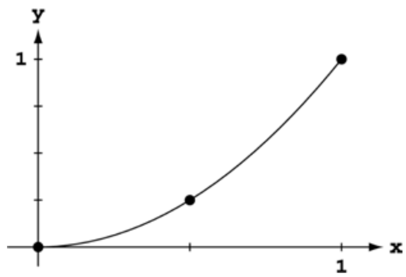
CHAPTER 5 INTEGRALS

5.1 AREA AND ESTIMATING WITH FINITE SUMS

1. $f(x) = x^2$



Since f is increasing on $[0, 1]$, we use left endpoints to obtain lower sums and right endpoints to obtain upper sums.

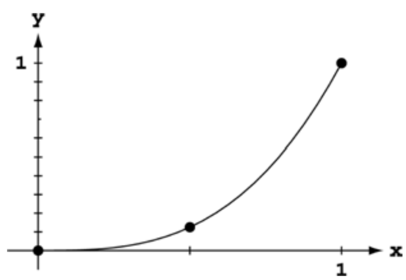
(a) $\Delta x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\Delta x = \frac{i}{2} \Rightarrow$ a lower sum is $\sum_{i=0}^{1}\left(\frac{i}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{2}\left(0^2 + \left(\frac{1}{2}\right)^2\right) = \frac{1}{8}$

(b) $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\Delta x = \frac{i}{4} \Rightarrow$ a lower sum is $\sum_{i=0}^{3}\left(\frac{i}{4}\right)^2 \cdot \frac{1}{4} = \frac{1}{4}\left(0^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2\right) = \frac{1}{4} \cdot \frac{7}{8} = \frac{7}{32}$

(c) $\Delta x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\Delta x = \frac{i}{2} \Rightarrow$ an upper sum is $\sum_{i=1}^{2}\left(\frac{i}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + 1^2\right) = \frac{5}{8}$

(d) $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\Delta x = \frac{i}{4} \Rightarrow$ an upper sum is $\sum_{i=1}^{4}\left(\frac{i}{4}\right)^2 \cdot \frac{1}{4} = \frac{1}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2\right) = \frac{1}{4} \cdot \left(\frac{30}{16}\right) = \frac{15}{32}$

2. $f(x) = x^3$



Since f is increasing on $[0, 1]$, we use left endpoints to obtain lower sums and right endpoints to obtain upper sums.

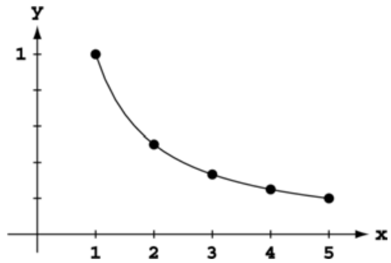
(a) $\Delta x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\Delta x = \frac{i}{2} \Rightarrow$ a lower sum is $\sum_{i=0}^{1}\left(\frac{i}{2}\right)^3 \cdot \frac{1}{2} = \frac{1}{2}\left(0^3 + \left(\frac{1}{2}\right)^3\right) = \frac{1}{16}$

(b) $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\Delta x = \frac{i}{4} \Rightarrow$ a lower sum is $\sum_{i=0}^{3}\left(\frac{i}{4}\right)^3 \cdot \frac{1}{4} = \frac{1}{4}\left(0^3 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3\right) = \frac{36}{256} = \frac{9}{64}$

(c) $\Delta x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\Delta x = \frac{i}{2} \Rightarrow$ an upper sum is $\sum_{i=1}^{2}\left(\frac{i}{2}\right)^3 \cdot \frac{1}{2} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^3 + 1^3\right) = \frac{1}{2} \cdot \frac{9}{8} = \frac{9}{16}$

(d) $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\Delta x = \frac{i}{4} \Rightarrow$ an upper sum is $\sum_{i=1}^{4}\left(\frac{i}{4}\right)^3 \cdot \frac{1}{4} = \frac{1}{4}\left(\left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3 + 1^3\right) = \frac{100}{256} = \frac{25}{64}$

3. $f(x) = \frac{1}{x}$



Since f is decreasing on $[1, 5]$, we use left endpoints to obtain upper sums and right endpoints to obtain lower sums.

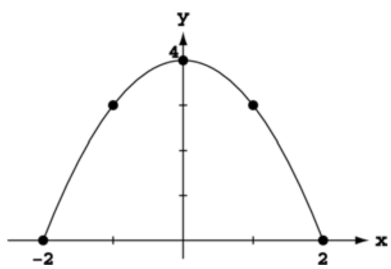
(a) $\Delta x = \frac{5-1}{2} = 2$ and $x_i = 1 + i\Delta x = 1 + 2i \Rightarrow$ a lower sum is $\sum_{i=1}^{2} \frac{1}{x_i} \cdot 2 = 2\left(\frac{1}{3} + \frac{1}{5}\right) = \frac{16}{15}$

(b) $\Delta x = \frac{5-1}{4} = 1$ and $x_i = 1 + i\Delta x = 1 + i \Rightarrow$ a lower sum is $\sum_{i=1}^{4} \frac{1}{x_i} \cdot 1 = 1\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{77}{60}$

(c) $\Delta x = \frac{5-1}{2} = 2$ and $x_i = 1 + i\Delta x = 1 + 2i \Rightarrow$ an upper sum is $\sum_{i=0}^{1} \frac{1}{x_i} \cdot 2 = 2\left(1 + \frac{1}{3}\right) = \frac{8}{3}$

(d) $\Delta x = \frac{5-1}{4} = 1$ and $x_i = 1 + i\Delta x = 1 + i \Rightarrow$ an upper sum is $\sum_{i=0}^{3} \frac{1}{x_i} \cdot 1 = 1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{25}{12}$

4. $f(x) = 4 - x^2$



Since f is increasing on $[-2, 0]$ and decreasing on $[0, 2]$, we use left endpoints on $[-2, 0]$ and right endpoints on $[0, 2]$ to obtain lower sums and use right endpoints on $[-2, 0]$ and left endpoints on $[0, 2]$ to obtain upper sums.

(a) $\Delta x = \frac{2-(-2)}{2} = 2$ and $x_i = -2 + i\Delta x = -2 + 2i \Rightarrow$ a lower sum is $2 \cdot (4 - (-2)^2) + 2 \cdot (4 - 2^2) = 0$

(b) $\Delta x = \frac{2-(-2)}{4} = 1$ and $x_i = -2 + i\Delta x = -2 + i \Rightarrow$ a lower sum is $\sum_{i=0}^{1} (4 - (x_i)^2) \cdot 1 + \sum_{i=3}^{4} (4 - (x_i)^2) \cdot 1$

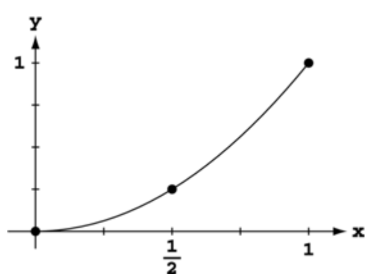
$= 1((4 - (-2)^2) + (4 - (-1)^2) + (4 - 1^2) + (4 - 2^2)) = 6$

(c) $\Delta x = \frac{2-(-2)}{2} = 2$ and $x_i = -2 + i\Delta x = -2 + 2i \Rightarrow$ an upper sum is $2 \cdot (4 - (0)^2) + 2 \cdot (4 - 0^2) = 16$

(d) $\Delta x = \frac{2-(-2)}{4} = 1$ and $x_i = -2 + i\Delta x = -2 + i \Rightarrow$ an upper sum is $\sum_{i=1}^{2} (4 - (x_i)^2) \cdot 1 + \sum_{i=2}^{3} (4 - (x_i)^2) \cdot 1$

$= 1((4 - (-1)^2) + (4 - 0^2) + (4 - 0^2) + (4 - 1^2)) = 14$

5. $f(x) = x^2$



Using 2 rectangles $\Rightarrow \Delta x = \frac{1-0}{2} = \frac{1}{2}$

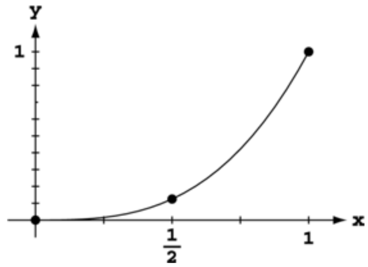
$\Rightarrow \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right) = \frac{1}{2}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right) = \frac{10}{32} = \frac{5}{16}$

Using 4 rectangles $\Rightarrow \Delta x = \frac{1-0}{4} = \frac{1}{4}$

$\Rightarrow \frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right)$

$= \frac{1}{4}\left(\left(\frac{1}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{5}{8}\right)^2 + \left(\frac{7}{8}\right)^2\right) = \frac{21}{64}$

6. $f(x) = x^3$



Using 2 rectangles $\Rightarrow \Delta x = \frac{1-0}{2} = \frac{1}{2}$

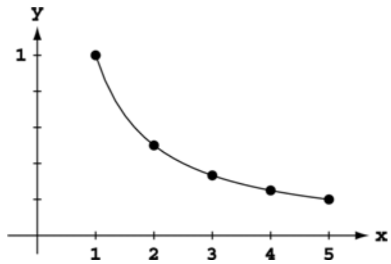
$\Rightarrow \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right) = \frac{1}{2}\left(\left(\frac{1}{4}\right)^3 + \left(\frac{3}{4}\right)^3\right) = \frac{28}{2\cdot 64} = \frac{7}{32}$

Using 4 rectangles $\Rightarrow \Delta x = \frac{1-0}{4} = \frac{1}{4}$

$\Rightarrow \frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right)$

$= \frac{1}{4}\left(\frac{1^3+3^3+5^3+7^3}{8^3}\right) = \frac{496}{4\cdot 8^3} = \frac{124}{8^3} = \frac{31}{128}$

7. $f(x) = \frac{1}{x}$



Using 2 rectangles $\Rightarrow \Delta x = \frac{5-1}{2} = 2 \Rightarrow 2(f(2) + f(4))$

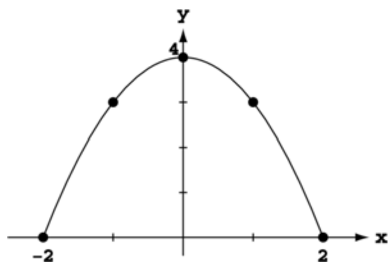
$= 2\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{3}{2}$

Using 4 rectangles $\Rightarrow \Delta \text{x} = \frac{5-1}{4} = 1$

$\Rightarrow 1\left(f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right) + f\left(\frac{9}{2}\right)\right) = 1\left(\frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9}\right)$

$= \frac{1488}{3\cdot 5\cdot 7\cdot 9} = \frac{496}{5\cdot 7\cdot 9} = \frac{496}{315}$

8. $f(x) = 4 - x^2$



Using 2 rectangles $\Rightarrow \Delta x = \frac{2-(-2)}{2} = 2$

$\Rightarrow 2(f(-1) + f(1)) = 2(3+3) = 12$

Using 4 rectangles $\Rightarrow \Delta x = \frac{2-(-2)}{4} = 1$

$\Rightarrow 1\left(f\left(-\frac{3}{2}\right) + f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right)\right)$

$= 1\left(\left(4 - \left(-\frac{3}{2}\right)^2\right) + \left(4 - \left(-\frac{1}{2}\right)^2\right) + \left(4 - \left(\frac{1}{2}\right)^2\right) + \left(4 - \left(\frac{3}{2}\right)^2\right)\right)$

$= 16 - \left(\frac{9}{4}\cdot 2 + \frac{1}{4}\cdot 2\right) = 16 - \frac{10}{2} = 11$

9. (a) $D \approx (0)(1) + (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) = 87$ centimeters

(b) $D \approx (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) + (0)(1) = 87$ centimeters

10. (a) $D \approx (1)(300) + (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300)$ $+ (1.0)(300) + (1.8)(300) + (1.5)(300) + (1.2)(300) = 5220$ meters (NOTE: 5 minutes = 300 seconds)

(b) $D \approx (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300) + (1.0)(300)$ $+ (1.8)(300) + (1.5)(300) + (1.2)(300) + (0)(300) = 4920$ meters (NOTE: 5 minutes = 300 seconds)

11. (a) $D \approx (0)(10) + (15)(10) + (5)(10) + (12)(10) + (10)(10) + (15)(10) + (12)(10) + (5)(10) + (7)(10)$ $+ (12)(10) + (15)(10) + (10)(10) = 1180 \text{ m} = 1.18 \text{ km}$

(b) $D \approx (15)(10) + (5)(10) + (12)(10) + (10)(10) + (15)(10) + (12)(10) + (5)(10) + (7)(10) + (12)(10)$ $+ (15)(10) + (10)(10) + (12)(10) = 1300 \text{ m} = 1.3 \text{ km}$

12. (a) The distance traveled will be the area under the curve. We will use the approximate velocities at the midpoints of each time interval to approximate this area using rectangles. Thus,

$D \approx (32)(0.001) + (82)(0.001) + (116)(0.001) + (143)(0.001) + (164)(0.001) + (181)(0.001) + (194)(0.001)$ $+ (207)(0.001) + (216)(0.001) + (224)(0.001) \approx 1.56 \text{ km}$

(b) Roughly, after 0.0063 hours, the car would have gone 0.78 km, where 0.0060 hours = 22.7 s. At 22.7 s, the velocity was approximately 192 km/h.

13. (a) Because the acceleration is decreasing, an upper estimate is obtained using left endpoints in summing acceleration $\cdot\Delta t$. Thus, $\Delta t = 1$ and speed $\approx [9.8 + 5.944 + 3.605 + 2.187 + 1.326](1) = 22.862$ m/s
(b) Using right endpoints we obtain a lower estimate: speed $\approx [5.944 + 3.605 + 2.187 + 1.326 + 0.805] =$ 13.867 m/s
(c) Upper estimates for the speed at each second are:

t	0	1	2	3	4	5
v	0	9.8	15.744	19.349	21.536	22.862

Thus, the distance fallen when $t = 3$ seconds is $s \approx [9.8 + 15.744 + 19.349](1) = 44.893$ m.

14. (a) The speed is a decreasing function of time $\Rightarrow$ right endpoints give a lower estimate for the height (distance) attained. Also

t	0	1	2	3	4	5
v	122.5	112.7	102.9	93.1	83.3	73.5

gives the time-velocity table by subtracting the constant $g = 9.8$ from the speed at each time increment $\Delta t = 1\,\text{s}$. Thus, the speed ≈ 73.5 m/s after 5 seconds.
(b) A lower estimate for height attained is $h \approx [112.7 + 102.9 + 93.1 + 83.3 + 73.5](1) = 465.5$ m.

15. Partition $[0, 2]$ into the four subintervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of these subintervals are $m_1 = 0.25$, $m_2 = 0.75$, $m_3 = 1.25$, and $m_4 = 1.75$. The heights of the four approximating rectangles are $f(m_1) = (0.25)^3 = \frac{1}{64}$, $f(m_2) = (0.75)^3 = \frac{27}{64}$, $f(m_3) = (1.25)^3 = \frac{125}{64}$, and $f(m_4) = (1.75)^3 = \frac{343}{64}$
Notice that the average value is approximated by $\frac{1}{2}\left[\left(\frac{1}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{7}{4}\right)^3\left(\frac{1}{2}\right)\right] = \frac{31}{16}$
$= \frac{1}{\text{length of } [0,2]} \cdot \begin{bmatrix} \text{approximate area under} \\ \text{curve } f(x) = x^3 \end{bmatrix}$. We use this observation in solving the next several exercises.

16. Partition $[1,9]$ into the four subintervals $[1, 3]$, $[3, 5]$, $[5, 7]$, and $[7, 9]$. The midpoints of these subintervals are $m_1 = 2$, $m_2 = 4$, $m_3 = 6$, and $m_4 = 8$. The heights of the four approximating rectangles are $f(m_1) = \frac{1}{2}$, $f(m_2) = \frac{1}{4}$, $f(m_3) = \frac{1}{6}$, and $f(m_4) = \frac{1}{8}$. The width of each rectangle is $\Delta x = 2$. Thus,
Area $\approx 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 2\left(\frac{1}{6}\right) + 2\left(\frac{1}{8}\right) = \frac{25}{12} \Rightarrow$ average value $\approx \frac{\text{area}}{\text{length of } [1,9]} = \frac{\left(\frac{25}{12}\right)}{8} = \frac{25}{96}$.

17. Partition $[0, 2]$ into the four subintervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of the subintervals are $m_1 = 0.25$, $m_2 = 0.75$, $m_3 = 1.25$, and $m_4 = 1.75$. The heights of the four approximating rectangles are $f(m_1) = \frac{1}{2} + \sin^2\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$, $f(m_2) = \frac{1}{2} + \sin^2\frac{3\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$, $f(m_3) = \frac{1}{2} + \sin^2\frac{5\pi}{4}$
$= \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$, and $f(m_4) = \frac{1}{2} + \sin^2\frac{7\pi}{4} = \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$. The width of each rectangle is $\Delta x = \frac{1}{2}$.
Thus, Area $\approx (1 + 1 + 1 + 1)\left(\frac{1}{2}\right) = 2 \Rightarrow$ average value $\approx \frac{\text{area}}{\text{length of } [0, 2]} = \frac{2}{2} = 1$.

18. Partition $[0, 4]$ into the four subintervals $[0, 1]$, $[1, 2]$, $[2, 3]$, and $[3, 4]$. The midpoints of the subintervals are $m_1 = \frac{1}{2}$, $m_2 = \frac{3}{2}$, $m_3 = \frac{5}{2}$, and $m_4 = \frac{7}{2}$. The heights of the four approximating rectangles are

$f(m_1)=1-\left(\cos\left(\frac{\pi\left(\frac{1}{2}\right)}{4}\right)\right)^4=1-\left(\cos\left(\frac{\pi}{8}\right)\right)^4=0.27145$ (to 5 decimal places), $f(m_2)=1-\left(\cos\left(\frac{\pi\left(\frac{3}{2}\right)}{4}\right)\right)^4$ $=1-\left(\cos\left(\frac{3\pi}{8}\right)\right)^4=0.97855$, $f(m_3)=1-\left(\cos\left(\frac{\pi\left(\frac{5}{2}\right)}{4}\right)\right)^4=1-\left(\cos\left(\frac{5\pi}{8}\right)\right)^4=0.97855$, and $f(m_4)=1-\left(\cos\left(\frac{\pi\left(\frac{7}{2}\right)}{4}\right)\right)^4=1-\left(\cos\left(\frac{7\pi}{8}\right)\right)^4=0.27145$. The width of each rectangle is $\Delta x=1$. Thus, Area $\approx(0.27145)(1)+(0.97855)(1)+(0.97855)(1)+(0.27145)(1)=2.5\Rightarrow$ average value $\approx\frac{\text{area}}{\text{length of }[0,4]}=\frac{2.5}{4}=\frac{5}{8}$

19. Since the leakage is increasing, an upper estimate uses right endpoints and a lower estimate uses left endpoints:
   (a) upper estimate $=(70)(1)+(97)(1)+(136)(1)+(190)(1)+(265)(1)=758$ liters,
   lower estimate $=(50)(1)+(70)(1)+(97)(1)+(136)(1)+(190)(1)=543$ liters.
   (b) upper estimate $=(70+97+136+190+265+369+516+720)=2363$ liters,
   lower estimate $=(50+70+97+136+190+265+369+516)=1693$ liters.
   (c) worst case: $2363+720t=25{,}000\Rightarrow t\approx 31.4$ hours;
   best case: $1693+720t=25{,}000\Rightarrow t\approx 32.4$ hours

20. Since the pollutant release increases over time, an upper estimate uses right endpoints and a lower estimate uses left endpoints;
   (a) upper estimate $=(0.2)(30)+(0.25)(30)+(0.27)(30)+(0.34)(30)+(0.45)(30)+(0.52)(30)=60.9$ tons
   lower estimate $=(0.05)(30)+(0.2)(30)+(0.25)(30)+(0.27)(30)+(0.34)(30)+(0.45)(30)=46.8$ tons
   (b) Using the lower (best case) estimate: $46.8+(0.52)(30)+(0.63)(30)+(0.70)(30)+(0.81)(30)=126.6$ tons, so near the end of September 125 tons of pollutants will have been released.

21. (a) The diagonal of the square has length 2, so the side length is $\sqrt{2}$. Area $=\left(\sqrt{2}\right)^2=2$
   (b) Think of the octagon as a collection of 16 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{16}=\frac{\pi}{8}$.
   Area $=16\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right)=4\sin\frac{\pi}{4}=2\sqrt{2}\approx 2.828$
   (c) Think of the 16-gon as a collection of 32 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{32}=\frac{\pi}{16}$.
   Area /
   (d) Each area is less than the area of the circle, π. As n increase, the area approaches π.

22. (a) Each of the isosceles triangles is made up of two right triangles having hypotenuse 1 and an acute angle measuring $\frac{2\pi}{2n}=\frac{\pi}{n}$ The area of each isosceles triangle is $A_T=2\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{n}\right)\left(\cos\frac{\pi}{n}\right)=\frac{1}{2}\sin\frac{2\pi}{n}$.
   (b) The area of the polygon is $A_P=nA_T=\frac{n}{2}\sin\frac{2\pi}{n}$, so $\lim\limits_{n\to\infty}\frac{n}{2}\sin\frac{2\pi}{n}=\lim\limits_{n\to\infty}\pi\cdot\frac{\sin\frac{2\pi}{n}}{\left(\frac{2\pi}{n}\right)}=\pi$
   (c) Multiply each area by r^2.
   $A_T=\frac{1}{2}r^2\sin\frac{2\pi}{n}$
   $A_P=\frac{n}{2}r^2\sin\frac{2\pi}{n}$
   $\lim\limits_{n\to\infty}A_P=\pi r^2$

23-26. Example CAS commands:

Maple:

```
with( Student[Calculus 1] );
f := x -> sin(x);
a := 0;
b := Pi;
Plot( f(x), x=a..b, title="#23(a) (Section 5.1)" );
N := [ 100, 200, 1000 ];                                    # (b)
for n in N do
  Xlist := [ a+1.*(b-a)/n*i $ i=0..n ];
  Ylist := map( f, Xlist );
end do:
for n in N do                                               # (c)
  Avg[n] := evalf(add(y,y=Ylist)/nops(Ylist));
end do;
avg := FunctionAverage( f(x), x=a..b, output=value );
evalf( avg );
FunctionAverage(f(x),x=a..b, output=plot);                  # (d)
fsolve( f(x)=avg, x=0.5 );
fsolve( f(x)=avg, x=2.5 );
fsolve( f(x)=Avg[1000], x=0.5 );
fsolve( f(x)=Avg[1000], x=2.5 );
```

Mathematica: (assigned function and values for a and b may vary):
Symbols for π, →, powers, roots, fractions, etc. are available in Palettes.
Never insert a space between the name of a function and its argument.

```
Clear[x]
f[x_]:=x Sin[1/x]
{a, b}={π/4, π}
Plot[f[x],{x, a, b}]
```

The following code computes the value of the function for each interval midpoint and then finds the average. Each sequence of commands for a different value of n (number of subdivisions) should be placed in a separate cell.

```
n =100; dx = (b – a) /n;
values = Table[N[f[x]],{x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
n =200; dx = (b – a) /n;
values = Table[N[f[x]],{x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
n =1000; dx = (b – a) /n;
values=Table[N[f[x]],{x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
FindRoot[f[x] == average,{x, a}]
```

5.2 SIGMA NOTATION AND LIMITS OF FINITE SUMS

1. $\sum_{k=1}^{2} \frac{6k}{k+1} = \frac{6(1)}{1+1} + \frac{6(2)}{2+1} = \frac{6}{2} + \frac{12}{3} = 7$

2. $\sum_{k=1}^{3} \frac{k-1}{k} = \frac{1-1}{1} + \frac{2-1}{2} + \frac{3-1}{3} = 0 + \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

3. $\sum_{k=1}^{4} \cos k\pi = \cos(1\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) = -1 + 1 - 1 + 1 = 0$

4. $\sum_{k=1}^{5} \sin k\pi = \sin(1\pi) + \sin(2\pi) + \sin(3\pi) + \sin(4\pi) + \sin(5\pi) = 0 + 0 + 0 + 0 + 0 = 0$

5. $\sum_{k=1}^{3} (-1)^{k+1} \sin\frac{\pi}{k} = (-1)^{1+1} \sin\frac{\pi}{1} + (-1)^{2+1} \sin\frac{\pi}{2} + (-1)^{3+1} \sin\frac{\pi}{3} = 0 - 1 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-2}{2}$

6. $\sum_{k=1}^{4} (-1)^k \cos k\pi = (-1)^1 \cos(1\pi) + (-1)^2 \cos(2\pi) + (-1)^3 \cos(3\pi) + (-1)^4 \cos(4\pi) = -(-1) + 1 - (-1) + 1 = 4$

7. (a) $\sum_{k=1}^{6} 2^{k-1} = 2^{1-1} + 2^{2-1} + 2^{3-1} + 2^{4-1} + 2^{5-1} + 2^{6-1} = 1 + 2 + 4 + 8 + 16 + 32$

(b) $\sum_{k=0}^{5} 2^k = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 1 + 2 + 4 + 8 + 16 + 32$

(c) $\sum_{k=1}^{4} 2^{k+1} = 2^{-1+1} + 2^{0+1} + 2^{1+1} + 2^{2+1} + 2^{3+1} + 2^{4+1} = 1 + 2 + 4 + 8 + 16 + 32$

All of them represent $1 + 2 + 4 + 8 + 16 + 32$

8. (a) $\sum_{k=1}^{6} (-2)^{k-1} = (-2)^{1-1} + (-2)^{2-1} + (-2)^{3-1} + (-2)^{4-1} + (-2)^{5-1} + (-2)^{6-1} = 1 - 2 + 4 - 8 + 16 - 32$

(b) $\sum_{k=0}^{5} (-1)^k 2^k = (-1)^0 2^0 + (-1)^1 2^1 + (-1)^2 2^2 + (-1)^3 2^3 + (-1)^4 2^4 + (-1)^5 2^5 = 1 - 2 + 4 - 8 + 16 - 32$

(c) $\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2} = (-1)^{-2+1} 2^{-2+2} + (-1)^{-1+1} 2^{-1+2} + (-1)^{0+1} 2^{0+2} + (-1)^{1+1} 2^{1+2} + (-1)^{2+1} 2^{2+2} + (-1)^{3+1} 2^{3+2}$
$= -1 + 2 - 4 + 8 - 16 + 32;$

(a) and (b) represent $1 - 2 + 4 - 8 + 16 - 32$; (c) is not equivalent to the other two

9. (a) $\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1} = \frac{(-1)^{2-1}}{2-1} + \frac{(-1)^{3-1}}{3-1} + \frac{(-1)^{4-1}}{4-1} = -1 + \frac{1}{2} - \frac{1}{3}$

(b) $\sum_{k=0}^{2} \frac{(-1)^k}{k+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} = 1 - \frac{1}{2} + \frac{1}{3}$

(c) $\sum_{k=-1}^{1} \frac{(-1)^k}{k+2} = \frac{(-1)^{-1}}{-1+2} + \frac{(-1)^0}{0+2} + \frac{(-1)^1}{1+2} = -1 + \frac{1}{2} - \frac{1}{3}$

(a) and (c) are equivalent; (b) is not equivalent to the other two.

10. (a) $\sum_{k=1}^{4} (k-1)^2 = (1-1)^2 + (2-1)^2 + (3-1)^2 + (4-1)^2 = 0 + 1 + 4 + 9$

(b) $\sum_{k=-1}^{3} (k+1)^2 = (-1+1)^2 + (0+1)^2 + (1+1)^2 + (2+1)^2 + (3+1)^2 = 0 + 1 + 4 + 9 + 16$

(c) $\sum_{k=-3}^{-1} k^2 = (-3)^2 + (-2)^2 + (-1)^2 = 9 + 4 + 1$

(a) and (c) are equivalent to each other; (b) is not equivalent to the other two.

11. $\sum_{k=1}^{6} k$

12. $\sum_{k=1}^{4} k^2$

13. $\sum_{k=1}^{4} \frac{1}{2^k}$

14. $\sum_{k=1}^{5} 2k$

15. $\sum_{k=1}^{5} (-1)^{k+1} \frac{1}{k}$

16. $\sum_{k=1}^{5} (-1)^k \frac{k}{5}$

17. (a) $\sum_{k=1}^{n} 3a_k = 3\sum_{k=1}^{n} a_k = 3(-5) = -15$

(b) $\sum_{k=1}^{n} \frac{b_k}{6} = \frac{1}{6}\sum_{k=1}^{n} b_k = \frac{1}{6}(6) = 1$

(c) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = -5 + 6 = 1$

(d) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k = -5 - 6 = -11$

(e) $\sum_{k=1}^{n} (b_k - 2a_k) = \sum_{k=1}^{n} b_k - 2\sum_{k=1}^{n} a_k = 6 - 2(-5) = 16$

18. (a) $\sum_{k=1}^{n} 8a_k = 8\sum_{k=1}^{n} a_k = 8(0) = 0$

(b) $\sum_{k=1}^{n} 250b_k = 250\sum_{k=1}^{n} b_k = 250(1) = 250$

(c) $\sum_{k=1}^{n} (a_k + 1) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} 1 = 0 + n = n$

(d) $\sum_{k=1}^{n} (b_k - 1) = \sum_{k=1}^{n} b_k - \sum_{k=1}^{n} 1 = 1 - n$

19. (a) $\sum_{k=1}^{10} k = \frac{10(10+1)}{2} = 55$

(b) $\sum_{k=1}^{10} k^2 = \frac{10(10+1)(2(10)+1)}{6} = 385$

(c) $\sum_{k=1}^{10} k^3 = \left[\frac{10(10+1)}{2}\right]^2 = 55^2 = 3025$

20. (a) $\sum_{k=1}^{13} k = \frac{13(13+1)}{2} = 91$

(b) $\sum_{k=1}^{13} k^2 = \frac{13(13+1)(2(13)+1)}{6} = 819$

(c) $\sum_{k=1}^{13} k^3 = \left[\frac{13(13+1)}{2}\right]^2 = 91^2 = 8281$

21. $\sum_{k=1}^{7} -2k = -2\sum_{k=1}^{7} k = -2\left(\frac{7(7+1)}{2}\right) = -56$

22. $\sum_{k=1}^{5} \frac{\pi k}{15} = \frac{\pi}{15}\sum_{k=1}^{5} k = \frac{\pi}{15}\left(\frac{5(5+1)}{2}\right) = \pi$

23. $\sum_{k=1}^{6} (3 - k^2) = \sum_{k=1}^{6} 3 - \sum_{k=1}^{6} k^2 = 3(6) - \frac{6(6+1)(2(6)+1)}{6} = -73$

24. $\sum_{k=1}^{6} (k^2 - 5) = \sum_{k=1}^{6} k^2 - \sum_{k=1}^{6} 5 = \frac{6(6+1)(2(6)+1)}{6} - 5(6) = 61$

25. $\sum_{k=1}^{5} k(3k + 5) = \sum_{k=1}^{5} (3k^2 + 5k) = 3\sum_{k=1}^{5} k^2 + 5\sum_{k=1}^{5} k = 3\left(\frac{5(5+1)(2(5)+1)}{6}\right) + 5\left(\frac{5(5+1)}{2}\right) = 240$

26. $\sum_{k=1}^{7} k(2k + 1) = \sum_{k=1}^{7} (2k^2 + k) = 2\sum_{k=1}^{7} k^2 + \sum_{k=1}^{7} k = 2\left(\frac{7(7+1)(2(7)+1)}{6}\right) + \frac{7(7+1)}{2} = 308$

27. $\sum\limits_{k=1}^{5} \frac{k^3}{225} + \left(\sum\limits_{k=1}^{5} k\right)^3 = \frac{1}{225} \sum\limits_{k=1}^{5} k^3 + \left(\sum\limits_{k=1}^{5} k\right)^3 = \frac{1}{225}\left(\frac{5(5+1)}{2}\right)^2 + \left(\frac{5(5+1)}{2}\right)^3 = 3376$

28. $\left(\sum\limits_{k=1}^{7} k\right)^2 - \sum\limits_{k=1}^{7} \frac{k^3}{4} = \left(\sum\limits_{k=1}^{7} k\right)^2 - \frac{1}{4} \sum\limits_{k=1}^{7} k^3 = \left(\frac{7(7+1)}{2}\right)^2 - \frac{1}{4}\left(\frac{7(7+1)}{2}\right)^2 = 588$

29. (a) $\sum\limits_{k=1}^{7} 3 = 3(7) = 21$ (b) $\sum\limits_{k=1}^{500} 7 = 7(500) = 3500$

(c) Let $j = k-2 \Rightarrow k = j+2$; if $k = 3 \Rightarrow j = 1$ and if $k = 264 \Rightarrow j = 262 \Rightarrow \sum\limits_{k=3}^{264} 10 = \sum\limits_{j=1}^{262} 10 = 10(262) = 2620$

30. (a) Let $j = k-8 \Rightarrow k = j+8$; if $k = 9 \Rightarrow j = 1$ and if $k = 36 \Rightarrow j = 28 \Rightarrow \sum\limits_{k=9}^{36} k = \sum\limits_{j=1}^{28} (j+8) = \sum\limits_{j=1}^{28} j + \sum\limits_{j=1}^{28} 8$
$= \frac{28(28+1)}{2} + 8(28) = 630$

(b) Let $j = k-2 \Rightarrow k = j+2$; if $k = 3 \Rightarrow j = 1$ and if $k = 17 \Rightarrow j = 15 \Rightarrow \sum\limits_{k=3}^{17} k^2 = \sum\limits_{j=1}^{15} (j+2)^2$
$= \sum\limits_{j=1}^{15} (j^2 + 4j + 4) = \sum\limits_{j=1}^{15} j^2 + \sum\limits_{j=1}^{15} 4j + \sum\limits_{j=1}^{15} 4 = \frac{15(15+1)(2(15)+1)}{6} + 4 \cdot \frac{15(15+1)}{2} + 4(15) = 1240 + 480 + 60 = 1780$

(c) Let $j = k-17 \Rightarrow k = j+17$; if $k = 18 \Rightarrow j = 1$ and if $k = 71 \Rightarrow j = 54 \Rightarrow \sum\limits_{k=3}^{71} k(k-1)$
$= \sum\limits_{j=1}^{54} (j+17)((j+17)-1) = \sum\limits_{j=1}^{54} (j^2 + 33j + 272) = \sum\limits_{j=1}^{54} j^2 + \sum\limits_{j=1}^{54} 33j + \sum\limits_{j=1}^{54} 272$
$= \frac{54(54+1)(2(54)+1)}{6} + 33 \cdot \frac{54(54+1)}{2} + 272(54) = 53955 + 49005 + 14688 = 117648$

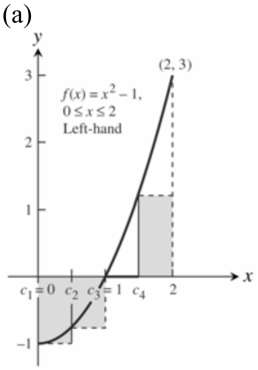
31. (a) $\sum\limits_{k=1}^{n} 4 = 4n$ (b) $\sum\limits_{k=1}^{n} c = cn$

(c) $\sum\limits_{k=1}^{n} (k-1) = \sum\limits_{k=1}^{n} k - \sum\limits_{k=1}^{n} 1 = \frac{n(n+1)}{2} - n = \frac{n^2-n}{2}$

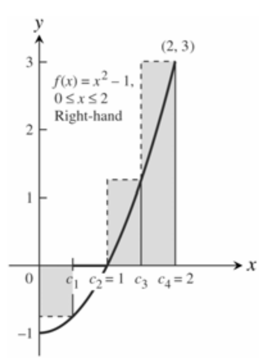
32. (a) $\sum\limits_{k=1}^{n} \left(\frac{1}{n} + 2n\right) = \left(\frac{1}{n} + 2n\right)n = 1 + 2n^2$ (b) $\sum\limits_{k=1}^{n} \frac{c}{n} = \frac{c}{n} \cdot n = c$

(c) $\sum\limits_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{n+1}{2n}$

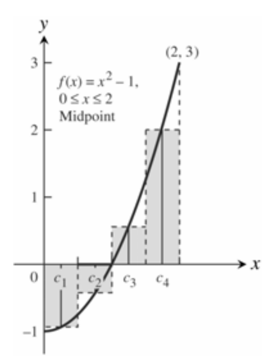
33. (a)



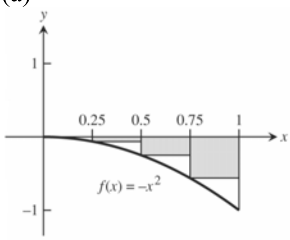
(b)



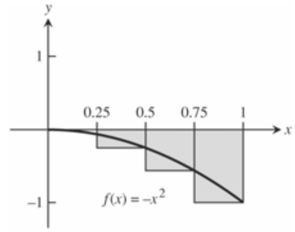
(c)



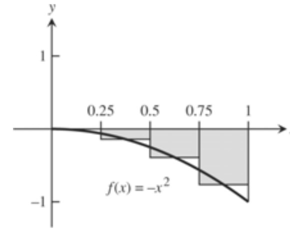
34. (a)



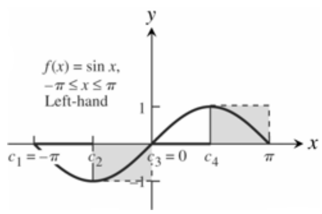
(b)



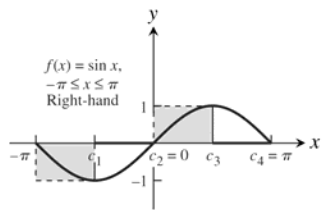
(c)



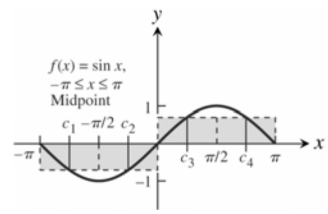
35. (a)



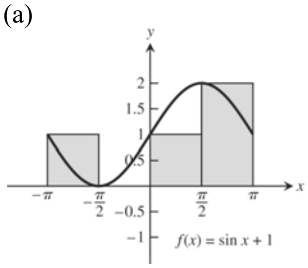
(b)



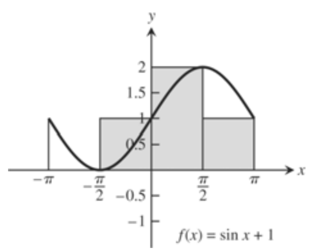
(c)



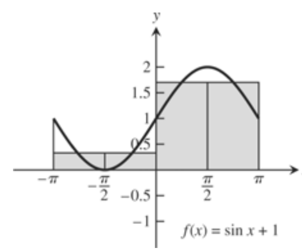
36. (a)



(b)



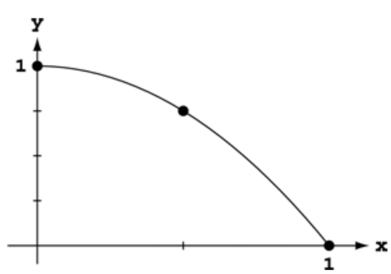
(c)



37. $|x_1 - x_0| = |1.2 - 0| = 1.2$, $|x_2 - x_1| = |1.5 - 1.2| = 0.3$, $|x_3 - x_2| = |2.3 - 1.5| = 0.8$, $|x_4 - x_3| = |2.6 - 2.3| = 0.3$, and $|x_5 - x_4| = |3 - 2.6| = 0.4$; the largest is $\|P\| = 1.2$.

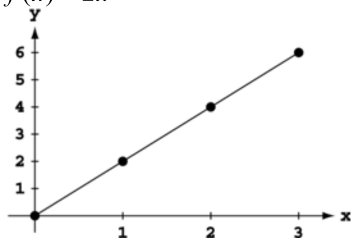
38. $|x_1 - x_0| = |-1.6 - (-2)| = 0.4$, $|x_2 - x_1| = |-0.5 - (-1.6)| = 1.1$, $|x_3 - x_2| = |0 - (-0.5)| = 0.5$, $|x_4 - x_3| = |0.8 - 0| = 0.8$, and $|x_5 - x_4| = |1 - 0.8| = 0.2$; the largest is $\|P\| = 1.1$.

39. $f(x) = 1 - x^2$



Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\Delta x = \frac{i}{n}$. The right-hand sum is $\sum_{i=1}^{n}\left(1 - c_i^2\right)\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \left(\frac{i}{n}\right)^2\right) = \frac{1}{n^3}\sum_{i=1}^{n}\left(n^2 - i^2\right)$
$= \frac{n^3}{n^3} - \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 1 - \frac{n(n+1)(2n+1)}{6n^3} = 1 - \frac{2n^3+3n^2+n}{6n^3}$
$= 1 - \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}$. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1 - c_i^2\right)\frac{1}{n}$
$= \lim_{n\to\infty}\left(1 - \frac{2-\frac{3}{n}+\frac{1}{n^2}}{6}\right) = 1 - \frac{1}{3} = \frac{2}{3}$

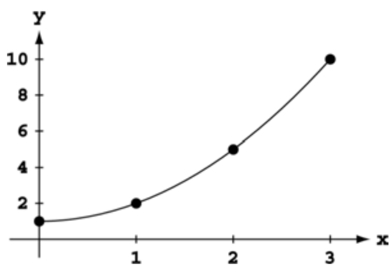
40. $f(x) = 2x$



Let $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and $c_i = i\Delta x = \frac{3i}{n}$. The right-hand sum is $\sum_{i=1}^{n} 2c_i\left(\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{6i}{n}\cdot\frac{3}{n} = \frac{18}{n^2}\sum_{i=1}^{n} i = \frac{18}{n^2}\cdot\frac{n(n+1)}{2} = \frac{9n^2+9n}{n^2}$.

Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6i}{n}\cdot\frac{3}{n} = \lim_{n\to\infty}\frac{9n^2+9n}{n^2} = \lim_{n\to\infty}\left(9+\frac{9}{n}\right) = 9.$

41. $f(x) = x^2 + 1$



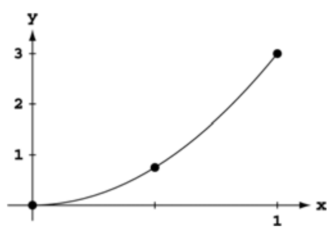
Let $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and $c_i = i\Delta x = \frac{3i}{n}$. The right-hand sum is $\sum_{i=1}^{n}\left(c_i^2+1\right)\frac{3}{n} = \sum_{i=1}^{n}\left(\left(\frac{3i}{n}\right)^2+1\right)\frac{3}{n} = \frac{3}{n}\sum_{i=1}^{n}\left(\frac{9i^2}{n^2}+1\right)$

$= \frac{27}{n}\sum_{i=1}^{n} i^2 + \frac{3}{n}\cdot n = \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)+3 = \frac{9(2n^3+3n^2+n)}{2n^3}+3$

$= \frac{18+\frac{27}{n}+\frac{9}{n^2}}{2}+3$. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(c_i^2+1\right)\frac{3}{n}$

$= \lim_{n\to\infty}\left(\frac{18+\frac{27}{n}+\frac{9}{n^2}}{2}+3\right) = 9+3 = 12.$

42. $f(x) = 3x^2$

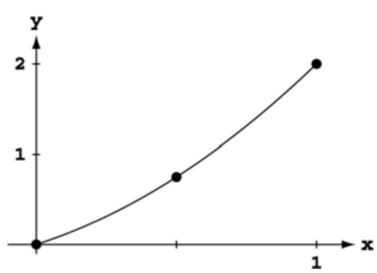


Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\Delta x = \frac{i}{n}$. The right-hand sum is $\sum_{i=1}^{n} 3c_i^2\left(\frac{1}{n}\right) = \sum_{i=1}^{n} 3\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = \frac{3}{n^3}\sum_{i=1}^{n} i^2 = \frac{3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$

$= \frac{2n^3+3n^2+n}{2n^3} = \frac{2+\frac{3}{n}+\frac{1}{n^2}}{2}$. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n} 3c_i^2\left(\frac{1}{n}\right)$

$= \lim_{n\to\infty}\left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{2}\right) = \frac{2}{2} = 1.$

43. $f(x) = x + x^2 = x(1+x)$



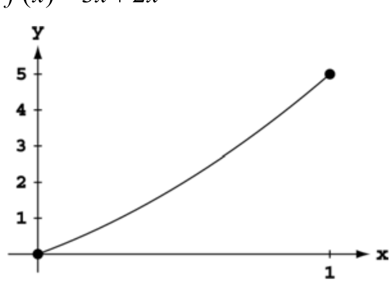
Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\Delta x = \frac{i}{n}$. The right-hand sum is $\sum_{i=1}^{n}\left(c_i+c_i^2\right)\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{i}{n}+\left(\frac{i}{n}\right)^2\right)\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2$

$= \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{n^2+n}{2n^2}+\frac{2n^3+3n^2+n}{6n^3}$

$= \frac{1+\frac{1}{n}}{2}+\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}$. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(c_i+c_i^2\right)\frac{1}{n}$

$= \lim_{n\to\infty}\left[\left(\frac{1+\frac{1}{n}}{2}\right)+\left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}\right)\right] = \frac{1}{2}+\frac{2}{6} = \frac{5}{6}.$

44. $f(x) = 3x + 2x^2$



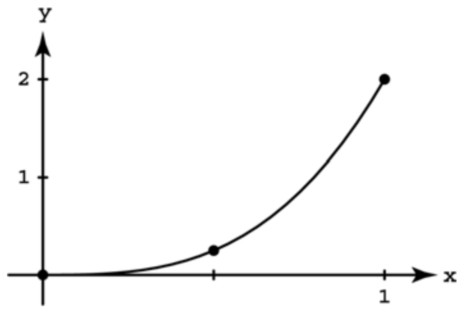
Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\Delta x = \frac{i}{n}$. The right-hand sum is $\sum_{i=1}^{n}\left(3c_i+2c_i^2\right)\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{3i}{n}+2\left(\frac{i}{n}\right)^2\right)\frac{1}{n} = \frac{3}{n^2}\sum_{i=1}^{n} i + \frac{2}{n^3}\sum_{i=1}^{n} i^2$

$= \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{2}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{3n^2+3n}{2n^2}+\frac{2n^2+3n+1}{3n^2}$

$= \frac{3+\frac{3}{n}}{2}+\frac{2+\frac{3}{n}+\frac{1}{n^2}}{3}$. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(3c_i+2c_i^2\right)\frac{1}{n}$

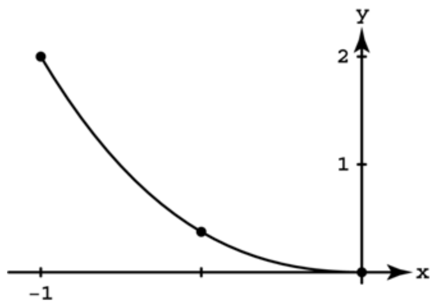
$$= \lim_{n\to\infty}\left[\left(\frac{3+\frac{3}{n}}{2}\right)+\left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{3}\right)\right] = \frac{3}{2}+\frac{2}{3} = \frac{13}{6}.$$

45. $f(x) = 2x^3$



Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\Delta x = \frac{i}{n}$. The right-hand sum is $\sum_{i=1}^{n}\left(2c_i^3\right)\frac{1}{n} = \sum_{i=1}^{n}\left(2\left(\frac{i}{n}\right)^3\right)\frac{1}{n} = \frac{2}{n^4}\sum_{i=1}^{n} i^3 = \frac{2}{n^4}\left(\frac{n(n+1)}{2}\right)^2$
$= \frac{2n^2(n^2+2n+1)}{4n^4} = \frac{n^2+2n+1}{2n^2} = \frac{1+\frac{2}{n}+\frac{1}{n^2}}{2}$. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(2c_i^3\right)\frac{1}{n}$
$= \lim_{n\to\infty}\left[\frac{1+\frac{2}{n}+\frac{1}{n^2}}{2}\right] = \frac{1}{2}.$

46. $f(x) = x^2 - x^3$



Let $\Delta x = \frac{0-(-1)}{n} = \frac{1}{n}$ and $c_i = -1 + i\Delta x = -1 + \frac{i}{n}$.
The right-hand sum is $\sum_{i=1}^{n}\left(c_i^2 - c_i^3\right)\frac{1}{n}$
$= \sum_{i=1}^{n}\left(\left(-1+\frac{i}{n}\right)^2 - \left(-1+\frac{i}{n}\right)^3\right)\frac{1}{n} = \sum_{i=1}^{n}\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right)\frac{1}{n}$
$= \sum_{i=1}^{n}\left(\frac{2}{n} - \frac{5i}{n^2} + \frac{4i^2}{n^3} - \frac{i^3}{n^4}\right) = \sum_{i=1}^{n}\frac{2}{n} - \frac{5}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^4}\sum_{i=1}^{n} i^3$
$= \frac{2}{n}(n) - \frac{5}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{4}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2$
$= 2 - \frac{5n+5}{2n} + \frac{4n^2+6n+2}{3n^2} - \frac{n^2+2n+1}{4n^2} = 2 - \frac{5+\frac{5}{n}}{2} + \frac{4+\frac{6}{n}+\frac{2}{n^2}}{3} - \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}$. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(c_i^2 - c_i^3\right)\frac{1}{n}$
$= \lim_{n\to\infty}\left[2 - \frac{5+\frac{5}{n}}{2} + \frac{4+\frac{6}{n}+\frac{2}{n^2}}{3} - \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}\right] = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}.$

5.3 THE DEFINITE INTEGRAL

1. $\int_0^2 x^2\,dx$
2. $\int_{-1}^0 2x^3\,dx$
3. $\int_{-7}^5 (x^2 - 3x)\,dx$
4. $\int_1^4 \frac{1}{x}\,dx$
5. $\int_2^3 \frac{1}{1-x}\,dx$
6. $\int_0^1 \sqrt{4-x^2}\,dx$
7. $\int_{-\pi/4}^0 (\sec x)\,dx$
8. $\int_0^{\pi/4} (\tan x)\,dx$
9. (a) $\int_2^2 g(x)\,dx = 0$
   (b) $\int_5^1 g(x)\,dx = -\int_1^5 g(x)\,dx = -8$
   (c) $\int_1^2 3f(x)\,dx = 3\int_1^2 f(x)\,dx = 3(-4) = -12$
   (d) $\int_2^5 f(x)\,dx = \int_1^5 f(x)\,dx - \int_1^2 f(x)\,dx = 6 - (-4) = 10$
   (e) $\int_1^5 [f(x) - g(x)]\,dx = \int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 6 - 8 = -2$
   (f) $\int_1^5 [4f(x) - g(x)]\,dx = 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 4(6) - 8 = 16$

10. (a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2(-1) = 2$

(b) $\int_7^9 [f(x)+h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 5+4=9$

(c) $\int_7^9 [2f(x)-3h(x)]\,dx = 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx = 2(5)-3(4) = -2$

(d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = -(-1) = 1$

(e) $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx = -1-5 = -6$

(f) $\int_9^7 [h(x)-f(x)]\,dx = \int_7^9 [f(x)-h(x)]\,dx = \int_7^9 f(x)\,dx - \int_7^9 h(x)\,dx = 5-4=1$

11. (a) $\int_1^2 f(u)\,du = \int_1^2 f(x)\,dx = 5$

(b) $\int_1^2 \sqrt{3}f(z)\,dz = \sqrt{3}\int_1^2 f(z)\,dz = 5\sqrt{3}$

(c) $\int_2^1 f(t)\,dt = -\int_1^2 f(t)\,dt = -5$

(d) $\int_1^2 [-f(x)]\,dx = -\int_1^2 f(x)\,dx = -5$

12. (a) $\int_0^{-3} g(t)\,dt = -\int_{-3}^0 g(t)\,dt = -\sqrt{2}$

(b) $\int_{-3}^0 g(u)\,du = \int_{-3}^0 g(t)\,dt = \sqrt{2}$

(c) $\int_{-3}^0 [-g(x)]\,dx = -\int_{-3}^0 g(x)\,dx = -\sqrt{2}$

(d) $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr = \frac{1}{\sqrt{2}}\int_{-3}^0 g(t)\,dt = \left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}) = 1$

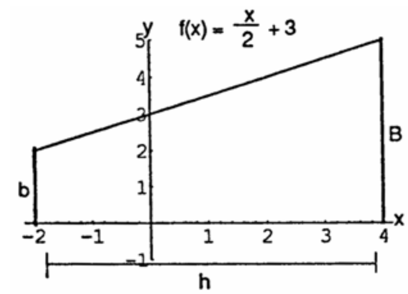
13. (a) $\int_3^4 f(z)\,dz = \int_0^4 f(z)\,dz - \int_0^3 f(z)\,dz = 7-3 = 4$

(b) $\int_4^3 f(t)\,dt = -\int_3^4 f(t)\,dt = -4$

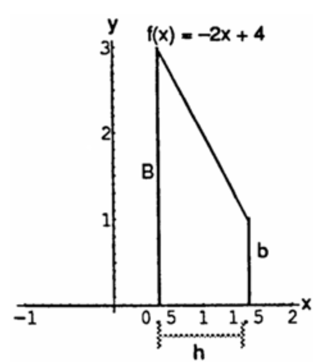
14. (a) $\int_1^3 h(r)\,dr = \int_{-1}^3 h(r)\,dr - \int_{-1}^1 h(r)\,dr = 6-0 = 6$

(b) $-\int_3^1 h(u)\,du = -\left(-\int_1^3 h(u)\,du\right) = \int_1^3 h(u)\,du = 6$

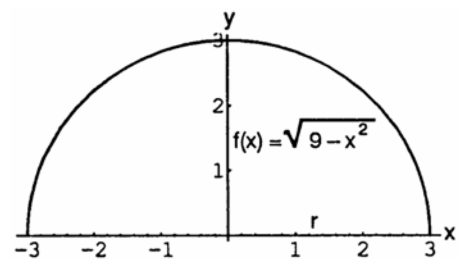
15. The area of the trapezoid is $A = \frac{1}{2}(B+b)h$
$= \frac{1}{2}(5+2)(6) = 21 \Rightarrow \int_{-2}^4 \left(\frac{x}{2}+3\right)\,dx = 21$ square units



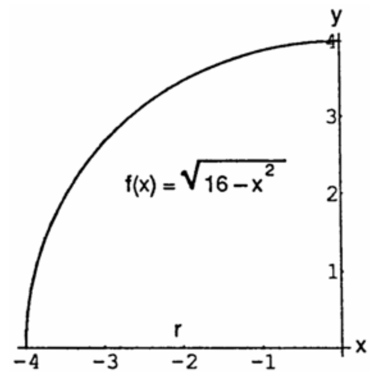
16. The area of the trapezoid is $A = \frac{1}{2}(B+b)h$
$= \frac{1}{2}(3+1)(1) = 2 \Rightarrow \int_{1/2}^{3/2} (-2x+4)\,dx = 2$ square units



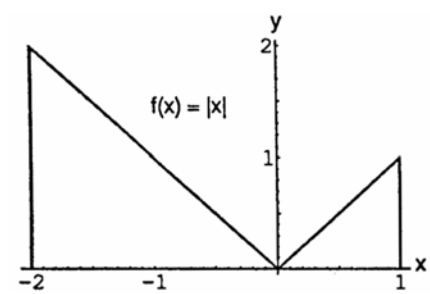
17. The area of the semicircle is $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2 = \frac{9}{2}\pi \Rightarrow \int_{-3}^{3}\sqrt{9-x^2}\,dx = \frac{9}{2}\pi$ square units



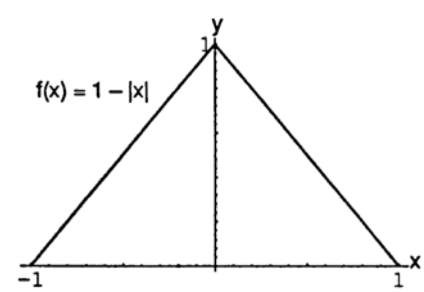
18. The graph of the quarter circle is $A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(4)^2 = 4\pi \Rightarrow \int_{-4}^{0}\sqrt{16-x^2}\,dx = 4\pi$ square units



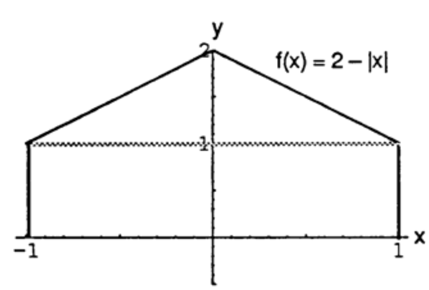
19. The area of the triangle on the left is $A = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$. The area of the triangle on the right is $A = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$. Then, the total area is 2.5 $\Rightarrow \int_{-2}^{1}|x|\,dx = 2.5$ square units



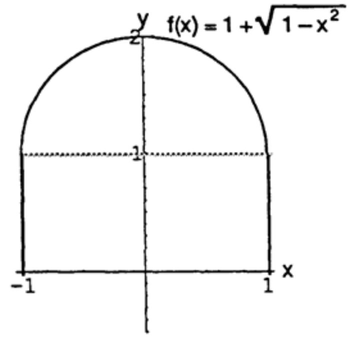
20. The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1 \Rightarrow \int_{-1}^{1}(1-|x|)\,dx = 1$ square unit



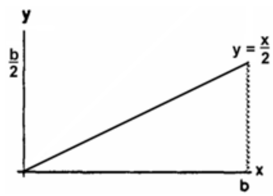
21. The area of the triangular peak is $A = \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1$. The area of the rectangular base is $S = \ell w = (2)(1) = 2$. Then the total area is $3 \Rightarrow \int_{-1}^{1}(2-|x|)\,dx = 3$ square units



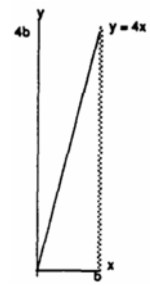
22. $y = 1 + \sqrt{1-x^2} \Rightarrow y - 1 = \sqrt{1-x^2} \Rightarrow (y-1)^2 = 1 - x^2$ $\Rightarrow x^2 + (y-1)^2 = 1$, a circle with center (0, 1) and radius of $1 \Rightarrow y = 1 + \sqrt{1-x^2}$ is the upper semicircle. The area of this semicircle is $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$. The area of the rectangular base is $A = \ell w = (2)(1) = 2$. Then the total area is $2 + \frac{\pi}{2} \Rightarrow \int_{-1}^{1}\left(1+\sqrt{1-x^2}\right) dx = 2 + \frac{\pi}{2}$ square units



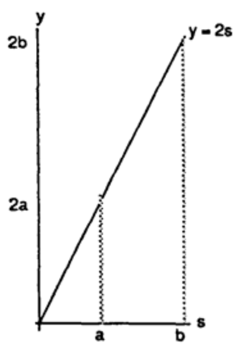
23. $\int_0^b \frac{x}{2}\, dx = \frac{1}{2}(b)\left(\frac{b}{2}\right) = \frac{b^2}{4}$



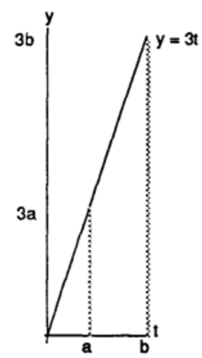
24. $\int_0^b 4x\, dx = \frac{1}{2}b(4b) = 2b^2$



25. $\int_a^b 2s\, ds = \frac{1}{2}b(2b) - \frac{1}{2}a(2a) = b^2 - a^2$



26. $\int_a^b 3t\, dt = \frac{1}{2}b(3b) - \frac{1}{2}a(3a) = \frac{3}{2}(b^2 - a^2)$



27. (a) $\int_{-2}^{2}\sqrt{4-x^2}\, dx = \frac{1}{2}[\pi(2)^2] = 2\pi$

(b) $\int_0^2 \sqrt{4-x^2}\, dx = \frac{1}{4}[\pi(2)^2] = \pi$

28. (a) $\int_{-1}^{0}\left(3x+\sqrt{1-x^2}\right) dx = \int_{-1}^{0} 3x\, dx + \int_{-1}^{0}\sqrt{1-x^2}\, dx = -\frac{1}{2}[(1)(3)] + \frac{1}{4}[\pi(1)^2] = \frac{\pi}{4} - \frac{3}{2}$

(b) $\int_{-1}^{1}\left(3x+\sqrt{1-x^2}\right) dx = \int_{-1}^{0} 3x\, dx + \int_0^1 3x\, dx + \int_{-1}^{1}\sqrt{1-x^2}\, dx = -\frac{1}{2}[(1)(3)] + \frac{1}{2}[(1)(3)] + \frac{1}{2}[\pi(1)^2] = \frac{\pi}{2}$

29. $\int_1^{\sqrt{2}} x\, dx = \frac{\left(\sqrt{2}\right)^2}{2} - \frac{(1)^2}{2} = \frac{1}{2}$

30. $\int_{0.5}^{2.5} x\, dx = \frac{(2.5)^2}{2} - \frac{(0.5)^2}{2} = 3$

31. $\int_{\pi}^{2\pi} \theta\, d\theta = \frac{(2\pi)^2}{2} - \frac{\pi^2}{2} = \frac{3\pi^2}{2}$

32. $\int_{\sqrt{2}}^{5\sqrt{2}} r\, dr = \frac{\left(5\sqrt{2}\right)^2}{2} - \frac{\left(\sqrt{2}\right)^2}{2} = 24$

33. $\int_0^{\sqrt[3]{7}} x^2\ dx = \frac{\left(\sqrt[3]{7}\right)^3}{3} = \frac{7}{3}$

34. $\int_0^{0.3} s^2\ ds = \frac{(0.3)^3}{3} = 0.009$

35. $\int_0^{1/2} t^2\ dt = \frac{\left(\frac{1}{2}\right)^3}{3} = \frac{1}{24}$

36. $\int_0^{\pi/2} \theta^2\ d\theta = \frac{\left(\frac{\pi}{2}\right)^3}{3} = \frac{\pi^3}{24}$

37. $\int_a^{2a} x\ dx = \frac{(2a)^2}{2} - \frac{a^2}{2} = \frac{3a^2}{2}$

38. $\int_a^{\sqrt{3}a} x\ dx = \frac{\left(\sqrt{3}a\right)^2}{2} - \frac{a^2}{2} = a^2$

39. $\int_0^{\sqrt[3]{b}} x^2\ dx = \frac{\left(\sqrt[3]{b}\right)^3}{3} = \frac{b}{3}$

40. $\int_0^{3b} x^2\ dx = \frac{(3b)^3}{3} = 9b^3$

41. $\int_3^1 7\ dx = 7(1-3) = -14$

42. $\int_0^2 5x\ dx = 5\int_0^2 x\ dx = 5\left[\frac{2^2}{2} - \frac{0^2}{2}\right] = 10$

43. $\int_0^2 (2t-3)\ dt = 2\int_1^1 t\ dt - \int_0^2 3\ dt = 2\left[\frac{2^2}{2} - \frac{0^2}{2}\right] - 3(2-0) = 4 - 6 = -2$

44. $\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right) dt = \int_0^{\sqrt{2}} t\ dt - \int_0^{\sqrt{2}} \sqrt{2}\ dt = \left[\frac{\left(\sqrt{2}\right)^2}{2} - \frac{0^2}{2}\right] - \sqrt{2}\left[\sqrt{2} - 0\right] = 1 - 2 = -1$

45. $\int_2^1 \left(1 + \frac{z}{2}\right) dz = \int_2^1 1\ dz + \int_2^1 \frac{z}{2}\ dz = \int_2^1 1\ dz - \frac{1}{2}\int_1^2 z\ dz = 1[1-2] - \frac{1}{2}\left[\frac{2^2}{2} - \frac{1^2}{2}\right] = -1 - \frac{1}{2}\left(\frac{3}{2}\right) = -\frac{7}{4}$

46. $\int_3^0 (2z-3)\ dz = \int_3^0 2z\ dz - \int_3^0 3\ dz = -2\int_0^3 z\ dz - \int_3^0 3\ dz = -2\left[\frac{3^2}{2} - \frac{0^2}{2}\right] - 3[0-3] = -9 + 9 = 0$

47. $\int_1^2 3u^2\ du = 3\int_1^2 u^2\ du = 3\left[\int_0^2 u^2\ du - \int_0^1 u^2\ du\right] = 3\left(\left[\frac{2^3}{3} - \frac{0^3}{3}\right] - \left[\frac{1^3}{3} - \frac{0^3}{3}\right]\right) = 3\left[\frac{2^3}{3} - \frac{1^3}{3}\right] = 3\left(\frac{7}{3}\right) = 7$

48. $\int_{1/2}^1 24u^2\ du = 24\int_{1/2}^1 u^2\ du = 24\left[\int_0^1 u^2\ du - \int_0^{1/2} u^2\ du\right] = 24\left[\frac{1^3}{3} - \frac{\left(\frac{1}{2}\right)^3}{3}\right] = 24\left[\frac{\left(\frac{7}{8}\right)}{3}\right] = 7$

49. $\int_0^2 (3x^2 + x - 5)\ dx = 3\int_0^2 x^2\ dx + \int_0^2 x\ dx - \int_0^2 5\ dx = 3\left[\frac{2^3}{3} - \frac{0^3}{3}\right] + \left[\frac{2^2}{2} - \frac{0^2}{2}\right] - 5[2-0] = (8+2) - 10 = 0$

50. $\int_1^0 (3x^2 + x - 5)\ dx = -\int_0^1 (3x^2 + x - 5)\ dx = -\left[3\int_0^1 x^2 dx + \int_0^1 x\ dx - \int_0^1 5\ dx\right] = -\left[3\left(\frac{1^3}{3} - \frac{0^3}{3}\right) + \left(\frac{1^2}{2} - \frac{0^2}{2}\right) - 5(1-0)\right]$
$= -\left(\frac{3}{2} - 5\right) = \frac{7}{2}$

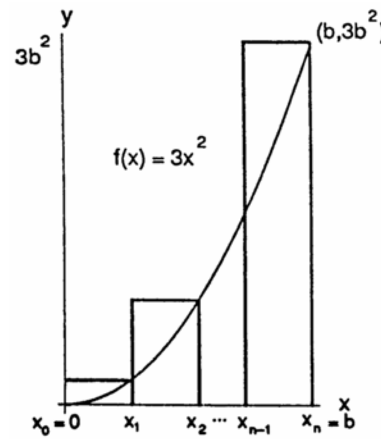
51. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0, x_1 = \Delta x, x_2 = 2\Delta x, \dots,$ $x_{n-1} = (n-1)\Delta x, x_n = n\Delta x = b$. Let the c_k's be the right endpoints of the subintervals $\Rightarrow c_1 = x_1, c_2 = x_2$, and so on. The rectangles defined have areas:

$$f(c_1)\Delta x = f(\Delta x)\Delta x = 3(\Delta x)^2\Delta x = 3(\Delta x)^3$$
$$f(c_2)\Delta x = f(2\Delta x)\Delta x = 3(2\Delta x)^2\Delta x = 3(2)^2(\Delta x)^3$$
$$f(c_3)\Delta x = f(3\Delta x)\Delta x = 3(3\Delta x)^2\Delta x = 3(3)^2(\Delta x)^3$$
$$\vdots$$
$$f(c_n)\Delta x = f(n\Delta x)\Delta x = 3(n\Delta x)^2\Delta x = 3(n)^2(\Delta x)^3$$

Then $S_n = \sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} 3k^2(\Delta x)^3$

$= 3(\Delta x)^3 \sum_{k=1}^{n} k^2 = 3\left(\frac{b^3}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)$

$= \frac{b^3}{2}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) \Rightarrow \int_0^b 3x^2\,dx = \lim_{n\to\infty}\frac{b^3}{2}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) = b^3.$



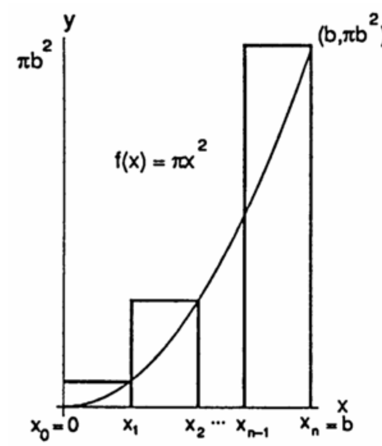
52. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0,\ x_1 = \Delta x,\ x_2 = 2\Delta x, \dots,$ $x_{n-1} = (n-1)\Delta x,\ x_n = n\Delta x = b$. Let the c_k's be the right endpoints of the subintervals $\Rightarrow c_1 = x_1,\ c_2 = x_2$, and so on. The rectangles defined have areas:

$$f(c_1)\Delta x = f(\Delta x)\Delta x = \pi(\Delta x)^2\Delta x = \pi(\Delta x)^3$$
$$f(c_2)\Delta x = f(2\Delta x)\Delta x = \pi(2\Delta x)^2\Delta x = \pi(2)^2(\Delta x)^3$$
$$f(c_3)\Delta x = f(3\Delta x)\Delta x = \pi(3\Delta x)^2\Delta x = \pi(3)^2(\Delta x)^3$$
$$\vdots$$
$$f(c_n)\Delta x = f(n\Delta x)\Delta x = \pi(n\Delta x)^2\Delta x = \pi(n)^2(\Delta x)^3$$

Then $S_n = \sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} \pi k^2(\Delta x)^3 = \pi(\Delta x)^3\sum_{k=1}^{n} k^2$

$= \pi\left(\frac{b^3}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{\pi b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$

$\Rightarrow \int_0^b \pi x^2\,dx = \lim_{n\to\infty}\frac{\pi b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) = \frac{\pi b^3}{3}.$



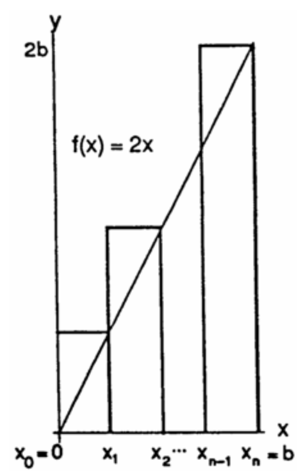
53. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0, x_1 = \Delta x,\ x_2 = 2\Delta x, \dots,$ $x_{n-1} = (n-1)\Delta x,\ x_n = n\Delta x = b$. Let the c_k's be the right endpoints of the subintervals $\Rightarrow c_1 = x_1, c_2 = x_2$, and so on. The rectangles defined have areas:

$$f(c_1)\Delta x = f(\Delta x)\Delta x = 2(\Delta x)(\Delta x) = 2(\Delta x)^2$$
$$f(c_2)\Delta x = f(2\Delta x)\Delta x = 2(2\Delta x)(\Delta x) = 2(2)(\Delta x)^2$$
$$f(c_3)\Delta x = f(3\Delta x)\Delta x = 2(3\Delta x)(\Delta x) = 2(3)(\Delta x)^2$$
$$\vdots$$
$$f(c_n)\Delta x = f(n\Delta x)\Delta x = 2(n\Delta x)(\Delta x) = 2(n)(\Delta x)^2$$

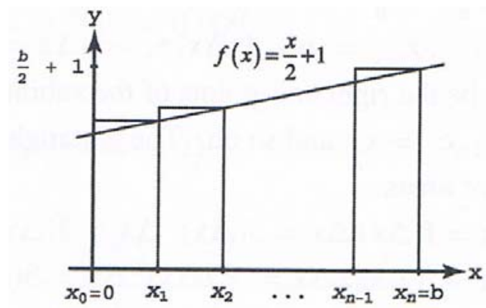
Then $S_n = \sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} 2k(\Delta x)^2$

$= 2(\Delta x)^2\sum_{k=1}^{n} k = 2\left(\frac{b^2}{n^2}\right)\left(\frac{n(n+1)}{2}\right) = b^2\left(1+\frac{1}{n}\right)$

$\Rightarrow \int_0^b 2x\,dx = \lim_{n\to\infty} b^2\left(1+\frac{1}{n}\right) = b^2.$



54. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0, x_1 = \Delta x, x_2 = 2\Delta x, \ldots,$ $x_{n-1} = (n-1)\Delta x,\ x_n = n\Delta x = b.$ Let the c_k 's be the right endpoints of the subintervals $\Rightarrow c_1 = x_1,\ c_2 = x_2$, and so on. The rectangles defined have areas:



$f(c_1)\Delta x = f(\Delta x)\Delta x = \left(\frac{\Delta x}{2}+1\right)(\Delta x) = \frac{1}{2}(\Delta x)^2 + \Delta x$

$f(c_2)\Delta x = f(2\Delta x)\Delta x = \left(\frac{2\Delta x}{2}+1\right)(\Delta x) = \frac{1}{2}(2)(\Delta x)^2 + \Delta x$

$f(c_3)\Delta x = f(3\Delta x)\Delta x = \left(\frac{3\Delta x}{2}+1\right)(\Delta x) = \frac{1}{2}(3)(\Delta x)^2 + \Delta x$

$\vdots$

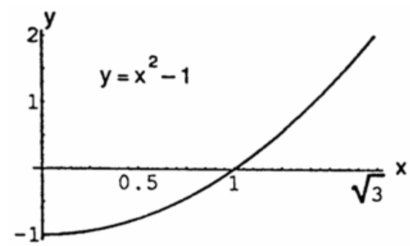
$f(c_n)\Delta x = f(n\Delta x)\Delta x = \left(\frac{n\Delta x}{2}+1\right)(\Delta x) = \frac{1}{2}(n)(\Delta x)^2 + \Delta x$

Then $S_n = \sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n}\left(\frac{1}{2}k(\Delta x)^2 + \Delta x\right) = \frac{1}{2}(\Delta x)^2 \sum_{k=1}^{n} k + \Delta x \sum_{k=1}^{n} 1 = \frac{1}{2}\left(\frac{b^2}{n^2}\right)\left(\frac{n(n+1)}{2}\right) + \left(\frac{b}{n}\right)(n)$

$= \frac{1}{4}b^2\left(1+\frac{1}{n}\right) + b \Rightarrow \int_0^b \left(\frac{x}{2}+1\right)dx = \lim_{n\to\infty}\left(\frac{1}{4}b^2\left(1+\frac{1}{n}\right)+b\right) = \frac{1}{4}b^2 + b.$

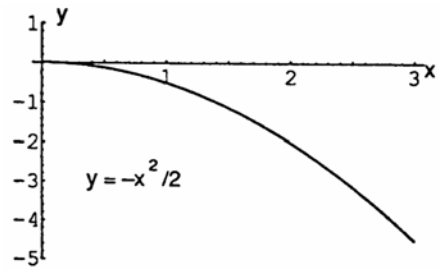
55. $\text{av}(f) = \left(\frac{1}{\sqrt{3}-0}\right)\int_0^{\sqrt{3}}(x^2-1)\,dx = \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} x^2 dx - \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} 1\,dx$

$= \frac{1}{\sqrt{3}}\left(\frac{\left(\sqrt{3}\right)^3}{3}\right) - \frac{1}{\sqrt{3}}\left(\sqrt{3}-0\right) = 1-1 = 0.$



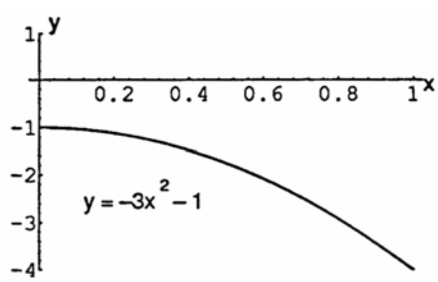
56. $\text{av}(f) = \left(\frac{1}{3-0}\right)\int_0^3\left(-\frac{x^2}{2}\right)dx = \frac{1}{3}\left(-\frac{1}{2}\right)\int_0^3 x^2 dx$

$= -\frac{1}{6}\left(\frac{3^3}{3}\right) = -\frac{3}{2}.$



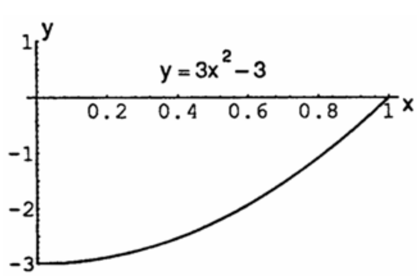
57. $\text{av}(f) = \left(\frac{1}{1-0}\right)\int_0^1(-3x^2-1)\,dx = -3\int_0^1 x^2 dx - \int_0^1 1\,dx$

$= -3\left(\frac{1^3}{3}\right) - (1-0) = -2.$

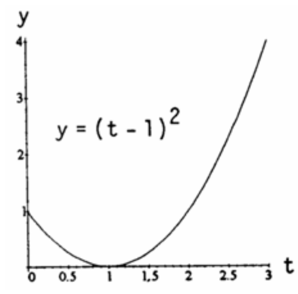


58. $\text{av}(f) = \left(\frac{1}{1-0}\right)\int_0^1(3x^2-3)\,dx = 3\int_0^1 x^2 dx - \int_0^1 3\,dx$

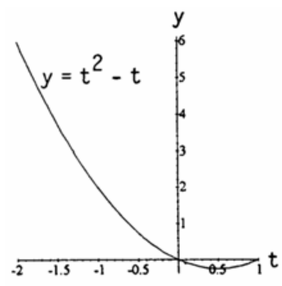
$= 3\left(\frac{1^3}{3}\right) - 3(1-0) = -2.$



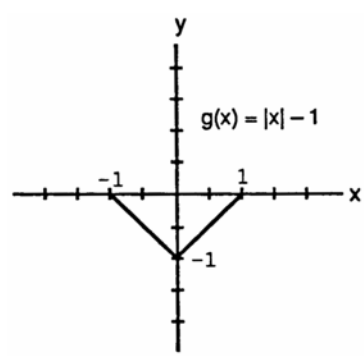
59. $\text{av}(f) = \left(\frac{1}{3-0}\right)\int_0^3 (t-1)^2 dt = \frac{1}{3}\int_0^3 t^2 dt - \frac{2}{3}\int_0^3 t\ dt + \frac{1}{3}\int_0^3 1\ dt$
$= \frac{1}{3}\left(\frac{3^3}{3}\right) - \frac{2}{3}\left(\frac{3^2}{2} - \frac{0^2}{2}\right) + \frac{1}{3}(3-0) = 1.$



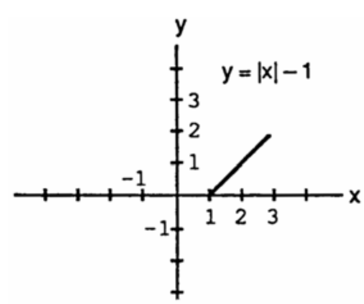
60. $\text{av}(f) = \left(\frac{1}{1-(-2)}\right)\int_{-2}^1 \left(t^2 - t\right) dt = \frac{1}{3}\int_{-2}^1 t^2 dt - \frac{1}{3}\int_{-2}^1 t\ dt$
$= \frac{1}{3}\int_0^1 t^2 dt - \frac{1}{3}\int_0^{-2} t^2\ dt - \frac{1}{3}\left(\frac{1^2}{2} - \frac{(-2)^2}{2}\right)$
$= \frac{1}{3}\left(\frac{1^3}{3}\right) - \frac{1}{3}\left(\frac{(-2)^3}{3}\right) + \frac{1}{2} = \frac{3}{2}.$



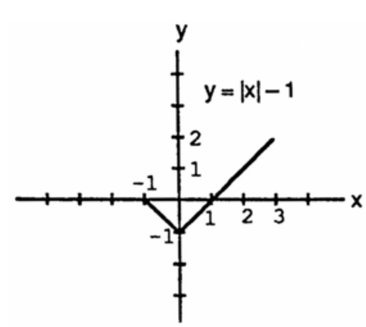
61. (a) $\text{av}(g) = \left(\frac{1}{1-(-1)}\right)\int_{-1}^1 (|x|-1) dx$
$= \frac{1}{2}\int_{-1}^0 (-x-1)\ dx + \frac{1}{2}\int_0^1 (x-1)\ dx$
$= -\frac{1}{2}\int_{-1}^0 x\ dx - \frac{1}{2}\int_{-1}^0 1\ dx + \frac{1}{2}\int_0^1 x\ dx - \frac{1}{2}\int_0^1 1\ dx$
$= -\frac{1}{2}\left(\frac{0^2}{2} - \frac{(-1)^2}{2}\right) - \frac{1}{2}(0-(-1)) + \frac{1}{2}\left(\frac{1^2}{2} - \frac{0^2}{2}\right) - \frac{1}{2}(1-0)$
$= -\frac{1}{2}.$



(b) $\text{av}(g) = \left(\frac{1}{3-1}\right)\int_1^3 (|x|-1)\ dx = \frac{1}{2}\int_1^3 (x-1)\ dx$
$= \frac{1}{2}\int_1^3 x\ dx - \frac{1}{2}\int_1^3 1\ dx = \frac{1}{2}\left(\frac{3^2}{2} - \frac{1^2}{2}\right) - \frac{1}{2}(3-1) = 1.$

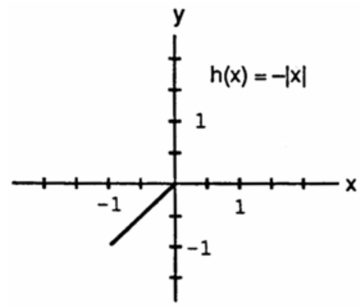


(c) $\text{av}(g) = \left(\frac{1}{3-1(-1)}\right)\int_{-1}^3 (|x|-1)\ dx$
$= \frac{1}{4}\int_{-1}^1 (|x|-1)\ dx + \frac{1}{4}\int_1^3 (|x|-1)\ dx$
$= \frac{1}{4}(-1+2) = \frac{1}{4}$ (see parts (a) and (b) above).



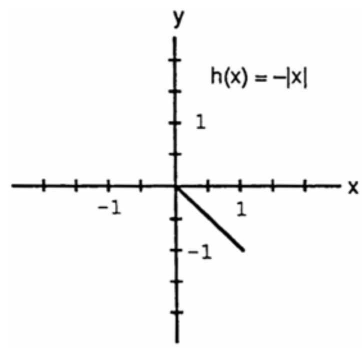
62. (a) $\text{av}(h)=\left(\frac{1}{0-(-1)}\right)\int_{-1}^{0}-|\,x|\;dx=\int_{-1}^{0}-(-x)\;dx$

$=\int_{-1}^{0}x\;dx=\frac{0^2}{2}-\frac{(-1)^2}{2}=-\frac{1}{2}.$



(b) $\text{av}(h)=\left(\frac{1}{1-0}\right)\int_{0}^{1}-|\,x\,|\,dx=-\int_{0}^{1}x\;dx$

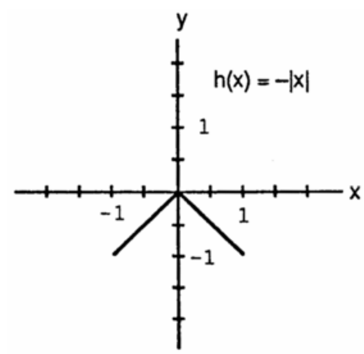
$=-\left(\frac{1^2}{2}-\frac{0^2}{2}\right)=-\frac{1}{2}.$



(c) $\text{av}(h)=\left(\frac{1}{1-(-1)}\right)\int_{-1}^{1}-|x|\;dx$

$=\frac{1}{2}\left(\int_{-1}^{0}-|x|\;dx+\int_{0}^{1}-|x|\;dx\right)$

$=\frac{1}{2}\left(-\frac{1}{2}+\left(-\frac{1}{2}\right)\right)=-\frac{1}{2}$(see parts (a) and (b) above).



63. Consider the partition P that subdivides the interval $[a, b]$ into n subintervals of width $\triangle x=\frac{b-a}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is $P=\left\{a,\ a+\frac{b-a}{n},\ a+\frac{2(b-a)}{n},\ldots,\ a+\frac{n(b-a)}{n}\right\}$ and $c_k=a+\frac{k(b-a)}{n}$. We get the Riemann sum $\sum_{k=1}^{n}f(c_k)\triangle x=\sum_{k=1}^{n}c\cdot\frac{b-a}{n}=\frac{c(b-a)}{n}\sum_{k=1}^{n}1=\frac{c(b-a)}{n}\cdot n=c(b-a).$ As $n\to\infty$ and $\|P\|\to 0$ this expression remains $c(b-a)$. Thus, $\int_{a}^{b}c\;dx=c(b-\ a)$.

64. Consider the partition P that subdivides the interval $[0, 2]$ into n subintervals of width $\triangle x=\frac{2-0}{n}=\frac{2}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is $P=\left\{0,\frac{2}{n},2\cdot\frac{2}{n},\ldots,n\cdot\frac{2}{n}=2\right\}$ and $c_k=k\cdot\frac{2}{n}=\frac{2k}{n}$. We get the Riemann sum
$\sum_{k=1}^{n}f(c_k)\triangle x=\sum_{k=1}^{n}\left[2\left(\frac{2k}{n}\right)+1\right]\cdot\frac{2}{n}=\frac{2}{n}\sum_{k=1}^{n}\left(\frac{4k}{n}+1\right)=\frac{8}{n^2}\sum_{k=1}^{n}k+\frac{2}{n}\sum_{k=1}^{n}1=\frac{8}{n^2}\cdot\frac{n(n+1)}{2}+\frac{2}{n}\cdot n=\frac{4(n+1)}{n}+2$. As $n\to\infty$ and $\|P\|\to 0$ the expression $\frac{4(n+1)}{n}+2$ has the value $4+2=6$. Thus, $\int_{0}^{2}(2x+1)\;dx=6.$

65. Consider the partition P that subdivides the interval $[a, b]$ into n subintervals of width $\triangle x = \frac{b-a}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is $P = \left\{a, a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}, \ldots, a+\frac{n(b-a)}{n}\right\}$ and $c_k = a + \frac{k(b-a)}{n}$. We get the Riemann sum $\sum_{k=1}^{n} f(c_k)\triangle x = \sum_{k=1}^{n} c_k^2\left(\frac{b-a}{n}\right) = \frac{b-a}{n}\sum_{k=1}^{n}\left(a+\frac{k(b-a)}{n}\right)^2$

$= \frac{b-a}{n}\sum_{k=1}^{n}\left(a^2 + \frac{2ak(b-a)}{n} + \frac{k^2(b-a)^2}{n^2}\right) = \frac{b-a}{n}\left(\sum_{k=1}^{n} a^2 + \frac{2a(b-a)}{n}\sum_{k=1}^{n} k + \frac{(b-a)^2}{n^2}\sum_{k=1}^{n} k^2\right)$

$= \frac{b-a}{n}\cdot na^2 + \frac{2a(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = (b-a)a^2 + a(b-a)^2\cdot\frac{n+1}{n} + \frac{(b-a)^3}{6}\cdot\frac{(n+1)(2n+1)}{n^2}$

$= (b-a)a^2 + a(b-a)^2\cdot\frac{1+\frac{1}{n}}{1} + \frac{(b-a)^3}{6}\cdot\frac{2+\frac{3}{n}+\frac{1}{n^2}}{1}$ As $n \to \infty$ and $\|P\| \to 0$ this expression has value $(b-a)a^2 + a(b-a)^2\cdot 1 + \frac{(b-a)^3}{6}\cdot 2 = ba^2 - a^3 + ab^2 - 2a^2b + a^3 + \frac{1}{3}(b^3 - 3b^2a + 3ba^2 - a^3) = \frac{b^3}{3} - \frac{a^3}{3}$. Thus, $\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}$.

66. Consider the partition P that subdivides the interval $[-1, 0]$ into n subintervals of width $\triangle x = \frac{0-(-1)}{n} = \frac{1}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is $P = \left\{-1, -1+\frac{1}{n}, -1+2\cdot\frac{1}{n}, \ldots, -1+n\cdot\frac{1}{n} = 0\right\}$ and $c_k = -1 + k\cdot\frac{1}{n} = -1+\frac{k}{n}$. We get the Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n}\left(\left(-1+\frac{k}{n}\right) - \left(-1+\frac{k}{n}\right)^2\right)\cdot\frac{1}{n}$

$= \frac{1}{n}\sum_{k=1}^{n}\left(-1+\frac{k}{n} - 1 + \frac{2k}{n} - \left(\frac{k}{n}\right)^2\right) = -\frac{2}{n}\sum_{k=1}^{n} 1 + \frac{3}{n^2}\sum_{k=1}^{n} k - \frac{1}{n^3}\sum_{k=1}^{n} k^2 = -\frac{2}{n}\cdot n + \frac{3}{n^2}\cdot\frac{n(n+1)}{2} - \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

$= -2 + \frac{3(n+1)}{2n} - \frac{(n+1)(2n+1)}{6n^2}$. As $n \to \infty$ and $\| P \| \to 0$ this expression has value $-2 + \frac{3}{2} - \frac{1}{3} = -\frac{5}{6}$.

Thus, $\int_{-1}^{0}(x - x^2)\ dx = -\frac{5}{6}$.

67. Consider the partition P that subdivides the interval $[-1, 2]$ into n subintervals of width $\Delta x = \frac{2-(-1)}{n} = \frac{3}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is $P = \left\{-1, -1+\frac{3}{n}, -1+2\cdot\frac{3}{n}, \ldots, -1+n\cdot\frac{3}{n} = 2\right\}$ and $c_k = -1 + k\cdot\frac{3}{n} = -1+\frac{3k}{n}$. We get the Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n}\left(3\left(-1+\frac{3k}{n}\right)^2 - 2\left(-1+\frac{3k}{n}\right)+1\right)\cdot\frac{3}{n}$

$= \frac{3}{n}\sum_{k=1}^{n}\left(3 - \frac{18k}{n} + \frac{27k^2}{n^2} + 2 - \frac{6k}{n} + 1\right) = \frac{18}{n}\sum_{k=1}^{n} 1 - \frac{72}{n^2}\sum_{k=1}^{n} k + \frac{81}{n^3}\sum_{k=1}^{n} k^2 = \frac{18}{n}\cdot n - \frac{72}{n^2}\cdot\frac{n(n+1)}{2} + \frac{81}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

$= 18 - \frac{36(n+1)}{n} + \frac{27(n+1)(2n+1)}{2n^2}$. As $n \to \infty$ and $\| P \| \to 0$ this expression has value $18 - 36 + 27 = 9$.

Thus, $\int_{-1}^{2}(3x^2 - 2x + 1)dx = 9$.

68. Consider the partition P that subdivides the interval $[-1, 1]$ into n subintervals of width $\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is $P = \left\{-1, -1+\frac{2}{n}, -1+2\cdot\frac{2}{n}, \ldots, -1+n\cdot\frac{2}{n} = 1\right\}$ and $c_k = -1 + k\cdot\frac{2}{n} = -1 + \frac{2k}{n}$. We get the Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} c_k^3\left(\frac{2}{n}\right) = \frac{2}{n}\sum_{k=1}^{n}\left(-1+\frac{2k}{n}\right)^3$

$= \frac{2}{n}\sum_{k=1}^{n}\left(-1 + \frac{6k}{n} - \frac{12k^2}{n^2} + \frac{8k^3}{n^3}\right) = \frac{2}{n}\left(-\sum_{k=1}^{n} 1 + \frac{6}{n}\sum_{k=1}^{n} k - \frac{12}{n^2}\sum_{k=1}^{n} k^2 + \frac{8}{n^3}\sum_{k=1}^{n} k^3\right)$

$= -\frac{2}{n}\cdot n + \frac{12}{n^2}\cdot\frac{n(n+1)}{2} - \frac{24}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4}\cdot\left(\frac{n(n+1)}{2}\right)^2 = -2 + 6\cdot\frac{n+1}{n} - 4\cdot\frac{(n+1)(2n+1)}{n^2} + 4\cdot\frac{(n+1)^2}{n^2} = -2 + 6\cdot\frac{1+\frac{1}{n}}{1}$

$= -2 + 6\cdot\frac{1+\frac{1}{n}}{1} - 4\cdot\frac{2+\frac{3}{n}+\frac{1}{n^2}}{1} + 4\cdot\frac{1+\frac{2}{n}+\frac{1}{n^2}}{1}$. As $n\to\infty$ and $\|P\|\to 0$ this expression has value $-2+6-8+4=0$. Thus, $\int_{-1}^{1} x^3\,dx = 0$.

69. Consider the partition P that subdivides the interval $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is $P = \left\{a,\, a+\frac{b-a}{n},\, a+\frac{2(b-a)}{n},\ldots,\, a+\frac{n(b-a)}{n} = b\right\}$ and $c_k = a + \frac{k(b-a)}{n}$. We get the Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} c_k^3\left(\frac{b-a}{n}\right) = \frac{b-a}{n}\sum_{k=1}^{n}\left(a+\frac{k(b-a)}{n}\right)^3$

$= \frac{b-a}{n}\sum_{k=1}^{n}\left(a^3 + \frac{3a^2k(b-a)}{n} + \frac{3ak^2(b-a)^2}{n^2} + \frac{k^3(b-a)^3}{n^3}\right) = \frac{b-a}{n}\left(\sum_{k=1}^{n} a^3 + \frac{3a^2(b-a)}{n}\sum_{k=1}^{n} k + \frac{3a(b-a)^2}{n^2}\sum_{k=1}^{n} k^2 + \frac{(b-a)^3}{n^3}\sum_{k=1}^{n} k^3\right)$

$= \frac{b-a}{n}\cdot na^3 + \frac{3a^2(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2} + \frac{3a(b-a)^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{(b-a)^4}{n^4}\cdot\left(\frac{n(n+1)}{2}\right)^2$

$= (b-a)a^3 + \frac{3a^2(b-a)^2}{2}\cdot\frac{n+1}{n} + \frac{a(b-a)^3}{2}\cdot\frac{(n+1)(2n+1)}{n^2} + \frac{(b-a)^4}{4}\cdot\frac{(n+1)^2}{n^2}$

$= (b-a)a^3 + \frac{3a^2(b-a)^2}{2}\cdot\frac{1+\frac{1}{n}}{1} + \frac{a(b-a)^3}{2}\cdot\frac{2+\frac{3}{n}+\frac{1}{n^2}}{1} + \frac{(b-a)^4}{4}\cdot\frac{1+\frac{2}{n}+\frac{1}{n^2}}{1}$. As $n\to\infty$ and $\|P\|\to 0$ this expression has value $(b-a)a^3 + \frac{3a^2(b-a)^2}{2} + a(b-a)^3 + \frac{(b-a)^4}{4} = \frac{b^4}{4} - \frac{a^4}{4}$. Thus, $\int_a^b x^3\,dx = \frac{b^4}{4} - \frac{a^4}{4}$.

70. Consider the partition P that subdivides the interval $[0, 1]$ into n subintervals of width $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is $P = \left\{0,\, 0+\frac{1}{n},\, 0+2\cdot\frac{1}{n},\ldots,\, 0+n\cdot\frac{1}{n} = 1\right\}$ and $c_k = 0 + k\cdot\frac{1}{n} = \frac{k}{n}$. We get the Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n}\left(3c_k - c_k^3\right)\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{k=1}^{n}\left(3\cdot\frac{k}{n} - \left(\frac{k}{n}\right)^3\right)$

$= \frac{1}{n}\left(\frac{3}{n}\sum_{k=1}^{n} k - \frac{1}{n^3}\sum_{k=1}^{n} k^3\right) = \frac{3}{n^2}\cdot\frac{n(n+1)}{2} - \frac{1}{n^4}\cdot\left(\frac{n(n+1)}{2}\right)^2 = \frac{3}{2}\cdot\frac{n+1}{n} - \frac{1}{4}\cdot\frac{(n+1)^2}{n^2} = \frac{3}{2}\cdot\frac{1+\frac{1}{n}}{1} - \frac{1}{4}\cdot\frac{1+\frac{2}{n}+\frac{1}{n^2}}{1}$. As $n\to\infty$ and $\|P\|\to 0$ this expression has value $\frac{3}{2} - \frac{1}{4} = \frac{5}{4}$. Thus, $\int_0^1 (3x - x^3)\,dx = \frac{5}{4}$.

71. To find where $x - x^2 \ge 0$, let $x - x^2 = 0 \Rightarrow x(1-x) = 0 \Rightarrow x = 0$ or $x = 1$. If $0 < x < 1$, then $0 < x - x^2 \Rightarrow a = 0$ and $b = 1$ maximize the integral.

72. To find where $x^4 - 2x^2 \le 0$, let $x^4 - 2x^2 = 0 \Rightarrow x^2(x^2 - 2) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{2}$. By the sign graph, $++++++\underset{-\sqrt{2}}{0}--\underset{0}{0}--\underset{\sqrt{2}}{0}+++++++$, we can see that $x^4 - 2x^2 \le 0$ on $\left[-\sqrt{2}, \sqrt{2}\right] \Rightarrow a = -\sqrt{2}$ and $b = \sqrt{2}$ minimize the integral.

73. $f(x) = \frac{1}{1+x^2}$ is decreasing on $[0, 1] \Rightarrow$ maximum value of f occurs at $0 \Rightarrow \max f = f(0) = 1$; minimum value of f occurs at $1 \Rightarrow \min f = f(1) = \frac{1}{1+1^2} = \frac{1}{2}$. Therefore, $(1-0)\min f \le \int_0^1 \frac{1}{1+x^2}\,dx \le (1-0)\max f$ $\Rightarrow \frac{1}{2} \le \int_0^1 \frac{1}{1+x^2}\,dx \le 1$. That is, an upper bound $= 1$ and a lower bound $= \frac{1}{2}$.

74. See Exercise 73 above. On $[0, 0.5]$, $\max f = \frac{1}{1+0^2} = 1$, $\min f = \frac{1}{1+(0.5)^2} = 0.8$. Therefore $(0.5-0)\ \min f \le \int_0^{0.5} f(x)\,dx \le (0.5-0)\ \max f \Rightarrow \frac{2}{5} \le \int_0^{0.5} \frac{1}{1+x^2}\,dx \le \frac{1}{2}$. On $[0.5, 1]$, $\max f = \frac{1}{1+(0.5)^2} = 0.8$ and $\min f = \frac{1}{1+1^2} = 0.5$. Therefore $(1-0.5)\ \min f \le \int_{0.5}^{1} \frac{1}{1+x^2}\,dx \le (1-0.5)\ \max f \Rightarrow \frac{1}{4} \le \int_{0.5}^{1} \frac{1}{1+x^2}\,dx \le \frac{2}{5}$. Then $\frac{1}{4} + \frac{2}{5} \le \int_0^{0.5} \frac{1}{1+x^2}\,dx + \int_{0.5}^{1} \frac{1}{1+x^2}\,dx \le \frac{1}{2} + \frac{2}{5} \Rightarrow \frac{13}{20} \le \int_0^1 \frac{1}{1+x^2}\,dx \le \frac{9}{10}$.

75. $-1 \le \sin\left(x^2\right) \le 1$ for all $x \Rightarrow (1-0)(-1) \le \int_0^1 \sin\left(x^2\right)dx \le (1-0)(1)$ or $\int_0^1 \sin x^2 dx \le 1 \Rightarrow \int_0^1 \sin x^2 dx$ cannot equal 2.

76. $f(x) = \sqrt{x+8}$ is increasing on $[0, 1] \Rightarrow \max f = f(1) = \sqrt{1+8} = 3$ and $\min f = f(0) = \sqrt{0+8} = 2\sqrt{2}$. Therefore, $(1-0)\ \min f \le \int_0^1 \sqrt{x+8}\,dx \le (1-0)\ \max f \Rightarrow 2\sqrt{2} \le \int_0^1 \sqrt{x+8}\,dx \le 3$.

77. If $f(x) \ge 0$ on $[a, b]$, then $\min f \ge 0$ and $\max f \ge 0$ on $[a, b]$. Now, $(b-a)\ \min f \le \int_a^b f(x)\,dx \le (b-a)\ \max f$. Then $b \ge a \Rightarrow b-a \ge 0 \Rightarrow (b-a)\ \min f \ge 0 \Rightarrow \int_a^b f(x)\,dx \ge 0$.

78. If $f(x) \le 0$ on $[a, b]$, then $\min f \le 0$ and $\max f \le 0$. Now, $(b-a)\ \min f \le \int_a^b f(x)\,dx \le (b-a)\ \max f$. Then $b \ge a \Rightarrow b-a \ge 0 \Rightarrow (b-a)\ \max f \le 0 \Rightarrow \int_a^b f(x)\,dx \le 0$.

79. $\sin x \le x$ for $x \ge 0 \Rightarrow \sin x - x \le 0$ for $x \ge 0 \Rightarrow \int_0^1 (\sin x - x)\,dx \le 0$ (see Exercise 78) $\Rightarrow \int_0^1 \sin x\,dx - \int_0^1 x\,dx \le 0$ $\Rightarrow \int_0^1 \sin x\,dx \le \int_0^1 x\,dx \Rightarrow \int_0^1 \sin x\,dx \le \left(\frac{1^2}{2} - \frac{0^2}{2}\right) \Rightarrow \int_0^1 \sin x\,dx \le \frac{1}{2}$. Thus an upper bound is $\frac{1}{2}$.

80. $\sec x \ge 1 + \frac{x^2}{2}$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \sec x - \left(1 + \frac{x^2}{2}\right) \ge 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \int_0^1 \left[\sec x - \left(1 + \frac{x^2}{2}\right)\right] dx \ge 0$ (see Exercise 77) since $[0, 1]$ is contained in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \int_0^1 \sec x\,dx - \int_0^1 \left(1 + \frac{x^2}{2}\right) dx \ge 0 \Rightarrow \int_0^1 \sec x\,dx \ge \int_0^1 \left(1 + \frac{x^2}{2}\right) dx$ $\Rightarrow \int_0^1 \sec x\,dx \ge \int_0^1 1\,dx + \frac{1}{2}\int_0^1 x^2 dx \Rightarrow \int_0^1 \sec x\,dx \ge (1-0) + \frac{1}{2}\left(\frac{1^3}{3}\right) \Rightarrow \int_0^1 \sec x\,dx \ge \frac{7}{6}$. Thus a lower bound is $\frac{7}{6}$.

81. Yes, for the following reasons: $\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$ is a constant K. Thus $\int_a^b \mathrm{av}(f)\,dx = \int_a^b K\,dx = K(b-a) \Rightarrow \int_a^b \mathrm{av}(f)\,dx = (b-a)K = (b-a)\cdot\frac{1}{b-a}\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$.

82. All three rules hold. The reasons: On any interval $[a, b]$ on which f and g are integrable, we have:

(a) $\mathrm{av}(f+g) = \frac{1}{b-a}\int_a^b [f(x)+g(x)]dx = \frac{1}{b-a}\left[\int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right] = \frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{b-a}\int_a^b g(x)\,dx$ $= \mathrm{av}(f) + \mathrm{av}(g)$

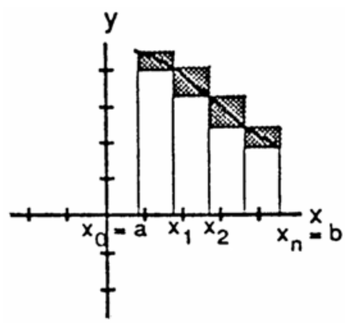
(b) $\mathrm{av}(kf) = \frac{1}{b-a}\int_a^b kf(x)\,dx = \frac{1}{b-a}\left[k\int_a^b f(x)dx\right] = k\left[\frac{1}{b-a}\int_a^b f(x)\,dx\right] = k\ \mathrm{av}(f)$

(c) $\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b g(x)\,dx$ since $f(x) \le g(x)$ on $[a, b]$, and $\frac{1}{b-a}\int_a^b g(x)\,dx = \text{av}(g)$. Therefore, $\text{av}(f) \le \text{av}(g)$.

83. (a) $U = \max_1 \Delta x + \max_2 \Delta x + \ldots + \max_n \Delta x$ where $\max_1 = f(x_1)$, $\max_2 = f(x_2), \ldots, \max_n = f(x_n)$ since f is increasing on $[a, b]$; $L = \min_1 \Delta x + \min_2 \Delta x + \ldots + \min_n \Delta x$ where $\min_1 = f(x_0)$, $\min_2 = f(x_1), \ldots,$ $\min_n = f(x_{n-1})$ since f is increasing on $[a, b]$. Therefore
$U - L = (\max_1 - \min_1)\Delta x + (\max_2 - \min_2)\Delta x + \ldots + (\max_n - \min_n)\Delta x$
$= (f(x_1) - f(x_0))\Delta x + (f(x_2) - f(x_1))\Delta x + \ldots + (f(x_n) - f(x_{n-1}))\Delta x = (f(x_n) - f(x_0))\,\Delta x$
$= (f(b) - f(a))\,\Delta x.$

(b) $U = \max_1 \Delta x_1 + \max_2 \Delta x_2 + \ldots + \max_n \Delta x_n$ where $\max_1 = f(x_1)$, $\max_2 = f(x_2), \ldots, \max_n = f(x_n)$ since f is increasing on $[a, b]$; $L = \min_1 \Delta x_1 + \min_2 \Delta x_2 + \ldots + \min_n \Delta x_n$ where $\min_1 = f(x_0)$, $\min_2 = f(x_1), \ldots,$ $\min_n = f(x_{n-1})$ since f is increasing on $[a, b]$. Therefore
$U - L = (\max_1 - \min_1)\,\Delta x_1 + (\max_2 - \min_2)\Delta x_2 + \ldots + (\max_n - \min_n)\,\Delta x_n$
$= (f(x_1) - f(x_0))\,\Delta x_1 + (f(x_2) - f(x_1))\Delta x_2 + \ldots + (f(x_n) - f(x_{n-1}))\,\Delta x_n$
$\le (f(x_1) - f(x_0))\,\Delta x_{\max} + (f(x_2) - f(x_1))\Delta x_{\max} + \ldots + (f(x_n) - f(x_{n-1}))\,\Delta x_{\max}$. Then
$U - L \le (f(x_n) - f(x_0))\,\Delta x_{\max} = (f(b) - f(a))\,\Delta x_{\max} = |f(b) - f(a)|\Delta x_{\max}$ since $f(b) \ge f(a)$. Thus
$\lim_{\|P\|\to 0}(U - L) = \lim_{\|P\|\to 0}(f(b) - f(a))\,\Delta x_{\max} = 0$, since $\Delta x_{\max} = \|P\|$.

84. (a) $U = \max_1 \Delta x + \max_2 \Delta x + \ldots + \max_n \Delta x$ where $\max_1 = f(x_0)$, $\max_2 = f(x_1), \ldots, \max_n = f(x_{n-1})$ since f is decreasing on $[a, b]$;
$L = \min_1 \Delta x + \min_2 \Delta x + \ldots + \min_n \Delta x$ where $\min_1 = f(x_1)$, $\min_2 = f(x_2), \ldots, \min_n = f(x_n)$ since f is decreasing on $[a, b]$. Therefore
$U - L = (\max_1 - \min_1)\Delta x + (\max_2 - \min_2)\,\Delta x + \ldots + (\max_n - \min_n)\,\Delta x$
$= (f(x_0) - f(x_1))\,\Delta x + (f(x_1) - f(x_2))\,\Delta x + \ldots + (f(x_{n-1}) - f(x_n))\,\Delta x$
$= (f(x_0) - f(x_n))\,\Delta x = (f(a) - f(b))\,\Delta x.$



(b) $U = \max_1 \Delta x_1 + \max_2 \Delta x_2 + \ldots + \max_n \Delta x_n$ where $\max_1 = f(x_0)$, $\max_2 = f(x_1), \ldots, \max_n = f(x_{n-1})$ since f is decreasing on $[a, b]$; $L = \min_1 \Delta x_1 + \min_2 \Delta x_2 + \ldots + \min_n \Delta x_n$ where $\min_1 = f(x_1)$, $\min_2 = f(x_2), \ldots, \min_n = f(x_n)$ since f is decreasing on $[a, b]$. Therefore
$U - L = (\max_1 - \min_1)\Delta x_1 + (\max_2 - \min_2)\Delta x_2 + \ldots + (\max_n - \min_n)\Delta x_n$
$= (f(x_0) - f(x_1))\Delta x_1 + (f(x_1) - f(x_2))\Delta x_2 + \ldots + (f(x_{n-1}) - f(x_n))\Delta x_n \le (f(x_0) - f(x_n))\Delta x_{\max}$
$= (f(a) - f(b)\Delta x_{\max} = |f(b) - f(a)|\,\Delta x_{\max}$ since $f(b) \le f(a)$. Thus
$\lim_{\|P\|\to 0}(U - L) = \lim_{\|P\|\to 0}|f(b) - f(a)|\,\Delta x_{\max} = 0$, since $\Delta x_{\max} = \|P\|$.

85. (a) Partition $\left[0, \frac{\pi}{2}\right]$ into n subintervals, each of length $\Delta x = \frac{\pi}{2n}$ with points $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x, \ldots, x_n = n\Delta x = \frac{\pi}{2}$. Since $\sin x$ is increasing on $\left[0, \frac{\pi}{2}\right]$, the upper sum U is the sum of the areas of the circumscribed rectangles of areas $f(x_1)\Delta x = (\sin \Delta x)\Delta x$, $f(x_2)\Delta x = (\sin 2\Delta x)\Delta x, \ldots,$ $f(x_n)\Delta x = (\sin n\Delta x)\Delta x.$

Then $U = (\sin \Delta x + \sin 2\Delta x + \ldots + \sin n\Delta x)\Delta x = \left[\frac{\cos\frac{\Delta x}{2} - \cos\left(\left(n+\frac{1}{2}\right)\Delta x\right)}{2\sin\frac{\Delta x}{2}}\right]\Delta x = \left[\frac{\cos\frac{\pi}{4n} - \cos\left(\left(n+\frac{1}{2}\right)\frac{\pi}{2n}\right)}{2\sin\frac{\pi}{4n}}\right]\left(\frac{\pi}{2n}\right)$

$= \frac{\pi\left(\cos\frac{\pi}{4n} - \cos\left(\frac{\pi}{2}+\frac{\pi}{4n}\right)\right)}{4n\sin\frac{\pi}{4n}} = \frac{\cos\frac{\pi}{4n} - \cos\left(\frac{\pi}{2}+\frac{\pi}{4n}\right)}{\left(\frac{\sin\frac{\pi}{4n}}{\frac{\pi}{4n}}\right)}$

(b) The area is $\int_0^{\pi/2} \sin x\, dx = \lim_{n\to\infty} \frac{\cos\frac{\pi}{4n} - \cos\left(\frac{\pi}{2}+\frac{\pi}{4n}\right)}{\left(\frac{\sin\frac{\pi}{4n}}{\frac{\pi}{4n}}\right)} = \frac{1-\cos\frac{\pi}{2}}{1} = 1.$

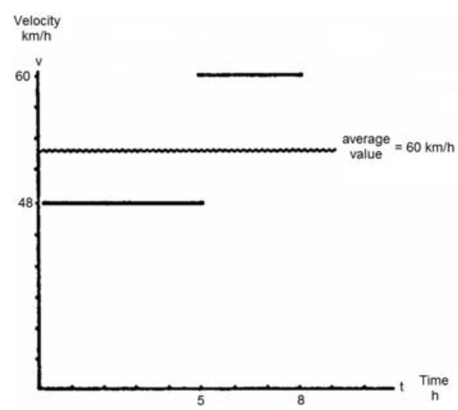
86. (a) The area of the shaded region is $\sum_{i=1}^{n} \Delta x_i \cdot m_i$ which is equal to L.

(b) The area of the shaded region is $\sum_{i=1}^{n} \Delta x_i \cdot M_i$ which is equal to U.

(c) The area of the shaded region is the difference in the areas of the shaded regions shown in the second part of the figure and the first part of the figure. Thus this area is $U - L$.

87. By Exercise 86, $U - L = \sum_{i=1}^{n} \Delta x_i \cdot M_i - \sum_{i=1}^{n} \Delta x_i \cdot m_i$ where $M_i = \max\,\{f(x) \text{ on the } i\text{th subinterval}\}$ and $m_i = \min\,\{f(x) \text{ on } i\text{th subinterval}\}$. Thus $U - L = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i < \sum_{i=1}^{n} \epsilon \cdot \Delta x_i$ provided $\Delta x_i < \delta$ for each $i = 1, \ldots, n$. Since $\sum_{i=1}^{n} \epsilon \cdot \Delta x_i = \epsilon \sum_{i=1}^{n} \Delta x_i = \epsilon(b-a)$ the result, $U - L < \epsilon(b-a)$ follows.

88. The car drove the first 240 kilometers in 5 hours and the second 240 kilometers in 3 hours, which means it drove 480 kilometers in 8 hours, for an average value of $\frac{480}{8}$ km/h = 60 km/h. In terms of average value of functions, the function whose average value we seek is $v(t) = \begin{cases} 48, & 0 \le t \le 5 \\ 80, & 5 \le t \le 8 \end{cases}$, and the average value is $\frac{(48)(5)+(80)(3)}{8} = 60.$



89-94. Example CAS commands:

Maple:

```
with( plots );
with( Student[Calculus1] );
f := x -> 1-x;
a := 0;
b := 1;
N :=[4, 10, 20, 50];
P := [seq( RiemannSum( f(x), x=a..b, partition=n, method=random, output=plot ), n=N )]:
display( P, insequence=true);
```

95-98. Example CAS commands:

Maple:

```
with( Student[Calculus1] );
f := x -> sin(x);
a := 0;
b := Pi;
plot( f(x), x=a..b, title="#95(a) (Section 5.3)" );
N := [ 100, 200, 1000 ];                                # (b)
for n in N do
  Xlist := [ a+1.*(b-a)/n*i $ i=0..n ];
  Ylist := map( f, Xlist );
end do:
for n in N do                                           # (c)
  Avg[n] := evalf(add(y,y=Ylist)/nops(Ylist));
end do;
avg := FunctionAverage( f(x), x=a..b, output=value );
evalf( avg );
FunctionAverage(f(x),x=a..b, output=plot);              # (d)
fsolve( f(x)=avg, x=0.5 );
fsolve( f(x)=avg, x=2.5 );
fsolve( f(x)=Avg[1000], x=0.5 );
fsolve( f(x)=Avg[1000], x=2.5 );
```

95-98. Example CAS commands:

Mathematica: (assigned function and values for a, b, and n may vary)

Sums of rectangles evaluated at left-hand endpoints can be represented and evaluated by this set of commands

```
Clear[x, f, a, b, n]
{a, b}={0, π}; n =10; dx = (b – a)/n;
f = Sin[x]^2;
xvals=Table[N[x],{x, a, b – dx, dx}];
yvals = f /.x → xvals;
boxes = MapThread[Line[{{#1, 0},{#1, #3},{#2, #3},{#2, 0}]&,{xvals, xvals + dx, yvals}];
Plot[f, {x, a, b}, Epilog → boxes];
Sum[yvals[[i]] dx, {i, 1, Length[yvals]}]//N
```

Sums of rectangles evaluated at right-hand endpoints can be represented and evaluated by this set of commands.

```
Clear[x, f, a, b, n]
{a, b}={0, π}; n =10; dx = (b – a)/n;
f = Sin[x]^2;
xvals =Table[N[x], {x, a + dx, b, dx}];
yvals = f /.x → xvals;
boxes = MapThread[Line[{{#1, 0},{#1, #3},{#2, #3},{#2, 0}]&,{xvals, – dx,xvals, yvals}];
Plot[f, {x, a, b}, Epilog → boxes];
Sum[yvals[[i]] dx, {i, 1, Length[yvals]}]//N
```

Sums of rectangles evaluated at midpoints can be represented and evaluated by this set of commands.

```
Clear[x, f, a, b, n]
{a, b}={0, π}; n =10; dx = (b – a)/n;
f = Sin[x]^2;
xvals =Table[N[x], {x, a + dx/2, b – dx/2, dx}];
yvals = f /.x → xvals;
boxes = MapThread[Line[{{#1, 0},{#1, #3},{#2, #3},{#2, 0}]&,{xvals, – dx/2, xvals + dx/2, yvals}];
Plot[f, {x, a, b},Epilog → boxes];
Sum[yvals[[i]] dx, {i, 1, Length[yvals]}]//N
```

5.4 THE FUNDAMENTAL THEOREM OF CALCULUS

1. $\int_0^2 x(x-3)\,dx = \int_0^2 (x^2-3x)\,dx = \left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_0^2 = \left(\frac{(2)^3}{3}-\frac{3(2)^2}{2}\right)-\left(\frac{(0)^3}{3}-\frac{3(0)^2}{2}\right) = -\frac{10}{3}$

2. $\int_{-1}^1 \left(x^2-2x+3\right)dx = \left[\frac{x^3}{3}-x^2+3x\right]_{-1}^1 = \left(\frac{(1)^3}{3}-(1)^2+3(1)\right)-\left(\frac{(-1)^3}{3}-(-1)^2+3(-1)\right) = \frac{20}{3}$

3. $\int_{-2}^2 \frac{3}{(x+3)^4}\,dx = -\frac{1}{(x+3)^3}\Bigg]_{-2}^2 = \left(-\frac{1}{(5)^3}-\left(-\frac{1}{(1)^3}\right)\right) = 1-\frac{1}{125} = \frac{124}{125}$

4. $\int_{-1}^1 x^{299}\,dx = \frac{x^{300}}{300}\Bigg]_{-1}^1 = \frac{1}{300}\left((1)^{300}-(-1)^{300}\right) = \frac{1}{300}(1-1) = 0$

5. $\int_1^4 \left(3x^2-\frac{x^3}{4}\right)dx = \left[x^3-\frac{x^4}{16}\right]_1^4 = \left(\left(4^3-\frac{4^4}{16}\right)-\left(1^3-\frac{1^4}{16}\right)\right) = \left(64-16-1+\frac{1}{16}\right) = \frac{753}{16}$

6. $\int_{-2}^3 \left(x^3-2x+3\right)dx = \left[\frac{x^4}{4}-x^2+3x\right]_1^4$

$$= \left(\frac{3^4}{4}-3^2+3(3)\right)-\left(\frac{(-2)^4}{4}-(-2)^2+3(-2)\right) = \frac{81}{4}+6 = \frac{105}{4}$$

7. $\int_0^1 \left(x^2+\sqrt{x}\right)dx = \left[\frac{x^3}{3}+\frac{2}{3}x^{3/2}\right]_0^1 = \left(\frac{1}{3}+\frac{2}{3}\right)-0 = 1$

8. $\int_1^{32} x^{-6/5}\,dx = \left[-5x^{-1/5}\right]_1^{32} = \left(-\frac{5}{2}\right)-(-5) = \frac{5}{2}$

9. $\int_0^{\pi/3} 2\sec^2 x\,dx = [2\tan x]_0^{\pi/3} = \left(2\tan\left(\frac{\pi}{3}\right)\right)-(2\tan 0) = 2\sqrt{3}-0 = 2\sqrt{3}$

10. $\int_0^{\pi}(1+\cos x)\,dx=[x+\sin x]_0^{\pi}=(\pi+\sin\pi)-(0+\sin 0)=\pi$

11. $\int_{\pi/4}^{3\pi/4}\csc\theta\cot\theta\,d\theta=[-\csc\theta]_{\pi/4}^{3\pi/4}=\left(-\csc\left(\frac{3\pi}{4}\right)\right)-\left(-\csc\left(\frac{\pi}{4}\right)\right)=-\sqrt{2}-\left(-\sqrt{2}\right)=0$

12. $\int_0^{\pi/3}4\frac{\sin u}{\cos^2 u}du=\left.\frac{4}{\cos u}\right]_0^{\pi/3}=\left(\frac{4}{(1/2)}-\frac{4}{1}\right)=4$

13. $\int_{\pi/2}^{0}\frac{1+\cos 2t}{2}\,dt=\int_{\pi/2}^{0}\left(\frac{1}{2}+\frac{1}{2}\cos 2t\right)dt=\left[\frac{1}{2}t+\frac{1}{4}\sin 2t\right]_{\pi/2}^{0}=\left(\frac{1}{2}(0)+\frac{1}{4}\sin 2(0)\right)-\left(\frac{1}{2}\left(\frac{\pi}{2}\right)+\frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right)=-\frac{\pi}{4}$

14. $\int_{-\pi/3}^{\pi/3}\sin^2 t\,dt$ Use the double angle formula $\cos 2t=1-2\sin^2 t$ which implies that $\sin^2 t=\frac{1-\cos(2t)}{2}$.

$$\int_{-\pi/3}^{\pi/3}\sin^2 t\,dt=\int_{-\pi/3}^{\pi/3}\frac{1-\cos 2t}{2}dt=\left[\frac{t}{2}-\frac{\sin 2t}{4}\right]_{-\pi/3}^{\pi/3}$$
$$=\left(\frac{\pi}{6}-\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right)-\left(-\frac{\pi}{6}-\frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right)\right)=\frac{\pi}{3}-\frac{\sqrt{3}}{4}$$

15. $\int_0^{\pi/4}\tan^2 x\,dx=\int_0^{\pi/4}(\sec^2 x-1)\,dx=[\tan x-x]_0^{\pi/4}=\left(\tan\left(\frac{\pi}{4}\right)-\frac{\pi}{4}\right)-(\tan(0)-0)=1-\frac{\pi}{4}$

16. $\int_0^{\pi/6}(\sec x+\tan x)^2\,dx=\int_0^{\pi/6}(\sec^2 x+2\sec x\tan x+\tan^2 x)\,dx=\int_0^{\pi/6}(2\sec^2 x+2\sec x\tan x-1)\,dx$
$=[2\tan x+2\sec x-x]_0^{\pi/6}\left(2\tan\left(\frac{\pi}{6}\right)+2\sec\left(\frac{\pi}{6}\right)-\left(\frac{\pi}{6}\right)\right)-(2\tan 0+2\sec 0-0)=2\sqrt{3}-\frac{\pi}{6}-2$

17. $\int_0^{\pi/8}\sin 2x\,dx=\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/8}=\left(-\frac{1}{2}\cos 2\left(\frac{\pi}{8}\right)\right)-\left(-\frac{1}{2}\cos 2(0)\right)=\frac{2-\sqrt{2}}{4}$

18. $\int_{-\pi/3}^{-\pi/4}\left(4\sec^2 t+\frac{\pi}{t^2}\right)dt=\int_{-\pi/3}^{-\pi/4}(4\sec^2 t+\pi t^{-2})\,dt=\left[4\tan t-\frac{\pi}{t}\right]_{-\pi/3}^{-\pi/4}$
$=\left(4\tan\left(-\frac{\pi}{4}\right)-\frac{\pi}{\left(-\frac{\pi}{4}\right)}\right)-\left(4\tan\left(\frac{\pi}{3}\right)-\frac{\pi}{\left(-\frac{\pi}{3}\right)}\right)=(4(-1)+4)-\left(4\left(-\sqrt{3}\right)+3\right)=4\sqrt{3}-3$

19. $\int_1^{-1}(r+1)^2dr=\int_1^{-1}(r^2+2r+1)\,dr=\left[\frac{r^3}{3}+r^2+r\right]_1^{-1}=\left(\frac{(-1)^3}{3}+(-1)^2+(-1)\right)-\left(\frac{1^3}{3}+1^2+1\right)=-\frac{8}{3}$

20. $\int_{-\sqrt{3}}^{\sqrt{3}}(t+1)(t^2+4)\,dt=\int_{-\sqrt{3}}^{\sqrt{3}}(t^3+t^2+4t+4)\,dt=\left[\frac{t^4}{4}+\frac{t^3}{3}+2t^2+4t\right]_{-\sqrt{3}}^{\sqrt{3}}$
$=\left(\frac{\left(\sqrt{3}\right)^4}{4}+\frac{\left(\sqrt{3}\right)^3}{3}+2(\sqrt{3})^2+4\sqrt{3}\right)-\left(\frac{\left(-\sqrt{3}\right)^4}{4}+\frac{\left(-\sqrt{3}\right)^3}{3}+2(-\sqrt{3})^2+4(-\sqrt{3})\right)=10\sqrt{3}$

21. $\int_{\sqrt{2}}^{1}\left(\frac{u^7}{2}-\frac{1}{u^5}\right)du=\int_{\sqrt{2}}^{1}\left(\frac{u^7}{2}-u^{-5}\right)du=\left[\frac{u^8}{16}+\frac{1}{4u^4}\right]_{\sqrt{2}}^{1}=\left(\frac{1^8}{16}+\frac{1}{4(1)^4}\right)-\left(\frac{(\sqrt{2})^8}{16}+\frac{1}{4\left(\sqrt{2}\right)^4}\right)=-\frac{3}{4}$

22. $\int_{-3}^{-1}\frac{y^5-2y}{y^3}\,dy=\int_{-3}^{-1}(y^2-2y^{-2})\,dy=\left[\frac{y^3}{3}+2y^{-1}\right]_{-3}^{-1}=\left(\frac{(-1)^3}{3}+\frac{2}{(-1)}\right)-\left(\frac{(-3)^3}{3}+\frac{2}{(-3)}\right)=\frac{22}{3}$

23. $\int_{1}^{\sqrt{2}}\frac{s^2+\sqrt{s}}{s^2}\,ds=\int_{1}^{\sqrt{2}}(1+s^{-3/2})\,ds=\left[s-\frac{2}{\sqrt{s}}\right]_{1}^{\sqrt{2}}=\left(\sqrt{2}-\frac{2}{\sqrt{\sqrt{2}}}\right)-\left(1-\frac{2}{\sqrt{1}}\right)=\sqrt{2}-2^{3/4}+1=\sqrt{2}-\sqrt[4]{8}+1$

24. $\int_{1}^{8}\frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}\,dx=\int_{1}^{8}\frac{2x^{1/3}-x+2-x^{2/3}}{x^{1/3}}\,dx=\int_{1}^{8}(2-x^{2/3}+2x^{-1/3}-x^{1/3})\,dx=\left[2x-\frac{3}{5}x^{5/3}+3x^{2/3}-\frac{3}{4}x^{4/3}\right]_{1}^{3}$
$=\left(2(8)-\frac{3}{5}(8)^{5/3}+3(8)^{2/3}-\frac{3}{4}(8)^{4/3}\right)-\left(2(1)-\frac{3}{5}(1)^{5/3}+3(1)^{2/3}-\frac{3}{4}(1)^{4/3}\right)=-\frac{137}{20}$

25. $\int_{\pi/2}^{\pi}\frac{\sin 2x}{2\sin x}\,dx=\int_{\pi/2}^{\pi}\frac{2\sin x\cos x}{2\sin x}\,dx=\int_{\pi/2}^{\pi}\cos x\,dx=\left[\sin x\right]_{\pi/2}^{\pi}=(\sin(\pi))-\left(\sin\left(\frac{\pi}{2}\right)\right)=-1$

26. $\int_{0}^{\pi/3}(\cos x+\sec x)^2\,dx=\int_{0}^{\pi/3}(\cos^2 x+2+\sec^2 x)\,dx=\int_{0}^{\pi/3}\left(\frac{\cos 2x+1}{2}+2+\sec^2 x\right)dx$
$=\int_{0}^{\pi/3}\left(\frac{1}{2}\cos 2x+\frac{5}{2}+\sec^2 x\right)dx=\left[\frac{1}{4}\sin 2x+\frac{5}{2}x+\tan x\right]_{0}^{\pi/3}$
$=\left(\frac{1}{4}\sin 2\left(\frac{\pi}{3}\right)+\frac{5}{2}\left(\frac{\pi}{3}\right)+\tan\left(\frac{\pi}{3}\right)\right)-\left(\frac{1}{4}\sin 2(0)+\frac{5}{2}(0)+\tan(0)\right)=\frac{5\pi}{6}+\frac{9\sqrt{3}}{8}$

27. $\int_{-4}^{4}|x|\,dx=\int_{-4}^{0}|x|\,dx+\int_{0}^{4}|x|\,dx=-\int_{-4}^{0}x\,dx+\int_{0}^{4}x\,dx=\left[-\frac{x^2}{2}\right]_{-4}^{0}+\left[\frac{x^2}{2}\right]_{0}^{4}=\left(-\frac{0^2}{2}+\frac{(-4)^2}{2}\right)+\left(\frac{4^2}{2}-\frac{0^2}{2}\right)=16$

28. $\int_{0}^{\pi}\frac{1}{2}\left(\cos x+|\cos x|\right)dx=\int_{0}^{\pi/2}\frac{1}{2}(\cos x+\cos x)\,dx+\int_{\pi/2}^{\pi}\frac{1}{2}(\cos x-\cos x)\,dx=\int_{0}^{\pi/2}\cos x\,dx=[\sin x]_{0}^{\pi/2}$
$=\sin\frac{\pi}{2}-\sin 0=1$

29. $\int_{0}^{\sqrt{\pi/2}}x\cos x^2\,dx=\frac{1}{2}\sin x^2\Big|_{0}^{\sqrt{\pi/2}}=\frac{1}{2}\left(\sin\frac{\pi}{2}-\sin 0\right)=\frac{1}{2}$

30. $\int_{1}^{\pi^2}\frac{\sin\sqrt{x}}{x}dx=-2\cos\sqrt{x}\Big|_{1}^{\pi^2}=-2\left(-1-\cos 1\right)=2+2\cos 1$

31. $\int_{2}^{5}\frac{x}{\sqrt{1+x^2}}\,dx=\int_{2}^{5}x(1+x^2)^{1/2}\,dx=\sqrt{1+x^2}\Big|_{2}^{5}=\sqrt{26}-\sqrt{5}$

32. $\int_{0}^{\pi/3}\sin^2 x\cos x\,dx=\int_{0}^{\pi/3}(\sin x)^2\cos x\,dx=\frac{1}{3}(\sin x)^3\Big|_{0}^{\pi/3}=\frac{1}{3}\sin^3\left(\frac{\pi}{3}\right)-\frac{1}{3}\sin^3(0)=\frac{\sqrt{3}}{8}$

33. (a) $\int_0^{\sqrt{x}} \cos t\, dt = [\sin t]_0^{\sqrt{x}} = \sin\sqrt{x} - \sin 0 = \sin\sqrt{x} \Rightarrow \frac{d}{dx}\left(\int_0^{\sqrt{x}} \cos t\, dt\right)$
$= \frac{d}{dx}(\sin\sqrt{x}) = \cos\sqrt{x}\left(\frac{1}{2}x^{-1/2}\right) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$

(b) $\frac{d}{dx}\left(\int_0^{\sqrt{x}} \cos t\, dt\right) = (\cos\sqrt{x})\left(\frac{d}{dx}(\sqrt{x})\right) = (\cos\sqrt{x})\left(\frac{1}{2}x^{-1/2}\right) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$

34. (a) $\int_1^{\sin x} 3t^2\, dt = [t^3]_1^{\sin x} = \sin^3 x - 1 \Rightarrow \frac{d}{dx}\left(\int_1^{\sin x} 3t^2\, dt\right) = \frac{d}{dx}(\sin^3 x - 1) = 3\sin^2 x \cos x$

(b) $\frac{d}{dx}\left(\int_1^{\sin x} 3t^2 dt\right) = (3\sin^2 x)\left(\frac{d}{dx}(\sin x)\right) = 3\sin^2 x \cos x$

35. (a) $\int_0^{t^4} \sqrt{u}\, du = \int_0^{t^4} u^{1/2}\, du = \left[\frac{2}{3}u^{3/2}\right]_0^{t^4} = \frac{2}{3}(t^4)^{3/2} - 0 = \frac{2}{3}t^6 \Rightarrow \frac{d}{dt}\left(\int_0^{t^4} \sqrt{u}\, du\right) = \frac{d}{dt}\left(\frac{2}{3}t^6\right) = 4t^5$

(b) $\frac{d}{dt}\left(\int_0^{t^4} \sqrt{u}\, du\right) = \sqrt{t^4}\left(\frac{d}{dt}(t^4)\right) = t^2(4t^3) = 4t^5$

36. (a) $\int_0^{\tan\theta} \sec^2 y\, dy = [\tan y]_0^{\tan\theta} = \tan(\tan\theta) - 0 = \tan(\tan\theta) \Rightarrow \frac{d}{d\theta}\left(\int_0^{\tan\theta} \sec^2 y\, dy\right)$
$= \frac{d}{d\theta}(\tan(\tan\theta)) = (\sec^2(\tan\theta))\sec^2\theta$

(b) $\frac{d}{d\theta}\left(\int_0^{\tan\theta} \sec^2 y\, dy\right) = (\sec^2(\tan\theta))\left(\frac{d}{d\theta}(\tan\theta)\right) = (\sec^2(\tan\theta))\sec^2\theta$

37. (a) $\int_0^{x^3} t^{-2/3}\, dt = 3t^{1/3}\Big|_0^{x^3} = 3(x - 0) = 3x \Rightarrow \frac{d}{dx}\left(\int_0^{x^3} t^{-2/3}\, dt\right) = \frac{d}{dx}(3x) = 3$

(b) $\frac{d}{dx}\left(\int_0^{x^3} t^{-2/3}\, dt\right) = \left(x^3\right)^{-2/3}\left(\frac{d}{dx}x^3\right) = \left(x^{-2}\right)\left(3x^2\right) = 3$

38. (a) $\int_1^{\sqrt{t}}\left(x^4 + \frac{3}{x^3}\right)dx = \frac{x^5}{5} - \frac{3}{2x^2}\Big|_1^{\sqrt{t}} = \frac{1}{5}t^{5/2} + \frac{3}{2}t^{-1} + \frac{13}{10}$
$\Rightarrow \frac{d}{dt}\int_0^{\sqrt{t}}\left(x^4 + \frac{3}{x^3}\right)dx = \frac{d}{dt}\left(\frac{1}{5}t^{5/2} - \frac{3}{2}t^{-1} + \frac{13}{10}\right) = \frac{1}{5}\cdot\frac{5}{2}t^{3/2} + \frac{3}{2}t^{-2} = \frac{1}{2}t^{3/2} + \frac{3}{2}t^{-2}$

(b) $\frac{d}{dt}\int_1^{\sqrt{t}}\left(x^4 + \frac{3}{x^3}\right)dx = \left(t^2 + \frac{3}{t^{3/2}}\right)\cdot\frac{d}{dt}\left(\sqrt{t}\right) = \left(t^2 + \frac{3}{t^{3/2}}\right)\cdot\frac{1}{2}t^{-1/2} = \frac{1}{2}t^{3/2} + \frac{3}{2}t^{-2}$

39. $y = \int_0^x \sqrt{1+t^2}\, dt \Rightarrow \frac{dy}{dx} = \sqrt{1+x^2}$

40. $y = \int_1^x \frac{1}{t}\, dt \Rightarrow \frac{dy}{dx} = \frac{1}{x},\ x > 0$

41. $y = \int_{\sqrt{x}}^0 \sin t^2 dt = -\int_0^{\sqrt{x}} \sin t^2\, dt \Rightarrow \frac{dy}{dx} = -\left(\sin(\sqrt{x})^2\right)\left(\frac{d}{dx}(\sqrt{x})\right) = -(\sin x)\left(\frac{1}{2}x^{-1/2}\right) = -\frac{\sin x}{2\sqrt{x}}$

42. $y = x\int_2^{x^2} \sin t^3 dt \Rightarrow \frac{dy}{dx} = x\cdot\frac{d}{dx}\left(\int_2^{x^2} \sin t^3 dt\right) + 1\cdot\int_2^{x^2} \sin t^3 dt = x\cdot\sin(x^2)^3\,\frac{d}{dx}(x^2) + \int_2^{x^2} \sin t^3 dt$

$= 2x^2 \sin x^6 + \int_2^{x^2} \sin t^3 dt$

43. $y = \int_{-1}^{x} \frac{t^2}{t^2+4}\,dt - \int_{3}^{x} \frac{t^2}{t^2+4}\,dt \Rightarrow \frac{dy}{dx} = \frac{x^2}{x^2+4} - \frac{x^2}{x^2+4} = 0$

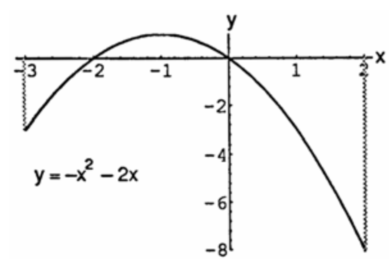
44. $y = \left(\int_0^x (t^3+1)^{10}dt\right)^3 \Rightarrow \frac{dy}{dx} = 3\left(\int_0^x (t^3+1)^{10}dt\right)^2 \frac{d}{dx}\left(\int_0^x (t^3+1)^{10}dt\right) = 3(x^3+1)^{10}\left(\int_0^x (t^3+1)^{10}dt\right)^2$

45. $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, |x| < \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 x}}\left(\frac{d}{dx}(\sin x)\right) = \frac{1}{\sqrt{\cos^2 x}}(\cos x) = \frac{\cos x}{|\cos x|} = \frac{\cos x}{\cos x} = 1 \text{ since } |x| < \frac{\pi}{2}$

46. $y = \int_0^{\tan x} \frac{dt}{1+t^2} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{1+\tan^2 x}\right)\left(\frac{d}{dx}(\tan x)\right) = \left(\frac{1}{\sec^2 x}\right)\left(\sec^2 x\right) = 1$

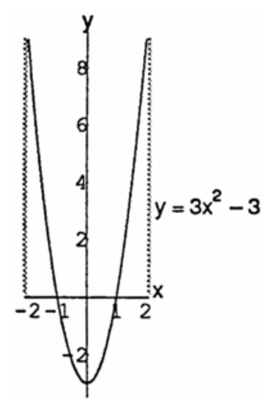
47. $-x^2 - 2x = 0 \Rightarrow -x(x+2) = 0 \Rightarrow x = 0$ or $x = -2$;

$$\text{Area} = -\int_{-3}^{-2}(-x^2-2x)dx + \int_{-2}^{0}(-x^2-2x)dx - \int_0^2(-x^2-2x)dx$$
$$= -\left[-\frac{x^3}{3} - x^2\right]_{-3}^{-2} + \left[-\frac{x^3}{3} - x^2\right]_{-2}^{0} - \left[-\frac{x^3}{3} - x^2\right]_{0}^{2}$$
$$= -\left(\left(-\frac{(-2)^3}{3} - (-2)^2\right) - \left(-\frac{(-3)^3}{3} - (-3)^2\right)\right) + \left(\left(-\frac{0^3}{3} - 0^2\right) - \left(-\frac{(-2)^3}{3} - (-2)^2\right)\right) - \left(\left(-\frac{2^3}{3} - 2^2\right) - \left(-\frac{0^3}{3} - 0^2\right)\right) = \frac{28}{3}$$



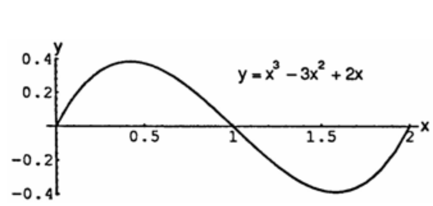
48. $3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$;
because of symmetry about the y-axis,

$$\text{Area} = 2\left(-\int_0^1 (3x^2-3)dx + \int_1^2 (3x^2-3)dx\right)$$
$$2\left(-[x^3-3x]_0^1 + [x^3-3x]_1^2\right)$$
$$= 2[-((1^3-3(1)) - (0^3-3(0))) + ((2^3-3(2)) - (1^3-3(1))] = 2(6) = 12$$



49. $x^3 - 3x^2 + 2x = 0 \Rightarrow x(x^2 - 3x + 2) = 0$
$\Rightarrow x(x-2)(x-1) = 0 \Rightarrow x = 0, 1,$ or 2;

$$\text{Area} = \int_0^1 (x^3-3x^2+2x)dx - \int_1^2 (x^3-3x^2+2x)dx$$
$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 = \left(\frac{1^4}{4} - 1^3 + 1^2\right) - \left(\frac{0^4}{4} - 0^3 + 0^2\right) - \left[\left(\frac{2^4}{4} - 2^3 + 2^2\right) - \left(\frac{1^4}{4} - 1^3 + 1^2\right)\right] = \frac{1}{2}$$



50. $x^{1/3}-x=0 \Rightarrow x^{1/3}\left(1-x^{2/3}\right)=0 \Rightarrow x^{1/3}=0$ or $1-x^{2/3}=0 \Rightarrow x=0$ or $1=x^{2/3} \Rightarrow x=0$ or $1=x^2 \Rightarrow x=0$ or $x=\pm 1$;

Area $=-\int_{-1}^{0}(x^{1/3}-x)\,dx+\int_{0}^{1}(x^{1/3}-x)\,dx-\int_{1}^{8}(x^{1/3}-x)\,dx$

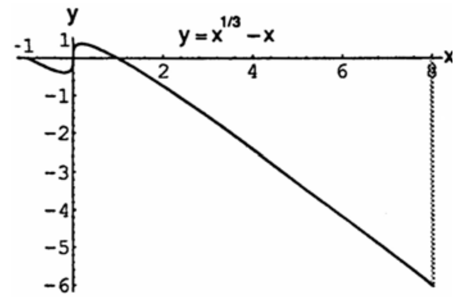
$=-\left[\frac{3}{4}x^{4/3}-\frac{x^2}{2}\right]_{-1}^{0}+\left[\frac{3}{4}x^{4/3}-\frac{x^2}{2}\right]_{0}^{1}-\left[\frac{3}{4}x^{4/3}-\frac{x^2}{2}\right]_{1}^{8}$

$=-\left[\left(\frac{3}{4}(0)^{4/3}-\frac{0^2}{2}\right)-\left(\frac{3}{4}(-1)^{4/3}-\frac{(-1)^2}{2}\right)\right]$

$+\left[\left(\frac{3}{4}(1)^{4/3}-\frac{1^2}{2}\right)-\left(\frac{3}{4}(0)^{4/3}-\frac{0^2}{2}\right)\right]$

$-\left[\left(\frac{3}{4}(8)^{4/3}-\frac{8^2}{2}\right)-\left(\frac{3}{4}(1)^{4/3}-\frac{1^2}{2}\right)\right]$

$=\frac{1}{4}+\frac{1}{4}-\left(-20-\frac{3}{4}+\frac{1}{2}\right)=\frac{83}{4}$



51. The area of the rectangle bounded by the lines $y=2$, $y=0$, $x=\pi$, and $x=0$ is 2π. The area under the curve $y=1+\cos x$ on $[0,\pi]$ is $\int_0^{\pi}(1+\cos x)\,dx=[x+\sin x]_0^{\pi}=(\pi+\sin\pi)-(0+\sin 0)=\pi$. Therefore the area of the shaded region is $2\pi-\pi=\pi$.

52. The area of the rectangle bounded by the lines by the lines $x=\frac{\pi}{6}$, $x=\frac{5\pi}{6}$, $y=\sin\frac{\pi}{6}=\frac{1}{2}=\sin\frac{5\pi}{6}$, and $y=0$ is $\frac{1}{2}\left(\frac{5\pi}{6}-\frac{\pi}{6}\right)=\frac{\pi}{3}$. The area under the curve $y=\sin x$ on $\left[\frac{\pi}{6},\frac{5\pi}{6}\right]$ is $\int_{\pi/6}^{5\pi/6}\sin x\,dx=[-\cos x]_{\pi/6}^{5\pi/6}$ $=\left(-\cos\frac{5\pi}{6}\right)-\left(-\cos\frac{\pi}{6}\right)=-\left(-\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{2}=\sqrt{3}$. Therefore the area of the shaded region is $\sqrt{3}-\frac{\pi}{3}$.

53. On $\left[-\frac{\pi}{4},0\right]$: The area of the rectangle bounded by the lines $y=\sqrt{2}$, $y=0$, $\theta=0$, and $\theta=-\frac{\pi}{4}$ is $\sqrt{2}\left(\frac{\pi}{4}\right)$ $=\frac{\pi\sqrt{2}}{4}$. The area between the curve $y=\sec\theta\tan\theta$ and $y=0$ is $-\int_{-\pi/4}^{0}\sec\theta\tan\theta\,d\theta=[-\sec\theta]_{-\pi/4}^{0}$ $=(-\sec 0)-\left(-\sec\left(-\frac{\pi}{4}\right)\right)=\sqrt{2}-1$. Therefore the area of the shaded region on $\left[-\frac{\pi}{4},0\right]$ is $\frac{\pi\sqrt{2}}{4}+(\sqrt{2}-1)$.
On $\left[0,\frac{\pi}{4}\right]$: The area of the rectangle bounded by $\theta=\frac{\pi}{4}$, $\theta=0$, $y=\sqrt{2}$, and $y=0$ is $\sqrt{2}\left(\frac{\pi}{4}\right)=\frac{\pi\sqrt{2}}{4}$. The area under the curve $y=\sec\theta\tan\theta$ is $\int_0^{\pi/4}\sec\theta\tan\theta d\theta=[\sec\theta]_0^{\pi/4}=\sec\frac{\pi}{4}-\sec 0=\sqrt{2}-1$. Therefore the area of the shaded region on $\left[0,\frac{\pi}{4}\right]$ is $\frac{\pi\sqrt{2}}{4}-(\sqrt{2}-1)$. Thus, the area of the total shaded region is $\left(\frac{\pi\sqrt{2}}{4}+\sqrt{2}-1\right)+\left(\frac{\pi\sqrt{2}}{4}-\sqrt{2}+1\right)=\frac{\pi\sqrt{2}}{2}$.

54. The area of the rectangle bounded by the lines $y=2$, $y=0$, $t=-\frac{\pi}{4}$, and $t=1$ is $2\left(1-\left(-\frac{\pi}{4}\right)\right)=2+\frac{\pi}{2}$. The area under the curve $y=\sec^2 t$ on $\left[-\frac{\pi}{4},0\right]$ is $\int_{-\pi/4}^{0}\sec^2 t\,dt=[\tan t]_{-\pi/4}^{0}=\tan 0-\tan\left(-\frac{\pi}{4}\right)=1$. The area under the curve $y=1-t^2$ on $[0,1]$ is $\int_0^1(1-t^2)\,dt=\left[t-\frac{t^2}{3}\right]_0^1=\left(1-\frac{1^3}{3}\right)-\left(0-\frac{0^3}{3}\right)=\frac{2}{3}$. Thus, the total area under the curves on $\left[-\frac{\pi}{4},1\right]$ is $1+\frac{2}{3}=\frac{5}{3}$. Therefore the area of the shaded region is $\left(2+\frac{\pi}{2}\right)-\frac{5}{3}=\frac{1}{3}+\frac{\pi}{2}$.

55. $y=\int_{\pi}^{x}\frac{1}{t}\,dt-3 \Rightarrow \frac{dy}{dx}=\frac{1}{x}$ and $y(\pi)=\int_{\pi}^{\pi}\frac{1}{t}dt-3=0-3=-3 \Rightarrow$ (d) is a solution to this problem.

56. $y=\int_{-1}^{x}\sec t\,dt+4\Rightarrow\frac{dy}{dx}=\sec x$ and $y(-1)=\int_{-1}^{-1}\sec t\,dt+4=0+4=4\Rightarrow$ (c) is a solution to this problem.

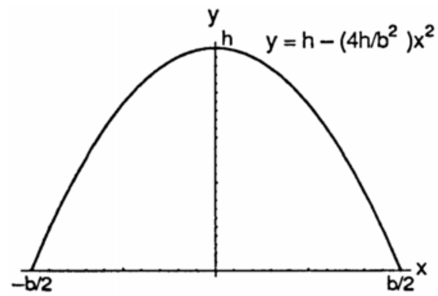
57. $y=\int_{0}^{x}\sec t\,dt+4\Rightarrow\frac{dy}{dx}=\sec x$ and $y(0)=\int_{0}^{0}\sec t\,dt+4=0+4=4\Rightarrow$ (b) is a solution to this problem.

58. $y=\int_{1}^{x}\frac{1}{t}\,dt-3\Rightarrow\frac{dy}{dx}=\frac{1}{x}$ and $y(1)=\int_{1}^{1}\frac{1}{t}\,dt-3=0-3=-3\Rightarrow$ (a) is a solution to this problem.

59. $y=\int_{2}^{x}\sec t\,dt+3$

60. $y=\int_{1}^{x}\sqrt{1+t^2}\,dt-2$

61. Area $=\int_{-b/2}^{b/2}\left(h-\left(\frac{4h}{b^2}\right)x^2\right)dx=\left[hx-\frac{4hx^3}{3b^2}\right]_{-b/2}^{b/2}$
$=\left(h\left(\frac{b}{2}\right)-\frac{4h\left(\frac{b}{2}\right)^2}{3b^2}\right)-\left(h\left(-\frac{b}{2}\right)-\frac{4h\left(-\frac{b}{2}\right)^2}{3b^2}\right)$
$=\left(\frac{bh}{2}-\frac{bh}{6}\right)-\left(-\frac{bh}{2}+\frac{bh}{6}\right)=bh-\frac{bh}{3}=\frac{2}{3}\,bh$



62. $k>0\Rightarrow$ one arch of $y=\sin kx$ will occur over the interval $\left[0,\frac{\pi}{k}\right]\Rightarrow$ the area $=\int_{0}^{\pi/k}\sin kx\,dx=\left[-\frac{1}{k}\cos kx\right]_{0}^{\pi/k}$
$=-\frac{1}{k}\cos\left(k\left(\frac{\pi}{k}\right)\right)-\left(-\frac{1}{k}\cos(0)\right)=\frac{2}{k}$

63. $\frac{dc}{dx}=\frac{1}{2\sqrt{x}}=\frac{1}{2}x^{-1/2}\Rightarrow c=\int_{0}^{x}\frac{1}{2}t^{-1/2}dt=[t^{1/2}]_0^x=\sqrt{x};\ \ c(100)-c(1)=\sqrt{100}-\sqrt{1}=\$9.00$

64. $r=\int_{0}^{3}\left(2-\frac{2}{(x+1)^2}\right)dx=2\int_{0}^{3}\left(1-\frac{1}{(x+1)^2}\right)dx=2\left[x-\left(\frac{-1}{x+1}\right)\right]_0^3=2\left[\left(3+\frac{1}{(3+1)}\right)-\left(0+\frac{1}{(0+1)}\right)\right]$
$=2\left[3\frac{1}{4}-1\right]=2\left(2\frac{1}{4}\right)=4.5$ or \$4500

65. (a) $t=0\Rightarrow T=30-2\sqrt{25-0}=20°\text{C};\ t=16\Rightarrow T=30-2\sqrt{25-16}=24°\text{C};$
$t=25\Rightarrow T=30-2\sqrt{25-25}=30°\text{C}$

(b) average temperature $=\frac{1}{25-0}\int_{0}^{25}\left(30-2\sqrt{25-t}\right)dt=\frac{1}{25}\left[30t+4/3(25-t)^{3/2}\right]_0^{25}$
$=\frac{1}{25}\left(30(25)+4/3(25-25)^{3/2}\right)-\frac{1}{25}\left(30(0)+4/3(25-0)^{3/2}\right)=23.33°\text{C}$

66. (a) $t=0\Rightarrow H=\sqrt{0+1}+5(0)^{1/3}=1$ m; $t=4\Rightarrow H=\sqrt{4+1}+5(4)^{1/3}=\sqrt{5}+5\sqrt[3]{4}\approx10.17$ m;
$t=8\Rightarrow H=\sqrt{8+1}+5(8)^{1/3}=13$ m

(b) average height $=\frac{1}{8-0}\int_{0}^{8}\left(\sqrt{t+1}+5t^{1/3}\right)dt=\frac{1}{8}\left[\frac{2}{3}(t+1)^{3/2}+\frac{15}{4}t^{4/3}\right]_0^8$
$=\frac{1}{8}\left(\frac{2}{3}(8+1)^{3/2}+\frac{15}{4}(8)^{4/3}\right)-\frac{1}{8}\left(\frac{2}{3}(0+1)^{3/2}+\frac{15}{4}(0)^{4/3}\right)=\frac{29}{3}\approx9.67$ m

67. $\int_{1}^{x}f(t)\,dt=x^2-2x+1\Rightarrow f(x)=\frac{d}{dx}\int_{1}^{x}f(t)\,dt=\frac{d}{dx}(x^2-2x+1)=2x-2$

68. $\int_0^x f(t)\,dt = x\cos\pi x \Rightarrow f(x) = \frac{d}{dx}\int_0^x f(t)\,dt = \cos\pi x - \pi x\sin\pi x \Rightarrow f(4) = \cos\pi(4) - \pi(4)\sin\pi(4) = 1$

69. $f(x) = 2 - \int_2^{x+1}\frac{9}{1+t}\,dt \Rightarrow f'(x) = -\frac{9}{1+(x+1)} = \frac{-9}{x+2} \Rightarrow f'(1) = -3;\ f(1) = 2 - \int_2^{1+1}\frac{9}{1+t}\,dt = 2 - 0 = 2;$
$L(x) = -3(x-1) + f(1) = -3(x-1) + 2 = -3x + 5$

70. $g(x) = 3 + \int_1^{x^2}\sec(t-1)\,dt \Rightarrow g'(x) = (\sec(x^2-1))(2x) = 2x\sec(x^2-1) \Rightarrow g'(-1) = 2(-1)\sec((-1)^2-1) = -2;$
$g(-1) = 3 + \int_1^{(-1)^2}\sec(t-1)\,dt = 3 + \int_1^1\sec(t-1)\,dt = 3 + 0 = 3;$
$L(x) = -2(x-(-1)) + g(-1) = -2(x+1) + 3 = -2x + 1$

71. (a) True: since f is continuous, g is differentiable by Part 1 of the Fundamental Theorem of Calculus.
(b) True: g is continuous because it is differentiable.
(c) True: since $g'(1) = f(1) = 0$.
(d) False, since $g''(1) = f'(1) > 0$.
(e) True, since $g'(1) = 0$ and $g''(1) = f'(1) > 0$.
(f) False: $g''(x) = f'(x) > 0$, so g'' never changes sign.
(g) True, since $g'(1) = f(1) = 0$ and $g'(x) = f(x)$ is an increasing function of x (because $f'(x) > 0$).

72. Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$ be any partition of $[a, b]$ and left F be any antiderivative of f.
(a) $\sum_{i=1}^{n}[F(x_i) - F(x_{i-1})]$
$= [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + [F(x_3) - F(x_2)] + \cdots + [F(x_{n-1}) - F(x_{n-2})] + [F(x_n) - F(x_{n-1})]$
$= -F(x_0) + F(x_1) - F(x_1) + F(x_2) - F(x_2) + \cdots + F(x_{n-1}) - F(x_{n-1}) + F(x_n)$
$= F(x_n) - F(x_0) = F(b) - F(a)$
(b) Since F is any antiderivative of f on $[a, b] \Rightarrow F$ is differentiable of $[a, b] \Rightarrow F$ is continuous on $[a, b]$. Consider any subinterval $[x_{i-1}, x_i]$ in $[a, b]$, then by the Mean Value Theorem there is at least one number c_i in (x_{i-1}, x_i) such that $[F(x_i) - F(x_{i-1})] = F'(c_i)(x_i - x_{i-1}) = f(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x_i$.
Thus $F(b) - F(a) = \sum_{i=1}^{n}[F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(c_i)\Delta x_i$.
(c) Taking the limit of $F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x_i$ we obtain $\lim_{\|P\|\to 0}(F(b) - F(a)) = \lim_{\|P\|\to 0}\left(\sum_{i=1}^{n} f(c_i)\Delta x_i\right)$
$\Rightarrow F(b) - F(a) = \int_a^b f(x)\,dx$

73. (a) $v = \frac{ds}{dt} = \frac{d}{dt}\int_0^t f(x)\,dx = f(t) \Rightarrow v(5) = f(5) = 2$ m/s
(b) $a = \frac{df}{dt}$ is negative since the slope of the tangent line at $t = 5$ is negative
(c) $s = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(3) = \frac{9}{2}$ m since the integral is the area of the triangle formed by $y = f(x)$, the x-axis and $x = 3$
(d) $t = 6$ since from $t = 6$ to $t = 9$, the region lies below the x-axis
(e) At $t = 4$ and $t = 7$, since there are horizontal tangents there
(f) Toward the origin between $t = 6$ and $t = 9$ since the velocity is negative on this interval. Away from the origin between $t = 0$ and $t = 6$ since the velocity is positive there.
(g) Right or positive side, because the integral of f from 0 to 9 is positive, there being more area above the x-axis than below it.

74. If the marginal cost is $\frac{x^2}{1000} - \frac{x}{2} + 115$, by the net change theorem the production cost is $p(x) = \int_0^x \frac{t^2}{1000} - \frac{t}{2} + 115\,dt = \frac{1}{3000}x^3 - \frac{1}{4}x^2 + 115x.$ Thus the average cost per unit for 600 units is $\frac{p(600)}{600} = 85.$

75–78. Example CAS commands:

Maple:

```
with( plots );
f := x -> x^3-4*x^2+3*x;
a := 0;
b := 4;
F := unapply( int(f(t),t=a..x), x );                    # (a)
p1 := plot( [f(x),F(x)], x=a..b, legend=["y = f(x)","y = F(x)"], title="#75(a) (Section 5.4)" ):
p1;
dF := D(F);                                             # (b)
q1 := solve( dF(x)=0, x );
pts1 := [ seq( [x,f(x)], x=remove(has,evalf([q1]),I) ) ];
p2 := plot( pts1, style=point, color=blue, symbolsize=18, symbol=diamond, legend="(x,f(x))
          where F'(x)=0" ):
display( [p1, p2], title="75(b) (Section 5.4)" );
incr := solve( dF(x)>0, x );                            # (c)
decr := solve( dF(x)<0, x );
df := D(f);                                             # (d)
p3 := plot( [df(x),F(x)], x=a..b, legend=["y = f'(x)","y = F(x)"], title="#75(d) (Section 5.4)" ):
p3;
q2 := solve( df(x)=0, x );
pts2 := [ seq( [x,F(x)], x=remove(has,evalf([q2]),I) ) ];
p4 := plot( pts2, style=point, color=blue, symbolsize=18, symbol=diamond, legend="(x,f(x))
          where f'(x)=0" ):
display( [p3,p4], title="75(d) (Section 5.4)" );
```

79-82. Example CAS commands:

Maple:

```
a := 1;
u := x -> x^2;
f := x -> sqrt(1-x^2);
F := unapply( int( f(t),t=a..u(x) ), x );
dF := D(F);                                                  # (b)
cp := solve( dF(x)=0, x );
solve( dF(x)>0, x );
solve( dF(x)<0, x );
```

```
d2F := D(dF);                                   # (c)
solve( d2F(x)=0, x );
 plot( F(x), x=-1..1, title="#79(d) (Section 5.4)" );
```

83. Example CAS commands:

Maple:

```
f := `f`;
q1 := Diff( Int( f(t), t=a..u(x) ), x );
d1 := value( q1 );
```

84. Example CAS commands:

Maple:

```
f := `f`;
q2 := Diff( Int( f(t), t=a..u(x) ), x,x );
value( q2 );
```

75-84. Example CAS commands:

Mathematica: (assigned function and values for a, and b may vary)
For transcendental functions the FindRoot is needed instead of the Solve command.
The Map command executes FindRoot over a set of initial guesses
Initial guesses will vary as the functions vary.

```
Clear[x, f, F]
{a, b}={0, 2π}; f[x_] = Sin[2x] Cos[x/3]
F[x_] = Integrate[f[t],{t, a, x}]
Plot[{f[x], F[x]},{x, a, b}]
x/.Map[FindRoot[F'[x]==0, {x, #}] &, {2, 3, 5, 6}]
x/.Map[FindRoot[f'[x]==0, {x, #}] &, {1, 2, 4, 5, 6}]
```

Slightly alter above commands for $79-84$.

```
Clear[x, f, F, u]
a=0; f[x_] = x^2 - 2x - 3
u[x_] = 1 - x^2
F[x_] = Integrate[f[t], {t, a, u(x)}]
x/.Map[FindRoot[F'[x]==0, {x, #}] &, {1, 2, 3, 4}]
x/.Map[FindRoot[F''[x]==0, {x, #}] &, {1, 2, 3, 4}]
```

After determining an appropriate value for b, the following can be entered

```
b = 4;
Plot[{F[x],{x, a, b}]
```

5.5 INDEFINITE INTEGRALS AND THE SUBSTITUTION METHOD

1. Let $u = 2x+4 \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}du = dx$

$$\int 2(2x+4)^5 dx = \int 2u^5 \tfrac{1}{2}du = \int u^5 du = \tfrac{1}{6}u^6 + C = \tfrac{1}{6}(2x+4)^6 + C$$

2. Let $u = 7x-1 \Rightarrow du = 7\,dx \Rightarrow \frac{1}{7}du = dx$
$\int 7\sqrt{7x-1}\,dx = \int 7(7x-1)^{1/2}\,dx = \int 7u^{1/2}\frac{1}{7}\,du = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(7x-1)^{3/2} + C$

3. Let $u = x^2+5 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$
$\int 2x(x^2+5)^{-4}dx = \int 2u^{-4}\frac{1}{2}du = \int u^{-4}du = -\frac{1}{3}u^{-3} + C = -\frac{1}{3}(x^2+5)^{-3} + C$

4. Let $u = x^4+1 \Rightarrow du = 4x^3\,dx \Rightarrow \frac{1}{4}du = x^3\,dx$
$\int \frac{4x^3}{(x^4+1)^2}dx = \int 4x^3(x^4+1)^{-2}dx = \int 4u^{-2}\frac{1}{4}du = \int u^{-2}du = -u^{-1} + C = \frac{-1}{x^4+1} + C$

5. Let $u = 3x^2+4x \Rightarrow du = (6x+4)dx = 2(3x+2)dx \Rightarrow \frac{1}{2}du = (3x+2)dx$
$\int (3x+2)(3x^2+4x)^4dx = \int u^4\frac{1}{2}du = \frac{1}{2}\int u^4du = \frac{1}{10}u^5 + C = \frac{1}{10}(3x^2+4x)^5 + C$

6. Let $u = 1+\sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx$
$\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx = \int (1+\sqrt{x})^{1/3}\frac{1}{\sqrt{x}}\,dx = \int u^{1/3}2\,du = 2\int u^{1/3}du = 2\cdot\frac{3}{4}u^{4/3} + C = \frac{3}{2}(1+\sqrt{x})^{4/3} + C$

7. Let $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$
$\int \sin 3x\,dx = \int \frac{1}{3}\sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos 3x + C$

8. Let $u = 2x^2 \Rightarrow du = 4x\,dx \Rightarrow \frac{1}{4}du = x\,dx$
$\int x\sin(2x^2)\,dx = \int \frac{1}{4}\sin u\,du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos 2x^2 + C$

9. Let $u = 2t \Rightarrow du = 2\,dt \Rightarrow \frac{1}{2}du = dt$
$\int \sec 2t\tan 2t\,dt = \int \frac{1}{2}\sec u\tan u\,du = \frac{1}{2}\sec u + C = \frac{1}{2}\sec 2t + C$

10. Let $u = 1-\cos\frac{t}{2} \Rightarrow du = \frac{1}{2}\sin\frac{t}{2}\,dt \Rightarrow 2\,du = \sin\frac{t}{2}\,dt$
$\int \left(1-\cos\frac{t}{2}\right)^2\left(\sin\frac{t}{2}\right)dt = \int 2u^2du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(1-\cos\frac{t}{2}\right)^3 + C$

11. Let $u = 1-r^3 \Rightarrow du = -3r^2dr \Rightarrow -3du = 9r^2dr$
$\int \frac{9r^2dr}{\sqrt{1-r^3}} = \int -3u^{-1/2}du = -3(2)u^{1/2} + C = -6(1-r^3)^{1/2} + C$

12. Let $u = y^4+4y^2+1 \Rightarrow du = (4y^3+8y)\,dy \Rightarrow 3\,du = 12\,(y^3+2y)\,dy$ $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy$
$= \int 3u^2\,du = u^3 + C = (y^4+4y^2+1)^3 + C$

13. Let $u = x^{3/2}-1 \Rightarrow du = \frac{3}{2}x^{1/2}dx \Rightarrow \frac{2}{3}du = \sqrt{x}\,dx$
$\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx = \int \frac{2}{3}\sin^2 u\,du = \frac{2}{3}\left(\frac{u}{2}-\frac{1}{4}\sin 2u\right) + C = \frac{1}{3}(x^{3/2}-1) - \frac{1}{6}\sin(2x^{3/2}-2) + C$

14. Let $u = -\frac{1}{x} \Rightarrow du = \frac{1}{x^2}dx$
$\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx = \int \cos^2(-u)\,du = \int \cos^2(u)\,du = \left(\frac{u}{2}+\frac{1}{4}\sin 2u\right) + C = -\frac{1}{2x} + \frac{1}{4}\sin\left(-\frac{2}{x}\right) + C = -\frac{1}{2x} - \frac{1}{4}\sin\left(\frac{2}{x}\right) + C$

15. (a) Let $u = \cot 2\theta \Rightarrow du = -2\csc^2 2\theta\, d\theta \Rightarrow -\frac{1}{2}du = \csc^2 2\theta\, d\theta$
$\int \csc^2 2\theta \cot 2\theta\, d\theta = -\int \frac{1}{2} u\, du = -\frac{1}{2}\left(\frac{u^2}{2}\right) + C = -\frac{u^2}{4} + C = -\frac{1}{4}\cot^2 2\theta + C$
(b) Let $u = \csc 2\theta \Rightarrow du = -2\csc 2\theta \cot 2\theta\, d\theta \Rightarrow -\frac{1}{2}du = \csc 2\theta \cot 2\theta\, d\theta$
$\int \csc^2 2\theta \cot 2\theta\, d\theta = \int -\frac{1}{2} u\, du = -\frac{1}{2}\left(\frac{u^2}{2}\right) + C = -\frac{u^2}{4} + C = -\frac{1}{4}\csc^2 2\theta + C$

16. (a) Let $u = 5x + 8 \Rightarrow du = 5\, dx \Rightarrow \frac{1}{5}du = dx$
$\int \frac{dx}{\sqrt{5x+8}} = \int \frac{1}{5}\left(\frac{1}{\sqrt{u}}\right) du = \frac{1}{5}\int u^{-1/2} du = \frac{1}{5}(2u^{1/2}) + C = \frac{2}{5}u^{1/2} + C = \frac{2}{5}\sqrt{5x+8} + C$
(b) Let $u = \sqrt{5x+8} \Rightarrow du = \frac{1}{2}(5x+8)^{-1/2}(5)\, dx \Rightarrow \frac{2}{5}du = \frac{dx}{\sqrt{5x+8}}$
$\int \frac{dx}{\sqrt{5x+8}} = \int \frac{2}{5}\, du = \frac{2}{5}u + C = \frac{2}{5}\sqrt{5x+8} + C$

17. Let $u = 3 - 2s \Rightarrow du = -2\, ds \Rightarrow -\frac{1}{2}du = ds$
$\int \sqrt{3-2s}\, ds = \int \sqrt{u}\left(-\frac{1}{2}du\right) = -\frac{1}{2}\int u^{1/2} du = \left(-\frac{1}{2}\right)\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{3}(3-2s)^{3/2} + C$

18. Let $u = 5s + 4 \Rightarrow du = 5\, ds \Rightarrow \frac{1}{5}du = ds$
$\int \frac{1}{\sqrt{5s+4}}\, ds = \int \frac{1}{\sqrt{u}}\left(\frac{1}{5}du\right) = \frac{1}{5}\int u^{-1/2} du = \left(\frac{1}{5}\right)(2u^{1/2}) + C = \frac{2}{5}\sqrt{5s+4} + C$

19. Let $u = 1 - \theta^2 \Rightarrow du = -2\theta\, d\theta \Rightarrow -\frac{1}{2}du = \theta\, d\theta$
$\int \theta \sqrt[4]{1-\theta^2}\, d\theta = \int \sqrt[4]{u}\left(-\frac{1}{2}du\right) = -\frac{1}{2}\int u^{1/4} du = \left(-\frac{1}{2}\right)\left(\frac{4}{5}u^{5/4}\right) + C = -\frac{2}{5}(1-\theta^2)^{5/4} + C$

20. Let $u = 7 - 3y^2 \Rightarrow du = -6y\, dy \Rightarrow -\frac{1}{2}du = 3y\, dy$
$\int 3y\sqrt{7-3y^2}\, dy = \int \sqrt{u}\left(-\frac{1}{2}du\right) = -\frac{1}{2}\int u^{1/2} du = \left(-\frac{1}{2}\right)\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{3}(7-3y^2)^{3/2} + C$

21. Let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\, dx \Rightarrow 2\, du = \frac{1}{\sqrt{x}}\, dx$
$\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\, dx = \int \frac{2\, du}{u^2} = -\frac{2}{u} + C = \frac{-2}{1+\sqrt{x}} + C$

22. Let $u = \sin x \Rightarrow du = \cos x\, dx$
$\int \sqrt{\sin x}\left(1 - \sin^2 x\right)\cos x\, dx = \int \left(u^{1/2} - u^{5/2}\right) du = \frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2} + C = \frac{2}{3}\sin^{3/2} x - \frac{2}{7}\sin^{7/2} x + C$

23. Let $u = 3x + 2 \Rightarrow du = 3dx \Rightarrow \frac{1}{3}du = dx$
$\int \sec^2(3x+2)\, dx = \int (\sec^2 u)\left(\frac{1}{3}du\right) = \frac{1}{3}\int \sec^2 u\, du = \frac{1}{3}\tan u + C = \frac{1}{3}\tan(3x+2) + C$

24. Let $u = \tan x \Rightarrow du = \sec^2 x\, dx$
$\int \tan^2 x \sec^2 x\, dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$

25. Let $u = \sin\left(\frac{x}{3}\right) \Rightarrow du = \frac{1}{3}\cos\left(\frac{x}{3}\right)dx \Rightarrow 3\, du = \cos\left(\frac{x}{3}\right)dx$
$\int \sin^5\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)dx = \int u^5(3\, du) = 3\left(\frac{1}{6}u^6\right) + C = \frac{1}{2}\sin^6\left(\frac{x}{3}\right) + C$

26. Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx \Rightarrow 2\,du = \sec^2\left(\frac{x}{2}\right)dx$
$\int \tan^7\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx = \int u^7(2\,du) = 2\left(\frac{1}{8}u^8\right) + C = \frac{1}{4}\tan^8\left(\frac{x}{2}\right) + C$

27. Let $u = \frac{r^3}{18} - 1 \Rightarrow du = \frac{r^2}{6}dr \Rightarrow 6\,du = r^2 dr$
$\int r^2\left(\frac{r^3}{18} - 1\right)^5 dr = \int u^5(6\,du) = 6\int u^5 du = 6\left(\frac{u^6}{6}\right) + C = \left(\frac{r^3}{18} - 1\right)^6 + C$

28. Let $u = 7 - \frac{r^5}{10} \Rightarrow du = -\frac{1}{2}r^4 dr \Rightarrow -2\,du = r^4 dr$
$\int r^4\left(7 - \frac{r^5}{10}\right)^3 dr = \int u^3(-2\,du) = -2\int u^3 du = -2\left(\frac{u^4}{4}\right) + C = -\frac{1}{2}\left(7 - \frac{r^5}{10}\right)^4 + C$

29. Let $u = x^{3/2} + 1 \Rightarrow du = \frac{3}{2}x^{1/2}dx \Rightarrow \frac{2}{3}du = x^{1/2}\,dx$
$\int x^{1/2}\sin(x^{3/2} + 1)\,dx = \int(\sin u)\left(\frac{2}{3}du\right) = \frac{2}{3}\int \sin u\,du = \frac{2}{3}(-\cos u) + C = -\frac{2}{3}\cos(x^{3/2} + 1) + C$

30. Let $u = \csc\left(\frac{v-\pi}{2}\right) \Rightarrow du = -\frac{1}{2}\csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv \Rightarrow -2du = \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$
$\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv = \int -2du = -2u + C = -2\csc\left(\frac{v-\pi}{2}\right) + C$

31. Let $u = \cos(2t+1) \Rightarrow du = -2\sin(2t+1)\,dt \Rightarrow -\frac{1}{2}du = \sin(2t+1)\,dt$
$\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt = \int -\frac{1}{2}\frac{du}{u^2} = \frac{1}{2u} + C = \frac{1}{2\cos(2t+1)} + C$

32. Let $u = \sec z \Rightarrow du = \sec z\tan z\,dz$
$\int \frac{\sec z\tan z}{\sqrt{\sec z}}\,dz = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}du = 2u^{1/2} + C = 2\sqrt{\sec z} + C$

33. Let $u = \frac{1}{t} - 1 = t^{-1} - 1 \Rightarrow du = -t^{-2}dt \Rightarrow -du = \frac{1}{t^2}\,dt$
$\int \frac{1}{t^2}\cos\left(\frac{1}{t} - 1\right)dt = \int(\cos u)(-du) = -\int \cos u\,du = -\sin u + C = -\sin\left(\frac{1}{t} - 1\right) + C$

34. Let $u = \sqrt{t} + 3 = t^{1/2} + 3 \Rightarrow du = \frac{1}{2}t^{-1/2}dt \Rightarrow 2du = \frac{1}{\sqrt{t}}dt$
$\int \frac{1}{\sqrt{t}}\cos(\sqrt{t} + 3)\,dt = \int(\cos u)(2\,du) = 2\int \cos u\,du = 2\sin u + C = 2\sin(\sqrt{t} + 3) + C$

35. Let $u = \sin\frac{1}{\theta} \Rightarrow du = \left(\cos\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right)d\theta \Rightarrow -du = \frac{1}{\theta^2}\cos\frac{1}{\theta}\,d\theta$
$\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta = \int -u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\sin^2\frac{1}{\theta} + C$

36. Let $u = \csc\sqrt{\theta} \Rightarrow du = \left(-\csc\sqrt{\theta}\cot\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)d\theta \Rightarrow -2du = \frac{1}{\sqrt{\theta}}\cot\sqrt{\theta}\csc\sqrt{\theta}\,d\theta$
$\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta = \int \frac{1}{\sqrt{\theta}}\cot\sqrt{\theta}\csc\sqrt{\theta}\,d\theta = \int -2\,du = -2u + C = -2\csc\sqrt{\theta} + C = -\frac{2}{\sin\sqrt{\theta}} + C$

37. Let $u = 1 + x \Rightarrow x = u - 1 \Rightarrow dx = du$
$\int \frac{x}{\sqrt{1+x}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du = \int\left(u^{1/2} - u^{-1/2}\right)du = \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C$

38. Let $u = 1 - \frac{1}{x} \Rightarrow du = \frac{1}{x^2}\,dx$
$\int \sqrt{\frac{x-1}{x^5}}\,dx = \int \frac{1}{x^2}\sqrt{\frac{x-1}{x}}\,dx = \int \frac{1}{x^2}\sqrt{1 - \frac{1}{x}}\,dx = \int \sqrt{u}\,du = \int u^{1/2}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(1 - \frac{1}{x}\right)^{3/2} + C$

39. Let $u = 2 - \frac{1}{x} \Rightarrow du = \frac{1}{x^2}\,dx$

$\int \frac{1}{x^2}\sqrt{2 - \frac{1}{x}}\,dx = \int \sqrt{u}\,du = \int u^{1/2}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(2 - \frac{1}{x}\right)^{3/2} + C$

40. Let $u = 1 - \frac{1}{x^2} \Rightarrow du = \frac{2}{x^3}\,dx \Rightarrow \frac{1}{2}du = \frac{1}{x^3}\,dx$

$\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx = \int \frac{1}{x^3}\sqrt{1 - \frac{1}{x^2}}\,dx = \int \sqrt{u}\,\frac{1}{2}\,du = \frac{1}{2}\int u^{1/2}du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}\left(1 - \frac{1}{x^2}\right)^{3/2} + C$

41. Let $u = 1 - \frac{3}{x^3} \Rightarrow du = \frac{9}{x^4}\,dx \Rightarrow \frac{1}{9}du = \frac{1}{x^4}\,dx$

$\int \sqrt{\frac{x^3-3}{x^{11}}}\,dx = \int \frac{1}{x^4}\sqrt{\frac{x^3-3}{x^3}}\,dx = \int \frac{1}{x^4}\sqrt{1 - \frac{3}{x^3}}\,dx = \int \sqrt{u}\,\frac{1}{9}\,du = \frac{1}{9}\int u^{1/2}du = \frac{2}{27}u^{3/2} + C = \frac{2}{27}\left(1 - \frac{3}{x^3}\right)^{3/2} + C$

42. Let $u = x^3 - 1 \Rightarrow du = 3x^2dx \Rightarrow \frac{1}{3}du = x^2dx$

$\int \sqrt{\frac{x^4}{x^3-1}}\,dx = \int \frac{x^2}{\sqrt{x^3-1}}\,dx = \int \frac{1}{\sqrt{u}}\,\frac{1}{3}\,du = \frac{1}{3}\int u^{-1/2}du = \frac{2}{3}u^{1/2} + C = \frac{2}{3}(x^3 - 1)^{3/2} + C$

43. Let $u = x - 1$. Then $du = dx$ and $x = u + 1$. Thus $\int x(x-1)^{10}dx = \int (u+1)u^{10}du = \int (u^{11} + u^{10})du$
$= \frac{1}{12}u^{12} + \frac{1}{11}u^{11} + C = \frac{1}{12}(x-1)^{12} + \frac{1}{11}(x-1)^{11} + C$

44. Let $u = 4 - x$. Then $du = -1\,dx$ and $(-1)du = dx$ and $x = 4 - u$. Thus $\int x\sqrt{4-x}dx = \int (4-u)\sqrt{u}(-1)\,du$
$= \int (4-u)(-u^{1/2})\,du = \int (u^{3/2} - 4u^{1/2})\,du = \frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + C = \frac{2}{5}(4-x)^{5/2} - \frac{8}{3}(4-x)^{3/2} + C$

45. Let $u = 1 - x$. Then $du = -1\,dx$ and $(-1)du = dx$ and $x = 1 - u$. Thus $\int (x+1)^2(1-x)^5\,dx$
$= \int (2-u)^2u^5(-1)\,du = \int (-u^7 + 4u^6 - 4u^5)\,du = -\frac{1}{8}u^8 + \frac{4}{7}u^7 - \frac{2}{3}u^6 + C = -\frac{1}{8}(1-x)^8 + \frac{4}{7}(1-x)^7 - \frac{2}{3}(1-x)^6 + C$

46. Let $u = x - 5$. Then $du = dx$ and $x = u + 5$. Thus $\int (x+5)(x-5)^{1/3}dx = \int (u+10)u^{1/3}du = \int (u^{4/3} + 10u^{1/3})\,du$
$= \frac{3}{7}u^{7/3} + \frac{15}{2}u^{4/3} + C = \frac{3}{7}(x-5)^{7/3} + \frac{15}{2}(x-5)^{4/3} + C$

47. Let $u = x^2 + 1$. Then $du = 2x\,dx$ and $\frac{1}{2}du = x\,dx$ and $x^2 = u - 1$. Thus $\int x^3\sqrt{x^2+1}\,dx = \int (u-1)\frac{1}{2}\sqrt{u}\,du$
$= \frac{1}{2}\int (u^{3/2} - u^{1/2})\,du = \frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C = \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + C = \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C$

48. Let $u = x^3 + 1 \Rightarrow du = 3x^2dx$ and $x^3 = u - 1$. So $\int 3x^5\sqrt{x^3+1}\,dx = \int (u-1)\sqrt{u}\,du = \int (u^{3/2} - u^{1/2})\,du$
$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2} + C$

49. Let $u = x^2 - 4 \Rightarrow du = 2x\,dx$ and $\frac{1}{2}du = x\,dx$. Thus $\int \frac{x}{(x^2-4)^3}\,dx = \int (x^2-4)^{-3}x\,dx = \int u^{-3}\,\frac{1}{2}\,du = \frac{1}{2}\int u^{-3}du$
$= -\frac{1}{4}u^{-2} + C = -\frac{1}{4}(x^2-4)^{-2} + C$

50. Let $u = 2x - 1 \Rightarrow x = \frac{1}{2}(u+1) \Rightarrow dx = \frac{1}{2}du$. Thus $\int \frac{x}{(2x-1)^{2/3}} = \int \frac{\frac{1}{2}(u+1)}{u^{2/3}}\left(\frac{1}{2}\,du\right) = \frac{1}{4}\int \left(u^{1/3} + u^{-2/3}\right)du$
$= \frac{1}{4}\left(\frac{3}{4}u^{4/3} + 3u^{1/3}\right) + C = \frac{3}{16}(2x-1)^{4/3} + \frac{3}{4}(2x-1)^{1/3} + C$

51. (a) Let $u=\tan x \Rightarrow du=\sec^2 x\,dx$; $v=u^3 \Rightarrow dv=3u^2\,du \Rightarrow 6dv=18u^2\,du$; $w=2+v \Rightarrow dw=dv$

$\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx=\int \frac{18u^2}{(2+u^3)^2}\,du=\int \frac{6\,dv}{(2+v)^2}=\int \frac{6\,dw}{w^2}=6\int w^{-2}dw=-6w^{-1}+C=-\frac{6}{2+v}+C$

$=-\frac{6}{2+u^3}+C=-\frac{6}{2+\tan^3 x}+C$

(b) Let $u=\tan^3 x \Rightarrow du=3\tan^2 x\sec^2 x\,dx \Rightarrow 6\,du=18\tan^2 x\sec^2 x\,dx$; $v=2+u \Rightarrow dv=du$

$\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx=\int \frac{6\,du}{(2+u)^2}=\int \frac{6\,dv}{v^2}=-\frac{6}{v}+C=-\frac{6}{2+u}+C=-\frac{6}{2+\tan^3 x}+C$

(c) Let $u=2+\tan^3 x \Rightarrow du=3\tan^2 x\sec^2 x\,dx \Rightarrow 6\,du=18\tan^2 x\sec^2 x\,dx$

$\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx=\int \frac{6\,du}{u^2}=-\frac{6}{u}+C=-\frac{6}{2+\tan^3 x}+C$

52. (a) Let $u=x-1 \Rightarrow du=dx$; $v=\sin u \Rightarrow dv=\cos u\,du$; $w=1+v^2 \Rightarrow dw=2v\,dv \Rightarrow \frac{1}{2}dw=v\,dv$

$\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx=\int \sqrt{1+\sin^2 u}\,\sin u\,\cos u\,du=\int v\sqrt{1+v^2}\,dv$

$=\int \frac{1}{2}\sqrt{w}\,dw=\frac{1}{3}w^{3/2}+C=\frac{1}{3}(1+v^2)^{3/2}+C=\frac{1}{3}(1+\sin^2 u)^{3/2}+C=\frac{1}{3}(1+\sin^2(x-1))^{3/2}+C$

(b) Let $u=\sin(x-1) \Rightarrow du=\cos(x-1)\,dx$; $v=1+u^2 \Rightarrow dv=2u\,du \Rightarrow \frac{1}{2}dv=u\,du$

$\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx=\int u\sqrt{1+u^2}\,du=\int \frac{1}{2}\sqrt{v}\,dv=\int \frac{1}{2}v^{1/2}dv$

$=\left(\frac{1}{2}\left(\frac{2}{3}\right)v^{3/2}\right)+C=\frac{1}{3}v^{3/2}+C=\frac{1}{3}(1+u^2)^{3/2}+C=\frac{1}{3}(1+\sin^2(x-1))^{3/2}+C$

(c) Let $u=1+\sin^2(x-1) \Rightarrow du=2\sin(x-1)\cos(x-1)\,dx \Rightarrow \frac{1}{2}du=\sin(x-1)\cos(x-1)\,dx$

$\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx=\int \frac{1}{2}\sqrt{u}\,du=\int \frac{1}{2}u^{1/2}du=\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)+C=\frac{1}{3}(1+\sin^2(x-1))^{3/2}+C$

53. Let $u=3(2r-1)^2+6 \Rightarrow du=6(2r-1)(2)\,dr \Rightarrow \frac{1}{12}du=(2r-1)dr$; $v=\sqrt{u} \Rightarrow dv=\frac{1}{2\sqrt{u}}du \Rightarrow \frac{1}{6}dv=\frac{1}{12\sqrt{u}}du$

$\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr=\int\left(\frac{\cos\sqrt{u}}{\sqrt{u}}\right)\left(\frac{1}{12}du\right)=\int(\cos v)\left(\frac{1}{6}dv\right)=\frac{1}{6}\sin v+C=\frac{1}{6}\sin\sqrt{u}+C$

$=\frac{1}{6}\sin\sqrt{3(2r-1)^2+6}+C$

54. Let $u=\cos\sqrt{\theta} \Rightarrow du=\left(-\sin\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)d\theta \Rightarrow -2du=\frac{\sin\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$

$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta=\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\sqrt{\cos^3\sqrt{\theta}}}\,d\theta=\int \frac{-2du}{u^{3/2}}=-2\int u^{-3/2}du=-2(-2u^{-1/2})+C=\frac{4}{\sqrt{u}}+C=\frac{4}{\sqrt{\cos\sqrt{\theta}}}+C$

55. Let $u=3t^2-1 \Rightarrow du=6t\,dt \Rightarrow 2du=12t\,dt$

$s=\int 12t(3t^2-1)^3dt=\int u^3(2\,du)=2\left(\frac{1}{4}u^4\right)+C=\frac{1}{2}u^4+C=\frac{1}{2}(3t^2-1)^4+C$;

$s=3$ when $t=1 \Rightarrow 3=\frac{1}{2}(3-1)^4+C \Rightarrow 3=8+C \Rightarrow C=-5 \Rightarrow s=\frac{1}{2}(3t^2-1)^4-5$

56. Let $u=x^2+8 \Rightarrow du=2x\,dx \Rightarrow 2\,du=4x\,dx$

$y=\int 4x(x^2+8)^{-1/3}dx=\int u^{-1/3}(2du)=2\left(\frac{3}{2}u^{2/3}\right)+C=3u^{2/3}+C=3(x^2+8)^{2/3}+C$;

$y=0$ when $x=0 \Rightarrow 0=3(8)^{2/3}+C \Rightarrow C=-12 \Rightarrow y=3(x^2+8)^{2/3}-12$

57. Let $u=t+\frac{\pi}{12} \Rightarrow du=dt$

$s=\int 8\sin^2\left(t+\frac{\pi}{12}\right)dt=\int 8\sin^2 u\,du=8\left(\frac{u}{2}-\frac{1}{4}\sin 2u\right)+C=4\left(t+\frac{\pi}{12}\right)-2\sin\left(2t+\frac{\pi}{6}\right)+C$;

$s=8$ when $t=0 \Rightarrow 8=4\left(\frac{\pi}{12}\right)-2\sin\left(\frac{\pi}{6}\right)+C \Rightarrow C=8-\frac{\pi}{3}+1=9-\frac{\pi}{3}$
$\Rightarrow s=4\left(t+\frac{\pi}{12}\right)-2\sin\left(2t+\frac{\pi}{6}\right)+9-\frac{\pi}{3}=4t-2\sin\left(2t+\frac{\pi}{6}\right)+9$

58. Let $u=\frac{\pi}{4}-\theta \Rightarrow -du=d\theta$
$r=\int 3\cos^2\left(\frac{\pi}{4}-\theta\right)d\theta=-\int 3\cos^2 u\,du=-3\left(\frac{u}{2}+\frac{1}{4}\sin 2u\right)+C=-\frac{3}{2}\left(\frac{\pi}{4}-\theta\right)-\frac{3}{4}\sin\left(\frac{\pi}{2}-2\theta\right)+C;$
$r=\frac{\pi}{8}$ when $\theta=0 \Rightarrow \frac{\pi}{8}=-\frac{3\pi}{8}-\frac{3}{4}\sin\frac{\pi}{2}+C \Rightarrow C=\frac{\pi}{2}+\frac{3}{4} \Rightarrow r=-\frac{3}{2}\left(\frac{\pi}{4}-\theta\right)-\frac{3}{4}\sin\left(\frac{\pi}{2}-2\theta\right)+\frac{\pi}{2}+\frac{3}{4}$
$\Rightarrow r=\frac{3}{2}\theta-\frac{3}{4}\sin\left(\frac{\pi}{2}-2\theta\right)+\frac{\pi}{8}+\frac{3}{4} \Rightarrow r=\frac{3}{2}\theta-\frac{3}{4}\cos 2\theta+\frac{\pi}{8}+\frac{3}{4}$

59. Let $u=2t-\frac{\pi}{2} \Rightarrow du=2\,dt \Rightarrow -2\,du=-4\,dt$
$\frac{ds}{dt}=\int -4\sin\left(2t-\frac{\pi}{2}\right)dt=\int(\sin u)(-2\,du)=2\cos u+C_1=2\cos\left(2t-\frac{\pi}{2}\right)+C_1;$
at $t=0$ and $\frac{ds}{dt}=100$ we have $100=2\cos\left(-\frac{\pi}{2}\right)+C_1 \Rightarrow C_1=100 \Rightarrow \frac{ds}{dt}=2\cos\left(2t-\frac{\pi}{2}\right)+100$
$\Rightarrow s=\int\left(2\cos\left(2t-\frac{\pi}{2}\right)+100\right)dt=\int(\cos u+50)\,du=\sin u+50u+C_2=\sin\left(2t-\frac{\pi}{2}\right)+50\left(2t-\frac{\pi}{2}\right)+C_2;$
at $t=0$ and $s=0$ we have $0=\sin\left(-\frac{\pi}{2}\right)+50\left(-\frac{\pi}{2}\right)+C_2 \Rightarrow C_2=1+25\pi$
$\Rightarrow s=\sin\left(2t-\frac{\pi}{2}\right)+100t-25\pi+(1+25\pi) \Rightarrow s=\sin\left(2t-\frac{\pi}{2}\right)+100t+1$

60. Let $u=\tan 2x \Rightarrow du=2\sec^2 2x\,dx \Rightarrow 2du=4\sec^2 2x\,dx;\ \ v=2x \Rightarrow dv=2dx \Rightarrow \frac{1}{2}dv=dx$
$\frac{dy}{dx}=\int 4\sec^2 2x\tan 2x\,dx=\int u(2\,du)=u^2+C_1=\tan^2 2x+C_1;$
at $x=0$ and $\frac{dy}{dx}=4$ we have $4=0+C_1 \Rightarrow C_1=4 \Rightarrow \frac{dy}{dx}=\tan^2 2x+4=(\sec^2 2x-1)+4=\sec^2 2x+3$
$\Rightarrow y=\int(\sec^2 2x+3)\,dx=\int(\sec^2 v+3)\left(\frac{1}{2}dv\right)=\frac{1}{2}\tan v+\frac{3}{2}v+C_2=\frac{1}{2}\tan 2x+3x+C_2;$
at $x=0$ and $y=-1$ we have $-1=\frac{1}{2}(0)+0+C_2 \Rightarrow C_2=-1 \Rightarrow y=\frac{1}{2}\tan 2x+3x-1$

61. Let $u=2t \Rightarrow du=2\,dt \Rightarrow 3\,du=6dt$
$s=\int 6\sin 2t\,dt=\int(\sin u)(3\,du)=-3\cos u+C=-3\cos 2t+C;$
at $t=0$ and $s=0$ we have $0=-3\cos 0+C \Rightarrow C=3 \Rightarrow s=3-3\cos 2t \Rightarrow s\left(\frac{\pi}{2}\right)=3-3\cos(\pi)=6\ m$

62. Let $u=\pi t \Rightarrow du=\pi dt \Rightarrow \pi\,du=\pi^2 dt$
$v=\int \pi^2\cos\pi t\,dt=\int(\cos u)(\pi\,du)=\pi\sin u+C_1=\pi\sin(\pi t)+C_1;$
at $t=0$ and $v=8$ we have $8=\pi(0)+C_1 \Rightarrow C_1=8 \Rightarrow v=\frac{ds}{dt}=\pi\sin(\pi t)+8 \Rightarrow s=\int(\pi\sin(\pi t)+8)\,dt$
$=\int\sin u\,du+8t+C_2=-\cos(\pi t)+8t+C_2$; at $t=0$ and $s=0$ we have $0=-1+C_2 \Rightarrow C_2=1$
$\Rightarrow s=8t-\cos(\pi t)+1 \Rightarrow s(1)=8-\cos\pi+1=10\ m$

5.6 DEFINITE INTEGRAL SUBSTITUTIONS AND THE AREA BETWEEN CURVES

1. (a) Let $u=y+1 \Rightarrow du=dy;\ \ y=0 \Rightarrow u=1,\ y=3 \Rightarrow u=4$
$\int_0^3\sqrt{y+1}\,dy=\int_1^4 u^{1/2}du=\left[\frac{2}{3}u^{3/2}\right]_1^4=\left(\frac{2}{3}\right)(4)^{3/2}-\left(\frac{2}{3}\right)(1)^{3/2}=\left(\frac{2}{3}\right)(8)-\left(\frac{2}{3}\right)(1)=\frac{14}{3}$
(b) Use the same substitution for u as in part (a); $y=-1 \Rightarrow u=0,\ y=0 \Rightarrow u=1$
$\int_{-1}^0\sqrt{y+1}\,dy=\int_0^1 u^{1/2}\,du=\left[\frac{2}{3}u^{3/2}\right]_0^1=\left(\frac{2}{3}\right)(1)^{3/2}-0=\frac{2}{3}$

2. (a) Let $u=1-r^2 \Rightarrow du=-2r\,dr \Rightarrow -\frac{1}{2}du=r\,dr$; $r=0 \Rightarrow u=1$, $r=1 \Rightarrow u=0$

$$\int_0^1 r\sqrt{1-r^2}\,dr=\int_1^0 -\frac{1}{2}\sqrt{u}\,du=\left[-\frac{1}{3}u^{3/2}\right]_1^0=0-\left(-\frac{1}{3}\right)(1)^{3/2}=\frac{1}{3}$$

(b) Use the same substitution for u as in part (a); $r=-1 \Rightarrow u=0$, $r=1 \Rightarrow u=0$

$$\int_{-1}^1 r\sqrt{1-r^2}\,dr=\int_0^0 -\frac{1}{2}\sqrt{u}\,du=0$$

3. (a) Let $u=\tan x \Rightarrow du=\sec^2 x\,dx$; $x=0 \Rightarrow u=0$, $x=\frac{\pi}{4} \Rightarrow u=1$

$$\int_0^{\pi/4}\tan x\sec^2 x\,dx=\int_0^1 u\,du=\left[\frac{u^2}{2}\right]_0^1=\frac{1^2}{2}-0=\frac{1}{2}$$

(b) Use the same substitution as in part (a); $x=-\frac{\pi}{4} \Rightarrow u=-1$, $x=0 \Rightarrow u=0$

$$\int_{-\pi/4}^0 \tan x\sec^2 x\,dx=\int_{-1}^0 u\,du=\left[\frac{u^2}{2}\right]_{-1}^0=0-\frac{1}{2}=-\frac{1}{2}$$

4. (a) Let $u=\cos x \Rightarrow du=-\sin x\,dx \Rightarrow -du=\sin x\,dx$; $x=0 \Rightarrow u=1$, $x=\pi \Rightarrow u=-1$

$$\int_0^\pi 3\cos^2 x\sin x\,dx=\int_1^{-1}-3u^2du=[-u^3]_1^{-1}=-(-1)^3-(-(1)^3)=2$$

(b) Use the same substitution as in part (a); $x=2\pi \Rightarrow u=1$, $x=3\pi \Rightarrow u=-1$

$$\int_{2\pi}^{3\pi}3\cos^2 x\sin x\,dx=\int_1^{-1}-3u^2\,du=2$$

5. (a) $u=1+t^4 \Rightarrow du=4t^3dt \Rightarrow \frac{1}{4}du=t^3dt$; $t=0 \Rightarrow u=1$, $t=1 \Rightarrow u=2$

$$\int_0^1 t^3(1+t^4)^3\,dt=\int_1^2 \frac{1}{4}u^3\,du=\left[\frac{u^4}{16}\right]_1^2=\frac{2^4}{16}-\frac{1^4}{16}=\frac{15}{16}$$

(b) Use the same substitution as in part (a); $t=-1 \Rightarrow u=2$, $t=1 \Rightarrow u=2$

$$\int_{-1}^1 t^3(1+t^4)^3\,dt=\int_2^2 \frac{1}{4}u^3\,du=0$$

6. (a) Let $u=t^2+1 \Rightarrow du=2t\,dt \Rightarrow \frac{1}{2}du=t\,dt$; $t=0 \Rightarrow u=1$, $t=\sqrt{7} \Rightarrow u=8$

$$\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt=\int_1^8 \frac{1}{2}u^{1/3}\,du=\left[\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)u^{4/3}\right]_1^8=\left(\frac{3}{8}\right)(8)^{4/3}-\left(\frac{3}{8}\right)(1)^{4/3}=\frac{45}{8}$$

(b) Use the same substitution as in part (a); $t=-\sqrt{7} \Rightarrow u=8$, $t=0 \Rightarrow u=1$

$$\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}dt=\int_8^1 \frac{1}{2}u^{1/3}\,du=-\int_1^8 \frac{1}{2}u^{1/3}\,du=-\frac{45}{8}$$

7. (a) Let $u=4+r^2 \Rightarrow du=2r\,dr \Rightarrow \frac{1}{2}du=r\,dr$; $r=-1 \Rightarrow u=5$, $r=1 \Rightarrow u=5$

$$\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr=5\int_5^5 \frac{1}{2}u^{-2}\,du=0$$

(b) Use the same substitution as in part (a); $r=0 \Rightarrow u=4$, $r=1 \Rightarrow u=5$

$$\int_0^1 \frac{5r}{(4+r^2)^2}\,dr=5\int_4^5 \frac{1}{2}u^{-2}\,du=5\left[-\frac{1}{2}u^{-1}\right]_4^5=5\left(-\frac{1}{2}(5)^{-1}\right)-5\left(-\frac{1}{2}(4)^{-1}\right)=\frac{1}{8}$$

8. (a) Let $u=1+v^{3/2} \Rightarrow du=\frac{3}{2}v^{1/2}\,dv \Rightarrow \frac{20}{3}\,du=10\sqrt{v}\,dv$; $v=0 \Rightarrow u=1$, $v=1 \Rightarrow u=2$

$$\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv=\int_1^2 \frac{1}{u^2}\left(\frac{20}{3}\,du\right)=\frac{20}{3}\int_1^2 u^{-2}\,du=-\frac{20}{3}\left[\frac{1}{u}\right]_1^2=-\frac{20}{3}\left[\frac{1}{2}-\frac{1}{1}\right]=\frac{10}{3}$$

(b) Use the same substitution as in part (a); $v=1 \Rightarrow u=2$, $v=4 \Rightarrow u=1+4^{3/2}=9$

$$\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv=\int_2^9 \frac{1}{u^2}\left(\frac{20}{3}\,du\right)=-\frac{20}{3}\left[\frac{1}{u}\right]_2^9=-\frac{20}{3}\left(\frac{1}{9}-\frac{1}{2}\right)=-\frac{20}{3}\left(-\frac{7}{18}\right)=\frac{70}{27}$$

9. (a) Let $u=x^2+1 \Rightarrow du=2x\,dx \Rightarrow 2\,du=4x\,dx$; $x=0 \Rightarrow u=1$, $x=\sqrt{3} \Rightarrow u=4$

$$\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx=\int_1^4 \frac{2}{\sqrt{u}}\,du=\int_1^4 2u^{-1/2}\,du=[4u^{1/2}]_1^4=4(4)^{1/2}-4(1)^{1/2}=4$$

(b) Use the same substitution as in part (a); $x=-\sqrt{3} \Rightarrow u=4$, $x=\sqrt{3} \Rightarrow u=4$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx=\int_4^4 \frac{2}{\sqrt{u}}\,du=0$$

10. (a) Let $u=x^4+9 \Rightarrow du=4x^3\,dx \Rightarrow \frac{1}{4}du=x^3\,dx$; $x=0 \Rightarrow u=9$, $x=1 \Rightarrow u=10$

$$\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx=\int_9^{10} \frac{1}{4}u^{-1/2}\,du=\left[\frac{1}{4}(2)u^{1/2}\right]_9^{10}=\frac{1}{2}(10)^{1/2}-\frac{1}{2}(9)^{1/2}=\frac{\sqrt{10}-3}{2}$$

(b) Use the same substitution as in part (a); $x=-1 \Rightarrow u=10$, $x=0 \Rightarrow u=9$

$$\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx=\int_{10}^9 \frac{1}{4}u^{-1/2}\,du=-\int_9^{10} \frac{1}{4}u^{-1/2}\,du=\frac{3-\sqrt{10}}{2}$$

11. (a) Let $u=4+5t \Rightarrow t=\frac{1}{5}(u-4)$, $dt=\frac{1}{5}du$; $t=0 \Rightarrow u=4$, $t=1 \Rightarrow u=9$.

$$\int_0^1 t\sqrt{4+5t}\,dt=\frac{1}{25}\int_4^9 (u-4)\sqrt{u}\,du=\frac{1}{25}\int_4^9\left(u^{3/2}-4u^{1/2}\right)du$$

$$=\frac{1}{25}\left[\frac{2}{5}u^{5/2}-\frac{8}{3}u^{3/2}\right]_4^9=\frac{1}{25}\left(\left(\frac{2}{5}(243)-\frac{8}{3}(27)\right)-\left(\frac{2}{5}(32)-\frac{8}{3}(8)\right)\right)=\frac{506}{375}$$

(b) Use the same substitution as in (a); $t=1 \Rightarrow u=9$, $t=9 \Rightarrow u=49$.

$$\int_1^9 t\sqrt{4+5t}\,dt=\frac{1}{25}\int_9^{49}\left(u^{3/2}-4u^{1/2}\right)du=\frac{1}{25}\left[\frac{2}{5}u^{5/2}-\frac{8}{3}u^{3/2}\right]_9^{49}$$

$$=\frac{1}{25}\left(\left(\frac{2}{5}(16{,}807)-\frac{8}{3}(343)\right)-\left(\frac{2}{5}(243)-\frac{8}{3}(27)\right)\right)=\frac{86{,}744}{375}$$

12. (a) Let $u=1-\cos 3t \Rightarrow du=3\sin 3t\,dt \Rightarrow \frac{1}{3}\,du=\sin 3t\,dt$; $t=0 \Rightarrow u=0$, $t=\frac{\pi}{6} \Rightarrow u=1-\cos\frac{\pi}{2}=1$

$$\int_0^{\pi/6}(1-\cos 3t)\sin 3t\,dt=\int_0^1 \frac{1}{3}u\,du=\left[\frac{1}{3}\left(\frac{u^2}{2}\right)\right]_0^1=\frac{1}{6}(1)^2-\frac{1}{6}(0)^2=\frac{1}{6}$$

(b) Use the same substitution as in part (a); $t=\frac{\pi}{6} \Rightarrow u=1$, $t=\frac{\pi}{3} \Rightarrow u=1-\cos\pi=2$

$$\int_{\pi/6}^{\pi/3}(1-\cos 3t)\sin 3t\,dt=\int_1^2 \frac{1}{3}u\,du=\left[\frac{1}{3}\left(\frac{u^2}{2}\right)\right]_1^2=\frac{1}{6}(2)^2-\frac{1}{6}(1)^2=\frac{1}{2}$$

13. (a) Let $u=4+3\sin z \Rightarrow du=3\cos z\,dz \Rightarrow \frac{1}{3}du=\cos z\,dz$; $z=0 \Rightarrow u=4$, $z=2\pi \Rightarrow u=4$

$$\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz=\int_4^4 \frac{1}{\sqrt{u}}\left(\frac{1}{3}du\right)=0$$

(b) Use the same substitution as in part (a); $z=-\pi \Rightarrow u=4+3\sin(-\pi)=4$, $z=\pi \Rightarrow u=4$

$$\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz=\int_4^4 \frac{1}{\sqrt{u}}\left(\frac{1}{3}\,du\right)=0$$

14. (a) Let $u = 2 + \tan \frac{t}{2} \Rightarrow du = \frac{1}{2}\sec^2 \frac{t}{2}\,dt \Rightarrow 2\,du = \sec^2 \frac{t}{2}\,dt$; $t = \frac{-\pi}{2} \Rightarrow u = 2 + \tan\left(\frac{-\pi}{4}\right) = 1$, $t = 0 \Rightarrow u = 2$

$$\int_{-\pi/2}^{0}\left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt = \int_1^2 u(2\,du) = [u^2]_1^2 = 2^2 - 1^2 = 3$$

(b) Use the same substitution as in part (a); $t = \frac{-\pi}{2} \Rightarrow u = 1$, $t = \frac{\pi}{2} \Rightarrow u = 3$

$$\int_{-\pi/2}^{\pi/2}(2+\tan\tfrac{t}{2})\sec^2\tfrac{t}{2}\,dt = 2\int_1^3 u\,du = [u^2]_1^3 = 3^2 - 1^2 = 8$$

15. Let $u = t^5 + 2t \Rightarrow du = (5t^4 + 2)\,dt$; $t = 0 \Rightarrow u = 0$, $t = 1 \Rightarrow u = 3$

$$\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt = \int_0^3 u^{1/2}\,du = \left[\tfrac{2}{3}u^{3/2}\right]_0^3 = \tfrac{2}{3}(3)^{3/2} - \tfrac{2}{3}(0)^{3/2} = 2\sqrt{3}$$

16. Let $u = 1 + \sqrt{y} \Rightarrow du = \frac{dy}{2\sqrt{y}}$; $y = 1 \Rightarrow u = 2$, $y = 4 \Rightarrow u = 3$

$$\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2} = \int_2^3 \frac{1}{u^2}\,du = \int_2^3 u^{-2}\,du = [-u^{-1}]_2^3 = \left(-\tfrac{1}{3}\right) - \left(-\tfrac{1}{2}\right) = \tfrac{1}{6}$$

17. Let $u = \cos 2\theta \Rightarrow du = -2\sin 2\theta\,d\theta \Rightarrow -\frac{1}{2}\,du = \sin 2\theta\,d\theta$; $\theta = 0 \Rightarrow u = 1$, $\theta = \frac{\pi}{6} \Rightarrow u = \cos 2\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$\int_0^{\pi/6}\cos^{-3}2\theta\sin 2\theta\,d\theta = \int_1^{1/2} u^{-3}\left(-\tfrac{1}{2}du\right) = -\tfrac{1}{2}\int_1^{1/2} u^{-3}\,du = \left[-\tfrac{1}{2}\left(\tfrac{u^{-2}}{-2}\right)\right]_1^{1/2} = \frac{1}{4\left(\frac{1}{2}\right)^2} - \frac{1}{4(1)^2} = \frac{3}{4}$$

18. Let $u = \tan\left(\frac{\theta}{6}\right) \Rightarrow du = \frac{1}{6}\sec^2\left(\frac{\theta}{6}\right)d\theta \Rightarrow 6\,du = \sec^2\left(\frac{\theta}{6}\right)d\theta$; $\theta = \pi \Rightarrow u = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, $\theta = \frac{3\pi}{2} \Rightarrow u = \tan\frac{\pi}{4} = 1$

$$\int_\pi^{3\pi/2}\cot^5\left(\tfrac{\theta}{6}\right)\sec^2\left(\tfrac{\theta}{6}\right)d\theta = \int_{1/\sqrt{3}}^1 u^{-5}(6\,du) = \left[6\left(\tfrac{u^{-4}}{-4}\right)\right]_{1/\sqrt{3}}^1 = \left[-\frac{3}{2u^4}\right]_{1/\sqrt{3}}^1 = -\frac{3}{2(1)^4} - \left(-\frac{3}{2\left(\frac{1}{\sqrt{3}}\right)^4}\right) = 12$$

19. Let $u = 5 - 4\cos t \Rightarrow du = 4\sin t\,dt \Rightarrow \frac{1}{4}du = \sin t\,dt$; $t = 0 \Rightarrow u = 5 - 4\cos 0 = 1$, $t = \pi \Rightarrow u = 5 - 4\cos\pi = 9$

$$\int_0^\pi 5(5-4\cos t)^{1/4}\sin t\,dt = \int_1^9 5u^{1/4}\left(\tfrac{1}{4}du\right) = \tfrac{5}{4}\int_1^9 u^{1/4}\,du = \left[\tfrac{5}{4}\left(\tfrac{4}{5}u^{5/4}\right)\right]_1^9 = 9^{5/4} - 1 = 3^{5/2} - 1$$

20. Let $u = 1 - \sin 2t \Rightarrow du = -2\cos 2t\,dt \Rightarrow -\frac{1}{2}du = \cos 2t\,dt$; $t = 0 \Rightarrow u = 1$, $t = \frac{\pi}{4} \Rightarrow u = 0$

$$\int_0^{\pi/4}(1-\sin 2t)^{3/2}\cos 2t\,dt = \int_1^0 -\tfrac{1}{2}u^{3/2}\,du = \left[-\tfrac{1}{2}\left(\tfrac{2}{5}u^{5/2}\right)\right]_1^0 = \left(-\tfrac{1}{5}(0)^{5/2}\right) - \left(-\tfrac{1}{5}(1)^{5/2}\right) = \tfrac{1}{5}$$

21. Let $u = 4y - y^2 + 4y^3 + 1 \Rightarrow du = (4 - 2y + 12y^2)\,dy$; $y = 0 \Rightarrow u = 1$, $y = 1 \Rightarrow u = 4(1) - (1)^2 + 4(1)^3 + 1 = 8$

$$\int_0^1 (4y - y^2 + 4y^3 + 1)^{-2/3}(12y^2 - 2y + 4)\,dy = \int_1^8 u^{-2/3}\,du = [3u^{1/3}]_1^8 = 3(8)^{1/3} - 3(1)^{1/3} = 3$$

22. Let $u = y^3 + 6y^2 - 12y + 9 \Rightarrow du = (3y^2 + 12y - 12)\,dy \Rightarrow \frac{1}{3}du = (y^2 + 4y - 4)\,dy$; $y = 0 \Rightarrow u = 9$, $y = 1 \Rightarrow u = 4$

$$\int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2}(y^2 + 4y - 4)\,dy = \int_9^4 \tfrac{1}{3}u^{-1/2}\,du = \left[\tfrac{1}{3}(2u^{1/2})\right]_9^4 = \tfrac{2}{3}(4)^{1/2} - \tfrac{2}{3}(9)^{1/2} = \tfrac{2}{3}(2-3) = -\tfrac{2}{3}$$

23. Let $u = \theta^{3/2} \Rightarrow du = \frac{3}{2}\theta^{1/2}d\theta \Rightarrow \frac{2}{3}du = \sqrt{\theta}\,d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \sqrt[3]{\pi^2} \Rightarrow u = \pi$

$$\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta = \int_0^\pi \cos^2 u\left(\tfrac{2}{3}du\right) = \left[\tfrac{2}{3}\left(\tfrac{u}{2} + \tfrac{1}{4}\sin 2u\right)\right]_0^\pi = \tfrac{2}{3}\left(\tfrac{\pi}{2} + \tfrac{1}{4}\sin 2\pi\right) - \tfrac{2}{3}(0) = \tfrac{\pi}{3}$$

24. Let $u = 1 + \frac{1}{t} \Rightarrow du = -t^{-2} dt$; $t = -1 \Rightarrow u = 0$, $t = -\frac{1}{2} \Rightarrow u = -1$

$\int_{-1}^{-1/2} t^{-2} \sin^2\left(1+\frac{1}{t}\right) dt = \int_0^{-1} -\sin^2 u\, du = \left[-\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right)\right]_0^{-1} = -\left[\left(-\frac{1}{2} - \frac{1}{4}\sin(-2)\right) - \left(\frac{0}{2} - \frac{1}{4}\sin 0\right)\right]$

$= \frac{1}{2} - \frac{1}{4}\sin 2$

25. Let $u = 4 - x^2 \Rightarrow du = -2x\, dx \Rightarrow -\frac{1}{2} du = x\, dx$; $x = -2 \Rightarrow u = 0$, $x = 0 \Rightarrow u = 4$, $x = 2 \Rightarrow u = 0$

$A = -\int_{-2}^{0} x\sqrt{4-x^2}\, dx + \int_0^2 x\sqrt{4-x^2}\, dx = -\int_0^4 -\frac{1}{2}u^{1/2}\, du + \int_4^0 -\frac{1}{2}u^{1/2}\, du = 2\int_0^4 \frac{1}{2}u^{1/2}\, du = \int_0^4 u^{1/2}\, du$

$= \left[\frac{2}{3}u^{3/2}\right]_0^4 = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{16}{3}$

26. Let $u = 1 - \cos x \Rightarrow du = \sin x\, dx$; $x = 0 \Rightarrow u = 0$, $x = \pi \Rightarrow u = 2$

$\int_0^{\pi} (1-\cos x)\sin x\, dx = \int_0^2 u\, du = \left[\frac{u^2}{2}\right]_0^2 = \frac{2^2}{2} - \frac{0^2}{2} = 2$

27. Let $u = 1 + \cos x \Rightarrow du = -\sin x\, dx \Rightarrow -du = \sin x\, dx$; $x = -\pi \Rightarrow u = 1 + \cos(-\pi) = 0$, $x = 0 \Rightarrow u = 1 + \cos 0 = 2$

$A = -\int_{-\pi}^{0} 3(\sin x)\sqrt{1+\cos x}\, dx = -\int_0^2 3u^{1/2}(-du) = 3\int_0^2 u^{1/2}\, du = \left[2u^{3/2}\right]_0^2 = 2(2)^{3/2} - 2(0)^{3/2} = 2^{5/2}$

28. Let $u = \pi + \pi\sin x \Rightarrow du = \pi\cos x\, dx \Rightarrow \frac{1}{\pi}\, du = \cos x\, dx$; $x = -\frac{\pi}{2} \Rightarrow u = \pi + \pi\sin\left(-\frac{\pi}{2}\right) = 0$, $x = 0 \Rightarrow u = \pi$

Because of symmetry about $x = -\frac{\pi}{2}$, $A = 2\int_{-\pi/2}^{0} \frac{\pi}{2}(\cos x)(\sin(\pi + \pi\sin x))\, dx = 2\int_0^{\pi} \frac{\pi}{2}(\sin u)\left(\frac{1}{\pi}\, du\right)$

$= \int_0^{\pi} \sin u\, du = [-\cos u]_0^{\pi} = (-\cos\pi) - (-\cos 0) = 2$

29. For the sketch given, $a = 0$, $b = \pi$; $f(x) - g(x) = 1 - \cos^2 x = \sin^2 x = \frac{1-\cos 2x}{2}$;

$A = \int_0^{\pi} \frac{(1-\cos 2x)}{2}\, dx = \frac{1}{2}\int_0^{\pi}(1-\cos 2x)\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}[(\pi - 0) - (0 - 0)] = \frac{\pi}{2}$

30. For the sketch given, $a = -\frac{\pi}{3}$, $b = \frac{\pi}{3}$; $f(t) - g(t) = \frac{1}{2}\sec^2 t - (-4\sin^2 t) = \frac{1}{2}\sec^2 t + 4\sin^2 t$;

$A = \int_{-\pi/3}^{\pi/3}\left(\frac{1}{2}\sec^2 t + 4\sin^2 t\right) dt = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\sec^2 t\, dt + 4\int_{-\pi/3}^{\pi/3}\sin^2 t\, dt = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\sec^2 t\, dt + 4\int_{-\pi/3}^{\pi/3}\frac{(1-\cos 2t)}{2}\, dt$

$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\sec^2 t\, dt + 2\int_{-\pi/3}^{\pi/3}(1-\cos 2t)\, dt = \frac{1}{2}[\tan t]_{-\pi/3}^{\pi/3} + 2\left[t - \frac{\sin 2t}{2}\right]_{-\pi/3}^{\pi/3} = \sqrt{3} + 4\cdot\frac{\pi}{3} - \sqrt{3} = \frac{4\pi}{3}$

31. For the sketch given, $a = -2$, $b = 2$; $f(x) - g(x) = 2x^2 - (x^4 - 2x^2) = 4x^2 - x^4$;

$A = \int_{-2}^{2}(4x^2 - x^4)\, dx = \left[\frac{4x^2}{3} - \frac{x^5}{5}\right]_{-2}^{2} = \left(\frac{32}{3} - \frac{32}{5}\right) - \left[-\frac{32}{3} - \left(-\frac{32}{5}\right)\right] = \frac{64}{3} - \frac{64}{5} = \frac{320-192}{15} = \frac{128}{15}$

32. For the sketch given, $c = 0$, $d = 1$; $f(y) - g(y) = y^2 - y^3$;

$A = \int_0^1 (y^2 - y^3)\, dy = \int_0^1 y^2\, dy - \int_0^1 y^3\, dy = \left[\frac{y^3}{3}\right]_0^1 - \left[\frac{y^4}{4}\right]_0^1 = \frac{(1-0)}{3} - \frac{(1-0)}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

33. For the sketch given, $c = 0$, $d = 1$; $f(y) - g(y) = (12y^2 - 12y^3) - (2y^2 - 2y) = 10y^2 - 12y^3 + 2y$;
$A = \int_0^1 (10y^2 - 12y^3 + 2y)\, dy = \int_0^1 10y^2\, dy - \int_0^1 12y^3\, dy + \int_0^1 2y\, dy = \left[\frac{10}{3}y^3\right]_0^1 - \left[\frac{12}{4}y^4\right]_0^1 + \left[\frac{2}{2}y^2\right]_0^1$
$= \left(\frac{10}{3} - 0\right) - (3 - 0) + (1 - 0) = \frac{4}{3}$

34. For the sketch given, $a = -1$, $b = 1$; $f(x) - g(x) = x^2 - (-2x^4) = x^2 + 2x^4$;
$A = \int_{-1}^{1} (x^2 + 2x^4)\, dx = \left[\frac{x^3}{3} + \frac{2x^5}{5}\right]_{-1}^{1} = \left(\frac{1}{3} + \frac{2}{5}\right) - \left[-\frac{1}{3} + \left(-\frac{2}{5}\right)\right] = \frac{2}{3} + \frac{4}{5} = \frac{10+12}{15} = \frac{22}{15}$

35. We want the area between the line $y = 1$, $0 \le x \le 2$, and the curve $y = \frac{x^2}{4}$, *minus* the area of a triangle (formed by $y = x$ and $y = 1$) with base 1 and height 1. Thus, $A = \int_0^2 \left(1 - \frac{x^2}{4}\right) dx - \frac{1}{2}(1)(1) = \left[x - \frac{x^3}{12}\right]_0^2 - \frac{1}{2}$
$= \left(2 - \frac{8}{12}\right) - \frac{1}{2} = 2 - \frac{2}{3} - \frac{1}{2} = \frac{5}{6}$

36. We want the area between the x-axis and the curve $y = x^2$, $0 \le x \le 1$ *plus* the area of a triangle (formed by $x = 1$, $x + y = 2$, and the x-axis) with base 1 and height 1. Thus, $A = \int_0^1 x^2\, dx + \frac{1}{2}(1)(1) = \left[\frac{x^3}{3}\right]_0^1 + \frac{1}{2} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

37. AREA = $A1 + A2$
A1: For the sketch given, $a = -3$ and we find b by solving the equations $y = x^2 - 4$ and $y = -x^2 - 2x$ simultaneously for x: $x^2 - 4 = -x^2 - 2x \Rightarrow 2x^2 + 2x - 4 = 0 \Rightarrow 2(x+2)(x-1) \Rightarrow x = -2$ or $x = 1$ so
$b = -2$: $f(x) - g(x) = (x^2 - 4) - (-x^2 - 2x) = 2x^2 + 2x - 4 \Rightarrow \text{A1} = \int_{-3}^{-2} (2x^2 + 2x - 4)\, dx$
$= \left[\frac{2x^3}{3} + \frac{2x^2}{2} - 4x\right]_{-3}^{-2} = \left(-\frac{16}{3} + 4 + 8\right) - (-18 + 9 + 12) = 9 - \frac{16}{3} = \frac{11}{3}$;
A2: For the sketch given, $a = -2$ and $b = 1$: $f(x) - g(x) = (-x^2 - 2x) - (x^2 - 4) = -2x^2 - 2x + 4$
$\Rightarrow \text{A2} = -\int_{-2}^{1} (2x^2 + 2x - 4)\, dx = -\left[\frac{2x^3}{3} + x^2 - 4x\right]_{-2}^{1} = -\left(\frac{2}{3} + 1 - 4\right) + \left(-\frac{16}{3} + 4 + 8\right)$
$= -\frac{2}{3} - 1 + 4 - \frac{16}{3} + 4 + 8 = 9$;
Therefore, AREA = $\text{A1} + \text{A2} = \frac{11}{3} + 9 = \frac{38}{3}$

38. AREA = $\text{A1} + \text{A2}$
A1: For the sketch given, $a = -2$ and $b = 0$: $f(x) - g(x) = (2x^3 - x^2 - 5x) - (-x^2 + 3x) = 2x^3 - 8x$
$\Rightarrow \text{A1} = \int_{-2}^{0} (2x^3 - 8x)\, dx = \left[\frac{2x^4}{4} - \frac{8x^2}{2}\right]_{-2}^{0} = 0 - (8 - 16) = 8$;
A2: For the sketch given, $a = 0$ and $b = 2$: $f(x) - g(x) = (-x^2 + 3x) - (2x^3 - x^2 - 5x) = 8x - 2x^3$
$\Rightarrow \text{A2} = \int_0^2 (8x - 2x^3)\, dx = \left[\frac{8x^2}{2} - \frac{2x^4}{4}\right]_0^2 = (16 - 8) = 8$;
Therefore, AREA = $A1 + A2 = 16$

39. AREA = $\text{A1} + \text{A2} + \text{A3}$
A1: For the sketch given, $a = -2$ and $b = -1$: $f(x) - g(x) = (-x + 2) - (4 - x^2) = x^2 - x - 2$
$\Rightarrow \text{A1} = \int_{-2}^{-1} (x^2 - x - 2)\, dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^{-1} = \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(-\frac{8}{3} - \frac{4}{2} + 4\right) = \frac{7}{3} - \frac{1}{2} = \frac{14-3}{6} = \frac{11}{6}$;

A2: For the sketch given, $a=-1$ and $b=2$: $f(x)-g(x)=(4-x^2)-(-x+2)=-(x^2-x-2)$

$$\Rightarrow \text{A2} = -\int_{-1}^{2}(x^2-x-2)\,dx = -\left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-1}^{2} = -\left(\frac{8}{3}-\frac{4}{2}-4\right)+\left(-\frac{1}{3}-\frac{1}{2}+2\right) = -3+8-\frac{1}{2} = \frac{9}{2};$$

A3: For the sketch given, $a=2$ and $b=3$: $f(x)-g(x)=(-x+2)-(4-x^2)=x^2-x-2$

$$\Rightarrow \text{A3} = \int_{2}^{3}(x^2-x-2)\,dx = \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{2}^{3} = \left(\frac{27}{3}-\frac{9}{2}-6\right)-\left(\frac{8}{3}-\frac{4}{2}-4\right) = 9-\frac{9}{2}-\frac{8}{3};$$

Therefore, AREA $= \text{A1}+A2+A3 = \frac{11}{6}+\frac{9}{2}+\left(9-\frac{9}{2}-\frac{8}{3}\right) = 9-\frac{5}{6} = \frac{49}{6}$

40. AREA $= \text{A1}+\text{A2}+\text{A3}$

A1: For the sketch given, $a=-2$ and $b=0$: $f(x)-g(x)=\left(\frac{x^3}{3}-x\right)-\frac{x}{3}=\frac{x^3}{3}-\frac{4}{3}x=\frac{1}{3}(x^3-4x)$

$$\Rightarrow \text{A1} = \frac{1}{3}\int_{-2}^{0}(x^3-4x)\,dx = \frac{1}{3}\left[\frac{x^4}{4}-2x^2\right]_{-2}^{0} = 0-\frac{1}{3}(4-8) = \frac{4}{3};$$

A2: For the sketch given, $a=0$ and we find b by solving the equations $y=\frac{x^3}{3}-x$ and $y=\frac{x}{3}$ simultaneously for x: $\frac{x^3}{3}-x=\frac{x}{3} \Rightarrow \frac{x^3}{3}-\frac{4}{3}x=0 \Rightarrow \frac{x}{3}(x-2)(x+2)=0 \Rightarrow x=-2$, $x=0$, or $x=2$ so $b=2$: $f(x)-g(x)$

$$= \frac{x}{3}-\left(\frac{x^3}{3}-x\right) = -\frac{1}{3}(x^3-4x) \Rightarrow \text{ A2} = -\frac{1}{3}\int_{0}^{2}(x^3-4x)\,dx = \frac{1}{3}\int_{0}^{2}(4x-x^3) = \frac{1}{3}\left[2x^2-\frac{x^4}{4}\right]_{0}^{2} = \frac{1}{3}(8-4) = \frac{4}{3};$$

A3: For the sketch given, $a=2$ and $b=3$: $f(x)-g(x)=\left(\frac{x^3}{3}-x\right)-\frac{x}{3}=\frac{1}{3}(x^3-4x)$

$$\Rightarrow \text{A3} = \frac{1}{3}\int_{2}^{3}(x^3-4x)\,dx = \frac{1}{3}\left[\frac{x^4}{4}-2x^2\right]_{2}^{3} = \frac{1}{3}\left[\left(\frac{81}{4}-2\cdot 9\right)-\left(\frac{16}{4}-8\right)\right] = \frac{1}{3}\left(\frac{81}{4}-14\right) = \frac{25}{12};$$

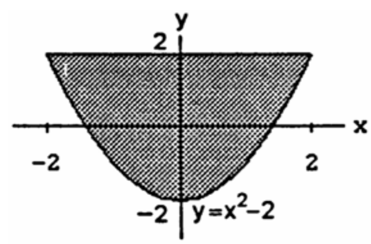
Therefore, AREA $= \text{A1}+\text{A2}+\text{A3} = \frac{4}{3}+\frac{4}{3}+\frac{25}{12} = \frac{32+25}{12} = \frac{19}{4}$

41. $a=-2$, $b=2$;

$f(x)-g(x)=2-(x^2-2)=4-x^2$

$$\Rightarrow A = \int_{-2}^{2}(4-x^2)\,dx = \left[4x-\frac{x^3}{3}\right]_{-2}^{2} = \left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)$$

$$= 2\cdot\left(\frac{24}{3}-\frac{8}{3}\right) = \frac{32}{3}$$

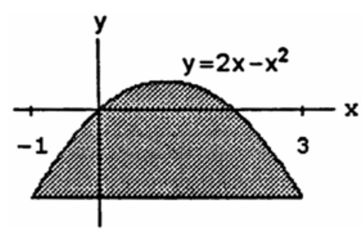


42. $a=-1, b=3$;

$f(x)-g(x)=(2x-x^2)-(-3)=2x-x^2+3$

$$\Rightarrow A = \int_{-1}^{3}(2x-x^2+3)dx = \left[x^2-\frac{x^3}{3}+3x\right]_{-1}^{3}$$

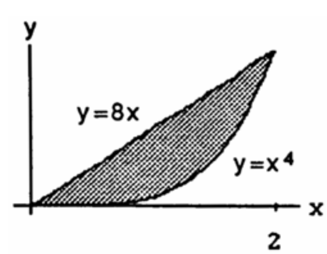
$$= \left(9-\frac{27}{3}+9\right)-\left(1+\frac{1}{3}-3\right) = 11-\frac{1}{3} = \frac{32}{3}$$



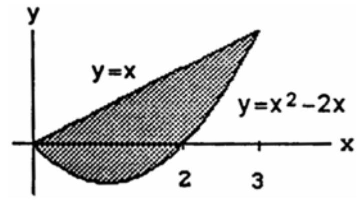
43. $a=0, b=2$;

$f(x)-g(x)=8x-x^4 \Rightarrow A=\int_{0}^{2}(8x-x^4)dx$

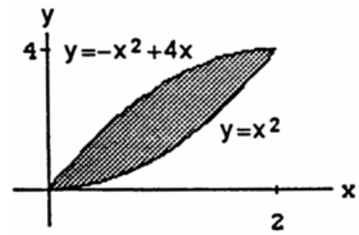
$$= \left[\frac{8x^2}{2}-\frac{x^5}{5}\right]_{0}^{2} = 16-\frac{32}{5} = \frac{80-32}{5} = \frac{48}{5}$$



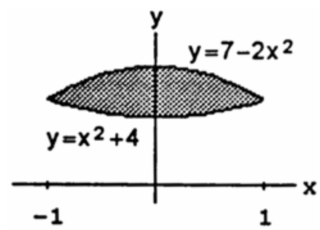
44. Limits of integration: $x^2-2x=x \Rightarrow x^2-3x=0$
$\Rightarrow x(x-3)=0 \Rightarrow a=0$ and $b=3$;
$f(x)-g(x)=x-(x^2-2x)=3x-x^2$
$\Rightarrow A=\int_0^3 (3x-x^2)dx=\left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3=\frac{27}{2}-9=\frac{27-18}{2}=\frac{9}{2}$



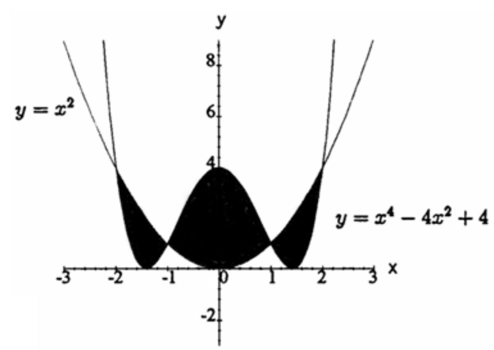
45. Limits of integration: $x^2=-x^2+4x \Rightarrow 2x^2-4x=0$
$\Rightarrow 2x(x-2)=0 \Rightarrow a=0$ and $b=2$;
$f(x)-g(x)=(-x^2+4x)-x^2=-2x^2+4x$
$\Rightarrow A=\int_0^2 (-2x^2+4x)dx=\left[\frac{-2x^3}{3}+\frac{4x^2}{2}\right]_0^2$
$=-\frac{16}{3}+\frac{16}{2}=\frac{-32+48}{6}=\frac{8}{3}$



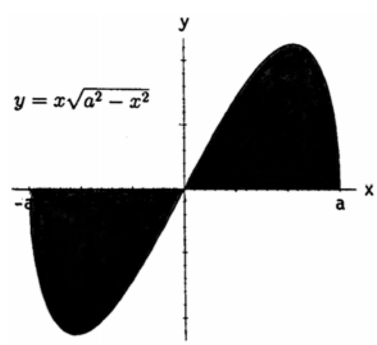
46. Limits of integration: $7-2x^2=x^2+4 \Rightarrow 3x^2-3=0$
$\Rightarrow 3(x-1)(x+1)=0 \Rightarrow a=-1$ and $b=1$;
$f(x)-g(x)=(7-2x^2)-(x^2+4)=3-3x^2$
$\Rightarrow A=\int_{-1}^1 (3-3x^2)dx=3\left[x-\frac{x^3}{3}\right]_{-1}^1=3\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]$
$=6\left(\frac{2}{3}\right)=4$



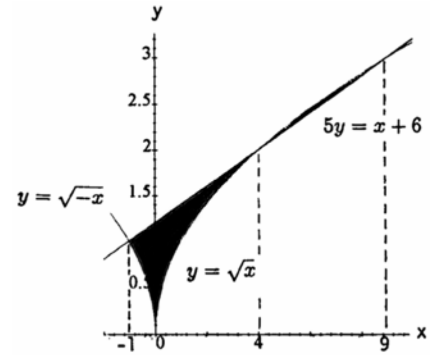
47. Limits of integration: $x^4-4x^2+4=x^2 \Rightarrow x^4-5x^2+4=0$
$\Rightarrow (x^2-4)(x^2-1)=0 \Rightarrow (x+2)(x-2)(x+1)(x-1)=0$
$\Rightarrow x=-2,-1,1,2;\ f(x)-g(x)=(x^4-4x^2+4)-x^2$
$=x^4-5x^2+4$ and
$g(x)-f(x)=x^2-(x^4-4x^2+4)=-x^4+5x^2-4$
$\Rightarrow A=\int_{-2}^{-1}\left(-x^4+5x^2-4\right)dx+\int_{-1}^{1}(x^4-5x^2+4)\,dx$
$+\int_1^2(-x^4+5x^2-4)\,dx$
$=\left[-\frac{x^5}{5}+\frac{5x^3}{3}-4x\right]_{-2}^{-1}+\left[\frac{x^5}{5}-\frac{5x^3}{3}+4x\right]_{-1}^{1}+\left[\frac{-x^5}{5}+\frac{5x^3}{3}-4x\right]_1^2$
$=\left(\frac{1}{5}-\frac{5}{3}+4\right)-\left(\frac{32}{5}-\frac{40}{3}+8\right)+\left(\frac{1}{5}-\frac{5}{3}+4\right)-\left(-\frac{1}{5}+\frac{5}{3}-4\right)+\left(-\frac{32}{5}+\frac{40}{3}-8\right)-\left(-\frac{1}{5}+\frac{5}{3}-4\right)=-\frac{60}{5}+\frac{60}{3}=\frac{300-180}{15}=8$



48. Limits of integration: $x\sqrt{a^2-x^2}=0 \Rightarrow x=0$ or
$\sqrt{a^2-x^2}=0 \Rightarrow x=0$ or $a^2-x^2=0 \Rightarrow x=-a,0,a$;
$A=\int_{-a}^0 -x\sqrt{a^2-x^2}\,dx+\int_0^a x\sqrt{a^2-x^2}\,dx$
$=\frac{1}{2}\left[\frac{2}{3}(a^2-x^2)^{3/2}\right]_{-a}^0-\frac{1}{2}\left[\frac{2}{3}(a^2-x^2)^{3/2}\right]_0^a$
$=\frac{1}{3}(a^2)^{3/2}-\left[-\frac{1}{3}(a^2)^{3/2}\right]=\frac{2a^3}{3}$

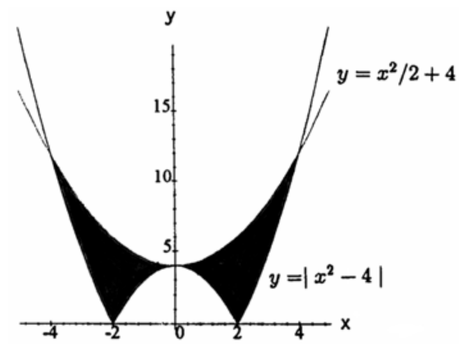


49. Limits of integration: $y=\sqrt{|x|}=\begin{cases}\sqrt{-x}, & x\le 0\\ \sqrt{x}, & x\ge 0\end{cases}$ and
$5y=x+6$ or $y=\frac{x}{5}+\frac{6}{5}$; for $x\le 0$; $\sqrt{-x}=\frac{x}{5}+\frac{6}{5}$
$\Rightarrow 5\sqrt{-x}=x+6 \Rightarrow 25(-x)=x^2+12x+36$
$\Rightarrow x^2+37x+36=0 \Rightarrow (x+1)(x+36)=0$
$\Rightarrow x=-1,-36$ (but $x=-36$ is not a solution);
for $x\ge 0: 5\sqrt{x}=x+6 \Rightarrow 25x=x^2+12x+36$
$\Rightarrow x^2-13x+36=0 \Rightarrow (x-4)(x-9)=0$
$\Rightarrow x=4,9$; there are three intersection points and

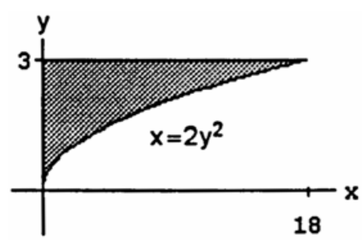


$$A=\int_{-1}^{0}\left(\tfrac{x+6}{5}-\sqrt{-x}\right)dx+\int_{0}^{4}\left(\tfrac{x+6}{5}-\sqrt{x}\right)dx+\int_{4}^{9}\left(\sqrt{x}-\tfrac{x+6}{5}\right)dx$$
$$=\left[\tfrac{(x+6)^2}{10}+\tfrac{2}{3}(-x)^{3/2}\right]_{-1}^{0}+\left[\tfrac{(x+6)^2}{10}-\tfrac{2}{3}x^{3/2}\right]_{0}^{4}+\left[\tfrac{2}{3}x^{3/2}-\tfrac{(x+6)^2}{10}\right]_{4}^{9}$$
$$=\left(\tfrac{36}{10}-\tfrac{25}{10}-\tfrac{2}{3}\right)+\left(\tfrac{100}{10}-\tfrac{2}{3}\cdot 4^{3/2}-\tfrac{36}{10}+0\right)+\left(\tfrac{2}{3}\cdot 9^{3/2}-\tfrac{225}{10}-\tfrac{2}{3}\cdot 4^{3/2}+\tfrac{100}{10}\right)=-\tfrac{50}{10}+\tfrac{20}{3}=\tfrac{5}{3}$$

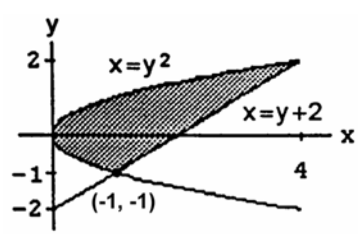
50. Limits of integration: $y=|x^2-4|=\begin{cases}x^2-4, & x\le -2 \text{ or } x\ge 2\\ 4-x^2, & -2\le x\le 2\end{cases}$
for $x\le -2$ and $x\ge 2: x^2-4=\frac{x^2}{2}+4$
$\Rightarrow 2x^2-8=x^2+8 \Rightarrow x^2=16 \Rightarrow x\pm 4$; for $-2\le x\le 2$:
$4-x^2=\frac{x^2}{2}+4 \Rightarrow 8-2x^2=x^2+8 \Rightarrow x^2=0 \Rightarrow x=0$; by symmetry of the graph,
$$A=2\int_{0}^{2}\left[\left(\tfrac{x^2}{2}+4\right)-(4+x^2)\right]dx$$
$$+2\int_{2}^{4}\left[\left(\tfrac{x^2}{2}+4\right)-\left(x^2-4\right)\right]dx=2\left[\tfrac{x^3}{2}\right]_{0}^{2}+2\left[8x-\tfrac{x^3}{6}\right]_{2}^{4}$$
$$=2\left(\tfrac{8}{2}-0\right)+2\left(32-\tfrac{64}{6}-16+\tfrac{8}{6}\right)=40-\tfrac{56}{3}=\tfrac{64}{3}$$



51. Limits of integration: $c=0$ and $d=3$;
$f(y)-g(y)=2y^2-0=2y^2$
$$\Rightarrow A=\int_{0}^{3}2y^2\,dy=\left[\tfrac{2y^3}{3}\right]_{0}^{3}=2\cdot 9=18$$

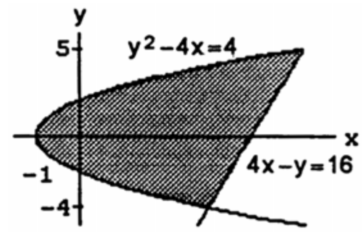


52. Limits of integration: $y^2=y+2 \Rightarrow (y+1)(y-2)=0$
$\Rightarrow c=-1$ and $d=2$; $f(y)-g(y)=(y+2)-y^2$
$$\Rightarrow A=\int_{-1}^{2}(y+2-y^2)\,dy=\left[\tfrac{y^2}{2}+2y-\tfrac{y^3}{3}\right]_{-1}^{2}$$
$$=\left(\tfrac{4}{2}+4-\tfrac{8}{3}\right)-\left(\tfrac{1}{2}-2+\tfrac{1}{3}\right)=6-\tfrac{8}{3}-\tfrac{1}{2}+2-\tfrac{1}{3}=\tfrac{9}{2}$$



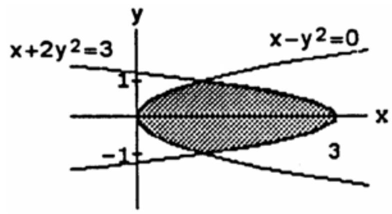
53. Limits of integration: $4x = y^2 - 4$ and $4x = 16 + y \Rightarrow y^2 - 4 = 16 + y \Rightarrow y^2 - y - 20 = 0$ $\Rightarrow (y-5)(y+4) = 0 \Rightarrow c = -4$ and $d = 5$; $f(y) - g(y) = \left(\frac{16+y}{4}\right) - \left(\frac{y^2-4}{4}\right) = \frac{-y^2+y+20}{4}$

$$\Rightarrow A = \frac{1}{4}\int_{-4}^{5}(-y^2 + y + 20)\,dy = \frac{1}{4}\left[-\frac{y^3}{3} + \frac{y^2}{2} + 20y\right]_{-4}^{5}$$
$$= \frac{1}{4}\left(-\frac{125}{3} + \frac{25}{2} + 100\right) - \frac{1}{4}\left(\frac{64}{3} + \frac{16}{2} - 80\right)$$
$$= \frac{1}{4}\left(-\frac{189}{3} + \frac{9}{2} + 180\right) = \frac{243}{8}$$



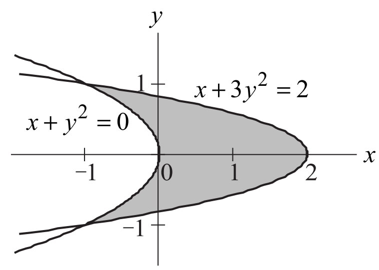
54. Limits of integration: $x = y^2$ and $x = 3 - 2y^2$ $\Rightarrow y^2 = 3 - 2y^2 \Rightarrow 3y^2 - 3 = 0 \Rightarrow 3(y-1)(y+1) = 0$ $\Rightarrow c = -1$ and $d = 1$; $f(y) - g(y) = (3 - 2y^2) - y^2$ $= 3 - 3y^2 = 3(1 - y^2) \Rightarrow A = 3\int_{-1}^{1}(1 - y^2)\,dy$

$$= 3\left[y - \frac{y^3}{3}\right]_{-1}^{1} = 3\left(1 - \frac{1}{3}\right) - 3\left(-1 + \frac{1}{3}\right) = 3 \cdot 2\left(1 - \frac{1}{3}\right) = 4$$



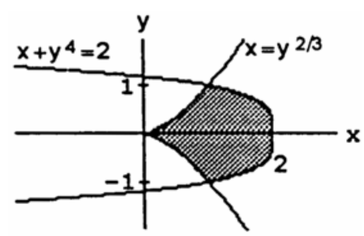
55. Limits of integration: $x = -y^2$ and $x = 2 - 3y^2$ $\Rightarrow -y^2 = 2 - 3y^2 \Rightarrow 2y^2 - 2 = 0$ $\Rightarrow 2(y-1)(y+1) = 0 \Rightarrow c = -1$ and $d = 1$; $f(y) - g(y) = (2 - 3y^2) - (-y^2) = 2 - 2y^2 = 2(1 - y^2)$

$$\Rightarrow A = 2\int_{-1}^{1}(1 - y^2)\,dy = 2\left[y - \frac{y^3}{3}\right]_{-1}^{1}$$
$$= 2\left(1 - \frac{1}{3}\right) - 2\left(-1 + \frac{1}{3}\right) = 4\left(\frac{2}{3}\right) = \frac{8}{3}$$

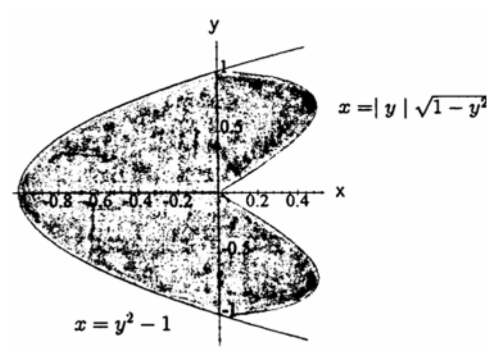


56. Limits of integration: $x = y^{2/3}$ and $x = 2 - y^4 \Rightarrow y^{2/3} = 2 - y^4 \Rightarrow c = -1$ and $d = 1$; $f(y) - g(y) = (2 - y^4) - y^{2/3} \Rightarrow A = \int_{-1}^{1}(2 - y^4 - y^{2/3})\,dy$

$$= \left[2y - \frac{y^5}{5} - \frac{3}{5}y^{5/3}\right]_{-1}^{1} = \left(2 - \frac{1}{5} - \frac{3}{5}\right) - \left(-2 + \frac{1}{5} + \frac{3}{5}\right)$$
$$= 2\left(2 - \frac{1}{5} - \frac{3}{5}\right) = \frac{12}{5}$$



57. Limits of integration: $x = y^2 - 1$ and $x = |y|\sqrt{1-y^2}$ $\Rightarrow y^2 - 1 = |y|\sqrt{1-y^2} \Rightarrow y^4 - 2y^2 + 1 = y^2(1 - y^2)$ $\Rightarrow y^4 - 2y^2 + 1 = y^2 - y^4 \Rightarrow 2y^4 - 3y^2 + 1 = 0$ $\Rightarrow (2y^2 - 1)(y^2 - 1) = 0 \Rightarrow 2y^2 - 1 = 0$ or $y^2 - 1 = 0 \Rightarrow y^2 = \frac{1}{2}$ or $y^2 = 1 \Rightarrow y = \pm\frac{\sqrt{2}}{2}$ or $y = \pm 1$.
Substitution shows that $\frac{\pm\sqrt{2}}{2}$ are not solutions $\Rightarrow y = \pm 1$;
for $-1 \le y \le 0$, $f(x) - g(x) = -y\sqrt{1-y^2} - (y^2 - 1)$ $= 1 - y^2 - y(1 - y^2)^{1/2}$, and by symmetry of the graph,



$$A=2\int_{-1}^{0}\left[1-y^2-y(1-y^2)^{1/2}\right]dy=2\int_{-1}^{0}(1-y^2)\,dy-2\int_{-1}^{0}y(1-y^2)^{1/2}\,dy$$

$$=2\left[y-\frac{y^3}{3}\right]_{-1}^{0}+2\left(\frac{1}{2}\right)\left[\frac{2(1-y^2)^{3/2}}{3}\right]_{-1}^{0}=2\left[(0-0)-\left(-1+\frac{1}{3}\right)\right]+\left(\frac{2}{3}-0\right)=2$$

58. AREA = A1 + A2

Limits of integration: $x=2y$ and

$x=y^3-y^2 \Rightarrow y^3-y^2-2y=0$

$\Rightarrow y(y^2-y-2)=y(y+1)(y-2)=0 \Rightarrow y=-1,0,2$:

for $-1\le y\le 0, f(y)-g(y)=y^3-y^2-2y$

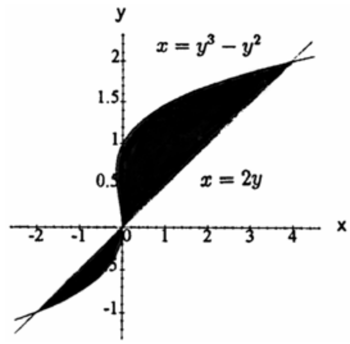
$\Rightarrow \text{A1}=\int_{-1}^{0}(y^3-y^2-2y)\,dy=\left[\frac{y^4}{4}-\frac{y^3}{3}-y^2\right]_{-1}^{0}$

$=0-\left(\frac{1}{4}+\frac{1}{3}-1\right)=\frac{5}{12}$;

for $0\le y\le 2, f(y)-g(y)=2y-y^3+y^2$

$\Rightarrow \text{A2}=\int_{0}^{2}(2y-y^3+y^2)\,dy=\left[y^2-\frac{y^4}{4}+\frac{y^3}{3}\right]_{0}^{2}$

$=\left(4-\frac{16}{4}+\frac{8}{3}\right)-0=\frac{8}{3}$; Therefore, $\text{A1}+\text{A2}=\frac{5}{12}+\frac{8}{3}=\frac{37}{12}$



59. Limits of integration: $y=-4x^2+4$ and $y=x^4-1$

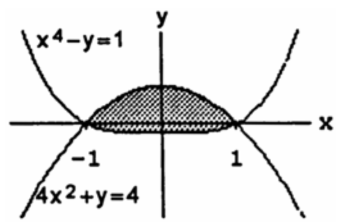
$\Rightarrow x^4-1=-4x^2+4 \Rightarrow x^4+4x^2-5=0$

$\Rightarrow (x^2+5)(x-1)(x+1)=0 \Rightarrow a=-1$ and $b=1$;

$f(x)-g(x)=-4x^2+4-x^4+1=-4x^2-x^4+5$

$\Rightarrow A=\int_{-1}^{1}(-4x^2-x^4+5)\,dx=\left[-\frac{4x^3}{3}-\frac{x^5}{5}+5x\right]_{-1}^{1}$

$=\left(-\frac{4}{3}-\frac{1}{5}+5\right)-\left(\frac{4}{3}+\frac{1}{5}-5\right)=2\left(-\frac{4}{3}-\frac{1}{5}+5\right)=\frac{104}{15}$



60. Limits of integration: $y=x^3$ and $y=3x^2-4$

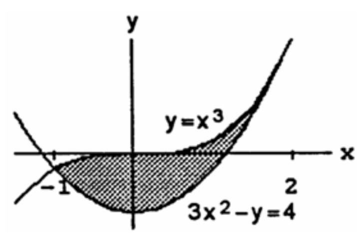
$\Rightarrow x^3-3x^2+4=0 \Rightarrow (x^2-x-2)(x-2)=0$

$\Rightarrow (x+1)(x-2)^2=0 \Rightarrow a=-1$ and $b=2$;

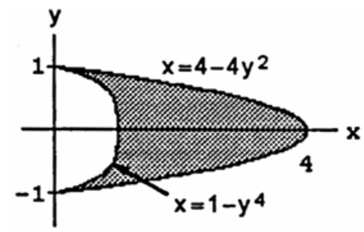
$f(x)-g(x)=x^3-(3x^2-4)=x^3-3x^2+4$

$\Rightarrow A=\int_{-1}^{2}(x^3-3x^2+4)\,dx=\left[\frac{x^4}{4}-\frac{3x^3}{3}+4x\right]_{-1}^{2}$

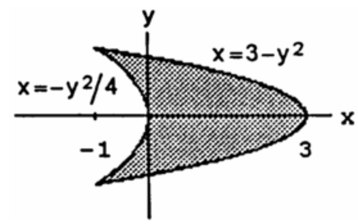
$=\left(\frac{16}{4}-\frac{24}{3}+8\right)-\left(\frac{1}{4}+1-4\right)=\frac{27}{4}$



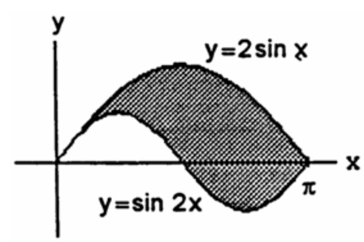
61. Limits of integration: $x = 4 - 4y^2$ and $x = 1 - y^4$
$\Rightarrow 4 - 4y^2 = 1 - y^4 \Rightarrow y^4 - 4y^2 + 3 = 0$
$\Rightarrow \left(y - \sqrt{3}\right)\left(y + \sqrt{3}\right)(y-1)(y+1) = 0 \Rightarrow c = -1$ and $d = 1$
since $x \ge 0$; $f(y) - g(y) = (4 - 4y^2) - (1 - y^4)$
$= 3 - 4y^2 + y^4 \Rightarrow A = \int_{-1}^{1} (3 - 4y^2 + y^4)\, dy$
$= \left[3y - \frac{4y^3}{3} + \frac{y^5}{5}\right]_{-1}^{1} = 2\left(3 - \frac{4}{3} + \frac{1}{5}\right) = \frac{56}{15}$



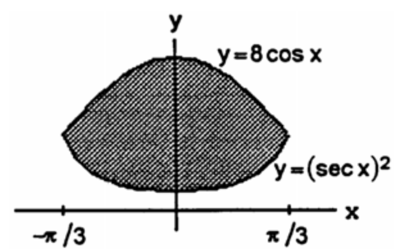
62. Limits of integration: $x = 3 - y^2$ and $x = -\frac{y^2}{4}$
$\Rightarrow 3 - y^2 = -\frac{y^2}{4} \Rightarrow \frac{3y^2}{4} - 3 = 0 \Rightarrow \frac{3}{4}(y-2)(y+2) = 0$
$\Rightarrow c = -2$ and $d = 2$; $f(y) - g(y) = (3 - y^2) - \left(\frac{-y^2}{4}\right)$
$= 3\left(1 - \frac{y^2}{4}\right) \Rightarrow A = 3\int_{-2}^{2} \left(1 - \frac{y^2}{4}\right) dy = 3\left[y - \frac{y^3}{12}\right]_{-2}^{2}$
$= 3\left[\left(2 - \frac{8}{12}\right) - \left(-2 + \frac{8}{12}\right)\right] = 3\left(4 - \frac{16}{12}\right) = 12 - 4 = 8$



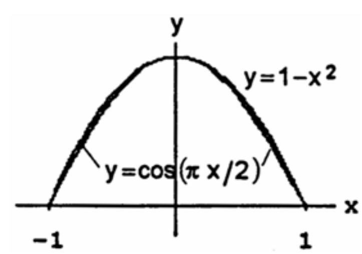
63. $a = 0, b = \pi$; $f(x) - g(x) = 2\sin x - \sin 2x$
$\Rightarrow A = \int_0^{\pi} (2\sin x - \sin 2x)dx = \left[-2\cos x + \frac{\cos 2x}{2}\right]_0^{\pi}$
$= \left[-2(-1) + \frac{1}{2}\right] - \left(-2 \cdot 1 + \frac{1}{2}\right) = 4$



64. $a = -\frac{\pi}{3}, b = \frac{\pi}{3}$; $f(x) - g(x) = 8\cos x - \sec^2 x$
$\Rightarrow A = \int_{-\pi/3}^{\pi/3} (8\cos x - \sec^2 x)dx = [8\sin x - \tan x]_{-\pi/3}^{\pi/3}$
$= \left(8 \cdot \frac{\sqrt{3}}{2} - \sqrt{3}\right) - \left(-8 \cdot \frac{\sqrt{3}}{2} + \sqrt{3}\right) = 6\sqrt{3}$



65. $a = -1, b = 1$; $f(x) - g(x) = (1 - x^2) - \cos\left(\frac{\pi x}{2}\right)$
$\Rightarrow A = \int_{-1}^{1} \left[1 - x^2 - \cos\left(\frac{\pi x}{2}\right)\right] dx$
$= \left[x - \frac{x^3}{3} - \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_{-1}^{1} = \left(1 - \frac{1}{3} - \frac{2}{\pi}\right) - \left(-1 + \frac{1}{3} + \frac{2}{\pi}\right)$
$= 2\left(\frac{2}{3} - \frac{2}{\pi}\right) = \frac{4}{3} - \frac{4}{\pi}$



66. $A = A1 + A2$

$a_1 = -1, b_1 = 0$ and $a_2 = 0, b_2 = 1;$

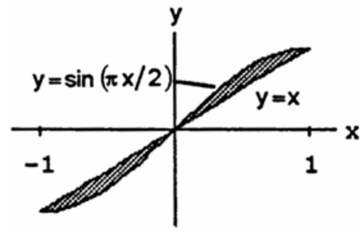
$f_1(x) - g_1(x) = x - \sin\left(\frac{\pi x}{2}\right)$ and $f_2(x) - g_2(x)$

$= \sin\left(\frac{\pi x}{2}\right) - x \Rightarrow$ by symmetry about the origin,

$A_1 + A_2 = 2A_1 \Rightarrow A = 2\int_0^1 \left[\sin\left(\frac{\pi x}{2}\right) - x\right] dx$

$= 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}\right]_0^1 = 2\left[\left(-\frac{2}{\pi}\cdot 0 - \frac{1}{2}\right) - \left(-\frac{2}{\pi}\cdot 1 - 0\right)\right]$

$= 2\left(\frac{2}{\pi} - \frac{1}{2}\right) = 2\left(\frac{4-\pi}{2\pi}\right) = \frac{4-\pi}{\pi}$

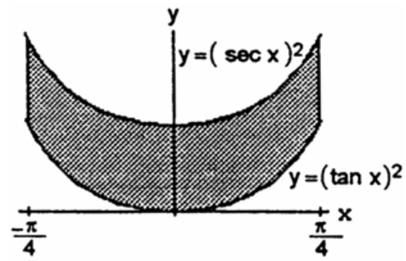


67. $a = -\frac{\pi}{4}, b = \frac{\pi}{4}; f(x) - g(x) = \sec^2 x - \tan^2 x$

$\Rightarrow A = \int_{-\pi/4}^{\pi/4} (\sec^2 x - \tan^2 x)\, dx$

$= \int_{-\pi/4}^{\pi/4} [\sec^2 x - (\sec^2 x - 1)]\, dx$

$= \int_{-\pi/4}^{\pi/4} 1 \cdot dx = [x]_{-\pi/4}^{\pi/4} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

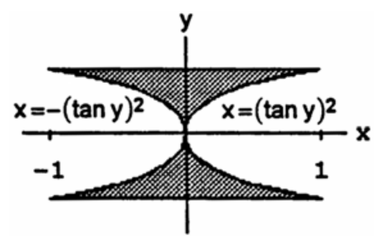


68. $c = -\frac{\pi}{4}, d = \frac{\pi}{4}; f(y) - g(y)$

$= \tan^2 y - (-\tan^2 y) = 2\tan^2 y = 2(\sec^2 y - 1)$

$\Rightarrow A = \int_{-\pi/4}^{\pi/4} 2\left(\sec^2 y - 1\right) dy = 2[(\tan y - y)]_{-\pi/4}^{\pi/4}$

$= 2\left[\left(1 - \frac{\pi}{4}\right) - \left(-1 + \frac{\pi}{4}\right)\right] = 4\left(1 - \frac{\pi}{4}\right) = 4 - \pi$

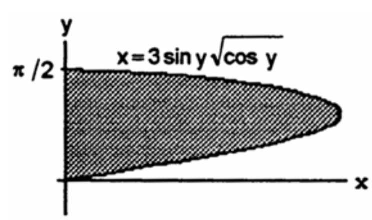


69. $c = 0, d = \frac{\pi}{2}; f(y) - g(y)$

$= 3\sin y\sqrt{\cos y} - 0 = 3\sin y\sqrt{\cos y}$

$\Rightarrow A = 3\int_0^{\pi/2} \sin y\sqrt{\cos y}\, dy = -3\left[\frac{2}{3}(\cos y)^{3/2}\right]_0^{\pi/2}$

$= -2(0 - 1) = 2$

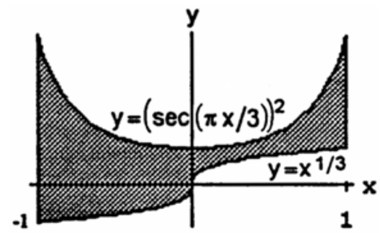


70. $a=-1, b=1;\ f(x)-g(x)=\sec^2\left(\frac{\pi x}{3}\right)-x^{1/3}$

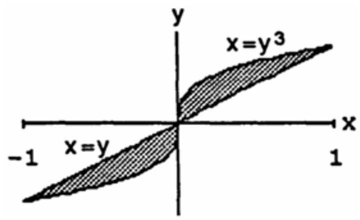
$\Rightarrow A=\int_{-1}^{1}\left[\sec^2\left(\frac{\pi x}{3}\right)-x^{1/3}\right]dx$

$=\left[\frac{3}{\pi}\tan\left(\frac{\pi x}{3}\right)-\frac{3}{4}x^{4/3}\right]_{-1}^{1}$

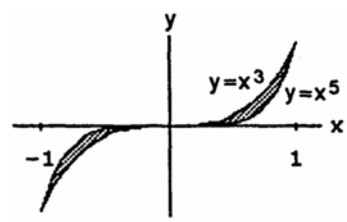
$=\left(\frac{3}{\pi}\sqrt{3}-\frac{3}{4}\right)-\left[\frac{3}{\pi}\left(-\sqrt{3}\right)-\frac{3}{4}\right]=\frac{6\sqrt{3}}{\pi}$



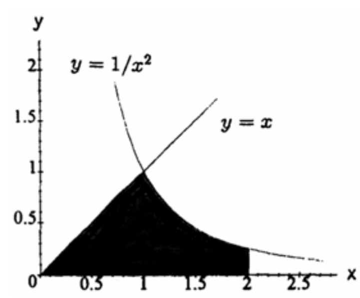
71. $A=A_1+A_2$
Limits of integration: $x=y^3$ and $x=y \Rightarrow y=y^3$
$\Rightarrow y^3-y=0 \Rightarrow y(y-1)(y+1)=0 \Rightarrow c_1=-1, d_1=0$
and $c_2=0, d_2=1;\ f_1(y)-g_1(y)=y^3-y$ and
$f_2(y)-g_2(y)=y-y^3 \Rightarrow$ by symmetry
about the origin, $A_1+A_2=2A_2 \Rightarrow A=$
$2\int_0^1 (y-y^3)\,dy=2\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1=2\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{1}{2}$



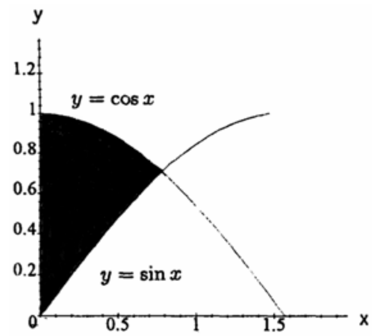
72. $A=A_1+A_2$
Limits of integration: $y=x^3$ and $y=x^5 \Rightarrow x^3=x^5$
$\Rightarrow x^5-x^3=0 \Rightarrow x^3(x-1)(x+1)=0 \Rightarrow a_1=-1, b_1=0$
and $a_2=0, b_2=1;\ f_1(x)-g_1(x)=x^3-x^5$ and
$f_2(x)-g_2(x)=x^5-x^3 \Rightarrow$ by symmetry about the
origin, $A_1+A_2=2A_2 \Rightarrow A=2\int_0^1 (x^3-x^5)\,dx$
$=2\left[\frac{x^4}{4}-\frac{x^6}{6}\right]_0^1=2\left(\frac{1}{4}-\frac{1}{6}\right)=\frac{1}{6}$



73. $A=A_1+A_2$
Limits of integration: $y=x$ and $y=\frac{1}{x^2} \Rightarrow x=\frac{1}{x^2},\ x\neq 0$
$\Rightarrow x^3=1 \Rightarrow x=1,\ f_1(x)-g_1(x)=x-0=x$
$\Rightarrow A_1=\int_0^1 x\,dx=\left[\frac{x^2}{2}\right]_0^1=\frac{1}{2};\ f_2(x)-g_2(x)=\frac{1}{x^2}-0$
$=x^{-2} \Rightarrow A_2=\int_1^2 x^{-2}\,dx=\left[\frac{-1}{x}\right]_1^2=-\frac{1}{2}+1=\frac{1}{2};$
$A=A_1+A_2=\frac{1}{2}+\frac{1}{2}=1$

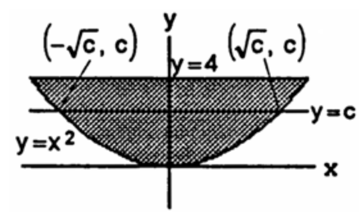


74. Limits of integration: $\sin x = \cos x \Rightarrow x = \frac{\pi}{4} \Rightarrow a = 0$ and $b = \frac{\pi}{4}$; $f(x) - g(x) = \cos x - \sin x$
$\Rightarrow A = \int_0^{\pi/4} (\cos x - \sin x)\, dx = [\sin x + \cos x]_0^{\pi/4}$
$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1) = \sqrt{2} - 1$



75. (a) The coordinates of the points of intersection of the line and parabola are $c = x^2 \Rightarrow x = \pm\sqrt{c}$ and $y = c$

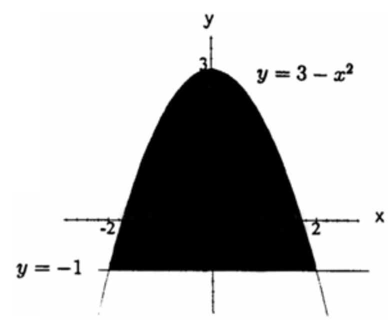
(b) $f(y) - g(y) = \sqrt{y} - \left(-\sqrt{y}\right) = 2\sqrt{y} \Rightarrow$ the area of the lower section is, $A_L = \int_0^c [f(y) - g(y)]\, dy$ $= 2\int_0^c \sqrt{y}\, dy = 2\left[\frac{2}{3} y^{3/2}\right]_0^c = \frac{4}{3} c^{3/2}$. The area of the entire shaded region can be found by setting $c = 4: A = \left(\frac{4}{3}\right) 4^{3/2} = \frac{4 \cdot 8}{3} = \frac{32}{3}$. Since we want c to divide the region into subsections of equal area we have $A = 2A_L \Rightarrow \frac{32}{3} = 2\left(\frac{4}{3} c^{3/2}\right) \Rightarrow c = 4^{2/3}$

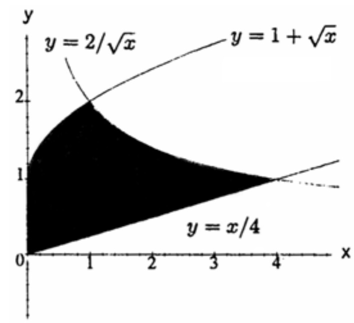


(c) $f(x) - g(x) = c - x^2 \Rightarrow A_L = \int_{-\sqrt{c}}^{\sqrt{c}} [f(x) - g(x)]\, dx = \int_{-\sqrt{c}}^{\sqrt{c}} (c - x^2)\, dx = \left[cx - \frac{x^3}{3}\right]_{-\sqrt{c}}^{\sqrt{c}} = 2\left[c^{3/2} - \frac{c^{3/2}}{3}\right]$ $= \frac{4}{3} c^{3/2}$. Again, the area of the whole shaded region can be found by setting $c = 4 \Rightarrow A = \frac{32}{3}$. From the condition $A = 2A_L$, we get $\frac{4}{3} c^{3/2} = \frac{32}{3} \Rightarrow c = 4^{2/3}$ as in part (b).

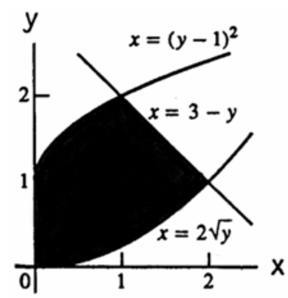
76. (a) Limits of integration: $y = 3 - x^2$ and $y = -1$
$\Rightarrow 3 - x^2 = -1 \Rightarrow x^2 = 4 \Rightarrow a = -2$ and $b = 2$;
$f(x) - g(x) = (3 - x^2) - (-1) = 4 - x^2$
$\Rightarrow A = \int_{-2}^{2} (4 - x^2)\, dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$
$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3} = \frac{32}{3}$

(b) Limits of integration: let $x = 0$ in $y = 3 - x^2$
$\Rightarrow y = 3$; $f(y) - g(y) = \sqrt{3-y} - \left(-\sqrt{3-y}\right)$
$= 2(3-y)^{1/2} \Rightarrow A = 2\int_{-1}^{3} (3-y)^{1/2}\, dy = -2 \int_{-1}^{3} (3-y)^{1/2}(-1)\, dy = (-2)\left[\frac{2(3-y)^{3/2}}{3}\right]_{-1}^{3}$
$= \left(-\frac{4}{3}\right)\left[0 - (3+1)^{3/2}\right] = \left(\frac{4}{3}\right)(8) = \frac{32}{3}$





77. Limits of integration: $y = 1 + \sqrt{x}$ and $y = \frac{2}{\sqrt{x}}$
$\Rightarrow 1 + \sqrt{x} = \frac{2}{\sqrt{x}}, x \neq 0 \Rightarrow \sqrt{x} + x = 2 \Rightarrow x = (2 - x)^2$
$\Rightarrow x = 4 - 4x + x^2 \Rightarrow x^2 - 5x + 4 = 0$
$\Rightarrow (x-4)(x-1) = 0 \Rightarrow x = 1, 4$ (but $x = 4$ does not satisfy the equation); $y = \frac{2}{\sqrt{x}}$ and $y = \frac{x}{4} \Rightarrow \frac{2}{\sqrt{x}} = \frac{x}{4}$
$\Rightarrow 8 = x\sqrt{x} \Rightarrow 64 = x^3 \Rightarrow x = 4$. Therefore,
AREA $= A_1 + A_2$: $f_1(x) - g_1(x) = \left(1 + x^{1/2}\right) - \frac{x}{4}$
$\Rightarrow A_1 = \int_0^1 \left(1 + x^{1/2} - \frac{x}{4}\right) dx = \left[x + \frac{2}{3}x^{3/2} - \frac{x^2}{8}\right]_0^1 = \left(1 + \frac{2}{3} - \frac{1}{8}\right) - 0 = \frac{37}{24}$; $f_2(x) - g_2(x) = 2x^{-1/2} - \frac{x}{4}$
$\Rightarrow A_2 = \int_1^4 \left(2x^{-1/2} - \frac{x}{4}\right) dx = \left[4x^{1/2} - \frac{x^2}{8}\right]_1^4 = \left(4 \cdot 2 - \frac{16}{8}\right) - \left(4 - \frac{1}{8}\right) = 4 - \frac{15}{8} = \frac{17}{8}$; Therefore,
AREA $= A_1 + A_2 = \frac{37}{24} + \frac{17}{8} = \frac{37+51}{24} = \frac{88}{24} = \frac{11}{3}$



78. Limits of integration: $(y-1)^2 = 3 - y$
$\Rightarrow y^2 - 2y + 1 = 3 - y \Rightarrow y^2 - y - 2 = 0$
$\Rightarrow (y-2)(y+1) = 0 \Rightarrow y = 2$ since $y > 0$; also,
$2\sqrt{y} = 3 - y \Rightarrow 4y = 9 - 6y + y^2 \Rightarrow y^2 - 10y + 9 = 0$
$\Rightarrow (y-9)(y-1) = 0 \Rightarrow y = 1$ since $y = 9$ does not satisfy the equation;
AREA $= A_1 + A_2$

$f_1(y) - g_1(y) = 2\sqrt{y} - 0 = 2y^{1/2} \Rightarrow A_1 = 2\int_0^1 y^{1/2}\, dy = 2\left[\frac{2y^{3/2}}{3}\right]_0^1 = \frac{4}{3}$;

$f_2(y) - g_2(y) = (3 - y) - (y-1)^2 \Rightarrow A_2 = \int_1^2 [3 - y - (y-1)^2]\, dy = \left[3y - \frac{1}{2}y^2 - \frac{1}{3}(y-1)^3\right]_1^2$
$= \left(6 - 2 - \frac{1}{3}\right) - \left(3 - \frac{1}{2} + 0\right) = 1 - \frac{1}{3} + \frac{1}{2} = \frac{7}{6}$. Therefore, $A_1 + A_2 = \frac{4}{3} + \frac{7}{6} = \frac{15}{6} = \frac{5}{2}$

79. Area between parabola and $y = a^2$: $A = 2\int_0^a (a^2 - x^2)\, dx = 2\left[a^2x - \frac{1}{3}x^3\right]_0^a = 2\left(a^3 - \frac{a^3}{3}\right) - 0 = \frac{4a^3}{3}$;
Area of triangle AOC: $\frac{1}{2}(2a)(a^2) = a^3$; limit of ratio $= \lim_{a \to 0^+} \frac{a^3}{\left(\frac{4a^3}{3}\right)} = \frac{3}{4}$ which is independent of a.

80. $A = \int_a^b 2f(x)\, dx - \int_a^b f(x)\, dx = 2\int_a^b f(x)\, dx - \int_a^b f(x)\, dx = \int_a^b f(x)\, dx = 4$

81. Neither one; they are both zero. Neither integral takes into account the changes in the formulas for the region's upper and lower bounding curves at $x = 0$. The area of the shaded region is actually
$A = \int_{-1}^0 [-x - (x)]\, dx + \int_0^1 [x - (-x)]\, dx = \int_{-1}^0 -2x\, dx + \int_0^1 2x\, dx = 2.$

82. It is sometimes true. It is true if $f(x) \geq g(x)$ for all x between a and b. Otherwise it is false. If the graph of f lies below the graph of g for a portion of the interval of integration, the integral over that portion will be negative and the integral over $[a, b]$ will be less than the area between the curves (see Exercise 71).

83. Let $u = 2x \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}\,du = dx$; $x = 1 \Rightarrow u = 2$, $x = 3 \Rightarrow u = 6$

$$\int_1^3 \frac{\sin 2x}{x}\,dx = \int_2^6 \frac{\sin u}{\left(\frac{u}{2}\right)}\left(\frac{1}{2}\,du\right) = \int_2^6 \frac{\sin u}{u}\,du = [F(u)]_2^6 = F(6) - F(2)$$

84. Let $u = 1 - x \Rightarrow du = -dx \Rightarrow -du = dx$; $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 0$

$$\int_0^1 f(1-x)\,dx = \int_1^0 f(u)(-du) = -\int_1^0 f(u)\,du = \int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$$

85. (a) Let $u = -x \Rightarrow du = -dx$; $x = -1 \Rightarrow u = 1$, $x = 0 \Rightarrow u = 0$

f odd $\Rightarrow f(-x) = -f(x)$. Then $\int_{-1}^0 f(x)dx = \int_1^0 f(-u)(-du) = \int_1^0 -f(u)\,(-du) = \int_1^0 f(u)\,du$

$= -\int_0^1 f(u)\,du = -3$

(b) Let $u = -x \Rightarrow du = -dx$; $x = -1 \Rightarrow u = 1$, $x = 0 \Rightarrow u = 0$

f even $\Rightarrow f(-x) = f(x)$. Then $\int_{-1}^0 f(x)dx = \int_1^0 f(-u)(-du) = -\int_1^0 f(u)du = \int_0^1 f(u)\,du = 3$

86. (a) Consider $\int_{-a}^0 f(x)\,dx$ when f is odd. Let $u = -x \Rightarrow du = -dx \Rightarrow -du = dx$ and $x = -a \Rightarrow u = a$ and $x = 0 \Rightarrow u = 0$. Thus $\int_{-a}^0 f(x)\,dx = \int_a^0 -f(-u)\,du = \int_a^0 f(u)\,du = -\int_0^a f(u)\,du = -\int_0^a f(x)\,dx$. Thus

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^a f(x)\,dx = \int_0^a f(x)\,dx = 0.$$

(b) $\int_{-\pi/2}^{\pi/2} \sin x\,dx = [-\cos x]_{-\pi/2}^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos\left(-\frac{\pi}{2}\right) = 0 + 0 = 0.$

87. Let $u = a - x \Rightarrow du = -dx$; $x = 0 \Rightarrow u = a$, $x = a \Rightarrow u = 0$

$$I = \int_0^a \frac{f(x)\,dx}{f(x)+f(a-x)} = \int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}(-du) = \int_0^a \frac{f(a-u)\,du}{f(u)+f(a-u)} = \int_0^a \frac{f(a-x)\,dx}{f(x)+f(a-x)}$$

$$\Rightarrow I + I = \int_0^a \frac{f(x)\,dx}{f(x)+f(a-x)} + \int_0^a \frac{f(a-x)\,dx}{f(x)+f(a-x)} = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx = \int_0^a dx = [x]_0^a = a - 0 = a.$$

Therefore, $2I = a \Rightarrow I = \frac{a}{2}$.

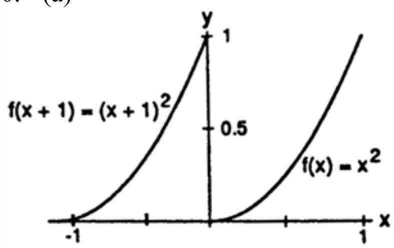
88. Let $u = \frac{xy}{t} \Rightarrow du = -\frac{xy}{t^2}dt \Rightarrow -\frac{t}{xy}\,du = \frac{1}{t}\,dt \Rightarrow -\frac{1}{u}du = \frac{1}{t}dt$; $t = x \Rightarrow u = y$, $t = xy \Rightarrow u = 1$. Therefore,

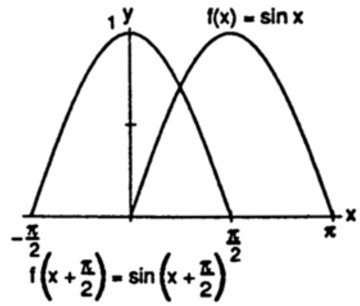
$$\int_x^{xy} \frac{1}{t}\,dt = \int_y^1 -\frac{1}{u}\,du = -\int_y^1 \frac{1}{u}\,du = \int_1^y \frac{1}{u}\,du = \int_1^y \frac{1}{t}\,dt$$

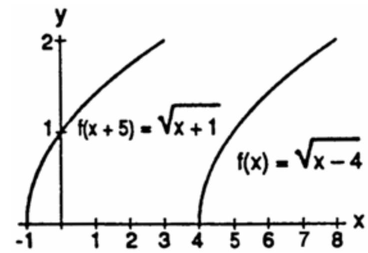
89. Let $u = x + c \Rightarrow du = dx$; $x = a - c \Rightarrow u = a$, $x = b - c \Rightarrow u = b$

$$\int_{a-c}^{b-c} f(x+c)\,dx = \int_a^b f(u)\,du = \int_a^b f(x)\,dx$$

90. (a) (b) (c)







91-94. Example CAS commands:

Maple:

```
f := x -> x^3/3-x^2/2-2*x+1/3;
g := x -> x-1;
plot( [f(x),g(x)], x=-5..5, legend=["y = f(x)","y = g(x)"], title="#91(a) (Section 5.6)" );
q1 := [ -5, -2, 1, 4 ];                    # (b)
q2 := [seq( fsolve( f(x)=g(x), x=q1[i]..q1[i+1] ), i=1..nops(q1)-1 )];
for i from 1 to nops(q2)-1 do              # (c)
  area[i] := int( abs(f(x)-g(x)),x=q2[i]..q2[i+1] );
end do;
add( area[i], i=1..nops(q2)-1 );           # (d)
```

Mathematica: (assigned functions may vary)

```
Clear[x, f, g]
f[x_] = x^2 Cos[x]
g[x_] = x^3 − x
Plot[{f[x], g[x]}, {x, −2, 2}]
```

After examining the plots, the initial guesses for FindRoot can be determined.

```
pts = x/.Map[FindRoot[f[x]==g[x],{x, #}]&, {−1, 0, 1}]
i1=NIntegrate[f[x] − g[x], {x, pts[[1]], pts[[2]]}]
i2=NIntegrate [f[x] − g[x], {x, pts[[2]], pts[[3]]}]
i1 + i2
```

CHAPTER 5 PRACTICE EXERCISES

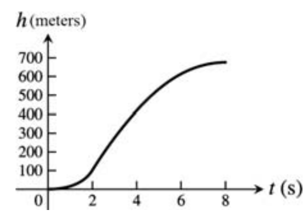
1. (a) Each time subinterval is of length $\Delta t = 0.4$ s. The distance traveled over each subinterval, using the midpoint rule, is $\Delta h = \frac{1}{2}(v_i + v_{i+1})\Delta t$, where v_i is the velocity at the left endpoint and v_{i+1} the velocity at the right endpoint of the subinterval. We then add Δh to the height attained so far at the left endpoint v_i to arrive at the height associated with velocity v_{i+1} at the right endpoint. Using this methodology we build the following table based on the figure in the text:

t (s)	0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	4.4	4.8	5.2	5.6	6.0
v (m/s)	0	10	25	55	100	190	180	165	150	140	130	115	105	90	76	65
h (m)	0	2	9	25	56	114	188	257	320	378	432	481	525	564	592	620.2

t (s)	6.4	6.8	7.2	7.6	8.0
v (m/s)	50	37	25	12	0
h (m)	643.2	660.6	672	679.4	681.8

NOTE: Your table values may vary slightly from ours depending on the v-values you read from the graph. Remember that some shifting of the graph occurs in the printing process.
The total height attained is about 680 m.

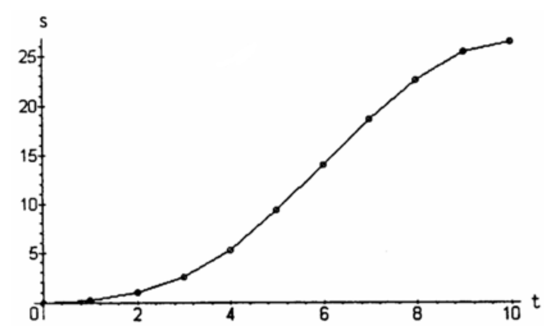
(b) The graph is based on the table in part (a).



2. (a) Each time subinterval is of length $\Delta t = 1$ s. The distance traveled over each subinterval, using the midpoint rule, is $\Delta s = \frac{1}{2}(v_i + v_{i+1})\,\Delta t$, where v_i is the velocity at the left, and v_{i+1} the velocity at the right, endpoint of the subinterval. We then add Δs to the distance attained so far at the left endpoint v_i to arrive at the distance associated with velocity v_{i+1} at the right endpoint. Using this methodology we build the table given below based on the figure in the text, obtaining approximately 26 m for the total distance traveled:

t (s)	0	1	2	3	4	5	6	7	8	9	10
v (m/s)	0	0.5	1.2	2	3.4	4.5	4.8	4.5	3.5	2	0
s (m)	0	0.25	1.1	2.7	5.4	9.35	14	18.65	22.65	25.4	26.4

(b) The graph shows the distance traveled by the moving body as a function of time for $0 \le t \le 10$.



3. (a) $\sum_{k=1}^{10} \frac{a_k}{4} = \frac{1}{4}\sum_{k=1}^{10} a_k = \frac{1}{4}(-2) = -\frac{1}{2}$

(b) $\sum_{k=1}^{10} (b_k - 3a_k) = \sum_{k=1}^{10} b_k - 3\sum_{k=1}^{10} a_k = 25 - 3(-2) = 31$

(c) $\sum_{k=1}^{10} (a_k + b_k - 1) = \sum_{k=1}^{10} a_k + \sum_{k=1}^{10} b_k - \sum_{k=1}^{10} 1 = -2 + 25 - (1)(10) = 13$

(d) $\sum_{k=1}^{10} \left(\frac{5}{2} - b_k\right) = \sum_{k=1}^{10} \frac{5}{2} - \sum_{k=1}^{10} b_k = \frac{5}{2}(10) - 25 = 0$

4. (a) $\sum_{k=1}^{20} 3a_k = 3\sum_{k=1}^{20} a_k = 3(0) = 0$

(b) $\sum_{k=1}^{20} (a_k + b_k) = \sum_{k=1}^{20} a_k + \sum_{k=1}^{20} b_k = 0 + 7 = 7$

(c) $\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7}\right) = \sum_{k=1}^{20} \frac{1}{2} - \frac{2}{7}\sum_{k=1}^{20} b_k = \frac{1}{2}(20) - \frac{2}{7}(7) = 8$

(d) $\sum_{k=1}^{20} (a_k - 2) = \sum_{k=1}^{20} a_k - \sum_{k=1}^{20} 2 = 0 - 2(20) = -40$

5. Let $u = 2x - 1 \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}du = dx$; $x = 1 \Rightarrow u = 1$, $x = 5 \Rightarrow u = 9$

$\int_1^5 (2x-1)^{-1/2}\,dx = \int_1^9 u^{-1/2}\left(\frac{1}{2}du\right) = \left[u^{1/2}\right]_1^9 = 3 - 1 = 2$

6. Let $u = x^2 - 1 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$; $x = 1 \Rightarrow u = 0$, $x = 3 \Rightarrow u = 8$

$\int_1^3 x(x^2-1)^{1/3}\,dx = \int_0^8 u^{1/3}\left(\frac{1}{2}du\right) = \left[\frac{3}{8}u^{4/3}\right]_0^8 = \frac{3}{8}(16 - 0) = 6$

7. Let $u = \frac{x}{2} \Rightarrow 2\,du = dx$; $x = -\pi \Rightarrow u = -\frac{\pi}{2}$, $x = 0 \Rightarrow u = 0$

$\int_{-\pi}^0 \cos\left(\frac{x}{2}\right)dx = \int_{-\pi/2}^0 (\cos u)(2\,du) = [2\sin u]_{-\pi/2}^0 = 2\sin 0 - 2\sin\left(-\frac{\pi}{2}\right) = 2(0 - (-1)) = 2$

8. Let $u = \sin x \Rightarrow du = \cos x\,dx$; $x = 0 \Rightarrow u = 0$, $x = \frac{\pi}{2} \Rightarrow u = 1$

$\int_0^{\pi/2} (\sin x)(\cos x)\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$

9. (a) $\int_{-2}^2 f(x)\,dx = \frac{1}{3}\int_{-2}^2 3\,f(x)\,dx = \frac{1}{3}(12) = 4$

(b) $\int_2^5 f(x)\,dx = \int_{-2}^5 f(x)\,dx - \int_{-2}^2 f(x)\,dx = 6 - 4 = 2$

(c) $\int_5^{-2} g(x)\,dx = -\int_{-2}^5 g(x)\,dx = -2$

(d) $\int_{-2}^5 (-\pi\,g(x))\,dx = -\pi\int_{-2}^5 g(x)\,dx = -\pi(2) = -2\pi$

(e) $\int_{-2}^5 \left(\frac{f(x)+g(x)}{5}\right)dx = \frac{1}{5}\int_{-2}^5 f(x)\,dx + \frac{1}{5}\int_{-2}^5 g(x)\,dx = \frac{1}{5}(6) + \frac{1}{5}(2) = \frac{8}{5}$

10. (a) $\int_0^2 g(x)\,dx = \frac{1}{7}\int_0^2 7\,g(x)\,dx = \frac{1}{7}(7) = 1$

(b) $\int_1^2 g(x)\,dx = \int_0^2 g(x)\,dx - \int_0^1 g(x)\,dx = 1 - 2 = -1$

(c) $\int_2^0 f(x)\,dx = -\int_0^2 f(x)\,dx = -\pi$

(d) $\int_0^2 \sqrt{2}\,f(x)\,dx = \sqrt{2}\int_0^2 f(x)\,dx = \sqrt{2}(\pi) = \pi\sqrt{2}$

(e) $\int_0^2 [g(x) - 3\,f(x)]\,dx = \int_0^2 g(x)\,dx - 3\int_0^2 f(x)\,dx = 1 - 3\pi$

11. $x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 3$ or $x = 1$;

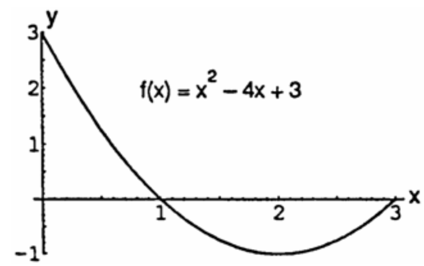
Area $= \int_0^1 (x^2 - 4x + 3)\,dx - \int_1^3 (x^2 - 4x + 3)\,dx$

$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3$

$= \left[\left(\frac{1^3}{3} - 2(1)^2 + 3(1)\right) - 0\right]$

$\quad - \left[\left(\frac{3^3}{3} - 2(3^2) + 3(3)\right) - \left(\frac{1^3}{3} - 2(1)^2 + 3(1)\right)\right]$

$= \left(\frac{1}{3} + 1\right) - \left[0 - \left(\frac{1}{3} + 1\right)\right] = \frac{8}{3}$



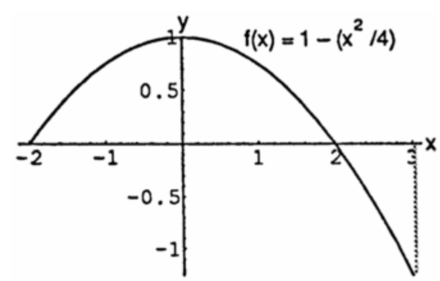
12. $1 - \frac{x^2}{4} = 0 \Rightarrow 4 - x^2 - 0 \Rightarrow x = \pm 2$;

Area $= \int_{-2}^2 \left(1 - \frac{x^2}{4}\right) dx - \int_2^3 \left(1 - \frac{x^2}{4}\right) dx$

$= \left[x - \frac{x^3}{12}\right]_{-2}^2 - \left[x - \frac{x^3}{12}\right]_2^3$

$= \left[\left(2 - \frac{2^3}{12}\right) - \left(-2 - \frac{(-2)^3}{12}\right)\right] - \left[\left(3 - \frac{3^3}{12}\right) - \left(2 - \frac{2^3}{12}\right)\right]$

$= \left[\frac{4}{3} - \left(-\frac{4}{3}\right)\right] - \left(\frac{3}{4} - \frac{4}{3}\right) = \frac{13}{4}$



13. $5 - 5x^{2/3} = 0 \Rightarrow 1 - x^{2/3} = 0 \Rightarrow x = \pm 1$;

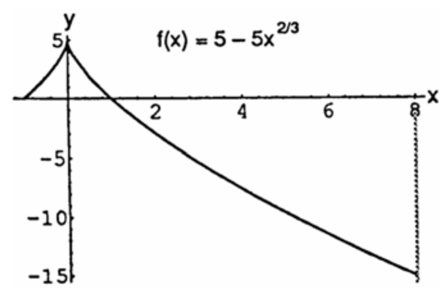
Area $= \int_{-1}^1 (5 - 5x^{2/3})\,dx - \int_1^8 (5 - 5x^{2/3})\,dx$

$= \left[5x - 3x^{5/3}\right]_{-1}^1 - \left[5x - 3x^{5/3}\right]_1^8$

$= \left[\left(5(1) - 3(1)^{5/3}\right) - \left(5(-1) - 3(-1)^{5/3}\right)\right]$

$\quad - \left[\left(5(8) - 3(8)^{5/3}\right) - \left(5(1) - 3(1)^{5/3}\right)\right]$

$= [2 - (-2)] - [(40 - 96) - 2] = 62$



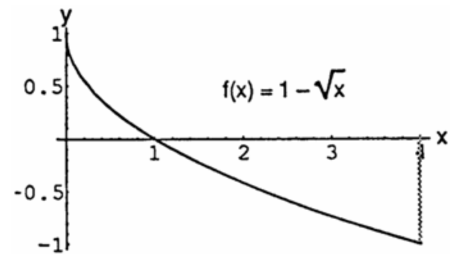
14. $1-\sqrt{x}=0 \Rightarrow x=1$;

Area $=\int_0^1 (1-\sqrt{x})\,dx - \int_1^4 (1-\sqrt{x})\,dx$

$=\left[x-\frac{2}{3}x^{3/2}\right]_0^1 - \left[x-\frac{2}{3}x^{3/2}\right]_1^4$

$=\left[\left(1-\frac{2}{3}(1)^{3/2}\right)-0\right]-\left[\left(4-\frac{2}{3}(4)^{3/2}\right)-\left(1-\frac{2}{3}(1)^{3/2}\right)\right]$

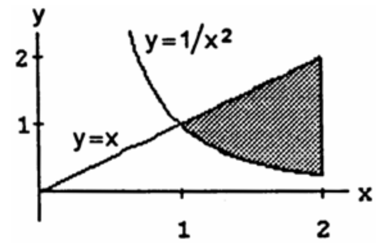
$=\frac{1}{3}-\left[\left(4-\frac{16}{3}\right)-\frac{1}{3}\right]=2$



15. $f(x)=x,\ g(x)=\frac{1}{x^2},\ a=1,\ b=2$

$\Rightarrow A=\int_a^b [f(x)-g(x)]\,dx$

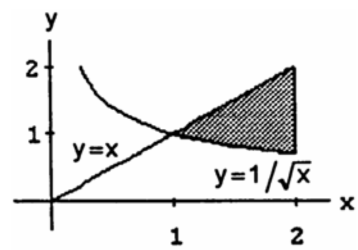
$=\int_1^2 \left(x-\frac{1}{x^2}\right)dx=\left[\frac{x^2}{2}+\frac{1}{x}\right]_1^2=\left(\frac{4}{2}+\frac{1}{2}\right)-\left(\frac{1}{2}+1\right)=1$



16. $f(x)=x,\ g(x)=\frac{1}{\sqrt{x}},\ a=1,\ b=2\ =\int_a^b [f(x)-g(x)]\,dx$

$\Rightarrow A=\int_1^2 \left(x-\frac{1}{\sqrt{x}}\right)dx=\left[\frac{x^2}{2}-2\sqrt{x}\right]_1^2$

$=\left(\frac{4}{2}-2\sqrt{2}\right)-\left(\frac{1}{2}-2\right)=\frac{7-4\sqrt{2}}{2}$



17. $f(x)=(1-\sqrt{x})^2,\ g(x)=0,\ a=0,\ b=1 \Rightarrow A=\int_a^b [f(x)-g(x)]\,dx=\ \int_0^1 (1-\sqrt{x})^2\ dx\ =\int_0^1 (1-2\sqrt{x}+x)\,dx$

$=\int_0^1 (1-2x^{1/2}+x)\,dx=\left[x-\frac{4}{3}x^{3/2}+\frac{x^2}{2}\right]_0^1=1-\frac{4}{3}+\frac{1}{2}=\frac{1}{6}(6-8+3)=\frac{1}{6}$

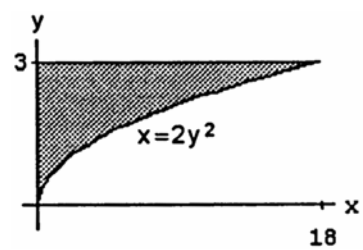
18. $f(x)=(1-x^3)^2,\ g(x)=0,\ a=0,\ b=1 \Rightarrow A=\int_a^b [f(x)-g(x)]\,dx=\ \int_0^1 (1-x^3)^2 dx=\ \int_0^1 (1-2x^3+x^6)\,dx$

$=\left[x-\frac{x^4}{2}+\frac{x^7}{7}\right]_0^1=1-\frac{1}{2}+\frac{1}{7}=\frac{9}{14}$

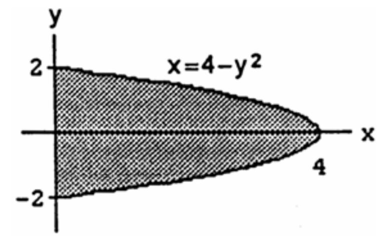
19. $f(y)=2y^2,\ g(y)=0,\ c=0,\ d=3$

$\Rightarrow A=\int_c^d [f(y)-g(y)]\,dy=\int_0^3 (2y^2-0)\,dy$

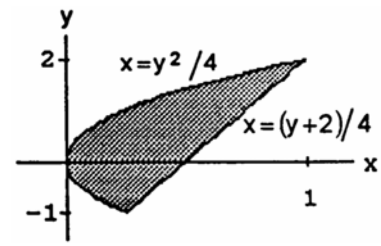
$=2\int_0^3 y^2\ dy=\frac{2}{3}[y^3]_0^3=18$



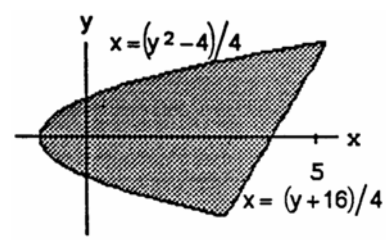
20. $f(y)=4-y^2,\ g(y)=0,\ c=-2,\ d=2$
$\Rightarrow A=\int_c^d [f(y)-g(y)]\,dy=\int_{-2}^{2}(4-y^2)\,dy$
$=\left[4y-\frac{y^3}{3}\right]_{-2}^{2}=2\left(8-\frac{8}{3}\right)=\frac{32}{3}$



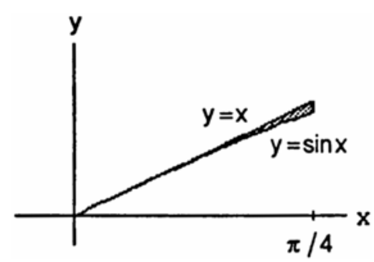
21. Let us find the intersection points: $\frac{y^2}{4}=\frac{y+2}{4}$
$\Rightarrow y^2-y-2=0\Rightarrow (y-2)(y+1)=0\Rightarrow y=-1$ or
$y=2\Rightarrow c=-1,\ d=2;\ f(y)=\frac{y+2}{4},\ g(y)=\frac{y^2}{4}$
$\Rightarrow A=\int_c^d [f(y)-g(y)]\,dy=\int_{-1}^{2}\left(\frac{y+2}{4}-\frac{y^2}{4}\right)dy$
$=\frac{1}{4}\int_{-1}^{2}(y+2-y^2)\,dy=\frac{1}{4}\left[\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_{-1}^{2}$
$=\frac{1}{4}\left[\left(\frac{4}{2}+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)\right]=\frac{9}{8}$



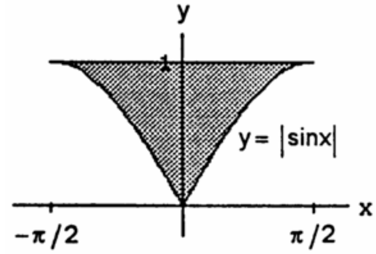
22. Let us find the intersection points: $\frac{y^2-4}{4}=\frac{y+16}{4}$
$\Rightarrow y^2-y-20=0\Rightarrow (y-5)(y+4)=0\Rightarrow y=-4$ or
$y=5\Rightarrow c=-4,\ d=5;\ f(y)=\frac{y+16}{4},\ g(y)=\frac{y^2-4}{4}$
$\Rightarrow A=\int_c^d [f(y)-g(y)]\,dy=\int_{-4}^{5}\left(\frac{y+16}{4}-\frac{y^2-4}{4}\right)dy$
$=\frac{1}{4}\int_{-4}^{5}(y+20-y^2)\,dy=\frac{1}{4}\left[\frac{y^2}{2}+20y-\frac{y^3}{3}\right]_{-4}^{5}$
$=\frac{1}{4}\left[\left(\frac{25}{2}+100-\frac{125}{3}\right)-\left(\frac{16}{2}-80+\frac{64}{3}\right)\right]$
$=\frac{1}{4}\left(\frac{9}{2}+180-63\right)=\frac{1}{4}\left(\frac{9}{2}+117\right)=\frac{1}{8}(9+234)=\frac{243}{8}$



23. $f(x)=x,\ g(x)=\sin x,\ a=0,\ b=\frac{\pi}{4}$
$\Rightarrow A=\int_a^b [f(x)-g(x)]\,dx=\int_0^{\pi/4}(x-\sin x)\,dx$
$=\left[\frac{x^2}{2}+\cos x\right]_0^{\pi/4}=\left(\frac{\pi^2}{32}+\frac{\sqrt{2}}{2}\right)-1$



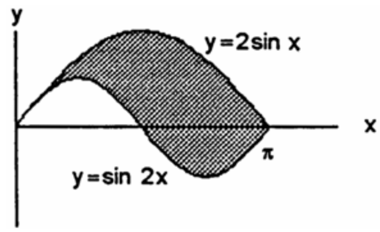
24. $f(x)=1,\ g(x)=|\sin x|,\ a=-\frac{\pi}{2},\ b=\frac{\pi}{2}$
$\Rightarrow A=\int_a^b [f(x)-g(x)]\,dx+\int_{-\pi/2}^{\pi/2}(1-|\sin x|)\,dx$
$=\int_{-\pi/2}^{0}(1+\sin x)\,dx+\int_0^{\pi/2}(1-\sin x)\,dx$
$=2\int_0^{\pi/2}(1-\sin x)\,dx=2[x+\cos x]_0^{\pi/2}$
$=2\left(\frac{\pi}{2}-1\right)=\pi-2$



25. $a=0, b=\pi, f(x)-g(x)=2\sin x-\sin 2x$

$\Rightarrow A=\int_0^{\pi}(2\sin x-\sin 2x)\,dx=\left[-2\cos x+\frac{\cos 2x}{2}\right]_0^{\pi}$

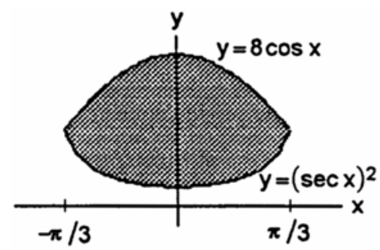
$=\left[-2\cdot(-1)+\frac{1}{2}\right]-\left(-2\cdot 1+\frac{1}{2}\right)=4$



26. $a=-\frac{\pi}{3}, b=\frac{\pi}{3}, f(x)-g(x)=8\cos x-\sec^2 x$

$\Rightarrow A=\int_{-\pi/3}^{\pi/3}(8\cos x-\sec^2 x)\,dx=[8\sin x-\tan x]_{-\pi/3}^{\pi/3}$

$=\left(8\cdot\frac{\sqrt{3}}{2}-\sqrt{3}\right)-\left(-8\cdot\frac{\sqrt{3}}{2}+\sqrt{3}\right)=6\sqrt{3}$

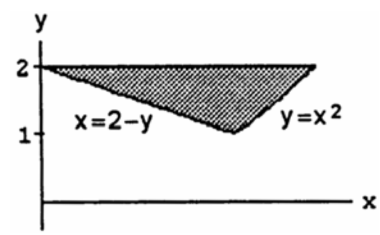


27. $f(y)=\sqrt{y}, g(y)=2-y, c=1, d=2$

$\Rightarrow A=\int_c^d[f(y)-g(y)]\,dy=\int_1^2[\sqrt{y}-(2-y)]\,dy$

$=\int_1^2\left(\sqrt{y}-2+y\right)dy=\left[\frac{2}{3}y^{3/2}-2y+\frac{y^2}{2}\right]_1^2$

$=\left(\frac{4}{3}\sqrt{2}-4+2\right)-\left(\frac{2}{3}-2+\frac{1}{2}\right)=\frac{4}{3}\sqrt{2}-\frac{7}{6}=\frac{8\sqrt{2}-7}{6}$

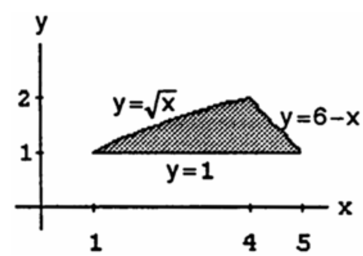


28. $f(y)=6-y, g(y)=y^2, c=1, d=2$

$\Rightarrow A=\int_c^d[f(y)-g(y)]\,dy=\int_1^2(6-y-y^2)\,dy$

$=\left[6y-\frac{y^2}{2}-\frac{y^3}{3}\right]_1^2=\left(12-2-\frac{8}{3}\right)-\left(6-\frac{1}{2}-\frac{1}{3}\right)$

$=4-\frac{7}{3}+\frac{1}{2}=\frac{24-14+3}{6}=\frac{13}{6}$

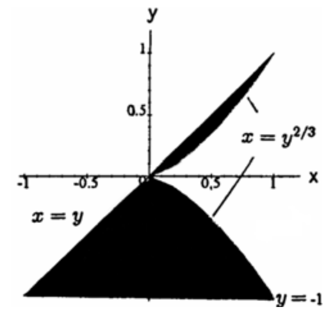


29. $f(x)=x^3-3x^2=x^2(x-3)\Rightarrow f'(x)=3x^2-6x=3x(x-2)\Rightarrow f'=+++\underset{0}{|}----\underset{2}{|}+++\Rightarrow f(0)=0$ is a maximum and $f(2)=-4$ is a minimum. $A=-\int_0^3(x^3-3x^2)\,dx=-\left[\frac{x^4}{4}-x^3\right]_0^3=-\left(\frac{81}{4}-27\right)=\frac{27}{4}$

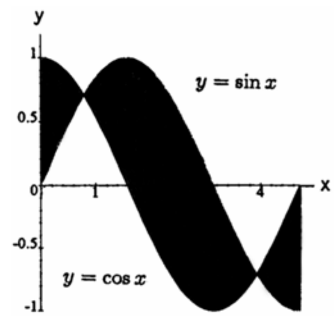
30. $A=\int_0^a(a^{1/2}-x^{1/2})^2\,dx=\int_0^a(a-2\sqrt{a}x^{1/2}+x)\,dx=\left[ax-\frac{4}{3}\sqrt{a}x^{3/2}+\frac{x^2}{2}\right]_0^a=a^2-\frac{4}{3}\sqrt{a}\cdot a\sqrt{a}+\frac{a^2}{2}$

$=a^2\left(1-\frac{4}{3}+\frac{1}{2}\right)=\frac{a^2}{6}(6-8+3)=\frac{a^2}{6}$

31. The area above the x-axis is $A_1 = \int_0^1 (y^{2/3} - y)\, dy$
$= \left[\frac{3y^{5/3}}{5} - \frac{y^2}{2}\right]_0^1 = \frac{1}{10}$; the area below the x-axis is
$A_2 = \int_{-1}^0 (y^{2/3} - y)\, dy = \left[\frac{3y^{5/3}}{5} - \frac{y^2}{2}\right]_{-1}^0 = \frac{11}{10} \Rightarrow$ the total area is $A_1 + A_2 = \frac{6}{5}$



32. $A = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$
$+ \int_{5\pi/4}^{3\pi/2} (\cos x - \sin x)\, dx$
$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{5\pi/4}$
$+ [\sin x + \cos x]_{5\pi/4}^{3\pi/2}$
$= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)\right] + \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right]$
$+ \left[(-1+0) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right] = \frac{8\sqrt{2}}{2} - 2 = 4\sqrt{2} - 2$



33. $y = x^2 + \int_1^x \frac{1}{t}\, dt \Rightarrow \frac{dy}{dx} = 2x + \frac{1}{x} \Rightarrow \frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}$; $y(1) = 1 + \int_1^1 \frac{1}{t} = 1$ and $y'(1) = 2 + 1 = 3$

34. $y = \int_0^x (1 + 2\sqrt{\sec t})\, dt \Rightarrow \frac{dy}{dx} = 1 + 2\sqrt{\sec x} \Rightarrow \frac{d^2y}{dx^2} = 2\left(\frac{1}{2}\right)(\sec x)^{-1/2}(\sec x \tan x) = \sqrt{\sec x}(\tan x)$;
$x = 0 \Rightarrow y = \int_0^0 (1 + 2\sqrt{\sec t})\, dt = 0$ and $x = 0 \Rightarrow \frac{dy}{dx} = 1 + 2\sqrt{\sec 0} = 3$

35. $y = \int_5^x \frac{\sin t}{t}\, dt - 3 \Rightarrow \frac{dy}{dx} = \frac{\sin x}{x}$; $x = 5 \Rightarrow y = \int_5^5 \frac{\sin t}{t}\, dt - 3 = -3$

36. $y = \int_{-1}^x \sqrt{2 - \sin^2 t}\, dt + 2$ so that $\frac{dy}{dx} = \sqrt{2 - \sin^2 x}$; $x = -1 \Rightarrow y = \int_{-1}^{-1} \sqrt{2 - \sin^2 t}\, dt + 2 = 2$

37. Let $t = \sin x \Rightarrow dt = -\cos x\, dx \Rightarrow -dt = \cos x\, dx$
$\int 8(\sin x)^{-3/2} \cos x\, dx = 8\int t^{-3/2}(-dt) = -8 \int t^{-3/2}\, dt = -8\left(\frac{t^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right) + C = 16t^{-1/2} + C = 16(\sin x)^{-1/2} + C$

38. Let $t = \tan x \Rightarrow dt = \sec^2 x\, dx$
$\int (\tan x)^{-5/2} \sec^2 x\, dx = \int t^{-5/2} dt = \frac{t^{-\frac{5}{2}+1}}{-\frac{5}{2}+1} + C = -\frac{2}{3} t^{-3/2} + C = -\frac{2}{3} \tan^{-3/2} x + C$

39. $\int [6\theta - 1 - 4\sin(4\theta + 1)] d\theta = 6\int \theta d\theta - \int 1 d\theta - 4\int \sin(4\theta + 1) d\theta = 6\frac{\theta^2}{2} - \theta + 4\frac{\cos(4\theta + 1)}{4} + C$
$= 3\theta^2 - \theta + \cos(4\theta + 1) + C$

40. Let $t = 3\theta + \pi \Rightarrow dt = 3d\theta \Rightarrow \frac{1}{3}dt = d\theta$

$$\int\left(\frac{2}{\sqrt{3\theta+\pi}} - 3\csc^2(3\theta+\pi)\right)d\theta = \int\left(\frac{2}{\sqrt{t}} - 3\csc^2 t\right)\left(\frac{1}{3}dt\right) = \frac{1}{3}\int(2t^{-1/2} - 3\csc^2 t)\,dt = \frac{2}{3}\left(\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + \cot t + C$$

$$= \frac{4}{3}\sqrt{t} + \cot t + C = \frac{4}{3}\sqrt{3\theta+\pi} + \cot t + C$$

41. $\int\left(x - \frac{5}{x}\right)\left(x + \frac{5}{x}\right)dx = \int\left(x^2 - \frac{25}{x^2}\right)dx = \int x^2 dx - 25\int x^{-2}dx = \frac{x^3}{3} - 25\left(\frac{x^{-2+1}}{-2+1}\right) + C = \frac{x^3}{3} + 25\frac{1}{x} + C = \frac{x^3}{3} + \frac{25}{x} + C$

42. $\int\frac{(t-2)^2+2}{t^5}dt = \int\left[\frac{t^2-4t+4+2}{t^5}\right]dt = \int\frac{t^2-4t+6}{t^5}dt = \int\left(\frac{1}{t^3} - \frac{4}{t^4} + \frac{6}{t^5}\right)dt = \int t^{-3}dt - 4\int t^{-4}dt + 6\int t^{-5}dt$

$$= \frac{t^{-3+1}}{-3+1} - 4\frac{t^{-4+1}}{-4+1} + 6\frac{t^{-5+1}}{-5+1} + C = \frac{t^{-2}}{-2} - 4\frac{t^{-3}}{-3} + 6\frac{t^{-4}}{-4} + C = \frac{4}{3t^3} - \frac{3}{2t^4} - \frac{1}{2t^2} + C$$

43. Let $6t^{3/2} = p \Rightarrow 6\cdot\frac{3}{2}t^{\frac{3}{2}-1}dt = dp \Rightarrow 9t^{1/2}dt = dp \Rightarrow \sqrt{t}dt = \frac{dp}{9}$

$$\int\sqrt{t}\cos\left(6t^{3/2}\right)dt = \int\cos\frac{dp}{9} = \frac{1}{9}\sin p + C = \frac{1}{9}\sin(6t^{3/2}) + C$$

44. Let $1 - \csc\theta = t \Rightarrow -(-\csc\theta\cot\theta)d\theta = dt \Rightarrow (\csc\theta\cot\theta)d\theta = dt$

$$\int\csc\theta\cot\theta\sqrt{1-\csc\theta}\,d\theta = \int\sqrt{t}dt = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{3}t^{3/2} + C = \frac{2}{3}(1-\csc\theta)^{3/2} + C$$

45. $\int_{-2}^{2}(4x^3 - 2x + 9)\,dx = \left[\frac{x^4}{4} - 2\frac{x^2}{2} + 9x\right]_{-2}^{2} = [((2)^4 - (2)^2 + 9(2)) - ((-2)^4 - (-2)^2 + 9(-2))]$

$$= [(16 - 4 + 18) - (16 - 4 - 18)] = [(30 - (-6)] = 36$$

46. $\int_0^3(6s^5 - 9s^2 + 7)\,ds = \left[6\frac{s^6}{6} - 9\frac{s^3}{3} + 7s\right]_0^3 = [((3)^6 - 3(3)^3 + 7(3)) - (0)] = 729 - 81 + 21 = 669$

47. $\int_1^3\frac{9}{x^3}dx = 9\frac{x^{-3+1}}{-3+1}\Big]_1^3 = 9\frac{x^{-2}}{-2}\Big]_1^3 = -\frac{9}{2}[(3)^{-2} - (1)^{-2}] = -\frac{9}{2}\left[\frac{1}{9} - 1\right] = -\frac{9}{2}\left(\frac{-8}{9}\right) = 4$

48. $\int_1^8 x^{-4/3}\,dx = \frac{x^{-\frac{4}{3}+1}}{-\frac{4}{3}+1}\Bigg]_1^8 = \frac{x^{-\frac{1}{3}}}{-\frac{1}{3}}\Bigg]_1^8 = -3[(8)^{-1/3} - (1)^{-1/3}] = -3\left(\frac{1}{2} - 1\right) = -3\left(-\frac{1}{2}\right) = \frac{3}{2}$

49. $\int_4^9\frac{dx}{x\sqrt{x}} = \int_4^9 x^{-3/2}dx = \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\Bigg]_4^9 = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\Bigg]_4^9 = -2[(9)^{-1/2} - (4)^{-1/2}] = -2\left(\frac{1}{3} - \frac{1}{2}\right) = -2\left(-\frac{1}{6}\right) = \frac{1}{3}$

50. Let $5-\sqrt{x}=t \Rightarrow -\frac{1}{2\sqrt{x}}dx=dt \Rightarrow \frac{1}{\sqrt{x}}dx=-2dt$

$$\int \frac{(5-\sqrt{x})^{1/3}}{\sqrt{x}}dx=\int t^{1/3}(-2dt)=-2\frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1}=-2\frac{t^{4/3}}{\frac{4}{3}}=-\frac{3}{2}(5-\sqrt{x})^{4/3}$$

$$\therefore \int_1^9 \frac{(5-\sqrt{x})^{1/3}}{\sqrt{x}}dx=-\frac{3}{2}(5-\sqrt{x})^{4/3}\Big]_1^9=\left[\left(-\frac{3}{2}(5-\sqrt{9})^{4/3}\right)-\left(-\frac{3}{2}(5-\sqrt{1})^{4/3}\right)\right]=\left[\left(-\frac{3}{2}(2)^{4/3}\right)-\left(-\frac{3}{2}(4)^{4/3}\right)\right]$$

$$=\left[\left(-\frac{3}{2}2\sqrt[3]{2}\right)-\left(-\frac{3}{2}4^{4/3}\right)\right]=\left[\left(-\frac{3}{2}2\sqrt[3]{2}+\frac{3}{2}4\sqrt[3]{4}\right)\right]=3\left(2\sqrt[3]{4}-\sqrt[3]{2}\right)$$

51. Let $4x+3=t \Rightarrow 4dx=dt \Rightarrow dx \Rightarrow \frac{dt}{4}$ $\therefore \int \frac{72\,dx}{(4x+3)^3}=\int \frac{72\,\frac{dt}{4}}{(t)^3}=\int t^{-3}dt=18\frac{t^{-3+1}}{-3+1}=-9t^{-2}=-9(4x+3)^{-2}$

$$\int_0^1 \frac{72\,dx}{(4x+3)^3}=-9(4x+3)^{-2}\Big]_0^1=-9\left[(4(1)+3)^{-2}-(4(0)+3)^{-2}\right]=-9\left[(7)^{-2}-(3)^{-2}\right]=-9\left[\frac{1}{49}-\frac{1}{9}\right]=-\frac{9}{49}+1=\frac{40}{49}$$

52. Let $9+7r=t \Rightarrow 7dr=dt \Rightarrow dr=\frac{dt}{7}$

$$\int \frac{dr}{\sqrt[3]{(9+7r)^2}}=\int \frac{dt/7}{\sqrt[3]{t^2}}=\int t^{-2/3}dt=\frac{1}{7}\frac{t^{-\frac{2}{3}+1}}{-\frac{2}{3}+1}=\frac{1}{7}\frac{t^{1/3}}{\frac{1}{3}}=\frac{3}{7}t^{1/3}+C=\frac{3}{7}(9+7r)^{1/3}+C$$

$$\therefore \int_0^2 \frac{dr}{\sqrt[3]{(9+7r)^2}}=\frac{3}{7}(9+7r)^{1/3}\Big]_0^2=\frac{3}{7}\left[(23)^{1/3}-(9)^{1/3}\right]$$

53. Let $t=1-x^{3/4} \Rightarrow dt=-\frac{3}{4}x^{\frac{3}{4}-1}dx \Rightarrow -\frac{4}{3}dt=x^{-\frac{1}{4}}dx$

$$\int x^{-1/4}\left(1-x^{3/4}\right)^{1/3}dx=\int t^{1/3}\left(\frac{-4}{3}\right)dt=\frac{-4}{3}\frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1}=-t^{4/3}=-\left(1-x^{3/4}\right)^{4/3}+C$$

$$\therefore \int_{1/2}^1 x^{-1/4}\left(1-x^{3/4}\right)^{1/3}dx=-\left(1-x^{3/4}\right)^{4/3}\Big]_{1/2}^1=\left[0-\left(1-\frac{1}{2}^{3/4}\right)^{4/3}\right]=\left(1-\frac{1}{2}^{3/4}\right)^{4/3}$$

54. Let $t=1+5x^5 \Rightarrow dt=5\cdot 5x^4dx \Rightarrow \frac{dt}{25}=x^4dx$

$$\int x^4(1+5x^5)^{-5/2}dx=\int t^{-5/2}\frac{dt}{25}=\frac{1}{25}\int t^{-5/2}=\frac{1}{25}\frac{t^{-\frac{5}{2}+1}}{-\frac{5}{2}+1}=-\frac{2}{75}t^{-3/2}=-\frac{2}{75}(1+5x^5)^{-3/2}+C$$

$$\int_0^{3/2} x^4(1+5x^5)^{-5/2}dx=-\frac{2}{75}(1+5x^5)^{-3/2}\Big]_0^{3/2}=-\frac{2}{75}\left[\left(1+5\left(\frac{3}{2}\right)^5\right)^{-3/2}-(1)^{-3/2}\right]=-\frac{2}{75}\left[\left(\frac{32}{1247}\right)^{3/2}-1\right]$$

55. $\int_0^{\pi}\cos^2 8r\,dr=\int_0^{\pi}\frac{1+\cos 2(8r)}{2}dr=\frac{1}{2}\int_0^{\pi}(1+\cos 16r)dr=\frac{1}{2}\left(r+\frac{\sin 16r}{16}\right)\Big]_0^{\pi}=\frac{1}{2}\left(\pi+\frac{\sin 16\pi}{16}-0\right)=\frac{\pi}{2}$

56. $\int_0^{\pi/4}\sin^2\left(2t+\frac{\pi}{4}\right)dt=\int_0^{\pi/4}\frac{1-\cos 2\left(2t+\frac{\pi}{4}\right)}{2}dt=\frac{1}{2}\left[t-\frac{\sin\left(4t+\frac{\pi}{2}\right)}{4}\right]_0^{\pi/4}=\frac{1}{2}\left[\left(\frac{\pi}{4}-\frac{\sin\left(4\frac{\pi}{4}+\frac{\pi}{2}\right)}{4}\right)-\left(0-\frac{\sin\left(\frac{\pi}{2}\right)}{4}\right)\right]$

$$=\frac{1}{2}\left(\frac{\pi}{4}-\frac{\sin\left(\frac{3\pi}{2}\right)}{4}\right)-\left(-\frac{1}{4}\right)=\frac{1}{2}\left[\frac{\pi}{4}+\frac{1}{4}+\frac{1}{4}\right]=\frac{1}{2}\left(\frac{\pi+2}{4}\right)=\frac{\pi+2}{8}$$

57. $\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2}\right) - (0+0)\right] = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

58. $\int_{\pi/3}^{5\pi/6} \sin^2\theta\, d\theta = \int_{\pi/3}^{5\pi/6} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/3}^{5\pi/6} = \frac{1}{2}\left[\left(\frac{5\pi}{6} - \frac{\sin 2\frac{5\pi}{6}}{2}\right) - \left(\frac{\pi}{3} - \frac{\sin 2\frac{\pi}{3}}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{5\pi}{6} + \frac{\sqrt{3}}{4} - \frac{\pi}{3} + \frac{\sqrt{3}}{4}\right] = \frac{1}{2}\left(\frac{\pi}{2} + \frac{\sqrt{3}}{2}\right) = \frac{\pi}{4} + \frac{\sqrt{3}}{4}$

59. $\int_{\pi}^{2\pi} \tan^2\frac{x}{6}\, dx = \int_{\pi}^{2\pi}\left(\sec^2\frac{x}{6} - 1\right) dx = \left[\frac{\tan\frac{x}{6}}{\frac{1}{6}} - x\right]_{\pi}^{2\pi} = \left[\left(6\tan\frac{2\pi}{6} - 2\pi\right) - \left(6\tan\frac{\pi}{6} - \pi\right)\right]$

$= 6\cdot\sqrt{3} - \pi - \frac{6}{\sqrt{3}} = \frac{12-\sqrt{3}\pi}{\sqrt{3}} = 4\sqrt{3} - \pi$

60. $\int_{-\pi}^{\pi} \cot^2\frac{\theta}{3}\, d\theta = \int_{-\pi}^{\pi}\left(\operatorname{cosec}^2\frac{\theta}{3} - 1\right) d\theta = \left[\frac{-\cot\frac{\theta}{3}}{\frac{1}{3}} - \theta\right]_{-\pi}^{\pi} = \left[\left(-3\cot\frac{\pi}{3} - \pi\right) - \left(-3\cot\left(-\frac{\pi}{3}\right) - (-\pi)\right)\right]$

$= -3\left(\frac{1}{\sqrt{3}}\right) - \pi - \frac{3}{\sqrt{3}} - \pi = -2\sqrt{3} - 2\pi$

61. $\int_{-\pi/3}^{0} \sin x\cos x\, dx = \int_{-\pi/3}^{0} \frac{\sin 2x}{2}\, dx = \frac{1}{2}\left[\frac{-\cos 2x}{2}\right]_{-\pi/3}^{0} = \frac{1}{2}\left[\left(\frac{-\cos 0}{2}\right) - \left(\frac{-\cos 2\left(-\frac{\pi}{3}\right)}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{-1}{2} + \frac{1/2}{2}\right] = \frac{1}{2}\left(\frac{-1}{2} + \frac{1}{4}\right) = \frac{1}{2}\left(\frac{-1}{4}\right) = \frac{-1}{8}$

62. $\int_{\pi/3}^{5\pi/6} \sec z\tan z\, dz = [-\sec z]_{\pi/3}^{5\pi/6} = \sec\frac{5\pi}{6} - \sec\frac{\pi}{3} = -\frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} = -\frac{4}{\sqrt{3}}$

63. Let $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\int 7(\cos x)^{5/2}\sin x\, dx = -\int 7t^{5/2} dt = -7\frac{t^{\frac{5}{2}+1}}{\frac{5}{2}+1} = -7\frac{2}{7}t^{\frac{7}{2}} = -2(\cos x)^{7/2} + C$

$\int_0^{\pi/2} 7(\cos x)^{5/2}\sin x\, dx = \left[-2(\cos x)^{7/2}\right]_0^{\pi/2} = -2\left(\cos\frac{\pi}{2}\right)^{7/2} + 2(\cos 0)^{7/2} = 2$

64. Let $\cos 4x = t \Rightarrow -\sin 4x\cdot 4\, dx = dt$

$\int 12\cos^2 4x\sin 4x\, dx = \int 3t^2(-dt) = -3\frac{t^3}{3} = -\cos^3 4x$

$\int_{-\pi/3}^{\pi/3} 12\cos^2 4x\sin 4x\, dx = \left[-\cos^3 4x\right]_{-\pi/3}^{\pi/3} = -\cos^3\frac{4\pi}{3} + \cos^3 4\left(\frac{-\pi}{3}\right) = -\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^3 = 0$

65. Let $4 + 5\cos^2 x = t \Rightarrow -10\cos x\sin x\, dx = dt \Rightarrow -5\cos x\sin x\, dx = \frac{dt}{2}$

$\int \frac{5\cos x\sin x}{\sqrt{4+5\cos^2 x}}\, dx = -\frac{1}{2}\int\frac{dt}{\sqrt{t}} = -\frac{1}{2}\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = -\frac{1}{2}\frac{t^{1/2}}{1/2} = -\sqrt{4+5\cos^2 x}$

$\int_0^{\pi/2} \frac{5\cos x\sin x}{\sqrt{4+5\cos^2 x}}\, dx = \left[-\sqrt{4+5\cos^2 x}\right]_0^{\pi/2} = -\left[\sqrt{4+5\cos^2\left(\frac{\pi}{2}\right)} - \sqrt{4+5\cos^2 0}\right] = -\left[\sqrt{4} - \sqrt{9}\right] = -(2-3) = 1$

66. Let $8+19\tan x=t \Rightarrow 19\sec^2 x\,dx=dt \Rightarrow \sec^2 x\,dx=\frac{dt}{19}$

$\int \frac{\sec^2 x}{(8+19\tan x)^{4/3}}dx=\frac{1}{19}\int t^{-4/3}dx=\frac{1}{19}\frac{t^{-\frac{4}{3}+1}}{-\frac{4}{3}+1}=\frac{1}{19}\frac{t^{-1/3}}{-1/3}=-\frac{3}{19}(8+19\tan x)^{-1/3}$

$\int_0^{\pi/4}\frac{\sec^2 x}{(8+19\tan x)^{4/3}}dx=\left[-\frac{3}{19}(8+19\tan x)^{-1/3}\right]_0^{\pi/4}=-\frac{3}{19}\left[\left(8+19\tan\frac{\pi}{4}\right)^{-1/3}-(8+19\tan 0)^{-1/3}\right]$

$=-\frac{3}{19}\left[\frac{1}{\sqrt[3]{27}}-\frac{1}{\sqrt[3]{8}}\right]=-\frac{3}{19}\left[\frac{1}{3}-\frac{1}{2}\right]=\frac{1}{38}$

67. (a) $\text{av}(f)=\frac{1}{1-(-1)}\int_{-1}^{1}(mx+b)\,dx=\frac{1}{2}\left[\frac{mx^2}{2}+bx\right]_{-1}^{1}=\frac{1}{2}\left[\left(\frac{m(1)^2}{2}+b(1)\right)-\left(\frac{m(-1)^2}{2}+b(-1)\right)\right]=\frac{1}{2}(2b)=b$

(b) $\text{av}(f)=\frac{1}{k-(-k)}\int_{-k}^{k}(mx+b)\,dx=\frac{1}{2k}\left[\frac{mx^2}{2}+bx\right]_{-k}^{k}=\frac{1}{2k}\left[\left(\frac{m(k)^2}{2}+b(k)\right)-\left(\frac{m(-k)^2}{2}+b(-k)\right)\right]$

$=\frac{1}{2k}(2bk)=b$

68. (a) $y_{\text{av}}=\frac{1}{3-0}\int_0^3\sqrt{3x}\,dx=\frac{1}{3}\int_0^3\sqrt{3}\,x^{1/2}\,dx=\frac{\sqrt{3}}{3}\left[\frac{2}{3}x^{3/2}\right]_0^3=\frac{\sqrt{3}}{3}\left[\frac{2}{3}(3)^{3/2}-\frac{2}{3}(0)^{3/2}\right]=\frac{\sqrt{3}}{3}(2\sqrt{3})=2$

(b) $y_{\text{av}}=\frac{1}{a-0}\int_0^a\sqrt{ax}\,dx=\frac{1}{a}\int_0^a\sqrt{a}\,x^{1/2}\,dx=\frac{\sqrt{a}}{a}\left[\frac{2}{3}x^{3/2}\right]_0^a=\frac{\sqrt{a}}{a}\left(\frac{2}{3}(a)^{3/2}-\frac{2}{3}(0)^{3/2}\right)=\frac{\sqrt{a}}{a}\left(\frac{2}{3}a\sqrt{a}\right)=\frac{2}{3}a$

69. $f'_{av}=\frac{1}{b-a}\int_a^b f'(x)\,dx=\frac{1}{b-a}[f(x)]_a^b=\frac{1}{b-a}[f(b)-f(a)]=\frac{f(b)-f(a)}{b-a}$ so the average value of f' over $[a, b]$ is the slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$, which is the average rate of change of f over $[a, b]$.

70. Yes, because the average value of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$. If the length of the interval is 2, then $b-a=2$ and the average value of the function is $\frac{1}{2}\int_a^b f(x)\,dx$.

71. We want to evaluate

$\frac{1}{365-0}\int_0^{365}f(x)\,dx=\frac{1}{365}\int_0^{365}\left(20\sin\left[\frac{2\pi}{365}(x-101)\right]-4\right)dx=\frac{20}{365}\int_0^{365}\sin\left[\frac{2\pi}{365}(x-101)\right]dx+\frac{-4}{365}\int_0^{365}dx$

Notice that the period of $y=\sin\left[\frac{2\pi}{365}(x-101)\right]$ is $\frac{2\pi}{\frac{2\pi}{365}}=365$ and that we are integrating this function over an interval of length 365. Thus the value of $\frac{20}{365}\int_0^{365}\sin\left[\frac{2\pi}{365}(x-101)\right]dx+\frac{-4}{365}\int_0^{365}dx$ is $\frac{20}{365}\cdot 0+\frac{-4}{365}\cdot 365=-4$.

72. $\frac{1}{675-20}\int_{20}^{675}(8.27+10^{-5}(26T-1.87T^2))\,dT=\frac{1}{655}\left[8.27T+\frac{26T^2}{2\cdot 10^5}-\frac{1.87T^3}{3\cdot 10^5}\right]_{20}^{675}$

$=\frac{1}{655}\left(\left[8.27(675)+\frac{26(675)^2}{2\cdot 10^5}-\frac{1.87(675)^3}{3\cdot 10^5}\right]-\left[8.27(20)+\frac{26(20)^2}{2\cdot 10^5}-\frac{1.87(20)^3}{3\cdot 10^5}\right]\right)$

$\approx\frac{1}{655}(3724.44-165.40)=5.43=$ the average value of C_v on $[20, 675]$. To find the temperature T at which $C_v=5.43$, solve $5.43=8.27+10^{-5}(26T-1.87T^2)$ for T. We obtain $1.87T^2-26T-284000=0$ $\Rightarrow T=\frac{26\pm\sqrt{(26)^2-4(1.87)(-284000)}}{2(1.87)}=\frac{26\pm\sqrt{2124996}}{3.74}$. So $T=-382.82$ or $T=396.72$. Only $T=396.72$ lies in the interval $[20, 675]$, so $T=396.72°$C.

73. $\frac{dy}{dx} = \sqrt{2+\cos^3 x}$

74. $\frac{dy}{dx} = \sqrt{2+\cos^3(7x^2)} \cdot \frac{d}{dx}(7x^2) = 14x\sqrt{2+\cos^3(7x^2)}$

75. $\frac{dy}{dx} = \frac{d}{dx}\left(-\int_1^x \frac{6}{3+t^4}\, dt\right) = -\frac{6}{3+x^4}$

76. $\frac{dy}{dx} = \frac{d}{dx}\left(\int_{\sec x}^{2} \frac{1}{t^2+1}\, dt\right) = -\frac{d}{dx}\left(\int_2^{\sec x} \frac{1}{t^2+1}\, dt\right) = -\frac{1}{\sec^2 x+1}\frac{d}{dx}(\sec x) = -\frac{\sec x\tan x}{1+\sec^2 x}$

77. Yes. The function f, being differentiable on $[a, b]$, is then continuous on $[a, b]$. The Fundamental Theorem of Calculus says that every continuous function on $[a, b]$ is the derivative of a function on $[a, b]$.

78. The second part of the Fundamental Theorem of Calculus states that if $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$, then $\int_a^b f(x)\, dx = F(b) - F(a)$. In particular, if $F(x)$ is an antiderivative of $\sqrt{1+x^4}$ on $[0, 1]$, then $\int_0^1 \sqrt{1+x^4}\, dx = F(1) - F(0)$.

79. $y = \int_x^1 \sqrt{1+t^2}\, dt = -\int_1^x \sqrt{1+t^2}\, dt \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[-\int_1^x \sqrt{1+t^2}\, dt\right] = -\frac{d}{dx}\left[\int_1^x \sqrt{1+t^2}\, dt\right] = -\sqrt{1+x^2}$

80. $y = \int_{\cos x}^0 \frac{1}{1-t^2}\, dt = -\int_0^{\cos x} \frac{1}{1-t^2}\, dt \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[-\int_0^{\cos x} \frac{1}{1-t^2}\, dt\right] = -\frac{d}{dx}\left[\int_0^{\cos x} \frac{1}{1-t^2}\, dt\right]$
$= -\left(\frac{1}{1-\cos^2 x}\right)\left(\frac{d}{dx}(\cos x)\right) = -\left(\frac{1}{\sin^2 x}\right)(-\sin x) = \frac{1}{\sin x} = \csc x$

81. We estimate the area A using midpoints of the vertical intervals, and we will estimate the width of the parking lot on each interval by averaging the widths at top and bottom. This gives the estimate $A \approx 5\cdot\left(\frac{0+12}{2} + \frac{12+18}{2} + \frac{18+17}{2} + \frac{17+16.5}{2} + \frac{16.5+18}{2} + \frac{18+21}{2} + \frac{21+22}{2} + \frac{22+14}{2}\right) \approx 657.5\ \text{m}^2$. The cost is $\text{Area}\cdot(\$21/\text{m}^2) \approx (657.5\ \text{m}^2)(\$21/\text{m}^2) = \$13{,}897.50 \Rightarrow$ the job cannot be done for \$10,000.

82. (a) Before the chute opens for A, $a = -9.8\ \text{m/s}^2$. Since the helicopter is hovering $v_0 = 0$ m/s $\Rightarrow v = \int -9.8\, dt = -9.8t + v_0 = -9.8t$. Then $s_0 = 2000\ \text{m} \Rightarrow s = \int -9.8t\, dt = -4.9t^2 + s_0 = -4.9t^2 + 2000$. At $t = 4$ s, $s = -4.9(4)^2 + 2000 = 1921.6$ m when A's chute opens;

(b) For B, $s_0 = 2200$ m, $v_0 = 0$, $a = -9.8\ \text{m/s}^2 \Rightarrow v = \int -9.8\, dt = -9.8t + v_0 = -9.8t \Rightarrow s = \int -9.8t\, dt = -4.9t^2 + s_0 = -4.9t^2 + 2200$. At $t = 13$ s, $s = -4.9(13)^2 + 2200 = 1371.9$ m when B's chute opens;

(c) After the chutes open, $v = -4.9\ \text{m/s} \Rightarrow s = \int -4.9\, dt = -4.9t + s_0$. For A, $s_0 = 1921.6$ m and for B, $s_0 = 1371.9$ m. Therefore, for A, $s = -4.9t + 1921.6$ and for B, $s = -4.9t + 1371.9$. When they hit the ground, $s = 0 \Rightarrow$ for A, $0 = -4.9t + 1921.6 \Rightarrow t = \frac{1921.6}{4.9} = 392$ seconds, and for B, $0 = -4.9t + 1371.9 \Rightarrow t = \frac{1371.9}{4.9} = 280$ seconds to hit the ground after the chutes open, Since B's chutes opens 58 seconds after A's opens $\Rightarrow$ B hits the ground first.

CHAPTER 5 ADDITIONAL AND ADVANCED EXERCISES

1. (a) Yes, because $\int_0^1 f(x)\,dx = \frac{1}{7}\int_0^1 7f(x)\,dx = \frac{1}{7}(7) = 1$

   (b) No. For example, $\int_0^1 8x\,dx = [4x^2]_0^1 = 4$, but $\int_0^1 \sqrt{8x}\,dx = \left[2\sqrt{2}\left(\frac{x^{3/2}}{\frac{3}{2}}\right)\right]_0^1 = \frac{4\sqrt{2}}{3}\left(1^{3/2} - 0^{3/2}\right) = \frac{4\sqrt{2}}{3} \neq \sqrt{4}$

2. (a) True: $\int_5^2 f(x)\,dx = -\int_2^5 f(x)\,dx = -3$

   (b) True: $\int_{-2}^5 [f(x)+g(x)]\,dx = \int_{-2}^5 f(x)\,dx + \int_{-2}^5 g(x)\,dx = \int_{-2}^2 f(x)\,dx + \int_2^5 f(x)\,dx + \int_{-2}^5 g(x)\,dx$ $= 4+3+2 = 9$

   (c) False: $\int_{-2}^5 f(x)dx = 4+3 = 7 > 2 = \int_{-2}^5 g(x)\,dx \Rightarrow \int_{-2}^5 [f(x)-g(x)]\,dx > 0 \Rightarrow \int_{-2}^5 [g(x)-f(x)]\,dx < 0.$ On the other hand, $f(x) \le g(x) \Rightarrow [g(x)-f(x)] \ge 0 \Rightarrow \int_{-2}^5 [g(x)-f(x)]\,dx \ge 0$ which is a contradiction.

3. $y = \frac{1}{a}\int_0^x f(t)\sin a(x-t)\,dt = \frac{1}{a}\int_0^x f(t)\sin ax\cos at\,dt - \frac{1}{a}\int_0^x f(t)\cos ax\sin at\,dt$
$= \frac{\sin ax}{a}\int_0^x f(t)\cos at\,dt - \frac{\cos ax}{a}\int_0^x f(t)\sin at\,dt \Rightarrow \frac{dy}{dx}$
$= \cos ax\left(\int_0^x f(t)\cos at\,dt\right) + \frac{\sin ax}{a}\left(\frac{d}{dx}\int_0^x f(t)\cos at\,dt\right) + \sin ax\int_0^x f(t)\sin at\,dt - \frac{\cos ax}{a}\left(\frac{d}{dx}\int_0^x f(t)\sin at\,dt\right)$
$= \cos ax\int_0^x f(t)\cos at\,dt + \frac{\sin ax}{a}(f(x)\cos ax) + \sin ax\int_0^x f(t)\sin at\,dt - \frac{\cos ax}{a}(f(x)\sin ax)$
$\Rightarrow \frac{dy}{dx} = \cos ax\int_0^x f(t)\cos at\,dt + \sin ax\int_0^x f(t)\sin at\,dt.$ Next, $\frac{d^2y}{dx^2} =$
$-a\sin ax\int_0^x f(t)\cos at\,dt + (\cos ax)\left(\frac{d}{dx}\int_0^x f(t)\cos at\,dt\right) + a\cos ax\int_0^x f(t)\sin at\,dt + (\sin ax)\left(\frac{d}{dx}\int_0^x f(t)\sin at\,dt\right)$
$= -a\sin ax\int_0^x f(t)\cos at\,dt + (\cos ax)f(x)\cos ax + a\cos ax\int_0^x f(t)\sin at\,dt + (\sin ax)f(x)\sin ax$
$= -a\sin ax\int_0^x f(t)\cos at\,dt + a\cos ax\int_0^x f(t)\sin at\,dt + f(x).$ Therefore, $y'' + a^2y$
$= a\cos ax\int_0^x f(t)\sin at\,dt - a\sin ax\int_0^x f(t)\cos at\,dt + f(x) + a^2\left(\frac{\sin ax}{a}\int_0^x f(t)\cos at\,dt - \frac{\cos ax}{a}\int_0^x f(t)\sin at\,dt\right)$
$= f(x).$ Note also that $y'(0) = y(0) = 0.$

4. $x = \int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt \Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}\int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt = \frac{d}{dy}\left[\int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt\right]\left(\frac{dy}{dx}\right)$ from the chain rule
$\Rightarrow 1 = \frac{1}{\sqrt{1+4y^2}}\left(\frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx} = \sqrt{1+4y^2}$. Then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\sqrt{1+4y^2}\right) = \frac{d}{dy}\left(\sqrt{1+4y^2}\right)\left(\frac{dy}{dx}\right)$
$= \frac{1}{2}\left(1+4y^2\right)^{-1/2}(8y)\left(\frac{dy}{dx}\right) = \frac{4y\left(\frac{dy}{dx}\right)}{\sqrt{1+4y^2}} = \frac{4y\left(\sqrt{1+4y^2}\right)}{\sqrt{1+4y^2}} = 4y.$ Thus $\frac{d^2y}{dx^2} = 4y$, and the constant of proportionality is 4.

5. (a) $\int_0^{x^2} f(t)\,dt = x\cos\pi x \Rightarrow \frac{d}{dx}\int_0^{x^2} f(t)\,dt = \cos\pi x - \pi x\sin\pi x \Rightarrow f(x^2)(2x) = \cos\pi x - \pi x\sin\pi x$

$\Rightarrow f(x^2) = \frac{\cos\pi x - \pi x\sin\pi x}{2x}$. Thus, $x = 2 \Rightarrow f(4) = \frac{\cos 2\pi - 2\pi\sin 2\pi}{4} = \frac{1}{4}$

(b) $\int_0^{f(x)} t^2\,dt = \left[\frac{t^3}{3}\right]_0^{f(x)} = \frac{1}{3}\left(f(x)\right)^3 \Rightarrow \frac{1}{3}\left(f(x)\right)^3 = x\cos\pi x \Rightarrow \left(f(x)\right)^3 = 3x\cos\pi x \Rightarrow f(x) = \sqrt[3]{3x\cos\pi x}$

$\Rightarrow f(4) = \sqrt[3]{3(4)\cos 4\pi} = \sqrt[3]{12}$

6. $\int_0^a f(x)\,dx = \frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$. Let $F(a) = \int_0^a f(t)\,dt \Rightarrow f(a) = F'(a)$. Now $F(a) = \frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$

$\Rightarrow f(a) = F'(a) = a + \frac{1}{2}\sin a + \frac{a}{2}\cos a - \frac{\pi}{2}\sin a \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{1}{2}\sin\frac{\pi}{2} + \frac{\left(\frac{\pi}{2}\right)}{2}\cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2} + \frac{1}{2} - \frac{\pi}{2} = \frac{1}{2}$

7. $\int_1^b f(x)\,dx = \sqrt{b^2+1} - \sqrt{2} \Rightarrow f(b) = \frac{d}{db}\int_1^b f(x)\,dx = \frac{1}{2}(b^2+1)^{-1/2}(2b) = \frac{b}{\sqrt{b^2+1}} \Rightarrow f(x) = \frac{x}{\sqrt{x^2+1}}$

8. The derivative of the left side of the equation is: $\frac{d}{dx}\left[\int_0^x\left[\int_0^u f(t)\,dt\right]du\right] = \int_0^x f(t)\,dt$; the derivative of the right side of the equation is: $\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] = \frac{d}{dx}\int_0^x f(u)\,x\,du - \frac{d}{dx}\int_0^x u\,f(u)\,du$

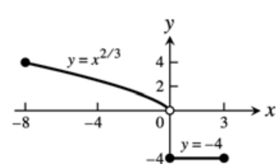
$= \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\int_0^x u\,f(u)\,du = \int_0^x f(u)\,du + x\left[\frac{d}{dx}\int_0^x f(u)\,du\right] - x\,f(x) = \int_0^x f(u)\,du + x\,f(x) - x\,f(x)$

$= \int_0^x f(u)\,du$. Since each side has the same derivative, they differ by a constant, and since both sides equal 0 when $x = 0$, the constant must be 0. Therefore, $\int_0^x\left[\int_0^u f(t)\,dt\right]du = \int_0^x f(u)(x-u)\,du$.

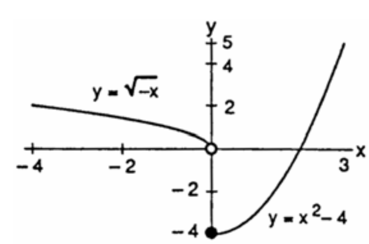
9. $\frac{dy}{dx} = 3x^2 + 2 \Rightarrow y = \int(3x^2+2)\,dx = x^3 + 2x + C$. Then $(1, -1)$ lies on the curve $\Rightarrow 1^3 + 2(1) + C = -1$
$\Rightarrow C = -4 \Rightarrow y = x^3 + 2x - 4$

10. The acceleration due to gravity downward is -9.8 m/s^2 $\Rightarrow v = \int -9.8\,dt = -9.8t + v_0$, where v_0 is the initial velocity $\Rightarrow v = -9.8t + 9.8 \Rightarrow s = \int(-9.8t + 9.8)\,dt = -4.9t^2 + 9.8t + C$. If the release point, at $t = 0$, is $s = 0$, then $C = 0 \Rightarrow s = -4.9t^2 + 9.8t$. Then $s = 5.2 \Rightarrow 5.2 = -4.9t^2 + 9.8t \Rightarrow 4.9t^2 - 9.8t + 5.2 = 0$. The discriminant of this quadratic equation is -5.88 which says there is no real time when $s = 5.2$ m. You had better duck.

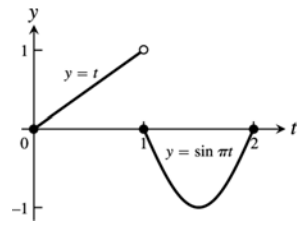
11. $\int_{-8}^3 f(x)\,dx = \int_{-8}^0 x^{2/3}dx + \int_0^3 -4\,dx$
$= \left[\frac{3}{5}x^{5/3}\right]_{-8}^0 + [-4x]_0^3$
$= \left(0 - \frac{3}{5}(-8)^{5/3}\right) + (-4(3) - 0) = \frac{96}{5} - 12 = \frac{36}{5}$



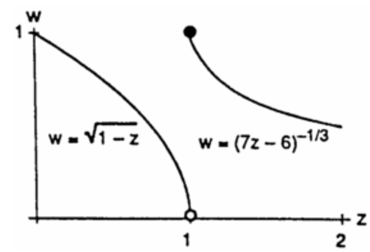
12. $\int_{-4}^3 f(x)\,dx = \int_{-4}^0 \sqrt{-x}\,dx + \int_0^3 (x^2-4)\,dx$
$= \left[-\frac{2}{3}(-x)^{3/2}\right]_{-4}^0 + \left[\frac{x^3}{3} - 4x\right]_0^3$
$= \left[0 - \left(-\frac{2}{3}(4)^{3/2}\right)\right] + \left[\left(\frac{3^3}{3} - 4(3)\right) - 0\right] = \frac{16}{3} - 3 = \frac{7}{3}$



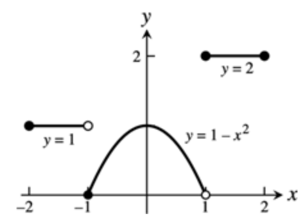
13. $\int_0^2 g(t)\,dt = \int_0^1 t\,dt + \int_1^2 \sin \pi t\,dt$
$= \left[\frac{t^2}{2}\right]_0^1 + \left[-\frac{1}{\pi}\cos \pi t\right]_1^2$
$= \left(\frac{1}{2} - 0\right) + \left[-\frac{1}{\pi}\cos 2\pi - \left(-\frac{1}{\pi}\cos \pi\right)\right] = \frac{1}{2} - \frac{2}{\pi}$



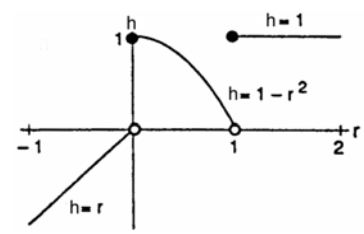
14. $\int_0^2 h(z)\,dz = \int_0^1 \sqrt{1-z}\,dz + \int_1^2 (7z-6)^{-1/3}\,dz$
$= \left[-\frac{2}{3}(1-z)^{3/2}\right]_0^1 + \left[\frac{3}{14}(7z-6)^{2/3}\right]_1^2$
$= \left[-\frac{2}{3}(1-1)^{3/2} - \left(-\frac{2}{3}(1-0)^{3/2}\right)\right]$
$+ \left[\frac{3}{14}(7(2)-6)^{2/3} - \frac{3}{14}(7(1)-6)^{2/3}\right]$
$= \frac{2}{3} + \left(\frac{6}{7} - \frac{3}{14}\right) = \frac{55}{42}$



15. $\int_{-2}^2 f(x)\,dx = \int_{-2}^{-1} dx + \int_{-1}^1 (1-x^2)\,dx + \int_1^2 2\,dx$
$= [x]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^1 + [2x]_1^2$
$= (-1-(-2)) + \left[\left(1 - \frac{1^3}{3}\right) - \left(-1 - \frac{(-1)^3}{3}\right)\right] + [2(2) - 2(1)]$
$= 1 + \frac{2}{3} - \left(-\frac{2}{3}\right) + 4 - 2 = \frac{13}{3}$



16. $\int_{-1}^2 h(r)\,dr = \int_{-1}^0 r\,dr + \int_0^1 (1-r^2)\,dr + \int_1^2 dr$
$= \left[\frac{r^2}{2}\right]_{-1}^0 + \left[r - \frac{r^3}{3}\right]_0^1 + [r]_1^2$
$= \left(0 - \frac{(-1)^2}{2}\right) + \left(\left(1 - \frac{1^3}{3}\right) - 0\right) + (2-1) = -\frac{1}{2} + \frac{2}{3} + 1 = \frac{7}{6}$



17. Ave. value $= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\left[\int_0^1 x\,dx + \int_1^2 (x-1)\,dx\right] = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^2$
$= \frac{1}{2}\left[\left(\frac{1^2}{2} - 0\right) + \left(\frac{2^2}{2} - 2\right) - \left(\frac{1^2}{2} - 1\right)\right] = \frac{1}{2}$

18. Ave. value $= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3-0}\int_0^3 f(x)\,dx = \frac{1}{3}\left[\int_0^1 dx + \int_1^2 0\,dx + \int_2^3 dx\right] = \frac{1}{3}[1 - 0 + 0 + 3 - 2] = \frac{2}{3}$

19. Let $f(x) = x^5$ on $[0, 1]$. Partition $[0, 1]$ into n subintervals with $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Then $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ are the right-hand endpoints of the subintervals. Since f is increasing on $[0, 1]$, $U = \sum_{j=1}^{\infty}\left(\frac{j}{n}\right)^5\left(\frac{1}{n}\right)$ is the upper sum for $f(x) = x^5$ on $[0, 1] \Rightarrow \lim_{n\to\infty}\sum_{j=1}^{\infty}\left(\frac{j}{n}\right)^5\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \ldots + \left(\frac{n}{n}\right)^5\right] = \lim_{n\to\infty}\left[\frac{1^5+2^5+\ldots+n^5}{n^6}\right] = \int_0^1 x^5\,dx = \left[\frac{x^6}{6}\right]_0^1 = \frac{1}{6}$

20. Let $f(x) = x^3$ on $[0, 1]$. Partition $[0, 1]$ into n subintervals with $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Then $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ are the right-hand endpoints of the subintervals. Since f is increasing on $[0, 1]$, $U = \sum_{j=1}^{\infty} \left(\frac{j}{n}\right)^3 \left(\frac{1}{n}\right)$ is the upper sum for $f(x) = x^3$ on $[0, 1] \Rightarrow \lim_{n\to\infty} \sum_{j=1}^{\infty} \left(\frac{j}{n}\right)^3 \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^3 + \left(\frac{2}{n}\right)^3 + \ldots + \left(\frac{n}{n}\right)^3\right] = \lim_{n\to\infty}\left[\frac{1^3+2^3+\ldots+n^3}{n^4}\right] = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

21. Let $y = f(x)$ on $[0, 1]$. Partition $[0, 1]$ into n subintervals with $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Then $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ are the right-hand endpoints of the subintervals. Since f is continuous on $[0, 1]$, $\sum_{j=1}^{\infty} f\left(\frac{j}{n}\right)\left(\frac{1}{n}\right)$ is a Riemann sum of $y = f(x)$ on $[0, 1] \Rightarrow \lim_{n\to\infty} \sum_{j=1}^{\infty} f\left(\frac{j}{n}\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f\left(\frac{n}{n}\right)\right] = \int_0^1 f(x)\,dx$

22. (a) $\lim_{n\to\infty} \frac{1}{n^2}[2+4+6+\ldots+2n] = \lim_{n\to\infty} \frac{1}{n}\left[\frac{2}{n}+\frac{4}{n}+\frac{6}{n}+\ldots+\frac{2n}{n}\right] = \int_0^1 2x\,dx = [x^2]_0^1 = 1$, where $f(x) = 2x$ on $[0, 1]$

(b) $\lim_{n\to\infty} \frac{1}{n^{16}}[1^{15}+2^{15}+\ldots+n^{15}] = \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^{15}+\left(\frac{2}{n}\right)^{15}+\ldots+\left(\frac{n}{n}\right)^{15}\right] = \int_0^1 x^{15}\,dx = \left[\frac{x^{16}}{16}\right]_0^1 = \frac{1}{16}$, where $f(x) = x^{15}$ on $[0, 1]$

(c) $\lim_{n\to\infty} \frac{1}{n}\left[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \ldots + \sin\frac{n\pi}{n}\right] = \int_0^1 \sin n\pi\,dx = \left[-\frac{1}{\pi}\cos\pi x\right]_0^1 = -\frac{1}{\pi}\cos\pi - \left(-\frac{1}{\pi}\cos 0\right) = \frac{2}{\pi}$, where $f(x) = \sin\pi x$ on $[0, 1]$

(d) $\lim_{n\to\infty} \frac{1}{n^{17}}\left[1^{15}+2^{15}+\ldots+n^{15}\right] = \left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{1}{n^{16}}[1^{15}+2^{15}+\ldots+n^{15}]\right) = \left(\lim_{n\to\infty}\frac{1}{n}\right)\int_0^1 x^{15}dx = 0\left(\frac{1}{16}\right) = 0$ (see part (b) above)

(e) $\lim_{n\to\infty} \frac{1}{n^{15}}\left[1^{15}+2^{15}+\ldots+n^{15}\right] = \lim_{n\to\infty} \frac{n}{n^{16}}[1^{15}+2^{15}+\ldots+n^{15}]$
$= \left(\lim_{n\to\infty} n\right)\left(\lim_{n\to\infty}\frac{1}{n^{16}}[1^{15}+2^{15}+\ldots+n^{15}]\right) = \left(\lim_{n\to\infty} n\right)\int_0^1 x^{15}dx = \infty$ (see part (b) above)

23. (a) Let the polygon be inscribed in a circle of radius r. If we draw a radius from the center of the circle (and the polygon) to each vertex of the polygon, we have n isosceles triangles formed (the equal sides are equal to r, the radius of the circle) and a vertex angle of θ_n where $\theta_n = \frac{2\pi}{n}$. The area of each triangle is $A_n = \frac{1}{2}r^2\sin\theta_n \Rightarrow$ the area of the polygon is $A = nA_n = \frac{nr^2}{2}\sin\theta_n = \frac{nr^2}{2}\sin\frac{2\pi}{n}$.

(b) $\lim_{n\to\infty} A = \lim_{n\to\infty}\frac{nr^2}{2}\sin\frac{2\pi}{n} = \lim_{n\to\infty}\frac{n\pi r^2}{2\pi}\sin\frac{2\pi}{n} = \lim_{n\to\infty}\left(\pi r^2\right)\frac{\sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)} = \left(\pi r^2\right)\lim_{2\pi/n\to\infty}\frac{\sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)} = \pi r^2$

24. Partition $[0, 1]$ into n subintervals, each of length $\Delta x = \frac{1}{n}$ with the points $x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_n = \frac{n}{n} = 1$. The inscribed rectangles so determined have areas $f(x_0)\,\Delta x = (0)^2\,\Delta x$, $f(x_1)\Delta x = \left(\frac{1}{n}\right)^2\Delta x$, $f(x_2)\,\Delta x = \left(\frac{2}{n}\right)^2\Delta x, \ldots, f(x_{n-1}) = \left(\frac{n-1}{n}\right)^2\Delta x$. The sum of these areas is
$S_n = \left(0^2 + \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{n-1}{n}\right)^2\right)\Delta x = \left(\frac{1^2}{n^2}+\frac{2^2}{n^2}+\ldots+\frac{(n-1)^2}{n^2}\right)\frac{1}{n} = \frac{1^2}{n^3}+\frac{2^2}{n^3}+\ldots+\frac{(n-1)^2}{n^3}$. Then
$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1^2}{n^3}+\frac{2^2}{n^3}+\ldots+\frac{(n-1)^2}{n^3}\right) = \int_0^1 x^2dx = \frac{1^3}{3} = \frac{1}{3}$.

25. (a) $g(1)=\int_1^1 f(t)\,dt=0$

(b) $g(3)=\int_1^3 f(t)\,dt=-\frac{1}{2}(2)(1)=-1$

(c) $g(-1)=\int_1^{-1} f(t)\,dt=-\int_{-1}^{1} f(t)\,dt=-\frac{1}{4}(\pi 2^2)=-\pi$

(d) $g'(x)=f(x)=0 \Rightarrow x=-3, 1, 3$ and the sign chart for $g'(x)=f(x)$ is $\underset{-3}{|}+++\underset{1}{|}---\underset{3}{|}+++$. So g has a relative maximum at $x=1$.

(e) $g'(-1)=f(-1)=2$ is the slope and $g(-1)=\int_1^{-1} f(t)dt=-\pi$, by (c). Thus the equation is $y+\pi=2(x+1)$ $\Rightarrow y=2x+2-\pi$.

(f) $g''(x)=f'(x)=0$ at $x=-1$ and $g''(x)=f'(x)$ is negative on $(-3,-1)$ and positive on $(-1,1)$ so there is an inflection point for g at $x=-1$. We notice that $g''(x)=f'(x)<0$ for x on $(-1,2)$ and $g''(x)=f'(x)>0$ for x on $(2,4)$, even though $g''(2)$ does not exist, g has a tangent line at $x=2$, so there is an inflection point at $x=2$.

(g) g is continuous on $[-3,4]$ and so it attains its absolute maximum and minimum values on this interval. We saw in (d) that $g'(x)=0 \Rightarrow x=-3, 1, 3$. We have that $g(-3)=\int_1^{-3} f(t)\,dt=-\int_{-3}^{1} f(t)\,dt=-\frac{\pi 2^2}{2}=-2\pi$

$g(1)=\int_1^1 f(t)\,dt=0 \qquad g(3)=\int_1^3 f(t)\,dt=-1 \qquad g(4)=\int_1^4 f(t)\,dt=-1+\frac{1}{2}\cdot 1\cdot 1=-\frac{1}{2}$

Thus, the absolute minimum is -2π and the absolute maximum is 0. Thus, the range is $[-2\pi, 0]$.

26. $y=\sin x+\int_x^{\pi}\cos 2t\,dt+1=\sin x-\int_{\pi}^{x}\cos 2t\,dt+1 \Rightarrow y'=\cos x-\cos(2x)$; when $x=\pi$ we have $y'=\cos\pi-\cos(2\pi)=-1-1=-2$. And $y''=-\sin x+2\sin(2x)$; when $x=\pi$, $y=\sin\pi+\int_x^{\pi}\cos 2t\,dt+1=0+0+1=1$.

27. $f(x)=\int_{1/x}^{x}\frac{1}{t}\,dt \Rightarrow f'(x)=\frac{1}{x}\left(\frac{dx}{dx}\right)-\left(\frac{1}{\frac{1}{x}}\right)\left(\frac{d}{dx}\left(\frac{1}{x}\right)\right)=\frac{1}{x}-x\left(-\frac{1}{x^2}\right)=\frac{1}{x}+\frac{1}{x}=\frac{2}{x}$

28. $f(x)=\int_{\cos x}^{\sin x}\frac{1}{1-t^2}\,dt \Rightarrow f'(x)=\left(\frac{1}{1-\sin^2 x}\right)\left(\frac{d}{dx}(\sin x)\right)-\left(\frac{1}{1-\cos^2 x}\right)\left(\frac{d}{dx}(\cos x)\right)=\frac{\cos x}{\cos^2 x}+\frac{\sin x}{\sin^2 x}=\frac{1}{\cos x}+\frac{1}{\sin x}$

29. $g(y)=\int_{\sqrt{y}}^{2\sqrt{y}}\sin t^2\,dt \Rightarrow g'(y)=\left(\sin\left(2\sqrt{y}\right)^2\right)\left(\frac{d}{dy}\left(2\sqrt{y}\right)\right)-\left(\sin\left(\sqrt{y}\right)^2\right)\left(\frac{d}{dy}\left(\sqrt{y}\right)\right)=\frac{\sin 4y}{\sqrt{y}}-\frac{\sin y}{2\sqrt{y}}$

30. $f(x)=\int_x^{x+3} t(5-t)\,dt \Rightarrow f'(x)=(x+3)(5-(x+3))\left(\frac{d}{dx}(x+3)\right)-x(5-x)\left(\frac{dx}{dx}\right)=(x+3)(2-x)-x(5-x)$ $=6-x-x^2-5x+x^2=6-6x$. Thus $f'(x)=0 \Rightarrow 6-6x=0 \Rightarrow x=1$. Also, $f''(x)=-6<0 \Rightarrow x=1$ gives a maximum.