Review of the last lecture

Key concepts and/or techniques:

- Conditional distribution
  Motivation: it is a probability distribution that describes the distribution of probability of events of a RV given the occurrence of a particular event.

  For example, the conditional pmf of X given $Y = y$ is

  $$g(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad x \in \overline{S_X}(y)$$

  provided that $f_Y(y) > 0$.

- Conditional mathematical expectations
  The conditional expectation of $g(Y)$ given $X = x$

  $$E(g(Y)|X = x) = \sum_{y \in \overline{S_Y}(x)} g(y)h(y|x)$$

Review of the last lecture

[Conditional pmf]

Conditional pmf of X given $Y = y$ is defined by

$$g(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad x \in \overline{S_X}(y)$$

provided that $f_Y(y) > 0$.

Similarly, conditional pmf of Y given that $X = x$ is defined by

$$h(y|x) = \frac{f(x,y)}{f_X(x)}, \quad y \in \overline{S_Y}(x)$$

provided that $f_X(x) > 0$.

Review of the last lecture

Conditional pmf

Conditional pmf is a well-defined pmf

1. $h(y|x) > 0$
2. $\sum_{y \in \overline{S_Y}(x)} h(y|x) = 1$
3. for $A \subseteq \overline{S_Y}(x)$

$$P(Y \in A|X = x) = \sum_{y \in A} h(y|x)$$

Review of the last lecture

[Conditional Mathematical Expectation]

- Let $g(Y)$ be a function of Y.

  Then the conditional expectation of $g(Y)$ given $X = x$

$$E(g(Y)|X = x) = \sum_{y \in \overline{S_Y}(x)} g(y)h(y|x)$$

- When $g(Y) = Y$, conditional mean

$$E(Y|X = x) = \sum_{y \in \overline{S_Y}(x)} yh(y|x)$$

- When $g(Y) = [Y - E(Y|X = x)]^2$, conditional variance

$$Var(Y|X = x) \stackrel{\Delta}{=} E\{[Y - E(Y|X = x)]^2|X = x\}$$

Review of the last lecture

Let X and Y be two continuous random variables and (X, Y) be a pair of RVs with their range denoted by $\overline{S} \subseteq R^2$. Then (X, Y) or X and Y is said to be a bivariate continuous RV.

[Road map]

To study the bivariate continuous random variable

discrete RV $\longrightarrow$	continuous RV	Mathematical expectations
pmf $\longrightarrow$	pdf	mean
joint pmf $\longrightarrow$	joint pdf	variance
marginal pmf $\longrightarrow$	marginal pdf	covariance
conditional pmf $\longrightarrow$	conditional pdf	correlation coefficient

STA2001 Probability and Statistics (I)

Lecture 17

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Joint pdf

Definition

The joint pdf of two continuous RVs X and Y is a function $f(x,y) : \overline{S} \to (0,\infty)$ with the following properties:

1. $f(x,y) > 0, (x,y) \in \overline{S}$
2. $\iint_{\overline{S}} f(x,y)dxdy = 1$
3. For $A \subseteq \overline{S}$,

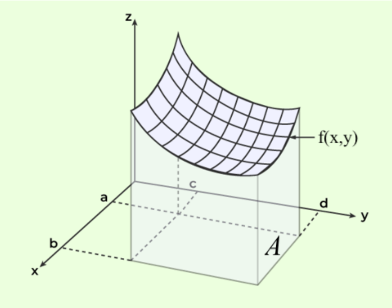
$$\begin{aligned} P((X,Y) \in A) &\triangleq P(\{(X,Y) \in A\}) \\ &= \iint_A f(x,y)dxdy \end{aligned}$$

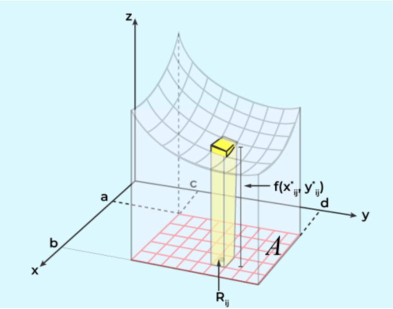
Remarks

Recall the geometric interpretation of double integral:

$$P((X,Y)\in A)=\iint_A f(x,y)dxdy$$

calculates the volume of the solid under the surface $z=f(x,y)$ over the region A in the the xy-plane.

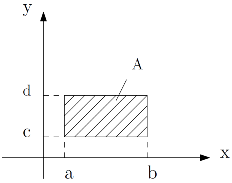




Remarks

- Very often, we extend the definition domain of $f(x, y)$ from $\overline{S}$ to $R \times R$ by letting $f(x, y) = 0$, for $(x, y) \notin \overline{S}$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$

- Bottom line:



If the set A is rectangular with its line segments parallel to the coordinate axes, i.e.,

$$A = \{(x, y) | a \leq x \leq b, c \leq y \leq d\},$$

then the double integral becomes

$$P((X, Y) \in A) = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

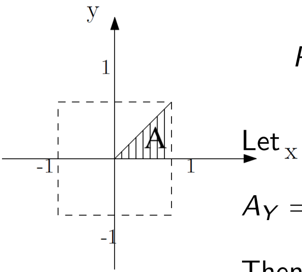
Remarks

- General case:

Let

$$A_X = \{x|(x,y) \in A\}, A_Y(x) = \{y|(x,y) \in A\} \text{ for } x \in A_X$$

Then



$$P((X,Y) \in A) = \int_{A_X} \int_{A_Y(x)} f(x,y)dy\,dx$$

Let

$$A_Y = \{y|(x,y) \in A\}, A_X(y) = \{x|(x,y) \in A\} \text{ for } y \in A_Y$$

Then

$$P((X,Y) \in A) = \int_{A_Y} \int_{A_X(y)} f(x,y)dx\,dy$$

Marginal pdf

Definition

The marginal pdf of X, $f_X(x) : \overline{S_X} \to (0, \infty)$

$$f_X(x) = \int_{\overline{S_Y}(x)} f(x, y) dy$$

$$\overline{S_Y}(x) = \{y | (x, y) \in \overline{S}\} \text{ for } x \in \overline{S_X}$$

The marginal pdf of Y, $f_Y(y) : \overline{S_Y} \to (0, \infty)$

$$f_Y(y) = \int_{\overline{S_X}(y)} f(x, y) dx$$

$$\overline{S_X}(y) = \{x | (x, y) \in \overline{S}\} \text{ for } y \in \overline{S_Y}$$

Example 1, page 156

Question

Let X and Y have the joint pdf

$$f(x,y) = \frac{3}{2}x^2(1-|y|), \quad -1 < x < 1, \quad -1 < y < 1$$

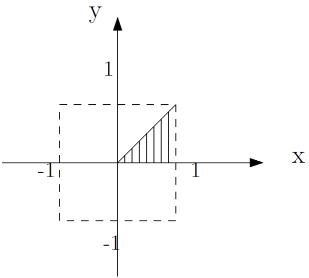
$$\Longrightarrow \begin{cases} \overline{S_X} = \{x| \quad -1 < x < 1\} \\ \overline{S_Y} = \{y| \quad -1 < y < 1\} \\ \overline{S} = \{(x,y)| -1 < x < 1, -1 < y < 1\} \end{cases}$$

Q1: Let $A = \{(x,y)|0 < x < 1, 0 < y < x\}$. What is the probability of A?

Q2: What is the marginal pdf of X and Y?

Q3: What is the expectation of X?

Example 1, page 156



Q1:

$$
\begin{aligned}
P(A) &= \int_0^1 \int_0^x \frac{3}{2}x^2(1-|y|)dydx \\
&= \int_0^1 \frac{3}{2}x^2(x-\frac{1}{2}x^2)dx = \frac{3}{2}\cdot\frac{1}{4}x^4\Big|_0^1 - \frac{3}{4}\cdot\frac{1}{5}x^5\Big|_0^1 \\
&= \int_0^1 \int_y^1 \frac{3}{2}x^2(1-|y|)dxdy \\
&= \int_0^1 \frac{3}{2}\cdot\frac{1}{3}x^3\Big|_y^1 (1-|y|)dy \\
&= \frac{9}{40}
\end{aligned}
$$

Example 1, page 156

Q2:

$$\begin{aligned}
\text{For } x \in \overline{S_X}, f_X(x) &= \int_{\overline{S_Y}(x)} f(x,y)dy \\
&= \int_{-1}^{1} \frac{3}{2}x^2(1-|y|)dy = \frac{3}{2}x^2(2+(-1)) = \frac{3}{2}x^2 \\
\text{For } y \in \overline{S_Y}, f_Y(y) &= \int_{\overline{S_X}(y)} f(x,y)dx \\
&= \int_{-1}^{1} \frac{3}{2}x^2(1-|y|)dx = \frac{3}{2}(1-|y|)\frac{1}{3}x^3\Big|_{-1}^{1} \\
&= 1-|y|
\end{aligned}$$

Example 1, page 156

Q3:

$$E(X) = \int_{\overline{S_X}} x f_X(x) dx = \int_{-1}^{1} x \frac{3}{2} x^2 dx = 0$$

$$E(X) = \int \int_{\overline{S}} x f(x, y) dx dy = \int_{-1}^{1} \int_{-1}^{1} \frac{3}{2} x^3 (1 - |y|) dx dy$$
$$= \int_{-1}^{1} \frac{3}{2} x^3 dx \int_{-1}^{1} (1 - |y|) dy = 0$$

Mathematical Expectation

Definition

Let $g(X,Y)$ be a function of X and Y, whose joint pdf $f(x,y) : \overline{S} \to (0,\infty)$. Then

$$\boxed{E[g(X,Y)] = \iint_{\overline{S}} g(x,y) f(x,y)\,dxdy}$$

- $g(X,Y) = X \to$ mean of X

$$\begin{aligned}
E[X] &= \iint_{\overline{S}} x f(x,y)\,dxdy \\
&= \int_{\overline{S_X}} x \int_{\overline{S_{Y}(x)}} f(x,y)\,dydx \\
&= \int_{\overline{S_X}} x f_X(x)\,dx
\end{aligned}$$

Mathematical Expectation

Definition

- $g(X, Y) = (X - E[X])^2 \rightarrow$ variance of X

$$\begin{aligned} Var[X] &= \iint_{\overline{S}} (x - E[X])^2 f(x, y)\,dxdy \\ &= \int_{\overline{S_X}} (x - E[X])^2 \int_{\overline{S_Y(x)}} f(x, y)\,dydx \\ &= \int_{\overline{S_X}} (x - E[X])^2 f_X(x)\,dx \end{aligned}$$

Example 2, page 155

Question

Let X and Y have the joint pdf

$$f(x,y) = (\frac{4}{3})(1-xy) \quad \text{with} \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

Q1 Find the marginal pdfs of X and Y?

Q2 Find the expectation of X?

Q3 Find the variance of X

Example 2, page 155

Question

Let X and Y have the joint pdf

$$f(x,y) = (\frac{4}{3})(1 - xy) \quad \text{with} \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

- Q1 Find the marginal pdfs of X and Y?
- Q2 Find the expectation of X?
- Q3 Find the variance of X

Q1 :

$$f_X(x) = \int_{\overline{S_Y}(x)} f(x,y)dy = \int_0^1 \frac{4}{3}(1 - xy)dy$$
$$= \frac{4}{3} - \frac{4}{3}x\frac{1}{2}y^2\Big|_0^1 = \frac{4}{3}(1 - \frac{1}{2}x)$$

Example 2, page 155

$$f_Y(y) = \int_{\overline{S_X(y)}} f(x,y)dx$$
$$= \int_0^1 \frac{4}{3}(1-xy)dx = \frac{4}{3}(1-\frac{1}{2}y)$$

Q2:

$$E[X] = \int_{\overline{S_X}} xf_X(x)dx = \int_0^1 x\frac{4}{3}(1-\frac{1}{2}x)dx$$
$$= \frac{4}{3}\cdot\frac{1}{2}x^2\Big|_0^1 - \frac{4}{6}\cdot\frac{1}{3}x^3\Big|_0^1 = \frac{4}{9}$$

Q3:

$$Var[X] = \int_{\overline{S_X}} (x-E[X])^2 f_X(x)dx = \int_0^1 (x-\frac{4}{9})^2\frac{4}{3}(1-\frac{1}{2}x)dx$$

Independent Continuous RVs

Definition

Two continuous RVs X and Y are independent if

$$f(x,y) = f_X(x)f_Y(y), \qquad x \in \overline{S_X}, \quad y \in \overline{S_Y}$$

If X and Y are not independent, then we say X and Y are dependent.

When X and Y are independent,

$$\overline{S} = \overline{S_X} \times \overline{S_Y}. \quad \overline{S} \text{ is said to be rectangular}$$

which is a necessary condition for independence of X and Y.

Example 2 — revisited

Note that

$$f(x,y) = (\frac{4}{3})(1 - xy) \quad \text{with} \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

$$f_X(x) = \frac{4}{3}(1 - \frac{1}{2}x)$$
$$f_Y(y) = \frac{4}{3}(1 - \frac{1}{2}y)$$

Since $f(x,y) \neq f_X(x)f_Y(y)$, X and Y are NOT independent.

Covariance and Correlation Coefficient

Definition

$$\begin{aligned} Cov(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

$$E(XY) = \iint_S xyf(x, y)dxdy.$$

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}, \quad Var(X) > 0, Var(Y) > 0.$$

Conditional pdf

Definition

Let X and Y have a joint pdf $f(x,y) : \overline{S} \to (0,\infty)$ and marginal pdf $f_X(x) : \overline{S_X} \to (0,\infty)$ and $f_Y(y) : \overline{S_Y} \to (0,\infty)$.

The conditional pdf of Y, given that $X = x$ are

$$h(y|x) = \frac{f(x,y)}{f_X(x)} \quad \text{for} \quad f_X(x) > 0, y \in \overline{S_Y}(x)$$

$$\text{For } A \subseteq \overline{S_Y}(x), \quad P(Y \in A | X = x) = \int_{y \in A} h(y|x)dy.$$

Conditional pdf

Definition

Let X and Y have a joint pdf $f(x, y) : \overline{S} \to (0, \infty)$ and marginal pdf $f_X(x) : \overline{S_X} \to (0, \infty)$ and $f_Y(y) : \overline{S_Y} \to (0, \infty)$.

The conditional pdf of Y, given that $X = x$ are

$$h(y|x) = \frac{f(x, y)}{f_X(x)} \quad \text{for} \quad f_X(x) > 0, y \in \overline{S_Y}(x)$$

$$\text{For } A \subseteq \overline{S_Y}(x), \quad P(Y \in A|X = x) = \int_{y \in A} h(y|x)dy.$$

The conditional pdf of X, given that $Y = y$ are

$$g(x|y) = \frac{f(x, y)}{f_Y(y)} \quad \text{for} \quad f_Y(y) > 0, x \in \overline{S_X}(y)$$

$$\text{For } A \subseteq \overline{S_X}(y),\ P(X \in A|Y = y) = \int_{x \in A} g(x|y)dx.$$

Conditional mathematical expectation

Definition

The conditional mathematical expectation of a function of Y, $g(Y)$, given that $X = x$ is

$$E(g(Y)|X = x) = \int_{\overline{S_Y}(x)} g(y)h(y|x)dy$$

The conditional mean and variance of Y, given that $X = x$ are

$$\begin{aligned} E(Y|X = x) &= \int_{\overline{S_Y}(x)} yh(y|x)dy \\ Var(Y|X = x) &= E\{[Y - E(Y|X = x)]^2|X = x\} \\ &= \int_{\overline{S_Y}(x)} [y - E(Y|X = x)]^2 h(y|x)dy \\ &= E[Y^2|X = x] - [E(Y|X = x)]^2 \end{aligned}$$

Example 3, page 157

Question

Let X and Y be two continuous RVs with

$$f(x,y) = 2, \quad 0 \le x \le y \le 1$$

$$\overline{S} = \{(x,y)|0 \le x \le y \le 1\}, \overline{S_X} = \overline{S_Y} = [0,1]$$

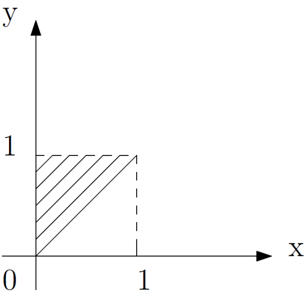
Q1: $f_X(x), f_Y(y), E(X), E(Y)$?

Q2: $h(y|x), E(Y|X=x), Var(Y|X=x)$?

Q3: $P(\frac{3}{4} \le Y \le \frac{7}{8}|X = \frac{1}{4})$?

Example 3, page 157

Q1:



$$f_X(x) = \int_{\overline{S_Y(x)}} f(x,y)dy = \int_x^1 2dy = 2(1-x), 0 \le x \le 1$$

$$f_Y(y) = \int_{\overline{S_X(y)}} f(x,y)dx = \int_0^y 2dy = 2y, \quad 0 \le y \le 1$$

$$E(X) = \int_{\overline{S_X}} xf_X(x)dx = \int_0^1 x2(1-x) = \frac{1}{3}$$

$$E(Y) = \int_{\overline{S_Y}} yf_Y(y)dy = \int_0^1 2y^2dy = \frac{2}{3}$$

Example 3, page 157

Q2:

$$
\begin{aligned}
h(y|x) &= \frac{f(x,y)}{f_X(x)} = \frac{1}{1-x}, 0 \le x \le y \le 1. \\
E(Y|X=x) &= \int_{\overline{S_Y}(x)} y h(y|x) dy = \int_x^1 y \frac{1}{1-x} dy \\
&= \frac{1}{2(1-x)}(1-x^2) = \frac{1}{2}(1+x) \\
Var(Y|X=x) &= \int_{\overline{S_Y}(x)} [y - \frac{1}{2}(1+x)]^2 h(y|x) dy \\
&= \int_x^1 \frac{1}{1-x}[y - \frac{1}{2}(1+x)]^2 dy \\
&= \frac{1}{3}\frac{1}{1-x}[y - \frac{1}{2}(1+x)]^3 \Big|_x^1 = \frac{1}{12}(1-x)^2
\end{aligned}
$$

Example 3, page 157

Q3:

$$P(\frac{3}{4} \leq Y \leq \frac{7}{8}|X = \frac{1}{4}) = \int_{\frac{3}{4}}^{\frac{7}{8}} h(y|\frac{1}{4})dy = \frac{4}{3}(\frac{7}{8} - \frac{3}{4}) = \frac{1}{6}$$