

# MAT1001 Midterm Examination

Saturday, October 29, 2022

Time: 9:30 - 11:30 AM

## Notes and Instructions

1. *No books, no notes, no dictionaries, and no calculators.*
2. *The total score of this examination is 100.*
3. *There are **11** questions (with parts) in total.*
4. *The symbol  $[N]$  at the beginning of a question indicates that the question is worth  $N$  points.*
5. *Answer all questions on the **answer book**.*
6. *Show your intermediate steps **except Questions 1 and 2** — answers without intermediate steps will receive minimal (or even no) marks.*



## MAT1001 Midterm Questions

1. [10] True or False (in general)? No explanation is required.

**T** (i) If  $f(4) = -\pi$ ,  $f(8) = \sqrt{2}$ , and  $f$  is differentiable everywhere, then the equation  $f(x) = 0$  must have at least one solution on the interval  $(4, 8)$ .

**F** (ii) Suppose that  $y = f(x)$  is increasing and differentiable on the real line. Then for any  $x$ , if  $\Delta x > 0$  then  $\Delta y > dy$ .

**T** (iii) If  $f(x) < g(x)$  for all  $x \in (-1, 1)$ , and the limits of  $f$  and  $g$  both exist as  $x$  approaches 0, then we have  $\lim_{x \rightarrow 0} f(x) < \lim_{x \rightarrow 0} g(x)$ .

**T** (iv) The function  $y = \frac{\sin(x)}{x}$  has a horizontal asymptote as  $x \rightarrow \infty$ .

**T** (v) Suppose that  $g(x) \leq f(x) \leq h(x)$  for all  $x > 0$ , and the limits of  $g$  and  $h$  both exist as  $x$  approaches 1. Then  $\lim_{x \rightarrow 1} f(x)$  must exist.

2. [16] Short questions: no explanation is required.

(i) Supposing a function  $y = f(x)$  defined on the real line has a second derivative  $f''(x)$  that is negative for  $x < 0$  and is positive for  $x > 0$ . However, both  $f'(x)$  and  $f''(x)$  fail to exist at  $x = 0$ , and there is a vertical tangent there. Which one of the following statements **must** be correct?

- A) The point  $x = 0$  gives a local maximum.
- B) The point  $x = 0$  gives a local minimum.
- C) The point  $(0, f(0))$  gives an inflection point.**
- D) The point  $x = 0$  is a discontinuity.

(ii) The function  $f(x) = |x - 3|$  is differentiable over which of the following domains?

- A)  $(-4, 4)$
- B)  $(-5, 1)$**
- C)  $(-\infty, +\infty)$
- D)  $(-1, 5)$

$$\begin{aligned}
 & 2x \sin^3(\phi) + x^2 \\
 & u = \phi x \\
 & \sin^3 u \\
 & \frac{d}{dx} \sin^3 u \\
 & = 3 \sin^2 u \cdot \frac{du}{dx} \\
 & = 3 \sin^2(\phi x) \cdot \phi \\
 & = 3 \phi \sin^2(\phi x)
 \end{aligned}$$

- (iii) Compute  $f'(x)$ , where  $f(x) = x^2 \sin^3(4x)$ . No need to simplify.
- (iv) Suppose that  $F(x) = f(x)g(x)$ , where  $f$  and  $g$  have derivatives of all orders. Express  $F''$  in terms of  $f$ ,  $g$ , and their first and second derivatives.
- (v) Assume that  $u = u(x)$ ,  $v = v(x)$  and  $w = w(x)$  are three functions of  $x$ , with  $w(x)$  never zero. Compute the derivative of  $\frac{u(x) + v^2(x)}{w(x)}$  with respect of  $x$  in terms of  $u, v, w, u', v'$ , and  $w'$ . No need to simplify.
- (vi) A particle vibrates horizontally on the  $x$ -axis. Its position at time  $t$  is given by  $x(t) = 8 \sin t$ , where  $t$  is in seconds and  $x$  is in centimeters. Find the velocity and acceleration of the particle at time  $t = \frac{2\pi}{3}$ .
- (vii) For (vi) above, in what direction is the particle moving at  $t = \frac{2\pi}{3}$ ?
- (viii) Find the function  $y = f(x)$  defined for  $x > 0$  that satisfies  $y' = \frac{1 + x^3}{x^2}$  and  $y(2) = 1$ .

3. [4+4] Consider the equation  $x^3 + 3x + 1 = 0$ .

- (i) Prove that the equation has a solution  $x = c$  with  $|c| < 1$ .
- (ii) Consider using Newton's method to estimate the solution. Starting with  $x_0 = 0$ , find  $x_1$  and  $x_2$ . Express your answers in the form  $n/m$  where  $n$  and  $m$  are integers.

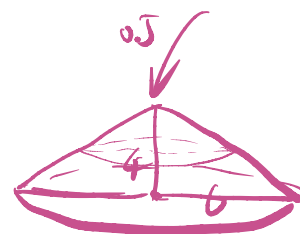
4. [4+4+4] Evaluate the following limits. Use only methods and theories from Chapters 2, 3, or 4 in the textbook.

- (i)  $\lim_{x \rightarrow 0} \frac{8x}{3 \sin(x) - x}$
- (ii)  $\lim_{x \rightarrow -\infty} \sqrt{x^2 + 3x} - \sqrt{x^2 - 2x}$
- (iii)  $\lim_{x \rightarrow 9} \frac{\sin(\sqrt{x} - 3)}{x - 9}$

5. [4+4] The biomass  $B(t)$  of a fish population is the total mass of the members of the population at time  $t$ . Let  $N(t)$  denote the number of individuals in the population at time  $t$ , and let  $M(t)$  denote the average mass of a fish at time  $t$ .

- (i) Express  $B(t)$  and  $B'(t)$  using the information above.
- (ii) In the case of the *black moor goldfish*, breeding occurs continually. Suppose that at time  $t = 4$  weeks the population is 820 fish and is growing at a rate of 50 fish per week, while the average mass is 1.2 g and is increasing at a rate of 0.1 g/week. At what rate is the biomass increasing when  $t = 4$ ?

6. [6] Suppose that there is a conical tank standing pointing up. The radius of the base is  $R = 6$  m and the height is  $H = 4$  m. Let the water run in the tank from the top at the rate of  $0.5 \text{ m}^3/\text{min}$ . How fast is the water level raising when the water level is half the height of the tank (2 m)?



$$V = \dots$$

$$\frac{dV}{dt} = 0.5 = \dots \checkmark$$

7. [6] Suppose that a particle is travelling on the orbit  $\frac{x^2}{4} + \frac{y^2}{9} = 1$ . If the velocity of the particle in the  $x$ -direction is  $\frac{dx}{dt} = 6$  (m/s) when  $x = 1$  m and  $y > 0$ , find the velocity of the particle in the  $y$ -direction (which is  $\frac{dy}{dt}$ ).

8. [4+3] Suppose that  $f$  is differentiable everywhere, with  $f(-6) = 4$  and  $f'(x) \leq 3$  for all  $x$ .

说明是考求导的问题, 否则就说是 continuous }

- (i) What is the largest possible value for  $f(0)$ ? Prove your claim using a theorem in Chapter 4.

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- (ii) Find an example of such  $f$  that gives the largest value of  $f(0)$ .

9. [4+3+3] Let  $f(x) = \cos^2(x) - 2\sin(x)$  with domain  $D = [0, 2\pi]$ .

- (i) Find the closed interval(s) where  $f(x)$  is strictly increasing, then find those where  $f(x)$  is strictly decreasing.
- (ii) Find all the critical points of  $f(x)$  in the interior of  $D$ .
- (iii) Identify the  $x$ -positions of all the local maxima and local minima of  $f(x)$  in  $D$  (for each  $x$ , state whether it gives a local min or a local max).

10. [4+2+6] Consider the function  $f(x) = x^3 - 6x^2 + 6x$  defined on  $[0, \infty)$ .

- (i) Find all interval(s) where  $f$  is concave up, then find those where  $f$  is concave down. (You can include the endpoints for your intervals.)
- (ii) Find all inflection points. (State their  $x$ -coordinates.)
- (iii) Sketch the graph  $y = f(x)$ . Indicate the following features on the graph : concavity, the  $x$ -positions of all the local and global (absolute) extrema, and the  $x$ -intercepts (zero crossings). For each extremum, indicate its type (i.e., local or global, max or min.)

11. [5] Suppose that  $f(x)$  is continuous for  $x \in [-1, 1]$  and  $f(0) = 3$ . Determine the value of the following limit, or explain why it does not exist:

$$\lim_{x \rightarrow 0} \frac{\sqrt[3]{1 + f(x) \sin(x)} - 1}{x}.$$

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sqrt[3]{1 + f(x) \sin(x)} - 1}{x} \\ &= \lim_{x \rightarrow 0} \frac{f(x) \sin(x) / 3}{x} \end{aligned}$$