overline{S} \to (0, 1]$$

Consider a function $g(X, Y)$ of X and Y.

Then the expectation of $g(X, Y)$ is

$$E[g(X, Y)] = \sum_{(x,y) \in \overline{S}} g(x, y) f(x, y)$$

Mathematical Expectation

$$\text{When } g(X,Y) = X, \quad E[X] \text{ is the mean of } X$$

$$\text{When } g(X,Y) = (X - E[X])^2,$$

$$E[(X - E[X])^2] \text{ is the variance of } X$$

Mathematical Expectation

$$\text{When } g(X,Y) = X, \quad E[X] \text{ is the mean of } X$$
$$\text{When } g(X,Y) = (X - E[X])^2,$$
$$E[(X - E[X])^2] \text{ is the variance of } X$$

There are seemingly two ways to calculate $E[X]$:

$$\begin{cases} E[X] = \overbrace{\sum_{x \in \overline{S_X}} x f_x(x)}^{\text{Marginal pmf}} \\ E[X] = \underbrace{\sum_{(x,y) \in \overline{S}} x f(x,y)}_{\text{Joint pmf}} \end{cases}$$

Mathematical Expectation

$$\text{When } g(X,Y) = X, \quad E[X] \text{ is the mean of } X$$

$$\text{When } g(X,Y) = (X - E[X])^2,$$

$$E[(X - E[X])^2] \text{ is the variance of } X$$

There seems two ways to calculate $E[X]$:

$$\text{Equivalent}\begin{cases} E(X) = \overbrace{\sum_{x \in \overline{S_X}} x f_x(x)}^{\text{Marginal pmf}} \\ E(X) = \underbrace{\sum_{(x,y) \in \overline{S}} x f(x,y)}_{\text{Joint pmf}} = \sum_{x \in \overline{S_X}} x \underbrace{\sum_{y \in \overline{S_Y}(x)} f(x,y)}_{= f_x(x)} \end{cases}$$

Example 1, [Page 134] —– Revisited

Question

Recall that X and Y are discrete RVs with joint pmf

$f(x, y) : \overline{S} \to (0, 1]$ with $\overline{S_X} = \overline{S_Y} = \{1, 2, 3, 4\}$

$$f(x, y) = \begin{cases} \frac{2}{16} & 1 \leq x < y \leq 4 \\ \frac{1}{16} & 1 \leq x = y \leq 4 \end{cases}$$

What is $E[X + Y]$?

Example 1, [Page 134] —– Revisited

$$
\begin{aligned}
E(X+Y) &= \sum_{(x,y)\in\overline{S}} (x+y)f(x,y) \\
&= \sum_{1\leq x=y\leq 4} (x+y)\frac{1}{16} + \sum_{1\leq x<y\leq 4} (x+y)\frac{2}{16} \\
&= \sum_{x=1}^{4}(2x)\frac{1}{16} + \sum_{x=1}^{4} \sum_{y\in\overline{S_Y}(x),x<y} (x+y)\frac{2}{16}
\end{aligned}
$$

Note: the expectation is w.r.t all random variable, i.e. X and Y.

Section 4.2 The correlation coefficient

Motivation

Study the relation between two RVs (random phenomena)

Covariance of X and Y

Definition

Let X and Y be RVs with joint pmf $f(x, y) : \overline{S} \to (0, 1]$
Take

$$\begin{aligned} g(X, Y) &= (X - E(X))(Y - E(Y)) \\ Cov(X, Y) &\triangleq E[(X - E(X))(Y - E(Y))] \\ &= \sum_{(x,y)\in\overline{S}} (x - E(X))(y - E(Y))f(x, y) \end{aligned}$$

Moreover, we have

$$Cov(X, Y) \triangleq E(XY) - E(X)E(Y)$$

Covariance of X and Y

- $Cov(X, Y) = E(XY) - E(X)E(Y)$

  When $Cov(X, Y) = 0$, X and Y are uncorrelated.

  When $Cov(X, Y) > 0$, X and Y are positively correlated.

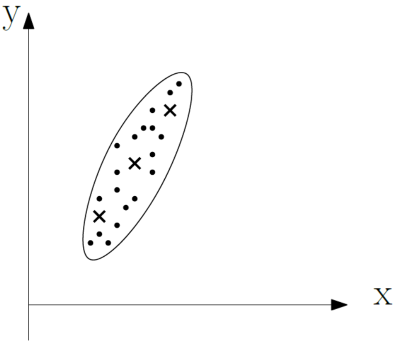
  When $Cov(X, Y) < 0$, X and Y are negatively correlated.

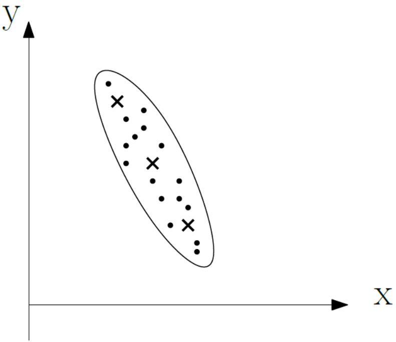
- Interpretation: Roughly speaking, a positive or negative covariance indicate that the values of $X - E(X)$ and $Y - E(Y)$ obtained in a single experiment "tend" to have the same or the opposite sign respectively.

$$Cov(X, Y) = \sum_{(x,y)\in\overline{S}} (x - E(X))(y - E(Y))f(x, y)$$

Example 1 [Positively Correlated and Negatively Correlated RVs]

Assume that X and Y are uniformly distributed over the ellipses.

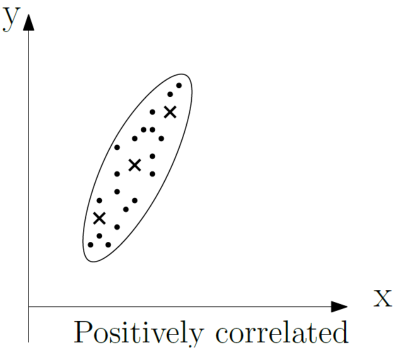




Question: which figure shows that X and Y are positively correlated?

Example 1 [Positively Correlated and Negatively Correlated RVs]

Assume that X and Y are uniformly distributed over the ellipses.



Independence $\Rightarrow$ Uncorrelation

- If X and Y are independent, we have

$$f(x,y) = f_X(x)f_Y(y) \Rightarrow \overline{S} = \overline{S_X} \times \overline{S_Y}$$

$$E(XY) = \sum_{(x,y)\in\overline{S}} xyf(x,y) = \sum_{x\in\overline{S_X}} \sum_{y\in\overline{S_Y}} xyf_X(x)f_Y(y)$$

$$= \sum_{x\in\overline{S_X}} xf_X(x) \left[\sum_{y\in\overline{S_Y}} yf_Y(y)\right] = E(X)E(Y)$$

Independence $\Rightarrow$ Uncorrelation

Therefore

$$Cov(X, Y) = E(XY) - E(X)E(Y) = 0$$

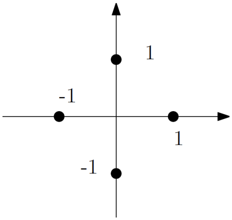
Independence of two RVs $\Rightarrow$ uncorrelation of two RVs.

However, the converse is not true , i.e, there exist X and Y which are uncorrelated but not independent.

Example 2 [Uncorrelation $\not\Rightarrow$ Independence]

Question

Let (X, Y) be a bivariate RV that takes values $(1, 0), (0, 1)$, $(-1, 0)$,$(0, -1)$, each with probability $\frac{1}{4}$, as shown in the figure below



Question 1

What are the marginal pmf of X and Y?

Question 2

What is $Cov(X, Y)$?

Question 3

Are X and Y independent?

Example 2 [Uncorrelation $\nRightarrow$ Independence]

To find marginal pmf of X and Y, $\overline{S_X} = \overline{S_Y} = \{-1, 0, 1\}$

$$f_X(x) = \begin{cases} \frac{1}{4}, & x = 1 \\ \frac{1}{2}, & x = 0 \\ \frac{1}{4}, & x = -1 \end{cases} \quad f_Y(y) = \begin{cases} \frac{1}{4}, & y = 1 \\ \frac{1}{2}, & y = 0 \\ \frac{1}{4}, & y = -1 \end{cases}$$

$$Cov(X, Y) = E(XY) - E(X)E(Y) = 0 - 0 \times 0 = 0$$

which shows that X and Y are uncorrelated.

$$f_X(0)f_Y(1) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} \neq f(0, 1) = \frac{1}{4}$$

which shows that X and Y are NOT independent.

Correlation Coefficient

Definition

The correlation coefficient of X and Y that have nonzero variance is defined as

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

- Interpretation : $\rho > 0$ (or $\rho < 0$) indicate the values of $X - E[X]$ and $Y - E[Y]$ "tend" to have the same (or negative, respectively) sign.

Properties of the Correlation Coefficient

- It is a normalized version of $Cov(X, Y)$ and in fact $-1 \leq \rho(X, Y) \leq 1$.
- $\rho = 1$ (resp. $\rho = -1$) if and only if there exists a positive (resp. negative) constant c such that

$$Y - E(Y) = c(X - E(X)),$$

and the size of $|\rho|$ provides a normalized measure of the extent to which this is true.

Proof for Properties of Correlation Coefficient (1/3)

To prove $|\rho(X, Y)| \leq 1$ is equivalent to prove that

$$\text{Cov}(X, Y)^2 \leq Var(X)Var(Y)$$

Proof for Properties of Correlation Coefficient (1/3)

To prove $|\rho(X, Y)| \leq 1$ is equivalent to prove that

$$\text{Cov}(X, Y)^2 \leq Var(X)Var(Y)$$

To this goal, we consider

$$E((V + tW)^2) \geq 0,$$

where $t \in \mathbb{R}$, $V = X - E(X)$, $W = Y - E(Y)$. Then we have

$$\begin{aligned} E((V + tW)^2) &= E(V^2 + 2tVW + W^2) \\ &= E(V^2) + 2tE(VW) + t^2E(W^2) \geq 0 \end{aligned}$$

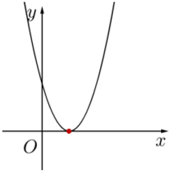
Proof for Properties of Correlation Coefficient (2/3)

Noting that

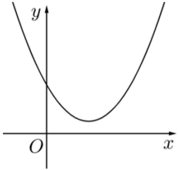
$$E(V^2) = Var(X), E(W^2) = Var(Y), E(VW) = \text{Cov}(X, Y)$$

yields that for $t \in \mathbb{R}$,

$$E((V + tW)^2) = Var(X) + 2\text{Cov}(X, Y)t + Var(Y)t^2 \geq 0$$



One Solution



No Real Solution

Since the above equation is true for any $t \in \mathbb{R}$, it must hold that

$$4\text{Cov}(X, Y)^2 - 4Var(X)Var(Y) \leq 0, \text{ i.e., } \text{Cov}(X, Y)^2 \leq Var(X)Var(Y)$$

which implies that $|\rho(X, Y)| \leq 1$.

Proof for Properties of Correlation Coefficient (3/3)

When $\mathrm{Cov}(X,Y)^2 - Var(X)Var(Y) = 0$, $|\rho(X,Y)| = 1$ implying

$$
\begin{aligned}
E((V+tW)^2) &= Var(X) + 2\mathrm{Cov}(X,Y)t + Var(Y)t^2 = 0 \\
&= Var(X) \pm 2\sqrt{Var(X)}\sqrt{Var(Y)}t + Var(Y)t^2 = 0 \\
&= (\sqrt{Var(X)} \pm \sqrt{Var(Y)}t)^2 = 0
\end{aligned}
$$

$$t^* = \mp \frac{\sqrt{Var(X)}}{\sqrt{Var(Y)}}.$$

Inserting the above t^* back in $E((V+tW)^2)$ yields

$$E((V+t^*W)^2) = 0 \implies V = -t^*W$$

$$X - E(X) = \pm \frac{\sqrt{Var(X)}}{\sqrt{Var(Y)}}(Y - E(Y)),$$

where $\rho = 1$ (resp. $\rho = -1$) corresponds to $+$ (resp. $-$).

Example

Question

Consider n independent tosses of a coin with probability of a head equal to p. Let X and Y be the number of heads and of tails in the n tosses, respectively. Calculate the correlation coefficient of X and Y.

Example

Question

Consider n independent tosses of a coin with probability of a head equal to p. Let X and Y be the number of heads and of tails in the n tosses, respectively. Calculate the correlation coefficient of X and Y.

$$X + Y = n \quad \Rightarrow E[X] + E[Y] = n \Rightarrow X - E(X) = -[Y - E(Y)]$$

Example

Question

Consider n independent tosses of a coin with probability of a head equal to p. Let X and Y be the number of heads and of tails in the n tosses, respectively. Calculate the correlation coefficient of X and Y.

$$X + Y = n \quad \Rightarrow E[X] + E[Y] = n \Rightarrow X - E(X) = -[Y - E(Y)]$$

$$\begin{aligned}
Cov(X, Y) &= E[(X - E(X))(Y - E(Y))] \\
&= -E[(Y - E(Y))^2] = -Var(Y) \\
Var(X) &= E[(X - E(X))^2] = E[(Y - E(Y))^2] = Var(Y) \\
\rho(X, Y) &= \frac{-Var(Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{-Var(Y)}{\sqrt{Var(Y)}\sqrt{Var(Y)}} = -1
\end{aligned}$$