1. Let $f(x,y) = 4/3$, $0 < x < 1$, $x^3 < y < 1$, zero elsewhere.

   (a) (2 points) Sketch the region where $f(x,y) > 0$.

   (b) (2 points) Find $P(X > Y)$.

   (c) (3 points) Find $E(X)$ and $E(Y)$.

   (d) (3 points) Find $\mathrm{Cov}(X,Y)$.

Solution: Please refer to Exercise (4.4-7).

(a) See Assignment

(b)

$$\begin{aligned} P(X > Y) &= 1 - P(X \le Y) \\ &= 1 - \int_0^1 \int_x^1 \frac{4}{3}\, dy\, dx \\ &= 1 - \int_0^1 \frac{4}{3}(1-x)\, dx \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

(c)

$$\begin{aligned} p_X(x) &= \int_{x^3}^1 \frac{4}{3} dy = \frac{4}{3}(1-x^3),\ 0 < x < 1 \\ p_Y(y) &= \int_0^{y^{\frac{1}{3}}} \frac{4}{3} dx = \frac{4}{3} y^{\frac{1}{3}},\ 0 < y < 1 \\ E(X) &= \int_0^1 x \cdot \frac{4}{3}(1-x^3) dx = \frac{2}{5} \\ E(Y) &= \int_0^1 y \cdot \frac{4}{3} y^{\frac{1}{3}} dy = \frac{4}{7}. \end{aligned}$$

(d)

$$\begin{aligned} E(XY) &= \int_0^1 \int_{x^3}^1 \frac{4}{3} xy dy dx = \frac{1}{4}, \\ Cov(X,Y) &= E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{2}{5} \cdot \frac{4}{7} = \frac{3}{140}. \end{aligned}$$

2. Die A has 4 red and 2 white faces, whereas die B has 1 red and 5 white faces. A fair coin is flipped only once at the beginning of the game. If it lands on heads, the game continues with die A; if it lands on tails, then die B is to be used.

1. Let $f(x, y) = 4/3$, $0 < x < 1$, $x^3 < y < 1$, zero elsewhere.

   (a) (2 points) Sketch the region where $f(x, y) > 0$.

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Solution: Please refer to Exercise (4.4-7).

(a) See Assignment

(b)

$$\begin{aligned} P(X > Y) &= 1 - P(X \leq Y) \\ &= 1 - \int_0^1 \int_x^1 \frac{4}{3}\, dy\, dx \\ &= 1 - \int_0^1 \frac{4}{3}(1 - x)\, dx \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

(c)

$$\begin{aligned} p_X(x) &= \int_{x^3}^1 \frac{4}{3} dy = \frac{4}{3}(1 - x^3),\ 0 < x < 1 \\ p_Y(y) &= \int_0^{y^{\frac{1}{3}}} \frac{4}{3} dx = \frac{4}{3} y^{\frac{1}{3}},\ 0 < y < 1 \\ E(X) &= \int_0^1 x \cdot \frac{4}{3}(1 - x^3) dx = \frac{2}{5} \\ E(Y) &= \int_0^1 y \cdot \frac{4}{3} y^{\frac{1}{3}} dy = \frac{4}{7}. \end{aligned}$$

(d)

$$\begin{aligned} E(XY) &= \int_0^1 \int_{x^3}^1 \frac{4}{3} xy dy dx = \frac{1}{4}, \\ Cov(X, Y) &= E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{2}{5} \cdot \frac{4}{7} = \frac{3}{140}. \end{aligned}$$

2. Die A has 4 red and 2 white faces, whereas die B has 1 red and 5 white faces. A fair coin is flipped only once at the beginning of the game. If it lands on heads, the game continues with die A; if it lands on tails, then die B is to be used.

(a) (3 points) Find the probability that the first throw of the die results in red.

(b) (3 points) If the first two throws result in red, what is the probability that die A is used?

(c) (4 points) If the first two throws result in red, what is the probability of red at the third throw?

Solution: Solution:
Please refer to Assignment 2 Let A denote the event that die A is used. Let R_i denote the event that the i-th roll results in red.

(a) We know that $P(R_1 \mid A) = 4/6, P(R_1 \mid A') = 1/6, P(A) = 1/2$, hence

$$\begin{aligned} P(R_1) &= P(R_1 \mid A) P(A) + P\left(R_1 \mid A'\right) P\left(A'\right) \\ &= \frac{4}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2} = \frac{5}{12} \end{aligned}$$

(b) By Baye's Law we have

$$\begin{aligned} P(A \mid R_1 \cap R_2) &= \frac{P(R_1 \cap R_2 \mid A) P(A)}{P(R_1 \cap R_2)} \\ &= \frac{P(R_1 \cap R_2 \mid A) P(A)}{P(R_1 \cap R_2 \mid A) P(A) + P(R_1 \cap R_2 \mid A') P(A')} \end{aligned}$$

R_1 and R_2 are conditionally independent, conditioning on A, hence

$$P(R_1 \cap R_2 \mid A) = P(R_1 \mid A) P(R_2 \mid A) = \left(\frac{4}{6}\right)^2 = \frac{4}{9}$$

and

$$P\left(R_1 \cap R_2 \mid A'\right) = P\left(R_1 \mid A'\right) P\left(R_2 \mid A'\right) = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$$

Hence

$$P(A \mid R_1 \cap R_2) = \frac{(4/9) \cdot (1/2)}{(4/9) \cdot (1/2) + (1/36) \cdot (1/2)} = \frac{16}{17}$$

(c) We have

$$\begin{aligned} P(R_3 \mid R_1 R_2) &= \frac{P(R_1 R_2 R_3)}{P(R_1 R_2)} \\ &= \frac{P(R_1 R_2 R_3 \mid A) P(A) + P\left(R_1 R_2 R_3 \mid A'\right) P\left(A'\right)}{P(R_1 R_2 \mid A) P(A) + P(R_1 R_2 \mid A') P(A')} \end{aligned}$$

R_1, R_2, R_3 are conditionally mutually independent, conditioning on A, hence

$$
\begin{aligned}
&P(R_3 \mid R_1R_2) \\
&= \frac{P(R_1 \mid A)\,P(R_2 \mid A)\,P(R_3 \mid A)\,P(A) + P(R_1 \mid A')\,P(R_2 \mid A')\,P(R_3 \mid A')\,P(A')}{P(R_1 \mid A)\,P(R_2 \mid A)\,P(A) + P(R_1 \mid A')\,P(R_2 \mid A')\,P(A')} \\
&= \frac{(4/6)^3 \cdot (1/2) + (1/6)^3 \cdot (1/2)}{(4/6)^2 \cdot (1/2) + (1/6)^2 \cdot (1/2)} \\
&= \frac{65}{102}
\end{aligned}
$$

Alternative Answer:

By Law of total probability (conditional version) we have

$$P(R_3 \mid R_1R_2) = P(R_3 \mid R_1R_2A)\,P(A \mid R_1R_2) + P\left(R_3 \mid R_1R_2A'\right)P\left(A' \mid R_1R_2\right)$$

Condition on A, R_1, R_2, R_3 are independent, hence $P(R_3 \mid R_1R_2A) = P(R_3 \mid A)$. Similarly, $P\left(R_3 \mid R_1R_2A'\right) = P\left(R_3 \mid A'\right)$

It is computed in the next question that

$$P(A \mid R_1R_2) = \frac{16}{17}$$

Thus

$$
\begin{aligned}
P(R_3 \mid R_1R_2) &= P(R_3 \mid A)\,P(A \mid R_1R_2) + P\left(R_3 \mid A'\right)P\left(A' \mid R_1R_2\right) \\
&= \frac{4}{6} \cdot \frac{16}{17} + \frac{1}{6} \cdot \left(1 - \frac{16}{17}\right) = \frac{65}{102}
\end{aligned}
$$

3. A car dealer sells X cars each day and always tries to sell an extended warranty on each of these cars. (In our opinion, most of these warranties are not good deals.) Let Y be the number of extended warranties sold; then $Y \leq X$. The joint pmf of X and Y is given by

$$f(x,y) = c(x+1)(4-x)(y+1)(3-y), \quad x = 0,1,2,3, y = 0,1,2, \text{ with } \ y \leq x.$$

(a) (1 points) Find the value of c. $\sum_{(x,y)} f(x,y) = \sum_{y \leq x} f(x,y) = f(3,0) + f(3,1)$ $f(3,2) + f(3,3)$ $+ f(2,0) + f(2,1)$ $+ f(2,2)$ $+ f(1,0) + f(1,1)$ $+ f(0,0)$

(b) (2 points) Record the marginal pmfs $f_X(x)$ and $f_Y(y)$ in the margins.

(c) (1 points) Are X and Y independent?

(d) (2 points) Compute μ_X and σ_X^2.

(e) (2 points) Compute μ_Y and σ_Y^2.

(f) (1 points) Compute $\mathrm{Cov}(X, Y)$.

(g) (1 points) Determine ρ, the correlation coefficient.

Solution: Please refer to Exercise 4.2-9.

(a) (1 points)

$$\sum\sum_{x,y\in S} f(x,y) = 1$$

for $y \leq x, x = 0, 1, 2, 3, y = 0, 1, 2,,$ thus
$f(0,0)+f(1,1)+f(1,0)+f(2,0)+f(2,1)+f(2,2)+f(3,0)+f(3,1)+f(3,2) = 1.$
thus $c = \frac{1}{154}$

(b) (2 points)

$$f_X(x) = \begin{cases} \frac{6}{77}, x = 0 \\ \frac{3}{11}, x = 1 \\ \frac{30}{77}, x = 2 \\ \frac{20}{77}, x = 3 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{30}{77}, y = 0 \\ \frac{32}{11}, y = 1 \\ \frac{15}{77}, y = 2 \end{cases}$$

(c) (1 points) Then space is not rectangular which means that X and Y are not independent.

(d) (2 points)

$$\mu_X = \sum_{x=0}^{3} xf(x) = 0 \times \frac{6}{77} + 1 \times \frac{3}{11} + 2 \times \frac{30}{77} + 3 \times \frac{20}{77} = \frac{141}{77}$$

$$\sigma_X^2 = \mathbb{E}(x^2) - [\mathbb{E}(x)]^2 = \frac{4836}{5929}$$

(e) (2 points)

$$\mu_Y = \sum_{y=0}^{3} yf(y) = 1 \times \frac{32}{77} + 2 \times \frac{15}{77} = \frac{62}{77}$$

$$\sigma_Y^2 = \mathbb{E}(y^2) - [\mathbb{E}(y)]^2 = \frac{3240}{5929}$$

(f) (1 points)

$$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mu_X\mu_Y = \frac{1422}{5929}$$

(g) (1 points)

$$\rho = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\sigma_X^2}\sqrt{\sigma_X^2}} = \frac{79}{24180}\sqrt{12090}$$

4. (a) (6 points) Let X and Y has the joint pdf

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}q(x,y)\right\},$$

$$q(x,y) = \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right].$$

Here, $|\rho| < 1$, $\mu_X, \mu_Y \in \mathbb{R}$, and $\sigma_X^2, \sigma_Y^2 > 0$. Prove that ρ is the correlation coefficient of X and Y.

(b) (4 points) Let (X,Y) be the height in centimeters of couples in a population, where X represents the height of the female and Y represents the height of the male. Suppose that (X,Y) follow a bivariate distribution with parameters $\mu_X = 165, \mu_Y = 172$, $\sigma_X = 3$, $\sigma_Y = 4$ and $\rho = 0.2$. The male of a couple is observed to be 180 cm tall.

(i) (2 points) Find the mean and variance of $X|Y = 180$.

(ii) (2 points) What is the probability that female of the couple is less than 162 cm in height given that the male of a couple is observed to be 180 cm tall?

Solution:

(a)

$$\begin{aligned}
\mathrm{E}(XY) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf(x,y)\mathrm{d}y\,\mathrm{d}x = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf_X(x)h(y\mid x)\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} xf_X(x)\left(\int_{-\infty}^{+\infty} yh(y\mid x)\mathrm{d}y\right)\mathrm{d}x
\end{aligned}$$

Since $h(y\mid x) = N\left(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x-\mu_X), (1-\rho^2)\sigma_Y^2\right)$,

$$\int_{-\infty}^{+\infty} yh(y\mid x)\mathrm{d}y = \mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x-\mu_X).$$

Then we have

$$\begin{aligned}
\mathrm{E}(XY) &= \int_{-\infty}^{+\infty} xf_X(x)\left(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x-\mu_X)\right)\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \mu_Y xf_X(x)\mathrm{d}x + \int_{-\infty}^{+\infty} \frac{\sigma_Y}{\sigma_X}\rho x(x-\mu_X)f_X(x)\mathrm{d}x
\end{aligned}$$

Note that

$$\int_{-\infty}^{+\infty} x f_X(x)\mu_Y \, \mathrm{d}x = \mu_X \mu_Y$$

and

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{\sigma_Y}{\sigma_X}\rho x \left(x-\mu_X\right) f_X(x)\mathrm{d}x \\
&\quad= \int_{-\infty}^{+\infty} \frac{\sigma_Y}{\sigma_X}\rho \left(x-\mu_X\right)^2 f_X(x)\mathrm{d}x + \int_{-\infty}^{+\infty} \frac{\sigma_Y}{\sigma_X}\rho\mu_X \left(x-\mu_X\right) f_X(x)\mathrm{d}x \\
&\quad= \frac{\sigma_Y}{\sigma_X}\rho\sigma_X^2 + \frac{\sigma_Y}{\sigma_X}\rho\mu_X^2 - \frac{\sigma_Y}{\sigma_X}\rho\mu_X^2 \\
&\quad= \rho\sigma_X\sigma_Y
\end{aligned}$$

Therefore, we have

$$\mathrm{E}(XY) = \mu_X\mu_Y + \rho\sigma_X\sigma_Y$$

Finally, note that

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = \rho\sigma_X\sigma_Y \\
\mathrm{Corr}(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = \rho
\end{aligned}$$

we complete the proof.

(b) (i) $X|Y=y$ is a normal random variable with mean

$$\mu_x + \rho\sigma_x(y-\mu_y)/\sigma_y$$

and variance

$$\sigma_x^2(1-\rho^2).$$

Plugging in the information available, we see that $X|Y=180$ has mean

$$165 + 0.2 * 3 * (180-172)/4 = 166.2$$

and variance

$$3^2 * (1-0.04) = 9 * 0.96 = 8.64$$

(ii) Thus, $X|Y=180$ is normal with mean 166.5 and standard deviation 8.64. Then

$$\mathrm{P}(X<162|Y=180) = \mathrm{P}(Z<\frac{162-166.2}{\sqrt{8.64}}) = \mathrm{P}(Z<-1.428869) = 7.7\%.$$

5. (a) (7 points) If X is $b(100, 0.1)$, find the approximate value of $P(12 \leq X \leq 14)$,

   (i) (3 points) by using the normal approximation.

   (ii) (2 points) by using the Poisson approximation (list the expression and do not need to do the calculation).

   (iii) (2 points) by using the pmf of binomial distribution (list the expression and do not need to do the calculation).

   (b) (3 points) Let X_1 and X_2 be two independent random variables. Let X_1 and $Y = X_1 + X_2$ be $\chi^2(10)$ and $\chi^2(25)$, respectively. Show that X_2 is a Chi-square distribution and find $P(X_2 < 25)$ by using the provided table.

Solution: Please refer to Assignment 10 (5.7-12) and (5.4-22). (i) Since $X \sim b(100, 0.1)$, the mean and variance of X can be easily obtained by

$$E(X) = np = 100 \times 0.1 = 10$$
$$\text{Var}(X) = npq = 100 \times 0.1 \times 0.9 = 9$$

By normal approximation (i.e. using CLT),

$$\begin{aligned} P(12 \leq X \leq 14) &= P\left(\frac{12 - 1/2 - 10}{3} \leq \frac{X - 10}{3} \leq \frac{14 + 1/2 - 10}{3}\right) \\ &\approx P(0.5 \leq Z \leq 1.5) \\ &= \Phi(1.5) - \Phi(0.5) \\ &= 0.9332 - 0.6915 \\ &= 0.2417 \end{aligned}$$

Note that we have applied half-unit correction for continuity here as X is a discrete random variable.

(ii) Let $\lambda \approx np = 100 \times 0.1 = 10$. The Poisson approximation is given by (directly compute it using the Poisson cdf, or using Table III in Appendix B),

$$P(12 \leq X \leq 14) = \sum_{x=12}^{14} \frac{e^{10} 10^x}{x!} = 0.2198$$

(iii) We compute it directly

$$P(12 \leq X \leq 14) = \sum_{x=12}^{14} \binom{100}{x} (0.1)^x (0.9)^{100-x} = 0.2198$$

(b) We use the mgfs of Y and X_1 to compute it, let $r = 25$ and $r_1 = 10$.

$$\begin{aligned} \frac{1}{(1-2t)^{r/2}} &= E\left(e^{tY}\right) \\ &= E\left(e^{tX_1} e^{tX_2}\right) \\ &= E\left(e^{tX_1}\right) E\left(e^{tX_2}\right) \\ &= \frac{1}{(1-2t)^{r_1/2}} M_{X_2}(t) \end{aligned}$$

Solve this equation yields

$$M_{X_2}(t) = \frac{1}{(1-2t)^{(r-r_1)/2}}, \quad \forall t < 1/2$$

we know that $X_2 \sim \chi^2(15)$.
By checking the table, we find $P(X_2 < 25) = 0.95$

6. Let Y_1, Y_2, Y_3 be independent random variables which have the Bernoulli distribution with the probability of success p.
   (a) (3 points) Define a new random variable $Z = Y_1+Y_2+Y_3$. What's the distribution of Z? Show with details.
   (b) For $k = 1, 2$, let

$$X_k = \begin{cases} 1, & Y_1 + Y_2 + Y_3 = k, \\ -1 & Y_1 + Y_2 + Y_3 \neq k. \end{cases}$$

   i. (3 points) Find the joint pmf of X_1, X_2.
   ii. (2 points) Find the marginal pmfs of X_1 and X_2, respectively.
   iii. (2 points) Compute $\text{Cov}(X_1 - X_2, X_2)$.

Solution: Please refer to Assignment 7

(a) For independent Y_1, Y_2, Y_3, we have

$$\text{E}(e^{tZ}) = \text{E}(e^{t(Y_1+Y_2+Y_3)}) = \text{E}(e^{tY_1})\text{E}(e^{tY_2})\text{E}(e^{tY_3}) = (1-p+pe^t)^3$$

Z has a binomial distribution $Z = Y_1 + Y_2 + Y_3 \sim b(3, p)$

(b) i. Let $f(X_1, X_2)$ be the joint pmf of X_1 and X_2,then we have

$$\begin{aligned} f(-1,-1) &= P\{X_1 = -1, X_2 = -1\} \\ &= P\{Y_1 + Y_2 + Y_3 \neq 1, Y_1 + Y_2 + Y_3 \neq 2\} \\ &= P\{Y_1 + Y_2 + Y_3 = 0\} + P\{Y_1 + Y_2 + Y_3 = 3\} \\ &= (1-p)^3 + p^3 \end{aligned}$$

$$\begin{aligned} f(-1,1) &= P\{X_1 = -1, X_2 = 1\} \\ &= P\{Y_1 + Y_2 + Y_3 \neq 1, Y_1 + Y_2 + Y_3 = 2\} \\ &= \binom{3}{2} p^2(1-p) \\ &= 3p^2(1-p) \end{aligned}$$

$$
\begin{aligned}
f(1,-1) &= P\{X_1 = 1, X_2 = -1\} \\
&= P\{Y_1 + Y_2 + Y_3 = 1, Y_1 + Y_2 + Y_3 \neq 2\} \\
&= P\{Y_1 + Y_2 + Y_3 = 1\} \\
&= \binom{3}{1} p(1-p)^2 = 3p(1-p)^2
\end{aligned}
$$

$$
\begin{aligned}
f(1,1) &= P\{X_1 = 1, X_2 = 1\} \\
&= P\{Y_1 + Y_2 + Y_3 = 1, Y_1 + Y_2 + Y_3 = 2\} = 0
\end{aligned}
$$

ii. Then we could get the marginal pmf of X_k as follows:

$$
f_{X1}(x) = \begin{cases} 1 - 3p + 6p^2 - 3p^3, & X_1 = -1, \\ 3p(1-p)^2, & X_1 = 1. \end{cases}
$$

$$
f_{X2}(x) = \begin{cases} 1 - 3p^2 + 3p^3, & X_2 = -1, \\ 3p^2(1-p), & X_2 = 1. \end{cases}
$$

iii.

$$
\begin{aligned}
\mathrm{Cov}(X_1 - X_2, X_2) &= \mathrm{Cov}(X_1, X_2) - \mathrm{Cov}(X_2, X_2) \\
&= \mathrm{E}(X_1 X_2) - \mathrm{E}(X_1)\mathrm{E}(X_2) - \mathrm{Var}(X_2) \\
&= -12p^2 - 24p^3 + 144p^4 - 180p^5 + 72p^6
\end{aligned}
$$

7. Suppose that X is an exponential random variable with pdf $\lambda e^{-\lambda x}$, and let $a > 0$ be a constant. Define the random variable $Y = \max(a, X)$.
   (a) (3 points) Find the cumulative density function of Y. Is Y an exponential random variable?
   (b) (3 points) Suppose X' is another random variable and X, X' are i.i.d.. Prove that $\min(X, X')$ is exponentially distributed.
   (c) (4 points) Suppose that we have n independent observations $X_1, X_2, \ldots, X_n$ from an exponential distribution with parameter λ, and we want to estimate the parameter λ using the sample mean $\overline{X}$. Use Chebyshev's inequality to derive an upper bound on the probability $\mathrm{P}(|\overline{X} - \lambda| \geq \frac{\lambda}{2})$.

$\mathrm{Var}(X_i) = \lambda^2$.

Solution:

(a) (3 points) To find the cumulative density function of Y, we start by noting that $\mathrm{P}(Y > y) = \mathrm{P}(a > y) + \mathrm{P}(X > y \cap a \leq y)$. Since X is an exponential random variable with parameter λ, we have $\mathrm{P}(X > y) = e^{-\lambda y}$ and $\mathrm{P}(X \leq y) = 1 - e^{-\lambda y}$. Therefore,

$$\mathrm{P}(Y > y) = \begin{cases} e^{-\lambda y}, & \text{if } y \geq a \\ 1, & \text{if } y < a \end{cases}$$

Y is not an exponential random variable since its range is $[a, \infty)$.

(b) (3 points) We have for any $x > 0$,

$$\begin{aligned} \mathrm{P}(\min(X, X') > x) &= \mathrm{P}(X > x, X' > x) \\ &= \mathrm{P}(X > x)\mathrm{P}(X' > x) \\ &= e^{-\lambda x} \cdot e^{-\lambda x} = e^{-2\lambda x}. \end{aligned}$$

Therefore, $\min(X, X')$ is an exponential random variable.

(c) (4 points) Using Chebyshev's inequality, we have:

$$\begin{aligned} \mathrm{P}(|\overline{X} - \lambda| \geq \frac{\lambda}{2}) &= \cancel{\mathrm{P}(|\overline{X} - \lambda|^2 \geq \frac{\lambda^2}{4})} \\ &\leq \frac{\mathrm{Var}(\overline{X})}{(\frac{\lambda}{2})^2} \qquad \mathrm{Var}(\overline{X}) = \frac{\mathrm{Var}(X)}{n} \\ &= \frac{\mathrm{Var}(X)}{n(\frac{\lambda}{2})^2} \\ &= \frac{2}{n\lambda^2} \end{aligned}$$

Therefore, an upper bound on the probability that our estimate $\overline{X}$ is more than $\frac{\lambda}{2}$ is $\frac{2}{n\lambda^2}$.

8. (a) (6 points) Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. Bernoulli random variables with parameter $1/2$. Let

$$Y_n := \frac{\sum_{i=1}^{n^2} X_i - (n^2/2)}{(n/2)}.$$

Let $\mathbb{Q}$ be the set of rational numbers.

(i) (3 points) Show that $Y_n \xrightarrow{d} Y$, where $Y \sim \mathcal{N}(0, 1)$.

Hint: If a sequence of random variables $Z_n \xrightarrow{d} Z$ and $\tilde{Z}_{\tilde{n}}$ is a subsequence of Z_n, then $\tilde{Z}_{\tilde{n}} \xrightarrow{d} Z$.

(ii) (3 points) Find $\mathrm{P}(Y_n \in \mathbb{Q})$ and $\mathrm{P}(Y \in \mathbb{Q})$. Does $\mathrm{P}(Y_n \in \mathbb{Q})$ converge to $\mathrm{P}(Y \in \mathbb{Q})$?

(b) (4 points) Let $X_n \xrightarrow{d} X$ where $X \equiv x$ is a constant random variable. Prove that $X_n \xrightarrow{p} X$.

Note that $\xrightarrow{d}$ is the convergence in distribution and $\xrightarrow{p}$ is the convergence in probability.

Solution:

(a) (i)Observe that $Y_n = Z_{n^2}$, where

$$Z_n := \frac{\sum_{i=1}^{n} X_i - n/2}{\sqrt{n}/2} = \frac{\sum_{i=1}^{n} (X_i - \mathbb{E}[X_i])}{\sqrt{n \operatorname{Var}[X_i]}}$$

By Central Limit Theorem, $Z_n \xrightarrow{D} Y$. Hence, as a subsequence of $Z_n, Y_n \xrightarrow{D} Y$.

(ii) Bernoulli random variables are integer valued, hence $\frac{\sum_{i=1}^{n^2} X_i - (n^2/2)}{(n/2)}$ is always a rational number. We have $\mathrm{P}(Y_n \in \mathbb{Q}) = 1$ for all n.

$Y \sim \mathcal{N}(0,1)$ is a continuous random variable. For any rational number q, we have $\mathrm{P}(Y = q) = 0$.

The set of rational numbers is countable. Hence by countable additivity of probability measures we have

$$\mathrm{P}(Y \in \mathbb{Q}) = \sum_{q \in \mathbb{Q}} \mathrm{P}(Y = q) = 0$$

We observe that $\mathrm{P}(Y_n \in \mathbb{Q})$ does not converge to $\mathrm{P}(Y \in \mathbb{Q})$.

(b) Fix $\varepsilon > 0$. We have

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{P}(|X_n - X| \le \varepsilon) &= \lim_{n\to\infty} \mathrm{P}(|X_n - x| \le \varepsilon) \\
&= \lim_{n\to\infty} \mathrm{P}(x - \varepsilon \le X_n \le x + \varepsilon) \\
&= \lim_{n\to\infty} F_{X_n}(x+\varepsilon) - F_{X_n}(x - \varepsilon) \\
&= F_X(x+\varepsilon) - F_X(x-\varepsilon) \\
&= \mathrm{P}(x - \varepsilon < X \le x + \varepsilon) = 1
\end{aligned}$$

9. (a) (4 points) Suppose (X, Y) is a bivariate normal random variable with parameters $\mu_X, \mu_Y, \sigma_X, \sigma_Y$ and ρ, show that $X + Y$ is normally distributed. Find it's mean and variance.

(b) (4 points) Suppose (X, Y) is as given in (a). Calculate the correlation coefficient between X and $X + Y$.

(c) (2 points) Come up with an example of a bivariate random variable (X, Y) which is not jointly normal, yet both marginals X and Y are normally distributed. Specify the joint pdf and verify your answer.

Solution:

(a) (4 points) Let X, Y be bivariate normal random variables with parameters $\mu_X, \mu_Y, \sigma_X, \sigma_Y$, and ρ. The joint moment-generating function of X, Y is given by:

$$M_{X,Y}(t_1, t_2) = E[e^{t_1X+t_2Y}] = e^{t_1\mu_X+t_2\mu_Y+\frac{1}{2}(t_1^2\sigma_X^2+t_2^2\sigma_Y^2+2\rho t_1t_2\sigma_X\sigma_Y)} \tag{1}$$

Now, let $Z = X + Y$. We want to find the moment-generating function of Z:

$$M_Z(t) = E[e^{t(X+Y)}] = E[e^{tX+tY}] \tag{2}$$

To find $M_Z(t)$, we need to evaluate the joint moment-generating function $M_{X,Y}(t_1, t_2)$ at $t_1 = t_2 = t$:

$$M_Z(t) = M_{X,Y}(t, t) = e^{t(\mu_X+\mu_Y)+\frac{1}{2}(t^2(\sigma_X^2+\sigma_Y^2+2\rho\sigma_X\sigma_Y))} \tag{3}$$

The resulting moment-generating function of Z is in the form of a normal distribution with mean $(\mu_X+\mu_Y)$ and variance $(\sigma_X^2+\sigma_Y^2+2\rho\sigma_X\sigma_Y)$. Therefore, $Z = X + Y$ is normally distributed with:

$$\mu_Z = \mu_X + \mu_Y \tag{4}$$

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y \tag{5}$$

(b) (4 points) Let (X, Y) be a bivariate normal distribution with parameters $\mu_X, \mu_Y, \sigma_X, \sigma_Y$, and ρ. Let $Z = X + Y$. We want to compute the correlation coefficient between X and Z.

Step 1: Find the covariance between X and Z.

$$\mathrm{Cov}(X, Z) = \mathrm{Cov}(X, X + Y) = \mathrm{Cov}(X, X) + \mathrm{Cov}(X, Y) \tag{6}$$

$$\mathrm{Cov}(X, X) = \mathrm{Var}(X) = \sigma_X^2 \tag{7}$$

$$\mathrm{Cov}(X, Y) = \rho\sigma_X\sigma_Y \tag{8}$$

$$\mathrm{Cov}(X, Z) = \sigma_X^2 + \rho\sigma_X\sigma_Y \tag{9}$$

Step 2: Find the standard deviation of Z.

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y \tag{10}$$

$$\sigma_Z = \sqrt{\sigma_Z^2} \tag{11}$$

Step 3: Calculate the correlation coefficient between X and Z.

$$\rho_{XZ} = \frac{\text{Cov}(X, Z)}{\sigma_X\sigma_Z} = \frac{\sigma_X^2 + \rho\sigma_X\sigma_Y}{\sigma_X\sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}} \tag{12}$$

Thus, the correlation coefficient between X and Z is given by the above expression.

(c) (2 points) For example, $f(x, y) = \begin{cases} \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y^2)}{2\sigma^2}} & \text{if } xy \geq 0, \\ 0 & \text{if } xy < 0 \end{cases}$

10. Suppose the joint distribution of two discrete random variables X, Y is given as

$$P_{XY}(n,m) = \frac{\lambda^n e^{-\lambda}}{n!} \binom{n}{m} p^m (1-p)^{n-m}$$

where $0 \le m \le n$ and $0 \le p \le 1$.

(a) (2 points) Find the marginal pmf $P_Y(m)$.

(b) (2 points) Find the marginal pmf $P_X(n)$.

(c) (3 points) Find the conditional pmf $P_{X|Y}(n|m)$.

(d) (3 points) Find the expectation $E[XY]$.

Hint: $\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$

Solution:

(a) Since we have

$$P_{XY}(n,m) = \frac{(p\lambda)^m e^{-\lambda}}{m!} \left(\frac{\lambda^{n-m}(1-p)^{n-m}}{(n-m)!}\right),$$

then the marginal pmf is

$$\begin{aligned} P_Y(m) &= \frac{(p\lambda)^m e^{-\lambda}}{m!} \sum_{n=m}^{\infty} \left(\frac{\lambda^{n-m}(1-p)^{n-m}}{(n-m)!}\right) \\ &= \frac{(p\lambda)^m e^{-\lambda}}{m!} \sum_{n=0}^{\infty} \left(\frac{\lambda^n (1-p)^n}{(n)!}\right) \\ &= \frac{(p\lambda)^m e^{-\lambda}}{m!} e^{\lambda(1-p)} \\ &= \frac{(p\lambda)^m e^{-\lambda p}}{m!}, \quad m = 0, 1, \cdots \end{aligned}$$

(b) Since we have

$$P_{XY}(n,m) = \frac{\lambda^n e^{-\lambda}}{n!} \binom{n}{m} p^m (1-p)^{n-m}$$

then the marginal pmf is

$$\begin{aligned} P_X(n) &= \frac{\lambda^n e^{-\lambda}}{n!} \sum_{m=0}^{n} \binom{n}{m} p^m (1-p)^{n-m} \\ &= \frac{\lambda^n e^{-\lambda}}{n!}, \quad n = 0, 1, \cdots \end{aligned}$$

where the summation equals 1 as the binominal distribution.

(c) The conditional distribution of X given Y is

$$P_{X|Y}(n \mid m) = \frac{P_{XY}(n,m)}{P_Y(m)} = \frac{1}{P_Y(m)} \frac{\lambda^n e^{-\lambda}}{n!} \binom{n}{m} p^m (1-p)^{n-m}$$

$$= \underbrace{\frac{\lambda^m e^{-\lambda}}{P_Y(m)m!}}_{C} \times \frac{(\lambda(1-p))^{n-m}}{(n-m)!}, \quad n = m, m+1, \cdots$$

Note that C is the normalization factor which does not depend on n. Based on the form of $P_{X|Y}(n \mid m)$, and the fact that $\sum_{n=m}^{\infty} P_{X|Y}(n \mid m) = 1$, we have

$$C = e^{-\lambda(1-p)}$$

that is $P_{X|Y}$ is a shifted Poisson distribution.

(d) If Z is a Poisson random variable with parameter $\lambda(1-p)$, then conditioned on $Y = m$ we have $X = Z + m$. Hence, $E[Z + m \mid Y = m] = \lambda(1-p) + m$.

Then the expectation is

$$\begin{aligned}
E[XY] &= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} mnp_{X|Y}(n|m)p_Y(m) \\
&= \sum_{m=0}^{\infty} mp_Y(m) \sum_{n=m}^{\infty} np_{X|Y}(n|m) \\
&= \sum_{m=0}^{\infty} mp_Y(m) E[X|Y=m] \\
&= \sum_{m=0}^{\infty} m \frac{(p\lambda)^m e^{-\lambda p}}{m!} (m + \lambda - \lambda p) \\
&= \sum_{m=0}^{\infty} m^2 \frac{(p\lambda)^m e^{-\lambda p}}{m!} + (\lambda - \lambda p) \sum_{m=0}^{\infty} m \frac{(p\lambda)^m e^{-\lambda p}}{m!} \\
&= E[Y^2] + (\lambda - \lambda p) E[Y] \\
&= \lambda p + (\lambda p)^2 + (\lambda - \lambda p)\lambda p \\
&= \lambda + \lambda^2 p
\end{aligned}$$

where Y is a Poisson distribution.