STA2001 Assignment 6

1. (3.2-7). Find the moment-generating function for the gamma distribution with parameters α and θ. Hint: In the integral representing $E\left(e^{tX}\right)$, change variables by letting $y = (1-\theta t)x/\theta$, where $1-\theta t > 0$.

   Solution: Let $f(x)$ be the pdf of the given Gamma distribution, then by definition of the MGF,

$$\begin{aligned}
M(t) &= E\left[e^{tX}\right] \\
&= \int_0^{+\infty} e^{tx} f(x)dx \\
&= \int_0^{+\infty} e^{tx} \cdot \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha}dx \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^{+\infty} x^{\alpha-1}e^{-x/\theta}e^{tx}dx \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^{+\infty} x^{\alpha-1}e^{-x(1-\theta t)/\theta}dx \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^{+\infty} \left(\frac{\theta u}{1-\theta t}\right)^{\alpha-1} e^{-u}\frac{\theta}{(1-\theta t)}du \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)\theta^\alpha(1-\theta t)^\alpha}\int_0^{+\infty} u^{\alpha-1}e^{-u}du \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)\theta^\alpha(1-\theta t)^\alpha}\Gamma(\alpha) \\
&= \frac{1}{(1-\theta t)^\alpha}
\end{aligned}$$

   where we let $u = (1-\theta t)x/\theta$ in the 6th equality and thus $du = \frac{(1-\theta t)}{\theta}dx$. The final results is obtained by noticing that $\Gamma(\alpha) := \int_0^{+\infty} x^{\alpha-1}e^{-x}dx$.

2. (3.2-9). If the moment-generating function of a random variable W is

$$M(t) = (1-7t)^{-20}$$

   find the pdf, mean and the variance of W.

   Solution:
   According to $M(t) = (1-7t)^{-20}$, we know that W has gamma distribution with $\theta = 7$, $\alpha = 20$. Therefore, the pdf should be

$$f(x) = \frac{1}{\Gamma(20)\cdot 7^{20}}x^{19}e^{-\frac{x}{7}}$$

   and

$$E(X) = \alpha\theta = 140$$
$$\mathrm{Var}(X) = \alpha\theta^2 = 980$$

3. (3.2-11). If X is $\chi^2(17)$, find
   (a) $P(X < 7.564)$
   (b) $P(X > 27.59)$

(c) $P(6.408 < X < 27.59)$
(d) $\chi^2_{0.95}(17)$
(e) $\chi^2_{0.025}(17)$

Solution:

Since X is $\chi^2(17)$ and degree of freedom is $r = 17$, we can check the chi-square distribution table and find that

(a) $P(X < 7.564) = 0.025$

(b) $P(X > 27.59) = 1 - P(X \leq 27.59) = 1 - 0.95 = 0.05$

(c) $P(6.408 < X < 27.59) = P(X < 27.59) - P(X < 6.408) = 0.95 - 0.01 = 0.94$

(d) $\chi^2_{0.95}(17) = 8.672$

(e) $\chi^2_{0.025}(17) = 30.19$

4. (3.2-22). Let X have a logistic distribution with pdf

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty.$$

Show that

$$Y = \frac{1}{1+e^{-X}}$$

has a $U(0,1)$ distribution.

Hint: Find $G(y) = P(Y \leq y) = P\left(\frac{1}{1+e^{-X}} \leq y\right)$, where $0 < y < 1$.

Solution:

We aim to use the the cdf of X to compute the cdf of Y according to the hint. So we compute the cdf of X first, which is obtained by

$$F_X(x) = \int_{-\infty}^{x} \frac{e^{-w}}{(1+e^{-w})^2}\, dw = \frac{1}{1+e^{-x}}, \quad -\infty < x < \infty$$

Note that by definition of Y and $-\infty < x < \infty$, we must have $0 < Y < 1$.

$$\begin{aligned}
G(y) &= P\left(\frac{1}{1+e^{-X}} \leq y\right) \\
&= P\left(1 + e^{-X} \geq \frac{1}{y}\right) \\
&= P\left(X \leq -\ln\left(\frac{1}{y} - 1\right)\right) \\
&= F_X\left(-\ln\left(\frac{1}{y} - 1\right)\right) \\
&= \frac{1}{1+e^{\ln(1/y-1)}} \\
&= y
\end{aligned}$$

where $0 < y < 1$. This shows that $Y \sim U(0,1)$.

5. (3.3-2). If Z is $N(0,1)$, find

(a) $P(0 \le Z \le 0.87)$
(b) $P(-2.64 \le Z \le 0)$
(c) $P(-2.13 \le Z \le -0.56)$
(d) $P(|Z| > 1.39)$
(e) $P(Z < -1.62)$
(f) $P(|Z| > 1)$
(g) $P(|Z| > 2)$
(h) $P(|Z| > 3)$

Solution:

As $Z \sim N(0, 1)$, we can check the table to get the desired results.

(a) $P(0 \le Z \le 0.87) = P(Z \le 0.87) - P(Z \le 0) = 0.8078 - 0.5 = 0.3078$

(b) $P(-2.64 \le Z \le 0) = P(Z \le 0) - P(Z \le -2.64) = P(Z \le 2.64) - P(Z \le 0) = 0.9959 - 0.5 = 0.4959$

(c) $P(-2.13 \le Z \le -0.56) = P(0.56 \le Z \le 2.13) = P(Z \le 2.13) - P(Z \le 0.56) = 0.9834 - 0.7123 = 0.27$

(d) $P(|Z| > 1.39) = 1 - P(-1.39 < Z < 1.39) = 1 - 2P(0 < Z < 1.39) = 1 - 2 \times (0.9177 - 0.5) = 0.16$

(e) $P(Z < -1.62) = 1 - P(Z < 1.62) = 1 - 0.9474 = 0.0526$

(f) $P(|Z| > 1) = 1 - P(-1 < Z < 1) = 1 - 2P(0 < Z < 1) = 1 - 2 \times 0.3413 = 0.3174$

(g) $P(|Z| > 2) = 1 - P(-2 < Z < 2) = 1 - 2P(0 < Z < 2) = 1 - 2 \times 0.4772 = 0.0456$

(h) $P(|Z| > 3) = 1 - P(-3 < Z < 3) = 1 - 2P(0 < Z < 3) = 1 - 2 \times 0.4987 = 0.0026$

6. (3.3-3). If Z is $N(0, 1)$, find values of c such that

(a) $P(Z \ge c) = 0.025$
(b) $P(|Z| \le c) = 0.95$
(c) $P(Z > c) = 0.05$
(d) $P(|Z| \le c) = 0.90$

Solution:

As $Z \sim N(0, 1)$, we can check the table to get the desired results.

(a) $P(Z \ge c) = 1 - P(Z < c) = 0.025$, so $P(Z < c) = 1 - 0.025 = 0.975$, and we find $c = 1.96$.

(b) $P(|Z| \le c) = 2P(0 \le Z \le c) = 2(P(Z \le c) - P(Z \le 0)) = 0.95$, so $P(Z \le c) = 0.95 \div 2 + 0.5 = 0.975$, and we find $c = 1.96$.

(c) $P(Z > c) = 1 - P(Z < c) = 0.05$, so $P(Z < c) = 1 - 0.05 = 0.95$, and we find $c = 1.645$.

(d) $P(|Z| \le c) = 2P(0 \le Z \le C) = 2(P(Z \le c) - P(Z \le 0)) = 0.90$, so $P(Z \le c) = 0.90 \div 2 + 0.5 = 0.95$, and we find $c = 1.64$.

7. (3.3-5). If X is normally distributed with a mean of 6 and a variance of 25, find

(a) $P(6 \le X \le 12)$
(b) $P(0 \le X \le 8)$
(c) $P(-2 < X \le 0)$
(d) $P(X > 21)$
(e) $P(|X - 6| < 5)$
(f) $P(|X - 6| < 10)$
(g) $P(|X - 6| < 15)$
(h) $P(|X - 6| < 12.41)$

Solution:

Using the fact that $Z = \dfrac{X-\mu_X}{\sigma_X} \sim N(0,1)$ and checking the normal distribution table, we can find the following answers.

(a) $P(6 \le X \le 12) = P\left(\frac{6-6}{5} \le Z \le \frac{12-6}{5}\right) = P(0 \le Z \le 1.2) = P(Z \le 1.2) - P(Z \le 0) = 0.3849$

(b) $P(0 \le X \le 8) = P\left(\frac{0-6}{5} \le Z \le \frac{8-6}{5}\right) = P(-1.2 \le Z \le 0.4) = P(Z \le 0.4) - P(Z \le -1.2) = P(Z \le 0.4) - (1 - P(Z \le 1.2)) = 0.5403$

(c) $P(-2 \le X \le 0) = P\left(\frac{-2-6}{5} \le Z \le \frac{0-6}{5}\right) = P(-1.6 \le Z \le -1.2) = P(1.2 \le Z \le 1.6) = P(Z \le 1.6) - P(Z \le 1.2) = 0.0603$

(d) $P(X > 21) = P\left(Z > \frac{21-6}{5}\right) = P(Z > 3) = 1 - P(Z \le 3) = 1 - 0.9887 = 0.0013$

(e) $P(|X-6| < 5) = P\left(\frac{|X-6|}{5} < \frac{5}{5}\right) = P(|Z| < 1) = 2(P(Z < 1) - P(Z < 0)) = 0.6826$

(f) $P(|X-6| < 10) = P\left(\frac{|X-6|}{5} < \frac{10}{5}\right) = P(|Z| < 2) = 2(P(Z < 2) - P(Z < 0)) = 0.9544$

(g) $P(|X-6| < 15) = P\left(\frac{|X-6|}{5} < \frac{15}{5}\right) = P(|Z| < 3) = 2(P(Z < 3) - P(Z < 0)) = 0.9974$

(h) $P(|X-6| < 12.41) = P\left(\frac{|X-6|}{5} < \frac{12.41}{5}\right) = P(|Z| < 2.48) = 2(P(Z < 2.48) - P(Z < 0)) = 0.9868$

8. Let X be a continuous random variable with probability density function (pdf) defined as follows

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, x \in (-\infty, \infty).$$

(a) Derive the moment generating function for X and show that $\mathrm{E}(X) = 0$, $\mathrm{Var}(X) = 1$.

(b) Define a random variable $Y = aX + b$. What is the distribution of Y? Show the proof in details.

(c) What is $E((Y-b)^{177})$?

(d) Prove that $\mathrm{Var}(X) = 1$ without using the moment generating function.

Solution:

(a) The derivation and the proof for $\mathrm{E}(X) = 0$, $\mathrm{Var}(X) = 1$ can be found in the end of lecture note 11.

(b) Using the moment generating function technique, we have

$$\begin{aligned} \mathrm{E}(\exp(tY)) &= \mathrm{E}(\exp(taX + tb)) = \mathrm{E}(\exp(taX)\exp(tb)) \\ &= \mathrm{E}(\exp(taX)\exp(tb)) = \exp(\frac{1}{2}a^2t^2)\exp(tb) \\ &= \exp(tb + \frac{1}{2}a^2t^2) \end{aligned}$$

Therefore, $Y \sim N(b, a)$.

(c) Since $Y - b = aX$, $\mathrm{E}((Y-b)^{177}) = \mathrm{E}(b^{177}X^{177}) = b^{177}\mathrm{E}(X^{177})$.

$$\mathrm{E}(X^{177}) = \int_{-\infty}^{\infty} x^{177} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = -\int_{-\infty}^{\infty} z^{177} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = -\mathrm{E}(X^{177})$$

Therefore, $\mathrm{E}((Y-b)^{177}) = 0$.

(d)

$$\begin{aligned} \mathrm{Var}(X) = E(X^2) &= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\ &= 2\int_{0}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \end{aligned}$$

Let $z = x^2$. Then we have

$$\text{Var}(X) = \frac{1}{\sqrt{2\pi}} \int_0^\infty z^{\frac{1}{2}} e^{-\frac{1}{2}z} dz$$

Let $y = \frac{1}{2}z$. Then

$$\begin{aligned}
\text{Var}(X) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty 2^{\frac{3}{2}} y^{\frac{1}{2}} e^{-y} dy \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty y^{\frac{1}{2}} e^{-y} dy = \frac{2}{\sqrt{\pi}}(-1) \int_0^\infty y^{\frac{1}{2}} de^{-y} \\
&= \frac{2}{\sqrt{\pi}}(-1) \left\{ y^{\frac{1}{2}} e^{-y}|_0^\infty - \frac{1}{2} \int_0^\infty y^{-\frac{1}{2}} e^{-y} dy \right\} \\
&= \frac{2}{\sqrt{\pi}} \frac{1}{2} \int_0^\infty y^{-\frac{1}{2}} e^{-y} dy \\
&= \frac{1}{\sqrt{\pi}} \Gamma(\frac{1}{2}) = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1
\end{aligned}$$

9. (3.3-10). If X is $N(\mu, \sigma^2)$, show that the distribution of $Y = aX + b$ is $N(a\mu + b, a^2\sigma^2)$ $a \neq 0$.
Hint: Find the cdf $P(Y \leq y)$ of Y, and in the resulting integral, let $w = ax + b$ or, equivalently, $x = (w - b)/a$.

Solution:

Let's first consider the case $a > 0$.

$$\begin{aligned}
G(y) &= P(Y \leq y) \\
&= P(aX + b \leq y) \\
&= P\left(X \leq \frac{y - b}{a}\right) \\
&= \int_{-\infty}^{(y-b)/a} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx \\
&= \int_{-\infty}^{y} \frac{1}{a\sigma\sqrt{2\pi}} e^{-(w-b-a\mu)^2/2\sigma^2 a^2} dw
\end{aligned}$$

where we let $w = ax + b$ in the last equality and thus $dw = adx$.

For $a < 0$, the derivation is quite similar, except that the standard deviation should be $-a\sigma$.

The final result shows that it is indeed a normal cumulative distribution function of $N(b + a\mu, \sigma^2 a^2)$.

10. (3.3-14). The strength X of a certain material is such that its distribution is found by $X = e^Y$, where Y is $N(10, 1)$. Find the cdf and pdf of X, and compute $P(10,000 < X < 20,000)$.
Note: $F(x) = P(X \leq x) = P(e^Y \leq x) = P(Y \leq \ln x)$ so that the random variable X is said to have a lognormal distribution.

Solution:

WLOG, let $Y \sim N(\mu, \sigma^2)$, then the pdf and cdf of Y are given by

$$\begin{aligned}
f_Y(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2} \\
F_Y(y) &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(z-\mu)^2/2\sigma^2} dz
\end{aligned}$$

Then, the cdf of X can be computed by

$$F_X(x) = P(X \le x) = P\left(e^Y \le x\right) = P(Y \le \ln x) = F_Y(\ln x) = \int_{-\infty}^{\ln x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(z-\mu)^2/2\sigma^2} dz$$

The pdf of X is the derivative of the cdf of X,

$$\begin{aligned}
f_X(x) &= F'_X(x) \\
&= \frac{d}{dx}\left(\int_{-\infty}^{\ln x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(z-\mu)^2/2\sigma^2} dz\right) \\
&= \frac{d}{dx}\left(\int_{0}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\ln t-\mu)^2/2\sigma^2} \frac{1}{t}\, dt\right) \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\ln x-\mu)^2/2\sigma^2} \left(\frac{1}{x}\right) \\
&= \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\ln x-\mu)^2/2\sigma^2}
\end{aligned}$$

where we have used $t = e^z$ in the 3rd equality and thus $dz = 1/u \cdot du$, and the derivative is obtained by the Fundamental theorem of calculus (or you can finish these two steps in one trial by using the Leibniz integral rule).

Finally we set $\mu = 10$ and $\sigma = 1$,

$$\begin{aligned}
F_X(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\ln x} e^{-(z-10)^2/2} dz \\
f_X(x) &= \frac{1}{x\sqrt{2\pi}} e^{-(\ln x-10)^2/2}
\end{aligned}$$

To compute $P(10000 < X < 20000)$, we write

$$\begin{aligned}
P(10000 < X < 20000) &= P(X \le 20000) - P(X \le 10000) \\
&= P\left(e^Y \le 20000\right) - P\left(e^Y \le 10000\right) \\
&= P(Y \le \ln 20000) - P(Y \le \ln 10000) \\
&= P(Z < -0.10) - P(Z < -0.79) \\
&= 1 - P(Z < 0.10) - (1 - P(Z < 0.79)) \\
&= 1 - 0.5398 - (1 - 0.7852) \\
&= 0.2454
\end{aligned}$$

where we have used the fact $Z = \dfrac{Y-\mu}{\sigma} \sim N(0,1)$.