# Review of Last Lecture

Key concepts and/or techniques:

1. Convergence of sequence of RVs
   - convergence in distribution
2. CLT: from random samples drawn from a distribution with finite mean and finite variance $\sigma^2 > 0$

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

3. Application of CLT:
   - $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ can be approximated by $N(0,1)$.
   - $\overline{X}$ can be approximated by $N(\mu, \frac{\sigma^2}{n})$.
   - $\sum_{i=1}^{n} X_i$ can be approximated by $N(n\mu, n\sigma^2)$.
4. Histogram of and Approximation for discrete distribution

# Review of Last Lecture

**Convergence in distribution**

A sequence of random variables $Z_1$, $Z_2$, ... is said to converge in distribution, or converge weakly, or converge in law to a random variable $Z$, denoted by $Z_n \xrightarrow{d} Z$, if

$$\lim_{n\to\infty} F_n(z) = F(z),$$

for every number $z \in R$ at which $F(z)$ is continuous, where $F_n(z)$ and $F(z)$ are the cdfs of random variables $Z_n$ and $Z$, respectively.

Note: convergence of sequence of numbers.

# Review of Last Lecture

**CLT**

Let $\overline{X}$ be the sample mean of the random sample of size $n$, $X_1$,$X_2$ ,$\cdots$,$X_n$ from a distribution with a finite mean $\mu$ and a finite nonzero variance $\sigma^2$, then as $n \to \infty$, the random variable $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ converge in distribution to $N(0,1)$.

Practical use of CLT: for large $n$,

- $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ can be approximated by $N(0,1)$.
- $\overline{X}$ can be approximated by $N(\mu, \frac{\sigma^2}{n})$.
- $\sum_{i=1}^{n} X_i$ can be approximated by $N(n\mu, n\sigma^2)$.

# Review of Last Lecture

For large $n$, the probabilities of events of $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$, $\overline{X}$ and $\sum_{i=1}^{n} X_i$ can be calculated approximately by treating them as if they are $N(0,1)$, $N(\mu, \frac{\sigma^2}{n})$, and $N(n\mu, n\sigma^2)$, respectively, and by looking up tables of normal distributions.

Recall that if $Y \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(a \leq Y \leq b) &= P(\frac{a-\mu}{\sigma} \leq \frac{Y-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}) \\
&= \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})
\end{aligned}
$$

where $\Phi(\cdot)$ is the cdf of $N(0,1)$

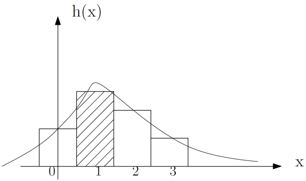
# Review of Last Lecture

**Histogram of and Approximation for discrete distribution**

Consider a discrete RV $Y$ with pmf $f(y) : \overline{S} \to (0, 1]$ with $\overline{S} = \{0, 1, \cdots, n\}$.Then the histogram for $Y$ is

$$h(y) = f(k), y \in (k - \frac{1}{2}, k + \frac{1}{2}), k = 0, 1, \cdots, n$$

The area below the histogram corresponds to probability, which make the histogram has similar property as the pdf of continuous distribution.




If it is possible to find a continuous distribution with pdf "close" to the histogram of the discrete distribution, then we can compute the probability of discrete distribution approximately by using the continuous distribution.

# STA2001 Probability and Statistics I

## Lecture 24

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# Half-unit correction for continuity

Now, let $Y = \sum_{i=1}^{n} X_i$, where $X_1, \cdots, X_n$ are i.i.d. random sample drawn from discrete distribution with mean $\mu$ and variance $\sigma^2$, then

| | | |
|---|---|---|
| $P(Y = k)$ | $\approx$ | $P(k - \frac{1}{2} < Y < k + \frac{1}{2})$ |
| discrete RV | | approximated by continuous RV |
| pmf $f(y)$ | | by CLT for large $n$, $Y = \sum_{i=1}^{n} X_i$ can be approximated by $N(n\mu, n\sigma^2)$ in the sense that the pdf of the normal distribution is close to the histogram of $Y$ |
| hard to calculate | | easy to calculate |

# Binomial distribution

Let $X_1, \cdots, X_n$ be a random sample of size $n$ from Bernoulli distribution $b(1, p)$, whose mean is $p$ and variance $p(1-p)$.

# Binomial distribution

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Then

$$Y = \sum_{i=1}^{n} X_i \sim b(n, p)$$

with mean $np$ and variance $np(1-p)$.

# Binomial distribution

Let $X_1, \cdots, X_n$ be a random sample of size $n$ from Bernoulli distribution $b(1, p)$, whose mean is $p$ and variance $p(1-p)$.

Then

$$Y = \sum_{i=1}^{n} X_i \sim b(n, p)$$

with mean $np$ and variance $np(1-p)$.

To calculate $P(Y = k)$ by definition, i.e.,
$P(Y = k) = \left( \begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k}$ is complicated.

# Binomial distribution

Now we try to calculate $P(Y = k)$ by CLT,

$$\frac{Y/n - p}{\sqrt{p(1-p)/n}} \xrightarrow{d} N(0, 1)$$

For sufficiently large $n$ , $Y$ can be approximated by $N(np, np(1-p))$ and thus probability for $b(n, p)$ can be approximated by that for $N(np, np(1-p))$.

# Binomial distribution

$$P(Y = k) \quad \approx \quad P(k - \tfrac{1}{2} < Y < k + \tfrac{1}{2})$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

$$f(k) = \tfrac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \qquad Y \sim N(np, np(1-p)) \text{ for large n}$$

$$\begin{aligned} P(k - \frac{1}{2} < Y < k + \frac{1}{2}) &= P\left( \frac{k - \frac{1}{2} - np}{\sqrt{np(1-p)}} < \frac{Y - np}{\sqrt{np(1-p)}} < \frac{k + \frac{1}{2} - np}{\sqrt{np(1-p)}} \right) \\ &= \Phi\left( \frac{k + \frac{1}{2} - np}{\sqrt{np(1-p)}} \right) - \Phi\left( \frac{k - \frac{1}{2} - np}{\sqrt{np(1-p)}} \right), \end{aligned}$$

$\Phi(\cdot)$ is the cdf for $N(0, 1)$

# Example 1, page 216

**Question**

Assume $Y \sim b(10, 0.5)$. $\quad Q : P(3 \leq Y < 6)$?

# Example 1, page 216

**Question**

Assume $Y \sim b(10, 0.5)$. $\quad Q : P(3 \leq Y < 6)$?

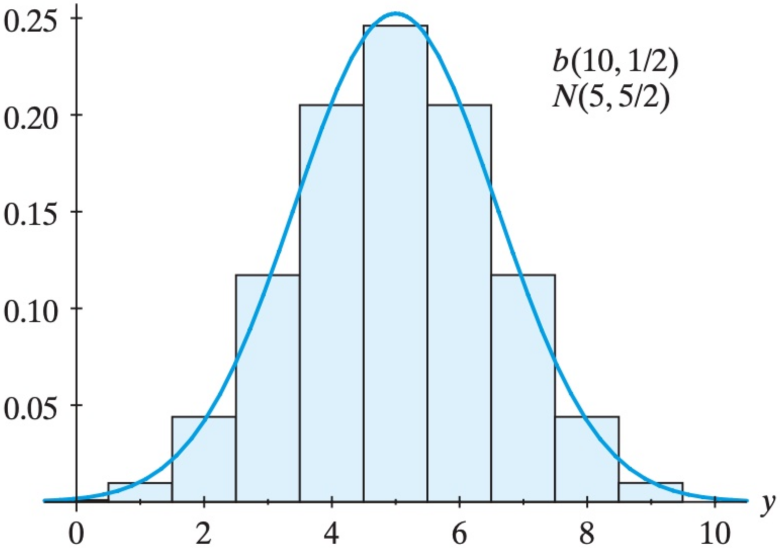
1. By definition,

$$P(3 \leq Y < 6) = \sum_{k=3}^{5} P(Y = k) = \sum_{k=3}^{5} f(k) = 0.5683$$

2. By CLT, $Y = \sum_{i=1}^{10} X_i, \quad X_1, \cdots, X_{10}$ are i.i.d. from $b(1, \frac{1}{2})$

$$Y \quad \text{approximately} \quad N(np, np(1-p)) = N(5, 2.5)$$

# Example 1, page 216



# Example 1, page 216

$$P(3 \le Y < 6) = \sum_{k=3}^{5} P(Y = k) \approxeq \sum_{k=3}^{5} P(k - \frac{1}{2} < Y < k + \frac{1}{2})$$

$$= P(2.5 < Y < 5.5) = P(\frac{2.5 - 5}{\sqrt{2.5}} < \frac{Y - 5}{\sqrt{2.5}} < \frac{5.5 - 5}{\sqrt{2.5}})$$

$$= \Phi(0.316) - \Phi(-1.581)$$

$$\approxeq 0.6240 - 0.0570 = 0.5670.$$

# Example 2, page 217

Let $X_1, X_2, \cdots, X_{20}$ be a random sample of size 20 drawn from Poisson distribution with mean $\lambda = 1$. Then

Q1. what is the distribution of $Y = \sum_{i=1}^{20} X_i$?

Q2. find $P(16 < Y \leq 21)$ approximately?

# Example 2, page 217

Let $X_1, X_2, \cdots, X_{20}$ be a random sample of size 20 drawn from Poisson distribution with mean $\lambda = 1$. Then

Q1. what is the distribution of $Y = \sum_{i=1}^{20} X_i$?

Q2. find $P(16 < Y \leq 21)$ approximately?

Q1: Recall that the mgf of Poisson distribution with mean $\lambda > 0$ is

$$M(t) = e^{\lambda(e^t - 1)}, \quad t \in \mathbb{R}$$

Then by Theorem 5.4-1, the mgf of $Y$, $M_Y(t)$ takes the form of

$$M_Y(t) = \prod_{i=1}^{20} e^{(e^t - 1)} = e^{20(e^t - 1)},$$

implying that $Y$ has a Poisson distribution with mean and variance both equal to 20.

# Example 2, page 217

Q2: Let $X_1, X_2, \cdots, X_n$ be a random sample of size $n$ from the Poisson distribution with mean $\lambda = 1$. Then by CLT,

$$\frac{Y/n - 1}{1/\sqrt{n}} \xrightarrow{d} N(0,1).$$

Therefore, for large $n$, $Y$ can be approximated by $N(n, n)$.

When $n = 20$,

$$\begin{aligned}
P(16 < Y \leq 21) &= P(17 \leq Y \leq 21) \\
&\approxeq P(16.5 \leq Y \leq 21.5) \\
&= P\left(\frac{16.5 - 20}{\sqrt{20}} \leq \frac{Y - 20}{\sqrt{20}} \leq \frac{21.5 - 20}{\sqrt{20}}\right) \\
&\approxeq \Phi(0.335) - \Phi(-0.783) = 0.4142
\end{aligned}$$

One may also try $P(16 < Y \leq 21) = \sum_{x=17}^{21} \frac{20^x e^{-20}}{x!} = 0.4226$

# Section 5.8 Chebyshev's Inequality and Convergence in Probability

# Motivation

**CLT**: given a random sample of size $n$, say $X_1, \cdots, X_n$, from a random distribution (the distribution may be unknown) with its mean and variance known, it is possible to compute approximately the probability of events for $\overline{X}$ or $\sum_{i=1}^{n} X_i$.

# Motivation

**CLT**: given a random sample of size $n$, say $X_1, \cdots, X_n$, from a random distribution (the distribution may be unknown) with its mean and variance known, it is possible to compute approximately the probability of events for $\overline{X}$ or $\sum_{i=1}^{n} X_i$.

**Chebyshev's inequality**: given a random distribution (the distribution may be unknown) with its mean and variance known, it is possible to compute approximately the probability of events for a random variable $X$ whose distribution is the given distribution.

# Theorem 5.8-1 [Chebyshev's inequality]

**Theorem 5.8-1(Chebyshev's inequality)**

If a RV $X$ has a finite mean $\mu$ and finite nonzero variance $\sigma^2$, then for every $k \geq 1$,

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

# Proof of Theorem 5.8-1

Consider the discrete RV case. Let $f(x) : \overline{S} \to (0,1]$ be the pmf.

$$\begin{aligned}\sigma^2 = E[(X-\mu)^2] &= \sum_{x\in\overline{S}} (x-\mu)^2 f(x) \\ &= \sum_{x\in A} (x-\mu)^2 f(x) + \sum_{x\in A'} (x-\mu)^2 f(x)\end{aligned}$$

where

$$A = \{x \| x-\mu| \geq k\sigma\}$$

# Proof of Theorem 5.8-1

Since

$$\sum_{x \in A'} (x-\mu)^2 f(x) \geq 0$$

$$\sigma^2 \geq \sum_{x \in A} (x-\mu)^2 f(x) \geq k^2\sigma^2 \sum_{x \in A} f(x) = k^2\sigma^2 P(X \in A)$$

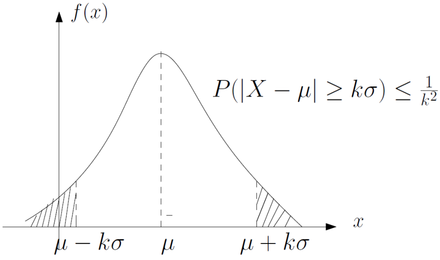
# Corollary 5.8-1, Page 222

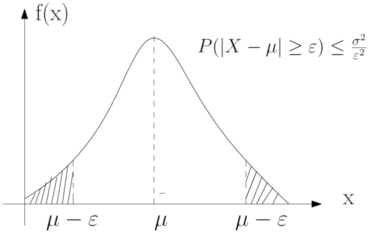
**Corollary 5.8-1(Chebyshev's inequality)**

If a RV $X$ has a finite mean $\mu$ and finite nonzero variance $\sigma^2$, then for any $\varepsilon > 0$,

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

# Graphical Interpretation of Cheybshev's Inequality




This links to the interpretation of $\sigma^2$: a measure of dispersion of the values that $X$ can take with respect to its mean $\mu$.

# Example 1, page 222

Let $X$ be a RV with mean $\mu = 25$ and variance $\sigma^2 = 16$.

Question: Find a lower bound for $P(17 < X < 33)$ and an upper bound for $P(|X - 25| \geq 12)$.

# Example 1, page 222

Let $X$ be a RV with mean $\mu = 25$ and variance $\sigma^2 = 16$.

Question: Find a lower bound for $P(17 < X < 33)$ and an upper bound for $P(|X - 25| \geq 12)$.

Lower bound for:

$$P(17 < X < 33) = 1 - P(|X - \mu| \geq 2\sigma) \geq 1 - \frac{1}{4}$$

Upper bound for:

$$P(|X - 25| \geq 12) = P(|X - \mu| \geq 3\sigma) \leq \frac{1}{9}$$

Note: $X$ is arbitrary!

# Convergence in Probability

**Definition**

A sequence of RVs $Z_1, Z_2, \cdots$, is said to converge in probability to a RV $Z$, often denoted by, $Z_n \xrightarrow{p} Z$, if for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|Z_n - Z| \geq \varepsilon) = 0.$$

# Example 1

Assume that $Z_n$ has an exponential distribution with $\theta = 1/n$, $n = 1, 2, \cdots$. Then show that $Z_n \xrightarrow{p} 0$, i.e., the sequence of RVs $Z_1, Z_2, \cdots$, converges in probability to $Z = 0$.

# Example 1

Assume that $Z_n$ has an exponential distribution with $\theta = 1/n$, $n = 1, 2, \cdots$. Then show that $Z_n \xrightarrow{p} 0$, i.e., the sequence of RVs $Z_1, Z_2, \cdots$, converges in probability to $Z = 0$.

For any $\epsilon > 0$,

$$\begin{aligned}
\lim_{n\to\infty} P\big(|Z_n - 0| \geq \epsilon\big) &= \lim_{n\to\infty} P\big(Z_n \geq \epsilon\big) \\
&= \lim_{n\to\infty} e^{-n\epsilon} \\
&= 0,
\end{aligned}$$

where the first equation is true because $Z_n \geq 0$, and the second equation is obtained by using the fact that $Z_n$ has an exponential distribution with $\theta = 1/n$.

# Theorem [Law of Large Number]

**Theorem (Law of Large Numbers)**

Let $X_1, X_2, \cdots, X_n$ be a random sample of size $n$ drawn from a distribution with finite mean $\mu$ and finite nonzero variance, and let $\overline{X}$ be the sample mean. Then $\overline{X}$ converges in probability to $\mu$, i.e., for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\left|\overline{X} - \mu\right| \geq \varepsilon\right) = 0$$

# Proof of Law of Large Number

Note that

$$E(\overline{X}) = \mu, \quad Var(\overline{X}) = \frac{1}{n}\sigma^2$$

By the Chebyshev's inequality, i.e., Corollary 5.8-1, for every $\varepsilon > 0$.

$$P(|\overline{X} - \mu| \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\frac{\sigma^2}{n}$$

# Proof of Law of Large Number

Taking limits on both sides yield

$$0 \leq \lim_{n\to\infty} P(|\overline{X} - \mu| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{\varepsilon^2 n} = 0$$

$$\Rightarrow \lim_{n\to\infty} P(|\overline{X} - \mu| \geq \varepsilon) = 0$$

or equivalently $\quad \lim_{n\to\infty} P(|\overline{X} - \mu| < \varepsilon) = 1$