FIN2010 Financial Management
Time value of money I

Agenda

- What is the time value of money (TVM)?
- TVM of a single cash flow
  - APR and EAR
- TVM of cash flow streams

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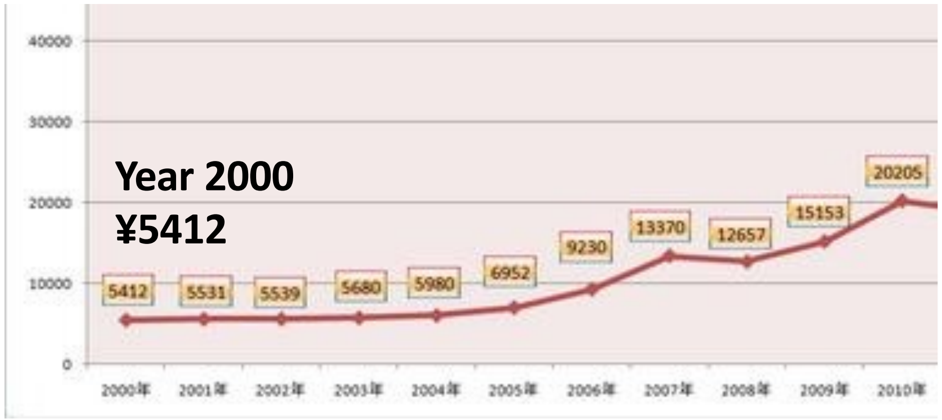
Time Value of Money

- Money has a time value
  - A dollar today is worth more than a dollar tomorrow

Property Price (/m²): Residential: Shenzhen 2000-2010 & 2011-2022



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  - If I had 1 million in 2000 and used it to buy an apartment in Shenzhen, I would have assets worth of ~10 million by 2022.

Key Concepts

- Present Value – what a cash flow is worth at the beginning of an investment period

- Future Value – what a cash flow is worth in the future

- Interest rate – “exchange rate” between earlier money and later money
  - Discount rate
  - Cost of capital
  - Opportunity cost of capital
  - Required return

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Future Values

- Suppose you invest 1,000 for one year at 5% per year. What is the future value in one year?
  - Interest = 1,000*5% = 50
  - Value in one year = initial investment + interest = 1,000 + 50 = 1,050
  - Future value (FV) = 1,000(1 + 5%) = 1,050

Future Values

- Suppose you leave the money in for another year. How much will you have two years from now?
  - **Compound interest**: interest is earned on both
    - the initial investment
    - the interest from previous period
  - FV with compound interest= 1050(1+0.05) = 1,102.50
  - **Simple interest**: interest is earned only on
    - the initial investment
  - FV with simple interest = 1,000 + 50 + 50 = 1,100
  - The extra 2.50 comes from the interest of .05(50) = 2.50 earned on the first interest payment

When do We Use Simple/Compound Interest?

In reality,

- Simple interest occurs if we do not reinvest the interest
  - Is rarely used.
  - E.g., Chinese banks pay simple interest on timed deposits
    - If you deposit 1000 into 工商银行 for 5 years, you will get $1000(1 + 2.65\% * 5) = 1132.5$
- Compound interest occurs if we do reinvest the interest
  - Is commonly used
  - E.g., if you owe money, your credit card balance compounds every day

(2023) 国有银行活期、定期存款利率表（整存整取）

序号	银行名称	活期	三个月	半年	一年	二年	三年	五年
1	中国银行	0.25	1.25	1.45	1.65	2.15	2.60	2.65
2	农业银行	0.25	1.25	1.45	1.65	2.15	2.60	2.65
3	工商银行	0.25	1.25	1.45	1.65	2.15	2.60	2.65
4	建设银行	0.25	1.25	1.45	1.65	2.15	2.60	2.65
5	交通银行	0.25	1.25	1.45	1.65	2.15	2.60	2.65
6	邮储银行	0.25	1.25	1.46	1.68	2.15	2.60	2.65

单位：% 数据来源：各大银行官网、APP、微信公众号等

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Future Values: General Formula

- $FV = PV(1 + r)^t$
  - FV = future value
  - PV = present value
  - r = per period interest rate
  - t = number of periods

- Suppose you deposit 1,000 in a bank today for five years. The banks pays <u>compound</u> interest of 2.75% per year. How much will you have in five years?
  - $FV = 1,000(1+2.75\%)^5 = 1,145.27$
- What is the effect of compounding?
  - Simple interest = 1000 + 1000*5*2.75% = 1137.5
  - Compounding added 7.77 to the value of the investment

Interpretation of PV and FV

- PV and FV help us establish equivalence between cash now and cash in the future.

- Therefore, we can use PV and FV to compare cash flow at different points in time.
  - If the interest rate is 3%, will you prefer $1{,}000$ today or $1{,}145.27$ in five years?
  - FV of $1{,}000 = 1{,}000 * (1 + 3\%)^5 = 1{,}159.27$.
  - Today 5 years later

    $1{,}000$ = $1{,}159.27$

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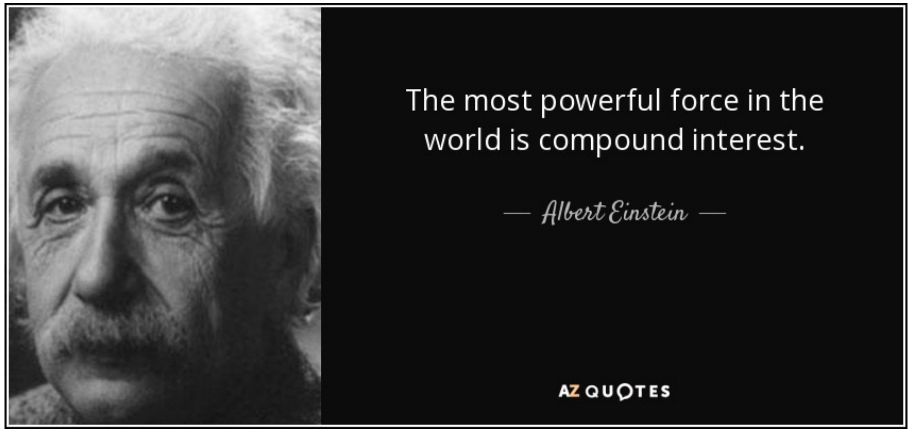
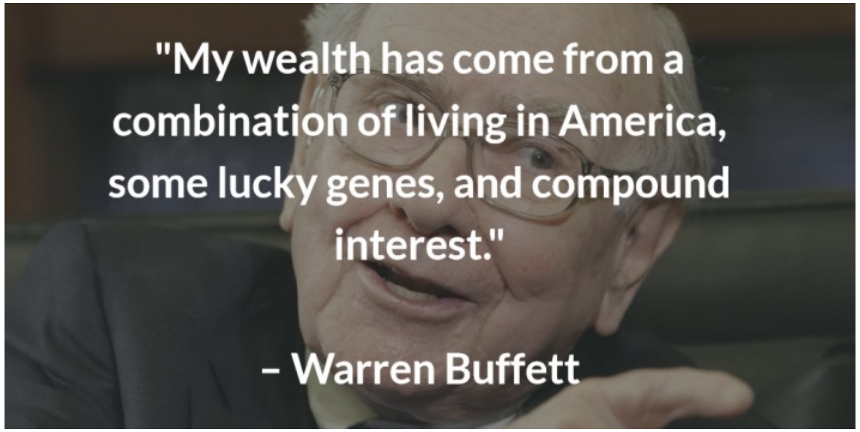
    $1{,}145.27$
  - Therefore, $1{,}000$ today is better than $1{,}145.27$ in five years.

The Power of Compounding





- Between 1965 and 2022, Berkshire Hathaway (CEO: Warren Buffett) earned an average return of 19.9% per year.
- Suppose you invested $1000 with Warren Buffet at the beginning of 1965, how much did you have at the end of 2022?
  - $\$1000 * (1 + 19.9\%)^{58} = \$37{,}283{,}151.9$
  - When it comes to investment, stay cool and let time be your best friend!

The Power of Compounding

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- Not understanding the compounding interest has cost many people fortune and even lives!

Solve for PV, t, r

- Start with the basic equation: $FV = PV(1+r)^t$
- $PV = FV\ (1+r)^{-t}$
- $R = \left(\frac{FV}{PV}\right)^{1/t} - 1$
- $t = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1+r)}$

Present Values-Example

- Suppose you need $95,000 to pay your tuition next year. If you can earn 7% annually, how much do you need to invest today?
  - $PV = 95{,}000(1.07)^{-1} = 88785.05$

Discount Rate-Example

- Li Ka-shing is famous for "hoarding" land for profit. His company bought 1.64 million m^2 of land (Yangjiashan) in Chongqing for 2.45B in 2007. His company offered to sell 1.03 million m^2 of the land (mostly undeveloped) for 20B in 2018. What is the annual appreciate rate of the land? Assume the per m^2 land price is the same for the entire area.

  - Original cost of land for 1.03 million m^2 = $2.45\text{B} * \frac{1.03}{1.64} = 1.54\text{B}$
  - $r = \left(\frac{20}{1.54}\right)^{1/11} - 1 = 26.3\%$

Number of Periods – Example

- Suppose you want to buy a new house. You currently have \$15,000, and you need \$20,000 for the down payment. You can earn 7.5% per year. How long does it take to save enough for the down payment?

  - $t = \ln(20{,}000/15{,}000)/\ln(1.075) = 3.98$ years

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Annual Percentage Rate (APR)

- Interests can be <u>charged</u> in any frequency
  - Credit card: daily
  - Home mortgage: monthly
  - Bonds: semiannually

- Interest rate is required by law to be <u>quoted</u> in annual frequency, the so-called annual percentage rate (APR)
  - APR = per period rate * the number of periods per year (m)
  - So, per period rate = APR / number of periods per year (m)
  - $FV = PV * \left(1 + \frac{APR}{m}\right)^{m*n}$

    where n is the number of years

APR vs. Period Rate

- What is the APR if the monthly rate is 0.5%?
  - 0.5% * 12 = 6%

- A 6% bond pays interests semiannually. What is the amount of each interest?
  - 6% / 2 = 3%

- Currently the home mortgage rate is 9%, and interest is charged monthly. What is the interest rate per month?
  - 9% / 12 = 0.75%

Effective Annual Rate (EAR) vs. APR

Annual Percentage Rate (APR) for Purchases	18.99% when you open your account. This APR will vary with the market based on the Prime Rate.
APR for Balance Transfers	**18.99%**. This APR will vary with the market based on the Prime Rate.
APR for Cash Advances	**25.24%**. This APR will vary with the market based on the Prime Rate.
Penalty APR and When it Applies	Up to **29.99%**, based on your creditworthiness. This APR will vary with the market based on the Prime Rate. This APR may be applied to your account if you: 1. Make a late payment; 2. Go over your credit limit; or 3. Make a payment that is returned. **How Long Will the Penalty APR Apply?** If your APRs are increased for any of these reasons, the Penalty APRs may apply indefinitely.

If compounded annually, **annual** interest for a $100 debt is $18.99

If compounded semiannually, **annual** interest is $100*$[(1+0.1899/2)^2 -1]$= $19.89

If compounded monthly, **annual** interest is $100*$[(1+0.1899/12)^{12} -1]$= $20.73

If compounded daily, annual interest is $100*$[(1+0.1899/365)^{365} -1]$= $20.91

- **Effective annual rate**: the actual interest rate after accounting for compounding that occurs during the year

$$1 + EAR = \left(1 + \frac{APR}{m}\right)^m \longrightarrow EAR = \left(1 + \frac{APR}{m}\right)^m - 1$$

where m is the compounding frequency

EAR Example—P2P Lending

- $APR = 30\% * 52 = 1560\%$
- $EAR = (1 + 30\%)^{52} - 1 = 841499.4$

EAR-Continuous Compounding

- Sometimes investments or loans are figured based on continuous compounding. In other words, m goes to infinity
- $EAR = \lim_{m \to \infty} \left(1 + \frac{APR}{m}\right)^m - 1 = e^{APR} - 1$

- Example: What is the effective annual rate of 7% compounded continuously?
  - $EAR = e^{7\%} - 1 = 7.25\%$

When should I use EAR vs. APR?

- $FV = PV(1 + r)^t$
  - r should be the effective rate in each period

Assume you need to calculate FV in n years:

- If you are given an APR:
  - APR always comes in a pair: {APR, compounding frequency}
  - APR only tells you the interest rate in each compounding period
    Compounding period interest $= APR/m$
  - For a period of any other lengths, do not use APR in this way. Instead, express time in terms of numbers of compounding periods:

$$FV = PV\left(1 + \frac{APR}{m}\right)^{nm}$$

- If you are given an EAR:
  - $FV = PV(1 + EAR)^n$, n expressed in terms of years

EAR vs. APR Example

12% APR, monthly compounding ↔ 12.68% EAR

- What is the relationship between the two numbers?
  - $1 + 12.68\% = (1 + 12\%/12)^{12}$
- What is the interest rate for a 2-month period?
  - Using APR: 1-month interest rate $r_{1m} = 1\%$,
    so $r_{2m} = (1 + r_{1m})^2 - 1 = 2.01\%$
  - Using EAR: $r_{2m} = (1 + 12.68\%)^{2/12} - 1 = 2.01\%$
- What is the interest rate for a half-month period?
  - Using APR: so $r_{0.5m} = (1 + r_{1m})^{1/2} - 1 = 0.499\%$
  - Using EAR: $r_{0.5m} = (1 + 12.68)^{1/24} - 1 = 0.499\%$

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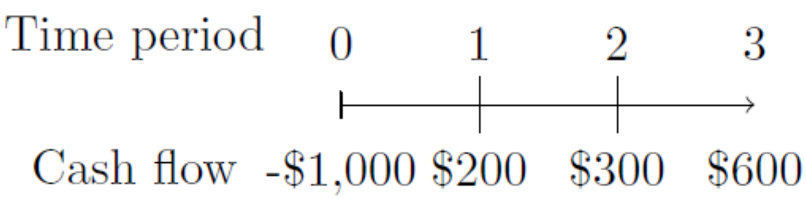
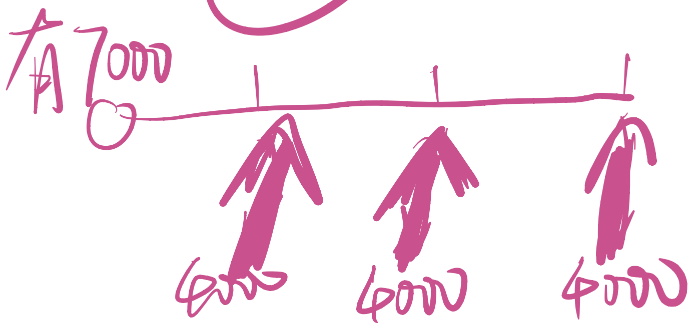
Cash Flows Streams

- Basic principle:
  – The PV (FV) of a stream of cash flows equals to the sum of PV (FV) of individual cash flows.
- A useful tool: timeline
  – A linear representation of the timing and amount of cash flows.
  – Lend out $1,000 today and then receive $200 in the first period, $300 in the second period, $600 in the third period.



Cash Flows Streams– FV Example

- You think you will be able to deposit $4,000 at the end of each of the next three years in a bank account paying 8 percent interest.
  - You currently have $7,000 in the account.
  - How much will you have in three years?
  - How much will you have in four years?



- Find the value at year 3 of each cash flow and add them together
  - Today (year 0): FV = $7000(1.08)^3$ = 8,817.98
  - Year 1: FV = $4,000(1.08)^2$ = 4,665.60
  - Year 2: FV = 4,000(1.08) = 4,320
  - Year 3: value = 4,000
  - Total value in 3 years = 8,817.98 + 4,665.60 + 4,320 + 4,000 = 21,803.58

  - Value at year 4 = 21,803.58(1.08) = 23,547.87

Cash Flows Streams– PV Example

- You are offered an investment that will give you $200 in one year, $400 the next year, $600 the following year, and $800 at the end of the fourth year. You can earn 12% on your investment. What is the PV of this cash flow streams?

- Find the PV of each cash flows and add them
  - Year 1 CF: $200 / (1.12)^1 = 178.57$
  - Year 2 CF: $400 / (1.12)^2 = 318.88$
  - Year 3 CF: $600 / (1.12)^3 = 427.07$
  - Year 4 CF: $800 / (1.12)^4 = 508.41$
  - Total PV = 178.57 + 318.88 + 427.07 + 508.41 = 1,432.93

Standardized Cash Flow Streams

- Annuity – finite series of equal payments that occur at regular intervals
  - If the first payment occurs at the end of the period, it is called an *ordinary annuity*
  - If the first payment occurs at the beginning of the period, it is called an *annuity due*
    - Examples: rent payment, tuition payment

- Perpetuity – infinite series of payments that occur at regular intervals
  - Equal payments: level perpetuity
  - Payments grow at a constant rate: growth perpetuity
    - Example: stock dividend

Review – Geometric Series

Sum of geometric series $a + ax + ax^2 + \cdots + ax^{n-1}$:

$$sum = a\,\frac{1 - x^n}{1 - x}$$

Notes:

- a : the first term of the series
- n: the number of terms in total
- x: common ratio

Infinite geometric series $a + ax + ax^2 + \cdots$:

$$sum = \frac{a}{1 - x}, |x| < 1$$

Standardized Cash Flow Streams-Examples

a.

Time period	0	1	2	3	4	5	6
Cash flow		\$100	\$100	\$100	\$100	\$100	\$100

b.

Time period	0	1	2	3	4	5	6
Cash flow	\$100	\$100	\$100	\$100	\$100	\$100	

c.

Time period	0	1	2	3	4	5	6
Cash flow		\$100	\$200	\$100	\$100	\$100	\$100

d.

Time period	0	1	2	3	4	5	6
Cash flow		\$100		\$100	\$100	\$100	\$100

e.

Time period	0	1	2	3	...	t	...
Cash flow		\$100	\$100	\$100		\$100	...

f.

Time period	0	1	2	3	...
Cash flow		100	200	400	...

a. An ordinary annuity from 0 to 6. An annuity due from 1-7
b. An annuity due from 0-6
c. Complex cash flows from 0-6. An annuity due from 3-7
d. Complex cash flows from 0-6. An ordinary annuity from 2-6
e. Level perpetuity.
f. Growth perpetuity

Annuity

t periods, the amount of each cash flow(PMT), periodic interest rate(r), number of periods per year (m).

Time period	0	1	2	3	...	t-1	t
Cash flow		PMT	PMT	PMT	...	PMT	PMT

$$
\begin{aligned}
FV &= PMT(1+r)^{t-1} + PMT(1+r)^{t-2} + ... + PMT(1+r)^{0} \\
&= PMT[(1+r)^{t-1} + (1+r)^{t-2} + ... + (1+r)^{0}] \quad \text{Inside the bracket: a geometric sequence} \\
&= PMT[\frac{(1+r)^{t}-1}{r}]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{PV} &= \mathrm{PMT} * \frac{1}{(1+r)^1} + \mathrm{PMT} * \frac{1}{(1+r)^2} + \mathrm{PMT} * \frac{1}{(1+r)^3} + ... + \mathrm{PMT} * \frac{1}{(1+r)^t} \\
&= \mathrm{PMT}[\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + ... + \frac{1}{(1+r)^t}] : \quad \text{Geometric sequence} \\
&= \mathrm{PMT} * \frac{1 - \frac{1}{(1+r)^t}}{r}
\end{aligned}
$$

Annuities Due

Time period	0	1	2	3	...	t-1	t
Cash flow	PMT	PMT	PMT	PMT	...	PMT	

$$
\begin{aligned}
\mathrm{FV} &= \mathrm{PMT} * (1+r)^t + \mathrm{PMT} * (1+r)^{t-1} + \mathrm{PMT} * (1+r)^{t-2} + \ldots + \mathrm{PMT} * (1+r)^1 \\
&= \mathrm{PMT}[(1+r)^t + (1+r)^{t-1} + (1+r)^{t-2} + \ldots + (1+r)^1] \\
&= \mathrm{PMT} * \frac{(1+r)^t - 1}{r}(1+r) \\
&= \mathrm{FV}_{OA}(1+r)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{PV} &= \mathrm{PMT} * \frac{1}{(1+r)^0} + \mathrm{PMT} * \frac{1}{(1+r)^1} + \mathrm{PMT} * \frac{1}{(1+r)^2} + \ldots + \mathrm{PMT} * \frac{1}{(1+r)^{t-1}} \\
&= \mathrm{PMT}[\frac{1}{(1+r)^0} + \frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^{t-1}}] \\
&= \mathrm{PMT} * \frac{1 - \frac{1}{(1+r)^t}}{r}(1+r) \\
&= \mathrm{PV}_{OA}(1+r)
\end{aligned}
$$

OA stands for ordinary annuity that are of the same PMT, r and t

Perpetuity

- **Level perpetuity**

Time period	0	1	2	3	...
Cash flow		PMT	PMT	PMT	...

$$
\begin{aligned}
\mathrm{PV} &= \mathrm{PMT} * \frac{1}{(1+r)^1} + \mathrm{PMT} * \frac{1}{(1+r)^2} + \mathrm{PMT} * \frac{1}{(1+r)^3} + \ldots \\
&= \mathrm{PMT}[\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots] \\
&= \mathrm{PMT} * \frac{1}{r}
\end{aligned}
$$

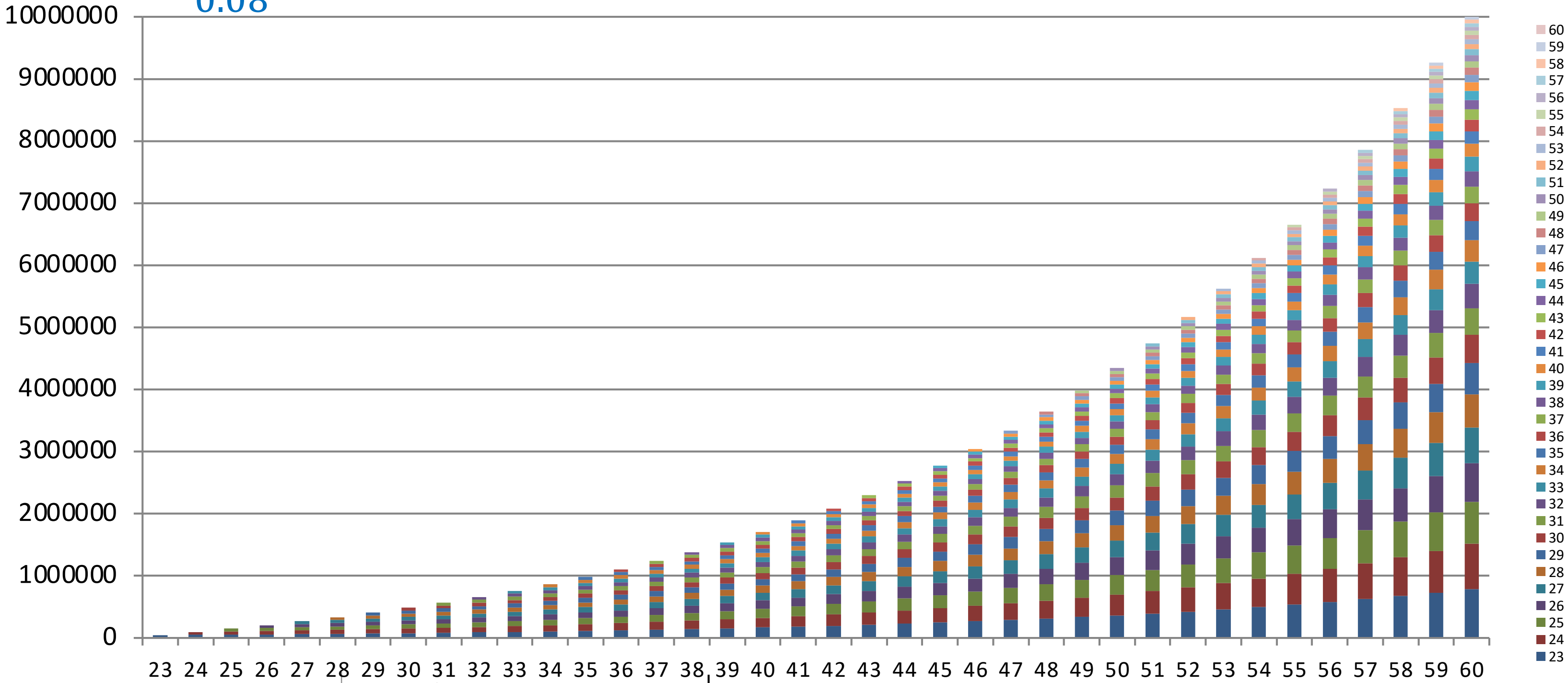
- **Growth perpetuity**

Time period	0	1	2	3	...
Cash flow		PMT_1	$\mathrm{PMT}_1(1+g)^1$	$\mathrm{PMT}_1(1+g)^2$	

$$
\begin{aligned}
\mathrm{PV} &= \mathrm{PMT} * \frac{1}{(1+r)^1} + \mathrm{PMT}(1+g) * \frac{1}{(1+r)^2} + \mathrm{PMT}(1+g)^2 * \frac{1}{(1+r)^3} + \ldots \\
&= \mathrm{PMT}[\frac{(1+g)^0}{(1+r)^1} + \frac{(1+g)^1}{(1+r)^2} + \frac{(1+g)^2}{(1+r)^3} + \ldots] \\
&= \mathrm{PMT} * \frac{1}{r-g} \quad (\text{Note that it has to be } g < r)
\end{aligned}
$$

Example—Retirement Planning

- Suppose you want to have 10 million when you retire at the age 60. You plan to save x on every birthday from 23 to 59. Annual interest rate = 8%.
  - $x \cdot 1.08^{37} + x \cdot 1.08^{36} + \cdots + x \cdot 1.08^{1} = 10{,}000{,}000$
  - $x \dfrac{1.08^{37}-1}{0.08}(1.08) = 10{,}000{,}000 \Rightarrow x = 45{,}596$



Some Takeaways

- Wealth grows exponentially
- Savings of 1.23M in total over the course of 37 years will grow to 10M
- Savings are much more valuable when you are young
  - The first saving will grow 17.25 times after 37 years
  - The first 8 (out of 37) savings accounts for 48% of terminal wealth
- You only reach the half-way mark when you are 52, 29 years into the savings plan

Summary

- For a single cash flow: $FV = PV * \left(1 + \frac{APR}{m}\right)^{m*n}$
  - APR: how interest rate is quoted. Equals to per period interest times m (#periods per year)
  - EAR: actual interest accrued in a year
- For cash flow streams: FV (or PV) is the sum of FV (or PV) of individual cash flows
- Standardized cash flow streams
  - Ordinary annuity: $PV = PMT\frac{1-(1+r)^{-n}}{r}$
  - Annuities due: $PV = PMT\left(1 + \frac{1-(1+r)^{-n+1}}{r}\right) = PMT(1+r)\frac{1-(1+r)^{-n}}{r}$
  - Level perpetuity: $PV = PMT/r$
  - Growth perpetuity: $PV = \frac{PMT}{r-g}$