MAT1002 Lecture 14, Tuesday, Mar/14/2023

Outline

- Partial derivatives (14.3)
- Differentiability (14.3)

Partial Derivatives for Two-Variable Functions

Definition

Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. The partial derivative of f at (x_0, y_0) with respect to x, denoted by $\frac{\partial}{\partial x} f(x_0, y_0)$, is defined by

Also: $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$

$$\frac{\partial}{\partial x} f(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

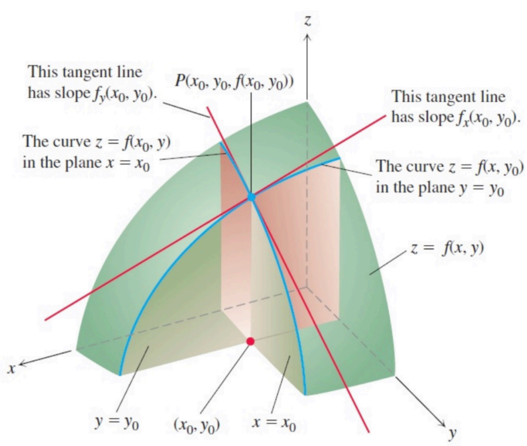
$\frac{d}{dx} g(x_0)$, where $g(x) := g_{y_0}(x) := f(x, y_0)$

and the partial derivative of f at (x_0, y_0) with respect to y, denoted by $\frac{\partial}{\partial y} f(x_0, y_0)$, is defined by

$$\frac{\partial}{\partial y} f(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Alternative notations for $\frac{\partial}{\partial x} f$ and $\frac{\partial}{\partial y} f$ include:

$$\frac{\partial f}{\partial x},\ f_x,\ D_x f \text{ and } D_1 f; \quad \frac{\partial f}{\partial y},\ f_y,\ D_y f \text{ and } D_2 f.$$



e.g. 1. Climbing mountain

2. Cobb-Douglas production function $P(x,y) = kx^{\alpha}y^{1-\alpha}$

x: labour time ; y: capital invested.

Partial Derivatives for n-Variable Functions

Definition
More generally, if $f : D \to \mathbb{R}$ is a function with $D \subseteq \mathbb{R}^n$, and

vector notation, $f(\vec{a}) := f(a_1, \dots, a_n)$

$$\vec{a} = \langle a_1, a_2, \dots, a_n \rangle \in D,$$

then the partial derivative of f at $\vec{a}$ with respect to the i-th coordinate (or i-th variable), denoted by $\frac{\partial}{\partial x_i} f(\vec{a})$, is defined to be the following limit:

$$\lim_{h \to 0} \frac{f(a_1, \dots, a_{i-1}, a_i + h, a_{i+1}, \dots, a_n) - f(a_1, \dots, a_i, \dots, a_n)}{h}.$$

$\lim_{h \to 0} \frac{f(\vec{a} + h\vec{e_i}) - f(\vec{a})}{h}$,

where $\vec{e_i}$ is the ith standard unit vector

$\langle 0, \dots, 0, 1, 0, \dots, 0 \rangle$

ith component

e.g. $\frac{\partial}{\partial z} f(a,b,c) = \lim_{h \to 0} \frac{f(a,b,c+h) - f(a,b,c)}{h}$.

To compute the partial derivative of f with respect to the i-th variable x_i, just treat all other variables as constants and differentiate f with respect to x_i.

Example

$f_x(x,y) = 2x + 3y$, $f_y(x,y) = 3x + 1$

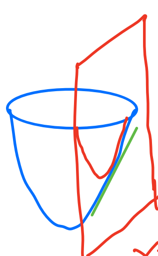
(a) For the function defined by $f(x, y) := x^2 + 3xy + y - 1$, find

$$\frac{\partial}{\partial x} f(4, -5) \quad \text{and} \quad \frac{\partial}{\partial y} f(4, -5).$$

$= -7$ $\qquad = 13$

(b) Find the slope of tangent to the parabola, obtained by intersecting the surface $z = \underbrace{x^2 + y^2}_{f(x,y)}$ with the plane $x = 1$, at the point $(1, 2, 5)$.

$$\text{Slope} = \frac{\partial}{\partial y} f(1,2) = 2y \Big|_{(x,y)=(1,2)} = 4.$$



If f is a function with two variables, then f_x and f_y are also functions with two variables, so they can have partial derivatives. These are the second-order partial derivatives of f, whose notations are given as follows:

$$\frac{\partial^2 f}{\partial x^2} := f_{xx} := (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),$$

$$\frac{\partial^2 f}{\partial y \partial x} := f_{xy} := (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$

$$\frac{\partial^2 f}{\partial x \partial y} := f_{yx} := (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial^2 f}{\partial y^2} := f_{yy} := (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

e.g. $\frac{\partial^3 f}{\partial x \partial y \partial z} = f_{zyx}$ ✓

These notations extend to functions with three or more variables.

Exercise

Compute the four second-order derivatives of the function f defined by

$$f(x, y) := x \cos y + ye^x.$$

In the exercise above, it turns out that $f_{xy} = f_{yx}$. This is not a coincidence.

Mixed Derivative Theorem (Clairaut's Theorem): If f_x, f_y, f_{xy}, and f_{yx} are defined on some open disk containing (a,b), and f_{xy} and f_{yx} are both continuous at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

This is easily satisfied by elementary functions as long as (a,b) is in the domain after differentiation

Proof [Optional] : See appendix A9 of book.

Example

Find $\frac{\partial^2 w}{\partial x \partial y}$ if $w = xy + e^y(y^2+1)^{-1}$.

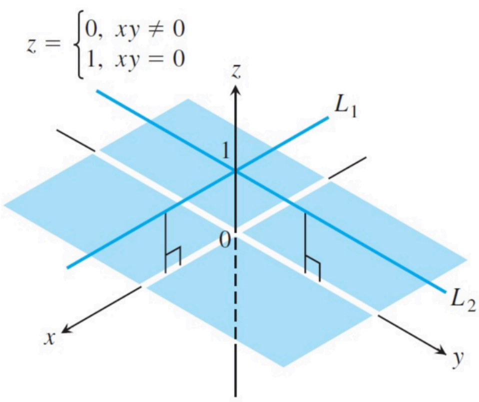
Ans : 1.

The Mixed Derivative Theorem extends to functions with more variables and higher order derivatives, as long as all these derivatives are defined in the open ball centered at the point and are cts:

$$f_{xxyz} = f_{xyzx} = f_{zxxy} = \cdots$$

Existence of Partial Derivatives Does Not Imply Continuity

Even if all the partial derivatives of a function f exists at a point, it does not mean that the function is continuous at that point. One example is shown in the following figure.



Another example is given below.

Example

The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) := \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

has both partial derivatives at $(0, 0)$. However, it is not continuous at $(0, 0)$, since we have shown that

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

does not exist.

It turns out that having all partial derivatives exist is not the same as being differentiable.

Differentiability

Consider a one-variable function $y = f(x)$, and suppose that f is differentiable at x_0. Then

Error of standard lin. approximation

$$\varepsilon := \varepsilon(\Delta x) = \frac{f(x_0+\Delta x) - \left(f(x_0) + f'(x_0)\Delta x\right)}{\Delta x} \to 0$$

as $\Delta x \to 0$. In other words, there exists $\varepsilon := \varepsilon(\Delta x)$ s.t.

$$\Delta y = f(x_0+\Delta x) - f(x_0) = f'(x_0)\Delta x + \varepsilon \Delta x, \quad \text{where}$$

$\varepsilon \to 0$ as $\Delta x \to 0$. (*)

(*) is essentially saying that f has a "tangent line" at $x = x_0$, which gives a "good" approximation for points "near" $x = x_0$.

Intuitively, we want a two-variable function f to be differentiable at (x_0, y_0) if the graph of f has a "tangent plane" at (x_0, y_0) that gives a "good" approximation for points (x, y) "near" (x_0, y_0). Formally, the definition can be extended from (*).

Def: A function $z=f(x,y)$ is differentiable at $(x_0,y_0)\in D$ if both f_x and f_y exist at (x_0,y_0), and

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$

satisfies

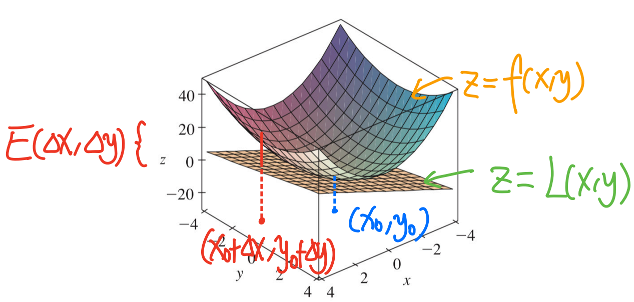
$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

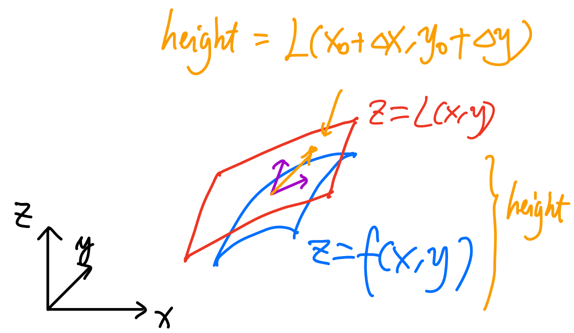
for some $\varepsilon_1 := \varepsilon_1(\Delta x,\Delta y)$ and $\varepsilon_2 := \varepsilon_2(\Delta x,\Delta y)$ such that $\varepsilon_1 \to 0$ and $\varepsilon_2\to 0$ as $(\Delta x,\Delta y)\to(0,0)$. We say f is differentiable on D if f is differentiable at all points in D.

Q: What does it have to do with tangent plane?

- Note that if $z=L(x,y)$ is the "tangent plane" to the graph of $z=f(x,y)$ at (x_0,y_0), then intuition suggests that

$$L(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0)+f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y,$$





So $\underline{f(x_0+\Delta x, y_0+\Delta y) - L(x_0+\Delta x, y_0+\Delta y)} =: E(\Delta x, \Delta y)$

$$= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y$$

$$= \Delta z - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y$$

Hence, in the def. of differentiability, it says that the error of the tangent plane approximation should equal

$$\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

for some ε_1 & ε_2 with ε_1 & $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

<u>Exercise (optional)</u> Show that if $E(\Delta x, \Delta y) := \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ where ε_1 & $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$, then

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{E(\Delta x, \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0 .$$

<u>e.g.</u> The function $f(x,y) = xy$ is differentiable at all (x_0, y_0), since

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y$$

$$= (x_0+\Delta x)(y_0+\Delta y) - x_0 y_0 - y_0 \Delta x - x_0 \Delta y$$

$$= \Delta x \Delta y = 0\Delta x + (\Delta x)\Delta y,$$ where $\Delta x \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

In practice, definition may be hard to use for determining differentiability. The following condition is more convenient to use.

<u>Theorem</u> If f_x and f_y are both continuous on an open disk centred at (a,b), then f is differentiable at (a,b).

<u>Proof</u> [Optional] : See appendix A9 of book.

By this theorem, for $f(x,y) = xy$, since $f_x = y$ and $f_y = x$ are both continuous, we see that f is differentiable.

If f is differentiable at (x_0, y_0), then as $(\Delta x, \Delta y) \to (0,0)$, $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y \to 0$, so $\Delta z \to 0$ by def, i.e.,

$$f(x_0 + \Delta x, y_0 + \Delta y) \to f(x_0, y_0).$$

In other words,

This can be formally proven using def ("δ-ε").

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(\Delta x, \Delta y)\to(0,0)} f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0),$$

So f is continuous at (x_0, y_0). This gives the "familiar" theorem.

Theorem If a two-variable function f is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

Differentiability of a three variable function $w = f(x, y, z)$ is defined similarly: f is differentiable at (x_0, y_0, z_0) if f_x, f_y and f_z all exist at (x_0, y_0, z_0), and

$$\Delta w = f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z$$
$$+ \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z$$

for some functions ε_1, ε_2 and ε_3 (of Δx, Δy and Δz) satisfying $\varepsilon_i \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$, $\forall i \in \{1, 2, 3\}$.

The theorems above involving differentiability also hold for functions with three (or even more) variables.