MAT1002 Lecture 18, Tuesday, Mar/28/2023

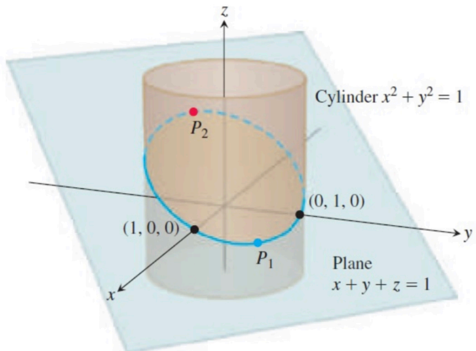
Outline

- Optimization with two equality constraints (14.8)
- Results related to second partials (14.9)
- Taylor's theorem (14.9)
- Double integrals on rectangles: definition (15.1)

Two Equality Constraints

Example (e.g. 14.8.5)

The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.



That is : maximize/minimize $\sqrt{x^2+y^2+z^2}$

subject to $x+y+z=1$ and $x^2+y^2=1$.

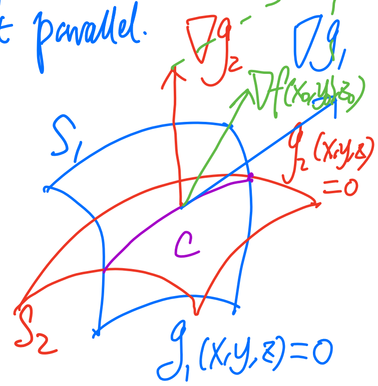
This is an optimization problem with two equality constraints:

$$\text{Max/min } f(x,y,z)$$
$$\text{s.t. } g_1(x,y,z)=0 \text{ and } g_2(x,y,z)=0,$$

that is, finding extrema of f on the set $S_1 \cap S_2$, where

$$\underbrace{S_i = \{(x,y,z) : g_i(x,y,z)=0\}}_{\text{a surface}}, \quad i \in \{1,2\}.$$

If f has a local extremum at $P_0 \in S_1 \cap S_2$ relative to points in $S_1 \cap S_2$, then $\nabla f(P_0)$ must be $\perp$ to $C = S_1 \cap S_2$. Since ∇g_1 and ∇g_2 are also $\perp$ to C, we have ∇f lying in the plane spanned by $\nabla g_1(P_0)$ and $\nabla g_2(P_0)$. Assuming they are $\neq \vec{0}$, not parallel.

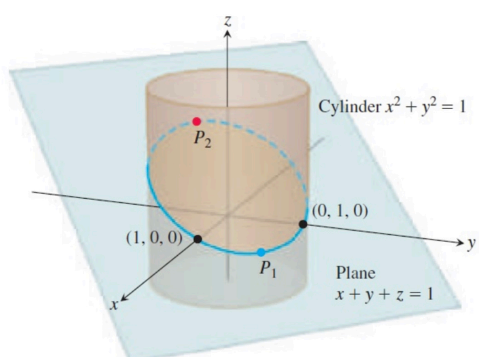


This means that the method of Lagrange multipliers can still be used, but the system to be solved is the following instead:

$$\begin{cases} \nabla f(x,y,z) = \lambda_1 \nabla g_1(x,y,z) + \lambda_2 \nabla g_2(x,y,z) \\ g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases}.$$

Example (e.g. 14.8.5)

The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.



Ans: Farthest: $\cancel{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1-\sqrt{2}\right) \text{ and }}$ $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1+\sqrt{2}\right)$.

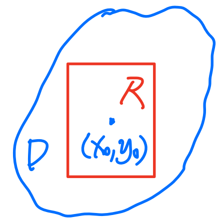
Closest: $(0,1,0)$ and $(1,0,0)$.

Proof of Some Results Regarding Second-Order Partials

1. Error of Standard Linear Approximation

Result in short: $|E(x,y)| \le \frac{1}{2}M(|x-x_0|+|y-y_0|)^2$, where M bounds the absolutely values of all second partials on R.

(All second partials are cts.) ($E(x,y) = f(x,y) - L(x,y)$.)



Proof: • Let $h := \Delta x$ and $k := \Delta y$ be "small changes".

• Define $F(t) := f(\underbrace{x_0+th}_{x(t)}, \underbrace{y_0+tk}_{y(t)})$, $t \in [0,1]$.

• By Taylor's theorem, since f is cts on $[0,1]$ & diff'able on $(0,1)$,

$$\exists c \in (0,1): \quad F(1) = F(0) + F'(0)(1-0) + \frac{1}{2}F''(c)(1-0)^2 \quad ①$$
$$= F(0) + F'(0) + \frac{1}{2}F''(c).$$

• $F'(t) = f_x x'(t) + f_y y'(t)$ (Chain rule, F diff'able.)

$$= f_x h + f_y k.$$

• Since all second partials of f are cts, F' is diff'able, so

$$F''(t) = h\frac{d}{dt}f_x + k\frac{d}{dt}f_y$$
$$= h(f_{xx}x'(t) + f_{xy}y'(t)) + k(f_{yx}x'(t) + f_{yy}y'(t))$$
$$= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2$$

· Now (1) $\Rightarrow$

$$f(x_0+h, y_0+k) = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)\,h + f_y(x_0,y_0)\,k}_{L(x_0+h,\, y_0+k)}$$

$$\underbrace{\frac{1}{2}\left(f_{xx} h^2 + 2 f_{xy} hk + f_{yy} k^2\right)\Big|_{(x,y)=(x_0+ch,\, y_0+ck)}}_{E(x_0+h,\, y_0+k)}$$

· In short notation, and by triangle inequality,

Evaluated at (x_0+ch, y_0+ck) for some $c \in (0,1)$

$$|E(x,y)| = \frac{1}{2}\left|f_{xx}(x-x_0)^2 + 2 f_{xy}(x-x_0)(y-y_0) + f_{yy}(y-y_0)^2\right|$$

$$\leq \frac{1}{2}\left(|f_{xx}|(x-x_0)^2 + 2|f_{xy}||x-x_0||y-y_0| + |f_{yy}|(y-y_0)^2\right)$$

$\leq M$

$$\leq \frac{1}{2} M \left(|x-x_0| + |y-y_0|\right)^2.$$

$\square$

2. The Second Derivative Test

Theorem (Second Derivative Test)

Let f be a function whose second partial derivatives are all continuous on a open ~~ball~~ disk centered at (a, b). Suppose that $\nabla f(a,b) = \vec{0}$ (so (a, b) is a critical point of f). Let

$$H := H(a,b) := f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2.$$

- *If $H > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a, b).*
- *If $H > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a, b).*
- *If $H < 0$, then f has no local extremum at (a, b); that is, (a, b) is a saddle point ~~o~~ of f.*

Proof of this is optional.

Proof: • Consider $h := \Delta x$ and $k := \Delta y$ as "small changes" from (a,b).

• From the first proof above, we saw that

$$f(a+h, b+k) = f(a,b) + f_x(a,b)h + f_y(a,b)k$$
$$\frac{1}{2}\left(f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2\right)\Big|_{(x,y)=(a+ch,\, b+ck)}$$

for some $c \in (0,1)$.

• Since $\nabla f(a,b) = (0,0)$,

$$f(a+h, b+k) - f(a,b) = \frac{1}{2}\left(f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2\right)\Big|_{(x,y)=(a+ch,\, b+ck)} \quad (1)$$

$$=: Q(c,h,k) =: Q.$$

Case 1: $H(a,b) = (f_{xx}f_{yy} - f_{xy}^2)(a,b) > 0$, $f_{xx}(a,b) > 0$.

- By continuity of second partials, $\exists \delta > 0$ s.t. $\forall (x_0, y_0) \in B_\delta(a,b)$, $H(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0$.
- Consider (h,k) such that $0 < \sqrt{h^2+k^2} < \delta$. Then

$$Q = \frac{1}{2}\left(f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2\right)\Big|_{(x,y)=(\underbrace{a+ch}_{x_0 :=},\, \underbrace{b+ck}_{y_0 :=})}.$$

- $$f_{xx}(x_0,y_0)\,Q = \frac{1}{2}\left(f_{xx}^2h^2 + 2f_{xx}f_{xy}hk + f_{xx}f_{yy}k^2\right)\Big|_{(x,y)=(x_0,y_0)}$$

$$= \frac{1}{2}\left[(f_{xx}h + f_{xy}k)^2 + \underbrace{(f_{xx}f_{yy} - f_{xy}^2)}_{H(x_0,y_0)>0}k^2\right]\Big|_{(x,y)=(x_0,y_0)} > 0.$$

(since $(h,k) \neq (0,0)$)

- Since $f_{xx}(x_0,y_0) > 0$, we have $Q > 0$. By ①,

$$f(a+h, b+k) > f(a,b) \quad \forall (h,k) \text{ with } 0 < \sqrt{h^2+k^2} < \delta,$$

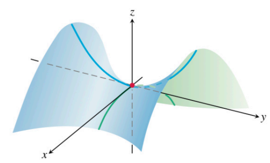
so (a,b) is a local minimum of f.

Case 2: $H(a,b) = (f_{xx}f_{yy} - f_{xy}^2)(a,b) > 0$, $f_{xx}(a,b) < 0$.

Similarly, we get $Q < 0$ and that (a,b) is a local maximum of f.

Case 3: $H(a,b) = (f_{xx}f_{yy} - f_{xy}^2)(a,b) < 0$

- We show that $\exists$ direction $\vec{w}$ and $\vec{v}$ such that f has a local min at (a,b) while restricted to $\vec{w}$ and has a local max at (a,b) while restricted to $\vec{v}$. This would show that (a,b) is a saddle point of f.



- Let $\vec{u} := \langle h, k \rangle$ be a unit vector. Consider

$$F(t) := f(a+th,\ b+tk).$$

Then $F'(0) = (f_x h + f_y k)\big|_{(x,y)=(a,b)} = 0$, and

$$F''(0) = f_{xx} h^2 + 2 f_{xy} hk + f_{yy} k^2 \big|_{(x,y)=(a,b)}.$$

- Define $g(t) := \underbrace{f_{xx}}_{A} t^2 + \underbrace{2 f_{xy}}_{B}\, t + \underbrace{f_{yy}}_{C}$. Suppose $f_{xx}(a,b) \neq 0$.

← Evaluated at (a,b).

Then $B^2 - 4AC = 4 f_{xy}^2 - 4 f_{xx} f_{yy} = -4(f_{xx} f_{yy} - f_{xy}^2) > 0$.

This means that $g(t)$ has two distinct real roots, and $g(t)$ is sometimes >0 and sometimes <0 (depending on t). ②

- If $k \neq 0$, then $g\left(\frac{h}{k}\right) = f_{xx} \frac{h^2}{k^2} + 2 f_{xy} \frac{h}{k} + f_{yy}$

$$\Rightarrow \quad k^2 g\left(\frac{h}{k}\right) = f_{xx} h^2 + 2 f_{xy} hk + f_{yy} k^2 = F''(0). \quad ③$$

- By ② and ③, we may pick different combinations of (h,k) so that $\underbrace{F''(0) > 0}_{\text{Concave up, local min in one direction}}$ on one and $\underbrace{F''(0) < 0}_{\text{Concave down, local max in another}}$ on the other.

- If $f_{xx}(a,b) = 0$, then $g(t) = 2 f_{xy} t + f_{yy}$ with $f_{xy}(a,b) \neq 0$.

The rest is similar. Sometimes >0 and Sometimes <0. $\square$

Taylor's Theorem (Two-Variable Functions)

with cts second-order partials

We saw that if we are given f and a fixed (a,b), and let

$$F(t) := f(a+th,\, b+tk),$$

then

$$F'(t) = f_x h + f_y k = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f \Big|_{(a+th,\, b+tk)},$$

$$F''(t) = f_{xx} h^2 + 2 f_{xy} hk + f_{yy} k^2 = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f \Big|_{(a+th,\, b+tk)}.$$

More generally, if all n^{th}-order partials of f are cts on an open region containing (a,b), then

$$F^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f \Big|_{(a+th,\, b+tk)},$$

e.g. $F^{(4)}(t) = h^4 f_{xxxx} + 4h^3 k f_{xxxy} + 6h^2k^2 f_{xxyy} + 4hk^3 f_{xyyy} + k^4 f_{yyyy}$.

One variable Taylor's theorem suggests that

$$F(1) = F(0) + F'(0) + \frac{1}{2!}F^{(2)}(0) + \cdots + \frac{1}{n!}F^{(n)}(0) + \frac{1}{(n+1)!}F^{(n+1)}(c)$$

for some $c \in (0,1)$. This translates to the following <u>two-variable version of Taylor's theorem</u>: if $f(x,y)$ has continuous partials up to and including $(n+1)^{th}$ order in an open disk (or rectangular region) R containing (a,b), then $\forall$ points in R,

$$f(a+h,\, b+k) = \left(\sum_{i=0}^{n} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f \Big|_{(a,b)}\right) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1}\Big|_{(a+ch,\, b+ck)}$$

for some $c \in (0,1)$.

Using Taylor expansion

EXAMPLE ~~1~~ 14.9.1 Find a quadratic approximation to $f(x, y) = \sin x \sin y$ near the origin. How accurate is the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$?

Sol.

$$\begin{aligned} f(0,0) &= \sin x \sin y|_{(0,0)} = 0, & f_{xx}(0,0) &= -\sin x \sin y|_{(0,0)} = 0, \\ f_x(0,0) &= \cos x \sin y|_{(0,0)} = 0, & f_{xy}(0,0) &= \cos x \cos y|_{(0,0)} = 1, \\ f_y(0,0) &= \sin x \cos y|_{(0,0)} = 0, & f_{yy}(0,0) &= -\sin x \sin y|_{(0,0)} = 0, \end{aligned}$$

- $f(x,y) \approx f(0,0) + (x f_x + y f_y)\big|_{(0,0)} + \frac{1}{2}(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy})\big|_{(0,0)}$

  $= xy$

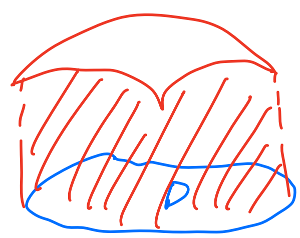
- $E(x,y) = \frac{1}{6}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xxy} + y^3 f_{yyy})\big|_{(cx,cy)}$

- Computing all third-order partials shows that their absolute values are all ≤ 1.

- Hence $|E(x,y)| \leq \frac{1}{6}((0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3) \leq 0.00134$.

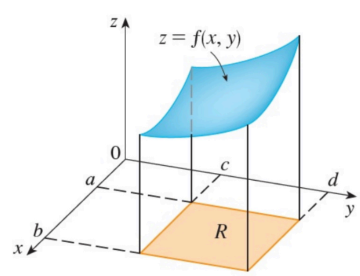
Double Integrals on Rectangles

What is the volume of this solid?



$z = f(x,y) \geq 0$

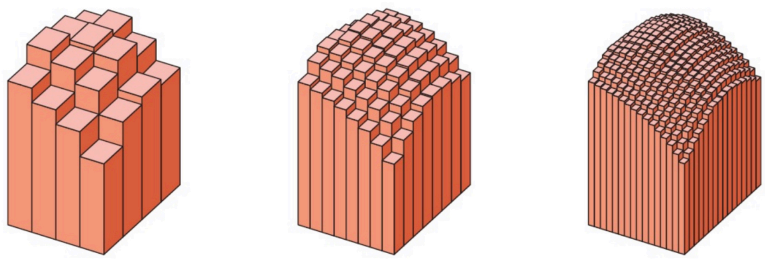
First look at case where D is a rectangle.

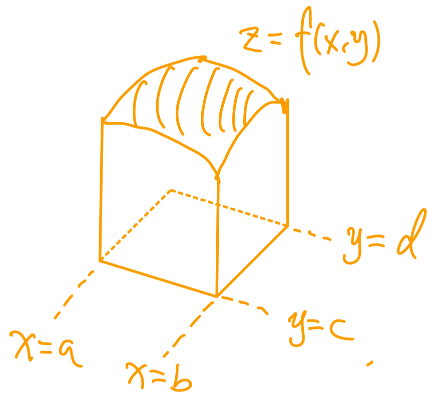


Let f be a two-variable nonnegative function defined on the rectangle

$$R := [a, b] \times [c, d] = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}.$$

To approximate the volume of the solid lying between the xy-plane and the graph of f, we can use rectangular solids.





More precisely:

- Take a partition $\{x_0, x_1, \ldots, x_m\}$ of $[a, b]$ and a partition $\{y_0, y_1, \ldots, y_n\}$ of $[c, d]$; they together form a partition of R, and break R into mn rectangles R_{ij}.
- Let ΔA_{ij} be the area of R_{ij}, and let (x_{ij}^*, y_{ij}^*) be a point in R_{ij}.
- Then $f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$ is the volume of the rectangular solid with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$.
- The total volume is then approximated by the sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}.$$

The sum above is called a Riemann sum of f.

Definition

Let P be a partition of a rectangle R. The norm of P, denoted by $\|P\|$, is the largest width of the sub-rectangles in the partition P. For a function f defined on the rectangle R, if the limit of the Riemann sums

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

exists as $\|P\| \to 0$, then f is said to be integrable (on R), and its limit is denoted by the double integral

$$\iint_R f(x, y)\, dA.$$

Remark

- Continuous functions are integrable; that is, if f is continuous on R, then
$$\iint_R f(x,y)\,dA$$
exists. In fact, if f is bounded on R, and is only discontinuous at finitely many points or on a finite union of smooth curves, then f is also integrable.
- If f is a nonnegative function defined on R, then the volume of the solid lying between the graph of f and the xy-plane is defined by
$$V := \iint_R f(x,y)\,dA.$$