STA2001 Probability and Statistics (I)

Lecture 2

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Review

- Random experiment, Sample space, Event and An event has occurred
- Set Theory
- $P(A) = \lim_{n\to\infty} \frac{\mathcal{N}(A)}{n}$
- Probability function is a function that assigns $P(A)$ to an event A, $A \subseteq S$

  1. $P(A) \geq 0$
  2. $P(S) = 1$
  3. $A_1, A_2, \cdots$ are countable and mutually exclusive events

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

Section 1.2 Method of Enumeration (Permutation and Combination)

Motivation

Why enumeration?

For some cases, to define and calculate $P(A)$ can be converted to count the number of outcomes in $A \rightarrow$ counting techniques.

Assumption 1: S contains m possible outcomes

$$e_k, \quad k = 1, 2, \cdots, m, \quad i.e., \quad S = \{e_1, e_2, \cdots, e_m\}.$$

Assumption 2: The m outcomes are "equally likely"

$$P(\{e_k\}) = \frac{1}{m}, \quad k = 1, \cdots, m.$$

Extension of rolling die example.
S={1,2,3,4,5,6}, P({k})=$\frac{1}{6}$, k=1,$\cdots$,6.

Motivation

Then

$$P(A) = \frac{N(A)}{N(S)},$$

where $N(X)$ is the number of outcomes in $X \subseteq S$.

- It can be verified $P(A)$ is a well-defined probability function that satisfies the probability axioms.
- To calculate $P(A) \Leftrightarrow$ to count the number of elements in A and in S under Assumptions 1& 2 $\Rightarrow$ links to the counting techniques, e.g., the method of enumeration.

Counting Techniques

Problem

To develop techniques for counting the number of outcomes associated with the events of random experiments:

- permutation
- combination
- distinguishable permutation

Assumption: a random experiment can be done by a sequential implementation of two or more sub-experiments.

Multiplication Principle

Problem

Consider that an experiment E can be done by a sequential implementation of 2 sub-experiments E_1 and E_2.

$$\rightarrow \boxed{\text{Experiment } E_1} \rightarrow n_1 \text{ outcomes}$$

$$\rightarrow \boxed{\text{Experiment } E_2} \rightarrow n_2 \text{ outcomes}$$

$$\rightarrow \underbrace{\boxed{\text{Experiment } E_1} \rightarrow \boxed{\text{Experiment } E_2}}_{\text{sequential implementation}} \rightarrow n_1 n_2 \text{ possible outcomes}$$

Example 1

E: Test drugs A, B and placebo on rats.

E_1: select a rat from the cage which is either male or female,

$n_1 = 2$

E_2: for each selected rat either drug A, drug B or placebo, $n_2 = 3$

In total there are $n_1 \cdot n_2 = 2 \times 3 = 6$ outcomes.

Then the outcomes for the experiment are denoted by

ordered pair: $\begin{matrix} (F,A), & (F,B), & (F,P) \\ (M,A), & (M,B), & (M,P) \end{matrix}$ in total $6 = 2 \times 3$

Permutation of n objects

Problem

Consider that n positions are to be filled with n different objects.

The task can be handled by multiplication principle.

$$\rightarrow \quad \boxed{\text{position 1}} \quad \rightarrow \quad \boxed{\text{pos.2}} \quad \rightarrow \quad \cdots \quad \rightarrow \quad \boxed{\text{pos.n}}$$

$$n \qquad\qquad n-1 \qquad\qquad\qquad 1$$

in total $n! = n(n-1)\cdots 2 \cdot 1$ arrangements $(0! = 1)$

Definition: each of the $n!$ arrangements of n different objects is called **a permutation of n objects**

Permutation of n objects taken r at a time

Problem

Consider that only r positions are to be filled with objects selected from n different objects.

By multiplication principle

$$\rightarrow \boxed{\text{pos.1}} \rightarrow \boxed{\text{pos.2}} \rightarrow \cdots \rightarrow \boxed{\text{pos.r}} \rightarrow$$

$$n \qquad\qquad n-1 \qquad\qquad\qquad n-r+1$$

in total ${}_nP_r = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$ arrangements.

Definition: Each of the ${}_nP_r$ arrangements is called **a permutation of n objects taken r at a time**.

Example 2

The number of possible 4-English letter words with different letters

$$_{26}P_4 = 26 \times 25 \times 24 \times 23 = \frac{26!}{22!}$$

Ordered Sample and Sampling

Definition[Ordered sample of size r]

If r objects are selected from a set of n objects and if the order of selection is noted, then the selected set of r objects is called **ordered sample of size r.**

Definition[Sampling with replacement]

Occurs when an object is selected and then replaced before the next object is selected (n^r).

Definition[Sampling without replacement]

Occurs when an object is not replaced after it has been selected (${}_nP_r$).

Example 2 (Revisited)

The number of 4-letter words with different letters

$$_{26}P_4 \longrightarrow \text{sampling without replacement}$$

The number of 4-letter words which can have the same letters

$$26^4 \longrightarrow \text{sampling with replacement}$$

Combination of n objects taken r at a time

Motivation

Sometimes, the order of selection is not important and we are only interested in the number of subsets of size r, i.e., **unordered sample of size** r, taken from a set of n different objects.

Instead to solve the problem in a direct way, we solve the problem in an indirect way and **we consider permutation of n objects taken r at a time by multiplication principle**.

Combination of n objects taken r at a time

1. $\rightarrow \boxed{\text{pos.1}} \rightarrow \boxed{\text{pos.2}} \rightarrow \cdots \rightarrow \boxed{\text{pos.r}} \rightarrow {}_nP_r$

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2. $\rightarrow \underset{X}{\boxed{\text{unordered subset of size r}}} \rightarrow \underset{r!}{\boxed{\text{permutation of r objects}}}$

$$\Rightarrow X \times r! =_n P_r \Rightarrow X = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!} \triangleq {}_nC_r$$
$$= \binom{n}{r} = \binom{n}{n-r} = {}_nC_{n-r}$$

Definition: Each of the ${}_nC_r$ unordered subsets is called **a combination of n objects taken r at a time**.

Example 3

$$ {}_5P_2 = 5 \times 4. $$

Alternatively,

$$ \binom{5}{2} \times 2! = \frac{5!}{3!2!} \times 2! = 5 \times 4 $$

Example 4

The number of possible 5-card hands drawn from a deck of 52 playing cards is

$$_{52}C_5 = \binom{52}{5}$$

- The number $\binom{n}{r}$ is often called binomial coefficients, because in binomial expansion

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} = (a+b)(a+b)\cdots(a+b)$$

Distinguishable Permutation of objects of two types

Motivation

Consider permutation of n objects of two types: r of one type and $(n-r)$ of the other type.

Instead to solve the problem in a direct way, we solve the problem in an indirect way and **we consider permutation of n different objects by multiplication principle**.

Distinguishable Permutation

1. $\rightarrow \boxed{\text{pos.1}} \rightarrow \boxed{\text{pos.2}} \rightarrow \cdots \rightarrow \boxed{\text{pos.n}} \rightarrow \quad n!$

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2. $\rightarrow \quad \boxed{\text{permute } n \text{ objects of two types}} \quad \rightarrow$

X

$\boxed{\text{permute } r \text{ objects of one type}} \quad \rightarrow$

$r!$

$\boxed{\text{permute } (n-r) \text{ objects of the other type}}$

$(n-r)!$

$$n! = X \cdot r! \cdot (n-r)! \Rightarrow X =_n C_r = \binom{n}{r}$$

Definition: Each of the ${}_nC_r$ permutations of n objects of two types

with r of one type and $(n-r)$ of the other type.

Example

Question

Flip a coin 10 times and the sequence of heads and tails is observed. What is the number of possible 10 tuples with 4 heads and 6 tails?

Example

Question

Flip a coin 10 times and the sequence of heads and tails is observed. What is the number of possible 10 tuples with 4 heads and 6 tails?

The number of possible 10 tuples with 4 heads and 6 tails is $\binom{10}{4}$

because it is a distinguishable permutation of 10 objects of two

types: 4 of one type and 6 of the other type.

Distinguishable permutation of objects of m types

Consider a set of n objects of m types:

n_1 of one type, n_2 of one type, $\cdots$, n_m of one type, where

$n_1 + n_2 + \cdots + n_m = n$

What's the number of distinguishable permutation of these n objects?

Distinguishable permutation of objects of m types

1. permutation of n different objects $n!$

   $\rightarrow \boxed{\text{pos.1}} \rightarrow \boxed{\text{pos.2}} \rightarrow \cdots \rightarrow \boxed{\text{pos.n}} \rightarrow \qquad \text{n!}$

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2. $$\begin{array}{cccc} \rightarrow & \boxed{\text{permute } n \text{ objects of } m \text{ types}} & \rightarrow \\ & X & \\ & \boxed{\text{permute } n_1 \text{ objects of type 1}} & \rightarrow \\ & n_1! & \\ & \vdots & \\ & \boxed{\text{permute } n_m \text{ objects of type } m} & \rightarrow \\ & n_m! & \end{array}$$

   $$n! = X \cdot n_1! \cdots n_m! \Rightarrow X = \frac{n!}{n_1! \cdots n_m!}$$