10.7

8. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(x+2)n}{n+1}\right| < 1$ $\quad\therefore\ -3<x<-1 \quad r=1$

when $x=-3$ we have $\sum\frac{1}{n}$ diverge

when $x=-1$ we have $\sum(-1)^n\frac{1}{n}$ conditional converge

$\therefore$ radius 1, convergence in $-3<x\le-1$

absolute converge : $-3<x<-1$

conditional converge : $x=-1$

22. $\lim_{n\to\infty}$

$\left|\frac{n+1}{n}\right| = \left|\frac{n\cdot 3^2(x-2)}{(n+1)}\right| < 1$

$\frac{17}{9}<x<\frac{19}{9}$

when $x=\frac{17}{9} \Rightarrow \sum\frac{1}{3n}$ diverge

$x=\frac{19}{9} \Rightarrow \sum(-1)^n\frac{1}{3n}$ converge conditional

converge : $x\in[\frac{17}{9},\frac{19}{9}]$ $\quad r=\frac{1}{9}$

absu converge : $x\in(\frac{17}{9},\frac{19}{9})$ $\quad$ conditional cvg : $x=\frac{19}{9}$

28. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{2(n+1)(x-1)}{(n)}\right| < 1$

$\frac{1}{2}<x<\frac{3}{2}$

when $x=\frac{1}{2} \Rightarrow \sum(n+1)$ diverge

when $x=\frac{3}{2} \Rightarrow \sum(-1)^n(n+1)$ conditional converge

conditional converge : $x=\frac{3}{2}$ $\quad R=\frac{1}{2}$

absu converge : $x\in(\frac{1}{2},\frac{3}{2})$ $\quad\therefore$ cvg : $x\in(\frac{1}{2},\frac{3}{2}]$

29. Since $\int_2^{\infty} \frac{1}{n \ln^2 n} dn = \int_2^{\infty} \frac{1}{\ln^2 n} d\ln n = \lim_{b \to \infty} \left[-\frac{1}{\ln n} \right]_2^b = \frac{1}{\ln 2}$

$\therefore \sum \frac{1}{n (\ln n)^2}$ converge

$\therefore \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(n+1)(\ln(n+1))^2}{n(\ln n)^2} \right| |x| < 1$

$x \in (-1, 1)$

when $x = -1$ or 1

abs converge : $x \in [-1, 1]$ $\quad r = 1$

no conditional converge interval

30. $\int_2^{\infty} \frac{1}{n \ln n} dn = \lim_{b\to\infty} [\ln(\ln n)]_2^b \to \infty$

$\therefore \sum \frac{1}{n \ln n}$ diverge

$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(n+1)\ln(n+1)}{n \ln n} \right| |x| < 1$

$x \in (-1, 1)$ but $\sum \frac{1}{n \ln n}$ converge

$\therefore$ conditional converge interval $x = \pm 1$

abs converge : $x \in (-1, 1)$ $\quad r = 1$ $\quad \therefore$ cvg : $x \in [-1, 1]$

36. $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1} - \sqrt{n}} \right| |x-3| < 1$

$\xi_1 \in [n+1, n+2]$, $\xi_2 \in [n, n+1]$ $\left| \frac{\frac{1}{2\sqrt{\xi_1}}[n+2-n-1]}{\frac{1}{2\sqrt{\xi_2}}(n+1-n)} \right| |x-3| < 1 \Rightarrow 2 < x < 4$

when $x = 2$ $\sum (-1)^n (\sqrt{n+1} - \sqrt{n})$ conditional converge

$\therefore$ abs converge : $x \in (2, 4)$

conditional converge : $x=2$ $R=1$

cvg: $x \in [2,4)$

37 $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{n+1}{3(n+1)}x\right| < 1$ $|x|<3$ $\therefore r=3$

46. $\sum (\ln x)^n = \frac{1-(\ln x)^n}{1-\ln x}$ $\therefore -1<\ln x<1$

$\therefore x \in (\frac{1}{e}, e)$

48. $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^2-1}{2}\right| < 1$

$-\sqrt{3} < x < \sqrt{3}$ $r=\sqrt{3}$

sum is $\frac{1}{1-\frac{x^2-1}{2}} = \frac{2}{3-x^2}$

50.

a $f(x) = \frac{7}{3}\left(1+\frac{1}{3}x+\left(\frac{x}{3}\right)^2+\left(\frac{x}{3}\right)^3+ \cdots\right)$

$\left|\frac{1}{3}x\right| < 1$ $|x|<3$

b. $g(x) = \frac{3}{2}\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\cdots\right)$

$\left|\frac{x}{2}\right|<1$ $|x|<2$

51. $g(x) = \frac{3}{x-2} = \frac{3}{3-[-(x-5)]}$

$= \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n (x-5)^n$

$\left|\frac{x-5}{3}\right| < 1$ $2<x<8$

53. $\left|\frac{x-3}{2}\right| < 1$ $\quad 1 < x < 5$

$$S = \sum_{n=0}^{\infty} (-\frac{1}{2})^n (x-3)^n = \frac{1}{1+\frac{x-3}{2}} = \frac{1}{x-1}$$

$$S' = \sum_{n=0}^{\infty} n(-\frac{1}{2})^n (x-3)^{n-1}$$

converge when $1 < x < 5$

$$S' = \frac{-2}{(x-1)^2}$$

diverge when $x = 1$

54. $\int f(x)\,dx = x - (x-3)^2 + \frac{1}{12}(x-3)^3 + \cdots + (-\frac{1}{2})^n \frac{1}{n+1}(x-3)^{n+1} \cdots$

$= \int \frac{2}{x-1}\,dx = 2\ln|x-1| + C$ $\quad$ when $x=1$ $\sum \frac{-2}{n+1}$ diverge

$\therefore$ cvg: $x \in (1,5]$ $\quad$ when $x=3$ $\quad 3 = 2\ln 2 + C$ $\quad x=5$ $\sum \frac{(-1)^n 2}{n+1}$ converge

$C = 3 - 2\ln 2$

68:

$\ln|\sec x + \tan x| = \int \sec x\,dx$

$= x + \frac{x^3}{6} + \frac{1}{24}x^5 + \frac{61}{5040}x^7 + \frac{277}{72576}x^9 \cdots + C$

a. $x=0$ $\quad C=0$ $\quad \ln|\sec x| = \frac{x^2}{2} + \frac{x^4}{12} \cdots\cdots$

converge when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

b. $\sec x \tan x = (\sec x)' = x + \frac{5}{31}x^3 + \frac{61}{120}x^5 + \frac{277}{1008}x^7 \cdots$

converge when $x \in (\frac{-\pi}{2}, \frac{\pi}{2})$

10.8

4. $P_3 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

10. $(1-x)^{\frac{1}{2}}$ $\qquad P_3 = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

14. $f(x) = \frac{2+x}{1-x}$ $\qquad f'(x) = \frac{1-x+2+x}{(1-x)^2} = \frac{3}{(1-x)^2}$

$f^{(k)}(0) = 3(-1)^{k-1} k!$

$$\frac{2+x}{1-x} = \sum_{n=1}^{\infty} (-1)^{n-1} 3x^n + 2$$

24. $f(1)=3$

$f'(x)=6x^2+2x+3 \qquad f'(1)=11$

$f''(x)=12x+2 \qquad f''(1)=14$

$f'''(x)=12 \qquad f'''(1)=12$

$P=3+11(x-1)+7(x-1)^2+2(x-1)^3$

31. $f(\frac{\pi}{4})=-1$

$f'(x)=-2\sin(2x+\frac{\pi}{2})$

$f''(x)=-4\cos(2x+\frac{\pi}{2})$

$f'''(x)=8\sin(2x+\frac{\pi}{2})$

$f''''(x)=16\cos(2x+\frac{\pi}{2})$

$\cdots \quad f^{2n}(\frac{\pi}{4})=(-1)^n 2^{2n} \qquad f^{(2n+1)}(\frac{\pi}{4})=0$

$\Rightarrow \cos(2x+\frac{\pi}{2})=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}(x-\frac{\pi}{4})^{2n}$

37. $f(x)=e^x \qquad f(a)=e^a$

$f^{k}(x)=e^x \qquad f^{k}(a)=e^a$

$\Rightarrow e^x=e^a+\frac{e^a}{1!}x+\frac{e^a}{2!}x^2+\frac{e^a}{3!}x^3+\cdots$

$=e^a\sum_{n=0}^{\infty}\frac{1}{n!}x^n$

40. $E'(x)=f'(x)-g'(x) \qquad E(a)=f(a)-g(a)=0$

$E^{k}(x)=f^{k}(x)-g^{k}(x)$

$\lim_{x\to a}\frac{E(x)}{(x-a)^n}=E^n(a)=E^{k}(a)\frac{1}{(x-a)^{n-k}}=0$

$\therefore \forall k \in [0, n] \quad E^k(a) = 0$

$\Leftrightarrow \quad f^k(x) = g^k(x)$

$\therefore g(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

10.9

4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(2k-1)!} x^{2k-1}$

$\sin(\frac{\pi}{2}x) = \frac{\pi}{2}x \cancel{+} - \frac{1}{3!}(\frac{\pi}{2}x)^3 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(2k-1)!}(\frac{\pi}{2}x)^{2k-1}$

8. $\tan^{-1}x = \int \frac{1}{1+x^2}dx = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$

$\tan^{-1}(3x^4) = 3x^4 - \frac{1}{3}(3x^4)^3 + \frac{1}{5}(3x^4)^5 + \cdots$

18. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}[1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6 \cdots]$

23. $\tan^{-1}x^2 = x^2 - \frac{1}{3}(x^2)^3 + \frac{1}{5}(x^2)^5 + \cdots$

$x\tan^{-1}x^2 = x^3 - \frac{1}{3}x^8 + \frac{1}{5}x^{11} + \cdots + \frac{1}{n}x^{2n+1}$

34. $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$

$\sin(\tan^{-1}x) = \tan^{-1}x - \frac{1}{6}(\tan^{-1}x)^3 \cdots$

$= x - \frac{1}{2}x^3 + \frac{3}{8}x^5 - \frac{5}{16}x^7 + \cdots$

47. a. $\because f'(a) = 0$ $\therefore f(x) = f(a) + \frac{f''(c_2)}{2}(x-a)^2$

$f''(a) = f''(c_2) \leq 0$

$\therefore f(x) = -Ax^2 + Bx + C$ $(A > 0)$

when $x \to a$

$\therefore$ a local maximum

In the same way

$f(x) = Ax^2 + Bx + C$ $(A > 0)$ $x \to a$

when $f'' \geq 0$ $\therefore$ a local minimum

49. a. $f'(x) = k(1+x)^{k-1}$ $f''(x) = k(k-1)(1+x)^{k-2}$

$P = 1 + kx + \frac{k(k-1)}{2}x^2$

b. $|R_2(x)| = \left|\frac{3\cdot2\cdot1}{3!}x^3\right| < \frac{1}{100} \Rightarrow 0 < x < \frac{1}{100^{\frac{1}{3}}}$

$\Rightarrow 0 < x < 0.2154$

51. If $f(x) = \sum a_n x^n$

$f^{(k)}(0) = k! \cdot a_k \Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$

$\therefore$ It is itself as the statement

32.

a. $\because f(x)$ is even $\therefore$ any odd-order $= 0$

$\therefore f$ contains only even powers

b. $\because f(x)$ is odd $\therefore$ any even-order $= 0$

$\therefore f$ contains only odd powers