

CHAPTER 8 TECHNIQUES OF INTEGRATION

8.1 USING BASIC INTEGRATION FORMULAS

1. $\int_0^1 \frac{16x}{8x^2+2} dx$

$$u = 8x^2 + 2 \quad du = 16x dx$$

$$u = 2 \text{ when } x = 0, \quad u = 10 \text{ when } x = 1$$

$$\begin{aligned} \int_0^1 \frac{16x}{8x^2+2} dx &= \int_2^{10} \frac{1}{u} du = \ln|u| \Big|_2^{10} \\ &= \ln 10 - \ln 2 = \ln 5 \end{aligned}$$

2. $\int \frac{x^2}{x^2+1} dx$

Use long division to write the integrand as $1 - \frac{1}{x^2+1}$.

$$\begin{aligned} \int \frac{x^2}{x^2+1} dx &= \int 1 dx - \int \frac{1}{x^2+1} dx \\ &= x - \tan^{-1} x + C \end{aligned}$$

3. $\int (\sec x - \tan x)^2 dx$

$$\begin{aligned} \text{Expand the integrand: } (\sec x - \tan x)^2 &= \sec^2 x - 2 \sec x \tan x + \tan^2 x \\ &= \sec^2 x - 2 \sec x \tan x + (\sec^2 x - 1) \\ &= 2 \sec^2 x - 2 \sec x \tan x - 1 \end{aligned}$$

$$\begin{aligned} \int (\sec x - \tan x)^2 dx &= 2 \int \sec^2 x dx - 2 \int \sec x \tan x dx - \int 1 dx \\ &= 2 \tan x - 2 \sec x - x + C \end{aligned}$$

We have used Formulas 8 and 10 from Table 8.1.

4. $\int_{\pi/4}^{\pi/3} \frac{1}{\cos^2 x \tan x} dx$

$$u = \tan x \quad du = \sec^2 x dx = \frac{1}{\cos^2 x} dx$$

$$u = 1 \text{ when } x = \pi/4, \quad u = \sqrt{3} \text{ when } x = \pi/3$$

$$\begin{aligned} \int_{\pi/4}^{\pi/3} \frac{1}{\cos^2 x \tan x} dx &= \int_1^{\sqrt{3}} \frac{1}{u} du = \ln|u| \Big|_1^{\sqrt{3}} \\ &= \ln \sqrt{3} - \ln 1 = \frac{1}{2} \ln 3 \end{aligned}$$

$$5. \int \frac{1-x}{\sqrt{1-x^2}} dx$$

Write as the sum of two integrals:

$$\int \frac{1-x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} dx - \int \frac{x}{\sqrt{1-x^2}} dx$$

For the first integral use Formula 18 in Table 8.1 with $a = 1$.

For the second:

$$u = 1 - x^2 \quad du = -2x dx$$

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int \frac{1}{u^{1/2}} du \\ &= -\sqrt{u} = -\sqrt{1-x^2} \end{aligned}$$

$$\text{So } \int \frac{1-x}{\sqrt{1-x^2}} dx = \sin^{-1} x + \sqrt{1-x^2} + C$$

$$6. \int \frac{1}{x-\sqrt{x}} dx$$

$$u = \sqrt{x} - 1 \quad du = \frac{1}{2\sqrt{x}} dx$$

$$\begin{aligned} \int \frac{1}{x-\sqrt{x}} dx &= 2 \int \frac{1}{u} du \\ &= 2 \ln|u| + C = 2 \ln|\sqrt{x} - 1| + C \end{aligned}$$

$$7. \int \frac{e^{-\cot z}}{\sin^2 z} dz$$

$$u = -\cot z \quad du = -\csc^2 z dz = \frac{1}{\sin^2 z} dz$$

$$\begin{aligned} \int \frac{e^{-\cot z}}{\sin^2 z} dz &= \int e^{-u} du \\ &= e^{-u} + C = e^{-\cot z} + C \end{aligned}$$

$$8. \int \frac{2^{\ln z^3}}{16z} dz$$

$$u = \ln z^3 = 3 \ln z \quad du = \frac{3}{z} dz$$

Using Formula 5 in Table 8.1,

$$\begin{aligned} \int \frac{2^{\ln z^3}}{16z} dz &= \frac{1}{48} \int 2^u du \\ &= \frac{2^u}{48 \ln 2} + C = \frac{2^{\ln z^3}}{48 \ln 2} + C \end{aligned}$$

9. $\int \frac{1}{e^z + e^{-z}} dz$

Multiply the integrand by $\frac{e^z}{e^z}$.

$$\int \frac{1}{e^z + e^{-z}} dz = \int \frac{e^z}{e^{2z} + 1} dx$$

$$u = e^z \quad du = e^z du$$

$$\begin{aligned} \int \frac{e^z}{e^{2z} + 1} dx &= \int \frac{1}{u^2 + 1} du \\ &= \tan^{-1} u + C = \tan^{-1} e^z + C \end{aligned}$$

10. $\int_1^2 \frac{8}{x^2 - 2x + 2} dx$

$$u = x - 1 \quad du = dx$$

$$u = 0 \text{ when } x = 1, u = 1 \text{ when } x = 2$$

$$\begin{aligned} \int_1^2 \frac{8}{x^2 - 2x + 2} dx &= 8 \int_0^1 \frac{1}{u^2 + 1} du \\ &= 8 \tan^{-1} u \Big|_0^1 = 8 \left(\frac{\pi}{4} - 0 \right) = 2\pi \end{aligned}$$

11. $\int_{-1}^0 \frac{4}{1 + (2x + 1)^2} dx$

$$u = 2x + 1 \quad du = 2dx$$

$$u = -1 \text{ when } x = -1, u = 1 \text{ when } x = 0$$

$$\begin{aligned} \int_{-1}^0 \frac{4}{1 + (2x + 1)^2} dx &= 2 \int_{-1}^1 \frac{1}{1 + u^2} du \\ &= 2 \tan^{-1} u \Big|_{-1}^1 = 2 \left(\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right) = \pi \end{aligned}$$

12. $\int_{-1}^3 \frac{4x^2 - 7}{2x + 3} dx$

Use long division to write the integrand as $2x - 3 + \frac{2}{2x + 3}$.

$$\int_{-1}^3 \frac{4x^2 - 7}{2x + 3} dx = \int_{-1}^3 2x dx - \int_{-1}^3 3 dx + \int_{-1}^3 \frac{2}{2x + 3} dx$$

$$\int_{-1}^3 2x dx - \int_{-1}^3 3 dx = x^2 \Big|_{-1}^3 - 3x \Big|_{-1}^3 = 8 - 12 = -4$$

For the last integral,

$$u = 2x + 3 \quad du = 2 dx$$

$$u = 1 \text{ when } x = -1, u = 9 \text{ when } x = 3$$

$$\begin{aligned}\int_{-1}^3 \frac{2}{2x+3} dx &= \int_1^9 \frac{1}{u} du \\ &= \left| \ln u \right|_1^9 = \ln 9 - \ln 1 = 2 \ln 3\end{aligned}$$

$$\text{So } \int_{-1}^3 \frac{4x^2 - 7}{2x+3} dx = -4 + 2 \ln 3$$

$$13. \int \frac{1}{1 - \sec t} dt$$

Multiply the integrand by $\frac{1 + \sec t}{1 + \sec t}$.

$$\begin{aligned}\frac{1}{1 - \sec t} \cdot \frac{1 + \sec t}{1 + \sec t} &= \frac{-1 - \sec t}{\tan^2 t} = -\cot^2 t - \frac{\cos t}{\sin^2 t} = 1 - \csc^2 t - \frac{\cos t}{\sin^2 t} \\ \int \frac{1}{1 - \sec t} dt &= \int 1 dt - \int \csc^2 t dt - \int \frac{\cos t}{\sin^2 t} dt \\ &= t + \cot t + \csc t + C\end{aligned}$$

Here we have used Formula 9 in Table 8.1 for the second integral, and the substitution $u = \sin t$, $du = \cos t dt$ for the third integral, which gives it the form $\int \frac{1}{u^2} du = -\frac{1}{u} = -\frac{1}{\sin t}$.

$$14. \int \csc t \sin 3t dt$$

Write $\sin 3t$ as $\sin(2t + t)$ and expand.

$$\begin{aligned}\csc t \sin 3t &= \frac{\cos 2t \sin t + (2 \sin t \cos t) \cos t}{\sin t} \\ &= \cos 2t + 2 \cos^2 t = 2 \cos 2t + 1\end{aligned}$$

$$\begin{aligned}\int \csc t \sin 3t dt &= \int 2 \cos 2t dt + \int 1 dt \\ &= \sin 2t + t + C\end{aligned}$$

$$15. \int_0^{\pi/4} \frac{1 + \sin \theta}{\cos^2 \theta} d\theta$$

Split into two integrals.

$$\begin{aligned}\int_0^{\pi/4} \frac{1 + \sin \theta}{\cos^2 \theta} d\theta &= \int_0^{\pi/4} \frac{1}{\cos^2 \theta} d\theta + \int_0^{\pi/4} \frac{\sin \theta}{\cos^2 \theta} d\theta \\ &= \int_0^{\pi/4} \sec^2 \theta d\theta + \int_0^{\pi/4} \frac{\sin \theta}{\cos^2 \theta} d\theta \\ &= [\tan \theta + \sec \theta]_0^{\pi/4} = (1 + \sqrt{2}) - (0 + 1) = \sqrt{2}\end{aligned}$$

The second integral is evaluated with the substitution $u = \cos \theta$ $du = -\sin \theta d\theta$, which gives

$$\int \frac{\sin \theta}{\cos^2 \theta} d\theta = -\int \frac{1}{u^2} du = \frac{1}{u} = \frac{1}{\cos \theta}.$$

$$16. \int \frac{1}{\sqrt{2\theta - \theta^2}} d\theta$$

Write the integrand as $\frac{1}{\sqrt{1 - (\theta - 1)^2}}$. With $u = \theta - 1$, $du = d\theta$,

$$\begin{aligned} \int \frac{1}{\sqrt{2\theta - \theta^2}} d\theta &= \int \frac{1}{\sqrt{1 - (\theta - 1)^2}} d\theta \\ &= \int \frac{1}{\sqrt{1 - u^2}} du = \sin^{-1} u + C = \sin^{-1}(\theta - 1) + C \end{aligned}$$

We have used Formula 18 in Table 8.1 with $a = 1$.

$$17. \int \frac{\ln y}{y + 4 \ln^2 y} dy$$

Write the integrand as $\frac{\ln y}{y} \cdot \frac{1}{1 + 4 \ln^2 y}$.

$$u = 1 + 4 \ln^2 y \quad du = \frac{8 \ln y}{y} dy$$

$$\begin{aligned} \int \frac{\ln y}{y + 4 \ln^2 y} dy &= \int \frac{\ln y}{y} \cdot \frac{1}{1 + 4 \ln^2 y} dy \\ &= \frac{1}{8} \int \frac{1}{u} du = \frac{1}{8} \ln |u| + C = \frac{1}{8} \ln(1 + 4 \ln^2 y) + C \end{aligned}$$

Note that the argument of the logarithm is positive, so we don't need absolute value bars.

$$18. \int \frac{2\sqrt{y}}{2\sqrt{y}} dy$$

$$u = \sqrt{y} \quad du = \frac{1}{2\sqrt{y}} dy$$

Using Formula 5 in Table 8.1,

$$\begin{aligned} \int \frac{2\sqrt{y}}{2\sqrt{y}} dy &= \int 2^u du \\ &= \frac{1}{\ln 2} 2^u + C = \frac{1}{\ln 2} 2\sqrt{y} + C \end{aligned}$$

$$19. \int \frac{1}{\sec \theta + \tan \theta} d\theta$$

Multiply the integrand by $\frac{\cos \theta}{\cos \theta}$.

$$\int \frac{1}{\sec \theta + \tan \theta} \cdot \frac{\cos \theta}{\cos \theta} d\theta = \int \frac{\cos \theta}{1 + \sin \theta} d\theta$$

$$u = 1 + \sin \theta \quad du = \cos \theta d\theta$$

$$\begin{aligned}\int \frac{\cos \theta}{1 + \sin \theta} d\theta &= \int \frac{1}{u} du = \ln|u| + C \\ &= \ln(1 + \sin \theta) + C\end{aligned}$$

We can discard the absolute value because $1 + \sin \theta$ is never negative.

$$20. \int \frac{1}{t\sqrt{3+t^2}} dt$$

Use Formula 5 in Table 7.10, with $a = \sqrt{3}$.

$$\int \frac{1}{t\sqrt{3+t^2}} dt = -\frac{1}{\sqrt{3}} \operatorname{csch}^{-1} \left| \frac{t}{\sqrt{3}} \right| + C$$

$$21. \int \frac{4t^3 - t^2 + 16t}{t^2 + 4} dt$$

Use long division to write the integrand as $4t - 1 + \frac{4}{t^2 + 4}$.

$$\begin{aligned}\int \frac{4t^3 - t^2 + 16t}{t^2 + 4} dt &= \int 4t dt - \int 1 dt + 4 \int \frac{1}{t^2 + 4} dt \\ &= 2t^2 - t + 2 \tan^{-1} \left(\frac{t}{2} \right) + C\end{aligned}$$

To evaluate the third integral we used Formula 19 in Table 8.1 with $a = 2$.

$$22. \int \frac{x + 2\sqrt{x-1}}{2x\sqrt{x-1}} dx$$

Split into two integrals.

$$\begin{aligned}\int \frac{x + 2\sqrt{x-1}}{2x\sqrt{x-1}} dx &= \int \frac{1}{2\sqrt{x-1}} dx + \int \frac{1}{x} dx \\ &= \sqrt{x-1} + \ln|x| + C\end{aligned}$$

For the first integral we used $u = \sqrt{x-1}$, $du = \frac{1}{2\sqrt{x-1}} dx$, $\int du = u + C$

$$23. \int_0^{\pi/2} \sqrt{1 - \cos \theta} d\theta$$

Multiply the integrand by $\frac{\sqrt{1 + \cos \theta}}{\sqrt{1 + \cos \theta}}$.

$$\int_0^{\pi/2} \sqrt{1 - \cos \theta} d\theta = \int_0^{\pi/2} \frac{\sqrt{1 - \cos^2 \theta}}{\sqrt{1 + \cos \theta}} d\theta = \int_0^{\pi/2} \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta.$$

(Note that when $0 \leq \theta \leq \pi/2$, $\sin \theta \geq 0$ so $\sqrt{\sin^2 \theta} = \sin \theta$.)

$$u = 1 + \cos \theta \quad du = -\sin \theta d\theta$$

$$u = 2 \text{ when } \theta = 0, \quad u = 1 \text{ when } \theta = \pi/2$$

$$\int_0^{\pi/2} \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta = -\int_2^1 \frac{1}{\sqrt{u}} du = \int_1^2 \frac{1}{\sqrt{u}} du = 2\sqrt{u} \Big|_1^2 = 2\sqrt{2} - 2$$

24. $\int (\sec t + \cot t)^2 dt$

Expand the integrand:

$$\begin{aligned} (\sec t + \cot t)^2 &= \sec^2 t + 2 \sec t \cot t + \cot^2 t \\ &= \sec^2 t + 2 \sec t \cot t + \csc^2 t - 1 \end{aligned}$$

$$\begin{aligned} \int (\sec t + \cot t)^2 dt &= \int \sec^2 t dt + 2 \int \csc t dt + \int \csc^2 t dt - \int 1 dt \\ &= \tan t - 2 \ln |\csc t + \cot t| - \cot t - t + C \end{aligned}$$

We have used Formulas 8, 9 and 15 from Table 8.1.

25. $\int \frac{1}{\sqrt{e^{2y} - 1}} dy$

Multiply the integrand by $\frac{e^y}{e^y}$.

$$\begin{aligned} \int \frac{1}{\sqrt{e^{2y} - 1}} dy &= \int \frac{e^y}{e^y \sqrt{e^{2y} - 1}} dy; \quad u = e^y \quad du = e^y dy \\ \int \frac{e^y}{e^y \sqrt{e^{2y} - 1}} dy &= \int \frac{1}{u \sqrt{u^2 - 1}} du \\ &= \sec^{-1} |u| + C = \sec^{-1} e^y + C \end{aligned}$$

We have used Formula 20 in Table 8.1.

26. $\int \frac{6}{\sqrt{y}(1+y)} dy$

$$u = \sqrt{y} \quad du = \frac{1}{2\sqrt{y}} dy$$

$$\begin{aligned} \int \frac{6}{\sqrt{y}(1+y)} dy &= 12 \int \frac{1}{1+u^2} du \\ &= 12 \tan^{-1} \sqrt{y} + C \end{aligned}$$

27. $\int \frac{2}{x\sqrt{1-4\ln^2 x}} dx$

$$u = 2 \ln x \quad du = \frac{2}{x} dx$$

$$\begin{aligned} \int \frac{2}{x\sqrt{1-4\ln^2 x}} dx &= \int \frac{1}{\sqrt{1-u^2}} du \\ &= \sin^{-1} u + C = \sin^{-1} (2 \ln x) + C \end{aligned}$$

$$28. \int \frac{1}{(x-2)\sqrt{x^2-4x+3}} dx$$

$$u = x - 2 \quad du = dx$$

$$\begin{aligned} \int \frac{1}{(x-2)\sqrt{x^2-4x+3}} dx &= \int \frac{1}{u\sqrt{u^2-1}} du \\ &= \sec^{-1}|u| + C = \sec^{-1}|x-2| + C \end{aligned}$$

We have used Formula 20 in Table 8.1 with $a = 1$.

$$29. \int (\csc x - \sec x)(\sin x + \cos x) dx$$

Expand the integrand and separate into two integrals.

$$(\csc x - \sec x)(\sin x + \cos x) = 1 + \cot x - \tan x - 1 = \cot x - \tan x$$

$$\begin{aligned} \int (\csc x - \sec x)(\sin x + \cos x) dx &= \int \cot x dx - \int \tan x dx \\ &= \ln|\sin x| - \ln|\sec x| + C = \ln|\sin x| + \ln|\cos x| + C \end{aligned}$$

We have used Formulas 12 and 13 from Table 8.1.

$$30. \int 3 \sinh\left(\frac{x}{2} + \ln 5\right) dx$$

$$u = \frac{x}{2} + \ln 5 \quad du = \frac{1}{2} dx$$

$$\begin{aligned} \int 3 \sinh\left(\frac{x}{2} + \ln 5\right) dx &= 6 \int \sinh u du \\ &= 6 \cosh u + C = 6 \cosh\left(\frac{x}{2} + \ln 5\right) + C \end{aligned}$$

$$31. \int_{\sqrt{2}}^3 \frac{2x^3}{x^2-1} dx$$

Use long division to write the integrand as $2x + \frac{2x}{x^2-1}$.

$$\int_{\sqrt{2}}^3 \frac{2x^3}{x^2-1} dx = \int_{\sqrt{2}}^3 \left(2x + \frac{2x}{x^2-1}\right) dx = \int_{\sqrt{2}}^3 2x dx + \int_{\sqrt{2}}^3 \frac{2x}{x^2-1} dx$$

For the second integral we use $u = x^2$, $du = 2x dx$.

$$\begin{aligned} \int_{\sqrt{2}}^3 2x dx + \int_{\sqrt{2}}^3 \frac{2x}{x^2-1} dx &= x^2 \Big|_{\sqrt{2}}^3 + \ln|x^2-1| \Big|_{\sqrt{2}}^3 \\ &= (9-2) + (\ln 8 - \ln 1) \\ &= 7 + \ln 8 \approx 9.079 \end{aligned}$$

32. $\int_{-1}^1 \sqrt{1+x^2} \sin x \, dx$ is the integral of an odd function over an interval symmetric to 0, so its value is 0.

33. $\int_{-1}^0 \sqrt{\frac{1+y}{1-y}} \, dy$

Multiply the integrand by $\frac{\sqrt{1+y}}{\sqrt{1+y}}$ and split the indefinite integral into a sum.

$$\begin{aligned} \int \sqrt{\frac{1+y}{1-y}} \, dy &= \int \frac{1+y}{\sqrt{1-y^2}} \, dy = \int \frac{1}{\sqrt{1-y^2}} \, dy + \int \frac{y}{\sqrt{1-y^2}} \, dy \\ &= \sin^{-1} y - \sqrt{1-y^2} + C \end{aligned}$$

The first integral is Formula 18 in Section 8.1, and for the second we use the substitution $u = 1 - y^2$, $du = -2y \, dy$. So

$$\begin{aligned} \int_{-1}^0 \sqrt{\frac{1+y}{1-y}} \, dy &= \left[\sin^{-1} y - \sqrt{1-y^2} \right]_{-1}^0 \\ &= (0-1) - \left(-\frac{\pi}{2} - 0 \right) = \frac{\pi}{2} - 1 \end{aligned}$$

34. $\int e^{z+e^z} \, dz$

Write the integrand as $e^z e^{e^z}$ and use the substitution $u = e^z$, $du = e^z \, dz$.

$$\begin{aligned} \int e^{z+e^z} \, dz &= \int e^z e^{e^z} \, dz = \int e^u \, du \\ &= e^u + C = e^{e^z} + C \end{aligned}$$

35. $\int \frac{7}{(x-1)\sqrt{x^2-2x-48}} \, dx$

$$u = x-1, \quad du = dx; \quad x^2 - 2x - 48 = u^2 - 7^2$$

We use Formula 20 in Table 8.1.

$$\begin{aligned} \int \frac{7}{(x-1)\sqrt{x^2-2x-48}} \, dx &= \int \frac{7}{u\sqrt{u^2-7^2}} \, du \\ &= \frac{1}{7} \left(7 \sec^{-1} \left| \frac{u}{7} \right| \right) + C = \sec^{-1} \left| \frac{x-1}{7} \right| + C \end{aligned}$$

$$36. \int \frac{1}{(2x+1)\sqrt{4x+4x^2}} dx$$

$$u = 2x+1, \quad du = 2dx; \quad 4x+4x^2 = u^2 - 1^2$$

We use Formula 20 in Table 8.1.

$$\begin{aligned} \int \frac{1}{(2x+1)\sqrt{4x+4x^2}} dx &= \frac{1}{2} \int \frac{1}{u\sqrt{u^2-1^2}} du \\ &= \frac{1}{2} \sec^{-1}|u| + C = \frac{1}{2} \sec^{-1}|2x+1| + C \end{aligned}$$

$$37. \int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5} d\theta$$

Use long division to write the integrand as $\theta^2 - \theta + 1 + \frac{5}{2\theta - 5}$.

$$\begin{aligned} \int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5} d\theta &= \int \theta^2 d\theta - \int \theta d\theta + \int 1 d\theta + \int \frac{5}{2\theta - 5} d\theta \\ &= \frac{1}{3}\theta^3 - \frac{1}{2}\theta^2 + \theta + \frac{5}{2} \ln|2\theta - 5| + C \end{aligned}$$

In the last integral we have used the substitution $u = 2\theta - 5$, $du = 2 d\theta$.

$$38. \int \frac{1}{\cos \theta - 1} d\theta$$

Multiply the integrand by $\frac{\cos \theta + 1}{\cos \theta + 1}$.

$$\begin{aligned} \frac{1}{\cos \theta - 1} \frac{\cos \theta + 1}{\cos \theta + 1} &= \frac{\cos \theta + 1}{\cos^2 \theta - 1} = -\frac{1 + \cos \theta}{\sin^2 \theta} = -\csc^2 \theta - \csc \theta \cot \theta \\ \int (-\csc^2 \theta - \csc \theta \cot \theta) d\theta &= -\int \csc^2 \theta d\theta - \int \csc \theta \cot \theta d\theta \\ &= \cot \theta + \csc \theta + C \end{aligned}$$

We have used Formulas 9 and 11 from Table 8.1.

$$39. \int \frac{1}{1+e^x} dx$$

Use one step of long division to write the integrand as $1 - \frac{e^x}{1+e^x}$.

$$\int \frac{1}{1+e^x} dx = \int 1 dx - \int \frac{e^x}{1+e^x} dx = x - \ln(1+e^x) + C$$

For the second integral we have used the substitution $u = 1+e^x$, $du = e^x dx$. Note that $1+e^x$ is always positive.

$$40. \int \frac{\sqrt{x}}{1+x^3} dx$$

$$u = x^{3/2}, \quad du = \frac{3}{2} x^{1/2} dx$$

$$\begin{aligned} \int \frac{\sqrt{x}}{1+x^3} dx &= \frac{2}{3} \int \frac{1}{1+u^2} du \\ &= \frac{2}{3} \tan^{-1} u + C = \frac{2}{3} \tan^{-1} x^{3/2} + C \end{aligned}$$

$$\begin{aligned} 41. \text{ The area is } \int_{-\pi/4}^{\pi/4} (2 \cos x - \sec x) dx &= \left[2 \sin x - \ln |\sec x + \tan x| \right]_{-\pi/4}^{\pi/4} \\ &= \left(\sqrt{2} - \ln |\sqrt{2} + 1| \right) - \left(-\sqrt{2} - \ln |\sqrt{2} - 1| \right) \\ &= 2\sqrt{2} + \ln \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right) = 2\sqrt{2} - \ln(3 + 2\sqrt{2}) \approx 1.066 \end{aligned}$$

$$42. \text{ The volume using the washer method is } \pi \int_{-\pi/4}^{\pi/4} (4 \cos^2 x - \sec^2 x) dx.$$

Split into two integrals; for the first write $4 \cos^2 x$ as $2(1 + \cos 2x)$ and for the second use Formula 8 in Table 8.1.

$$\begin{aligned} \pi \int_{-\pi/4}^{\pi/4} (4 \cos^2 x - \sec^2 x) dx &= \pi \int_{-\pi/4}^{\pi/4} 4 \cos^2 x dx - \pi \int_{-\pi/4}^{\pi/4} \sec^2 x dx \\ &= \pi \int_{-\pi/4}^{\pi/4} 2(1 + \cos 2x) dx - \pi \int_{-\pi/4}^{\pi/4} \sec^2 x dx \\ &= \pi [2x + \sin 2x]_{-\pi/4}^{\pi/4} - \pi \tan x \Big|_{-\pi/4}^{\pi/4} \\ &= \pi \left(\left(\frac{\pi}{2} + 1 \right) - \left(-\frac{\pi}{2} - 1 \right) \right) - \pi(1 - (-1)) = \pi^2 \end{aligned}$$

$$43. \text{ For } y = \ln(\cos x), \quad dy/dx = -\tan x. \text{ The arc length is given by}$$

$$\int_0^{\pi/3} \sqrt{1 + (-\tan x)^2} dx = \int_0^{\pi/3} \sec x dx \text{ since } \sec x \text{ is positive on the interval of integration.}$$

$$\begin{aligned} \int_0^{\pi/3} \sec x dx &= \ln |\sec x + \tan x| \Big|_0^{\pi/3} \\ &= \ln(2 + \sqrt{3}) - \ln(1 + 0) = \ln(2 + \sqrt{3}) \end{aligned}$$

$$44. \text{ For } y = \ln(\sec x), \quad dy/dx = \tan x. \text{ The arc length is given by}$$

$$\int_0^{\pi/4} \sqrt{1 + (\tan x)^2} dx = \int_0^{\pi/4} \sec x dx \text{ since } \sec x \text{ is positive on the interval of integration.}$$

$$\begin{aligned} \int_0^{\pi/4} \sec x dx &= \ln |\sec x + \tan x| \Big|_0^{\pi/4} \\ &= \ln(\sqrt{2} + 1) - \ln(1 + 0) = \ln(\sqrt{2} + 1) \end{aligned}$$

45. Since secant is an even function and the domain is symmetric to 0,
- $\bar{x} = 0$
- .

For the y -coordinate:

$$\begin{aligned}\bar{y} &= \frac{\frac{1}{2} \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx}{\int_{-\pi/4}^{\pi/4} \sec x \, dx} = \frac{\left. \frac{1}{2} \tan x \right|_{-\pi/4}^{\pi/4}}{\left. \ln |\sec x + \tan x| \right|_{-\pi/4}^{\pi/4}} \\ &= \frac{1}{\left(\ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1) \right)} \\ &= \frac{1}{\ln\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right)} = \frac{1}{\ln(3 + 2\sqrt{2})} \approx 0.567\end{aligned}$$

46. Since both cosecant and the domain are symmetric around
- $\pi/2$
- ,
- $\bar{x} = \pi/2$
- .

$$\begin{aligned}\bar{y} &= \frac{\frac{1}{2} \int_{\pi/6}^{5\pi/6} \csc^2 x \, dx}{\int_{\pi/6}^{5\pi/6} \csc x \, dx} = \frac{\left. -\frac{1}{2} \cot x \right|_{\pi/6}^{5\pi/6}}{\left. -\ln |\csc x + \cot x| \right|_{\pi/6}^{5\pi/6}} \\ &= \frac{-\frac{1}{2}(\sqrt{3} - (-\sqrt{3}))}{-\left(\ln(2 + \sqrt{3}) - \ln(2 - \sqrt{3})\right)} \\ &= \frac{\sqrt{3}}{\ln\left(\frac{2 + \sqrt{3}}{2 - \sqrt{3}}\right)} = \frac{\sqrt{3}}{\ln(7 + 4\sqrt{3})} \approx 0.658\end{aligned}$$

47. $\int (1 + 3x^3) e^{x^3} dx = x e^{x^3} + C$

48. $\int \frac{1}{1 + \sin^2 x} dx$

Multiply the integrand by $\frac{\sec^2 x}{\sec^2 x}$.

$$\int \frac{1}{1 + \sin^2 x} dx = \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x} dx = \int \frac{\sec^2 x}{1 + 2 \tan^2 x} dx$$

$$u = \tan x, \quad du = \sec^2 x \, dx$$

$$\int \frac{\sec^2 x}{1 + 2 \tan^2 x} dx = \int \frac{1}{1 + 2u^2} du$$

$$v = \sqrt{2}u, \quad dv = \sqrt{2} \, du$$

$$\begin{aligned}\int \frac{1}{1 + 2u^2} du &= \frac{1}{\sqrt{2}} \int \frac{1}{1 + v^2} dv \\ &= \frac{1}{\sqrt{2}} \tan^{-1} v + C \\ &= \frac{1}{\sqrt{2}} \tan^{-1} (\sqrt{2} \tan x) + C\end{aligned}$$

49. $\int x^7 \sqrt{x^4 + 1} \, dx$

$$u = x^4 + 1, \, du = 4x^3 dx; \quad x^7 dx = \frac{u-1}{4} du$$

$$\begin{aligned} \int x^7 \sqrt{x^4 + 1} \, dx &= \frac{1}{4} \int (u-1) \sqrt{u} \, du \\ &= \frac{1}{4} \int u^{3/2} \, du - \frac{1}{4} \int u^{1/2} \, du \\ &= \frac{1}{10} u^{5/2} - \frac{1}{6} u^{3/2} + C \\ &= \frac{1}{30} u^{3/2} (3u-5) + C = \frac{1}{30} (x^4 + 1)^{3/2} (3x^4 - 2) + C \end{aligned}$$

50. $\int \left((x^2 - 1)(x + 1) \right)^{-2/3} dx$

The easiest substitution to use is probably $u = \frac{x-1}{x+1}$, $du = \frac{2}{(1+x)^2} dx$.

The integral can be written as

$$\begin{aligned} \int \frac{1}{\left(\frac{x-1}{x+1} \right)^{2/3} (x+1)^2} dx &= \frac{1}{2} \int u^{-2/3} du \\ &= \frac{3}{2} u^{1/3} + C = \frac{3}{2} \left(\frac{x-1}{x+1} \right)^{1/3} + C \end{aligned}$$

8.2 INTEGRATION BY PARTS

1. $u = x, \, du = dx; \, dv = \sin \frac{x}{2} \, dx, \, v = -2 \cos \frac{x}{2};$

$$\int x \sin \frac{x}{2} \, dx = -2x \cos \frac{x}{2} - \int \left(-2 \cos \frac{x}{2} \right) dx = -2x \cos \left(\frac{x}{2} \right) + 4 \sin \left(\frac{x}{2} \right) + C$$

2. $u = \theta, \, du = d\theta; \, dv = \cos \pi \theta \, d\theta, \, v = \frac{1}{\pi} \sin \pi \theta;$

$$\int \theta \cos \pi \theta \, d\theta = \frac{\theta}{\pi} \sin \pi \theta - \int \frac{1}{\pi} \sin \pi \theta \, d\theta = \frac{\theta}{\pi} \sin \pi \theta + \frac{1}{\pi^2} \cos \pi \theta + C$$

3. $\cos t$

$$t^2 \xrightarrow{(+)} \sin t$$

$$2t \xrightarrow{(-)} -\cos t$$

$$2 \xrightarrow{(+)} -\sin t$$

$$0 \qquad \int t^2 \cos t \, dt = t^2 \sin t + 2t \cos t - 2 \sin t + C$$

$$\begin{array}{rcl}
 4. & \sin x & \\
 x^2 & \xrightarrow{(+)} & -\cos x \\
 2x & \xrightarrow{(-)} & -\sin x \\
 2 & \xrightarrow{(+)} & \cos x
 \end{array}$$

$$0 \quad \int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

$$5. \quad u = \ln x, \, du = \frac{dx}{x}; \, dv = x \, dx, \, v = \frac{x^2}{2};$$

$$\int_1^2 x \ln x \, dx = \left[\frac{x^2}{2} \ln x \right]_1^2 - \int_1^2 \frac{x^2}{2} \frac{dx}{x} = 2 \ln 2 - \left[\frac{x^2}{4} \right]_1^2 = 2 \ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

$$6. \quad u = \ln x, \, du = \frac{dx}{x}; \, dv = x^3 \, dx, \, v = \frac{x^4}{4};$$

$$\int_1^e x^3 \ln x \, dx = \left[\frac{x^4}{4} \ln x \right]_1^e - \int_1^e \frac{x^4}{4} \frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16} \right]_1^e = \frac{3e^4 + 1}{16}$$

$$7. \quad u = x, \, du = dx; \, dv = e^x \, dx, \, v = e^x;$$

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$$

$$8. \quad u = x, \, du = dx; \, dv = e^{3x} \, dx, \, v = \frac{1}{3} e^{3x};$$

$$\int x e^{3x} \, dx = \frac{x}{3} e^{3x} - \frac{1}{3} \int e^{3x} \, dx = \frac{x}{3} e^{3x} - \frac{1}{9} e^{3x} + C$$

$$9. \quad e^{-x}$$

$$x^2 \xrightarrow{(+)} -e^{-x}$$

$$2x \xrightarrow{(-)} e^{-x}$$

$$2 \xrightarrow{(+)} -e^{-x}$$

$$0 \quad \int x^2 e^{-x} \, dx = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

$$10. \quad e^{2x}$$

$$x^2 - 2x + 1 \xrightarrow{(+)} \frac{1}{2} e^{2x}$$

$$2x - 2 \xrightarrow{(-)} \frac{1}{4} e^{2x}$$

$$2 \xrightarrow{(+)} \frac{1}{8} e^{2x}$$

$$\begin{aligned}
 0 \quad \int (x^2 - 2x + 1) e^{2x} \, dx &= \frac{1}{2} (x^2 - 2x + 1) e^{2x} - \frac{1}{4} (2x - 2) e^{2x} + \frac{1}{4} e^{2x} + C \\
 &= \left(\frac{1}{2} x^2 - \frac{3}{2} x + \frac{5}{4} \right) e^{2x} + C
 \end{aligned}$$

$$11. \quad u = \tan^{-1} y, \, du = \frac{dy}{1+y^2}; \, dv = dy, \, v = y;$$

$$\int \tan^{-1} y \, dy = y \tan^{-1} y - \int \frac{y \, dy}{(1+y^2)} = y \tan^{-1} y - \frac{1}{2} \ln(1+y^2) + C = y \tan^{-1} y - \ln \sqrt{1+y^2} + C$$

$$12. \quad u = \sin^{-1} y, \quad du = \frac{dy}{\sqrt{1-y^2}}; \quad dv = dy, \quad v = y;$$

$$\int \sin^{-1} y \, dy = y \sin^{-1} y - \int \frac{y \, dy}{\sqrt{1-y^2}} = y \sin^{-1} y + \sqrt{1-y^2} + C$$

$$13. \quad u = x, \quad du = dx; \quad dv = \sec^2 x \, dx, \quad v = \tan x;$$

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx = x \tan x + \ln |\cos x| + C$$

$$14. \quad \int 4x \sec^2 2x \, dx; [y = 2x, \, dy = 2dx] \rightarrow \int y \sec^2 y \, dy = y \tan y - \int \tan y \, dy = y \tan y - \ln |\sec y| + C \\ = 2x \tan 2x - \ln |\sec 2x| + C$$

$$15. \quad \begin{array}{r} e^x \\ x^3 \end{array} \xrightarrow{(+)} e^x$$

$$3x^2 \xrightarrow{(-)} e^x$$

$$6x \xrightarrow{(+)} e^x$$

$$6 \xrightarrow{(-)} e^x$$

$$0 \quad \int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = (x^3 - 3x^2 + 6x - 6)e^x + C$$

$$16. \quad \begin{array}{r} e^{-p} \\ p^4 \end{array} \xrightarrow{(+)} -e^{-p}$$

$$4p^3 \xrightarrow{(-)} e^{-p}$$

$$12p^2 \xrightarrow{(+)} -e^{-p}$$

$$24p \xrightarrow{(-)} e^{-p}$$

$$24 \xrightarrow{(+)} -e^{-p}$$

$$0 \quad \int p^4 e^{-p} \, dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24p e^{-p} - 24e^{-p} + C \\ = (-p^4 - 4p^3 - 12p^2 - 24p - 24)e^{-p} + C$$

$$17. \quad \begin{array}{r} e^x \\ x^2 - 5x \end{array} \xrightarrow{(+)} e^x$$

$$2x - 5 \xrightarrow{(-)} e^x$$

$$2 \xrightarrow{(+)} e^x$$

$$0 \quad \int (x^2 - 5x) e^x \, dx = (x^2 - 5x) e^x - (2x - 5) e^x + 2e^x + C = x^2 e^x - 7x e^x + 7e^x + C \\ = (x^2 - 7x + 7) e^x + C$$

$$\begin{array}{rcl}
 18. & & e^r \\
 & r^2 + r + 1 & \xrightarrow{(+)} e^r \\
 & 2r + 1 & \xrightarrow{(-)} e^r \\
 & 2 & \xrightarrow{(+)} e^r \\
 0 & & \int (r^2 + r + 1) e^r dr = (r^2 + r + 1) e^r - (2r + 1) e^r + 2e^r + C \\
 & & = \left[(r^2 + r + 1) - (2r + 1) + 2 \right] e^r + C = (r^2 - r + 2) e^r + C
 \end{array}$$

$$\begin{array}{rcl}
 19. & & e^x \\
 & x^5 & \xrightarrow{(+)} e^x \\
 & 5x^4 & \xrightarrow{(-)} e^x \\
 & 20x^3 & \xrightarrow{(+)} e^x \\
 & 60x^2 & \xrightarrow{(-)} e^x \\
 & 120x & \xrightarrow{(+)} e^x \\
 & 120 & \xrightarrow{(-)} e^x \\
 0 & & \int x^5 e^x dx = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120e^x + C \\
 & & = (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) e^x + C
 \end{array}$$

$$\begin{array}{rcl}
 20. & & e^{4t} \\
 & t^2 & \xrightarrow{(+)} \frac{1}{4} e^{4t} \\
 & 2t & \xrightarrow{(-)} \frac{1}{16} e^{4t} \\
 & 2 & \xrightarrow{(+)} \frac{1}{64} e^{4t} \\
 0 & & \int t^2 e^{4t} dt = \frac{t^2}{4} e^{4t} - \frac{2t}{16} e^{4t} + \frac{2}{64} e^{4t} + C = \frac{t^2}{4} e^{4t} - \frac{t}{8} e^{4t} + \frac{1}{32} e^{4t} + C \\
 & & = \left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32} \right) e^{4t} + C
 \end{array}$$

$$\begin{aligned}
 21. \quad I &= \int e^\theta \sin \theta \, d\theta; [u = \sin \theta, du = \cos \theta \, d\theta; dv = e^\theta \, d\theta, v = e^\theta] \Rightarrow I = e^\theta \sin \theta - \int e^\theta \cos \theta \, d\theta; \\
 &[u = \cos \theta, du = -\sin \theta \, d\theta; dv = e^\theta \, d\theta, v = e^\theta] \Rightarrow I = e^\theta \sin \theta - \left(e^\theta \cos \theta + \int e^\theta \sin \theta \, d\theta \right) \\
 &= e^\theta \sin \theta - e^\theta \cos \theta - I + C' \Rightarrow 2I = (e^\theta \sin \theta - e^\theta \cos \theta) + C' \Rightarrow I = \frac{1}{2} (e^\theta \sin \theta - e^\theta \cos \theta) + C, \text{ where } C = \frac{C'}{2} \\
 &\text{is another arbitrary constant}
 \end{aligned}$$

$$\begin{aligned}
 22. \quad I &= \int e^{-y} \cos y \, dy; [u = \cos y, du = -\sin y \, dy; dv = e^{-y} \, dy, v = -e^{-y}] \\
 &\Rightarrow I = -e^{-y} \cos y - \int (-e^{-y})(-\sin y) \, dy = -e^{-y} \cos y - \int e^{-y} \sin y \, dy; \\
 &[u = \sin y, du = \cos y \, dy; dv = e^{-y} \, dy, v = -e^{-y}] \Rightarrow I = -e^{-y} \cos y - \left(-e^{-y} \sin y - \int (-e^{-y}) \cos y \, dy \right)
 \end{aligned}$$

$$= -e^{-y} \cos y + e^{-y} \sin y - I + C' \Rightarrow 2I = e^{-y} (\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2} (e^{-y} \sin y - e^{-y} \cos y) + C, \text{ where } C = \frac{C'}{2} \text{ is another arbitrary constant}$$

23. $I = \int e^{2x} \cos 3x \, dx; [u = \cos 3x; \, du = -3 \sin 3x \, dx; \, dv = e^{2x} \, dx; \, v = \frac{1}{2} e^{2x}]$
 $\Rightarrow I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x \, dx; [u = \sin 3x, \, du = 3 \cos 3x, \, dv = e^{2x} \, dx; \, v = \frac{1}{2} e^{2x}]$
 $\Rightarrow I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \left(\frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \int e^{2x} \cos 3x \, dx \right) = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} I + C'$
 $\Rightarrow \frac{13}{4} I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x + C' \Rightarrow I = \frac{e^{2x}}{13} (3 \sin 3x + 2 \cos 3x) + C, \text{ where } C = \frac{4}{13} C'$
24. $\int e^{-2x} \sin 2x \, dx; [y = 2x, \, du = 2 \, dx] \rightarrow \frac{1}{2} \int e^{-y} \sin y \, dy = I; [u = \sin y, \, du = \cos y \, dy; \, dv = e^{-y} \, dy, \, v = -e^{-y}]$
 $\Rightarrow I = \frac{1}{2} \left(-e^{-y} \sin y + \int e^{-y} \cos y \, dy \right) [u = \cos y, \, du = -\sin y; \, dv = e^{-y} \, dy, \, v = -e^{-y}]$
 $\Rightarrow I = -\frac{1}{2} e^{-y} \sin y + \frac{1}{2} \left(-e^{-y} \cos y - \int (-e^{-y}) (-\sin y) \, dy \right) = -\frac{1}{2} e^{-y} (\sin y + \cos y) - I + C'$
 $\Rightarrow 2I = -\frac{1}{2} e^{-y} (\sin y + \cos y) + C' \Rightarrow I = -\frac{1}{4} e^{-y} (\sin y + \cos y) + C = -\frac{e^{-2x}}{4} (\sin 2x + \cos 2x) + C, \text{ where } C = \frac{C'}{2}$
25. $\int e^{\sqrt{3s+9}} \, ds; \left[\begin{array}{l} 3s+9 = x^2 \\ ds = \frac{2}{3} x \, dx \end{array} \right] \rightarrow \int e^x \cdot \frac{2}{3} x \, dx = \frac{2}{3} \int x e^x \, dx; [u = x, \, du = dx; \, dv = e^x \, dx, \, v = e^x];$
 $\frac{2}{3} \int x e^x \, dx = \frac{2}{3} \left(x e^x - \int e^x \, dx \right) = \frac{2}{3} (x e^x - e^x) + C = \frac{2}{3} (\sqrt{3s+9} e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}) + C$
26. $u = x, \, du = dx; \, dv = \sqrt{1-x} \, dx, \, v = -\frac{2}{3} \sqrt{(1-x)^3};$
 $\int_0^1 x \sqrt{1-x} \, dx = \left[-\frac{2}{3} x \sqrt{(1-x)^3} \right]_0^1 + \frac{2}{3} \int_0^1 \sqrt{(1-x)^3} \, dx = 0 + \frac{2}{3} \left[-\frac{2}{5} (1-x)^{5/2} \right]_0^1 = \frac{4}{15}$
27. $u = x, \, du = dx; \, dv = \tan^2 x \, dx, \, v = \int \tan^2 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{1-\cos^2 x}{\cos^2 x} \, dx = \int \frac{dx}{\cos^2 x} - \int dx = \tan x - x;$
 $\int_0^{\pi/3} x \tan^2 x \, dx = \left[x (\tan x - x) \right]_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x) \, dx = \frac{\pi}{3} \left(\sqrt{3} - \frac{\pi}{3} \right) + \left[\ln |\cos x| + \frac{x^2}{2} \right]_0^{\pi/3}$
 $= \frac{\pi}{3} \left(\sqrt{3} - \frac{\pi}{3} \right) + \ln \frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}$
28. $u = \ln(x+x^2), \, du = \frac{(2x+1) \, dx}{x+x^2}; \, dv = dx, \, v = x; \int \ln(x+x^2) \, dx = x \ln(x+x^2) - \int \frac{2x+1}{x(x+1)} \cdot x \, dx$
 $= x \ln(x+x^2) - \int \frac{(2x+1) \, dx}{x+1} = x \ln(x+x^2) - \int \left(2 - \frac{1}{x+1} \right) dx = x \ln(x+x^2) - 2x + \ln|x+1| + C$
29. $\int \sin(\ln x) \, dx; \left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x} \, dx \\ dx = e^u \, du \end{array} \right] \rightarrow \int (\sin u) e^u \, du. \text{ From Exercise 21, } \int (\sin u) e^u \, du = e^u \left(\frac{\sin u - \cos u}{2} \right) + C$
 $= \frac{1}{2} [-x \cos(\ln x) + x \sin(\ln x)] + C$

$$30. \int z(\ln z)^2 dz; \left[\begin{array}{l} u = \ln z \\ du = \frac{1}{z} dz \\ dz = e^u du \end{array} \right] \rightarrow \int e^u \cdot u^2 \cdot e^u du = \int e^{2u} \cdot u^2 du;$$

$$u^2 \xrightarrow{(+)} \frac{1}{2} e^{2u}$$

$$2u \xrightarrow{(-)} \frac{1}{4} e^{2u}$$

$$2 \xrightarrow{(+)} \frac{1}{8} e^{2u}$$

$$0 \quad \int u^2 e^{2u} du = \frac{u^2}{2} e^{2u} - \frac{u}{2} e^{2u} + \frac{1}{4} e^{2u} + C = \frac{e^{2u}}{4} (2u^2 - 2u + 1) + C$$

$$= \frac{z^2}{4} [2(\ln z)^2 - 2 \ln z + 1] + C$$

$$31. \int x \sec x^2 dx \left[\text{Let } u = x^2, du = 2x dx \Rightarrow \frac{1}{2} du = x dx \right] \rightarrow \int x \sec x^2 dx = \frac{1}{2} \int \sec u du = \frac{1}{2} \ln |\sec u + \tan u| + C$$

$$= \frac{1}{2} \ln |\sec x^2 + \tan x^2| + C$$

$$32. \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx \left[\text{Let } u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2du = \frac{1}{\sqrt{x}} dx \right] \rightarrow \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = 2 \int \cos u du = 2 \sin u + C = 2 \sin \sqrt{x} + C$$

$$33. \int x(\ln x)^2 dx; \left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \\ dx = e^u du \end{array} \right] \rightarrow \int e^u \cdot u^2 \cdot e^u du = \int e^{2u} \cdot u^2 du;$$

$$u^2 \xrightarrow{(+)} \frac{1}{2} e^{2u}$$

$$2u \xrightarrow{(-)} \frac{1}{4} e^{2u}$$

$$2 \xrightarrow{(+)} \frac{1}{8} e^{2u}$$

$$0 \quad \int u^2 e^{2u} du = \frac{u^2}{2} e^{2u} - \frac{u}{2} e^{2u} + \frac{1}{4} e^{2u} + C = \frac{e^{2u}}{4} (2u^2 - 2u + 1) + C$$

$$= \frac{x^2}{4} [2(\ln x)^2 - 2 \ln x + 1] + C = \frac{x^2}{2} (\ln x)^2 - \frac{x^2}{2} \ln x + \frac{x^2}{4} + C$$

$$34. \int \frac{1}{x(\ln x)^2} dx \left[\text{Let } u = \ln x, du = \frac{1}{x} dx \right] \rightarrow \int \frac{1}{x(\ln x)^2} dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$$

$$35. u = \ln x, du = \frac{1}{x} dx; dv = \frac{1}{x^2} dx, v = -\frac{1}{x};$$

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$36. \int \frac{(\ln x)^3}{x} dx \left[\text{Let } u = \ln x, du = \frac{1}{x} dx \right] \rightarrow \int \frac{(\ln x)^3}{x} dx = \int u^3 du = \frac{1}{4} u^4 + C = \frac{1}{4} (\ln x)^4 + C$$

$$37. \int x^3 e^{x^4} dx \left[\text{Let } u = x^4, du = 4x^3 dx \Rightarrow \frac{1}{4} du = x^3 dx \right] \rightarrow \int x^3 e^{x^4} dx = \frac{1}{4} \int e^u du = \frac{1}{4} e^u + C = \frac{1}{4} e^{x^4} + C$$

$$38. u = x^3, du = 3x^2 dx; dv = x^2 e^{x^3} dx, v = \frac{1}{3} e^{x^3};$$

$$\int x^5 e^{x^3} dx = \int x^3 e^{x^3} x^2 dx = \frac{1}{3} x^3 e^{x^3} - \frac{1}{3} \int e^{x^3} 3x^2 dx = \frac{1}{3} x^3 e^{x^3} - \frac{1}{3} e^{x^3} + C$$

$$39. u = x^2, du = 2x dx; dv = \sqrt{x^2 + 1} x dx, v = \frac{1}{3} (x^2 + 1)^{3/2};$$

$$\int x^3 \sqrt{x^2 + 1} dx = \frac{1}{3} x^2 (x^2 + 1)^{3/2} - \frac{1}{3} \int (x^2 + 1)^{3/2} 2x dx = \frac{1}{3} x^2 (x^2 + 1)^{3/2} - \frac{2}{15} (x^2 + 1)^{5/2} + C$$

$$40. \int x^2 \sin x^3 dx \left[\text{Let } u = x^3, du = 3x^2 dx \Rightarrow \frac{1}{3} du = x^2 dx \right] \rightarrow \int x^2 \sin x^3 dx = \frac{1}{3} \int \sin u du = -\frac{1}{3} \cos u + C$$

$$= -\frac{1}{3} \cos x^3 + C$$

$$41. u = \sin 3x, du = 3 \cos 3x dx; dv = \cos 2x dx, v = \frac{1}{2} \sin 2x;$$

$$\int \sin 3x \cos 2x dx = \frac{1}{2} \sin 3x \sin 2x - \frac{3}{2} \int \cos 3x \sin 2x dx$$

$$u = \cos 3x, du = -3 \sin 3x dx; dv = \sin 2x dx, v = -\frac{1}{2} \cos 2x;$$

$$\int \sin 3x \cos 2x dx = \frac{1}{2} \sin 3x \sin 2x - \frac{3}{2} \left[-\frac{1}{2} \cos 3x \cos 2x - \frac{3}{2} \int \sin 3x \cos 2x dx \right]$$

$$= \frac{1}{2} \sin 3x \sin 2x + \frac{3}{4} \cos 3x \cos 2x + \frac{9}{4} \int \sin 3x \cos 2x dx$$

$$\Rightarrow -\frac{5}{4} \int \sin 3x \cos 2x dx = \frac{1}{2} \sin 3x \sin 2x + \frac{3}{4} \cos 3x \cos 2x$$

$$\Rightarrow \int \sin 3x \cos 2x dx = -\frac{2}{5} \sin 3x \sin 2x - \frac{3}{5} \cos 3x \cos 2x + C$$

$$42. u = \sin 2x, du = 2 \cos 2x dx; dv = \cos 4x dx, v = \frac{1}{4} \sin 4x;$$

$$\int \sin 2x \cos 4x dx = \frac{1}{4} \sin 2x \sin 4x - \frac{1}{2} \int \cos 2x \sin 4x dx$$

$$u = \cos 2x, du = -2 \sin 2x dx; dv = \sin 4x dx, v = -\frac{1}{4} \cos 4x;$$

$$\int \sin 2x \cos 4x dx = \frac{1}{4} \sin 2x \sin 4x - \frac{1}{2} \left[-\frac{1}{4} \cos 2x \cos 4x - \frac{1}{2} \int \sin 2x \cos 4x dx \right]$$

$$= \frac{1}{4} \sin 2x \sin 4x + \frac{1}{8} \cos 2x \cos 4x + \frac{1}{4} \int \sin 2x \cos 4x dx$$

$$\Rightarrow \frac{3}{4} \int \sin 2x \cos 4x dx = \frac{1}{4} \sin 2x \sin 4x + \frac{1}{8} \cos 2x \cos 4x$$

$$\Rightarrow \int \sin 2x \cos 4x dx = \frac{1}{3} \sin 2x \sin 4x + \frac{1}{6} \cos 2x \cos 4x + C$$

$$43. \int \sqrt{x} \ln x dx \left[\text{Let } u = \ln x, du = \frac{1}{x} dx, dv = \sqrt{x} dx, v = \frac{2}{3} x^{3/2} \right]$$

$$\int \sqrt{x} \ln x dx = \frac{2}{3} x^{3/2} \ln x - \frac{2}{3} \int \sqrt{x} dx$$

$$= \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2} + C = \frac{2}{9} x^{3/2} (3 \ln x - 2) + C$$

$$44. \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx \left[\text{Let } u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2du = \frac{1}{\sqrt{x}} dx \right] \rightarrow \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^u du = 2e^u + C = 2e^{\sqrt{x}} + C$$

$$45. \int \cos \sqrt{x} dx; \left[\begin{array}{l} y = \sqrt{x} \\ dy = \frac{1}{2\sqrt{x}} dx \\ dx = 2y dy \end{array} \right] \rightarrow \int \cos y \cdot 2y dy = \int 2y \cos y dy;$$

$$u = 2y, du = 2dy; dv = \cos y dy, v = \sin y;$$

$$\int 2y \cos y dy = 2y \sin y - \int 2 \sin y dy = 2y \sin y + 2 \cos y + C = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C$$

$$46. \int \sqrt{x} e^{\sqrt{x}} dx; \left[\begin{array}{l} y = \sqrt{x} \\ dy = \frac{1}{2\sqrt{x}} dx \\ dx = 2y dy \end{array} \right] \rightarrow \int y e^y \cdot 2y dy = \int 2y^2 e^y dy;$$

$$2y^2 \xrightarrow{(+)} e^y$$

$$4y \xrightarrow{(-)} e^y$$

$$4 \xrightarrow{(+)} e^y$$

$$0 \qquad \int 2y^2 e^y dy = 2y^2 e^y - 4y e^y + 4e^y + C = 2x e^{\sqrt{x}} - 4\sqrt{x} e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$$

$$47. \qquad \sin 2\theta$$

$$\theta^2 \xrightarrow{(+)} -\frac{1}{2} \cos 2\theta$$

$$2\theta \xrightarrow{(-)} -\frac{1}{4} \sin 2\theta$$

$$2 \xrightarrow{(+)} \frac{1}{8} \cos 2\theta$$

$$0 \qquad \int_0^{\pi/2} \theta^2 \sin 2\theta d\theta = \left[-\frac{\theta^2}{2} \cos 2\theta + \frac{\theta}{2} \sin 2\theta + \frac{1}{4} \cos 2\theta \right]_0^{\pi/2} \\ = \left[-\frac{\pi^2}{8} \cdot (-1) + \frac{\pi}{4} \cdot 0 + \frac{1}{4} \cdot (-1) \right] - \left[0 + 0 + \frac{1}{4} \cdot 1 \right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2 - 4}{8}$$

$$48. \qquad \cos 2x$$

$$x^3 \xrightarrow{(+)} \frac{1}{2} \sin 2x$$

$$3x^2 \xrightarrow{(-)} -\frac{1}{4} \cos 2x$$

$$6x \xrightarrow{(+)} -\frac{1}{8} \sin 2x$$

$$6 \xrightarrow{(-)} \frac{1}{16} \cos 2x$$

$$0 \qquad \int_0^{\pi/2} x^3 \cos 2x dx = \left[\frac{x^3}{2} \sin 2x + \frac{3x^2}{4} \cos 2x - \frac{3x}{4} \sin 2x - \frac{3}{8} \cos 2x \right]_0^{\pi/2} \\ = \left[\frac{\pi^3}{16} \cdot 0 + \frac{3\pi^2}{16} \cdot (-1) - \frac{3\pi}{8} \cdot 0 - \frac{3}{8} \cdot (-1) \right] - \left[0 + 0 - 0 - \frac{3}{8} \cdot 1 \right] = -\frac{3\pi^2}{16} + \frac{3}{4} = \frac{3(4 - \pi^2)}{16}$$

$$49. \quad u = \sec^{-1} t, \quad du = \frac{dt}{t\sqrt{t^2-1}}; \quad dv = t \, dt, \quad v = \frac{t^2}{2};$$

$$\begin{aligned} \int_{2/\sqrt{3}}^2 t \sec^{-1} t \, dt &= \left[\frac{t^2}{2} \sec^{-1} t \right]_{2/\sqrt{3}}^2 - \int_{2/\sqrt{3}}^2 \left(\frac{t^2}{2} \right) \frac{dt}{t\sqrt{t^2-1}} = \left(2 \cdot \frac{\pi}{3} - \frac{2}{3} \cdot \frac{\pi}{6} \right) - \int_{2/\sqrt{3}}^2 \frac{t \, dt}{2\sqrt{t^2-1}} \\ &= \frac{5\pi}{9} - \left[\frac{1}{2} \sqrt{t^2-1} \right]_{2/\sqrt{3}}^2 = \frac{5\pi}{9} - \frac{1}{2} \left(\sqrt{3} - \sqrt{\frac{4}{3}-1} \right) = \frac{5\pi}{9} - \frac{1}{2} \left(\sqrt{3} - \frac{\sqrt{3}}{3} \right) = \frac{5\pi}{9} - \frac{\sqrt{3}}{3} = \frac{5\pi-3\sqrt{3}}{9} \end{aligned}$$

$$50. \quad u = \sin^{-1}(x^2), \quad du = \frac{2x \, dx}{\sqrt{1-x^4}}; \quad dv = 2x \, dx, \quad v = x^2;$$

$$\begin{aligned} \int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) \, dx &= \left[x^2 \sin^{-1}(x^2) \right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \cdot \frac{2x \, dx}{\sqrt{1-x^4}} = \left(\frac{1}{2} \right) \left(\frac{\pi}{6} \right) + \frac{1}{2} \int_0^{1/\sqrt{2}} (1-x^4)^{-1/2} (4x^3) \, dx \\ &= \frac{\pi}{12} + \left[\sqrt{1-x^4} \right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi+6\sqrt{3}-12}{12} \end{aligned}$$

$$51. \quad \int x \tan^{-1} x \, dx \quad \left[\text{Let } u = \tan^{-1} x, \quad du = \frac{1}{1+x^2} dx, \quad dv = x \, dx, \quad v = \frac{x^2}{2} \right]$$

$$\begin{aligned} \int x \tan^{-1} x \, dx &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx \\ &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) \, dx \\ &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C \\ &= \frac{1}{2} (x^2 + 1) \tan^{-1} x - \frac{x}{2} + C \end{aligned}$$

$$52. \quad \int x^2 \tan^{-1} \left(\frac{x}{2} \right) \, dx \quad \left[\text{Let } u = \tan^{-1} \frac{x}{2}, \quad du = \frac{1/2}{1+(x/2)^2} dx, \quad dv = x^2 \, dx, \quad v = \frac{x^3}{3} \right]$$

$$\begin{aligned} \int x^2 \tan^{-1} \left(\frac{x}{2} \right) \, dx &= \frac{x^3}{3} \tan^{-1} \frac{x}{2} - \frac{1}{3} \int \frac{\frac{1}{2} x^3}{1 + \frac{x^2}{4}} \, dx \\ &= \frac{x^3}{3} \tan^{-1} \frac{x}{2} - \frac{1}{3} \int \left(2x - \frac{2x}{1 + \frac{x^2}{4}} \right) \, dx \\ &= \frac{x^3}{3} \tan^{-1} \frac{x}{2} - \frac{1}{3} x^2 + \frac{1}{3} \int \left(\frac{2x}{1 + \frac{x^2}{4}} \right) \, dx \end{aligned}$$

In the remaining integral, let $w = 1 + \frac{x^2}{4}$, $dw = \frac{x}{2} dx$.

$$\frac{1}{3} \int \left(\frac{2x}{1 + \frac{x^2}{4}} \right) dx = \frac{1}{3} \int \frac{4}{w} dw = \frac{4}{3} \ln |w| = \frac{4}{3} \ln \left(1 + \frac{x^2}{4} \right)$$

Thus the original integral is equal to

$$\frac{x^3}{3} \tan^{-1} \frac{x}{2} - \frac{1}{3} x^2 + \frac{4}{3} \ln \left(1 + \frac{x^2}{4} \right) + C$$

53. (a) $u = x, du = dx; dv = \sin x \, dx, v = -\cos x;$

$$S_1 = \int_0^\pi x \sin x \, dx = [-x \cos x]_0^\pi + \int_0^\pi \cos x \, dx = \pi + [\sin x]_0^\pi = \pi$$

$$(b) \quad S_2 = -\int_0^{2\pi} x \sin x \, dx = -\left[[-x \cos x]_\pi^{2\pi} + \int_\pi^{2\pi} \cos x \, dx \right] = -\left[-3\pi + [\sin x]_\pi^{2\pi} \right] = 3\pi$$

$$(c) \quad S_3 = \int_{2\pi}^{3\pi} x \sin x \, dx = [-x \cos x]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} \cos x \, dx = 5\pi + [\sin x]_{2\pi}^{3\pi} = 5\pi$$

$$(d) \quad S_{n+1} = (-1)^{n+1} \int_{n\pi}^{(n+1)\pi} x \sin x \, dx = (-1)^{n+1} \left[[-x \cos x]_{n\pi}^{(n+1)\pi} + [\sin x]_{n\pi}^{(n+1)\pi} \right]$$

$$= (-1)^{n+1} \left[-(n+1)\pi(-1)^n + n\pi(-1)^{n+1} \right] + 0 = (2n+1)\pi$$

54. (a) $u = x, du = dx; dv = \cos x \, dx, v = \sin x;$

$$S_1 = -\int_{\pi/2}^{3\pi/2} x \cos x \, dx = -\left[[x \sin x]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin x \, dx \right] = -\left(-\frac{3\pi}{2} - \frac{\pi}{2} \right) - [\cos x]_{\pi/2}^{3\pi/2} = 2\pi$$

$$(b) \quad S_2 = \int_{3\pi/2}^{5\pi/2} x \cos x \, dx = [x \sin x]_{3\pi/2}^{5\pi/2} - \int_{3\pi/2}^{5\pi/2} \sin x \, dx = \left[\frac{5\pi}{2} - \left(-\frac{3\pi}{2} \right) \right] - [\cos x]_{3\pi/2}^{5\pi/2} = 4\pi$$

$$(c) \quad S_3 = -\int_{5\pi/2}^{7\pi/2} x \cos x \, dx = -\left[[x \sin x]_{5\pi/2}^{7\pi/2} - \int_{5\pi/2}^{7\pi/2} \sin x \, dx \right] = -\left(-\frac{7\pi}{2} - \frac{5\pi}{2} \right) - [\cos x]_{5\pi/2}^{7\pi/2} = 6\pi$$

$$(d) \quad S_n = (-1)^n \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} x \cos x \, dx = (-1)^n \left[[x \sin x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} - \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \sin x \, dx \right]$$

$$= (-1)^n \left[\frac{(2n+1)\pi}{2} (-1)^n - \frac{(2n-1)\pi}{2} (-1)^{n-1} \right] - [\cos x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} = \frac{1}{2} (2n\pi + \pi + 2n\pi - \pi) = 2n\pi$$

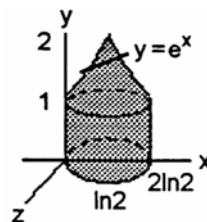
55. $V = \int_0^{\ln 2} 2\pi (\ln 2 - x) e^x \, dx$

$$= 2\pi \ln 2 \int_0^{\ln 2} e^x \, dx - 2\pi \int_0^{\ln 2} x e^x \, dx$$

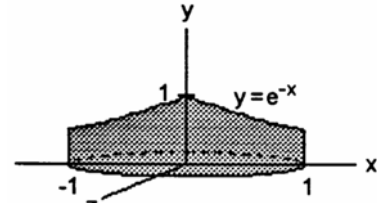
$$= (2\pi \ln 2) \left[e^x \right]_0^{\ln 2} - 2\pi \left(\left[x e^x \right]_0^{\ln 2} - \int_0^{\ln 2} e^x \, dx \right)$$

$$= 2\pi \ln 2 - 2\pi \left(2 \ln 2 - \left[e^x \right]_0^{\ln 2} \right) = -2\pi \ln 2 + 2\pi$$

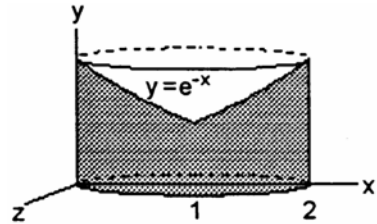
$$= 2\pi(1 - \ln 2)$$



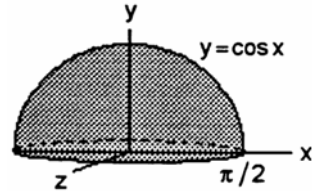
$$\begin{aligned}
 56. \quad (a) \quad V &= \int_0^1 2\pi x e^{-x} dx = 2\pi \left(\left[-x e^{-x} \right]_0^1 + \int_0^1 e^{-x} dx \right) \\
 &= 2\pi \left(-\frac{1}{e} + \left[-e^{-x} \right]_0^1 \right) = 2\pi \left(-\frac{1}{e} - \frac{1}{e} + 1 \right) = 2\pi - \frac{4\pi}{e}
 \end{aligned}$$



$$\begin{aligned}
 (b) \quad V &= \int_0^1 2\pi(1-x)e^{-x} dx; \\
 u &= 1-x, du = -dx; dv = e^{-x} dx, v = -e^{-x}; \\
 V &= 2\pi \left[\left[(1-x)(-e^{-x}) \right]_0^1 - \int_0^1 e^{-x} dx \right] \\
 &= 2\pi \left[[0 - 1(-1)] + \left[e^{-x} \right]_0^1 \right] = 2\pi \left(1 + \frac{1}{e} - 1 \right) = \frac{2\pi}{e}
 \end{aligned}$$



$$\begin{aligned}
 57. \quad (a) \quad V &= \int_0^{\pi/2} 2\pi x \cos x dx = 2\pi \left(\left[x \sin x \right]_0^{\pi/2} - \int_0^{\pi/2} \sin x dx \right) \\
 &= 2\pi \left(\frac{\pi}{2} + \left[\cos x \right]_0^{\pi/2} \right) = 2\pi \left(\frac{\pi}{2} + 0 - 1 \right) = \pi(\pi - 2)
 \end{aligned}$$



$$\begin{aligned}
 (b) \quad V &= \int_0^{\pi/2} 2\pi \left(\frac{\pi}{2} - x \right) \cos x dx; \quad u = \frac{\pi}{2} - x, du = -dx; dv = \cos x dx, v = \sin x; \\
 V &= 2\pi \left[\left(\frac{\pi}{2} - x \right) \sin x \right]_0^{\pi/2} + 2\pi \int_0^{\pi/2} \sin x dx = 0 + 2\pi \left[-\cos x \right]_0^{\pi/2} = 2\pi(0 + 1) = 2\pi
 \end{aligned}$$

$$58. \quad (a) \quad V = \int_0^{\pi} 2\pi x(x \sin x) dx;$$

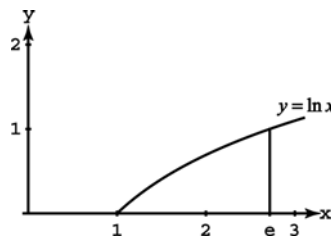
$$\begin{array}{rcl}
 & \sin x & \\
 x^2 & \xrightarrow{(+)} & -\cos x \\
 2x & \xrightarrow{(-)} & -\sin x \\
 2 & \xrightarrow{(+)} & \cos x \\
 0 & &
 \end{array}$$

$$0 \Rightarrow V = 2\pi \int_0^{\pi} x^2 \sin x dx = 2\pi \left[-x^2 \cos x + 2x \sin x + 2 \cos x \right]_0^{\pi} = 2\pi(\pi^2 - 4)$$

$$\begin{aligned}
 (b) \quad V &= \int_0^{\pi} 2\pi(\pi - x)x \sin x dx = 2\pi^2 \int_0^{\pi} x \sin x dx - 2\pi \int_0^{\pi} x^2 \sin x dx = 2\pi^2 \left[-x \cos x + \sin x \right]_0^{\pi} - (2\pi^3 - 8\pi) \\
 &= 8\pi
 \end{aligned}$$

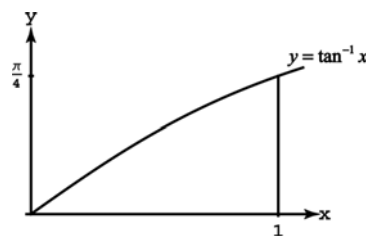
$$\begin{aligned}
 59. \quad (a) \quad A &= \int_1^e \ln x dx = \left[x \ln x \right]_1^e - \int_1^e dx \\
 &= (e \ln e - 1 \ln 1) - [x]_1^e = e - (e - 1) = 1
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad V &= \int_1^e \pi(\ln x)^2 dx = \pi \left(\left[x(\ln x)^2 \right]_1^e - \int_1^e 2 \ln x dx \right) \\
 &= \pi \left(\left(e(\ln e)^2 - 1(\ln 1)^2 \right) - \left([2x \ln x]_1^e - \int_1^e 2 dx \right) \right) \\
 &= \pi \left(e - \left((2e \ln e - 2(1) \ln 1) - [2x]_1^e \right) \right) = \pi \left(e - (2e - (2e - 2)) \right) = \pi(e - 2)
 \end{aligned}$$



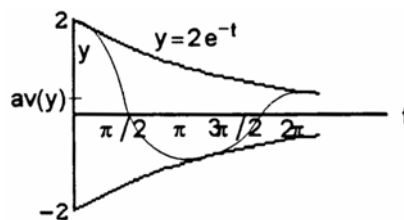
$$\begin{aligned}
 \text{(c)} \quad V &= \int_1^e 2\pi(x+2) \ln x \, dx = 2\pi \int_1^e (x+2) \ln x \, dx = 2\pi \left(\left[\left(\frac{1}{2}x^2 + 2x \right) \ln x \right]_1^e - \int_1^e \left(\frac{1}{2}x + 2 \right) dx \right) \\
 &= 2\pi \left(\left(\frac{1}{2}e^2 + 2e \right) \ln e - \left(\frac{1}{2} + 2 \right) \ln 1 - \left[\left(\frac{1}{4}x^2 + 2x \right) \right]_1^e \right) = 2\pi \left(\left(\frac{1}{2}e^2 + 2e \right) - \left(\left(\frac{1}{4}e^2 + 2e \right) - \frac{9}{4} \right) \right) = \frac{\pi}{2}(e^2 + 9) \\
 \text{(d)} \quad M &= \int_1^e \ln x \, dx = 1 \text{ (from part (a))}; \quad \bar{x} = \frac{1}{1} \int_1^e x \ln x \, dx = \left[\frac{1}{2}x^2 \ln x \right]_1^e - \int_1^e \frac{1}{2}x \, dx \\
 &= \left(\frac{1}{2}e^2 \ln e - \frac{1}{2}(1)^2 \ln 1 \right) - \left[\frac{1}{4}x^2 \right]_1^e = \frac{1}{2}e^2 - \left(\frac{1}{4}e^2 - \frac{1}{4}(1)^2 \right) = \frac{1}{4}(e^2 + 1); \\
 \bar{y} &= \frac{1}{1} \int_1^e \frac{1}{2}(\ln x)^2 \, dx = \frac{1}{2} \left(\left[x(\ln x)^2 \right]_1^e - \int_1^e 2 \ln x \, dx \right) = \frac{1}{2} \left(\left(e(\ln e)^2 - 1 \cdot (\ln 1)^2 \right) - \left([2x \ln x]_1^e - \int_1^e 2 \, dx \right) \right) \\
 &= \frac{1}{2} \left(e - \left(2e \ln e - 2(1) \ln 1 \right) - [2x]_1^e \right) = \frac{1}{2}(e - 2e + 2e - 2) = \frac{1}{2}(e - 2) \Rightarrow (\bar{x}, \bar{y}) = \left(\frac{e^2+1}{4}, \frac{e-2}{2} \right) \text{ is the centroid.}
 \end{aligned}$$

$$\begin{aligned}
 60. \text{ (a)} \quad A &= \int_0^1 \tan^{-1} x \, dx = \left[x \tan^{-1} x \right]_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx \\
 &= \left(\tan^{-1} 1 - 0 \right) - \frac{1}{2} \left[\ln(1+x^2) \right]_0^1 \\
 &= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \ln 1) = \frac{\pi}{4} - \frac{1}{2} \ln 2
 \end{aligned}$$

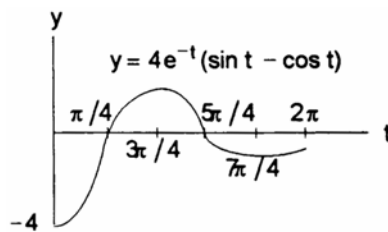


$$\begin{aligned}
 \text{(b)} \quad V &= \int_0^1 2\pi x \tan^{-1} x \, dx = 2\pi \left(\left[\frac{x^2}{2} \tan^{-1} x \right]_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} \, dx \right) = 2\pi \left(\frac{1}{2} \tan^{-1} 1 - 0 - \frac{1}{2} \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx \right) \\
 &= 2\pi \left(\frac{\pi}{8} - \frac{1}{2} \left[x - \tan^{-1} x \right]_0^1 \right) = 2\pi \left(\frac{\pi}{8} - \frac{1}{2} (1 - \tan^{-1} 1 - (0 - 0)) \right) = 2\pi \left(\frac{\pi}{8} - \frac{1}{2} \left(1 - \frac{\pi}{4} \right) \right) = \frac{\pi(\pi-2)}{2}
 \end{aligned}$$

$$\begin{aligned}
 61. \quad av(y) &= \frac{1}{2\pi} \int_0^{2\pi} 2e^{-t} \cos t \, dt = \frac{1}{\pi} \left[e^{-t} \left(\frac{\sin t - \cos t}{2} \right) \right]_0^{2\pi} \\
 \text{(see Exercise 22)} &\Rightarrow av(y) = \frac{1}{2\pi} (1 - e^{-2\pi})
 \end{aligned}$$



$$\begin{aligned}
 62. \quad av(y) &= \frac{1}{2\pi} \int_0^{2\pi} 4e^{-t} (\sin t - \cos t) \, dt \\
 &= \frac{2}{\pi} \int_0^{2\pi} e^{-t} \sin t \, dt - \frac{2}{\pi} \int_0^{2\pi} e^{-t} \cos t \, dt \\
 &= \frac{2}{\pi} \left[e^{-t} \left(\frac{-\sin t - \cos t}{2} \right) - e^{-t} \left(\frac{\sin t - \cos t}{2} \right) \right]_0^{2\pi} \\
 &= \frac{2}{\pi} \left[-e^{-t} \sin t \right]_0^{2\pi} = 0
 \end{aligned}$$



$$\begin{aligned}
 63. \quad I &= \int x^n \cos x \, dx; [u = x^n, du = nx^{n-1} \, dx; dv = \cos x \, dx, v = \sin x] \\
 &\Rightarrow I = x^n \sin x - \int nx^{n-1} \sin x \, dx
 \end{aligned}$$

$$64. \quad I = \int x^n \sin x \, dx; [u = x^n, du = nx^{n-1} \, dx; dv = \sin x \, dx, v = -\cos x] \\ \Rightarrow I = -x^n \cos x + \int nx^{n-1} \cos x \, dx$$

$$65. \quad I = \int x^n e^{ax} \, dx; [u = x^n, du = nx^{n-1} \, dx; dv = e^{ax} \, dx, v = \frac{1}{a} e^{ax}] \\ \Rightarrow I = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, a \neq 0$$

$$66. \quad I = \int (\ln x)^n \, dx; [u = (\ln x)^n, du = \frac{n(\ln x)^{n-1}}{x} \, dx; dv = 1 \, dx, v = x] \\ \Rightarrow I = x(\ln x)^n - \int n(\ln x)^{n-1} \, dx$$

$$67. \quad u = (\ln x)^n, du = \frac{n}{x} (\ln x)^{n-1} \, dx, dv = x^m \, dx, v = \frac{x^{m+1}}{m+1} \\ uv = \frac{1}{m+1} x^{m+1} (\ln x)^{n-1} \text{ and } \int v \, du = \frac{n}{m+1} \int x^m (\ln x)^{n-1} \, dx$$

$$68. \quad \text{First to show that } \int_0^{\pi/2} \cos^n x \, dx = \int_0^{\pi/2} \sin^n x \, dx \text{ note that } \cos x \text{ over the interval } [0, \pi/2] \text{ is the reflection of } \sin x \text{ over the same interval around the line } x = \pi/4.$$

Each iteration of the reduction formula in Example 5 for the definite integral produces an expression like

$$\left[\frac{(\cos^{n-1} x)(\sin x)}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x \, dx$$

The evaluation on the left will be 0 as long as $n \geq 2$, and factors of the form $\frac{n}{n-1}$ accumulate in front of the integral on the right. When the initial n is even, the last iteration will have $n = 2$ and the remaining integral will be $\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n}$. When the initial n is odd the last iteration will have $n = 3$ and the remaining integral will be $\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \int_0^{\pi/2} \cos x \, dx = 1 \cdot \frac{2 \cdot 4 \cdots (n-1)}{3 \cdots n}$.

$$69. \quad \int_a^b (x-a) f(x) \, dx; \left[u = x-a, du = dx; dv = f(x) \, dx, v = \int_b^x f(t) \, dt = -\int_x^b f(t) \, dt \right] \\ = \left[(x-a) \int_b^x f(t) \, dt \right]_a^b - \int_a^b \left(\int_b^x f(t) \, dt \right) dx = \left((b-a) \int_b^b f(t) \, dt - (a-a) \int_b^a f(t) \, dt \right) - \int_a^b \left(-\int_x^b f(t) \, dt \right) dx \\ = 0 + \int_a^b \left(\int_x^b f(t) \, dt \right) dx = \int_a^b \left(\int_x^b f(t) \, dt \right) dx$$

$$70. \quad \int \sqrt{1-x^2} \, dx; \left[u = \sqrt{1-x^2}, du = \frac{-x}{\sqrt{1-x^2}} \, dx; dv = dx, v = x \right] \\ = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}} \, dx = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}} \, dx = x\sqrt{1-x^2} - \left(\int \frac{1-x^2}{\sqrt{1-x^2}} \, dx - \int \frac{1}{\sqrt{1-x^2}} \, dx \right) \\ = x\sqrt{1-x^2} - \int \sqrt{1-x^2} \, dx + \int \frac{1}{\sqrt{1-x^2}} \, dx$$

$$\begin{aligned}\Rightarrow \int \sqrt{1-x^2} \, dx &= x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}} \, dx - \int \sqrt{1-x^2} \, dx \Rightarrow 2 \int \sqrt{1-x^2} \, dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}} \, dx \\ \Rightarrow \int \sqrt{1-x^2} \, dx &= \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \, dx + C\end{aligned}$$

$$71. \int \sin^{-1} x \, dx = x \sin^{-1} x - \int \sin y \, dy = x \sin^{-1} x + \cos y + C = x \sin^{-1} x + \cos(\sin^{-1} x) + C$$

$$72. \int \tan^{-1} x \, dx = x \tan^{-1} x - \int \tan y \, dy = x \tan^{-1} x + \ln |\cos y| + C = x \tan^{-1} x + \ln |\cos(\tan^{-1} x)| + C$$

$$\begin{aligned}73. \int \sec^{-1} x \, dx &= x \sec^{-1} x - \int \sec y \, dy = x \sec^{-1} x - \ln |\sec y + \tan y| + C \\ &= x \sec^{-1} x - \ln \left| \sec(\sec^{-1} x) + \tan(\sec^{-1} x) \right| + C = x \sec^{-1} x - \ln \left| x + \sqrt{x^2 - 1} \right| + C\end{aligned}$$

$$74. \int \log_2 x \, dx = x \log_2 x - \int 2^y \, dy = x \log_2 x - \frac{2^y}{\ln 2} + C = x \log_2 x - \frac{x}{\ln 2} + C$$

$$75. \text{ Yes, } \cos^{-1} x \text{ is the angle whose cosine is } x \text{ which implies } \sin(\cos^{-1} x) = \sqrt{1-x^2}.$$

$$76. \text{ Yes, } \tan^{-1} x \text{ is the angle whose tangent is } x \text{ which implies } \sec(\tan^{-1} x) = \sqrt{1+x^2}.$$

$$77. (a) \int \sinh^{-1} x \, dx = x \sinh^{-1} x - \int \sinh y \, dy = x \sinh^{-1} x - \cosh y + C = x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C;$$

$$\text{check: } d \left[x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C \right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \sinh(\sinh^{-1} x) \frac{1}{\sqrt{1+x^2}} \right] dx = \sinh^{-1} x \, dx$$

$$(b) \int \sinh^{-1} x \, dx = x \sinh^{-1} x - \int x \left(\frac{1}{\sqrt{1+x^2}} \right) dx = x \sinh^{-1} x - \frac{1}{2} \int (1+x^2)^{-1/2} 2x \, dx = x \sinh^{-1} x - (1+x^2)^{1/2} + C$$

$$\text{check: } d \left[x \sinh^{-1} x - (1+x^2)^{1/2} + C \right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}} \right] dx = \sinh^{-1} x \, dx$$

$$78. (a) \int \tanh^{-1} x \, dx = x \tanh^{-1} x - \int \tanh y \, dy = x \tanh^{-1} x - \ln |\cosh y| + C$$

$$= x \tanh^{-1} x - \ln |\cosh(\tanh^{-1} x)| + C;$$

$$\text{check: } d \left[x \tanh^{-1} x - \ln |\cosh(\tanh^{-1} x)| + C \right] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{\sinh(\tanh^{-1} x)}{\cosh(\tanh^{-1} x)} \frac{1}{1-x^2} \right] dx$$

$$= \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2} \right] dx = \tanh^{-1} x \, dx$$

$$(b) \int \tanh^{-1} x \, dx = x \tanh^{-1} x - \int \frac{x}{1-x^2} \, dx = x \tanh^{-1} x - \frac{1}{2} \int \frac{2x}{1-x^2} \, dx = x \tanh^{-1} x + \frac{1}{2} \ln |1-x^2| + C$$

$$\text{check: } d \left[x \tanh^{-1} x + \frac{1}{2} \ln |1-x^2| + C \right] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2} \right] dx = \tanh^{-1} x \, dx$$

8.3 TRIGONOMETRIC INTEGRALS

1. $\int \cos 2x \, dx = \frac{1}{2} \int \cos 2x \cdot 2 \, dx = \frac{1}{2} \sin 2x + C$
2. $\int_0^{\pi} 3 \sin \frac{x}{3} \, dx = 9 \int_0^{\pi} \sin \frac{x}{3} \cdot \frac{1}{3} \, dx = 9 \left[-\cos \frac{x}{3} \right]_0^{\pi} = 9 \left(-\cos \frac{\pi}{3} + \cos 0 \right) = 9 \left(-\frac{1}{2} + 1 \right) = \frac{9}{2}$
3. $\int \cos^3 x \sin x \, dx = -\int \cos^3 x (-\sin x) \, dx = -\frac{1}{4} \cos^4 x + C$
4. $\int \sin^4 2x \cos 2x \, dx = \frac{1}{2} \int \sin^4 2x \cos 2x \cdot 2 \, dx = \frac{1}{10} \sin^5 2x + C$
5. $\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx = \int \sin x \, dx - \int \cos^2 x \sin x \, dx = -\cos x + \frac{1}{3} \cos^3 x + C$
6. $\int \cos^3 4x \, dx = \int \cos^2 4x \cos 4x \, dx = \frac{1}{4} \int (1 - \sin^2 4x) \cos 4x \cdot 4 \, dx = \frac{1}{4} \int \cos 4x \cdot 4 \, dx - \frac{1}{4} \int \sin^2 4x \cos 4x \cdot 4 \, dx$
 $= \frac{1}{4} \sin 4x - \frac{1}{12} \sin^3 4x + C$
7. $\int \sin^5 x \, dx = \int (\sin^2 x)^2 \sin x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx = \int (1 - 2\cos^2 x + \cos^4 x) \sin x \, dx$
 $= \int \sin x \, dx - \int 2\cos^2 x \sin x \, dx + \int \cos^4 x \sin x \, dx = -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$
8. $\int_0^{\pi} \sin^5 \left(\frac{x}{2} \right) dx$ (using Exercise 7) $= \int_0^{\pi} \sin \left(\frac{x}{2} \right) dx - \int_0^{\pi} 2 \cos^2 \left(\frac{x}{2} \right) \sin \left(\frac{x}{2} \right) dx + \int_0^{\pi} \cos^4 \left(\frac{x}{2} \right) \sin \left(\frac{x}{2} \right) dx$
 $= \left[-2 \cos \left(\frac{x}{2} \right) + \frac{4}{3} \cos^3 \left(\frac{x}{2} \right) - \frac{2}{5} \cos^5 \left(\frac{x}{2} \right) \right]_0^{\pi} = (0) - \left(-2 + \frac{4}{3} - \frac{2}{5} \right) = \frac{16}{15}$
9. $\int \cos^3 x \, dx = \int (\cos^2 x) \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx = \int \cos x \, dx - \int \sin^2 x \cos x \, dx = \sin x - \frac{1}{3} \sin^3 x + C$
10. $\int_0^{\pi/6} 3 \cos^5 3x \, dx = \int_0^{\pi/6} (\cos^2 3x)^2 \cos 3x \cdot 3 \, dx = \int_0^{\pi/6} (1 - \sin^2 3x)^2 \cos 3x \cdot 3 \, dx$
 $= \int_0^{\pi/6} (1 - 2\sin^2 3x + \sin^4 3x) \cos 3x \cdot 3 \, dx$
 $= \int_0^{\pi/6} \cos 3x \cdot 3 \, dx - 2 \int_0^{\pi/6} \sin^2 3x \cos 3x \cdot 3 \, dx + \int_0^{\pi/6} \sin^4 3x \cos 3x \cdot 3 \, dx = \left[\sin 3x - 2 \frac{\sin^3 3x}{3} + \frac{\sin^5 3x}{5} \right]_0^{\pi/6}$
 $= \left(1 - \frac{2}{3} + \frac{1}{5} \right) - (0) = \frac{8}{15}$
11. $\int \sin^3 x \cos^3 x \, dx = \int \sin^3 x \cos^2 x \cos x \, dx = \int \sin^3 x (1 - \sin^2 x) \cos x \, dx = \int \sin^3 x \cos x \, dx - \int \sin^5 x \cos x \, dx$
 $= \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C$

12. $\int \cos^3 2x \sin^5 2x \, dx = \frac{1}{2} \int \cos^3 2x \sin^5 2x \cdot 2 \, dx = \frac{1}{2} \int \cos 2x \cos^2 2x \sin^5 2x \cdot 2 \, dx$
 $= \frac{1}{2} \int (1 - \sin^2 2x) \sin^5 2x \cos 2x \cdot 2 \, dx = \frac{1}{2} \int \sin^5 2x \cos 2x \cdot 2 \, dx - \frac{1}{2} \int \sin^7 2x \cos 2x \cdot 2 \, dx$
 $= \frac{1}{12} \sin^6 2x - \frac{1}{16} \sin^8 2x + C$
13. $\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx = \frac{1}{2} \int (1 + \cos 2x) \, dx = \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x \, dx = \frac{1}{2} \int dx + \frac{1}{4} \int \cos 2x \cdot 2 \, dx$
 $= \frac{1}{2} x + \frac{1}{4} \sin 2x + C$
14. $\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \frac{1 - \cos 2x}{2} \, dx = \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx = \frac{1}{2} \int_0^{\pi/2} dx - \frac{1}{2} \int_0^{\pi/2} \cos 2x \, dx$
 $= \frac{1}{2} \int_0^{\pi/2} dx - \frac{1}{4} \int_0^{\pi/2} \cos 2x \cdot 2 \, dx = \left[\frac{1}{2} x - \frac{1}{4} \sin 2x \right]_0^{\pi/2} = \left(\frac{1}{2} \left(\frac{\pi}{2} \right) - \frac{1}{4} \sin 2 \left(\frac{\pi}{2} \right) \right) - \left(\frac{1}{2} (0) - \frac{1}{4} \sin 2(0) \right)$
 $= \left(\frac{\pi}{4} - 0 \right) - (0 - 0) = \frac{\pi}{4}$
15. $\int_0^{\pi/2} \sin^7 y \, dy = \int_0^{\pi/2} \sin^6 y \sin y \, dy = \int_0^{\pi/2} (1 - \cos^2 y)^3 \sin y \, dy$
 $= \int_0^{\pi/2} \sin y \, dy - 3 \int_0^{\pi/2} \cos^2 y \sin y \, dy + 3 \int_0^{\pi/2} \cos^4 y \sin y \, dy - \int_0^{\pi/2} \cos^6 y \sin y \, dy$
 $= \left[-\cos y + 3 \frac{\cos^3 y}{3} - 3 \frac{\cos^5 y}{5} + \frac{\cos^7 y}{7} \right]_0^{\pi/2} = (0) - \left(-1 + 1 - \frac{3}{5} + \frac{1}{7} \right) = \frac{16}{35}$
16. $\int 7 \cos^7 t \, dt$ (using Exercise 15) $= 7 \left[\int \cos t \, dt - 3 \int \sin^2 t \cos t \, dt + 3 \int \sin^4 t \cos t \, dt - \int \sin^6 t \cos t \, dt \right]$
 $= 7 \left(\sin t - 3 \frac{\sin^3 t}{3} + 3 \frac{\sin^5 t}{5} - \frac{\sin^7 t}{7} \right) + C = 7 \sin t - 7 \sin^3 t + \frac{21}{5} \sin^5 t - \sin^7 t + C$
17. $\int_0^{\pi} 8 \sin^4 x \, dx = 8 \int_0^{\pi} \left(\frac{1 - \cos 2x}{2} \right)^2 \, dx = 2 \int_0^{\pi} (1 - 2 \cos 2x + \cos^2 2x) \, dx$
 $= 2 \int_0^{\pi} dx - 2 \int_0^{\pi} \cos 2x \cdot 2 \, dx + 2 \int_0^{\pi} \frac{1 + \cos 4x}{2} \, dx = [2x - 2 \sin 2x]_0^{\pi} + \int_0^{\pi} dx + \int_0^{\pi} \cos 4x \, dx$
 $= 2\pi + \left[x + \frac{1}{4} \sin 4x \right]_0^{\pi} = 2\pi + \pi = 3\pi$
18. $\int 8 \cos^4 2\pi x \, dx = 8 \int \left(\frac{1 + \cos 4\pi x}{2} \right)^2 \, dx = 2 \int (1 + 2 \cos 4\pi x + \cos^2 4\pi x) \, dx = 2 \int dx + 4 \int \cos 4\pi x \, dx + 2 \int \frac{1 + \cos 8\pi x}{2} \, dx$
 $= 3 \int dx + 4 \int \cos 4\pi x \, dx + \int \cos 8\pi x \, dx = 3x + \frac{1}{\pi} \sin 4\pi x + \frac{1}{8\pi} \sin 8\pi x + C$
19. $\int 16 \sin^2 x \cos^2 x \, dx = 16 \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right) \, dx = 4 \int (1 - \cos^2 2x) \, dx = 4 \int dx - 4 \int \left(\frac{1 + \cos 4x}{2} \right) \, dx$
 $= 4x - 2 \int dx - 2 \int \cos 4x \, dx = 4x - 2x - \frac{1}{2} \sin 4x + C = 2x - \frac{1}{2} \sin 4x + C = 2x - \sin 2x \cos 2x + C$
 $= 2x - 2 \sin x \cos x (2 \cos^2 x - 1) + C = 2x - 4 \sin x \cos^3 x + 2 \sin x \cos x + C$

20.
$$\begin{aligned}\int_0^\pi 8 \sin^4 y \cos^2 y \, dy &= 8 \int_0^\pi \left(\frac{1-\cos 2y}{2} \right)^2 \left(\frac{1+\cos 2y}{2} \right) dy = \int_0^\pi dy - \int_0^\pi \cos 2y \, dy - \int_0^\pi \cos^2 2y \, dy + \int_0^\pi \cos^3 2y \, dy \\ &= \left[y - \frac{1}{2} \sin 2y \right]_0^\pi - \int_0^\pi \left(\frac{1+\cos 4y}{2} \right) dy + \int_0^\pi (1 - \sin^2 2y) \cos 2y \, dy \\ &= \pi - \frac{1}{2} \int_0^\pi dy - \frac{1}{2} \int_0^\pi \cos 4y \, dy + \int_0^\pi \cos 2y \, dy - \int_0^\pi \sin^2 2y \cos 2y \, dy \\ &= \pi + \left[-\frac{1}{2} y - \frac{1}{8} \sin 4y + \frac{1}{2} \sin 2y - \frac{1}{2} \cdot \frac{\sin^3 2y}{3} \right]_0^\pi = \pi - \frac{\pi}{2} = \frac{\pi}{2}\end{aligned}$$
21.
$$\int 8 \cos^3 2\theta \sin 2\theta \, d\theta = 8 \left(-\frac{1}{2} \right) \frac{\cos^4 2\theta}{4} + C = -\cos^4 2\theta + C$$
22.
$$\begin{aligned}\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta &= \int_0^{\pi/2} \sin^2 2\theta (1 - \sin^2 2\theta) \cos 2\theta \, d\theta \\ &= \int_0^{\pi/2} \sin^2 2\theta \cos 2\theta \, d\theta - \int_0^{\pi/2} \sin^4 2\theta \cos 2\theta \, d\theta = \left[\frac{1}{2} \cdot \frac{\sin^3 2\theta}{3} - \frac{1}{2} \cdot \frac{\sin^5 2\theta}{5} \right]_0^{\pi/2} = 0\end{aligned}$$
23.
$$\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}} \, dx = \int_0^{2\pi} \left| \sin \frac{x}{2} \right| dx = \int_0^{2\pi} \sin \frac{x}{2} \, dx = \left[-2 \cos \frac{x}{2} \right]_0^{2\pi} = 2 + 2 = 4$$
24.
$$\int_0^\pi \sqrt{1-\cos 2x} \, dx = \int_0^\pi \sqrt{2} |\sin x| \, dx = \int_0^\pi \sqrt{2} \sin x \, dx = \left[-\sqrt{2} \cos x \right]_0^\pi = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$$
25.
$$\int_0^\pi \sqrt{1-\sin^2 t} \, dt = \int_0^\pi |\cos t| \, dt = \int_0^{\pi/2} \cos t \, dt - \int_{\pi/2}^\pi \cos t \, dt = [\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^\pi = 1 - 0 - 0 + 1 = 2$$
26.
$$\int_0^\pi \sqrt{1-\cos^2 \theta} \, d\theta = \int_0^\pi |\sin \theta| \, d\theta = \int_0^\pi \sin \theta \, d\theta = [-\cos \theta]_0^\pi = 1 + 1 = 2$$
27.
$$\begin{aligned}\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}} \, dx &= \int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}} \frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}} \, dx = \int_{\pi/3}^{\pi/2} \frac{\sin^2 x \sqrt{1+\cos x}}{\sqrt{1-\cos^2 x}} \, dx = \int_{\pi/3}^{\pi/2} \frac{\sin^2 x \sqrt{1+\cos x}}{\sqrt{\sin^2 x}} \, dx \\ &= \int_{\pi/3}^{\pi/2} \sin x \sqrt{1+\cos x} \, dx = \left[-\frac{2}{3} (1+\cos x)^{3/2} \right]_{\pi/3}^{\pi/2} = -\frac{2}{3} \left(1+\cos \left(\frac{\pi}{2} \right) \right)^{3/2} + \frac{2}{3} \left(1+\cos \left(\frac{\pi}{3} \right) \right)^{3/2} = -\frac{2}{3} + \frac{2}{3} \left(\frac{3}{2} \right)^{3/2} \\ &= \sqrt{\frac{3}{2}} - \frac{2}{3}\end{aligned}$$
28.
$$\begin{aligned}\int_0^{\pi/6} \sqrt{1+\sin x} \, dx &= \int_0^{\pi/6} \frac{\sqrt{1+\sin x}}{1} \frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}} \, dx = \int_0^{\pi/6} \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}} \, dx = \int_0^{\pi/6} \frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}} \, dx = \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin x}} \, dx \\ &= \left[-2(1-\sin x)^{1/2} \right]_0^{\pi/6} = -2\sqrt{1-\sin \left(\frac{\pi}{6} \right)} + 2\sqrt{1-\sin 0} = -2\sqrt{\frac{1}{2}} + 2\sqrt{1} = 2 - \sqrt{2}\end{aligned}$$
29.
$$\begin{aligned}\int_{5\pi/6}^\pi \frac{\cos^4 x}{\sqrt{1-\sin x}} \, dx &= \int_{5\pi/6}^\pi \frac{\cos^4 x}{\sqrt{1-\sin x}} \frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}} \, dx = \int_{5\pi/6}^\pi \frac{\cos^4 x \sqrt{1+\sin x}}{\sqrt{1-\sin^2 x}} \, dx = \int_{5\pi/6}^\pi \frac{\cos^4 x \sqrt{1+\sin x}}{\sqrt{\cos^2 x}} \, dx \\ &= \int_{5\pi/6}^\pi \frac{\cos^4 x \sqrt{1+\sin x}}{-\cos x} \, dx = -\int_{5\pi/6}^\pi \cos^3 x \sqrt{1+\sin x} \, dx = -\int_{5\pi/6}^\pi \cos x (1-\sin^2 x) \sqrt{1+\sin x} \, dx \\ &= -\int_{5\pi/6}^\pi \cos x \sqrt{1+\sin x} \, dx + \int_{5\pi/6}^\pi \cos x \sin^2 x \sqrt{1+\sin x} \, dx;\end{aligned}$$

- $$\left[\text{Let } u = 1 + \sin x \Rightarrow u - 1 = \sin x \Rightarrow du = \cos x \, dx, x = \frac{5\pi}{6} \Rightarrow u = 1 + \sin\left(\frac{5\pi}{6}\right) = \frac{3}{2}, x = \pi \Rightarrow u = 1 + \sin \pi = 1 \right]$$
- $$= \left[-\frac{2}{3}(1 + \sin x)^{3/2} \right]_{5\pi/6}^{\pi} + \int_{3/2}^1 (u-1)^2 \sqrt{u} \, du = \left[-\frac{2}{3}(1 + \sin x)^{3/2} \right]_{5\pi/6}^{\pi} + \int_{3/2}^1 (u^{5/2} - 2u^{3/2} + \sqrt{u}) \, du$$
- $$= \left(-\frac{2}{3}(1 + \sin \pi)^{3/2} + \frac{2}{3} \left(1 + \sin\left(\frac{5\pi}{6}\right) \right)^{3/2} \right) + \left[\frac{2}{7} u^{7/2} - \frac{4}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right]_{3/2}^1$$
- $$= \left(-\frac{2}{3} + \frac{2}{3} \left(\frac{3}{2} \right)^{3/2} \right) + \left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3} \right) - \left(\frac{2}{7} \left(\frac{3}{2} \right)^{7/2} - \frac{4}{5} \left(\frac{3}{2} \right)^{5/2} + \frac{2}{3} \left(\frac{3}{2} \right)^{3/2} \right) = \frac{4}{5} \left(\frac{3}{2} \right)^{5/2} - \frac{2}{7} \left(\frac{3}{2} \right)^{7/2} - \frac{18}{35}$$
30. $\int_{\pi/2}^{7\pi/12} \sqrt{1 - \sin 2x} \, dx = \int_{\pi/2}^{7\pi/12} \frac{\sqrt{1 - \sin 2x}}{1} \frac{\sqrt{1 + \sin 2x}}{\sqrt{1 + \sin 2x}} \, dx = \int_{\pi/2}^{7\pi/12} \frac{\sqrt{1 - \sin^2 2x}}{\sqrt{1 + \sin 2x}} \, dx = \int_{\pi/2}^{7\pi/12} \frac{\sqrt{\cos^2 2x}}{\sqrt{1 + \sin 2x}} \, dx$
- $$= \int_{\pi/2}^{7\pi/12} \frac{-\cos 2x}{\sqrt{1 + \sin 2x}} \, dx = \left[-\sqrt{1 + \sin 2x} \right]_{\pi/2}^{7\pi/12} = -\sqrt{1 + \sin 2\left(\frac{7\pi}{12}\right)} + \sqrt{1 + \sin 2\left(\frac{\pi}{2}\right)} = -\sqrt{\frac{1}{2}} + 1 = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$
31. $\int_0^{\pi/2} \theta \sqrt{1 - \cos 2\theta} \, d\theta = \int_0^{\pi/2} \theta \sqrt{2} |\sin \theta| \, d\theta = \sqrt{2} \int_0^{\pi/2} \theta \sin \theta \, d\theta = \sqrt{2} [-\theta \cos \theta + \sin \theta]_0^{\pi/2} = \sqrt{2}(1) = \sqrt{2}$
32. $\int_{-\pi}^{\pi} (1 - \cos^2 t)^{3/2} \, dt = \int_{-\pi}^{\pi} (\sin^2 t)^{3/2} \, dt = \int_{-\pi}^{\pi} |\sin^3 t| \, dt = -\int_{-\pi}^0 \sin^3 t \, dt + \int_0^{\pi} \sin^3 t \, dt$
- $$= -\int_{-\pi}^0 (1 - \cos^2 t) \sin t \, dt + \int_0^{\pi} (1 - \cos^2 t) \sin t \, dt$$
- $$= -\int_{-\pi}^0 \sin t \, dt + \int_{-\pi}^0 \cos^2 t \sin t \, dt + \int_0^{\pi} \sin t \, dt - \int_0^{\pi} \cos^2 t \sin t \, dt$$
- $$= \left[\cos t - \frac{\cos^3 t}{3} \right]_{-\pi}^0 + \left[-\cos t + \frac{\cos^3 t}{3} \right]_0^{\pi} = \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) + \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) = \frac{8}{3}$$
33. $\int \sec^2 x \tan x \, dx = \int \tan x \sec^2 x \, dx = \frac{1}{2} \tan^2 x + C$
34. $\int \sec x \tan^2 x \, dx = \int \sec x \tan x \tan x \, dx; u = \tan x, du = \sec^2 x \, dx, dv = \sec x \tan x \, dx, v = \sec x;$
- $$= \sec x \tan x - \int \sec^3 x \, dx = \sec x \tan x - \int \sec^2 x \sec x \, dx = \sec x \tan x - \int (\tan^2 x + 1) \sec x \, dx$$
- $$= \sec x \tan x - \left(\int \tan^2 x \sec x \, dx + \int \sec x \, dx \right) = \sec x \tan x - \ln |\sec x + \tan x| - \int \tan^2 x \sec x \, dx$$
- $$\Rightarrow \int \sec x \tan^2 x \, dx = \sec x \tan x - \ln |\sec x + \tan x| - \int \tan^2 x \sec x \, dx$$
- $$\Rightarrow 2 \int \tan^2 x \sec x \, dx = \sec x \tan x - \ln |\sec x + \tan x|$$
- $$\Rightarrow \int \tan^2 x \sec x \, dx = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C$$
35. $\int \sec^3 x \tan x \, dx = \int \sec^2 x \sec x \tan x \, dx = \frac{1}{3} \sec^3 x + C$
36. $\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \tan^2 x \sec x \tan x \, dx = \int \sec^2 x (\sec^2 x - 1) \sec x \tan x \, dx$
- $$= \int \sec^4 x \sec x \tan x \, dx - \int \sec^2 x \sec x \tan x \, dx = \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$$
37. $\int \sec^2 x \tan^2 x \, dx = \int \tan^2 x \sec^2 x \, dx = \frac{1}{3} \tan^3 x + C$

38. $\int \sec^4 x \tan^2 x \, dx = \int \sec^2 x \tan^2 x \sec^2 x \, dx = \int (\tan^2 x + 1) \tan^2 x \sec^2 x \, dx$
 $= \int \tan^4 x \sec^2 x \, dx + \int \tan^2 x \sec^2 x \, dx = \frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C$
39. $\int_{-\pi/3}^0 2 \sec^3 x \, dx$; $u = \sec x$, $du = \sec x \tan x \, dx$, $dv = \sec^2 x \, dx$, $v = \tan x$;
 $\int_{-\pi/3}^0 2 \sec^3 x \, dx = [2 \sec x \tan x]_{-\pi/3}^0 - 2 \int_{-\pi/3}^0 \sec x \tan^2 x \, dx = 2 \cdot 1 \cdot 0 - 2 \cdot 2 \cdot (-\sqrt{3}) - 2 \int_{-\pi/3}^0 \sec x (\sec^2 x - 1) \, dx$
 $= 4\sqrt{3} - 2 \int_{-\pi/3}^0 \sec^3 x \, dx + 2 \int_{-\pi/3}^0 \sec x \, dx$;
 $2 \int_{-\pi/3}^0 2 \sec^3 x \, dx = 4\sqrt{3} + [2 \ln |\sec x + \tan x|]_{-\pi/3}^0 \Rightarrow 2 \int_{-\pi/3}^0 2 \sec^3 x \, dx = 4\sqrt{3} + 2 \ln |1 + 0| - 2 \ln |2 - \sqrt{3}|$
 $= 4\sqrt{3} - 2 \ln(2 - \sqrt{3}) \Rightarrow \int_{-\pi/3}^0 2 \sec^3 x \, dx = 2\sqrt{3} - \ln(2 - \sqrt{3})$
40. $\int e^x \sec^3(e^x) \, dx$; $u = \sec(e^x)$, $du = \sec(e^x) \tan(e^x) e^x \, dx$, $dv = \sec^2(e^x) e^x \, dx$, $v = \tan(e^x)$;
 $\int e^x \sec^3(e^x) \, dx = \sec(e^x) \tan(e^x) - \int \sec(e^x) \tan^2(e^x) e^x \, dx$
 $= \sec(e^x) \tan(e^x) - \int \sec(e^x) (\sec^2(e^x) - 1) e^x \, dx$
 $= \sec(e^x) \tan(e^x) - \int \sec^3(e^x) e^x \, dx + \int \sec(e^x) e^x \, dx$
 $2 \int e^x \sec^3(e^x) \, dx = \sec(e^x) \tan(e^x) + \ln |\sec(e^x) + \tan(e^x)| + C$
 $\int e^x \sec^3(e^x) \, dx = \frac{1}{2} [\sec(e^x) \tan(e^x) + \ln |\sec(e^x) + \tan(e^x)|] + C$
41. $\int \sec^4 \theta \, d\theta = \int (1 + \tan^2 \theta) \sec^2 \theta \, d\theta = \int \sec^2 \theta \, d\theta + \int \tan^2 \theta \sec^2 \theta \, d\theta = \tan \theta + \frac{1}{3} \tan^3 \theta + C$
 $= \tan \theta + \frac{1}{3} \tan \theta (\sec^2 \theta - 1) + C = \frac{1}{3} \tan \theta \sec^2 \theta + \frac{2}{3} \tan \theta + C$
42. $\int 3 \sec^4(3x) \, dx = \int (1 + \tan^2(3x)) \sec^2(3x) 3 \, dx = \int \sec^2(3x) 3 \, dx + \int \tan^2(3x) \sec^2(3x) 3 \, dx$
 $= \tan(3x) + \frac{1}{3} \tan^3(3x) + C$
43. $\int_{\pi/4}^{\pi/2} \csc^4 \theta \, d\theta = \int_{\pi/4}^{\pi/2} (1 + \cot^2 \theta) \csc^2 \theta \, d\theta = \int_{\pi/4}^{\pi/2} \csc^2 \theta \, d\theta + \int_{\pi/4}^{\pi/2} \cot^2 \theta \csc^2 \theta \, d\theta = \left[-\cot \theta - \frac{\cot^3 \theta}{3} \right]_{\pi/4}^{\pi/2}$
 $= (0) - \left(-1 - \frac{1}{3} \right) = \frac{4}{3}$
44. $\int \sec^6 x \, dx = \int \sec^4 x \sec^2 x \, dx = \int (\tan^2 x + 1)^2 \sec^2 x \, dx = \int (\tan^4 x + 2 \tan^2 x + 1) \sec^2 x \, dx$
 $= \int \tan^4 x \sec^2 x \, dx + 2 \int \tan^2 x \sec^2 x \, dx + \int \sec^2 x \, dx = \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x + C$
45. $\int 4 \tan^3 x \, dx = 4 \int (\sec^2 x - 1) \tan x \, dx = 4 \int \sec^2 x \tan x \, dx - 4 \int \tan x \, dx = 4 \frac{\tan^2 x}{2} - 4 \ln |\sec x| + C$
 $= 2 \tan^2 x - 4 \ln |\sec x| + C = 2 \tan^2 x - 2 \ln |\sec^2 x| + C = 2 \tan^2 x - 2 \ln (1 + \tan^2 x) + C$

46. $\int_{-\pi/4}^{\pi/4} 6 \tan^4 x \, dx = 6 \int_{-\pi/4}^{\pi/4} (\sec^2 x - 1) \tan^2 x \, dx = 6 \int_{-\pi/4}^{\pi/4} \sec^2 x \tan^2 x \, dx - 6 \int_{-\pi/4}^{\pi/4} \tan^2 x \, dx$
 $= 6 \int_{-\pi/4}^{\pi/4} \sec^2 x \tan^2 x \, dx - 6 \int_{-\pi/4}^{\pi/4} (\sec^2 x - 1) \, dx = \left[6 \frac{\tan^3 x}{3} \right]_{-\pi/4}^{\pi/4} - 6 \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx + 6 \int_{-\pi/4}^{\pi/4} dx$
 $= 2(1 - (-1)) - [6 \tan x]_{-\pi/4}^{\pi/4} + [6x]_{-\pi/4}^{\pi/4} = 4 - 6(1 - (-1)) + \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi - 8$
47. $\int \tan^5 x \, dx = \int \tan^4 x \tan x \, dx = \int (\sec^2 x - 1)^2 \tan x \, dx = \int (\sec^4 x - 2\sec^2 x + 1) \tan x \, dx$
 $= \int \sec^4 x \tan x \, dx - 2 \int \sec^2 x \tan x \, dx + \int \tan x \, dx$
 $= \int \sec^3 x \sec x \tan x \, dx - 2 \int \sec x \sec x \tan x \, dx + \int \tan x \, dx = \frac{1}{4} \sec^4 x - \sec^2 x + \ln |\sec x| + C$
 $= \frac{1}{4} (\tan^2 x + 1)^2 - (\tan^2 x + 1) + \ln |\sec x| + C = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \ln |\sec x| + C$
48. $\int \cot^6 2x \, dx = \int \cot^4 2x \cot^2 2x \, dx = \int \cot^4 2x (\csc^2 2x - 1) \, dx = \int \cot^4 2x \csc^2 2x \, dx - \int \cot^4 2x \, dx$
 $= \int \cot^4 2x \csc^2 2x \, dx - \int \cot^2 2x \cot^2 2x \, dx = \int \cot^4 2x \csc^2 2x \, dx - \int \cot^2 2x (\csc^2 2x - 1) \, dx$
 $= \int \cot^4 2x \csc^2 2x \, dx - \int \cot^2 2x \csc^2 2x \, dx + \int \cot^2 2x \, dx$
 $= \int \cot^4 2x \csc^2 2x \, dx - \int \cot^2 2x \csc^2 2x \, dx + \int (\csc^2 2x - 1) \, dx$
 $= \int \cot^4 2x \csc^2 2x \, dx - \int \cot^2 2x \csc^2 2x \, dx + \int \csc^2 2x \, dx - \int dx = -\frac{1}{10} \cot^5 2x + \frac{1}{6} \cot^3 2x - \frac{1}{2} \cot 2x - x + C$
49. $\int_{\pi/6}^{\pi/3} \cot^3 x \, dx = \int_{\pi/6}^{\pi/3} (\csc^2 x - 1) \cot x \, dx = \int_{\pi/6}^{\pi/3} \csc^2 x \cot x \, dx - \int_{\pi/6}^{\pi/3} \cot x \, dx = \left[-\frac{\cot^2 x}{2} + \ln |\csc x| \right]_{\pi/6}^{\pi/3}$
 $= -\frac{1}{2} \left(\frac{1}{3} - 3 \right) + \left(\ln \frac{2}{\sqrt{3}} - \ln 2 \right) = \frac{4}{3} - \ln \sqrt{3}$
50. $\int 8 \cot^4 t \, dt = 8 \int (\csc^2 t - 1) \cot^2 t \, dt = 8 \int \csc^2 t \cot^2 t \, dt - 8 \int \cot^2 t \, dt = -\frac{8}{3} \cot^3 t - 8 \int (\csc^2 t - 1) \, dt$
 $= -\frac{8}{3} \cot^3 t + 8 \cot t + 8t + C$
51. $\int \sin 3x \cos 2x \, dx = \frac{1}{2} \int (\sin x + \sin 5x) \, dx = -\frac{1}{2} \cos x - \frac{1}{10} \cos 5x + C$
52. $\int \sin 2x \cos 3x \, dx = \frac{1}{2} \int (\sin(-x) + \sin 5x) \, dx = \frac{1}{2} \int (-\sin x + \sin 5x) \, dx = \frac{1}{2} \cos x - \frac{1}{10} \cos 5x + C$
53. $\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos 0 - \cos 6x) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos 6x \, dx = \frac{1}{2} \left[x - \frac{1}{12} \sin 6x \right]_{-\pi}^{\pi}$
 $= \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi$
54. $\int_0^{\pi/2} \sin x \cos x \, dx = \frac{1}{2} \int_0^{\pi/2} (\sin 0 + \sin 2x) \, dx = \frac{1}{2} \int_0^{\pi/2} \sin 2x \, dx = -\frac{1}{4} [\cos 2x]_0^{\pi/2} = -\frac{1}{4} (-1 - 1) = \frac{1}{2}$
55. $\int \cos 3x \cos 4x \, dx = \frac{1}{2} \int (\cos(-x) + \cos 7x) \, dx = \frac{1}{2} \int (\cos x + \cos 7x) \, dx = \frac{1}{2} \sin x + \frac{1}{14} \sin 7x + C$

56. $\int_{-\pi/2}^{\pi/2} \cos 7x \cos x \, dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (\cos 6x + \cos 8x) \, dx = \frac{1}{2} \left[\frac{1}{6} \sin 6x + \frac{1}{8} \sin 8x \right]_{-\pi/2}^{\pi/2} = 0$
57. $\int \sin^2 \theta \cos 3\theta \, d\theta = \int \frac{1-\cos 2\theta}{2} \cos 3\theta \, d\theta = \frac{1}{2} \int \cos 3\theta \, d\theta - \frac{1}{2} \int \cos 2\theta \cos 3\theta \, d\theta$
 $= \frac{1}{2} \int \cos 3\theta \, d\theta - \frac{1}{2} \int \frac{1}{2} (\cos(2-3)\theta + \cos(2+3)\theta) \, d\theta = \frac{1}{2} \int \cos 3\theta \, d\theta - \frac{1}{4} \int (\cos(-\theta) + \cos 5\theta) \, d\theta$
 $= \frac{1}{2} \int \cos 3\theta \, d\theta - \frac{1}{4} \int \cos \theta \, d\theta - \frac{1}{4} \int \cos 5\theta \, d\theta = \frac{1}{6} \sin 3\theta - \frac{1}{4} \sin \theta - \frac{1}{20} \sin 5\theta + C$
58. $\int \cos^2 2\theta \sin \theta \, d\theta = \int (2\cos^2 \theta - 1)^2 \sin \theta \, d\theta = \int (4\cos^4 \theta - 4\cos^2 \theta + 1) \sin \theta \, d\theta$
 $= \int 4\cos^4 \theta \sin \theta \, d\theta - \int 4\cos^2 \theta \sin \theta \, d\theta + \int \sin \theta \, d\theta = -\frac{4}{5} \cos^5 \theta + \frac{4}{3} \cos^3 \theta - \cos \theta + C$
59. $\int \cos^3 \theta \sin 2\theta \, d\theta = \int \cos^3 \theta (2\sin \theta \cos \theta) \, d\theta = 2 \int \cos^4 \theta \sin \theta \, d\theta = -\frac{2}{5} \cos^5 \theta + C$
60. $\int \sin^3 \theta \cos 2\theta \, d\theta = \int \sin^2 \theta \cos 2\theta \sin \theta \, d\theta = \int (1-\cos^2 \theta)(2\cos^2 \theta - 1) \sin \theta \, d\theta$
 $= \int (-2\cos^4 \theta + 3\cos^2 \theta - 1) \sin \theta \, d\theta = -2 \int \cos^4 \theta \sin \theta \, d\theta + 3 \int \cos^2 \theta \sin \theta \, d\theta - \int \sin \theta \, d\theta$
 $= \frac{2}{5} \cos^5 \theta - \cos^3 \theta + \cos \theta + C$
61. $\int \sin \theta \cos \theta \cos 3\theta \, d\theta = \frac{1}{2} \int 2\sin \theta \cos \theta \cos 3\theta \, d\theta = \frac{1}{2} \int \sin 2\theta \cos 3\theta \, d\theta$
 $= \frac{1}{2} \int \frac{1}{2} (\sin(2-3)\theta + \sin(2+3)\theta) \, d\theta = \frac{1}{4} \int (\sin(-\theta) + \sin 5\theta) \, d\theta = \frac{1}{4} \int (-\sin \theta + \sin 5\theta) \, d\theta$
 $= \frac{1}{4} \cos \theta - \frac{1}{20} \cos 5\theta + C$
62. $\int \sin \theta \sin 2\theta \sin 3\theta \, d\theta = \int \frac{1}{2} (\cos(1-2)\theta - \cos(1+2)\theta) \sin 3\theta \, d\theta = \frac{1}{2} \int (\cos(-\theta) - \cos 3\theta) \sin 3\theta \, d\theta$
 $= \frac{1}{2} \int \sin 3\theta \cos \theta \, d\theta - \frac{1}{2} \int \sin 3\theta \cos 3\theta \, d\theta = \frac{1}{2} \int \frac{1}{2} (\sin(3-1)\theta + \sin(3+1)\theta) \, d\theta - \frac{1}{4} \int 2\sin 3\theta \cos 3\theta \, d\theta$
 $= \frac{1}{4} \int (\sin 2\theta + \sin 4\theta) \, d\theta - \frac{1}{4} \int \sin 6\theta \, d\theta = -\frac{1}{8} \cos 2\theta - \frac{1}{16} \cos 4\theta + \frac{1}{24} \cos 6\theta + C$
63. $\int \frac{\sec^3 x}{\tan x} \, dx = \int \frac{\sec^2 x \sec x}{\tan x} \, dx = \int \frac{(\tan^2 x + 1) \sec x}{\tan x} \, dx = \int \frac{\tan^2 x \sec x}{\tan x} \, dx + \int \frac{\sec x}{\tan x} \, dx = \int \tan x \sec x \, dx + \int \csc x \, dx$
 $= \sec x - \ln |\csc x + \cot x| + C$
64. $\int \frac{\sin^3 x}{\cos^4 x} \, dx = \int \frac{\sin^2 x \sin x}{\cos^4 x} \, dx = \int \frac{(1-\cos^2 x) \sin x}{\cos^4 x} \, dx = \int \frac{\sin x}{\cos^4 x} \, dx - \int \frac{\cos^2 x \sin x}{\cos^4 x} \, dx = \int \sec^3 x \tan x \, dx - \int \sec x \tan x \, dx$
 $= \int \sec^2 x \sec x \tan x \, dx - \int \sec x \tan x \, dx = \frac{1}{3} \sec^3 x - \sec x + C$
65. $\int \frac{\tan^2 x}{\csc x} \, dx = \int \frac{\sin^2 x}{\cos^2 x} \sin x \, dx = \int \frac{(1-\cos^2 x)}{\cos^2 x} \sin x \, dx = \int \frac{1}{\cos^2 x} \sin x \, dx - \int \frac{\cos^2 x}{\cos^2 x} \sin x \, dx$
 $= \int \sec x \tan x \, dx - \int \sin x \, dx = \sec x + \cos x + C$
66. $\int \frac{\cot x}{\cos^2 x} \, dx = \int \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x} \, dx = \int \frac{2}{2\sin x \cos x} \, dx = \int \frac{2}{\sin 2x} \, dx = \int \csc 2x \, 2dx = -\ln |\csc 2x + \cot 2x| + C$

$$67. \int x \sin^2 x \, dx = \int x \frac{1 - \cos 2x}{2} \, dx = \frac{1}{2} \int x \, dx - \frac{1}{2} \int x \cos 2x \, dx \quad \left[u = x, du = dx, dv = \cos 2x \, dx, v = \frac{1}{2} \sin 2x \right]$$

$$= \frac{1}{4} x^2 - \frac{1}{2} \left[\frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \, dx \right] = \frac{1}{4} x^2 - \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x + C$$

$$68. \int x \cos^3 x \, dx = \int x \cos^2 x \cos x \, dx = \int x (1 - \sin^2 x) \cos x \, dx = \int x \cos x \, dx - \int x \sin^2 x \cos x \, dx;$$

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x; \quad [u = x, du = dx, dv = \cos x \, dx, v = \sin x]$$

$$\int x \sin^2 x \cos x \, dx = \frac{1}{3} x \sin^3 x - \int \frac{1}{3} \sin^3 x \, dx; \quad [u = x, du = dx, dv = \sin^2 x \cos x \, dx, v = \frac{1}{3} \sin^3 x]$$

$$= \frac{1}{3} x \sin^3 x - \frac{1}{3} \int (1 - \cos^2 x) \sin x \, dx = \frac{1}{3} x \sin^3 x - \frac{1}{3} \int \sin x \, dx + \frac{1}{3} \int \cos^2 x \sin x \, dx$$

$$= \frac{1}{3} x \sin^3 x + \frac{1}{3} \cos x - \frac{1}{9} \cos^3 x;$$

$$\Rightarrow \int x \cos x \, dx - \int x \sin^2 x \cos x \, dx = (x \sin x + \cos x) - \left(\frac{1}{3} x \sin^3 x + \frac{1}{3} \cos x - \frac{1}{9} \cos^3 x \right) + C$$

$$= x \sin x - \frac{1}{3} x \sin^3 x + \frac{2}{3} \cos x + \frac{1}{9} \cos^3 x + C$$

$$69. y = \ln(\sec x); y' = \frac{\sec x \tan x}{\sec x} = \tan x; (y')^2 = \tan^2 x; \int_0^{\pi/4} \sqrt{1 + \tan^2 x} \, dx = \int_0^{\pi/4} |\sec x| \, dx$$

$$= [\ln |\sec x + \tan x|]_0^{\pi/4} = \ln(\sqrt{2} + 1) - \ln(0 + 1) = \ln(\sqrt{2} + 1)$$

$$70. M = \int_{-\pi/4}^{\pi/4} \sec x \, dx = [\ln |\sec x + \tan x|]_{-\pi/4}^{\pi/4} = \ln(\sqrt{2} + 1) - \ln|\sqrt{2} - 1| = \ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1};$$

$$\bar{y} = \frac{1}{\ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1}} \int_{-\pi/4}^{\pi/4} \frac{\sec^2 x}{2} \, dx = \frac{1}{2 \ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1}} [\tan x]_{-\pi/4}^{\pi/4} = \frac{1}{2 \ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1}} (1 - (-1)) = \frac{1}{\ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1}} \Rightarrow (\bar{x}, \bar{y}) = \left(0, \left(\ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)^{-1} \right)$$

$$71. V = \pi \int_0^{\pi} \sin^2 x \, dx = \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx = \frac{\pi}{2} \int_0^{\pi} dx - \frac{\pi}{2} \int_0^{\pi} \cos 2x \, dx = \frac{\pi}{2} [x]_0^{\pi} - \frac{\pi}{4} [\sin 2x]_0^{\pi}$$

$$= \frac{\pi}{2} (\pi - 0) - \frac{\pi}{4} (0 - 0) = \frac{\pi^2}{2}$$

$$72. A = \int_0^{\pi} \sqrt{1 + \cos 4x} \, dx = \int_0^{\pi} \sqrt{2} |\cos 2x| \, dx = \sqrt{2} \int_0^{\pi/4} \cos 2x \, dx - \sqrt{2} \int_{\pi/4}^{3\pi/4} \cos 2x \, dx + \sqrt{2} \int_{3\pi/4}^{\pi} \cos 2x \, dx$$

$$= \frac{\sqrt{2}}{2} [\sin 2x]_0^{\pi/4} - \frac{\sqrt{2}}{2} [\sin 2x]_{\pi/4}^{3\pi/4} + \frac{\sqrt{2}}{2} [\sin 2x]_{3\pi/4}^{\pi} = \frac{\sqrt{2}}{2} (1 - 0) - \frac{\sqrt{2}}{2} (-1 - 1) + \frac{\sqrt{2}}{2} (0 + 1) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$$

$$73. M = \int_0^{2\pi} (x + \cos x) \, dx = \left[\frac{1}{2} x^2 + \sin x \right]_0^{2\pi} = \left(\frac{1}{2} (2\pi)^2 + \sin(2\pi) \right) - \left(\frac{1}{2} (0)^2 + \sin(0) \right) = 2\pi^2;$$

$$\bar{x} = \frac{1}{2\pi^2} \int_0^{2\pi} x(x + \cos x) \, dx = \frac{1}{2\pi^2} \int_0^{2\pi} (x^2 + x \cos x) \, dx = \frac{1}{2\pi^2} \int_0^{2\pi} x^2 \, dx + \frac{1}{2\pi^2} \int_0^{2\pi} x \cos x \, dx$$

$$[u = x, du = dx, dv = \cos x \, dx, v = \sin x]$$

$$= \frac{1}{6\pi^2} [x^3]_0^{2\pi} + \frac{1}{2\pi^2} \left([x \sin x]_0^{2\pi} - \int_0^{2\pi} \sin x \, dx \right) = \frac{1}{6\pi^2} (8\pi^3 - 0) + \frac{1}{2\pi^2} \left(2\pi \sin 2\pi - 0 - \int_0^{2\pi} \sin x \, dx \right)$$

$$= \frac{4\pi}{3} + \frac{1}{2\pi^2} [\cos x]_0^{2\pi} = \frac{4\pi}{3} + \frac{1}{2\pi^2} (\cos 2\pi - \cos 0) = \frac{4\pi}{3} + 0 = \frac{4\pi}{3}; \quad \bar{y} = \frac{1}{2\pi^2} \int_0^{2\pi} \frac{1}{2} (x + \cos x)^2 \, dx$$

$$= \frac{1}{4\pi^2} \int_0^{2\pi} (x^2 + 2x \cos x + \cos^2 x) \, dx = \frac{1}{4\pi^2} \int_0^{2\pi} x^2 \, dx + \frac{1}{2\pi^2} \int_0^{2\pi} x \cos x \, dx + \frac{1}{4\pi^2} \int_0^{2\pi} \cos^2 x \, dx$$

$$\begin{aligned}
&= \frac{1}{12\pi^2} \left[x^3 \right]_0^{2\pi} + \frac{1}{2\pi^2} \left[x \sin x + \cos x \right]_0^{2\pi} + \frac{1}{4\pi^2} \int_0^{2\pi} \frac{\cos 2x+1}{2} dx = \frac{2\pi}{3} + 0 + \frac{1}{8\pi^2} \int_0^{2\pi} \cos 2x dx + \frac{1}{8\pi^2} \int_0^{2\pi} dx \\
&= \frac{2\pi}{3} + \frac{1}{16\pi^2} \left[\sin 2x \right]_0^{2\pi} + \frac{1}{8\pi^2} \left[x \right]_0^{2\pi} = \frac{2\pi}{3} + 0 + \frac{1}{4\pi} = \frac{8\pi^2+3}{12\pi} \Rightarrow \text{The centroid is } \left(\frac{4\pi}{3}, \frac{8\pi^2+3}{12\pi} \right).
\end{aligned}$$

$$\begin{aligned}
74. \quad V &= \int_0^{\pi/3} \pi (\sin x + \sec x)^2 dx = \pi \int_0^{\pi/3} (\sin^2 x + 2 \sin x \sec x + \sec^2 x) dx \\
&= \pi \int_0^{\pi/3} \sin^2 x dx + \pi \int_0^{\pi/3} 2 \tan x dx + \pi \int_0^{\pi/3} \sec^2 x dx = \pi \int_0^{\pi/3} \frac{1-\cos 2x}{2} dx + 2\pi [\ln |\sec x|]_0^{\pi/3} + \pi [\tan x]_0^{\pi/3} \\
&= \frac{\pi}{2} \int_0^{\pi/3} dx - \frac{\pi}{2} \int_0^{\pi/3} \cos 2x dx + 2\pi (\ln |\sec \frac{\pi}{3}| - \ln |\sec 0|) + \pi (\tan \frac{\pi}{3} - \tan 0) \\
&= \frac{\pi}{2} [x]_0^{\pi/3} - \frac{\pi}{4} [\sin 2x]_0^{\pi/3} + 2\pi \ln 2 + \pi\sqrt{3} = \frac{\pi}{2} \left(\frac{\pi}{3} - 0 \right) - \frac{\pi}{4} (\sin 2(\frac{\pi}{3}) - \sin 2(0)) + 2\pi \ln 2 + \pi\sqrt{3} \\
&= \frac{\pi^2}{6} - \frac{\pi\sqrt{3}}{8} + 2\pi \ln 2 + \pi\sqrt{3} = \frac{\pi(4\pi+21\sqrt{3}-48\ln 2)}{24}
\end{aligned}$$

8.4 TRIGONOMETRIC SUBSTITUTIONS

$$1. \quad x = 3 \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \frac{3 d\theta}{\cos^2 \theta}, 9 + x^2 = 9(1 + \tan^2 \theta) = 9 \sec^2 \theta \Rightarrow \frac{1}{\sqrt{9+x^2}} = \frac{1}{3|\sec \theta|} = \frac{|\cos \theta|}{3} = \frac{\cos \theta}{3};$$

because $\cos \theta > 0$ when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$;

$$\int \frac{dx}{\sqrt{9+x^2}} = 3 \int \frac{\cos \theta d\theta}{3 \cos^2 \theta} = \int \frac{d\theta}{\cos \theta} = \ln |\sec \theta + \tan \theta| + C' = \ln \left| \frac{\sqrt{9+x^2}}{3} + \frac{x}{3} \right| + C' = \ln \left| \sqrt{9+x^2} + x \right| + C$$

$$\begin{aligned}
2. \quad \int \frac{3 dx}{\sqrt{1+9x^2}}; [3x = u, 3 dx = du] &\rightarrow \int \frac{du}{\sqrt{1+u^2}}; u = \tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}, du = \frac{dt}{\cos^2 t}, \sqrt{1+u^2} = |\sec t| = \sec t; \\
\int \frac{du}{\sqrt{1+u^2}} &= \int \frac{dt}{\cos^2 t (\sec t)} = \int \sec t dt = \ln |\sec t + \tan t| + C = \ln \left| \sqrt{u^2+1} + u \right| + C = \ln \left| \sqrt{1+9x^2} + 3x \right| + C
\end{aligned}$$

$$3. \quad \int_{-2}^2 \frac{dx}{(4+x^2)} = \left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_{-2}^2 = \frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} (-1) = \left(\frac{1}{2} \right) \left(\frac{\pi}{4} \right) - \left(\frac{1}{2} \right) \left(-\frac{\pi}{4} \right) = \frac{\pi}{4}$$

$$4. \quad \int_0^2 \frac{dx}{8+2x^2} = \frac{1}{2} \int_0^2 \frac{dx}{4+x^2} = \frac{1}{2} \left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_0^2 = \frac{1}{2} \left(\frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} 0 \right) = \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \left(\frac{\pi}{4} \right) - 0 = \frac{\pi}{16}$$

$$5. \quad \int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1} \frac{x}{3} \right]_0^{3/2} = \sin^{-1} \frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

$$6. \quad \int_0^{1/\sqrt{2}} \frac{2 dx}{\sqrt{1-4x^2}}; [t = 2x, dt = 2 dx] \rightarrow \int_0^{1/\sqrt{2}} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1} t \right]_0^{1/\sqrt{2}} = \sin^{-1} \frac{1}{\sqrt{2}} - \sin^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\begin{aligned}
7. \quad t &= 5 \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dt = 5 \cos \theta d\theta, \sqrt{25-t^2} = 5 \cos \theta; \\
\int \sqrt{25-t^2} dt &= \int (5 \cos \theta)(5 \cos \theta) d\theta = 25 \int \cos^2 \theta d\theta = 25 \int \frac{1+\cos 2\theta}{2} d\theta = 25 \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) + C \\
&= \frac{25}{2} (\theta + \sin \theta \cos \theta) + C = \frac{25}{2} \left[\sin^{-1} \left(\frac{t}{5} \right) + \left(\frac{t}{5} \right) \left(\frac{\sqrt{25-t^2}}{5} \right) \right] + C = \frac{25}{2} \sin^{-1} \left(\frac{t}{5} \right) + \frac{t\sqrt{25-t^2}}{2} + C
\end{aligned}$$

8. $t = \frac{1}{3} \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dt = \frac{1}{3} \cos \theta d\theta, \sqrt{1-9t^2} = \cos \theta;$

$$\int \sqrt{1-9t^2} dt = \frac{1}{3} \int (\cos \theta)(\cos \theta) d\theta = \frac{1}{3} \int \cos^2 \theta d\theta = \frac{1}{6} (\theta + \sin \theta \cos \theta) + C = \frac{1}{6} \left[\sin^{-1}(3t) + 3t\sqrt{1-9t^2} \right] + C$$
9. $x = \frac{7}{2} \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = \frac{7}{2} \sec \theta \tan \theta d\theta, \sqrt{4x^2-49} = \sqrt{49 \sec^2 \theta - 49} = 7 \tan \theta;$

$$\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{\left(\frac{7}{2} \sec \theta \tan \theta\right) d\theta}{7 \tan \theta} = \frac{1}{2} \int \sec \theta d\theta = \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = \frac{1}{2} \ln \left| \frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7} \right| + C$$
10. $x = \frac{3}{5} \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = \frac{3}{5} \sec \theta \tan \theta d\theta, \sqrt{25x^2-9} = \sqrt{9 \sec^2 \theta - 9} = 3 \tan \theta;$

$$\int \frac{5 dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5} \sec \theta \tan \theta\right) d\theta}{3 \tan \theta} = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C = \ln \left| \frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3} \right| + C$$
11. $y = 7 \sec \theta, 0 < \theta < \frac{\pi}{2}, dy = 7 \sec \theta \tan \theta d\theta, \sqrt{y^2-49} = 7 \tan \theta;$

$$\begin{aligned} \int \frac{\sqrt{y^2-49}}{y} dy &= \int \frac{(7 \tan \theta)(7 \sec \theta \tan \theta) d\theta}{7 \sec \theta} = 7 \int \tan^2 \theta d\theta = 7 \int (\sec^2 \theta - 1) d\theta = 7(\tan \theta - \theta) + C \\ &= 7 \left[\frac{\sqrt{y^2-49}}{7} - \sec^{-1} \left(\frac{y}{7} \right) \right] + C \end{aligned}$$
12. $y = 5 \sec \theta, 0 < \theta < \frac{\pi}{2}, dy = 5 \sec \theta \tan \theta d\theta, \sqrt{y^2-25} = 5 \tan \theta;$

$$\begin{aligned} \int \frac{\sqrt{y^2-25}}{y^3} dy &= \int \frac{(5 \tan \theta)(5 \sec \theta \tan \theta) d\theta}{125 \sec^3 \theta} = \frac{1}{5} \int \tan^2 \theta \cos^2 \theta d\theta = \frac{1}{5} \int \sin^2 \theta d\theta = \frac{1}{10} \int (1 - \cos 2\theta) d\theta \\ &= \frac{1}{10} (\theta - \sin \theta \cos \theta) + C = \frac{1}{10} \left[\sec^{-1} \left(\frac{y}{5} \right) - \left(\frac{\sqrt{y^2-25}}{y} \right) \left(\frac{5}{y} \right) \right] + C = \left[\frac{\sec^{-1} \left(\frac{y}{5} \right)}{10} - \frac{\sqrt{y^2-25}}{2y^2} \right] + C \end{aligned}$$
13. $x = \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = \sec \theta \tan \theta d\theta, \sqrt{x^2-1} = \tan \theta;$

$$\int \frac{dx}{x^2 \sqrt{x^2-1}} = \int \frac{\sec \theta \tan \theta d\theta}{\sec^2 \theta \tan \theta} = \int \frac{d\theta}{\sec \theta} = \sin \theta + C = \frac{\sqrt{x^2-1}}{x} + C$$
14. $x = \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = \sec \theta \tan \theta d\theta, \sqrt{x^2-1} = \tan \theta;$

$$\begin{aligned} \int \frac{2 dx}{x^3 \sqrt{x^2-1}} &= \int \frac{2 \tan \theta \sec \theta d\theta}{\sec^3 \theta \tan \theta} = 2 \int \cos^2 \theta d\theta = 2 \int \left(\frac{1+\cos 2\theta}{2} \right) d\theta = \theta + \sin \theta \cos \theta + C \\ &= \theta + \tan \theta \cos^2 \theta + C = \sec^{-1} x + \sqrt{x^2-1} \left(\frac{1}{x} \right)^2 + C = \sec^{-1} x + \frac{\sqrt{x^2-1}}{x^2} + C \end{aligned}$$
15. $u = 9-x^2 \Rightarrow du = -2x dx \Rightarrow -\frac{1}{2} du = x dx;$

$$\int \frac{x dx}{\sqrt{9-x^2}} = -\frac{1}{2} \int \frac{1}{\sqrt{u}} du = -\sqrt{u} + C = -\sqrt{9-x^2} + C$$

16. $x = 2 \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = 2 \sec^2 \theta d\theta, 4 + x^2 = 4 \sec^2 \theta;$

$$\begin{aligned}\int \frac{x^2 dx}{4+x^2} &= \int \frac{(4 \tan^2 \theta)(2 \sec^2 \theta) d\theta}{4 \sec^2 \theta} = \int 2 \tan^2 \theta d\theta = 2 \left(\int \sec^2 \theta - 1 \right) d\theta = 2 \int \sec^2 \theta d\theta - 2 \int d\theta \\ &= 2 \tan \theta - 2\theta + C = x - 2 \tan^{-1} \left(\frac{x}{2} \right) + C\end{aligned}$$

17. $x = 2 \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \frac{2 d\theta}{\cos^2 \theta}, \sqrt{x^2 + 4} = \frac{2}{\cos \theta};$

$$\begin{aligned}\int \frac{x^3 dx}{\sqrt{x^2 + 4}} &= \int \frac{(8 \tan^3 \theta)(\cos \theta) d\theta}{\cos^2 \theta} = 8 \int \frac{\sin^3 \theta d\theta}{\cos^4 \theta} = 8 \int \frac{(\cos^2 \theta - 1)(-\sin \theta) d\theta}{\cos^4 \theta}; [t = \cos \theta, dt = -\sin \theta d\theta] \\ &\rightarrow 8 \int \frac{t^2 - 1}{t^4} dt = 8 \int \left(\frac{1}{t^2} - \frac{1}{t^4} \right) dt = 8 \left(-\frac{1}{t} + \frac{1}{3t^3} \right) + C = 8 \left(-\sec \theta + \frac{\sec^3 \theta}{3} \right) + C = 8 \left(-\frac{\sqrt{x^2 + 4}}{2} + \frac{(x^2 + 4)^{3/2}}{8 \cdot 3} \right) + C \\ &= \frac{1}{3} (x^2 + 4)^{3/2} - 4\sqrt{x^2 + 4} + C = \frac{1}{3} (x^2 - 8)\sqrt{x^2 + 4} + C\end{aligned}$$

18. $x = \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \sec^2 \theta d\theta, \sqrt{x^2 + 1} = \sec \theta;$

$$\int \frac{dx}{x^2 \sqrt{x^2 + 1}} = \int \frac{\sec^2 \theta d\theta}{\tan^2 \theta \sec \theta} = \int \frac{\cos \theta d\theta}{\sin^2 \theta} = -\frac{1}{\sin \theta} + C = \frac{-\sqrt{x^2 + 1}}{x} + C$$

19. $w = 2 \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dw = 2 \cos \theta d\theta, \sqrt{4 - w^2} = 2 \cos \theta;$

$$\int \frac{8 dw}{w^2 \sqrt{4 - w^2}} = \int \frac{8 \cdot 2 \cos \theta d\theta}{4 \sin^2 \theta \cdot 2 \cos \theta} = 2 \int \frac{d\theta}{\sin^2 \theta} = -2 \cot \theta + C = \frac{-2\sqrt{4 - w^2}}{w} + C$$

20. $w = 3 \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dw = 3 \cos \theta d\theta, \sqrt{9 - w^2} = 3 \cos \theta;$

$$\begin{aligned}\int \frac{\sqrt{9 - w^2}}{w^2} dw &= \int \frac{3 \cos \theta \cdot 3 \cos \theta d\theta}{9 \sin^2 \theta} = \int \cot^2 \theta d\theta = \int \left(\frac{1 - \sin^2 \theta}{\sin^2 \theta} \right) d\theta = \int (\csc^2 \theta - 1) d\theta \\ &= -\cot \theta - \theta + C = -\frac{\sqrt{9 - w^2}}{w} - \sin^{-1} \left(\frac{w}{3} \right) + C\end{aligned}$$

21. $\int \sqrt{\frac{x+1}{1-x}} dx$ Multiply the integrand by $\sqrt{\frac{1+x}{1+x}}$.

$$\int \sqrt{\frac{x+1}{1-x}} dx = \int \frac{x+1}{\sqrt{1-x^2}} dx \quad \text{where } -1 < x < 1$$

$x = \sin \theta, dx = \cos \theta d\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ so that $\cos \theta > 0$ and $\sqrt{1-x^2} = \cos \theta.$

$$\begin{aligned}\int \frac{x+1}{\sqrt{1-x^2}} dx &= \int \frac{\sin \theta + 1}{\cos \theta} \cos \theta d\theta \\ &= \int (\sin \theta + 1) d\theta = \theta - \cos \theta + C \\ &= \sin^{-1} x - \sqrt{1-x^2} + C\end{aligned}$$

22. $u = x^2 - 4 \Rightarrow du = 2x dx \Rightarrow \frac{1}{2} du = x dx;$

$$\int x \sqrt{x^2 - 4} dx = \frac{1}{2} \int \sqrt{u} du = \frac{1}{3} u^{3/2} + C = \frac{1}{3} (x^2 - 4)^{3/2} + C$$

23. $x = \sin \theta, 0 \leq \theta \leq \frac{\pi}{3}, dx = \cos \theta d\theta, (1 - x^2)^{3/2} = \cos^3 \theta;$

$$\int_0^{\sqrt{3}/2} \frac{4x^2 dx}{(1-x^2)^{3/2}} = \int_0^{\pi/3} \frac{4 \sin^2 \theta \cos \theta d\theta}{\cos^3 \theta} = 4 \int_0^{\pi/3} \left(\frac{1 - \cos^2 \theta}{\cos^2 \theta} \right) d\theta = 4 \int_0^{\pi/3} (\sec^2 \theta - 1) d\theta = 4 [\tan \theta - \theta]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$$

24. $x = 2 \sin \theta, 0 \leq \theta \leq \frac{\pi}{6}, dx = 2 \cos \theta d\theta, (4 - x^2)^{3/2} = 8 \cos^3 \theta;$

$$\int_0^1 \frac{dx}{(4-x^2)^{3/2}} = \int_0^{\pi/6} \frac{2 \cos \theta d\theta}{8 \cos^3 \theta} = \frac{1}{4} \int_0^{\pi/6} \frac{d\theta}{\cos^2 \theta} = \frac{1}{4} [\tan \theta]_0^{\pi/6} = \frac{\sqrt{3}}{12} = \frac{1}{4\sqrt{3}}$$

25. $x = \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = \sec \theta \tan \theta d\theta, (x^2 - 1)^{3/2} = \tan^3 \theta;$

$$\int \frac{dx}{(x^2 - 1)^{3/2}} = \int \frac{\sec \theta \tan \theta d\theta}{\tan^3 \theta} = \int \frac{\cos \theta d\theta}{\sin^2 \theta} = -\frac{1}{\sin \theta} + C = -\frac{x}{\sqrt{x^2 - 1}} + C$$

26. $x = \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = \sec \theta \tan \theta d\theta, (x^2 - 1)^{5/2} = \tan^5 \theta;$

$$\int \frac{x^2 dx}{(x^2 - 1)^{5/2}} = \int \frac{\sec^2 \theta \sec \theta \tan \theta d\theta}{\tan^5 \theta} = \int \frac{\cos \theta d\theta}{\sin^4 \theta} = -\frac{1}{3 \sin^3 \theta} + C = -\frac{x^3}{3(x^2 - 1)^{3/2}} + C$$

27. $x = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \cos \theta d\theta, (1 - x^2)^{3/2} = \cos^3 \theta;$

$$\int \frac{(1-x^2)^{3/2} dx}{x^6} = \int \frac{\cos^3 \theta \cos \theta d\theta}{\sin^6 \theta} = \int \cot^4 \theta \csc^2 \theta d\theta = -\frac{\cot^5 \theta}{5} + C = -\frac{1}{5} \left(\frac{\sqrt{1-x^2}}{x} \right)^5 + C$$

28. $x = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \cos \theta d\theta, (1 - x^2)^{1/2} = \cos \theta;$

$$\int \frac{(1-x^2)^{1/2} dx}{x^4} = \int \frac{\cos \theta \cos \theta d\theta}{\sin^4 \theta} = \int \cot^2 \theta \csc^2 \theta d\theta = -\frac{\cot^3 \theta}{3} + C = -\frac{1}{3} \left(\frac{\sqrt{1-x^2}}{x} \right)^3 + C$$

29. $x = \frac{1}{2} \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \frac{1}{2} \sec^2 \theta d\theta, (4x^2 + 1)^2 = \sec^4 \theta;$

$$\int \frac{8dx}{(4x^2 + 1)^2} = \int \frac{8(\frac{1}{2} \sec^2 \theta) d\theta}{\sec^4 \theta} = 4 \int \cos^2 \theta d\theta = 2(\theta + \sin \theta \cos \theta) + C = 2 \tan^{-1} 2x + \frac{4x}{4x^2 + 1} + C$$

30. $t = \frac{1}{3} \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dt = \frac{1}{3} \sec^2 \theta d\theta, 9t^2 + 1 = \sec^2 \theta;$

$$\int \frac{6dt}{(9t^2 + 1)^2} = \int \frac{6(\frac{1}{3} \sec^2 \theta) d\theta}{\sec^4 \theta} = 2 \int \cos^2 \theta d\theta = \theta + \sin \theta \cos \theta + C = \tan^{-1} 3t + \frac{3t}{9t^2 + 1} + C$$

31. $u = x^2 - 1 \Rightarrow du = 2x dx \Rightarrow \frac{1}{2} du = x dx$;

$$\int \frac{x^3}{x^2-1} dx = \int \left(x + \frac{x}{x^2-1} \right) dx = \int x dx + \int \frac{x}{x^2-1} dx = \frac{1}{2} x^2 + \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} x^2 + \frac{1}{2} \ln |u| + C = \frac{1}{2} x^2 + \frac{1}{2} \ln |x^2 - 1| + C$$
32. $u = 25 + 4x^2 \Rightarrow du = 8x dx \Rightarrow \frac{1}{8} du = x dx$;

$$\int \frac{x}{25+4x^2} dx = \frac{1}{8} \int \frac{1}{u} du = \frac{1}{8} \ln |u| + C = \frac{1}{8} \ln (25 + 4x^2) + C$$
33. $v = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dv = \cos \theta d\theta, (1-v^2)^{5/2} = \cos^5 \theta$;

$$\int \frac{v^2 dv}{(1-v^2)^{5/2}} = \int \frac{\sin^2 \theta \cos \theta d\theta}{\cos^5 \theta} = \int \tan^2 \theta \sec^2 \theta d\theta = \frac{\tan^3 \theta}{3} + C = \frac{1}{3} \left(\frac{v}{\sqrt{1-v^2}} \right)^3 + C$$
34. $r = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dr = \cos \theta d\theta, (1-r^2)^{5/2} = \cos^5 \theta$;

$$\int \frac{(1-r^2)^{5/2} dr}{r^8} = \int \frac{\cos^5 \theta \cos \theta d\theta}{\sin^8 \theta} = \int \cot^6 \theta \csc^2 \theta d\theta = -\frac{\cot^7 \theta}{7} + C = -\frac{1}{7} \left[\frac{\sqrt{1-r^2}}{r} \right]^7 + C$$
35. Let $e^t = 3 \tan \theta, t = \ln(3 \tan \theta), \tan^{-1} \left(\frac{1}{3} \right) \leq \theta \leq \tan^{-1} \left(\frac{4}{3} \right), dt = \frac{\sec^2 \theta}{\tan \theta} d\theta, \sqrt{e^{2t} + 9} = \sqrt{9 \tan^2 \theta + 9} = 3 \sec \theta$;

$$\begin{aligned} \int_0^{\ln 4} \frac{e^t dt}{\sqrt{e^{2t} + 9}} &= \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3 \tan \theta \sec^2 \theta d\theta}{\tan \theta \cdot 3 \sec \theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec \theta d\theta = [\ln |\sec \theta + \tan \theta|]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \\ &= \ln \left(\frac{5}{3} + \frac{4}{3} \right) - \ln \left(\frac{\sqrt{10}}{3} + \frac{1}{3} \right) = \ln 9 - \ln (1 + \sqrt{10}) \end{aligned}$$
36. Let $e^t = \tan \theta, t = \ln(\tan \theta), \tan^{-1} \left(\frac{3}{4} \right) \leq \theta \leq \tan^{-1} \left(\frac{4}{3} \right), dt = \frac{\sec^2 \theta}{\tan \theta} d\theta, 1 + e^{2t} = 1 + \tan^2 \theta = \sec^2 \theta$;

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t dt}{(1+e^{2t})^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan \theta) \left(\frac{\sec^2 \theta}{\tan \theta} \right) d\theta}{\sec^3 \theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos \theta d\theta = [\sin \theta]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$$
37. $\int_{1/12}^{1/4} \frac{2 dt}{\sqrt{t+4t\sqrt{t}}}$; $\left[u = 2\sqrt{t}, du = \frac{1}{\sqrt{t}} dt \right] \rightarrow \int_{1/\sqrt{3}}^1 \frac{2 du}{1+u^2}; u = \tan \theta, \frac{\pi}{6} \leq \theta \leq \frac{\pi}{4}, du = \sec^2 \theta d\theta, 1+u^2 = \sec^2 \theta$;

$$\int_{1/\sqrt{3}}^1 \frac{2 du}{1+u^2} = \int_{\pi/6}^{\pi/4} \frac{2 \sec^2 \theta d\theta}{\sec^2 \theta} = [2\theta]_{\pi/6}^{\pi/4} = 2 \left(\frac{\pi}{4} - \frac{\pi}{6} \right) = \frac{\pi}{6}$$
38. $y = e^{\tan \theta}, 0 \leq \theta \leq \frac{\pi}{4}, dy = e^{\tan \theta} \sec^2 \theta d\theta, \sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2 \theta} = \sec \theta$;

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan \theta} \sec^2 \theta d\theta}{e^{\tan \theta} \sec \theta} = \int_0^{\pi/4} \sec \theta d\theta = [\ln |\sec \theta + \tan \theta|]_0^{\pi/4} = \ln(1 + \sqrt{2})$$
39. $x = \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = \sec \theta \tan \theta d\theta, \sqrt{x^2-1} = \sqrt{\sec^2 \theta-1} = \tan \theta$;

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec \theta \tan \theta d\theta}{\sec \theta \tan \theta} = \theta + C = \sec^{-1} x + C$$

40. $x = \tan \theta, dx = \sec^2 \theta d\theta, 1 + x^2 = \sec^2 \theta;$

$$\int \frac{dx}{x^2 + 1} = \int \frac{\sec^2 \theta d\theta}{\sec^2 \theta} = \theta + C = \tan^{-1} x + C$$

41. $x = \sec \theta, dx = \sec \theta \tan \theta d\theta, \sqrt{x^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta;$

$$\int \frac{x dx}{\sqrt{x^2 - 1}} = \int \frac{\sec \theta \sec \theta \tan \theta d\theta}{\tan \theta} = \int \sec^2 \theta d\theta = \tan \theta + C = \sqrt{x^2 - 1} + C$$

42. $x = \sin \theta, dx = \cos \theta d\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \sqrt{1 - x^2} = \sqrt{1 - \sin^2 \theta} = \cos \theta;$

$$\int \frac{dx}{\sqrt{1 - x^2}} = \int \frac{\cos \theta d\theta}{\cos \theta} = \theta + C = \sin^{-1} x + C$$

43. Let $x^2 = \tan \theta, 0 \leq \theta < \frac{\pi}{2}, 2x dx = \sec^2 \theta d\theta \Rightarrow x dx = \frac{1}{2} \sec^2 \theta d\theta; \sqrt{1 + x^4} = \sqrt{1 + \tan^2 \theta} = \sec \theta$

$$\int \frac{x dx}{\sqrt{1 + x^4}} = \frac{1}{2} \int \frac{\sec^2 \theta d\theta}{\sec \theta} = \frac{1}{2} \int \sec \theta d\theta = \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = \frac{1}{2} \ln \left| \sqrt{1 + x^4} + x^2 \right| + C$$

44. Let $\ln x = \sin \theta, -\frac{\pi}{2} \leq \theta < 0$ or $0 < \theta \leq \frac{\pi}{2}, \frac{1}{x} dx = \cos \theta d\theta, \sqrt{1 - (\ln x)^2} = \cos \theta$

$$\begin{aligned} \int \frac{\sqrt{1 - (\ln x)^2}}{x \ln x} dx &= \int \frac{\cos^2 \theta}{\sin \theta} d\theta = \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta = \int \csc \theta d\theta - \int \sin \theta d\theta = -\ln |\csc \theta + \cot \theta| + \cos \theta + C \\ &= -\ln \left| \frac{1}{\ln x} + \frac{\sqrt{1 - (\ln x)^2}}{\ln x} \right| + \sqrt{1 - (\ln x)^2} + C = -\ln \left| \frac{1 + \sqrt{1 - (\ln x)^2}}{\ln x} \right| + \sqrt{1 - (\ln x)^2} + C \end{aligned}$$

45. Let $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u du \Rightarrow \int \sqrt{\frac{4-x}{x}} dx = \int \sqrt{\frac{4-u^2}{u^2}} 2u du = 2 \int \sqrt{4-u^2} du;$

$$u = 2 \sin \theta, du = 2 \cos \theta d\theta, 0 < \theta \leq \frac{\pi}{2}, \sqrt{4-u^2} = 2 \cos \theta$$

$$\begin{aligned} 2 \int \sqrt{4-u^2} du &= 2 \int (2 \cos \theta)(2 \cos \theta) d\theta = 8 \int \cos^2 \theta d\theta = 8 \int \frac{1 + \cos 2\theta}{2} d\theta = 4 \int d\theta + 4 \int \cos 2\theta d\theta \\ &= 4\theta + 2 \sin 2\theta + C = 4\theta + 4 \sin \theta \cos \theta + C = 4 \sin^{-1} \left(\frac{u}{2} \right) + 4 \left(\frac{u}{2} \right) \left(\frac{\sqrt{4-u^2}}{2} \right) + C = 4 \sin^{-1} \left(\frac{\sqrt{x}}{2} \right) + \sqrt{x} \sqrt{4-x} + C \\ &= 4 \sin^{-1} \left(\frac{\sqrt{x}}{2} \right) + \sqrt{4x-x^2} + C \end{aligned}$$

46. Let $u = x^{3/2} \Rightarrow x = u^{2/3} \Rightarrow dx = \frac{2}{3} u^{-1/3} du$

$$\int \sqrt{\frac{x}{1-x^3}} dx = \int \frac{u^{2/3}}{1-(u^{2/3})^3} \left(\frac{2}{3} u^{-1/3} \right) du = \int \frac{u^{1/3}}{\sqrt{1-u^2}} \left(\frac{2}{3u^{1/3}} \right) du = \frac{2}{3} \int \frac{1}{\sqrt{1-u^2}} du = \frac{2}{3} \sin^{-1} u + C = \frac{2}{3} \sin^{-1} (x^{3/2}) + C$$

47. Let $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u du \Rightarrow \int \sqrt{x} \sqrt{1-x} dx = \int u \sqrt{1-u^2} 2u du = 2 \int u^2 \sqrt{1-u^2} du;$

$$u = \sin \theta, du = \cos \theta d\theta, -\frac{\pi}{2} < \theta \leq \frac{\pi}{2}, \sqrt{1-u^2} = \cos \theta$$

$$\begin{aligned} 2 \int u^2 \sqrt{1-u^2} du &= 2 \int \sin^2 \theta \cos \theta \cos \theta d\theta = 2 \int \sin^2 \theta \cos^2 \theta d\theta = \frac{1}{2} \int \sin^2 2\theta d\theta = \frac{1}{2} \int \frac{1 - \cos 4\theta}{2} d\theta \\ &= \frac{1}{4} \int d\theta - \frac{1}{4} \int \cos 4\theta d\theta = \frac{1}{4} \theta - \frac{1}{16} \sin 4\theta + C = \frac{1}{4} \theta - \frac{1}{8} \sin 2\theta \cos 2\theta + C = \frac{1}{4} \theta - \frac{1}{4} \sin \theta \cos \theta (2 \cos^2 \theta - 1) + C \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\theta - \frac{1}{2}\sin\theta\cos^3\theta + \frac{1}{4}\sin\theta\cos\theta + C = \frac{1}{4}\sin^{-1}u - \frac{1}{2}u(1-u^2)^{3/2} - \frac{1}{4}u\sqrt{1-u^2} + C \\
&= \frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}(1-x)^{3/2} - \frac{1}{4}\sqrt{x}\sqrt{1-x} + C
\end{aligned}$$

48. Let $w = \sqrt{x-1} \Rightarrow w^2 = x-1 \Rightarrow 2w dw = dx \Rightarrow \int \frac{\sqrt{x-2}}{\sqrt{x-1}} dx = \int \frac{\sqrt{w^2-1}}{w} 2w dw = 2 \int \sqrt{w^2-1} dw$

$w = \sec\theta, dx = \sec\theta \tan\theta d\theta, 0 < \theta < \frac{\pi}{2}, \sqrt{w^2-1} = \tan\theta$

$2 \int \sqrt{w^2-1} dw = 2 \int \tan\theta \sec\theta \tan\theta d\theta; u = \tan\theta, du = \sec^2\theta d\theta, dv = \sec\theta \tan\theta d\theta, v = \sec\theta$

$2 \int \tan\theta \sec\theta \tan\theta d\theta = 2 \sec\theta \tan\theta - 2 \int \sec^3\theta d\theta = 2 \sec\theta \tan\theta - 2 \int \sec^2\theta \sec\theta d\theta$

$= 2 \sec\theta \tan\theta - 2 \int (\tan^2\theta + 1) \sec\theta d\theta = 2 \sec\theta \tan\theta - 2 \left(\int \tan^2\theta \sec\theta d\theta + \int \sec\theta d\theta \right)$

$= 2 \sec\theta \tan\theta - 2 \ln|\sec\theta + \tan\theta| - 2 \int \tan^2\theta \sec\theta d\theta$

$\Rightarrow 2 \int \tan^2\theta \sec\theta d\theta = \sec\theta \tan\theta - \ln|\sec\theta + \tan\theta| + C = w\sqrt{w^2-1} - \ln\left|w + \sqrt{w^2-1}\right| + C$

$= \sqrt{x-1}\sqrt{x-2} - \ln|\sqrt{x-1} + \sqrt{x-2}| + C$

49. $x \frac{dy}{dx} = \sqrt{x^2-4}; dy = \sqrt{x^2-4} \frac{dx}{x}; y = \int \frac{\sqrt{x^2-4}}{x} dx;$

$x = 2\sec\theta, 0 < \theta < \frac{\pi}{2}, dx = 2\sec\theta \tan\theta d\theta, \sqrt{x^2-4} = 2\tan\theta$

$\rightarrow y = \int \frac{(2\tan\theta)(2\sec\theta \tan\theta) d\theta}{2\sec\theta} = 2 \int \tan^2\theta d\theta = 2 \int (\sec^2\theta - 1) d\theta = 2(\tan\theta - \theta) + C$

$= 2 \left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right) \right] + C; x = 2 \text{ and } y = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow y = 2 \left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\frac{x}{2} \right]$

50. $\sqrt{x^2-9} \frac{dy}{dx} = 1, dy = \frac{dx}{\sqrt{x^2-9}}; y = \int \frac{dx}{\sqrt{x^2-9}}; x = 3\sec\theta, 0 < \theta < \frac{\pi}{2}, dx = 3\sec\theta \tan\theta d\theta, \sqrt{x^2-9} = 3\tan\theta$

$\rightarrow y = \int \frac{3\sec\theta \tan\theta d\theta}{3\tan\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C; x = 5 \text{ and } y = \ln 3$

$\Rightarrow \ln 3 = \ln 3 + C \Rightarrow C = 0 \Rightarrow y = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|$

51. $(x^2+4) \frac{dy}{dx} = 3, dy = \frac{3dx}{x^2+4}; y = 3 \int \frac{dx}{x^2+4} = \frac{3}{2} \tan^{-1} \frac{x}{2} + C; x = 2 \text{ and } y = 0 \Rightarrow 0 = \frac{3}{2} \tan^{-1} 1 + C \Rightarrow C = -\frac{3\pi}{8}$

$\Rightarrow y = \frac{3}{2} \tan^{-1}\left(\frac{x}{2}\right) - \frac{3\pi}{8}$

52. $(x^2+1)^2 \frac{dy}{dx} = \sqrt{x^2+1}, dy = \frac{dx}{(x^2+1)^{3/2}}; x = \tan\theta, dx = \sec^2\theta d\theta, (x^2+1)^{3/2} = \sec^3\theta;$

$y = \int \frac{\sec^2\theta d\theta}{\sec^3\theta} = \int \cos\theta d\theta = \sin\theta + C = \tan\theta \cos\theta + C = \frac{\tan\theta}{\sec\theta} + C = \frac{x}{\sqrt{x^2+1}} + C; x = 0 \text{ and } y = 1$

$\Rightarrow 1 = 0 + C \Rightarrow C = 1 \Rightarrow y = \frac{x}{\sqrt{x^2+1}} + 1$

$$53. \quad A = \int_0^3 \frac{\sqrt{9-x^2}}{3} dx; \quad x = 3 \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad dx = 3 \cos \theta d\theta, \quad \sqrt{9-x^2} = \sqrt{9-9 \sin^2 \theta} = 3 \cos \theta;$$

$$A = \int_0^{\pi/2} \frac{3 \cos \theta \cdot 3 \cos \theta d\theta}{3} = 3 \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{3}{2} [\theta + \sin \theta \cos \theta]_0^{\pi/2} = \frac{3\pi}{4}$$

$$54. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \pm b \sqrt{1 - \frac{x^2}{a^2}}; \quad A = 4 \int_0^a b \sqrt{1 - \frac{x^2}{a^2}} dx = 4b \int_0^a \sqrt{1 - \frac{x^2}{a^2}} dx$$

$$\left[x = a \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, dx = a \cos \theta d\theta, \sqrt{1 - \frac{x^2}{a^2}} = \cos \theta, x = 0 = a \sin \theta \Rightarrow \theta = 0, x = a = a \sin \theta \Rightarrow \theta = \frac{\pi}{2} \right]$$

$$4b \int_0^a \sqrt{1 - \frac{x^2}{a^2}} dx = 4b \int_0^{\pi/2} \cos \theta (a \cos \theta) d\theta = 4ab \int_0^{\pi/2} \cos^2 \theta d\theta = 4ab \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta$$

$$= 2ab \int_0^{\pi/2} d\theta + 2ab \int_0^{\pi/2} \cos 2\theta d\theta = 2ab [\theta]_0^{\pi/2} + ab [\sin 2\theta]_0^{\pi/2} = 2ab \left(\frac{\pi}{2} - 0 \right) + ab (\sin \pi - \sin 0) = \pi ab$$

$$55. \quad (a) \quad A = \int_0^{1/2} \sin^{-1} x dx \left[u = \sin^{-1} x, du = \frac{1}{\sqrt{1-x^2}} dx, dv = dx, v = x \right]$$

$$= \left[x \sin^{-1} x \right]_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}} dx = \left(\frac{1}{2} \sin^{-1} \frac{1}{2} - 0 \right) + \left[\sqrt{1-x^2} \right]_0^{1/2} = \frac{\pi+6\sqrt{3}-12}{12}$$

$$(b) \quad M = \int_0^{1/2} \sin^{-1} x dx = \frac{\pi+6\sqrt{3}-12}{12};$$

$$\bar{x} = \frac{1}{\frac{\pi+6\sqrt{3}-12}{12}} \int_0^{1/2} x \sin^{-1} x dx = \frac{12}{\pi+6\sqrt{3}-12} \int_0^{1/2} x \sin^{-1} x dx \left[u = \sin^{-1} x, du = \frac{1}{\sqrt{1-x^2}} dx, dv = x dx, v = \frac{1}{2} x^2 \right]$$

$$= \frac{12}{\pi+6\sqrt{3}-12} \left(\left[\frac{1}{2} x^2 \sin^{-1} x \right]_0^{1/2} - \frac{1}{2} \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} dx \right)$$

$$\left[x = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \cos \theta d\theta, \sqrt{1-x^2} = \cos \theta, x = 0 = \sin \theta \Rightarrow \theta = 0, x = \frac{1}{2} = \sin \theta \Rightarrow \theta = \frac{\pi}{6} \right]$$

$$= \frac{12}{\pi+6\sqrt{3}-12} \left(\left(\frac{1}{2} \left(\frac{1}{2} \right)^2 \sin^{-1} \left(\frac{1}{2} \right) - 0 \right) - \frac{1}{2} \int_0^{\pi/6} \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta \right) = \frac{12}{\pi+6\sqrt{3}-12} \left(\frac{\pi}{48} - \frac{1}{2} \int_0^{\pi/6} \sin^2 \theta d\theta \right)$$

$$= \frac{12}{\pi+6\sqrt{3}-12} \left(\frac{\pi}{48} - \frac{1}{2} \int_0^{\pi/6} \frac{1 - \cos 2\theta}{2} d\theta \right) = \frac{12}{\pi+6\sqrt{3}-12} \left(\frac{\pi}{48} - \frac{1}{4} \int_0^{\pi/6} d\theta + \frac{1}{4} \int_0^{\pi/6} \cos 2\theta d\theta \right)$$

$$= \frac{12}{\pi+6\sqrt{3}-12} \left(\frac{\pi}{48} + \left[-\frac{\theta}{4} + \frac{1}{8} \sin 2\theta \right]_0^{\pi/6} \right) = \frac{3\sqrt{3}-\pi}{4(\pi+6\sqrt{3}-12)};$$

$$\bar{y} = \frac{1}{\frac{\pi+6\sqrt{3}-12}{12}} \int_0^{1/2} \frac{1}{2} (\sin^{-1} x)^2 dx \left[u = (\sin^{-1} x)^2, du = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} dx, dv = dx, v = x \right]$$

$$= \frac{6}{\pi+6\sqrt{3}-12} \left(\left[x (\sin^{-1} x)^2 \right]_0^{1/2} - \int_0^{1/2} \frac{2x \sin^{-1} x}{\sqrt{1-x^2}} dx \right) \left[u = \sin^{-1} x, du = \frac{1}{\sqrt{1-x^2}} dx, dv = \frac{2x}{\sqrt{1-x^2}} dx, v = -2\sqrt{1-x^2} \right]$$

$$= \frac{6}{\pi+6\sqrt{3}-12} \left(\left(\frac{1}{2} \left(\sin^{-1} \left(\frac{1}{2} \right) \right)^2 - 0 \right) + \left[2\sqrt{1-x^2} \sin^{-1} x \right]_0^{1/2} - \int_0^{1/2} \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} dx \right)$$

$$= \frac{6}{\pi+6\sqrt{3}-12} \left(\frac{\pi^2}{72} + \left(2\sqrt{1-\left(\frac{1}{2}\right)^2} \sin^{-1} \left(\frac{1}{2} \right) - 0 \right) - [2x]_0^{1/2} \right) = \frac{6}{\pi+6\sqrt{3}-12} \left(\frac{\pi^2}{72} + \frac{\pi\sqrt{3}}{6} - 1 \right) = \frac{\pi^2+12\pi\sqrt{3}-72}{12(\pi+6\sqrt{3}-12)}$$

$$\begin{aligned}
 56. \quad V &= \int_0^1 \pi \left(\sqrt{x \tan^{-1} x} \right)^2 dx = \pi \int_0^1 x \tan^{-1} x \, dx \quad \left[u = \tan^{-1} x, du = \frac{1}{1+x^2} dx, dv = x \, dx, v = \frac{1}{2} x^2 \right] \\
 &= \pi \left(\left[\frac{1}{2} x^2 \tan^{-1} x \right]_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} dx \right) = \pi \left(\left(\frac{1}{2} \tan^{-1} 1 - 0 \right) - \frac{1}{2} \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx \right) = \pi \left(\frac{\pi}{8} - \frac{1}{2} \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx \right) \\
 &= \pi \left(\frac{\pi}{8} - \frac{1}{2} \int_0^1 dx + \frac{1}{2} \int_0^1 \frac{1}{1+x^2} dx \right) = \pi \left(\frac{\pi}{8} + \left[-\frac{1}{2} x + \frac{1}{2} \tan^{-1} x \right]_0^1 \right) = \pi \left(\frac{\pi}{8} + \left(-\frac{1}{2} + \frac{1}{2} \tan^{-1} 1 + 0 - 0 \right) \right) = \frac{\pi(\pi-2)}{4}
 \end{aligned}$$

$$\begin{aligned}
 57. \quad (a) \quad &\text{Integration by parts: } u = x^2, du = 2x \, dx, dv = x\sqrt{1-x^2} \, dx, v = -\frac{1}{3}(1-x^2)^{3/2} \\
 &\int x^3 \sqrt{1-x^2} \, dx = -\frac{1}{3} x^2 (1-x^2)^{3/2} + \frac{1}{3} \int (1-x^2)^{3/2} 2x \, dx = -\frac{1}{3} x^2 (1-x^2)^{3/2} - \frac{2}{15} (1-x^2)^{5/2} + C \\
 (b) \quad &\text{Substitution: } u = 1-x^2 \Rightarrow x^2 = 1-u \Rightarrow du = -2x \, dx \Rightarrow -\frac{1}{2} du = x \, dx \\
 &\int x^3 \sqrt{1-x^2} \, dx = \int x^2 \sqrt{1-x^2} x \, dx = -\frac{1}{2} \int (1-u) \sqrt{u} \, du = -\frac{1}{2} \int (\sqrt{u} - u^{3/2}) \, du = -\frac{1}{3} u^{3/2} + \frac{1}{5} u^{5/2} + C \\
 &= -\frac{1}{3} (1-x^2)^{3/2} + \frac{1}{5} (1-x^2)^{5/2} + C \\
 (c) \quad &\text{Trig substitution: } x = \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, dx = \cos \theta \, d\theta, \sqrt{1-x^2} = \cos \theta \\
 &\int x^3 \sqrt{1-x^2} \, dx = \int \sin^3 \theta \cos \theta \cos \theta \, d\theta = \int \sin^2 \theta \cos^2 \theta \sin \theta \, d\theta = \int (1-\cos^2 \theta) \cos^2 \theta \sin \theta \, d\theta \\
 &= \int \cos^2 \theta \sin \theta \, d\theta - \int \cos^4 \theta \sin \theta \, d\theta = -\frac{1}{3} \cos^3 \theta + \frac{1}{5} \cos^5 \theta + C = -\frac{1}{3} (1-x^2)^{3/2} + \frac{1}{5} (1-x^2)^{5/2} + C
 \end{aligned}$$

$$\begin{aligned}
 58. \quad (a) \quad &\text{The slope of the line tangent to } y = f(x) \text{ is given by } f'(x). \text{ Consider the triangle whose hypotenuse is} \\
 &\text{the 10 m rope, the length of the base is } x \text{ and height } h = \sqrt{100-x^2}. \text{ The slope of the tangent line is also} \\
 &-\frac{\sqrt{100-x^2}}{x}, \text{ thus } f'(x) = -\frac{\sqrt{100-x^2}}{x}.
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad f(x) &= \int -\frac{\sqrt{100-x^2}}{x} dx \quad \left[x = 10 \sin \theta, 0 < \theta \leq \frac{\pi}{2}, dx = 10 \cos \theta \, d\theta, \sqrt{100-x^2} = 10 \cos \theta \right] \\
 &= -\int \frac{10 \cos \theta}{10 \sin \theta} 10 \cos \theta \, d\theta = -10 \int \frac{\cos^2 \theta}{\sin \theta} \, d\theta = -10 \int \frac{(1-\sin^2 \theta)}{\sin \theta} \, d\theta = -10 \int \csc \theta \, d\theta + 10 \int \sin \theta \, d\theta \\
 &= 10 \ln |\csc \theta + \cot \theta| - 10 \cos \theta + C = 10 \ln \left| \frac{10}{x} + \frac{\sqrt{100-x^2}}{x} \right| - \sqrt{100-x^2} + C; \\
 f(10) &= 0 \Rightarrow 0 = 10 \ln \left| \frac{10}{10} + \frac{\sqrt{100-10^2}}{10} \right| - \sqrt{100-10^2} + C = C \Rightarrow f(x) = 10 \ln \left| \frac{10}{x} + \frac{\sqrt{100-x^2}}{x} \right| - \sqrt{100-x^2}
 \end{aligned}$$

8.5 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

$$\begin{aligned}
 1. \quad \frac{5x-13}{(x-3)(x-2)} &= \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x-13 = A(x-2) + B(x-3) = (A+B)x - (2A+3B) \\
 \Rightarrow \left. \begin{aligned} A+B &= 5 \\ 2A+3B &= 13 \end{aligned} \right\} &\Rightarrow -B = (10-13) \Rightarrow B = 3 \Rightarrow A = 2; \text{ thus, } \frac{5x-13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}
 \end{aligned}$$

$$2. \quad \frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x-7 = A(x-1) + B(x-2) = (A+B)x - (A+2B)$$

$$\Rightarrow \left. \begin{aligned} A+B &= 5 \\ A+2B &= 7 \end{aligned} \right\} \Rightarrow B=2 \Rightarrow A=3; \text{ thus, } \frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$$

$$3. \quad \frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x+4 = A(x+1) + B = Ax + (A+B) \Rightarrow \left. \begin{aligned} A &= 1 \\ A+B &= 4 \end{aligned} \right\} \Rightarrow A=1 \text{ and } B=3; \text{ thus,}$$

$$\frac{x+4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$$

$$4. \quad \frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x+2 = A(x-1) + B = Ax + (-A+B) \Rightarrow \left. \begin{aligned} A &= 2 \\ -A+B &= 2 \end{aligned} \right\}$$

$$\Rightarrow A=2 \text{ and } B=4; \text{ thus, } \frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$$

$$5. \quad \frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1) + B(z-1) + Cz^2 \Rightarrow z+1 = (A+C)z^2 + (-A+B)z - B \Rightarrow \left. \begin{aligned} A+C &= 0 \\ -A+B &= 1 \\ -B &= 1 \end{aligned} \right\}$$

$$\Rightarrow B=-1 \Rightarrow A=-2 \Rightarrow C=2; \text{ thus, } \frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$$

$$6. \quad \frac{z}{z^3-z^2-6z} = \frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2} \Rightarrow 1 = A(z+2) + B(z-3) = (A+B)z + (2A-3B) \Rightarrow \left. \begin{aligned} A+B &= 0 \\ 2A-3B &= 1 \end{aligned} \right\}$$

$$\Rightarrow -5B=1 \Rightarrow B=-\frac{1}{5} \Rightarrow A=\frac{1}{5}; \text{ thus, } \frac{z}{z^3-z^2-6z} = \frac{\frac{1}{5}}{z-3} + \frac{-\frac{1}{5}}{z+2}$$

$$7. \quad \frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6} \text{ (after long division); } \frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2} \Rightarrow 5t+2 = A(t-2) + B(t-3)$$

$$= (A+B)t + (-2A-3B) \Rightarrow \left. \begin{aligned} A+B &= 5 \\ -2A-3B &= 2 \end{aligned} \right\} \Rightarrow -B=(10+2)=12 \Rightarrow B=-12 \Rightarrow A=17; \text{ thus,}$$

$$\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$$

$$8. \quad \frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2(t^2+9)} \text{ (after long division); } \frac{-9t^2+9}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}$$

$$\Rightarrow -9t^2+9 = At(t^2+9) + B(t^2+9) + (Ct+D)t^2 = (A+C)t^3 + (B+D)t^2 + 9At + 9B$$

$$\Rightarrow \left. \begin{aligned} A+C &= 0 \\ B+D &= -9 \\ 9A &= 0 \\ 9B &= 9 \end{aligned} \right\} \Rightarrow A=0 \Rightarrow C=0; B=1 \Rightarrow D=-10; \text{ thus, } \frac{t^4+9}{t^4+9t^2} = 1 + \frac{1}{t^2} + \frac{-10}{t^2+9}$$

$$9. \quad \frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x) + B(1-x); x=1 \Rightarrow A=\frac{1}{2}; x=-1 \Rightarrow B=\frac{1}{2};$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2} \int \frac{dx}{1-x} + \frac{1}{2} \int \frac{dx}{1+x} = \frac{1}{2} [\ln|1+x| - \ln|1-x|] + C$$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + Bx; x=0 \Rightarrow A = \frac{1}{2}; x=-2 \Rightarrow B = -\frac{1}{2};$
 $\int \frac{dx}{x^2+2x} = \frac{1}{2} \int \frac{dx}{x} - \frac{1}{2} \int \frac{dx}{x+2} = \frac{1}{2} [\ln|x| - \ln|x+2|] + C$
11. $\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1) + B(x+6); x=1 \Rightarrow B = \frac{5}{7}; x=-6 \Rightarrow A = \frac{-2}{-7} = \frac{2}{7};$
 $\int \frac{x+4}{x^2+5x-6} dx = \frac{2}{7} \int \frac{dx}{x+6} + \frac{5}{7} \int \frac{dx}{x-1} = \frac{2}{7} \ln|x+6| + \frac{5}{7} \ln|x-1| + C = \frac{1}{7} \ln|(x+6)^2(x-1)^5| + C$
12. $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3) + B(x-4); x=3 \Rightarrow B = \frac{7}{-1} = -7; x=4 \Rightarrow A = \frac{9}{1} = 9;$
 $\int \frac{2x+1}{x^2-7x+12} dx = 9 \int \frac{dx}{x-4} - 7 \int \frac{dx}{x-3} = 9 \ln|x-4| - 7 \ln|x-3| + C = \ln \left| \frac{(x-4)^9}{(x-3)^7} \right| + C$
13. $\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3); y=-1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4}; y=3 \Rightarrow A = \frac{3}{4};$
 $\int_4^8 \frac{y dy}{y^2-2y-3} = \frac{3}{4} \int_4^8 \frac{dy}{y-3} + \frac{1}{4} \int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4} \ln|y-3| + \frac{1}{4} \ln|y+1| \right]_4^8 = \left(\frac{3}{4} \ln 5 + \frac{1}{4} \ln 9 \right) - \left(\frac{3}{4} \ln 1 + \frac{1}{4} \ln 5 \right)$
 $= \frac{1}{2} \ln 5 + \frac{1}{2} \ln 3 = \frac{\ln 15}{2}$
14. $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By; y=0 \Rightarrow A=4; y=-1 \Rightarrow B = \frac{3}{-1} = -3;$
 $\int_{1/2}^1 \frac{y+4}{y^2+y} dy = 4 \int_{1/2}^1 \frac{dy}{y} - 3 \int_{1/2}^1 \frac{dy}{y+1} = [4 \ln|y| - 3 \ln|y+1|]_{1/2}^1 = (4 \ln 1 - 3 \ln 2) - (4 \ln \frac{1}{2} - 3 \ln \frac{3}{2})$
 $= \ln \frac{1}{8} - \ln \frac{1}{16} + \ln \frac{27}{8} = \ln \left(\frac{27}{8} \cdot \frac{1}{8} \cdot 16 \right) = \ln \frac{27}{4}$
15. $\frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2); t=0 \Rightarrow A = -\frac{1}{2}; t=-2 \Rightarrow B = \frac{1}{6};$
 $t=1 \Rightarrow C = \frac{1}{3}; \int \frac{dt}{t^3+t^2-2t} = -\frac{1}{2} \int \frac{dt}{t} + \frac{1}{6} \int \frac{dt}{t+2} + \frac{1}{3} \int \frac{dt}{t-1} = -\frac{1}{2} \ln|t| + \frac{1}{6} \ln|t+2| + \frac{1}{3} \ln|t-1| + C$
16. $\frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) + Cx(x+2); x=0 \Rightarrow A = \frac{3}{-8}; x=-2 \Rightarrow B = \frac{1}{16};$
 $x=2 \Rightarrow C = \frac{5}{16}; \int \frac{x+3}{2x^3-8x} dx = -\frac{3}{8} \int \frac{dx}{x} + \frac{1}{16} \int \frac{dx}{x+2} + \frac{5}{16} \int \frac{dx}{x-2} = -\frac{3}{8} \ln|x| + \frac{1}{16} \ln|x+2| + \frac{5}{16} \ln|x-2| + C$
 $= \frac{1}{16} \ln \left| \frac{(x-2)^5(x+2)}{x^6} \right| + C$
17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1) + B = Ax + (A+B)$
 $\Rightarrow A=3, A+B=2 \Rightarrow A=3, B=-1; \int_0^1 \frac{x^3 dx}{x^2+2x+1} = \int_0^1 (x-2) dx + 3 \int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2}$
 $= \left[\frac{x^2}{2} - 2x + 3 \ln|x+1| + \frac{1}{x+1} \right]_0^1 = \left(\frac{1}{2} - 2 + 3 \ln 2 + \frac{1}{2} \right) - (1) = 3 \ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1) + B = Ax + (-A+B)$
 $\Rightarrow A=3, -A+B=-2 \Rightarrow A=3, B=1$; $\int_{-1}^0 \frac{x^3 dx}{x^2-2x+1} = \int_{-1}^0 (x+2) dx + 3 \int_{-1}^0 \frac{dx}{x-1} + \int_{-1}^0 \frac{dx}{(x-1)^2}$
 $= \left[\frac{x^2}{2} + 2x + 3 \ln|x-1| - \frac{1}{x-1} \right]_{-1}^0 = \left(0 + 0 + 3 \ln 1 - \frac{1}{(-1)} \right) - \left(\frac{1}{2} - 2 + 3 \ln 2 - \frac{1}{(-2)} \right) = 2 - 3 \ln 2$
19. $\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2$;
 $x = -1 \Rightarrow C = \frac{1}{4}$; $x = 1 \Rightarrow D = \frac{1}{4}$; coefficient of $x^3 = A+B \Rightarrow A+B=0$; constant $= A-B+C+D$
 $\Rightarrow A-B+C+D=1 \Rightarrow A-B=\frac{1}{2}$; thus, $A=\frac{1}{4} \Rightarrow B=-\frac{1}{4}$;
 $\int \frac{dx}{(x^2-1)^2} = \frac{1}{4} \int \frac{dx}{x+1} - \frac{1}{4} \int \frac{dx}{x-1} + \frac{1}{4} \int \frac{dx}{(x+1)^2} + \frac{1}{4} \int \frac{dx}{(x-1)^2} = \frac{1}{4} \ln \left| \frac{x+1}{x-1} \right| - \frac{x}{2(x^2-1)} + C$
20. $\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$; $x = -1 \Rightarrow C = -\frac{1}{2}$;
 $x = 1 \Rightarrow A = \frac{1}{4}$; coefficient of $x^2 = A+B \Rightarrow A+B=1 \Rightarrow B = \frac{3}{4}$; $\int \frac{x^2 dx}{(x-1)(x^2+2x+1)}$
 $= \frac{1}{4} \int \frac{dx}{x-1} + \frac{3}{4} \int \frac{dx}{x+1} - \frac{1}{2} \int \frac{dx}{(x+1)^2} = \frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+1| + \frac{1}{2(x+1)} + C = \frac{\ln|(x-1)(x+1)^3|}{4} + \frac{1}{2(x+1)} + C$
21. $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1)$; $x = -1 \Rightarrow A = \frac{1}{2}$; coefficient of $x^2 = A+B$
 $\Rightarrow A+B=0 \Rightarrow B = -\frac{1}{2}$; constant $= A+C \Rightarrow A+C=1 \Rightarrow C = \frac{1}{2}$; $\int_0^1 \frac{dx}{(x+1)(x^2+1)} = \frac{1}{2} \int_0^1 \frac{dx}{x+1} + \frac{1}{2} \int_0^1 \frac{(-x+1)}{x^2+1} dx$
 $= \left[\frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \tan^{-1} x \right]_0^1 = \left(\frac{1}{2} \ln 2 - \frac{1}{4} \ln 2 + \frac{1}{2} \tan^{-1} 1 \right) - \left(\frac{1}{2} \ln 1 - \frac{1}{4} \ln 1 + \frac{1}{2} \tan^{-1} 0 \right)$
 $= \frac{1}{4} \ln 2 + \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{(\pi+2 \ln 2)}{8}$
22. $\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Rightarrow 3t^2+t+4 = A(t^2+1) + (Bt+C)t$; $t=0 \Rightarrow A=4$; coefficient of $t^2 = A+B \Rightarrow A+B=3$
 $\Rightarrow B=-1$; coefficient of $t = C \Rightarrow C=1$; $\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t} dt = 4 \int_1^{\sqrt{3}} \frac{dt}{t} + \int_1^{\sqrt{3}} \frac{(-t+1)}{t^2+1} dt$
 $= \left[4 \ln|t| - \frac{1}{2} \ln(t^2+1) + \tan^{-1} t \right]_1^{\sqrt{3}} = \left(4 \ln \sqrt{3} - \frac{1}{2} \ln 4 + \tan^{-1} \sqrt{3} \right) - \left(4 \ln 1 - \frac{1}{2} \ln 2 + \tan^{-1} 1 \right)$
 $= 2 \ln 3 - \ln 2 + \frac{\pi}{3} + \frac{1}{2} \ln 2 - \frac{\pi}{4} = 2 \ln 3 - \frac{1}{2} \ln 2 + \frac{\pi}{12} = \ln \left(\frac{9}{\sqrt{2}} \right) + \frac{\pi}{12}$
23. $\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2} \Rightarrow y^2+2y+1 = (Ay+B)(y^2+1) + Cy+D = Ay^3 + By^2 + (A+C)y + (B+D)$
 $\Rightarrow A=0, B=1; A+C=2 \Rightarrow C=2; B+D=1 \Rightarrow D=0$; $\int \frac{y^2+2y+1}{(y^2+1)^2} dy = \int \frac{1}{y^2+1} dy + 2 \int \frac{y}{(y^2+1)^2} dy$
 $= \tan^{-1} y - \frac{1}{y^2+1} + C$

$$24. \frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2} \Rightarrow 8x^2+8x+2 = (Ax+B)(4x^2+1) + Cx+D = 4Ax^3+4Bx^2+(A+C)x+(B+D);$$

$$A=0, B=2; A+C=8 \Rightarrow C=8; B+D=2 \Rightarrow D=0; \int \frac{8x^2+8x+2}{(4x^2+1)^2} dx = 2 \int \frac{dx}{4x^2+1} + 8 \int \frac{x dx}{(4x^2+1)^2}$$

$$= \tan^{-1} 2x - \frac{1}{4x^2+1} + C$$

$$25. \frac{2s+2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3}$$

$$\Rightarrow 2s+2 = (As+B)(s-1)^3 + C(s^2+1)(s-1)^2 + D(s^2+1)(s-1) + E(s^2+1)$$

$$= As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B + C(s^4-2s^3+2s^2-2s+1) + D(s^3-s^2+s-1) + E(s^2+1)$$

$$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$$

$$\left. \begin{array}{l} A+C=0 \\ -3A+B-2C+D=0 \\ \Rightarrow 3A-3B+2C-D+E=0 \\ -A+3B-2C+D=2 \\ -B+C-D+E=2 \end{array} \right\} \text{summing all equations} \Rightarrow 2E=4 \Rightarrow E=2;$$

$$\text{summing eqs (2) and (3)} \Rightarrow -2B+2=0 \Rightarrow B=1; \text{summing eqs (3) and (4)} \Rightarrow 2A+2=2 \Rightarrow A=0;$$

$$C=0 \text{ from eq (1); then } -1+0-D+2=2 \text{ from eq (5)} \Rightarrow D=-1;$$

$$\int \frac{2s+2}{(s^2+1)(s-1)^3} ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2 \int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1} s + C$$

$$26. \frac{s^4+81}{s(s^2+9)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \Rightarrow s^4+81 = A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s$$

$$= A(s^4+18s^2+81) + (Bs^4+Cs^3+9Bs^2+9Cs) + Ds^2+Es$$

$$= (A+B)s^4 + Cs^3 + (18A+9B+D)s^2 + (9C+E)s + 81A \Rightarrow 81A=81 \text{ or } A=1; A+B=1 \Rightarrow B=0; C=0;$$

$$9C+E=0 \Rightarrow E=0; 18A+9B+D=0 \Rightarrow D=-18; \int \frac{s^4+81}{s(s^2+9)^2} ds = \int \frac{ds}{s} - 18 \int \frac{s ds}{(s^2+9)^2} = \ln |s| + \frac{9}{s^2+9} + C$$

$$27. \frac{x^2-x+2}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \Rightarrow x^2-x+2 = A(x^2+x+1) + (Bx+C)(x-1) = (A+B)x^2 + (A-B+C)x + (A-C)$$

$$\Rightarrow A+B=1, A-B+C=-1, A-C=2 \Rightarrow \text{adding eq(2) and eq(3)} \Rightarrow 2A-B=1, \text{ add this equation to eq (1)}$$

$$\Rightarrow 3A=2 \Rightarrow A=\frac{2}{3} \Rightarrow B=1-A=\frac{1}{3} \Rightarrow C=-1-A+B=-\frac{4}{3}; \int \frac{x^2-x+2}{x^3-1} dx = \int \left(\frac{2/3}{x-1} + \frac{(1/3)x-4/3}{x^2+x+1} \right) dx$$

$$= \frac{2}{3} \int \frac{1}{x-1} dx + \frac{1}{3} \int \frac{x-4}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx \left[u = x + \frac{1}{2} \Rightarrow u - \frac{1}{2} = x \Rightarrow du = dx \right]$$

$$= \frac{2}{3} \int \frac{1}{x-1} dx + \frac{1}{3} \int \frac{u-\frac{9}{2}}{u^2+\frac{3}{4}} du = \frac{2}{3} \int \frac{1}{x-1} dx + \frac{1}{3} \int \frac{u}{u^2+\frac{3}{4}} du - \frac{3}{2} \int \frac{1}{u^2+\frac{3}{4}} du$$

$$= \frac{2}{3} \ln |x-1| + \frac{1}{6} \ln \left| \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} \right| - \frac{3}{\sqrt{3}} \tan^{-1} \left(\frac{x+\frac{1}{2}}{\sqrt{3}/2} \right) + C = \frac{2}{3} \ln |x-1| + \frac{1}{6} \ln |x^2+x+1| - \sqrt{3} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + C$$

28. $\frac{1}{x^4+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1} \Rightarrow 1 = A(x+1)(x^2-x+1) + Bx(x^2-x+1) + (Cx+D)x(x+1)$
 $= (A+B+C)x^3 + (-B+C+D)x^2 + (B+D)x + A \Rightarrow A=1, B+D=0 \Rightarrow D=-B, -B+C+D=0$
 $\Rightarrow -2B+C=0 \Rightarrow C=2B, A+B+C=0 \Rightarrow 1+B+2B=0 \Rightarrow B=-\frac{1}{3} \Rightarrow C=-\frac{2}{3} \Rightarrow D=\frac{1}{3};$
 $\int \frac{1}{x^4+x} dx = \int \left(\frac{1}{x} - \frac{1/3}{x+1} + \frac{(-2/3)x+1/3}{x^2-x+1} \right) dx = \int \frac{1}{x} dx - \frac{1}{3} \int \frac{1}{x+1} dx - \frac{1}{3} \int \frac{2x-1}{x^2-x+1} dx$
 $= \ln|x| - \frac{1}{3} \ln|x+1| - \frac{1}{3} \ln|x^2-x+1| + C$
29. $\frac{x^2}{x^4-1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \Rightarrow x^2 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x-1)(x+1)$
 $= (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x - A+B-D \Rightarrow A+B+C=0, -A+B+D=1,$
 $A+B-C=0, -A+B-D=0 \Rightarrow$ adding eq (1) to eq (3) gives $2A+2B=0$, adding eq (2) to eq (4) gives
 $-2A+2B=1$, adding these two equations gives $4B=1 \Rightarrow B=\frac{1}{4}$, using $2A+2B=0 \Rightarrow A=-\frac{1}{4}$, using
 $-A+B-D=0 \Rightarrow D=\frac{1}{2}$, and using $A+B-C=0 \Rightarrow C=0; \int \frac{x^2}{x^4-1} dx = \int \left(\frac{-1/4}{x+1} + \frac{1/4}{x-1} + \frac{1/2}{x^2+1} \right) dx$
 $= -\frac{1}{4} \int \frac{1}{x+1} dx + \frac{1}{4} \int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{1}{x^2+1} dx = -\frac{1}{4} \ln|x+1| + \frac{1}{4} \ln|x-1| + \frac{1}{2} \tan^{-1} x + C = \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| + \frac{1}{2} \tan^{-1} x + C$
30. $\frac{x^2+x}{x^4-3x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1} \Rightarrow x^2+x = A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x-2)(x+2)$
 $= (A+B+C)x^3 + (2A-2B+D)x^2 + (A+B-4C)x + 2A-2B-4D \Rightarrow A+B+C=0, 2A-2B+D=1,$
 $A+B-4C=1, 2A-2B-4D=0 \Rightarrow$ subtracting eq (1) from eq (3) gives $-5C=1 \Rightarrow C=-\frac{1}{5}$, subtracting
eq (2) from eq (4) gives $-5D=-1 \Rightarrow D=\frac{1}{5}$, substituting for C in eq (1) gives $A+B=\frac{1}{5}$, and substituting
for D in eq (4) gives $2A-2B=\frac{4}{5} \Rightarrow A-B=\frac{2}{5}$, adding this equation to the previous equation gives
 $2A=\frac{3}{5} \Rightarrow A=\frac{3}{10} \Rightarrow B=-\frac{1}{10}; \int \frac{x^2+x}{x^4-3x^2-4} dx = \int \left(\frac{3/10}{x-2} - \frac{1/10}{x+2} + \frac{(-1/5)x+1/5}{x^2+1} \right) dx$
 $= \frac{3}{10} \int \frac{1}{x-2} dx - \frac{1}{10} \int \frac{1}{x+2} dx - \frac{1}{5} \int \frac{x}{x^2+1} dx + \frac{1}{5} \int \frac{1}{x^2+1} dx = \frac{3}{10} \ln|x-2| - \frac{1}{10} \ln|x+2| - \frac{1}{10} \ln|x^2+1| + \frac{1}{5} \tan^{-1} x + C$
31. $\frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3+5\theta^2+8\theta+4 = (A\theta+B)(\theta^2+2\theta+2) + C\theta+D$
 $= A\theta^3 + (2A+B)\theta^2 + (2A+2B+C)\theta + (2B+D) \Rightarrow A=2; 2A+B=5 \Rightarrow B=1; 2A+2B+C=8 \Rightarrow C=2;$
 $2B+D=4 \Rightarrow D=2; \int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} d\theta = \int \frac{2\theta+1}{\theta^2+2\theta+2} d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2} d\theta$
 $= \int \frac{(2\theta+2)d\theta}{\theta^2+2\theta+2} - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{(2\theta+2)d\theta}{(\theta^2+2\theta+2)^2} = \ln(\theta^2+2\theta+2) - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2}$
 $= \ln(\theta^2+2\theta+2) - \tan^{-1}(\theta+1) - \frac{1}{\theta^2+2\theta+2} + C$
32. $\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{(\theta^2+1)^2} + \frac{E\theta+F}{(\theta^2+1)^3}$
 $\Rightarrow \theta^4-4\theta^3+2\theta^2-3\theta+1 = (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + E\theta+F$
 $= (A\theta+B)(\theta^4+2\theta^2+1) + (C\theta^3+D\theta^2+C\theta+D) + E\theta+F$

$$\begin{aligned}
&= (A\theta^5 + B\theta^4 + 2A\theta^3 + 2B\theta^2 + A\theta + B) + (C\theta^3 + D\theta^2 + C\theta + D) + E\theta + F \\
&= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + (B+D+F) \Rightarrow A=0; B=1; \\
&2A+C=-4 \Rightarrow C=-4; 2B+D=2 \Rightarrow D=0; A+C+E=-3 \Rightarrow E=1; B+D+F=1 \Rightarrow F=0; \\
&\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3} d\theta = \int \frac{d\theta}{\theta^2 + 1} - 4 \int \frac{\theta d\theta}{(\theta^2 + 1)^2} + \int \frac{\theta d\theta}{(\theta^2 + 1)^3} = \tan^{-1} \theta + 2(\theta^2 + 1)^{-1} - \frac{1}{4}(\theta^2 + 1)^{-2} + C
\end{aligned}$$

$$\begin{aligned}
33. \quad &\frac{2x^3 - 2x^2 + 1}{x^2 - x} = 2x + \frac{1}{x^2 - x} = 2x + \frac{1}{x(x-1)}; \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + Bx; x=0 \Rightarrow A=-1; x=1 \Rightarrow B=1; \\
&\int \frac{2x^3 - 2x^2 + 1}{x^2 - x} = \int 2x dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C
\end{aligned}$$

$$\begin{aligned}
34. \quad &\frac{x^4}{x^2 - 1} = (x^2 + 1) + \frac{1}{x^2 - 1} = (x^2 + 1) + \frac{1}{(x+1)(x-1)}; \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x+1); \\
&x=-1 \Rightarrow A=-\frac{1}{2}; x=1 \Rightarrow B=\frac{1}{2}; \int \frac{x^4}{x^2 - 1} dx = \int (x^2 + 1) dx - \frac{1}{2} \int \frac{dx}{x+1} + \frac{1}{2} \int \frac{dx}{x-1} \\
&= \frac{1}{3}x^3 + x - \frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + C = \frac{x^3}{3} + x + \frac{1}{2} \ln\left|\frac{x-1}{x+1}\right| + C
\end{aligned}$$

$$\begin{aligned}
35. \quad &\frac{9x^3 - 3x + 1}{x^3 - x^2} = 9 + \frac{9x^2 - 3x + 1}{x^2(x-1)} \text{ (after long division)}; \frac{9x^2 - 3x + 1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \\
&\Rightarrow 9x^2 - 3x + 1 = Ax(x-1) + B(x-1) + Cx^2; x=1 \Rightarrow C=7; x=0 \Rightarrow B=-1; A+C=9 \Rightarrow A=2; \\
&\int \frac{9x^3 - 3x + 1}{x^3 - x^2} dx = \int 9 dx + 2 \int \frac{dx}{x} - \int \frac{dx}{x^2} + 7 \int \frac{dx}{x-1} = 9x + 2 \ln|x| + \frac{1}{x} + 7 \ln|x-1| + C
\end{aligned}$$

$$\begin{aligned}
36. \quad &\frac{16x^3}{4x^2 - 4x + 1} = (4x + 4) + \frac{12x - 4}{4x^2 - 4x + 1}; \frac{12x - 4}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} \Rightarrow 12x - 4 = A(2x-1) + B \\
&\Rightarrow A=6; -A+B=-4 \Rightarrow B=2; \int \frac{16x^3}{4x^2 - 4x + 1} dx = 4 \int (x+1) dx + 6 \int \frac{dx}{2x-1} + 2 \int \frac{dx}{(2x-1)^2} \\
&= 2(x+1)^2 + 3 \ln|2x-1| - \frac{1}{2x-1} + C_1 = 2x^2 + 4x + 3 \ln|2x-1| - (2x-1)^{-1} + C, \text{ where } C = 2 + C_1
\end{aligned}$$

$$\begin{aligned}
37. \quad &\frac{y^4 + y^2 - 1}{y^3 + y} = y - \frac{1}{y(y^2 + 1)}; \frac{1}{y(y^2 + 1)} = \frac{A}{y} + \frac{By+C}{y^2 + 1} \Rightarrow 1 = A(y^2 + 1) + (By+C)y = (A+B)y^2 + Cy + A \\
&\Rightarrow A=1; A+B=0 \Rightarrow B=-1; C=0; \int \frac{y^4 + y^2 - 1}{y^3 + y} dy = \int y dy - \int \frac{dy}{y} + \int \frac{y dy}{y^2 + 1} = \frac{y^2}{2} - \ln|y| + \frac{1}{2} \ln(1 + y^2) + C
\end{aligned}$$

$$\begin{aligned}
38. \quad &\frac{2y^4}{y^3 - y^2 + y - 1} = 2y + 2 + \frac{2}{y^3 - y^2 + y - 1}; \frac{2}{y^3 - y^2 + y - 1} = \frac{2}{(y^2 + 1)(y-1)} = \frac{A}{y-1} + \frac{By+C}{y^2 + 1} \\
&\Rightarrow 2 = A(y^2 + 1) + (By+C)(y-1) = (Ay^2 + A) + (By^2 + Cy - By - C) = (A+B)y^2 + (-B+C)y + (A-C) \\
&\Rightarrow A+B=0, -B+C=0 \text{ or } C=B, A-C=A-B=2 \Rightarrow A=1, B=-1, C=-1; \\
&\int \frac{2y^4}{y^3 - y^2 + y - 1} dy = 2 \int (y+1) dy + \int \frac{dy}{y-1} - \int \frac{y}{y^2 + 1} dy - \int \frac{dy}{y^2 + 1} = (y+1)^2 + \ln|y-1| - \frac{1}{2} \ln(y^2 + 1) - \tan^{-1} y + C_1 \\
&= y^2 + 2y + \ln|y-1| - \frac{1}{2} \ln(y^2 + 1) - \tan^{-1} y + C, \text{ where } C = C_1 + 1
\end{aligned}$$

$$39. \quad \int \frac{e^t dt}{e^{2t} + 3e^t + 2}; [e^t = y, e^t dt = dy] \rightarrow \int \frac{dy}{y^2 + 3y + 2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2} = \ln\left|\frac{y+1}{y+2}\right| + C = \ln\left(\frac{e^t + 1}{e^t + 2}\right) + C$$

40. $\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1} dt = \int \frac{e^{3t} + 2e^t - 1}{e^{2t} + 1} e^t dt; [y = e^t, dy = e^t dt] \rightarrow \int \frac{y^3 + 2y - 1}{y^2 + 1} dy = \int \left(y + \frac{y-1}{y^2 + 1} \right) dy = \frac{y^2}{2} + \int \frac{y}{y^2 + 1} dy - \int \frac{dy}{y^2 + 1}$
 $= \frac{y^2}{2} + \frac{1}{2} \ln(y^2 + 1) - \tan^{-1} y + C = \frac{1}{2} e^{2t} + \frac{1}{2} \ln(e^{2t} + 1) - \tan^{-1}(e^t) + C$
41. $\int \frac{\cos y dy}{\sin^2 y + \sin y - 6}; [\sin y = t, \cos y dy = dt] \rightarrow \int \frac{dt}{t^2 + t - 6} = \frac{1}{5} \int \left(\frac{1}{t-2} - \frac{1}{t+3} \right) dt = \frac{1}{5} \ln \left| \frac{t-2}{t+3} \right| + C = \frac{1}{5} \ln \left| \frac{\sin y - 2}{\sin y + 3} \right| + C$
42. $\int \frac{\sin \theta d\theta}{\cos^2 \theta + \cos \theta - 2}; [\cos \theta = y, -\sin \theta d\theta = dy] \rightarrow -\int \frac{dy}{y^2 + y - 2} = \frac{1}{3} \int \frac{dy}{y+2} - \frac{1}{3} \int \frac{dy}{y-1} = \frac{1}{3} \ln \left| \frac{y+2}{y-1} \right| + C = \frac{1}{3} \ln \left| \frac{\cos \theta + 2}{\cos \theta - 1} \right| + C$
 $= -\frac{1}{3} \ln \left| \frac{\cos \theta - 1}{\cos \theta + 2} \right| + C$
43. $\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2 + 1)(x-2)^2} dx = \int \frac{\tan^{-1}(2x)}{4x^2 + 1} dx - 3 \int \frac{x}{(x-2)^2} dx = \frac{1}{2} \int \tan^{-1}(2x) \frac{2dx}{4x^2 + 1} - 3 \int \frac{dx}{x-2} - 6 \int \frac{dx}{(x-2)^2}$
 $= \frac{1}{4} (\tan^{-1} 2x)^2 - 3 \ln |x-2| + \frac{6}{x-2} + C$
44. $\int \frac{(x+1)^2 \tan^{-1}(3x) + 9x^3 + x}{(9x^2 + 1)(x+1)^2} dx = \int \frac{\tan^{-1}(3x)}{9x^2 + 1} dx + \int \frac{x}{(x+1)^2} dx = \frac{1}{3} \int \tan^{-1}(3x) \frac{3dx}{9x^2 + 1} + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2}$
 $= \frac{1}{6} (\tan^{-1} 3x)^2 + \ln |x+1| + \frac{1}{x+1} + C$
45. $\int \frac{1}{x^{3/2} - \sqrt{x}} dx = \int \frac{1}{\sqrt{x}(x-1)} dx; [\text{Let } u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2du = \frac{1}{\sqrt{x}} dx] \rightarrow \int \frac{2}{u^2 - 1} du;$
 $\frac{2}{u^2 - 1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A+B=0, -A+B=2 \Rightarrow B=1 \Rightarrow A=-1;$
 $\int \frac{2}{u^2 - 1} du = \int \left(\frac{-1}{u+1} + \frac{1}{u-1} \right) du = -\int \frac{1}{u+1} du + \int \frac{1}{u-1} du = -\ln |u+1| + \ln |u-1| + C = \ln \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| + C$
46. $\int \frac{1}{(x^{1/3} - 1)\sqrt{x}} dx; [\text{Let } x = u^6 \Rightarrow dx = 6u^5 du] \rightarrow \int \frac{1}{(u^2 - 1)u^3} 6u^5 du = \int \frac{6u^2}{u^2 - 1} du = \int \left(6 + \frac{6}{u^2 - 1} \right) du = 6 \int du + \int \frac{6}{u^2 - 1} du;$
 $\frac{6}{u^2 - 1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 6 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A+B=0, -A+B=6 \Rightarrow B=3 \Rightarrow A=-3;$
 $6 \int du + \int \frac{6}{u^2 - 1} du = 6u + \int \left(\frac{-3}{u+1} + \frac{3}{u-1} \right) du = 6u - 3 \int \frac{1}{u+1} du + 3 \int \frac{1}{u-1} du = 6u - 3 \ln |u+1| + 3 \ln |u-1| + C$
 $= 6x^{1/6} + 3 \ln \left| \frac{x^{1/6} - 1}{x^{1/6} + 1} \right| + C$
47. $\int \frac{\sqrt{x+1}}{x} dx; [\text{Let } x+1 = u^2 \Rightarrow dx = 2u du] \rightarrow \int \frac{u}{u^2 - 1} 2u du = \int \frac{2u^2}{u^2 - 1} du = \int \left(2 + \frac{2}{u^2 - 1} \right) du = 2 \int du + \int \frac{2}{u^2 - 1} du;$
 $\frac{2}{u^2 - 1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A+B=0, -A+B=2 \Rightarrow B=1 \Rightarrow A=-1;$
 $2 \int du + \int \frac{2}{u^2 - 1} du = 2u + \int \left(\frac{-1}{u+1} + \frac{1}{u-1} \right) du = 2u - \int \frac{1}{u+1} du + \int \frac{1}{u-1} du = 2u - \ln |u+1| + \ln |u-1| + C$
 $= 2\sqrt{x+1} + \ln \left| \frac{\sqrt{x+1} - 1}{\sqrt{x+1} + 1} \right| + C$

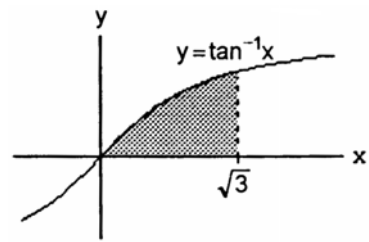
48. $\int \frac{1}{x\sqrt{x+9}} dx$; [Let $x+9 = u^2 \Rightarrow dx = 2u du$] $\rightarrow \int \frac{1}{(u^2-9)u} 2u du = \int \frac{2}{u^2-9} du$; $\frac{2}{u^2-9} = \frac{A}{u-3} + \frac{B}{u+3}$
 $\Rightarrow 2 = A(u+3) + B(u-3) = (A+B)u + 3A-3B \Rightarrow A+B=0, 3A-3B=2 \Rightarrow A=\frac{1}{3} \Rightarrow B=-\frac{1}{3}$;
 $\int \frac{2}{u^2-9} du = \int \left(\frac{1/3}{u-3} - \frac{1/3}{u+3} \right) du = \frac{1}{3} \int \frac{1}{u-3} du - \frac{1}{3} \int \frac{1}{u+3} du = \frac{1}{3} \ln |u-3| - \frac{1}{3} \ln |u+3| + C = \frac{1}{3} \ln \left| \frac{\sqrt{x+9}-3}{\sqrt{x+9}+3} \right| + C$
49. $\int \frac{1}{x(x^4+1)} dx = \int \frac{x^3}{x^4(x^4+1)} dx$; [Let $u = x^4 \Rightarrow du = 4x^3 dx$] $\rightarrow \frac{1}{4} \int \frac{1}{u(u+1)} du$; $\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$
 $\Rightarrow 1 = A(u+1) + Bu = (A+B)u + A \Rightarrow A=1 \Rightarrow B=-1$;
 $\frac{1}{4} \int \frac{1}{u(u+1)} du = \frac{1}{4} \int \left(\frac{1}{u} - \frac{1}{u+1} \right) du = \frac{1}{4} \int \frac{1}{u} du - \frac{1}{4} \int \frac{1}{u+1} du = \frac{1}{4} \ln |u| - \frac{1}{4} \ln |u+1| + C = \frac{1}{4} \ln \left(\frac{x^4}{x^4+1} \right) + C$
50. $\int \frac{1}{x^6(x^5+4)} dx = \int \frac{x^4}{x^{10}(x^5+4)} dx$; [Let $u = x^5 \Rightarrow du = 5x^4 dx$] $\rightarrow \frac{1}{5} \int \frac{1}{u^2(u+4)} du$; $\frac{1}{u^2(u+4)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+4}$
 $\Rightarrow 1 = Au(u+4) + B(u+4) + Cu^2 = (A+C)u^2 + (4A+B)u + 4B \Rightarrow A+C=0, 4A+B=0, 4B=1 \Rightarrow B=\frac{1}{4}$
 $\Rightarrow A=-\frac{1}{16} \Rightarrow C=\frac{1}{16}$; $\frac{1}{5} \int \frac{1}{u^2(u+4)} du = \frac{1}{5} \int \left(-\frac{1/16}{u} + \frac{1/4}{u^2} + \frac{1/16}{u+4} \right) du = -\frac{1}{80} \int \frac{1}{u} du + \frac{1}{20} \int \frac{1}{u^2} du + \frac{1}{80} \int \frac{1}{u+4} du$
 $= -\frac{1}{80} \ln |u| - \frac{1}{20u} + \frac{1}{80} \ln |u+4| + C = -\frac{1}{80} \ln |x^5| - \frac{1}{20x^5} + \frac{1}{80} \ln |x^5+4| + C = \frac{1}{80} \ln \left| \frac{x^5+4}{x^5} \right| - \frac{1}{20x^5} + C$
51. $(t^2-3t+2) \frac{dx}{dt} = 1$; $x = \int \frac{dt}{t^2-3t+2} = \int \frac{dt}{t-2} - \int \frac{dt}{t-1} = \ln \left| \frac{t-2}{t-1} \right| + C$; $\frac{t-2}{t-1} = Ce^x$; $t=3$ and $x=0 \Rightarrow \frac{1}{2} = C$
 $\Rightarrow \frac{t-2}{t-1} = \frac{1}{2} e^x \Rightarrow x = \ln \left| 2 \left(\frac{t-2}{t-1} \right) \right| = \ln |t-2| - \ln |t-1| + \ln 2$
52. $(3t^4+4t^2+1) \frac{dx}{dt} = 2\sqrt{3}$; $x = 2\sqrt{3} \int \frac{dt}{3t^4+4t^2+1} = \sqrt{3} \int \frac{dt}{t^2+\frac{1}{3}} - \sqrt{3} \int \frac{dt}{t^2+1} = 3 \tan^{-1}(\sqrt{3}t) - \sqrt{3} \tan^{-1} t + C$; $t=1$ and
 $x = \frac{-\pi\sqrt{3}}{4} \Rightarrow -\frac{\sqrt{3}\pi}{4} = \pi - \frac{\sqrt{3}}{4} \pi + C \Rightarrow C = -\pi \Rightarrow x = 3 \tan^{-1}(\sqrt{3}t) - \sqrt{3} \tan^{-1} t - \pi$
53. $(t^2+2t) \frac{dx}{dt} = 2x+2$; $\frac{1}{2} \int \frac{dx}{x+1} = \int \frac{dt}{t^2+2t} \Rightarrow \frac{1}{2} \ln |x+1| = \frac{1}{2} \int \frac{dt}{t} - \frac{1}{2} \int \frac{dt}{t+2} \Rightarrow \ln |x+1| = \ln \left| \frac{t}{t+2} \right| + C$; $t=1$ and
 $x=1 \Rightarrow \ln 2 = \ln \frac{1}{3} + C \Rightarrow C = \ln 2 + \ln 3 = \ln 6 \Rightarrow \ln |x+1| = \ln 6 \left| \frac{t}{t+2} \right| \Rightarrow x+1 = \frac{6t}{t+2} \Rightarrow x = \frac{6t}{t+2} - 1, t > 0$
54. $(t+1) \frac{dx}{dt} = x^2+1 \Rightarrow \int \frac{dx}{x^2+1} = \int \frac{dt}{t+1} \Rightarrow \tan^{-1} x = \ln |t+1| + C$; $t=0$ and $x=0 \Rightarrow \tan^{-1} 0 = \ln |1| + C$
 $\Rightarrow C = \tan^{-1} 0 = 0 \Rightarrow \tan^{-1} x = \ln |t+1| \Rightarrow x = \tan(\ln(t+1)), t > -1$
55. $V = \pi \int_{0.5}^{2.5} y^2 dx = \pi \int_{0.5}^{2.5} \frac{9}{3x-x^2} dx = 3\pi \left(\int_{0.5}^{2.5} \left(-\frac{1}{x-3} + \frac{1}{x} \right) dx \right) = \left[3\pi \ln \left| \frac{x}{x-3} \right| \right]_{0.5}^{2.5} = 3\pi \ln 25$
56. $V = 2\pi \int_0^1 xy dx = 2\pi \int_0^1 \frac{2x}{(x+1)(2-x)} dx = 4\pi \int_0^1 \left[-\frac{1}{3} \left(\frac{1}{x+1} \right) + \frac{2}{3} \left(\frac{1}{2-x} \right) \right] dx = \left[-\frac{4\pi}{3} (\ln |x+1| + 2 \ln |2-x|) \right]_0^1 = \frac{4\pi}{3} (\ln 2)$

$$57. \quad A = \int_0^{\sqrt{3}} \tan^{-1} x \, dx = \left[x \tan^{-1} x \right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2} \, dx$$

$$= \frac{\pi\sqrt{3}}{3} - \left[\frac{1}{2} \ln(x^2 + 1) \right]_0^{\sqrt{3}} = \frac{\pi\sqrt{3}}{3} - \ln 2;$$

$$\bar{x} = \frac{1}{A} \int_0^{\sqrt{3}} x \tan^{-1} x \, dx = \frac{1}{A} \left(\left[\frac{1}{2} x^2 \tan^{-1} x \right]_0^{\sqrt{3}} - \frac{1}{2} \int_0^{\sqrt{3}} \frac{x^2}{1+x^2} \, dx \right)$$

$$= \frac{1}{A} \left[\frac{\pi}{2} - \left[\frac{1}{2} (x - \tan^{-1} x) \right]_0^{\sqrt{3}} \right] = \frac{1}{A} \left(\frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{\pi}{6} \right) = \frac{1}{A} \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \cong 1.10$$



$$58. \quad A = \int_3^5 \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} \, dx = 3 \int_3^5 \frac{dx}{x} - \int_3^5 \frac{dx}{x+3} + 2 \int_3^5 \frac{dx}{x-1} = \left[3 \ln|x| - \ln|x+3| + 2 \ln|x-1| \right]_3^5 = \ln \frac{125}{9};$$

$$\bar{x} = \frac{1}{A} \int_3^5 \frac{x(4x^2 + 13x - 9)}{x^3 + 2x^2 - 3x} \, dx = \frac{1}{A} \left([4x]_3^5 + 3 \int_3^5 \frac{dx}{x+3} + 2 \int_3^5 \frac{dx}{x-1} \right) = \frac{1}{A} (8 + 11 \ln 2 - 3 \ln 6) \cong 3.90$$

$$59. \quad (a) \quad \frac{dx}{dt} = kx(N-x) \Rightarrow \int \frac{dx}{x(N-x)} = \int k \, dt \Rightarrow \frac{1}{N} \int \frac{dx}{x} + \frac{1}{N} \int \frac{dx}{N-x} = \int k \, dt \Rightarrow \frac{1}{N} \ln \left| \frac{x}{N-x} \right| = kt + C;$$

$$k = \frac{1}{250}, N = 1000, t = 0 \text{ and } x = 2 \Rightarrow \frac{1}{1000} \ln \left| \frac{2}{998} \right| = C \Rightarrow \frac{1}{1000} \ln \left| \frac{x}{1000-x} \right| = \frac{t}{250} + \frac{1}{1000} \ln \left(\frac{1}{499} \right)$$

$$\Rightarrow \ln \left| \frac{499x}{1000-x} \right| = 4t \Rightarrow \frac{499x}{1000-x} = e^{4t} \Rightarrow 499x = e^{4t}(1000-x) \Rightarrow (499 + e^{4t})x = 1000e^{4t} \Rightarrow x = \frac{1000e^{4t}}{499 + e^{4t}}$$

$$(b) \quad x = \frac{1}{2} N = 500 \Rightarrow 500 = \frac{1000e^{4t}}{499 + e^{4t}} \Rightarrow 500 \cdot 499 + 500e^{4t} = 1000e^{4t} \Rightarrow e^{4t} = 499 \Rightarrow t = \frac{1}{4} \ln 499 \approx 1.55 \text{ days}$$

$$60. \quad \frac{dx}{dt} = k(a-x)(b-x) \Rightarrow \frac{dx}{(a-x)(b-x)} = k \, dt$$

$$(a) \quad a = b: \int \frac{dx}{(a-x)^2} = \int k \, dt \Rightarrow \frac{1}{a-x} = kt + C; t = 0 \text{ and } x = 0 \Rightarrow \frac{1}{a} = C \Rightarrow \frac{1}{a-x} = kt + \frac{1}{a}$$

$$\Rightarrow \frac{1}{a-x} = \frac{akt+1}{a} \Rightarrow a-x = \frac{a}{akt+1} \Rightarrow x = a - \frac{a}{akt+1} = \frac{a^2 kt}{akt+1}$$

$$(b) \quad a \neq b: \int \frac{dx}{(a-x)(b-x)} = \int k \, dt \Rightarrow \frac{1}{b-a} \int \frac{dx}{a-x} - \frac{1}{b-a} \int \frac{dx}{b-x} = \int k \, dt \Rightarrow \frac{1}{b-a} \ln \left| \frac{b-x}{a-x} \right| = kt + C; t = 0 \text{ and}$$

$$x = 0 \Rightarrow \frac{1}{b-a} \ln \frac{b}{a} = C \Rightarrow \ln \left| \frac{b-x}{a-x} \right| = (b-a)kt + \ln \left(\frac{b}{a} \right) \Rightarrow \frac{b-x}{a-x} = \frac{b}{a} e^{(b-a)kt} \Rightarrow x = \frac{ab[1 - e^{(b-a)kt}]}{a - be^{(b-a)kt}}$$

8.6 INTEGRAL TABLES AND COMPUTER ALGEBRA SYSTEMS

$$1. \quad \int \frac{dx}{x\sqrt{x-3}} = \frac{2}{\sqrt{3}} \tan^{-1} \sqrt{\frac{x-3}{3}} + C$$

(We used FORMULA 29(b) with $a = 1$, $b = 3$)

$$2. \quad \int \frac{dx}{x\sqrt{x+4}} = \frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{x+4} - \sqrt{4}}{\sqrt{x+4} + \sqrt{4}} \right| + C = \frac{1}{2} \ln \left| \frac{\sqrt{x+4} - 2}{\sqrt{x+4} + 2} \right| + C$$

(We used FORMULA 20(a) with $a = 1$, $b = 4$)

$$\begin{aligned}
 3. \quad \int \frac{x dx}{\sqrt{x-2}} &= \int \frac{(x-2)dx}{\sqrt{x-2}} + 2 \int \frac{dx}{\sqrt{x-2}} = \int (\sqrt{x-2})^1 dx + 2 \int (\sqrt{x-2})^{-1} dx \\
 &= \left(\frac{2}{1}\right) \frac{(\sqrt{x-2})^3}{3} + 2 \left(\frac{2}{1}\right) \frac{(\sqrt{x-2})^1}{1} = \sqrt{x-2} \left[\frac{2(x-2)}{3} + 4 \right] + C \\
 &\quad \text{(We used FORMULA 11 with } a=1, b=-2, n=1 \text{ and } a=1, b=-2, n=-1 \text{)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \int \frac{x dx}{(2x+3)^{3/2}} &= \frac{1}{2} \int \frac{(2x+3)dx}{(2x+3)^{3/2}} - \frac{3}{2} \int \frac{dx}{(2x+3)^{3/2}} = \frac{1}{2} \int \frac{dx}{\sqrt{2x+3}} - \frac{3}{2} \int \frac{dx}{(\sqrt{2x+3})^3} \\
 &= \frac{1}{2} \int (\sqrt{2x+3})^{-1} dx - \frac{3}{2} \int (\sqrt{2x+3})^{-3} dx = \left(\frac{1}{2}\right) \left(\frac{2}{2}\right) \frac{(\sqrt{2x+3})^1}{1} - \left(\frac{3}{2}\right) \left(\frac{2}{2}\right) \frac{(\sqrt{2x+3})^{-1}}{(-1)} + C \\
 &= \frac{1}{2\sqrt{2x+3}} (2x+3+3) + C = \frac{(x+3)}{\sqrt{2x+3}} + C \\
 &\quad \text{(We used FORMULA 11 with } a=2, b=3, n=-1 \text{ and } a=2, b=3, n=-3 \text{)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \int x\sqrt{2x-3} dx &= \frac{1}{2} \int (2x-3)\sqrt{2x-3} dx + \frac{3}{2} \int \sqrt{2x-3} dx = \frac{1}{2} \int (\sqrt{2x-3})^3 dx + \frac{3}{2} \int (\sqrt{2x-3})^1 dx \\
 &= \left(\frac{1}{2}\right) \left(\frac{2}{2}\right) \frac{(\sqrt{2x-3})^5}{5} + \left(\frac{3}{2}\right) \left(\frac{2}{2}\right) \frac{(\sqrt{2x-3})^3}{3} + C = \frac{(2x-3)^{3/2}}{2} \left[\frac{2x-3}{5} + 1 \right] + C = \frac{(2x-3)^{3/2}(x+1)}{5} + C \\
 &\quad \text{(We used FORMULA 11 with } a=2, b=-3, n=3 \text{ and } a=2, b=-3, n=1 \text{)}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \int x(7x+5)^{3/2} dx &= \frac{1}{7} \int (7x+5)(7x+5)^{3/2} dx - \frac{5}{7} \int (7x+5)^{3/2} dx = \frac{1}{7} \int (\sqrt{7x+5})^5 dx - \frac{5}{7} \int (\sqrt{7x+5})^3 dx \\
 &= \left(\frac{1}{7}\right) \left(\frac{2}{7}\right) \frac{(\sqrt{7x+5})^7}{7} - \left(\frac{5}{7}\right) \left(\frac{2}{7}\right) \frac{(\sqrt{7x+5})^5}{5} + C = \frac{(7x+5)^{5/2}}{49} \left[\frac{2(7x+5)}{7} - 2 \right] + C = \frac{(7x+5)^{5/2}}{49} \left(\frac{14x-4}{7} \right) + C \\
 &\quad \text{(We used FORMULA 11 with } a=7, b=5, n=5 \text{ and } a=7, b=5, n=3 \text{)}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \int \frac{\sqrt{9-4x}}{x^2} dx &= -\frac{\sqrt{9-4x}}{x} + \frac{(-4)}{2} \int \frac{dx}{x\sqrt{9-4x}} + C \\
 &\quad \text{(We used FORMULA 14 with } a=-4, b=9 \text{)} \\
 &= -\frac{\sqrt{9-4x}}{x} - 2 \left(\frac{1}{\sqrt{9}} \right) \ln \left| \frac{\sqrt{9-4x}-\sqrt{9}}{\sqrt{9-4x}+\sqrt{9}} \right| + C \\
 &\quad \text{(We used FORMULA 29(a) with } a=-4, b=9 \text{)} \\
 &= \frac{-\sqrt{9-4x}}{x} - \frac{2}{3} \ln \left| \frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \int \frac{dx}{x^2\sqrt{4x-9}} &= -\frac{\sqrt{4x-9}}{(-9)x} + \frac{4}{18} \int \frac{dx}{x\sqrt{4x-9}} + C \\
 &\quad \text{(We used FORMULA 15 with } a=4, b=-9 \text{)} \\
 &= \frac{\sqrt{4x-9}}{9x} + \left(\frac{2}{9}\right) \left(\frac{2}{\sqrt{9}}\right) \tan^{-1} \sqrt{\frac{4x-9}{9}} + C \\
 &\quad \text{(We used FORMULA 29(b) with } a=4, b=9 \text{)} \\
 &= \frac{\sqrt{4x-9}}{9x} + \frac{4}{27} \tan^{-1} \sqrt{\frac{4x-9}{9}} + C
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \int x\sqrt{4x-x^2} \, dx &= \int x\sqrt{2 \cdot 2x-x^2} \, dx = \frac{(x+2)(2x-3 \cdot 2)\sqrt{2 \cdot 2x-x^2}}{6} + \frac{2^3}{2} \sin^{-1}\left(\frac{x-2}{2}\right) + C \\
 &= \frac{(x+2)(2x-6)\sqrt{4x-x^2}}{6} + 4 \sin^{-1}\left(\frac{x-2}{2}\right) + C = \frac{(x+2)(x-3)\sqrt{4x-x^2}}{3} + 4 \sin^{-1}\left(\frac{x-2}{2}\right) + C \\
 &\quad (\text{We used FORMULA 51 with } a = 2)
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \int \frac{\sqrt{x-x^2}}{x} \, dx &= \int \frac{\sqrt{2 \cdot \frac{1}{2}x-x^2}}{x} \, dx = \sqrt{2 \cdot \frac{1}{2}x-x^2} + \frac{1}{2} \sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right) + C = \sqrt{x-x^2} + \frac{1}{2} \sin^{-1}(2x-1) + C \\
 &\quad (\text{We used FORMULA 52 with } a = \frac{1}{2})
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \int \frac{dx}{x\sqrt{7+x^2}} &= \int \frac{dx}{x\sqrt{(\sqrt{7})^2+x^2}} = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{7}+\sqrt{(\sqrt{7})^2+x^2}}{x} \right| + C = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{7}+\sqrt{7+x^2}}{x} \right| + C \\
 &\quad (\text{We used FORMULA 26 with } a = \sqrt{7})
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \int \frac{dx}{x\sqrt{7-x^2}} &= \int \frac{dx}{x\sqrt{(\sqrt{7})^2-x^2}} = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{7}+\sqrt{(\sqrt{7})^2-x^2}}{x} \right| + C = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{7}+\sqrt{7-x^2}}{x} \right| + C \\
 &\quad (\text{We used FORMULA 34 with } a = \sqrt{7})
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \int \frac{\sqrt{4-x^2}}{x} \, dx &= \int \frac{\sqrt{2^2-x^2}}{x} \, dx = \sqrt{2^2-x^2} - 2 \ln \left| \frac{2+\sqrt{2^2-x^2}}{x} \right| + C = \sqrt{4-x^2} - 2 \ln \left| \frac{2+\sqrt{4-x^2}}{x} \right| + C \\
 &\quad (\text{We used FORMULA 31 with } a = 2)
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \int \frac{\sqrt{x^2-4}}{x} \, dx &= \int \frac{\sqrt{x^2-2^2}}{x} \, dx = \sqrt{x^2-2^2} - 2 \sec^{-1} \left| \frac{x}{2} \right| + C = \sqrt{x^2-4} - 2 \sec^{-1} \left| \frac{x}{2} \right| + C \\
 &\quad (\text{We used FORMULA 42 with } a = 2)
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \int e^{2t} \cos 3t \, dt &= \frac{e^{2t}}{2^2+3^2} (2 \cos 3t + 3 \sin 3t) + C = \frac{e^{2t}}{13} (2 \cos 3t + 3 \sin 3t) + C \\
 &\quad (\text{We used FORMULA 108 with } a = 2, b = 3)
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \int e^{-3t} \sin 4t \, dt &= \frac{e^{-3t}}{(-3)^2+4^2} (-3 \sin 4t - 4 \cos 4t) + C = \frac{e^{-3t}}{25} (-3 \sin 4t - 4 \cos 4t) + C \\
 &\quad (\text{We used FORMULA 107 with } a = -3, b = 4)
 \end{aligned}$$

$$\begin{aligned}
 17. \quad \int x \cos^{-1} x \, dx &= \int x^1 \cos^{-1} x \, dx = \frac{x^{1+1}}{1+1} \cos^{-1} x + \frac{1}{1+1} \int \frac{x^{1+1} dx}{\sqrt{1-x^2}} = \frac{x^2}{2} \cos^{-1} x + \frac{1}{2} \int \frac{x^2 dx}{\sqrt{1-x^2}} \\
 &\quad (\text{We used FORMULA 100 with } a = 1, n = 1) \\
 &= \frac{x^2}{2} \cos^{-1} x + \frac{1}{2} \left(\frac{1}{2} \sin^{-1} x \right) - \frac{1}{2} \left(\frac{1}{2} x \sqrt{1-x^2} \right) + C = \frac{x^2}{2} \cos^{-1} x + \frac{1}{4} \sin^{-1} x - \frac{1}{4} x \sqrt{1-x^2} + C \\
 &\quad (\text{We used FORMULA 33 with } a = 1)
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \int x \tan^{-1} x \, dx &= \int x^1 \tan^{-1}(1x) \, dx = \frac{x^{1+1}}{1+1} \tan^{-1}(1x) - \frac{1}{1+1} \int \frac{x^{1+1} dx}{1+(1)^2 x^2} = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2 dx}{1+x^2} \\
 &\quad (\text{We used FORMULA 101 with } a = 1, n = 1)
 \end{aligned}$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx \quad (\text{after long division})$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{1}{1+x^2} dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C = \frac{1}{2} \left((x^2 + 1) \tan^{-1} x - x \right) + C$$

$$19. \int x^2 \tan^{-1} x \, dx = \frac{x^{2+1}}{2+1} \tan^{-1} x - \frac{1}{2+1} \int \frac{x^{2+1}}{1+x^2} dx = \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx$$

(We used FORMULA 101 with $a = 1, n = 2$)

$$\int \frac{x^3}{1+x^2} dx = \int x \, dx - \int \frac{x \, dx}{1+x^2} = \frac{x^2}{2} - \frac{1}{2} \ln(1+x^2) + C \Rightarrow \int x^2 \tan^{-1} x \, dx = \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \ln(1+x^2) + C$$

$$20. \int \frac{\tan^{-1} x}{x^2} dx = \int x^{-2} \tan^{-1} x \, dx = \frac{x^{(-2+1)}}{(-2+1)} \tan^{-1} x - \frac{1}{(-2+1)} \int \frac{x^{(-2+1)}}{1+x^2} dx = \frac{x^{-1}}{(-1)} \tan^{-1} x + \int \frac{x^{-1}}{1+x^2} dx$$

(We used FORMULA 101 with $a = 1, n = -2$)

$$\int \frac{x^{-1} dx}{1+x^2} = \int \frac{dx}{x(1+x^2)} = \int \frac{dx}{x} - \int \frac{x \, dx}{1+x^2} = \ln|x| - \frac{1}{2} \ln(1+x^2) + C \Rightarrow \int \frac{\tan^{-1} x}{x^2} dx = -\frac{1}{x} \tan^{-1} x + \ln|x| - \frac{1}{2} \ln(1+x^2) + C$$

$$21. \int \sin 3x \cos 2x \, dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$$

(We used FORMULA 62(a) with $a = 3, b = 2$)

$$22. \int \sin 2x \cos 3x \, dx = -\frac{\cos 5x}{10} + \frac{\cos x}{2} + C$$

(We used FORMULA 62(a) with $a = 2, b = 3$)

$$23. \int 8 \sin 4t \sin \frac{t}{2} \, dt = \frac{8}{7} \sin\left(\frac{7t}{2}\right) - \frac{8}{9} \sin\left(\frac{9t}{2}\right) + C = 8 \left[\frac{\sin\left(\frac{7t}{2}\right)}{7} - \frac{\sin\left(\frac{9t}{2}\right)}{9} \right] + C$$

(We used FORMULA 62(b) with $a = 4, b = \frac{1}{2}$)

$$24. \int \sin \frac{t}{3} \sin \frac{t}{6} \, dt = 3 \sin\left(\frac{t}{6}\right) - \sin\left(\frac{t}{2}\right) + C$$

(We used FORMULA 62(b) with $a = \frac{1}{3}, b = \frac{1}{6}$)

$$25. \int \cos \frac{\theta}{3} \cos \frac{\theta}{4} \, d\theta = 6 \sin\left(\frac{\theta}{12}\right) + \frac{6}{7} \sin\left(\frac{7\theta}{12}\right) + C$$

(We used FORMULA 62(c) with $a = \frac{1}{3}, b = \frac{1}{4}$)

$$26. \int \cos \frac{\theta}{2} \cos 7\theta \, d\theta = \frac{1}{13} \sin\left(\frac{13\theta}{2}\right) + \frac{1}{15} \sin\left(\frac{15\theta}{2}\right) + C = \frac{\sin\left(\frac{13\theta}{2}\right)}{13} + \frac{\sin\left(\frac{15\theta}{2}\right)}{15} + C$$

(We used FORMULA 62(c) with $a = \frac{1}{2}, b = 7$)

$$27. \int \frac{x^3+x+1}{(x^2+1)^2} dx = \int \frac{x \, dx}{x^2+1} + \int \frac{dx}{(x^2+1)^2} = \frac{1}{2} \int \frac{2x \, dx}{x^2+1} + \int \frac{dx}{(x^2+1)^2} = \frac{1}{2} \ln(x^2+1) + \frac{x}{2(1+x^2)} + \frac{1}{2} \tan^{-1} x + C$$

(For the second integral we used FORMULA 17 with $a = 1$)

$$28. \int \frac{x^2+6x}{(x^2+3)^2} dx = \int \frac{dx}{x^2+3} + \int \frac{6x dx}{(x^2+3)^2} - \int \frac{3 dx}{(x^2+3)^2} = \int \frac{dx}{x^2+(\sqrt{3})^2} + 3 \int \frac{2x dx}{(x^2+3)^2} - 3 \int \frac{dx}{[x^2+(\sqrt{3})^2]^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{x^2+3} - 3 \left(\frac{x}{2(\sqrt{3})^2((\sqrt{3})^2+x^2)} + \frac{1}{2(\sqrt{3})^3} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) \right) + C$$

(For the first integral we used FORMULA 16 with $a = \sqrt{3}$; for the third integral we used FORMULA 17 with $a = \sqrt{3}$)

$$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{x^2+3} - \frac{x}{2(x^2+3)} + C$$

$$29. \int \sin^{-1} \sqrt{x} dx; \begin{cases} u = \sqrt{x} \\ x = u^2 \\ dx = 2u du \end{cases} \rightarrow 2 \int u^1 \sin^{-1} u du = 2 \left(\frac{u^{1+1}}{1+1} \sin^{-1} u - \frac{1}{1+1} \int \frac{u^{1+1}}{\sqrt{1-u^2}} du \right) = u^2 \sin^{-1} u - \int \frac{u^2 du}{\sqrt{1-u^2}}$$

(We used FORMULA 99 with $a = 1, n = 1$)

$$= u^2 \sin^{-1} u - \left(\frac{1}{2} \sin^{-1} u - \frac{1}{2} u \sqrt{1-u^2} \right) + C = \left(u^2 - \frac{1}{2} \right) \sin^{-1} u + \frac{1}{2} u \sqrt{1-u^2} + C$$

(We used FORMULA 33 with $a = 1$)

$$= \left(x - \frac{1}{2} \right) \sin^{-1} \sqrt{x} + \frac{1}{2} \sqrt{x-x^2} + C$$

$$30. \int \frac{\cos^{-1} \sqrt{x}}{\sqrt{x}} dx; \begin{cases} u = \sqrt{x} \\ x = u^2 \\ dx = 2u du \end{cases} \rightarrow \int \frac{\cos^{-1} u}{u} \cdot 2u du = 2 \int \cos^{-1} u du = 2 \left(u \cos^{-1} u - \frac{1}{1} \sqrt{1-u^2} \right) + C$$

(We used FORMULA 97 with $a = 1$)

$$= 2 \left(\sqrt{x} \cos^{-1} \sqrt{x} - \sqrt{1-x} \right) + C$$

$$31. \int \frac{\sqrt{x}}{\sqrt{1-x}} dx; \begin{cases} u = \sqrt{x} \\ x = u^2 \\ dx = 2u du \end{cases} \rightarrow \int \frac{u \cdot 2u}{\sqrt{1-u^2}} du = 2 \int \frac{u^2}{\sqrt{1-u^2}} du = 2 \left(\frac{1}{2} \sin^{-1} u - \frac{1}{2} u \sqrt{1-u^2} \right) + C = \sin^{-1} u - u \sqrt{1-u^2} + C$$

(We used FORMULA 33 with $a = 1$)

$$= \sin^{-1} \sqrt{x} - \sqrt{x} \sqrt{1-x} + C = \sin^{-1} \sqrt{x} - \sqrt{x-x^2} + C$$

$$32. \int \frac{\sqrt{2-x}}{\sqrt{x}} dx; \begin{cases} u = \sqrt{x} \\ x = u^2 \\ dx = 2u du \end{cases} \rightarrow \int \frac{\sqrt{2-u^2}}{u} \cdot 2u du = 2 \int \sqrt{(\sqrt{2})^2 - u^2} du = 2 \left[\frac{u}{2} \sqrt{(\sqrt{2})^2 - u^2} + \frac{(\sqrt{2})^2}{2} \sin^{-1} \left(\frac{u}{\sqrt{2}} \right) \right] + C$$

(We used FORMULA 29 with $a = \sqrt{2}$)

$$= u \sqrt{2-u^2} + 2 \sin^{-1} \left(\frac{u}{\sqrt{2}} \right) + C = \sqrt{2x-x^2} + 2 \sin^{-1} \sqrt{\frac{x}{2}} + C$$

$$33. \int (\cot t) \sqrt{1 - \sin^2 t} \, dt = \int \frac{\sqrt{1 - \sin^2 t} (\cos t) \, dt}{\sin t}; \left[\begin{array}{l} u = \sin t \\ du = \cos t \, dt \end{array} \right] \rightarrow \int \frac{\sqrt{1 - u^2} \, du}{u} = \sqrt{1 - u^2} - \ln \left| \frac{1 + \sqrt{1 - u^2}}{u} \right| + C$$

(We used FORMULA 31 with $a = 1$)

$$= \sqrt{1 - \sin^2 t} - \ln \left| \frac{1 + \sqrt{1 - \sin^2 t}}{\sin t} \right| + C$$

$$34. \int \frac{dt}{(\tan t) \sqrt{4 - \sin^2 t}} = \int \frac{\cos t \, dt}{(\sin t) \sqrt{4 - \sin^2 t}}; [u = \sin t, du = \cos t \, dt] \rightarrow \int \frac{du}{u \sqrt{4 - u^2}} = -\frac{1}{2} \ln \left| \frac{2 + \sqrt{4 - u^2}}{u} \right| + C$$

(We used FORMULA 34 with $a = 2$)

$$= -\frac{1}{2} \ln \left| \frac{2 + \sqrt{4 - \sin^2 t}}{\sin t} \right| + C$$

$$35. \int \frac{dy}{y \sqrt{3 + (\ln y)^2}}; \left[\begin{array}{l} u = \ln y \\ y = e^u \\ dy = e^u \, du \end{array} \right] \rightarrow \int \frac{e^u \, du}{e^u \sqrt{3 + u^2}} = \int \frac{du}{\sqrt{3 + u^2}} = \ln \left| u + \sqrt{3 + u^2} \right| + C = \ln \left| \ln y + \sqrt{3 + (\ln y)^2} \right| + C$$

(We used FORMULA 20 with $a = \sqrt{3}$)

$$36. \int \tan^{-1} \sqrt{y} \, dy; \left[\begin{array}{l} t = \sqrt{y} \\ y = t^2 \\ dy = 2t \, dt \end{array} \right] \rightarrow 2 \int t \tan^{-1} t \, dt = 2 \left[\frac{t^2}{2} \tan^{-1} t - \frac{1}{2} \int \frac{t^2}{1 + t^2} \, dt \right] = t^2 \tan^{-1} t - \int \frac{t^2}{1 + t^2} \, dt$$

(We used FORMULA 101 with $n = 1, a = 1$)

$$= t^2 \tan^{-1} t - \int \frac{t^2 + 1}{t^2 + 1} \, dt + \int \frac{dt}{1 + t^2} = t^2 \tan^{-1} t - t + \tan^{-1} t + C = y \tan^{-1} \sqrt{y} + \tan^{-1} \sqrt{y} - \sqrt{y} + C$$

$$37. \int \frac{1}{\sqrt{x^2 + 2x + 5}} \, dx = \int \frac{1}{\sqrt{(x+1)^2 + 4}} \, dx; [t = x+1, dt = dx] \rightarrow \int \frac{1}{\sqrt{t^2 + 4}} \, dt = \ln \left| t + \sqrt{t^2 + 4} \right| + C$$

(We used FORMULA 20 with $a = 2$)

$$= \ln \left| (x+1) + \sqrt{(x+1)^2 + 4} \right| + C = \ln \left| (x+1) + \sqrt{x^2 + 2x + 5} \right| + C$$

$$38. \int \frac{x^2}{\sqrt{x^2 - 4x + 5}} \, dx = \int \frac{x^2}{\sqrt{(x-2)^2 + 1}} \, dx; \left[\begin{array}{l} t = x-2 \\ dt = dx \end{array} \right] \rightarrow \int \frac{(t+2)^2}{\sqrt{t^2 + 1}} \, dt = \int \frac{t^2 + 4t + 2}{\sqrt{t^2 + 1}} \, dt = \int \frac{t^2}{\sqrt{t^2 + 1}} \, dt + \int \frac{4t}{\sqrt{t^2 + 1}} \, dt + \int \frac{2}{\sqrt{t^2 + 1}} \, dt$$

(We used FORMULA 25 with $a = 1$)

(We used FORMULA 20 with $a = 1$)

$$= \left[-\frac{1}{2} \ln \left| t + \sqrt{t^2 + 1} \right| + \frac{t \sqrt{t^2 + 1}}{2} \right] + 4 \sqrt{t^2 + 1} + \left[4 \ln \left| t + \sqrt{t^2 + 1} \right| \right] + C$$

$$= -\frac{1}{2} \ln \left| (x-2) + \sqrt{(x-2)^2 + 1} \right| + \frac{(x-2) \sqrt{(x-2)^2 + 1}}{2} + 4 \sqrt{(x-2)^2 + 1} + 4 \ln \left| (x-2) + \sqrt{(x-2)^2 + 1} \right| + C$$

$$= \frac{7}{2} \ln \left| (x-2) + \sqrt{x^2 - 4x + 5} \right| + \frac{(x+6) \sqrt{x^2 - 4x + 5}}{2} + C$$

39. $\int \sqrt{5-4x-x^2} dx = \int \sqrt{9-(x+2)^2} dx; [t = x+2, dt = dx] \rightarrow \int \sqrt{9-t^2} dt = \frac{t}{2}\sqrt{9-t^2} + \frac{3^2}{2}\sin^{-1}\left(\frac{t}{3}\right) + C$
 (We used FORMULA 29 with $a = 3$)
 $= \frac{x+2}{2}\sqrt{9-(x+2)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + C = \frac{x+2}{2}\sqrt{5-4x-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + C$
40. $\int x^2\sqrt{2x-x^2} dx = \int x^2\sqrt{1-(x-1)^2} dx; [t = x-1, dt = dx] \rightarrow \int (t+1)^2\sqrt{1-t^2} dt = \int (t^2+2t+1)\sqrt{1-t^2} dt$
 $= \int t^2\sqrt{1-t^2} dt + \int 2t\sqrt{1-t^2} dt + \int \sqrt{1-t^2} dt$
 (We used FORMULA 30 with $a = 1$) (We used FORMULA 29 with $a = 1$)
 $= \left[\frac{1^4}{8}\sin^{-1}\left(\frac{t}{1}\right) - \frac{1}{8}t\sqrt{1-t^2}(1^2-2t^2) \right] - \frac{2}{3}(1-t^2)^{3/2} + \left[\frac{t}{2}\sqrt{1-t^2} + \frac{1^2}{2}\sin^{-1}\left(\frac{t}{1}\right) \right] + C$
 $= \frac{1}{8}\sin^{-1}(x-1) - \frac{1}{8}(x-1)\sqrt{1-(x-1)^2}(1^2-2(x-1)^2) - \frac{2}{3}(1-(x-1)^2)^{3/2} + \frac{x-1}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1) + C$
 $= \frac{5}{8}\sin^{-1}(x-1) - \frac{2}{3}(2x-x^2)^{3/2} + \frac{x-1}{8}\sqrt{2x-x^2}(2x^2-4x+5) + C$
41. $\int \sin^5 2x dx = -\frac{\sin^4 2x \cos 2x}{5 \cdot 2} + \frac{5-1}{5} \int \sin^3 2x dx = -\frac{\sin^4 2x \cos 2x}{10} + \frac{4}{5} \left[-\frac{\sin^2 2x \cos 2x}{3 \cdot 2} + \frac{3-1}{3} \int \sin 2x dx \right]$
 (We used FORMULA 60 with $a = 2, n = 5$ and $a = 2, n = 3$)
 $= -\frac{\sin^4 2x \cos 2x}{10} - \frac{8}{15} \sin^2 2x \cos 2x + \frac{8}{15} \left(-\frac{1}{2} \right) \cos 2x + C = -\frac{\sin^4 2x \cos 2x}{10} - \frac{2 \sin^2 2x \cos 2x}{15} - \frac{4 \cos 2x}{15} + C$
42. $\int 8 \cos^4 2\pi t dt = 8 \left(\frac{\cos^3 2\pi t \sin 2\pi t}{4 \cdot 2\pi} + \frac{4-1}{4} \int \cos^2 2\pi t dt \right)$
 (We used FORMULA 61 with $a = 2\pi, n = 4$)
 $= \frac{\cos^3 2\pi t \sin 2\pi t}{\pi} + 6 \left[\frac{t}{2} + \frac{\sin(2 \cdot 2\pi \cdot t)}{4 \cdot 2\pi} \right] + C$
 (We used FORMULA 59 with $a = 2\pi$)
 $= \frac{\cos^3 2\pi t \sin 2\pi t}{\pi} + 3t + \frac{3 \sin 4\pi t}{4\pi} + C = \frac{\cos^3 2\pi t \sin 2\pi t}{\pi} + \frac{3 \cos 2\pi t \sin 2\pi t}{2\pi} + 3t + C$
43. $\int \sin^2 2\theta \cos^3 2\theta d\theta = \frac{\sin^3 2\theta \cos^2 2\theta}{2(2+3)} + \frac{3-1}{3+2} \int \sin^2 2\theta \cos 2\theta d\theta$
 (We used FORMULA 69 with $a = 2, m = 3, n = 2$)
 $= \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{2}{5} \int \sin^2 2\theta \cos 2\theta d\theta = \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{2}{5} \left[\frac{1}{2} \int \sin^2 2\theta (\cos 2\theta) 2d\theta \right] = \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{\sin^3 2\theta}{15} + C$
44. $\int 2 \sin^2 t \sec^4 t dt = \int 2 \sin^2 t \cos^{-4} t dt = 2 \left(-\frac{\sin t \cos^{-3} t}{2-4} + \frac{2-1}{2-4} \int \cos^{-4} t dt \right)$
 (We used FORMULA 68 with $a = 1, n = 2, m = -4$)
 $= \sin t \cos^{-3} t - \int \cos^{-4} t dt = \sin t \cos^{-3} t - \int \sec^4 t dt = \sin t \cos^{-3} t - \left(\frac{\sec^2 t \tan t}{4-1} + \frac{4-2}{4-1} \int \sec^2 t dt \right)$
 (We used FORMULA 92 with $a = 1, n = 4$)
 $= \sin t \cos^{-3} t - \left(\frac{\sec^2 t \tan t}{3} \right) - \frac{2}{3} \tan t + C = \frac{2}{3} \sec^2 t \tan t - \frac{2}{3} \tan t + C = \frac{2}{3} \tan t (\sec^2 t - 1) + C = \frac{2}{3} \tan^3 t + C$
 An easy way to find the integral using substitution:
 $\int 2 \sin^2 t \cos^{-4} t dt = \int 2 \tan^2 t \sec^2 t dt = 2 \int (\tan t)^2 \sec^2 t dt = \frac{2}{3} \tan^3 t + C$

$$45. \int 4 \tan^3 2x \, dx = 4 \left(\frac{\tan^2 2x}{2 \cdot 2} - \int \tan 2x \, dx \right) = \tan^2 2x - 4 \int \tan 2x \, dx$$

(We used FORMULA 86 with $n = 3, a = 2$)

$$= \tan^2 2x - \frac{4}{2} \ln |\sec 2x| + C = \tan^2 2x - 2 \ln |\sec 2x| + C$$

$$46. \int 8 \cot^4 t \, dt = 8 \left(-\frac{\cot^3 t}{3} - \int \cot^2 t \, dt \right)$$

(We used FORMULA 87 with $a = 1, n = 4$)

$$= 8 \left(-\frac{1}{3} \cot^3 t + \cot t + t \right) + C$$

(We used FORMULA 85 with $a = 1$)

$$47. \int 2 \sec^3 \pi x \, dx = 2 \left[\frac{\sec \pi x \tan \pi x}{\pi(3-1)} + \frac{3-2}{3-1} \int \sec \pi x \, dx \right]$$

(We used FORMULA 92 with $n = 3, a = \pi$)

$$= \frac{1}{\pi} \sec \pi x \tan \pi x + \frac{1}{\pi} \ln |\sec \pi x + \tan \pi x| + C$$

(We used FORMULA 88 with $a = \pi$)

$$48. \int 3 \sec^4 3x \, dx = 3 \left[\frac{\sec^2 3x \tan 3x}{3(4-1)} + \frac{4-2}{4-1} \int \sec^2 3x \, dx \right]$$

(We used FORMULA 92 with $n = 4, a = 3$)

$$= \frac{\sec^2 3x \tan 3x}{3} + \frac{2}{3} \tan 3x + C$$

(We used FORMULA 90 with $a = 3$)

$$49. \int \csc^5 x \, dx = -\frac{\csc^3 x \cot x}{5-1} + \frac{5-2}{5-1} \int \csc^3 x \, dx = -\frac{\csc^3 x \cot x}{4} + \frac{3}{4} \left(-\frac{\csc x \cot x}{3-1} + \frac{3-2}{3-1} \int \csc x \, dx \right)$$

(We used FORMULA 93 with $n = 5, a = 1$ and $n = 3, a = 1$)

$$= -\frac{1}{4} \csc^3 x \cot x - \frac{3}{8} \csc x \cot x - \frac{3}{8} \ln |\csc x + \cot x| + C$$

(We used FORMULA 89 with $a = 1$)

$$50. \int 16x^3 (\ln x)^2 \, dx = 16 \left[\frac{x^4 (\ln x)^2}{4} - \frac{2}{4} \int x^3 \ln x \, dx \right] = 16 \left[\frac{x^4 (\ln x)^2}{4} - \frac{1}{2} \left[\frac{x^4 (\ln x)}{4} - \frac{1}{4} \int x^3 \, dx \right] \right]$$

(We used FORMULA 110 with $a = 1, n = 3, m = 2$ and $a = 1, n = 3, m = 1$)

$$= 16 \left(\frac{x^4 (\ln x)^2}{4} - \frac{x^4 (\ln x)}{8} + \frac{x^4}{32} \right) + C = 4x^4 (\ln x)^2 - 2x^4 \ln x + \frac{x^4}{2} + C$$

$$51. \int e^t \sec^3 (e^t - 1) \, dt; \left[x = e^t - 1, dx = e^t \, dt \right] \rightarrow \int \sec^3 x \, dx = \frac{\sec x \tan x}{3-1} + \frac{3-2}{3-1} \int \sec x \, dx$$

(We used FORMULA 92 with $a = 1, n = 3$)

$$= \frac{\sec x \tan x}{2} + \frac{1}{2} \ln |\sec x + \tan x| + C = \frac{1}{2} \left[\sec(e^t - 1) \tan(e^t - 1) + \ln |\sec(e^t - 1) + \tan(e^t - 1)| \right] + C$$

$$52. \int \frac{\csc^3 \sqrt{\theta}}{\sqrt{\theta}} d\theta; \begin{bmatrix} t = \sqrt{\theta} \\ \theta = t^2 \\ d\theta = 2t dt \end{bmatrix} \rightarrow 2 \int \csc^3 t dt = 2 \left[-\frac{\csc t \cot t}{3-1} + \frac{3-2}{3-1} \int \csc t dt \right] = 2 \left[-\frac{\csc t \cot t}{2} - \frac{1}{2} \ln |\csc t + \cot t| \right] + C$$

(We used FORMULA 93 with $a = 1, n = 3$)

$$= -\csc \sqrt{\theta} \cot \sqrt{\theta} - \ln |\csc \sqrt{\theta} + \cot \sqrt{\theta}| + C$$

$$53. \int_0^1 2\sqrt{x^2 + 1} dx; [x = \tan t, dx = \sec^2 t dt] \rightarrow 2 \int_0^{\pi/4} \sec t \cdot \sec^2 t dt = 2 \int_0^{\pi/4} \sec^3 t dt$$

$$= 2 \left[\frac{\sec t \cdot \tan t}{3-1} + \frac{3-2}{3-1} \int_0^{\pi/4} \sec t dt \right]$$

(We used FORMULA 92 with, $n = 3, a = 1$)

$$= \left[\sec t \cdot \tan t + \ln |\sec t + \tan t| \right]_0^{\pi/4} = \sqrt{2} + \ln(\sqrt{2} + 1)$$

$$54. \int_0^{\sqrt{3}/2} \frac{dy}{(1-y^2)^{5/2}}; [y = \sin x, dy = \cos x dx] \rightarrow \int_0^{\pi/3} \frac{\cos x dx}{\cos^5 x} = \int_0^{\pi/3} \sec^4 x dx = \left[\frac{\sec^2 x \tan x}{4-1} \right]_0^{\pi/3} + \frac{4-2}{4-1} \int_0^{\pi/3} \sec^2 x dx$$

(We used FORMULA 92 with $a = 1, n = 4$)

$$= \left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3} \tan x \right]_0^{\pi/3} = \left(\frac{4}{3} \right) \sqrt{3} + \left(\frac{2}{3} \right) \sqrt{3} = 2\sqrt{3}$$

$$55. \int_1^2 \frac{(r^2-1)^{3/2}}{r} dr; [r = \sec \theta, dr = \sec \theta \tan \theta d\theta] \rightarrow \int_0^{\pi/3} \frac{\tan^3 \theta}{\sec \theta} (\sec \theta \tan \theta) d\theta = \int_0^{\pi/3} \tan^4 \theta d\theta$$

$$= \left[\frac{\tan^3 \theta}{4-1} \right]_0^{\pi/3} - \int_0^{\pi/3} \tan^2 \theta d\theta = \left[\frac{\tan^3 \theta}{3} - \tan \theta + \theta \right]_0^{\pi/3} = \frac{3\sqrt{3}}{3} - \sqrt{3} + \frac{\pi}{3} = \frac{\pi}{3}$$

(We used FORMULA 86 with $a = 1, n = 4$ and FORMULA 84 with $a = 1$)

$$56. \int_0^{1/\sqrt{3}} \frac{dt}{(t^2+1)^{7/2}}; [t = \tan \theta, dt = \sec^2 \theta d\theta] \rightarrow \int_0^{\pi/6} \frac{\sec^2 \theta d\theta}{\sec^7 \theta} = \int_0^{\pi/6} \cos^5 \theta d\theta$$

$$= \left[\frac{\cos^4 \theta \sin \theta}{5} \right]_0^{\pi/6} + \left(\frac{5-1}{5} \right) \int_0^{\pi/6} \cos^3 \theta d\theta = \left[\frac{\cos^4 \theta \sin \theta}{5} \right]_0^{\pi/6} + \frac{4}{5} \left[\frac{\cos^2 \theta \sin \theta}{3} \right]_0^{\pi/6} + \left(\frac{3-1}{3} \right) \int_0^{\pi/6} \cos \theta d\theta$$

$$= \left[\frac{\cos^4 \theta \sin \theta}{5} + \frac{4}{15} \cos^2 \theta \sin \theta + \frac{8}{15} \sin \theta \right]_0^{\pi/6}$$

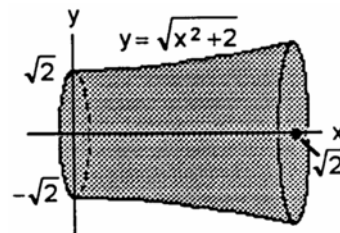
(We used FORMULA 61 with $a = 1, n = 5$ and $a = 1, n = 3$)

$$= \frac{\left(\frac{\sqrt{3}}{2}\right)^4 \left(\frac{1}{2}\right)}{5} + \left(\frac{4}{15}\right) \left(\frac{\sqrt{3}}{2}\right)^2 \left(\frac{1}{2}\right) + \left(\frac{8}{15}\right) \left(\frac{1}{2}\right) = \frac{9}{160} + \frac{1}{10} + \frac{4}{15} = \frac{3 \cdot 9 + 48 + 32 \cdot 4}{480} = \frac{203}{480}$$

$$57. S = \int_0^{\sqrt{2}} 2\pi y \sqrt{1+(y')^2} dx$$

$$= 2\pi \int_0^{\sqrt{2}} \sqrt{x^2 + 2} \sqrt{1 + \frac{x^2}{x^2 + 2}} dx$$

$$= 2\sqrt{2}\pi \int_0^{\sqrt{2}} \sqrt{x^2 + 1} dx$$



$$= 2\sqrt{2}\pi \left[\frac{x\sqrt{x^2+1}}{2} + \frac{1}{2} \ln \left| x + \sqrt{x^2+1} \right| \right]_0^{\sqrt{2}}$$

(We used FORMULA 21 with $a = 1$)

$$= \sqrt{2}\pi \left[\sqrt{6} + \ln(\sqrt{2} + \sqrt{3}) \right] = 2\pi\sqrt{3} + \pi\sqrt{2} \ln(\sqrt{2} + \sqrt{3})$$

$$58. \quad L = \int_0^{\sqrt{3}/2} \sqrt{1+(2x)^2} dx = 2 \int_0^{\sqrt{3}/2} \sqrt{\frac{1}{4} + x^2} dx = 2 \left[\frac{x}{2} \sqrt{\frac{1}{4} + x^2} + \left(\frac{1}{4}\right) \left(\frac{1}{2}\right) \ln \left(x + \sqrt{\frac{1}{4} + x^2} \right) \right]_0^{\sqrt{3}/2}$$

(We used FORMULA 2 with $a = \frac{1}{2}$)

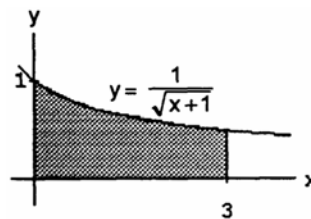
$$= \left[\frac{x}{2} \sqrt{1+4x^2} + \frac{1}{4} \ln \left(x + \frac{1}{2} \sqrt{1+4x^2} \right) \right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}}{4} \sqrt{1+4\left(\frac{3}{4}\right)} + \frac{1}{4} \ln \left(\frac{\sqrt{3}}{2} + \frac{1}{2} \sqrt{1+4\left(\frac{3}{4}\right)} \right) - \frac{1}{4} \ln \frac{1}{2}$$

$$= \frac{\sqrt{3}}{4} (2) + \frac{1}{4} \ln \left(\frac{\sqrt{3}}{2} + 1 \right) + \frac{1}{4} \ln 2 = \frac{\sqrt{3}}{2} + \frac{1}{4} \ln(\sqrt{3} + 2)$$

$$59. \quad A = \int_0^3 \frac{dx}{\sqrt{x+1}} = \left[2\sqrt{x+1} \right]_0^3 = 2;$$

$$\bar{x} = \frac{1}{A} \int_0^3 \frac{x dx}{\sqrt{x+1}} = \frac{1}{A} \int_0^3 \sqrt{x+1} dx - \frac{1}{A} \int_0^3 \frac{dx}{\sqrt{x+1}}$$

$$= \frac{1}{2} \cdot \frac{2}{3} \left[(x+1)^{3/2} \right]_0^3 - 1 = \frac{4}{3};$$



(We used FORMULA 11 with $a=1, b=1, n=1$ and $a=1, b=1, n=-1$)

$$\bar{y} = \frac{1}{2A} \int_0^3 \frac{dx}{x+1} = \frac{1}{4} [\ln(x+1)]_0^3 = \frac{1}{4} \ln 4 = \frac{1}{2} \ln 2 = \ln \sqrt{2}$$

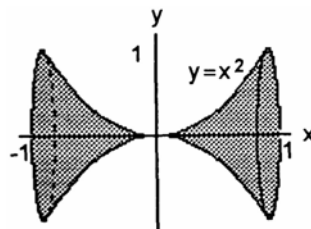
$$60. \quad M_y = \int_0^3 x \left(\frac{36}{2x+3} \right) dx = 18 \int_0^3 \frac{2x+3}{2x+3} dx - 54 \int_0^3 \frac{dx}{2x+3} = [18x - 27 \ln |2x+3|]_0^3 = 18 \cdot 3 - 27 \ln 9 - (-27 \ln 3)$$

$$= 54 - 27 \cdot 2 \ln 3 + 27 \ln 3 = 54 - 27 \ln 3$$

$$61. \quad S = 2\pi \int_{-1}^1 x^2 \sqrt{1+4x^2} dx; \quad [u = 2x, du = 2 dx]$$

$$\rightarrow \frac{\pi}{4} \int_{-2}^2 u^2 \sqrt{1+u^2} du$$

$$= \frac{\pi}{4} \left[\frac{u}{8} (1+2u^2) \sqrt{1+u^2} - \frac{1}{8} \ln(u + \sqrt{1+u^2}) \right]_{-2}^2$$



(We used FORMULA 22 with $a = 1$)

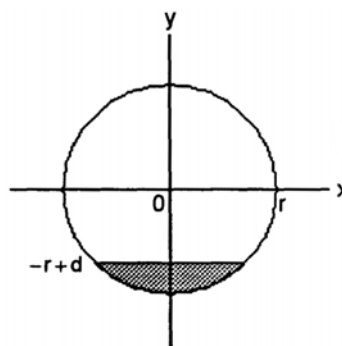
$$= \frac{\pi}{4} \left[\frac{2}{8} (1+2 \cdot 4) \sqrt{1+4} - \frac{1}{8} \ln(2 + \sqrt{1+4}) + \frac{2}{8} (1+2 \cdot 4) \sqrt{1+4} + \frac{1}{8} \ln(-2 + \sqrt{1+4}) \right]$$

$$= \frac{\pi}{4} \left[\frac{9}{2} \sqrt{5} - \frac{1}{8} \ln \left(\frac{2+\sqrt{5}}{-2+\sqrt{5}} \right) \right] \approx 7.62$$

62. (a) The volume of the filled part equals the length of the tank times the area of the shaded region shown in the accompanying figure. Consider a layer of gasoline of thickness dy located at height y where $-r < y < -r + d$. The width of this layer is $2\sqrt{r^2 - y^2}$. Therefore,

$$A = 2 \int_{-r}^{-r+d} \sqrt{r^2 - y^2} dy \text{ and}$$

$$V = L \cdot A = 2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2} dy$$



$$(b) \quad 2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2} dy = 2L \left[\frac{y\sqrt{r^2 - y^2}}{2} + \frac{r^2}{2} \sin^{-1} \frac{y}{r} \right]_{-r}^{-r+d}$$

(We used FORMULA 29 with $a = r$)

$$= 2L \left[\left(\frac{d-r}{2} \right) \sqrt{2rd - d^2} + \frac{r^2}{2} \sin^{-1} \left(\frac{d-r}{r} \right) + \frac{r^2}{2} \left(\frac{\pi}{2} \right) \right] = 2L \left[\left(\frac{d-r}{2} \right) \sqrt{2rd - d^2} + \left(\frac{r^2}{2} \right) \left(\sin^{-1} \left(\frac{d-r}{r} \right) + \frac{\pi}{2} \right) \right]$$

63. The integrand $f(x) = \sqrt{x - x^2}$ is nonnegative, so the integral is maximized by integrating over the function's entire domain, which runs from $x = 0$ to $x = 1$

$$\Rightarrow \int_0^1 \sqrt{x - x^2} dx = \int_0^1 \sqrt{2 \cdot \frac{1}{2} x - x^2} dx = \left[\frac{\left(\frac{x-1}{2} \right)}{2} \sqrt{2 \cdot \frac{1}{2} x - x^2} + \frac{\left(\frac{1}{2} \right)^2}{2} \sin^{-1} \left(\frac{x-\frac{1}{2}}{\frac{1}{2}} \right) \right]_0^1$$

(We used FORMULA 48 with $a = \frac{1}{2}$)

$$= \left[\frac{\left(\frac{x-1}{2} \right)}{2} \sqrt{x - x^2} + \frac{1}{8} \sin^{-1} (2x - 1) \right]_0^1 = \frac{1}{8} \cdot \frac{\pi}{2} - \frac{1}{8} \left(-\frac{\pi}{2} \right) = \frac{\pi}{8}$$

64. The integrand is maximized by integrating $g(x) = x\sqrt{2x - x^2}$ over the largest domain on which g is nonnegative, namely $[0, 2]$

$$\Rightarrow \int_0^2 x\sqrt{2x - x^2} dx = \left[\frac{(x+1)(2x-3)\sqrt{2x-x^2}}{6} + \frac{1}{2} \sin^{-1} (x-1) \right]_0^2$$

(We used FORMULA 51 with $a = 1$)

$$= \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \left(-\frac{\pi}{2} \right) = \frac{\pi}{2}$$

CAS EXPLORATIONS

65. Example CAS commands:

Maple:

$$q1 := \text{Int}(x \cdot \ln(x), x); \quad \# (a)$$

$$q1 = \text{value}(q1);$$

$$q2 := \text{Int}(x^2 \cdot \ln(x), x); \quad \# (b)$$

$$q2 = \text{value}(q2);$$

$$q3 := \text{Int}(x^3 \cdot \ln(x), x); \quad \# (c)$$

```

q3 = value( q3 );
q4 := Int( x^4*ln(x), x );          # (d)
q4 = value( q4 );
q5 := Int( x^n*ln(x), x );          # (e)
q6 = value( q5 );
q7 := simplify(q6) assuming n::integer;
q5 = collect( factor(q7), ln(x) );

```

66. Example CAS commands:

Maple:

```

q1 := Int( ln(x)/x, x );             # (a)
q1 = value( q1 );
q2 := Int( ln(x)/x^2, x );           # (b)
q2 = value( q2 );
q3 := Int( ln(x)/x^3, x );           # (c)
q3 = value( q3 );
q4 := Int( ln(x)/x^4, x );           # (d)
q4 = value( q4 );
q5 := Int( ln(x)/x^n, x );           # (e)
q6 := value( q5 );
q7 := simplify(q6) assuming n::integer;
q5 = collect( factor(q7), ln(x) );

```

67. Example CAS commands:

Maple:

```

q := Int( sin(x)^n/sin(x)^n+cos(x)^n, x=0..Pi/2 ); # (a)
q = value( q );
q1 := eval( q, n=1 );                 # (b)
q1 = value( q1 );
for N in [1,2,3,5,7] do
    q1 := eval( q, n=N );
    print( q1 = evalf(q1) );
end do;
qq1 := PDEtools[dchange]( x=Pi/2-u, q, [u] );      # (c)
qq2 := subs( u=x, qq1 );
qq3 := q + q = q + qq2;
qq4 := combine( qq3 );

```

```
qq5 := value( qq4 );
simplify( qq5/2 );
```

65-67. Example CAS commands:

Mathematica: (functions may vary)

In Mathematica, the natural log is denoted by Log rather than Ln, Log base 10 is Log[x, 10]

Mathematica does not include an arbitrary constant when computing an indefinite integral,

```
Clear[x, f, n]
f[x_]:=Log[x]/x^n
Integrate[f[x], x]
```

For exercise 67, Mathematica cannot evaluate the integral with arbitrary n . It does evaluate the integral (value is $\pi/4$ in each case) for small values of n , but for large values of n , it identifies this integral as Indeterminate

$$65. \quad (e) \quad \int x^n \ln x \, dx = \frac{x^{n+1} \ln x}{n+1} - \frac{1}{n+1} \int x^n \, dx, \quad n \neq -1$$

(We used FORMULA 110 with $a = 1, m = 1$)

$$= \frac{x^{n+1} \ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}}{n+1} \left(\ln x - \frac{1}{n+1} \right) + C$$

$$66. \quad (e) \quad \int x^{-n} \ln x \, dx = \frac{x^{-n+1} \ln x}{-n+1} - \frac{1}{(-n+1)} \int x^{-n} \, dx, \quad n \neq 1$$

(We used FORMULA 110 with $a = 1, m = 1, n = -n$)

$$= \frac{x^{1-n} \ln x}{1-n} - \frac{1}{1-n} \left(\frac{x^{1-n}}{1-n} \right) + C = \frac{x^{1-n}}{1-n} \left(\ln x - \frac{1}{1-n} \right) + C$$

67. (a) Neither MAPLE nor MATHEMATICA can find this integral for arbitrary n .
 (b) MAPLE and MATHEMATICA get stuck at about $n = 5$.
 (c) Let $x = \frac{\pi}{2} - u \Rightarrow dx = -du$; $x = 0 \Rightarrow u = \frac{\pi}{2}$, $x = \frac{\pi}{2} \Rightarrow u = 0$;

$$I = \int_0^{\pi/2} \frac{\sin^n x \, dx}{\sin^n x + \cos^n x} = \int_{\pi/2}^0 \frac{-\sin^n \left(\frac{\pi}{2} - u \right) du}{\sin^n \left(\frac{\pi}{2} - u \right) + \cos^n \left(\frac{\pi}{2} - u \right)} = \int_0^{\pi/2} \frac{\cos^n u \, du}{\cos^n u + \sin^n u} = \int_0^{\pi/2} \frac{\cos^n x \, dx}{\cos^n x + \sin^n x}$$

$$\Rightarrow I + I = \int_0^{\pi/2} \left(\frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} \right) dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

8.7 NUMERICAL INTEGRATION

1. $\int_1^2 x \, dx$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;

$$\sum mf(x_i) = 12 \Rightarrow T = \frac{1}{8}(12) = \frac{3}{2};$$

$$f(x) = x \Rightarrow f'(x) = 1 \Rightarrow f'' = 0$$

$$\Rightarrow M = 0 \Rightarrow |E_T| = 0$$

(b) $\int_1^2 x \, dx = \left[\frac{x^2}{2} \right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$

$$\Rightarrow |E_T| = \int_1^2 x \, dx - T = 0$$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;

$$\sum mf(x_i) = 18 \Rightarrow S = \frac{1}{12}(18) = \frac{3}{2};$$

$$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$$

(b) $\int_1^2 x \, dx = \frac{3}{2} \Rightarrow |E_S| = \int_1^2 x \, dx - S = \frac{3}{2} - \frac{3}{2} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	2	5/2
x_2	3/2	3/2	2	3
x_3	7/4	7/4	2	7/2
x_4	2	2	1	2

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	4	5
x_2	3/2	3/2	2	3
x_3	7/4	7/4	4	7
x_4	2	2	1	2

2. $\int_1^3 (2x-1) \, dx$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

$$\Rightarrow \frac{\Delta x}{2} = \frac{1}{4};$$

$$\sum mf(x_i) = 24 \Rightarrow T = \frac{1}{4}(24) = 6;$$

$$f(x) = 2x-1 \Rightarrow f'(x) = 2 \Rightarrow f'' = 0$$

$$\Rightarrow M = 0 \Rightarrow |E_T| = 0$$

(b) $\int_1^3 (2x-1) \, dx = \left[x^2 - x \right]_1^3 = (9-3) - (1-1) = 6 \Rightarrow |E_T| = \int_1^3 (2x-1) \, dx - T = 6 - 6 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	2	4
x_2	2	3	2	6
x_3	5/2	4	2	8
x_4	3	5	1	5

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;

$$\sum mf(x_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6;$$

$$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$$

(b) $\int_1^3 (2x-1) \, dx = 6 \Rightarrow |E_S| = \int_1^3 (2x-1) \, dx - S = 6 - 6 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	4	8
x_2	2	3	2	6
x_3	5/2	4	4	16
x_4	3	5	1	5

3. $\int_{-1}^1 (x^2 + 1) dx$

- I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;
 $\sum mf(x_i) = 11 \Rightarrow T = \frac{1}{4}(11) = 2.75$;
 $f(x) = x^2 + 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2$
 $\Rightarrow M = 2 \Rightarrow |E_T| \leq \frac{1-(-1)}{12} \left(\frac{1}{2}\right)^2 (2) = \frac{1}{12}$ or
 0.08333

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	-1	2	1	2
x_1	-1/2	5/4	2	5/2
x_2	0	1	2	2
x_3	1/2	5/4	2	5/2
x_4	1	2	1	2

(b) $\int_{-1}^1 (x^2 + 1) dx = \left[\frac{x^3}{3} + x \right]_{-1}^1 = \left(\frac{1}{3} + 1 \right) - \left(-\frac{1}{3} - 1 \right) = \frac{8}{3} \Rightarrow E_T = \int_{-1}^1 (x^2 + 1) dx - T = \frac{8}{3} - \frac{11}{4} = -\frac{1}{12}$
 $\Rightarrow |E_T| = \left| -\frac{1}{12} \right| \approx 0.08333$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{8}{3}} \right) \times 100 \approx 3\%$

- II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
 $\sum mf(x_i) = 16 \Rightarrow S = \frac{1}{6}(16) = \frac{8}{3} = 2.66667$;
 $f^3(x) = 0 \Rightarrow f(4)(x) = 0 \Rightarrow M = 0$
 $\Rightarrow |E_s| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	-1	2	1	2
x_1	-1/2	5/4	4	5
x_2	0	1	2	2
x_3	1/2	5/4	4	5
x_4	1	2	1	2

(b) $\int_{-1}^1 (x^2 + 1) dx = \left[\frac{x^3}{3} + x \right]_{-1}^1 = \frac{8}{3}$
 $\Rightarrow |E_s| = \int_{-1}^1 (x^2 + 1) dx - S = \frac{8}{3} - \frac{8}{3} = 0$

(c) $\frac{|E_s|}{\text{True Value}} \times 100 = 0\%$

4. $\int_{-2}^0 (x^2 - 1) dx$

- I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2}$
 $\Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$; $\sum mf(x_i) = 3 \Rightarrow T = \frac{1}{4}(3) = \frac{3}{4}$;
 $f(x) = x^2 - 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2$
 $\Rightarrow M = 2 \Rightarrow |E_T| \leq \frac{0-(-2)}{12} \left(\frac{1}{2}\right)^2 (2) = \frac{1}{12}$
 ≈ 0.08333

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	-2	3	1	3
x_1	-3/2	5/4	2	5/2
x_2	-1	0	2	0
x_3	-1/2	-3/4	2	-3/2
x_4	0	-1	1	-1

(b) $\int_{-2}^0 (x^2 - 1) dx = \left[\frac{x^3}{3} - x \right]_{-2}^0 = 0 - \left(-\frac{8}{3} + 2 \right) = \frac{2}{3} \Rightarrow E_T = \int_{-2}^0 (x^2 - 1) dx - T = \frac{2}{3} - \frac{3}{4} = -\frac{1}{12} \Rightarrow |E_T| = \frac{1}{12}$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{2}{3}} \right) \times 100 \approx 13\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;

$$\sum mf(x_i) = 4 \Rightarrow S = \frac{1}{6}(4) = \frac{2}{3}; \quad f^{(3)}(x) = 0$$

$$\Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_s| = 0$$

(b) $\int_{-2}^0 (x^2 - 1) dx = \frac{2}{3} \Rightarrow |E_s| = \left| \int_{-2}^0 (x^2 - 1) dx - S \right|$
 $= \frac{2}{3} - \frac{2}{3} = 0$

(c) $\frac{|E_s|}{\text{True Value}} \times 100 = 0\%$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	-2	3	1	3
x_1	-3/2	5/4	4	5
x_2	-1	0	2	0
x_3	-1/2	-3/4	4	-3
x_4	0	-1	1	-1

5. $\int_0^2 (t^3 + t) dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;

$$\sum mf(t_i) = 25 \Rightarrow T = \frac{1}{4}(25) = \frac{25}{4};$$

$$f(t) = t^3 + t \Rightarrow f'(t) = 3t^2 + 1 \Rightarrow f''(t) = 6t$$

$$\Rightarrow M = 12 = f''(2) \Rightarrow |E_T| \leq \frac{2-0}{12} \left(\frac{1}{2} \right)^2 (12) = \frac{1}{2}$$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	2	5/4
t_2	1	2	2	4
t_3	3/2	39/8	2	39/4
t_4	2	10	1	10

(b) $\int_0^2 (t^3 + t) dt = \left[\frac{t^4}{4} + \frac{t^2}{2} \right]_0^2 = \left(\frac{2^4}{4} + \frac{2^2}{2} \right) - 0 = 6 \Rightarrow |E_T| = \left| \int_0^2 (t^3 + t) dt - T \right| = 6 - \frac{25}{4} = -\frac{1}{4} \Rightarrow |E_T| = \frac{1}{4}$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{\frac{1}{4}}{6} \times 100 \approx 4\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;

$$\sum mf(t_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6;$$

$$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_s| = 0$$

(b) $\int_0^2 (t^3 + t) dt = 6 \Rightarrow |E_s| = \left| \int_0^2 (t^3 + t) dt - S \right|$
 $= 6 - 6 = 0$

(c) $\frac{|E_s|}{\text{True Value}} \times 100 = 0\%$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	4	5/2
t_2	1	2	2	4
t_3	3/2	39/8	4	39/2
t_4	2	10	1	10

6. $\int_{-1}^1 (t^3 + 1) dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;

$$\sum mf(t_i) = 8 \Rightarrow T = \frac{1}{4}(8) = 2;$$

$$f(t) = t^3 + 1 \Rightarrow f'(t) = 3t^2 \Rightarrow f''(t) = 6t$$

$$\Rightarrow M = 6 = f''(1) \Rightarrow |E_T| \leq \frac{1-(-1)}{12} \left(\frac{1}{2} \right)^2 (6) = \frac{1}{4}$$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	-1	0	1	0
t_1	-1/2	7/8	2	7/4
t_2	0	1	2	2
t_3	1/2	9/8	2	9/4
t_4	1	2	1	2

(b) $\int_{-1}^1 (t^3 + 1) dt = \left[\frac{t^4}{4} + t \right]_{-1}^1 = \left(\frac{1^4}{4} + 1 \right) - \left(\frac{(-1)^4}{4} + (-1) \right) = 2 \Rightarrow |E_T| = \left| \int_{-1}^1 (t^3 + 1) dt - T \right| = 2 - 2 = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;

$$\sum mf(t_i) = 12 \Rightarrow S = \frac{1}{6}(12) = 2;$$

$$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_s| = 0$$

(b) $\int_{-1}^1 (t^3 + 1) dt = 2 \Rightarrow |E_s| = \int_{-1}^1 (t^3 + 1) dt - S$
 $= 2 - 2 = 0$

(c) $\frac{|E_s|}{\text{True Value}} \times 100 = 0\%$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	-1	0	1	0
t_1	-1/2	7/8	4	7/2
t_2	0	1	2	2
t_3	1/2	9/8	4	9/2
t_4	1	2	1	2

7. $\int_1^2 \frac{1}{s^2} ds$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;

$$\sum mf(s_i) = \frac{179,573}{44,100} \Rightarrow T = \frac{1}{8} \left(\frac{179,573}{44,100} \right) = \frac{179,573}{352,800}$$

$$\approx 0.50899; f(s) = \frac{1}{s^2} \Rightarrow f'(s) = -\frac{2}{s^3}$$

$$\Rightarrow f''(s) = \frac{6}{s^4} \Rightarrow M = 6 = f''(1)$$

$$\Rightarrow |E_T| \leq \frac{2-1}{12} \left(\frac{1}{4} \right)^2 (6) = \frac{1}{32} = 0.03125$$

(b) $\int_1^2 \frac{1}{s^2} ds = \int_1^2 s^{-2} ds = \left[-\frac{1}{s} \right]_1^2 = -\frac{1}{2} - \left(-\frac{1}{1} \right) = \frac{1}{2} \Rightarrow E_T = \int_1^2 \frac{1}{s^2} ds - T = \frac{1}{2} - 0.50899 = -0.00899$

$$\Rightarrow |E_T| = 0.00899$$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.00899}{0.5} \times 100 \approx 2\%$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	2	32/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	2	32/49
s_4	2	1/4	1	1/4

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;

$$\sum mf(s_i) = \frac{264,821}{44,100} \Rightarrow S = \frac{1}{12} \left(\frac{264,821}{44,100} \right)$$

$$= \frac{264,821}{529,200} \approx 0.50042;$$

$$f^{(3)}(s) = -\frac{24}{s^5} \Rightarrow f^{(4)}(s) = \frac{120}{s^6} \Rightarrow M = 120$$

$$\Rightarrow |E_s| \leq \left| \frac{2-1}{180} \right| \left(\frac{1}{4} \right)^4 (120) = \frac{1}{384} \approx 0.00260$$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	4	64/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	4	64/49
s_4	2	1/4	1	1/4

(b) $\int_1^2 \frac{1}{s^2} ds = \frac{1}{2} \Rightarrow E_s = \int_1^2 \frac{1}{s^2} ds - S = \frac{1}{2} - 0.50042 = -0.00042 \Rightarrow |E_s| = 0.00042$

(c) $\frac{|E_s|}{\text{True Value}} \times 100 = \frac{0.0004}{0.5} \times 100 \approx 0.08\%$

8. $\int_2^4 \frac{1}{(s-1)^2} ds$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;

$$\sum mf(s_i) = \frac{1269}{450} \Rightarrow T = \frac{1}{4} \left(\frac{1269}{450} \right) = \frac{1269}{1800} = 0.70500;$$

$$f(s) = (s-1)^{-2} \Rightarrow f'(s) = \frac{-2}{(s-1)^3} \Rightarrow f''(s) = \frac{6}{(s-1)^4}$$

$$\Rightarrow |E_T| \leq \frac{4-2}{12} \left(\frac{1}{2} \right)^2 (6) = \frac{1}{4} = 0.25 \Rightarrow M = 6$$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	2	1	1	1
s_1	5/2	4/9	2	8/9
s_2	3	1/4	2	1/2
s_3	7/2	4/25	2	8/25
s_4	4	1/9	1	1/9

$$(b) \int_2^4 \frac{1}{(s-1)^2} ds = \left[\frac{-1}{(s-1)} \right]_2^4 = \left(\frac{-1}{4-1} \right) - \left(\frac{-1}{2-1} \right) = \frac{2}{3} \Rightarrow E_T = \int_2^4 \frac{1}{(s-1)^2} ds - T = \frac{2}{3} - 0.705 \approx -0.03833$$

$$\Rightarrow |E_T| \approx 0.03833$$

$$(c) \frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03833}{\left(\frac{2}{3}\right)} \times 100 \approx 6\%$$

$$\text{II. (a) For } n = 4, \Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6};$$

$$\sum mf(s_i) = \frac{1813}{450} \Rightarrow S = \frac{1}{6} \left(\frac{1813}{450} \right)$$

$$= \frac{1813}{2700} \approx 0.67148; f^{(3)}(s) = \frac{-24}{(s-1)^5}$$

$$\Rightarrow f^{(4)}(s) = \frac{120}{(s-1)^6} \Rightarrow M = 120$$

$$\Rightarrow |E_s| \leq \frac{4-2}{180} \left(\frac{1}{2} \right)^4 (120) = \frac{1}{12} \approx 0.08333$$

	s_i	$f(s_i)$	m	$mf(s_i)$
	s_0	2	1	1
	s_1	5/2	4	16/9
	s_2	3	2	1/2
	s_3	7/2	4	16/25
	s_4	4	1	1/9

$$(b) \int_2^4 \frac{1}{(s-1)^2} ds = \frac{2}{3} \Rightarrow E_s = \int_2^4 \frac{1}{(s-1)^2} ds - S \approx \frac{2}{3} - 0.67148 = -0.00481 \Rightarrow |E_s| \approx 0.00481$$

$$(c) \frac{|E_s|}{\text{True Value}} \times 100 = \frac{0.00481}{\left(\frac{2}{3}\right)} \times 100 \approx 1\%$$

$$9. \int_0^\pi \sin t \, dt$$

$$\text{I. (a) For } n = 4, \Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{8};$$

$$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.8284;$$

$$\Rightarrow T = \frac{\pi}{8} (2 + 2\sqrt{2}) \approx 1.89612; f(t) = \sin t$$

$$\Rightarrow f'(t) = \cos t \Rightarrow f''(t) = -\sin t \Rightarrow M = 1$$

$$\Rightarrow |E_T| \leq \frac{\pi-0}{12} \left(\frac{\pi}{4} \right)^2 (1) = \frac{\pi^3}{192} \approx 0.16149$$

	t_i	$f(t_i)$	m	$mf(t_i)$
	t_0	0	1	0
	t_1	$\pi/4$	2	$\sqrt{2}$
	t_2	$\pi/2$	2	2
	t_3	$3\pi/4$	2	$\sqrt{2}$
	t_4	π	1	0

$$(b) \int_0^\pi \sin t \, dt = [-\cos t]_0^\pi = (-\cos \pi) - (-\cos 0) = 2 \Rightarrow |E_T| = \int_0^\pi \sin t \, dt - T \approx 2 - 1.89612 = 0.10388$$

$$(c) \frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.10388}{2} \times 100 \approx 5\%$$

$$\text{II. (a) For } n = 4, \Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{12};$$

$$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.6569 \Rightarrow S = \frac{\pi}{12} (2 + 4\sqrt{2})$$

$$\approx 2.00456; f^{(3)}(t) = -\cos t \Rightarrow f^{(4)}(t) = \sin t$$

$$\Rightarrow M = 1 \Rightarrow |E_s| \leq \frac{\pi-0}{180} \left(\frac{\pi}{4} \right)^4 (1) \approx 0.00664$$

$$(b) \int_0^\pi \sin t \, dt = 2 \Rightarrow E_s = \int_0^\pi \sin t \, dt - S \approx 2 - 2.00456$$

$$= -0.00456 \Rightarrow |E_s| \approx 0.00456$$

$$(c) \frac{|E_s|}{\text{True Value}} \times 100 = \frac{0.00456}{2} \times 100 \approx 0\%$$

	t_i	$f(t_i)$	m	$mf(t_i)$
	t_0	0	1	0
	t_1	$\pi/4$	4	$2\sqrt{2}$
	t_2	$\pi/2$	2	2
	t_3	$3\pi/4$	4	$2\sqrt{2}$
	t_4	π	1	0

10. $\int_0^1 \sin \pi t dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;

$$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.828$$

$$\Rightarrow T = \frac{1}{8}(2 + 2\sqrt{2}) \approx 0.60355; f(t) = \sin \pi t$$

$$\Rightarrow f'(t) = \pi \cos \pi t \Rightarrow f''(t) = -\pi^2 \sin \pi t$$

$$\Rightarrow M = \pi^2 \Rightarrow |E_T| \leq \frac{1-0}{12} \left(\frac{1}{4}\right)^2 (\pi^2) \approx 0.05140$$

	t_i	$f(t_i)$	m	$mf(t_i)$
	t_0	0	1	0
	t_1	$1/4$	2	$\sqrt{2}$
	t_2	$1/2$	2	2
	t_3	$3/4$	2	$\sqrt{2}$
	t_4	1	1	0

(b) $\int_0^1 \sin \pi t dt = \left[-\frac{1}{\pi} \cos \pi t\right]_0^1 = \left(-\frac{1}{\pi} \cos \pi\right) - \left(-\frac{1}{\pi} \cos 0\right) = \frac{2}{\pi} \approx 0.63662 \Rightarrow |E_T| = \int_0^1 \sin \pi t dt - T$
 $\approx \frac{2}{\pi} - 0.60355 = 0.03307$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03307}{\left(\frac{2}{\pi}\right)} \times 100 \approx 5\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;

$$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.65685$$

$$\Rightarrow S = \frac{1}{12}(2 + 4\sqrt{2}) \approx 0.63807;$$

$$f^{(3)}(t) = -\pi^3 \cos \pi t \Rightarrow f^{(4)}(t) = \pi^4 \sin \pi t$$

$$\Rightarrow M = \pi^4 \Rightarrow |E_s| \leq \frac{1-0}{180} \left(\frac{1}{4}\right)^4 (\pi^4) \approx 0.00211$$

	t_i	$f(t_i)$	m	$mf(t_i)$
	t_0	0	1	0
	t_1	$1/4$	4	$2\sqrt{2}$
	t_2	$1/2$	2	2
	t_3	$3/4$	4	$2\sqrt{2}$
	t_4	1	1	0

(b) $\int_0^1 \sin \pi t dt = \frac{2}{\pi} \approx 0.63662 \Rightarrow E_s = \int_0^1 \sin \pi t dt - S \approx \frac{2}{\pi} - 0.63807 = -0.00145 \Rightarrow |E_s| \approx 0.00145$

(c) $\frac{|E_s|}{\text{True Value}} \times 100 = \frac{0.00145}{\left(\frac{2}{\pi}\right)} \times 100 \approx 0\%$

11. (a) $M = 0$ (see Exercise 1): Then $n = 1 \Rightarrow \Delta x = 1 \Rightarrow |E_T| = \frac{1}{12}(1)^2(0) = 0 < 10^{-4}$

(b) $M = 0$ (see Exercise 1): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = \frac{1}{2} \Rightarrow |E_s| = \frac{1}{180}\left(\frac{1}{2}\right)^4(0) = 0 < 10^{-4}$

12. (a) $M = 0$ (see Exercise 2): Then $n = 1 \Rightarrow \Delta x = 2 \Rightarrow |E_T| = \frac{2}{12}(2)^2(0) = 0 < 10^{-4}$

(b) $M = 0$ (see Exercise 2): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180}(1)^4(0) < 10^{-4}$

13. (a) $M = 2$ (see Exercise 3): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12}\left(\frac{2}{n}\right)^2(2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3}(10^4) \Rightarrow n > \sqrt{\frac{4}{3}(10^4)}$
 $\Rightarrow n > 115.4$, so let $n = 116$

(b) $M = 0$ (see Exercise 3): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

14. (a) $M = 2$ (see Exercise 4): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12}\left(\frac{2}{n}\right)^2(2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3}(10^4) \Rightarrow n > \sqrt{\frac{4}{3}(10^4)}$
 $\Rightarrow n > 115.4$, so let $n = 116$

(b) $M = 0$ (see Exercise 4): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

15. (a) $M = 12$ (see Exercise 5): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (12) = \frac{8}{n^2} < 10^{-4} \Rightarrow n^2 > 8(10^4) \Rightarrow n > \sqrt{8(10^4)}$
 $\Rightarrow n > 282.8$, so let $n = 283$
- (b) $M = 0$ (see Exercise 5): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$
16. (a) $M = 6$ (see Exercise 6): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4(10^4) \Rightarrow n > \sqrt{4(10^4)}$
 $= 200$, so let $n = 201$
- (b) $M = 0$ (Exercise 6): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$
17. (a) $M = 6$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_T| \leq \frac{1}{12} \left(\frac{1}{n}\right)^2 (6) = \frac{1}{2n^2} < 10^{-4} \Rightarrow n^2 > \frac{1}{2}(10^4) \Rightarrow n > \sqrt{\frac{1}{2}(10^4)}$
 $\Rightarrow n > 70.7$, so let $n = 71$
- (b) $M = 120$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_s| = \frac{1}{180} \left(\frac{1}{n}\right)^4 (120) = \frac{2}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{2}{3}(10^4)$
 $\Rightarrow n > \sqrt[4]{\frac{2}{3}(10^4)} \Rightarrow n = 9.04$, so let $n = 10$ (n must be even)
18. (a) $M = 6$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4(10^4) \Rightarrow n > \sqrt{4(10^4)}$
 $\Rightarrow n > 200$, so let $n = 201$
- (b) $M = 120$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_s| \leq \frac{2}{180} \left(\frac{2}{n}\right)^4 (120) = \frac{64}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{64}{3}(10^4)$
 $\Rightarrow n > \sqrt[4]{\frac{64}{3}(10^4)} \Rightarrow n > 21.5$, so let $n = 22$ (n must be even)
19. (a) $f(x) = \sqrt{x+1} \Rightarrow f'(x) = \frac{1}{2}(x+1)^{-1/2} \Rightarrow f''(x) = -\frac{1}{4}(x+1)^{-3/2} = -\frac{1}{4(\sqrt{x+1})^3} \Rightarrow M = \frac{1}{4(\sqrt{1})^3} = \frac{1}{4}$.
Then $\Delta x = \frac{3}{n} \Rightarrow |E_T| \leq \frac{3}{12} \left(\frac{3}{n}\right)^2 \left(\frac{1}{4}\right) = \frac{9}{16n^2} < 10^{-4} \Rightarrow n^2 > \frac{9}{16}(10^4) \Rightarrow n > \sqrt{\frac{9}{16}(10^4)} \Rightarrow n > 75$, so let
 $n = 76$
- (b) $f^{(3)}(x) = \frac{3}{8}(x+1)^{-5/2} \Rightarrow f^{(4)}(x) = -\frac{15}{16}(x+1)^{-7/2} = -\frac{15}{16(\sqrt{x+1})^7} \Rightarrow M = \frac{15}{16(\sqrt{1})^7} = \frac{15}{16}$. Then
 $\Delta x = \frac{3}{n} \Rightarrow |E_s| \leq \frac{3}{180} \left(\frac{3}{n}\right)^4 \left(\frac{15}{16}\right) = \frac{3^5(15)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(15)(10^4)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(15)(10^4)}{16(180)}} \Rightarrow n > 10.6$, so let
 $n = 12$ (n must be even)
20. (a) $f(x) = \frac{1}{\sqrt{x+1}} \Rightarrow f'(x) = -\frac{1}{2}(x+1)^{-3/2} \Rightarrow f''(x) = \frac{3}{4}(x+1)^{-5/2} = \frac{3}{4(\sqrt{x+1})^5} \Rightarrow M = \frac{3}{4(\sqrt{1})^5} = \frac{3}{4}$. Then
 $\Delta x = \frac{3}{n} \Rightarrow |E_T| \leq \frac{3}{12} \left(\frac{3}{n}\right)^2 \left(\frac{3}{4}\right) = \frac{3^4}{48n^2} < 10^{-4} \Rightarrow n^2 > \frac{3^4(10^4)}{48} \Rightarrow n > \sqrt{\frac{3^4(10^4)}{48}} \Rightarrow n > 129.9$, so let $n = 130$
- (b) $f^{(3)}(x) = -\frac{15}{8}(x+1)^{-7/2} \Rightarrow f^{(4)}(x) = \frac{105}{16}(x+1)^{-9/2} = \frac{105}{16(\sqrt{x+1})^9} \Rightarrow M = \frac{105}{16(\sqrt{1})^9} = \frac{105}{16}$. Then
 $\Delta x = \frac{3}{n} \Rightarrow |E_s| \leq \frac{3}{180} \left(\frac{3}{n}\right)^4 \left(\frac{105}{16}\right) = \frac{3^5(105)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(105)(10^4)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(105)(10^4)}{16(180)}} \Rightarrow n > 17.25$,
so let $n = 18$ (n must be even)

21. (a)
- $f(x) = \sin(x+1) \Rightarrow f'(x) = \cos(x+1) \Rightarrow f''(x) = -\sin(x+1) \Rightarrow M = 1$
- . Then

$$\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (1) = \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8(10^4)}{12} \Rightarrow n > \sqrt{\frac{8(10^4)}{12}} \Rightarrow n > 81.6, \text{ so let } n = 82$$

- (b)
- $f^{(3)}(x) = -\cos(x+1) \Rightarrow f^{(4)}(x) = \sin(x+1) \Rightarrow M = 1$
- . Then
- $\Delta x = \frac{2}{n} \Rightarrow |E_s| \leq \frac{2}{180} \left(\frac{2}{n}\right)^4 (1)$

$$= \frac{32}{180n^4} < 10^{-4} \Rightarrow n^4 > \frac{32(10^4)}{180} \Rightarrow n > \sqrt[4]{\frac{32(10^4)}{180}} \Rightarrow n > 6.49, \text{ so let } n = 8 \text{ (} n \text{ must be even)}$$

22. (a)
- $f(x) = \cos(x+\pi) \Rightarrow f'(x) = -\sin(x+\pi) \Rightarrow f''(x) = -\cos(x+\pi) \Rightarrow M = 1$
- . Then

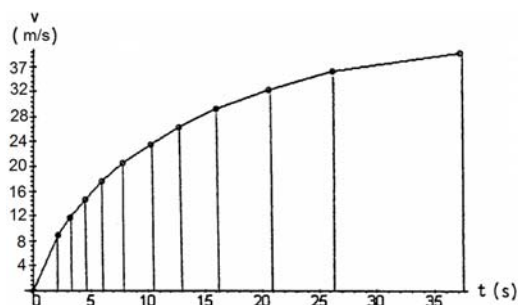
$$\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (1) = \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8(10^4)}{12} \Rightarrow n > \sqrt{\frac{8(10^4)}{12}} \Rightarrow n > 81.6, \text{ so let } n = 82$$

- (b)
- $f^{(3)}(x) = \sin(x+\pi) \Rightarrow f^{(4)}(x) = \cos(x+\pi) \Rightarrow M = 1$
- . Then
- $\Delta x = \frac{2}{n} \Rightarrow |E_s| \leq \frac{2}{180} \left(\frac{2}{n}\right)^4 (1)$

$$= \frac{32}{180n^4} < 10^{-4} \Rightarrow n^4 > \frac{32(10^4)}{180} \Rightarrow n > \sqrt[4]{\frac{32(10^4)}{180}} \Rightarrow n > 6.49, \text{ so let } n = 8 \text{ (} n \text{ must be even)}$$

- 23.
- $1(1.2+2(1.6)+2(1.8)+\cdots 2(2.5)+2.6) + 2(1.2+2(1.6)+2(1.8)+\cdots 2(2.5)+2.6) + \cdots 2(2.5)+2.6)(5) = 106.5 \text{ m}^3$

24. Use the conversion
- $36 \text{ km/h} = 10 \text{ m/s}$
- since time is measured in seconds. The distance traveled as the car accelerates from, say,
- $40 \text{ km/h} = 11.11 \text{ m/s}$
- to
- $50 \text{ km/h} = 13.89 \text{ m/s}$
- in
- $(4.5 - 3.2) = 1.3 \text{ s}$
- is the area of the trapezoid (see figure) associated with that time interval:
- $\frac{1}{2}(11.11 + 13.89)(1.3) = 16.25 \text{ m}$
- . The total distance traveled by the car is the sum of all these eleven trapezoids (using
- $\frac{\Delta t}{2}$
- and the table below):



$v(\text{km/h})$	0	30	40	50	60	70	80	90	100	110	120	130
$v(\text{m/s})$	0	8.33	11.11	13.89	16.67	19.44	22.22	25	27.78	30.56	33.33	36.11
$t(\text{s})$	0	2.2	3.2	4.5	5.9	7.8	10.2	12.7	16	20.6	26.2	37.1
$\Delta t/2$	0	1.1	0.5	0.65	0.7	0.95	1.2	1.25	1.65	2.3	2.8	5.45

$$(8.33)(1.1) + (19.44)(0.5) + (25)(0.65) + (30.56)(0.7) + (36.11)(0.95) + (41.67)(1.2) + (47.22)(1.25) + (52.78)(1.65) + (58.56)(2.3) + (63.89)(2.8) + (69.44)(5.45) = 978.5 \text{ m} \approx 0.9785 \text{ km}$$

25. Using Simpson's Rule,
- $\Delta x = 3 \Rightarrow \frac{\Delta x}{3} = 0.1$
- ;

$\sum my_i = 11.65 \Rightarrow \text{Cross Section Area} \approx 0.1(11.65) = 1.165 \text{ m}^2$ Let x be the length of the tank. Then the Volume $V = (\text{Cross Sectional Area}) x = 1.165x$.
Now 2000 kg gasoline at 673 kg/m^3
 $\Rightarrow V = \frac{2000}{673} = 2.97 \text{ m}^3$
 $\Rightarrow 2.97 = 1.165x \Rightarrow x \approx 2.55 \text{ m}$

	x_i	y_i	m	my_i
x_0	0	0.5	1	0.5
x_1	1	0.55	4	2.2
x_2	2	0.6	2	1.2
x_3	3	0.65	4	2.6
x_4	4	0.7	2	1.4
x_5	5	0.75	4	3.0
x_6	6	0.75	1	0.75

$$26. \frac{24}{2}[0.019 + 2(0.020) + 2(0.021) + \dots + 2(0.031) + 0.035] = 4.2 L$$

$$27. (a) |E_s| \leq \frac{b-a}{180} (\Delta x^4) M; n = 4 \Rightarrow \Delta x = \frac{\pi-0}{4} = \frac{\pi}{8}; |f^{(4)}| \leq 1 \Rightarrow M = 1 \Rightarrow |E_s| \leq \frac{(\frac{\pi}{2}-0)}{180} \left(\frac{\pi}{8}\right)^4 (1) \approx 0.00021$$

$$(b) \Delta x = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{24};$$

$$\sum mf(x_i) = 10.47208705$$

$$\Rightarrow S = \frac{\pi}{24}(10.47208705) \approx 1.37079$$

$$(c) \approx \left(\frac{0.00021}{1.37079}\right) \times 100 \approx 0.015\%$$

	x_i	$f(x_i)$	m	$mf(x_{1i})$
x_0	0	1	1	1
x_1	$\pi/8$	0.974495358	4	3.897981432
x_2	$\pi/4$	0.900316316	2	1.800632632
x_3	$3\pi/8$	0.784213303	4	3.136853212
x_4	$\pi/2$	0.636619772	1	0.636619772

$$28. (a) \Delta x = \frac{b-a}{n} = \frac{1-0}{10} = 0.1 \Rightarrow \operatorname{erf}(1) = \frac{2}{\sqrt{3}} \left(\frac{0.1}{3}\right) (y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 4y_9 + y_{10})$$

$$= \frac{2}{30\sqrt{\pi}} (e^0 + 4e^{-0.01} + 2e^{-0.04} + 4e^{-0.09} + \dots + 4e^{-0.81} + e^{-1}) \approx 0.843$$

$$(b) |E_s| \leq \frac{1-0}{180} (0.1)^4 (12) \approx 6.7 \times 10^{-6}$$

$$29. T = \frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + 2y_3 + \dots + 2y_{n-1} + y_n) \text{ where } \Delta x = \frac{b-a}{n} \text{ and } f \text{ is continuous on } [a, b]. \text{ So}$$

$$T = \frac{b-a}{n} \frac{(y_0 + y_1 + y_1 + y_2 + y_2 + \dots + y_{n-1} + y_{n-1} + y_n)}{2} = \frac{b-a}{n} \left(\frac{f(x_0) + f(x_1)}{2} + \frac{f(x_1) + f(x_2)}{2} + \dots + \frac{f(x_{n-1}) + f(x_n)}{2} \right).$$

Since f is continuous on each interval $[x_{k-1}, x_k]$, and $\frac{f(x_{k-1}) + f(x_k)}{2}$ is always between $f(x_{k-1})$ and $f(x_k)$, there is a point c_k in $[x_{k-1}, x_k]$ with $f(c_k) = \frac{f(x_{k-1}) + f(x_k)}{2}$; this is a consequence of the Intermediate Value Theorem.

Thus our sum is $\sum_{k=1}^n \left(\frac{b-a}{n}\right) f(c_k)$ which has the form $\sum_{k=1}^n \Delta x_k f(c_k)$ with $\Delta x_k = \frac{b-a}{n}$ for all k . This a Riemann Sum for f on $[a, b]$.

$$30. S = \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 2y_{n-2} + 4y_{n-1} + y_n) \text{ where } n \text{ is even, } \Delta x = \frac{b-a}{n} \text{ and } f \text{ is continuous on}$$

$$[a, b]. \text{ So } S = \frac{b-a}{n} \left(\frac{y_0 + 4y_1 + y_2}{3} + \frac{y_2 + 4y_3 + y_4}{3} + \frac{y_4 + 4y_5 + y_6}{3} + \dots + \frac{y_{n-2} + 4y_{n-1} + y_n}{3} \right)$$

$$= \frac{b-a}{\frac{n}{2}} \left(\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} + \frac{f(x_4) + 4f(x_5) + f(x_6)}{6} + \dots + \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right)$$

$\frac{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})}{6}$ is the average of the six values of the continuous function on the interval

$[x_{2k}, x_{2k+2}]$, so it is between the minimum and maximum of f on this interval. By the Extreme Value

Theorem for continuous functions, f takes on its maximum and minimum in this interval, so there are x_a and

x_b with $x_{2k} \leq x_a, x_b \leq x_{2k+2}$ and $f(x_a) \leq \frac{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})}{6} \leq f(x_b)$.

By the Intermediate Value Theorem, there is c_k in $[x_{2k}, x_{2k+2}]$ with $f(c_k) = \frac{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})}{6}$.

So our sum has the form $\sum_{k=1}^{n/2} \Delta x_k f(c_k)$ with $\Delta x_k = \frac{b-a}{(n/2)}$, a Riemann sum for f on $[a, b]$.

31. (a) $a = 1, e = \frac{1}{2} \Rightarrow \text{Length} = 4 \int_0^{\pi/2} \sqrt{1 - \frac{1}{4} \cos^2 t} dt$
 $= 2 \int_0^{\pi/2} \sqrt{4 - \cos^2 t} dt = \int_0^{\pi/2} f(t) dt$; use the Trapezoid Rule with $n = 10$
 $\Rightarrow \Delta t = \frac{b-a}{n} = \frac{(\frac{\pi}{2})-0}{10} = \frac{\pi}{20}$
 $\int_0^{\pi/2} \sqrt{4 - \cos^2 t} dt \approx \sum_{n=0}^{10} mf(x_n) = 37.3686183$
 $\Rightarrow T = \frac{\Delta t}{2} (37.3686183) = \frac{\pi}{40} (37.3686183)$
 $= 2.934924419$
 $\Rightarrow \text{Length} = 2(2.934924419) \approx 5.870$
- (b) $|f''(t)| < 1 \Rightarrow M = 1$
 $\Rightarrow |E_T| \leq \frac{b-a}{12} (\Delta t^2 M) \leq \frac{(\frac{\pi}{2})-0}{12} \left(\frac{\pi}{20}\right)^2 1 \leq 0.0032$

	x_i	$f(x_i)$	m	$mf(x_i)$	
	x_0	0	1.732050808	1	1.732050808
	x_1	$\pi/20$	1.739100843	2	3.478201686
	x_2	$\pi/10$	1.759400893	2	3.518801786
	x_3	$3\pi/20$	1.790560631	2	3.581121262
	x_4	$\pi/5$	1.82906848	2	3.658136959
	x_5	$\pi/4$	1.870828693	2	3.741657387
	x_6	$3\pi/10$	1.911676881	2	3.823353762
	x_7	$7\pi/20$	1.947791731	2	3.895583461
	x_8	$2\pi/5$	1.975982919	2	3.951965839
	x_9	$9\pi/20$	1.993872679	2	3.987745357
	x_{10}	$\pi/2$	2	1	2

32. $\Delta x = \frac{\pi-0}{8} = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{24}$;
 $\sum mf(x_i) = 29.184807792$
 $\Rightarrow S = \frac{\pi}{24} (29.18480779) \approx 3.82028$

	x_i	$f(x_i)$	m	$mf(x_i)$	
	x_0	0	1.414213562	1	1.414213562
	x_1	$\pi/8$	1.361452677	4	5.445810706
	x_2	$\pi/4$	1.224744871	2	2.449489743
	x_3	$3\pi/8$	1.070722471	4	4.282889883
	x_4	$\pi/2$	1	2	2
	x_5	$5\pi/8$	1.070722471	4	4.282889883
	x_6	$3\pi/4$	1.224744871	2	2.449489743
	x_7	$7\pi/8$	1.361452677	4	5.445810706
	x_8	π	1.414213562	1	1.414213562

33. The length of the curve $y = \sin\left(\frac{3\pi}{20}x\right)$ from 0 to 20 is: $L = \int_0^{20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$; $\frac{dy}{dx} = \frac{3\pi}{20} \cos\left(\frac{3\pi}{20}x\right)$
 $\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{9\pi^2}{400} \cos^2\left(\frac{3\pi}{20}x\right) \Rightarrow L = \int_0^{20} \sqrt{1 + \frac{9\pi^2}{400} \cos^2\left(\frac{3\pi}{20}x\right)} dx$. Using numerical integration we find
 $L \approx 21.07$ cm
34. First, we'll find the length of the cosine curve: $L = \int_{-7.5}^{7.5} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$; $\frac{dy}{dx} = -\frac{7.5\pi}{15} \sin\left(\frac{\pi x}{15}\right)$
 $\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{\pi^2}{4} \sin^2\left(\frac{\pi x}{15}\right) \Rightarrow L = \int_{-7.5}^{7.5} \sqrt{1 + \frac{\pi^2}{4} \sin^2\left(\frac{\pi x}{15}\right)} dx$. Using a numerical integrator we find
 $L \approx 21.9554$ m. Surface area is: $A = \text{length} \cdot \text{width} \approx (21.9554)(90) = 1,975.986$ m.
Cost = $26.11A = (26.11)(1,975.986) = \$51,593$. Answers may vary slightly, depending on the numerical integration used.

35. $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \cos^2 x \Rightarrow S = \int_0^\pi 2\pi(\sin x)\sqrt{1 + \cos^2 x} \, dx$; a numerical integration gives $S \approx 14.4$
36. $y = \frac{x^2}{4} \Rightarrow \frac{dy}{dx} = \frac{x}{2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4} \Rightarrow S = \int_0^2 2\pi\left(\frac{x^2}{4}\right)\sqrt{1 + \frac{x^2}{4}} \, dx$; a numerical integration gives $S \approx 5.28$
37. A calculator or computer numerical integrator yields $\sin^{-1} 0.6 \approx 0.643501109$.
38. A calculator or computer numerical integrator yields $\pi \approx 3.1415929$.
39. The amount of medication absorbed over a 12-h period is given by $\int_0^{12} (6 - \ln(2t^2 - 3t + 3)) \, dt$. A numerical integrator yields a value of 28.684 for this integral, so the amount of medication absorbed over a 12-h period is approximately 28.7 milligrams.
40. The average concentration of antihistamine over a 6-h period is given by $\frac{1}{6} \int_0^1 (12.5 - 4 \ln(t^2 - 3t + 4)) \, dt$. A numerical integrator yields a value of 6.078 for this integral, so the average concentration is approximately 6.1 grams per liter.

8.8 IMPROPER INTEGRALS

- $\int_0^\infty \frac{dx}{x^2+1} = \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b \rightarrow \infty} [\tan^{-1} x]_0^b = \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$
- $\int_1^\infty \frac{dx}{x^{1.001}} = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^{1.001}} = \lim_{b \rightarrow \infty} [-1000x^{-0.001}]_1^b = \lim_{b \rightarrow \infty} \left(\frac{-1000}{b^{0.001}} + 1000\right) = 1000$
- $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b \rightarrow 0^+} \int_b^1 x^{-1/2} dx = \lim_{b \rightarrow 0^+} [2x^{1/2}]_b^1 = \lim_{b \rightarrow 0^+} (2 - 2\sqrt{b}) = 2 - 0 = 2$
- $\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{b \rightarrow 4^-} \int_0^b (4-x)^{-1/2} dx = \lim_{b \rightarrow 4^-} [-2\sqrt{4-b} - (-2\sqrt{4})] = 0 + 4 = 4$
- $\int_{-1}^1 \frac{dx}{x^{2/3}} = \int_{-1}^0 \frac{dx}{x^{2/3}} + \int_0^1 \frac{dx}{x^{2/3}} = \lim_{b \rightarrow 0^-} [3x^{1/3}]_{-1}^b + \lim_{c \rightarrow 0^+} [3x^{1/3}]_c^1 = \lim_{b \rightarrow 0^-} [3b^{1/3} - 3(-1)^{1/3}] + \lim_{c \rightarrow 0^+} [3(1)^{1/3} - 3c^{1/3}] = (0 + 3) + (3 - 0) = 6$
- $\int_{-8}^1 \frac{dx}{x^{1/3}} = \int_{-8}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} = \lim_{b \rightarrow 0^-} \left[\frac{3}{2}x^{2/3}\right]_{-8}^b + \lim_{c \rightarrow 0^+} \left[\frac{3}{2}x^{2/3}\right]_c^1 = \lim_{b \rightarrow 0^-} \left[\frac{3}{2}b^{2/3} - \frac{3}{2}(-8)^{2/3}\right] + \lim_{c \rightarrow 0^+} \left[\frac{3}{2}(1)^{2/3} - \frac{3}{2}c^{2/3}\right] = \left[0 - \frac{3}{2}(4)\right] + \left(\frac{3}{2} - 0\right) = -\frac{9}{2}$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b \rightarrow 1^-} \left[\sin^{-1} x \right]_0^b = \lim_{b \rightarrow 1^-} \left(\sin^{-1} b - \sin^{-1} 0 \right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$
8. $\int_0^1 \frac{dr}{r^{0.999}} = \lim_{b \rightarrow 0^+} \left[1000r^{0.001} \right]_b^1 = \lim_{b \rightarrow 0^+} \left(1000 - 1000b^{0.001} \right) = 1000 - 0 = 1000$
9. $\int_{-\infty}^{-2} \frac{2 dx}{x^2-1} = \int_{-\infty}^{-2} \frac{dx}{x-1} - \int_{-\infty}^{-2} \frac{dx}{x+1} = \lim_{b \rightarrow -\infty} \left[\ln|x-1| \right]_b^{-2} - \lim_{b \rightarrow -\infty} \left[\ln|x+1| \right]_b^{-2} = \lim_{b \rightarrow -\infty} \left[\ln \left| \frac{x-1}{x+1} \right| \right]_b^{-2}$
 $= \lim_{b \rightarrow -\infty} \left(\ln \left| \frac{-3}{-1} \right| - \ln \left| \frac{b-1}{b+1} \right| \right) = \ln 3 - \ln \left(\lim_{b \rightarrow -\infty} \frac{b-1}{b+1} \right) = \ln 3 - \ln 1 = \ln 3$
10. $\int_{-\infty}^2 \frac{2 dx}{x^2+4} = \lim_{b \rightarrow -\infty} \left[\tan^{-1} \frac{x}{2} \right]_b^2 = \lim_{b \rightarrow -\infty} \left(\tan^{-1} 1 - \tan^{-1} \frac{b}{2} \right) = \frac{\pi}{4} - \left(-\frac{\pi}{2} \right) = \frac{3\pi}{4}$
11. $\int_2^{\infty} \frac{2 dv}{v^2-v} = \lim_{b \rightarrow \infty} \left[2 \ln \left| \frac{v-1}{v} \right| \right]_2^b = \lim_{b \rightarrow \infty} \left(2 \ln \left| \frac{b-1}{b} \right| - 2 \ln \left| \frac{2-1}{2} \right| \right) = 2 \ln(1) - 2 \ln \left(\frac{1}{2} \right) = 0 + 2 \ln 2 = \ln 4$
12. $\int_2^{\infty} \frac{2 dt}{t^2-1} = \lim_{b \rightarrow \infty} \left[\ln \left| \frac{t-1}{t+1} \right| \right]_2^b = \lim_{b \rightarrow \infty} \left(\ln \left| \frac{b-1}{b+1} \right| - \ln \left| \frac{2-1}{2+1} \right| \right) = \ln(1) - \ln \left(\frac{1}{3} \right) = 0 + \ln 3 = \ln 3$
13. $\int_{-\infty}^{\infty} \frac{2x dx}{(x^2+1)^2} = \int_{-\infty}^0 \frac{2x dx}{(x^2+1)^2} + \int_0^{\infty} \frac{2x dx}{(x^2+1)^2}; \left[\begin{array}{l} u = x^2+1 \\ du = 2x dx \end{array} \right] \rightarrow \int_{\infty}^1 \frac{du}{u^2} + \int_1^{\infty} \frac{du}{u^2} = \lim_{b \rightarrow \infty} \left[-\frac{1}{u} \right]_b^1 + \lim_{c \rightarrow \infty} \left[-\frac{1}{u} \right]_1^c$
 $= \lim_{b \rightarrow \infty} \left(-1 + \frac{1}{b} \right) + \lim_{c \rightarrow \infty} \left[-\frac{1}{c} - (-1) \right] = (-1+0) + (0+1) = 0$
14. $\int_{-\infty}^{\infty} \frac{x dx}{(x^2+4)^{3/2}} = \int_{-\infty}^0 \frac{x dx}{(x^2+4)^{3/2}} + \int_0^{\infty} \frac{x dx}{(x^2+4)^{3/2}}; \left[\begin{array}{l} u = x^2+4 \\ du = 2x dx \end{array} \right] \rightarrow \int_{\infty}^4 \frac{du}{2u^{3/2}} + \int_4^{\infty} \frac{du}{2u^{3/2}} = \lim_{b \rightarrow \infty} \left[-\frac{1}{\sqrt{u}} \right]_b^4 + \lim_{c \rightarrow \infty} \left[-\frac{1}{\sqrt{u}} \right]_4^c$
 $= \lim_{b \rightarrow \infty} \left(-\frac{1}{2} + \frac{1}{\sqrt{b}} \right) + \lim_{c \rightarrow \infty} \left(-\frac{1}{\sqrt{c}} + \frac{1}{2} \right) = \left(-\frac{1}{2} + 0 \right) + \left(0 + \frac{1}{2} \right) = 0$
15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}} d\theta; \left[\begin{array}{l} u = \theta^2+2\theta \\ du = 2(\theta+1)d\theta \end{array} \right] \rightarrow \int_0^3 \frac{du}{2\sqrt{u}} = \lim_{b \rightarrow 0^+} \int_b^3 \frac{du}{2\sqrt{u}} = \lim_{b \rightarrow 0^+} \left[\sqrt{u} \right]_b^3 = \lim_{b \rightarrow 0^+} \left(\sqrt{3} - \sqrt{b} \right) = \sqrt{3} - 0 = \sqrt{3}$
16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}} ds = \frac{1}{2} \int_0^2 \frac{2s ds}{\sqrt{4-s^2}} + \int_0^2 \frac{ds}{\sqrt{4-s^2}}; \left[\begin{array}{l} u = 4-s^2 \\ du = -2s ds \end{array} \right] \rightarrow -\frac{1}{2} \int_4^0 \frac{du}{\sqrt{u}} + \lim_{c \rightarrow 2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}}$
 $= \lim_{b \rightarrow 0^+} \int_b^4 \frac{du}{2\sqrt{u}} + \lim_{c \rightarrow 2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}} = \lim_{b \rightarrow 0^+} \left[\sqrt{u} \right]_b^4 + \lim_{c \rightarrow 2^-} \left[\sin^{-1} \frac{s}{2} \right]_0^c = \lim_{b \rightarrow 0^+} \left(2 - \sqrt{b} \right) + \lim_{c \rightarrow 2^-} \left(\sin^{-1} \frac{c}{2} - \sin^{-1} 0 \right)$
 $= (2-0) + \left(\frac{\pi}{2} - 0 \right) = \frac{4+\pi}{2}$
17. $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}; \left[\begin{array}{l} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{array} \right] \rightarrow \int_0^{\infty} \frac{2 du}{u^2+1} = \lim_{b \rightarrow \infty} \int_0^b \frac{2 du}{u^2+1} = \lim_{b \rightarrow \infty} \left[2 \tan^{-1} u \right]_0^b = \lim_{b \rightarrow \infty} \left(2 \tan^{-1} b - 2 \tan^{-1} 0 \right)$
 $= 2 \left(\frac{\pi}{2} \right) - 2(0) = \pi$

$$\begin{aligned}
 18. \quad \int_1^\infty \frac{dx}{x\sqrt{x^2-1}} &= \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{b \rightarrow 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c \rightarrow \infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}} = \lim_{b \rightarrow 1^+} \left[\sec^{-1} |x| \right]_b^2 + \lim_{c \rightarrow \infty} \left[\sec^{-1} |x| \right]_2^c \\
 &= \lim_{b \rightarrow 1^+} \left(\sec^{-1} 2 - \sec^{-1} b \right) + \lim_{c \rightarrow \infty} \left(\sec^{-1} c - \sec^{-1} 2 \right) = \left(\frac{\pi}{3} - 0 \right) + \left(\frac{\pi}{2} - \frac{\pi}{3} \right) = \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 19. \quad \int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1} v)} &= \lim_{b \rightarrow \infty} \left[\ln |1 + \tan^{-1} v| \right]_0^b = \lim_{b \rightarrow \infty} \left[\ln |1 + \tan^{-1} b| - \ln |1 + \tan^{-1} 0| \right] = \ln \left(1 + \frac{\pi}{2} \right) - \ln(1+0) \\
 &= \ln \left(1 + \frac{\pi}{2} \right)
 \end{aligned}$$

$$20. \quad \int_0^\infty \frac{16 \tan^{-1} x}{1+x^2} dx = \lim_{b \rightarrow \infty} \left[8 \left(\tan^{-1} x \right)^2 \right]_0^b = \lim_{b \rightarrow \infty} \left[8 \left(\tan^{-1} b \right)^2 - 8 \left(\tan^{-1} 0 \right)^2 \right] = 8 \left(\frac{\pi}{2} \right)^2 - 8(0) = 2\pi^2$$

$$\begin{aligned}
 21. \quad \int_{-\infty}^0 \theta e^\theta d\theta &= \lim_{b \rightarrow -\infty} \left[\theta e^\theta - e^\theta \right]_b^0 = \lim_{b \rightarrow -\infty} \left[(0 \cdot e^0 - e^0) - (be^b - e^b) \right] = -1 - \lim_{b \rightarrow -\infty} \left(\frac{b-1}{e^{-b}} \right) = -1 - \lim_{b \rightarrow -\infty} \left(\frac{1}{-e^{-b}} \right) \\
 &\quad \text{(l'Hôpital's rule for } \frac{\infty}{\infty} \text{ form)} \\
 &= -1 - 0 = -1
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \int_0^\infty 2e^{-\theta} \sin \theta d\theta &= \lim_{b \rightarrow \infty} \int_0^b 2e^{-\theta} \sin \theta d\theta = \lim_{b \rightarrow \infty} 2 \left[\frac{e^{-\theta}}{1+1} (-\sin \theta - \cos \theta) \right]_0^b \quad (\text{Formula 107 with } a = -1, b = 1) \\
 &= \lim_{b \rightarrow \infty} \left[\frac{-2(\sin b + \cos b)}{2e^b} + \frac{2(\sin 0 + \cos 0)}{2e^0} \right] = 0 + \frac{2(0+1)}{2} = 1
 \end{aligned}$$

$$23. \quad \int_{-\infty}^0 e^{-|x|} dx = \int_{-\infty}^0 e^x dx = \lim_{b \rightarrow -\infty} \left[e^x \right]_b^0 = \lim_{b \rightarrow -\infty} (1 - e^b) = (1 - 0) = 1$$

$$\begin{aligned}
 24. \quad \int_{-\infty}^\infty 2xe^{-x^2} dx &= \int_{-\infty}^0 2xe^{-x^2} dx + \int_0^\infty 2xe^{-x^2} dx = \lim_{b \rightarrow -\infty} \left[-e^{-x^2} \right]_b^0 + \lim_{c \rightarrow \infty} \left[-e^{-x^2} \right]_0^c \\
 &= \lim_{b \rightarrow -\infty} \left[-1 - (-e^{-b^2}) \right] + \lim_{c \rightarrow \infty} \left[-e^{-c^2} - (-1) \right] = (-1 - 0) + (0 + 1) = 0
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \int_0^1 x \ln x dx &= \lim_{b \rightarrow 0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} \right]_b^1 = \lim_{b \rightarrow 0^+} \left[\left(\frac{1}{2} \ln 1 - \frac{1}{4} \right) - \left(\frac{b^2}{2} \ln b - \frac{b^2}{4} \right) \right] = -\frac{1}{4} - \lim_{b \rightarrow 0^+} \left(\frac{\ln b}{\frac{2}{b^2}} \right) + 0 \\
 &= -\frac{1}{4} - \lim_{b \rightarrow 0^+} \left(\frac{\frac{1}{b}}{\frac{-4}{b^3}} \right) = -\frac{1}{4} + \lim_{b \rightarrow 0^+} \left(\frac{b^2}{4} \right) = -\frac{1}{4} + 0 = -\frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \int_0^1 (-\ln x) dx &= \lim_{b \rightarrow 0^+} \left[x - x \ln x \right]_b^1 = \lim_{b \rightarrow 0^+} \left[(1 - 1 \ln 1) - (b - b \ln b) \right] = 1 - 0 + \lim_{b \rightarrow 0^+} \frac{\ln b}{\left(\frac{1}{b} \right)} = 1 + \lim_{b \rightarrow 0^+} \left(\frac{\frac{1}{b}}{\left(\frac{-1}{b^2} \right)} \right) \\
 &= 1 - \lim_{b \rightarrow 0^+} b = 1 - 0 = 1
 \end{aligned}$$

$$27. \quad \int_0^2 \frac{ds}{\sqrt{4-s^2}} = \lim_{b \rightarrow 2^-} \left[\sin^{-1} \frac{s}{2} \right]_0^b = \lim_{b \rightarrow 2^-} \left[\sin^{-1} \frac{b}{2} - \sin^{-1} 0 \right] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$28. \int_0^1 \frac{4r \, dr}{\sqrt{1-r^4}} = \lim_{b \rightarrow 1^-} \left[2 \sin^{-1}(r^2) \right]_0^b = \lim_{b \rightarrow 1^-} \left[2 \sin^{-1}(b^2) - 2 \sin^{-1} 0 \right] = 2 \cdot \frac{\pi}{2} - 0 = \pi$$

$$29. \int_1^2 \frac{ds}{s\sqrt{s^2-1}} = \lim_{b \rightarrow 1^+} \left[\sec^{-1} s \right]_b^2 = \lim_{b \rightarrow 1^+} \left[\sec^{-1} 2 - \sec^{-1} b \right] = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

$$30. \int_2^4 \frac{dt}{t\sqrt{t^2-4}} = \lim_{b \rightarrow 2^+} \left[\frac{1}{2} \sec^{-1} \frac{t}{2} \right]_b^4 = \lim_{b \rightarrow 2^+} \left[\left(\frac{1}{2} \sec^{-1} \frac{4}{2} \right) - \frac{1}{2} \sec^{-1} \left(\frac{b}{2} \right) \right] = \frac{1}{2} \left(\frac{\pi}{3} \right) - \frac{1}{2} \cdot 0 = \frac{\pi}{6}$$

$$31. \int_{-1}^4 \frac{dx}{\sqrt{|x|}} = \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{dx}{\sqrt{-x}} + \lim_{c \rightarrow 0^+} \int_c^4 \frac{dx}{\sqrt{x}} = \lim_{b \rightarrow 0^-} \left[-2\sqrt{-x} \right]_{-1}^b + \lim_{c \rightarrow 0^+} \left[2\sqrt{x} \right]_c^4$$

$$= \lim_{b \rightarrow 0^-} \left[(-2\sqrt{-b}) - (-2\sqrt{-(-1)}) \right] + \lim_{c \rightarrow 0^+} \left[2\sqrt{4} - 2\sqrt{c} \right] = 0 + 2 + 2 \cdot 2 - 0 = 6$$

$$32. \int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{b \rightarrow 1^-} \left[-2\sqrt{1-x} \right]_0^b + \lim_{c \rightarrow 1^+} \left[2\sqrt{x-1} \right]_c^2$$

$$= \lim_{b \rightarrow 1^-} \left[(-2\sqrt{1-b}) - (-2\sqrt{1-0}) \right] + \lim_{c \rightarrow 1^+} \left[2\sqrt{2-1} - (2\sqrt{c-1}) \right] = 0 + 2 + 2 - 0 = 4$$

$$33. \int_{-1}^{\infty} \frac{d\theta}{\theta^2+5\theta+6} = \lim_{b \rightarrow \infty} \left[\ln \left| \frac{\theta+2}{\theta+3} \right| \right]_{-1}^b = \lim_{b \rightarrow \infty} \left[\ln \left| \frac{b+2}{b+3} \right| - \ln \left| \frac{-1+2}{-1+3} \right| \right] = 0 - \ln \left(\frac{1}{2} \right) = \ln 2$$

$$34. \int_0^{\infty} \frac{dx}{(x+1)(x^2+1)} = \lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \tan^{-1} x \right]_0^b = \lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln \left(\frac{x+1}{\sqrt{x^2+1}} \right) + \frac{1}{2} \tan^{-1} x \right]_0^b$$

$$= \lim_{b \rightarrow \infty} \left[\left(\frac{1}{2} \ln \left(\frac{b+1}{\sqrt{b^2+1}} \right) + \frac{1}{2} \tan^{-1} b \right) - \left(\frac{1}{2} \ln \frac{1}{\sqrt{1}} + \frac{1}{2} \tan^{-1} 0 \right) \right] = \frac{1}{2} \ln 1 + \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \ln 1 - \frac{1}{2} \cdot 0 = \frac{\pi}{4}$$

$$35. \int_0^{\pi/2} \tan \theta \, d\theta = \lim_{b \rightarrow (\pi/2)^-} \left[-\ln|\cos \theta| \right]_0^b = \lim_{b \rightarrow (\pi/2)^-} \left[-\ln|\cos b| + \ln 1 \right] = \lim_{b \rightarrow (\pi/2)^-} \left[-\ln|\cos b| \right] = +\infty, \text{ the integral diverges}$$

$$36. \int_0^{\pi/2} \cot \theta \, d\theta = \lim_{b \rightarrow 0^+} \left[\ln|\sin \theta| \right]_b^{\pi/2} = \lim_{b \rightarrow 0^+} \left[\ln 1 - \ln|\sin b| \right] = - \lim_{b \rightarrow 0^+} \left[\ln|\sin b| \right] = +\infty, \text{ the integral diverges}$$

$$37. \int_0^1 \frac{\ln x}{x^2} dx$$

$$\int_{1/3}^1 \frac{\ln x}{x^2} dx \text{ is bounded, so convergence is determined by } \int_0^{1/3} \frac{\ln x}{x^2} dx.$$

On $(0, 1/3]$, $\ln x < -1$ and $\frac{\ln x}{x^2} < -\frac{1}{x^2}$. Since $\int_0^{1/3} -\frac{1}{x^2} dx$ diverges to $-\infty$, so does $\int_0^{1/3} \frac{\ln x}{x^2} dx$ and

hence $\int_0^1 \frac{\ln x}{x^2} dx$ diverges.

38. Since $\int \frac{1}{x \ln x} dx = \ln(\ln x)$, $\int_1^2 \frac{1}{x \ln x} dx = \lim_{a \rightarrow 1^+} (\ln(\ln 2) - \ln(\ln a)) = \infty$; the integral diverges. (In this case we don't need a comparison test.)
39. $\int_0^{\ln 2} x^{-2} e^{-1/x} dx$; $\left[\frac{1}{x} = y\right] \rightarrow \int_{\infty}^{1/\ln 2} \frac{y^2 e^{-y} dy}{-y^3} = \int_{1/\ln 2}^{\infty} e^{-y} dy = \lim_{b \rightarrow \infty} [-e^{-y}]_{1/\ln 2}^b = \lim_{b \rightarrow \infty} [-e^{-b} - (-e^{-1/\ln 2})] = 0 + e^{-1/\ln 2} = e^{-1/\ln 2}$, so the integral converges.
40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$; $[y = \sqrt{x}] \rightarrow 2 \int_0^1 e^{-y} dy = 2 - \frac{2}{e}$, so the integral converges.
41. $\int_0^{\pi} \frac{dt}{\sqrt{t+\sin t}}$. Since for $0 \leq t \leq \pi$, $0 \leq \frac{1}{\sqrt{t+\sin t}} \leq \frac{1}{\sqrt{t}}$ and $\int_0^{\pi} \frac{dt}{\sqrt{t}}$ converges, then the original integral converges as well by the Direct Comparison Test.
42. $\int_0^1 \frac{dt}{t-\sin t}$; Let $f(t) = \frac{1}{t-\sin t}$ and $g(t) = \frac{1}{t^3}$, then $\lim_{t \rightarrow 0} \frac{f(t)}{g(t)} = \lim_{t \rightarrow 0} \frac{t^3}{t-\sin t} = \lim_{t \rightarrow 0} \frac{3t^2}{1-\cos t} = \lim_{t \rightarrow 0} \frac{6t}{\sin t} = \lim_{t \rightarrow 0} \frac{6}{\cos t} = 6$.
Now, $\int_0^1 \frac{dt}{t^3} = \lim_{b \rightarrow 0^+} \left[-\frac{1}{2t^2}\right]_b^1 = \lim_{b \rightarrow 0^+} \left[-\frac{1}{2} - \left(-\frac{1}{2b^2}\right)\right] = +\infty$, which diverges $\Rightarrow \int_0^1 \frac{dt}{t-\sin t}$ diverges by the Limit Comparison Test.
43. $\int_0^2 \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$ and $\int_0^1 \frac{dx}{1-x^2} = \lim_{b \rightarrow 1^-} \left[\frac{1}{2} \ln \left|\frac{1+x}{1-x}\right|\right]_0^b = \lim_{b \rightarrow 1^-} \left[\frac{1}{2} \ln \left|\frac{1+b}{1-b}\right| - 0\right] = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1-x^2}$ diverges as well.
44. $\int_0^2 \frac{dx}{1-x} = \int_0^1 \frac{dx}{1-x} + \int_1^2 \frac{dx}{1-x}$ and $\int_0^1 \frac{dx}{1-x} = \lim_{b \rightarrow 1^-} [-\ln(1-x)]_0^b = \lim_{b \rightarrow 1^-} [-\ln(1-b) - 0] = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1-x}$ diverges as well.
45. $\int_{-1}^1 \ln|x| dx = \int_{-1}^0 \ln(-x) dx + \int_0^1 \ln x dx$; $\int_0^1 \ln x dx = \lim_{b \rightarrow 0^+} [x \ln x - x]_b^1 = \lim_{b \rightarrow 0^+} [(1 \cdot 0 - 1) - (b \ln b - b)] = -1 - 0 = -1$; $\int_{-1}^0 \ln(-x) dx = -1 \Rightarrow \int_{-1}^1 \ln|x| dx = -2$ converges.
46. $\int_{-1}^1 (-x \ln|x|) dx = \int_{-1}^0 [-x \ln(-x)] dx + \int_0^1 (-x \ln x) dx = \lim_{b \rightarrow 0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_b^1 - \lim_{c \rightarrow 0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_c^1 = \lim_{b \rightarrow 0^+} \left[\left(\frac{1}{2} \ln 1 - \frac{1}{4}\right) - \left(\frac{b^2}{2} \ln b - \frac{b^2}{4}\right)\right] - \lim_{c \rightarrow 0^+} \left[\left(\frac{1}{2} \ln 1 - \frac{1}{4}\right) - \left(\frac{c^2}{2} \ln c - \frac{c^2}{4}\right)\right] = -\frac{1}{4} - 0 + \frac{1}{4} + 0 = 0 \Rightarrow$ the integral converges (see Exercise 25 for the limit calculations).
47. $\int_1^{\infty} \frac{dx}{1+x^3}$; $0 \leq \frac{1}{1+x^3} \leq \frac{1}{x^3}$ for $1 \leq x < \infty$ and $\int_1^{\infty} \frac{dx}{x^3}$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{1+x^3}$ converges by the Direct Comparison Test.

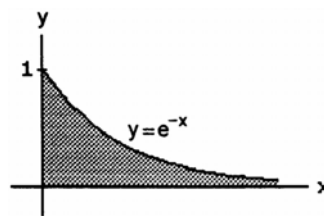
48. $\int_4^\infty \frac{dx}{\sqrt{x-1}}$; $\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{\sqrt{x-1}}\right)}{\left(\frac{1}{\sqrt{x}}\right)} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{x-1}} = \lim_{x \rightarrow \infty} \frac{1}{1-\frac{1}{\sqrt{x}}} = \frac{1}{1-0} = 1$ and $\int_4^\infty \frac{dx}{\sqrt{x}} = \lim_{b \rightarrow \infty} [2\sqrt{x}]_4^b = \infty$, which diverges
 $\Rightarrow \int_4^\infty \frac{dx}{\sqrt{x-1}}$ diverges by the Limit Comparison Test.
49. $\int_2^\infty \frac{dv}{\sqrt{v-1}}$; $\lim_{v \rightarrow \infty} \frac{\left(\frac{1}{\sqrt{v-1}}\right)}{\left(\frac{1}{\sqrt{v}}\right)} = \lim_{v \rightarrow \infty} \frac{\sqrt{v}}{\sqrt{v-1}} = \lim_{v \rightarrow \infty} \frac{1}{\sqrt{1-\frac{1}{v}}} = \frac{1}{\sqrt{1-0}} = 1$ and $\int_2^\infty \frac{dv}{\sqrt{v}} = \lim_{b \rightarrow \infty} [2\sqrt{v}]_2^b = \infty$, which diverges
 $\Rightarrow \int_2^\infty \frac{dv}{\sqrt{v-1}}$ diverges by the Limit Comparison Test.
50. $\int_0^\infty \frac{d\theta}{1+e^\theta}$; $0 \leq \frac{1}{1+e^\theta} \leq \frac{1}{e^\theta}$ for $0 \leq \theta < \infty$ and $\int_0^\infty \frac{d\theta}{e^\theta} = \lim_{b \rightarrow \infty} [-e^{-\theta}]_0^b = \lim_{b \rightarrow \infty} (-e^{-b} + 1) = 1 \Rightarrow \int_0^\infty \frac{d\theta}{e^\theta}$ converges
 $\Rightarrow \int_0^\infty \frac{d\theta}{1+e^\theta}$ by the Direct Comparison Test.
51. $\int_0^\infty \frac{dx}{\sqrt{x^6+1}} = \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{\sqrt{x^6+1}} < \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{x^3}$ and $\int_1^\infty \frac{dx}{x^3} = \lim_{b \rightarrow \infty} \left[-\frac{1}{2x^2}\right]_1^b = \lim_{b \rightarrow \infty} \left(-\frac{1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2}$
 $\Rightarrow \int_0^\infty \frac{dx}{\sqrt{x^6+1}}$ converges by the Direct Comparison Test.
52. $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$; $\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$; $\int_2^\infty \frac{1}{x} dx = \lim_{b \rightarrow \infty} [\ln b]_2^b = \infty$, which diverges
 $\Rightarrow \int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges by the Limit Comparison Test.
53. $\int_1^\infty \frac{\sqrt{x+1}}{x^2} dx$; $\lim_{x \rightarrow \infty} \frac{\left(\frac{\sqrt{x}}{x^2}\right)}{\left(\frac{\sqrt{x+1}}{x^2}\right)} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{x+1}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{x}}} = 1$; $\int_1^\infty \frac{\sqrt{x}}{x^2} dx = \int_1^\infty \frac{dx}{x^{3/2}} = \lim_{b \rightarrow \infty} [-2x^{-1/2}]_1^b = \lim_{b \rightarrow \infty} \left(\frac{-2}{\sqrt{b}} + 2\right) = 2 \Rightarrow \int_1^\infty \frac{\sqrt{x+1}}{x^2} dx$ converges by the Limit Comparison Test.
54. $\int_2^\infty \frac{x dx}{\sqrt{x^4-1}}$; $\lim_{x \rightarrow \infty} \frac{\left(\frac{x}{\sqrt{x^4-1}}\right)}{\left(\frac{x}{\sqrt{x^4}}\right)} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^4}}{\sqrt{x^4-1}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1-\frac{1}{x^4}}} = 1$; $\int_2^\infty \frac{x dx}{\sqrt{x^4}} = \int_2^\infty \frac{dx}{x} = \lim_{x \rightarrow \infty} [\ln x]_2^b = \infty$, which diverges
 $\Rightarrow \int_2^\infty \frac{x dx}{\sqrt{x^4-1}}$ diverges by the Limit Comparison Test.
55. $\int_\pi^\infty \frac{2+\cos x}{x} dx$; $0 < \frac{1}{x} \leq \frac{2+\cos x}{x}$ for $x \geq \pi$ and $\int_\pi^\infty \frac{dx}{x} = \lim_{b \rightarrow \infty} [\ln x]_\pi^b = \infty$, which diverges $\Rightarrow \int_\pi^\infty \frac{2+\cos x}{x} dx$ diverges by the Direct Comparison Test.

56. $\int_{\pi}^{\infty} \frac{1+\sin x}{x^2} dx$; $0 \leq \frac{1+\sin x}{x^2} \leq \frac{2}{x^2}$ for $x \geq \pi$ and $\int_{\pi}^{\infty} \frac{2}{x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{2}{x} \right]_{\pi}^b = \lim_{b \rightarrow \infty} \left(-\frac{2}{b} + \frac{2}{\pi} \right) = \frac{2}{\pi} \Rightarrow \int_{\pi}^{\infty} \frac{2}{x^2} dx$ converges $\Rightarrow \int_{\pi}^{\infty} \frac{1+\sin x}{x^2} dx$ converges by the Direct Comparison Test.
57. $\int_4^{\infty} \frac{2}{t^{3/2}-1} dt$; $\lim_{t \rightarrow \infty} \frac{t^{3/2}}{t^{3/2}-1} = 1$ and $\int_4^{\infty} \frac{2}{t^{3/2}} dt = \lim_{b \rightarrow \infty} \left[-4t^{-1/2} \right]_4^b = \lim_{b \rightarrow \infty} \left(\frac{-4}{\sqrt{b}} + 2 \right) = 2 \Rightarrow \int_4^{\infty} \frac{2}{t^{3/2}} dt$ converges $\Rightarrow \int_4^{\infty} \frac{2}{t^{3/2}-1} dt$ converges by the Limit Comparison Test.
58. $\int_2^{\infty} \frac{dx}{\ln x}$; $0 < \frac{1}{x} < \frac{1}{\ln x}$ for $x > 2$ and $\int_2^{\infty} \frac{dx}{x}$ diverges $\Rightarrow \int_2^{\infty} \frac{dx}{\ln x}$ diverges by the Direct Comparison Test.
59. $\int_1^{\infty} \frac{e^x}{x} dx$; $0 < \frac{1}{x} < \frac{e^x}{x}$ for $x > 1$ and $\int_1^{\infty} \frac{dx}{x}$ diverges $\Rightarrow \int_1^{\infty} \frac{e^x}{x} dx$ diverges by the Direct Comparison Test.
60. $\int_e^{\infty} \ln(\ln x) dx$; $[x = e^y] \rightarrow \int_e^{\infty} (\ln y)e^y dy$; $0 < \ln y < (\ln y)e^y$ for $y \geq e$ and $\int_e^{\infty} \ln y dy = \lim_{b \rightarrow \infty} [y \ln y - y]_e^b = \infty$, which diverges $\Rightarrow \int_e^{\infty} (\ln y)e^y dy$ diverges $\Rightarrow \int_e^{\infty} \ln(\ln x) dx$ diverges by the Direct Comparison Test.
61. $\int_1^{\infty} \frac{dx}{\sqrt{e^x-x}}$; $\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{\sqrt{e^x-x}}\right)}{\left(\frac{1}{\sqrt{e^x}}\right)} = \lim_{x \rightarrow \infty} \frac{\sqrt{e^x}}{\sqrt{e^x-x}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1-\frac{x}{e^x}}} = \frac{1}{\sqrt{1-0}} = 1$; $\int_1^{\infty} \frac{dx}{\sqrt{e^x}} = \int_1^{\infty} e^{-x/2} dx = \lim_{b \rightarrow \infty} \left[-2e^{-x/2} \right]_1^b$
 $= \lim_{b \rightarrow \infty} \left(-2e^{-b/2} + 2e^{-1/2} \right) = \frac{2}{\sqrt{e}} \Rightarrow \int_1^{\infty} e^{-x/2} dx$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{\sqrt{e^x-x}}$ converges by the Limit Comparison Test.
62. $\int_1^{\infty} \frac{dx}{e^x-2^x}$; $\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{e^x-2^x}\right)}{\left(\frac{1}{e^x}\right)} = \lim_{x \rightarrow \infty} \frac{e^x}{e^x-2^x} = \lim_{x \rightarrow \infty} \frac{1}{1-\left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = 1$ and $\int_1^{\infty} \frac{dx}{e^x} = \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_1^b = \lim_{b \rightarrow \infty} \left(-e^{-b} + e^{-1} \right) = \frac{1}{e}$
 $\Rightarrow \int_1^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{e^x-2^x}$ converges by the Limit Comparison Test.
63. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}} = 2 \int_0^{\infty} \frac{dx}{\sqrt{x^4+1}}$; $\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}} = \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{\infty} \frac{dx}{\sqrt{x^4+1}} < \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{\infty} \frac{dx}{x^2}$ and $\int_1^{\infty} \frac{dx}{x^2} = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b$
 $= \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$ converges by the Direct Comparison Test.
64. $\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}} = 2 \int_0^{\infty} \frac{dx}{e^x+e^{-x}}$; $0 < \frac{1}{e^x+e^{-x}} < \frac{1}{e^x}$ for $x > 0$; $\int_0^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow 2 \int_0^{\infty} \frac{dx}{e^x+e^{-x}}$ converges by the Direct Comparison Test.
65. (a) $\int_1^2 \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \rightarrow \int_0^{\ln 2} \frac{dt}{t^p} = \lim_{b \rightarrow 0^+} \left[\frac{1}{-p+1} t^{1-p} \right]_b^{\ln 2} = \lim_{b \rightarrow 0^+} \left[\frac{b^{1-p}}{p-1} + \frac{1}{1-p} (\ln 2)^{1-p} \right] \Rightarrow$ the integral converges for $p < 1$ and diverges for $p \geq 1$

- (b) $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \rightarrow \int_{\ln 2}^{\infty} \frac{dt}{t^p}$ and this integral is essentially the same as in Exercise 65(a): it converges for $p > 1$ and diverges for $p \leq 1$

66. $\int_0^{\infty} \frac{2x}{x^2+1} dx = \lim_{b \rightarrow \infty} \left[\ln(x^2+1) \right]_0^b = \lim_{b \rightarrow \infty} [\ln(b^2+1) - 0] = \lim_{b \rightarrow \infty} \ln(b^2+1) = \infty \Rightarrow$ the integral $\int_{-\infty}^{\infty} \frac{2x}{x^2+1} dx$ diverges. But $\lim_{b \rightarrow \infty} \int_{-\infty}^b \frac{2x}{x^2+1} dx = \lim_{b \rightarrow \infty} \left[\ln(x^2+1) \right]_{-b}^b = \lim_{b \rightarrow \infty} [\ln(b^2+1) - \ln(b^2+1)] = \lim_{b \rightarrow \infty} \ln\left(\frac{b^2+1}{b^2+1}\right) = \lim_{b \rightarrow \infty} (\ln 1) = 0$

67. $A = \int_0^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} [-e^{-x}]_0^b$
 $= \lim_{b \rightarrow \infty} [(-e^{-b}) - (-e^{-0})] = 0 + 1 = 1$



68. $\bar{x} = \frac{1}{A} \int_0^{\infty} x e^{-x} dx = \lim_{b \rightarrow \infty} [-x e^{-x} - e^{-x}]_0^b = \lim_{b \rightarrow \infty} [(-b e^{-b} - e^{-b}) - (-0 \cdot e^{-0} - e^{-0})] = 0 + 1 = 1$;
 $\bar{y} = \frac{1}{2A} \int_0^{\infty} (e^{-x})^2 dx = \frac{1}{2} \int_0^{\infty} e^{-2x} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-2x} \right]_0^b = \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-2b} + \frac{1}{2} \right] = 0 + \frac{1}{2} = \frac{1}{2}$

69. $V = \int_0^{\infty} 2\pi x e^{-x} dx = 2\pi \int_0^{\infty} x e^{-x} dx = 2\pi \lim_{b \rightarrow \infty} [-x e^{-x} - e^{-x}]_0^b = 2\pi \lim_{b \rightarrow \infty} [(-b e^{-b} - e^{-b}) - (-0 \cdot e^{-0} - e^{-0})] = 2\pi$

70. $V = \int_0^{\infty} \pi (e^{-x})^2 dx = \pi \int_0^{\infty} e^{-2x} dx = \pi \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-2x} \right]_0^b = \pi \lim_{b \rightarrow \infty} \left(-\frac{1}{2} e^{-2b} + \frac{1}{2} \right) = \frac{\pi}{2}$

71. $A = \int_0^{\pi/2} (\sec x - \tan x) dx = \lim_{b \rightarrow (\pi/2)^-} [\ln |\sec x + \tan x| - \ln |\sec x|]_0^b = \lim_{b \rightarrow (\pi/2)^-} [\ln |1 + \frac{\tan b}{\sec b}| - \ln |1 + 0|]$
 $= \lim_{b \rightarrow (\pi/2)^-} \ln |1 + \sin b| = \ln 2$

72. (a) $V = \int_0^{\pi/2} \pi \sec^2 x dx - \int_0^{\pi/2} \pi \tan^2 x dx = \pi \int_0^{\pi/2} (\sec^2 x - \tan^2 x) dx = \int_0^{\pi/2} \pi [\sec^2 x - (\sec^2 x - 1)] dx$
 $= \pi \int_0^{\pi/2} dx = \frac{\pi^2}{2}$

(b) $S_{\text{outer}} = \int_0^{\pi/2} 2\pi \sec x \sqrt{1 + \sec^2 x \tan^2 x} dx \geq \int_0^{\pi/2} 2\pi \sec x (\sec x \tan x) dx = \pi \lim_{b \rightarrow (\pi/2)^-} [\tan^2 x]_0^b =$
 $\pi \lim_{b \rightarrow (\pi/2)^-} [(\tan^2 b) - 0] = \pi \lim_{b \rightarrow (\pi/2)^-} [\tan^2 b] = \infty \Rightarrow S_{\text{outer}} \text{ diverges; } S_{\text{inner}} = \int_0^{\pi/2} 2\pi \tan x \sqrt{1 + \sec^4 x} dx \geq$
 $\int_0^{\pi/2} 2\pi \tan x \sec^2 x dx = \pi \lim_{b \rightarrow (\pi/2)^-} [\tan^2 x]_0^b = \pi \lim_{b \rightarrow (\pi/2)^-} [(\tan^2 b) - 0] = \pi \lim_{b \rightarrow (\pi/2)^-} [\tan^2 b] = \infty \Rightarrow S_{\text{inner}}$
 diverges

73. (a) $\int_0^1 \frac{1}{\sqrt{t}(1+t)} dt$

With $u = \sqrt{t}$ and $du = \frac{1}{2\sqrt{t}} dt$ the limits of integration are unchanged.

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{t}(1+t)} dt &= \int_0^1 \frac{2}{1+u^2} du \\ &= 2 \lim_{a \rightarrow 0^+} \left(\tan^{-1} 1 - \tan^{-1} a \right) \\ &= 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} \end{aligned}$$

(b) $\int_0^\infty \frac{1}{\sqrt{t}(1+t)} dt$

With $u = \sqrt{t}$ and $du = \frac{1}{2\sqrt{t}} dt$ the limits of integration are unchanged. We split the integral into two integrals, the first of which was evaluated in (a).

$$\begin{aligned} \int_0^\infty \frac{1}{\sqrt{t}(1+t)} dt &= \int_0^1 \frac{2}{1+u^2} du + \int_1^\infty \frac{2}{1+u^2} du \\ &= \frac{\pi}{2} + 2 \lim_{b \rightarrow \infty} \left(\tan^{-1} b - \tan^{-1} 1 \right) \\ &= \frac{\pi}{2} + 2 \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \pi \end{aligned}$$

74. Let c be any number in $(3, \infty)$.

$$\int_3^\infty \frac{1}{x\sqrt{x^2-9}} dx = \int_3^c \frac{1}{x\sqrt{x^2-9}} dx + \int_c^\infty \frac{1}{x\sqrt{x^2-9}} dx \quad \text{provided both integrals on the right converge.}$$

Formula 20 in Table 8.1 gives $\int \frac{1}{x\sqrt{x^2-9}} dx = \frac{1}{3} \sec^{-1} \left| \frac{x}{3} \right|$. (The definition of the inverse secant is given in Section 3.9.) Both integrals do converge:

$$\int_3^c \frac{1}{x\sqrt{x^2-9}} dx = \lim_{a \rightarrow 3^+} \left(\frac{1}{3} \sec^{-1} \left| \frac{c}{3} \right| - \frac{1}{3} \sec^{-1} \left| \frac{a}{3} \right| \right) = \frac{1}{3} \sec^{-1} \frac{c}{3}$$

$$\int_c^\infty \frac{1}{x\sqrt{x^2-9}} dx = \lim_{b \rightarrow \infty} \left(\frac{1}{3} \sec^{-1} \left| \frac{b}{3} \right| - \frac{1}{3} \sec^{-1} \left| \frac{c}{3} \right| \right) = \frac{\pi}{6} - \frac{1}{3} \sec^{-1} \frac{c}{3}$$

Thus $\int_3^\infty \frac{1}{x\sqrt{x^2-9}} dx = \frac{\pi}{6}$.

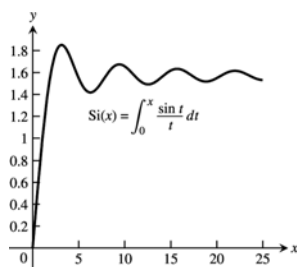
75. (a) $\int_3^\infty e^{-3x} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{3} e^{-3x} \right]_3^b = \lim_{b \rightarrow \infty} \left[\left(-\frac{1}{3} e^{-3b} \right) - \left(-\frac{1}{3} e^{-3 \cdot 3} \right) \right] = 0 + \frac{1}{3} \cdot e^{-9} = \frac{1}{3} e^{-9} \approx 0.0000411$
 < 0.000042 . Since $e^{-x^2} \leq e^{-3x}$ for $x > 3$, then $\int_3^\infty e^{-x^2} dx < 0.000042$ and therefore $\int_0^\infty e^{-x^2} dx$ can be replaced by $\int_0^3 e^{-x^2} dx$ without introducing an error greater than 0.000042.

(b) $\int_0^3 e^{-x^2} dx \approx 0.88621$

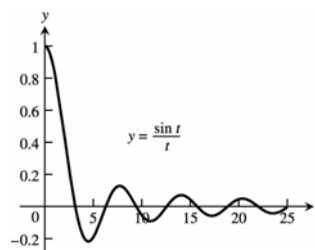
76. (a) $V = \int_1^\infty \pi \left(\frac{1}{x} \right)^2 dx = \pi \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = \pi \lim_{b \rightarrow \infty} \left[\left(-\frac{1}{b} \right) - \left(-\frac{1}{1} \right) \right] = \pi(0 + 1) = \pi$

- (b) When you take the limit to ∞ , you are no longer modeling the real world which is finite. The comparison step in the modeling process discussed in Section 4.2 relating the mathematical world to the real world fails to hold.

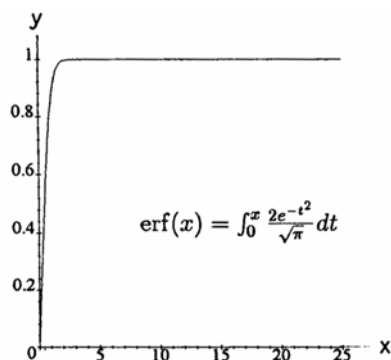
77. (a)



(b) $> \text{int}((\sin(t))/t, t = 0.. \text{infinity});$ (answer is $\frac{\pi}{2}$)

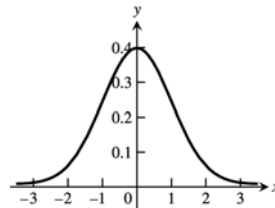


78. (a)



(b) $> f := 2 * \exp(-t^2) / \text{sqrt}(\text{Pi});$
 $> \text{int}(f, t = 0.. \text{infinity});$ (answer is 1)

79. (a) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$
 f is increasing on $(-\infty, 0]$,
 f is decreasing on $[0, \infty)$,
 f has a local maximum at $(0, f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)$



- (b) Maple commands:
 $> f := \exp(-x^2/2) / \text{sqrt}(2 * \text{pi});$
 $> \text{int}(f, x = -1..1);$ ≈ 0.683
 $> \text{int}(f, x = -2..2);$ ≈ 0.954
 $> \text{int}(f, x = -3..3);$ ≈ 0.997

(c) Part (b) suggests that as n increases, the integral approaches 1. We can take $\int_{-n}^n f(x) dx$ as close to 1 as we want by choosing $n > 1$ large enough. Also, we can make $\int_n^\infty f(x) dx$ and $\int_{-\infty}^{-n} f(x) dx$ as small as we want by choosing n large enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$.) Thus, $\int_n^\infty f(x) dx < \int_n^\infty e^{-x/2} dx$.

$$\int_n^\infty e^{-x/2} dx = \lim_{c \rightarrow \infty} \int_n^c e^{-x/2} dx = \lim_{c \rightarrow \infty} \left[-2e^{-x/2} \right]_n^c = \lim_{c \rightarrow \infty} \left[-2e^{-c/2} + 2e^{-n/2} \right] = 2e^{-n/2}$$

As $n \rightarrow \infty$, $2e^{-n/2} \rightarrow 0$, for large enough n , $\int_n^\infty f(x) dx$ is as small as we want.

Likewise for large enough n , $\int_{-\infty}^{-n} f(x) dx$ is as small as we want.

80. (a) The statement is true since $\int_{-\infty}^b f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^b f(x) dx$, $\int_b^\infty f(x) dx = \int_a^\infty f(x) dx - \int_a^b f(x) dx$ and $\int_a^b f(x) dx$ exists since $f(x)$ is integrable on every interval $[a, b]$.

$$\begin{aligned} \text{(b)} \quad \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx &= \int_{-\infty}^a f(x) dx + \int_a^b f(x) dx - \int_a^b f(x) dx + \int_a^\infty f(x) dx \\ &= \int_{-\infty}^b f(x) dx + \int_b^a f(x) dx + \int_a^\infty f(x) dx = \int_{-\infty}^b f(x) dx + \int_b^\infty f(x) dx \end{aligned}$$

81. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := 0..exp(1);
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=domain, y=-50..10, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
      thickness=[3,4,1,2,0], legend=["p= -2", "p= -1", "p= 0", "p= 1", "p= 2"], title="#81 (Section 8.8)" );
q1 := Int( f(x,p), x=domain );
q2 := value( q1 );
q3 := simplify( q2 ) assuming p>-1;
q4 := simplify( q2 ) assuming p<-1;
q5 := value( eval( q1, p=-1 ) );
i1 := q1 = piecewise( p<-1, q4, p=-1, q5, p>-1, q3 );
```

82. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := exp(1)..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=exp(1)..infinity, y=0..100, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
      thickness=[3,4,1,2,0], legend=["p= -2", "p= -1", "p= 0", "p= 1", "p= 2"], title="#82 (Section 8.8)" );
```

```

q6 := Int( f(x,p), x=domain );
q7 := value( q6 );
q8 := simplify( q7 ) assuming p>-1;
q9 := simplify( q7 ) assuming p<-1;
q10 := value( eval( q6, p=-1 ) );
i2 := q6 = piecewise( p<-1, q9, p=-1, q10, p>-1, q8 );

```

83. Example CAS commands:

Maple:

```

f := (x,p) -> x^p*ln(x);
domain := 0..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=0..10, y=-50..50, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
      thickness=[3,4,1,2,0], legend=["p= -2", "p = -1", "p = 0", "p = 1", "p = 2"], title="#83 (Section 8.8)" );
q11 := Int( f(x,p), x=domain );
q11 = lhs(i1+i2);
`` = rhs(i1+i2);
`` = piecewise( p<-1, q4+q9, p=-1, q5+q10, p>-1, q3+q8 );
`` = piecewise( p<-1, -infinity, p=-1, undefined, p>-1, infinity );

```

84. Example CAS commands:

Maple:

```

f := (x,p) -> x^p*ln(abs(x));
domain := -infinity..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=4..4, y=-20..10, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
      legend=["p= -2", "p = -1", "p = 0", "p = 1", "p = 2"], title="#84 (Section 8.8)" );
q12 := Int( f(x,p), x=domain );
q12p := Int( f(x,p), x=0..infinity );
q12n := Int( f(x,p), x=-infinity..0 );
q12 = q12p + q12n;
`` = simplify( q12p+q12n );

```

81-84. Example CAS commands:

Mathematica: (functions and domains may vary)

```

Clear[x, f, p]
f[x_]:= x^pLog[Abs[x]]
int = Integrate[f[x], {x, e, 100}]
int /. p -> 2.5

```

In order to plot the function, a value for p must be selected.

$$p = 3;$$

$$\text{Plot}[f[x], \{x, 2.72, 10\}]$$

85. Maple gives $\int_0^{2/\pi} \sin\left(\frac{1}{x}\right) dx = \frac{1}{\pi} \left(2 - \pi \cdot \text{Ci}\left(\frac{\pi}{2}\right) \right) \approx 0.16462$, where Ci is the cosine integral function defined by $\text{Ci}(t) = -\int_t^\infty \frac{\cos x}{x} dx$.

86. Maple gives $\int_0^{2/\pi} x \sin\left(\frac{1}{x}\right) dx = \frac{2}{\pi^2} + \frac{1}{2} \text{Si}\left(\frac{\pi}{2}\right) - \frac{\pi}{4} \approx 0.10276$, where Si is the sine integral function defined by $\text{Si}(t) = \int_0^t \frac{\sin x}{x} dx$.

8.9 PROBABILITY

1. $\int_4^8 \frac{1}{18} x dx = \frac{4}{3} \neq 1$; not a probability density.

2. $\int_0^2 \frac{1}{2} (2-x) dx = 1$; a probability density.

3.
$$\int_0^{\ln(1+\ln 2)/\ln 2} 2^x dx = \frac{2^x}{\ln 2} \Big|_0^{\ln(1+\ln 2)/\ln 2} = \left(\frac{1+\ln 2}{\ln 2} - \frac{1}{\ln 2} \right) = 1$$

This is a probability density.

4. $x-1$ is not nonnegative on $[0, 1+\sqrt{3}]$, so not a probability density.

5. $\int_1^\infty \frac{1}{x^2} dx = 1$; a probability density.

6.
$$\int_0^\infty \frac{8}{\pi(4+x^2)} dx = \lim_{b \rightarrow \infty} \left(\frac{4}{\pi} \tan^{-1}\left(\frac{x}{2}\right) \Big|_0^b \right) = \frac{4}{\pi} \cdot \frac{\pi}{2} = 2$$

This is not a probability density.

7. $\int_0^{\pi/4} 2 \cos 2x dx = 1$; a probability density.

8. $\int_0^e \frac{1}{x} dx$ diverges; not a probability density.
9. (a) The probability that a tire lasts between 25,000 and 32,000 kilometers
 (b) The probability that a tire lasts more than 30,000 kilometers
 (c) The probability that a tire lasts less than 20,000 kilometers
 (d) The probability that a tire lasts less than 15,000 kilometers
10. (a) $\int_{\pi}^{3\pi/2} \frac{1}{2\pi} dx + \int_{\pi/2}^{\pi} \frac{1}{2\pi} dx = 0.5$
 (b) $\int_2^{2\pi} \frac{1}{2\pi} dx = 1 - \frac{1}{\pi} \approx 0.682$
11. $\int_1^3 xe^{-x} dx = -(x+1)e^{-x} \Big|_1^3 = -4e^{-3} + 2e^{-1} \approx 0.537$
12. $\int_2^{15} \frac{\ln x}{x^2} dx = -\frac{\ln x + 1}{x} \Big|_2^{15} = -\frac{\ln 15}{15} - \frac{1}{15} + \frac{\ln 2}{2} + \frac{1}{2} \approx 0.599$
13. $\int_0^1 \frac{3}{2} x(2-x) dx = -\frac{1}{2} x^2(x-3) \Big|_{1/2}^1 = \frac{11}{16} \approx 0.688$
14. Using software to evaluate the Sine Integral we find $\int_{\frac{200}{1059}}^{\infty} \frac{\sin^2 \pi x}{\pi x^2} dx \approx 1.00004780741$ so the given function is very nearly a probability density over the given interval. Again using software we find that
- $$\int_{\frac{200}{1059}}^{\pi/6} \frac{\sin^2 \pi x}{\pi x^2} dx \approx 0.6732.$$
15. $\int_4^9 \frac{2}{x^3} dx = \frac{65}{1296} \approx 0.0502$
16. $\int_{\pi/6}^{\pi/4} \sin x dx = -\cos x \Big|_{\pi/6}^{\pi/4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \approx 0.159$
17. $\int_3^c \frac{1}{6} x dx = \frac{1}{12} c^2 - \frac{3}{4}$. Solving $\frac{1}{12} c^2 - \frac{3}{4} = 1$, we find $c = \sqrt{21}$.
18. $\int_c^{c+1} \frac{1}{x} dx = \ln(c+1) - \ln c = \ln\left(\frac{c+1}{c}\right)$. Solving $\ln\left(\frac{c+1}{c}\right) = 1$, we find $\frac{c+1}{c} = e$ and thus $c = \frac{1}{e-1}$.
19. $\int_0^c 4e^{-2x} dx = -2e^{-2c} + 2$. Solving $-2e^{-2c} + 2 = 1$, we find $c = \frac{1}{2} \ln 2$.

$$20. \int_0^5 cx\sqrt{25-x^2} dx = -\frac{1}{3}c(25-x^2)^{3/2} \Big|_0^5 = \frac{125}{3}c, \text{ so } c = \frac{3}{125}.$$

21. We will assume that the given function is to be a probability density over the whole real line.

$$\int_{-\infty}^{\infty} \frac{c}{1+x^2} dx = c\pi \text{ so we take } c = \frac{1}{\pi}. \text{ Then } \int_1^2 \frac{1}{\pi(1+x^2)} dx = \frac{\tan^{-1}x}{\pi} \Big|_1^2 = \frac{\tan^{-1}2}{\pi} - \frac{1}{4} \approx 0.10242.$$

$$22. \int_0^1 c\sqrt{x}(1-x) dx = c \left(\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} \right) \Big|_0^1 = \frac{4}{15}c, \text{ so } c = \frac{15}{4}. \text{ Then } \int_{1/4}^{1/2} \frac{15}{4}\sqrt{x}(1-x) dx = \frac{7}{16}\sqrt{2} - \frac{17}{64} \approx 0.353$$

$$23. \int_0^{\infty} e^{-cx} dx = \frac{1}{c} \lim_{b \rightarrow \infty} (-e^{-bcx} + 1) = \frac{1}{c}. \text{ Thus multiplying } e^{-cx} \text{ by } c \text{ produces a probability density on } [0, \infty).$$

$$\begin{aligned} 24. \text{Var}(X) &= \int_{-\infty}^{\infty} (X - \mu)^2 f(X) dX \\ &= \int_{-\infty}^{\infty} X^2 f(X) dX + \int_{-\infty}^{\infty} (-2X\mu) f(X) dX + \int_{-\infty}^{\infty} \mu^2 f(X) dX \\ \int_{-\infty}^{\infty} (-2X\mu) f(X) dX &= -2\mu \int_{-\infty}^{\infty} X f(X) dX = -2\mu^2 \\ \int_{-\infty}^{\infty} \mu^2 f(X) dX &= \mu^2 \int_{-\infty}^{\infty} f(X) dX = \mu^2(1) = \mu^2 \end{aligned}$$

$$\text{Thus } \text{Var}(X) = \int_{-\infty}^{\infty} X^2 f(X) dX - \mu^2.$$

$$25. \text{mean} = \int_0^4 x \left(\frac{1}{8}x \right) dx = \frac{8}{3}$$

To find the median we need to solve $\int_0^c \frac{1}{8}x dx = \frac{1}{16}c^2 = \frac{1}{2}$ for c . Thus the median is $\sqrt{8}$.

$$26. \text{mean} = \int_0^3 x \left(\frac{1}{9}x^2 \right) dx = \frac{9}{4}$$

To find the median we need to solve $\int_0^c \frac{1}{9}x^2 dx = \frac{1}{27}c^3 = \frac{1}{2}$ for c . Thus the median is $\frac{3}{2}2^{2/3} \approx 2.381$.

$$27. \text{mean} = \int_1^{\infty} x \left(\frac{2}{x^3} \right) dx = \lim_{b \rightarrow \infty} \left(-\frac{2}{x} \right) \Big|_1^b = 2$$

To find the median we need to solve $\int_1^c \frac{2}{x^3} dx = -\frac{1}{c^2} + 1 = \frac{1}{2}$ for c . Thus the median is $\sqrt{2}$.

$$28. \text{mean} = \int_1^e x \left(\frac{1}{x} \right) dx = e - 1 \approx 1.718$$

To find the median we need to solve $\int_1^c \frac{1}{x} dx = \ln c = \frac{1}{2}$ for c . Thus the median is $\sqrt{e} \approx 1.649$.

29. The exponential density with mean 1 is e^{-X} . The probability that the food is digested in less than 30 minutes is $\int_0^{1/2} e^{-X} dX = -e^{-1/2} + 1 \approx 0.3935$.
30. The exponential density with mean 4 is $(1/4)e^{-X/4}$. The probability that a flower is pollinated within 5 minutes is $\int_0^5 (1/4)e^{-X/4} dX = -e^{-5/4} + 1 \approx 0.7135$. Out of 1000 flowers we would expect 713 or 714 to be pollinated within 5 minutes.
31. The exponential density with mean 1200 is $(1/1200)e^{-X/1200}$.
- (a) The probability that a bulb will last less than 1000 hours is $\int_0^{1000} (1/1200)e^{-X/1200} dX = -e^{-5/6} + 1 \approx 0.5654$.
- (b) By Example 9 the median lifetime is $1200 \ln 2 \approx 831.8$ so the expected time until half the bulbs in a batch fail is 832 h.
32. To find the density, solve $\int_0^3 (1/c)e^{-X/c} dX = -e^{-3/c} + 1 = \frac{1}{3}$. Then $c = \frac{3}{\ln(3/2)} \approx 7.3989$, so the mean lifetime of the components is 7.4 years. The probability of failure within 1 year is $\int_0^1 \frac{\ln(3/2)}{3} e^{-\frac{X \ln(3/2)}{3}} dX = -\left(\frac{2}{3}\right)^{1/3} + 1 \approx 0.1264$.
33. To find the density, solve $\int_0^2 (1/c)e^{-X/c} dX = -e^{-2/c} + 1 = \frac{2}{5}$; $c = \frac{2}{\ln(5/3)}$. The probability that a hydra dies within 6 months, or half a year, is $\int_0^{1/2} \frac{\ln(5/3)}{2} e^{-\frac{X \ln(5/3)}{2}} dX = -\left(\frac{3}{5}\right)^{1/4} + 1 \approx 0.1199$, so we would expect $(0.12)(500) = 60$ hydra to die within the first six months.
34. To find the density, solve $\int_0^{50} (1/c)e^{-X/c} dX = -e^{-50/c} + 1 = \frac{3}{10}$; $c = \frac{50}{\ln(10/7)}$. The probability that a high-risk driver is involved in an accident in the first 80 days is $\int_0^{80} \frac{\ln(10/7)}{50} e^{-\frac{X \ln(10/7)}{50}} dX = -\left(\frac{7}{10}\right)^{8/5} + 1 \approx 0.4349$, so we would expect 43 or 44 out of 100 high-risk drivers to be involved in an accident in the first 80 days.
35. Using seconds as the time unit, the density is $(1/30)e^{-X/30}$.
- (a) $\int_0^{15} (1/30)e^{-X/30} dX = -e^{-1/2} + 1 \approx 0.393$
- (b) $\int_{60}^{\infty} (1/30)e^{-X/30} dX = e^{-2} \approx 0.135$
- (c) In a continuous distribution the probability of a particular number is 0.
- (d) The probability that a single customer waits less than 3 minutes is $-e^{-6} + 1 \approx 0.997521$. The probability that at least one customer out of 200 waits longer than 3 minutes is $1 - (0.997521)^{200} \approx 0.391 < 0.5$, so the most likely outcome is that all 200 are served within 3 minutes.

36. For parts (a) and (b) the density is $(1/16)e^{-X/16}$. For parts (c) and (d) the density is $(1/32)e^{-X/32}$.

$$(a) \int_{10}^{30} (1/16)e^{-X/16} dX = -e^{-15/8} + e^{-5/8} \approx 0.382$$

$$(b) \int_{25}^{\infty} (1/16)e^{-X/16} dX = e^{-25/16} \approx 0.210$$

$$(c) \int_{35}^{50} (1/32)e^{-X/32} dX = -e^{-25/16} + e^{-35/32} \approx 0.125$$

$$(d) \int_0^{20} (1/32)e^{-X/32} dX = -e^{-5/8} + 1 \approx 0.465$$

37. The expected payout per printer is $200 \int_0^1 (1/2)e^{-X/2} dX + 100 \int_1^2 (1/2)e^{-X/2} dX \approx \102.56 . Thus the expected refund total for 100 machines is \$10,256.

38. To find the density, solve $\int_0^2 \frac{1}{c} e^{-X/c} dX = -e^{-2/c} + 1 = \frac{1}{2}$, which gives $c = \frac{2}{\ln(2)}$. The probability of failure in the first year is $\int_0^1 \frac{\ln 2}{2} e^{-\frac{X \ln 2}{2}} dX = -\frac{\sqrt{2}}{2} + 1 \approx 0.293$. We expect $(150)(0.293) = 43.934$ or about 44 copiers to fail during the first year.

For Exercises 39–52, the density function is $f(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ with μ and σ as given in the solution.

39. $\mu = 4.19$, $\sigma = 0.72$

$$\int_{4.27}^{4.99} f(X) dX \approx 0.323; \text{ about 323 children}$$

$$\int_{3.83}^{4.32} f(X) dX \approx 0.262; \text{ about 262 children}$$

40. $\mu = 510$, $\sigma = 120$

$$\int_{444}^{\infty} f(X) dX = \frac{1}{2} + \int_{444}^{\mu} f(X) dX \approx 0.74593$$

41. $\mu = 55$, $\sigma = 4$

$$\int_0^{60} f(X) dX \approx 0.89435$$

42. $\mu = 35,000$, $\sigma = 6000$

$$(a) \int_{29,000}^{\infty} f(X) dX = \frac{1}{2} + \int_{29,000}^{\mu} f(X) dX \approx 0.84134; \quad (4000)(0.84134) \approx 3365 \text{ tires}$$

(b) We want to find L such that $\int_L^{\infty} f(X) dX = 0.9$. A CAS gives $L \approx 27,311$, so 90% of tires will have a lifetime of at least 27,311 km.

43. $\mu = 166$, $\sigma = 6$

$$(a) \int_{172}^{\infty} f(X) dX = \frac{1}{2} - \int_{\mu}^{172} f(X) dX \approx 0.159, \text{ or } 16\%.$$

$$(b) \int_{155}^{163} f(X) dX \approx 0.23832$$

44. $\mu = 81, \sigma = 7$

$$\int_{75}^{85} f(X) dX \approx 0.520; \text{ one would expect 52 of the babies to live to between 75 and 85.}$$

45. $\mu = 266, \sigma = 16; (36)(7) = 252, (40)(7) = 280$

$$\int_{252}^{280} f(X) dX \approx 0.6184; \text{ we would expect 618 of the women to have pregnancies lasting between 36 and 40 weeks.}$$

46. $\mu = 1400, \sigma = 100$

(a) $\int_{1325}^{1450} f(X) dX \approx 0.46484$

$$(b) \int_{1480}^{\infty} f(X) dX = \frac{1}{2} - \int_{\mu}^{1480} f(X) dX \approx 0.21186$$

$(500)(0.21186) \approx 106; \text{ we would expect about 106 males to have a brain weight exceeding 1480 g.}$

47. $\mu = 80, \sigma = 12$

$$\int_0^{70} f(X) dX \approx 0.20233$$

$(300)(0.20233) \approx 61; \text{ about 61 adults.}$

48. $\mu = 4.4, \sigma = 0.2$

$$\int_{4.3}^{4.45} f(X) dX \approx 0.29017$$

49. $\mu = 35, \sigma = 9$

$$\int_{40}^{\infty} f(X) dX = \frac{1}{2} - \int_{\mu}^{40} f(X) dX \approx 0.28926$$

About 289 shafts would need more than 45 grams of added weight.

50. $\mu = 300, \sigma = 50$

$$\int_{400}^{\infty} f(X) dX = \frac{1}{2} - \int_{\mu}^{400} f(X) dX \approx 0.02275$$

$$\int_0^{250} f(X) dX \approx 0.15866$$

$0.02275 + 0.15866 = 0.18141 \text{ or about } 18\%$

51. $\mu = (0.8)(2500) = 2000, \sigma = (0.4)(50) = 20$

(a) $\int_{1960}^{\infty} f(X) dX = \frac{1}{2} + \int_{1960}^{\mu} f(X) dX \approx 0.977$

(b) $\int_0^{1980} f(X) dX \approx 0.159$

(c) $\int_{1940}^{2020} f(X) dX \approx 0.840$

52. $\mu = (0.5)(400) = 200$, $\sigma = \frac{20}{2} = 10$

To improve the approximation to the binomial distribution we will modify the interval of integration. We give the true binomial values for comparison.

(a) $\int_{189.5}^{209.5} f(X) dX \approx 0.68208$; correct to 5 places

(b) $\int_0^{169.5} f(X) dX \approx 0.00114$; true value ≈ 0.00112

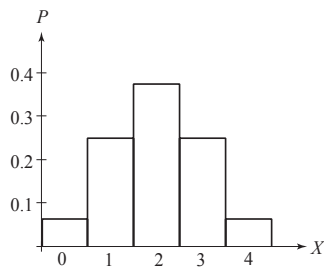
(c) $\int_{220.5}^{\infty} f(X) dX = \frac{1}{2} - \int_{\mu}^{220.5} f(X) dX \approx 0.02018$; true value ≈ 0.02012

(d) The value is very close to 0, in fact about 10^{-24} .

53. (a) and (b)

Outcome	X
HHHH	0
THHH	1
HTHH	1
HHTH	1
HHHT	1
TTHH	2
THTH	2
THHT	2
HTTH	2
HTHT	2
HHTT	2
TTTH	3
TTHT	3
THTT	3
HTTT	3
TTTT	4

(c)

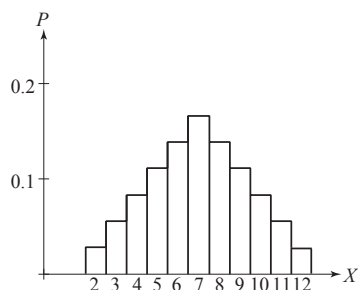


The probability of at least 2 heads is $\frac{1}{16}(1 + 4 + 6) = \frac{11}{16}$.

54. (a) The first die is listed first in each pair, the second die second.

$1 + 1 = 2$	$2 + 1 = 3$	$3 + 1 = 4$	$4 + 1 = 5$	$5 + 1 = 6$	$6 + 1 = 7$
$1 + 2 = 3$	$2 + 2 = 4$	$3 + 2 = 5$	$4 + 2 = 6$	$5 + 2 = 7$	$6 + 2 = 8$
$1 + 3 = 4$	$2 + 3 = 5$	$3 + 3 = 6$	$4 + 3 = 7$	$5 + 3 = 8$	$6 + 3 = 9$
$1 + 4 = 5$	$2 + 4 = 6$	$3 + 4 = 7$	$4 + 4 = 8$	$5 + 4 = 9$	$6 + 4 = 10$
$1 + 5 = 6$	$2 + 5 = 7$	$3 + 5 = 8$	$4 + 5 = 9$	$5 + 5 = 10$	$6 + 5 = 11$
$1 + 6 = 7$	$2 + 6 = 8$	$3 + 6 = 9$	$4 + 6 = 10$	$5 + 6 = 11$	$6 + 6 = 12$

(b)

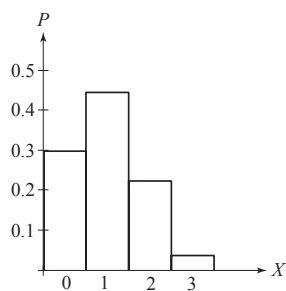


(c) $P(8) = \frac{5}{36}$

(d) $P(X \leq 5) = \frac{1+2+3+4}{36} = \frac{5}{18}$

$$P(X > 9) = \frac{3+2+1}{36} = \frac{1}{6}$$

55. (a) {LLL, LLD, LDL, DLL, LLU, LUL, ULL, LDD, DLD, DDL, LUU, ULU, UUL, DDU, DUD, UDD, DUU, UDU, UUD, LUD, LDU, ULD, UDL, DLU, DUL, DDD, UUU}

(b) We will assume that the three answers are equally likely, though other assumptions might be reasonable. The plots for the X = number of Ls, the number of Us and the number of Ds are identical.

(c) $P(\text{at least two L}) = \frac{1+3+3}{27} = \frac{7}{27} \approx 0.26$

(d)
$$P(\text{no more than one D}) = 1 - P(\text{at least two D})$$

$$= 1 - \frac{7}{27} = \frac{20}{27} \approx 0.74$$

56. The probability that both systems fail is 0.0148. Since the two systems have the same performance distribution, the failure probability for a single system is $\sqrt{0.0148} = 0.121655$. The success probability for a single system is $1 - \sqrt{0.0148}$ so the probability that both succeed is $(1 - \sqrt{0.0148})^2 = 0.771489$. The probability that one fails and one succeeds is $1 - 0.0148 - 0.771489 = 0.213711$. Since the events “main fails, backup succeeds” and “main succeeds, backup fails” have the same probability, the probability that only the main fails is $0.213711/2 = 0.106856$. Thus the probability that the main fails, either along with the backup or by itself, is $0.0148 + 0.106856 = 0.121656$.

CHAPTER 8 PRACTICE EXERCISES

1. $\int x \ln(x) dx = \ln(x) \int x dx - \int \left(\int x dx \right) \left(\frac{d}{dx} (\ln(x)) \right) dx = \ln(x) \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{x^2}{2} \ln(x) - \frac{x^2}{4} + C$
2. $\int x^2 \cos x dx = x^2 \int \cos x dx - \int \left(\int \cos x dx \right) \left(\frac{d}{dx} (x^2) \right) dx = x^2 (\sin x) - \int (\sin x)(2x) dx$
 $= x^2 \sin x - 2 \int x \sin x dx = x^2 \sin x - 2 \left[x \int \sin x dx - \int \left(\int \sin x dx \right) \left(\frac{d}{dx} (x) \right) dx \right]$
 $= x^2 \sin x - 2 \left[-x \cos x - \int -\cos x dx \right] = x^2 \sin x + 2x \cos x - 2 \sin x + C$
3. Put $\sin^{-1} 2x = t$, $2x = \sin t$, $2 dx = \cos t dt$, $dx = \frac{\cos t}{2} dt$;
 $\int \sin^{-1} 2x dx = \int t \frac{\cos t}{2} dt = \frac{1}{2} \left[t \int \cos t dt - \int \left(\int \cos t dt \right) \left(\frac{d}{dt} (t) \right) dt \right]$
 $= \frac{1}{2} [t \sin t + \cos t] + C = \frac{1}{2} [2x \sin^{-1} 2x + \cos(\sin^{-1} 2x)] + C$
4. Put $\tan^{-1} \left(\frac{x}{3} \right) = t \Rightarrow \frac{x}{3} = \tan t \Rightarrow \frac{dx}{3} = \sec^2 t dt \Rightarrow dx = 3 \sec^2 t dt$;
 $\int \tan^{-1} \left(\frac{x}{3} \right) dx = 3 \int t \sec^2 t dt = 3 \left[t \int \sec^2 t dt - \int \left(\int \sec^2 t dt \right) \left(\frac{d}{dt} (t) \right) dt \right] = 3 \left[t \tan t - \int \tan t dt \right]$
 $= 3 \left[t \tan t + \log |\cos t| \right] + C = 3 \left[\frac{x}{3} \tan^{-1} \left(\frac{x}{3} \right) + \log \left| \cos \left(\tan^{-1} \frac{x}{3} \right) \right| \right] + C$
5. $\int (1+x^2)e^x dx = (1+x^2) \int e^x dx - \int \left(\int e^x dx \right) \left(\frac{d}{dx} (1+x^2) \right) dx = (1+x^2)e^x - 2 \int x e^x dx$
 $= (1+x^2)e^x - 2xe^x + 2e^x + C$
6. $\int x^2 \cos(2-x) dx = x^2 \int \cos(2-x) dx - \int \left(\int \cos(2-x) dx \right) \left(\frac{d}{dx} (x^2) \right) dx = x^2 \left(\frac{\sin(2-x)}{-1} \right) - \int \left(\frac{\sin(2-x)}{-1} \right) (2x) dx$
 $= -x^2 \sin(2-x) + 2 \left[x \int \sin(2-x) dx - \int \left(\int \sin(2-x) dx \right) \left(\frac{d}{dx} (x) \right) dx \right]$
 $= -x^2 \sin(2-x) + 2x \cos(2-x) - 2 \int \cos(2-x) dx = -x^2 \sin(2-x) + 2x \cos(2-x) + 2 \sin(2-x) + C$
7. $\int e^x \sin 3x dx = \frac{e^x}{1^2+3^2} ((1) \sin 3x - 3 \cos 3x) + C = \frac{e^x}{10} (\sin 3x - 3 \cos 3x) + C$
8. $\int x \sin 2x \cos 2x dx = \int x \frac{\sin 2(2x)}{2} dx = \frac{1}{2} \int x \sin 4x dx = \frac{1}{2} \left[x \left(\frac{-\cos 4x}{4} \right) - \int \left(\frac{-\cos 4x}{4} \right) \left(\frac{d}{dx} (x) \right) dx \right]$
 $= \frac{-x \cos 4x}{8} + \frac{\sin 4x}{32} + C$

9. Consider $\frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)} \Rightarrow x = A(x-3) + B(x-2)$
 when $x = 2$, $2 = A(2-3) + B(2-2) \Rightarrow A = -2$
 when $x = 3$, $3 = A(3-3) + B(3-2) \Rightarrow B = 3$
 $\int \frac{x}{x^2-5x+6} dx = \int \frac{-2}{x-2} + \frac{3}{x-3} dx = -2 \ln(x-2) + 3 \ln(x-3) + C$
10. Consider $\frac{x}{x^2+4x-5} = \frac{x}{(x+5)(x-1)} = \frac{A}{(x+5)} + \frac{B}{(x-1)} \Rightarrow x = A(x-1) + B(x+5)$
 when $x = 1$, $1 = B(1+5) \Rightarrow B = 1/6$
 when $x = -5$, $-5 = A(-5-1) + B(-5+5) \Rightarrow A = 5/6$
 $\int \frac{x}{x^2+4x-5} dx = \int \frac{5/6}{x+5} + \frac{1/6}{x-1} dx = 5/6 \ln(x+5) + 1/6 \ln(x-1) + C$
11. Consider $\frac{1}{x(x-2)^2} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} \Rightarrow 1 = A(x-2)^2 + B(x-2) + Cx$
 when $x = 0$, $1 = 4A \Rightarrow A = 1/4$
 when $x = 2$, $1 = C(2) \Rightarrow C = 1/2$
 coefficient of x^2 , $0 = A + B \Rightarrow B = -1/4$
 $\int \frac{1}{x(x-2)^2} dx = \int \frac{1/4}{x} dx - \int \frac{1/4}{x-2} dx + \int \frac{1/2}{(x-2)^2} dx = \frac{1}{4} \ln x - \frac{1}{4} \ln |x-2| - \frac{1}{2(x-1)} + C$
12. Consider $\frac{x+3}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \Rightarrow x+3 = Ax(x-2) + B(x-2) + Cx^2$
 when $x = 0$, $3 = B(-2) \Rightarrow B = -3/2$
 when $x = 2$, $5 = C(2)^2 \Rightarrow C = 5/4$
 coefficient of x^2 , $0 = A + C \Rightarrow A = -5/4$
 $\int \frac{x+3}{x^2(x-2)} dx = \int \frac{-5/4}{x} dx + \int \frac{-3/2}{x^2} dx + \int \frac{5/4}{x-2} dx = -\frac{5}{4} \ln|x| + \frac{3}{2x} + \frac{5}{4} \ln|x-2| + C$
13. Put $\sin x = t$, $\cos x dx = dt \Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \frac{dt}{(1-t)(2-t)}$
 consider $\frac{1}{(1-t)(2-t)} = \frac{A}{1-t} + \frac{B}{2-t} \Rightarrow 1 = A(2-t) + B(1-t)$
 when $t = 1$, $1 = A(1) \Rightarrow A = 1$
 when $t = 2$, $1 = B(-1)^2 \Rightarrow B = -1$
 $\int \frac{dt}{(1-t)(2-t)} = \int \frac{1}{(1-t)} dt - \int \frac{1}{(2-t)} dt = -\ln|1-t| + \ln|2-t| + C$
 $\int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \frac{1}{1-\sin x} dx - \int \frac{1}{2-\sin x} dx = -\ln|1-\sin x| + \ln|2-\sin x| + C$
14. Put $\cos x = t$, $-\sin x dx = dt \Rightarrow \int \frac{\sin x}{\cos^2 x + 3 \cos x + 2} dx = \int \frac{-dt}{t^2 + 3t + 2} = -\int \frac{dt}{(t+2)(t+1)}$
 consider $\frac{1}{(t+2)(t+1)} = \frac{A}{t+2} + \frac{B}{t+1} \Rightarrow 1 = A(t+1) + B(t+2)$
 when $t = -2$, $1 = B(-1) \Rightarrow B = -1$
 when $t = -1$, $1 = A(1) \Rightarrow A = 1$
 $\int \frac{1}{(t+2)(t+1)} dt = \int \frac{1}{t+1} dt - \int \frac{1}{t+2} dt = \ln|t+1| - \ln|t+2| + C$
 $\int \frac{\sin x}{\cos^2 x + 3 \cos x + 2} dx = \ln|\cos x + 1| - \ln|\cos x + 2| + C$

15. Put $\cos x = t$, $-\sin x dx = dt \Rightarrow \int \frac{\sin x}{\cos^2 x + 3 \cos x + 2} dx = \int \frac{-dt}{t^2 + 3t + 2} = -\int \frac{dt}{(t+2)(t+1)}$
 $\frac{5x^2 + 7x + 9}{x^3 - x} = \frac{5x^2 + 7x + 9}{x(x^2 - 1)} = \frac{A}{x} + \frac{B}{x^2 - 1} \Rightarrow 5x^2 + 7x + 9 = A(x^2 - 1) + (Bx + C)(x)$
 when $x = 0$, $9 = A(-1) \Rightarrow A = -1/9$
 coefficient of x^2 , $5 = A + B \Rightarrow B = 5 - A \Rightarrow B = 5 - (-1/9) = \frac{46}{9}$
 coefficient of x , $7 = C$
 $\int \frac{5x^2 + 7x + 9}{x^3 - x} dx = \int \frac{-1/9}{x} dx + \int \frac{46x + 7}{x^2 - 1} dx = \frac{-1}{9} \ln |x| + \frac{46}{9} \frac{1}{2} \ln |x^2 - 1| + 7 \ln \left| \frac{x-1}{x+1} \right| + C$
16. $\int \frac{9x}{x^3 + 9x} dx = 9 \int \frac{x}{x(x^2 + 9)} dx = 9 \int \frac{1}{x^2 + 9} dx = 9 \frac{1}{3} \tan^{-1} \frac{x}{3} + C = 3 \tan^{-1} \frac{x}{3} + C$
17. Consider $\frac{x-5}{x(x^2-4)} = \frac{A}{x} + \frac{Bx+C}{x^2-4} \Rightarrow x-5 = Ax(x^2-4) + (B+C)(x)$
 when $x = 0$, $-5 = A(-4) \Rightarrow A = 5/4$
 coefficient of x^2 , $0 = A + B \Rightarrow B = -A \Rightarrow B = -5/4$
 coefficient of x , $C = 1$
 $\int \frac{x-5}{3x^3-12x} dx = \frac{1}{3} \int \frac{x-5}{x(x^2-4)} dx = \frac{1}{3} \left[\int \frac{5/4}{x} dx + \int \frac{-5/4 x}{x^2-4} dx + \int \frac{1}{x^2-4} dx \right] = \frac{1}{3} \left[\frac{5}{4} \ln |x| - \frac{5}{8} \ln |x^2 - 4| + \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| \right] + C$
18. Consider $\frac{5x+9}{(x-2)(x+3)(x+4)} = \frac{A}{x-2} + \frac{B}{x+3} + \frac{C}{x+4}$
 $\Rightarrow 5x + 9 = A(x+3)(x+4) + B(x-2)(x+4) + C(x-2)(x+3)$
 when $x = 2$, $5(2) + 9 = A(2+3)(2+4) \Rightarrow 19 = A(30) \Rightarrow A = \frac{19}{30}$
 when $x = -3$, $5(-3) + 9 = A(-3-2)(-3+4) \Rightarrow 6 = B(-5) \Rightarrow B = \frac{-6}{5}$
 when $x = -4$, $5(-4) + 9 = A(-4-2)(-4+3) \Rightarrow -11 = C(6) \Rightarrow C = \frac{-6}{11}$
 coefficient of x^2 , $0 = A + B \Rightarrow B = -A \Rightarrow B = -5/4$
 coefficient of x , $C = 1$
 $\int \frac{5x+9}{(x-2)(x+3)(x+4)} dx = \int \frac{19/30}{x-2} dx + \int \frac{-6/5}{x+3} dx + \int \frac{-6/11}{x+4} dx = \frac{19}{30} \ln |x-2| - \frac{6}{5} \ln |x+3| - \frac{6}{11} \ln |x+4| + C$
19. $\int \frac{d\theta}{\theta^4 + 5\theta^2 + 4} = \int \frac{d\theta}{(\theta^2 + 4)(\theta^2 + 1)}$
 consider $\frac{1}{(\theta^2 + 4)(\theta^2 + 1)} = \frac{1}{(p+4)(p+1)}$ by replacing θ^2 by p
 $\frac{1}{(p+4)(p+1)} = \frac{A}{p+4} + \frac{B}{p+1} \Rightarrow 1 = A(p+1) + B(p+4)$
 when $p = -1$, $1 = B(-1+4) \Rightarrow B = \frac{1}{3}$
 when $p = -4$, $1 = A(-4+1) \Rightarrow A = \frac{-1}{3}$
 $\int \frac{d\theta}{\theta^4 + 5\theta^2 + 4} = \int \frac{d\theta}{(\theta^2 + 4)(\theta^2 + 1)} = \int \frac{-1/3}{\theta^2 + 4} d\theta + \int \frac{1/3}{\theta^2 + 1} d\theta = \frac{-1}{3} \frac{1}{2} \tan^{-1} \left(\frac{\theta}{2} \right) + \frac{1}{2} \tan^{-1} \theta + C$
20. Put $t^2 = p \Rightarrow 2tdt = dp \Rightarrow tdt = \frac{dp}{2}$
 $\int \frac{t dt}{t^4 + t^2 - 2} = \int \frac{dp/2}{p^2 + p - 2} = \frac{1}{2} \int \frac{dp}{(p+2)(p-1)}$

$$\text{consider } \frac{1}{(p+2)(p-1)} = \frac{A}{p+2} + \frac{B}{p-1} \Rightarrow 1 = A(p-1) + B(p+2)$$

$$\text{when } p = 1, \quad 1 = B(1+2) \Rightarrow B = \frac{1}{3}$$

$$\text{when } p = -2, \quad 1 = A(-2-1) \Rightarrow A = -\frac{1}{3}$$

$$\begin{aligned} \therefore \frac{1}{2} \int \frac{dp}{(p+2)(p-1)} &= \frac{1}{2} \left[\int \frac{-1/3}{p+2} dp + \int \frac{1/3}{p-1} dp \right] = \frac{1}{2} \left[-\frac{1}{3} \ln|p+2| + \frac{1}{3} \ln|p-1| \right] + C \\ &= \frac{1}{2} \left[-\frac{1}{3} \ln|t^2+2| + \frac{1}{3} \ln|t^2-1| \right] + C \end{aligned}$$

$$21. \int \frac{x^3+x^2}{x^2+x-2} dx = \int \left(x + \frac{2x}{x^2+x-2} \right) dx = \int x dx + \frac{2}{3} \int \frac{dx}{x-1} + \frac{4}{3} \int \frac{dx}{x+2} = \frac{x^2}{2} + \frac{4}{3} \ln|x+2| + \frac{2}{3} \ln|x-1| + C$$

$$22. \int \frac{x^3+1}{x^3-x} dx = \int \left(1 + \frac{x+1}{x^3-x} \right) dx = \int \left[1 + \frac{1}{x(x-1)} \right] dx = \int dx + \int \frac{dx}{x-1} - \int \frac{dx}{x} = x + \ln|x-1| - \ln|x| + C$$

$$23. \int \frac{x^3+4x^2}{x^2+4x+3} dx = \int \left(x - \frac{3x}{x^2+4x+3} \right) dx = \int x dx + \frac{3}{2} \int \frac{dx}{x+1} - \frac{9}{2} \int \frac{dx}{x+3} = \frac{x^2}{2} - \frac{9}{2} \ln|x+3| + \frac{3}{2} \ln|x+1| + C$$

$$\begin{aligned} 24. \int \frac{2x^3+x^2-21x+24}{x^2+2x-8} dx &= \int \left[(2x-3) + \frac{x}{x^2+2x-8} \right] dx = \int (2x-3) dx + \frac{1}{3} \int \frac{dx}{x-2} + \frac{2}{3} \int \frac{dx}{x+4} \\ &= x^2 - 3x + \frac{2}{3} \ln|x+4| + \frac{1}{3} \ln|x-2| + C \end{aligned}$$

$$25. \int \frac{dx}{x(3\sqrt{x+1})}; \begin{cases} u = \sqrt{x+1} \\ du = \frac{dx}{2\sqrt{x+1}} \\ dx = 2u du \end{cases} \rightarrow \frac{2}{3} \int \frac{u du}{(u^2-1)u} = \frac{1}{3} \int \frac{du}{u-1} - \frac{1}{3} \int \frac{du}{u+1} = \frac{1}{3} \ln|u-1| - \frac{1}{3} \ln|u+1| + C = \frac{1}{3} \ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + C$$

$$26. \int \frac{dx}{x(1+\sqrt[3]{x})}; \begin{cases} u = \sqrt[3]{x} \\ du = \frac{dx}{3x^{2/3}} \\ dx = 3u^2 du \end{cases} \rightarrow \int \frac{3u^2 du}{u^3(1+u)} = 3 \int \frac{du}{u(1+u)} = 3 \ln \left| \frac{u}{u+1} \right| + C = 3 \ln \left| \frac{\sqrt[3]{x}}{1+\sqrt[3]{x}} \right| + C$$

$$27. \int \frac{ds}{e^s-1}; \begin{cases} u = e^s - 1 \\ du = e^s ds \\ ds = \frac{du}{u+1} \end{cases} \rightarrow \int \frac{du}{u(u+1)} = -\int \frac{du}{u+1} + \int \frac{du}{u} = \ln \left| \frac{u}{u+1} \right| + C = \ln \left| \frac{e^s-1}{e^s} \right| + C = \ln|1-e^{-s}| + C$$

$$28. \int \frac{ds}{\sqrt{e^s+1}}; \begin{cases} u = \sqrt{e^s+1} \\ du = \frac{e^s ds}{2\sqrt{e^s+1}} \\ ds = \frac{2u du}{u^2-1} \end{cases} \rightarrow \int \frac{2u du}{u(u^2-1)} = 2 \int \frac{du}{(u+1)(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u+1} = \ln \left| \frac{u-1}{u+1} \right| + C = \ln \left| \frac{\sqrt{e^s+1}-1}{\sqrt{e^s+1}+1} \right| + C$$

$$29. (a) \int \frac{y dy}{\sqrt{16-y^2}} = -\frac{1}{2} \int \frac{-2y dy}{\sqrt{16-y^2}} = -\sqrt{16-y^2} + C$$

$$(b) \int \frac{y dy}{\sqrt{16-y^2}}; [y = 4 \sin x] \rightarrow 4 \int \frac{\sin x \cos x dx}{\cos x} = -4 \cos x + C = -\frac{4\sqrt{16-y^2}}{4} + C = -\sqrt{16-y^2} + C$$

30. (a) $\int \frac{x dx}{\sqrt{4+x^2}} = \frac{1}{2} \int \frac{2x dx}{\sqrt{4+x^2}} = \sqrt{4+x^2} + C$
 (b) $\int \frac{x dx}{\sqrt{4+x^2}}; [x = 2 \tan y] \rightarrow \int \frac{2 \tan y \cdot 2 \sec^2 y dy}{2 \sec y} = 2 \int \sec y \tan y dy = 2 \sec y + C = \sqrt{4+x^2} + C$
31. (a) $\int \frac{x dx}{4-x^2} = -\frac{1}{2} \int \frac{(-2x) dx}{4-x^2} = -\frac{1}{2} \ln |4-x^2| + C$
 (b) $\int \frac{x dx}{4-x^2}; [x = 2 \sin \theta] \rightarrow \int \frac{2 \sin \theta \cdot 2 \cos \theta d\theta}{4 \cos^2 \theta} = \int \tan \theta d\theta = -\ln |\cos \theta| + C = -\ln \left(\frac{\sqrt{4-x^2}}{2} \right) + C$
 $= -\frac{1}{2} \ln |4-x^2| + C$
32. (a) $\int \frac{t dt}{\sqrt{4t^2-1}} = \frac{1}{8} \int \frac{8t dt}{\sqrt{4t^2-1}} = \frac{1}{4} \sqrt{4t^2-1} + C$
 (b) $\int \frac{t dt}{\sqrt{4t^2-1}}; [t = \frac{1}{2} \sec \theta] \rightarrow \int \frac{\frac{1}{2} \sec \theta \tan \theta \cdot \frac{1}{2} \sec \theta d\theta}{\tan \theta} = \frac{1}{4} \int \sec^2 \theta d\theta = \frac{\tan \theta}{4} + C = \frac{\sqrt{4t^2-1}}{4} + C$
33. $\int \frac{x dx}{9-x^2}; \left[\begin{array}{l} u = 9-x^2 \\ du = -2x dx \end{array} \right] \rightarrow -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln |u| + C = \ln \frac{1}{\sqrt{u}} + C = \ln \frac{1}{\sqrt{9-x^2}} + C$
34. $\int \frac{dx}{x(9-x^2)} = \frac{1}{9} \int \frac{dx}{x} + \frac{1}{18} \int \frac{dx}{3-x} - \frac{1}{18} \int \frac{dx}{3+x} = \frac{1}{9} \ln |x| - \frac{1}{18} \ln |3-x| - \frac{1}{18} \ln |3+x| + C = \frac{1}{9} \ln |x| - \frac{1}{18} \ln |9-x^2| + C$
35. $\int \frac{dx}{9-x^2} = \frac{1}{6} \int \frac{dx}{3-x} + \frac{1}{6} \int \frac{dx}{3+x} = -\frac{1}{6} \ln |3-x| + \frac{1}{6} \ln |3+x| + C = \frac{1}{6} \ln \left| \frac{x+3}{x-3} \right| + C$
36. $\int \frac{dx}{\sqrt{9-x^2}}; \left[\begin{array}{l} x = \sin \theta \\ dx = \cos \theta d\theta \end{array} \right] \rightarrow \int \frac{\cos \theta}{3 \cos \theta} d\theta = \int d\theta = \theta + C = \sin^{-1} \frac{x}{3} + C$
37. $\int \sin^5 x \cos^6 x dx = \frac{\sin^{5-1} x \cos^{6+1} x}{5+6} + \frac{5-1}{5+6} \int \sin^{5-2} x \cos^6 x dx = \frac{\sin^4 x \cos^7 x}{11} + \frac{4}{11} \int \sin^3 x \cos^6 x dx$
 $= \frac{\sin^4 x \cos^7 x}{11} + \frac{4}{11} \left[\frac{\sin^{3-1} x \cos^{6+1} x}{3+6} + \frac{3-1}{3+6} \int \sin^{3-2} x \cos^6 x dx \right]$
 $= \frac{\sin^4 x \cos^7 x}{11} + \frac{4}{99} \sin^2 x \cos^7 x + \frac{8}{99} \left[\int \sin x \cos^6 x dx \right]$
 $= \frac{\sin^4 x \cos^7 x}{11} + \frac{4}{99} \sin^2 x \cos^7 x + \frac{8}{99} \left(-\frac{\cos^7 x}{7} \right) + C$
38. $\int \cos^3 x \sin^3 x dx = \frac{\sin^{3-1} x \cos^{3+1} x}{3+3} + \frac{3-1}{3+3} \int \sin^{3-2} x \cos^4 x dx = \frac{\sin^2 x \cos^4 x}{6} + \frac{2}{6} \int \sin x \cos^4 x dx$
 $= \frac{\sin^2 x \cos^4 x}{6} - \frac{1}{3} \frac{\cos^5 x}{5} + C = \frac{\sin^2 x \cos^4 x}{6} - \frac{\cos^5 x}{15} + C$
39. Put $\tan x = t \Rightarrow \sec^2 x dx = dt$
 $\int \tan^6 x \sec^2 x dx = \int t^6 dt = \frac{t^7}{7} + C = \frac{\tan^7 x}{7} + C$

$$40. \int \tan^3 x \sec^3 x \, dx = \int (\sec^2 x - 1) \sec^2 x \cdot \sec x \cdot \tan x \, dx = \int \sec^4 x \cdot \sec x \cdot \tan x \, dx - \int \sec^2 x \cdot \sec x \cdot \tan x \, dx$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

$$41. \int \sin 7\theta \cos 5\theta \, d\theta = \frac{1}{2} \int \sin(7\theta + 5\theta) + \sin(7\theta - 5\theta) \, d\theta = \frac{1}{2} \int \sin(12\theta) + \frac{1}{2} \int \sin(2\theta) \, d\theta = \frac{1}{2} \left[-\frac{\cos 12\theta}{12} - \frac{\cos 2\theta}{2} \right] + C$$

$$42. \text{ Put } \cot \theta = t \Rightarrow -\operatorname{cosec}^2 \theta \, d\theta = dt$$

$$\int \csc^2 \theta \cot^3 \theta \, d\theta = -\int t^3 \, dt = -\frac{t^4}{4} + C = -\frac{\cot^4 \theta}{4} + C$$

$$43. \int \sqrt{1 - \cos\left(\frac{t}{2}\right)} \, dt = \int \sqrt{2} \sin \frac{t}{4} \, dt = \sqrt{2} \left(\frac{-\cos\left(\frac{t}{4}\right)}{1/4} \right) + C = -4\sqrt{2} \cos\left(\frac{t}{4}\right) + C = -4\sqrt{1 + \cos\left(\frac{t}{2}\right)} + C$$

$$44. \text{ Put } e^\theta = t \Rightarrow e^\theta \, d\theta = dt$$

$$\int e^\theta \sqrt{1 - \cos^2 e^\theta} \, d\theta = \int \sqrt{1 - \cos^2 t} \, dt = \int \sin t \, dt = -\cos t + C = -\cos e^\theta + C$$

$$45. |E_s| \leq \frac{3-1}{180} (\Delta x)^4 M \text{ where } \Delta x = \frac{3-1}{n} = \frac{2}{n}; \, f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4}$$

$$\Rightarrow f^{(4)}(x) = 24x^{-5} \text{ which is decreasing on } [1, 3] \Rightarrow \text{maximum of } f^{(4)}(x) \text{ on } [1, 3] \text{ is}$$

$$f^{(4)}(1) = 24 \Rightarrow M = 24. \text{ Then } |E_s| \leq 0.0001 \Rightarrow \left(\frac{3-1}{180}\right) \left(\frac{2}{n}\right)^4 (24) \leq 0.0001 \Rightarrow \left(\frac{768}{180}\right) \left(\frac{1}{n^4}\right) \leq 0.0001$$

$$\Rightarrow \frac{1}{n^4} \leq (0.0001) \left(\frac{180}{768}\right) \Rightarrow n^4 \geq 10,000 \left(\frac{768}{180}\right) \Rightarrow n \geq 14.37 \Rightarrow n \geq 16 \text{ (n must be even)}$$

$$46. |E_T| \leq \frac{1-0}{12} (\Delta x)^2 M \text{ where } \Delta x = \frac{1-0}{n} = \frac{1}{n}; \, 0 \leq f''(x) \leq 8 \Rightarrow M = 8. \text{ Then } |E_T| \leq 10^{-3} \Rightarrow \frac{1}{12} \left(\frac{1}{n}\right)^2 (8) \leq 10^{-3}$$

$$\Rightarrow \frac{2}{3n^2} \leq 10^{-3} \Rightarrow \frac{3n^2}{2} \geq 1000 \Rightarrow n^2 \geq \frac{2000}{3} \Rightarrow n \geq 25.82 \Rightarrow n \geq 26$$

$$47. \Delta x = \frac{b-a}{n} = \frac{\pi-0}{6} = \frac{\pi}{6} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{12};$$

$$\sum_{i=0}^6 m f(x_i) = 12 \Rightarrow T = \left(\frac{\pi}{12}\right)(12) = \pi;$$

	x_i	$f(x_i)$	m	$m f(x_i)$
x_0	0	0	1	0
x_1	$\pi/6$	1/2	2	1
x_2	$\pi/3$	3/2	2	3
x_3	$\pi/2$	2	2	4
x_4	$2\pi/3$	3/2	2	3
x_5	$5\pi/6$	1/2	2	1
x_6	π	0	1	0

$$\sum_{i=0}^6 mf(x_i) = 18 \text{ and } \frac{\Delta x}{3} = \frac{\pi}{18}$$

$$\Rightarrow S = \left(\frac{\pi}{18}\right)(18) = \pi.$$

	x_i	$f(x_i)$	m	$mf(x_i)$
	x_0	0	1	0
	x_1	$\pi/6$	4	2
	x_2	$\pi/3$	2	3
	x_3	$\pi/2$	4	8
	x_4	$2\pi/3$	2	3
	x_5	$5\pi/6$	4	2
	x_6	π	1	0

48. $|f^{(4)}(x)| \leq 3 \Rightarrow M = 3; \Delta x = \frac{2-1}{n} = \frac{1}{n}$. Hence $|E_s| \leq 10^{-5} \Rightarrow \left(\frac{2-1}{180}\right)\left(\frac{1}{n}\right)^4 (3) \leq 10^{-5} \Rightarrow \frac{1}{60n^4} \leq 10^{-5} \Rightarrow n^4 \geq \frac{10^5}{60}$
 $\Rightarrow n \geq 6.38 \Rightarrow n \geq 8$ (n must be even)

49. $y_{av} = \frac{1}{365-0} \int_0^{365} \left[20 \sin\left(\frac{2\pi}{365}(x-101)\right) - 4 \right] dx = \frac{1}{365} \left[-20 \left(\frac{365}{2\pi} \cos\left(\frac{2\pi}{365}(x-101)\right) - 4x \right) \right]_0^{365}$
 $= \frac{1}{365} \left[\left(-20 \left(\frac{365}{2\pi} \right) \cos\left[\frac{2\pi}{365}(365-101)\right] - 4(365) \right) - \left(-20 \left(\frac{365}{2\pi} \right) \cos\left[\frac{2\pi}{365}(0-101)\right] - 4(0) \right) \right]$
 $= -\frac{20}{2\pi} \cos\left(\frac{2\pi}{365}(264)\right) - 4 + \frac{20}{2\pi} \cos\left(\frac{2\pi}{365}(-101)\right) = -\frac{20}{2\pi} \left(\cos\left(\frac{2\pi}{365}(264)\right) - \cos\left(\frac{2\pi}{365}(-101)\right) \right) - 4$
 $\approx -\frac{20}{2\pi} (0.16705 - 0.16705) - 4 = -4^\circ\text{C}$

50. $av(C_v) = \frac{1}{675-20} \int_{20}^{675} \left[8.27 + 10^{-5} (26T - 1.87T^2) \right] dT = \frac{1}{655} \left[8.27T + \frac{13}{10^5} T^2 - \frac{0.62333}{10^5} T^3 \right]_{20}^{675}$
 $\approx \frac{1}{655} [(5582.25 + 59.23125 - 1917.03194) - (165.4 + 0.052 - 0.04987)] \approx 5.434;$
 $8.27 + 10^{-5} (26T - 1.87T^2) = 5.434 \Rightarrow 1.87T^2 - 26T - 283,600 = 0 \Rightarrow T \approx \frac{26 + \sqrt{676 + 4(1.87)(283,600)}}{2(1.87)} \approx 396.45^\circ\text{C}$

51. (a) Each interval is 5 min = $\frac{1}{12}$ hour. $\frac{1}{24} [2.5 + 2(2.4) + 2(2.3) + \dots + 2(2.4) + 2.3] = \frac{29}{12} \approx 2.42$ liters
 (b) $(60 \text{ km/h}) \left(\frac{12}{29} \text{ hours/liter} \right) \approx 24.83 \text{ km/liter}$

52. Using the Simpson's rule, $\Delta x = 5 \Rightarrow \frac{\Delta x}{3} = \frac{5}{3}$;
 $\sum mf(x_i) = 401 \Rightarrow \text{Area} \approx (401)\left(\frac{5}{3}\right) = 668.3 \text{ m}^2$;
 The cost is $\text{Area} \cdot (\$21/\text{m}^2) \approx (668.3^2)(\$21/\text{m}^2)$
 $= \$14,035.00 \Rightarrow$ the job cannot be done for \$11,000.

	x_i	$f(x_i)$	m	$mf(x_i)$
	x_0	0	1	0
	x_1	5	4	48
	x_2	10	2	36
	x_3	15	4	68
	x_4	20	2	33
	x_5	25	4	72
	x_6	30	2	42
	x_7	35	4	88
	x_8	40	1	14

53. $\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{b \rightarrow 3^-} \int_0^b \frac{dx}{\sqrt{9-x^2}} = \lim_{b \rightarrow 3^-} \left[\sin^{-1}\left(\frac{x}{3}\right) \right]_0^b = \lim_{b \rightarrow 3^-} \left[\sin^{-1}\left(\frac{b}{3}\right) - \sin^{-1}\left(\frac{0}{3}\right) \right] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$
54. $\int_0^1 \ln x \, dx = \lim_{b \rightarrow 0^+} [x \ln x - x]_b^1 = \lim_{b \rightarrow 0^+} [(1 \cdot \ln 1 - 1) - (b \ln b - b)] = -1 - \lim_{b \rightarrow 0^+} \frac{\ln b}{\left(\frac{1}{b}\right)} = -1 - \lim_{b \rightarrow 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)} = -1 + 0 = -1$
55. $\int_{-1}^1 \frac{dy}{y^{2/3}} = \int_{-1}^0 \frac{dy}{y^{2/3}} + \int_0^1 \frac{dy}{y^{2/3}} = 2 \int_0^1 \frac{dy}{y^{2/3}} = 2 \cdot 3 \lim_{b \rightarrow 0^+} \left[y^{1/3} \right]_b^1 = 6 \lim_{b \rightarrow 0^+} [1 - b^{1/3}] = 6$
56. $\int_{-2}^{\infty} \frac{d\theta}{(\theta+1)^{3/5}} = \int_{-2}^{-1} \frac{d\theta}{(\theta+1)^{3/5}} + \int_{-1}^2 \frac{d\theta}{(\theta+1)^{3/5}} + \int_2^{\infty} \frac{d\theta}{(\theta+1)^{3/5}}$ converges if each integral converges, but $\lim_{\theta \rightarrow \infty} \frac{\theta^{3/5}}{(\theta+1)^{3/5}} = 1$
 and $\int_2^{\infty} \frac{d\theta}{\theta^{3/5}}$ diverges $\Rightarrow \int_{-2}^{\infty} \frac{d\theta}{(\theta+1)^{3/5}}$ diverges
57. $\int_3^{\infty} \frac{2 \, du}{u^2 - 2u} = \int_3^{\infty} \frac{du}{u-2} - \int_3^{\infty} \frac{du}{u} = \lim_{b \rightarrow \infty} \left[\ln \left| \frac{u-2}{u} \right| \right]_3^b = \lim_{b \rightarrow \infty} \left[\ln \left| \frac{b-2}{b} \right| - \ln \left| \frac{3-2}{3} \right| \right] = 0 - \ln \left(\frac{1}{3} \right) = \ln 3$
58. $\int_1^{\infty} \frac{3v-1}{4v^3-v^2} \, dv = \int_1^{\infty} \left(\frac{1}{v} + \frac{1}{v^2} - \frac{4}{4v-1} \right) \, dv = \lim_{b \rightarrow \infty} \left[\ln v - \frac{1}{v} - \ln(4v-1) \right]_1^b = \lim_{b \rightarrow \infty} \left[\ln \left(\frac{b}{4b-1} \right) - \frac{1}{b} - (\ln 1 - 1 - \ln 3) \right]$
 $= \ln \frac{1}{4} + 1 + \ln 3 = 1 + \ln \frac{3}{4}$
59. $\int_0^{\infty} x^2 e^{-x} \, dx = \lim_{b \rightarrow \infty} \left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right]_0^b = \lim_{b \rightarrow \infty} \left[(-b^2 e^{-b} - 2b e^{-b} - 2e^{-b}) - (-2) \right] = 0 + 2 = 2$
60. $\int_{-\infty}^0 x e^{3x} \, dx = \lim_{b \rightarrow -\infty} \left[\frac{x}{3} e^{3x} - \frac{1}{9} e^{3x} \right]_b^0 = \lim_{b \rightarrow -\infty} \left[-\frac{1}{9} - \left(\frac{b}{3} e^{3b} - \frac{1}{9} e^{3b} \right) \right] = -\frac{1}{9} - 0 = -\frac{1}{9}$

61. $\int_{-\infty}^{\infty} \frac{dx}{4x^2+9} = 2 \int_0^{\infty} \frac{dx}{4x^2+9} = \frac{1}{2} \int_0^{\infty} \frac{dx}{x^2+\frac{9}{4}} = \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{2}{3} \tan^{-1} \left(\frac{2x}{3} \right) \right]_0^b = \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{2}{3} \tan^{-1} \left(\frac{2b}{3} \right) - \frac{2}{3} \tan^{-1}(0) \right]$
 $= \frac{1}{2} \left[\left(\frac{2}{3} \cdot \frac{\pi}{2} \right) - 0 \right] = \frac{\pi}{6}$
62. $\int_{-\infty}^{\infty} \frac{4dx}{x^2+16} = 2 \int_0^{\infty} \frac{4dx}{x^2+16} = 2 \lim_{b \rightarrow \infty} \left[\tan^{-1} \left(\frac{x}{4} \right) \right]_0^b = 2 \lim_{b \rightarrow \infty} \left[\tan^{-1} \left(\frac{b}{4} \right) - \tan^{-1}(0) \right] = 2 \left[\left(\frac{\pi}{2} \right) - 0 \right] = \pi$
63. $\lim_{\theta \rightarrow \infty} \frac{\theta}{\sqrt{\theta^2+1}} = 1$ and $\int_6^{\infty} \frac{d\theta}{\theta}$ diverges $\Rightarrow \int_6^{\infty} \frac{d\theta}{\sqrt{\theta^2+1}}$ diverges
64. $I = \int_0^{\infty} e^{-u} \cos u \, du = \lim_{b \rightarrow \infty} \left[-e^{-u} \cos u \right]_0^b - \int_0^{\infty} e^{-u} \sin u \, du = 1 + \lim_{b \rightarrow \infty} \left[e^{-u} \sin u \right]_0^b - \int_0^{\infty} (e^{-u}) \cos u \, du$
 $\Rightarrow I = 1 + 0 - I \Rightarrow 2I = 1 \Rightarrow I = \frac{1}{2}$ converges
65. $\int_1^{\infty} \frac{\ln z}{z} \, dz = \int_1^e \frac{\ln z}{z} \, dz + \int_e^{\infty} \frac{\ln z}{z} \, dz = \left[\frac{(\ln z)^2}{2} \right]_1^e + \lim_{b \rightarrow \infty} \left[\frac{(\ln z)^2}{2} \right]_e^b = \left(\frac{1^2}{2} - 0 \right) + \lim_{b \rightarrow \infty} \left[\frac{(\ln b)^2}{2} - \frac{1}{2} \right] = \infty \Rightarrow$ diverges
66. $0 < \frac{e^{-t}}{\sqrt{t}} \leq e^{-t}$ for $t \geq 1$ and $\int_1^{\infty} e^{-t} \, dt$ converges $\Rightarrow \int_1^{\infty} \frac{e^{-t}}{\sqrt{t}} \, dt$ converges
67. $\int_{-\infty}^{\infty} \frac{2 \, dx}{e^x + e^{-x}} = 2 \int_0^{\infty} \frac{2 \, dx}{e^x + e^{-x}} < \int_0^{\infty} \frac{4 \, dx}{e^x}$ converges $\Rightarrow \int_{-\infty}^{\infty} \frac{2 \, dx}{e^x + e^{-x}}$ converges
68. $\int_{-\infty}^{\infty} \frac{dx}{x^2(1+e^x)} = \int_{-\infty}^{-1} \frac{dx}{x^2(1+e^x)} + \int_{-1}^0 \frac{dx}{x^2(1+e^x)} + \int_0^1 \frac{dx}{x^2(1+e^x)} + \int_1^{\infty} \frac{dx}{x^2(1+e^x)}$
 $\lim_{x \rightarrow 0} \left[\frac{\left(\frac{1}{x^2} \right)}{\left[\frac{1}{x^2(1+e^x)} \right]} \right] = \lim_{x \rightarrow 0} \frac{x^2(1+e^x)}{x^2} = \lim_{x \rightarrow 0} (1+e^x) = 2$ and $\int_0^1 \frac{dx}{x^2}$ diverges $\Rightarrow \int_0^1 \frac{dx}{x^2(1+e^x)}$ diverges $\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2(1+e^x)}$ diverges
69. $\int \frac{x \, dx}{1+\sqrt{x}}; \begin{cases} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{cases} \rightarrow \int \frac{u^2 \cdot 2u \, du}{1+u} = \int \left(2u^2 - 2u + 2 - \frac{2}{1+u} \right) du = \frac{2}{3}u^3 - u^2 + 2u - 2 \ln |1+u| + C$
 $= \frac{2x^{3/2}}{3} - x + 2\sqrt{x} - 2 \ln(1+\sqrt{x}) + C$
70. $\int \frac{x^3+2}{4-x^2} \, dx = -\int \left(x + \frac{4x+2}{x^2-4} \right) dx = -\int x \, dx - \frac{3}{2} \int \frac{dx}{x+2} - \frac{5}{2} \int \frac{dx}{x-2} = -\frac{x^2}{2} - \frac{3}{2} \ln |x+2| - \frac{5}{2} \ln |x-2| + C$
71. $\int \sqrt{2x-x^2} \, dx; \quad x-1 = \sin \theta \quad dx = \cos \theta \, d\theta$

For the integrand to be nonnegative x must be between 0 and 2 so $x-1$ is between -1 and 1 and we can take θ between $-\pi/2$ and $\pi/2$, where cosine is nonnegative. Thus

$$\sqrt{2x-x^2} = \sqrt{1-(x-1)^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta.$$

$$\begin{aligned}
 \int \sqrt{2x-x^2} \, dx &= \int \cos \theta \cos \theta \, d\theta = \int \cos^2 \theta \, d\theta = \frac{1}{2} \int (1 + \cos 2\theta) \, d\theta \\
 &= \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta + C = \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta + C \\
 &= \frac{1}{2} \sin^{-1}(x-1) + \frac{1}{2} (x-1) \sqrt{2x-x^2} + C
 \end{aligned}$$

$$72. \int \frac{dx}{\sqrt{-2x-x^2}} = \int \frac{dx}{\sqrt{1-(x+1)^2}} = \sin^{-1}(x+1) + C$$

$$\begin{aligned}
 73. \int \frac{2-\cos x + \sin x}{\sin^2 x} \, dx &= \int 2 \csc^2 x \, dx - \int \frac{\cos x}{\sin^2 x} \, dx + \int \csc x \, dx = -2 \cot x + \frac{1}{\sin x} - \ln |\csc x + \cot x| + C \\
 &= -2 \cot x + \csc x - \ln |\csc x + \cot x| + C
 \end{aligned}$$

$$74. \int \sin^2 \theta \cos^5 \theta \, d\theta$$

$$\int \sin^2 \theta \cos^5 \theta \, d\theta = \int \sin^2 \theta (1 - \sin^2 \theta)^2 \cos \theta \, d\theta; \quad u = \sin \theta \quad du = \cos \theta \, d\theta$$

$$\begin{aligned}
 \int \sin^2 \theta (1 - \sin^2 \theta)^2 \cos \theta \, d\theta &= \int (\sin^2 \theta - 2 \sin^4 \theta + \sin^6 \theta) \cos \theta \, d\theta \\
 &= \int (u^2 - 2u^4 + u^6) \, du = \frac{u^3}{3} - \frac{2}{5} u^5 + \frac{1}{7} u^7 + C \\
 &= \frac{\sin^3 \theta}{3} - \frac{2}{5} \sin^5 \theta + \frac{1}{7} \sin^7 \theta + C
 \end{aligned}$$

$$75. \int \frac{9 \, dv}{81-v^4} = \frac{1}{2} \int \frac{dv}{v^2+9} + \frac{1}{12} \int \frac{dv}{3-v} + \frac{1}{12} \int \frac{dv}{3+v} = \frac{1}{12} \ln \left| \frac{3+v}{3-v} \right| + \frac{1}{6} \tan^{-1} \frac{v}{3} + C$$

$$76. \int_2^\infty \frac{dx}{(x-1)^2} = \lim_{b \rightarrow \infty} \left[\frac{1}{1-x} \right]_2^b = \lim_{b \rightarrow \infty} \left[\frac{1}{1-b} - (-1) \right] = 0 + 1 = 1$$

$$77. \cos(2\theta + 1)$$

$$\theta \xrightarrow{(+)} \frac{1}{2} \sin(2\theta + 1)$$

$$1 \xrightarrow{(-)} -\frac{1}{4} \cos(2\theta + 1)$$

$$0 \Rightarrow \int \theta \cos(2\theta + 1) \, d\theta = \frac{\theta}{2} \sin(2\theta + 1) + \frac{1}{4} \cos(2\theta + 1) + C$$

$$78. \int \frac{x^3 \, dx}{x^2-2x+1} = \int \left(x + 2 + \frac{3x-2}{x^2-2x+1} \right) dx = \int (x+2) \, dx + 3 \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} = \frac{x^2}{2} + 2x + 3 \ln |x-1| - \frac{1}{x-1} + C$$

$$79. \int \frac{\sin 2\theta \, d\theta}{(1+\cos 2\theta)^2} = -\frac{1}{2} \int \frac{(-2 \sin 2\theta) \, d\theta}{(1+\cos 2\theta)^2} = \frac{1}{2(1+\cos 2\theta)} + C = \frac{1}{4} \sec^2 \theta + C$$

$$80. \int_{\pi/4}^{\pi/2} \sqrt{1+\cos 4x} \, dx = -\sqrt{2} \int_{\pi/4}^{\pi/2} \cos 2x \, dx = \left[-\frac{\sqrt{2}}{2} \sin 2x \right]_{\pi/4}^{\pi/2} = \frac{\sqrt{2}}{2}$$

81. $\int \frac{x \, dx}{\sqrt{2-x}}; \left[\begin{array}{l} y = 2-x \\ dy = -dx \end{array} \right] \rightarrow -\int \frac{(2-y) \, dy}{\sqrt{y}} = \frac{2}{3} y^{3/2} - 4y^{1/2} + C = \frac{2}{3} (2-x)^{3/2} - 4(2-x)^{1/2} + C$
 $= 2 \left[\frac{(\sqrt{2-x})^3}{3} - 2\sqrt{2-x} \right] + C$
82. $\int \frac{\sqrt{1-v^2}}{v^2} \, dv; [v = \sin \theta] \rightarrow \int \frac{\cos \theta \cdot \cos \theta \, d\theta}{\sin^2 \theta} = \int \frac{(1-\sin^2 \theta) \, d\theta}{\sin^2 \theta} = \int \csc^2 \theta \, d\theta - \int d\theta = \cot \theta - \theta + C$
 $= -\sin^{-1} v - \frac{\sqrt{1-v^2}}{v} + C$
83. $\int \frac{dy}{y^2-2y+2} = \int \frac{dy}{(y-1)^2+1} = \tan^{-1}(y-1) + C$
84. $\int \frac{x \, dx}{\sqrt{8-2x^2-x^4}} = \frac{1}{2} \int \frac{(2x) \, dx}{\sqrt{9-(x^2+1)^2}} = \frac{1}{2} \sin^{-1} \left(\frac{x^2+1}{3} \right) + C$
85. $\int \frac{z+1}{z^2(z^2+4)} \, dz = \frac{1}{4} \int \left(\frac{1}{z} + \frac{1}{z^2} - \frac{z+1}{z^2+4} \right) dz = \frac{1}{4} \ln |z| - \frac{1}{4z} - \frac{1}{8} \ln(z^2+4) - \frac{1}{8} \tan^{-1} \frac{z}{2} + C$
86. $\int x^2(x-1)^{1/3} \, dx; \quad u = x-1 \quad du = dx \quad x^2 = (u+1)^2$
 $\int x^2(x-1)^{1/3} \, dx = \int (u^2+2u+1) \cdot u^{1/3} \, du$
 $= \int (u^{7/3} + 2u^{4/3} + u^{1/3}) \, du$
 $= \frac{3}{10} u^{10/3} + \frac{6}{7} u^{7/3} + \frac{3}{4} u^{4/3} + C$
 $= \frac{3}{10} (x-1)^{10/3} + \frac{6}{7} (x-1)^{7/3} + \frac{3}{4} (x-1)^{4/3} + C$
87. $\int \frac{t \, dt}{\sqrt{9-4t^2}} = -\frac{1}{8} \int \frac{(-8t) \, dt}{\sqrt{9-4t^2}} = -\frac{1}{4} \sqrt{9-4t^2} + C$
88. $u = \tan^{-1} x, \, du = \frac{dx}{1+x^2}; \quad dv = \frac{dx}{x^2}, \, v = -\frac{1}{x}; \quad \int \frac{\tan^{-1} x \, dx}{x^2} = -\frac{1}{x} \tan^{-1} x + \int \frac{dx}{x(1+x^2)} = -\frac{1}{x} \tan^{-1} x + \int \frac{dx}{x} - \int \frac{x \, dx}{1+x^2}$
 $= -\frac{1}{x} \tan^{-1} x + \ln |x| - \frac{1}{2} \ln(1+x^2) + C = -\frac{\tan^{-1} x}{x} + \ln |x| - \ln \sqrt{1+x^2} + C$
89. $\int \frac{e^t \, dt}{e^{2t}+3e^t+2}; [e^t = x] \rightarrow \int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln |x+1| - \ln |x+2| + C = \ln \left| \frac{x+1}{x+2} \right| + C = \ln \left(\frac{e^t+1}{e^t+2} \right) + C$
90. $\int \tan^3 t \, dt = \int (\tan t) (\sec^2 t - 1) \, dt = \frac{\tan^2 t}{2} - \int \tan t \, dt = \frac{\tan^2 t}{2} - \ln |\sec t| + C$

$$91. \int_1^\infty \frac{\ln y \, dy}{y^3}; \left[\begin{array}{l} x = \ln y \\ dx = \frac{dy}{y} \\ dy = e^x dx \end{array} \right] \rightarrow \int_0^\infty \frac{x \cdot e^x}{e^{3x}} dx = \int_0^\infty x e^{-2x} dx = \lim_{b \rightarrow \infty} \left[-\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} \right]_0^b$$

$$= \lim_{b \rightarrow \infty} \left[\left(-\frac{b}{2e^{2b}} - \frac{1}{4e^{2b}} \right) - \left(0 - \frac{1}{4} \right) \right] = \frac{1}{4}$$

$$92. \int y^{3/2} (\ln y)^2 \, dy; \quad u = (\ln y)^2, \, du = \frac{2 \ln y}{y} dy, \, dv = y^{3/2} dy, \, v = \frac{2}{5} y^{5/2}$$

$$\int y^{3/2} (\ln y)^2 \, dy = \frac{2}{5} y^{5/2} (\ln y)^2 - \frac{4}{5} \int y^{3/2} \ln y \, dy$$

$$\text{Now we compute } \int y^{3/2} \ln y \, dy: \quad u = \ln y, \, du = \frac{1}{y} dy, \, dv = y^{3/2} dy, \, v = \frac{2}{5} y^{5/2}$$

$$\int y^{3/2} \ln y \, dy = \frac{2}{5} y^{5/2} \ln y - \frac{2}{5} \int y^{3/2} dy$$

$$= \frac{2}{5} y^{5/2} \ln y - \frac{4}{5} y^{5/2}$$

$$\int y^{3/2} (\ln y)^2 \, dy = \frac{2}{5} y^{5/2} (\ln y)^2 - \frac{4}{5} \int y^{3/2} \ln y \, dy$$

$$= \frac{2}{5} y^{5/2} (\ln y)^2 - \frac{4}{5} \left(\frac{2}{5} y^{5/2} \ln y - \frac{4}{5} y^{5/2} \right)$$

$$= y^{5/2} \left(\frac{2}{5} (\ln y)^2 - \frac{8}{25} \ln y + \frac{16}{125} \right) + C$$

$$93. \int e^{\ln \sqrt{x}} dx = \int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + C$$

$$94. \int e^\theta \sqrt{3+4e^\theta} \, d\theta; \left[\begin{array}{l} u = 4e^\theta \\ du = 4e^\theta \, d\theta \end{array} \right] \rightarrow \frac{1}{4} \int \sqrt{3+u} \, du = \frac{1}{4} \cdot \frac{2}{3} (3+u)^{3/2} + C = \frac{1}{6} (3+4e^\theta)^{3/2} + C$$

$$95. \int \frac{\sin 5t \, dt}{1+(\cos 5t)^2}; \left[\begin{array}{l} u = \cos 5t \\ du = -5 \sin 5t \, dt \end{array} \right] \rightarrow -\frac{1}{5} \int \frac{du}{1+u^2} = -\frac{1}{5} \tan^{-1} u + C = -\frac{1}{5} \tan^{-1} (\cos 5t) + C$$

$$96. \int \frac{dv}{\sqrt{e^{2v}-1}}; \left[\begin{array}{l} x = e^v \\ dx = e^v \, dv \end{array} \right] \rightarrow \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C = \sec^{-1} (e^v) + C$$

$$97. \int \frac{dr}{1+\sqrt{r}}; \left[\begin{array}{l} u = \sqrt{r} \\ du = \frac{dr}{2\sqrt{r}} \end{array} \right] \rightarrow \int \frac{2u \, du}{1+u} = \int \left(2 - \frac{2}{1+u} \right) du = 2u - 2 \ln |1+u| + C = 2\sqrt{r} - 2 \ln (1+\sqrt{r}) + C$$

$$98. \int \frac{4x^3-20x}{x^4-10x^2+9} dx = \int \frac{(4x^3-20x)dx}{x^4-10x^2+9} = \ln |x^4-10x^2+9| + C$$

$$99. \int \frac{x^3}{1+x^2} dx = \int \left(x - \frac{x}{1+x^2} \right) dx = \int x dx - \frac{1}{2} \int \frac{2x}{1+x^2} dx = \frac{1}{2} x^2 - \frac{1}{2} \ln(1+x^2) + C$$

$$100. \int \frac{x^2}{1+x^3} dx = \frac{1}{3} \int \frac{3x^2}{1+x^3} dx = \frac{1}{3} \ln|1+x^3| + C$$

$$101. \int \frac{1+x^2}{1+x^3} dx; \quad \frac{1+x^2}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2} \Rightarrow 1+x^2 = A(1-x+x^2) + (Bx+C)(1+x) \\ = (A+B)x^2 + (-A+B+C)x + (A+C) \Rightarrow A+B=1, -A+B+C=0, A+C=1 \Rightarrow A=\frac{2}{3}, B=\frac{1}{3}, C=\frac{1}{3};$$

$$\int \frac{1+x^2}{1+x^3} dx = \int \left(\frac{2/3}{1+x} + \frac{(1/3)x+1/3}{1-x+x^2} \right) dx = \frac{2}{3} \int \frac{1}{1+x} dx + \frac{1}{3} \int \frac{x+1}{1-x+x^2} dx = \frac{2}{3} \int \frac{1}{1+x} dx + \frac{1}{3} \int \frac{x+1}{\frac{3}{4} + \left(x-\frac{1}{2}\right)^2} dx;$$

$$\left[\begin{array}{l} u = x - \frac{1}{2} \\ du = dx \end{array} \right] \rightarrow \frac{1}{3} \int \frac{u+\frac{3}{2}}{\frac{3}{4}+u^2} du = \frac{1}{3} \int \frac{u}{\frac{3}{4}+u^2} du + \frac{1}{2} \int \frac{1}{\frac{3}{4}+u^2} du = \frac{1}{6} \ln \left| \frac{3}{4} + u^2 \right| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{u}{\sqrt{3}/2} \right)$$

$$= \frac{1}{6} \ln \left| \frac{3}{4} + \left(x - \frac{1}{2}\right)^2 \right| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x-\frac{1}{2}}{\sqrt{3}/2} \right) = \frac{1}{6} \ln |1-x+x^2| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right)$$

$$\Rightarrow \frac{2}{3} \int \frac{1}{1+x} dx + \frac{1}{3} \int \frac{x+1}{1-x+x^2} dx = \frac{2}{3} \ln |1+x| + \frac{1}{6} \ln |1-x+x^2| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right) + C$$

$$102. \int \frac{1+x^2}{(1+x)^3} dx; \quad \left[\begin{array}{l} u = 1+x \\ du = dx \end{array} \right] \rightarrow \int \frac{1+(u-1)^2}{u^3} du = \int \frac{u^2-2u+2}{u^3} du = \int \frac{1}{u} du - \int \frac{2}{u^2} du + \int \frac{2}{u^3} du = \ln |u| + \frac{2}{u} - \frac{1}{u^2} + C \\ = \ln |1+x| + \frac{2}{1+x} - \frac{1}{(1+x)^2} + C$$

$$103. \int \sqrt{x} \sqrt{1+\sqrt{x}} dx; \quad \left[\begin{array}{l} w = \sqrt{x} \Rightarrow w^2 = x \\ 2w dw = dx \end{array} \right] \rightarrow \int 2w^2 \sqrt{1+w} dw \\ \begin{array}{l} \sqrt{1+w} \\ 2w^2 \xrightarrow{(+)} \frac{2}{3}(1+w)^{3/2} \\ 4w \xrightarrow{(-)} \frac{4}{15}(1+w)^{5/2} \\ 4 \xrightarrow{(+)} \frac{8}{105}(1+w)^{7/2} \\ 0 \end{array} \Rightarrow \int 2w^2 \sqrt{1+w} dw = \frac{4}{3} w^2(1+w)^{3/2} - \frac{16}{15} w(1+w)^{5/2} + \frac{32}{105} (1+w)^{7/2} + C \\ = \frac{4}{3} x(1+\sqrt{x})^{3/2} - \frac{16}{15} \sqrt{x}(1+\sqrt{x})^{5/2} + \frac{32}{105} (1+\sqrt{x})^{7/2} + C$$

$$104. \int \sqrt{1+\sqrt{1+x}} dx; \quad \left[\begin{array}{l} w = \sqrt{1+x} \Rightarrow w^2 = 1+x \\ 2w dw = dx \end{array} \right] \rightarrow \int 2w \sqrt{1+w} dw; \\ \left[u = 2w, du = 2dw, dv = \sqrt{1+w} dw, v = \frac{2}{3}(1+w)^{3/2} \right] \\ \int 2w \sqrt{1+w} dw = \frac{4}{3} w(1+w)^{3/2} - \int \frac{4}{3} (1+w)^{3/2} dw = \frac{4}{3} w(1+w)^{3/2} - \frac{8}{15} (1+w)^{5/2} + C \\ = \frac{4}{3} \sqrt{1+x} (1+\sqrt{1+x})^{3/2} - \frac{8}{15} (1+\sqrt{1+x})^{5/2} + C$$

$$105. \int \frac{1}{\sqrt{x}\sqrt{1+x}} dx; \left[\begin{array}{l} u = \sqrt{x} \Rightarrow u^2 = x \\ 2u du = dx \end{array} \right] \rightarrow \int \frac{2}{\sqrt{1+u^2}} du; \left[\begin{array}{l} u = \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, du = \sec^2 \theta d\theta, \sqrt{1+u^2} = \sec \theta \end{array} \right]$$

$$\int \frac{2}{\sqrt{1+u^2}} du = \int \frac{2 \sec^2 \theta}{\sec \theta} d\theta = \int 2 \sec \theta d\theta = 2 \ln |\sec \theta + \tan \theta| + C = 2 \ln \left| \sqrt{1+u^2} + u \right| + C = 2 \ln \left| \sqrt{1+x} + \sqrt{x} \right| + C$$

$$106. \int_0^{1/2} \sqrt{1+\sqrt{1-x^2}} dx;$$

$$\left[\begin{array}{l} x = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \cos \theta d\theta, \sqrt{1-x^2} = \cos \theta, x = 0 = \sin \theta \Rightarrow \theta = 0, x = \frac{1}{2} = \sin \theta \Rightarrow \theta = \frac{\pi}{6} \end{array} \right]$$

$$\rightarrow \int_0^{\pi/6} \sqrt{1+\cos \theta} \cos \theta d\theta = \int_0^{\pi/6} \frac{\sqrt{1-\cos^2 \theta}}{\sqrt{1-\cos \theta}} \cos \theta d\theta = \int_0^{\pi/6} \frac{\sin \theta \cos \theta}{\sqrt{1-\cos \theta}} d\theta = \lim_{c \rightarrow 0^+} \int_c^{\pi/6} \frac{\sin \theta \cos \theta}{\sqrt{1-\cos \theta}} d\theta;$$

$$\left[\begin{array}{l} u = \cos \theta, du = -\sin \theta d\theta, dv = \frac{\sin \theta}{\sqrt{1-\cos \theta}} d\theta, v = 2(1-\cos \theta)^{1/2} \end{array} \right]$$

$$= \lim_{c \rightarrow 0^+} \left[\left[2 \cos \theta (1-\cos \theta)^{1/2} \right]_c^{\pi/6} + \int_c^{\pi/6} 2(1-\cos \theta)^{1/2} \sin \theta d\theta \right]$$

$$= \lim_{c \rightarrow 0^+} \left[\left(2 \cos \left(\frac{\pi}{6} \right) \left(1 - \cos \left(\frac{\pi}{6} \right) \right)^{1/2} - 2 \cos c (1-\cos c)^{1/2} \right) + \left[\frac{4}{3} (1-\cos \theta)^{3/2} \right]_c^{\pi/6} \right]$$

$$= \lim_{c \rightarrow 0^+} \left[\sqrt{3} \left(1 - \frac{\sqrt{3}}{2} \right)^{1/2} - 2 \cos c (1-\cos c)^{1/2} + \left(\frac{4}{3} \left(1 - \cos \left(\frac{\pi}{6} \right) \right)^{3/2} - \frac{4}{3} (1-\cos c)^{3/2} \right) \right]$$

$$= \lim_{c \rightarrow 0^+} \left[\sqrt{3} \left(1 - \frac{\sqrt{3}}{2} \right)^{1/2} - 2 \cos c (1-\cos c)^{1/2} + \frac{4}{3} \left(1 - \frac{\sqrt{3}}{2} \right)^{3/2} - \frac{4}{3} (1-\cos c)^{3/2} \right]$$

$$= \sqrt{3} \left(1 - \frac{\sqrt{3}}{2} \right)^{1/2} + \frac{4}{3} \left(1 - \frac{\sqrt{3}}{2} \right)^{3/2} = \left(1 - \frac{\sqrt{3}}{2} \right)^{1/2} \left(\frac{4+\sqrt{3}}{3} \right) = \frac{(4+\sqrt{3})\sqrt{2-\sqrt{3}}}{3\sqrt{2}}$$

$$107. \int \frac{\ln x}{x+x \ln x} dx = \int \frac{\ln x}{x(1+\ln x)} dx; \left[\begin{array}{l} u = 1 + \ln x \\ du = \frac{1}{x} dx \end{array} \right] \rightarrow \int \frac{u-1}{u} du = \int du - \int \frac{1}{u} du = u - \ln |u| + C$$

$$= (1 + \ln x) - \ln |1 + \ln x| + C = \ln x - \ln |1 + \ln x| + C$$

$$108. \int \frac{1}{x \ln x \cdot \ln(\ln x)} dx; \left[\begin{array}{l} u = \ln(\ln x) \\ du = \frac{1}{x \ln x} dx \end{array} \right] \rightarrow \int \frac{1}{u} du = \ln |u| + C = \ln |\ln(\ln x)| + C$$

$$109. \int \frac{x^{\ln x} \ln x}{x} dx; \left[\begin{array}{l} u = x^{\ln x} \Rightarrow \ln u = \ln x^{\ln x} = (\ln x)^2 \Rightarrow \frac{1}{u} du = \frac{2 \ln x}{x} dx \Rightarrow du = \frac{2u \ln x}{x} dx = \frac{2x^{\ln x} \ln x}{x} dx \end{array} \right]$$

$$\rightarrow \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln x^{\ln x} + C$$

$$110. \int (\ln x)^{\ln x} \left[\frac{1}{x} + \frac{\ln(\ln x)}{x} \right] dx; \left[\begin{array}{l} u = (\ln x)^{\ln x} \Rightarrow \ln u = \ln(\ln x)^{\ln x} = (\ln x) \ln(\ln x) \Rightarrow \frac{1}{u} du = \left(\frac{\ln x}{x \ln x} + \frac{\ln(\ln x)}{x} \right) dx \end{array} \right]$$

$$\Rightarrow du = u \left[\frac{1}{x} + \frac{\ln(\ln x)}{x} \right] dx = (\ln x)^{\ln x} \left[\frac{1}{x} + \frac{\ln(\ln x)}{x} \right] dx \rightarrow \int du = u + C = (\ln x)^{\ln x} + C$$

$$111. \int \frac{1}{x\sqrt{1-x^4}} dx = \int \frac{x}{x^2\sqrt{1-x^4}} dx; \left[x^2 = \sin \theta, 0 \leq \theta < \frac{\pi}{2}, 2x dx = \cos \theta d\theta, \sqrt{1-x^4} = \cos \theta \right]$$

$$\rightarrow \frac{1}{2} \int \frac{\cos \theta}{\sin \theta \cos \theta} d\theta = \frac{1}{2} \int \csc \theta d\theta = -\frac{1}{2} \ln |\csc \theta + \cot \theta| + C = -\frac{1}{2} \ln \left| \frac{1}{x^2} + \frac{\sqrt{1-x^4}}{x^2} \right| + C = -\frac{1}{2} \ln \left| \frac{1+\sqrt{1-x^4}}{x^2} \right| + C$$

$$112. \int \frac{\sqrt{1-x}}{x} dx; \left[u = \sqrt{1-x} \Rightarrow u^2 = 1-x \Rightarrow 2u du = -dx \right] \rightarrow \int \frac{-2u^2}{1-u^2} du = \int \frac{2u^2}{u^2-1} du = \int \left(2 + \frac{2}{u^2-1} \right) du;$$

$$\frac{2}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} \Rightarrow 2 = A(u+1) + B(u-1) = (A+B)u + A-B \Rightarrow A+B=0, A-B=2 \Rightarrow A=1 \Rightarrow B=-1;$$

$$\int \left(2 + \frac{2}{u^2-1} \right) du = \int 2 du + \int \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du = 2u + \ln |u-1| - \ln |u+1| + C = 2\sqrt{1-x} + \frac{1}{2} \ln \left| \frac{\sqrt{1-x}-1}{\sqrt{1-x}+1} \right| + C$$

$$113. (a) \int_0^a f(a-x) dx; [u = a-x \Rightarrow du = -dx, x=0 \Rightarrow u=a, x=a \Rightarrow u=0] \rightarrow -\int_a^0 f(u) du = \int_0^a f(u) du, \text{ which}$$

is the same integral as $\int_0^a f(x) dx$.

$$(b) \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{\sin(\frac{\pi}{2}-x)}{\sin(\frac{\pi}{2}-x) + \cos(\frac{\pi}{2}-x)} dx = \int_0^{\pi/2} \frac{\sin(\frac{\pi}{2})\cos x - \cos(\frac{\pi}{2})\sin x}{\sin(\frac{\pi}{2})\cos x - \cos(\frac{\pi}{2})\sin x + \cos(\frac{\pi}{2})\cos x + \sin(\frac{\pi}{2})\sin x} dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \Rightarrow 2 \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow 2 \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{2} \Rightarrow \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$$

$$114. \int \frac{\sin x}{\sin x + \cos x} dx = \int \frac{\sin x + \cos x - \cos x + \sin x - \sin x}{\sin x + \cos x} dx = \int \frac{\sin x + \cos x}{\sin x + \cos x} dx + \int \frac{-\cos x + \sin x}{\sin x + \cos x} dx + \int \frac{-\sin x}{\sin x + \cos x} dx$$

$$= \int dx - \int \frac{\cos x - \sin x}{\sin x + \cos x} dx - \int \frac{\sin x}{\sin x + \cos x} dx = x - \ln |\sin x + \cos x| - \int \frac{\sin x}{\sin x + \cos x} dx$$

$$\Rightarrow 2 \int \frac{\sin x}{\sin x + \cos x} dx = x - \ln |\sin x + \cos x| \Rightarrow \int \frac{\sin x}{\sin x + \cos x} dx = \frac{x}{2} - \frac{1}{2} \ln |\sin x + \cos x| + C$$

$$115. \int \frac{\sin^2 x}{1 + \sin^2 x} dx = \int \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}} dx = \int \frac{\tan^2 x}{\sec^2 x + \tan^2 x} dx = \int \frac{\tan^2 x + \sec^2 x - \sec^2 x}{\sec^2 x + \tan^2 x} dx = \int \frac{\tan^2 x + \sec^2 x}{\sec^2 x + \tan^2 x} dx - \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x} dx$$

$$= \int dx - \int \frac{\sec^2 x}{1 + 2 \tan^2 x} dx = x - \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + C$$

$$116. \int \frac{1 - \cos x}{1 + \cos x} dx = \int \frac{(1 - \cos x)^2}{1 - \cos^2 x} dx = \int \frac{1 - 2 \cos x + \cos^2 x}{\sin^2 x} dx = \int \frac{1}{\sin^2 x} dx - \int \frac{2 \cos x}{\sin^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x} dx$$

$$= \int \csc^2 x dx - 2 \int \csc x \cot x dx + \int \cot^2 x dx = -\cot x + 2 \csc x + \int (\csc^2 x - 1) dx = -2 \cot x + 2 \csc x - x + C$$

CHAPTER 8 ADDITIONAL AND ADVANCED EXERCISES

$$1. u = (\sin^{-1} x)^2, du = \frac{2 \sin^{-1} x dx}{\sqrt{1-x^2}}; dv = dx, v = x;$$

$$\int (\sin^{-1} x)^2 dx = x (\sin^{-1} x)^2 - \int \frac{2x \sin^{-1} x dx}{\sqrt{1-x^2}}; u = \sin^{-1} x, du = \frac{dx}{\sqrt{1-x^2}}; dv = -\frac{2x dx}{\sqrt{1-x^2}}, v = 2\sqrt{1-x^2};$$

$$-\int \frac{2x \sin^{-1} x \, dx}{\sqrt{1-x^2}} = 2 \left(\sin^{-1} x \right) \sqrt{1-x^2} - \int 2 \, dx = 2 \left(\sin^{-1} x \right) \sqrt{1-x^2} - 2x + C; \text{ therefore}$$

$$\int \left(\sin^{-1} x \right)^2 \, dx = x \left(\sin^{-1} x \right)^2 + 2 \left(\sin^{-1} x \right) \sqrt{1-x^2} - 2x + C$$

$$2. \quad \frac{1}{x} = \frac{1}{x},$$

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1},$$

$$\frac{1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)},$$

$$\frac{1}{x(x+1)(x+2)(x+3)} = \frac{1}{6x} - \frac{1}{2(x+1)} + \frac{1}{2(x+2)} - \frac{1}{6(x+3)},$$

$$\frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{24x} - \frac{1}{6(x+1)} + \frac{1}{4(x+2)} - \frac{1}{6(x+3)} + \frac{1}{24(x+4)}$$

$$\Rightarrow \text{the following pattern: } \frac{1}{x(x+1)(x+2) \cdots (x+m)} = \sum_{k=0}^m \frac{(-1)^k}{(k!)(m-k)!(x+k)};$$

$$\text{therefore } \int \frac{dx}{x(x+1)(x+2) \cdots (x+m)} = \sum_{k=0}^m \left[\frac{(-1)^k}{(k!)(m-k)!} \ln |x+k| \right] + C$$

$$3. \quad u = \sin^{-1} x, \, du = \frac{dx}{\sqrt{1-x^2}}; \, dv = x \, dx, \, v = \frac{x^2}{2}; \quad \int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \int \frac{x^2 dx}{2\sqrt{1-x^2}};$$

$$\left[\begin{array}{l} x = \sin \theta \\ dx = \cos \theta \, d\theta \end{array} \right] \rightarrow \int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \int \frac{\sin^2 \theta \cos \theta \, d\theta}{2 \cos \theta} = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \sin^2 \theta \, d\theta$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) + C = \frac{x^2}{2} \sin^{-1} x + \frac{\sin \theta \cos \theta - \theta}{4} + C = \frac{x^2}{2} \sin^{-1} x + \frac{x\sqrt{1-x^2} - \sin^{-1} x}{4} + C$$

$$4. \quad \int \sin^{-1} \sqrt{y} \, dy; \quad \left[\begin{array}{l} z = \sqrt{y} \\ dz = \frac{dy}{2\sqrt{y}} \end{array} \right] \rightarrow \int 2z \sin^{-1} z \, dz; \text{ from Exercise 3, } \int z \sin^{-1} z \, dz = \frac{z^2 \sin^{-1} z}{2} + \frac{z\sqrt{1-z^2} - \sin^{-1} z}{4} + C$$

$$\Rightarrow \int \sin^{-1} \sqrt{y} \, dy = y \sin^{-1} \sqrt{y} + \frac{\sqrt{y}\sqrt{1-y} - \sin^{-1} \sqrt{y}}{2} + C = y \sin^{-1} \sqrt{y} + \frac{\sqrt{y-y^2}}{2} - \frac{\sin^{-1} \sqrt{y}}{2} + C$$

$$5. \quad \int \frac{dt}{t-\sqrt{1-t^2}}; \quad \left[\begin{array}{l} t = \sin \theta \\ dt = \cos \theta \, d\theta \end{array} \right] \rightarrow \int \frac{\cos \theta \, d\theta}{\sin \theta - \cos \theta} = \int \frac{d\theta}{\tan \theta - 1}; \quad \left[\begin{array}{l} u = \tan \theta \\ du = \sec^2 \theta \, d\theta \\ d\theta = \frac{du}{u^2+1} \end{array} \right] \rightarrow \int \frac{du}{(u-1)(u^2+1)}$$

$$= \frac{1}{2} \int \frac{du}{u-1} - \frac{1}{2} \int \frac{du}{u^2+1} - \frac{1}{2} \int \frac{u \, du}{u^2+1} = \frac{1}{2} \ln \left| \frac{u-1}{\sqrt{u^2+1}} \right| - \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \ln \left| \frac{\tan \theta - 1}{\sec \theta} \right| - \frac{1}{2} \theta + C$$

$$= \frac{1}{2} \ln \left(t - \sqrt{1-t^2} \right) - \frac{1}{2} \sin^{-1} t + C$$

$$6. \quad \int \frac{1}{x^4+4} \, dx = \int \frac{1}{(x^2+2)^2 - 4x^2} \, dx = \int \frac{1}{(x^2+2x+2)(x^2-2x+2)} \, dx = \frac{1}{16} \int \left[\frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1} - \frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1} \right] \, dx$$

$$= \frac{1}{16} \ln \left| \frac{x^2+2x+2}{x^2-2x+2} \right| + \frac{1}{8} \left[\tan^{-1}(x+1) + \tan^{-1}(x-1) \right] + C$$

$$7. \lim_{x \rightarrow \infty} \int_{-x}^x \sin t \, dt = \lim_{x \rightarrow \infty} [-\cos t]_{-x}^x = \lim_{x \rightarrow \infty} [-\cos x + \cos(-x)] = \lim_{x \rightarrow \infty} (-\cos x + \cos x) = \lim_{x \rightarrow \infty} 0 = 0$$

$$8. \lim_{x \rightarrow 0^+} \int_x^1 \frac{\cos t}{t^2} \, dt; \quad \lim_{t \rightarrow 0^+} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{\cos t}{t^2}\right)} = \lim_{t \rightarrow 0^+} \frac{1}{\cos t} = 1 \Rightarrow \lim_{x \rightarrow 0^+} \int_x^1 \frac{\cos t}{t^2} \, dt \text{ diverges since } \int_0^1 \frac{dt}{t^2} \text{ diverges; thus}$$

$\lim_{x \rightarrow 0^+} x \int_x^1 \frac{\cos t}{t^2} \, dt$ is an indeterminate $0 \cdot \infty$ form and we apply l'Hôpital's rule:

$$\lim_{x \rightarrow 0^+} x \int_x^1 \frac{\cos t}{t^2} \, dt = \lim_{x \rightarrow 0^+} \frac{-\int_1^x \frac{\cos t}{t^2} \, dt}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{-\left(\frac{\cos x}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \rightarrow 0^+} \cos x = 1$$

$$9. \lim_{n \rightarrow \infty} \sum_{k=1}^n \ln \sqrt{1 + \frac{k}{n}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \ln \left(1 + k \left(\frac{1}{n}\right)\right) \left(\frac{1}{n}\right) = \int_0^1 \ln(1+x) \, dx; \quad \left[\begin{array}{l} u = 1+x, \, du = dx \\ x=0 \Rightarrow u=1, \, x=1 \Rightarrow u=2 \end{array} \right]$$

$$\rightarrow \int_1^2 \ln u \, du = [u \ln u - u]_1^2 = (2 \ln 2 - 2) - (\ln 1 - 1) = 2 \ln 2 - 1 = \ln 4 - 1$$

$$10. \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2 - k^2}} = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \left(\frac{n}{\sqrt{n^2 - k^2}} \right) \left(\frac{1}{n} \right) = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \left(\frac{1}{\sqrt{1 - \left[k \left(\frac{1}{n}\right)\right]^2}} \right) \left(\frac{1}{n} \right) = \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \left[\sin^{-1} x \right]_0^1 = \frac{\pi}{2}$$

$$11. \frac{dy}{dx} = \sqrt{\cos 2x} \Rightarrow 1 + \left(\frac{dy}{dx} \right)^2 = 1 + \cos 2x = 2 \cos^2 x; \quad L = \int_0^{\pi/4} \sqrt{1 + \left(\sqrt{\cos 2t} \right)^2} \, dt = \sqrt{2} \int_0^{\pi/4} \sqrt{\cos^2 t} \, dt$$

$$= \sqrt{2} [\sin t]_0^{\pi/4} = 1$$

$$12. \frac{dy}{dx} = \frac{-2x}{1-x^2} \Rightarrow 1 + \left(\frac{dy}{dx} \right)^2 = \frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2} = \frac{1+2x^2+x^4}{(1-x^2)^2} = \left(\frac{1+x^2}{1-x^2} \right)^2; \quad L = \int_0^{1/2} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx = \int_0^{1/2} \left(\frac{1+x^2}{1-x^2} \right) \, dx$$

$$= \int_0^{1/2} \left(-1 + \frac{2}{1-x^2} \right) \, dx = \int_0^{1/2} \left(-1 + \frac{1}{1+x} + \frac{1}{1-x} \right) \, dx = \left[-x + \ln \left| \frac{1+x}{1-x} \right| \right]_0^{1/2} = \left(-\frac{1}{2} + \ln 3 \right) - (0 + \ln 1) = \ln 3 - \frac{1}{2}$$

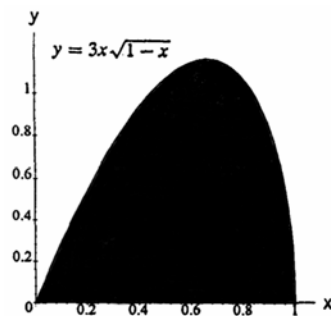
$$13. V = \int_a^b 2\pi \left(\begin{array}{c} \text{shell} \\ \text{radius} \end{array} \right) \left(\begin{array}{c} \text{shell} \\ \text{height} \end{array} \right) \, dx = \int_0^1 2\pi xy \, dx$$

$$= 6\pi \int_0^1 x^2 \sqrt{1-x} \, dx; \quad \left[u = 1-x, \, du = -dx, \, x^2 = (1-u)^2 \right]$$

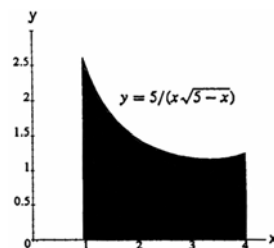
$$\rightarrow -6\pi \int_1^0 (1-u)^2 \sqrt{u} \, du = 6\pi \int_0^1 (u^{1/2} - 2u^{3/2} + u^{5/2}) \, du$$

$$= 6\pi \left[\frac{2}{3} u^{3/2} - \frac{4}{5} u^{5/2} + \frac{2}{7} u^{7/2} \right]_0^1$$

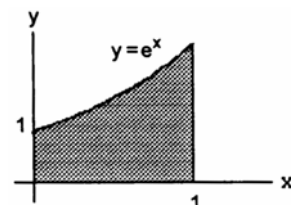
$$= 6\pi \left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7} \right) = 6\pi \left(\frac{70-84+30}{105} \right) = 6\pi \left(\frac{16}{105} \right) = \frac{32\pi}{35}$$



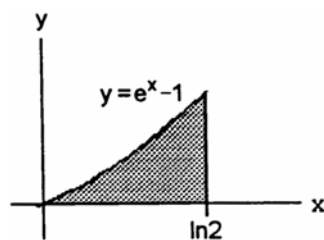
$$\begin{aligned}
 14. \quad V &= \int_a^b \pi y^2 dx = \pi \int_1^4 \frac{25 dx}{x^2(5-x)} = \pi \int_1^4 \left(\frac{1}{x} + \frac{5}{x^2} + \frac{1}{5-x} \right) dx \\
 &= \pi \left[\ln \left| \frac{x}{5-x} \right| - \frac{5}{x} \right]_1^4 = \pi \left(\ln 4 - \frac{5}{4} \right) - \pi \left(\ln \frac{1}{4} - 5 \right) \\
 &= \frac{15\pi}{4} + 2\pi \ln 4
 \end{aligned}$$



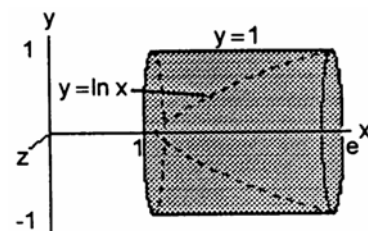
$$\begin{aligned}
 15. \quad V &= \int_a^b 2\pi \left(\begin{matrix} \text{shell} \\ \text{radius} \end{matrix} \right) \left(\begin{matrix} \text{shell} \\ \text{height} \end{matrix} \right) dx \\
 &= \int_0^1 2\pi x e^x dx = 2\pi \left[x e^x - e^x \right]_0^1 = 2\pi
 \end{aligned}$$



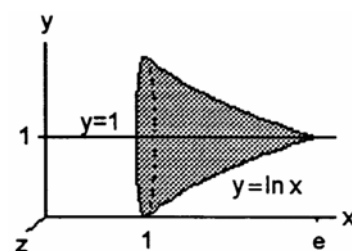
$$\begin{aligned}
 16. \quad V &= \int_0^{\ln 2} 2\pi (\ln 2 - x)(e^x - 1) dx \\
 &= 2\pi \int_0^{\ln 2} \left[(\ln 2)e^x - \ln 2 - x e^x + x \right] dx \\
 &= 2\pi \left[(\ln 2)e^x - (\ln 2)x - x e^x + e^x + \frac{x^2}{2} \right]_0^{\ln 2} \\
 &= 2\pi \left[2\ln 2 - (\ln 2)^2 - 2\ln 2 + 2 + \frac{(\ln 2)^2}{2} \right] - 2\pi (\ln 2 + 1) \\
 &= 2\pi \left[-\frac{(\ln 2)^2}{2} - \ln 2 + 1 \right]
 \end{aligned}$$



$$\begin{aligned}
 17. \quad (a) \quad V &= \int_1^e \pi \left[1 - (\ln x)^2 \right] dx \\
 &= \pi \left[x - x(\ln x)^2 \right]_1^e + 2\pi \int_1^e \ln x dx \\
 &\quad \text{(FORMULA 110)} \\
 &= \pi \left[x - x(\ln x)^2 + 2(x \ln x - x) \right]_1^e \\
 &= \pi \left[-x - x(\ln x)^2 + 2x \ln x \right]_1^e \\
 &= \pi [-e - e + 2e - (-1)] = \pi
 \end{aligned}$$



$$\begin{aligned}
 (b) \quad V &= \int_1^e \pi (1 - \ln x)^2 dx = \pi \int_1^e \left[1 - 2\ln x + (\ln x)^2 \right] dx \\
 &= \pi \left[x - 2(x \ln x - x) + x(\ln x)^2 \right]_1^e - 2\pi \int_1^e \ln x dx \\
 &= \pi \left[x - 2(x \ln x - x) + x(\ln x)^2 - 2(x \ln x - x) \right]_1^e \\
 &= \pi \left[5x - 4x \ln x + x(\ln x)^2 \right]_1^e \\
 &= \pi [(5e - 4e + e) - (5)] = \pi(2e - 5)
 \end{aligned}$$



$$18. (a) \quad V = \pi \int_0^1 \left[(e^y)^2 - 1 \right] dy = \pi \int_0^1 (e^{2y} - 1) dy = \pi \left[\frac{e^{2y}}{2} - y \right]_0^1 = \pi \left[\frac{e^2}{2} - 1 - \left(-\frac{1}{2} \right) \right] = \frac{\pi(e^2 - 3)}{2}$$

$$(b) \quad V = \pi \int_0^1 (e^y - 1)^2 dy = \pi \int_0^1 (e^{2y} - 2e^y + 1) dy = \pi \left[\frac{e^{2y}}{2} - 2e^y + y \right]_0^1 = \pi \left[\left(\frac{e^2}{2} - 2e + 1 \right) - \left(-\frac{1}{2} - 2 \right) \right]$$

$$= \pi \left(\frac{e^2}{2} - 2e + \frac{5}{2} \right) = \frac{\pi(e^2 - 4e + 5)}{2}$$

$$19. (a) \quad \lim_{x \rightarrow 0^+} x \ln x = 0 \Rightarrow \lim_{x \rightarrow 0^+} f(x) = 0 = f(0) \Rightarrow f \text{ is continuous}$$

$$(b) \quad V = \int_0^2 \pi x^2 (\ln x)^2 dx; \quad \left[u = (\ln x)^2, du = (2 \ln x) \frac{dx}{x}; dv = x^2 dx, v = \frac{x^3}{3} \right]$$

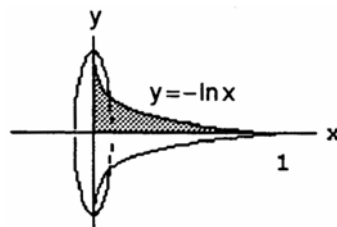
$$\rightarrow \pi \left(\lim_{b \rightarrow 0^+} \left[\frac{x^3}{3} (\ln x)^2 \right]_b^2 - \int_0^2 \left(\frac{x^3}{3} \right) (2 \ln x) \frac{dx}{x} \right) = \pi \left[\left(\frac{8}{3} \right) (\ln 2)^2 - \left(\frac{2}{3} \right) \lim_{b \rightarrow 0^+} \left[\frac{x^3}{3} \ln x - \frac{x^3}{9} \right]_b^2 \right]$$

$$= \pi \left[\frac{8(\ln 2)^2}{3} - \frac{16(\ln 2)}{9} + \frac{16}{27} \right]$$

$$20. \quad V = \int_0^1 \pi (-\ln x)^2 dx = \pi \int_0^1 (\ln x)^2 dx$$

$$= \pi \left(\lim_{b \rightarrow 0^+} \left[x (\ln x)^2 \right]_b^1 - 2 \int_0^1 \ln x dx \right)$$

$$= -2\pi \lim_{b \rightarrow 0^+} [x \ln x - x]_b^1 = 2\pi$$



$$21. \quad M = \int_1^e \ln x dx = [x \ln x - x]_1^e = (e - e) - (0 - 1) = 1;$$

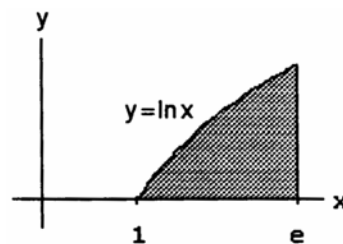
$$M_x = \int_1^e (\ln x) \left(\frac{\ln x}{2} \right) dx = \frac{1}{2} \int_1^e (\ln x)^2 dx$$

$$= \frac{1}{2} \left(\left[x (\ln x)^2 \right]_1^e - 2 \int_1^e \ln x dx \right) = \frac{1}{2} (e - 2);$$

$$M_y = \int_1^e x \ln x dx = \left[\frac{x^2 \ln x}{2} \right]_1^e - \frac{1}{2} \int_1^e x dx$$

$$= \frac{1}{2} \left[x^2 \ln x - \frac{x^2}{2} \right]_1^e = \frac{1}{2} \left[\left(e^2 - \frac{e^2}{2} \right) + \frac{1}{2} \right] = \frac{1}{4} (e^2 + 1);$$

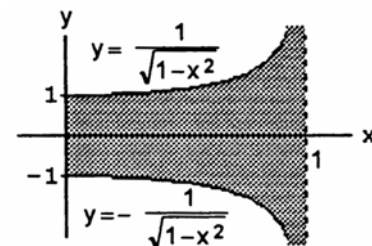
$$\text{therefore, } \bar{x} = \frac{M_y}{M} = \frac{e^2 + 1}{4} \text{ and } \bar{y} = \frac{M_x}{M} = \frac{e - 2}{2}$$



$$22. \quad M = \int_0^1 \frac{2x dx}{\sqrt{1-x^2}} = \left[2 \sin^{-1} x \right]_0^1 = \pi;$$

$$M_y = \int_0^1 \frac{2x dx}{\sqrt{1-x^2}} = 2 \left[-\sqrt{1-x^2} \right]_0^1 = 2; \text{ therefore,}$$

$$\bar{x} = \frac{M_y}{M} = \frac{2}{\pi} \text{ and } \bar{y} = 0 \text{ by symmetry}$$



$$\begin{aligned}
23. \quad L &= \int_1^e \sqrt{1 + \frac{1}{x^2}} dx = \int_1^e \frac{\sqrt{x^2 + 1}}{x} dx; \quad \left[\begin{array}{l} x = \tan \theta \\ dx = \sec^2 \theta d\theta \end{array} \right] \rightarrow L = \int_{\pi/4}^{\tan^{-1} e} \frac{\sec \theta \cdot \sec^2 \theta d\theta}{\tan \theta} = \int_{\pi/4}^{\tan^{-1} e} \frac{(\sec \theta)(\tan^2 \theta + 1)}{\tan \theta} d\theta \\
&= \int_{\pi/4}^{\tan^{-1} e} (\tan \theta \sec \theta + \csc \theta) d\theta = [\sec \theta - \ln |\csc \theta + \cot \theta|]_{\pi/4}^{\tan^{-1} e} \\
&= \left(\sqrt{1 + e^2} - \ln \left| \frac{\sqrt{1 + e^2}}{e} + \frac{1}{e} \right| \right) - \left[\sqrt{2} - \ln(1 + \sqrt{2}) \right] = \sqrt{1 + e^2} - \ln \left(\frac{\sqrt{1 + e^2}}{e} + \frac{1}{e} \right) - \sqrt{2} + \ln(1 + \sqrt{2})
\end{aligned}$$

$$\begin{aligned}
24. \quad y = \ln x \Rightarrow 1 + \left(\frac{dx}{dy} \right)^2 &= 1 + x^2 \Rightarrow S = 2\pi \int_c^d x \sqrt{1 + x^2} dy \Rightarrow S = 2\pi \int_0^1 e^y \sqrt{1 + e^{2y}} dy; \\
\left[\begin{array}{l} u = e^y \\ du = e^y dy \end{array} \right] &\rightarrow S = 2\pi \int_1^e \sqrt{1 + u^2} du; \quad \left[\begin{array}{l} u = \tan \theta \\ du = \sec^2 \theta d\theta \end{array} \right] \rightarrow 2\pi \int_{\pi/4}^{\tan^{-1} e} \sec \theta \cdot \sec^2 \theta d\theta \\
&= 2\pi \left(\frac{1}{2} \right) [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|]_{\pi/4}^{\tan^{-1} e} = \pi \left[\left(\sqrt{1 + e^2} \right) e + \ln \left| \sqrt{1 + e^2} + e \right| \right] - \pi \left[\sqrt{2} \cdot 1 + \ln(\sqrt{2} + 1) \right] \\
&= \pi \left[e \sqrt{1 + e^2} + \ln \left(\frac{\sqrt{1 + e^2} + e}{\sqrt{2} + 1} \right) - \sqrt{2} \right]
\end{aligned}$$

$$\begin{aligned}
25. \quad S &= 2\pi \int_{-1}^1 f(x) \sqrt{1 + [f'(x)]^2} dx; \quad f(x) = (1 - x^{2/3})^{3/2} \Rightarrow [f'(x)]^2 + 1 = \frac{1}{x^{2/3}} \Rightarrow S = 2\pi \int_{-1}^1 (1 - x^{2/3})^{3/2} \cdot \frac{dx}{\sqrt{x^{2/3}}} \\
&= 4\pi \int_0^1 (1 - x^{2/3})^{3/2} \left(\frac{1}{x^{1/3}} \right) dx; \quad \left[\begin{array}{l} u = x^{2/3} \\ du = \frac{2}{3} \frac{dx}{x^{1/3}} \end{array} \right] \rightarrow 4 \cdot \frac{3}{2} \pi \int_0^1 (1 - u)^{3/2} du = -6\pi \int_0^1 (1 - u)^{3/2} (-1) du \\
&= -6\pi \cdot \frac{2}{5} \left[(1 - u)^{5/2} \right]_0^1 = \frac{12\pi}{5}
\end{aligned}$$

$$\begin{aligned}
26. \quad y &= \int_1^x \sqrt{\sqrt{t} - 1} dt \Rightarrow \frac{dy}{dx} = \sqrt{\sqrt{x} - 1} \Rightarrow L = \int_1^{16} \sqrt{1 + (\sqrt{\sqrt{x} - 1})^2} dx = \int_1^{16} \sqrt{1 + \sqrt{x} - 1} dx = \int_1^{16} \sqrt{x} dx = \int_1^{16} \sqrt[4]{x} dx = \left[\frac{4}{5} x^{5/4} \right]_1^{16} \\
&= \frac{4}{5} (16)^{5/4} - \frac{4}{5} (1)^{5/4} = \frac{124}{5}
\end{aligned}$$

$$\begin{aligned}
27. \quad \int_1^\infty \left(\frac{ax}{x^2 + 1} - \frac{1}{2x} \right) dx &= \lim_{b \rightarrow \infty} \int_1^b \left(\frac{ax}{x^2 + 1} - \frac{1}{2x} \right) dx = \lim_{b \rightarrow \infty} \left[\frac{a}{2} \ln(x^2 + 1) - \frac{1}{2} \ln x \right]_1^b = \lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln \frac{(x^2 + 1)^a}{x} \right]_1^b \\
&= \lim_{b \rightarrow \infty} \frac{1}{2} \left[\ln \frac{(b^2 + 1)^a}{b} - \ln 2^a \right]; \quad \lim_{b \rightarrow \infty} \frac{(b^2 + 1)^a}{b} > \lim_{b \rightarrow \infty} \frac{b^{2a}}{b} = \lim_{b \rightarrow \infty} b^{2(a - \frac{1}{2})} = \infty \text{ if } a > \frac{1}{2} \Rightarrow \text{the improper integral} \\
&\text{diverges if } a > \frac{1}{2}; \text{ for } a = \frac{1}{2}: \lim_{b \rightarrow \infty} \frac{\sqrt{b^2 + 1}}{b} = \lim_{b \rightarrow \infty} \sqrt{1 + \frac{1}{b^2}} = 1 \Rightarrow \lim_{b \rightarrow \infty} \frac{1}{2} \left[\ln \frac{(b^2 + 1)^{1/2}}{b} - \ln 2^{1/2} \right] \\
&= \frac{1}{2} \left(\ln 1 - \frac{1}{2} \ln 2 \right) = -\frac{\ln 2}{4}; \text{ if } a < \frac{1}{2}: 0 \leq \lim_{b \rightarrow \infty} \frac{(b^2 + 1)^a}{b} < \lim_{b \rightarrow \infty} \frac{(b + 1)^{2a}}{b + 1} = \lim_{b \rightarrow \infty} (b + 1)^{2a - 1} = 0 \\
&\Rightarrow \lim_{b \rightarrow \infty} \ln \frac{(b^2 + 1)^a}{b} = -\infty \Rightarrow \text{the improper integral diverges if } a < \frac{1}{2}; \text{ in summary, the improper integral} \\
&\int_1^\infty \left(\frac{ax}{x^2 + 1} - \frac{1}{2x} \right) dx \text{ converges only when } a = \frac{1}{2} \text{ and has the value } -\frac{\ln 2}{4}
\end{aligned}$$

28. $G(x) = \lim_{b \rightarrow \infty} \int_0^b e^{-xt} dt = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} e^{-xt} \right]_0^b = \lim_{b \rightarrow \infty} \left(\frac{1 - e^{-xb}}{x} \right) = \frac{1-0}{x} = \frac{1}{x}$ if $x > 0 \Rightarrow xG(x) = x\left(\frac{1}{x}\right) = 1$ if $x > 0$
29. $A = \int_1^\infty \frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \leq 1$. Thus, $p \leq 1$ for infinite area. The volume of the solid of revolution about the x -axis is $V = \int_1^\infty \pi \left(\frac{1}{x^p} \right)^2 dx = \pi \int_1^\infty \frac{dx}{x^{2p}}$ which converges if $2p > 1$ and diverges if $2p \leq 1$. Thus we want $p > \frac{1}{2}$ for finite volume. In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $\frac{1}{2} < p \leq 1$.
30. The area is given by the integral $A = \int_0^1 \frac{dx}{x^p}$;
- $p = 1$: $A = \lim_{b \rightarrow 0^+} [\ln x]_b^1 = -\lim_{b \rightarrow 0^+} \ln b = \infty$, diverges;
- $p > 1$: $A = \lim_{b \rightarrow 0^+} \left[x^{1-p} \right]_b^1 = 1 - \lim_{b \rightarrow 0^+} b^{1-p} = -\infty$, diverges;
- $p < 1$: $A = \lim_{b \rightarrow 0^+} \left[x^{1-p} \right]_b^1 = 1 - \lim_{b \rightarrow 0^+} b^{1-p} = 1 - 0$, converges; thus, $p \geq 1$ for infinite area.
- The volume of the solid of revolution about the x -axis is $V_x = \pi \int_0^1 \frac{dx}{x^{2p}}$ which converges if $2p < 1$ or $p < \frac{1}{2}$, and diverges if $p \geq \frac{1}{2}$. Thus, V_x is infinite whenever the area is infinite ($p \geq 1$). The volume of the solid of revolution about the y -axis is $V_y = \pi \int_1^\infty [R(y)]^2 dy = \pi \int_1^\infty \frac{dy}{y^{2/p}}$ which converges if $\frac{2}{p} > 1 \Leftrightarrow p < 2$ (see Exercise 29). In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $1 \leq p < 2$, as described above.

31. See the generalization proved in 32.

$$\begin{aligned}
 32. \quad 0 &\leq \int_0^a \left(f'(x) + x - \frac{a}{2} \right) dx \\
 &= \int_0^a (f'(x))^2 dx + \int_0^a (2x - a)f'(x) dx + \int_0^a \left(x - \frac{a}{2} \right)^2 dx
 \end{aligned}$$

The last integral is $\frac{1}{3} \left(x - \frac{a}{2} \right)^3 \Big|_0^a = \frac{a^3}{12}$.

Using integration by parts with $u = 2x - a$, $du = 2dx$, $dv = f'(x)$, $v = f(x)$, and the fact that $f(a) = f(0) = b$, the second integral is $(2x - a)f(x) \Big|_0^a - 2 \int_0^a f(x) dx = 2ab - 2 \int_0^a f(x) dx$. Thus

$$\int_0^a (f'(x))^2 dx \geq 2 \int_0^a f(x) dx - \left(2ab + \frac{a^3}{12} \right).$$

$$\begin{array}{ll}
 33. & e^{2x} \xrightarrow{(+)} \cos 3x \\
 & 2e^{2x} \xrightarrow{(-)} \frac{1}{3} \sin 3x \\
 & 4e^{2x} \xrightarrow{(+)} -\frac{1}{9} \cos 3x
 \end{array}$$

$$I = \frac{e^{2x}}{3} \sin 3x + \frac{2e^{2x}}{9} \cos 3x - \frac{4}{9} I \Rightarrow \frac{13}{9} I = \frac{e^{2x}}{9} (3 \sin 3x + 2 \cos 3x) \Rightarrow I = \frac{e^{2x}}{13} (3 \sin 3x + 2 \cos 3x) + C$$

$$\begin{array}{ll}
 34. & e^{3x} \xrightarrow{(+)} \sin 4x \\
 & 3e^{3x} \xrightarrow{(-)} -\frac{1}{4} \cos 4x \\
 & 9e^{3x} \xrightarrow{(+)} -\frac{1}{16} \sin 4x
 \end{array}$$

$$I = -\frac{e^{3x}}{4} \cos 4x + \frac{3e^{3x}}{16} \sin 4x - \frac{9}{16} I \Rightarrow \frac{25}{16} I = \frac{e^{3x}}{16} (3 \sin 4x - 4 \cos 4x) \Rightarrow I = \frac{e^{3x}}{25} (3 \sin 4x - 4 \cos 4x) + C$$

$$\begin{array}{ll}
 35. & \sin 3x \xrightarrow{(+)} \sin x \\
 & 3 \cos 3x \xrightarrow{(-)} -\cos x \\
 & -9 \sin 3x \xrightarrow{(+)} -\sin x
 \end{array}$$

$$I = -\sin 3x \cos x + 3 \cos 3x \sin x + 9I \Rightarrow -8I = -\sin 3x \cos x + 3 \cos 3x \sin x \Rightarrow I = \frac{\sin 3x \cos x - 3 \cos 3x \sin x}{8} + C$$

$$\begin{array}{ll}
 36. & \cos 5x \xrightarrow{(+)} \sin 4x \\
 & -5 \sin 5x \xrightarrow{(-)} -\frac{1}{4} \cos 4x \\
 & -25 \cos 5x \xrightarrow{(+)} -\frac{1}{16} \sin 4x
 \end{array}$$

$$\begin{aligned}
 I &= -\frac{1}{4} \cos 5x \cos 4x - \frac{5}{16} \sin 5x \sin 4x + \frac{25}{16} I \Rightarrow -\frac{9}{16} I = -\frac{1}{4} \cos 5x \cos 4x - \frac{5}{16} \sin 5x \sin 4x \\
 &\Rightarrow I = \frac{1}{9} (4 \cos 5x \cos 4x + 5 \sin 5x \sin 4x) + C
 \end{aligned}$$

$$\begin{array}{ll}
 37. & e^{ax} \xrightarrow{(+)} \sin bx \\
 & ae^{ax} \xrightarrow{(-)} -\frac{1}{b} \cos bx \\
 & a^2 e^{ax} \xrightarrow{(+)} -\frac{1}{b^2} \sin bx
 \end{array}$$

$$I = -\frac{e^{ax}}{b} \cos bx + \frac{ae^{ax}}{b^2} \sin bx - \frac{a^2}{b^2} I \Rightarrow \left(\frac{a^2 + b^2}{b^2} \right) I = \frac{e^{ax}}{b^2} (a \sin bx - b \cos bx) \Rightarrow I = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$$

$$\begin{array}{ll}
 38. & e^{ax} \xrightarrow{(+)} \cos bx \\
 & ae^{ax} \xrightarrow{(-)} \frac{1}{b} \sin bx \\
 & a^2 e^{ax} \xrightarrow{(+)} -\frac{1}{b^2} \cos bx
 \end{array}$$

$$I = \frac{e^{ax}}{b} \sin bx + \frac{ae^{ax}}{b^2} \cos bx - \frac{a^2}{b^2} I \Rightarrow \left(\frac{a^2 + b^2}{b^2} \right) I = \frac{e^{ax}}{b^2} (a \cos bx + b \sin bx) \Rightarrow I = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C$$

39. $\ln(ax) \xrightarrow{(+)} 1$
 $\frac{1}{x} \xrightarrow{(-)} x$

$$I = x \ln(ax) - \int \left(\frac{1}{x} \right) x \, dx = x \ln(ax) - x + C$$

40. $\ln(ax) \xrightarrow{(+)} x^2$
 $\frac{1}{x} \xrightarrow{(-)} \frac{1}{3} x^3$

$$I = \frac{1}{3} x^3 \ln(ax) - \int \left(\frac{1}{x} \right) \left(\frac{x^3}{3} \right) dx = \frac{1}{3} x^3 \ln(ax) - \frac{1}{9} x^3 + C$$

41. $\int \frac{dx}{1 - \sin x} = \int \frac{\left(\frac{2dz}{1+z^2} \right)}{1 - \left(\frac{2z}{1+z^2} \right)} = \int \frac{2 \, dz}{(1-z)^2} = \frac{2}{1-z} + C = \frac{2}{1 - \tan\left(\frac{x}{2}\right)} + C$

42. $\int \frac{dx}{1 + \sin x + \cos x} = \int \frac{\left(\frac{2dz}{1+z^2} \right)}{1 + \left(\frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2} \right)} = \int \frac{2 \, dz}{1+z^2+2z+1-z^2} = \int \frac{dz}{1+z} = \ln|1+z| + C = \ln \left| \tan\left(\frac{x}{2}\right) + 1 \right| + C$

43. $\int_0^{\pi/2} \frac{dx}{1 + \sin x} = \int_0^1 \frac{\left(\frac{2dz}{1+z^2} \right)}{1 + \left(\frac{2z}{1+z^2} \right)} = \int_0^1 \frac{2 \, dz}{(1+z)^2} = -\left[\frac{2}{1+z} \right]_0^1 = -(1-2) = 1$

44. $\int_{\pi/3}^{\pi/2} \frac{dx}{1 - \cos x} = \int_{1/\sqrt{3}}^1 \frac{\left(\frac{2dz}{1+z^2} \right)}{1 - \left(\frac{1-z^2}{1+z^2} \right)} = \int_{1/\sqrt{3}}^1 \frac{dz}{z^2} = \left[-\frac{1}{z} \right]_{1/\sqrt{3}}^1 = \sqrt{3} - 1$

45. $\int_0^{\pi/2} \frac{d\theta}{2 + \cos \theta} = \int_0^1 \frac{\left(\frac{2 \, dz}{1+z^2} \right)}{2 + \left(\frac{1-z^2}{1+z^2} \right)} = \int_0^1 \frac{2 \, dz}{2+2z^2+1-z^2} = \int_0^1 \frac{2 \, dz}{z^2+3} = \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{z}{\sqrt{3}} \right]_0^1 = \frac{2}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}} = \frac{\sqrt{3}\pi}{9}$

46. $\int_{\pi/2}^{2\pi/3} \frac{\cos \theta \, d\theta}{\sin \theta \cos \theta + \sin \theta} = \int_1^{\sqrt{3}} \frac{\left(\frac{1-z^2}{1+z^2} \right) \left(\frac{2 \, dz}{1+z^2} \right)}{\left[\frac{2z(1-z^2)}{(1+z^2)^2} + \left(\frac{2z}{1+z^2} \right) \right]} = \int_1^{\sqrt{3}} \frac{2(1-z^2) \, dz}{2z-2z^3+2z+2z^3} = \int_1^{\sqrt{3}} \frac{1-z^2}{2z} \, dz = \left[\frac{1}{2} \ln z - \frac{z^2}{4} \right]_1^{\sqrt{3}}$
 $= \left(\frac{1}{2} \ln \sqrt{3} - \frac{3}{4} \right) - \left(0 - \frac{1}{4} \right) = \frac{\ln 3}{4} - \frac{1}{2} = \frac{1}{4} (\ln 3 - 2) = \frac{1}{2} (\ln \sqrt{3} - 1)$

47. $\int \frac{dt}{\sin t - \cos t} = \int \frac{\left(\frac{2dz}{1+z^2} \right)}{\left(\frac{2z}{1+z^2} - \frac{1-z^2}{1+z^2} \right)} = \int \frac{2 \, dz}{2z-1+z^2} = \int \frac{2 \, dz}{(z+1)^2 - 2} = \frac{1}{\sqrt{2}} \ln \left| \frac{z+1-\sqrt{2}}{z+1+\sqrt{2}} \right| + C = \frac{1}{\sqrt{2}} \ln \left| \frac{\tan\left(\frac{t}{2}\right)+1-\sqrt{2}}{\tan\left(\frac{t}{2}\right)+1+\sqrt{2}} \right| + C$

48. $\int \frac{\cos t \, dt}{1 - \cos t} = \int \frac{\left(\frac{1-z^2}{1+z^2} \right) \left(\frac{2 \, dz}{1+z^2} \right)}{1 - \left(\frac{1-z^2}{1+z^2} \right)} = \int \frac{2(1-z^2) \, dz}{(1+z^2)^2 - (1+z^2)(1-z^2)} = \int \frac{2(1-z^2) \, dz}{(1+z^2)(1+z^2-1+z^2)} = \int \frac{(1-z^2) \, dz}{(1+z^2)z^2} = \int \frac{dz}{z^2(1+z^2)} - \int \frac{dz}{1+z^2}$
 $= \int \frac{dz}{z^2} - 2 \int \frac{dz}{z^2+1} = -\frac{1}{z} - 2 \tan^{-1} z + C = -\cot\left(\frac{t}{2}\right) - t + C$

$$49. \int \sec \theta \, d\theta = \int \frac{d\theta}{\cos \theta} = \int \frac{\left(\frac{2 \, dz}{1+z^2}\right)}{\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2 \, dz}{1-z^2} = \int \frac{2 \, dz}{(1+z)(1-z)} = \int \frac{dz}{1+z} + \int \frac{dz}{1-z} = \ln |1+z| - \ln |1-z| + C = \ln \left| \frac{1+\tan(\frac{\theta}{2})}{1-\tan(\frac{\theta}{2})} \right| + C$$

$$50. \int \csc \theta \, d\theta = \int \frac{d\theta}{\sin \theta} = \int \frac{\left(\frac{2 \, dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}\right)} = \int \frac{dz}{z} = \ln |z| + C = \ln \left| \tan \frac{\theta}{2} \right| + C$$

$$51. (a) \Gamma(1) = \int_0^\infty e^{-t} \, dt = \lim_{b \rightarrow \infty} \int_0^b e^{-t} \, dt = \lim_{b \rightarrow \infty} \left[-e^{-t} \right]_0^b = \lim_{b \rightarrow \infty} \left[-\frac{1}{e^b} - (-1) \right] = 0 + 1 = 1$$

$$(b) \, u = t^x, \, du = x t^{x-1} \, dt; \, dv = e^{-t} \, dt, \, v = -e^{-t}; \, x = \text{fixed positive real}$$

$$\Rightarrow \Gamma(x+1) = \int_0^\infty t^x e^{-t} \, dt = \lim_{b \rightarrow \infty} \left[-t^x e^{-t} \right]_0^b + x \int_0^\infty t^{x-1} e^{-t} \, dt = \lim_{b \rightarrow \infty} \left(-\frac{b^x}{e^b} + 0^x e^0 \right) + x \Gamma(x) = x \Gamma(x)$$

$$(c) \Gamma(n+1) = n \Gamma(n) = n! :$$

$$n = 0 : \Gamma(0+1) = \Gamma(1) = 0!;$$

$$n = k : \text{Assume } \Gamma(k+1) = k! \quad \text{for some } k > 0;$$

$$n = k+1 : \Gamma(k+1+1) = (k+1) \Gamma(k+1) \quad \text{from part (b)}$$

$$= (k+1)k! \quad \text{induction hypothesis}$$

$$= (k+1)! \quad \text{definition of factorial}$$

Thus, $\Gamma(n+1) = n \Gamma(n) = n!$ for every positive integer n .

$$52. (a) \Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \text{ and } n \Gamma(n) = n! \Rightarrow n! \approx n \left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{n}} = \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$$

(b)

n	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$	calculator
10	3598695.619	3628800
20	2.4227868×10^{18}	2.432902×10^{18}
30	2.6451710×10^{32}	2.652528×10^{32}
40	8.1421726×10^{47}	8.1591528×10^{47}
50	3.0363446×10^{64}	3.0414093×10^{64}
60	8.3094383×10^{81}	8.3209871×10^{81}

(c)

n	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi} e^{1/12n}$	calculator
10	3598695.619	3628810.051	3628800