MAT1001 Final Examination

Tuesday, December 20, 2022

Time: 6:30 - 9:30 PM

**Notes and Instructions**

*1. No books, no notes, no dictionaries, and no calculators.*

*2. The total score of this examination is* ***148****.*

*3. There are* ***15*** *questions (with parts) in total.*

*4. The symbol [****N****] at the beginning of a question indicates that the question is worth N points.*

*5. Answer all questions on the* ***answer book****.*

*6. Show your intermediate steps* ***except Questions 1, 2, and 3*** *— answers without intermediate steps will receive minimal (or even no) marks.*

*7. The symbols* arcsin, arctan*, etc., denote the inverse trigonometric functions* $sin^{-1}$, $tan^{-1}$, *etc. .*

**MAT1001 Final Examination Questions**

1. [**12**] True or False (in general)? No explanation is required.

For parts (i) to (iv), the functions $f(x)$ and $g(x)$ are assumed to be positive for sufficiently large $x$, and we consider $x \to \infty$.

(i) If $f = o(g)$, then $f = O(g)$.

(ii) If $f$ and $g$ grow at the same rate, then $f = O(g)$ and $g = O(f)$.

(iii) The function $\log_2$ grows faster than $\log_{10}$.

(iv) $\ln(\ln(x)) = O(\ln(x))$.

(v) If $f$ and $g$ are both increasing and differentiable on $(-\infty, \infty)$, then the composite function $f \circ g$ is also increasing and differentiable on $(-\infty, \infty)$.

(vi) The function $f(x) = x - \sin x$ is increasing on $(-\infty, \infty)$.

2. [**15**] Short Questions. No explanation is required.

(i) The equation $x^3 - 3y + 1 = e^y$ determines $y$ as a function of $x$ implicitly. Find $dy/dx$ at $x = 0$. (Note that $(x, y) = (0, 0)$ satisfies the equation.)

(ii) Fill in the blank: $\operatorname{arccot}'(x) =$ ________.

(iii) Find $dy/dx$, where $y = x^{\sqrt{x}}$.

(iv) Find $f'(x)$, where $f(x) = \displaystyle\int_{\cos(x)}^{1} \sqrt{1 - t^2}\, dt$.

(v) Decompose the following fraction into partial fractions:

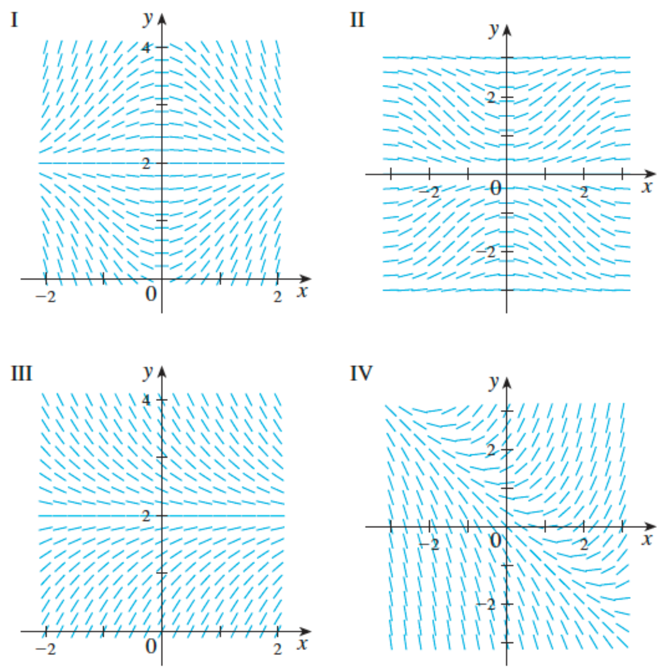
$$\frac{1}{(x-1)^2(x-2)}.$$

$$\| $$

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

3. [**4**] Match the following differential equations (1) to (4) with their slope fields (I) to (IV). No justification is required

(1) $y' = \sin x \sin y$  (2) $y' = 2 - y$  (3) $y' = x(2 - y)$  (4) $y' = x + y - 1$




4. [**4**] Find an approximated value for arctan(1.01) by using standard linear approximation centred at $x = 1$. State your final answer in three decimal places (e.g., $e \approx 2.718$).

5. [**6+3**] Consider the integral

$$\int_0^{1.2} e^{x^2}\, dx$$

and supposing we use the Trapezoidal Rule and Simpson's Rule to approximate it with the number of intervals $n = 6$.

(i) Write down the expressions of the two approximated values ($T$ and $S$) in the form of sums. No simplification is required.

(ii) Will the Trapezoidal Rule over-estimate or under-estimate the true value of the integral? Briefly explain why.

6. [**3+3**] Consider the function $f(x) = 2x^3 - 3x^2 + 4$ defined on $D = [1, \infty)$.

(i) Prove that $f^{-1}$ exists by showing that $f$ is injective (one-to-one).

(ii) Find $(f^{-1})'(4)$.

7. [**6**] Find the area enclosed by the curves $y = 2x^2 + x - 9$ and $y = x - 1$.

8. [**16**] Evaluate the following limits, or explain why they do not exist.

(i) $\lim\limits_{x \to 0^+} (1 + \sin(4x))^{\cot x}$

(ii) $\lim\limits_{n \to \infty} \left( \dfrac{n}{n^2 + 1^2} + \dfrac{n}{n^2 + 2^2} + \ldots + \dfrac{n}{n^2 + (n-1)^2} + \dfrac{n}{n^2 + n^2} \right)$

(iii) $\lim\limits_{x \to \infty} \left( \dfrac{a^x - 1}{(a-1)x} \right)^{\frac{1}{x}}$, where $a \in (0, 1)$ is a constant.

(iv) $\lim\limits_{x \to 0} \dfrac{x - \displaystyle\int_0^x e^{t^2}\, dt}{x^2 \sin(2x)}$

9. [**24**] Find the following integrals.

(i) $\displaystyle\int \frac{e^{1/x}}{x^2}\, dx$

(ii) $\displaystyle\int_0^{1/2} \frac{x^2}{\sqrt{1 - x^2}}\, dx$

(iii) $\displaystyle\int_0^1 \arccos x \, dx$

(iv) $\displaystyle\int_{-3}^1 \frac{x + 8}{x^2 + 6x + 13}\, dx$

(v) $\displaystyle\int_0^{\pi/4} \tan^5(x)\, dx$

(vi) $\displaystyle\int_0^\infty \frac{1 + x^2}{1 + x^4}\, dx \qquad \Big( \textit{hint}\text{: consider } u = x - \tfrac{1}{x}. \Big)$

10. [**6**] Find the solution to the initial value problem

$$xy' = y + x^2 \sin x, \quad x > 0, \quad y(\pi) = 0.$$

11. [**6**] You have a certain area $A$ of material to make a cylindrical can of radius $r$ and height $h$. The material of area $A$ includes the top, the bottom and the side of the cylinder. Find the values of $r$ and $h$ so that the volume of the cylinder is maximized.

12. [**6**] Find the length of the curve given by

$$y = \int_0^x \sqrt{\cos(2t)}\, dt, \quad 0 \le x \le \frac{\pi}{2}.$$

13. [**6+5+4**] The curve given by

$$y = \arcsin\left(\frac{x}{a}\right), \quad 0 \le x \le a,$$

is revolved around the $y$-axis to form an open bowl facing up. Starting from time $t = 0$, water is let in from the top of the bowl at a volume rate of $r(t)$ given by

$$r(t) = Ae^{-\frac{t}{k}}, \quad t \ge 0.$$

Here, $A$, $a$, and $k$ are all positive constants.

(i) Find the volume of the bowl.

(ii) For a given pair of values of $A$ and $k$, find the value of $a$ so that the bowl will become full (but not overflow) as time $t$ goes to infinity. Use this value of $a$ in part (iii) below.

(iii) Find the time point $t_0$ when the water level in the bowl is at $y = \pi/4$.

14. [**2+5**] Consider a plate formed by the region enclosed by the curves

$$y = \ln(bx), \quad b > 0,$$

the vertical line $x = a$, where $a > 0$, and the horizontal $x$-axis. It is immersed in water with a depth of $H$, i.e., the water surface is at $y = H$. The weight density of the water is $w$. Assume that $ab = e$, where $e$ is Euler's number.

(i) Find the smallest value $L$ of $H$ so that the plate is totally immersed in the water.

(ii) Compute the force $F$ exerted against one side of the plate when it is completely immersed in the water with the depth equal to the smallest value $L$ above.

15. [**12**] For each of the following improper integrals, determine whether it converges or diverges.

(i) $\displaystyle\int_2^\infty \frac{e^{-x}}{x}\,dx$

(ii) $\displaystyle\int_1^2 \frac{x}{x^3-1}\,dx$

(iii) $\displaystyle\int_0^{\pi/2} \tan^2(\theta)\,d\theta$