16.6

Surface Integrals of Scalar Functions

In Exercises 1–8, integrate the given function over the given surface.

1. Parabolic cylinder $G(x, y, z) = x$, over the parabolic cylinder $y = x^2, 0 \le x \le 2, 0 \le z \le 3$

1. Let the parametrization be $\mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$$= 2x\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1} \Rightarrow \iint_S G(x, y, z)\, d\sigma = \int_0^3 \int_0^2 x\sqrt{4x^2+1}\, dx\, dz = \int_0^3 \left[\tfrac{1}{12}\left(4x^2+1\right)^{3/2}\right]_0^2 dz$$

$$= \int_0^3 \tfrac{1}{12}\left(17\sqrt{17}-1\right) dz = \tfrac{17\sqrt{17}-1}{4}$$

6. Cone $F(x, y, z) = z - x$, over the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$

6. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} = r\sqrt{2};\ z = r \text{ and } x = r\cos\theta$$

$$\Rightarrow F(x, y\, z) = r - r\cos\theta \Rightarrow \iint_S F(x, y, z)\, d\sigma = \int_0^{2\pi} \int_0^1 (r - r\cos\theta)\left(r\sqrt{2}\right) dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \int_0^1 (1-\cos\theta)r^2\, dr\, d\theta = \tfrac{2\pi\sqrt{2}}{3}$$

14. Integrate $G(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0$, $x = 1$, and $z = 0$.

14. $f(x, y, z) = y^2 + 4z = 16 \Rightarrow \nabla f = 2y\mathbf{j} + 4\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2+16} = 2\sqrt{y^2+4}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 4$

$$\Rightarrow d\sigma = \tfrac{2\sqrt{y^2+4}}{4}\, dx\, dy \Rightarrow \iint_S G\, d\sigma = \int_{-4}^4 \int_0^1 \left(x\sqrt{y^2+4}\right)\left(\tfrac{\sqrt{y^2+4}}{2}\right) dx\, dy = \int_{-4}^4 \int_0^1 \tfrac{x\left(y^2+4\right)}{2}\, dx\, dy$$

$$= \int_{-4}^4 \tfrac{1}{4}\left(y^2+4\right) dy = \tfrac{1}{2}\left[\tfrac{y^3}{3} + 4y\right]_0^4 = \tfrac{1}{2}\left(\tfrac{64}{3}+16\right) = \tfrac{56}{3}$$

Finding Flux or Surface Integrals of Vector Fields

In Exercises 19–28, use a parametrization to find the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ across the surface in the specified direction.

19. Parabolic cylinder $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward (normal away from the x-axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0, x = 1$, and $z = 0$

19. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \left(4 - y^2\right)\mathbf{k},\ 0 \le x \le 1, -2 \le y \le 2;\ z = 0 \Rightarrow 0 = 4 - y^2$

$$\Rightarrow y = \pm\ 2;\ \mathbf{r}_x = \mathbf{i} \text{ and } \mathbf{r}_y = \mathbf{j} - 2y\mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} |\mathbf{r}_x \times \mathbf{r}_y|\, dy\, dx$$

$$= (2xy - 3z)\, dy\, dx = \left[2xy - 3\left(4 - y^2\right)\right] dy\, dx \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^1 \int_{-2}^2 \left(2xy + 3y^2 - 12\right) dy\, dx$$

$$= \int_0^1 \left[xy^2 + y^3 - 12y\right]_{-2}^2 dx = \int_0^1 (-32)\, dx = -32$$

26. **Cone** $\mathbf{F} = y^2\mathbf{i} + xz\mathbf{j} - \mathbf{k}$ outward (normal away from the z-axis) through the cone $z = 2\sqrt{x^2 + y^2},\ 0 \le z \le 2$

26. Let the parametrization be $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k},\ 0 \le r \le 1$ (since $0 \le z \le 2$) and $0 \le \theta \le 2\pi$

$$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2 \end{vmatrix}$$

$$= (2r\cos\theta)\mathbf{i} + (2r\sin\theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|} |\mathbf{r}_\theta \times \mathbf{r}_r|\, d\theta\, dr$$

$$= \left(2r^3 \sin^2\theta\cos\theta + 4r^3\cos\theta\sin\theta + r\right) d\theta\, dr \text{ since } \mathbf{F} = \left(r^2\sin^2\theta\right)\mathbf{i} + \left(2r^2\cos\theta\right)\mathbf{j} - \mathbf{k}$$

$$\Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \int_0^1 \left(2r^3\sin^2\theta\cos\theta + 4r^3\cos\theta\sin\theta + r\right) dr\, d\theta = \int_0^{2\pi} \left(\tfrac{1}{2}\sin^2\theta\cos\theta + \cos\theta\sin\theta + \tfrac{1}{2}\right) d\theta$$

$$= \left[\tfrac{1}{6}\sin^3\theta + \tfrac{1}{2}\sin^2\theta + \tfrac{1}{2}\theta\right]_0^{2\pi} = \pi$$

In Exercises 29 and 30, find the surface integral of the field $\mathbf{F}$ over the portion of the given surface in the specified direction.

29. $\mathbf{F}(x, y, z) = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

S: rectangular surface $z = 0,\quad 0 \le x \le 2,\quad 0 \le y \le 3,$ direction $\mathbf{k}$

29. $g(x, y, z) = z,\ \mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1 \text{ and } |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R (\mathbf{F} \cdot \mathbf{k})\, dA$

$$= \int_0^2 \int_0^3 3\, dy\, dx = 18$$

37. Find the flux of the field $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$.

37. $g(x, y, z) = y^2 + z = 4 \Rightarrow \nabla g = 2y\mathbf{j} + \mathbf{k} \Rightarrow |\nabla g| = \sqrt{4y^2 + 1} \Rightarrow \mathbf{n} = \frac{2y\mathbf{j} + \mathbf{k}}{\sqrt{4y^2+1}} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{2xy - 3z}{\sqrt{4y^2+1}};$

$$\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2 + 1}\, dA \Rightarrow \text{Flux} = \iint_R \left(\frac{2xy - 3z}{\sqrt{4y^2+1}}\right)\sqrt{4y^2 + 1}\, dA = \iint_R (2xy - 3z)\, dA;$$

$$z = 0 \text{ and } z = 4 - y^2 \Rightarrow y^2 = 4 \Rightarrow \text{Flux} = \iint_R \left[2xy - 3\left(4 - y^2\right)\right] dA = \int_0^1 \int_{-2}^2 \left(2xy - 12 + 3y^2\right) dy\, dx$$

$$= \int_0^1 \left[xy^2 - 12y + y^3\right]_{-2}^2 dx = \int_0^1 (-32)\, dx = -32$$

16.7

Using Stokes' Theorem to Find Line Integrals

In Exercises 1–6, use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

1. $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$

C: The ellipse $4x^2 + y^2 = 4$ in the xy-plane, counterclockwise when viewed from above

1. $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (2-0)\mathbf{k} = 2\mathbf{k}$ and $\mathbf{n} = \mathbf{k} \Rightarrow \operatorname{curl} \mathbf{F} \cdot \mathbf{n} = 2 \Rightarrow d\sigma = dx\, dy$

$\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R 2\, dA = 2(\text{Area of the ellipse}) = 4\pi$

4. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + z^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

4. $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} = (2y-2z)\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k}$ and $\mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$

$\Rightarrow \operatorname{curl} \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{3}}(2y - 2z + 2z - 2x + 2x - 2y) = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S 0\, d\sigma = 0$

Integral of the Curl Vector Field

7. Let $\mathbf{n}$ be the outer unit normal of the elliptical shell

$$S: \quad 4x^2 + 9y^2 + 36z^2 = 36, \qquad z \geq 0,$$

and let

$$\mathbf{F} = y\mathbf{i} + x^2\mathbf{j} + (x^2 + y^4)^{3/2} \sin e^{\sqrt{xyz}}\,\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

(*Hint:* One parametrization of the ellipse at the base of the shell is $x = 3\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$.)

7. $x = 3\cos t$ and $y = 2\sin t \Rightarrow \mathbf{F} = (2\sin t)\mathbf{i} + \left(9\cos^2 t\right)\mathbf{j} + \left(9\cos^2 t + 16\sin^4 t\right)\sin e^{\sqrt{(6\sin t\cos t)(0)}}\mathbf{k}$ at the base of the shell; $\mathbf{r} = (3\cos t)\mathbf{i} + (2\sin t)\mathbf{j} \Rightarrow d\mathbf{r} = (-3\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -6\sin^2 t + 18\cos^3 t$

$\Rightarrow \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\left(-6\sin^2 t + 18\cos^3 t\right)dt = \left[-3t + \frac{3}{2}\sin 2t + 6(\sin t)\left(\cos^2 t + 2\right)\right]_0^{2\pi} = -6\pi$

Stokes' Theorem for Parametrized Surfaces

In Exercises 13–18, use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

14. $\mathbf{F} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x + z)\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (9 - r^2)\mathbf{k}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$

14. $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x+z \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$; $\mathbf{r}_r \times \mathbf{r}_\theta = \left(2r^2\cos\theta\right)\mathbf{i} + \left(2r^2\sin\theta\right)\mathbf{j} + r\mathbf{k}$ and

$\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = (\nabla\times\mathbf{F})\cdot(\mathbf{r}_r\times\mathbf{r}_\theta)\,dr\,d\theta$ (see Exercise 13 above)

$\Rightarrow \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^3\left(-2r^2\cos\theta - 4r^2\sin\theta - 2r\right)dr\,d\theta = \int_0^{2\pi}\left[-\frac{2}{3}r^3\cos\theta - \frac{4}{3}r^3\sin\theta - r^2\right]_0^3 d\theta$

$= \int_0^{2\pi}(-18\cos\theta - 36\sin\theta - 9)\,d\theta = -9(2\pi) = -18\pi$

15. $\mathbf{F} = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

15. $\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y & 2y^3z & 3z\end{vmatrix}=-2y^3\mathbf{i}+0\mathbf{j}-x^2\mathbf{k};\ \mathbf{r}_r\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & \sin\theta & 1\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix}=(-r\cos\theta)\mathbf{i}-(r\sin\theta)\mathbf{j}+r\mathbf{k}$ and

$\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma=(\nabla\times\mathbf{F})\cdot(\mathbf{r}_r\times\mathbf{r}_\theta)\,dr\,d\theta$ (see Exercise 13 above) $\Rightarrow\iint_S\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma=\iint_R\left(2ry^3\cos\theta-rx^2\right)dr\,d\theta$

$=\int_0^{2\pi}\int_0^1\left(2r^4\sin^3\theta\cos\theta-r^3\cos^2\theta\right)dr\,d\theta=\int_0^{2\pi}\left(\frac{2}{5}\sin^3\theta\cos\theta-\frac{1}{4}\cos^2\theta\right)d\theta=\left[\frac{1}{10}\sin^4\theta-\frac{1}{4}\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\right]_0^{2\pi}$

$=-\frac{\pi}{4}$

21. Zero circulation Use Equation (8) and Stokes' Theorem to show that the circulations of the following fields around the boundary of any smooth orientable surface in space are zero.

a. $\mathbf{F}=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$ **b.** $\mathbf{F}=\nabla(xy^2z^3)$

c. $\mathbf{F}=\nabla\times(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$ **d.** $\mathbf{F}=\nabla f$

21. (a) $\mathbf{F}=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}\Rightarrow\text{curl }\mathbf{F}=\mathbf{0}\Rightarrow\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint_S\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma=\iint_S 0\,d\sigma=0$

(b) Let $f(x,y,z)=x^2y^2z^3\Rightarrow\nabla\times\mathbf{F}=\nabla\times\nabla f=\mathbf{0}\Rightarrow\text{curl }\mathbf{F}=\mathbf{0}\Rightarrow\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint_S\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma=\iint_S 0\,d\sigma=0$

(c) $\mathbf{F}=\nabla\times(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\mathbf{0}\Rightarrow\nabla\times\mathbf{F}=\mathbf{0}\Rightarrow\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint_S\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma=\iint_S 0\,d\sigma=0$

(d) $\mathbf{F}=\nabla f\Rightarrow\nabla\times\mathbf{F}=\nabla\times\nabla f=\mathbf{0}\Rightarrow\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint_S\nabla\times\mathbf{F}\cdot\mathbf{n}\ \ d\sigma=\iint_S 0\,d\sigma=0$

16.8

Calculating Flux Using the Divergence Theorem

In Exercises 5–16, use the Divergence Theorem to find the outward flux of **F** across the boundary of the region D.

5. Cube $\mathbf{F}=(y-x)\mathbf{i}+(z-y)\mathbf{j}+(y-x)\mathbf{k}$

D: The cube bounded by the planes $x=\pm1$, $y=\pm1$, and $z=\pm1$

5. $\frac{\partial}{\partial x}(y-x)=-1,\ \frac{\partial}{\partial y}(z-y)=-1,\ \frac{\partial}{\partial z}(y-x)=0\Rightarrow\nabla\cdot\mathbf{F}=-2\Rightarrow\text{Flux}=\int_{-1}^1\int_{-1}^1\int_{-1}^1-2\,dx\,dy\,dz=-2\left(2^3\right)=-16$

10. Cylindrical can $\mathbf{F} = (6x^2 + 2xy)\mathbf{i} + (2y + x^2z)\mathbf{j} + 4x^2y^3\mathbf{k}$

D: The region cut from the first octant by the cylinder $x^2 + y^2 = 4$ and the plane $z = 3$

10. $\frac{\partial}{\partial x}\left(6x^2+2xy\right)=12x+2y, \frac{\partial}{\partial y}\left(2y+x^2z\right)=2, \frac{\partial}{\partial z}\left(4x^2y^3\right)=0 \Rightarrow \nabla\cdot\mathbf{F}=12x+2y+2$

$\Rightarrow \text{Flux} = \iiint\limits_D (12x+2y+2)dV = \int_0^3\int_0^{\pi/2}\int_0^2 (12r\cos\theta+2r\sin\theta+2)\, r\, dr\, d\theta\, dz$

$= \int_0^3\int_0^{\pi/2}\left(32\cos\theta+\frac{16}{3}\sin\theta+4\right)d\theta\, dz = \int_0^3\left(32+2\pi+\frac{16}{3}\right)dz = 112+6\pi$

13. Thick sphere $\mathbf{F} = \sqrt{x^2 + y^2 + z^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

D: The region $1 \le x^2 + y^2 + z^2 \le 2$

13. Let $\rho=\sqrt{x^2+y^2+z^2}$. Then $\frac{\partial\rho}{\partial x}=\frac{x}{\rho}, \frac{\partial\rho}{\partial y}=\frac{y}{\rho}, \frac{\partial\rho}{\partial z}=\frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}(\rho x)=\left(\frac{\partial\rho}{\partial x}\right)x+\rho=\frac{x^2}{\rho}+\rho,$

$\frac{\partial}{\partial y}(\rho y)=\left(\frac{\partial\rho}{\partial y}\right)y+\rho=\frac{y^2}{\rho}+\rho, \frac{\partial}{\partial z}(\rho z)=\left(\frac{\partial\rho}{\partial z}\right)z+\rho=\frac{z^2}{\rho}+\rho \Rightarrow \nabla\cdot\mathbf{F}=\frac{x^2+y^2+z^2}{\rho}+3\rho=4\rho,$ since

$\rho=\sqrt{x^2+y^2+z^2} \Rightarrow \text{Flux} = \iiint\limits_D 4\rho\, dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}}(4\rho)\left(\rho^2\sin\phi\right)d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}3\sin\phi\, d\phi\, d\theta$

$= \int_0^{2\pi}6\, d\theta = 12\pi$

19. Let $\mathbf{F} = (y\cos 2x)\mathbf{i} + (y^2\sin 2x)\mathbf{j} + (x^2y + z)\mathbf{k}$. Is there a vector field $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$? Explain your answer.

19. For the field $\mathbf{F}=(y\cos 2x)\mathbf{i}+(y^2\sin 2x)\mathbf{j}+(x^2y+x)\mathbf{k}$, $\nabla\cdot\mathbf{F}=-2y\sin 2x+2y\sin 2x+1=1$. If $\mathbf{F}$ were the curl of a field $\mathbf{A}$ whose component functions have continuous second partial derivatives, then we would have $\text{div}\,\mathbf{F}=\text{div}(\text{curl}\,\mathbf{A})=\nabla\cdot(\nabla\times\mathbf{A})=0$. Since $\text{div}\,\mathbf{F}=1$, $\mathbf{F}$ is not the curl of such a field.

22. Maximum flux Among all rectangular solids defined by the inequalities $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le 1$, find the one for which the total flux of $\mathbf{F} = (-x^2 - 4xy)\mathbf{i} - 6yz\mathbf{j} + 12z\mathbf{k}$ outward through the six sides is greatest. What *is* the greatest flux?

22. $\nabla\cdot\mathbf{F}=-2x-4y-6z+12 \Rightarrow \text{Flux} = \int_0^a\int_0^b\int_0^1(-2x-4y-6z+12)\, dz\, dy\, dx = \int_0^a\int_0^b(-2x-4y+9)\, dy\, dx =$

$\int_0^a\left(-2xb-2b^2+9b\right)dx = -a^2b-2ab^2+9ab = ab(-a-2b+9) = f(a,b); \frac{\partial f}{\partial a}=-2ab-2b^2+9b$ and

$\frac{\partial f}{\partial b}=-a^2-4ab+9a$ so that $\frac{\partial f}{\partial a}=0$ and $\frac{\partial f}{\partial b}=0 \Rightarrow b(-2a-2b+9)=0$ and $a(-a-4b+9)=0 \Rightarrow b=0$ or

$-2a-2b+9=0$, and $a=0$ or $-a-4b+9=0$. Now $b=0$ or $a=0 \Rightarrow \text{Flux}=0; -2a-2b+9=0$ and

$-a-4b+9=0 \Rightarrow 3a-9=0 \Rightarrow a=3 \Rightarrow b=\frac{3}{2}$ so that $f\left(3,\frac{3}{2}\right)=\frac{27}{2}$ is the maximum flux.

28. Harmonic functions A function $f(x, y, z)$ is said to be *harmonic* in a region D in space if it satisfies the Laplace equation

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

throughout D.

a. Suppose that f is harmonic throughout a bounded region D enclosed by a smooth surface S and that $\mathbf{n}$ is the chosen unit normal vector on S. Show that the integral over S of $\nabla f \cdot \mathbf{n}$, the derivative of f in the direction of $\mathbf{n}$, is zero.

b. Show that if f is harmonic on D, then

$$\iint_S f \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D |\nabla f|^2\, dV.$$

28. (a) From the Divergence Theorem, $\iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \nabla f\, dV = \iiint_D \left(\nabla^2 f\right) dV = \iiint_D 0\, dV = 0$

(b) From the Divergence Theorem, $\iint_S f \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot f \nabla f\, dV$. Now,

$$f\nabla f = \left(f \frac{\partial f}{\partial x}\right)\mathbf{i} + \left(f \frac{\partial f}{\partial y}\right)\mathbf{j} + \left(f \frac{\partial f}{\partial z}\right)\mathbf{k} \Rightarrow \nabla \cdot f \nabla f = \left[f \frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial x}\right)^2\right] + \left[f \frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial y}\right)^2\right] + \left[f \frac{\partial^2 f}{\partial z^2} + \left(\frac{\partial f}{\partial x}\right)^2\right]$$

$= f\nabla^2 f + |\nabla f|^2 = 0 + |\nabla f|^2$ since f is harmonic $\Rightarrow \iint_S f \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D |\nabla f|^2\, dV$, as claimed.

29. Green's first formula Suppose that f and g are scalar functions with continuous first- and second-order partial derivatives throughout a region D that is bounded by a closed piecewise smooth surface S. Show that

$$\iint_S f \nabla g \cdot \mathbf{n}\, d\sigma = \iiint_D (f \nabla^2 g + \nabla f \cdot \nabla g)\, dV. \qquad (10)$$

Equation (10) is **Green's first formula**. (*Hint:* Apply the Divergence Theorem to the field $\mathbf{F} = f \nabla g$.)

29. $\iint\limits_S f\nabla g\cdot\mathbf{n}\,d\sigma = \iiint\limits_D \nabla\cdot f\nabla g\,dV = \iiint\limits_D \nabla\cdot\left(f\frac{\partial g}{\partial x}\mathbf{i}+f\frac{\partial g}{\partial y}\mathbf{j}+f\frac{\partial g}{\partial z}\mathbf{k}\right)dV$

$= \iiint\limits_D\left(f\frac{\partial^2 g}{\partial x^2}+\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial y^2}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial z^2}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)dV$

$= \iiint\limits_D\left[f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)\right]dV = \iiint\limits_D\left(f\nabla^2 g+\nabla f\cdot\nabla g\right)dV$

30. Green's second formula (*Continuation of Exercise 29.*) Interchange f and g in Equation (10) to obtain a similar formula. Then subtract this formula from Equation (10) to show that

$$\iint\limits_S (f\nabla g - g\nabla f)\cdot\mathbf{n}\,d\sigma = \iiint\limits_D (f\nabla^2 g - g\nabla^2 f)\,dV. \qquad (11)$$

This equation is **Green's second formula.**

30. By Exercise 29, $\iint\limits_S f\nabla g\cdot\mathbf{n}\,d\sigma = \iiint\limits_D\left(f\nabla^2 g+\nabla f\cdot\nabla g\right)dV$ and by interchanging the roles of f and g, $\iint\limits_S g\nabla f\cdot\mathbf{n}\,d\sigma = \iiint\limits_D\left(g\nabla^2 f+\nabla g\cdot\nabla f\right)dV$. Subtracting the second equation from the first yields: $\iint\limits_S (f\nabla g - g\nabla f)\cdot\mathbf{n}\,d\sigma = \iiint\limits_D\left(f\nabla^2 g - g\nabla^2 f\right)dV$ since $\nabla f\cdot\nabla g = \nabla g\cdot\nabla f$.