STA2001 Probability and statistical Inference I

Assignment 1

Deadline: 23:59 pm, Jan 19th, Friday

1. 1.1-6. If $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$, find
(a) $P(A \cup B)$
(b) $P(A \cap B')$
(c) $P(A' \cup B')$

Solution:
(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.3 = 0.6$
(b) $A = A \cup \emptyset = A \cup (B \cap B') = (A \cap B) \cup (A \cap B')$
Due to the fact that $(A \cap B)$ and $(A \cap B')$ are mutually exclusive, We have $P(A) = P(A \cap B) + P(A \cap B')$. Thus, $P(A \cap B') = P(A) - P(A \cap B) = 0.4 - 0.3 = 0.1$
(c) De Morgan's Laws
$P(A' \cup B') = P(A \cap B)' = 1 - P(A \cap B) = 1 - 0.3 = 0.7$

2. 1.1-9. Roll a fair six-sided die three times. Let $A_1 = \{1 \text{ or } 2 \text{ on the first roll}\}$, $A_2 = \{3 \text{ or } 4 \text{ on the second roll}\}$, and $A_3 = \{5 \text{ or } 6 \text{ on the third roll}\}$. It is given that $P(A_i) = 1/3$, i = 1, 2, 3; $P(A_i \cap A_j) = (1/3)^2$, $i \neq j$; and $P(A_1 \cap A_2 \cap A_3) = (1/3)^3$.
(a) Use Theorem 1.1-6 to find $P(A_1 \cup A_2 \cup A_3)$.
(b) Show that $P(A_1 \cup A_2 \cup A_3) = 1 - (1 - 1/3)^3$.

Solution:
(a) $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - (\frac{1}{3})^2 - (\frac{1}{3})^2 - (\frac{1}{3})^2 + (\frac{1}{3})^3 = \frac{19}{27}$

(b) Note that $A_1 \cup A_2 \cup A_3 = (A_1' \cap A_2' \cap A_3')'$ Since $P(A_i) = \frac{1}{3}, i = 1, 2, 3 P(A_i') = 1 - \frac{1}{3}, i = 1, 2, 3$; And$A_i', i = 1, 2, 3$ are independent events, $P(A_1' \cap A_2' \cap A_3') = P(A_1')P(A_2)'P(A_3') = (1 - \frac{1}{3})^3$
Thus, $P(A_1 \cup A_2 \cup A_3) = 1 - P(A_1' \cap A_2' \cap A_3') = 1 - (1 - \frac{1}{3})^3$.

3. 1.1-10. Prove Theorem 1.1-6.

Theorem 1.1-6 If A, B, and C are any three events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Solution: Since $A \cup B \cup C = A \cup (B \cup C)$.
By theorem 1.1-5, we have

$$\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\
&= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) \cup P(A \cap C)] \\
&= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)] \\
&= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)
\end{aligned}$$

4. 1.1-13. Divide a line segment into two parts by selecting a point at random. Use your intuition to assign a probability to the event that the longer segment is at least two times longer than the shorter segment.

Solution:
Let the length of the segment AB is L, and C is the point where we select.
Let the length of AC=X
We need to ensure that the longer segment is at least two times longer than the shorter segment,
$x \geq 2(L - x)$ or $2x \leq L - x$,
Then we get $3x \geq 2L$ or $3x \leq L$
Thus the probability of the event is $\frac{\{c|3x \geq 2L \text{or} 3x \leq L\}}{\{X|x \in AB\}} = \frac{2}{3}$.

5. 1.2-11. Three students (S) and six faculty members (F) are on a panel discussing a new college policy.
(a) In how many different ways can the nine participants be lined up at a table in the front of the auditorium?
(b) How many lineups are possible, considering only the labels S and F?
(c) For each of the nine participants, you are to decide whether the participant did a good job or a poor job stating his or her opinion of the new policy; that is, give each of the nine participants a grade of G or P. How many different "scorecards" are possible?

Solution:

(a) Nine participants are different individuals, so they could be arranged in 9!=362880 ways
(b) Separating nine participants into two types (Students and faculty members), we don't consider the inner order of students and faculty members, but consider the order between students and faculty members, so we use distinguishable permutation, that is

$$\frac{9!}{3! * 6!} = 84$$

(c) Each participants could be given a grade of G or P, so it should be $2^9 = 512$ outcomes.

6. 1.2-13. A bridge hand is found by taking 13 cards at random and without replacement from a deck of 52 playing cards. Find the probability of drawing each of the following hands.
(a) One in which there are 5 spades, 4 hearts, 3 diamonds, and 1 club.
(b) One in which there are 5 spades, 4 hearts, 2 diamonds, and 2 clubs.
(c) One in which there are 5 spades, 4 hearts, 1 diamond, and 3 clubs.
(d) Suppose you are dealt 5 cards of one suit, 4 cards of another. Would the probability of having the other suits split 3 and 1 be greater than the probability of having them split 2 and 2?

Solution:
(a) $P(A) = \frac{\binom{13}{5}\binom{13}{4}\binom{13}{3}\binom{13}{1}}{\binom{52}{13}} = 0.00539$
(b) $P(B) = \frac{\binom{13}{5}\binom{13}{4}\binom{13}{2}\binom{13}{2}}{\binom{52}{13}} = 0.00882$
(c) $P(C) = \frac{\binom{13}{5}\binom{13}{4}\binom{13}{1}\binom{13}{3}}{\binom{52}{13}} = 0.00539$
(d) Yes. Suppose we need to split heart suit and diamond suit, there are 13C1*13C3 cases of "1 heart and 3 diamonds", and 13C3*13C1 cases of "3 hearts and 1 diamond", and 13C2*13C2 cases of "2 hearts and 2 diamonds".
Since we have

$$\binom{13}{1}\binom{13}{3} + \binom{13}{3}\binom{13}{1} \geq \binom{13}{2}\binom{13}{2}$$

The probability of splitting suits into 1 and 3 should be greater than 2 and 2.

7. 1.2-14. A bag of 36 dum-dum pops (suckers) contains up to 10 flavors. That is, there are from 0 to 36 suckers of each of 10 flavors in the bag. How many different flavor combinations are possible?

   Solution:
   It is an unordered permutation with replacement problem.
   Since the pops (suckers) are not distinguishable while the flavors are distinguishable, we have:

$$\binom{r+n-1}{r,(n-1)} = \binom{36+10-1}{36,(10-1)} = \frac{45!}{36!*9!}$$

8. 1.2-17. A poker hand is defined as drawing 5 cards at random without replacement from a deck of 52 playing cards. Find the probability of each of the following poker hands:
   (a) Four of a kind (four cards of equal face value and one card of a different value).
   (b) Full house (one pair and one triple of cards with equal face value).
   (c) Three of a kind (three equal face values plus two cards of different values).
   (d) Two pairs (two pairs of equal face value plus one card of a different value).
   (e) One pair (one pair of equal face value plus three cards of different values).

   Solution:

   (a)First, we pick 1 face value from $A \sim K$ in 13 ways. Then we need to pick one card in the remaining cards with different face values from the first pick.

$$P(A) = \frac{\binom{13}{1}\binom{52-4}{1}}{\binom{52}{5}} = 0.00024$$

   (b)First we choose 1 face value to have 3 cards and then another face value to have 2 cards.

$$P(B) = \frac{\binom{13}{1}\binom{4}{3}\binom{13-1}{1}\binom{4}{2}}{\binom{52}{5}} = 0.00144$$

   (c)First we choose 1 face value to have 3 cards and then another two face values to have 1 card respectively.

$$P(C) = \frac{\binom{13}{1}\binom{4}{3}\binom{13-1}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = 0.0211$$

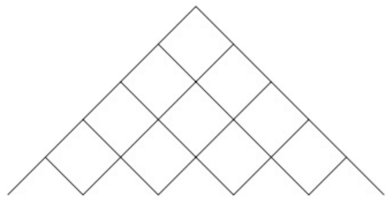
   (d)First we choose 2 face values to have 2 cards respectively and then pick the last one from the remaining cards with a different face value.

$$P(D) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{52-4-4}{1}}{\binom{52}{5}} = 0.0475$$

(e)First we choose 1 face value to have 2 cards and then pick three cards with different face values.

$$P(E) = \frac{\binom{13}{1}\binom{4}{2}\binom{13-1}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = 0.423$$

9. Skiers at the top of the mountain have a variety of choices as they head down the trails. Assume that at each intersection, a skier is equally likely to go left or right. Find the percent of skiers who end up at B and C, respectively.



Solution:

At each turn there is the choice of going left or right. The total number of paths down the mountain is 2^5. To end up at point B the skier must make 1 left turns and 4 right turns, and to end up at point C the skier must make 2 left turns and 3 right turns. The proportion of skiers who end up at B is then

$$P(B) = \frac{\binom{5}{1}}{2^5} = \frac{5}{32},$$

and the proportion of skiers who end up at C is

$$P(C) = \frac{\binom{5}{2}}{2^5} = \frac{10}{32}.$$

10. Codewords from the alphabet $\{0, 1, 2, 3\}$ are called legitimate if they have an even number of 0's. What's the probability of a codeword of length k is legitimatic?

Hint: First show the recursion $a_k = 2a_{k-1} + 4^{k-1}$, where a_k is the number of legitimate codewords of length k.

Solution:
We first show the recursion $a_k = 2a_{k-1} + 4^{k-1}$. If there are a_k legitimate codewords of length k, there are $4k - a_k$ illegitimate codewords of length k. Legitimate codewords of length k come in two types: those that end in 1, 2, 3 begin with a legitimate codeword of length $k-1$, those that end in 0 begin with an illegitimate codeword of length $k-1$. Hence we have $a_k = 3a_{k-1} + (4^{k-1} - a_{k-1}) = 2a_{k-1} + 4^{k-1}$, with initial condition is $a_1 = 3$. It is not hard to find

$$2a_k - 4^k = 2(2a_{k-1} - 4^{k-1})$$

Hence the solution is $a_k = (2^k + 4^k)/2$. Thus, the probability is

$$P(\text{legitimate}) = \frac{1}{2} + \frac{1}{2^{k+1}}$$