STA2001 Tutorial 12

1. 5.4-10. Let X equal the outcome when a fair four-sided die that has its faces numbered 0, 1, 2, and 3 is rolled. Let Y equal the outcome when a fair four-sided die that has its faces numbered 0, 4, 8, and 12 is rolled.

   (a) Define the mgf of X.

   (b) Define the mgf of Y.

   (c) Let $W = X + Y$, the sum when the pair of dice is rolled. Find the mgf of W.

   (d) Give the pmf of W; that is, determine $P(W = w), w = 0, 1, \cdots, 15$, from the mgf of W.

   Solution:

   (a) The pmf of X is given by

   $$f(x) = \frac{1}{4}, \qquad x = 0, 1, 2, 3.$$

   Thus,

   $$M_X(t) = E(e^{tX}) = \sum_x e^{tx} f(x) = \frac{1}{4}(e^{0t} + e^{1t} + e^{2t} + e^{3t})$$

   (b) The pmf of Y is given by

   $$g(y) = \frac{1}{4}, \qquad y = 0, 4, 8, 12.$$

   Thus,

   $$M_Y(t) = E(e^{tY}) = \sum_y e^{ty} g(y) = \frac{1}{4}(e^{0t} + e^{4t} + e^{8t} + e^{12t})$$

   (c) Note that X and Y are independent, using the property of mgf, we have

   $$\begin{aligned} M_W(t) &= E(e^{tW}) \\ &= E\left[e^{t(X+Y)}\right] \\ &= E[e^{tX}]E[e^{tY}] \\ &= \frac{1}{16}\left(e^{0t} + e^{1t} + e^{2t} + e^{3t}\right)\left(e^{0t} + e^{4t} + e^{8t} + e^{12t}\right) \\ &= \frac{1}{16}\left(e^{0t} + e^{1t} + e^{2t} + \cdots + e^{15t}\right) \end{aligned}$$

(d) From (c), we know that

$$M_W(t) = \sum_{w \in S} f(w) e^{tw} = \frac{1}{16}\left(e^{0t} + e^{1t} + e^{2t} + \cdots + e^{15t}\right)$$

where $f(w) = P(W = w)$ is the pmf of W, and $w \in S := \{0, 1, 2, \cdots, 15\}$. Thus, the coefficient of e^{tw} is exactly the probability $P(W = w)$,

$$P(W = w) = \frac{1}{16}, \quad w = 0, 1, 2, \cdots, 15.$$

The question also illustrates that, given a moment generating function, it uniquely determines a probability distribution.

2. 5.4-20. The time X in minutes of a visit to a cardiovascular disease specialist by a patient is modeled by a gamma pdf with $\alpha = 1.5$ and $\theta = 10$. Suppose that you are such a patient and have four patients ahead of you. Assuming independence, what integral gives the probability that you will wait more than 90 minutes?

Solution:

Let X_i be the time (in minutes) of a visit to a disease specialist by the i-th patient, and we know that X_i follows a Gamma distribution with $\alpha = 1.5$ and $\theta = 10$

Therefore, $Y := X_1 + X_2 + X_3 + X_4$ has a gamma distribution with $\alpha = 6$, $\theta = 10$ (you can use the mgf technique and assumption of independence to prove this).

Integrating the corresponding gamma pdf over the interval $[90, \infty)$ gives the desired probability,

$$P(Y > 90) = \int_{90}^{\infty} \frac{1}{\Gamma(6)10^6} y^{6-1} e^{-y/10}\, dy$$

Note that computing the this integral is not an easy task.

Moreover, if the question asks us to compute the probability (an actual number), then we could transform this problem into computing the a cumulative probability of some Poisson distribution as we did before.

Since $\theta = 10$, $\lambda = 1/\theta = 0.1$, and we have

$$\begin{aligned} P(Y > 90) &= P\,(\text{fewer than 6 occurences in } [0, 90]) \\ &= \sum_{k=0}^{5} \frac{(90\lambda)^k e^{-90\lambda}}{k!} \\ &= 0.1157 \end{aligned}$$

Note that the "occurrences" here does not refer to the patients, instead it refers the number of exponential random variables that we used to construct this gamma random variable (read Chapter 3.2 in the textbook).

3. 5.5-10. A consumer buys n light bulbs, each of which has a lifetime that has a mean of 800 hours, a standard deviation of 100 hours, and a normal distribution. A light bulb is replaced by another as soon as it burns out. Assuming independence of the lifetimes, find the smallest n so that the succession of light bulbs produces light for at least $10,000$ hours with a probability of 0.90.

   Solution:

   Let X_i be the lifetime of the i-th light bulbs, then we know that $X_i \sim N(800, 100^2)$ and they are independent. Therefore, the sum of the life time of n light bulbs is defined as

   $$Y := \sum_{i=1}^{n} X_i \sim N(800n, 100^2 n)$$

   Thus,

   $$\begin{aligned} & P(Y \geq 10000) = 0.9 \\ \Leftrightarrow\ & P\left(\frac{Y - 800n}{100\sqrt{n}} \geq \frac{10000 - 800n}{100\sqrt{n}}\right) = 0.9 \\ \Leftrightarrow\ & P\left(Z \geq \frac{10000 - 800n}{100\sqrt{n}}\right) = 0.9 \\ \Leftrightarrow\ & 1 - P\left(Z < \frac{10000 - 800n}{100\sqrt{n}}\right) = 0.9 \\ \Leftrightarrow\ & 1 - \Phi\left(\frac{10000 - 800n}{100\sqrt{n}}\right) = 0.9 \\ \Leftrightarrow\ & \Phi\left(\frac{10000 - 800n}{100\sqrt{n}}\right) = 0.1 \end{aligned}$$

   where Z is the standard normal random variable with cdf Φ.

   Check the normal distribution table, we have

   $$\frac{10000 - 800n}{100\sqrt{n}} = -1.282$$

   Note that this is an quadratic equation with one variable, and we could solve for n and obtain $n = 13.08$.

   In this question, the accurate result for n is required, and we could round the number to $n^* = 14$ or $n^* = 13$ optionally. This is because in this question it requires to find "...with a probability of 0.90", not "...with a probability of at lest (or at most) 0.90", which will be different for the latter.

4. 5.6-2. Let $Y = X_1 + X_2 + \cdots + X_{15}$ be the sum of a random sample of size 15 from the distribution whose pdf is $f(x) = (3/2)x^2$, $-1 < x < 1$. Using the pdf of Y, we find that $P(-0.3 \leq Y \leq 1.5) = 0.22788$. Use the central limit theorem to approximate this probability.

   Solution:

   The expectation and variance of X are given by

$$
\begin{aligned}
E(X) &= \int_{-1}^{1} x \cdot \frac{3}{2}x^2 dx = 0 \\
\mathrm{Var}(X) = E(X^2) - (E(x))^2 &= \int_{-1}^{1} \frac{3}{2}x^2 \cdot x^2 dx = \frac{3}{5}
\end{aligned}
$$

   Since $X_1, X_2, \ldots, X_{15}$ is a random sample from the same distribution, they are i.i.d. Therefore,

$$
\begin{aligned}
E(Y) &= 15 \times E(X) = 0 \\
\mathrm{Var}(Y) &= 15 \times \mathrm{Var}(X) = 9
\end{aligned}
$$

   By central limit theorem,

$$
\begin{aligned}
P(-0.3 \leq Y \leq 1.5) &= P\left(\frac{-0.3 - 0}{\sqrt{9}} \leq \frac{Y - 0}{\sqrt{9}} \leq \frac{1.5 - 0}{\sqrt{9}}\right) \\
&\approx P(-0.1 \leq Z \leq 0.5) \\
&= \Phi(0.5) - \Phi(-0.1) \\
&= \Phi(0.5) - (1 - \Phi(0.1)) \\
&= \Phi(0.5) + \Phi(0.1) - 1 \\
&= 0.2313
\end{aligned}
$$

   where Z is the standard normal random variable with cdf Φ. The final result is obtained by checking the normal distribution table.

   Compared to probability obtained by using the pdf of Y, the approximated result by central limit theorem is quite accurate.