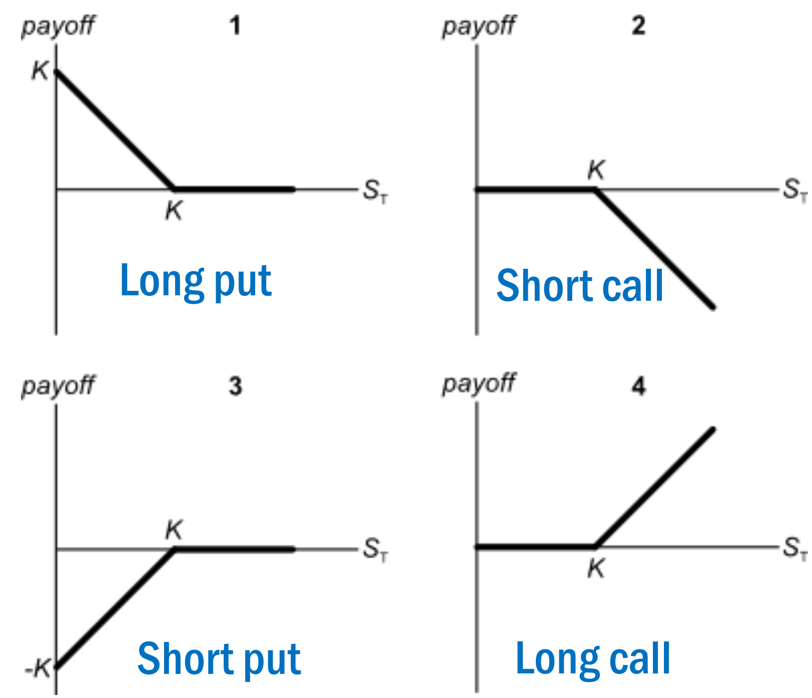
FIN2010 Financial Management
Lecture 12: Derivative Pricing

Review—Derivatives

- Forward and future: contracts that lock in the future price
  - Forward: no capital commitment, flexible, illiquid, high counterparty risk
  - Future: exchange-traded, standardized, liquid, low counterparty risk
    - Buyers and sellers deposit margin (money deposited at the exchange as a guarantee of payment)
- Option: the right (not obligation) to execute a trade at a fixed price in the future
  - Call: the right to buy
  - Put: the right to sell
- Swap: a contract that exchange two cash flows
- CDS: an insurance contract of bonds



Agenda

- Pricing by no arbitrage
- Forward and future pricing
  - Forward price
  - Carry
  - Future price
- Option pricing
  - Put-call parity
  - Binomial tree model
  - Black-Scholes-Merton model

Agenda

- Pricing by no arbitrage
- Forward and future pricing
  - Forward price
  - Carry
  - Future price
- Option pricing
  - Put-call parity
  - Binomial tree model
  - Black-Scholes-Merton model

Law of One Price (LOOP)

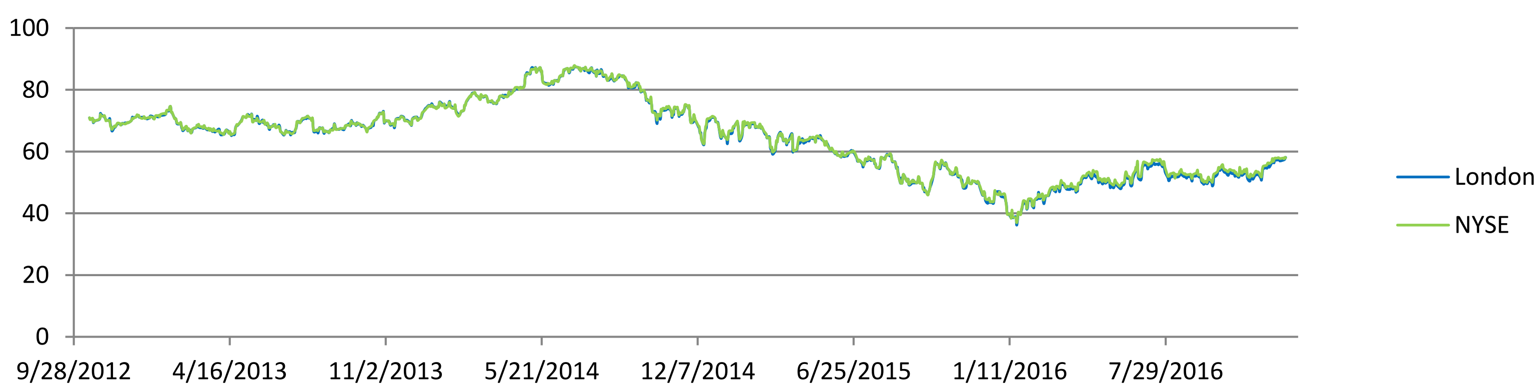
- The same good must have the same price at all places
  - May not always hold for consumers' goods

China: ￥98 (JD.com)

US: $3.99

Why?

  - But works very well for financial assets (Shell, RDS-B and RDSB-L)



LOOP for Derivatives

- If two strategies have the same payoff under every possible future state, they should have the same initial price
  - Asset A: pays $1 if the close price of S&P 500 is from 2500 to 2600
  - Asset B: pays $1 if the close price of S&P 500 is from 2500 to 2550
  - Asset C: pays $1 if the close price of S&P 500 is from 2550 to 2600
  - We should expect $P_A = P_B + P_C$

Arbitrage

An arbitrage opportunity exists if one of the following strategy exists:

1. (1) Require no initial funds: $CF_0 \ = 0$
   (2) Never lose money in any future state: $CF_T \ \geq 0$
   (3) Have strictly positive payoff in some states with positive probability: $\exists state\ S\ \ s.t. P(S) > 0, and\ CF_T^S > 0$
   No initial cost for future positive payoff
2. (1) Getting paid today: $CF_0 \ > 0$
   (2) Never lose money in any future state: $CF_T \ \geq 0$
   No future liability for initial positive payoff

– In essence, arbitrage is getting something for nothing

Example—Different Location

1. In Shell is traded at $70 in NYSE and $72 in London

Action	Cash flow at t=0
Sell in London	+72
Buy in NYSE	-70

What would the arbitrage do to the price in the two markets?

Example—Bond Replication

2. Three bonds:
   - A treasury note has 1 year remaining till maturity and 2% coupon rate. Current priced at 100.
   - 6-month zero coupon T-bill is priced at 99.2
   - 1-year zero coupon T-bill is priced at 98.2.

CF from $100 par value

Treasury note:	-100	+1	+101

CF from

$1 par value 6-month bill +:	-99.2%*1	+1	
$101 par value 1-year bill:	-98.2%*101		+101
	Now	6 month later	One year later

Future cash flows are the same from the two cash flow streams. But the current costs are not equal: 100 < 99.2%*1 + 98.2%*101 = 100.174.

There is an arbitrage opportunity!

How should I Create an Arbitrage?

- Find two strategies that give me the exact same payoff
- Compare the costs of the two
- Long the cheaper one, short the more expensive one

Action	CF now	CF at month 6	CF at year 1
Buy 100 Note	-100	+1	+101
Sell 1 6m bill	+0.992	-1	
Sell 101 1yr bill	+99.182		-101
Total	+0.174	0	0

- An arbitrage profit of 0.174 today.
- Such arbitrage activities will eventually drive away the price differences.

Agenda

- Pricing by no arbitrage
- Forward and future pricing
  - Forward price
  - Carry
  - Future price
- Option pricing
  - Put-call parity
  - Binomial tree model
  - Black-Scholes-Merton model

Pricing Forward Contracts

Intuition: if two strategy generate the same outcomes, they should have the same prices.

Consider a buyer that wishes to have 1 barrel of oil in one year:

- The buyer can buy 1 barrel for $52 now, and hold it for a year
- Or the buyer can sign a 1-year forward contract on 1 barrel of crude oil with forward price F
  (1) Instead of spending the $52 now, the buyer can put it in bank for year
  (2) After a year, the buyer has $52(1+ r_f), and use the proceeds to buy the oil

- The buyer should be indifferent between these choices. Therefore, we should expect $F_t = S_0(1 + r_f)^t$
- Question: If you long a forward contract, you agree to buy the underlying at expiration with price F_t. Do you profit from this forward when $S_t > F_t$ or $S_t < F_t$?

What if Forward Price is Different?

- Suppose Apple's stock is traded at $160 per share now. A 1-year forward price is $162. Suppose the 1 year risk free rate is 1%. How do we devise an arbitrage?
  - 160 *(1+1%) = 161.6
  - $F_t > S_0(1+r_f)^t$: there is an arbitrage opportunity
  - How to arbitrage? Again: buy the cheap one and sell the expensive one.

Action	Cash flow at t=0	Cash flow at t=1yr
Buy Apple	- 160	0
Sell forward	0	+ 162
Borrow cash	+ 160	- 160*(1+1%)
Total	0	+ 0.4

Carry

- When replicating the forward contract, one needs to buy/short the underlying now. What if the underlying asset generates payoff in the interim?
  - Crude oil: storage cost (negative payoff)
  - Stock: may pay dividends
  - Bonds: coupon payments

- These interim payoffs are called **carry**
- Forward pricing with carry:
  - $F_t = S_0(1 + r_f)^t - FV_t(Carry)$

Example - Stock Forward with Dividends

- Suppose Apple is trade at $160 per share now. Apple plans to pay $1 dividend a year later, right before the futures contract expires. A 1-year forward price for Apple is $162. Suppose the 1 year interest rate is 1%. How should be the forward price?

- With dividends
  - $F_t = S_0(1 + r_f)^t - FV_t(Dividend)$
  - $F_t = 160\ (1 + 1\%)^1 - 1 = 160.6$

Foreign Exchange (Forex) - Notations

- USD/JPY=113 means: 1 USD = 113 JPY
  - AAA/BBB = K means 1 unit of AAA currency is equal to K units of BBB currency
  - Long AAA/BBB pair means using K units of BBB to buy 1 unit of AAA
- When you long USD/JPY forward, you buy USD using JPY at expiration
  - You profit if USD/JPY goes up (USD appreciates), because with the forward contract, you have locked in a lower rate.
  - If you have positive cash flow in JPY at date t, you can hedge by longing USD/JPY

Forex Forward Rate

- One year interest rate is 1.5% in the US, and 0.1% in Japan. Currently the exchange rate is 1USD=113JPY. What should the one year forward rate of USD/JPY be?
- **Solutions:**
- If an investor convert one JPY to USD now, how much will he have in a year?

$$\frac{1}{S_0} * (1 + r_{USD})^t$$

- If an investor hold the JPY for a year and convert a year later, how much will he have in a year?

$$1 * \left(1 + r_{JPY}\right)^t * \frac{1}{F_t}$$

- According to the LOOF, two payoffs should be the same

$$F_t = S_0 \left(\frac{1 + r_{JPY}}{1 + r_{USD}}\right)^t$$

t =1, so $F_t = S_0 \left(\frac{1+r_{JPY}}{1+r_{USD}}\right)$ = 111.44

- Covered interest rate parity: $F_t = S_0 \left(\frac{1+r_{BBB}}{1+r_{AAA}}\right)^t$, where S_0 and F_t are expressed in AAA/BBB

Future Price

- Future pricing is exactly the same as forward pricing
  - Only difference is that the gains/losses of forward contracts are only realized at expiration, while those of future contracts are realized daily (marked-to-market)
- Suppose I long 1 contract (1,00 barrel) of 9-30-2020 crude oil future in 3-1-2020. Oil will be delivered on 9-30-2020. The interest rate is 2% per year. Storage cost for oil is $0.1 per year per barrel. For the sake of simplicity, suppose storage cost is paid at the beginning. Margin requirement is $2250.

Date	Today	Next Month	At Expiration
Spot Price	63	64	65
Theoretical Future Price	$= (63 + 0.1 * 7/12)(1.02)^{7/12}$ $= 63.79$	$= (64 + 0.1 * 6/12)(1.02)^{6/12}$ $= 64.69$	$65
Initial Margin	$2250	$2250	$3150
Gain/Loss		$0.9*1000	+$0.31*1000
Ending Balance of Margin		$3150	$3460

The total cost for acquiring the 1,00 barrel of oil = $65*1000 - $3460 + 2250= $63,790. That is $63.79/barrel.

Agenda

- Pricing by no arbitrage
- Forward and future pricing
  - Forward price
  - Carry
  - Future price
- Option pricing
  - Put-call parity
  - Binomial tree model
  - Black-Scholes-Merton model

Put and Call Options

- Two types of options
  - European options: the option can be exercised only at the expiration date of the option. Our focus!
  - American options: the option can be exercised at any time before the expiration date.
- Recall the payoff of call and put options at expiration date:

	Payoff			
	Long Call	Long Put	Short Call	Short Put
$S_T \leq K$	0	$K-S_T$	0	$S_T - K$
$S_T > K$	$S_T - K$	0	$K-S_T$	0

- What if we long a call option and short a put option with same strike and maturity?
  - Total payoff = $S_T - K$

S_T: spot price of the underlying asset at expiration date
K: strike price

Put-call Parity

- At expiration, payoff of long call + short put = S_T – K
- The payoff is the same as holding a stock and a debt with par value of K (i.e., you borrow and promise to pay K at maturity)
  – Stock value = S_T , debt to pay = K
  – Total payoff = S_T - K
- LOOP: the payoff in the future is the same → the cost now should also be the same
  – Costs for long call + short put = C – P
  – Costs for long stock + borrow = $S_0 - \frac{K}{(1+r_f)^t}$
- Therefore: $C - P = S_0 - \frac{K}{(1+rf)^t}$
  – Put-call parity: a principle that defines the relationship between the price of European put options and European call options of the same class, that is, with the same underlying asset, strike price, and expiration date.

Put-call Parity with Dividend

- What if the stock pays dividend before the maturity date?
- At expiration, payoff of long call + short put = S_T – K
- The payoff is the same as holding a stock and a debt with par value (K + FV(Div))
  - Stock value = S_T + FV(Div), debt to pay = K + FV(Div)
  - Total payoff = S_T - K
- The payoff in the future is the same → the cost now should also be the same
  - Costs for long call + short put = C – P
  - Costs for long stock + borrow = $S_0 - \frac{K}{(1+rf)^t} - \frac{FV(Div)}{(1+rf)^t}$
- Therefore: C – P = $S_0 - \frac{K}{(1+rf)^t} - \frac{FV(Div)}{(1+rf)^t}$ = S_0 –PV(K)-PV(Div)

Agenda

- Pricing by no arbitrage
- Forward and future pricing
  - Forward price
  - Carry
  - Future price
- Option pricing
  - Put-call parity
  - Binomial tree model
  - Black-Scholes-Merton model （not required）

Risk Neutral Probability

- Suppose an asset A has the following payoff in t year:

	Probability	Payoff
Up states	P_u	X_u
Down states	P_d	X_d

$X_G > X_B$

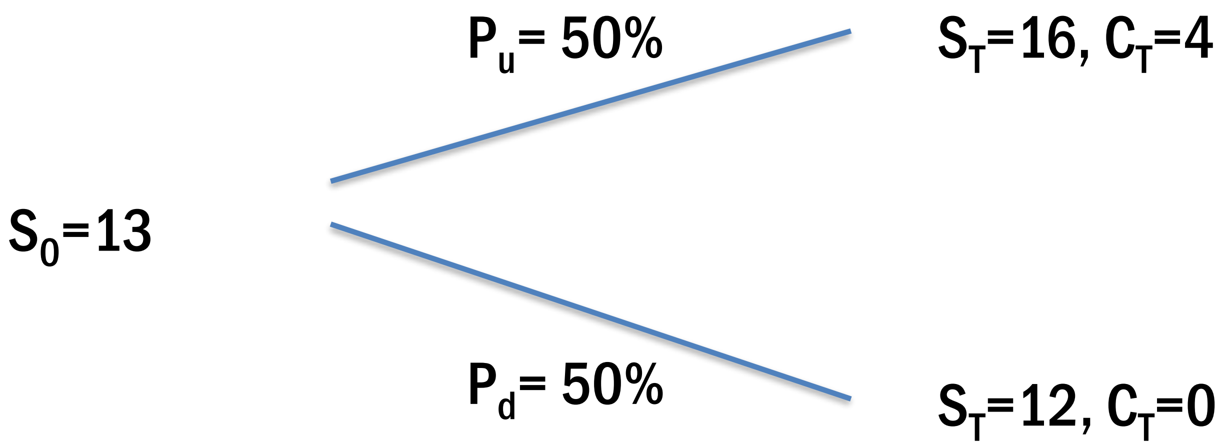
- What is the fair value of the asset A now?
  - Should it be $\frac{P_u * X_u + P_d * X_d}{(1+r_f)^t}$?
  - No, we are risk averse! Should be $\frac{P_u * X_u + P_d * X_d}{(1+r)^t}$, where r = r_f + risk premium.
  - $\frac{P_u * X_u + P_d * X_d}{(1+r)^t} < \frac{P_u * X_u + P_d * X_d}{(1+r_f)^t}$
- Can find a set of λ_u and λ_d such that:
  - $\frac{P_u * X_u + P_d * X_d}{(1+r)^t} = \frac{\lambda_u * P_u * X_u + \lambda_d * P_d * X_d}{(1+r_f)^t}$
  - $\lambda_u * P_u + \lambda_d * P_d = 1$

Risk Neutral Probability

- We call P_u and P_d physical probability.
- We call $\lambda_u * P_u$ and $\lambda_d * P_d$ risk neutral probabilities (RNP)
  - Risk neutral probabilities : probabilities of possible future outcomes which have been adjusted for risk.
  - $\frac{P_u * X_u + P_d * X_d}{(1+r)^t} = \frac{\lambda_u * P_u * X_u + \lambda_d * P_d * X_d}{(1+r_f)^t} < \frac{P_u * X_u + P_d * X_d}{(1+r_f)^t} \qquad (1)$
  - Where $RNP_u = \lambda_u * P_u,\ RNP_d = \lambda_d * P_d$
- Interpretation of λ: how much an investor is willing to pay for \$1 of payoff in the future.
- For this equation (1) to hold, it must be that $\lambda_d > \lambda_u$
  - A risk averse investor is willing to pay more for payoffs in bad states.
  - Why? \$1 is more important to us when we are poor (in down states) than when we are rich (in up states)!

Binomial Tree

- Consider an option on Ford with strike price K=12 and expires in a month. Assume interest rate is 0, and the price of Ford will be either 12 or 16 in a month with 50-50 chance. For simplicity, assume risk free interest rate is 0% per month. What is the value of the option (C_0)?



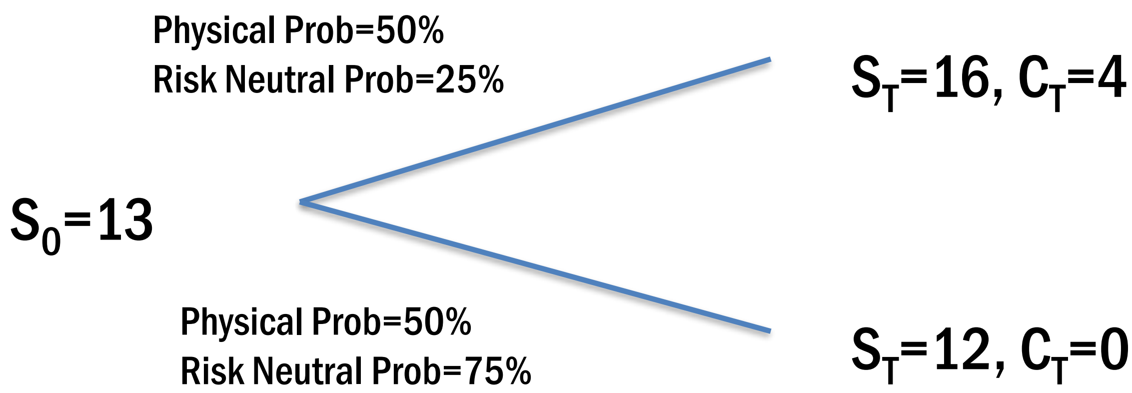
Does $C_0 = \frac{50\%*4+50\%*0}{(1+0\%)} = 50\% * 4 + 50\% * 0$?

No! $C_0 = \frac{RNP_u*4+RNP_d*0}{(1+0\%)} = RNP_u * 4 + RNP_d * 0$

C_T: the value of call option at maturity date
C_0: the value of call option now

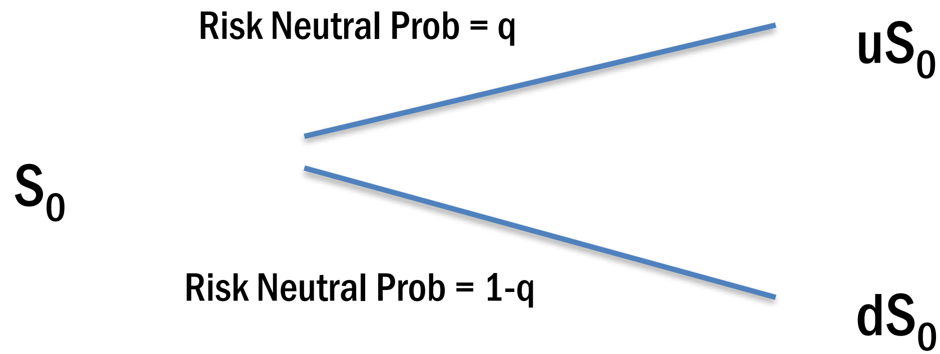
How do We Find the Risk Neutral Probability?

- Use the prices of underlying assets



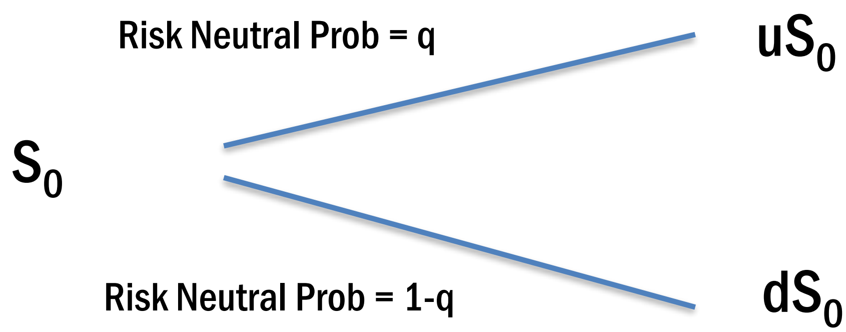
– $13 = S_0 = 25\%(16) + 75\%(12)$
– With risk neutral probability, we can get the price of the option:
$C_0 = 25\%(4) + 75\%(0) = 1$

How do We Find the Risk Neutral Probability?



- A general case:
- S_0: spot price of a stock now
- uS_0: spot prices of a stock in the future in an up market.
- dS_0: spot price of a stock in the future in a down market.
- We want $S_0 = \frac{quS_0 + (1-q)dS_0}{(1+r_f)^1}$, so $q = \frac{1+r_f-d}{u-d}$

General Case of Binomial Tree (One-period)



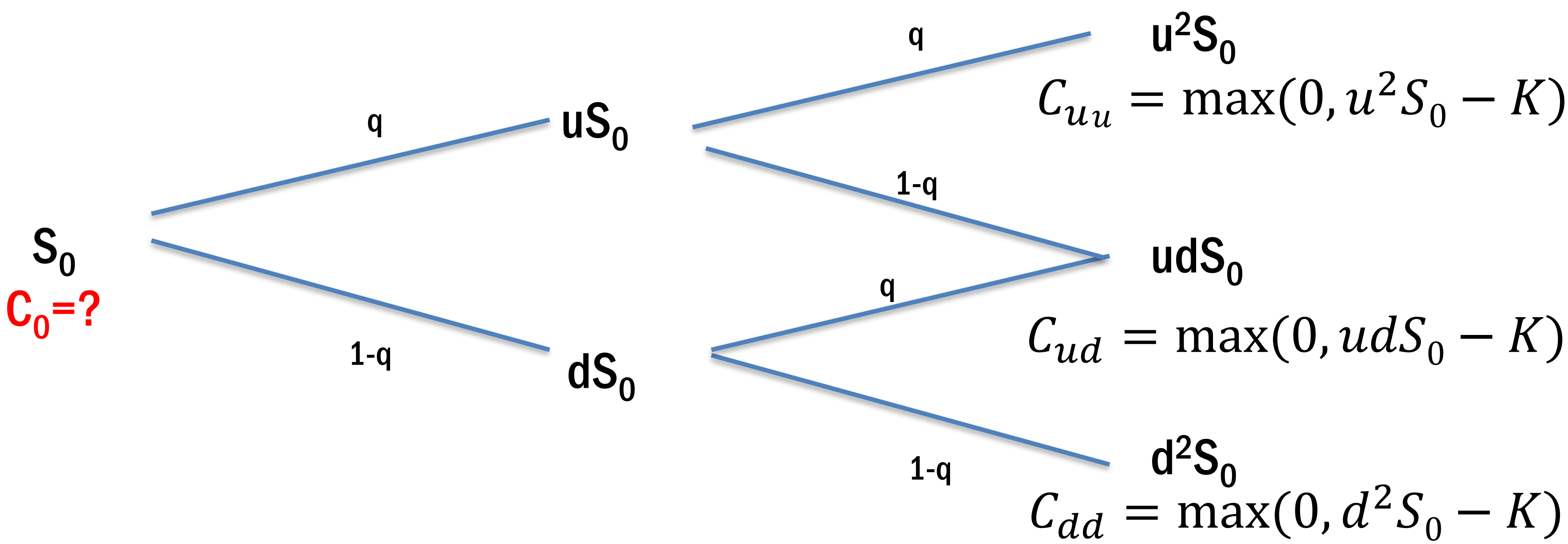
$$C_u = \max(0, uS_0 - K)$$
$$P_u = \max(K - uS, 0)$$

$$C_d = \max(0, dS_0 - K)$$
$$P_d = \max(K - dS_0, 0)$$

1. Find risk neutral probability using stock prices $q = \frac{1+r_f-d}{u-d}$
2. Calculate the payoff of the option in each state: C_u and C_d
3. Calculate the price of the call option:

$$C_0 = \frac{qC_U + (1-q)C_d}{(1+r_f)^1}, \quad P_0 = \frac{qP_U + (1-q)P_d}{(1+r_f)^1}$$

General Case of Binomial Tree (Two-period)



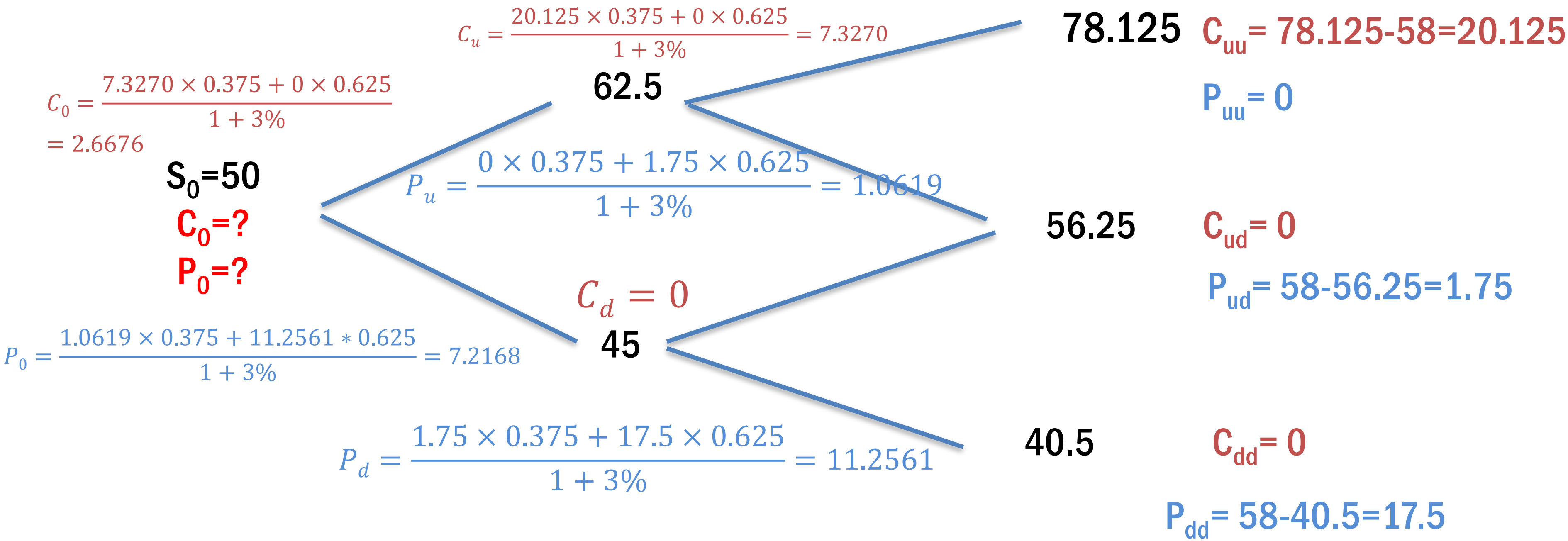
1. Find risk neutral probability for the up state $q = \frac{1+rf-d}{u-d}$
2. Calculate the payoff of the option in each state: C_{uu}, C_{ud}, and C_{dd}
3. Calculate $C_u = \frac{qC_{uu} + (1-q)C_{ud}}{(1+r_f)^1}$, $C_d = \frac{qC_{ud} + (1-q)C_{dd}}{(1+r_f)^1}$
4. Calculate the initial price of the option:

$$C_0 = \frac{qC_u + (1-q)C_d}{(1+r_f)^1} = \frac{q^2C_{uu} + 2q(1-q)C_{ud} + (1-q)^2C_{dd}}{(1+rf)^2}$$

What about the value of put options?

Example

- People expect stock A's future prices to be as follows in the next two months:



- Assume stock A's current price is 50, and the risk free rate is 3%/month. What should be the price of a call option with a strike price of \$58? What about a put option with a strike price of \$58?

Solution:

- $u = \frac{78.125}{62.5} = 1.25 \quad d = \frac{56.125}{62.5} = 0.898$
- $q = \frac{1+rf-d}{u-d}$ =0.375. 1-q= 0.625

Black-Scholes-Merton Model (Not required)

- A continuous version of the binomial tree model.
- Assumptions:
  - Stock price follows a Geometric Brownian motion: $\mathrm{d}S_{\mathrm{t}} = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$
    - Continuous time, take the number of periods in binomial tree to infinity
    - σ is the volatility of the underlying asset
  - Risk free portfolio earns continuous rate r: $\mathrm{d}B_{\mathrm{t}} = rB_t\mathrm{d}t$
  - No transaction cost or trading constraint
- Call Price

$$C(S_0) = N(d_1)S_0 - N(d_2)Ke^{-rT}$$

  - Where $N(.)$ is the CDF of the standard normal distribution
  - $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right], d_2 = d_1 - \sigma\sqrt{T}$
- Put Price

$$P(S_0) = N(-d_2)Ke^{-rT} - N(-d_1)S_0$$

Option Greeks

		Calls	Puts
Delta	$\frac{\partial C}{\partial S}$	$N(d_1)$	$-N(-d_1) = N(d_1) - 1$
Gamma	$\frac{\partial^2 C}{\partial S^2}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$	
Vega	$\frac{\partial C}{\partial \sigma}$	$SN'(d_1)\sqrt{T-t}$	
Theta	$\frac{\partial C}{\partial t}$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$
Rho	$\frac{\partial C}{\partial r}$	$K(T-t)e^{-r(T-t)}N(d_2)$	$-K(T-t)e^{-r(T-t)}N(-d_2)$

- Delta: how options prices changes when the S_0 changes
  - Positive for call, negative for put
- Vega: how option prices changes when volatility changes
  - The more volatile the underlying, the more expensive the options
- Theta: how price changes over time
  - Option loses value over its life span

Summary

- Principle for pricing derivatives: the law of one price and no arbitrage condition
- Futures/forward:
  - No carry
  - With carry
- Option:
  - Risk neutral probability
  - Binomial tree method
  - Black Scholes model (not required)