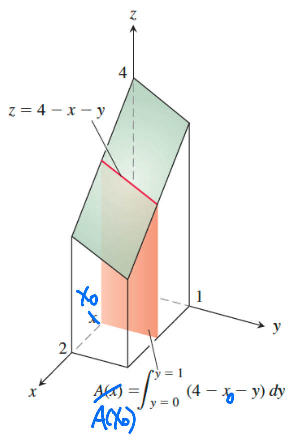
MAT1002 Lecture 19, Thursday, Mar/30/2023

Outline

- Double integrals on rectangles: computation (15.1)
- Double integrals on general bounded regions (15.2)
- Double integrals in polar coordinates (15.4)

Q: How to compute $\iint_R f(x,y)\,dA$?

e.g. Find the volume under the plane $z = 4 - x - y$ and over R: $0 \le x \le 2$, $0 \le y \le 1$ in the xy-plane



Approach 1: $\cdot$ Fix $x_0 \in [0,2]$

- Consider thin "plate" on the left, whose area is

$$A(x_0) = \int_0^1 (4 - x_0 - y)\,dy$$

- If the thin place has thickness Δx, then it has volume approximately equal to $A(x_0)\Delta x$.
- Using integrals, we have

$$V = \int_0^2 A(x)\,dx = \underbrace{\int_0^2 \int_0^1 (4 - x - y)\,dy\,dx}_{\text{iterated integral}}.$$

One variable integrals, a few times $\longleftarrow$ iterated integral

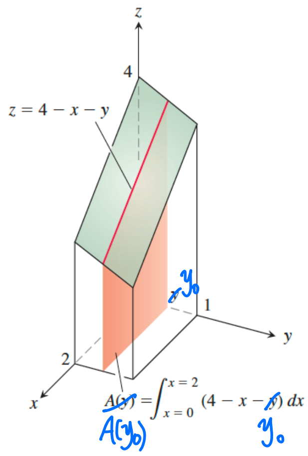
- $$V = \int_0^2 \left[\left(4y - xy - \frac{1}{2}y^2\right)\Big|_{y=0}^{1} \right] dx = \int_0^2 \left(4 - x - \frac{1}{2}\right) dx$$

$$= \left(\frac{7}{2}x - \frac{1}{2}x^2\right)\Big|_{x=0}^{2} = 7 - 2 = 5.$$

Approach 2:

- Fix $y_0 \in [0,1]$. Using a similar idea, we have

$$A(y_0) = \int_0^2 (4-x-y_0)\,dx$$



- $V = \int_0^1 A(y)\,dy = \int_0^1 \int_0^2 (4-x-y)\,dx\,dy$

$$= \int_0^1 \left[\left(4x - \frac{1}{2}x^2 - yx\right)\Big|_{x=0}^{2} \right] dy$$

$$= \int_0^1 (8-2-2y)\,dy = (6y - y^2)\Big|_{y=0}^{1} = 5$$

Fubini's theorem states that the pattern above persists as long as the integrand is continuous on R.

Theorem (Fubini's Theorem) (for rectangles)

If f is continuous on $R := [a,b] \times [c,d]$, then

iterated integrals

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Consequences: if f is cts on R:

- Double integrals can be computed using iterated integrals.
- Order does not matter.

Example

Evaluate $\iint_R y\sin(xy)\, dA$, where $R = [1,2] \times [0,\pi]$. Ans: 0

Note: If the integrand has the form $f(x,y) = g(x)h(y)$, then

$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$ (Why?)

e.g. (15.1, ex 13) $\int_1^4 \int_1^e \frac{\ln x}{xy}\, dx\, dy =$ ________ $= \ln 2$.

Double Integrals on General Bounded Regions

Let D be a closed and bounded region in $\mathbb{R}^2$, and suppose that f is a function defined on D. Let R be a rectangle enclosing D, and define a new function F on R by

$D \subseteq R$

$$F(x,y) := \begin{cases} f(x,y), & \text{if } (x,y) \in D; \\ 0, & \text{if } (x,y) \in R \setminus D. \end{cases}$$

We define the double integral of f over D by

$$\iint_D f(x,y)\, dA := \iint_R F(x,y)\, dA.$$

Again, if f is nonnegative and continuous on D, then we define the volume of the solid lying between the xy-plane and the graph of f to be $\iint_D f(x,y)\, dA$.

Type I and Type II Regions

Definition

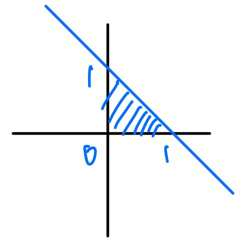
A plane region D is said to be of type I if it has the form

$$\{(x, y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\},$$

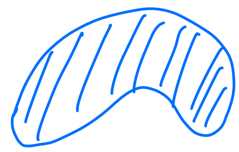
where g_1 and g_2 are continuous functions of x, and D is said to be of type II if it has the form

$$\{(x, y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\},$$

where h_1 and h_2 are continuous functions of y.



Type I or II?



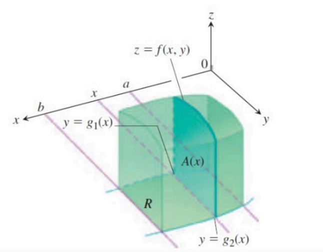
Theorem (Fubini's Theorem) (Type-I Regions)

If f is continuous on a type I region D, where

$$D = \{(x, y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\},$$

then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$



Theorem (Fubini's Theorem) (Type-II Regions)

If f is continuous on a type II region D, where

$$D = \{(x, y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\},$$

then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example

(a) Evaluate the double integral $\iint_D x + 2y \, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

(b) Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2) \, dy dx$.

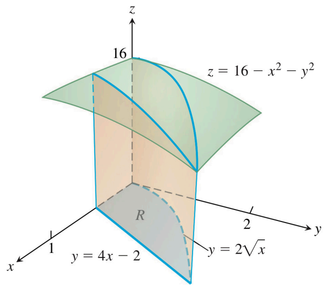
Ans:

(a) $32/15$

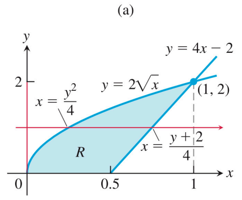
(b) $\frac{1}{2}(1-\cos 1)$

<u>Remark</u> Example (b) above suggests that one may simplify an integral by rewriting a type-I region as a type-II region (or vice versa).

EXAMPLE ~~4~~ 15.2.4 Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.



(a)



(b)

Write the double integral in two ways:

1. ______________

2. ______________

Properties of Double Integrals

Theorem

Assuming that the integrals exist, the following hold:

(a) $\iint_D (f(x,y) \pm g(x,y))\, dA = \iint_D f(x,y)\, dA \pm \iint_D g(x,y)\, dA.$

(b) $\iint_D cf(x,y)\, dA = c \iint_D f(x,y)\, dA.$

(c) *If* $f(x,y) \geq g(x,y)$ *for all* $(x,y) \in D$, *then*

$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA.$$

In particular, if $f(x,y) \geq 0$ $\forall (x,y) \in D$, then $\iint_D f(x,y)\, dA \geq 0$.

(d) *If* $D = D_1 \cup D_2$, *where* D_1 *and* D_2 *do not overlap except possibly on their boundaries, then*

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA.$$

Areas and Average Values

The idea of using double integrals to represent volumes motivates the following definitions.

Definition (usually closed and bounded)

The area $A(D)$ of a plane region D is defined by

$$A(D) := \iint_D dA := \iint_D 1\, dA.$$

The average value of a function f over D is

$$\frac{1}{A(D)} \iint_D f(x,y)\, dA.$$

Example

Find the area of the region D enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Ans: $9/2$.

Regions in Polar Coordinates

Consider integrating over the region

$$D = \{(x, y) : 1 \leq x^2 + y^2 \leq 4,\ y \geq 0\}.$$

It could be tedious to describe D as a union of type I/type II regions, but it can be described very naturally using polar coordinates; it corresponds to

$$R = \{(r, \theta) : 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}.$$

In general, suppose that D is a region on the xy-plane which can be described in polar coordinates by

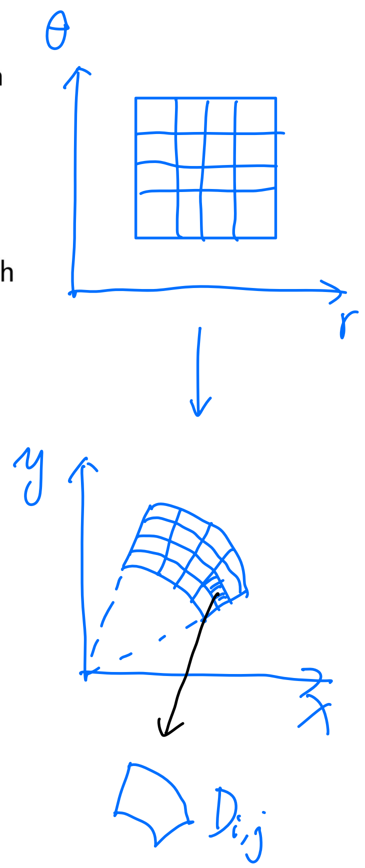
$$0 \leq a \leq r \leq b, \quad \alpha \leq \theta \leq \beta;$$

- Partition this into polar subrectangles R_{ij} of equal size, which corresponds to a region D_{ij} on the xy-plane.
- Let ΔA_{ij} be the area of D_{ij}.
- Pick a point (r_i^*, θ_j^*) in R_{ij}; we may pick its center, i.e.,

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \quad \text{and} \quad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j).$$

- Computing area shows that $\Delta A_{ij} = r_i^* \Delta r \Delta \theta$, which yields

$$\begin{aligned} &\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*) r_i^* \Delta r \Delta \theta. \end{aligned}$$



Theorem
If f is continuous on a region D which has a polar description

$$0 \le a \le r \le b, \quad \alpha \le \theta \le \beta,$$

where $0 \le \beta - \alpha \le 2\pi$, *then*

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

e.g. Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx.$$

Ans: $\pi/8$.

The following is a more general version of the theorem above.

Theorem
If f is continuous on a region D which has a polar description

$$\alpha \le \theta \le \beta, \quad 0 \le h_1(\theta) \le r \le h_2(\theta),$$

where $0 \le \beta - \alpha \le 2\pi$, *then*

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

Moral of story: dA in xy-coordinates becomes $r\,dr\,d\theta$ in polar coordinates.

e.g. Use a double integral to find the area enclosed by one loop of the four-leaved rose given by $r = \cos 2\theta$.

Ans: $\pi/8$.

Example
Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Ans: $3\pi/2$.