MAT1002 Lecture 21, Thursday, Apr/06/2023

Outline

- Change of variable for triple integrals (15.8)
- Spherical coordinates (15.7)
- Line integrals of real-valued functions (16.1)

Change of Variable Formula (Triple Integrals)

There is a similar change of variables formula for triple integrals. Let T be a transformation given by

$$x = g(u, v, w), \quad y = h(u, v, w) \quad \text{and} \quad z = k(u, v, w),$$

which maps a region R in the uvw-space onto a region E in the xyz-space. The Jacobian of T is the following 3×3 determinant:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} := \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

Under assumptions similar to those in the theorem of change of variables for double integrals, the following formula holds for triple integrals:

$$\iiint_E f(x, y, z)\, dV$$
$$= \iiint_R f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

e.g. Evaluate $\int_0^3\int_0^4\int_{x=y/2}^{y/2+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz.$ $(=: I)$

Sol: Let $u=\frac{2x-y}{2}$, $v=\frac{z}{3}$, $w=\frac{y}{2}$.

$\Rightarrow x=u+w,\quad y=2w,\quad z=3v$

$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}1&0&1\\0&0&2\\0&3&0\end{vmatrix}=-6 \Rightarrow \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|=6.$

$0\le z\le 3 \iff 0\le v\le 1$

$0\le y\le 4 \iff 0\le w\le 2$

$\frac{y}{2}\le x\le \frac{y}{2}+1 \iff w\le u+w\le w+1 \iff 0\le u\le 1$

$$\therefore I=\int_0^1\int_0^1\int_0^2(u+v)\,6\,dw\,dv\,du=12\int_0^1\int_0^1(u+v)\,dv\,du$$

$$=12\int_0^1\left(uv+\frac{1}{2}v^2\right)\Big|_{v=0}^{1}du=12\int_0^1\left(u+\frac{1}{2}\right)du$$

$$=12\left(\frac{1}{2}u^2+\frac{1}{2}u\right)\Big|_{u=0}^{1}=12.$$

Spherical coordinates

Consider finding the mass of a closed ball B with radius a, centered at $(0,0,0)$:

$$\text{mass}(B) = \iiint_B \rho(x,y,z)\,dV = \underbrace{\int_0^{2\pi}\int_0^a \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a-x^2-y^2}} \rho(x,y,z)\,dz\,r\,dr\,d\theta}_{\text{Cylindrical}}$$

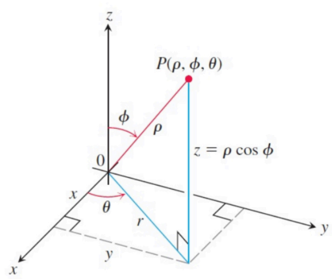
The inner integral is not friendly.

Spherical coordinates may help.

Given any point P in the xyz-space, we can represent the point by (ρ, ϕ, θ), where

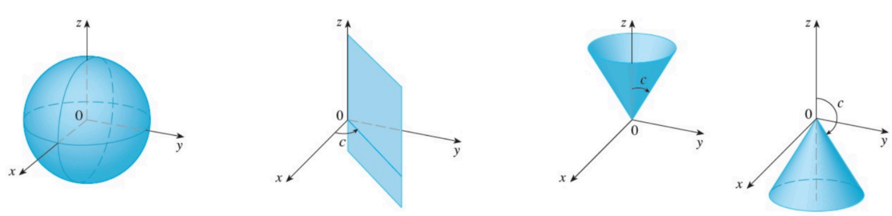
- ρ is the distance from P to the origin ($\rho \geq 0$);
- ϕ is the angle made from the positive z-axis to $\overrightarrow{OP}$ ($0 \leq \phi \leq \pi$), and;
- θ is the same as in cylindrical coordinates.

The coordinate system above in (ρ, ϕ, θ) is called the spherical coordinate system.



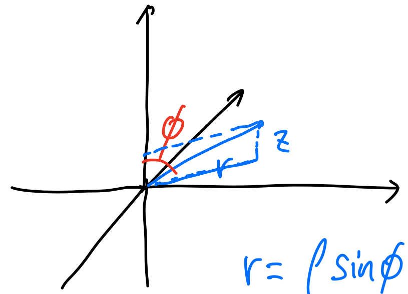
The figures below, from left to right, show the surfaces for:

1. $\rho = c$.
2. $\theta = c$.
3. $\phi = c$.



$0 < c < \pi/2$ $\quad$ $\pi/2 < c < \pi$

Conversion Between Coordinate Systems



Let r be as in cylindrical coordinates. Then

$$z = \rho\cos\phi \quad \text{and} \quad r = \rho\sin\phi,$$

$r = \rho \sin\phi$
$z = \rho \cos\phi$

from which we can deduce

$$\boxed{x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.}$$

By definition, we also have (given x, y & z)

$$\boxed{\rho = \sqrt{x^2 + y^2 + z^2}.}$$

Then we can find ϕ and then θ as well.

e.g. (a) Find a spherical coordinate equation for the sphere

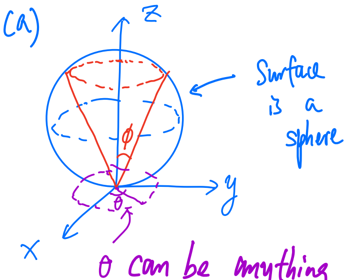
$$x^2 + y^2 + (z-1)^2 = 1.$$

(b) Find a spherical coordinate equation for the cone

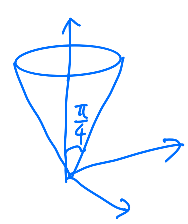
$$z = \sqrt{x^2 + y^2}.$$

Ans: (a) $\rho = 2\cos\phi$
$0 \le \phi \le \pi/2$.
(b) $\phi = \pi/4$.

(a)



(b)



Integration with Spherical Coordinates

When evaluating $\iiint_E f(x,y,z)\,dV$ with spherical substitutions

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi,$$

the absolute value of the Jacobian is

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \rho^2\sin\phi,$$

so the change of differentials becomes

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Check : $x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi,$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

Expand along Row 3

$$= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} - (-\rho\sin\phi) \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi \left(\rho^2 \left(\underbrace{\cancel{\cos\phi\sin\phi\cos^2\theta + \cos\phi\sin\phi\sin^2\theta}}^{\cos\phi\sin\phi} \right) \right)$$

$$+ \rho\sin\phi \left(\rho \left(\underbrace{\cancel{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta}}_{\sin^2\phi} \right) \right)$$

$$= \rho^2 \sin\phi (\cos^2\phi + \sin^2\phi) = \rho^2 \sin\phi.$$

Since $0 \le \phi \le \pi$, $\left| \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} \right| = \rho^2 \sin\phi.$

Example

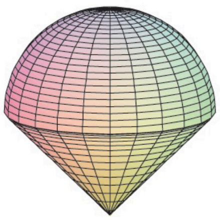
(a) Evaluate $\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where B is the closed ball centered at the origin with radius 1.

(b) Find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.

Ans:

(a) $\frac{4\pi}{3}(e-1)$.

(b) $\pi/8$.

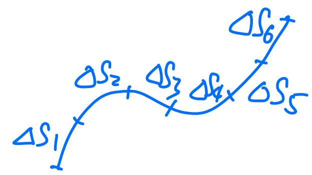


☺ An ice cream ☺

e.g. Find the volume of the solid E, where E is the intersection of solid $x^2+y^2 \le 1$ and solid $x^2+y^2+z^2 \le 4$.

Line Integrals of Real-Valued Functions

Suppose we would like to compute the mass of a wire C in the space with linear density $f(x, y, z)$. By breaking C into smaller pieces and picking points (x_k, y_k, z_k) from the ~~subintervals~~ subarcs, the mass is approximately



$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k.$$

This motivates the definition of line integrals.

Definition

If f is a real-valued function defined on a curve C, then the **line integral of f along C** ~~(with respect to arc length)~~ is

$$\int_C f(x, y, z)\, ds := \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k,$$

s is arc length parameter

provided that the limit exists.

P partition C into pieces

- For plane curve C, the line integral of $f(x,y)$ along C is $\int_C f(x,y)\,ds$, defined to be $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta s_k$.
- Line integrals can be defined similarly for an n-variable real-valued function $f(x_1, \dots, x_n)$ and curves C in $\mathbb{R}^n$.
- Will focus on $\mathbb{R}^2$ and $\mathbb{R}^3$.

Computation

- A function $\mathbf{r} : I \to \mathbb{R}^n$ is called smooth if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ for every t. A curve having a smooth parametrization is called a smooth curve. } Recall
- If C is given by a smooth parametrization

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \leq t \leq b,$$

and f is continuous on C, then $\int_C f\, ds$ exists, and

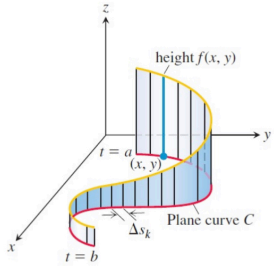
$$\int_C f(x,y,z)\, ds = \int_a^b f\left(x(t), y(t), z(t)\right) \underbrace{|\mathbf{r}'(t)|}_{\text{speed of movement}}\, dt$$
$$= \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)|\, dt.$$

- If $f \equiv 1$, then $\int_C f\, ds$ just gives the arc length of C.

- For curves in $\mathbb{R}^2$, it is similar:

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) |\vec{r}'(t)|\, dt = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)|\, dt.$$

- The line integral of a nonnegative function $f(x,y)$ along a plane curve can be interpreted as the area of the "fence" between the graph and the xy-plane, as indicated below.



(At least) Two meanings of $\int_C f(x,y)\, ds$ for nonnegative f:

- Mass of a wire.
- Area of a fence/wall.

e.g. Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of $x^2+y^2=1$.

Ans: $2\pi + \frac{2}{3}$.

Remark

If C is a piecewise smooth curve, that is, if C is a union of finitely many smooth curves $C_1, C_2, \ldots, C_n$, where the initial point of C_{i+1} is the terminal point of C_i, then

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds.$$