MAT1002 Lecture 3, Thursday, Jan/12/2023

Outline

- Comparison tests (10.4)
- Absolute convergence (10.5)
- Ratio test (10.5)
- Root test (10.5)

Comparison Tests (Series with eventually nonnegative terms)

Theorem ((Direct) Comparison Test)

Let $\sum a_n$ and $\sum b_n$ be series. Suppose that there exists an integer N such that $a_n \geq b_n \geq 0$ for all n satisfying $n \geq N$.

(i) *If $\sum a_n$ converges, then $\sum b_n$ converges.*

(ii) *If $\sum b_n$ diverges, then $\sum a_n$ diverges.*

Proof: (i) Suppose $\sum_{n=N}^{\infty} a_n = L$. For $k \geq N$, let

$$S_k := \sum_{n=N}^{k} b_n = b_N + \dots + b_k.$$

Then for all $k \geq N$,

$$0 \leq S_k = \sum_{n=N}^{k} b_n \leq \sum_{n=N}^{k} a_n \leq \sum_{n=N}^{\infty} a_n = L.$$

Hence $\{S_k\}_{k=N}^{\infty}$ is bounded, implying that $\sum_{n=N}^{\infty} b_n$ converges by the Monotonic Sequence Theorem.

(ii) is equivalent to (i). $\square$

e.g. 1 Determine the convergence of $\sum_{n=8}^{\infty} \frac{1}{3^n + n^{1/3}}$.

Sol. $\forall n \geq 8$, $0 < \frac{1}{3^n + n^{1/3}} < \left(\frac{1}{3}\right)^n$.

Since $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$ converges (geometric series with $|r| < 1$),

Series converges by (direct) comparison test.

e.g.2 What about the following series?

(a) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$; (b) $\sum_{n=0}^{\infty} \frac{1}{n!}$. ($\Leftarrow$ e.g. 10.4.1 (b))

Theorem (Limit Comparison Test)

Let $\sum a_n$ and $\sum b_n$ be series. Suppose that there exists an integer N such that $a_n > 0$ and $b_n > 0$ for all n satisfying $n \geq N$.

(i) *If $\lim_{n\to\infty} (a_n/b_n) = L > 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.* ($L \in \mathbb{R}$)

(ii) *If $\lim_{n\to\infty} (a_n/b_n) = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.*

(iii) *If $\lim_{n\to\infty} (a_n/b_n) = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

Proof: See Chapter 10.4 of the book. $\square$

e.g.3 Determine the convergence of the following series.

(a) $\sum_{n=4}^{\infty} \frac{1+n\ln n}{n^2+5}$ (b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^{5/4}}$ (c) $\sum_{n=48}^{\infty} \sin\frac{1}{n}$

(e.g. 10.4.2 (c))

Sol: (a) See the book.

(b) • Compare with $\frac{1}{n^{4.5/4}} = \frac{1}{n^{9/8}}$: $\frac{\ln n}{n^{5/4}} \Big/ \frac{1}{n^{9/8}} = \frac{\ln n}{n^{1/8}}$

• $\lim_{n\to\infty} \frac{\ln n}{n^{1/8}} = \lim_{x\to\infty} \frac{\ln x}{x^{1/8}} \overset{\text{L'Hô}}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{8}x^{-7/8}} = \lim_{x\to\infty} 8x^{-\frac{1}{8}} = 0.$

• Since $\sum_{n=1}^{\infty} \frac{1}{n^{9/8}}$ Converges (p-series with $p>1$), $\sum_n \frac{\ln n}{n^{5/4}}$ Converges.

(c) Divergent (details to be shown in class).

Now we shift our attention to series with both positive and negative terms, for which integral test and comparison tests may not work.

Absolute Convergence

Definition
A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

e.g. 4 $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely. Indeed,

$$0 < \left|\frac{\sin n}{n^2}\right| < \frac{1}{n^2}, \quad \forall n \geq 1 \ (n \in \mathbb{Z}),$$

So $\sum_n \left|\frac{\sin n}{n^2}\right|$ converges by comparison test.

Theorem (Absolute Convergence Test)
If $\sum |a_n|$ converges, then $\sum a_n$ converges. In other words, if $\sum a_n$ converges absolutely, then it converges.

Proof: Since $a_n + |a_n| = \begin{cases} 0, & \text{if } a_n < 0 \\ 2|a_n|, & \text{if } a_n \geq 0 \end{cases}$,

$$0 \leq a_n + |a_n| \leq 2|a_n|, \quad \forall n.$$

If $\sum |a_n|$ converges, then $\sum (a_n + |a_n|)$ converges by comparison test.

This means $\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ also converges. $\square$

<u>Remark</u>: The converse is false. As we will see, the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges, although $\sum_n \left| (-1)^{n+1} \frac{1}{n} \right| = \sum_n \frac{1}{n}$ diverges.

The ratio test and root test can be used to test for absolute convergence.

> **Theorem (Ratio Test)**
>
> *Let $\sum a_n$ be a series, and suppose that*
>
> $$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \in \mathbb{R} \cup \{\infty\}.$$
>
> *Then:*
>
> (i) *if $L < 1$, the series converges absolutely.*
>
> (ii) *if $L > 1$ or $L = \infty$, the series diverges.*
>
> (iii) *if $L = 1$, the test is inconclusive.*

<u>e.g.s</u> (a) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Since $\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)}{(n+1)} \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1} \right)^n = \frac{1}{(1+\frac{1}{n})^n} \xrightarrow{n \to \infty} \frac{1}{e} < 1,$

$\sum_n \frac{n!}{n^n}$ converges (absolutely).

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

Since $\frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2} \longrightarrow 4$ as $n\to\infty$,

$\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ diverges.

Proof of Ratio Test

(ii) If $L>1$ or $L=\infty$, then $\exists N$ s.t. $\forall n \geq N$,

$\left|\frac{a_{n+1}}{a_n}\right| > 1$, i.e., $|a_{n+1}| > |a_n|$, so

$$|a_N| < |a_{N+1}| < |a_{N+2}| < |a_{N+3}| < \cdots.$$

This means that $\lim_{n\to\infty} a_n \neq 0$, so $\sum a_n$ diverges.

(i) If $L<1$, let $r \in (L,1)$. Then $\exists N$ s.t. $\forall n \geq N$,

$\left|\frac{a_{n+1}}{a_n}\right| < r$. This means

Comparison with a geometric series

$$|a_{N+1}| < r|a_N|$$
$$|a_{N+2}| < r|a_{N+1}| < r(r|a_N|) = r^2|a_N|$$
$$\vdots$$
$$|a_{N+k}| < r|a_{N+k-1}| < r(r|a_{N+k-2}|) < \cdots < r^k|a_N|$$

Since $0 < |a_{N+k}| \leq r^k|a_N| \quad \forall k \geq 0$

and $\sum_{k=0}^{\infty} |a_N| r^k$ converges $(0 \le L < r < 1)$, we have

as a geometric series

$\sum_{k=0}^{\infty} |a_{N+k}| = \sum_{n=N}^{\infty} |a_n|$ converges.

so $\sum a_n$ cvgs absolutely.

(iii) For a p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $(p>0)$ $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^p}{(n+1)^p} \to 1^p = 1$ as $n \to \infty$. However, it may converge (say $p=2$) or diverge (say $p=1$). $\square$

Theorem (Root Test)

Let $\sum a_n$ be a series, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \in \mathbb{R} \cup \{\infty\}.$$

Then:

(i) *if $L < 1$, the series converges absolutely.*

(ii) *if $L > 1$ or $L = \infty$, the series diverges.*

(iii) *if $L = 1$, the test is inconclusive.*

e.g. 6 (a) $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} n/2^n, & \text{if } n \text{ is odd.} \\ 1/2^n, & \text{if } n \text{ is even.} \end{cases}$

- Note that it is not convenient to use the ratio test.
- Since

$$\sqrt[n]{|a_n|} = \begin{cases} \frac{\sqrt[n]{n}}{2}, & \text{if } n \text{ is odd} \\ \frac{1}{2}, & \text{if } n \text{ is even} \end{cases},$$

$\frac{1}{2} \le \sqrt[n]{|a_n|} \le \frac{\sqrt[n]{n}}{2}$, $\forall n$. Since $\frac{\sqrt[n]{n}}{2} \to \frac{1}{2}$, by Sandwich theorem,

$\sqrt[n]{|a_n|} \to \frac{1}{2} < 1$. By root test, $\sum a_n$ cvgs absolutely.

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

$\sqrt[n]{|a_n|} = \frac{2}{(\sqrt[n]{n})^3} \to 2$ as $n \to \infty$, so series diverges.

<u>Proof</u>: (ii) If $L > 1$ or $L = \infty$, then $\exists N$ s.t. $\forall n \geq N$,

$\sqrt[n]{|a_n|} > 1 \Rightarrow |a_n| > 1^n = 1 \Rightarrow \lim_{n\to\infty} a_n \neq 0$.

So $\sum_n a_n$ diverges.

(i) If $L < 1$, let $r \in (L, 1)$. Then $\exists N$ s.t. $\forall n \geq N$,

$\sqrt[n]{|a_n|} < r \Rightarrow |a_n| < r^n$. Since $0 \leq |a_n| < r^n$, $\forall n \geq N$,

and $\sum_{n=N}^{\infty} r^n$ converges, by comparison test, $\sum |a_n|$ converges.

(iii) Again, for any p-series $\sum_n \frac{1}{n^p}$, $(|a_n|)^{\frac{1}{n}} = \frac{1}{(n^{\frac{1}{n}})^p} \to 1$

as $n \to \infty$. But the p-series may converge ($p=2$) or diverge ($p=1$).

$\square$