MAT1002 Final Exam Reference Solution

1. (i) $\frac{\partial w}{\partial \theta} = -f_x \cdot \rho \sin\phi \sin\theta + f_y \cdot \rho \sin\phi \cos\theta$

(ii) $\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$

(iii) $\nabla f(1,1,1) = \langle 2,1,2 \rangle$, $\vec{u} = \frac{1}{\sqrt{5}} \langle 0, 2, -1 \rangle$

$$D_{\vec{u}} f(1,1,1) = \langle 2,1,2 \rangle \cdot \frac{1}{\sqrt{5}} \langle 0,2,-1 \rangle = 0$$

(iv) $Q(x,y) = 2 - x^2 - y^2$.

(v) F

(vi) $\int_{-1}^{0} \int_{0}^{y^2} \int_{0}^{1} dz\, dx\, dy$.

(vii) 2

(viii) $\frac{x}{\sqrt[3]{1+x}} = x(1+x)^{-\frac{1}{3}} = \binom{-\frac{1}{3}}{0} x + \binom{-\frac{1}{3}}{1} x^2 + \binom{-\frac{1}{3}}{2} x^3 + \binom{-\frac{1}{3}}{3} x^4$

$$= x - \frac{1}{3}x^2 + \frac{2}{9}x^3 - \frac{14}{81}x^4$$

(ix) F

2. (i) As $(x,y)\to(0,0)$, $\sin x+\cos y\to 1$ but $x+y\to 0$,

so limit does not exist (as a real number).

(ii) $$\left|\frac{x^3-y^3}{x^2+y^2}\right| = \left|\frac{r^3(\cos^3\theta-\sin^3\theta)}{r^2}\right| = r\left|\cos^3\theta-\sin^3\theta\right| \leq 2r \quad \forall\theta.$$

$$\Rightarrow \qquad -2r \leq \frac{x^3-y^3}{x^2+y^2} \leq 2r.$$

As $(x,y)\to(0,0)$, $r\to 0$, so $2r\to 0$. By squeeze, limit $=0$.

(Alternatively, $\left|\frac{x^3}{x^2+y^2}\right| \leq |x| \to 0$ as $(x,y)\to(0,0)$ and

$\left|\frac{y^3}{x^2+y^2}\right| \leq |y| \to 0$ as $(x,y)\to(0,0)$, so

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0.$$

This means $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0+0 = 0.$)

(iii) Since $\displaystyle\lim_{\substack{(r,\theta)\to(1,0)\\ \text{along } \theta=0}} \frac{r\sin\theta}{r^2-1} = 0$ and

$$\lim_{\substack{(r,\theta)\to(1,0)\\ \text{along } r=1+\theta}} \frac{r\sin\theta}{r^2-1} = \lim_{\theta\to 0} \frac{(1+\theta)\sin\theta}{(\theta+2)\,\theta} = \lim_{\theta\to 0}\left(\frac{1+\theta}{2+\theta}\right)\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{1}{2},$$

limit D.N.E. by the two-path test.

3. $f_x(x,y) = 0$ on $\mathbb{R}^2$.

$\Rightarrow f_{xx}(x,y) = 0, \quad D = \mathbb{R}^2.$

$\& \; f_{xy}(x,y) = 0, \quad D = \mathbb{R}^2.$

$$f_y(x,y) = \begin{cases} 3y^2, & y \geq 0 \\ -2y, & y < 0 \end{cases} \quad \text{on } \mathbb{R}^2$$

$$\Rightarrow f_{yy}(x,y) = \begin{cases} 6y, & y > 0 \\ -2, & y < 0 \end{cases}, \quad D = \mathbb{R}^2 \setminus \{(x,y) : y = 0\}$$

(or xy-plane with x-axis removed)

4. $\nabla g = \langle 2x, 4y, 2z \rangle \Rightarrow \nabla g(1,1,1) = \langle 2, 4, 2 \rangle$

$L: \begin{cases} x = 1 + 2t \\ y = 1 + 4t \\ z = 1 + 2t \end{cases}, \quad t \in \mathbb{R}$

$M: \ 2(x-1) + 4(y-1) + 2(z-1) = 0$

or $\ 2x + 4y + 2z = 8$.

5. <u>Method 1</u>: Minimize $x^2+y^2+z^2$

s.t. $2x+y-z=4$.

Solve $\begin{cases} \nabla f = \lambda \nabla g \\ 2x+y-z=4 \end{cases} \Leftrightarrow \begin{cases} 2x = \lambda \cdot 2 \\ 2y = \lambda \\ 2z = -\lambda \\ 2x+y-z=4 \end{cases}$

$\Rightarrow 2\lambda + \frac{\lambda}{2} + \frac{\lambda}{2} = 4 \Rightarrow \lambda = \frac{4}{3} \Rightarrow \begin{cases} x = 4/3 \\ y = 2/3 \\ z = -2/3 \end{cases}$

By geometric property, $(x,y,z) = (\frac{4}{3}, \frac{2}{3}, -\frac{2}{3})$ gives the desired minimum distance.

<u>Method 2</u>: Define $h(x,y) = x^2+y^2+(2x+y-4)^2$.

Then $\nabla h = \langle 2x + 2(2x+y-4)\cdot 2,\ 2y + 2(2x+y-4) \rangle$

$= \langle 10x+4y-16,\ 4x+4y-8 \rangle$,

$\nabla h = \langle 0,0 \rangle \Leftrightarrow (x,y) = (\frac{4}{3}, \frac{2}{3})$.

Since $h_{xx} = 10$, $h_{xy} = 4 = h_{yx}$, $h_{yy} = 4$,

$h_{xx}h_{yy} - h_{xy}^2 = 24 > 0$, $h_{xx} > 0$, $(x,y) = (\frac{4}{3}, \frac{2}{3})$ gives a local min. Since it is the only local min, it is the global min, so $(x,y,z) = (\frac{4}{3}, \frac{2}{3}, -\frac{2}{3})$ gives the desired minimum.

6. Minimize and maximize xyz^2 cts

s.t. $x^2+y^2+z^2=4$. closed and bdd

Solve $\begin{cases} \nabla f = \lambda \nabla g \\ x^2+y^2+z^2=4 \end{cases} \iff \begin{cases} yz^2 = 2x\lambda & ① \\ xz^2 = 2y\lambda & ② \\ 2xyz = 2z\lambda & ③ \\ x^2+y^2+z^2=4 & ④ \end{cases}$

First, note that any solution with $x=0$, $y=0$, or $z=0$ must give $T(x,y,z)=0$. Consider other solutions. Then ③ $\Rightarrow xy=\lambda$.

Now ①·② $\Rightarrow z^4 = 4\lambda^2 \Rightarrow z^4 = 4x^2y^2$

$\Rightarrow z^2 = \pm 2xy$ ⑤

$y=x$

$x^2+x^2+2x^2=4$

and ①÷② $\Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow y^2=x^2$

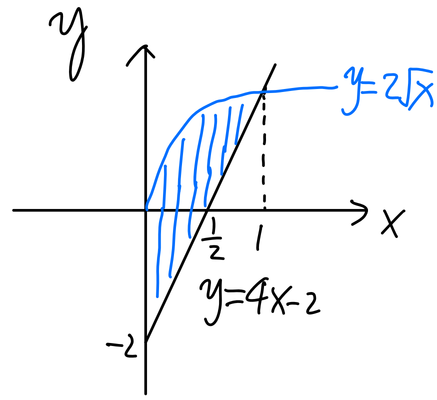
$\Rightarrow y = \pm x$ ⑥

Now ④, ⑤, ⑥ $\Rightarrow (x,y,z) \in \{(1,1,\pm\sqrt{2}), (-1,-1,\pm\sqrt{2}), (1,-1,\pm\sqrt{2}), (-1,1,\pm\sqrt{2})\}$.

Points with highest temperature: $(1,1,\pm\sqrt{2}), (-1,-1,\pm\sqrt{2})$, $T=2$;

Points with lowest temperature: $(1,-1,\pm\sqrt{2}), (-1,1,\pm\sqrt{2})$, $T=-2$.

7. (i) $\iint_R (2x-1-y)\, dA$



$$= \int_0^1 \int_{4x-2}^{2\sqrt{x}} (2x-1-y)\, dy\, dx$$

$$= \int_0^1 \left(2xy - y - \frac{1}{2}y^2 \Big|_{y=4x-2}^{2\sqrt{x}} \right) dx$$

$$= \int_0^1 \left(4x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - 2x - 2x(4x-2) + 4x - 2 + \frac{1}{2}(4x-2)^2 \right) dx$$

$$= \int_0^1 \left(4x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - 2x \cancel{- 8x^2} \cancel{+ 4x} \cancel{+ 4x} \cancel{- 2} \cancel{+ 8x^2} \cancel{+ 2} \cancel{- 8x} \right) dx$$

$$= \int_0^1 \left(4x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - 2x \right) dx = \frac{8}{5}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} - x^2 \Big|_{x=0}^{1}$$

$$= -\frac{11}{15}.$$

(ii) $$\int_0^\infty \frac{e^{-x} - e^{-2x}}{x}\, dx = \int_0^\infty \int_1^2 e^{-xy}\, dy\, dx$$

$$= \lim_{b\to\infty} \int_0^b \int_1^2 e^{-xy}\, dy\, dx = \lim_{b\to\infty} \int_1^2 \int_0^b e^{-xy}\, dx\, dy$$

$$= \lim_{b\to\infty} \int_1^2 \frac{1-e^{-by}}{y}\, dy = \int_1^2 \frac{1}{y}\, dy = \ln 2.$$

8. (i) $x^2+y^2+(z-1)^2=1$

$$\Leftrightarrow \rho^2-2\rho\cos\phi=0 \quad \Leftrightarrow \quad \rho=2\cos\phi$$

$$z=\sqrt{x^2+y^2} \quad \Leftrightarrow \quad \rho\cos\phi=\rho\sin\phi \quad \Leftrightarrow \quad \phi=\frac{\pi}{4}.$$

$$V(E)=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

(ii)
$$V(E)=2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{1}{3}\sin\phi\,\rho^3\right)\Big|_{\rho=0}^{2\cos\phi}d\phi$$

$$=\frac{2\pi}{3}\int_{\pi/4}^{\pi/2}8\sin\phi\cos^3\phi\, d\phi$$

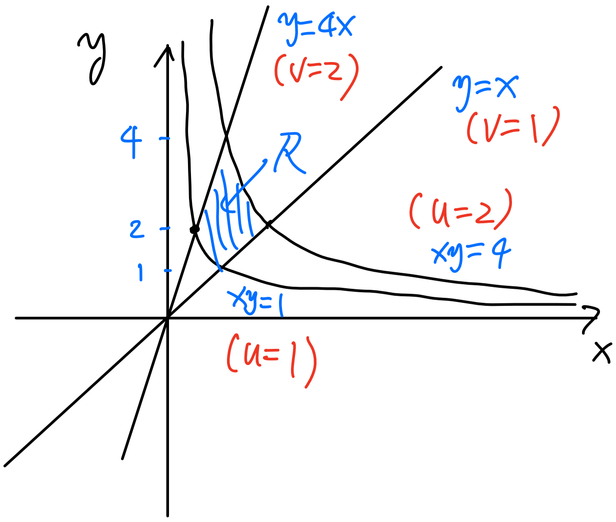
$$=\frac{16\pi}{3}\int_{\frac{\sqrt{2}}{2}}^{0}u^3(-du)$$

$$=\frac{16\pi}{3}\left(\frac{1}{4}u^4\Big|_0^{\frac{\sqrt{2}}{2}}\right)=\frac{\pi}{3}.$$

9. The integral sum is

$$\iint_R (x^2+y^2)\,dA,$$

where R is as shown on the right.



With $x=\frac{u}{v}$ & $y=uv$,

we have $\begin{cases} u^2 = xy \\ v^2 = y/x \end{cases}$.

One possible transformation is to consider u & v to be both positive; then the corresponding uv-region is

$$D: \{(u,v),\ 1 \le u \le 2,\ 1 \le v \le 2\}.$$

Now

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{u}{v} + \frac{u}{v} = \frac{2u}{v} > 0,$$

So $$\iint_R (x^2+y^2)\,dA = \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2v^2\right)\frac{2u}{v}\,du\,dv.$$

10. (i)

$M \quad N \quad P$

$x \quad y \quad z$

$M_y = N_x \Rightarrow -e^x \sin y = -ae^{ax} \sin y$

$\Rightarrow a = 1$

$M_z = P_x \Rightarrow \frac{1}{z} = \frac{b}{z} \Rightarrow b = 1$

(ii) Let $\nabla f = \vec{F} = \langle e^x \cos y + \ln z, -e^x \sin y, \frac{x}{z} \rangle$.

$f_x = e^x \cos y + \ln z \Rightarrow f = e^x \cos y + x \ln z + K(y,z)$ ①

$\Rightarrow f_y = -e^x \sin y + K_y(y,z)$

But $f_y = N = -e^x \sin y \Rightarrow K_y(y,z) = 0 \Rightarrow K(y,z) = L(z)$.

Now ① $\Rightarrow f_z = \frac{x}{z} + L'(z)$.

But $f_z = P = \frac{x}{z} \Rightarrow L'(z) = 0 \Rightarrow L(z) = C$.

Now ① $\Rightarrow f = e^x \cos y + x \ln z + C$, where C is any constant.

(iii) Since $\vec{F}$ is conservative,

$$\text{Work} = \int_{(2,\pi,1)}^{(0,3,2)} \vec{F} \cdot d\vec{r} = f(0,3,2) - f(2,\pi,1)$$

$$= \cos 3 - (e^2(-1)) = \cos(3) + e^2.$$

11. Circulation $= \oint_C \vec{F}\cdot d\vec{r} = \iint_R (N_x - M_y)\, dA = \iint_R 0\, dA = 0.$

$$\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\, ds = \iint_R (M_x + N_y)\, dA = \iint_R (2x+2y)\, dA$$

$$= 2\int_0^{\pi}\int_0^{a} r(\cos\theta + \sin\theta)\, r\, dr\, d\theta$$

$$= 2\left(\int_0^{\pi} (\cos\theta + \sin\theta)\, d\theta\right)\left(\int_0^{a} r^2\, dr\right)$$

$$= 2 \left(\sin\theta - \cos\theta\right)\Big|_0^{\pi} \ \frac{1}{3}a^3$$

$$= \frac{2}{3}a^3(1+1) = \frac{4}{3}a^3.$$

12. $\text{Flux} = \iint_{\substack{x^2+y^2+z^2=1 \\ z\geq 0}} \text{curl}\vec{A}\cdot\vec{n}\,d\sigma$

$\overset{\text{Stokes'}}{=} \oint_C \vec{A}\cdot d\vec{r} = \oint_C M dx + N dy + P dz \quad \left(C: \begin{array}{l} x=\cos t \\ y=\sin t, \\ z=0 \\ 0\leq t\leq 2\pi \end{array} \right)$

$= \int_0^{2\pi} \sin t(-\sin t\,dt) + e^0 \cos t\,dt$

$= \int_0^{2\pi} (\cos t - \sin^2 t)\,dt$

$= \int_0^{2\pi} \left(\cos t - \frac{1-\cos 2t}{2}\right) dt$

$= \left(\sin t - \frac{1}{2}t + \frac{1}{4}\sin 2t\right)\Big|_{t=0}^{2\pi}$

$= -\pi$

13. Flux $= \iint_S \vec{F} \cdot \vec{n} \, d\sigma$

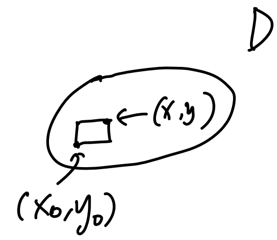
$$\overset{\text{div.}}{\underset{\text{thm}}{=}} \iiint_E \underbrace{\operatorname{div} \vec{F}}_{6x^2+3y^2+3y^2} \cdot dV \qquad E: \ 0 \le z \le 1-x^2-y^2, \ x^2+y^2 \le 1.$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6r^2 \cdot r \, dz \, dr \, d\theta$$

$$= 2\pi \cdot \int_0^1 (1-r^2) 6r^3 \, dr$$

$$= 2\pi \left(\frac{6}{4} r^4 - r^6 \right) \Big|_{r=0}^{1} = 2\pi \frac{1}{2} = \pi.$$

14. <u>Proof</u>: Fix any $(x,y)\in D$, and let $(x_0,y_0)\in D$ satisfy $x_0<x_1$ and $y_0<y_1$. Define F near (x,y) by



$$F(x',y') := \iint_{\substack{x_0\le u\le x' \\ y_0\le v\le y'}} f(u,v)\, dA$$

Fubini's thm
$$\begin{cases} = \displaystyle\int_{x_0}^{x'}\int_{y_0}^{y'} f(u,v)\, dv\, du \\ = \displaystyle\int_{y_0}^{y'}\int_{x_0}^{x'} f(u,v)\, du\, dv . \end{cases}$$

By assumption, $F\equiv 0$ near (x,y), so all partial derivatives of $F=0$ at (x,y). $(*)$

On the other hand,

$$\frac{\partial}{\partial x}F(x,y) = \frac{\partial}{\partial x}\int_{x_0}^{x}\int_{y_0}^{y} f(u,v)\, dv\, du \overset{\text{FTC}}{=} \int_{y_0}^{y} f(x,v)\, dv$$

$$\Rightarrow \frac{\partial}{\partial y}\frac{\partial}{\partial x}F(x,y) \overset{\text{FTC}}{=} f(x,y). \qquad (**)$$

By $(*)$ & $(**)$, $f(x,y)=0$. $\square$