CHAPTER 12 VECTORS AND THE GEOMETRY OF SPACE

12.1 THREE-DIMENSIONAL COORDINATE SYSTEMS

1. The line through the point $(2, 3, 0)$ parallel to the z-axis

2. The line though the point $(-1, 0, 0)$ parallel to the y-axis

3. The x-axis

4. The line through the point $(1, 0, 0)$ parallel to the z-axis

5. The circle $x^2 + y^2 = 4$ in the xy-plane

6. The circle $x^2 + y^2 = 4$ in the plane $z = -2$

7. The circle $x^2 + z^2 = 4$ in the xz-plane

8. The circle $y^2 + z^2 = 1$ in the yz-plane

9. The circle $y^2 + z^2 = 1$ in the yz-plane

10. The circle $x^2 + z^2 = 9$ in the plane $y = -4$

11. The circle $x^2 + y^2 = 16$ in the xy-plane

12. The circle $x^2 + z^2 = 3$ in the xz-plane

13. The ellipse formed by the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $z = y$.

14. The circle formed by the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $y = x$.

15. The parabola $y = x^2$ in the xy-plane.

16. The parabola $z = y^2$ in the plane $x = 1$.

17. (a) The first quadrant of the xy-plane (b) The fourth quadrant of the xy-plane

18. (a) The slab bounded by the planes $x = 0$ and $x = 1$
    (b) The square column bounded by the planes $x = 0, x = 1, y = 0, y = 1$
    (c) The unit cube in the first octant having one vertex at the origin

19. (a) The solid ball of radius 1 centered at the origin
    (b) The exterior of the sphere of radius 1 centered at the origin

20. (a) The circumference and interior of the circle $x^2 + y^2 = 1$ in the xy-plane
    (b) The circumference and interior of the circle $x^2 + y^2 = 1$ in the plane $z = 3$
    (c) A solid cylindrical column of radius 1 whose axis is the z-axis

21. (a) The solid enclosed between the sphere of radius 1 and radius 2 centered at the origin
    (b) The solid upper hemisphere of radius 1 centered at the origin

22. (a) The line $y = x$ in the xy-plane
    (b) The plane $y = x$ consisting of all points of the form (x, x, z)

23. (a) The region on or inside the parabola $y = x^2$ in the xy-plane and all points above this region.
    (b) The region on or to the left of the parabola $x = y^2$ in the xy-plane and all points above it that are 2 units or less away from the xy-plane.

24. (a) All the points the lie on the plane $z = 1 - y$.
    (b) All points that lie on the curve $z = y^3$ in the plane $x = -2$.

25. (a) $x = 3$ (b) $y = -1$ (c) $z = -2$

26. (a) $x = 3$ (b) $y = -1$ (c) $z = 2$

27. (a) $z = 1$ (b) $x = 3$ (c) $y = -1$

28. (a) $x^2 + y^2 = 4, z = 0$ (b) $y^2 + z^2 = 4, x = 0$ (c) $x^2 + z^2 = 4, y = 0$

29. (a) $x^2 + (y-2)^2 = 4, z = 0$ (b) $(y-2)^2 + z^2 = 4, x = 0$ (c) $x^2 + z^2 = 4, y = 2$

30. (a) $(x+3)^2 + (y-4)^2 = 1, z = 1$ (b) $(y-4)^2 + (z-1)^2 = 1, x = -3$ (c) $(x+3)^2 + (z-1)^2 = 1, y = 4$

31. (a) $y = 3, z = -1$ (b) $x = 1, z = -1$ (c) $x = 1, y = 3$

32. $\sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + (y-2)^2 + z^2} \Rightarrow x^2 + y^2 + z^2 = x^2 + (y-2)^2 + z^2 \Rightarrow y^2 = y^2 - 4y + 4 \Rightarrow y = 1$

33. $x^2 + y^2 + z^2 = 25, z = 3 \Rightarrow x^2 + y^2 = 16$ in the plane $z = 3$

34. $x^2 + y^2 + (z-1)^2 = 4$ and $x^2 + y^2 + (z+1)^2 = 4 \Rightarrow x^2 + y^2 + (z-1)^2 = x^2 + y^2 + (z+1)^2 \Rightarrow z = 0, x^2 + y^2 = 3$

35. $0 \le z \le 1$

36. $0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2$

37. $z \le 0$

38. $z = \sqrt{1 - x^2 - y^2}$

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39. (a) $(x-1)^2+(y-1)^2+(z-1)^2<1$ (b) $(x-1)^2+(y-1)^2+(z-1)^2>1$

40. $1\le x^2+y^2+z^2\le 4$

41. $|P_1P_2|=\sqrt{(3-1)^2+(3-1)^2+(0-1)^2}=\sqrt{9}=3$

42. $|P_1P_2|=\sqrt{(2+1)^2+(5-1)^2+(0-5)^2}=\sqrt{50}=5\sqrt{2}$

43. $|P_1P_2|=\sqrt{(4-1)^2+(-2-4)^2+(7-5)^2}=\sqrt{49}=7$

44. $|P_1P_2|=\sqrt{(2-3)^2+(3-4)^2+(4-5)^2}=\sqrt{3}$

45. $|P_1P_2|=\sqrt{(2-0)^2+(-2-0)^2+(-2-0)^2}=\sqrt{3\cdot 4}=2\sqrt{3}$

46. $|P_1P_2|=\sqrt{(0-5)^2+(0-3)^2+(0+2)^2}=\sqrt{38}$

47. center $(-2, 0, 2)$, radius $2\sqrt{2}$

48. center $\left(1, -\frac{1}{2}, -3\right)$, radius 5

49. center $\left(\sqrt{2}, \sqrt{2}, -\sqrt{2}\right)$, radius $\sqrt{2}$

50. center $\left(0, -\frac{1}{3}, \frac{1}{3}\right)$, radius $\frac{4}{3}$

51. $(x-1)^2+(y-2)^2+(z-3)^2=14$

52. $x^2+(y+1)^2+(z-5)^2=4$

53. $(x+1)^2+\left(y-\frac{1}{2}\right)^2+\left(z+\frac{2}{3}\right)^2=\frac{16}{81}$

54. $x^2+(y+7)^2+z^2=49$

55. $x^2+y^2+z^2+4x-4z=0\Rightarrow\left(x^2+4x+4\right)+y^2+\left(z^2-4z+4\right)=4+4\Rightarrow(x+2)^2+(y-0)^2+(z-2)^2=\left(\sqrt{8}\right)^2$
$\Rightarrow$ the center is at $(-2, 0, 2)$ and the radius is $\sqrt{8}$

56. $x^2+y^2+z^2-6y+8z=0\Rightarrow x^2+\left(y^2-6y+9\right)+\left(z^2+8z+16\right)=9+16\Rightarrow(x-0)^2+(y-3)^2+(z+4)^2=5^2$
$\Rightarrow$ the center is at $(0, 3, -4)$ and the radius is 5

57. $2x^2+2y^2+2z^2+x+y+z=9\Rightarrow x^2+\frac{1}{2}x+y^2+\frac{1}{2}y+z^2+\frac{1}{2}z=\frac{9}{2}$
$\Rightarrow\left(x^2+\frac{1}{2}x+\frac{1}{16}\right)+\left(y^2+\frac{1}{2}y+\frac{1}{16}\right)+\left(z^2+\frac{1}{2}z+\frac{1}{16}\right)=\frac{9}{2}+\frac{3}{16}\Rightarrow\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\left(\frac{5\sqrt{3}}{4}\right)^2$
$\Rightarrow$ the center is at $\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ and the radius is $\frac{5\sqrt{3}}{4}$

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58. $3x^2+3y^2+3z^2+2y-2z=9 \Rightarrow x^2+y^2+\frac{2}{3}y+z^2-\frac{2}{3}z=3 \Rightarrow x^2+\left(y^2+\frac{2}{3}y+\frac{1}{9}\right)+\left(z^2-\frac{2}{3}z+\frac{1}{9}\right)=3+\frac{2}{9}$
$\Rightarrow (x-0)^2+\left(y+\frac{1}{3}\right)^2+\left(z-\frac{1}{3}\right)^2=\left(\frac{\sqrt{29}}{3}\right)^2 \Rightarrow$ the center is at $\left(0, -\frac{1}{3}, \frac{1}{3}\right)$ and the radius is $\frac{\sqrt{29}}{3}$

59. (a) the distance between (x, y, z) and $(x, 0, 0)$ is $\sqrt{y^2+z^2}$
(b) the distance between (x, y, z) and $(0, y, 0)$ is $\sqrt{x^2+z^2}$
(c) the distance between (x, y, z) and $(0, 0, z)$ is $\sqrt{x^2+y^2}$

60. (a) the distance between (x, y, z) and $(x, y, 0)$ is z
(b) the distance between (x, y, z) and $(0, y, z)$ is x
(c) the distance between (x, y, z) and $(x, 0, z)$ is y

61. $|AB|=\sqrt{(1-(-1))^2+(-1-2)^2+(3-1)^2}=\sqrt{4+9+4}=\sqrt{17}$
$|BC|=\sqrt{(3-1)^2+(4-(-1))^2+(5-3)^2}=\sqrt{4+25+4}=\sqrt{33}$
$|CA|=\sqrt{(-1-3)^2+(2-4)^2+(1-5)^2}=\sqrt{16+4+16}=\sqrt{36}=6$
Thus the perimeter of triangle ABC is $\sqrt{17}+\sqrt{33}+6$.

62. $|PA|=\sqrt{(2-3)^2+(-1-1)^2+(3-2)^2}=\sqrt{1+4+1}=\sqrt{6}$
$|PB|=\sqrt{(4-3)^2+(3-1)^2+(1-2)^2}=\sqrt{1+4+1}=\sqrt{6}$
Thus P is equidistant from A and B.

63. $\sqrt{(x-x)^2+(y-(-1))^2+(z-z)^2}=\sqrt{(x-x)^2+(y-3)^2+(z-z)^2} \Rightarrow (y+1)^2=(y-3)^2 \Rightarrow 2y+1=-6y+9$
$\Rightarrow y=1$

64. $\sqrt{(x-0)^2+(y-0)^2+(z-2)^2}=\sqrt{(x-x)^2+(y-y)^2+(z-0)^2} \Rightarrow x^2+y^2+(z-2)^2=z^2$
$\Rightarrow x^2+y^2-4z+4=0 \Rightarrow z=\frac{x^2}{4}+\frac{y^2}{4}+1$

65. (a) Since the entire sphere is below the xy-plane, the point on the sphere closest to the xy-plane is the point at the top of the sphere, which occurs when $x=0$ and $y=3 \Rightarrow 0^2+(3-3)^2+(z+5)^2=4 \Rightarrow z=-5\pm 2$ $\Rightarrow z=-3 \Rightarrow (0, 3, -3)$.
(b) Both the center $(0, 3, -5)$ and the point $(0, 7, -5)$ lie in the plane $z=-5$, so the point on the sphere closest to $(0, 7, -5)$ should also be in the same plane. In fact it should lie on the line segment between $(0, 3, -5)$ and $(0, 7, -5)$, thus the point occurs when $x=0$ and $z=-5 \Rightarrow 0^2+(y-3)^2+(-5+5)^2=4$ $\Rightarrow y=3\pm 2 \Rightarrow y=5 \Rightarrow (0, 5, -5)$.

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66. $\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}=\sqrt{(x-0)^2+(y-4)^2+(z-0)^2}=\sqrt{(x-3)^2+(y-0)^2+(z-0)^2}$
$=\sqrt{(x-2)^2+(y-2)^2+(z+3)^2}$
$\Rightarrow x^2+y^2+z^2=x^2+y^2-8y+16+z^2=x^2-6x+9+y^2+z^2=x^2-4x+y^2-4y+z^2+6z+17$
Solve: $x^2+y^2+z^2=x^2+y^2-8y+16+z^2 \Rightarrow 0=-8y+16 \Rightarrow y=2$
Solve: $x^2+y^2+z^2=x^2-6x+9+y^2+z^2 \Rightarrow 0=-6x+9 \Rightarrow x=\frac{3}{2}$
Solve: $x^2+y^2+z^2=x^2-4x+y^2-4y+z^2+6z+17 \Rightarrow 0=-4x-4y+6z+17 \Rightarrow 0=-4\left(\frac{3}{2}\right)-4(2)+6z+17$
$\Rightarrow z=-\frac{1}{2} \Rightarrow \left(\frac{3}{2}, 2, -\frac{1}{2}\right)$

12.2 VECTORS

1. (a) $\langle 3(3), 3(-2)\rangle = \langle 9, -6\rangle$
   (b) $\sqrt{9^2+(-6)^2}=\sqrt{117}=3\sqrt{13}$

2. (a) $\langle -2(-2), -2(5)\rangle = \langle 4, -10\rangle$
   (b) $\sqrt{4^2+(-10)^2}=\sqrt{116}=2\sqrt{29}$

3. (a) $\langle 3+(-2), -2+5\rangle = \langle 1, 3\rangle$
   (b) $\sqrt{1^2+3^2}=\sqrt{10}$

4. (a) $\langle 3-(-2), -2-5\rangle = \langle 5, -7\rangle$
   (b) $\sqrt{5^2+(-7)^2}=\sqrt{74}$

5. (a) $2\mathbf{u}=\langle 2(3), 2(-2)\rangle=\langle 6, -4\rangle$
   $3\mathbf{v}=\langle 3(-2), 3(5)\rangle=\langle -6, 15\rangle$
   $2\mathbf{u}-3\mathbf{v}=\langle 6-(-6), -4-15\rangle=\langle 12, -19\rangle$
   (b) $\sqrt{12^2+(-19)^2}=\sqrt{505}$

6. (a) $-2\mathbf{u}=\langle -2(3), -2(-2)\rangle=\langle -6, 4\rangle$
   $5\mathbf{v}=\langle 5(-2), 5(5)\rangle=\langle -10, 25\rangle$
   $-2\mathbf{u}+5\mathbf{v}=\langle -6+(-10), 4+25\rangle=\langle -16, 29\rangle$
   (b) $\sqrt{(-16)^2+29^2}=\sqrt{1097}$

7. (a) $\frac{3}{5}\mathbf{u}=\left\langle \frac{3}{5}(3), \frac{3}{5}(-2)\right\rangle=\left\langle \frac{9}{5}, -\frac{6}{5}\right\rangle$
   $\frac{4}{5}\mathbf{v}=\left\langle \frac{4}{5}(-2), \frac{4}{5}(5)\right\rangle=\left\langle -\frac{8}{5}, 4\right\rangle$
   $\frac{3}{5}\mathbf{u}+\frac{4}{5}\mathbf{v}=\left\langle \frac{9}{5}+\left(-\frac{8}{5}\right), -\frac{6}{5}+4\right\rangle=\left\langle \frac{1}{5}, \frac{14}{5}\right\rangle$
   (b) $\sqrt{\left(\frac{1}{5}\right)^2+\left(\frac{14}{5}\right)^2}=\frac{\sqrt{197}}{5}$

8. (a) $-\frac{5}{13}\mathbf{u}=\left\langle -\frac{5}{13}(3), -\frac{5}{13}(-2)\right\rangle=\left\langle -\frac{15}{13}, \frac{10}{13}\right\rangle$
   $\frac{12}{13}\mathbf{v}=\left\langle \frac{12}{13}(-2), \frac{12}{13}(5)\right\rangle=\left\langle -\frac{24}{13}, \frac{60}{13}\right\rangle$
   $-\frac{5}{13}\mathbf{u}+\frac{12}{13}\mathbf{v}=\left\langle -\frac{15}{13}+\left(-\frac{24}{13}\right), \frac{10}{13}+\frac{60}{13}\right\rangle=\left\langle -3, \frac{70}{13}\right\rangle$
   (b) $\sqrt{(-3)^2+\left(\frac{70}{13}\right)^2}=\frac{\sqrt{6421}}{13}$

9. $\langle 2-1, -1-3\rangle=\langle 1, -4\rangle$

10. $\left\langle \frac{2+(-4)}{2}-0, \frac{-1+3}{2}-0\right\rangle=\langle -1, 1\rangle$

11. $\langle 0-2, 0-3\rangle=\langle -2, -3\rangle$

12. $\overrightarrow{AB}=\langle 2-1, 0-(-1)\rangle=\langle 1, 1\rangle$, $\overrightarrow{CD}=\langle -2-(-1), 2-3\rangle=\langle -1, -1\rangle$, $\overrightarrow{AB}+\overrightarrow{CD}=\langle 0, 0\rangle$

13. $\left\langle \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\right\rangle=\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$

14. $\left\langle \cos\left(-\frac{3\pi}{4}\right), \sin\left(-\frac{3\pi}{4}\right)\right\rangle=\left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$

15. This is the unit vector which makes an angle of $120°+90°=210°$ with the positive x-axis;
$\langle \cos 210°, \sin 210°\rangle=\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$

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16. $\langle \cos 135°, \sin 135° \rangle = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

17. $\overrightarrow{P_1P_2} = (2-5)\mathbf{i} + (9-7)\mathbf{j} + \left(-2-(-1)\right)\mathbf{k} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

18. $\overrightarrow{P_1P_2} = (-3-1)\mathbf{i} + (0-2)\mathbf{j} + (5-0)\mathbf{k} = -4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

19. $\overrightarrow{AB} = \left(-10-(-7)\right)\mathbf{i} + \left(8-(-8)\right)\mathbf{j} + (1-1)\mathbf{k} = -3\mathbf{i} + 16\mathbf{j}$

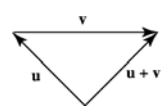
20. $\overrightarrow{AB} = (-1-1)\mathbf{i} + (4-0)\mathbf{j} + (5-3)\mathbf{k} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

21. $5\mathbf{u} - \mathbf{v} = 5\langle 1, 1, -1\rangle - \langle 2, 0, 3\rangle = \langle 5, 5, -5\rangle - \langle 2, 0, 3\rangle = \langle 5-2, 5-0, -5-3\rangle = \langle 3, 5, -8\rangle = 3\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$

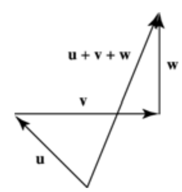
22. $-2\mathbf{u} + 3\mathbf{v} = -2\langle -1, 0, 2\rangle + 3\langle 1, 1, 1\rangle = \langle 2, 0, -4\rangle + \langle 3, 3, 3\rangle = \langle 5, 3, -1\rangle = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

23. The vector **v** is horizontal and 1 unit long. The vectors **u** and **w** are $\frac{11}{16}$ unit long. **w** is vertical and **u** makes a $45°$ angle with the horizontal. All vectors must be drawn to scale.

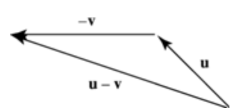
(a)



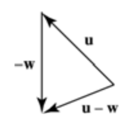
(b)



(c)

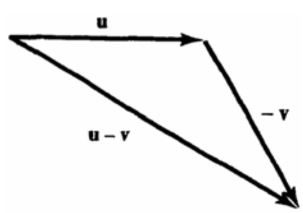


(d)

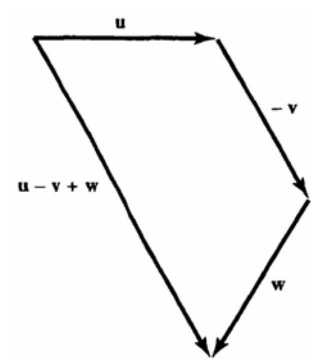


24. The angle between the vectors is $120°$ and vector **u** is horizontal. They are all 1 unit long. Draw to scale.

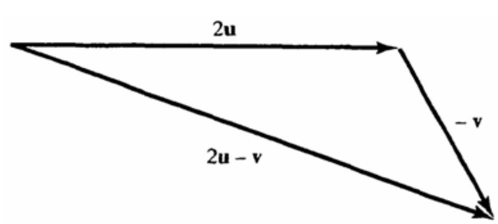
(a)



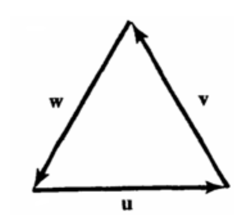
(b)



(c)



(d)



$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$

25. length $= |2\mathbf{i}+\mathbf{j}-2\mathbf{k}| = \sqrt{2^2+1^2+(-2)^2} = 3$, the direction is $\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k} \Rightarrow 2\mathbf{i}+\mathbf{j}-2\mathbf{k} = 3\left(\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}\right)$

26. length $= |9\mathbf{i}-2\mathbf{j}+6\mathbf{k}| = \sqrt{81+4+36} = 11$, the direction is $\frac{9}{11}\mathbf{i}-\frac{2}{11}\mathbf{j}+\frac{6}{11}\mathbf{k} \Rightarrow 9\mathbf{i}-2\mathbf{j}+6\mathbf{k} = 11\left(\frac{9}{11}\mathbf{i}-\frac{2}{11}\mathbf{j}+\frac{6}{11}\mathbf{k}\right)$

27. length $= |5\mathbf{k}| = \sqrt{25} = 5$, the direction is $\mathbf{k} \Rightarrow 5\mathbf{k} = 5(\mathbf{k})$

28. length $= \left|\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k}\right| = \sqrt{\frac{9}{25}+\frac{16}{25}} = 1$, the direction is $\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k} \Rightarrow \frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k} = 1\left(\frac{3}{4}\mathbf{i}+\frac{4}{5}\mathbf{k}\right)$

29. length $= \left|\frac{1}{\sqrt{6}}\mathbf{i}-\frac{1}{\sqrt{6}}\mathbf{j}-\frac{1}{\sqrt{6}}\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{1}{2}}$, the direction is $\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}$
$\Rightarrow \frac{1}{\sqrt{6}}\mathbf{i}-\frac{1}{\sqrt{6}}\mathbf{j}-\frac{1}{\sqrt{6}}\mathbf{k} = \sqrt{\frac{1}{2}}\left(\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}\right)$

30. length $= \left|\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{3}}\right)^2} = 1$, the direction is $\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}$
$10 \Rightarrow \frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k} = 1\left(\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}\right)$

31. (a) $2\mathbf{i}$ (b) $-\sqrt{3}\mathbf{k}$ (c) $\frac{3}{10}\mathbf{j}+\frac{2}{5}\mathbf{k}$ (d) $6\mathbf{i}-2\mathbf{j}+3\mathbf{k}$

32. (a) $-7\mathbf{j}$ (b) $-\frac{3\sqrt{2}}{5}\mathbf{i}-\frac{4\sqrt{2}}{5}\mathbf{k}$ (c) $\frac{1}{4}\mathbf{i}-\frac{1}{3}\mathbf{j}-\mathbf{k}$ (d) $\frac{a}{\sqrt{2}}\mathbf{i}+\frac{a}{\sqrt{3}}\mathbf{j}-\frac{a}{\sqrt{6}}\mathbf{k}$

33. $|\mathbf{v}| = \sqrt{12^2+5^2} = \sqrt{169} = 13$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{13}\mathbf{v} = \frac{1}{13}(12\mathbf{i}-5\mathbf{k}) \Rightarrow$ the desired vector is $\frac{7}{13}(12\mathbf{i}-5\mathbf{k})$

34. $|\mathbf{v}| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{3}}{2}$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow$ the desired vector is $-3\left(\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}\right)$
$= -\sqrt{3}\mathbf{i}+\sqrt{3}\mathbf{j}+\sqrt{3}\mathbf{k}$

35. (a) $3\mathbf{i}+4\mathbf{j}-5\mathbf{k} = 5\sqrt{2}\left(\frac{3}{5\sqrt{2}}\mathbf{i}+\frac{4}{5\sqrt{2}}\mathbf{j}-\frac{1}{\sqrt{2}}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{3}{5\sqrt{2}}\mathbf{i}+\frac{4}{5\sqrt{2}}\mathbf{j}-\frac{1}{\sqrt{2}}\mathbf{k}$
(b) the midpoint is $\left(\frac{1}{2}, 3, \frac{5}{2}\right)$

36. (a) $3\mathbf{i}-6\mathbf{j}+2\mathbf{k} = 7\left(\frac{3}{7}\mathbf{i}-\frac{6}{7}\mathbf{j}+\frac{2}{7}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{3}{7}\mathbf{i}-\frac{6}{7}\mathbf{j}+\frac{2}{7}\mathbf{k}$
(b) the midpoint is $\left(\frac{5}{2}, 1, 6\right)$

37. (a) $-\mathbf{i}-\mathbf{j}-\mathbf{k} = \sqrt{3}\left(-\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}\right) \Rightarrow$ the direction is $-\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}$
(b) the midpoint is $\left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right)$

38. (a) $2\mathbf{i}-2\mathbf{j}-2\mathbf{k} = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}$
(b) the midpoint is $(1, -1, -1)$

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39. $\overrightarrow{AB} = (5-a)\mathbf{i} + (1-b)\mathbf{j} + (3-c)\mathbf{k} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow 5-a=1, 1-b=4,$ and $3-c=-2 \Rightarrow a=4, b=-3,$ and $c=15 \Rightarrow A$ is the point $(4, -3, 5)$

40. $\overrightarrow{AB} = (a+2)\mathbf{i} + (b+3)\mathbf{j} + (c-6)\mathbf{k} = -7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k} \Rightarrow a+2=-7, b+3=3,$ and $c-6=8 \Rightarrow a=-9, b=0,$ and $c=14 \Rightarrow B$ is the point $(-9, 0, 14)$

41. $2\mathbf{i} + \mathbf{j} = a(\mathbf{i}+\mathbf{j}) + b(\mathbf{i}-\mathbf{j}) = (a+b)\mathbf{i} + (a-b)\mathbf{j} \Rightarrow a+b=2$ and $a-b=1 \Rightarrow 2a=3 \Rightarrow a=\frac{3}{2}$ and $b=a-1=\frac{1}{2}$

42. $\mathbf{i} - 2\mathbf{j} = a(2\mathbf{i}+3\mathbf{j}) + b(\mathbf{i}+\mathbf{j}) = (2a+b)\mathbf{i} + (3a+b)\mathbf{j} \Rightarrow 2a+b=1$ and $3a+b=-2 \Rightarrow a=-3$ and $b=1-2a=7 \Rightarrow \mathbf{u}_1 = a(2\mathbf{i}+3\mathbf{j}) = -6\mathbf{i} - 9\mathbf{j}$ and $\mathbf{u}_2 = b(\mathbf{i}+\mathbf{j}) = 7\mathbf{i} + 7\mathbf{j}$

43. $25°$ west of north is $90° + 25° = 115°$ north of east. $800\langle \cos 115°, \sin 115° \rangle \approx \langle -338.095, 725.046 \rangle$

44. Let $\mathbf{u} = \langle x, y \rangle$ represent the velocity of the plane alone, $\mathbf{v} = \langle 110\cos 60°, 110\sin 60° \rangle = \langle 55, 55\sqrt{3} \rangle$, and let the resultant $\mathbf{u} + \mathbf{v} = \langle 800, 0 \rangle$. Then $\langle x, y \rangle + \langle 55, 55\sqrt{3} \rangle = \langle 800, 0 \rangle \Rightarrow \langle x+55, y+55\sqrt{3} \rangle = \langle 800, 0 \rangle \Rightarrow x+55=800$ and $y + 55\sqrt{3} = 0 \Rightarrow x = 745$ and $y = -55\sqrt{3} \Rightarrow \mathbf{u} = \langle 745, -55\sqrt{3} \rangle \Rightarrow |\mathbf{u}| = \sqrt{745^2 + \left(-55\sqrt{3}\right)^2} \approx 751$ km/h, and $\tan\theta = \frac{-55\sqrt{3}}{745} \Rightarrow \theta \approx -7.3° \Rightarrow -7.3°$ south of east.

45. $\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 30°, |\mathbf{F}_1|\sin 30° \rangle = \left\langle -\frac{\sqrt{3}}{2}|\mathbf{F}_1|, \frac{1}{2}|\mathbf{F}_1| \right\rangle$, $\mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 45°, |\mathbf{F}_2|\sin 45° \rangle = \left\langle \frac{1}{\sqrt{2}}|\mathbf{F}_2|, \frac{1}{\sqrt{2}}|\mathbf{F}_2| \right\rangle$, and $\mathbf{w} = \langle 0, -100 \rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 100 \rangle \Rightarrow \left\langle -\frac{\sqrt{3}}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2|, \frac{1}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2| \right\rangle = \langle 0, 100 \rangle$ $\Rightarrow -\frac{\sqrt{3}}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2| = 0$ and $\frac{1}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2| = 100$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = \frac{\sqrt{6}}{2}|\mathbf{F}_1|$. Substituting this result into the second equation gives us: $\frac{1}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}\left(\frac{\sqrt{6}}{2}|\mathbf{F}_1|\right) = 100$ $\Rightarrow |\mathbf{F}_1| = \frac{200}{1+\sqrt{3}} \approx 73.205 \text{ N} \Rightarrow |\mathbf{F}_2| = \frac{100\sqrt{6}}{1+\sqrt{3}} \approx 89.658 \text{ N} \Rightarrow \mathbf{F}_1 \approx \langle -63.397, 36.603 \rangle$ and $\mathbf{F}_2 = \approx \langle 63.397, 63.397 \rangle$

46. $\mathbf{F}_1 = \langle -35\cos\alpha, 35\sin\alpha \rangle$, $\mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 60°, |\mathbf{F}_2|\sin 60° \rangle = \left\langle \frac{1}{2}|\mathbf{F}_2|, \frac{\sqrt{3}}{2}|\mathbf{F}_2| \right\rangle$, and $\mathbf{w} = \langle 0, -50 \rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 50 \rangle \Rightarrow \left\langle -35\cos\alpha + \frac{1}{2}|\mathbf{F}_2|, 35\sin\alpha + \frac{\sqrt{3}}{2}|\mathbf{F}_2| \right\rangle = \langle 0, 50 \rangle \Rightarrow -35\cos\alpha + \frac{1}{2}|\mathbf{F}_2| = 0$ and $35\sin\alpha + \frac{\sqrt{3}}{2}|\mathbf{F}_2| = 50$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = 70\cos\alpha$. Substituting this result into the second equation gives us: $35\sin\alpha + 35\sqrt{3}\cos\alpha = 50 \Rightarrow \sqrt{3}\cos\alpha = \frac{10}{7} - \sin\alpha$ $\Rightarrow 3\cos^2\alpha = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha \Rightarrow 3\left(1 - \sin^2\alpha\right) = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha \Rightarrow 196\sin^2\alpha - 140\sin\alpha - 47 = 0$ $\Rightarrow \sin\alpha = \frac{5 \pm 6\sqrt{2}}{14}$. Since $\alpha > 0 \Rightarrow \sin\alpha > 0 \Rightarrow \sin\alpha = \frac{5+6\sqrt{2}}{14} \Rightarrow \alpha \approx 74.42°$, and $|\mathbf{F}_2| = 70\cos\alpha \approx 18.81$ N.

47. $\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 40°, |\mathbf{F}_1|\sin 40° \rangle$, $\mathbf{F}_2 = \langle 100\cos 35°, 100\sin 35° \rangle$, and $\mathbf{w} = \langle 0, -w \rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, w \rangle$ $\Rightarrow \langle -|\mathbf{F}_1|\cos 40° + 100\cos 35°, |\mathbf{F}_1|\sin 40° + 100\sin 35° \rangle = \langle 0, w \rangle \Rightarrow -|\mathbf{F}_1|\cos 40° + 100\cos 35° = 0$ and

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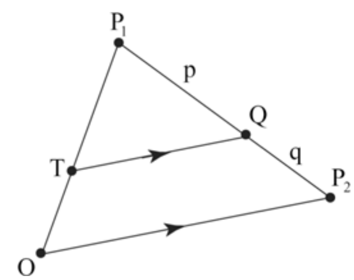
$|\mathbf{F}_1|\sin 40° + 100\sin 35° = w$. Solving the first equation for $|\mathbf{F}_1|$ results in: $|\mathbf{F}_1| = \frac{100\cos 35°}{\cos 40°} \approx 106.933$ N. Substituting this result into the second equation gives us: $w \approx 126.093$ N.

48. $\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos\alpha, |\mathbf{F}_1|\sin\alpha\rangle = \langle -75\cos\alpha, 75\sin\alpha\rangle$, $\mathbf{F}_2 = \langle |\mathbf{F}_2|\cos\beta, |\mathbf{F}_2|\sin\beta\rangle = \langle 75\cos\alpha, 75\sin\alpha\rangle$, and $\mathbf{w} = \langle 0, -25\rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 25\rangle \Rightarrow \langle -75\cos\alpha + 75\cos\alpha, 75\sin\alpha + 75\sin\alpha\rangle = \langle 0, 25\rangle \Rightarrow 150\sin\alpha = 25$ $\Rightarrow \alpha \approx 9.59°$.

49. (a) The tree is located at the tip of the vector $\overrightarrow{OP} = (5\cos 60°)\mathbf{i} + (5\sin 60°)\mathbf{j} = \frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j} \Rightarrow P = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$

(b) The telephone pole is located at the point Q, which is the tip of the vector $\overrightarrow{OP} + \overrightarrow{PQ}$ $= \left(\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}\right) + (10\cos 315°)\mathbf{i} + (10\sin 315°)\mathbf{j} = \left(\frac{5}{2} + \frac{10\sqrt{2}}{2}\right)\mathbf{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\mathbf{j} \Rightarrow Q = \left(\frac{5+10\sqrt{2}}{2}, \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)$

50. Let $t = \frac{q}{p+q}$ and $s = \frac{q}{p+q}$. Choose T on $\overline{OP_1}$ so that $\overline{TQ}$ is parallel to $\overline{OP_2}$, so that ΔTP_1Q is similar to ΔOP_1P_2. Then $\frac{|OT|}{|OP_1|} = t \Rightarrow \overrightarrow{OT} = t\,\overrightarrow{OP_1}$ so that $T = (t\,x_1, t\,y_1, t\,z_1)$. Also, $\frac{|TQ|}{|OP_2|} = s \Rightarrow \overrightarrow{TQ} = s\,\overrightarrow{OP_2} = s\langle x_2, y_2, z_2\rangle$. Letting $Q = (x, y, z)$, we have that $\overrightarrow{TQ} = \langle x - t x_1, y - t y_1, z - t z_1\rangle = s\langle x_2, y_2, z_2\rangle$



Thus $x = t x_1 + s x_2$, $y = t y_1 + s y_2$, $z = t z_1 + s z_2$.

(Note that if Q is the midpoint, then $\frac{p}{q} = 1$ and $t = s = \frac{1}{2}$ so that $x = \frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{x_1+x_2}{2}$, $y = \frac{y_1+y_2}{2}$, $z = \frac{z_1+z_2}{2}$ so that this result agrees with the midpoint formula.)

51. (a) the midpoint of AB is $M\left(\frac{5}{2}, \frac{5}{2}, 0\right)$ and $\overrightarrow{CM} = \left(\frac{5}{2} - 1\right)\mathbf{i} + \left(\frac{5}{2} - 1\right)\mathbf{j} + (0-3)\mathbf{k} = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}$

(b) the desired vector is $\left(\frac{2}{3}\right)\overrightarrow{CM} = \frac{2}{3}\left(\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}\right) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(c) the vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass $\Rightarrow$ the terminal point of $(\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is the point $(2, 2, 1)$, which is the location of the center of mass

52. The midpoint of AB is $M\left(\frac{3}{2}, 0, \frac{5}{2}\right)$ and $\left(\frac{2}{3}\right)\overrightarrow{CM} = \frac{2}{3}\left[\left(\frac{3}{2} + 1\right)\mathbf{i} + (0-2)\mathbf{j} + \left(\frac{5}{2} + 1\right)\mathbf{k}\right] = \frac{2}{3}\left(\frac{5}{2}\mathbf{i} - 2\mathbf{j} + \frac{7}{2}\mathbf{k}\right)$ $= \frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}$. The vector from the origin to the point of intersection of the medians is $\left(\frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}\right) + \overrightarrow{OC} = \left(\frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}\right) + (-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}$.

53. Without loss of generality we identify the vertices of the quadrilateral such that $A(0, 0, 0)$, $B(x_b, 0, 0)$, $C(x_c, y_c, 0)$ and $D(x_d, y_d, z_d) \Rightarrow$ the midpoint of AB is $M_{AB}\left(\frac{x_b}{2}, 0, 0\right)$, the midpoint of BC is $M_{BC}\left(\frac{x_b+x_c}{2}, \frac{y_c}{2}, 0\right)$, the midpoint of CD is $M_{CD}\left(\frac{x_c+x_d}{2}, \frac{y_c+y_d}{2}, \frac{z_d}{2}\right)$ and the midpoint of AD is $M_{AD}\left(\frac{x_d}{2}, \frac{y_d}{2}, \frac{z_d}{2}\right) \Rightarrow$ the midpoint of $M_{AB}M_{CD}$ is $\left(\frac{\frac{x_b}{2} + \frac{x_c + x_d}{2}}{2}, \frac{y_c+y_d}{4}, \frac{z_d}{4}\right)$ which is the same as the midpoint of $M_{AD}M_{BC} = \left(\frac{\frac{x_b + x_c}{2} + \frac{x_d}{2}}{2}, \frac{y_c+y_d}{4}, \frac{z_d}{4}\right)$.

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54. Let $V_1, V_2, V_3, \ldots, V_n$ be the vertices of a regular n-sided polygon and $\mathbf{v}_i$ denote the vector from the center to V_i for $i = 1, 2, 3, \ldots, n$. If $\mathbf{S} = \sum_{i=1}^{n} \mathrm{v}_i$ and the polygon is rotated through an angle of $\frac{i(2\pi)}{n}$ where $i = 1, 2, 3, \ldots, n$, then **S** would remain the same. Since the vector **S** does not change with these rotations we conclude that $\mathbf{S} = \mathbf{0}$.

55. Without loss of generality we can coordinatize the vertices of the triangle such that $A(0, 0)$, $B(b, 0)$ and $C(x_c, y_c) \Rightarrow$ a is located at $\left(\frac{b+x_c}{2}, \frac{y_c}{2}\right)$, b is at $\left(\frac{x_c}{2}, \frac{y_c}{2}\right)$ and c is at $\left(\frac{b}{2}, 0\right)$. Therefore, $\overrightarrow{Aa} = \left(\frac{b}{2} + \frac{x_c}{2}\right)\mathbf{i} + \left(\frac{y_c}{2}\right)\mathbf{j}$, $\overrightarrow{Bb} = \left(\frac{x_c}{2} - b\right)\mathbf{i} + \left(\frac{y_c}{2}\right)\mathbf{j}$, and $\overrightarrow{Cc} = \left(\frac{b}{2} - x_c\right)\mathbf{i} + (-y_c)\mathbf{j} \Rightarrow \overrightarrow{Aa} + \overrightarrow{Bb} + \overrightarrow{Cc} = \mathbf{0}$.

56. Let **u** be any unit vector in the plane. If **u** is positioned so that its initial point is at the origin and terminal point is at (x, y), then **u** makes an angle θ with **i**, measured in the counter-clockwise direction. Since $|\mathbf{u}| = 1$, we have that $x = \cos\theta$ and $y = \sin\theta$. Thus $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. Since **u** was assumed to be any unit vector in the plane, this holds for <u>every</u> unit vector in the plane.

12.3 THE DOT PRODUCT

<u>NOTE</u>: In Exercises 1-8 below we calculate $\mathrm{proj}_{\mathbf{v}}\ \mathbf{u}$ as the vector $\left(\frac{|\mathbf{u}|\cos\theta}{|\mathbf{v}|}\right)\mathbf{v}$, so the scalar multiplier of **v** is the number in column 5 divided by the number in column 2.

	$\mathbf{v}\cdot\mathbf{u}$	$\lvert\mathbf{v}\rvert$	$\lvert\mathbf{u}\rvert$	$\cos\theta$	$\lvert\mathbf{u}\rvert\cos\theta$	$\mathrm{proj}_{\mathbf{v}}\ \mathbf{u}$
1.	-25	5	5	-1	-5	$-2\mathbf{i}+4\mathbf{j}-\sqrt{5}\mathbf{k}$
2.	3	1	13	$\frac{3}{13}$	3	$3\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k}\right)$
3.	25	15	5	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{1}{9}(10\mathbf{i}+11\mathbf{j}-2\mathbf{k})$
4.	13	15	3	$\frac{13}{45}$	$\frac{13}{15}$	$\frac{13}{225}(2\mathbf{i}+10\mathbf{j}-11\mathbf{k})$
5.	2	$\sqrt{34}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}\sqrt{34}}$	$\frac{2}{\sqrt{34}}$	$\frac{1}{17}(5\mathbf{j}-3\mathbf{k})$
6.	$\sqrt{3}-\sqrt{2}$	$\sqrt{2}$	3	$\frac{\sqrt{3}-\sqrt{2}}{3\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j})$
7.	$10+\sqrt{17}$	$\sqrt{26}$	$\sqrt{21}$	$\frac{10+\sqrt{17}}{\sqrt{546}}$	$\frac{10+\sqrt{17}}{\sqrt{26}}$	$\frac{10+\sqrt{17}}{26}(5\mathbf{i}+\mathbf{j})$
8.	$\frac{1}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{1}{5}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{5}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle$

9. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(2)(1)+(1)(2)+(0)(-1)}{\sqrt{2^2+1^2+0^2}\sqrt{1^2+2^2+(-1)^2}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{5}\sqrt{6}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{30}}\right) \approx 0.75$ rad

10. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(2)(3)+(-2)(0)+(1)(4)}{\sqrt{2^2+(-2)^2+1^2}\sqrt{3^2+0^2+4^2}}\right) = \cos^{-1}\left(\frac{10}{\sqrt{9}\sqrt{25}}\right) = \cos^{-1}\left(\frac{2}{3}\right) \approx 0.84$ rad

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11. $\theta=\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)=\cos^{-1}\left(\frac{(\sqrt{3})(\sqrt{3})+(-7)(1)+(0)(-2)}{\sqrt{(\sqrt{3})^2+(-7)^2+0^2}\sqrt{(\sqrt{3})^2+(1)^2+(-2)^2}}\right)=\cos^{-1}\left(\frac{3-7}{\sqrt{52}\sqrt{8}}\right)=\cos^{-1}\left(\frac{-1}{\sqrt{26}}\right)\approx 1.77 \text{ rad}$

12. $\theta=\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)=\cos^{-1}\left(\frac{(1)(-1)+(\sqrt{2})(1)+(-\sqrt{2})(1)}{\sqrt{(1)^2+(\sqrt{2})^2+(-\sqrt{2})^2}\sqrt{(-1)^2+(1)^2+(1)^2}}\right)=\cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{3}}\right)=\cos^{-1}\left(\frac{-1}{\sqrt{15}}\right)\approx 1.83 \text{ rad}$

13. $\overrightarrow{AB}=\langle 3,1\rangle,\overrightarrow{BC}=\langle -1,-3\rangle,$ and $\overrightarrow{AC}=\langle 2,-2\rangle.\ \overrightarrow{BA}=\langle -3,-1\rangle,\overrightarrow{CB}=\langle 1,3\rangle,\overrightarrow{CA}=\langle -2,2\rangle.$

$\left|\overrightarrow{AB}\right|=\left|\overrightarrow{BA}\right|=\sqrt{10},\left|\overrightarrow{BC}\right|=\left|\overrightarrow{CB}\right|=\sqrt{10},\left|\overrightarrow{AC}\right|=\left|\overrightarrow{CA}\right|=2\sqrt{2},$

Angle at $A=\cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}\right)=\cos^{-1}\left(\frac{3(2)+1(-2)}{(\sqrt{10})(2\sqrt{2})}\right)=\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\approx 63.435°$

Angle at $B=\cos^{-1}\left(\frac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{\left|\overrightarrow{BC}\right|\left|\overrightarrow{BA}\right|}\right)=\cos^{-1}\left(\frac{(-1)(-3)+(-3)(-1)}{(\sqrt{10})(\sqrt{10})}\right)=\cos^{-1}\left(\frac{3}{5}\right)\approx 53.130°,$ and

Angle at $C=\cos^{-1}\left(\frac{\overrightarrow{CB}\cdot\overrightarrow{CA}}{\left|\overrightarrow{CB}\right|\left|\overrightarrow{CA}\right|}\right)=\cos^{-1}\left(\frac{1(-2)+3(2)}{(\sqrt{10})(2\sqrt{2})}\right)=\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\approx 63.435°$

14. $\overrightarrow{AC}=\langle 2,4\rangle$ and $\overrightarrow{BD}=\langle 4,-2\rangle.\ \overrightarrow{AC}\cdot\overrightarrow{BD}=2(4)+4(-2)=0,$ so the angle measures are all $90°$.

15. (a) $\cos\alpha=\frac{\mathbf{i}\cdot\mathbf{v}}{|\mathbf{i}||\mathbf{v}|}=\frac{a}{|\mathbf{v}|},\ \cos\beta=\frac{\mathbf{j}\cdot\mathbf{v}}{|\mathbf{j}||\mathbf{v}|}=\frac{b}{|\mathbf{v}|},\ \cos\gamma=\frac{\mathbf{k}\cdot\mathbf{v}}{|\mathbf{k}||\mathbf{v}|}=\frac{c}{|\mathbf{v}|}$ and

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\left(\frac{a}{|\mathbf{v}|}\right)^2+\left(\frac{b}{|\mathbf{v}|}\right)^2+\left(\frac{c}{|\mathbf{v}|}\right)^2=\frac{a^2+b^2+c^2}{|\mathbf{v}||\mathbf{v}|}=\frac{|\mathbf{v}||\mathbf{v}|}{|\mathbf{v}||\mathbf{v}|}=1$

(b) $|\mathbf{v}|=1\Rightarrow\cos\alpha=\frac{a}{|\mathbf{v}|}=a,\cos\beta=\frac{b}{|\mathbf{v}|}=b$ and $\cos\gamma=\frac{c}{|\mathbf{v}|}=c$ are the direction cosines of $\mathbf{v}$

16. $\mathbf{u}=10\mathbf{i}+2\mathbf{k}$ is parallel to the pipe in the north direction and $\mathbf{v}=10\mathbf{j}+\mathbf{k}$ is parallel to the pipe in the east direction. The angle between the two pipes is $\theta=\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)=\cos^{-1}\left(\frac{2}{\sqrt{104}\sqrt{101}}\right)\approx 1.55\text{ rad}\approx 88.88°.$

17. The sum of two vectors of equal length is *always* orthogonal to their difference, as we can see from the equation $(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_1-\mathbf{v}_2)=\mathbf{v}_1\cdot\mathbf{v}_1+\mathbf{v}_2\cdot\mathbf{v}_1-\mathbf{v}_1\cdot\mathbf{v}_2-\mathbf{v}_2\cdot\mathbf{v}_2=|\mathbf{v}_1|^2-|\mathbf{v}_2|^2=0$

18. $\overrightarrow{CA}\cdot\overrightarrow{CB}=(-\mathbf{v}+(-\mathbf{u}))\cdot(-\mathbf{v}+\mathbf{u})=\mathbf{v}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{u}=|\mathbf{v}|^2-|\mathbf{u}|^2=0$ because $|\mathbf{u}|=|\mathbf{v}|,$ since both equal the radius of the circle. Therefore, $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

19. Let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rhombus $\Rightarrow$ the diagonals are $\mathbf{d}_1=\mathbf{u}+\mathbf{v}$ and $\mathbf{d}_2=-\mathbf{u}+\mathbf{v}$
$\Rightarrow\mathbf{d}_1\cdot\mathbf{d}_2=(\mathbf{u}+\mathbf{v})\cdot(-\mathbf{u}+\mathbf{v})=-\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}=|\mathbf{v}|^2-|\mathbf{u}|^2=0$ because $|\mathbf{u}|=|\mathbf{v}|,$ since a rhombus has equal sides.

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20. Suppose the diagonals of a rectangle are perpendicular, and let **u** and **v** be the sides of a rectangle $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u}+\mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u}+\mathbf{v}$. Since the diagonals are perpendicular we have $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$ $\Leftrightarrow (\mathbf{u}+\mathbf{v})\cdot(-\mathbf{u}+\mathbf{v}) = -\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} = 0 \Leftrightarrow |\mathbf{v}|^2-|\mathbf{u}|^2 = 0 \Leftrightarrow (|\mathbf{v}|+|\mathbf{u}|)(|\mathbf{v}|-|\mathbf{u}|) = 0 \Leftrightarrow (|\mathbf{v}|+|\mathbf{u}|) = 0$ which is not possible, or $(|\mathbf{v}|-|\mathbf{u}|) = 0$ which is equivalent to $|\mathbf{v}| = |\mathbf{u}| \Rightarrow$ the rectangle is a square.

21. Clearly the diagonals of a rectangle are equal in length. What is not as obvious is the statement that equal diagonals happen only in a rectangle. We show this is true by letting the adjacent sides of a parallelogram be the vectors $(v_1\mathbf{i}+v_2\mathbf{j})$ and $(u_1\mathbf{i}+u_2\mathbf{j})$. The equal diagonals of the parallelogram are $\mathbf{d}_1 = (v_1\mathbf{i}+v_2\mathbf{j})+(u_1\mathbf{i}+u_2\mathbf{j})$ and $\mathbf{d}_2 = (v_1\mathbf{i}+v_2\mathbf{j})-(u_1\mathbf{i}+u_2\mathbf{j})$. Hence $|\mathbf{d}_1| = |\mathbf{d}_2| = |(v_1\mathbf{i}+v_2\mathbf{j})+(u_1\mathbf{i}+u_2\mathbf{j})|$ $= |(v_1\mathbf{i}+v_2\mathbf{j})-(u_1\mathbf{i}+u_2\mathbf{j})| \Rightarrow |(v_1+u_1)\mathbf{i}+(v_2+u_2)\mathbf{j}| = |(v_1-u_1)\mathbf{i}+(v_2-u_2)\mathbf{j}| \Rightarrow \sqrt{(v_1+u_1)^2+(v_2+u_2)^2}$ $= \sqrt{(v_1-u_1)^2+(v_2-u_2)^2} \Rightarrow v_1^2+2v_1u_1+u_1^2+v_2^2+2v_2u_2+u_2^2 = v_1^2-2v_1u_1+u_1^2+v_2^2-2v_2u_2+u_2^2$ $\Rightarrow 2(v_1u_1+v_2u_2) = -2(v_1u_1+v_2u_2) \Rightarrow v_1u_1+v_2u_2 = 0 \Rightarrow (v_1\mathbf{i}+v_2\mathbf{j})\cdot(u_1\mathbf{i}+u_2\mathbf{j}) = 0 \Rightarrow$ the vectors $(v_1\mathbf{i}+v_2\mathbf{j})$ and $(u_1\mathbf{i}+u_2\mathbf{j})$ are perpendicular and the parallelogram must be a rectangle.

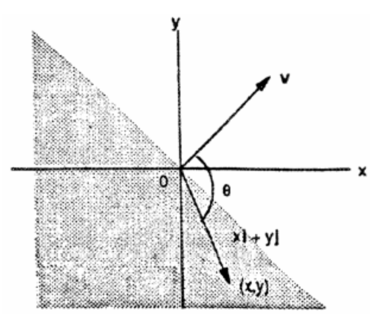
22. If $|\mathbf{u}| = |\mathbf{v}|$ and $\mathbf{u}+\mathbf{v}$ is the indicated diagonal, then $(\mathbf{u}+\mathbf{v})\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{u} = |\mathbf{u}|^2+\mathbf{v}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{v}+|\mathbf{v}|^2$ $= \mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} = (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} \Rightarrow$ the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{u}}{|\mathbf{u}+\mathbf{v}||\mathbf{u}|}\right)$ between the diagonal and **u** and the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{v}}{|\mathbf{u}+\mathbf{v}||\mathbf{v}|}\right)$ between the diagonal and **v** are equal because the inverse cosine function is one-to-one. Therefore, the diagonal bisects the angle between **u** and **v**.

23. horizontal component: $400\cos(8°) \approx 396$ m/s; vertical component: $400\sin(8°) \approx 55.7$ m/s

24. $|\mathbf{w}|\cos(33°-15°) = 2.5$ N, so $|\mathbf{w}| = \frac{2.5\text{ N}}{\cos 18°}$. Then $\mathbf{w} = \frac{2.5\text{ N}}{\cos 18°}\langle \cos 33°, \sin 33°\rangle \approx \langle 2.205, 1.432\rangle$

25. (a) Since $|\cos\theta| \le 1$, we have $|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}||\mathbf{v}||\cos\theta| \le |\mathbf{u}||\mathbf{v}|(1) = |\mathbf{u}||\mathbf{v}|$.
    (b) We have equality precisely when $|\cos\theta| = 1$ or when one or both of **u** and **v** is **0**. In the case of nonzero vectors, we have equality when $\theta = 0$ or π, i.e., when the vectors are parallel.

26. $(x\mathbf{i}+y\mathbf{j})\cdot\mathbf{v} = |x\mathbf{i}+y\mathbf{j}||\mathbf{v}|\cos\theta \le 0$ when $\frac{\pi}{2} \le \theta \le \pi$. This means (x, y) has to be a point whose position vector makes an angle with **v** that is a right angle or bigger.



27. $\mathbf{v}\cdot\mathbf{u}_1 = (a\mathbf{u}_1+b\mathbf{u}_2)\cdot\mathbf{u}_1 = a\mathbf{u}_1\cdot\mathbf{u}_1+b\mathbf{u}_2\cdot\mathbf{u}_1 = a|\mathbf{u}_1|^2+b(\mathbf{u}_2\cdot\mathbf{u}_1) = a(1)^2+b(0) = a$

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28. No, $\mathbf{v}_1$ need not equal $\mathbf{v}_2$. For example, $\mathbf{i}+\mathbf{j} \neq \mathbf{i}+2\mathbf{j}$ but $\mathbf{i}\cdot(\mathbf{i}+\mathbf{j}) = \mathbf{i}\cdot\mathbf{i}+\mathbf{i}\cdot\mathbf{j} = 1+0 = 1$ and $\mathbf{i}\cdot(\mathbf{i}+2\mathbf{j}) = \mathbf{i}\cdot\mathbf{i}+2\mathbf{i}\cdot\mathbf{j} = 1+2\cdot 0 = 1$.

29. $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} \Rightarrow \left(\mathbf{u}-\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right)\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right) = \mathbf{u}\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right)-\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right)\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)(\mathbf{u}\cdot\mathbf{v})-\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)^2(\mathbf{v}\cdot\mathbf{v})$

$= \frac{(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{v}|^2}-\frac{(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{v}|^4}|\mathbf{v}|^2 = 0$

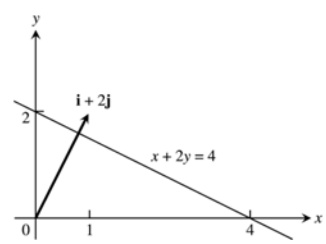
30. $\mathbf{F} = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i}-\mathbf{j} \Rightarrow \text{proj}_{\mathbf{v}}\mathbf{F} = \frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{5}{\left(\sqrt{10}\right)^2}(3\mathbf{i}-\mathbf{j}) = \frac{3}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}$, is the vector parallel to $\mathbf{v}$.

$\mathbf{F}-\text{proj}_{\mathbf{v}}\mathbf{F} = (2\mathbf{i}+\mathbf{j}-3\mathbf{k})-\left(\frac{3}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}\right) = \frac{1}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}-3\mathbf{k}$ is the vector orthogonal to $\mathbf{v}$.

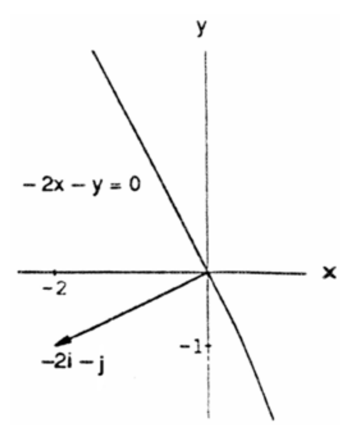
31. $P(x_1, y_1) = P\left(x_1, \frac{c}{b}-\frac{a}{b}x_1\right)$ and $Q(x_2, y_2) = Q\left(x_2, \frac{c}{b}-\frac{a}{b}x_2\right)$ and any two points P and Q on the line with $b \neq 0 \Rightarrow \overrightarrow{PQ} = (x_2-x_1)\mathbf{i}+\frac{a}{b}(x_1-x_2)\mathbf{j} \Rightarrow \overrightarrow{PQ}\cdot\mathbf{v} = \left[(x_2-x_1)\mathbf{i}+\frac{a}{b}(x_1-x_2)\mathbf{j}\right]\cdot(a\mathbf{i}+b\mathbf{j})$
$= a(x_2-x_1)+b\left(\frac{a}{b}\right)(x_1-x_2) = 0 \Rightarrow \mathbf{v}$ is perpendicular to $\overrightarrow{PQ}$ for $b \neq 0$. If $b = 0$, then $\mathbf{v} = a\mathbf{i}$ is perpendicular to the vertical line $ax = c$. Alternatively, the slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of the line $ax+by = c$ is $-\frac{a}{b}$, so the slopes are negative reciprocals $\Rightarrow$ the vector $\mathbf{v}$ and the line are perpendicular.

32. The slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of $bx-ay = c$ is $\frac{b}{a}$, provided that $a \neq 0$. If $a = 0$, then $\mathbf{v} = b\mathbf{j}$ is parallel to the vertical line $bx = c$. In either case, the vector $\mathbf{v}$ is parallel to the line $bx-ay = c$.

33. $\mathbf{v} = \mathbf{i}+2\mathbf{j}$ is perpendicular to the line $x+2y = c$;
$P(2, 1)$ on the line $\Rightarrow 2+2 = c \Rightarrow x+2y = 4$



34. $\mathbf{v} = -2\mathbf{i}-\mathbf{j}$ is perpendicular to the line $-2x-y = c$;
$P(-1, 2)$ on the line $\Rightarrow (-2)(-1)-2 = c$
$\Rightarrow -2x-y = 0$



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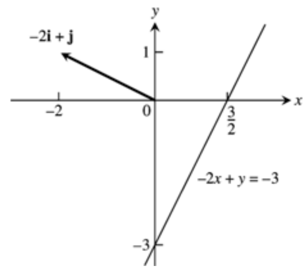
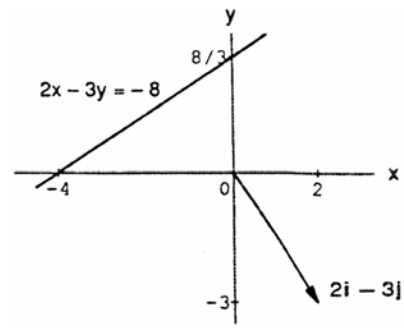
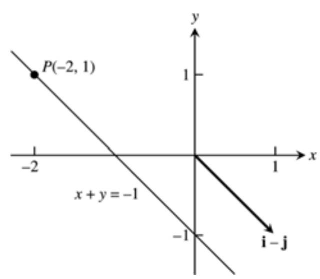
35. $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ is perpendicular to the line $-2x + y = c$;
$P(-2,-7)$ on the line $\Rightarrow (-2)(-2) - 7 = c$
$\Rightarrow -2x + y = -3$



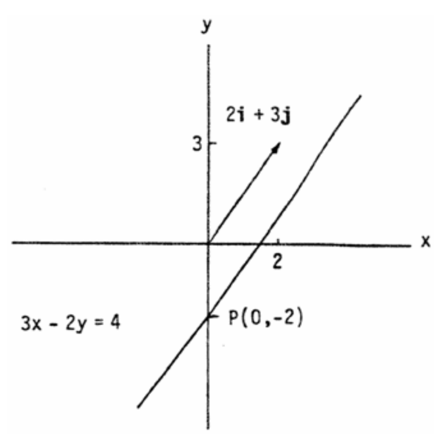
36. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ is perpendicular to the line $2x - 3y = c$;
$P(11,10)$ on the line $\Rightarrow (2)(11) - (3)(10) = c$
$\Rightarrow 2x - 3y = -8$



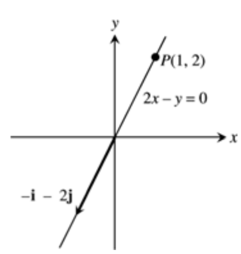
37. $\mathbf{v} = \mathbf{i} - \mathbf{j}$ is parallel to the line $-x - y = c$;
$P(-2,1)$ on the line $\Rightarrow -(-2) - 1 = c \Rightarrow -x - y = 1$
or $x + y = -1$.



38. $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ is parallel to the line $3x - 2y = c$;
$P(0,-2)$ on the line $\Rightarrow 0 - 2(-2) = c \Rightarrow 3x - 2y = 4$



39. $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$ is parallel to the line $-2x + y = c$;
$P(1,2)$ on the line $\Rightarrow -2(1) + 2 = c \Rightarrow -2x - y = 0$
or $2x - y = 0$.



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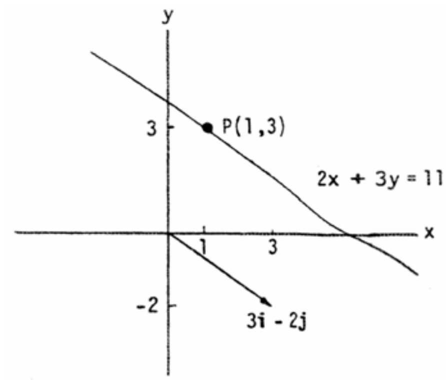
40. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ is parallel to the line $-2x - 3y = c$; $P(1,3)$ on the line $\Rightarrow (-2)(1) - (3)(3) = c$ $\Rightarrow -2x - 3y = -11$ or $2x + 3y = 11$



41. $P(0,0)$, $Q(1,1)$ and $\mathbf{F} = 5\mathbf{j} \Rightarrow \overrightarrow{PQ} = \mathbf{i} + \mathbf{j}$ and $\mathbf{W} = \mathbf{F} \cdot \overrightarrow{PQ} = (5\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) = 5 \text{ N} \cdot \text{m} = 5 \text{ J}$

42. $\mathbf{W} = |\mathbf{F}|$ (distance) $\cos\theta = (602{,}148 \text{ N})(605 \text{ km})(\cos 0) = 364{,}299{,}540 \text{ N} \cdot \text{km} = (364{,}299{,}540)(1000) \text{ N} \cdot \text{m}$ $= 3.6429954 \times 10^{11} \text{ J}$

43. $\mathbf{W} = |\mathbf{F}| \left|\overrightarrow{PQ}\right| \cos\theta = (200)(20)(\cos 30°) = 2000\sqrt{3} = 3464.10 \text{ N} \cdot \text{m} = 3464.10 \text{ J}$

44. $\mathbf{W} = |\mathbf{F}| \left|\overrightarrow{PQ}\right| \cos\theta = (1000)(5280)(\cos 60°) = 2{,}640{,}000 \text{ N} \cdot \text{m}$

In Exercises 45-50 we use the fact that $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is normal to the line $ax + by = c$.

45. $\mathbf{n}_1 = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{6-1}{\sqrt{10}\sqrt{5}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

46. $\mathbf{n}_1 = -\sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \sqrt{3}\mathbf{i} + \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

47. $\mathbf{n}_1 = \sqrt{3}\mathbf{i} - \mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \sqrt{3}\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{\sqrt{3}+\sqrt{3}}{\sqrt{4}\sqrt{4}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

48. $\mathbf{n}_1 = \mathbf{i} + \sqrt{3}\mathbf{j}$ and $\mathbf{n}_2 = \left(1 - \sqrt{3}\right)\mathbf{i} + \left(1 + \sqrt{3}\right)\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1-\sqrt{3}+\sqrt{3}+3}{\sqrt{1+3}\sqrt{1-2\sqrt{3}+3+1+2\sqrt{3}+3}}\right)$ $= \cos^{-1}\left(\frac{4}{2\sqrt{8}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

49. $\mathbf{n}_1 = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{3+4}{\sqrt{25}\sqrt{2}}\right) = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right) \approx 0.14$ rad

50. $\mathbf{n}_1 = 12\mathbf{i} + 5\mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{24-10}{\sqrt{169}\sqrt{8}}\right) = \cos^{-1}\left(\frac{14}{26\sqrt{2}}\right) \approx 1.18$ rad

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12.4 THE CROSS PRODUCT

1. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = 3\left(\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right) \Rightarrow$ length $=3$ and the direction is $\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$;

   $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -3\left(\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right) \Rightarrow$ length $=3$ and the direction is $-\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$

2. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 5(\mathbf{k}) \Rightarrow$ length $=5$ and the direction is $\mathbf{k}$

   $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -5(\mathbf{k}) \Rightarrow$ length $=5$ and the direction is $-\mathbf{k}$

3. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \mathbf{0} \Rightarrow$ length $=0$ and has no direction

   $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = \mathbf{0} \Rightarrow$ length $=0$ and has no direction

4. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = \mathbf{0} \Rightarrow$ length $=0$ and has no direction

   $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = \mathbf{0} \Rightarrow$ length $=0$ and has no direction

5. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = -6(\mathbf{k}) \Rightarrow$ length $=6$ and the direction is $-\mathbf{k}$

   $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = 6(\mathbf{k}) \Rightarrow$ length $=6$ and the direction is $\mathbf{k}$

6. $\mathbf{u}\times\mathbf{v} = (\mathbf{i}\times\mathbf{j})\times(\mathbf{j}\times\mathbf{k}) = \mathbf{k}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j} \Rightarrow$ length $=1$ and the direction is $\mathbf{j}$

   $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -\mathbf{j} \Rightarrow$ length $=1$ and the direction is $-\mathbf{j}$

7. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix} = 6\mathbf{i}-12\mathbf{k} \Rightarrow$ length $=6\sqrt{5}$ and the direction is $\frac{1}{\sqrt{5}}\mathbf{i}-\frac{2}{\sqrt{5}}\mathbf{k}$

   $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -\left(6\mathbf{i}-12\mathbf{k}\right) \Rightarrow$ length $=6\sqrt{5}$ and the direction is $-\frac{1}{\sqrt{5}}\mathbf{i}+\frac{2}{\sqrt{5}}\mathbf{k}$

8. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2\mathbf{i}-2\mathbf{j}+2\mathbf{k} \Rightarrow$ length $=2\sqrt{3}$ and the direction is $-\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}$

   $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -\left(-2\mathbf{i}-2\mathbf{j}+2\mathbf{k}\right) \Rightarrow$ length $=2\sqrt{3}$ and the direction is $\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}$

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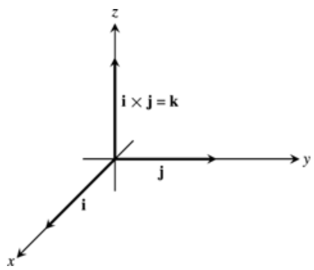
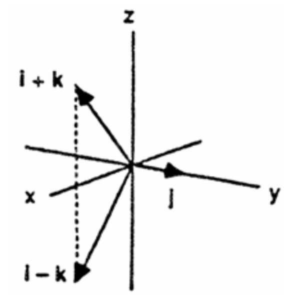
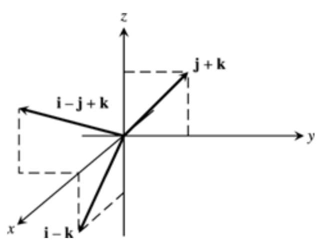
9. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix}=\mathbf{k}$



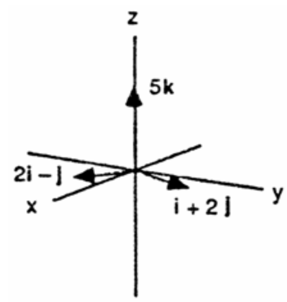
10. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -1\\ 0 & 1 & 0\end{vmatrix}=\mathbf{i}+\mathbf{k}$



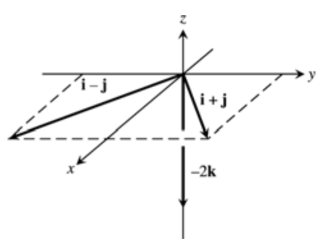
11. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -1\\ 0 & 1 & 1\end{vmatrix}=\mathbf{i}-\mathbf{j}+\mathbf{k}$



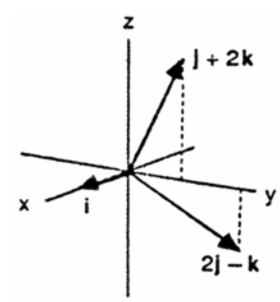
12. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 0\\ 1 & 2 & 0\end{vmatrix}=5\mathbf{k}$



13. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathrm{j} & \mathbf{k}\\ 1 & 1 & 0\\ 1 & -1 & 0\end{vmatrix}=-2\mathbf{k}$



14. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 2\\ 1 & 0 & 0\end{vmatrix}=2\mathbf{j}-\mathbf{k}$



15. (a) $\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix}=8\mathbf{i}+4\mathbf{j}+4\mathbf{k}\Rightarrow$ Area $=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\frac{1}{2}\sqrt{64+16+16}=2\sqrt{6}$

(b) $\mathbf{u}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}=\frac{1}{\sqrt{6}}\left(2\mathbf{i}+\mathbf{j}+\mathbf{k}\right)$

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16. (a) $\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 2\\ 2 & -2 & 0\end{vmatrix}=4\mathbf{i}+4\mathbf{j}-2\mathbf{k}\Rightarrow$ Area $=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\frac{1}{2}\sqrt{16+16+4}=3$

(b) $\mathbf{u}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}=\frac{1}{3}(2\mathbf{i}+2\mathbf{j}-\mathbf{k})$

17. (a) $\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 1\\ 1 & 1 & 0\end{vmatrix}=-\mathbf{i}+\mathbf{j}\Rightarrow$ Area $\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\frac{1}{2}\sqrt{1+1}=\frac{\sqrt{2}}{2}$

(b) $\mathbf{u}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}=\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})\Rightarrow-\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$

18. (a) $\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & -1\\ 1 & 0 & -2\end{vmatrix}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}\Rightarrow$ Area $=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\frac{1}{2}\sqrt{4+9+1}=\frac{\sqrt{14}}{2}$

(b) $\mathbf{u}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}=\frac{1}{\sqrt{14}}(2\mathbf{i}+3\mathbf{j}+\mathbf{k})$

19. If $\mathbf{u}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{v}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, and $\mathbf{w}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$, then $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$,

$(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}=\begin{vmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix}$ and $(\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$ which all have the same absolute value, since interchanging two rows in a determinant does not change its absolute value

$\Rightarrow$ the volume is $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|=abs\begin{vmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{vmatrix}=8$

20. $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|=abs\begin{vmatrix}1 & -1 & 1\\ 2 & 1 & -2\\ -1 & 2 & -1\end{vmatrix}=4$ (for details about verification, see Exercise 19)

21. $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|=abs\begin{vmatrix}2 & 1 & 0\\ 2 & -1 & 1\\ 1 & 0 & 2\end{vmatrix}=|-7|=7$ (for details about verification, see Exercise 19)

22. $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|=abs\begin{vmatrix}1 & 1 & -2\\ -1 & 0 & -1\\ 2 & 4 & -2\end{vmatrix}=8$ (for details about verification, see Exercise 19)

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23. (a) $\mathbf{u}\cdot\mathbf{v}=-6,\ \mathbf{u}\cdot\mathbf{w}=-81,\ \mathbf{v}\cdot\mathbf{w}=18\Rightarrow$ none are perpendicular

(b) $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\5&-1&1\\0&1&-5\end{vmatrix}\neq\mathbf{0},\ \mathbf{u}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\5&-1&1\\-15&3&-3\end{vmatrix}=\mathbf{0},\ \mathbf{v}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&-5\\-15&3&-3\end{vmatrix}\neq\mathbf{0}\Rightarrow\mathbf{u}$ and $\mathbf{w}$ are parallel

24. (a) $\mathbf{u}\cdot\mathbf{v}=0,\ \mathbf{u}\cdot\mathbf{w}=0,\ \mathbf{u}\cdot\mathbf{r}=-3\pi,\ \mathbf{v}\cdot\mathbf{w}=0,\ \mathbf{v}\cdot\mathbf{r}=0,\ \mathbf{w}\cdot\mathbf{r}=0\Rightarrow\mathbf{u}\perp\mathbf{v},\ \mathbf{u}\perp\mathbf{w},\ \mathbf{v}\perp\mathbf{w},\ \mathbf{v}\perp\mathbf{r}$ and $\mathbf{w}\perp\mathbf{r}$

(b) $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\-1&1&1\end{vmatrix}\neq\mathbf{0},\ \mathbf{u}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\1&0&1\end{vmatrix}\neq\mathbf{0},\ \mathbf{u}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\-\frac{\pi}{2}&-\pi&\frac{\pi}{2}\end{vmatrix}=\mathbf{0}$

$\mathbf{v}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&1&1\\1&0&1\end{vmatrix}\neq\mathbf{0},\ \mathbf{v}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&1&1\\-\frac{\pi}{2}&-\pi&\frac{\pi}{2}\end{vmatrix}\neq\mathbf{0},\ \mathbf{w}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&1\\-\frac{\pi}{2}&-\pi&\frac{\pi}{2}\end{vmatrix}\neq\mathbf{0}\Rightarrow\mathbf{u}$ and $\mathbf{r}$ are parallel

25. $\left|\overrightarrow{PQ}\times\mathbf{F}\right|=\left|\overrightarrow{PQ}\right||\mathbf{F}|\sin(60^\circ)=0.2\cdot15\cdot\frac{\sqrt{3}}{2}\,\mathrm{N}\cdot\mathrm{m}=1.5\sqrt{3}\ \mathrm{N}\cdot\mathrm{m}$

26. $\left|\overrightarrow{PQ}\times\mathbf{F}\right|=\left|\overrightarrow{PQ}\right||\mathbf{F}|\sin(135^\circ)=0.2\cdot15\cdot\frac{\sqrt{2}}{2}\ \mathrm{N}\cdot\mathrm{m}=1.5\sqrt{2}\ \mathrm{N}\cdot\mathrm{m}$

27. (a) true, $|\mathbf{u}|=\sqrt{a_1^2+a_2^2+a_3^2}=\sqrt{\mathbf{u}\cdot\mathbf{u}}$

(b) not always true, $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^2$

(c) true, $\mathbf{u}\times\mathbf{0}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\u_1&u_2&u_3\\0&0&0\end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$ and $\mathbf{0}\times\mathbf{u}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&0\\u_1&u_2&u_3\end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$

(d) true, $\mathbf{u}\times(-\mathbf{u})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\u_1&u_2&u_3\\-u_1&-u_2&-u_3\end{vmatrix}=\left(-u_2u_3+u_2u_3\right)\mathbf{i}-\left(-u_1u_3+u_1u_3\right)\mathbf{j}+\left(-u_1u_2+u_1u_2\right)\mathbf{k}=\mathbf{0}$

(e) not always true, $\mathbf{i}\times\mathbf{j}=\mathbf{k}\neq-\mathbf{k}=\mathbf{j}\times\mathbf{i}$ for example

(f) true, distributive property of the cross product

(g) true, $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}=\mathbf{u}\cdot(\mathbf{v}\times\mathbf{v})=\mathbf{u}\cdot\mathbf{0}=0$

(h) true, the volume of a parallelepiped with $\mathbf{u},\mathbf{v},$ and $\mathbf{w}$ along the three edges is the same whether the plane containing $\mathbf{u}$ and $\mathbf{v}$ or the plane containing $\mathbf{v}$ and $\mathbf{w}$ is used as the base plane, and the dot product is commutative.

28. (a) true, $\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+u_3v_3=v_1u_1+v_2u_2+v_3u_3=\mathbf{v}\cdot\mathbf{u}$

(b) true, $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}=-\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\v_1&v_2&v_3\\u_1&u_2&u_3\end{vmatrix}=-(\mathbf{v}\times\mathbf{u})$

(c) true, $(-\mathbf{u})\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-u_1&-u_2&-u_3\\v_1&v_2&v_3\end{vmatrix}=-\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}=-(\mathbf{u}\times\mathbf{v})$

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(d) true, $(c\mathbf{u})\cdot\mathbf{v} = (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 = u_1(cv_1) + u_2(cv_2) + u_3(cv_3) = \mathbf{u}\cdot(c\mathbf{v}) = c\left(u_1v_1 + u_2v_2 + u_3v_3\right)$
$= c(\mathbf{u}\cdot\mathbf{v})$

(e) true, $c(\mathbf{u}\times\mathbf{v}) = c\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (c\mathbf{u})\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ cv_1 & cv_2 & cv_3 \end{vmatrix} = \mathbf{u}\times(c\mathbf{v})$

(f) true, $\mathbf{u}\cdot\mathbf{u} = u_1^2 + u_2^2 + u_3^2 = \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 = |\mathbf{u}|^2$

(g) true, $(\mathbf{u}\times\mathbf{u})\cdot\mathbf{u} = \mathbf{0}\cdot\mathbf{u} = 0$

(h) true, $\mathbf{u}\times\mathbf{v}\perp\mathbf{u}$ and $\mathbf{u}\times\mathbf{v}\perp\mathbf{v} \Rightarrow (\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = \mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 0$

29. (a) $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v}$ (b) $(\mathbf{u}\times\mathbf{v})$ (c) $\left((\mathbf{u}\times\mathbf{v})\times\mathbf{w}\right)$ (d) $\left|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}\right|$
(e) $(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{w})$ (f) $|\mathbf{u}|\,\frac{\mathbf{v}}{|\mathbf{v}|}$

30. $(\mathbf{i}\times\mathbf{j})\times\mathbf{j} = \mathbf{k}\times\mathbf{j} = -\mathbf{i}$; $\mathbf{i}\times(\mathbf{j}\times\mathbf{j}) = \mathbf{i}\times\mathbf{0} = \mathbf{0}$. The cross product is not associative.

31. (a) yes, $\mathbf{u}\times\mathbf{v}$ and $\mathbf{w}$ are both vectors (b) no, $\mathbf{u}$ is a vector but $\mathbf{v}\cdot\mathbf{w}$ is a scalar
(c) yes, $\mathbf{u}$ and $\mathbf{u}\times\mathbf{w}$ are both vectors (d) no, $\mathbf{u}$ is a vector but $\mathbf{v}\cdot\mathbf{w}$ is a scalar

32. $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ is perpendicular to $\mathbf{u}\times\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v} \Rightarrow (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ is parallel to a vector in the plane of $\mathbf{u}$ and $\mathbf{v}$ which means it lies in the plane determined by $\mathbf{u}$ and $\mathbf{v}$. The situation is degenerate if $\mathbf{u}$ and $\mathbf{v}$ are parallel so $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ and the vectors do not determine a plane. Similar reasoning shows that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ lies in the plane of $\mathbf{v}$ and $\mathbf{w}$ provided $\mathbf{v}$ and $\mathbf{w}$ are nonparallel.

33. No, $\mathbf{v}$ need not equal $\mathbf{w}$. For example, $\mathbf{i}+\mathbf{j} \neq -\mathbf{i}+\mathbf{j}$, but $\mathbf{i}\times(\mathbf{i}+\mathbf{j}) = \mathbf{i}\times\mathbf{i} + \mathbf{i}\times\mathbf{j} = \mathbf{0}+\mathbf{k} = \mathbf{k}$ and $\mathbf{i}\times(-\mathbf{i}+\mathbf{j}) = \mathbf{i}\times(-\mathbf{i}) + \mathbf{i}\times\mathbf{j} = \mathbf{0}+\mathbf{k} = \mathbf{k}$.

34. Yes. If $\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{w}$ and $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$, then $\mathbf{u}\times(\mathbf{v}-\mathbf{w}) = \mathbf{0}$ and $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = 0$. Suppose now that $\mathbf{v}\neq\mathbf{w}$. Then $\mathbf{u}\times(\mathbf{v}-\mathbf{w}) = \mathbf{0}$ implies that $\mathbf{v}-\mathbf{w} = k\mathbf{u}$ for some real number $k\neq 0$. This in turn implies that $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{u}\cdot(k\mathbf{u}) = k|\mathbf{u}|^2 = 0$, which implies that $\mathbf{u} = \mathbf{0}$. Since $\mathbf{u}\neq\mathbf{0}$, it cannot be true that $\mathbf{v}\neq\mathbf{w}$, so $\mathbf{v} = \mathbf{w}$.

35. $\overrightarrow{AB} = -\mathbf{i}+\mathbf{j}$ and $\overrightarrow{AD} = -\mathbf{i}-\mathbf{j} \Rightarrow \overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 2\mathbf{k} \Rightarrow$ area $= \left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = 2$

36. $\overrightarrow{AB} = 7\mathbf{i}+3\mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i}+5\mathbf{j} \Rightarrow \overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 29\mathbf{k} \Rightarrow$ area $= \left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = 29$

37. $\overrightarrow{AB} = 3\mathbf{i}-2\mathbf{j}$ and $\overrightarrow{AD} = 5\mathbf{i}+\mathbf{j} \Rightarrow \overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 0 \\ 5 & 1 & 0 \end{vmatrix} = 13\mathbf{k} \Rightarrow$ area $= \left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = 13$

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38. $\overrightarrow{AB} = 7\mathbf{i} - 4\mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i} + 5\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -4 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 43\mathbf{k} \Rightarrow$ area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 43$

39. $\overrightarrow{AB} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{DC} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \Rightarrow \overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$; $\overrightarrow{BC} = 2\mathbf{i} - \mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i} - \mathbf{j} \Rightarrow \overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$. $\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 4 \\ 2 & -1 & 0 \end{vmatrix} = 4\mathbf{i} + 8\mathbf{j} - 7\mathbf{k} \Rightarrow$ area $= \left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{129}$

40. $\overrightarrow{AC} = \mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{DB} = \mathbf{i} + 4\mathbf{j} \Rightarrow \overrightarrow{AC}$ is parallel to $\overrightarrow{DB}$; $\overrightarrow{AD} = -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{CB} = -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \Rightarrow \overrightarrow{AD}$ is parallel to $\overrightarrow{CB}$. $\overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ -1 & 3 & 3 \end{vmatrix} = 12\mathbf{i} - 3\mathbf{j} + 7\mathbf{k} \Rightarrow$ area $= \left|\overrightarrow{AC} \times \overrightarrow{AD}\right| = \sqrt{202}$

41. $\overrightarrow{AB} = -2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{AC} = 3\mathbf{i} + \mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix} = -11\mathbf{k} \Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{11}{2}$

42. $\overrightarrow{AB} = 4\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{AC} = 3\mathbf{i} + 2\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix} = -4\mathbf{k} \Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 2$

43. $\overrightarrow{AB} = 6\mathbf{i} - 5\mathbf{j}$ and $\overrightarrow{AC} = 11\mathbf{i} - 5\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 0 \\ 11 & -5 & 0 \end{vmatrix} = 25\mathbf{k} \Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{25}{2}$

44. $\overrightarrow{AB} = 16\mathbf{i} - 5\mathbf{j}$ and $\overrightarrow{AC} = 4\mathbf{i} + 4\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 16 & -5 & 0 \\ 4 & 4 & 0 \end{vmatrix} = 84\mathbf{k} \Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 42$

45. $\overrightarrow{AB} = -\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{AC} = -\mathbf{i} - \mathbf{k} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{3}{2}$

46. $\overrightarrow{AB} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\overrightarrow{AC} = 3\mathbf{i} + 3\mathbf{k} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ 3 & 0 & 3 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{k} \Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{3\sqrt{2}}{2}$

47. $\overrightarrow{AB} = -\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{AC} = \mathbf{j} - 2\mathbf{k} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 0 & 1 & -2 \end{vmatrix} = -4\mathbf{i} - 2\mathbf{j} - \mathbf{k} \Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{\sqrt{21}}{2}$

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48. $\overrightarrow{AB}=\mathbf{i}+2\mathbf{j}, \overrightarrow{AC}=-3\mathbf{j}+2\mathbf{k}$ and $\overrightarrow{AD}=3\mathbf{i}-4\mathbf{j}+5\mathbf{k} \Rightarrow \left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\cdot\overrightarrow{AD}=\begin{vmatrix}1 & 2 & 0\\ 0 & -3 & 2\\ 3 & -4 & 5\end{vmatrix}=5$

$\Rightarrow$ volume $=\left|\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\cdot\overrightarrow{AD}\right|=5$

49. If $\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}$ and $\mathbf{B}=b_1\mathbf{i}+b_2\mathbf{j}$, then $\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & 0\\ b_1 & b_2 & 0\end{vmatrix}=\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{k}$ and the triangle's area is

$\frac{1}{2}|\mathbf{A}\times\mathbf{B}|=\pm\frac{1}{2}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\cdot$ The applicable sign is $(+)$ if the acute angle from $\mathbf{A}$ to $\mathbf{B}$ runs counterclockwise in the xy-plane, and $(-)$ if it runs clockwise, because the area must be a nonnegative number.

50. If $\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}, \mathbf{B}=b_1\mathbf{i}+b_2\mathbf{j}$, and $\mathbf{C}=c_1\mathbf{i}+c_2\mathbf{j}$, then the area of the triangle is $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\cdot$

Now, $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_1-a_1 & b_2-a_2 & 0\\ c_1-a_1 & c_2-a_2 & 0\end{vmatrix}=\begin{vmatrix}b_1-a_1 & b_2-a_2\\ c_1-a_1 & c_2-a_2\end{vmatrix}\mathbf{k}\Rightarrow\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$

$=\frac{1}{2}\left|(b_1-a_1)(c_2-a_2)-(c_1-a_1)(b_2-a_2)\right|=\frac{1}{2}\left|a_1(b_2-c_2)+a_2(c_1-b_1)+(b_1c_2-c_1b_2)\right|=\pm\frac{1}{2}\begin{vmatrix}a_1 & a_2 & 1\\ b_1 & b_2 & 1\\ c_1 & c_2 & 1\end{vmatrix}\cdot$

The applicable sign ensures the area formula gives a nonnegative number.

12.5 LINES AND PLANES IN SPACE

1. The direction $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $P(3,-4,-1)\Rightarrow x=3+t, y=-4+t, z=-1+t$

2. The direction $\overrightarrow{PQ}=-2\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ and $P(1,2,-1)\Rightarrow x=1-2t, y=2-2t, z=-1+2t$

3. The direction $\overrightarrow{PQ}=5\mathbf{i}+5\mathbf{j}-5\mathbf{k}$ and $P(-2,0,3)\Rightarrow x=-2+5t, y=5t, z=3-5t$

4. The direction $\overrightarrow{PQ}=-\mathbf{j}-\mathbf{k}$ and $P(1,2,0)\Rightarrow x=1, y=2-t, z=-t$

5. The direction $2\mathbf{j}+\mathbf{k}$ and $P(0,0,0)\Rightarrow x=0, y=2t, z=t$

6. The direction $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $P(3,-2,1)\Rightarrow x=3+2t, y=-2-t, z=1+3t$

7. The direction $\mathbf{k}$ and $P(1,1,1)\Rightarrow x=1, y=1, z=1+t$

8. The direction $3\mathbf{i}+7\mathbf{j}-5\mathbf{k}$ and $P(2,4,5)\Rightarrow x=2+3t, y=4+7t, z=5-5t$

9. The direction $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $P(0,-7,0)\Rightarrow x=t, y=-7+2t, z=2t$

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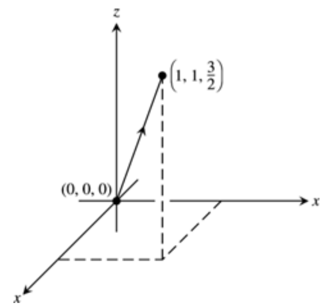
10. The direction is $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 3\\ 3 & 4 & 5\end{vmatrix}=-2\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ and $P(2,3,0)\Rightarrow x=2-2t,\ y=3+4t, z=-2t$

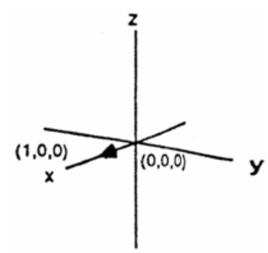
11. The direction $\mathbf{i}$ and $P(0,0,0)\Rightarrow x=t,\ y=0, z=0$

12. The direction $\mathbf{k}$ and $P(0,0,0)\Rightarrow x=0,\ y=0, z=t$

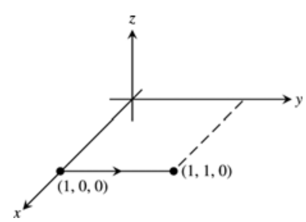
13. The direction $\overrightarrow{PQ}=\mathbf{i}+\mathbf{j}+\frac{3}{2}\mathbf{k}$ and $P(0,0,0)$
$\Rightarrow x=t, y=t,\ z=\frac{3}{2}t,$ where $0\le t\le 1$



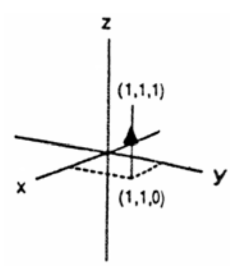
14. The direction $\overrightarrow{PQ}=\mathbf{i}$ and $P(0,0,0)$
$\Rightarrow x=t,\ y=0, z=0,$ where $0\le t\le 1$



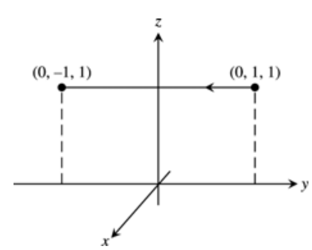
15. The direction $\overrightarrow{PQ}=\mathbf{j}$ and $P(1,1,0)$
$\Rightarrow x=1,\ y=1+t,\ z=0,$ where $-1\le t\le 0$



16. The direction $\overrightarrow{PQ}=\mathbf{k}$ and $P(1,1,0)$
$\Rightarrow x=1,\ y=1, z=t,$ where $0\le t\le 1$



17. The direction $\overrightarrow{PQ}=-2\mathbf{j}$ and $P(0,1,1)$
$\Rightarrow x=0,\ y=1-2t,\ z=1,$ where $0\le t\le 1$



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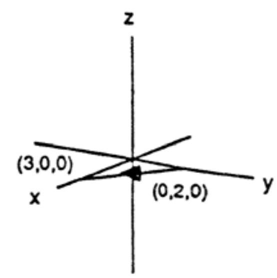
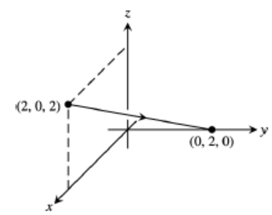
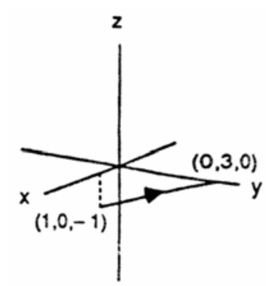
18. The direction $\overrightarrow{PQ} = 3\mathbf{i} - 2\mathbf{j}$ and $P(0, 2, 0)$
$\Rightarrow x = 3t,\ y = 2 - 2t,\ z = 0,$ where $0 \le t \le 1$



19. The direction $\overrightarrow{PQ} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $P(2, 0, 2)$
$\Rightarrow x = 2 - 2t,\ y = 2t,\ z = 2 - 2t,$ where $0 \le t \le 1$



20. The direction $\overrightarrow{PQ} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $P(1, 0, -1)$
$\Rightarrow x = 1 - t,\ y = 3t,\ z = -1 + t,$ where $0 \le t \le 1$



21. $3(x-0) + (-2)(y-2) + (-1)(z+1) = 0 \Rightarrow 3x - 2y - z = -3$

22. $3(x-1) + (1)(y+1) + (1)(z-3) = 0 \Rightarrow 3x + y + z = 5$

23. $\overrightarrow{PQ} = \mathbf{i} - \mathbf{j} + 3\mathbf{k},\ \overrightarrow{PS} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \Rightarrow \overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ is normal to the plane
$\Rightarrow 7(x-2) + (-5)(y-0) + (-4)(z-2) = 0 \Rightarrow 7x - 5y - 4z = 6$

24. $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k},\ \overrightarrow{PS} = -3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is normal to the plane
$\Rightarrow (-1)(x-1) + (-3)(y-5) + (1)(z-7) = 0 \Rightarrow x + 3y - z = 9$

25. $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k},\ P(2, 4, 5) \Rightarrow (1)(x-2) + (3)(y-4) + (4)(z-5) = 0 \Rightarrow x + 3y + 4z = 34$

26. $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + \mathbf{k},\ P(1, -2, 1) \Rightarrow (1)(x-1) + (-2)(y+2) + (1)(z-1) = 0 \Rightarrow x - 2y + z = 6$

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27. $\begin{cases} x = 2t+1 = s+2 \\ y = 3t+2 = 2s+4 \end{cases} \Rightarrow \begin{cases} 2t-s=1 \\ 3t-2s=2 \end{cases} \Rightarrow \begin{cases} 4t-2s=2 \\ 3t-2s=2 \end{cases} \Rightarrow t=0$ and $s=-1$; then $z = 4t+3 = -4s-1$ $\Rightarrow 4(0)+3 = (-4)(-1)-1$ is satisfied $\Rightarrow$ the lines intersect when $t=0$ and $s=-1 \Rightarrow$ the point of intersection is $x=1$, $y=2$, and $z=3$ or $P(1, 2, 3)$. A vector normal to the plane determined by these lines is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = -20\mathbf{i} + 12\mathbf{j} + \mathbf{k}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by $(-20)(x-1)+(12)(y-2)+(1)(z-3) = 0 \Rightarrow -20x+12y+z=7$.

28. $\begin{cases} x = t = 2s+2 \\ y = -t+2 = s+3 \end{cases} \Rightarrow \begin{cases} t-2s=2 \\ -t-s=1 \end{cases} \Rightarrow s=-1$ and $t=0$; then $z = t+1 = 5s+6 \Rightarrow 0+1 = 5(-1)+6$ is satisfied $\Rightarrow$ the lines do intersect when $s=-1$ and $t=0 \Rightarrow$ the point of intersection is $x=0$, $y=2$ and $z=1$ or $P(0, 2, 1)$. A vector normal to the plane determined by these lines is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, when $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by $(-6)(x-0)+(-3)(y-2)+(3)(z-1) = 0 \Rightarrow 6x+3y-3z=3$.

29. The cross product of $\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $-4\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ has the same direction as the normal to the plane $\Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} = 6\mathbf{j} + 6\mathbf{k}$. Select a point on either line, such as $P(-1, 2, 1)$. Since the lines are given to intersect, the desired plane is $0(x+1)+6(y-2)+6(z-1) = 0 \Rightarrow 6y+6z=18 \Rightarrow y+z=3$.

30. The cross product of $\mathbf{i}-3\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+\mathbf{j}+\mathbf{k}$ has the same direction as the normal to the plane $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. Select a point on either line, such as $P(0, 3, -2)$. Since the lines are given to intersect, the desired plane is $(-2)(x-0)+(-2)(y-3)+(4)(z+2) = 0 \Rightarrow -2x-2y+4z=-14$ $\Rightarrow x+y-2z=7$.

31. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ is a vector in the direction of the line of intersection of the planes $\Rightarrow 3(x-2)+(-3)(y-1)+3(z+1) = 0 \Rightarrow 3x-3y+3z=0 \Rightarrow x-y+z=0$ is the desired plane containing $P_0(2, 1, -1)$

32. A vector normal to the desired plane is $\overrightarrow{P_1P_2} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 4 & -1 & 2 \end{vmatrix} = -2\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}$; choosing $P_1(1, 2, 3)$ as a point on the plane $\Rightarrow (-2)(x-1)+(-12)(y-2)+(-2)(z-3) = 0 \Rightarrow -2x-12y-2z=-32 \Rightarrow x+6y+z=16$ is the desired plane

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33. $S(0,0,12)$, $P(0,0,0)$ and $\mathbf{v}=4\mathbf{i}-2\mathbf{j}+2\mathbf{k} \Rightarrow \overrightarrow{PS}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 0 & 12\\ 4 & -2 & 2\end{vmatrix}=24\mathbf{i}+48\mathbf{j}=24(\mathbf{i}+2\mathbf{j})$

$\Rightarrow d=\frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}=\frac{24\sqrt{1+4}}{\sqrt{16+4+4}}=\frac{24\sqrt{5}}{\sqrt{24}}=\sqrt{5\cdot 24}=2\sqrt{30}$ is the distance from S to the line

34. $S(0,0,0)$, $P(5,5,-3)$ and $\mathbf{v}=3\mathbf{i}+4\mathbf{j}-5\mathbf{k} \Rightarrow \overrightarrow{PS}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -5 & -5 & 3\\ 3 & 4 & -5\end{vmatrix}=13\mathbf{i}-16\mathbf{j}-5\mathbf{k}$

$\Rightarrow d=\frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}=\frac{\sqrt{169+256+25}}{\sqrt{9+16+25}}=\frac{\sqrt{450}}{\sqrt{50}}=\sqrt{9}=3$ is the distance from S to the line

35. $S(2,1,3)$, $P(2,1,3)$ and $\mathbf{v}=2\mathbf{i}+6\mathbf{j} \Rightarrow \overrightarrow{PS}\times\mathbf{v}=\mathbf{0} \Rightarrow d=\frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}=\frac{0}{\sqrt{40}}=0$ is the distance from S to the line (i.e., the point S lies on the line)

36. $S(2,1,-1)$, $P(0,1,0)$ and $\mathbf{v}=2\mathbf{i}+2\mathbf{j}+2\mathbf{k} \Rightarrow \overrightarrow{PS}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & -1\\ 2 & 2 & 2\end{vmatrix}=2\mathbf{i}-6\mathbf{j}+4\mathbf{k}$

$\Rightarrow d=\frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}=\frac{\sqrt{4+36+16}}{\sqrt{4+4+4}}=\frac{\sqrt{56}}{\sqrt{12}}=\sqrt{\frac{14}{3}}$ is the distance from S to the line

37. $S(3,-1,4)$, $P(4,3,-5)$ and $\mathbf{v}=-\mathbf{i}+2\mathbf{j}+3\mathbf{k} \Rightarrow \overrightarrow{PS}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & -4 & 9\\ -1 & 2 & 3\end{vmatrix}=-30\mathbf{i}-6\mathbf{j}-6\mathbf{k}$

$\Rightarrow d=\frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}=\frac{\sqrt{900+36+36}}{\sqrt{1+4+9}}=\frac{\sqrt{972}}{\sqrt{14}}=\frac{\sqrt{486}}{\sqrt{7}}=\frac{\sqrt{81\cdot 6}}{\sqrt{7}}=\frac{9\sqrt{42}}{7}$ is the distance from S to the line

38. $S(-1,4,3)$, $P(10,-3,0)$ and $\mathbf{v}=4\mathbf{i}+4\mathbf{k} \Rightarrow \overrightarrow{PS}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -11 & 7 & 3\\ 4 & 0 & 4\end{vmatrix}=28\mathbf{i}+56\mathbf{j}-28\mathbf{k}=28(\mathbf{i}+2\mathbf{j}-\mathbf{k})$

$\Rightarrow d=\frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}=\frac{28\sqrt{1+4+1}}{4\sqrt{1+1}}=7\sqrt{3}$ is the distance from S to the line

39. $S(2,-3,4)$, $x+2y+2z=13$ and $P(13,0,0)$ is on the plane $\Rightarrow \overrightarrow{PS}=-11\mathbf{i}-3\mathbf{j}+4\mathbf{k}$ and $\mathbf{n}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

$\Rightarrow d=\left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|=\left|\frac{-11-6+8}{\sqrt{1+4+4}}\right|=\left|\frac{-9}{\sqrt{9}}\right|=3$

40. $S(0,0,0)$, $3x+2y+6z=6$ and $P(2,0,0)$ is on the plane $\Rightarrow \overrightarrow{PS}=-2\mathbf{i}$ and $\mathbf{n}=3\mathbf{i}+2\mathbf{j}+6\mathbf{k}$

$\Rightarrow d=\left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|=\left|\frac{-6}{\sqrt{9+4+36}}\right|=\frac{6}{\sqrt{49}}=\frac{6}{7}$

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41. $S(0, 1, 1)$, $4y + 3z = -12$ and $P(0, -3, 0)$ is on the plane $\Rightarrow \overrightarrow{PS} = 4\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 4\mathbf{j} + 3\mathbf{k}$
$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{16+3}{\sqrt{16+9}}\right| = \frac{19}{5}$

42. $S(2, 2, 3)$, $2x + y + 2z = 4$ and $P(2, 0, 0)$ is on the plane $\Rightarrow \overrightarrow{PS} = 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{2+6}{\sqrt{4+1+4}}\right| = \frac{8}{3}$

43. $S(0, -1, 0)$, $2x + y + 2z = 4$ and $P(2, 0, 0)$ is on the plane $\Rightarrow \overrightarrow{PS} = -2\mathbf{i} - \mathbf{j}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-4-1+0}{\sqrt{4+1+4}}\right| = \frac{5}{3}$

44. $S(1, 0, -1)$, $-4x + y + z = 4$ and $P(-1, 0, 0)$ is on the plane $\Rightarrow \overrightarrow{PS} = 2\mathbf{i} - \mathbf{k}$ and $\mathbf{n} = -4\mathbf{i} + \mathbf{j} + \mathbf{k}$
$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-8-1}{\sqrt{16+1+1}}\right| = \frac{9}{\sqrt{18}} = \frac{3\sqrt{2}}{2}$

45. The point $P(1, 0, 0)$ is on the first plane and $S(10, 0, 0)$ is a point on the second plane $\Rightarrow \overrightarrow{PS} = 9\mathbf{i}$, and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the first plane $\Rightarrow$ the distance from S to the first plane is $d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{9}{\sqrt{1+4+36}}\right| = \frac{9}{\sqrt{41}}$, which is also the distance between the planes.

46. The line is parallel to the plane since $\mathbf{v} \cdot \mathbf{n} = \left(\mathbf{i} + \mathbf{j} - \frac{1}{2}\mathbf{k}\right) \cdot \left(\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}\right) = 1 + 2 - 3 = 0$. Also the point $S(1, 0, 0)$ when $t = -1$ lies on the line, and the point $P(10, 0, 0)$ lies on the plane $\Rightarrow \overrightarrow{PS} = -9\mathbf{i}$. The distance from S to the plane is $d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-9}{\sqrt{1+4+36}}\right| = \frac{9}{\sqrt{41}}$, which is also the distance from the line to the plane.

47. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{2+1}{\sqrt{2}\sqrt{9}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

48. $\mathbf{n}_1 = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{5-2-3}{\sqrt{27}\sqrt{14}}\right) = \cos^{-1}(0) = \frac{\pi}{2}$

49. $\mathbf{n}_1 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{4-4-2}{\sqrt{12}\sqrt{9}}\right) = \cos^{-1}\left(\frac{-1}{3\sqrt{3}}\right) \approx 1.76$ rad

50. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{3}\sqrt{1}}\right) \approx 0.96$ rad

51. $\mathbf{n}_1 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{2+4-1}{\sqrt{9}\sqrt{6}}\right) = \cos^{-1}\left(\frac{5}{3\sqrt{6}}\right) \approx 0.82$ rad

52. $\mathbf{n}_1 = 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{n}_2 = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{8+18}{\sqrt{25}\sqrt{49}}\right) = \cos^{-1}\left(\frac{26}{35}\right) \approx 0.73$ rad

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53. $2x-y+3z=6 \Rightarrow 2(1-t)-(3t)+3(1+t)=6 \Rightarrow -2t+5=6 \Rightarrow t=-\frac{1}{2} \Rightarrow x=\frac{3}{2}, y=-\frac{3}{2}$ and $z=\frac{1}{2}$ $\Rightarrow \left(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}\right)$ is the point

54. $6x+3y-4z=-12 \Rightarrow 6(2)+3(3+2t)-4(-2-2t)=-12 \Rightarrow 14t+29=-12 \Rightarrow t=-\frac{41}{14} \Rightarrow x=2, y=3-\frac{41}{7},$ and $z=-2+\frac{41}{7} \Rightarrow \left(2, -\frac{20}{7}, \frac{27}{7}\right)$ is the point

55. $x+y+z=2 \Rightarrow (1+2t)+(1+5t)+(3t)=2 \Rightarrow 10t+2=2 \Rightarrow t=0 \Rightarrow x=1, y=1$ and $z=0 \Rightarrow (1,1,0)$ is the point

56. $2x-3z=7 \Rightarrow 2(-1+3t)-3(5t)=7 \Rightarrow -9t-2=7 \Rightarrow t=-1 \Rightarrow x=-1-3, y=-2$ and $z=-5 \Rightarrow (-4,-2,-5)$ is the point

57. $\mathbf{n}_1=\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{n}_2=\mathbf{i}+\mathbf{j} \Rightarrow \mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 1\\ 1 & 1 & 0\end{vmatrix}=-\mathbf{i}+\mathbf{j},$ the direction of the desired line; $(1,1,-1)$ is on both planes $\Rightarrow$ the desired line is $x=1-t, y=1+t, z=-1$

58. $\mathbf{n}_1=3\mathbf{i}-6\mathbf{j}-2\mathbf{k}$ and $\mathbf{n}_2=2\mathbf{i}+\mathbf{j}-2\mathbf{k} \Rightarrow \mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -6 & -2\\ 2 & 1 & -2\end{vmatrix}=14\mathbf{i}+2\mathbf{j}+15\mathbf{k},$ the direction of the desired line; $(1,0,0)$ is on both planes $\Rightarrow$ the desired line is $x=1+14t, y=2t, z=15t$

59. $\mathbf{n}_1=\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ and $\mathbf{n}_2=\mathbf{i}+\mathbf{j}-2\mathbf{k} \Rightarrow \mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 4\\ 1 & 1 & -2\end{vmatrix}=6\mathbf{j}+3\mathbf{k},$ the direction of the desired line; $(4,3,1)$ is on both planes $\Rightarrow$ the desired line is $x=4, y=3+6t, z=1+3t$

60. $\mathbf{n}_1=5\mathbf{i}-2\mathbf{j}$ and $\mathbf{n}_2=4\mathbf{j}-5\mathbf{k} \Rightarrow \mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & -2 & 0\\ 0 & 4 & -5\end{vmatrix}=10\mathbf{i}+25\mathbf{j}+20\mathbf{k},$ the direction of the desired line; $(1,-3,1)$ is on both planes $\Rightarrow$ the desired line is $x=1+10t, y=-3+25t, z=1+20t$

61. <u>$L1$ & $L2$</u>: $\begin{cases} x=3+2t=1+4s \\ y=-1+4t=1+2s \end{cases} \Rightarrow \begin{cases} 2t-4s=-2 \\ 4t-2s=2 \end{cases} \Rightarrow \begin{cases} 2t-4s=-2 \\ 2t-s=1 \end{cases} \Rightarrow -3s=-3 \Rightarrow s=1$ and $t=1$ $\Rightarrow$ on $L1, z=1$ and on $L2, z=1 \Rightarrow L1$ and $L2$ intersect at $(5,3,1)$.

<u>$L2$ & $L3$</u>: The direction of $L2$ is $\frac{1}{6}(4\mathbf{i}+2\mathbf{j}+4\mathbf{k})=\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ which is the same as the direction $\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ of $L3$; hence $L2$ and $L3$ are parallel.

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<u>L1 & L3</u>: $\begin{cases} x = 3+2t = 3+2r \\ y = -1+4t = 2+r \end{cases} \Rightarrow \begin{cases} 2t-2r = 0 \\ 4t-r = 3 \end{cases} \Rightarrow \begin{cases} t-r = 0 \\ 4t-r = 3 \end{cases} \Rightarrow 3t = 3 \Rightarrow t = 1$ and $r = 1 \Rightarrow$ on $L1$, $z = 2$ while on $L3$, $z = 0 \Rightarrow$ $L1$ and $L2$ do not intersect. The direction of $L1$ is $\frac{1}{\sqrt{21}}(2\mathbf{i}+4\mathbf{j}-\mathbf{k})$ while the direction of $L3$ is $\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ and neither is a multiple of the other; hence $L1$ and $L3$ are skew.

62. <u>L1 & L2</u>: $\begin{cases} x = 1+2t = 2-s \\ y = -1-t = 3s \end{cases} \Rightarrow \begin{cases} 2t+s = 1 \\ -t-3s = 1 \end{cases} \Rightarrow -5s = 3 \Rightarrow s = -\frac{3}{5}$ and $t = \frac{4}{5} \Rightarrow$ on $L1$, $z = \frac{12}{5}$ while on $L2$, $z = 1-\frac{3}{5} = \frac{2}{5} \Rightarrow$ $L1$ and $L2$ do not intersect. The direction of $L1$ is $\frac{1}{\sqrt{14}}(2\mathbf{i}-\mathbf{j}+3\mathbf{k})$ while the direction of $L2$ is $\frac{1}{\sqrt{11}}(-\mathbf{i}+3\mathbf{j}+\mathbf{k})$ and neither is a multiple of the other; hence, $L1$ and $L2$ are skew.

<u>L2 & L3</u>: $\begin{cases} x = 2-s = 5+2r \\ y = 3s = 1-r \end{cases} \Rightarrow \begin{cases} -s-2r = 3 \\ 3s+r = 1 \end{cases} \Rightarrow 5s = 5 \Rightarrow s = 1$ and $r = -2 \Rightarrow$ on $L2$, $z = 2$ and on $L3$, $z = 2 \Rightarrow$ $L2$ and $L3$ intersect at $(1, 3, 2)$.

<u>L1 & L3</u>: $L1$ and $L3$ have the same direction $\frac{1}{\sqrt{14}}(2\mathbf{i}-\mathbf{j}+3\mathbf{k})$; hence $L1$ and $L3$ are parallel.

63. $x = 2+2t,\ y = -4-t,\ z = 7+3t;\ x = -2-t,\ y = -2+\frac{1}{2}t,\ z = 1-\frac{3}{2}t$

64. $1(x-4)-2(y-1)+1(z-5) = 0 \Rightarrow x-4-2y+2+z-5 = 0 \Rightarrow x-2y+z = 7;$
$-\sqrt{2}(x-3)+2\sqrt{2}(y+2)-\sqrt{2}(z-0) = 0 \Rightarrow -\sqrt{2}x+2\sqrt{2}y-\sqrt{2}z = -7\sqrt{2}$

65. $x = 0 \Rightarrow t = -\frac{1}{2},\ y = -\frac{1}{2},\ z = -\frac{3}{2} \Rightarrow \left(0, -\frac{1}{2}, -\frac{3}{2}\right);$ $y = 0 \Rightarrow t = -1,\ x = -1,\ z = -3 \Rightarrow (-1, 0, -3);$
$z = 0 \Rightarrow t = 0,\ x = 1,\ y = -1 \Rightarrow (1, -1, 0)$

66. The line contains $(0, 0, 3)$ and $\left(\sqrt{3}, 1, 3\right)$ because the projection of the line onto the xy-plane contains the origin and intersects the positive x-axis at a 30° angle. The direction of the line is $\sqrt{3}\mathbf{i}+\mathbf{j}+0\mathbf{k} \Rightarrow$ the line in question is $x = \sqrt{3}t,\ y = t,\ z = 3$.

67. With substitution of the line into the plane we have $2(1-2t)+(2+5t)-(-3t) = 8 \Rightarrow 2-4t+2+5t+3t = 8$ $\Rightarrow 4t+4 = 8 \Rightarrow t = 1 \Rightarrow$ the point $(-1, 7, -3)$ is contained in both the line and plane, so they are not parallel.

68. The planes are parallel when either vector $A_1\mathbf{i}+B_1\mathbf{j}+C_1\mathbf{k}$ or $A_2\mathbf{i}+B_2\mathbf{j}+C_2\mathbf{k}$ is a multiple of the other or when $(A_1\mathbf{i}+B_1\mathbf{j}+C_1\mathbf{k})\times(A_2\mathbf{i}+B_2\mathbf{j}+C_2\mathbf{k}) = \mathbf{0}$. The planes are perpendicular when their normals are perpendicular, or $(A_1\mathbf{i}+B_1\mathbf{j}+C_1\mathbf{k})\cdot(A_2\mathbf{i}+B_2\mathbf{j}+C_2\mathbf{k}) = 0$.

69. There are many possible answers. One is found as follows: eliminate t to get $t = x-1 = 2-y = \frac{z-3}{2}$ $\Rightarrow x-1 = 2-y$ and $2-y = \frac{z-3}{2} \Rightarrow x+y = 3$ and $2y+z = 7$ are two such planes.

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70. Since the plane passes through the origin, its general equation is of the form $Ax + By + Cz = 0$. Since it meets the plane *M* at a right angle, their normal vectors are perpendicular $\Rightarrow 2A + 3B + C = 0$. One choice satisfying this equation is $A = 1, B = -1$ and $C = 1 \Rightarrow x - y + z = 0$. Any plane $Ax + By + Cz = 0$ with $2A + 3B + C = 0$ will pass through the origin and be perpendicular to *M*.

71. The points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ are the *x*, *y*, and *z* intercepts of the plane. Since *a*, *b*, and *c* are all nonzero, the plane must intersect all three coordinate axes and cannot pass through the origin. Thus, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ describes all planes <u>except</u> those through the origin or parallel to a coordinate axis.

72. Yes. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors parallel to the lines, then $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$ is perpendicular to the lines.

73. (a) $\overrightarrow{EP} = c\overrightarrow{EP_1} \Rightarrow -x_0\mathbf{i} + y\mathbf{j} + z\mathbf{k} = c\left[\left(x_1 - x_0\right)\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}\right] \Rightarrow -x_0 = c\left(x_1 - x_0\right),\ y = cy_1$ and $z = cz_1$, where *c* is a positive real number

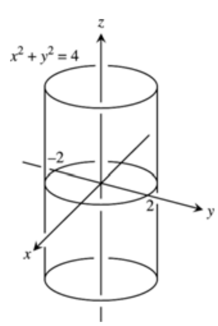
(b) $At\ x_1 = 0 \Rightarrow c = 1 \Rightarrow y = y_1$ and $z = z_1$; at $x_1 = x_0 \Rightarrow x_0 = 0,\ y = 0,\ z = 0$; $\lim\limits_{x_0 \to \infty} c = \lim\limits_{x_0 \to \infty} \frac{-x_0}{x_1 - x_0}$ $= \lim\limits_{x_0 \to \infty} \frac{-1}{-1} = 1 \Rightarrow c \to 1$ so that $y \to y_1$ and $z \to z_1$

74. The plane which contains the triangular plane is $x + y + z = 2$. The line containing the endpoints of the line segment is $x = 1 - t,\ y = 2t,\ z = 2t$. The plane and the line intersect at $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$. The visible section of the line segment is $\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1$ unit in length. The length of the line segment is $\sqrt{1^2 + 2^2 + 2^2} = 3 \Rightarrow \frac{2}{3}$ of the line segment is hidden from view.

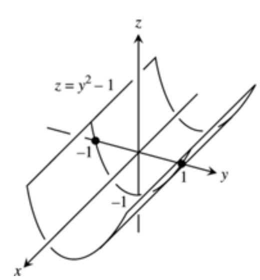
12.6 CYLINDERS AND QUADRIC SURFACES

1. *d*, ellipsoid
2. *i*, hyperboloid
3. *a*, cylinder
4. *g*, cone
5. *l*, hyperbolic paraboloid
6. *e*, paraboloid
7. *b*, cylinder
8. *j*, hyperboloid
9. *k*, hyperbolic paraboloid
10. *f*, paraboloid
11. *h*, cone
12. *c*, ellipsoid

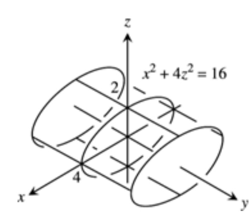
13. $x^2 + y^2 = 4$



14. $z = y^2 - 1$



15. $x^2 + 4z^2 = 16$



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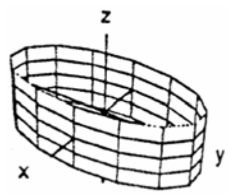
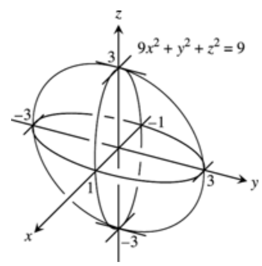
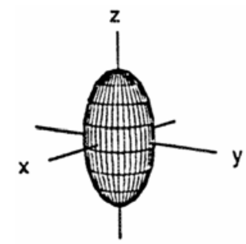
16. $4x^2 + y^2 = 36$



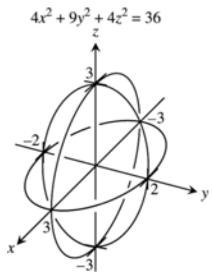
17. $9x^2 + y^2 + z^2 = 9$



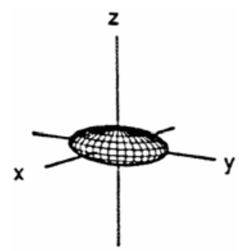
18. $4x^2 + 4y^2 + z^2 = 16$



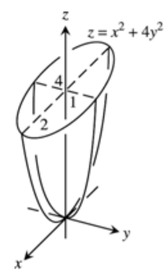
19. $4x^2 + 9y^2 + 4z^2 = 36$



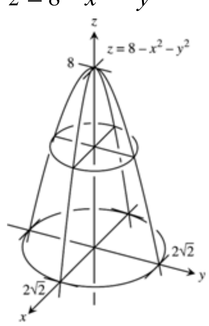
20. $9x^2 + 4y^2 + 36z^2 = 36$



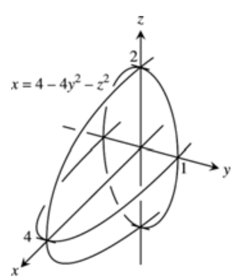
21. $x^2 + 4y^2 = z$



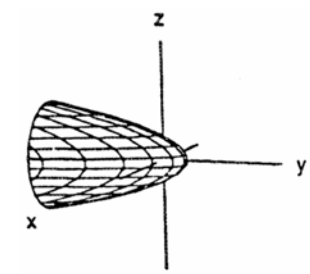
22. $z = 8 - x^2 - y^2$



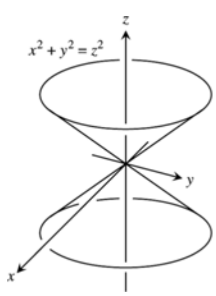
23. $x = 4 - 4y^2 - z^2$



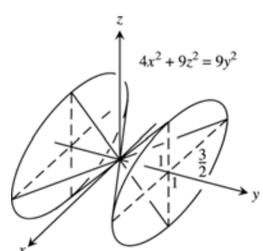
24. $y = 1 - x^2 - z^2$



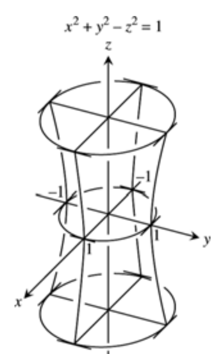
25. $x^2 + y^2 = z^2$



26. $4x^2 + 9z^2 = 9y^2$



27. $x^2 + y^2 - z^2 = 1$



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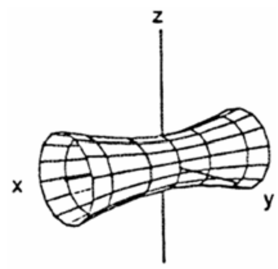
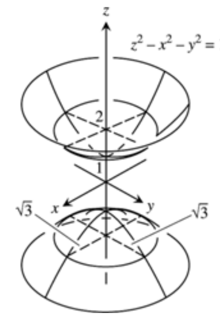
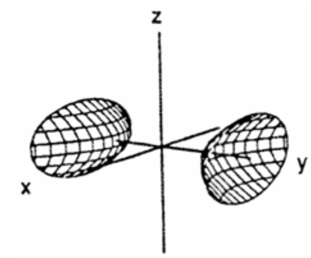
28. $y^2 + z^2 - x^2 = 1$



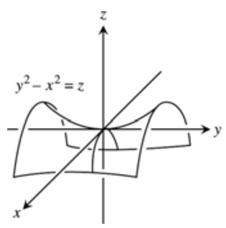
29. $z^2 - x^2 - y^2 = 1$



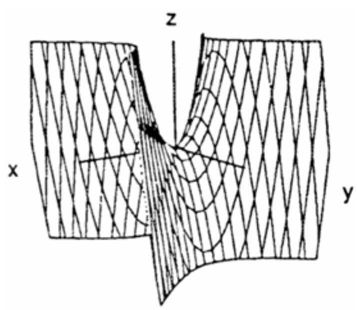
30. $\frac{y^2}{4} - \frac{x^2}{4} - z^2 = 1$



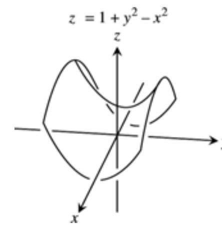
31. $y^2 - x^2 = z$



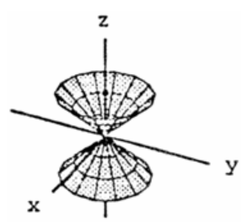
32. $x^2 - y^2 = z$



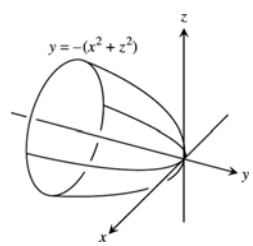
33. $z = 1 + y^2 - x^2$



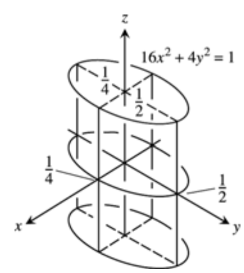
34. $4x^2 + 4y^2 = z^2$



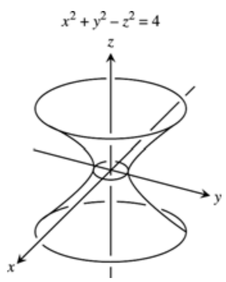
35. $y = -\left(x^2 + z^2\right)$



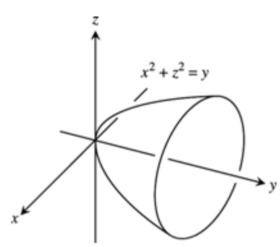
36. $16x^2 + 4y^2 = 1$



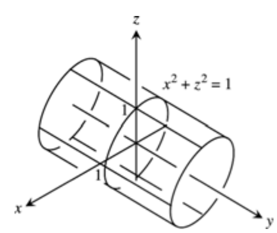
37. $x^2 + y^2 - z^2 = 4$



38. $x^2 + z^2 = y$



39. $x^2 + z^2 = 1$



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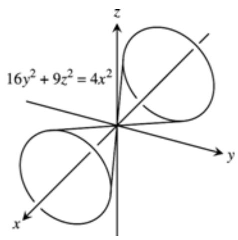
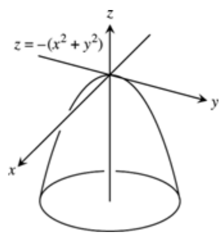
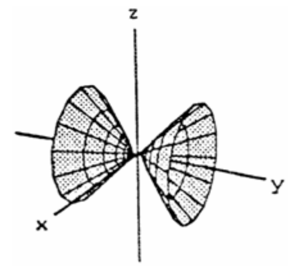
40. $16y^2+9z^2=4x^2$



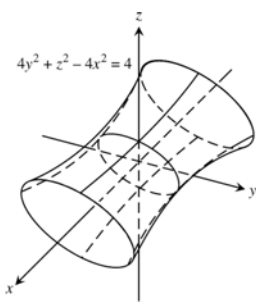
41. $z=-\left(x^2+y^2\right)$



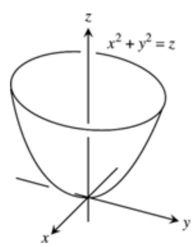
42. $y^2-x^2-z^2=1$



43. $4y^2+z^2-4x^2=4$



44. $x^2+y^2=z$



45. (a) If $x^2+\frac{y^2}{4}+\frac{z^2}{9}=1$ and $z=c$, then $x^2+\frac{y^2}{4}=\frac{9-c^2}{9} \Rightarrow \frac{x^2}{\left(\frac{9-c^2}{9}\right)}+\frac{y^2}{\left[\frac{4\left(9-c^2\right)}{9}\right]}=1 \Rightarrow A=ab\pi$

$=\pi\left(\frac{\sqrt{9-c^2}}{3}\right)\left(\frac{2\sqrt{9-c^2}}{3}\right)=\frac{2\pi\left(9-c^2\right)}{9}$

(b) From part (a), each slice has the area $\frac{2\pi\left(9-z^2\right)}{9}$, where $-3\le z\le 3$. Thus $V=2\int_0^3 \frac{2\pi}{9}\left(9-z^2\right)dz$

$=\frac{4\pi}{9}\int_0^3\left(9-z^2\right)dz=\frac{4\pi}{9}\left[9z-\frac{z^3}{3}\right]_0^3=\frac{4\pi}{9}(27-9)=8\pi$

(c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \Rightarrow \frac{x^2}{\left[\frac{a^2\left(c^2-z^2\right)}{c^2}\right]}+\frac{y^2}{\left[\frac{b^2\left(c^2-z^2\right)}{c^2}\right]}=1 \Rightarrow A=\pi\left(\frac{a\sqrt{c^2-z^2}}{c}\right)\left(\frac{b\sqrt{c^2-z^2}}{c}\right)$

$\Rightarrow V=2\int_0^c \frac{\pi ab}{c^2}\left(c^2-z^2\right)dz=\frac{2\pi ab}{c^2}\left[c^2z-\frac{z^3}{3}\right]_0^c=\frac{2\pi ab}{c^2}\left(\frac{2}{3}c^3\right)=\frac{4\pi abc}{3}$. Note that if $r=a=b=c$, then $V=\frac{4\pi r^3}{3}$, which is the volume of a sphere.

46. The ellipsoid has the form $\frac{x^2}{R^2}+\frac{y^2}{R^2}+\frac{z^2}{c^2}=1$. To determine c^2 we note that the point $(0, r, h)$ lies on the surface of the barrel. Thus, $\frac{r^2}{R^2}+\frac{h^2}{c^2}=1 \Rightarrow c^2=\frac{h^2R^2}{R^2-r^2}$. We calculate the volume by the disk method: $V=\pi\int_{-h}^{h} y^2\,dz$. Now, $\frac{y^2}{R^2}+\frac{z^2}{c^2}=1 \Rightarrow y^2=R^2\left(1-\frac{z^2}{c^2}\right)=R^2\left[1-\frac{z^2\left(R^2-r^2\right)}{h^2R^2}\right]=R^2-\left(\frac{R^2-r^2}{h^2}\right)z^2$

$\Rightarrow V=\pi\int_{-h}^{h}\left[R^2-\left(\frac{R^2-r^2}{h^2}\right)z^2\right]dz=\pi\left[R^2z-\frac{1}{3}\left(\frac{R^2-r^2}{h^2}\right)z^3\right]_{-h}^{h}=2\pi\left[R^2h-\frac{1}{3}\left(R^2-r^2\right)h\right]=2\pi\left(\frac{2R^2h}{3}+\frac{r^2h}{3}\right)$

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$= \frac{4}{3}\pi R^2 h + \frac{2}{3}\pi r^2 h,$ the volume of the barrel. If $r = R,$ then $V = 2\pi R^2 h$ which is the volume of a cylinder of radius R and height $2h$. If $r = 0$ and $h = R,$ then $V = \frac{4}{3}\pi R^3$ which is the volume of a sphere.

47. We calculate the volume by the slicing method, taking slices parallel to the xy-plane. For fixed z, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ gives the ellipse $\frac{x^2}{\left(\frac{za^2}{c}\right)} + \frac{y^2}{\left(\frac{zb^2}{c}\right)} = 1$. The area of this ellipse is $\pi\left(a\sqrt{\frac{z}{c}}\right)\left(b\sqrt{\frac{z}{c}}\right) = \frac{\pi abz}{c}$ (see Exercise 45a). Hence the volume is given by $V = \int_0^h \frac{\pi abz}{c}\,dz = \left[\frac{\pi abz^2}{2c}\right]_0^h = \frac{\pi abh^2}{c}$. Now the area of the elliptic base when $z = h$ is $A = \frac{\pi abh}{c}$, as determined previously. Thus, $V = \frac{\pi abh^2}{c} = \frac{1}{2}\left(\frac{\pi abh}{c}\right)h = \frac{1}{2}$ (base)(altitude), as claimed.

48. (a) For each fixed value of z, the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ results in a cross-sectional ellipse

$$\frac{x^2}{\left[\frac{a^2\left(c^2+z^2\right)}{c^2}\right]} + \frac{y^2}{\left[\frac{b^2\left(c^2+z^2\right)}{c^2}\right]} = 1.$$ The area of the cross-sectional ellipse (see Exercise 45a) is

$A(z) = \pi\left(\frac{a}{c}\sqrt{c^2+z^2}\right)\left(\frac{b}{c}\sqrt{c^2+z^2}\right) = \frac{\pi ab}{c^2}\left(c^2+z^2\right)$. The volume of the solid by the method of slices is

$$V = \int_0^h A(z)\,dz = \int_0^h \frac{\pi ab}{c^2}\left(c^2+z^2\right)dz = \frac{\pi ab}{c^2}\left[c^2 z + \frac{1}{3}z^3\right]_0^h = \frac{\pi ab}{c^2}\left(c^2 h + \frac{1}{3}h^3\right) = \frac{\pi abh}{3c^2}\left(3c^2+h^2\right)$$

(b) $A_0 = A(0) = \pi ab$ and $A_h = A(h) = \frac{\pi ab}{c^2}\left(c^2+h^2\right)$, from part (a) $\Rightarrow V = \frac{\pi abh}{3c^2}\left(3c^2+h^2\right)$

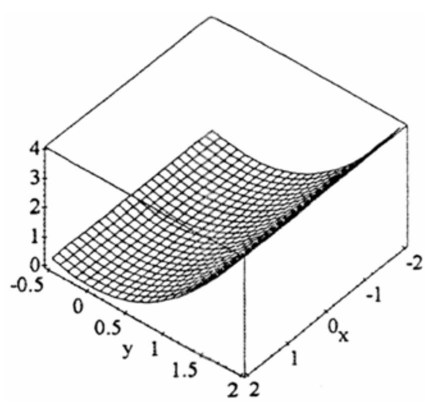
$$= \frac{\pi abh}{3}\left(2+1+\frac{h^2}{c^2}\right) = \frac{\pi abh}{3}\left(2+\frac{c^2+h^2}{c^2}\right) = \frac{h}{3}\left[2\pi ab + \frac{\pi ab}{c^2}\left(c^2+h^2\right)\right] = \frac{h}{3}\left(2A_0 + A_h\right)$$

(c) $A_m = A\left(\frac{h}{2}\right) = \frac{\pi ab}{c^2}\left(c^2+\frac{h^2}{4}\right) = \frac{\pi ab}{4c^2}\left(4c^2+h^2\right) \Rightarrow \frac{h}{6}\left(A_0 + 4A_m + A_h\right)$

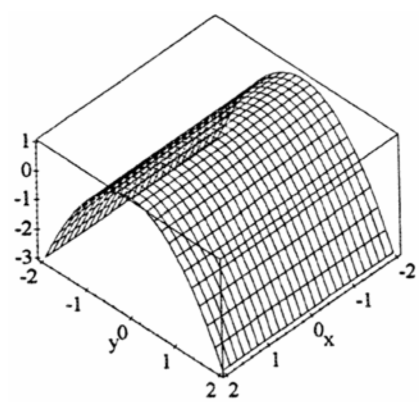
$$= \frac{h}{6}\left[\pi ab + \frac{\pi ab}{c^2}\left(4c^2+h^2\right) + \frac{\pi ab}{c^2}\left(c^2+h^2\right)\right] = \frac{\pi abh}{6c^2}\left(c^2+4c^2+h^2+c^2+h^2\right) = \frac{\pi abh}{6c^2}\left(6c^2+2h^2\right)$$

$= \frac{\pi abh}{3c^2}\left(3c^2+h^2\right) = V$ from part (a)

49. $z = y^2$



50. $z = 1 - y^2$



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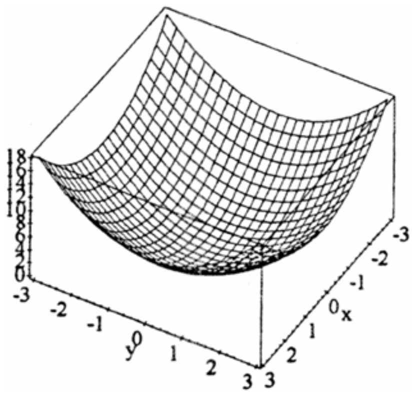
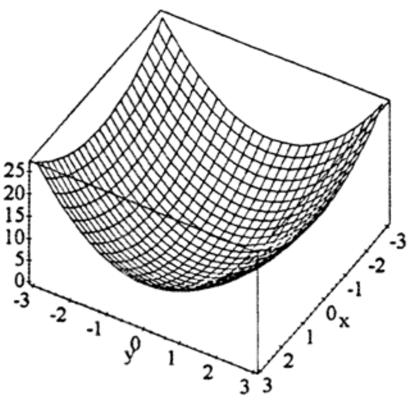
51. $z = x^2 + y^2$

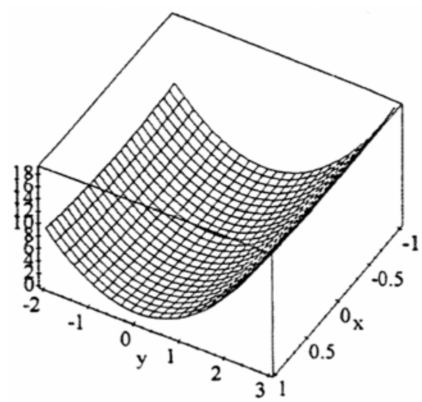


52. $z = x^2 + 2y^2$

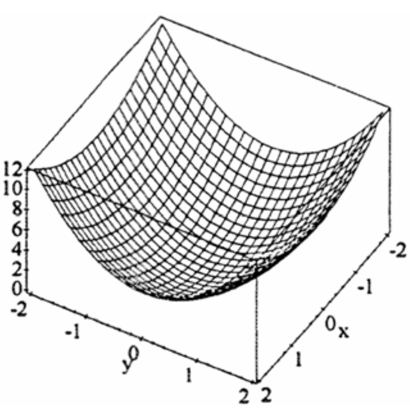
(a)



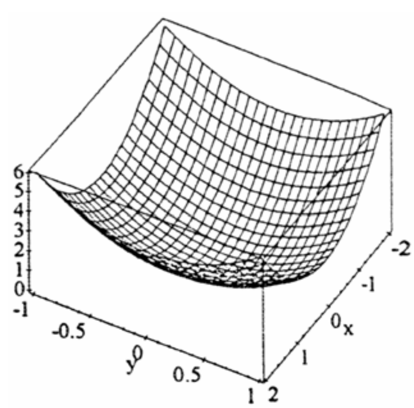
(b)



(c)



(d)



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53-58. Example CAS commands:

Maple:

```
with( plots );
eq := x^2/9 + y^2/36 = 1 - z^2/25;
implicitplot3d( eq, x=-3..3, y=-6..6, z=-5..5, scaling=constrained,
                shading=zhue, axes=boxed, title="#53 (Section 12.6)" );
```

Mathematica: (functions and domains may vary):

In the following chapter, you will consider contours or level curves for surfaces in three dimensions. For the purposes of plotting the functions of two variables expressed implicitly in this section, we will call upon the function **ContourPlot3D**. To insert the stated function, write all terms on the same side of the equal sign and the default contour equating that expression to zero will be plotted.

This built-in function requires the loading of a special graphics package.

```
<<Graphics`ContourPlot3D`
Clear[x, y, z]
ContourPlot3D[x^2/9 – y^2/16 – z^2/2 – 1, {x, –9, 9}, {y, –12, 12}, {z, –5, 5},
      Axes → True, AxesLabel → {x, y, z}, Boxed → False,
      PlotLabel → "Elliptic Hyperboloid of Two Sheets"]
```

Your identification of the plot may or may not be able to be done without considering the graph.

CHAPTER 12 PRACTICE EXERCISES

1. (a) $5u+9v=5\langle -6,7\rangle+9\langle 3,-7\rangle=\langle -30,35\rangle+\langle 27,-63\rangle=\langle -30+27,35-63\rangle=\langle -3-28\rangle$

   (b) Magnitude $=\sqrt{(-3)^2+(-28)^2}=\sqrt{793}$

2. (a) $u-v=\langle -6,7\rangle-\langle 3,-7\rangle=\langle -6-3,7+7\rangle=\langle -9,14\rangle$

   (b) Magnitude $=\sqrt{(-9)^2+(14)^2}=\sqrt{277}$

3. (a) $-7u=-7\langle -6,7\rangle=\langle 42,-49\rangle$

   (b) Magnitude $=\sqrt{(42)^2+(-49)^2}=\sqrt{4165}$

4. (a) $9v=9\langle 3,-7\rangle=\langle 27,-63\rangle$

   (b) Magnitude $=\sqrt{(27)^2+(-63)^2}=\sqrt{4698}$

5. Let the vector be $\langle a,b\rangle$ $a=|v|\cos\frac{3\pi}{4}=\sqrt{0^2+1^2}\left(\frac{-1}{\sqrt{2}}\right)=\frac{-1}{\sqrt{2}}$

   $b=|v|\sin\frac{3\pi}{4}=\sqrt{0^2+1^2}\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}$

   $\therefore$ The vector obtained by rotating $\langle 0,1\rangle$ through an angle of $\frac{3\pi}{4}$ is $\left\langle\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\rangle$.

6. The unit vector that makes an angle of $\frac{\pi}{3}$ radian with the positive x-axis is $\left\langle\cos\frac{\pi}{3},\sin\frac{\pi}{3}\right\rangle$

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7. $2\left(\frac{1}{\sqrt{4^2+1^2}}\right)(4\mathbf{i}-\mathbf{j})=\left(\frac{8}{\sqrt{17}}\mathbf{i}-\frac{2}{\sqrt{17}}\mathbf{j}\right)$

8. $-5\left(\frac{1}{\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}}\right)\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}\right)=-3\mathbf{i}-4\mathbf{j}$

9. Length $=\left|\sqrt{3}\hat{i}+\sqrt{3}\hat{j}\right|=\sqrt{(\sqrt{3})^2+(\sqrt{3})^2}=\sqrt{6}$

Direction $v=\frac{\vec{v}}{|\vec{v}|}=\frac{\left|\sqrt{3}\hat{i}+\sqrt{3}\hat{j}\right|}{\sqrt{6}}=\frac{\sqrt{3}\hat{i}+\sqrt{3}\hat{j}}{\sqrt{2}\sqrt{3}}=\frac{\hat{i}}{\sqrt{2}}+\frac{\hat{j}}{\sqrt{2}}$

10. Length $=\left|-3\hat{i}+2\hat{j}\right|=\sqrt{(-3)^2+(2)^2}=\sqrt{9+4}=\sqrt{13}$

Direction $v=\frac{\vec{v}}{|\vec{v}|}=\frac{-3\hat{i}+2\hat{j}}{\sqrt{13}}=\frac{-3}{\sqrt{13}}\hat{i}+\frac{2}{\sqrt{13}}\hat{j}$

11. Length $=\left|(-5\sin t)\hat{i}+(5\cos t)\hat{j}\right|=\sqrt{(-5\sin t)^2+(5\cos t)^2}=\sqrt{25(\sin^2 t+\cos^2 t)}=5$

Direction $v=\frac{\vec{v}}{|\vec{v}|}=\frac{(-5\sin t)\hat{i}+(5\cos t)\hat{j}}{5}=(-\sin t)\hat{i}+(\cos t)\hat{j}$

when $t=\frac{\pi}{4}$, $v=-\sin\frac{\pi}{4}\hat{i}+\cos\frac{\pi}{4}\hat{j}=-\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}$

12. Length $=\left|(e^t\sin t+e^t\cos t)\hat{i}+(e^t\cos t-e^t\sin t)\hat{j}\right|=\sqrt{(e^t\sin t+e^t\cos t)^2+(e^t\cos t-e^t\sin t)^2}$

$=\sqrt{e^{2t}(\sin^2 t+\cos^2 t+2\sin t\cos t+\cos^2 t+\sin^2 t-2\sin t\cos t)}=\sqrt{e^{2t}(2)}=e^t\sqrt{2}$

Direction $v=\frac{\vec{v}}{|\vec{v}|}=\frac{(e^t\sin t+e^t\cos t)\hat{i}+(e^t\cos t-e^t\sin t)\hat{j}}{e^t\sqrt{2}}$

when $t=\ln 3$, $v=\frac{\left(e^{\ln 3}\sin\ln 3+e^{\ln 3}\cos\ln 3\right)\hat{i}+\left(e^{\ln 3}\cos\ln 3-e^{\ln 3}\sin\ln 3\right)\hat{j}}{e^{\ln 3}\sqrt{2}}$

$v=\left(3\sin\ln 3+3\cos\ln 3\right)\hat{i}+\left(3\cos\ln 3-3\sin\ln 3\right)\hat{j}$

13. Length $=\sqrt{(4)^2+(5)^2+(-3)^2}=\sqrt{16+25+9}=\sqrt{50}=5\sqrt{2}$

Direction $v=\frac{\vec{v}}{|\vec{v}|}=\frac{4\hat{i}+5\hat{j}-3\hat{k}}{5\sqrt{2}}=\frac{4}{5\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}-\frac{3}{5\sqrt{2}}\hat{k}$

14. Length $=\sqrt{(1)^2+(-5)^2+(1)^2}=\sqrt{27}$

Direction $v=\frac{\vec{v}}{|\vec{v}|}=\frac{\hat{i}-5\hat{j}+\hat{k}}{\sqrt{27}}=\frac{1}{\sqrt{27}}\hat{i}-\frac{5}{\sqrt{27}}\hat{j}+\frac{1}{\sqrt{27}}\hat{k}$

15. Required vector $=8\cdot\frac{\vec{v}}{|\vec{v}|}=8\cdot\frac{3\hat{i}+2\hat{j}-\hat{k}}{\sqrt{(3)^2+(2)^2+(1)^2}}=8\cdot\frac{3\hat{i}+2\hat{j}-\hat{k}}{\sqrt{14}}$

16. Required vector $=6\cdot\frac{\vec{v}}{|\vec{v}|}=6\cdot\frac{\frac{2}{5}\hat{i}+\frac{3}{5}\hat{j}+\frac{4}{5}\hat{k}}{\sqrt{\left(\frac{2}{5}\right)^2+\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}}=6\cdot\frac{\frac{2}{5}\hat{i}+\frac{3}{5}\hat{j}+\frac{4}{5}\hat{k}}{\sqrt{\frac{29}{25}}}==6\cdot\frac{2\hat{i}+3\hat{j}+4\hat{k}}{\sqrt{29}}$

17. $|v|=\sqrt{(1)^2+(-1)^2}=\sqrt{2}$

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$|u| = \sqrt{(3)^2 + (-1)^2 + (5)^2} = \sqrt{35}$

$v \cdot u = (i - j) \cdot (3i - j + 5k) = (1)(3) + (-1)(-1) + (0)(5) = 3 + 1 = 4$

$u \cdot v = (3i - j + 5k) \cdot (i - j) = (3)(1) + (-1)(-1) + (5)(0) = 3 + 1 = 4$

$$v \times u = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 3 & -1 & 5 \end{vmatrix} = i(-5 - 0) - j(5 - 0) + k(-1 + 3) = -5i - 5j + 2k$$

$$u \times v = \begin{vmatrix} i & j & k \\ 3 & -1 & 5 \\ 1 & -1 & 0 \end{vmatrix} = i(0 + 5) - j(0 - 5) + k(-3 + 1) = 5i + 5j - 2k$$

$|v \times u| = \sqrt{(-5)^2 + (-5)^2 + (2)^2} = \sqrt{25 + 25 + 4} = \sqrt{54}$

The angle between v and u, $\theta = \cos^{-1} \frac{v \cdot u}{|v||u|} = \cos^{-1}\left(\frac{4}{\sqrt{2}\sqrt{35}}\right) = \cos^{-1}\left(\frac{2\sqrt{2}}{\sqrt{35}}\right)$

The scalar component of u in the direction of $v = |u| \cos\theta = \sqrt{35} \cos\left[\cos^{-1}\left(\frac{2\sqrt{2}}{\sqrt{35}}\right)\right] = \sqrt{35} \cdot \frac{2\sqrt{2}}{\sqrt{35}} = 2\sqrt{2}$

Vector projection of u onto $v = proj_v u = \left(\frac{u \cdot v}{|v|^2}\right) v = \left(\frac{4}{(\sqrt{2})^2}\right)(i - j) = 2(i - j)$

18. $|v| = \sqrt{(1)^2 + (-1)^2 + (-3)^2} = \sqrt{11}$

$|u| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$v \cdot u = (i - j - 3k) \cdot (-j - k) = (1)(0) + (-1)(-1) + (-3)(-1) = 0 + 1 + 3 = 4$

$u \cdot v = (-j - k) \cdot (i - j - 3k) = (0)(1) + (-1)(-1) + (-1)(-3) = 0 + 1 + 3 = 4$

$$v \times u = \begin{vmatrix} i & j & k \\ 1 & -1 & -3 \\ 0 & -1 & -1 \end{vmatrix} = i(1 - 3) - j(-1 - 0) + k(-1 - 0) = -2i + j - k$$

$$u \times v = \begin{vmatrix} i & j & k \\ 0 & -1 & -1 \\ 1 & -1 & -3 \end{vmatrix} = i(3 - 1) - j(0 + 1) + k(0 + 1) = 2i - j + k$$

$|v \times u| = \sqrt{(-2)^2 + (1)^2 + (-1)^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$

The angle between v and u, $\theta = \cos^{-1} \frac{v \cdot u}{|v||u|} = \cos^{-1}\left(\frac{4}{\sqrt{11}\sqrt{2}}\right) = \cos^{-1}\left(\frac{2\sqrt{2}}{\sqrt{11}}\right)$

The scalar component of u in the direction of $v = |u| \cos\theta = \sqrt{2} \cos\left[\cos^{-1}\left(\frac{2\sqrt{2}}{\sqrt{11}}\right)\right] = \sqrt{2} \cdot \frac{2\sqrt{2}}{\sqrt{11}} = \frac{4}{\sqrt{11}}$

Vector projection of u onto $v = proj_v u = \left(\frac{u \cdot v}{|v|^2}\right) v = \left(\frac{4}{(\sqrt{11})^2}\right)(i - j - 3k) = \frac{4}{11}(i - j - 3k)$

19. $proj_v u = \left(\frac{u \cdot v}{|v|^2}\right) v = \left[\frac{(i - j + 7k) \cdot (4i - j + k)}{\left(\sqrt{(4)^2 + (-1)^2 + (1)^2}\right)^2}\right] \cdot (4i - j + k) = \left(\frac{(1)(4) + (-1)(-1) + (7)(1)}{(\sqrt{18})^2}\right) \cdot (4i - j + k) = \frac{2}{3}(4i - j + k)$

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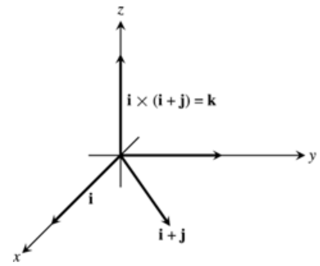
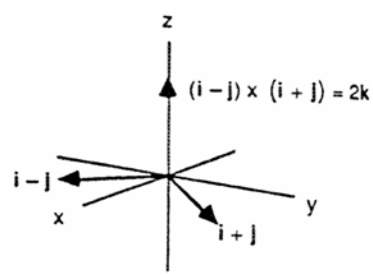
20. $proj_v u = \left(\frac{u\cdot v}{|v|^2}\right)v = \left[\frac{(j+5k)\cdot(i-j-k)}{\left(\sqrt{(1)^2+(-1)^2+(-1)^2}\right)^2}\right]\cdot(i-j-k) = \left(\frac{(1)(0)+(1)(-1)+(5)(-1)}{(\sqrt{3})^2}\right)\cdot(i-j-k)$

$= \frac{-6}{3}(i-j-k) = -2(i-j-k)$

21. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{k}$



22. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 2\mathbf{k}$



23. Let $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Then $|\mathbf{v} - 2\mathbf{w}|^2 = \left|(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) - 2(w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k})\right|^2$

$= \left|(v_1 - 2w_1)\mathbf{i} + (v_2 - 2w_2)\mathbf{j} + (v_3 - 2w_3)\mathbf{k}\right|^2 = \left(\sqrt{(v_1 - 2w_1)^2 + (v_2 - 2w_2)^2 + (v_3 - 2w_3)^2}\right)^2$

$= \left(v_1^2 + v_2^2 + v_3^2\right) - 4(v_1w_1 + v_2w_2 + v_3w_3) + 4\left(w_1^2 + w_2^2 + w_3^2\right) = |\mathbf{v}|^2 - 4\mathbf{v}\cdot\mathbf{w} + 4|\mathbf{w}|^2$

$|\mathbf{v}|^2 - 4|\mathbf{v}||\mathbf{w}|\cos\theta + 4|\mathbf{w}|^2 = 4 - 4(2)(3)\left(\cos\frac{\pi}{3}\right) + 36 = 40 - 24\left(\frac{1}{2}\right) = 40 - 12 = 28 \Rightarrow |\mathbf{v} - 2\mathbf{w}| = \sqrt{28} = 2\sqrt{7}$

24. $\mathbf{u}$ and $\mathbf{v}$ are parallel when $\mathbf{u}\times\mathbf{v} = \mathbf{0} \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -5 \\ -4 & -8 & a \end{vmatrix} = \mathbf{0} \Rightarrow (4a - 40)\mathbf{i} + (20 - 2a)\mathbf{j} + (0)\mathbf{k} = \mathbf{0}$

$\Rightarrow 4a - 40 = 0$ and $20 - 2a \Rightarrow a = 10$

25. (a) Area of the parallelogram $= \vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ 3 & 4 & -5 \end{vmatrix} = \hat{i}(5+4) - \hat{j}(-5+3) + \hat{k}(4+3) = 9\hat{i} + 2\hat{j} + 7\hat{k}$

$|\vec{u}\times\vec{v}| = \sqrt{9^2 + 2^2 + 7^2} = \sqrt{134}$ sq. units

(b) Volume of the parallelopiped $= \begin{vmatrix} 1 & -1 & -1 \\ 3 & 4 & -5 \\ 2 & 2 & -1 \end{vmatrix} = 1(-4+10) + 1(-3+10) + (-1)(6-8) = 6 + 7 + 2 = 15$

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26. (a) Area of the parallelogram $= \vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = \hat{i}(1+0) - \hat{j}(0) + \hat{k}(0) = \hat{i}$ $\qquad |\vec{u} \times \vec{v}| = \sqrt{1^2} = 1$

(b) Volume of the parallelopiped $= \begin{vmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 1$

27. The desired vector is $\mathbf{n} \times \mathbf{v}$ or $\mathbf{v} \times \mathbf{n}$ since $\mathbf{n} \times \mathbf{v}$ is perpendicular to both $\mathbf{n}$ and $\mathbf{v}$ and, therefore, also parallel to the plane.

28. If $a = 0$ and $b \neq 0$, then the line $by = c$ and $\mathbf{i}$ are parallel. If $a \neq 0$ and $b = 0$, then the line $ax = c$ and $\mathbf{j}$ are parallel. If a and b are both $\neq 0$, then $ax + by = c$ contains the points $\left(\frac{c}{a}, 0\right)$ and $\left(0, \frac{c}{b}\right) \Rightarrow$ the vector $ab\left(\frac{c}{a}\mathbf{i} - \frac{c}{b}\mathbf{j}\right) = c\left(b\mathbf{i} - a\mathbf{j}\right)$ and the line are parallel. Therefore, the vector $b\mathbf{i} - a\mathbf{j}$ is parallel to the line $ax + by = c$ in every case.

29. The line L passes through the point $P(0, 0, -1)$ parallel to $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\overrightarrow{PS} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} - (2+1)\mathbf{j} + (2+2)\mathbf{k} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, we find the distance $d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}}$ $= \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}$.

30. The line L passes through the point $P(2, 2, 0)$ parallel to $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\overrightarrow{PS} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} - (-2-1)\mathbf{j} + (-2-2)\mathbf{k} = \mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, we find the distance $d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}}$ $= \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}$.

31. Parametric equations for the line are $x = 1 - 3t,\ y = 2,\ z = 3 + 7t$.

32. The line is parallel to $\overrightarrow{PQ} = 0\mathbf{i} + \mathbf{j} - \mathbf{k}$ and contains the point $P(1, 2, 0) \Rightarrow$ parametric equations are $x = 1,\ y = 2 + t,\ z = -t$ for $0 \leq t \leq 1$.

33. The point P $(4, 0, 0)$ lies on the plane $x - y = 4$, and $\overrightarrow{PS} = (6-4)\mathbf{i} + 0\mathbf{j} + (-6+0)\mathbf{k} = 2\mathbf{i} - 6\mathbf{k}$ with $\mathbf{n} = \mathbf{i} - \mathbf{j}$ $\Rightarrow d = \frac{|\mathbf{n} \cdot \overrightarrow{PS}|}{|\mathbf{n}|} = \left|\frac{2+0+0}{\sqrt{1+1+0}}\right| = \frac{2}{\sqrt{2}} = \sqrt{2}$.

34. The point $P(0, 0, 2)$ lies on the plane $2x + 3y + z = 2$, and $\overrightarrow{PS} = (3-0)\mathbf{i} + (0-0)\mathbf{j} + (10-2)\mathbf{k} = 3\mathbf{i} + 8\mathbf{k}$ with $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow d = \frac{|\mathbf{n} \cdot \overrightarrow{PS}|}{|\mathbf{n}|} = \left|\frac{6+0+8}{\sqrt{4+9+1}}\right| = \frac{14}{\sqrt{14}} = \sqrt{14}$.

35. $P(3, -2, 1)$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow (2)(x-3) + (1)\left(y-(-2)\right) + (1)(z-1) = 0 \Rightarrow 2x + y + z = 5$

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36. $P(-1, 6, 0)$ and $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \Rightarrow (1)\big(x-(-1)\big) + (-2)(y-6) + (3)(z-0) = 0 \Rightarrow x - 2y + 3z = -13$

37. $P(1, -1, 2), Q(2, 1, 3)$ and $R(-1, 2, -1) \Rightarrow \overrightarrow{PQ} = \mathbf{i} + 2\mathbf{j} + \mathbf{k},\ \overrightarrow{PR} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ and

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{K} \\ 1 & 2 & 1 \\ -2 & 3 & -3 \end{vmatrix} = -9\mathbf{i} + \mathbf{j} + 7\mathbf{k} \text{ is normal to the plane } \Rightarrow (-9)(x-1) + (1)(y+1) + (7)(z-2) = 0$$

$\Rightarrow -9x + y + 7z = 4$

38. $P(1, 0, 0), Q(0, 1, 0)$ and $R(0, 0, 1) \Rightarrow \overrightarrow{PQ} = -\mathbf{i} + \mathbf{j},\ \overrightarrow{PR} = -\mathbf{i} + \mathbf{k}$ and $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is normal to the plane $\Rightarrow (1)(x-1) + (1)(y-0) + (1)(z-0) = 0 \Rightarrow x + y + z = 1$

39. $\left(0, -\frac{1}{2}, -\frac{3}{2}\right)$, since $t = -\frac{1}{2}, y = -\frac{1}{2}$ and $z = -\frac{3}{2}$ when $x = 0; (-1, 0, -3)$, since $t = -1, x = -1$ and $z = -3$ when $y = 0; (1, -1, 0)$, since $t = 0, x = 1$ and $y = -1$ when $z = 0$

40. $x = 2t, y = -t, z = -t$ represents a line containing the origin and perpendicular to the plane $2x - y - z = 4$; this line intersects the plane $3x - 5y + 2z = 6$ when t is the solution of $3(2t) - 5(-t) + 2(-t) = 6 \Rightarrow t = \frac{2}{3}$ $\Rightarrow \left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$ is the point of intersection

41. $\mathbf{n}_1 = \mathbf{i}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow$ the desired angle is $\cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

42. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \mathbf{j} + \mathbf{k} \Rightarrow$ the desired angle is $\cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

43. The direction of the line is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. Since the point $(-5, 3, 0)$ is on both planes, the desired line is $x = -5 + 5t, y = 3 - t, z = -3t$.

44. The direction of the intersection is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 5 & -2 & -1 \end{vmatrix} = -6\mathbf{i} - 9\mathbf{j} - 12\mathbf{k} = -3\big(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}\big)$ and is the same as the direction of the given line.

45. (a) The corresponding normals are $\mathbf{n}_1 = 3\mathbf{i} + 6\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and since $\mathbf{n}_1 \cdot \mathbf{n}_2 = (3)(2) + (0)(2) + (6)(-1) = 6 + 0 - 6 = 0$, we have that the plans are orthogonal

(b) The line of intersection is parallel to $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & 2 & -1 \end{vmatrix} = -12\mathbf{i} + 15\mathbf{j} + 6\mathbf{k}$. Now to find a point in the intersection, solve $\begin{cases} 3x + 6z = 1 \\ 2x + 2y - z = 3 \end{cases} \Rightarrow \begin{cases} 3x + 6z = 1 \\ 12x + 12y - 6z = 18 \end{cases} \Rightarrow 15x + 12y = 19 \Rightarrow x = 0$ and $y = \frac{19}{12}$ $\Rightarrow \left(0, \frac{19}{12}, \frac{1}{6}\right)$ is a point on the line we seek. Therefore, the line is $x = -12t, y = \frac{19}{12} + 15t$ and $z = \frac{1}{6} + 6t$.

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46. A vector in the direction of the plane's normal is $\mathbf{n}=\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 3 & 1\\ 1 & -1 & 2\end{vmatrix}=7\mathbf{i}-3\mathbf{j}-5\mathbf{k}$ and $P(1, 2, 3)$ on the plane $\Rightarrow 7(x-1)-3(y-2)-5(z-3)=0 \Rightarrow 7x-3y-5z=-14.$

47. Yes; $\mathbf{v}\cdot\mathbf{n}=(2\mathbf{i}-4\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}+0\mathbf{k})=2\cdot 2-4\cdot 1+1\cdot 0=0 \Rightarrow$ the vector is orthogonal to the plane's normal $\Rightarrow \mathbf{v}$ is parallel to the plane

48. $\mathbf{n}\cdot\overrightarrow{PP_0}>0$ represents the half-space of points lying on one side of the plane in the direction which the normal **n** points

49. A normal to the plane is $\mathbf{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & -1\\ 2 & -1 & 0\end{vmatrix}=-\mathbf{i}-2\mathbf{j}-2\mathbf{k} \Rightarrow$ the distance is

$$d=\left|\frac{\overrightarrow{AP}\cdot\mathbf{n}}{\mathbf{n}}\right|=\left|\frac{(\mathbf{i}+4\mathbf{j})\cdot(-\mathbf{i}-2\mathbf{j}-2\mathbf{k})}{\sqrt{1+4+4}}\right|=\left|\frac{-1-8+0}{3}\right|=3$$

50. $P(0, 0, 0)$ lies on the plane $2x+3y+5z=0,$ and $\overrightarrow{PS}=2\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ with $\mathbf{n}=2\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ $\Rightarrow d=\left|\frac{\mathbf{n}\cdot\overrightarrow{PS}}{|\mathbf{n}|}\right|=\left|\frac{4+6+15}{\sqrt{4+9+25}}\right|=\frac{25}{\sqrt{38}}$

51. $\mathbf{n}=2\mathbf{i}-\mathbf{j}-\mathbf{k}$ is normal to the plane $\Rightarrow \mathbf{n}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & -1\\ 1 & 1 & 1\end{vmatrix}=0\mathbf{i}-3\mathbf{j}+3\mathbf{k}=-3\mathbf{j}+3\mathbf{k}$ is orthogonal to **v** and parallel to the plane

52. The vector $\mathbf{B}\times\mathbf{C}$ is normal to the plane of $\mathbf{B}$ and $\mathbf{C} \Rightarrow \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is orthogonal to **A** and parallel to the plane of **B** and **C**:

$\mathbf{B}\times\mathbf{C}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 1\\ 1 & 1 & -2\end{vmatrix}=-5\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 1\\ -5 & 3 & -1\end{vmatrix}=-2\mathbf{i}-3\mathbf{j}+\mathbf{k}$

$\Rightarrow |\mathbf{A}\times(\mathbf{B}\times\mathbf{C})|=\sqrt{4+9+1}=\sqrt{14}$ and $\mathbf{u}=\frac{1}{\sqrt{47}}(-2\mathbf{i}-3\mathbf{j}+\mathbf{k})$ is the desired unit vector.

53. A vector parallel to the line of intersection is $\mathbf{v}=\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 1\\ 1 & -1 & 2\end{vmatrix}=5\mathbf{i}-\mathbf{j}-3\mathbf{k} \Rightarrow |\mathbf{v}|=\sqrt{25+1+9}=\sqrt{35}$

$\Rightarrow 2\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)=\frac{2}{\sqrt{35}}(5\mathbf{i}-\mathbf{j}-3\mathbf{k})$ is the desired vector.

54. The line containing $(0, 0, 0)$ normal to the plane is represented by $x=2t,\ y=-t,$ and $z=-t.$ This line intersects the plane $3x-5y+2z=6$ when $3(2t)-5(-t)+2(-t)=6 \Rightarrow t=\frac{2}{3} \Rightarrow$ the point is $\left(\frac{4}{3},-\frac{2}{3},-\frac{2}{3}\right).$

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55. The line is represented by $x = 3 + 2t, y = 2 - t,$ and $z = 1 + 2t$. It meets the plane $2x - y + 2z = -2$ when $2(3+2t) - (2-t) + 2(1+2t) = -2 \Rightarrow t = -\frac{8}{9} \Rightarrow$ the point is $\left(\frac{11}{9}, \frac{26}{9}, -\frac{7}{9}\right)$.

56. The direction of the intersection is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}||\mathbf{i}|}\right) = \cos^{-1}\left(\frac{3}{\sqrt{35}}\right) \approx 59.5°$

57. The intersection occurs when $(3+2t) + 3(2t) - t = -4 \Rightarrow t = -1 \Rightarrow$ the point is $(1, -2, -1)$. The required line must be perpendicular to both the given line and to the normal, and hence is parallel to $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 1 & 3 & -1 \end{vmatrix} = -5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \Rightarrow$ the line is represented by $x = 1 - 5t$, $y = -2 + 3t$, and $z = -1 + 4t$.

58. If $P(a, b, c)$ is a point on the line of intersection, then P lies in both planes $\Rightarrow a - 2b + c + 3 = 0$ and $2a - b - c + 1 = 0 \Rightarrow (a - 2b + c + 3) + k(2a - b - c + 1) = 0$ for all k.

59. The vector $\overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ \frac{26}{5} & 0 & -\frac{26}{5} \end{vmatrix} = \frac{26}{5}(2\mathbf{i} + 7\mathbf{j} + 2\mathbf{k})$ is normal to the plane and $A(-2, 0, -3)$ lies on the plane $\Rightarrow 2(x+2) + 7(y-0) + 2(z-(-3)) = 0 \Rightarrow 2x + 7y + 2z + 10 = 0$ is an equation of the plane.

60. Yes; the line's direction vector is $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ which is parallel to the line and also parallel to the normal $-4\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$ to the plane $\Rightarrow$ the line is orthogonal to the plane.

61. The vector $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -3 & 0 & 1 \end{vmatrix} = -\mathbf{i} - 11\mathbf{j} - 3\mathbf{k}$ is normal to the plane.

(a) No, the plane is not orthogonal to $\overrightarrow{PQ} \times \overrightarrow{PR}$.

(b) No, these equations represent a line, not a plane.

(c) No, the plane $(x+2) + 11(y-1) - 3z = 0$ has normal $\mathbf{i} + 11\mathbf{j} - 3\mathbf{k}$ which is not parallel to $\overrightarrow{PQ} \times \overrightarrow{PR}$.

(d) No, this vector equation is equivalent to the equations $3y + 3z = 3, 3x - 2z = -6,$ and $3x + 2y = -4$ $\Rightarrow x = -\frac{4}{3} - \frac{2}{3}t, y = t, z = 1 - t,$ which represents a line, not a plane.

(e) yes, this is a plane containing the point $R(-2, 1, 0)$ with normal $\overrightarrow{PQ} \times \overrightarrow{PR}$.

62. (a) The line through A and B is $x = 1 + t, y = -t, z = -1 + 5t$; the line through C and D must be parallel and is $L_1: x = 1 + t, y = 2 - t, z = 3 + 5t$. The line through B and C is $x = 1, y = 2 + 2s, z = 3 + 4s$; the line through A and D must be parallel and is $L_2: x = 2, y = -1 + 2s, z = 4 + 4s$. The lines L_1 and L_2 intersect at $D(2, 1, 8)$ where $t = 1$ and $s = 1$.

(b) $\cos\theta = \frac{(2\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+5\mathbf{k})}{\sqrt{20}\sqrt{27}} = \frac{3}{\sqrt{15}}$

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(c) $\left(\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\overrightarrow{BC}\cdot\overrightarrow{BC}}\right)\overrightarrow{BC} = \frac{18}{20}\overrightarrow{BC} = \frac{9}{5}(\mathbf{j}+2\mathbf{k})$ where $\overrightarrow{BA} = \mathbf{i}-\mathbf{j}+5\mathbf{k}$ and $\overrightarrow{BC} = 2\mathbf{j}+4\mathbf{k}$

(d) area $= \left|(2\mathbf{j}+4\mathbf{k})\times(\mathbf{i}-\mathbf{j}+5\mathbf{k})\right| = \left|14\mathbf{i}+4\mathbf{j}-2\mathbf{k}\right| = 6\sqrt{6}$

(e) From part (d), $\mathbf{n} = 14\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ is normal to the plane $\Rightarrow 14(x-1)+4(y-0)-2(z+1)=0$ $\Rightarrow 7x+2y-z=8.$

(f) From part (d), $\mathbf{n} = 14\mathbf{i}+4\mathbf{j}-2\mathbf{k} \Rightarrow$ the area of the projection on the yz-plane is $|\mathbf{n}\cdot\mathbf{i}| = 14$; the area of the projection on the xz-plane is $|\mathbf{n}\cdot\mathbf{j}| = 4$; and the area of the projection on the xy-plane is $|\mathbf{n}\cdot\mathbf{k}| = 2$.

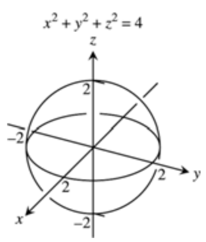
63. $\overrightarrow{AB} = -2\mathbf{i}+\mathbf{j}+\mathbf{k}, \overrightarrow{CD} = \mathbf{i}+4\mathbf{j}-\mathbf{k},$ and $\overrightarrow{AC} = 2\mathbf{i}+\mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 1 & 4 & -1 \end{vmatrix} = -5\mathbf{i}-\mathbf{j}-9\mathbf{k} \Rightarrow$ the distance is

$d = \left|\frac{(2\mathbf{i}+\mathbf{j})\cdot(-5\mathbf{i}-\mathbf{j}-9\mathbf{k})}{\sqrt{25+1+81}}\right| = \frac{11}{\sqrt{107}}$

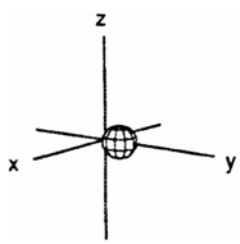
64. $\overrightarrow{AB} = -2\mathbf{i}+4\mathbf{j}-\mathbf{k}, \overrightarrow{CD} = \mathbf{i}-\mathbf{j}+2\mathbf{k},$ and $\overrightarrow{AC} = -3\mathbf{i}+3\mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 7\mathbf{i}+3\mathbf{j}-2\mathbf{k} \Rightarrow$ the distance is

$d = \left|\frac{(-3\mathbf{i}+3\mathbf{j})\cdot(7\mathbf{i}+3\mathbf{j}-2\mathbf{k})}{\sqrt{49+9+4}}\right| = \frac{12}{\sqrt{62}}$

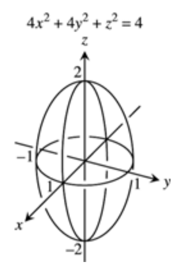
65. $x^2+y^2+z^2=4$



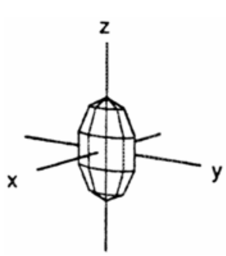
66. $x^2+(y-1)^2+z^2=1$



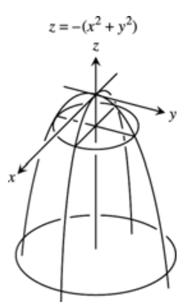
67. $4x^2+4y^2+z^2=4$



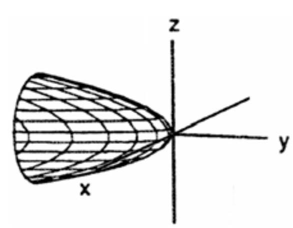
68. $36x^2+9y^2+4z^2=36$



69. $z=-\left(x^2+y^2\right)$



70. $y=-\left(x^2+z^2\right)$



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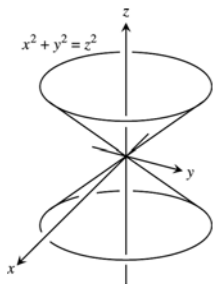
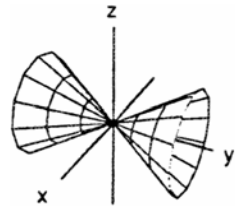
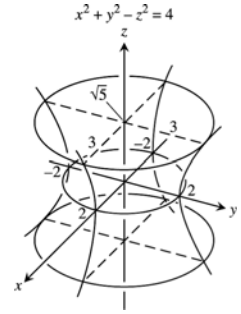
71. $x^2 + y^2 = z^2$



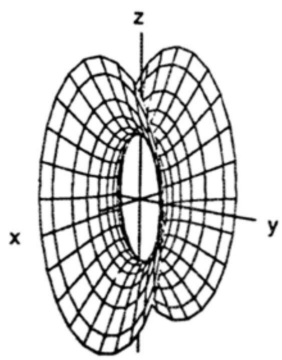
72. $x^2 + z^2 = y^2$



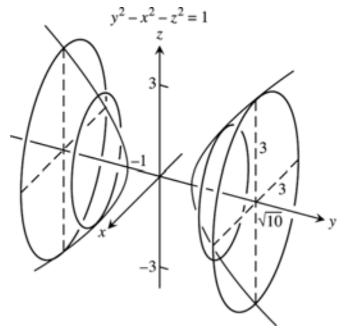
73. $x^2 - y^2 - z^2 = 4$



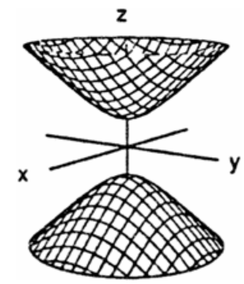
74. $4y^2 + z^2 - 4x^2 = 4$



75. $y^2 - x^2 - z^2 = 1$



76. $z^2 - x^2 - y^2 = 1$



CHAPTER 12 ADDITIONAL AND ADVANCED EXERCISES

1. Information from ship A indicates the submarine is now on the line $L_1: x = 4 + 2t, y = 3t, z = -\frac{1}{3}t;$ information from ship B indicates the submarine is now on the line $L_2: x = 18s, y = 5 - 6s, z = -s.$ The current position of the sub is $\left(6, 3, -\frac{1}{3}\right)$ and occurs when the lines intersect at $t = 1$ and $s = \frac{1}{3}$. The straight line path of the submarine contains both point $P\left(2, -1, -\frac{1}{3}\right)$ and $Q\left(6, 3, -\frac{1}{3}\right)$; the line representing this path is $L: x = 2 + 4t, y = -1 + 4t, z = -\frac{1}{3}$. The submarine traveled the distance between P and Q in 4 minutes $\Rightarrow$ a speed of $\frac{|\overrightarrow{PQ}|}{4} = \frac{\sqrt{32}}{4} = \sqrt{2}$ thousand m/min. In 20 minutes the submarine will move $20\sqrt{2}$ thousand m from Q along the line L $\Rightarrow 20\sqrt{2} = \sqrt{(2 + 4t - 6)^2 + (-1 + 4t - 3)^2 + 0^2} \Rightarrow 800 = 16(t-1)^2 + 16(t-1)^2 = 32(t-1)^2$ $\Rightarrow (t-1)^2 = \frac{800}{32} = 25 \Rightarrow t = 6 \Rightarrow$ the submarine will be located at $\left(26, 23, -\frac{1}{3}\right)$ in 20 minutes.

2. H_2 stops its flight when $6 + 110t = 446 \Rightarrow t = 4$ hours. After 6 hours, H_1 is at $P(246, 57, 9)$ while H_2 is at $(446, 13, 0)$. The distance between P and Q is $\sqrt{(246 - 446)^2 + (57 - 13)^2 + (9 - 0)^2} \approx 204.98$ kilometers. At 150 km/h, it would take about 1.37 hours for H_1 to reach H_2.

3. Torque $= \left|\overrightarrow{PQ} \times \mathbf{F}\right| \Rightarrow 20.4\,\mathrm{N \cdot m} = \left|\overrightarrow{PQ}\right| |\mathbf{F}| \sin\frac{\pi}{2} = 0.23\mathrm{m} \cdot |\mathbf{F}| \Rightarrow |\mathbf{F}| = 88.7\ \mathrm{N}$

4. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ be the vector from O to A and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ be the vector from O to B. The vector $\mathbf{v}$ orthogonal to $\mathbf{a}$ and $\mathbf{b} \Rightarrow \mathbf{v}$ is parallel to $\mathbf{b} \times \mathbf{a}$ (since the rotation is clockwise). Now $\mathbf{b} \times \mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$;

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$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow (2, 2, 2)$ is the center of the circular path $(1, 3, 2)$ takes $\Rightarrow$ radius $= \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2} \Rightarrow$ arc length per second covered by the point is $\frac{3}{2}\sqrt{2}$ units/s $= |\mathbf{v}|$ (velocity is constant). A unit vector in the direction of $\mathbf{v}$ is $\frac{\mathbf{b}\times\mathbf{a}}{|\mathbf{b}\times\mathbf{a}|} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k}$ $\Rightarrow \mathbf{v} = |\mathbf{v}|\left(\frac{\mathbf{b}\times\mathbf{a}}{|\mathbf{b}\times\mathbf{a}|}\right) = \frac{3}{2}\sqrt{2}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k}\right) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \sqrt{3}\mathbf{k}$

5. (a) By the Law of Cosines we have $\cos\alpha = \frac{3^2+5^2-4^2}{2(3)(5)} = \frac{3}{5}$ and $\cos\beta = \frac{4^2+5^2-3^2}{2(4)(5)} = \frac{4}{5} \Rightarrow \sin\alpha = \frac{4}{5}$ and $\sin\beta = \frac{3}{5}$ $\Rightarrow \mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos\alpha, |\mathbf{F}_1|\sin\alpha\rangle = \left\langle -\frac{3}{5}|\mathbf{F}_1|, \frac{4}{5}|\mathbf{F}_1|\right\rangle$, $\mathbf{F}_2 = \langle |\mathbf{F}_2|\cos\beta, |\mathbf{F}_2|\sin\beta\rangle = \left\langle \frac{4}{5}|\mathbf{F}_2|, \frac{3}{5}|\mathbf{F}_2|\right\rangle$, and $\mathbf{w} = \langle 0, -100\rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 100\rangle \Rightarrow \left\langle -\frac{3}{5}|\mathbf{F}_1| + \frac{4}{5}|\mathbf{F}_2|, \frac{4}{5}|\mathbf{F}_1| + \frac{3}{5}|\mathbf{F}_2|\right\rangle = \langle 0, 100\rangle \Rightarrow -\frac{3}{5}|\mathbf{F}_1| + \frac{4}{5}|\mathbf{F}_2| = 0$ and $\frac{4}{5}|\mathbf{F}_1| + \frac{3}{5}|\mathbf{F}_2| = 100$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = \frac{3}{4}|\mathbf{F}_1|$. Substituting this result into the second equation gives us: $\frac{4}{5}|\mathbf{F}_1| + \frac{9}{20}|\mathbf{F}_1| = 100 \Rightarrow |\mathbf{F}_1| = 80$ N $\Rightarrow |\mathbf{F}_2| = 60$ N $\Rightarrow \mathbf{F}_1 = \langle -48, 64\rangle$ and $\mathbf{F}_2 = \langle 48, 36\rangle$, and $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$ and $\beta = \tan^{-1}\left(\frac{3}{4}\right)$

(b) By the Law of Cosines we have $\cos\alpha = \frac{5^2+13^2-12^2}{2(5)(13)} = \frac{5}{13}$ and $\cos\beta = \frac{12^2+13^2-5^2}{2(12)(13)} = \frac{12}{13} \Rightarrow \sin\alpha = \frac{12}{13}$ and $\sin\beta = \frac{5}{13} \Rightarrow \mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos\alpha, |\mathbf{F}_1|\sin\alpha\rangle = \left\langle -\frac{5}{13}|\mathbf{F}_1|, \frac{12}{13}|\mathbf{F}_1|\right\rangle$, $\mathbf{F}_2 = \langle |\mathbf{F}_2|\cos\beta, |\mathbf{F}_2|\sin\beta\rangle$ $= \left\langle \frac{12}{13}|\mathbf{F}_2|, \frac{5}{13}|\mathbf{F}_2|\right\rangle$, and $\mathbf{w} = \langle 0, -200\rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 200\rangle \Rightarrow \left\langle -\frac{5}{13}|\mathbf{F}_1| + \frac{12}{13}|\mathbf{F}_2|, \frac{12}{13}|\mathbf{F}_1| + \frac{5}{13}|\mathbf{F}_2|\right\rangle$ $= \langle 0, 200\rangle \Rightarrow -\frac{5}{13}|\mathbf{F}_1| + \frac{12}{13}|\mathbf{F}_2| = 0$ and $\frac{12}{13}|\mathbf{F}_1| + \frac{5}{13}|\mathbf{F}_2| = 200$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = \frac{5}{12}|\mathbf{F}_1|$. Substituting this result into the second equation gives us: $\frac{12}{13}|\mathbf{F}_1| + \frac{25}{156}|\mathbf{F}_1| = 200$ $\Rightarrow |\mathbf{F}_1| = \frac{2400}{13} \approx 184.615$ N $\Rightarrow |\mathbf{F}_2| = \frac{1000}{13} \approx 76.923$ N $\Rightarrow \mathbf{F}_1 = \left\langle -\frac{12000}{1169}, \frac{28800}{1169}\right\rangle \approx \langle -71.006, 170.414\rangle$ and $\mathbf{F}_2 = \left\langle \frac{12000}{1169}, \frac{5000}{1169}\right\rangle \approx \langle 71.006, 29.586\rangle$.

6. (a) $\mathbf{T}_1 = \langle -|\mathbf{T}_1|\cos\alpha, |\mathbf{T}_1|\sin\alpha\rangle$, $\mathbf{T}_2 = \langle |\mathbf{T}_2|\cos\beta, |\mathbf{T}_2|\sin\beta\rangle$, and $\mathbf{w} = \langle 0, -w\rangle$. Since $\mathbf{T}_1 + \mathbf{T}_2 = \langle 0, w\rangle$ $\Rightarrow \langle -|\mathbf{T}_1|\cos\alpha + |\mathbf{T}_2|\cos\beta, |\mathbf{T}_1|\sin\alpha + |\mathbf{T}_2|\sin\beta\rangle = \langle 0, w\rangle \Rightarrow -|\mathbf{T}_1|\cos\alpha + |\mathbf{T}_2|\cos\beta = 0$ and $|\mathbf{T}_1|\sin\alpha + |\mathbf{T}_2|\sin\beta = w$. Solving the first equation for $|\mathbf{T}_2|$ results in: $|\mathbf{T}_2| = \frac{\cos\alpha}{\cos\beta}|\mathbf{T}_1|$. Substituting this result into the second equation gives us: $|\mathbf{T}_1|\sin\alpha + \frac{\cos\alpha\sin\beta}{\cos\beta}|\mathbf{T}_1| = w$ $\Rightarrow |\mathbf{T}_1| = \frac{w\cos\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta} = \frac{w\cos\beta}{\sin(\alpha+\beta)}$ and $|\mathbf{T}_2| = \frac{w\cos\beta}{\sin(\alpha+\beta)}$

(b) $\frac{d}{d\alpha}(|\mathbf{T}_1|) = \frac{d}{d\alpha}\left(\frac{w\cos\beta}{\sin(\alpha+\beta)}\right) = \frac{-w\cos\beta\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}$; $\frac{d}{d\alpha}(|\mathbf{T}_1|) = 0 \Rightarrow -w\cos\beta\cos(\alpha+\beta) = 0 \Rightarrow \cos(\alpha+\beta) = 0$ $\Rightarrow \alpha + \beta = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{2} - \beta$; $\frac{d^2}{d\alpha^2}(|\mathbf{T}_1|) = \frac{d}{d\alpha}\left(\frac{-w\cos\beta\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}\right) = \frac{w\cos\beta\left(\cos^2(\alpha+\beta)+1\right)}{\sin^3(\alpha+\beta)}$; $\left.\frac{d^2}{d\alpha^2}(|\mathbf{T}_1|)\right|_{\alpha=\frac{\pi}{2}-\beta} = w\cos\beta > 0 \Rightarrow$ local minimum when $\alpha = \frac{\pi}{2} - \beta$

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(c) $\frac{d}{d\beta}(|\mathbf{T}_2|) = \frac{d}{d\beta}\left(\frac{w\cos\alpha}{\sin(\alpha+\beta)}\right) = \frac{-w\cos\alpha\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}$; $\frac{d}{d\beta}(|\mathbf{T}_2|) = 0 \Rightarrow -w\cos\alpha\cos(\alpha+\beta) = 0$

$\Rightarrow \cos(\alpha+\beta) = 0 \Rightarrow \alpha+\beta = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{2} - \alpha$; $\frac{d^2}{d\beta^2}(|\mathbf{T}_2|) = \frac{d}{d\beta}\left(\frac{-w\cos\alpha\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}\right) = \frac{w\cos\alpha\left(\cos^2(\alpha+\beta)+1\right)}{\sin^3(\alpha+\beta)}$;

$\frac{d^2}{d\alpha^2}(|\mathbf{T}_2|)\Big|_{\beta=\frac{\pi}{2}-\alpha} = w\cos\alpha > 0 \Rightarrow$ local minimum when $\beta = \frac{\pi}{2} - \alpha$

7. (a) If $P(x, y, z)$ is a point in the plane determined by the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$, then the vectors $\overrightarrow{PP_1}, \overrightarrow{PP_2}$ and $\overrightarrow{PP_3}$ all lie in the plane. Thus $\overrightarrow{PP_1}\cdot\left(\overrightarrow{PP_2}\times\overrightarrow{PP_3}\right) = 0$

$\Rightarrow \begin{vmatrix} x_1-x & y_1-y & z_1-z \\ x_2-x & y_2-y & z_2-z \\ x_3-x & y_3-y & z_3-z \end{vmatrix} = 0$ by the determinant formula for the triple scalar product in Section 12.4.

(b) Subtract row 1 from rows 2, 3, and 4 and evaluate the resulting determinant (which has the same value as the given determinant) by cofactor expansion about column 4. This expansion is exactly the determinant in part (a) so we have all points $P(x, y, z)$ in the plane determined by $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$.

8. Let $L_1: x = a_1s+b_1, y = a_2s+b_2, z = a_3s+b_3$ and $L_2: x = c_1t+d_1, y = c_2t+d_2, z = c_3t+d_3$. If $L_1 \parallel L_2$, then for some k, $a_i = kc_i, i = 1, 2, 3$ and the determinant $\begin{vmatrix} a_1 & c_1 & b_1-d_1 \\ a_2 & c_2 & b_2-d_2 \\ a_3 & c_3 & b_3-d_3 \end{vmatrix} = \begin{vmatrix} kc_1 & c_1 & b_1-d_1 \\ kc_2 & c_2 & b_2-d_2 \\ kc_3 & c_3 & b_3-d_3 \end{vmatrix} = 0$, since the first column is a multiple of the second column. The lines L_1 and L_2 intersect if and only if the system

$\begin{cases} a_1s - c_1t + (b_1-d_1) = 0 \\ a_2s - c_2t + (b_2-d_2) = 0 \\ a_3s - c_3t + (b_3-d_3) = 0 \end{cases}$ has a nontrivial solution $\Leftrightarrow$ the determinant of the coefficients is zero.

9. (a) Place the tetrahedron so that A is at $(0, 0, 0)$, the point P is on the y-axis, and ΔABC lies in the xy-plane. Since ΔABC is an equilateral triangle, all the angles in the triangle are $60°$ and since AP bisects $BC \Rightarrow \Delta ABP$ is a $30°-60°-90°$ triangle. Thus the coordinates of P are $\left(0, \sqrt{3}, 0\right)$, the coordinates of B are $\left(1, \sqrt{3}, 0\right)$, and the coordinates of C are $\left(-1, \sqrt{3}, 0\right)$. Let the coordinates of D be given by (a, b, c). Since all of the faces are equilateral triangles $\Rightarrow$ all the angles in each of the triangles are $60° \Rightarrow \cos(\angle DAB) = \cos(60°) = \frac{\overrightarrow{AD}\cdot\overrightarrow{AB}}{|\overrightarrow{AD}||\overrightarrow{AB}|} = \frac{a+b\sqrt{3}}{(2)(2)} = \frac{1}{2} \Rightarrow a+b\sqrt{3} = 2$ and $\cos(\angle DAC) = \cos(60°)$

$= \frac{\overrightarrow{AD}\cdot\overrightarrow{AC}}{|\overrightarrow{AD}||\overrightarrow{AC}|} = \frac{-a+b\sqrt{3}}{(2)(2)} = \frac{1}{2} \Rightarrow -a+b\sqrt{3} = 2$. Add the two equations to obtain: $2b\sqrt{3} = 4 \Rightarrow b = \frac{2}{\sqrt{3}}$.

Substituting this value for b in the first equation gives us: $a + \left(\frac{2}{\sqrt{3}}\right)\sqrt{3} = 2 \Rightarrow a = 0$. Since

$|\overrightarrow{AD}| = \sqrt{a^2+b^2+c^2} = 2 \Rightarrow 0^2 + \left(\frac{2}{\sqrt{3}}\right)^2 + c^2 = 4 \Rightarrow c = \frac{2\sqrt{2}}{\sqrt{3}}$. Thus the coordinates of D are $\left(0, \frac{2}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}\right)$.

$\cos\theta = \cos(\angle DAP) = \frac{\overrightarrow{AD}\cdot\overrightarrow{AP}}{|\overrightarrow{AD}||\overrightarrow{AP}|} = \frac{2}{2\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \Rightarrow 57.74°$

(b) Since ΔABC lies in the xy-plane $\Rightarrow$ the normal to the face given by ΔABC is $\mathbf{n}_1 = \mathbf{k}$. The face given by ΔBCD is an adjacent face. The vectors $\overrightarrow{DB} = \mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{2\sqrt{2}}{\sqrt{3}}\mathbf{k}$ and $\overrightarrow{DC} = -\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{2\sqrt{2}}{\sqrt{3}}\mathbf{k}$ both lie in the

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plane containing ΔBCD. The normal to this plane is given by $\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \frac{1}{\sqrt{3}} & -\frac{2\sqrt{2}}{\sqrt{3}} \\ -1 & \frac{1}{\sqrt{3}} & -\frac{2\sqrt{2}}{\sqrt{3}} \end{vmatrix} = \frac{4\sqrt{2}}{\sqrt{3}}\mathbf{j} + \frac{2}{\sqrt{3}}\mathbf{k}.$

The angle θ between two adjacent faces is given by $\cos\theta = \cos(\angle DAP) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{2/\sqrt{3}}{(1)\left(6/\sqrt{3}\right)}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.53°.$

10. Extend $\overrightarrow{CD}$ to $\overrightarrow{CG}$ so that $\overrightarrow{CD} = \overrightarrow{DG}$. Then $\overrightarrow{CG} = t\overrightarrow{CF} = \overrightarrow{CB} + \overrightarrow{BG}$ and $t\overrightarrow{CF} = 3\overrightarrow{CE} + \overrightarrow{CA}$, since $ACBG$ is a parallelogram. If $t\overrightarrow{CF} - 3\overrightarrow{CE} = \overrightarrow{CA} = \mathbf{0}$, then $t - 3 - 1 = 0 \Rightarrow t = 4$, since F, E, and A are collinear. Therefore, $\overrightarrow{CG} = 4\overrightarrow{CF} \Rightarrow \overrightarrow{CD} = 2\overrightarrow{CF} \Rightarrow F$ is the midpoint of $\overline{CD}$.

11. If $Q(x, y)$ is a point on the line $ax + by = c$, then $\overrightarrow{P_1Q} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j}$, and $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is normal to the line. The distance is $\left|\text{proj}_{\mathbf{n}} \overrightarrow{P_1Q}\right| = \left|\frac{\left[(x-x_1)\mathbf{i}+(y-y_1)\mathbf{j}\right]\cdot(a\mathbf{i}+b\mathbf{j})}{\sqrt{a^2+b^2}}\right| = \frac{|a(x-x_1)+b(y-y_1)|}{\sqrt{a^2+b^2}} = \frac{|ax_1+by_1-c|}{\sqrt{a^2+b^2}}$, since $c = ax + by$.

12. (a) Let $Q(x, y, z)$ be any point on $Ax + By + Cz - D = 0$. Let $\overrightarrow{QP_1} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}$, and $\mathbf{n} = \frac{A\mathbf{i}+B\mathbf{j}+C\mathbf{k}}{\sqrt{A^2+B^2+C^2}}$. The distance is $\left|\text{proj}_{\mathbf{n}} \overrightarrow{QP_1}\right| = \left|\left[(x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}\right]\cdot\left(\frac{A\mathbf{i}+B\mathbf{j}+C\mathbf{k}}{\sqrt{A^2+B^2+C^2}}\right)\right|$
$= \frac{|Ax_1+By_1+Cz_1-(Ax+By+Cz)|}{\sqrt{A^2+B^2+C^2}} = \frac{|Ax_1+By_1+Cz_1-D|}{\sqrt{A^2+B^2+C^2}}.$

(b) Since both tangent planes are parallel, one-half of the distance between them is equal to the radius of the sphere, i. e., $r = \frac{1}{2}\frac{|3-9|}{\sqrt{1+1+1}} = \sqrt{3}$ (see also Exercise 12a). Clearly, the points $(1, 2, 3)$ and $(-1, -2, -3)$ are on the line containing the sphere's center. Hence, the line containing the center is $x = 1 + 2t$, $y = 2 + 4t$, $z = 3 + 6t$. The distance from the plane $x + y + z - 3 = 0$ to the center is $\sqrt{3}$ $\Rightarrow \frac{|(1+2t)+(2+4t)+(3+6t)-3|}{\sqrt{1+1+1}} = \sqrt{3}$ from part (a) $\Rightarrow t = 0 \Rightarrow$ the center is at $(1, 2, 3)$. Therefore an equation of the sphere is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 3$.

13. (a) If (x_1, y_1, z_1) is on the plane $Ax + By + Cz = D_1$, then the distance d between the planes is $d = \frac{|Ax_1+By_1+Cz_1-D_2|}{\sqrt{A^2+B^2+C^2}} = \frac{|D_1-D_2|}{|A\mathbf{i}+B\mathbf{j}+C\mathbf{k}|}$, since $Ax_1 + By_1 + Cz_1 = D_1$, by Exercise 12(a).

(b) $d = \frac{|12-6|}{\sqrt{4+9+1}} = \frac{6}{\sqrt{14}}$

(c) $\frac{|2(3)+(-1)(2)+2(-1)+4|}{\sqrt{14}} = \frac{|2(3)+(-1)(2)+2(-1)-D|}{\sqrt{14}} \Rightarrow D = 8 \text{ or } -4 \Rightarrow$ the desired plane is $2x - y + 2x = 8$

(d) Choose the point $(2, 0, 1)$ on the plane. Then $\frac{|3-D|}{\sqrt{6}} = 5 \Rightarrow D = 3 \pm 5\sqrt{6} \Rightarrow$ the desired planes are $x - 2y + z = 3 + 5\sqrt{6}$ and $x - 2y + z = 3 - 5\sqrt{6}$.

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14. Let $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{BC}$ and $D(x, y, z)$ be any point in the plane determined by A, B and C. Then the point D lies in this plane if and only if $\overrightarrow{AD} \cdot \mathbf{n} = 0 \Leftrightarrow \overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{BC}\right) = 0.$

15. $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the plane $x + 2y + 6z = 6$; $\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix} = 4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ is parallel to the plane and perpendicular to the plane of $\mathbf{v}$ and $\mathbf{n}$ $\Rightarrow \mathbf{w} = \mathbf{n} \times (\mathbf{v} \times \mathbf{n}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 6 \\ 4 & -5 & 1 \end{vmatrix} = 32\mathbf{i} + 23\mathbf{j} - 13\mathbf{k}$ is a vector parallel to the plane $x + 2y + 6z = 6$ in the direction of the projection vector $\operatorname{proj}_P \mathbf{v}$. Therefore,

$\operatorname{proj}_P \mathbf{v} = \operatorname{proj}_{\mathbf{w}} \mathbf{v} = \left(\mathbf{v} \cdot \frac{\mathbf{w}}{|\mathbf{w}|}\right)\frac{\mathbf{w}}{|\mathbf{w}|} = \left(\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\right)\mathbf{w} = \left(\frac{32+23-13}{32^2+23^2+13^2}\right)\mathbf{w} = \frac{42}{1722}\mathbf{w} = \frac{1}{41}\mathbf{w} = \frac{32}{41}\mathbf{i} + \frac{23}{41}\mathbf{j} - \frac{13}{41}\mathbf{k}$

16. $\operatorname{proj}_{\mathbf{z}} \mathbf{w} = -\operatorname{proj}_{\mathbf{z}} \mathbf{v}$ and $\mathbf{w} - \operatorname{proj}_{\mathbf{z}} \mathbf{w} = \mathbf{v} - \operatorname{proj}_{\mathbf{z}} \mathbf{v} \Rightarrow \mathbf{w} = (\mathbf{w} - \operatorname{proj}_{\mathbf{z}} \mathbf{w}) + \operatorname{proj}_{\mathbf{z}} \mathbf{w} = (\mathbf{v} - \operatorname{proj}_{\mathbf{z}} \mathbf{v}) + \operatorname{proj}_{\mathbf{z}} \mathbf{w}$
$= \mathbf{v} - 2\operatorname{proj}_{\mathbf{z}} \mathbf{v} = \mathbf{v} - 2\left(\frac{\mathbf{v}\cdot\mathbf{z}}{|\mathbf{z}|^2}\right)\mathbf{z}$

17. (a) $\mathbf{u} \times \mathbf{v} = 2\mathbf{i} \times 2\mathbf{j} = 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}; (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u} = 0\mathbf{v} - 0\mathbf{u} = \mathbf{0}$; $\mathbf{v} \times \mathbf{w} = 4\mathbf{i} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$;
$(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} = 0\mathbf{v} - 0\mathbf{w} = \mathbf{0}$

(b) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ -1 & 2 & -1 \end{vmatrix} = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$;

$(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - 2(\mathbf{i} - \mathbf{j} + \mathbf{k}) = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$;

$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & 4 & 5 \end{vmatrix} = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$;

$(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - (-1)(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$

(c) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -4 \\ 1 & 0 & 2 \end{vmatrix} = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;

$(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 4(2\mathbf{i} + \mathbf{j}) = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;

$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ -2 & -3 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$;

$(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 3(\mathbf{i} + 2\mathbf{k}) = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$

(d) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 2 & 4 & -2 \end{vmatrix} = -10\mathbf{i} - 10\mathbf{k}$;

$(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u} = 10(-\mathbf{i} - \mathbf{k}) - 0(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = -10\mathbf{i} - 10\mathbf{k}$;

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$$\mathbf{v}\times\mathbf{w}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 0 & -1\\ 2 & 4 & -2\end{vmatrix}=4\mathbf{i}-4\mathbf{j}-4\mathbf{k}\Rightarrow \mathbf{u}\times(\mathbf{v}\times\mathbf{w})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & -2\\ 4 & -4 & -4\end{vmatrix}=-12\mathbf{i}-4\mathbf{j}-8\mathbf{k};$$

$$(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}=10(-\mathbf{i}-\mathbf{k})-1(2\mathbf{i}+4\mathbf{j}-2\mathbf{k})=-12\mathbf{i}-4\mathbf{j}-8\mathbf{k}$$

18. (a) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})+\mathbf{v}\times(\mathbf{w}\times\mathbf{u})+\mathbf{w}\times(\mathbf{u}\times\mathbf{v})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}+(\mathbf{v}\cdot\mathbf{u})\mathbf{w}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}+(\mathbf{w}\cdot\mathbf{v})\mathbf{u}-(\mathbf{w}\cdot\mathbf{u})\mathbf{v}=\mathbf{0}$

(b) $[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{i})]\mathbf{i}+[(\mathbf{u}\cdot(\mathbf{v}\times\mathbf{j})]\mathbf{j}+[(\mathbf{u}\cdot(\mathbf{v}\times\mathbf{k})]\mathbf{k}=[(\mathbf{u}\times\mathbf{v})\cdot\mathbf{i}]\mathbf{i}+[(\mathbf{u}\times\mathbf{v})\cdot\mathbf{j}]\mathbf{j}+[(\mathbf{u}\times\mathbf{v})\cdot\mathbf{k}]\mathbf{k}=\mathbf{u}\times\mathbf{v}$

(c) $(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{r})=\mathbf{u}\cdot[\mathbf{v}\times(\mathbf{w}\times\mathbf{r})]=\mathbf{u}\cdot[(\mathbf{v}\cdot\mathbf{r})\mathbf{w}-(\mathbf{v}\cdot\mathbf{w})\mathbf{r}]=(\mathbf{u}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{r})-(\mathbf{u}\cdot\mathbf{r})(\mathbf{v}\cdot\mathbf{w})=\begin{vmatrix}\mathbf{u}\cdot\mathbf{w} & \mathbf{v}\cdot\mathbf{w}\\ \mathbf{u}\cdot\mathbf{r} & \mathbf{v}\cdot\mathbf{r}\end{vmatrix}$

19. The formula is always true; $\mathbf{u}\times[\mathbf{u}\times(\mathbf{u}\times\mathbf{v})]\cdot\mathbf{w}=\mathbf{u}\times[(\mathbf{u}\cdot\mathbf{v})\mathbf{u}-(\mathbf{u}\cdot\mathbf{u})\mathbf{v}]\cdot\mathbf{w}=[(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\times\mathbf{u}-(\mathbf{u}\cdot\mathbf{u})\mathbf{u}\times\mathbf{v}]\cdot\mathbf{w}$
$=-|\mathbf{u}|^2\,\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}=-|\mathbf{u}|^2\,\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$

20. If $\mathbf{u}=(\cos B)\mathbf{i}+(\sin B)\mathbf{j}$ and $\mathbf{v}=(\cos A)\mathbf{i}+(\sin A)\mathbf{j}$, where $A>B$, then $\mathbf{u}\times\mathbf{v}=[|\mathbf{u}||\mathbf{v}|\sin(A-B)]\mathbf{k}$
$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos B & \sin B & 0\\ \cos A & \sin A & 0\end{vmatrix}=(\cos B\sin A-\sin B\cos A)\mathbf{k}\Rightarrow\sin(A-B)=\cos B\sin A-\sin B\cos A,$ since
$|\mathbf{u}|=1$ and $|\mathbf{v}|=1$.

21. If $\mathbf{u}=a\mathbf{i}+b\mathbf{j}$ and $\mathbf{v}=c\mathbf{i}+d\mathbf{j}$, then $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos\theta\Rightarrow ac+bd=\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos\theta$
$\Rightarrow(ac+bd)^2=(a^2+b^2)(c^2+d^2)\cos^2\theta\Rightarrow(ac+bd)^2\le(a^2+b^2)(c^2+d^2)$, since $\cos^2\theta\le 1$.

22. If $\mathbf{u}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, then $\mathbf{u}\cdot\mathbf{u}=a^2+b^2+c^2\ge 0$ and $\mathbf{u}\cdot\mathbf{u}=0$ iff $a=b=c=0$.

23. $|\mathbf{u}+\mathbf{v}|^2=(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})=\mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v}\le|\mathbf{u}|^2+2|\mathbf{u}||\mathbf{v}|+|\mathbf{v}|^2=(|\mathbf{u}|+|\mathbf{v}|)^2\Rightarrow|\mathbf{u}+\mathbf{v}|\le|\mathbf{u}|+|\mathbf{v}|$

24. Let α denote the angle between **w** and **u**, and β the angle between **w** and **v**. Let $a=|\mathbf{u}|$ and $b=|\mathbf{v}|$.
Then $\cos\alpha=\frac{\mathbf{w}\cdot\mathbf{u}}{|\mathbf{w}||\mathbf{u}|}=\frac{(a\mathbf{v}+b\mathbf{u})\cdot\mathbf{u}}{|\mathbf{w}||\mathbf{u}|}=\frac{(a\mathbf{v}\cdot\mathbf{u}+b\mathbf{u}\cdot\mathbf{u})}{|\mathbf{w}||\mathbf{u}|}=\frac{(a\mathbf{v}\cdot\mathbf{u}+b\mathbf{u}\cdot\mathbf{u})}{|\mathbf{w}||\mathbf{u}|}=\frac{(a\mathbf{v}\cdot\mathbf{u}+ba^2)}{|\mathbf{w}|a}=\frac{\mathbf{v}\cdot\mathbf{u}+ba}{|\mathbf{w}|}$, and likewise, $\cos\beta=\frac{\mathbf{u}\cdot\mathbf{v}+ba}{|\mathbf{w}|}$.
Since the angle between **u** and **v** is always $\le\frac{\pi}{2}$ and $\cos\alpha=\cos\beta$, we have that $\alpha=\beta\Rightarrow\mathbf{w}$ bisects the angle between **u** and **v**.

25. $(|\mathbf{u}|\mathbf{v}+|\mathbf{v}|\mathbf{u})\cdot(|\mathbf{v}|\mathbf{u}-|\mathbf{u}|\mathbf{v})=|\mathbf{u}|\,\mathbf{v}\cdot|\mathbf{v}|\,\mathbf{u}+|\mathbf{v}|\,\mathbf{u}\cdot|\mathbf{v}|\,\mathbf{u}-|\mathbf{u}|\,\mathbf{v}\cdot|\mathbf{u}|\,\mathbf{v}-|\mathbf{v}|\,\mathbf{u}\cdot|\mathbf{u}|\,\mathbf{v}$
$=|\mathbf{v}|\,\mathbf{u}\cdot|\mathbf{u}|\,\mathbf{v}+|\mathbf{v}|^2\,\mathbf{u}\cdot\mathbf{u}-|\mathbf{u}|^2\,\mathbf{v}\cdot\mathbf{v}-|\mathbf{v}|\,\mathbf{u}\cdot|\mathbf{u}|\,\mathbf{v}=|\mathbf{v}|^2|\mathbf{u}|^2-|\mathbf{u}|^2|\mathbf{v}|^2=0$

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