MAT1002 Lecture 17, Thursday, Mar/23/2023

Outline

- Local extrema (14.8)
- Global extrema of continuous functions (14.7)
- Optimization with equality constraints (14.8)
  - ↳ Lagrange multiplier

Optimization

Local Extrema

Definition

Let $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, be a function, and let $(a_1, \ldots, a_n)$ be a point in D.

- We say that f has a local maximum at $(a_1, \ldots, a_n)$ if there exist an $r > 0$ such that

  (value)

  $$f(a_1, \ldots, a_n) \geq f(x_1, \ldots, x_n)$$

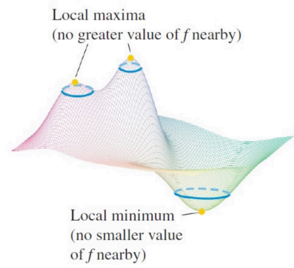
  for every $(x_1, \ldots, x_n) \in D \cap B_r(a_1, \ldots, a_n)$.

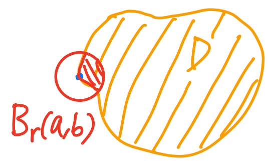
  open ball with radius r (disk if $n=2$)

- We say that f has a local minimum at $(a_1, \ldots, a_n)$ if there exist an $r > 0$ such that

  $$f(a_1, \ldots, a_n) \leq f(x_1, \ldots, x_n)$$

  for every $(x_1, \ldots, x_n) \in D \cap B_r(a_1, \ldots, a_n)$.





Caution: For $f(x,y) = x^2 + y^2$ defined on the closed disk

$$D = \{(x,y) : x^2 + y^2 \leq 1\},$$

f has local maxima at all (x,y) satisfying $x^2+y^2=1$, the boundary of D.

Local extrema are also called relative extrema.

Although many theorems for extrema (i.e., maxima or minima) hold for n-variable functions, we will focus on these theorems for functions with 2 or 3 variables.

Theorem (First Derivative Test)

For a function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^2$, suppose that:

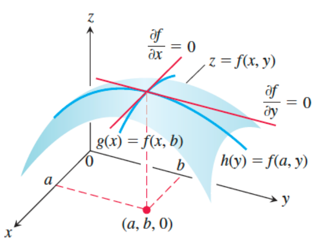
- *f has a local maximum or a local minimum at an interior point (a, b) of D, and;*
- *both $f_x(a, b)$ and $f_y(a, b)$ exist*

Then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. That is, $\nabla f(a, b) = \vec{0}$.

Proof: • Define $g(x) := f(x, b)$. Then $g(a)$ is a local extremum, where a is an interior point of the domain of g. By one-variable theory, $g'(a)$ D.N.E or $g'(a) = 0$.

• $g'(a) = \frac{\partial}{\partial x} f(a, b)$, which exists by assumption, so $g'(a) = 0$ and $f_x(a,b) = 0$.

• Similarly, $f_y(a,b) = 0$.



$\square$

Definition

An interior point (a, b) in the domain of a function f is called a critical point of f if either $\nabla f(a, b) = \vec{0}$ or at least one of $f_x(a, b)$ and $f_y(a, b)$ does not exist.

Another way to state the first derivative test is that if f has a local extremum at an interior point P_0 of the domain, then P_0 is a critical point. The converse is not true.

e.g. For $f(x,y) = y^2 - x^2$, $(0,0)$ is a critical point, but it does not give a local extremum.

Def: Let $f: D \to \mathbb{R}$ ($D \subseteq \mathbb{R}^2$) be differentiable. If (a,b) is a critical point of f that does not give a local extremum of f, then (a,b) is called a saddle point of f, and $(a, b, f(a,b))$ is called a saddle point of the surface $z = f(x,y)$.

Theorem (Second Derivative Test)

Let f be a function whose second partial derivatives are all continuous on a open ~~ball~~ disk centered at (a, b). Suppose that $\nabla f(a, b) = \vec{0}$ (so (a, b) is a critical point of f). Let

$$H := H(a, b) := f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2.$$

- *If $H > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).*
- *If $H > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).*
- *If $H < 0$, then f has no local extremum at (a, b); that is, (a, b) is a saddle point ~~o~~ of f.*

We postpone the proof to a later section.

Remark

If $H = 0$, the second derivative test gives no information: (a, b) could give a local maximum or a local minimum, or it could be a saddle point.

Example

Find all local extrema of the function

$$f(x, y) := x^4 + y^4 - 4xy + 1$$

defined on $\mathbb{R}^2$.

<u>Sol</u>: $(1,1)$ and $(-1,-1)$, both give a local minimum.

Global/Absolute Extrema

Definition

Let $(a_1, \dots, a_n)$ be a point in the domain D of a function f.

- We say that f has a global maximum (or an absolute maximum) at $(a_1, \dots, a_n)$ if

$$f(a_1, \dots, a_n) \geq f(x_1, \dots, x_n)$$

  for every $(x_1, \dots, x_n) \in D$.

- We say that f has a global minimum (or an absolute minimum) at $(a_1, \dots, a_n)$ if

$$f(a_1, \dots, a_n) \leq f(x_1, \dots, x_n)$$

  for every $(x_1, \dots, x_n) \in D$.

We saw in MAT1001 that a one-variable continuous function defined on a closed and bounded interval must attain its global extrema. The following is an analogous theorem for two-variable functions.

Theorem

Let $f : D \to \mathbb{R}$ be a continuous function, where D is a closed and bounded set in $\mathbb{R}^2$. Then f has a global maximum $f(x_1, y_1)$ and a global minimum $f(x_2, y_2)$ at some (x_1, y_1) and (x_2, y_2) in D.

In short, "continuous functions defined on closed and bounded sets attain their global extrema".

Remark: D may not contain the absolute (global) extrema if it is not closed, even if it is bounded. e.g.

$f(x,y) = x$, $D = \{(x,y) : x^2+y^2 < 1\}$.

Since, by definition, a global maximum (or minimum) must be ~~itself~~ a local maximum (or minimum), the following is a strategy for finding the global extrema of a continuous function defined a closed and bounded domain D.

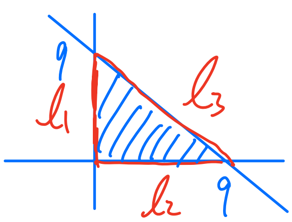
1. Find the values of f at the critical points (in the interior of the domain). all candidates for local extrema in interior of D
2. Find the local extrema of f on the boundary of D.
3. The largest value and the smallest value from step 1 and 2 are, respectively, the global maximum of f and the global minimum of f.

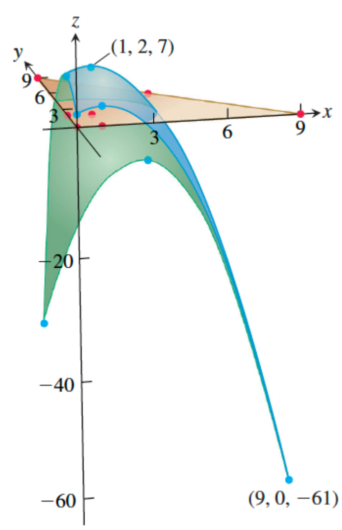
Example (e.g. 14.7.6)

Find the global maximum and global minimum of the function

$$f(x, y) := 2 + 2x + 4y - x^2 - y^2 \quad \leftarrow \text{cts}$$

defined on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$.





Sol (Outline):

- $f_x = 2-2x$, $f_y = 4-2y$;
  $f_x = 0 = f_y \iff (x,y) = (1,2)$; this is in interior of D.

- Boundary is $l_1 \cup l_2 \cup l_3$, where
  $l_1 := \{(0,y) : 0 \le y \le 9\}$, $l_2 := \{(x,0) : 0 \le x \le 9\}$,
  $l_3 := \{(x, 9-x) : 0 \le x \le 9\}$.

- On l_3: $f(x,y) = f(x, 9-x) = -43 + 16x - 2x^2 = p(x)$;
  $p'(x) = 0 \iff x = 4 \iff y = 5$, so
  $(x,y) = (4,5)$ is a candidate.
  At end points $x=0$ and $x=9$, we have $(0,9)$ and $(9,0)$ also being candidates.

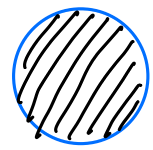
- By doing similar arithmetic for l_1 and l_2,
  we found the following candidates:
  l_1: $(0,0)$, $(0,2)$, $(0,9)$.
  l_2: $(0,0)$, $(1,0)$, $(9,0)$.

- Compare all candidates:
  Max $f(1,2) = 7$, $f(0,0) = 2$, $f(0,2) = 6$, $f(0,9) = -43$,
  $f(1,0) = 3$, Min $f(9,0) = -61$, $f(4,5) = -11$.

Optimization with Equality Constraints

Q: Given a mountain altitude function (cts) $f(x,y)$, what is the highest altitude given by points in the unit disk $x^2+y^2 \le 1$?

- Find crit. pts. in the interior of D, and find the maximum in the boundary, then compare.



D: $x^2+y^2\le 1$

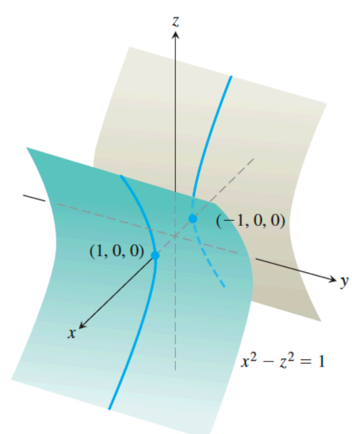
Notation

$$\text{Maximize } f(x,y)$$
$$\text{Subject to (s.t.) } x^2+y^2=1. \leftarrow \text{Boundary}$$

This means we are trying to find $\max_{(x,y)\in C} f(x,y)$, where

$$C=\{(x,y): x^2+y^2=1\}.$$

Another example Find all points on the surface $x^2-z^2-1=0$ that are closest to the origin.



Notation

$$\text{Minimize } \sqrt{x^2+y^2+z^2}$$
$$\text{Subject to } x^2-z^2-1=0.$$

Optimization with One Equality Constraint

Maximize (or minimize) $f(x_1, \ldots, x_n)$

Subject to (s.t.) $g(x_1, \ldots, x_n) = 0$

Consider Minimize $x^2+y^2+z^2$ s.t. $x^2-z^2-1=0$. S

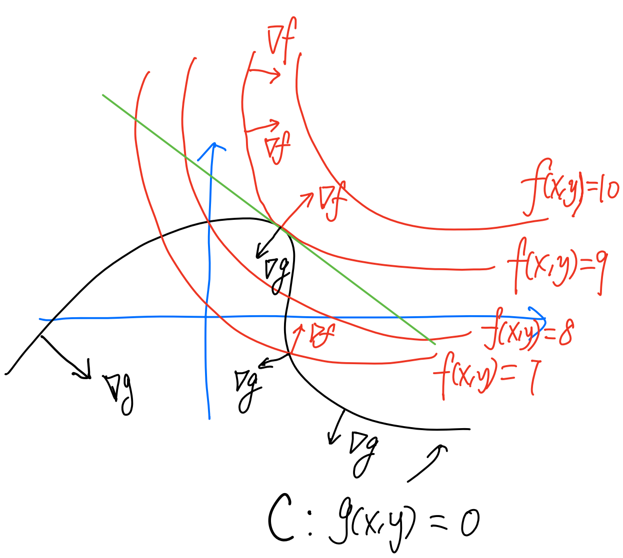
Critical pt method will not help, since for $f(x,y,z)=x^2+y^2+z^2$, $f_x = 2x$, $f_y = 2y$, $f_z = 2z$, so the only critical pt is $(x,y,z) = (0,0,0)$, which is not in S!

Need another method.

Method of Lagrange Multipliers

Consider a 2-variable case:

$$\begin{aligned} \text{Max} \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) = 0 \end{aligned}$$



Theorem (14.8.12, 2D Version)

Let $f(x,y)$ be a differentiable function on D, and let C given by

(Smooth) $\vec{r}(t) = \langle x(t), y(t) \rangle$, $t \in I$ i.e., $P_0 = (x(t_0), y(t_0))$,

be a curve contained in the interior of D. If P_0 is an interior point of C that gives a local extremum of f relative to points in C, then $\nabla f \perp \vec{r}\,'$ at P_0.

t_0 int. pt of I.

This thm is also true for $f(x,y,z)$ and $C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Proof: • Consider the values of f, but only on C.

• Let $g(t) := f(\underbrace{x(t), y(t)}_{\in C})$, and let $P_0 = (x(t_0), y(t_0))$.

• By assumption, $f(x(t_0), y(t_0))$ is a local maximum of f on C, so $g(t_0)$ is maximum on $(t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$. Hence $g'(t_0) = 0$.

• But $g'(t_0) = f_x(P_0)x'(t_0) + f_y(P_0)y'(t_0) = \nabla f(P_0) \cdot \vec{r}\,'(t_0)$, so

$$\nabla f(P_0) \cdot \vec{r}\,'(t_0) = 0.$$

$\square$

In particular, when f has a local extremum at a point P_0 (interior of C) relative to C, where C is $g(x,y) = 0$, then both ∇f and

∇g are $\perp$ to the curve C at P_0, so either:

- $\nabla g = \langle 0, 0 \rangle$, or;
- $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$ $(\nabla f \,//\, \nabla g)$.

This observation suggests the following method.

The Method of Lagrange Multipliers

$\min_{(x,y)\in C} f(x,y)$, where $C := \{(x,y) : g(x,y) = 0\}$.

Suppose that the minimum of $f(x, y)$ subject to the constraint $g(x, y) = 0$ exists, where f and g are differentiable. Assume that $\nabla g(x, y) \neq \vec{0}$ whenever $g(x, y) = 0$. The following is a procedure for finding the constrained minimum:

1. Find all values of x, y and λ that simultaneously satisfy
$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0.$$
2. Evaluate f at all the points (x, y) obtained in step 1. The smallest value is the desired minimum.

If the constrained maximum exists, then the largest value in step 2 is the maximum.

Example (e.g. 14.8.3)

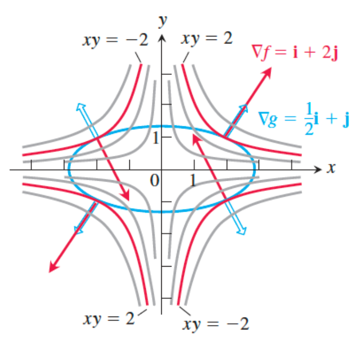
Find the maximum and minimum of the function

$$f(xy) := xy$$

defined on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Specify where the extrema occur on the ellipse.



Ans: Max at $(2,1)$ and $(-2,-1)$; Min at $(2,-1)$ and $(-2,1)$.

The method also works for functions with more variables.

e.g. Find all points on the surface $S: x^2 - z^2 - 1 = 0$ that are closest to origin, and find the minimum distance.

Ans: $(\pm 1, 0, 0)$; distance $= 1$.