

MAT1002 Lecture 20, Tuesday, Apr/04/2023

Outline

- Triple integrals (15.5)
- Cylindrical coordinates (15.7)
- Change of variable formula (15.8)

Triple integrals (Motivation: density and mass)

Let f be a three-variable function defined on the rectangular solid

$$R := [a, b] \times [c, d] \times [r, s].$$

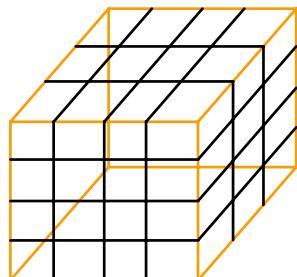
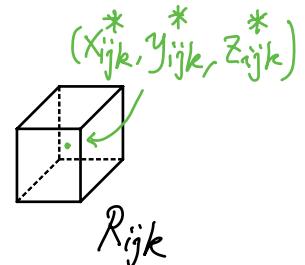
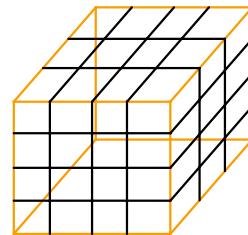
Then the **triple integral** of f over R is defined by

$$\iiint_R f(x, y, z) dV := \lim_{\|P\| \rightarrow 0} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk},$$

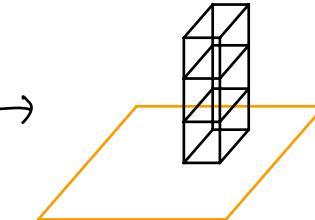
provided that the limit exists. The Riemann sum above is defined similarly to that defined ~~on Page 6.~~ for double integrals.

One way to think of triple integrals is that, if f is the density function, then $\iiint_R f(x, y, z) dV$ is the mass of the solid R .

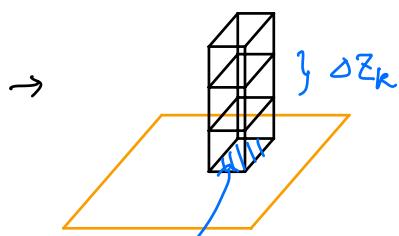
density at (x, y, z) , say kg/m^3 .



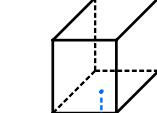
Projection of Solid onto
xy-plane



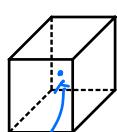
Consider a "pile".



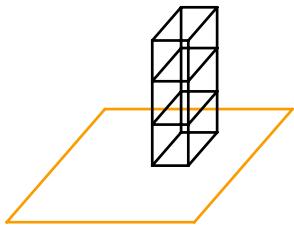
$$\begin{aligned} \text{Area} &= \Delta A_{ij} \\ &= \Delta x_i \Delta y_j \end{aligned}$$



Fix (x_{ij}^*, y_{ij}^*)
in the base
region

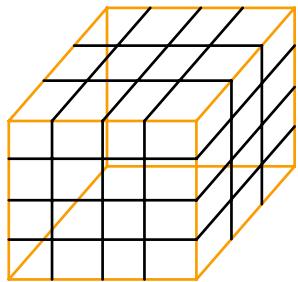


Pick height z_{ijk}^* .
Sample point
 $(x_{ij}^*, y_{ij}^*, z_{ijk}^*)$



Mass of this pile is

$$m_{ij} \approx \sum_{k=1}^n f(x_{ij}^*, y_{ij}^*, z_{ijk}^*) \Delta z_k \Delta A_{ij}$$



Mass of "Tofu" is

$$m = \sum_{i,j} m_{ij}$$

$$\approx \sum_{i,j} \left(\sum_k f(x_{ij}^*, y_{ij}^*, z_{ijk}^*) \Delta z_k \right) \Delta A_{ij}$$

- Taking limit as $\|P\| \rightarrow 0$ yields

\iint viewed as \iint

$$m = \iint_D \left(\int_r^s f(x,y,z) dz \right) dA \quad \leftarrow$$

If f is cts $\xrightarrow{} = \int_a^b \int_c^d \int_r^s f(x,y,z) dz dy dx.$ (two-variable Fubini's thm)

This is the three-variable Fubini's theorem.

Theorem (Fubini's Theorem) (Triple integrals)

If f is continuous on the rectangular solid

$$R = [a, b] \times [c, d] \times [r, s],$$

$x \quad y \quad z$

then

$$\iiint_R f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz.$$

And order
does not
matter.

Exercise

Evaluate $\iiint_R xyz^2 dV$, where $R = [0, 1] \times [-1, 2] \times [0, 3]$.

Ans: $= \int_0^1 \int_{-1}^2 \int_0^3 xyz^2 dz dy dx = 27/4.$

Remark Note that

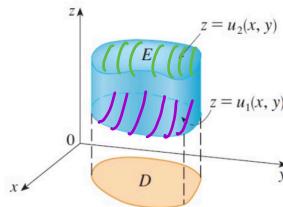
$$\int_a^b \int_c^d \int_r^s g_1(x) g_2(y) g_3(z) dz dy dx = (\int_a^b g_1(x) dx) (\int_c^d g_2(y) dy) (\int_r^s g_3(z) dz).$$

Q: How to compute $\iiint_E f(x, y, z) dV$ when E is not a rectangular solid?

Suppose that E has the form

$$\{(x, y, z) : (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\},$$

where D is the projection of E onto the xy -plane.



Then

$$\iiint_E f(x, y, z) dV = \iint_D \left(\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right) dA. \quad \text{Viewed as } \iint$$

Note that in the inner integral on the right-hand side, x and y are held fixed. If D is a type-I region, then E can be written as

$$\{(x, y, z) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}.$$

By Fubini's theorem for double integrals, the equation above becomes

$$\iiint_E f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz dy dx.$$

A similar strategy can be used when D is of type II or when E has other similar forms.

Example

- Evaluate $\iiint_E z dV$, where E is the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Ans. $\frac{1}{24}$.



Number Prefixes

Greek	Latin	Meaning	Examples
Mon-	uni-	1	monograph, monomial, monotheism, universe, uniform, unicorn, monotone
Di-	bi-, du-	2	biology, bilingual, binary, bimonthly, binoculars, duo, duet
Tri-	-tri	3	tricycle, triad, triathlon, triangle, tripod, triumvirate, triple
Tetra-	quadr-, quart-	4	tetrameter, quadrilateral, quadrangle, quadruple, quartet
Penta-	quin-	5	pentameter, pentagon, quintet, quintuplet
Hexa-	sext-	6	hexagon, hexameter, sextuplet, sextet
Hepta-	septem-, septi-	7	heptagon, heptameter, heptagon, septuagenarian
Octo-	octa-	8	octagon, octogenarian, octopus, octahedron
Ene-	novem-	9	novena
Deca-	deci-, decem-	10	decade, decagon, decahedron, decimal
Hemi-	semi-	Half	hemisphere, semicircle, semicolon, semifinal, semiannual
Poly-	multi-	Many	polygon, polygamy, polyester, polymer, polynomial
Hecto-	cent-, cente-	100	cent, centennial, centurion, centenary, cent
Kilo-	milli-, mille-	1000	kilogram, kilometer, kilobyte, milligram

Definition

The **volume** $V(E)$ of a solid E in \mathbb{R}^3 is defined by

$$V(E) := \iiint_E dV := \iiint_E 1 \, dV.$$

The **average value** of a function f over E is

$$\frac{1}{V(E)} \iiint_E f(x, y, z) \, dV.$$

Exercise

Use a triple integral to find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

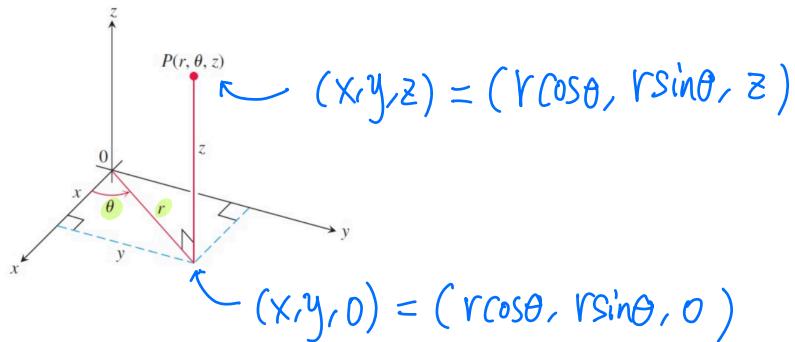
Ans: $\frac{1}{3}$.

Cylindrical Coordinates

Given any point P in the xyz -space, we can represent the point by (r, θ, z) , where

- r and θ are polar coordinates of the projection of P onto the xy -plane, and;
- z is the z -coordinate of P in the Cartesian coordinates.

The coordinate system above in (r, θ, z) is called the **cylindrical coordinate** system.



Cylindrical coordinates may be helpful in computing triple integrals when the solid E of integration has the form

$$E = \{(x, y, z) : (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\},$$

where D (the projection of E onto the xy -plane) is convenient to describe in polar coordinates. If D is given by

$$\alpha \leq \theta \leq \beta, \quad 0 \leq h_1(\theta) \leq r \leq h_2(\theta), \\ (\beta - \alpha \leq 2\pi)$$

then

$$\iiint_E f(x, y, z) dV = \iint_D \left(\int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z) dz \right) dA \\ = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(x, y)}^{u_2(x, y)} f(r \cos \theta, r \sin \theta, z) dz \, r dr d\theta$$

e.g. A solid E lies inside the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at a point $P(x,y)$ is $K\sqrt{x^2+y^2}$ for some constant K . Find $\text{mass}(E)$.

Ans $\frac{12\pi}{5}K$.

Exercise

Use cylindrical coordinates to evaluate the integral

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) dz dy dx.$$

Ans $16\pi/5$.

Moral of the story : $dV = r dz dr d\theta$ after changing from xyz-to cylindrical coordinates.

Change of Variable Formula (Double Integrals)

When discussing double integrals in polar coordinates last week, we saw that

$$\iint_D f(x,y) dA = \iint_R f(r \cos \theta, r \sin \theta) r dr d\theta,$$

where R is the polar region corresponding to the region D (which is in the xy -plane). Essentially, we made the substitution

$$x = r \cos \theta, \quad y = r \sin \theta,$$

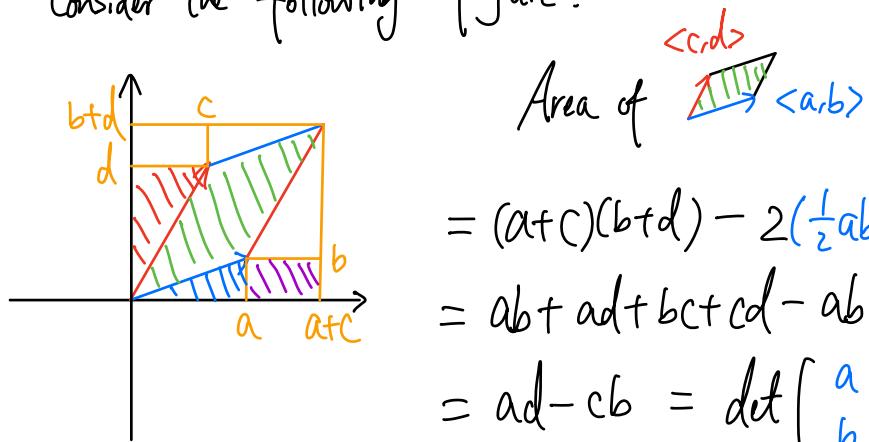
and replaced the differential dA with $r dr d\theta$. More generally, if we make the substitution

$$x = g(u, v), \quad y = h(u, v),$$

what would the formula become?

Geometric Properties of Determinants

Consider the following figure:



Fact :

$\left| \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right|$ ← Abs. values (↗)
 is the area of the parallelogram spanned by
 $\langle a, b \rangle$ and $\langle c, d \rangle$.

Another fact :

You can "take out" any constant from a row or from a column :

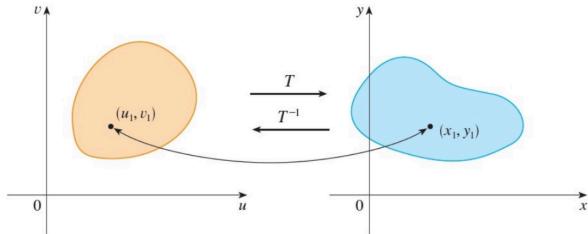
$$\det \begin{bmatrix} \lambda a & \lambda c \\ \lambda b & \lambda d \end{bmatrix} = \lambda \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \lambda a \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Suppose that

$$x = g(u, v) \quad \text{and} \quad y = h(u, v),$$

where g and h have continuous partial derivatives. We may understand this as a transformation T that maps a point in the uv -plane to a point in the xy -plane:

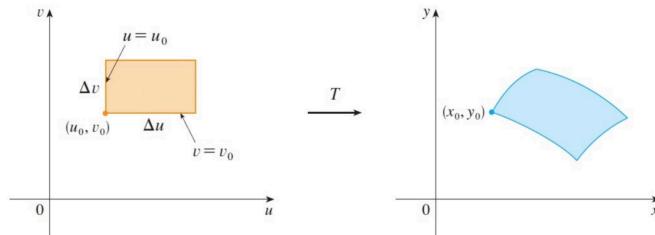
$$T(u, v) := (g(u, v), h(u, v)) = (x, y).$$



Let R be a region in the uv -plane, and let D be the image of R under the transformation T . That is,

$$D = T(R) := \{T(u, v) : (u, v) \in R\}.$$

Partition R into rectangles R_{ij} , and let D_{ij} be the image of R_{ij} under T .



Let $A(D_{ij})$ and $A(R_{ij})$ be the area of D_{ij} and R_{ij} , respectively.

When the norm of the partition is small,

$$A(D_{ij}) \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| A(R_{ij}) = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u_i \Delta v_j,$$

absolute value of
the Jacobian

where

Jacobian, which is a determinant

$$\left| \begin{array}{cc} \frac{\partial(x,y)}{\partial(u,v)} & = \begin{array}{c} \text{circled} \\ \text{blue} \end{array} \\ \hline \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}.$$

The determinant $\frac{\partial(x,y)}{\partial(u,v)}$ above is called the **Jacobian** of the transformation.

Details : Some Rij

The diagram shows a rectangular box with vertices at (u_0, v_0) . A blue arrow points from the top-left corner to a brace labeled ΔU indicating the width of the box. Another brace labeled ΔV indicates the height. To the right, a curved arrow labeled T maps the box to a new position, with the resulting coordinates labeled as $\langle x(u_0, v_0), y(u_0, v_0) \rangle$.

Fix \mathcal{V}_0 , let $\vec{r}_i(u) := \underline{T}(u, \mathcal{V}_0)$

Fix u_0 , let $\vec{r}_v(v) := T(u_0, v)$

$$\langle x(u_0, v), y(u_0, v) \rangle$$

$$\vec{J} = T(u_0, v_0 + \Delta v) - T(u_0, v_0) = \vec{r}_2(v_0 + \Delta v) - \vec{r}_2(v_0)$$

$$\approx \vec{r}'_2(v_0) \Delta v$$

$$\begin{aligned} & \tilde{T}(u_0 + \Delta u, v_0) - T(u_0, v_0) \\ &= \vec{r}_i(u_0 + \Delta u) - \vec{r}_i(u_0) \\ &\approx \vec{r}'_i(u_0) \Delta u \end{aligned}$$

$$\text{Hence } A(D_{ij}) \approx \text{Area} \left(\begin{array}{c} \text{red dashed line} \\ \text{blue arrows} \end{array} \right) = \left| \det \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix} \right|$$

$$= \left| \det \begin{bmatrix} \frac{\partial X}{\partial u} \Delta u & \frac{\partial X}{\partial v} \Delta v \\ \frac{\partial y}{\partial u} \Delta u & \frac{\partial y}{\partial v} \Delta v \end{bmatrix} \right| = \Delta u \Delta v \left| \det \begin{bmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right|$$

Evaluated at (u_0, v_0)

$\frac{\partial(x,y)}{\partial(u,v)}$ "Jacobian of T"

In terms of differentials, one can memorize the above as

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$$

Theorem (Change of Variables in Double Integrals)

Let f be continuous on a region D in the xy -plane. Suppose that

- the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ maps a region R in the uv -plane onto D , and;
- the transformation is one-to-one in the interior of R , and;
- all partial derivatives of g and h are continuous.

Then

$$\iint_D f(x, y) dA = \iint_R f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$$

e.g. (Polar transformation)

If a change of variables is given by the transformation

$$\underbrace{x = g(r, \theta) = r \cos \theta}_{\text{and}} \quad \underbrace{y = h(r, \theta) = r \sin \theta},$$

then it follows that

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \geq 0,$$

which gives

$$\iint_D f(x, y) dA = \iint_R f(r \cos \theta, r \sin \theta) r dr d\theta.$$

This is consistent with the formula we stated in Week 9.

last week

To satisfy the one-to-one condition, here we restrict to $r \geq 0$, i.e., domain of T is $\{(r, \theta) : r \geq 0, \theta \text{ is in an interval with length } = 2\pi\}$

Example

Evaluate the following integrals.

- (a) $\iint_D e^{(x+y)/(x-y)} dA$, where D is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$ and $(0, -1)$.

(b) $\int_1^2 \int_{1/y}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy$.

Ans

(a) $\frac{3}{4}(e - e^{-1})$

(b) $2e(e-2)$