

Review of Last Lecture

Key concepts and/or techniques:

Let X and Y be two continuous random variables and (X, Y) be a pair of RVs with their range denoted by $\bar{S} \subseteq \mathbb{R}^2$. Then (X, Y) or X and Y is said to be a bivariate continuous RV.

[Road map]

To study the bivariate continuous random variable

discrete RV \longrightarrow continuous RV	Mathematical expectations
pmf \longrightarrow pdf	mean
joint pmf \longrightarrow joint pdf	variance
marginal pmf \longrightarrow marginal pdf	covariance
conditional pmf \longrightarrow conditional pdf	correlation coefficient

Review of Last Lecture

[Marginal pdf]

The marginal pdf of X and Y are:

$$f_X(x) = \int_{\overline{S_Y}(x)} f(x, y) dy : \overline{S_X} \rightarrow (0, \infty)$$

$$\overline{S_Y}(x) = \{y | (x, y) \in \overline{S}\} \text{ for } x \in \overline{S_X}$$

$$f_Y(y) = \int_{\overline{S_X}(y)} f(x, y) dx : \overline{S_Y} \rightarrow (0, \infty)$$

$$\overline{S_X}(y) = \{x | (x, y) \in \overline{S}\} \text{ for } y \in \overline{S_Y}$$

Review of Last Lecture

[Covariance and Correlation Coefficient]

Let $g(X, Y)$ be a function of X and Y , whose joint pdf $f(x, y) : \bar{S} \rightarrow [0, \infty)$. Then

$$E[g(X, Y)] = \iint_{\bar{S}} g(x, y) f(x, y) dx dy$$

1. covariance $\text{Cov}(X, Y)$
2. correlation coefficient $\rho(X, Y)$

[Independent Variables]

Two continuous RVs X and Y are independent if

$$f(x, y) = f_X(x)f_Y(y), \quad x \in \bar{S}_X, \quad y \in \bar{S}_Y$$

Review of Last Lecture

[Conditional pdf and Conditional Expectation]

The conditional pdf, mean, and variance of Y , given that $X = x$ are

$$h(y|x) = \frac{f(x, y)}{f_X(x)} \quad \text{for } f_X(x) > 0, y \in \overline{S_Y}(x)$$

$$P(Y \in A|X = x) = \int_{y \in A} h(y|x) dy, A \subseteq \overline{S_Y}(x)$$

$$E(Y|X = x) = \int_{\overline{S_Y}(x)} yh(y|x) dy$$

$$\begin{aligned} \text{Var}(Y|X = x) &= E\{[Y - E(Y|X = x)]^2|X = x\} \\ &= \int_{\overline{S_Y}(x)} [y - E(Y|X = x)]^2 h(y|x) dy \\ &= E[Y^2|X = x] - [E(Y|X = x)]^2 \end{aligned}$$

STA2001 Probability and Statistics (I)

Lecture 18

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Section 4.5 Bivariate Normal distribution

Pdf of Bivariate Normal Distribution

Definition

Let X and Y be 2 continuous RVs and have the joint pdf

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}q(x, y)\right], x \in \mathbb{R}, y \in \mathbb{R},$$

$$q(x, y) = \frac{1}{1-\rho^2} \left[\left(\frac{x-\mu_X}{\sigma_X} \right)^2 - 2\rho \left(\frac{x-\mu_X}{\sigma_X} \right) \left(\frac{y-\mu_Y}{\sigma_Y} \right) + \left(\frac{y-\mu_Y}{\sigma_Y} \right)^2 \right] \geq 0$$

where $\mu_X, \mu_Y \in \mathbb{R}$, $\sigma_X, \sigma_Y > 0$ and $|\rho| < 1$. Then X and Y are said to be bivariate normally distributed.

Key components: Scaled exponential function with a quadratic and negative function as its exponent.

Properties of Bivariate Normal Distribution

1. Marginal pdf of X and Y are normal with

$$X \sim N(\mu_X, \sigma_X^2), Y \sim N(\mu_Y, \sigma_Y^2)$$

Properties of Bivariate Normal Distribution

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$$X \sim N(\mu_X, \sigma_X^2), Y \sim N(\mu_Y, \sigma_Y^2)$$

2. Conditional pdf of X given that $Y = y$ is normal with mean

$$\mu_X + \frac{\sigma_X}{\sigma_Y} \rho (y - \mu_Y)$$

and variance

$$(1 - \rho^2) \sigma_X^2$$

i.e.,

$$X|Y = y \sim N\left(\mu_X + \frac{\sigma_X}{\sigma_Y} \rho (y - \mu_Y), (1 - \rho^2) \sigma_X^2\right)$$

Moreover,

$$Y|X = x \sim N\left(\mu_Y + \frac{\sigma_Y}{\sigma_X} \rho (x - \mu_X), (1 - \rho^2) \sigma_Y^2\right)$$

Proof of the Two Properties

First, recall that

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy.$$

Clearly, it is equivalent to prove that

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{1}{2}\left(\frac{x - \mu_X}{\sigma_X}\right)^2\right).$$

To this goal, we arrange $f(x, y)$ as follows:

$$\begin{aligned} f(x, y) &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times \\ &\exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y - \mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x - \mu_X}{\sigma_X}\right)\right]^2 - \frac{1}{2}\left(\frac{x - \mu_X}{\sigma_X}\right)^2\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{1}{2}\left(\frac{x - \mu_X}{\sigma_X}\right)^2\right) \times h(y|x) \end{aligned}$$

Proof of the Two Properties

$$\begin{aligned}h(y|x) &= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right) \\&= \frac{1}{\sqrt{2\pi}\sqrt{(1-\rho^2)\sigma_Y^2}} \exp\left(-\frac{1}{2} \left[\frac{y - [\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)]}{\sqrt{(1-\rho^2)\sigma_Y^2}}\right]^2\right)\end{aligned}$$

Clearly, to complete the proof, we only need to show that

$$\int_{-\infty}^{\infty} h(y|x) dy = 1$$

which is true because $h(y|x)$ is the probability density function of $N(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x - \mu_X), (1 - \rho^2)\sigma_Y^2)$.

Actually, $h(y|x)$ is the conditional probability density function of Y given $X = x$ because

$$h(y|x) = \frac{f(x, y)}{f_X(x)}.$$

Properties of Bivariate Normal Distribution

3. Independence \iff Uncorrelation

- ▶ prove that ρ is indeed the correlation coefficient of X and Y
 - ▶ $\text{Cov}(X, Y) = \rho\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)} = \rho\sigma_X\sigma_Y$
 - ▶ $f(x, y) = h(y|x)f_X(x)$
- ▶ Uncorrelation means $\rho = 0 \implies$

$$f(x, y) = f_X(x)f_Y(y) \quad \text{means independence}$$

The proof is left as a question in the assignment.

Example 1, page 165

Question

Observe a group of college students. Let X and Y denote their grades in high school and in the 1st year in college, respectively, have a bivariate normal distribution with

$$\mu_X = 2.9, \quad \mu_Y = 2.4, \quad \sigma_X = 0.4, \quad \sigma_Y = 0.5 \quad \text{and} \quad \rho = 0.8$$

Find $P(2.1 < Y < 3.3)$ and $P(2.1 < Y < 3.3|X = 3.2)$.

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Find $P(2.1 < Y < 3.3)$ and $P(2.1 < Y < 3.3|X = 3.2)$.

Since $Y \sim N(\mu_Y, \sigma_Y^2) = N(2.4, 0.5^2)$, then

$$\begin{aligned} P(2.1 < Y < 3.3) &= P\left(\frac{2.1 - 2.4}{0.5} < \frac{Y - 2.4}{0.5} < \frac{3.3 - 2.4}{0.5}\right) \\ &= \Phi(1.8) - \Phi(-0.6) = 0.69 \end{aligned}$$

Example 1, page 165

Note that

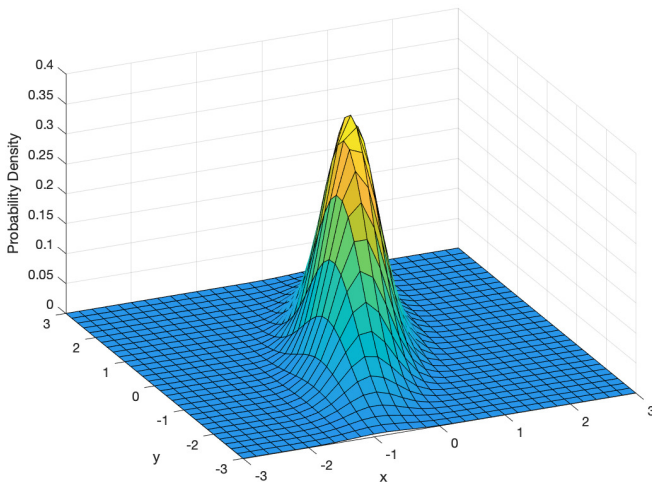
$$Y|X = x \sim N\left(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x - \mu_X), (1 - \rho^2)\sigma_Y^2\right)$$

Let $X = 3.2$. Then $Y|X = 3.2 \sim N(2.7, 0.3^2)$

$$\begin{aligned} &P(2.1 < y < 3.3|X = 3.2) \\ &= P\left(\frac{2.1 - 2.7}{0.3} < \frac{Y - 2.7}{0.3} < \frac{3.3 - 2.7}{0.3} \middle| X = 3.2\right) \\ &= \Phi(2) - \Phi(-2) = 0.95 \end{aligned}$$

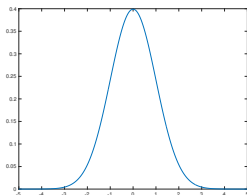
Geometrical interpretation [P.166]

Let $z = f(x, y)$ and draw it in $x - y - z$ 3-dimensional space



Geometrical interpretation [P.166]

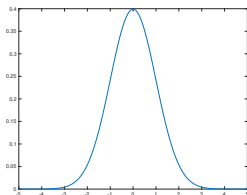
Let $z = f(x, y)$ and draw it in $x - y - z$ 3-dimensional space



1. Consider $z = f(x_0, y) = f_X(x_0)h(y|x_0)$
 - ▶ the intersection of the surface $z = f(x, y)$ with the plane $x = x_0$, which is parallel to the yz -plane
 - ▶ a bell-shaped curve and has the shape of a normal pdf

Geometrical interpretation [P.166]

Let $z = f(x, y)$ and draw it in $x - y - z$ 3-dimensional space



2. Consider $z = f(x, y_0) = f_Y(y_0)g(x|y_0)$

- ▶ the intersection of the surface $z = f(x, y)$ with the plane $y = y_0$, which is parallel to the xz -plane
- ▶ a bell-shaped curve and has the shape of a normal pdf

Geometrical interpretation [P.166]

3. Consider $z_0 = f(x, y)$ with $0 < z_0 < \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$
- ▶ the intersection of the surface $z = f(x, y)$ with the plane $z = z_0$, which is parallel to the xy -plane
 - ▶ an ellipse

$$\exp\left[-\frac{1}{2}q(x, y)\right] = z_0 \cdot 2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}$$

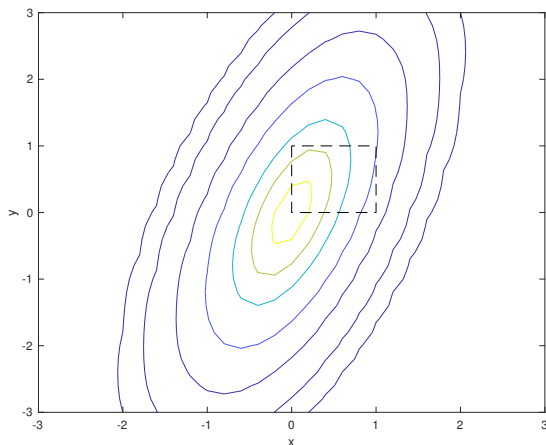
Taking logarithm yields

$$\begin{aligned}\left(\frac{x - \mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x - \mu_X}{\sigma_X}\right)\left(\frac{y - \mu_Y}{\sigma_Y}\right) + \left(\frac{y - \mu_Y}{\sigma_Y}\right)^2 \\ = -2(1 - \rho^2)\ln(z_0 2\pi\sigma_X\sigma_Y\sqrt{1 - \rho^2})\end{aligned}$$

which corresponds to an ellipse.

Geometrical interpretation [P.166]

We can draw these ellipses for different z_0 on the xy -plane, which are called level curves or contours.



Chapter 5. Distribution of Functions of Random Variables

Section 5.1 Function of one random variable

Function of One Random Variable

Question

Let X be a RV of either discrete or continuous type. Consider a function of X , say $Y = u(X)$. Then Y is also a RV and has its pmf or pdf.

How to compute the pmf or pdf of Y ?

In what follows, we consider the case where $Y = u(X)$ is a one-to-one mapping.

Discrete case

Let X be a discrete RV with pmf $f(x) : \overline{S_X} \rightarrow (0, 1]$, and $Y = u(X)$ be a one-to-one mapping with inverse $X = v(Y)$.

Then the pmf of Y , denoted by $g(y) : \overline{S_Y} \rightarrow (0, 1]$ is

- ▶ $\overline{S_Y} = \{y | y = u(x), x \in \overline{S_X}\}$
- ▶ for any $y \in \overline{S_Y}$,

$$g(y) = P(Y = y) = P(u(X) = y) = P(X = v(y)),$$

Since

$$P(X = x) = f(x), \quad g(y) = f[v(y)] \quad \text{for } y \in \overline{S_Y}$$

Example 1, page 177

Question

Let X have a Poisson distribution with $\lambda = 4$ and its pmf takes the form of

$$f(x) = \frac{4^x e^{-4}}{x!}, \quad x = 0, 1, 2, \dots$$

If $Y = \sqrt{X}$, what is the pmf $g(y)$ of Y ?

Example 1, page 177

Question

Let X have a Poisson distribution with $\lambda = 4$ and its pmf takes the form of

$$f(x) = \frac{4^x e^{-4}}{x!}, \quad x = 0, 1, 2, \dots$$

If $Y = \sqrt{X}$, what is the pmf $g(y)$ of Y ?

First, $\overline{S}_Y = \{0, 1, \sqrt{2}, \sqrt{3}, \dots\}$, and then

$$Y = u(X) = \sqrt{X} \implies X = v(Y) = Y^2$$

$$\begin{aligned} g(y) &= P(Y = y) = P(\sqrt{X} = y) = P(X = y^2) \\ &= f(y^2) = \frac{4^{y^2} e^{-4}}{(y^2)!}, \quad y = 0, 1, \sqrt{2}, \sqrt{3}, \dots \end{aligned}$$

Continuous Case: the idea

Question

Let X be a continuous RV with pdf $f(x) : [c_1, c_2] \rightarrow [0, \infty)$ and $Y = u(X)$ is a function of X .

Our goal is to calculate the pdf of Y , say $g(y)$.

For a continuous RV X , recall the relation between a pdf $f(x)$ and its corresponding cdf $F(x)$:

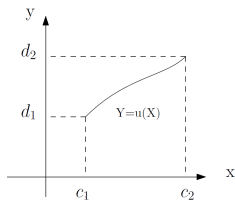
$$F(x) = \int_0^x f(t)dt \Rightarrow F'(x) = \frac{dF(x)}{dx} = f(x),$$

where $F(x)$ is differentiable.

Continuous Case: Case 1

- Case 1: $Y = u(X)$ is continuous, strictly increasing, has inverse function $X = v(Y)$, whose derivative $\frac{dv(y)}{dy}$ exists.

1. Determine the cdf of Y , $G(y)$, $y \in \overline{S_Y}$.

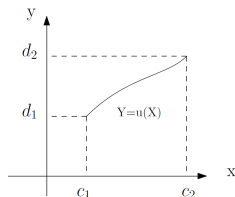


We first find the sample space of Y , $\overline{S_Y}$. Since $Y = u(X)$ is continuous and increasing, $\overline{S_Y} = [d_1, d_2]$ with $d_1 = u(c_1)$ and $d_2 = u(c_2)$.

$$G(y) = P(Y \leq y) = P(u(X) \leq y) = P(X \leq v(y)) = \int_{c_1}^{v(y)} f(x) dx$$

Continuous Case: Case 1

2. Determine the pdf of Y , $g(y)$, $y \in \overline{S_Y}$.



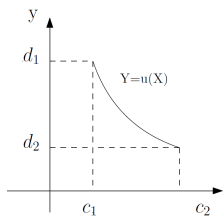
$$\begin{aligned} g(y) &= G'(y) = \frac{dG(y)}{dy} = f(v(y))v'(y) \\ &= f(v(y)) \frac{dv(y)}{dy} \\ &= f(v(y)) \left| \frac{dv(y)}{dy} \right|, \quad y \in \overline{S_Y} \end{aligned}$$

Continuous Case: Case 2

- Case 2: $Y = u(X)$ is continuous, strictly decreasing and has inverse function $X = v(Y)$, whose derivative $\frac{dv(y)}{dy}$ exists.

1. Determine the cdf of Y , $G(y)$, $y \in \overline{S_Y}$.

We first find the sample space of Y , $\overline{S_Y}$. Since $Y = u(X)$ is continuous and strictly decreasing $\overline{S_Y} = [d_2, d_1]$ with $d_1 = u(c_1)$ and $d_2 = u(c_2)$.



$$\begin{aligned} G(y) &= P(Y \leq y) = P(u(X) \leq y) = P(X \geq v(y)) \\ &= 1 - P(X \leq v(y)) = 1 - \int_{c_1}^{v(y)} f(x) dx \end{aligned}$$

Continuous Case: Case 2

2. Determine the pdf of Y , $g(y)$, $y \in \overline{S_Y}$.

$$\begin{aligned} g(y) &= G'(y) = \frac{dG(y)}{dy} = -f(v(y))v'(y) \\ &= -f(v(y))\frac{dv(y)}{dy} \\ &= f(v(y))\left|\frac{dv(y)}{dy}\right| \quad y \in \overline{S_Y} \end{aligned}$$

Summary: for both strictly increasing and decreasing cases,

$$g(y) = f(v(y))\left|\frac{dv(y)}{dy}\right|$$