CHAPTER 13 VECTOR-VALUED FUNCTIONS AND MOTION IN SPACE

13.1 CURVES IN SPACE AND THEIR TANGENTS

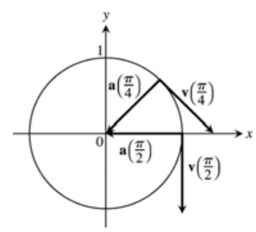
1. $x = t+1$ and $y = t^2 - 1 \Rightarrow y = (x-1)^2 - 1 = x^2 - 2x$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\,\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{a} = 2\mathbf{j}$ at $t = 1$

2. $x = \frac{t}{t+1}$ and $y = \frac{1}{t} \Rightarrow x = \frac{\frac{1}{y}}{\frac{1}{y}+1} = \frac{1}{1+y} \Rightarrow y = \frac{1}{x} - 1$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{1}{(t+1)^2}\mathbf{i} - \frac{1}{t^2}\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = -\frac{2}{(t+1)^3}\mathbf{i} + \frac{2}{t^3}\mathbf{j}$
$\Rightarrow \mathbf{v} = 4\mathbf{i} - 4\mathbf{j}$ and $\mathbf{a} = -16\mathbf{i} - 16\mathbf{j}$ at $t = -\frac{1}{2}$

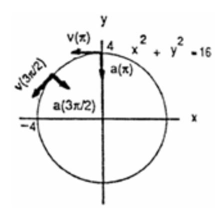
3. $x = e^t$ and $y = \frac{2}{9}e^{2t} \Rightarrow y = \frac{2}{9}x^2$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = e^t\mathbf{i} + \frac{4}{9}e^{2t}\mathbf{j} \Rightarrow \mathbf{a} = e^t\mathbf{i} + \frac{8}{9}e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{a} = 3\mathbf{i} + 8\mathbf{j}$ at $t = \ln 3$

4. $x = \cos 2t$ and $y = 3\sin 2t \Rightarrow x^2 + \frac{1}{9}y^2 = 1$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j}$
$\Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = (-4\cos 2t)\mathbf{i} + (-12\sin 2t)\mathbf{j} \Rightarrow \mathbf{v} = 6\mathbf{j}$ and $\mathbf{a} = -4\mathbf{i}$ at $t = 0$

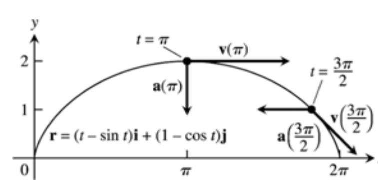
5. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j}$
$\Rightarrow$ for $t = \frac{\pi}{4}$, $\mathbf{v}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and $\mathbf{a}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$;
for $t = \frac{\pi}{2}$, $\mathbf{v}\left(\frac{\pi}{2}\right) = -\mathbf{j}$ and $\mathbf{a}\left(\frac{\pi}{2}\right) = -\mathbf{i}$



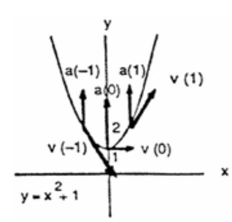
6. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(-2\sin\frac{t}{2}\right)\mathbf{i} + \left(2\cos\frac{t}{2}\right)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left(-\cos\frac{t}{2}\right)\mathbf{i} + \left(-\sin\frac{t}{2}\right)\mathbf{j}$
$\Rightarrow$ for $t = \pi$, $\mathbf{v}(\pi) = -2\mathbf{i}$ and $\mathbf{a}(\pi) = -\mathbf{j}$;
for $t = \frac{3\pi}{2}$, $\mathbf{v}\left(\frac{3\pi}{2}\right) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$ and $\mathbf{a}\left(\frac{3\pi}{2}\right) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$



7. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$
$\Rightarrow$ for $t = \pi$, $\mathbf{v}(\pi) = 2\mathbf{i}$ and $\mathbf{a}(\pi) = -\mathbf{j}$;
for $t = \frac{3\pi}{2}$, $\mathbf{v}\left(\frac{3\pi}{2}\right) = \mathbf{i} - \mathbf{j}$ and $\mathbf{a}\left(\frac{3\pi}{2}\right) = -\mathbf{i}$



8. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\,\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} \Rightarrow$ for $t = -1$, , $\mathbf{v}(-1) = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{a}(-1) = 2\mathbf{j}$; for $t = 0$, $\mathbf{v}(0) = \mathbf{i}$ and $\mathbf{a}(0) = 2\mathbf{j}$;
for $t = 1$, $\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{a}(1) = 2\mathbf{j}$



9. $\mathbf{r}=(t+1)\mathbf{i}+\left(t^2-1\right)\mathbf{j}+2t\mathbf{k} \Rightarrow \mathbf{v}=\frac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j}+2\mathbf{k} \Rightarrow \mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=2\mathbf{j}$; Speed: $|\mathbf{v}(1)|=\sqrt{1^2+(2(1))^2+2^2}=3$;

Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|}=\frac{\mathbf{i}+2(1)\mathbf{j}+2\mathbf{k}}{3}=\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}(1)=3\left(\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right)$

10. $\mathbf{r}=(1+t)\mathbf{i}+\frac{t^2}{\sqrt{2}}\mathbf{j}+\frac{t^3}{3}\mathbf{k} \Rightarrow \mathbf{v}=\frac{d\mathbf{r}}{dt}=\mathbf{i}+\frac{2t}{\sqrt{2}}\mathbf{j}+t^2\mathbf{k} \Rightarrow \mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=\frac{2}{\sqrt{2}}\mathbf{j}+2t\mathbf{k}$;

Speed: $|\mathbf{v}(1)|=\sqrt{1^2+\left(\frac{2(1)}{\sqrt{2}}\right)^2+\left(1^2\right)^2}=2$;

Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|}=\frac{\mathbf{i}+\frac{2(1)}{\sqrt{2}}\mathbf{j}+\left(1^2\right)\mathbf{k}}{2}=\frac{1}{2}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}+\frac{1}{2}\mathbf{k} \Rightarrow \mathbf{v}(1)=2\left(\frac{1}{2}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}+\frac{1}{2}\mathbf{k}\right)$

11. $\mathbf{r}=(2\cos t)\mathbf{i}+(3\sin t)\mathbf{j}+4t\mathbf{k} \Rightarrow \mathbf{v}=\frac{d\mathbf{r}}{dt}=(-2\sin t)\mathbf{i}+(3\cos t)\mathbf{j}+4\mathbf{k} \Rightarrow \mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=(-2\cos t)\mathbf{i}-(3\sin t)\mathbf{j}$;

Speed: $\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right|=\sqrt{\left(-2\sin\frac{\pi}{2}\right)^2+\left(3\cos\frac{\pi}{2}\right)^2+4^2}=2\sqrt{5}$;

Direction: $\frac{\mathbf{v}\left(\frac{\pi}{2}\right)}{\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right|}=\left(-\frac{2}{2\sqrt{5}}\sin\frac{\pi}{2}\right)\mathbf{i}+\left(\frac{3}{2\sqrt{5}}\cos\frac{\pi}{2}\right)\mathbf{j}+\frac{4}{2\sqrt{5}}\mathbf{k}=-\frac{1}{\sqrt{5}}\mathbf{i}+\frac{2}{\sqrt{5}}\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{2}\right)=2\sqrt{5}\left(-\frac{1}{\sqrt{5}}\mathbf{i}+\frac{2}{\sqrt{5}}\mathbf{k}\right)$

12. $\mathbf{r}=(\sec t)\mathbf{i}+(\tan t)\mathbf{j}+\frac{4}{3}t\mathbf{k} \Rightarrow \mathbf{v}=\frac{d\mathbf{r}}{dt}=\left(\sec t\tan t\right)\mathbf{i}+\left(\sec^2 t\right)\mathbf{j}+\frac{4}{3}\mathbf{k}$

$\Rightarrow \mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=\left(\sec t\tan^2 t+\sec^3 t\right)\mathbf{i}+\left(2\sec^2 t\tan t\right)\mathbf{j}$;

Speed: $\left|\mathbf{v}\left(\frac{\pi}{6}\right)\right|=\sqrt{\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6}\right)^2+\left(\sec^2\frac{\pi}{6}\right)^2+\left(\frac{4}{3}\right)^2}=2$;

Direction: $\frac{\mathbf{v}\left(\frac{\pi}{6}\right)}{\left|\mathbf{v}\left(\frac{\pi}{6}\right)\right|}=\frac{\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6}\right)\mathbf{i}+\left(\sec^2\frac{\pi}{6}\right)\mathbf{j}+\frac{4}{3}\mathbf{k}}{2}=\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{6}\right)=2\left(\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right)$

13. $\mathbf{r}=\left(2\ln(t+1)\right)\mathbf{i}+t^2\mathbf{j}+\frac{t^2}{2}\mathbf{k} \Rightarrow \mathbf{v}=\frac{d\mathbf{r}}{dt}=\left(\frac{2}{t+1}\right)\mathbf{i}+2t\mathbf{j}+t\mathbf{k} \Rightarrow \mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=\left[\frac{-2}{(t+1)^2}\right]\mathbf{i}+2\mathbf{j}+\mathbf{k}$;

Speed: $|\mathbf{v}(1)|=\sqrt{\left(\frac{2}{1+1}\right)^2+(2(1))^2+1^2}=\sqrt{6}$;

Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|}=\frac{\left(\frac{2}{1+1}\right)\mathbf{i}+2(1)\mathbf{j}+(1)\mathbf{k}}{\sqrt{6}}=\frac{1}{\sqrt{6}}\mathbf{i}+\frac{2}{\sqrt{6}}\mathbf{j}+\frac{1}{\sqrt{6}}\mathbf{k} \Rightarrow \mathbf{v}(1)=\sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i}+\frac{2}{\sqrt{6}}\mathbf{j}+\frac{1}{\sqrt{6}}\mathbf{k}\right)$

14. $\mathbf{r}=\left(e^{-t}\right)\mathbf{i}+\left(2\cos 3t\right)\mathbf{j}+\left(2\sin 3t\right)\mathbf{k} \Rightarrow \mathbf{v}=\frac{d\mathbf{r}}{dt}=\left(-e^{-1}\right)\mathbf{i}-\left(6\sin 3t\right)\mathbf{j}+\left(6\cos 3t\right)\mathbf{k}$

$\Rightarrow \mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=\left(e^{-t}\right)\mathbf{i}-\left(18\cos 3t\right)\mathbf{j}-\left(18\sin 3t\right)\mathbf{k}$;

Speed: $|\mathbf{v}(0)|=\sqrt{\left(-e^0\right)^2+\left(-6\sin 3(0)\right)^2+\left(6\cos 3(0)\right)^2}=\sqrt{37}$;

Direction: $\frac{\mathbf{v}(0)}{|\mathbf{v}(0)|}=\frac{\left(-e^0\right)\mathbf{i}-6\sin 3(0)\mathbf{j}+6\cos 3(0)\mathbf{k}}{\sqrt{37}}=-\frac{1}{\sqrt{37}}\mathbf{i}+\frac{6}{\sqrt{37}}\mathbf{k} \Rightarrow \mathbf{v}(0)=\sqrt{37}\left(-\frac{1}{\sqrt{37}}\mathbf{i}+\frac{6}{\sqrt{37}}\mathbf{k}\right)$

15. $\mathbf{v}=3\mathbf{i}+\sqrt{3}\mathbf{j}+2t\mathbf{k}$ and $\mathbf{a}=2\mathbf{k} \Rightarrow \mathbf{v}(0)=3\mathbf{i}+\sqrt{3}\mathbf{j}$ and $\mathbf{a}(0)=2\mathbf{k} \Rightarrow |\mathbf{v}(0)|=\sqrt{3^2+\left(\sqrt{3}\right)^2+0^2}=\sqrt{12}$ and

$|\mathbf{a}(0)|=\sqrt{2^2}=2$; $\mathbf{v}(0)\cdot\mathbf{a}(0)=0 \Rightarrow \cos\theta=0 \Rightarrow \theta=\frac{\pi}{2}$

16. $\mathbf{v}=\frac{\sqrt{2}}{2}\mathbf{i}+\left(\frac{\sqrt{2}}{2}-32t\right)\mathbf{j}$ and $\mathbf{a}=-32\mathbf{j} \Rightarrow \mathbf{v}(0)=\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j}$ and $\mathbf{a}(0)=-32\mathbf{j} \Rightarrow |\mathbf{v}(0)|=\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}=1$ and $|\mathbf{a}(0)|=\sqrt{(-32)^2}=32; \mathbf{v}(0)\cdot\mathbf{a}(0)=\left(\frac{\sqrt{2}}{2}\right)(-32)=-16\sqrt{2} \Rightarrow \cos\theta=\frac{-16\sqrt{2}}{1(32)}=-\frac{\sqrt{2}}{2} \Rightarrow \theta=\frac{3\pi}{4}$

17. $\mathbf{v}=\left(\frac{2t}{t^2+1}\right)\mathbf{i}+\left(\frac{1}{t^2+1}\right)\mathbf{j}+t\left(t^2+1\right)^{-1/2}\mathbf{k}$ and $\mathbf{a}=\left[\frac{-2t^2+2}{\left(t^2+1\right)^2}\right]\mathbf{i}-\left[\frac{2t}{\left(t^2+1\right)^2}\right]\mathbf{j}+\left[\frac{1}{\left(t^2+1\right)^{3/2}}\right]\mathbf{k} \Rightarrow \mathbf{v}(0)=\mathbf{j}$ and $\mathbf{a}(0)=2\mathbf{i}+\mathbf{k} \Rightarrow |\mathbf{v}(0)|=1$ and $|\mathbf{a}(0)|=\sqrt{2^2+1^2}=\sqrt{5}; \mathbf{v}(0)\cdot\mathbf{a}(0)=0 \Rightarrow \cos\theta=0 \Rightarrow \theta=\frac{\pi}{2}$

18. $\mathbf{v}=\frac{2}{3}(1+t)^{1/2}\mathbf{i}-\frac{2}{3}(1-t)^{1/2}\mathbf{j}+\frac{1}{3}\mathbf{k}$ and $a=\frac{1}{3}(1+t)^{-1/2}\mathbf{i}+\frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \mathbf{v}(0)=\frac{2}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{1}{3}\mathbf{k}$ and $\mathbf{a}(0)=\frac{1}{3}\mathbf{i}+\frac{1}{3}\mathbf{j} \Rightarrow |\mathbf{v}(0)|=\sqrt{\left(\frac{2}{3}\right)^2+\left(-\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2}=1$ and $|\mathbf{a}(0)|=\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}=\frac{\sqrt{2}}{3}$; $\mathbf{v}(0)\cdot\mathbf{a}(0)=\frac{2}{9}-\frac{2}{9}=0$ $\Rightarrow \cos\theta=0 \Rightarrow \theta=\frac{\pi}{2}$

19. $\mathbf{r}(t)=(\sin t)\mathbf{i}+\left(t^2-\cos t\right)\mathbf{j}+e^t\mathbf{k} \Rightarrow \mathbf{v}(t)=(\cos t)\mathbf{i}+(2t+\sin t)\mathbf{j}+e^t\mathbf{k}$; $t_0=0 \Rightarrow \mathbf{v}(t_0)=\mathbf{i}+\mathbf{k}$ and $\mathbf{r}(t_0)=P_0=(0,-1,1) \Rightarrow x=0+t=t,\ y=-1,$ and $z=1+t$ are parametric equations of the tangent line

20. $\mathbf{r}(t)=t^2\mathbf{i}+(2t-1)\mathbf{j}+t^3\mathbf{k} \Rightarrow \mathbf{v}(t)=2t\mathbf{i}+2\mathbf{j}+3t^2\mathbf{k}$; $t_0=2 \Rightarrow \mathbf{v}(2)=4\mathbf{i}+2\mathbf{j}+12\mathbf{k}$ and $\mathbf{r}(t_0)=P_0=(4,3,8)$ $\Rightarrow x=4+4t,\ y=3+2t,$ and $z=8+12t$ are parametric equations of the tangent line

21. $\mathbf{r}(t)=(\ln t)\mathbf{i}+\frac{t-1}{t+2}\mathbf{j}+(t\ln t)\mathbf{k} \Rightarrow \mathbf{v}(t)=\frac{1}{t}\mathbf{i}+\frac{3}{(t+2)^2}\mathbf{j}+(\ln t+1)\mathbf{k}$; $t_0=1 \Rightarrow \mathbf{v}(1)=\mathbf{i}+\frac{1}{3}\mathbf{j}+\mathbf{k}$ and $\mathbf{r}(t_0)=P_0=(0,0,0) \Rightarrow x=0+t=t,\ y=0+\frac{1}{3}t=\frac{1}{3}t,$ and $z=0+t=t$ are parametric equations of the tangent line

22. $\mathbf{r}(t)=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}+(\sin 2t)\mathbf{k} \Rightarrow \mathbf{v}(t)=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}+(2\cos 2t)\mathbf{k}$; $t_0=\frac{\pi}{2} \Rightarrow \mathbf{v}(t_0)=-\mathbf{i}-2\mathbf{k}$ and $\mathbf{r}(t_0)=P_0=(0,1,0) \Rightarrow x=0-t=-t,\ y=1,$ and $z=0-2t=-2t$ are parametric equations of the tangent line

23. (a) $\mathbf{v}(t)=-(\sin t)\mathbf{i}+(\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t)=-(\cos t)\mathbf{i}-(\sin t)\mathbf{j}$;
   (i) $|\mathbf{v}(t)|=\sqrt{(-\sin t)^2+(\cos t)^2}=1 \Rightarrow$ constant speed;
   (ii) $\mathbf{v}\cdot\mathbf{a}=(\sin t)(\cos t)-(\cos t)(\sin t)=0 \Rightarrow$ yes, orthogonal;
   (iii) counterclockwise movement;
   (iv) yes, $\mathbf{r}(0)=\mathbf{i}+0\mathbf{j}$
   (b) $\mathbf{v}(t)=-(2\sin 2t)\mathbf{i}+(2\cos 2t)\mathbf{j} \Rightarrow \mathbf{a}(t)=-(4\cos 2t)\mathbf{i}-(4\sin 2t)\mathbf{j}$;
   (i) $|\mathbf{v}(t)|=\sqrt{4\sin^2 2t+4\cos^2 2t}=2 \Rightarrow$ constant speed;
   (ii) $\mathbf{v}\cdot\mathbf{a}=8\sin 2t\cos 2t-8\cos 2t\sin 2t=0 \Rightarrow$ yes, orthogonal;
   (iii) counterclockwise movement;
   (iv) yes, $\mathbf{r}(0)=\mathbf{i}+0\mathbf{j}$
   (c) $\mathbf{v}(t)=-\sin\left(t-\frac{\pi}{2}\right)\mathbf{i}+\cos\left(t-\frac{\pi}{2}\right)\mathbf{j} \Rightarrow \mathbf{a}(t)=-\cos\left(t-\frac{\pi}{2}\right)\mathbf{i}-\sin\left(t-\frac{\pi}{2}\right)\mathbf{j}$;
   (i) $|\mathbf{v}(t)|=\sqrt{\sin^2\left(t-\frac{\pi}{2}\right)+\cos^2\left(t-\frac{\pi}{2}\right)}=1 \Rightarrow$ constant speed;

(ii) $\mathbf{v}\cdot\mathbf{a} = \sin\left(t-\frac{\pi}{2}\right)\cos\left(t-\frac{\pi}{2}\right) - \cos\left(t-\frac{\pi}{2}\right)\sin\left(t-\frac{\pi}{2}\right) = 0 \Rightarrow$ yes, orthogonal;
(iii) counterclockwise movement;
(iv) no, $\mathbf{r}(0) = 0\mathbf{i} - \mathbf{j}$ instead of $\mathbf{i} + 0\mathbf{j}$

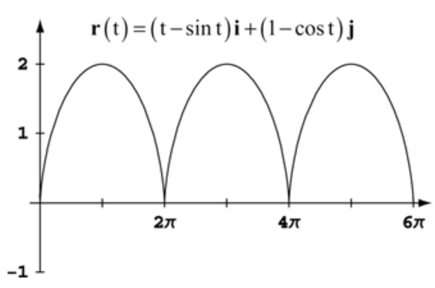
(d) $\mathbf{v}(t) = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} + (\sin t)\mathbf{j}$;
(i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow$ constant speed;
(ii) $\mathbf{v}\cdot\mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;
(iii) clockwise movement;
(iv) yes, $\mathbf{r}(0) = \mathbf{i} - 0\mathbf{j}$

(e) $\mathbf{v}(t) = -(2t\sin t)\mathbf{i} + (2t\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(2\sin t + 2t\cos t)\mathbf{i} + (2\cos t - 2t\sin t)\mathbf{j}$;
(i) $|\mathbf{v}(t)| = \sqrt{(-(2t\sin t))^2 + (2t\cos t)^2} = \sqrt{4t^2\left(\sin^2 t + \cos^2 t\right)} = 2|t| = 2t, t \ge 0 \Rightarrow$ variable speed;
(ii) $\mathbf{v}\cdot\mathbf{a} = 4\left(t\sin^2 t + t^2\sin t\cos t\right) + 4\left(t\cos^2 t - t^2\cos t\sin t\right) = 4t \ne 0$ in general $\Rightarrow$ not orthogonal in general;
(iii) counterclockwise movement;
(iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

24. Let $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ denote the position vector of the point $(2, 2, 1)$ and let, $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and $\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$. Then $\mathbf{r}(t) = \mathbf{p} + (\cos t)\mathbf{u} + (\sin t)\mathbf{v}$. Note that $(2, 2, 1)$ is a point on the plane and $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is normal to the plane. Moreover, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors with $\mathbf{u}\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel to the plane. Therefore, $\mathbf{r}(t)$ identifies a point that lies in the plane for each t. Also, for each t, $(\cos t)\mathbf{u} + (\sin t)\mathbf{v}$ is a unit vector. Starting at the point $\left(2 + \frac{1}{\sqrt{2}}, 2 - \frac{1}{\sqrt{2}}, 1\right)$ the vector $\mathbf{r}(t)$ traces out a circle of radius 1 and center $(2, 2, 1)$ in the plane $x + y - 2z = 2$.

25. The velocity vector is tangent to the graph of $y^2 = 2x$ at the point $(2, 2)$, has length 5, and a positive $\mathbf{i}$ component. Now, $y^2 = 2x \Rightarrow 2y\frac{dy}{dx} = 2 \Rightarrow \left.\frac{dy}{dx}\right|_{(2,2)} = \frac{2}{2\cdot 2} = \frac{1}{2} \Rightarrow$ the tangent vector lies in the direction of the vector $\mathbf{i} + \frac{1}{2}\mathbf{j} \Rightarrow$ the velocity vector is $\mathbf{v} = \frac{5}{\sqrt{1+\frac{1}{4}}}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \frac{5}{\left(\frac{\sqrt{5}}{2}\right)}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = 2\sqrt{5}\mathbf{i} + \sqrt{5}\mathbf{j}$

26. (a)



(b) $\mathbf{v} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $|\mathbf{v}|^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2\cos t \Rightarrow |\mathbf{v}|^2$ is at a max when $\cos t = -1 \Rightarrow t = \pi, 3\pi, 5\pi$, etc., and at these values of t, $|\mathbf{v}|^2 = 4 \Rightarrow \max|\mathbf{v}| = \sqrt{4} = 2$; $|\mathbf{v}|^2$ is at a min when $\cos t = 1 \Rightarrow t = 0, 2\pi, 4\pi$, etc., and at these values of t, $|\mathbf{v}|^2 = 0 \Rightarrow \min|\mathbf{v}| = 0$; $|\mathbf{a}|^2 = \sin^2 t + \cos^2 t = 1$ for every $t \Rightarrow \max|\mathbf{a}| = \min|\mathbf{a}| = \sqrt{1} = 1$

27. $\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \mathbf{r}\cdot\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt}\cdot\mathbf{r} = 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 2\cdot 0 = 0 \Rightarrow \mathbf{r}\cdot\mathbf{r}$ is a constant $\Rightarrow |\mathbf{r}| = \sqrt{\mathbf{r}\cdot\mathbf{r}}$ is constant

28. (a) $\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}) = \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}) + \mathbf{u}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{w}) = \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}) + \mathbf{u}\cdot\left(\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \mathbf{v}\times\frac{d\mathbf{w}}{dt}\right)$

$= \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}) + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \mathbf{u}\cdot\mathbf{v}\times\frac{d\mathbf{w}}{dt}$

(b) $\frac{d}{dt}\left[\mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)\right] = \frac{d\mathbf{r}}{dt}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right) + \mathbf{r}\cdot\left(\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}\right) + \mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right) = \mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right)$, since $\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = 0$ and $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{B}) = 0$ for any vectors $\mathbf{A}$ and $\mathbf{B}$

29. (a) $\mathbf{u} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow c\mathbf{u} = cf(t)\mathbf{i} + cg(t)\mathbf{j} + ch(t)\mathbf{k} \Rightarrow \frac{d}{dt}(c\mathbf{u}) = c\frac{df}{dt}\mathbf{i} + c\frac{dg}{dt}\mathbf{j} + c\frac{dh}{dt}\mathbf{k}$

$= c\left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right) = c\frac{d\mathbf{u}}{dt}$

(b) $\mathbf{u} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \Rightarrow f(t)\mathbf{u} = f(t)x(t)\mathbf{i} + f(t)y(t)\mathbf{j} + f(t)z(t)\mathbf{k}$

$\Rightarrow \frac{d}{dt}(f(t)\mathbf{u}) = \left[\frac{df}{dt}x(t) + f(t)\frac{dx}{dt}\right]\mathbf{i} + \left[\frac{df}{dt}y(t) + f(t)\frac{dy}{dt}\right]\mathbf{j} + \left[\frac{df}{dt}z(t) + f(t)\frac{dz}{dt}\right]\mathbf{k}$

$= \frac{df}{dt}\left[x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\right] + f(t)\left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right] = \frac{df}{dt}\mathbf{u} + f(t)\frac{d\mathbf{u}}{dt}$

30. Let $\mathbf{u} = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ and $\mathbf{v} = g_1(t)\mathbf{i} + g_2(t)\mathbf{j} + g_3(t)\mathbf{k}$.

Then $\mathbf{u} + \mathbf{v} = \left[f_1(t) + g_1(t)\right]\mathbf{i} + \left[f_2(t) + g_2(t)\right]\mathbf{j} + \left[f_3(t) + g_3(t)\right]\mathbf{k}$

$\Rightarrow \frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \left[f_1'(t) + g_1'(t)\right]\mathbf{i} + \left[f_2'(t) + g_2'(t)\right]\mathbf{j} + \left[f_3'(t) + g_3'(t)\right]\mathbf{k}$

$= \left[f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}\right] + \left[g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}\right] = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$;

$\mathbf{u} - \mathbf{v} = \left[f_1(t) - g_1(t)\right]\mathbf{i} + \left[f_2(t) - g_2(t)\right]\mathbf{j} + \left[f_3(t) - g_3(t)\right]\mathbf{k}$

$\Rightarrow \frac{d}{dt}(\mathbf{u} - \mathbf{v}) = \left[f_1'(t) - g_1'(t)\right]\mathbf{i} + \left[f_2'(t) - g_2'(t)\right]\mathbf{j} + \left[f_3'(t) - g_3'(t)\right]\mathbf{k}$

$= \left[f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}\right] - \left[g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}\right] = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}$

31. Suppose $\mathbf{r}$ is continuous at $t = t_0$. Then $\lim_{t\to t_0}\mathbf{r}(t) = \mathbf{r}(t_0) \Leftrightarrow \lim_{t\to t_0}\left[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}\right]$ $= f(t_0)\mathbf{i} + g(t_0)\mathbf{j} + h(t_0)\mathbf{k} \Leftrightarrow \lim_{t\to t_0} f(t) = f(t_0)$, $\lim_{t\to t_0} g(t) = g(t_0)$, and $\lim_{t\to t_0} h(t) = h(t_0) \Leftrightarrow f$, g, and h are continuous at $t = t_0$.

32. $$\lim_{t\to t_0}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)] = \lim_{t\to t_0}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ f_1(t) & f_2(t) & f_3(t)\\ g_1(t) & g_2(t) & g_3(t)\end{vmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \lim_{t\to t_0} f_1(t) & \lim_{t\to t_0} f_2(t) & \lim_{t\to t_0} f_3(t)\\ \lim_{t\to t_0} g_1(t) & \lim_{t\to t_0} g_2(t) & \lim_{t\to t_0} g_3(t)\end{vmatrix} = \lim_{t\to t_0}\mathbf{r}_1(t)\times\lim_{t\to t_0}\mathbf{r}_2(t)$$

$= \mathbf{A}\times\mathbf{B}$

33. $r'(t_0)$ exists $\Rightarrow f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j} + h'(t_0)\mathbf{k}$ exists $\Rightarrow f'(t_0), g'(t_0), h'(t_0)$ all exist $\Rightarrow f, g,$ and h are continuous at $t = t_0 \Rightarrow \mathbf{r}(t)$ is continuous at $t = t_0$

34. $\mathbf{u} = \mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ with a, b, c real constants $\Rightarrow \frac{d\mathbf{u}}{dt} = \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0$

35-38. Example CAS commands:

Maple:

```
> with( plots );
r := t -> [sin(t)-t*cos(t),cos(t)+t*sin(t),t^2];
t0 := 3*Pi/2;
lo := 0;
hi := 6*Pi;
P1 := spacecurve( r(t), t=lo..hi, axes=boxed, thickness=3 ):
display( P1, title="#35(a) (Section 13.1)" );
Dr := unapply( diff(r(t),t), t );                    # (b)
Dr(t0);                                              # (c)
q1 := expand( r(t0) + Dr(t0)*(t-t0) );
T := unapply( q1, t );
P2 := spacecurve( T(t), t=lo..hi, axes=boxed, thickness=3, color=black ):
display( [P1,P2], title="#35(d) (Section 13.1)" );
```

39-40. Example CAS commands:

Maple:

```
a := 'a'; b := 'b';
r := (a,b,t) -> [cos(a*t),sin(a*t),b*t];
Dr := unapply( diff(r(a,b,t),t), (a,b,t) );
t0 := 3*Pi/2;
q1 := expand( r(a,b,t0) + Dr(a,b,t0)*(t-t0) );
T := unapply( q1, (a,b,t) );
lo := 0;
hi := 4*Pi;
P := NULL:
for a in [ 1, 2, 4, 6 ] do
   P1 := spacecurve( r(a,1,t), t=lo..hi, thickness=3 ):
   P2 := spacecurve( T(a,1,t), t=lo..hi, thickness=3, color=black ):
   P := P, display( [P1,P2], axes=boxed, title=sprintf("#39 (Section 13.1)\n a=%a",a) );
end do:
display( [P], insequence=true );
```

35-40. Example CAS commands:

Mathematica: (assigned functions, parameters, and intervals will vary)
The x-y-z components for the curve are entered as a list of functions of t. The unit vectors **i**, **j**, **k** are not inserted.
If a graph is too small, highlight it and drag out a corner or side to make it larger.
Only the components of r[t] and values for t0, tmin, and tmax require alteration for each problem.

```
Clear[r, v, t, x, y, z]
r[t_]={ Sin[t] − t Cos[t], Cos[t] + t Sin[t], t^2}
t0= 3π/2; tmin= 0; tmax= 6π;
ParametricPlot3D[Evaluate[r[t]], {t, tmin, tmax}, AxesLabel → {x, y, z}];
v[t_]= r'[t]
tanline[t_]= v[t0] t + r[t0]
ParametricPlot3D[Evaluate[{r[t], tanline[t]}], {t, tmin, tmax}, AxesLabel → {x, y, z}];
```

For 39 and 40, the curve can be defined as a function of t, a, and b. Leave a space between a and t and b and t.

```
Clear[r, v, t, x, y, z, a, b]
r[t_,a_,b_]:={Cos[a t], Sin[a t], b t}
t0= 3π/2; tmin= 0; tmax= 4π;
v[t_,a_,b_]= D[r[t, a, b], t]
tanline[t_,a_,b_]= v[t0, a, b] t + r[t0, a, b]
pa1=ParametricPlot3D[Evaluate[{r[t, 1, 1], tanline[t, 1, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
pa2=ParametricPlot3D[Evaluate[{r[t, 2, 1], tanline[t, 2, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
pa4=ParametricPlot3D[Evaluate[{r[t, 4, 1], tanline[t, 4, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
pa6=ParametricPlot3D[Evaluate[{r[t, 6, 1], tanline[t, 6, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
Show[GraphicsRow[{pa1, pa2, pa4, pa6}]]
```

13.2 INTEGRALS OF VECTOR FUNCTIONS; PROJECTILE MOTION

1. $\int_0^1\left[t^3\mathbf{i}+7\mathbf{j}+(t+1)\mathbf{k}\right]dt=\left[\frac{t^4}{4}\right]_0^1\mathbf{i}+\left[7t\right]_0^1\mathbf{j}+\left[\frac{t^2}{2}+t\right]_0^1\mathbf{k}=\frac{1}{4}\mathbf{i}+7\mathbf{j}+\frac{3}{2}\mathbf{k}$

2. $\int_1^2\left[(6-6t)\mathbf{i}+3\sqrt{t}\,\mathbf{j}+\left(\frac{4}{t^2}\right)\mathbf{k}\right]dt=\left[6t-3t^2\right]_1^2\mathbf{i}+\left[2t^{3/2}\right]_1^2\mathbf{j}+\left[-4t^{-1}\right]_1^2\mathbf{k}=-3\mathbf{i}+\left(4\sqrt{2}-2\right)\mathbf{j}+2\mathbf{k}$

3. $\int_{-\pi/4}^{\pi/4}\left[(\sin t)\mathbf{i}+(1+\cos t)\mathbf{j}+\left(\sec^2 t\right)\mathbf{k}\right]dt=\left[-\cos t\right]_{-\pi/4}^{\pi/4}\mathbf{i}+\left[t+\sin t\right]_{-\pi/4}^{\pi/4}\mathbf{j}+\left[\tan t\right]_{-\pi/4}^{\pi/4}\mathbf{k}=\left(\frac{\pi+2\sqrt{2}}{2}\right)\mathbf{j}+2\mathbf{k}$

4. $\int_0^{\pi/3}\left[(\sec t\tan t)\mathbf{i}+(\tan t)\mathbf{j}+(2\sin t\cos t)\mathbf{k}\right]dt=\int_0^{\pi/3}\left[(\sec t\tan t)\mathbf{i}+(\tan t)\mathbf{j}+(\sin 2t)\mathbf{k}\right]dt$

$=\left[\sec t\right]_0^{\pi/3}\mathbf{i}+\left[-\ln(\cos t)\right]_0^{\pi/3}\mathbf{j}+\left[-\frac{1}{2}\cos 2t\right]_0^{\pi/3}\mathbf{k}=\mathbf{i}+(\ln 2)\mathbf{j}+\frac{3}{4}\mathbf{k}$

5. $\int_1^4\left(\frac{1}{t}\mathbf{i}+\frac{1}{5-t}\mathbf{j}+\frac{1}{2t}\mathbf{k}\right)dt=\left[\ln t\right]_1^4\mathbf{i}+\left[-\ln(5-t)\right]_1^4\mathbf{j}+\left[\frac{1}{2}\ln t\right]_1^4\mathbf{k}=(\ln 4)\mathbf{i}+(\ln 4)\mathbf{j}+(\ln 2)\mathbf{k}$

6. $\int_0^1 \left(\frac{2}{\sqrt{1-t^2}} \mathbf{i} + \frac{\sqrt{3}}{1+t^2} \mathbf{k} \right) dt = \left[2 \sin^{-1} t \right]_0^1 \mathbf{i} + \left[\sqrt{3} \tan^{-1} t \right]_0^1 \mathbf{k} = \pi \mathbf{i} + \frac{\pi\sqrt{3}}{4} \mathbf{k}$

7. $\int_0^1 \left(te^{t^2} \mathbf{i} + e^{-t} \mathbf{j} + \mathbf{k} \right) dt = \left[\frac{1}{2} e^{t^2} \right]_0^1 \mathbf{i} - \left[e^{-t} \right]_0^1 \mathbf{j} + \left[t \right]_0^1 \mathbf{k} = \frac{e-1}{2} \mathbf{i} + \frac{e-1}{e} \mathbf{i} + \mathbf{k}$

8. $\int_1^{\ln 3} \left(te^t \mathbf{i} + e^t \mathbf{j} + \ln t \, \mathbf{k} \right) dt = \left[te^t - e^t \right]_1^{\ln 3} \mathbf{i} - \left[e^t \right]_1^{\ln 3} \mathbf{j} + \left[t \ln t - t \right]_1^{\ln 3} \mathbf{k}$
$= 3(\ln 3 - 1) \mathbf{i} + (3 - e) \mathbf{j} + \left(\ln 3 (\ln(\ln 3) - 1) + 1 \right) \mathbf{k}$

9. $\int_0^{\pi/2} \left[(\cos t) \mathbf{i} - (\sin 2t) \mathbf{j} + \left(\sin^2 t \right) \mathbf{k} \right] dt = \int_0^{\pi/2} \left[(\cos t) \mathbf{i} - (\sin 2t) \mathbf{j} + \left(\frac{1}{2} - \frac{1}{2} \cos 2t \right) \mathbf{k} \right] dt$
$= \left[\sin t \right]_0^{\pi/2} \mathbf{i} + \left[\frac{1}{2} \cos t \right]_0^{\pi/2} \mathbf{j} + \left[\frac{1}{2} t - \frac{1}{4} \sin 2t \right]_0^{\pi/2} \mathbf{k} = \mathbf{i} - \mathbf{j} + \frac{\pi}{4} \mathbf{k}$

10. $\int_0^{\pi/4} \left[(\sec t) \mathbf{i} + \left(\tan^2 t \right) \mathbf{j} - (t \sin t) \mathbf{k} \right] dt = \int_0^{\pi/4} \left[(\sec t) \mathbf{i} + \left(\sec^2 t - 1 \right) \mathbf{j} - (t \sin t) \mathbf{k} \right] dt$
$= \left[\ln(\sec t + \tan t) \right]_0^{\pi/4} \mathbf{i} + \left[\tan t - t \right]_0^{\pi/4} \mathbf{j} + \left[t \cos t - \sin t \right]_0^{\pi/4} \mathbf{k} = \ln\left(1+\sqrt{2}\right) \mathbf{i} + \left(1 - \frac{\pi}{4}\right) \mathbf{j} + \left(\frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \mathbf{k}$

11. $\mathbf{r} = \int (-t\mathbf{i} - t\mathbf{j} - t\mathbf{k}) \, dt = -\frac{t^2}{2} \mathbf{i} - \frac{t^2}{2} \mathbf{j} - \frac{t^2}{2} \mathbf{k} + \mathbf{C}$; $\mathbf{r}(0) = 0\mathbf{i} - 0\mathbf{j} - 0\mathbf{k} + \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
$\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 1 \right) \mathbf{i} + \left(-\frac{t^2}{2} + 2 \right) \mathbf{j} + \left(-\frac{t^2}{2} + 3 \right) \mathbf{k}$

12. $\mathbf{r} = \int \left[(180t) \mathbf{i} + \left(180t - 16t^2 \right) \mathbf{j} \right] dt = 90t^2 \mathbf{i} + \left(90t^2 - \frac{16}{3} t^3 \right) \mathbf{j} + \mathbf{C}$; $\mathbf{r}(0) = 90(0)^2 \mathbf{i} + \left[90(0)^2 - \frac{16}{3}(0)^3 \right] \mathbf{j} + \mathbf{C} = 100\mathbf{j}$
$\Rightarrow \mathbf{C} = 100\mathbf{j} \Rightarrow \mathbf{r} = 90t^2 \mathbf{i} + \left(90t^2 - \frac{16}{3} t^3 + 100 \right) \mathbf{j}$

13. $\mathbf{r} = \int \left[\left(\frac{3}{2} (t+1)^{1/2} \right) \mathbf{i} + e^{-t} \mathbf{j} + \left(\frac{1}{t+1} \right) \mathbf{k} \right] dt = (t+1)^{3/2} \mathbf{i} - e^{-t} \mathbf{j} + \ln(t+1) \mathbf{k} + \mathbf{C}$;
$\mathbf{r}(0) = (0+1)^{3/2} \mathbf{i} - e^{-0} \mathbf{j} + \ln(0+1) \mathbf{k} + \mathbf{C} = \mathbf{k} \Rightarrow \mathbf{C} = -\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r} = \left[(t+1)^{3/2} - 1 \right] \mathbf{i} + \left(1 - e^{-t} \right) \mathbf{j} + \left[1 + \ln(t+1) \right] \mathbf{k}$

14. $\mathbf{r} = \int \left[\left(t^3 + 4t \right) \mathbf{i} + t\mathbf{j} + 2t^2 \mathbf{k} \right] dt = \left(\frac{t^4}{4} + 2t^2 \right) \mathbf{i} + \frac{t^2}{2} \mathbf{j} + \frac{2t^3}{3} \mathbf{k} + \mathbf{C}$; $\mathbf{r}(0) = \left(\frac{0^4}{4} + 2(0)^2 \right) \mathbf{i} + \frac{0^2}{2} \mathbf{j} + \frac{2(0)^3}{3} \mathbf{k} + \mathbf{C} = \mathbf{i} + \mathbf{j}$
$\Rightarrow \mathbf{C} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{r} = \left(\frac{t^4}{4} + 2t^2 + 1 \right) \mathbf{i} + \left(\frac{t^2}{2} + 1 \right) \mathbf{j} + \frac{2t^3}{3} \mathbf{k}$

15. $\frac{d\mathbf{r}}{dt} = \int (-32\mathbf{k}) \, dt = -32t\mathbf{k} + \mathbf{C}_1$; $\frac{d\mathbf{r}}{dt}(0) = 8\mathbf{i} + 8\mathbf{j} \Rightarrow -32(0)\mathbf{k} + \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j} \Rightarrow \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = 8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k}$;
$\mathbf{r} = \int (8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k}) \, dt = 8t\mathbf{i} + 8t\mathbf{j} - 16t^2 \mathbf{k} + \mathbf{C}_2$; $\mathbf{r}(0) = 100\mathbf{k} \Rightarrow 8(0)\mathbf{i} + 8(0)\mathbf{j} - 16(0)^2 \mathbf{k} + \mathbf{C}_2 = 100\mathbf{k}$
$\Rightarrow \mathbf{C}_2 = 100\mathbf{k} \Rightarrow \mathbf{r} = 8t\mathbf{i} + 8t\mathbf{j} + \left(100 - 16t^2 \right) \mathbf{k}$

16. $\frac{d\mathbf{r}}{dt} = \int -(\mathbf{i}+\mathbf{j}+\mathbf{k})\,dt = -(t\mathbf{i}+t\mathbf{j}+t\mathbf{k})+\mathbf{C}_1$; $\frac{d\mathbf{r}}{dt}(0)=\mathbf{0} \Rightarrow -(0\mathbf{i}+0\mathbf{j}+0\mathbf{k})+\mathbf{C}_1=\mathbf{0} \Rightarrow \mathbf{C}_1=\mathbf{0}$
$\Rightarrow \frac{d\mathbf{r}}{dt} = -(t\mathbf{i}+t\mathbf{j}+t\mathbf{k})$; $\mathbf{r} = \int -(t\mathbf{i}+t\mathbf{j}+t\mathbf{k})\,dt = -\left(\frac{t^2}{2}\mathbf{i}+\frac{t^2}{2}\mathbf{j}+\frac{t^2}{2}\mathbf{k}\right)+\mathbf{C}_2$; $\mathbf{r}(0)=10\mathbf{i}+10\mathbf{j}+10\mathbf{k}$
$\Rightarrow -\left(\frac{0^2}{2}\mathbf{i}+\frac{0^2}{2}\mathbf{j}+\frac{0^2}{2}\mathbf{k}\right)+\mathbf{C}_2 = 10\mathbf{i}+10\mathbf{j}+10\mathbf{k} \Rightarrow \mathbf{C}_2 = 10\mathbf{i}+10\mathbf{j}+10\mathbf{k}$
$\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2}+10\right)\mathbf{i}+\left(-\frac{t^2}{2}+10\right)\mathbf{j}+\left(-\frac{t^2}{2}+10\right)\mathbf{k}$

17. $\frac{d\mathbf{v}}{dt}=\mathbf{a}=3\mathbf{i}-\mathbf{j}+\mathbf{k} \Rightarrow \mathbf{v}(t)=3t\mathbf{i}-t\mathbf{j}+t\mathbf{k}+\mathbf{C}_1$; the particle travels in the direction of the vector $(4-1)\mathbf{i}+(1-2)\mathbf{j}+(4-3)\mathbf{k}=3\mathbf{i}-\mathbf{j}+\mathbf{k}$ (since it travels in a straight line), and at time $t=0$ it has speed 2
$\Rightarrow \mathbf{v}(0)=\frac{2}{\sqrt{9+1+1}}(3\mathbf{i}-\mathbf{j}+\mathbf{k})=\mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{v}(t)=\left(3t+\frac{6}{\sqrt{11}}\right)\mathbf{i}-\left(t+\frac{2}{\sqrt{11}}\right)\mathbf{j}+\left(t+\frac{2}{\sqrt{11}}\right)\mathbf{k}$
$\Rightarrow \mathbf{r}(t)=\left(\frac{3}{2}t^2+\frac{6}{\sqrt{11}}t\right)\mathbf{i}-\left(\frac{1}{2}t^2+\frac{2}{\sqrt{11}}t\right)\mathbf{j}+\left(\frac{1}{2}t^2+\frac{2}{\sqrt{11}}t\right)\mathbf{k}+\mathbf{C}_2$; $\mathbf{r}(0)=\mathbf{i}+2\mathbf{j}+3\mathbf{k}=\mathbf{C}_2$
$\Rightarrow \mathbf{r}(t)=\left(\frac{3}{2}t^2+\frac{6}{\sqrt{11}}t+1\right)\mathbf{i}-\left(\frac{1}{2}t^2+\frac{2}{\sqrt{11}}t-2\right)\mathbf{j}+\left(\frac{1}{2}t^2+\frac{2}{\sqrt{11}}t+3\right)\mathbf{k}=\left(\frac{1}{2}t^2+\frac{2}{\sqrt{11}}t\right)(3\mathbf{i}-\mathbf{j}+\mathbf{k})+(\mathbf{i}-2\mathbf{j}+3\mathbf{k})$

18. $\frac{d\mathbf{v}}{dt}=\mathbf{a}=2\mathbf{i}+\mathbf{j}+\mathbf{k} \Rightarrow \mathbf{v}(t)=2t\mathbf{i}+t\mathbf{j}+t\mathbf{k}+\mathbf{C}_1$; the particle travels in the direction of the vector $(3-1)\mathbf{i}+(0-(-1))\mathbf{j}+(3-2)\mathbf{k}=2\mathbf{i}+\mathbf{j}+\mathbf{k}$ (since it travels in a straight line), and at time $t=0$ it has speed 2
$\Rightarrow \mathbf{v}(0)=\frac{2}{\sqrt{4+1+1}}(2\mathbf{i}+\mathbf{j}+\mathbf{k})=\mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{v}(t)=\left(2t+\frac{4}{\sqrt{6}}\right)\mathbf{i}+\left(t+\frac{2}{\sqrt{6}}\right)\mathbf{j}+\left(t+\frac{2}{\sqrt{6}}\right)\mathbf{k}$
$\Rightarrow \mathbf{r}(t)=\left(t^2+\frac{6}{\sqrt{6}}t\right)\mathbf{i}+\left(\frac{1}{2}t^2+\frac{2}{\sqrt{6}}t\right)\mathbf{j}+\left(\frac{1}{2}t^2+\frac{2}{\sqrt{6}}t\right)\mathbf{k}+\mathbf{C}_2$; $\mathbf{r}(0)=\mathbf{i}-\mathbf{j}+2\mathbf{k}=\mathbf{C}_2$
$\Rightarrow \mathbf{r}(t)=\left(t^2+\frac{4}{\sqrt{6}}t+1\right)\mathbf{i}+\left(\frac{1}{2}t^2+\frac{2}{\sqrt{6}}t-1\right)\mathbf{j}+\left(\frac{1}{2}t^2+\frac{2}{\sqrt{6}}t+2\right)\mathbf{k}=\left(\frac{1}{2}t^2+\frac{2}{\sqrt{6}}t\right)(2\mathbf{i}+\mathbf{j}+\mathbf{k})+(\mathbf{i}-\mathbf{j}+2\mathbf{k})$

19. $x=(v_0\cos\alpha)t \Rightarrow (21\,\text{km})\left(\frac{1000\,\text{m}}{1\,\text{km}}\right)=(840\text{ m/s})(\cos 60°)t \Rightarrow t=\frac{21{,}000\,\text{m}}{(840\,\text{m/s})(\cos 60°)}=50$ seconds

20. $R=\frac{v_0^2}{g}\sin 2\alpha$ and maximum R occurs when $\alpha=45° \Rightarrow 24.5\text{ km}=\left(\frac{v_0^2}{9.8\text{m/s}^2}\right)(\sin 90°)$
$\Rightarrow v_0=\sqrt{(9.8)(24{,}500)\,\text{m}^2/\text{s}^2}=490$ m/s

21. (a) $t=\frac{2v_0\sin\alpha}{g}=\frac{2(500\text{ m/s})(\sin 45°)}{9.8\text{ m/s}^2}\approx 72.2$ seconds; $R=\frac{v_0^2}{g}\sin 2\alpha=\frac{(500\text{ m/s})^2}{9.8\text{ m/s}^2}(\sin 90°)\approx 25{,}510.2$ m

(b) $x=(v_0\cos\alpha)t \Rightarrow 5000\text{ m}=(500\text{ m/s})(\cos 45°)t \Rightarrow t=\frac{5000\text{ m}}{(500\text{ m/s})(\cos 45°)}\approx 14.14$ s; thus,
$y=(v_0\sin\alpha)t-\frac{1}{2}gt^2 \Rightarrow y\approx(500\text{ m/s})(\sin 45°)(14.14\,\text{s})-\frac{1}{2}(9.8\text{ m/s}^2)(14.14\,\text{s})^2\approx 4020$ m

(c) $y_{\max}=\frac{(v_0\sin\alpha)^2}{2g}=\frac{((500\,\text{m/s})(\sin 45°))^2}{2(9.8\,\text{m/s}^2)}\approx 6378\,\text{m}$

22. $y=y_0+(v_0\sin\alpha)t-\frac{1}{2}gt^2 \Rightarrow y=9.8\text{ m}+(9.8\text{ m/s})(\sin 30°)t-\frac{1}{2}(9.8\text{ m/s}^2)t^2 \Rightarrow y=9.8+4.9t-4.9t^2$;
the ball hits the ground when $y=0 \Rightarrow 0=9.8+4.9t-4.9t^2 \Rightarrow t=-1$ or $t=2 \Rightarrow t=2\,\text{s}$ since $t>0$; thus,
$x=(v_0\cos\alpha)t \Rightarrow x=(9.8\text{ m/s})(\cos 30°)t=9.8\left(\frac{\sqrt{3}}{2}\right)(2)\approx 16.97$ m

23. (a) $R = \frac{v_0^2}{g}\sin 2\alpha \Rightarrow 10\,\text{m} = \left(\frac{v_0^2}{9.8\,\text{m/s}^2}\right)(\sin 90°) \Rightarrow v_0^2 = 98\ \text{m}^2/\text{s}^2 \Rightarrow v_0 \approx 9.9\ \text{m/s};$

(b) $6\text{m} \approx \frac{(9.9\,\text{m/s}^2)}{9.8\,\text{m/s}^2}(\sin 2\alpha) \Rightarrow \sin 2\alpha \approx 0.59999 \Rightarrow 2\alpha \approx 36.87°$ or $143.12° \Rightarrow \alpha \approx 18.4°$ or $71.6°$

24. $v_0 = 5\times 10^6$ m/s and $x = 40\,\text{cm} = 0.4\,\text{m}$; thus $x = (v_0\cos\alpha)t \Rightarrow 0.4\ \text{m} = (5\times 10^6\ \text{m/s})(\cos 0°)t$
$\Rightarrow t = 0.08\times 10^{-6}\,\text{s} = 8\times 10^{-8}\text{s}$; also, $y = y_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2$
$\Rightarrow y = (5\times 10^6\ \text{m/s})(\sin 0°)(8\times 10^{-8}\,\text{s}) - \frac{1}{2}(9.8\,\text{m/s}^2)(8\times 10^{-8}\,\text{s})^2 = -3.136\times 10^{-14}\,\text{m}$ or $-3.136\times 10^{-12}\,\text{cm}.$
Therefore, it drops 3.136×10^{-12} cm.

25. $R = \frac{v_0^2}{g}\sin 2\alpha \Rightarrow 16{,}000\ \text{m} = \frac{(400\,\text{m/s})^2}{9.8\,\text{m/s}^2}\sin 2\alpha \Rightarrow \sin 2\alpha = 0.98 \Rightarrow 2\alpha \approx 78.5°$ or $2\alpha \approx 101.5° \Rightarrow \alpha \approx 39.3°$ or $50.7°$

26. (a) $R = \frac{(2v_0)^2}{g}\sin 2\alpha = \frac{4v_0^2}{g}\sin 2\alpha = 4\left(\frac{v_0^2}{g}\sin\alpha\right)$ or 4 times the original range.

(b) Now, let the initial range be $R = \frac{v_0^2}{g}\sin 2\alpha$. Then we want the factor p so that pv_0 will double the range $\Rightarrow \frac{(pv_0)^2}{g}\sin 2\alpha = 2\left(\frac{v_0^2}{g}\sin 2\alpha\right) \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}$ or about 141%. The same percentage will approximately double the height: $\frac{(pv_0\sin\alpha)^2}{2g} = \frac{2(v_0\sin\alpha)^2}{2g} \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}.$

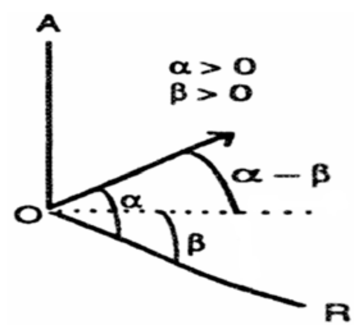
27. The projectile reaches its maximum height when its vertical component of velocity is zero
$\Rightarrow \frac{dy}{dt} = v_0\sin\alpha - gt = 0 \Rightarrow t = \frac{v_0\sin\alpha}{g} \Rightarrow y_{\max} = (v_0\sin\alpha)\left(\frac{v_0\sin\alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_0\sin\alpha}{g}\right)^2 = \frac{(v_0\sin\alpha)^2}{g} - \frac{(v_0\sin\alpha)^2}{2g}$
$= \frac{(v_0\sin\alpha)^2}{2g}$. To find the flight time we find the time when the projectile lands: $(v_0\sin\alpha)t - \frac{1}{2}gt^2 = 0$
$\Rightarrow t\left(v_0\sin\alpha - \frac{1}{2}gt\right) = 0 \Rightarrow t = 0$ or $t = \frac{2v_0\sin\alpha}{g}$. Since $t = 0$ is the time when the projectile is fired, then $t = \frac{2v_0\sin\alpha}{g}$ is the time when the projectile strikes the ground. The range is the value of the horizontal component when $t = \frac{2v_0\sin\alpha}{g} \Rightarrow R = x = (v_0\cos\alpha)\left(\frac{2v_0\sin\alpha}{g}\right) = \frac{v_0^2}{g}(2\sin\alpha\cos\alpha) = \frac{v_0^2}{g}\sin 2\alpha$. The range is largest when $2\alpha = 1 \Rightarrow \alpha = 45°$.

28. When marble A is located R units downrange, we have $x = (v_0\cos\alpha)t \Rightarrow R = (v_0\cos\alpha)t \Rightarrow t = \frac{R}{v_0\cos\alpha}$. At that time the height of marble A is $y = y_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2 = (v_0\sin\alpha)\left(\frac{R}{v_0\cos\alpha}\right) - \frac{1}{2}g\left(\frac{R}{v_0\cos\alpha}\right)^2$
$\Rightarrow y = R\tan\alpha - \frac{1}{2}g\left(\frac{R^2}{v_0^2\cos^2\alpha}\right)$. The height of marble B at the same time $t = \frac{R}{v_0\cos\alpha}$ seconds is
$h = R\tan\alpha - \frac{1}{2}gt^2 = R\tan\alpha - \frac{1}{2}g\left(\frac{R^2}{v_0^2\cos^2\alpha}\right)$. Since the heights are the same, the marbles collide regardless of the initial velocity v_0.

29. $\frac{d\mathbf{r}}{dt} = \int(-g\mathbf{j})dt = -gt\mathbf{j} + \mathbf{C}_1$ and $\frac{d\mathbf{r}}{dt}(0) = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j} \Rightarrow -g(0)\mathbf{j} - \mathbf{C}_1 = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j}$ $\Rightarrow \mathbf{C}_1 = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha - gt)\mathbf{j}$; $\mathbf{r} = \int[(v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha - gt)\mathbf{j}]dt$ $= (v_0 t\cos\alpha)\mathbf{i} + \left(v_0 t\sin\alpha - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{C}_2$ and $\mathbf{r}(0) = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow [v_0(0)\cos\alpha]\mathbf{i} + \left[v_0(0)\sin\alpha - \frac{1}{2}g(0)^2\right]\mathbf{j} + \mathbf{C}_2$ $= x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{C}_2 = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{r} = (x_0 + v_0 t\cos\alpha)\mathbf{i} + \left(y_0 + v_0 t\sin\alpha - \frac{1}{2}gt^2\right)\mathbf{j} \Rightarrow x = x_0 + v_0 t\cos\alpha$ and $y = y_0 + v_0 t\sin\alpha - \frac{1}{2}gt^2$

30. The maximum height is $y = \frac{(v_0\sin\alpha)^2}{2g}$ and this occurs for $x = \frac{v_0^2}{2g}\sin 2\alpha = \frac{v_0^2\sin\alpha\cos\alpha}{g}$. These equations describe parametrically the points on a curve in the xy-plane associated with the maximum heights on the parabolic trajectories in terms of the parameter (launch angle) α. Eliminating the parameter α, we have $x^2 = \frac{v_0^4\sin^2\alpha\cos^2\alpha}{g^2} = \frac{\left(v_0^4\sin^2\alpha\right)\left(1-\sin^2\alpha\right)}{g^2} = \frac{v_0^4\sin^2\alpha}{g^2} - \frac{v_0^4\sin^4\alpha}{g^2} = \frac{v_0^2}{g}(2y) - (2y)^2 \Rightarrow x^2 + 4y^2 - \left(\frac{2v_0^2}{g}\right)y = 0$ $\Rightarrow x^2 + 4\left[y^2 - \left(\frac{v_0^2}{2g}\right)y + \frac{v_0^4}{16g^2}\right] = \frac{v_0^4}{4g^2} \Rightarrow x^2 + 4\left(y - \frac{v_0^2}{4g}\right)^2 = \frac{v_0^4}{4g^2}$, where $x \ge 0$.

31. (a) At the time t when the projectile hits the line OR we have $\tan\beta = \frac{y}{x}$; $x = [v_0\cos(\alpha-\beta)]t$ and $y = [v_0\sin(\alpha-\beta)]t - \frac{1}{2}gt^2 < 0$ since R is below level ground. Therefore let $|y| = \frac{1}{2}gt^2 - [v_0\sin(\alpha-\beta)]t > 0$ so that



$\tan\beta = \frac{\left[\frac{1}{2}gt^2(v_0\sin(\alpha-\beta))t\right]}{[v_0\cos(\alpha-\beta)]t} = \frac{\left[\frac{1}{2}gt - v_0\sin(\alpha-\beta)\right]}{v_0\cos(\alpha-\beta)}$

$\Rightarrow v_0\cos(\alpha-\beta)\tan\beta = \frac{1}{2}gt - v_0\sin(\alpha-\beta)$

$\Rightarrow t = \frac{2v_0\sin(\alpha-\beta) + 2v_0\cos(\alpha-\beta)\tan\beta}{g}$,

which is the time when the projectile hits the downhill slope. Therefore,

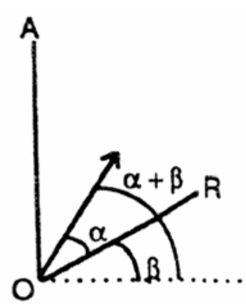
$x = [v_0\cos(\alpha-\beta)]\left[\frac{2v_0\sin(\alpha-\beta) + 2v_0\cos(\alpha-\beta)\tan\beta}{g}\right] = \frac{2v_0^2}{g}\left[\cos^2(\alpha-\beta)\tan\beta + \sin(\alpha-\beta)\cos(\alpha-\beta)\right]$.

If x is maximized, then OR is maximized: $\frac{dx}{d\alpha} = \frac{2v_0^2}{g}[-\sin 2(\alpha-\beta)\tan\beta + \cos 2(\alpha-\beta)] = 0$

$\Rightarrow -\sin 2(\alpha-\beta)\tan\beta + \cos 2(\alpha-\beta) = 0 \Rightarrow \tan\beta = \cot 2(\alpha-\beta) \Rightarrow 2(\alpha-\beta) = 90° - \beta$

$\Rightarrow \alpha - \beta = \frac{1}{2}(90° - \beta) \Rightarrow \alpha = \frac{1}{2}(90° + \beta) = \frac{1}{2}$ of $\angle AOR$.

(b) At the time t when the projectile hits OR we have $\tan\beta = \frac{y}{x}$; $x = [v_0\cos(\alpha+\beta)]t$ and $y = [v_0\sin(\alpha+\beta)]t - \frac{1}{2}gt^2$



$\Rightarrow \tan\beta = \frac{[v_0\sin(\alpha+\beta)]t - \frac{1}{2}gt^2}{[v_0\cos(\alpha+\beta)]t} = \frac{\left[v_0\sin(\alpha+\beta) - \frac{1}{2}gt\right]}{v_0\cos(\alpha+\beta)}$

$\Rightarrow v_0\cos(\alpha+\beta)\tan\beta = v_0\sin(\alpha+\beta) - \frac{1}{2}gt$

$\Rightarrow t = \frac{2v_0\sin(\alpha+\beta) - 2v_0\cos(\alpha+\beta)\tan\beta}{g}$, which is the time when the projectile hits the uphill slope.

Therefore, $x = \left[v_0 \cos(\alpha+\beta)\right]\left[\frac{2v_0 \sin(\alpha+\beta) - 2v_0 \cos(\alpha+\beta)\tan\beta}{g}\right]$

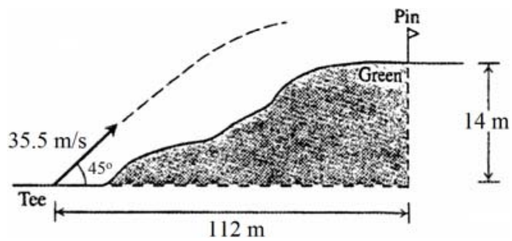
$= \frac{2v_0^2}{g}\left[\sin(\alpha+\beta)\cos(\alpha+\beta) - \cos^2(\alpha+\beta)\tan\beta\right]$. If x is maximized, then OR is maximized:

$\frac{dx}{d\alpha} = \frac{2v_0^2}{g}\left[\cos 2(\alpha+\beta) + \sin 2(\alpha+\beta)\tan\beta\right] = 0 \Rightarrow \cos 2(\alpha+\beta) + \sin 2(\alpha+\beta)\tan\beta = 0$

$\Rightarrow \cot 2(\alpha+\beta) + \tan\beta = 0 \Rightarrow \cot 2(\alpha+\beta) = -\tan\beta = \tan(-\beta) \Rightarrow 2(\alpha+\beta) = 90° - (-\beta) = 90° + \beta$

$\Rightarrow \alpha = \frac{1}{2}(90° - \beta) = \frac{1}{2}$ of $\angle AOR$. Therefore v_0 would bisect $\angle AOR$ for maximum range uphill.

32. $v_0 = 35.5$ m/s, $\alpha = 45°$, and $x = (v_0 \cos\alpha)t$
$\Rightarrow 112 = (35.5 \cos 45°)t \Rightarrow t \approx 4.46$ s;
also $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$
$\Rightarrow y = (35.5 \sin 45°)(4.46) - \frac{1}{2}(9.8)(4.46)^2 \approx 14.5$ m.
It will take the ball 4.46 s to travel 112 m. At that time the ball will be 14.5 m in the air and will hit the green past the pin.



33. (a) (Assuming that "x" is zero at the point of impact:)
$\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = (12\cos 27°)t$ and $y(t) = 1.3 + (12\sin 27°)t - 4.9t^2$.

(b) $y_{\max} = \frac{(v_0 \sin\alpha)^2}{2g} + 1.3 = \frac{(12\sin 27°)^2}{19.6} + 1.3 \approx 2.814$ m, which is reached at $t = \frac{v_0 \sin\alpha}{g} = \frac{12\sin 27°}{9.8}$
≈ 0.556 seconds.

(c) For the time, solve $y = 1.3 + (12\sin 27°)t - 4.9t^2 = 0$ for t, using the quadratic formula
$t = \frac{12\sin 27° + \sqrt{(-12\sin 27°)^2 + 25.48}}{9.8} \approx 1.31$s. Then the range is about $x(1.31) = (12\cos 27°)(1.31) \approx 14$ m.

(d) For the time, solve $y = 1.3 + (12\sin 27°)t - 4.9t^2 = 2.3$ for t, using the quadratic formula
$t = \frac{12\sin 27° \pm \sqrt{(-12\sin 27°)^2 - 19.6}}{9.8} \approx 0.232$ and 0.880 seconds. At those times the ball is about
$x(0.232) = (12\cos 27°)(0.232) \approx 2.48$ m and $x(0.880) = (12\cos 27°)(0.880) \approx 9.41$ m from the impact point, or about $14 - 2.48 \approx 11.52$ m and $14 - 9.41 \approx 4.59$ m from the landing spot.

(e) Yes. It changes things because the ball won't clear the net ($y_{\max} \approx 2.48$).

34. $x = x_0 + (v_0 \cos\alpha)t = 0 + (v_0 \cos 40°)t \approx 0.766\, v_0 t$ and $y = y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2 = 2 + (v_0 \sin 40°)t - 4.9t^2$
$\approx 2 + 0.643\, v_0 t - 4.9t^2$; now the shot went 22.63 m $\Rightarrow 22.63 = 0.766\, v_0 t \Rightarrow t \approx \frac{29.541}{v_0}$s; the shot lands when
$y = 0 \Rightarrow 0 = 2 + (0.643)(29.541) - 4.9\left(\frac{29.541}{v_0}\right)^2 \Rightarrow 0 \approx 20.989 - \frac{4276.2}{v_0^2} \Rightarrow v_0 \approx \sqrt{\frac{4276.2}{20.989}} \approx 14.27$ m/s, the shot's initial speed

35. Flight time $= 1$s and the measure of the angle of elevation is about $64°$ (using a protractor) so that
$t = \frac{2v_0 \sin\alpha}{g} \Rightarrow 1 = \frac{2v_0 \sin 64°}{9.8} \Rightarrow v_0 \approx 5.45$ m/s. Then $y_{\max} = \frac{(5.45 \sin 64°)^2}{2(9.8)} \approx 1.225$ m and $R = \frac{v_0^2}{g}\sin 2\alpha$
$\Rightarrow R = \frac{(5.45)^2}{9.8}\sin 128° \approx 2.39$ m $\Rightarrow$ the engine traveled about 2.39 m in 1s $\Rightarrow$ the engine velocity was about 2.39 m/s.

36. (a) $\mathbf{r}(t)=(x(t))\mathbf{i}+(y(t))\mathbf{j}$; where $x(t)=(40\cos 23°-4)t$ and $y(t)=0.8+(40\sin 23°)t-4.9t^2$.

(b) $y_{\max}=\frac{(v_0\sin\alpha)^2}{2g}+0.8=\frac{(40\sin 23°)^2}{19.6}+0.8\approx 13.26$ m, which is reached at $t=\frac{v_0\sin\alpha}{g}=\frac{40\sin 23°}{9.8}$ ≈ 1.595 seconds.

(c) For the time, solve $y=0.8+(40\sin 23°)t-4.9t^2=0$ for t, using the quadratic formula $t=\frac{40\sin 23°+\sqrt{(40\sin 23°)^2+15.68}}{9.8}\approx 3.24$ s. Then the range at $t\approx 3.24$ is about $x=(40\cos 23°-4)(3.24)\approx 106.34$ m.

(d) For the time, solve $y=0.8+(40\sin 23°)t-4.9t^2=6$ for t, using the quadratic formula $t=\frac{40\sin 23°+\sqrt{(40\sin 23°)^2-101.92}}{9.8}\approx 0.277$ and 2.812 seconds. At those times the ball is about $x(0.277)=(40\cos 23°-4)(0.277)\approx 9.09$ m from home plate and $x(2.812)=(40\cos 23°-4)(2.812)\approx 92.29$ m from home plate.

(e) Yes. According to part (d), the ball is still 6 m above the ground when it is 92.29 m from home plate.

37. $\frac{d^2\mathbf{r}}{dt^2}+k\frac{d\mathbf{r}}{dt}=-g\mathbf{j}\Rightarrow P(t)=k$ and $\mathbf{Q}(t)=-g\mathbf{j}\Rightarrow\int P(t)\,dt=kt\Rightarrow v(t)=e^{\int P(t)\,dt}=e^{kt}\Rightarrow\frac{d\mathbf{r}}{dt}=\frac{1}{v(t)}\int v(t)\mathbf{Q}(t)\,dt$

$=-ge^{-kt}\int e^{kt}\mathbf{j}\,dt=-ge^{-kt}\left[\frac{e^{kt}}{k}\mathbf{j}+\mathbf{C}_1\right]=-\frac{g}{k}\mathbf{j}+\mathbf{C}e^{-kt}$, where $\mathbf{C}=-g\mathbf{C}_1$; apply the initial condition:

$\frac{d\mathbf{r}}{dt}\Big|_{t=0}=(v_0\cos\alpha)\mathbf{i}+(v_0\sin\alpha)\mathbf{j}=-\frac{g}{k}\mathbf{j}+\mathbf{C}\Rightarrow\mathbf{C}=(v_0\cos\alpha)\mathbf{i}+\left(\frac{g}{k}+v_0\sin\alpha\right)\mathbf{j}$

$\Rightarrow\frac{d\mathbf{r}}{dt}=\left(v_0e^{-kt}\cos\alpha\right)\mathbf{i}+\left(-\frac{g}{k}+e^{-kt}\left(\frac{g}{k}+v_0\sin\alpha\right)\right)\mathbf{j},\quad\mathbf{r}=\int\left[\left(v_0e^{-kt}\cos\alpha\right)\mathbf{i}+\left(-\frac{g}{k}+e^{-kt}\left(\frac{g}{k}+v_0\sin\alpha\right)\right)\mathbf{j}\right]dt$

$=\left(-\frac{v_0}{k}e^{-kt}\cos\alpha\right)\mathbf{i}+\left(-\frac{gt}{k}-\frac{e^{-kt}}{k}\left(\frac{g}{k}+v_0\sin\alpha\right)\right)\mathbf{j}+\mathbf{C}_2$; apply the initial condition:

$\mathbf{r}(0)=\mathbf{0}=\left(-\frac{v_0}{k}\cos\alpha\right)\mathbf{i}+\left(-\frac{g}{k^2}-\frac{v_0\sin\alpha}{k}\right)\mathbf{j}+\mathbf{C}_2\Rightarrow\mathbf{C}_2=\left(\frac{v_0}{k}\cos\alpha\right)\mathbf{i}+\left(\frac{g}{k^2}+\frac{v_0\sin\alpha}{k}\right)\mathbf{j}$

$\Rightarrow\mathbf{r}(t)=\left(\frac{v_0}{k}\left(1-e^{-kt}\right)\cos\alpha\right)\mathbf{i}+\left(\frac{v_0}{k}\left(1-e^{-kt}\right)\sin\alpha+\frac{g}{k^2}\left(1-kt-e^{-kt}\right)\right)\mathbf{j}$

38. (a) $\mathbf{r}(t)=(x(t))\mathbf{i}+(y(t))\mathbf{j}$; where $x(t)=\left(\frac{50}{0.12}\right)\left(1-e^{-0.12t}\right)(\cos 20°)$ and $y(t)=1+\left(\frac{50}{0.12}\right)\left(1-e^{-0.12t}\right)(\sin 20°)+\left(\frac{9.8}{0.12^2}\right)\left(1-0.12t-e^{-0.12t}\right)$

(b) Solve graphically using a calculator or CAS: At $t\approx 1.584$ seconds the ball reaches a maximum height of about 14.12 m.

(c) Use a graphing calculator or CAS to find that $y=0$ when the ball has traveled for ≈ 3.341 seconds. The range is about $x(3.341)=\left(\frac{50}{0.12}\right)\left(1-e^{-0.12(3.341)}\right)(\cos 20°)\approx 129.32$ m.

(d) Use a graphing calculator or CAS to find that $y=9$ for $t\approx 0.583$ and 2.628 seconds, at which times the ball is about $x(0.583)\approx 26.46$ m and $x(2.628)\approx 105.90$ m from home plate.

(e) Yes, the batter has hit a home run since a graph of the trajectory shows that the ball is more than 6 m above the ground when it passes over the fence.

39. (a) $\int_a^b k\mathbf{r}(t)\,dt=\int_a^b\left[kf(t)\mathbf{i}+kg(t)\mathbf{j}+kh(t)\mathbf{k}\right]dt=\int_a^b\left[kf(t)\right]dt\,\mathbf{i}+\int_a^b\left[kg(t)\right]dt\,\mathbf{j}+\int_a^b\left[kh(t)\right]dt\,\mathbf{k}$

$=k\left(\int_a^b f(t)\,dt\,\mathbf{i}+\int_a^b g(t)\,dt\,\mathbf{j}+\int_a^b h(t)\,dt\,\mathbf{k}\right)=k\int_a^b\mathbf{r}(t)\,dt$

(b) $\int_a^b [\mathbf{r}_1(t) \pm \mathbf{r}_2(t)]\,dt = \int_a^b \left([f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}] \pm [f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}]\right) dt$

$= \int_a^b \left([f_1(t) \pm f_2(t)]\mathbf{i} + [g_1(t) \pm g_2(t)]\mathbf{j} + [h_1(t) \pm h_2(t)]\mathbf{k}\right) dt$

$= \int_a^b [f_1(t) \pm f_2(t)]\,dt\,\mathbf{i} + \int_a^b [g_1(t) \pm g_2(t)]\,dt\,\mathbf{j} + \int_a^b [h_1(t) \pm h_2(t)]\,dt\,\mathbf{k}$

$= \left[\int_a^b f_1(t)\,dt\,\mathbf{i} \pm \int_a^b f_2(t)\,dt\,\mathbf{i}\right] + \left[\int_a^b g_1(t)\,dt\,\mathbf{j} \pm \int_a^b g_2(t)\,dt\,\mathbf{j}\right] + \left[\int_a^b h_1(t)\,dt\,\mathbf{k} \pm \int_a^b h_2(t)\,dt\,\mathbf{k}\right]$

$= \int_a^b \mathbf{r}_1(t)\,dt \pm \int_a^b \mathbf{r}_2(t)\,dt$

(c) Let $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Then $\int_a^b \mathbf{C} \cdot \mathbf{r}(t)\,dt = \int_a^b [c_1 f(t) + c_2 g(t) + c_3 h(t)]\,dt$

$= c_1\int_a^b f(t)\,dt + c_2\int_a^b g(t)\,dt + c_3\int_a^b h(t)\,dt = \mathbf{C} \cdot \int_a^b \mathbf{r}(t)\,dt;$

$\int_a^b \mathbf{C} \times \mathbf{r}(t)\,dt = \int_a^b [c_2 h(t) - c_3 g(t)]\mathbf{i} + [c_3 f(t) - c_1 h(t)]\mathbf{j} + [c_1 g(t) - c_2 f(t)]\mathbf{k}\,dt$

$= \left[c_2\int_a^b h(t)\,dt - c_3\int_a^b g(t)\,dt\right]\mathbf{i} + \left[c_3\int_a^b f(t)\,dt - c_1\int_a^b h(t)\,dt\right]\mathbf{j} + \left[c_1\int_a^b g(t)\,dt - c_2\int_a^b f(t)\,dt\right]\mathbf{k}$

$= \mathbf{C} \times \int_a^b \mathbf{r}(t)\,dt$

40. (a) Let u and $\mathbf{r}$ be continuous on $[a, b]$. Then $\lim_{t \to t_0} u(t)\mathbf{r}(t) = \lim_{t \to t_0} [u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$

$= u(t_0)f(t_0)\mathbf{i} + u(t_0)g(t_0)\mathbf{j} + u(t_0)h(t_0)\mathbf{k} = u(t_0)\mathbf{r}(t_0) \Rightarrow u\mathbf{r}$ is continuous for every t_0 in $[a,b]$.

(b) Let u and $\mathbf{r}$ be differentiable. Then $\frac{d}{dt}(u\mathbf{r}) = \frac{d}{dt}[u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$

$= \left(\frac{du}{dt}f(t) + u(t)\frac{df}{dt}\right)\mathbf{i} + \left(\frac{du}{dt}g(t) + u(t)\frac{dg}{dt}\right)\mathbf{j} + \left(\frac{du}{dt}h(t) + u(t)\frac{dh}{dt}\right)\mathbf{k}$

$= [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]\frac{du}{dt} + u(t)\left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right) = \mathbf{r}\frac{du}{dt} + u\frac{d\mathbf{r}}{dt}$

41. (a) If $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$ have identical derivatives on I, then $\frac{d\mathbf{R}_1}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{dg_1}{dt}\mathbf{j} + \frac{dh_1}{dt}\mathbf{k} = \frac{df_2}{dt}\mathbf{i} + \frac{dg_2}{dt}\mathbf{j} + \frac{dh_2}{dt}\mathbf{k}$

$= \frac{d\mathbf{R}_2}{dt} \Rightarrow \frac{df_1}{dt} = \frac{df_2}{dt},\ \frac{dg_1}{dt} = \frac{dg_2}{dt},\ \frac{dh_1}{dt} = \frac{dh_2}{dt} \Rightarrow f_1(t) = f_2(t) + c_1,\ g_1(t) = g_2(t) + c_2,\ h_1(t) = h_2(t) + c_3$

$\Rightarrow f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k} = [f_2(t) + c_1]\mathbf{i} + [g_2(t) + c_2]\mathbf{j} + [h_2(t) + c_3]\mathbf{k} \Rightarrow \mathbf{R}_1(t) = \mathbf{R}_2(t) + \mathbf{C}$, where $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

(b) Let $\mathbf{R}(t)$ be an antiderivative of $\mathbf{r}(t)$ on I. Then $\mathbf{R}'(t) = \mathbf{r}(t)$. If $\mathbf{U}(t)$ is an antiderivative of $\mathbf{r}(t)$ on I, then $\mathbf{U}'(t) = \mathbf{r}(t)$. Thus $\mathbf{U}'(t) = \mathbf{R}'(t)$ on $I \Rightarrow \mathbf{U}(t) = \mathbf{R}(t) + \mathbf{C}$.

42. $\frac{d}{dt}\int_a^t \mathbf{r}(\tau)\,d\tau = \frac{d}{dt}\int_a^t [f(\tau)\mathbf{i} + g(\tau)\mathbf{j} + h(\tau)\mathbf{k}]\,d\tau = \frac{d}{dt}\int_a^t f(\tau)\,d\tau\mathbf{i} + \frac{d}{dt}\int_a^t g(\tau)\,d\tau\mathbf{j} + \frac{d}{dt}\int_a^t h(\tau)\,d\tau\mathbf{k}$

$= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \mathbf{r}(t)$. Since $\frac{d}{dt}\int_a^t \mathbf{r}(\tau)\,d\tau = \mathbf{r}(t)$, we have that $\int_a^t \mathbf{r}(\tau)\,d\tau$ is an antiderivative of $\mathbf{r}$.

If $\mathbf{R}$ is any antidervative of $\mathbf{r}$, then $\mathbf{R}(t) = \int_a^t \mathbf{r}(\tau)\,d\tau + \mathbf{C}$ by Exercise 41(b). Then $\mathbf{R}(a) = \int_a^a \mathbf{r}(\tau)\,d\tau + \mathbf{C}$

$= \mathbf{0} + \mathbf{C} \Rightarrow \mathbf{C} = \mathbf{R}(a) \Rightarrow \int_a^t \mathbf{r}(\tau)\,d\tau = \mathbf{R}(t) - \mathbf{C} = \mathbf{R}(t) - \mathbf{R}(a) \Rightarrow \int_a^b \mathbf{r}(\tau)\,d\tau = \mathbf{R}(b) - \mathbf{R}(a)$.

43. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = \left(\frac{1}{0.08}\right)\left(1 - e^{-0.08t}\right)(50\cos 20° - 5)$ and

$y(t) = 1 + \left(\frac{50}{0.08}\right)\left(1 - e^{-0.08t}\right)(\sin 20°) + \left(\frac{9.8}{0.08^2}\right)\left(1 - 0.08t - e^{-0.08t}\right)$

(b) Solve graphically using a calculator or CAS: At $t \approx 1.633$ seconds the ball reaches a maximum height of about 41.893 feet.

(c) Use a graphing calculator or CAS to find that $y = 0$ when the ball has traveled for ≈ 3.404 seconds. The range is about $x(3.404) = \left(\frac{1}{0.08}\right)\left(1 - e^{-0.08(3.404)}\right)(50\cos 20° - 5) \approx 125.11$ m.

(d) Use a graphing calculator or CAS to find that $y = 10$ for $t \approx 0.670$ and 2.622 seconds, at which times the ball is about $x(0.670) \approx 27.39$ m and $x(2.622) \approx 99.30$ m from home plate.

(e) No; the range is less than 120 m. To find the wind needed for a home run, first use the method of part (d) to find that $y = 6$ at $t \approx 0.327$ and 2.987 seconds. Then define

$x(w) = \left(\frac{1}{0.08}\right)\left(1 - e^{-0.08(2.716)}\right)(152\cos 20° + w)$, and solve $x(w) = 120$ to find $w \approx -1.8$ m/s.

44. $y_{\max} = \frac{(v_0 \sin\alpha)^2}{2g} \Rightarrow \frac{3}{4} y_{\max} = \frac{3(v_0 \sin\alpha)^2}{8g}$ and $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow \frac{3(v_0 \sin\alpha)^2}{8g} = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$

$\Rightarrow 3(v_0 \sin\alpha)^2 = (8gv_0 \sin\alpha)t - 4g^2t^2 \Rightarrow 4g^2t^2 - (8gv_0 \sin\alpha)t + 3(v_0 \sin\alpha)^2 = 0 \Rightarrow 2gt - 3v_0 \sin\alpha = 0$ or $2gt - v_0 \sin\alpha = 0 \Rightarrow t = \frac{3v_0 \sin\alpha}{2g}$ or $t = \frac{v_0 \sin\alpha}{2g}$. Since the time it takes to reach $y_{\max}$ is $t_{\max} = \frac{v_0 \sin\alpha}{g}$, then the time it takes the projectile to reach $\frac{3}{4}$ of $y_{\max}$ is the shorter time $t = \frac{v_0 \sin\alpha}{2g}$ or half the time it takes to reach the maximum height.

13.3 ARC LENGTH IN SPACE

1. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k} \Rightarrow \mathbf{v} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \sqrt{5}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + \left(\sqrt{5}\right)^2}$

$= \sqrt{4\sin^2 t + 4\cos^2 t + 5} = 3$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(-\frac{2}{3}\sin t\right)\mathbf{i} + \left(\frac{2}{3}\cos t\right)\mathbf{j} + \frac{\sqrt{5}}{3}\mathbf{k}$ and Length $= \int_0^{\pi} |\mathbf{v}|\,dt = \int_0^{\pi} 3\,dt$

$= [3t]_0^{\pi} = 3\pi$

2. $\mathbf{r} = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12\cos 2t)\mathbf{i} + (-12\sin 2t)\mathbf{j} + 5\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{(12\cos 2t)^2 + (-12\sin 2t)^2 + 5^2} = \sqrt{144\cos^2 2t + 144\sin^2 2t + 25} = 13$;

$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{12}{13}\cos 2t\right)\mathbf{i} - \left(\frac{12}{13}\sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k}$ and Length $= \int_0^{\pi} |\mathbf{v}|\,dt = \int_0^{\pi} 13\,dt = [13t]_0^{\pi} = 13\pi$

3. $\mathbf{r} = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + t^{1/2}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + \left(t^{1/2}\right)^2} = \sqrt{1+t}$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+t}}\mathbf{i} + \frac{\sqrt{t}}{\sqrt{1+t}}\mathbf{k}$ and

Length $= \int_0^{8} \sqrt{1+t}\,dt = \left[\frac{2}{3}(1+t)^{3/2}\right]_0^{8} = \frac{52}{3}$

4. $\mathbf{r} = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ and

Length $= \int_0^{3} \sqrt{3}\,dt = \left[\sqrt{3}t\right]_0^{3} = 3\sqrt{3}$

5. $\mathbf{r} = \left(\cos^3 t\right)\mathbf{j} + \left(\sin^3 t\right)\mathbf{k} \Rightarrow \mathbf{v} = \left(-3\cos^2 t \sin t\right)\mathbf{j} + \left(3\sin^2 t \cos t\right)\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{\left(-3\cos^2 t \sin t\right)^2 + \left(3\sin^2 t \cos t\right)^2} = \sqrt{\left(9\cos^2 t \sin^2 t\right)\left(\cos^2 t + \sin^2 t\right)} = 3|\cos t \sin t|$;

$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-3\cos^2 t \sin t}{3|\cos t \sin t|}\mathbf{j} + \frac{3\sin^2 t \cos t}{3|\cos t \sin t|}\mathbf{k} = (-\cos t)\mathbf{j} + (\sin t)\mathbf{k}$, if $0 \le t \le \frac{\pi}{2}$, and

Length $= \int_0^{\pi/2} 3|\cos t \sin t|\,dt = \int_0^{\pi/2} 3\cos t \sin t\,dt = \int_0^{\pi/2} \frac{3}{2}\sin 2t\,dt = \left[-\frac{3}{4}\cos 2t\right]_0^{\pi/2} = \frac{3}{2}$

6. $\mathbf{r} = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k} \Rightarrow \mathbf{v} = 18t^2\mathbf{i} - 6t^2\mathbf{j} - 9t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\left(18t^2\right)^2 + \left(-6t^2\right)^2 + \left(-9t^2\right)^2} = \sqrt{441t^4} = 21t^2$;

$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{18t^2}{21t^2}\mathbf{i} - \frac{6t^2}{21t^2}\mathbf{j} - \frac{9t^2}{21t^2}\mathbf{k} = \frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}$ and Length $= \int_1^2 21t^2 dt = \left[7t^3\right]_1^2 = 49$

7. $\mathbf{r} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \left(\sqrt{2}t^{1/2}\right)\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + \left(\sqrt{2t}\right)^2} = \sqrt{1 + t^2 + 2t} = \sqrt{(t+1)^2} = |t+1| = t+1$, if $t \ge 0$;

$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - t\sin t}{t+1}\right)\mathbf{i} + \left(\frac{\sin t + t\cos t}{t+1}\right)\mathbf{j} + \left(\frac{\sqrt{2}t^{1/2}}{t+1}\right)\mathbf{k}$ and Length $= \int_0^{\pi}(t+1)\,dt = \left[\frac{t^2}{2} + t\right]_0^{\pi} = \frac{\pi^2}{2} + \pi$

8. $\mathbf{r} = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (\sin t + t\cos t - \sin t)\mathbf{i} + (\cos t - t\sin t - \cos t)\mathbf{j}$

$= (t\cos t)\mathbf{i} - (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (-t\sin t)^2} = \sqrt{t^2} = |t| = t$ if $\sqrt{2} \le t \le 2$;

$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{t\cos t}{t}\right)\mathbf{i} - \left(\frac{t\sin t}{t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ and Length $= \int_{\sqrt{2}}^2 t\,dt = \left[\frac{t^2}{2}\right]_{\sqrt{2}}^2 = 1$

9. Let $P(t_0)$ denote the point. Then $\mathbf{v} = (5\cos t)\mathbf{i} - (5\sin t)\mathbf{j} + 12\mathbf{k}$ and $26\pi = \int_0^{t_0}\sqrt{25\cos^2 t + 25\sin^2 t + 144}\,dt$

$= \int_0^{t_0} 13\,dt = 13t_0 \Rightarrow t_0 = 2\pi$, and the point is $P(2\pi) = (5\sin 2\pi, 5\cos 2\pi, 24\pi) = (0, 5, 24\pi)$

10. Let $P(t_0)$ denote the point. Then $\mathbf{v} = (12\cos t)\mathbf{i} + (12\sin t)\mathbf{j} + 5\mathbf{k}$ and

$-13\pi = \int_0^{t_0}\sqrt{144\cos^2 t + 144\sin^2 t + 25}\,dt = \int_0^{t_0} 13\,dt = 13t_0 \Rightarrow t_0 = -\pi$, and the point is

$P(-\pi) = (12\sin(-\pi), -12\cos(-\pi), -5\pi) = (0, 12, -5\pi)$

11. $\mathbf{r} = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v} = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}$

$= \sqrt{25} = 5 \Rightarrow s(t) = \int_0^t 5\,d\tau = 5t \Rightarrow$ Length $= s\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$

12. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j}$

$= (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\cos t)^2} = \sqrt{t^2} = t$, since $\frac{\pi}{2} \le t \le \pi \Rightarrow s(t) = \int_0^t \tau\,d\tau = \frac{t^2}{2}$

$\Rightarrow$ Length $= s(\pi) - s\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{3\pi^2}{8}$

13. $\mathbf{r}=\left(e^t \cos t\right)\mathbf{i}+\left(e^t \sin t\right)\mathbf{j}+e^t\mathbf{k} \Rightarrow \mathbf{v}=\left(e^t \cos t-e^t \sin t\right)\mathbf{i}+\left(e^t \sin t+e^t \cos t\right)\mathbf{j}+e^t\mathbf{k}$

$\Rightarrow |\mathbf{v}|=\sqrt{\left(e^t \cos t-e^t \sin t\right)^2+\left(e^t \sin t+e^t \cos t\right)^2+\left(e^t\right)^2}=\sqrt{3e^{2t}}=\sqrt{3}e^t \Rightarrow s(t)=\int_0^t \sqrt{3}e^\tau \, d\tau=\sqrt{3}e^t-\sqrt{3}$

$\Rightarrow$ Length $=s(0)-s(-\ln 4)=0-\left(\sqrt{3}e^{-\ln 4}-\sqrt{3}\right)=\frac{3\sqrt{3}}{4}$

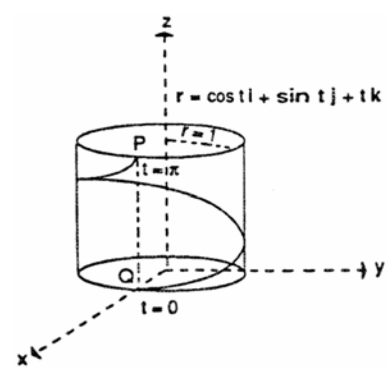
14. $\mathbf{r}=(1+2t)\mathbf{i}+(1+3t)\mathbf{j}+(6-6t)\mathbf{k} \Rightarrow \mathbf{v}=2\mathbf{i}+3\mathbf{j}-6\mathbf{k} \Rightarrow |\mathbf{v}|=\sqrt{2^2+3^2+(-6)^2}=7 \Rightarrow s(t)=\int_0^t 7\, d\tau=7t$

$\Rightarrow$ Length $=s(0)-s(-1)=0-(-7)=7$

15. $\mathbf{r}=\left(\sqrt{2}t\right)\mathbf{i}+\left(\sqrt{2}t\right)\mathbf{j}+\left(1-t^2\right)\mathbf{k} \Rightarrow \mathbf{v}=\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j}-2t\mathbf{k} \Rightarrow |\mathbf{v}|=\sqrt{\left(\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2+(-2t)^2}=\sqrt{4+4t^2}$

$=2\sqrt{1+t^2} \Rightarrow$ Length $=\int_0^1 2\sqrt{1+t^2}\, dt=\left[2\left(\frac{t}{2}\sqrt{1+t^2}+\frac{1}{2}\ln\left(t+\sqrt{1+t^2}\right)\right)\right]_0^1=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$

16. Let the helix make one complete turn from $t=0$ to $t=2\pi$. Note that the radius of the cylinder is 1 $\Rightarrow$ the circumference of the base is 2π. When $t=2\pi$, the point P is $(\cos 2\pi, \sin 2\pi, 2\pi)=(1, 0, 2\pi) \Rightarrow$ the cylinder is 2π units high. Cut the cylinder along PQ and flatten. The resulting rectangle has a width equal to the circumference of the cylinder $=2\pi$ and a height equal to 2π, the height of the cylinder. Therefore, the rectangle is a square and the portion of the helix from $t=0$ to $t=2\pi$ is its diagonal.



17. (a) $\mathbf{r}=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}+(1-\cos t)\mathbf{k},\ 0\le t\le 2\pi \Rightarrow x=\cos t,\ y=\sin t,\ z=1-\cos t$

$\Rightarrow x^2+y^2=\cos^2 t+\sin^2 t=1$, a right circular cylinder with the z-axis as the axis and radius $=1$. Therefore $P(\cos t, \sin t, 1-\cos t)$ lies on the cylinder $x^2+y^2=1$; $t=0 \Rightarrow P(1,0,0)$ is on the curve; $t=\frac{\pi}{2} \Rightarrow Q(0,1,1)$ is on the curve; $t=\pi \Rightarrow R(-1,0,2)$ is on the curve. Then $\overrightarrow{PQ}=-\mathbf{i}+\mathbf{j}+\mathbf{k}$ and

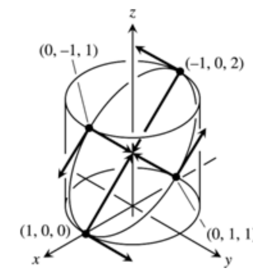
$\overrightarrow{PR}=-2\mathbf{i}+2\mathbf{k} \Rightarrow \overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{vmatrix}=2\mathbf{i}+2\mathbf{k}$ is a vector normal to the plane of P, Q, and R. Then

the plane containing P, Q, and R has an equation $2x+2z=2(1)+2(0)$ or $x+z=1$. Any point on the curve will satisfy this equation since $x+z=\cos t+(1-\cos t)=1$. Therefore, any point on the curve lies on the intersection of the cylinder $x^2+y^2=1$ and the plane $x+z=1 \Rightarrow$ the curve is an ellipse.

(b) $\mathbf{v}=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}+(\sin t)\mathbf{k} \Rightarrow |\mathbf{v}|=\sqrt{\sin^2 t+\cos^2 t+\sin^2 t}=\sqrt{1+\sin^2 t}$

$\Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}+(\sin t)\mathbf{k}}{\sqrt{1+\sin^2 t}} \Rightarrow \mathbf{T}(0)=\mathbf{j},\ \mathbf{T}\left(\frac{\pi}{2}\right)=\frac{-\mathbf{i}+\mathbf{k}}{\sqrt{2}},\ \mathbf{T}(\pi)=-\mathbf{j},\ \mathbf{T}\left(\frac{3\pi}{2}\right)=\frac{\mathbf{i}-\mathbf{k}}{\sqrt{2}}$

(c) $\mathbf{a} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} + (\cos t)\mathbf{k}$; $\mathbf{n} = \mathbf{i} + \mathbf{k}$ is normal to the plane $x + z = 1$
$\Rightarrow \mathbf{n} \cdot \mathbf{a} = -\cos t + \cos t = 0$
$\Rightarrow$ $\mathbf{a}$ is orthogonal to $\mathbf{n}$
$\Rightarrow \mathbf{a}$ is parallel to the plane;
$\mathbf{a}(0) = -\mathbf{i} + \mathbf{k},\ \mathbf{a}\left(\frac{\pi}{2}\right) = -\mathbf{j},\ \ \mathbf{a}(\pi) = \mathbf{i} - \mathbf{k},\ \mathbf{a}\left(\frac{3\pi}{2}\right) = \mathbf{j}$



(d) $|\mathbf{v}| = \sqrt{1 + \sin^2 t}$ (See part (b)) $\Rightarrow L = \int_0^{2\pi} \sqrt{1 + \sin^2 t}\ dt$

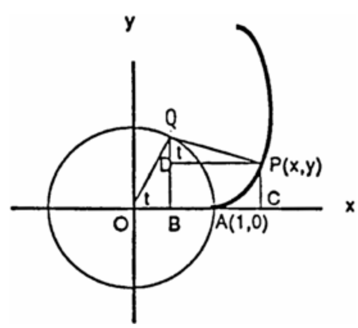
(e) $L \approx 7.64$ (by *Mathematica*)

18. (a) $\mathbf{r} = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (-4\sin 4t)\mathbf{i} + (4\cos 4t)\mathbf{j} + 4\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(-4\sin 4t)^2 + (4\cos 4t)^2 + 4^2} = \sqrt{32} = 4\sqrt{2} \Rightarrow$ Length $= \int_0^{\pi/2} 4\sqrt{2}\ dt = \left[4\sqrt{2}t\right]_0^{\pi/2} = 2\pi\sqrt{2}$

(b) $\mathbf{r} = \left(\cos \frac{t}{2}\right)\mathbf{i} + \left(\sin \frac{t}{2}\right)\mathbf{j} + \frac{t}{2}\mathbf{k} \Rightarrow \mathbf{v} = \left(-\frac{1}{2}\sin \frac{t}{2}\right)\mathbf{i} + \left(\frac{1}{2}\cos \frac{t}{2}\right)\mathbf{j} + \frac{1}{2}\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{\left(-\frac{1}{2}\sin \frac{t}{2}\right)^2 + \left(\frac{1}{2}\cos \frac{t}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{2} \Rightarrow$ Length $= \int_0^{4\pi} \frac{\sqrt{2}}{2}\ dt = \left[\frac{\sqrt{2}}{2}t\right]_0^{4\pi} = 2\pi\sqrt{2}$

(c) $\mathbf{r} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j} - \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (-\cos t)^2 + (-1)^2} = \sqrt{1+1}$
$= \sqrt{2} \Rightarrow$ Length $= \int_{-2\pi}^{0} \sqrt{2}\ dt = \left[\sqrt{2}t\right]_{-2\pi}^{0} = 2\pi\sqrt{2}$

19. $\angle PQB = \angle QOB = t$ and $PQ =$ arc $(AQ) = t$ since $PQ =$ length of the unwound string $=$ length of arc (AQ); thus $x = OB + BC = OB + DP = \cos t + t\sin t$, and $y = PC = QB - QD = \sin t - t\cos t$



20. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + t\cos t + \sin t)\mathbf{i} + (\cos t - (t(-\sin t) + \cos t))\mathbf{j}$
$= (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2} = |t| = t,\ t \geq 0$
$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{t\cos t}{t}\mathbf{i} + \frac{t\sin t}{t}\mathbf{j} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

21. $\mathbf{v} = \frac{d}{dt}(x_0 + tu_1)\mathbf{i} + \frac{d}{dt}(y_0 + tu_2)\mathbf{j} + \frac{d}{dt}(z_0 + tu_3)\mathbf{k} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = \mathbf{u}$, so
$s(t) = \int_0^t |\mathbf{v}|\,dt = \int_0^t |\mathbf{u}|\,d\tau = \int_0^t 1\,d\tau = t$

22. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \Rightarrow |\mathbf{v}(t)| = \sqrt{(1)^2 + (2t)^2 + \left(3t^2\right)^2} = \sqrt{1 + 4t^2 + 9t^4}$. $(0, 0, 0) \Rightarrow t = 0$ and $(2, 4, 8) \Rightarrow t = 2$. Thus $L = \int_0^2 |\mathbf{v}(t)|\,dt = \int_0^2 \sqrt{1 + 4t^2 + 9t^4}\ dt$. Using Simpson's rule with $n = 10$ and $\Delta x = \frac{2-0}{10} = 0.2$

$$\Rightarrow L \approx \tfrac{0.2}{3}\left(|\mathbf{v}(0)| + 4|\mathbf{v}(0.2)| + 2|\mathbf{v}(0.4)| + 4|\mathbf{v}(0.6)| + 2|\mathbf{v}(0.8)| + 4|\mathbf{v}(1)| + 2|\mathbf{v}(1.2)| + 4|\mathbf{v}(1.4)| + 2|\mathbf{v}(1.6)| + 4|\mathbf{v}(1.8)| + |\mathbf{v}(2)|\right)$$

$$\approx \tfrac{0.2}{3}\left(1 + 4(1.0837) + 2(1.3676) + 4(1.8991) + 2(2.6919) + 4(3.7417) + 2(5.0421) + 4(6.5890) + 2(8.3800) + 4(10.4134) + 12.6886\right) = \tfrac{0.2}{3}(143.5594) \approx 9.5706$$

13.4 CURVATURE AND NORMAL VECTORS OF A CURVE

1. $\mathbf{r} = t\mathbf{i} + \ln(\cos t)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + \left(\frac{-\sin t}{\cos t}\right)\mathbf{j} = \mathbf{i} - (\tan t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-\tan t)^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t$, since $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sec t}\right)\mathbf{i} - \left(\frac{\tan t}{\sec t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$ $\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sec t}\cdot 1 = \cos t$.

2. $\mathbf{r} = \ln(\sec t)\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \left(\frac{\sec t \tan t}{\sec t}\right)\mathbf{i} + \mathbf{j} = (\tan t)\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(\tan t)^2 + 1^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t$, since $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\tan t}{\sec t}\right)\mathbf{i} - \left(\frac{1}{\sec t}\right)\mathbf{j} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ $\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(\cos t)^2 + (\sin t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sec t}\cdot 1 = \cos t$.

3. $\mathbf{r} = (2t+3)\mathbf{i} + \left(5 - t^2\right)\mathbf{j} \Rightarrow \mathbf{v} = 2\mathbf{i} - 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + (-2t)^2} = 2\sqrt{1+t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{2\sqrt{1+t^2}}\mathbf{i} + \frac{-2t}{2\sqrt{1+t^2}}\mathbf{j}$ $= \frac{1}{\sqrt{1+t^2}}\mathbf{i} - \frac{t}{\sqrt{1+t^2}}\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = \frac{-t}{\left(\sqrt{1+t^2}\right)^3}\mathbf{i} - \frac{1}{\left(\sqrt{1+t^2}\right)^3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-t}{\left(\sqrt{1+t^2}\right)^3}\right)^2 + \left(-\frac{1}{\left(\sqrt{1+t^2}\right)^3}\right)^2} = \sqrt{\frac{1}{\left(1+t^2\right)^2}} = \frac{1}{1+t^2}$ $\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{-t}{\sqrt{1+t^2}}\mathbf{i} - \frac{1}{\sqrt{1+t^2}}\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2\sqrt{1+t^2}}\cdot\frac{1}{1+t^2} = \frac{1}{2\left(1+t^2\right)^{3/2}}$

4. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} \Rightarrow \mathbf{v} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2} = |t| = t$, since $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(t\cos t)\mathbf{i} + (t\sin t)\mathbf{j}}{t} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ $\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{t}\cdot 1 = \frac{1}{t}$

5. (a) $\kappa(x) = \frac{1}{|\mathbf{v}(x)|}\cdot\left|\frac{d\mathbf{T}(x)}{dt}\right|$. Now, $\mathbf{v} = \mathbf{i} + f'(x)\mathbf{j} \Rightarrow |\mathbf{v}(x)| = \sqrt{1 + [f'(x)]^2}$ $\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(1 + [f'(x)]^2\right)^{-1/2}\mathbf{i} + f'(x)\left(1 + [f'(x)]^2\right)^{-1/2}\mathbf{j}$. Thus $\frac{d\mathbf{T}}{dt}(x) = \frac{-f'(x)f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\mathbf{i} + \frac{f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\mathbf{j}$

$$\Rightarrow \left|\frac{d\mathbf{T}(x)}{dt}\right| = \sqrt{\left[\frac{-f'(x)f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\right]^2 + \left(\frac{f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\right)^2} = \sqrt{\frac{[f''(x)]^2\left(1+[f'(x)]^2\right)}{\left(1+[f'(x)]^2\right)^3}} = \frac{|f''(x)|}{\left|1+[f'(x)]^2\right|}$$

Thus $\kappa(x) = \frac{1}{\left(1+[f'(x)]^2\right)^{1/2}} \cdot \frac{|f''(x)|}{\left|1+[f'(x)]^2\right|} = \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}}$

(b) $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\cos x}\right)(-\sin x) = -\tan x \Rightarrow \frac{d^2y}{dx^2} = -\sec^2 x \Rightarrow \kappa = \frac{\left|-\sec^2 x\right|}{\left[1+(-\tan x)^2\right]^{3/2}} = \frac{\sec^2 x}{\left|\sec^3 x\right|}$

$= \frac{1}{\sec x} = \cos x$, since $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) Note that $f''(x) = 0$ at an inflection point.

6. (a) $\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\mathbf{i} + \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\mathbf{j}$

$$\frac{d\mathbf{T}}{dt} = \frac{\dot{y}(\dot{y}\ddot{x}-\dot{x}\ddot{y})}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}\mathbf{i} + \frac{\dot{x}(\dot{x}\ddot{y}-\dot{y}\ddot{x})}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left[\frac{\dot{y}(\dot{y}\ddot{x}-\dot{x}\ddot{y})}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}\right]^2 + \left[\frac{\dot{x}(\dot{x}\ddot{y}-\dot{y}\ddot{x})}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}\right]^2} = \sqrt{\frac{\left(\dot{y}^2+\dot{x}^2\right)(\dot{y}\ddot{x}-\dot{x}\ddot{y})^2}{\left(\dot{x}^2+\dot{y}^2\right)^3}} = \frac{|\dot{y}\ddot{x}-\dot{x}\ddot{y}|}{\left|\dot{x}^2+\dot{y}^2\right|};$$

$$\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}} \cdot \frac{|\dot{y}\ddot{x}-\dot{x}\ddot{y}|}{\left|\dot{x}^2+\dot{y}^2\right|} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}.$$

(b) $\mathbf{r}(t) = t\mathbf{i} + \ln(\sin t)\mathbf{j}$, $0 < t < \pi \Rightarrow x = t$ and $y = \ln(\sin t) \Rightarrow \dot{x} = 1$, $\ddot{x} = 0$; $\dot{y} = \frac{\cos t}{\sin t} = \cot t$, $\ddot{y} = -\csc^2 t$

$\Rightarrow \kappa = \frac{\left|-\csc^2 t - 0\right|}{\left(1+\cot^2 t\right)^{3/2}} = \frac{\csc^2 t}{\csc^3 t} = \sin t$

(c) $\mathbf{r}(t) = \tan^{-1}(\sinh t)\mathbf{i} + \ln(\cosh t)\mathbf{j} \Rightarrow x = \tan^{-1}(\sinh t)$ and $y = \ln(\cosh t) \Rightarrow \dot{x} = \frac{\cosh t}{1+\sinh^2 t} = \frac{1}{\cosh t}$

$= \operatorname{sech} t, \ddot{x} = -\operatorname{sech} t \tanh t; \dot{y} = \frac{\sinh t}{\cosh t} = \tanh t,\ \ddot{y} = \operatorname{sech}^2 t \Rightarrow \kappa = \frac{\left|\operatorname{sech}^3 t + \operatorname{sech} t \tanh^2 t\right|}{\left(\operatorname{sech}^2 t + \tanh^2 t\right)} = |\operatorname{sech} t| = \operatorname{sech} t$

7. (a) $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j}$ is tangent to the curve at the point $(f(t), g(t))$;
$\mathbf{n} \cdot \mathbf{v} = [-g'(t)\mathbf{i} + f'(t)\mathbf{j}] \cdot [f'(t)\mathbf{i} + g'(t)\mathbf{j}] = -g'(t)f'(t) + f'(t)g'(t) = 0$; $-\mathbf{n} \cdot \mathbf{v} = -(\mathbf{n} \cdot \mathbf{v}) = 0$; thus, $\mathbf{n}$ and $-\mathbf{n}$ are both normal to the curve at the point

(b) $\mathbf{r}(t) = t\mathbf{i} + e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2e^{2t}\mathbf{j} \Rightarrow \mathbf{n} = -2e^{2t}\mathbf{i} + \mathbf{j}$ points toward the concave side of the curve; $\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and

$|\mathbf{n}| = \sqrt{4e^{4t}+1} \Rightarrow \mathbf{N} = \frac{-2e^{2t}}{\sqrt{1+4e^{4t}}}\mathbf{i} + \frac{1}{\sqrt{1+4e^{4t}}}\mathbf{j}$

(c) $\mathbf{r}(t) = \sqrt{4-t^2}\,\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \frac{-t}{\sqrt{4-t^2}}\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n} = -\mathbf{i} - \frac{t}{\sqrt{4-t^2}}\mathbf{j}$ points toward the concave side of the curve;

$\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and $|\mathbf{n}| = \sqrt{1+\frac{t^2}{4-t^2}} = \frac{2}{\sqrt{4-t^2}} \Rightarrow \mathbf{N} = -\frac{1}{2}\left(\sqrt{4-t^2}\,\mathbf{i} + t\mathbf{j}\right)$

8. (a) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}t^3\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{n} = t^2\mathbf{i} - \mathbf{j}$ points toward the concave side of the curve when $t < 0$ and $-\mathbf{n} = -t^2\mathbf{i} + \mathbf{j}$ points toward the concave side when $t > 0 \Rightarrow \mathbf{N} = \frac{1}{\sqrt{1+t^4}}\left(t^2\mathbf{i} - \mathbf{j}\right)$ for $t < 0$ and

$\mathbf{N} = \frac{1}{\sqrt{1+t^4}}\left(-t^2\mathbf{i} + \mathbf{j}\right)$ for $t > 0$

(b) From part (a), $|\mathbf{v}| = \sqrt{1+t^4} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+t^4}}\mathbf{i} + \frac{t^2}{\sqrt{1+t^4}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{-2t^3}{\left(1+t^4\right)^{3/2}}\mathbf{i} + \frac{2t}{\left(1+t^4\right)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{4t^6+4t^2}{\left(1+t^4\right)^3}}$

$= \frac{2|t|}{1+t^4}$; $\mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{1+t^4}{2|t|}\left(\frac{-2t^3}{\left(1+t^4\right)^{3/2}}\mathbf{i} + \frac{2t}{\left(1+t^4\right)^{3/2}}\mathbf{j}\right) = \frac{-t^3}{|t|\sqrt{1+t^4}}\mathbf{i} + \frac{t}{|t|\sqrt{1+t^4}}\mathbf{j}$; $t \neq 0$. $\mathbf{N}$ does not exist at $t = 0$, where the curve has a point of inflection; $\left.\frac{d\mathbf{T}}{dt}\right|_{t=0} = 0$ so the curvature $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{ds} \cdot \frac{dt}{ds}\right| = 0$ at $t = 0 \Rightarrow \mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$ is undefined. Since $x = t$ and $y = \frac{1}{3}t^3 \Rightarrow y = \frac{1}{3}x^3$, the curve is the cubic power curve which is concave down for $x = t < 0$ and concave up for $x = t > 0$.

9. $\mathbf{r} = (3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (3\cos t)\mathbf{i} + (-3\sin t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(3\cos t)^2 + (-3\sin t)^2 + 4^2} = \sqrt{25} = 5$

$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{3}{5}\cos t\right)\mathbf{i} - \left(\frac{3}{5}\sin t\right)\mathbf{j} + \frac{4}{5}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{3}{5}\sin t\right)\mathbf{i} - \left(\frac{3}{5}\cos t\right)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{3}{5}\sin t\right)^2 + \left(-\frac{3}{5}\cos t\right)^2} = \frac{3}{5}$

$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$; $\kappa = \frac{1}{5} \cdot \frac{3}{5} = \frac{3}{25}$

10. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{v} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2}$

$= |t| = t$, if $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$, $t > 0 \Rightarrow \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $\kappa = \frac{1}{t} \cdot 1 = \frac{1}{t}$

11. $\mathbf{r} = \left(e^t \cos t\right)\mathbf{i} + \left(e^t \sin t\right)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{v} = \left(e^t \cos t - e^t \sin t\right)\mathbf{i} + \left(e^t \sin t + e^t \cos t\right)\mathbf{j}$

$\Rightarrow |\mathbf{v}| = \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2} = \sqrt{2e^{2t}} = e^t\sqrt{2}$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$

$\Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)^2 + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)^2} = 1$

$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{e^t\sqrt{2}} \cdot 1 = \frac{1}{e^t\sqrt{2}}$

12. $\mathbf{r} = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12\cos 2t)\mathbf{i} - (12\sin 2t)\mathbf{j} + 5\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(12\cos 2t)^2 + (-12\sin 2t)^2 + 5^2}$

$= \sqrt{169} = 13 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{12}{13}\cos 2t\right)\mathbf{i} - \left(\frac{12}{13}\sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{24}{13}\sin 2t\right)\mathbf{i} - \left(\frac{24}{13}\cos 2t\right)\mathbf{j}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{24}{13}\sin 2t\right)^2 + \left(-\frac{24}{13}\cos 2t\right)^2} = \frac{24}{13} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin 2t)\mathbf{i} - (\cos 2t)\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{13} \cdot \frac{24}{13} = \frac{24}{169}$.

13. $\mathbf{r}=\left(\frac{t^3}{3}\right)\mathbf{i}+\left(\frac{t^2}{2}\right)\mathbf{j},\ t>0 \Rightarrow \mathbf{v}=t^2\mathbf{i}+t\mathbf{j} \Rightarrow |\mathbf{v}|=\sqrt{t^4+t^2}=t\sqrt{t^2+1}$, since $t>0 \Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{t}{\sqrt{t^2+1}}\mathbf{i}+\frac{1}{\sqrt{t^2+1}}\mathbf{j}$

$\Rightarrow \frac{d\mathbf{T}}{dt}=\frac{1}{\left(t^2+1\right)^{3/2}}\mathbf{i}-\frac{t}{\left(t^2+1\right)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|=\sqrt{\left(\frac{1}{\left(t^2+1\right)^{3/2}}\right)^2+\left(\frac{-t}{\left(t^2+1\right)^{3/2}}\right)^2}=\sqrt{\frac{1+t^2}{\left(t^2+1\right)^3}}=\frac{1}{t^2+1}$

$\Rightarrow \mathbf{N}=\frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}=\frac{1}{\sqrt{t^2+1}}\mathbf{i}-\frac{t}{\sqrt{t^2+1}}\mathbf{j};\ \kappa=\frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right|=\frac{1}{t\sqrt{t^2+1}}\cdot\frac{1}{t^2+1}=\frac{1}{t\left(t^2+1\right)^{3/2}}$.

14. $\mathbf{r}=\left(\cos^3 t\right)\mathbf{i}+\left(\sin^3 t\right)\mathbf{j},\ 0<t<\frac{\pi}{2} \Rightarrow \mathbf{v}=\left(-3\cos^2 t\sin t\right)\mathbf{i}+\left(3\sin^2 t\cos t\right)\mathbf{j}$

$\Rightarrow |\mathbf{v}|=\sqrt{\left(-3\cos^2 t\sin t\right)^2+\left(3\sin^2 t\cos t\right)^2}=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}=3\cos t\sin t$, since $0<t<\frac{\pi}{2}$

$\Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=(-\cos t)\mathbf{i}+(\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}=(\sin t)\mathbf{i}+(\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|=\sqrt{\sin^2 t+\cos^2 t}=1$

$\Rightarrow \mathbf{N}=\frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}=(\sin t)\mathbf{i}+(\cos t)\mathbf{j};\ \kappa=\frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right|=\frac{1}{3\cos t\sin t}\cdot 1=\frac{1}{3\cos t\sin t}$.

15. $\mathbf{r}=t\mathbf{i}+\left(a\cosh\frac{t}{a}\right)\mathbf{j},\ a>0 \Rightarrow \mathbf{v}=\mathbf{i}+\left(\sinh\frac{t}{a}\right)\mathbf{j} \Rightarrow |\mathbf{v}|=\sqrt{1+\sinh^2\left(\frac{t}{a}\right)}=\sqrt{\cosh^2\left(\frac{t}{a}\right)}=\cosh\frac{t}{a}$

$\Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\left(\operatorname{sech}\frac{t}{a}\right)\mathbf{i}+\left(\tanh\frac{t}{a}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}=\left(-\frac{1}{a}\operatorname{sech}\frac{t}{a}\tanh\frac{t}{a}\right)\mathbf{i}+\left(\frac{1}{a}\operatorname{sech}^2\frac{t}{a}\right)\mathbf{j}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|=\sqrt{\frac{1}{a^2}\operatorname{sech}^2\left(\frac{t}{a}\right)\tanh^2\left(\frac{t}{a}\right)+\frac{1}{a^2}\operatorname{sech}^4\left(\frac{t}{a}\right)}=\frac{1}{a}\operatorname{sech}\left(\frac{t}{a}\right) \Rightarrow \mathbf{N}=\frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}=\left(-\tanh\frac{t}{a}\right)\mathbf{i}+\left(\operatorname{sech}\frac{t}{a}\right)\mathbf{j};$

$\kappa=\frac{1}{|v|}\cdot\left|\frac{d\mathbf{T}}{dt}\right|=\frac{1}{\cosh\frac{t}{a}}\cdot\frac{1}{a}\operatorname{sech}\left(\frac{t}{a}\right)=\frac{1}{a}\operatorname{sech}^2\left(\frac{t}{a}\right)$.

16. $\mathbf{r}=(\cosh t)\mathbf{i}-(\sinh t)\mathbf{j}+t\mathbf{k} \Rightarrow \mathbf{v}=(\sinh t)\mathbf{i}-(\cosh t)\mathbf{j}+\mathbf{k} \Rightarrow |\mathbf{v}|=\sqrt{\sinh^2 t+(-\cosh t)^2+1}=\sqrt{2}\cosh t$

$\Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}+\left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt}=\left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\mathbf{i}-\left(\frac{1}{\sqrt{2}}\operatorname{sech} t\tanh t\right)\mathbf{k}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|=\sqrt{\frac{1}{2}\operatorname{sech}^4 t+\frac{1}{2}\operatorname{sech}^2 t\tanh^2 t}=\frac{1}{\sqrt{2}}\operatorname{sech} t \Rightarrow \mathbf{N}=\frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}=(\operatorname{sech} t)\mathbf{i}-(\tanh t)\mathbf{k};$

$\kappa=\frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right|=\frac{1}{\sqrt{2}\cosh t}\cdot\frac{1}{\sqrt{2}}\operatorname{sech} t=\frac{1}{2}\operatorname{sech}^2 t$.

17. $y=ax^2 \Rightarrow y'=2ax \Rightarrow y''=2a$; from Exercise 5(a), $\kappa(x)=\frac{|2a|}{\left(1+4a^2x^2\right)^{3/2}}=|2a|\left(1+4a^2x^2\right)^{-3/2}$

$\Rightarrow \kappa'(x)=-\frac{3}{2}|2a|\left(1+4a^2x^2\right)^{-5/2}\left(8a^2x\right)$; thus, $\kappa'(x)=0 \Rightarrow x=0$. Now, $\kappa'(x)>0$ for $x<0$ and $\kappa'(x)<0$ for $x>0$ so that $\kappa(x)$ has an absolute maximum at $x=0$ which is the vertex of the parabola. Since $x=0$ is the only critical point for $\kappa(x)$, the curvature has no minimum value.

18. $\mathbf{r}=(a\cos t)\mathbf{i}+(b\sin t)\mathbf{j}\Rightarrow\mathbf{v}=(-a\sin t)\mathbf{i}+(b\cos t)\mathbf{j}\Rightarrow\mathbf{a}=(-a\cos t)\mathbf{i}-(b\sin t)\mathbf{j}$

$\Rightarrow \mathbf{v}\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -a\sin t & b\cos t & 0\\ -a\cos t & -b\sin t & 0\end{vmatrix}=ab\mathbf{k}\Rightarrow|\mathbf{v}\times\mathbf{a}|=|ab|=ab,$ since $a>b>0;\ \kappa(t)=\frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

$=ab\left(a^2\sin^2 t+b^2\cos^2 t\right)^{-3/2};\ \kappa'(t)=-\frac{3}{2}(ab)\left(a^2\sin^2 t+b^2\cos^2 t\right)^{-5/2}\left(2a^2\sin t\cos t-2b^2\sin t\cos t\right)$

$=-\frac{3}{2}(ab)\left(a^2-b^2\right)(\sin t)\left(a^2\sin^2 t+b^2\cos^2 t\right)^{-5/2}$; thus, $\kappa'(t)=0\Rightarrow\sin 2t=0\Rightarrow t=0,\pi$ identifying points on the major axis, or $t=\frac{\pi}{2},\frac{3\pi}{2}$ identifying points on the minor axis. Furthermore, $\kappa'(t)<0$ for $0<t<\frac{\pi}{2}$ and for $\pi<t<\frac{3\pi}{2};\ \kappa'(t)>0$ for $\frac{\pi}{2}<t<\pi$ and $\frac{3\pi}{2}<t<2\pi$. Therefore, the points associated with $t=0$ and $t=\pi$ on the major axis give absolute maximum curvature and the points associated with $t=\frac{\pi}{2}$ and $t=\frac{3\pi}{2}$ on the minor axis give absolute minimum curvature.

19. $\kappa=\frac{a}{a^2+b^2}\Rightarrow\frac{d\kappa}{da}=\frac{-a^2+b^2}{\left(a^2+b^2\right)^2};\ \frac{d\kappa}{da}=0\Rightarrow -a^2+b^2=0\Rightarrow a=\pm b\Rightarrow a=b$ since $a,b\ge 0$. Now, $\frac{d\kappa}{da}>0$ if $a<b$ and $\frac{d\kappa}{da}<0$ if $a>b\Rightarrow\kappa$ is at a maximum for $a=b$ and $\kappa(b)=\frac{b}{b^2+b^2}=\frac{1}{2b}$ is the maximum value of κ.

20. (a) From Example 5, the curvature of the helix $\mathbf{r}(t)=(a\cos t)\mathbf{i}+(a\sin t)\mathbf{j}+bt\mathbf{k},\ a,b\ge 0$ is $\kappa=\frac{a}{a^2+b^2}$; also $|\mathbf{v}|=\sqrt{a^2+b^2}$. For the helix $\mathbf{r}(t)=(3\cos t)\mathbf{i}+(3\sin t)\mathbf{j}+t\mathbf{k},\ 0\le t\le 4\pi,\ a=3$ and $b=1\Rightarrow\kappa=\frac{3}{3^2+1^2}=\frac{3}{10}$ and $|\mathbf{v}|=\sqrt{10}\Rightarrow K=\int_0^{4\pi}\frac{3}{10}\sqrt{10}\,dt=\left[\frac{3}{\sqrt{10}}t\right]_0^{4\pi}=\frac{12\pi}{\sqrt{10}}$

(b) $y=x^2\Rightarrow x=t$ and $y=t^2,\ -\infty<t<\infty\Rightarrow r(t)=t\mathbf{i}+t^2\mathbf{j}\Rightarrow\mathbf{v}=\mathbf{i}+2t\mathbf{j}\Rightarrow|\mathbf{v}|=\sqrt{1+4t^2}$;

$\mathbf{T}=\frac{1}{\sqrt{1+4t^2}}\mathbf{i}+\frac{2t}{\sqrt{1+4t^2}}\mathbf{j};\ \frac{d\mathbf{T}}{dt}=\frac{-4t}{\left(1+4t^2\right)^{3/2}}\mathbf{i}+\frac{2}{\left(1+4t^2\right)^{3/2}}\mathbf{j};\ \left|\frac{d\mathbf{T}}{dt}\right|=\sqrt{\frac{16t^2+4}{\left(1+4t^2\right)^3}}=\frac{2}{1+4t^2}$. Thus $\kappa=\frac{1}{\sqrt{1+4t^2}}\cdot\frac{2t}{1+4t^2}$

$=\frac{2}{\left(\sqrt{1+4t^2}\right)^3}$. Then $K=\int_{-\infty}^{\infty}\frac{2}{\left(\sqrt{1+4t^2}\right)^3}\left(\sqrt{1+4t^2}\right)dt=\int_{-\infty}^{\infty}\frac{2}{1+4t^2}\,dt=\lim\limits_{a\to-\infty}\int_a^0\frac{2}{1+4t^2}\,dt+\lim\limits_{b\to\infty}\int_0^b\frac{2}{1+4t^2}\,dt$

$=\lim\limits_{a\to-\infty}\left[\tan^{-1}2t\right]_a^0+\lim\limits_{b\to\infty}\left[\tan^{-1}2t\right]_0^b=\lim\limits_{a\to-\infty}\left(-\tan^{-1}2a\right)+\lim\limits_{b\to\infty}\left(\tan^{-1}2b\right)=\frac{\pi}{2}+\frac{\pi}{2}=\pi$

21. $\mathbf{r}=t\mathbf{i}+(\sin t)\mathbf{j}\Rightarrow\mathbf{v}=\mathbf{i}+(\cos t)\mathbf{j}\Rightarrow|\mathbf{v}|=\sqrt{1^2+(\cos t)^2}=\sqrt{1+\cos^2 t}\Rightarrow\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right|=\sqrt{1+\cos^2\left(\frac{\pi}{2}\right)}=1$;

$\mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{\mathbf{i}+(\cos t)\mathbf{j}}{\sqrt{1+\cos^2 t}}\Rightarrow\frac{d\mathbf{T}}{dt}=\frac{\sin t\cos t}{\left(1+\cos^2 t\right)^{3/2}}\mathbf{i}+\frac{-\sin t}{\left(1+\cos^2 t\right)^{3/2}}\mathbf{j}\Rightarrow\left|\frac{d\mathbf{T}}{dt}\right|=\frac{|\sin t|}{1+\cos^2 t};\ \left|\frac{d\mathbf{T}}{dt}\right|_{t=\frac{\pi}{2}}=\frac{\left|\sin\frac{\pi}{2}\right|}{1+\cos^2\left(\frac{\pi}{2}\right)}=\frac{1}{1}=1$. Thus $\kappa\left(\frac{\pi}{2}\right)=\frac{1}{1}\cdot 1=1\Rightarrow\rho=\frac{1}{1}=1$ and the center is $\left(\frac{\pi}{2},0\right)\Rightarrow\left(x-\frac{\pi}{2}\right)^2+y^2=1$

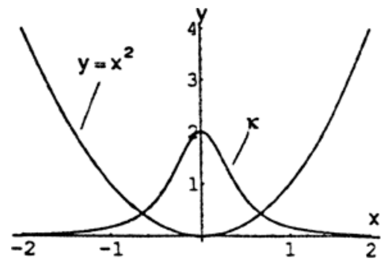
22. $\mathbf{r}=(2\ln t)\mathbf{i}-\left(t+\frac{1}{t}\right)\mathbf{j}\Rightarrow\mathbf{v}=\left(\frac{2}{t}\right)\mathbf{i}-\left(1-\frac{1}{t^2}\right)\mathbf{j}\Rightarrow|\mathbf{v}|=\sqrt{\frac{4}{t^2}+\left(1-\frac{1}{t^2}\right)^2}=\frac{t^2+1}{t^2}\Rightarrow\mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{2t}{t^2+1}\mathbf{i}-\frac{t^2-1}{t^2+1}\mathbf{j}$;

$\frac{d\mathbf{T}}{dt}=\frac{-2\left(t^2-1\right)}{\left(t^2+1\right)^2}\mathbf{i}-\frac{4t}{\left(t^2+1\right)^2}\mathbf{j}\Rightarrow\left|\frac{d\mathbf{T}}{dt}\right|=\sqrt{\frac{4\left(t^2-1\right)^2+16t^2}{\left(t^2+1\right)^4}}=\frac{2}{t^2+1}$. Thus $\kappa=\frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right|=\frac{t^2}{t^2+1}\cdot\frac{2}{t^2+1}=\frac{2t^2}{\left(t^2+1\right)^2}$

$\Rightarrow \kappa(1) = \frac{2}{2^2} = \frac{1}{2} \Rightarrow \rho = \frac{1}{\kappa} = 2$. The circle of curvature is tangent to the curve at $P(0, -2) \Rightarrow$ circle has same tangent as the curve $\Rightarrow \mathbf{v}(1) = 2\mathbf{i}$ is tangent to the circle $\Rightarrow$ the center lies on the y-axis. If $t \neq 1 (t > 0)$, then $(t-1)^2 > 0 \Rightarrow t^2 - 2t + 1 > 0 \Rightarrow t^2 + 1 > 2t \Rightarrow \frac{t^2+1}{t} > 2$ since $t > 0 \Rightarrow t + \frac{1}{t} > 2 \Rightarrow -\left(t + \frac{1}{t}\right) < -2 \Rightarrow y < -2$ on both sides of $(0, -2) \Rightarrow$ the curve is concave down $\Rightarrow$ center of circle of curvature is $(0, -4)$ $\Rightarrow x^2 + (y+4)^2 = 4$ is an equation of the circle of curvature

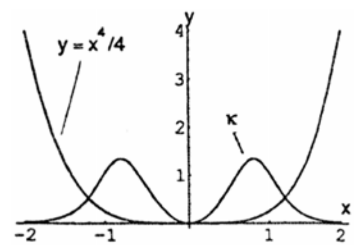
23. $y = x^2 \Rightarrow f'(x) = 2x$ and $f''(x) = 2$

$$\Rightarrow \kappa = \frac{|2|}{\left(1+(2x)^2\right)^{3/2}} = \frac{2}{\left(1+4x^2\right)^{3/2}}$$



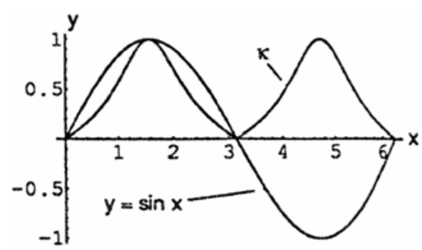
24. $y = \frac{x^4}{4} \Rightarrow f'(x) = x^3$ and $f''(x) = 3x^2$

$$\Rightarrow \kappa = \frac{\left|3x^2\right|}{\left(1+\left(x^3\right)^2\right)^{3/2}} = \frac{3x^2}{\left(1+x^6\right)^{3/2}}$$



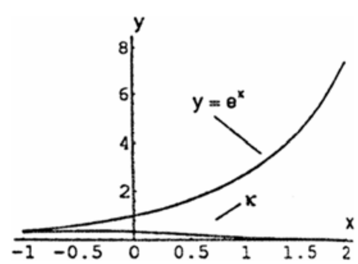
25. $y = \sin x \Rightarrow f'(x) = \cos x$ and $f''(x) = -\sin x$

$$\Rightarrow \kappa = \frac{|-\sin x|}{\left(1+\cos^2 x\right)^{3/2}} = \frac{|\sin x|}{\left(1+\cos^2 x\right)^{3/2}}$$



26. $y = e^x \Rightarrow f'(x) = e^x$ and $f''(x) = e^x$

$$\Rightarrow \kappa = \frac{|e^x|}{\left(1+\left(e^x\right)^2\right)^{3/2}} = \frac{e^x}{\left(1+e^{2x}\right)^{3/2}}$$



27. We will use the formula $\kappa = \frac{1}{|\mathbf{v}(a)|}\left|\frac{d\mathbf{T}}{dt}(a)\right|$ to find the curvature at the point $\left(a, a^2\right)$.

By Example 4 in Section 13.4,

$\mathbf{v}(t) = \sqrt{1+4t^2}$ and $\frac{d\mathbf{T}}{dt} = 2\left(1+4t^2\right)^{-3/2}(-2t\mathbf{i} + \mathbf{j})$.

At $t = a$ this gives $\kappa = \frac{1}{|\mathbf{v}(a)|}\left|\frac{d\mathbf{T}}{dt}(a)\right| = \frac{2}{\sqrt{1+4a^2}}\left(1+4a^2\right)^{-3/2}\sqrt{1+4a^2} = \frac{2}{\left(1+4a^2\right)^{3/2}}$. Thus the radius of the osculating circle is $r = \frac{1}{2}\left(1+4a^2\right)^{3/2}$. To show that the given formulas for center and radius are correct we must first show that the distance between $\left(a, a^2\right)$ and $\left(-4a^3, 3a^2+\frac{1}{2}\right)$ is r. This distance is $\sqrt{\left(-4a^3-a\right)^2+\left(3a^2+\frac{1}{2}-a^2\right)^2} = \sqrt{16a^6+12a^4+3a^2+\frac{1}{4}} = \frac{1}{2}\sqrt{\left(1+4a^2\right)^3}$ as required. Finally we must show that the line containing the points $\left(-4a^3, 3a^2+\frac{1}{2}\right)$ and $\left(a, a^2\right)$ is perpendicular to the tangent line at $\left(a, a^2\right)$, which has slope $2a$. This requires that $\dfrac{3a^2+\frac{1}{2}-a^2}{-4a^3-a} = -\dfrac{1}{2a}$, which is correct.

28. By Exercise 27, for $a = 1$, the center of the osculating circle is at $\left(-4, \frac{7}{2}\right)$ and its radius is $\frac{5\sqrt{5}}{2}$. A parametrization of this circle is $x(\theta) = -4 + \frac{5\sqrt{5}}{2}\cos\theta$, $y(\theta) = \frac{7}{2} + \frac{5\sqrt{5}}{2}\sin\theta$.

29-36. Example CAS commands:

Maple:

```
with( plots );
r := t -> [3*cos(t),5*sin(t)];
lo := 0;
hi := 2*Pi;
t0 := Pi/4;
P1 := plot( [r(t)[], t=lo..hi] ):
display( P1, scaling=constrained, title="#29(a) (Section 13.4)" );
CURVATURE := (x,y,t) -> simplify(abs(diff(x,t)*diff(y,t,t)-diff(y,t)*diff(x,t,t))/
                        (diff(x,t)^2+diff(y,t)^2)^(3/2));
kappa := eval(CURVATURE(r(t)[],t),t=t0);
UnitNormal := (x,y,t) ->expand( [-diff(y,t),diff(x,t)]/sqrt(diff(x,t)^2+diff(y,t)^2) );
N := eval( UnitNormal(r(t)[],t), t=t0 );
C := expand( r(t0) + N/kappa );
OscCircle := (x-C[1]^2+(y-C[2])^2 = 1/kappa^2;
evalf( OscCircle );
P2 := implicitplot( (x-C[1])^2+(y-C[2])^2 = 1/kappa^2, x=-7..4, y=-4..6, color=blue ):
display( [P1,P2], scaling=constrained, title="#27(e) (Section 13.4)" );
```

Mathematica: (assigned functions and parameters may vary)
In Mathematica, the dot product can be applied either with a period "." or with the word, "Dot".
Similarly, the cross product can be applied either with a very small "x" (in the palette next to the arrow) or with the word, "Cross". However, the Cross command assumes the vectors are in three dimensions.
For the purposes of applying the cross product command, we will define the position vector r as a three dimensional vector with zero for its z-component. For graphing, we will use only the first two components.

```
Clear[r, t, x, y]
r[t_]={3 Cos[t], 5 Sin[t] }
t0= π/4; tmin= 0; tmax= 2π;
r2[t_]= {r[t][[1]], r[t][[2]]}
pp=ParametricPlot[r2[t], {t, tmin, tmax}];
mag[v_]=Sqrt[v.v]
vel[t_]= r'[t]
speed[t_]=mag[vel[t]]
acc[t_]= vel'[t]
curv[t_]= mag[Cross[vel[t],acc[t]]]/speed[t]^3//Simplify
unittan[t_]= vel[t]/speed[t]//Simplify
unitnorm[t_]= unittan'[t] / mag[unittan'[t]]
ctr= r[t0] + (1 / curv[t0]) unitnorm[t0] //Simplify
{a,b}= {ctr[[1]], ctr[[2]]}
```

To plot the osculating circle, load a graphics package and then plot it, and show it together with the original curve.

```
<< Graphics`ImplicitPlot`
pc=ImplicitPlot[(x − a)^2 + (y − b)^2 == 1/curv[t0]^2, {x, −8, 8}, {y, −8, 8}]
radius=Graphics[Line[{{a, b}, r2[t0]}]]
Show[pp, pc, radius, AspectRatio → 1]
```

13.5 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

1. $\mathbf{r}=(a\cos t)\mathbf{i}+(a\sin t)\mathbf{j}+bt\mathbf{k} \Rightarrow \mathbf{v}=(-a\sin t)\mathbf{i}+(a\cos t)\mathbf{j}+b\mathbf{k} \Rightarrow |\mathbf{v}|=\sqrt{(-a\sin t)^2+(a\cos t)^2+b^2}$
$=\sqrt{a^2+b^2} \Rightarrow a_T=\frac{d}{dt}|\mathbf{v}|=0;\ \ \mathbf{a}=(-a\cos t)\mathbf{i}+(-a\sin t)\mathbf{j} \Rightarrow |\mathbf{a}|=\sqrt{(-a\cos t)^2+(-a\sin t)^2}=\sqrt{a^2}=|a|$
$\Rightarrow a_N=\sqrt{|\mathbf{a}|^2-a_T^2}=\sqrt{|\mathbf{a}|^2-0^2}=|\mathbf{a}|=|a| \Rightarrow \mathbf{a}=(0)\mathbf{T}+|a|\mathbf{N}=|a|\mathbf{N}$

2. $\mathbf{r}=(1+3t)\mathbf{i}+(t-2)\mathbf{j}-3t\mathbf{k} \Rightarrow \mathbf{v}=3\mathbf{i}+\mathbf{j}-3\mathbf{k} \Rightarrow |\mathbf{v}|=\sqrt{3^2+1^2+(-3)^2}=\sqrt{19} \Rightarrow a_T=\frac{d}{dt}|\mathbf{v}|=0;$
$\mathbf{a}=\mathbf{0} \Rightarrow a_N=\sqrt{|\mathbf{a}|^2-a_T^2}=0 \Rightarrow \mathbf{a}=(0)\mathbf{T}+(0)\mathbf{N}=\mathbf{0}$

3. $\mathbf{r}=(t+1)\mathbf{i}+2t\mathbf{j}+t^2\mathbf{k} \Rightarrow \mathbf{v}=\mathbf{i}+2\mathbf{j}+2t\mathbf{k} \Rightarrow |\mathbf{v}|=\sqrt{1^2+2^2+(2t)^2}=\sqrt{5+4t^2} \Rightarrow a_T=\frac{1}{2}\left(5+4t^2\right)^{-1/2}(8t)$
$=4t\left(5+4t^2\right)^{-1/2} \Rightarrow a_T(1)=\frac{4}{\sqrt{9}}=\frac{4}{3};\ \ \mathbf{a}=2\mathbf{k} \Rightarrow \mathbf{a}(1)=2\mathbf{k} \Rightarrow |\mathbf{a}(1)|=2$
$\Rightarrow a_N=\sqrt{|\mathbf{a}|^2-a_T^2}=\sqrt{2^2-\left(\frac{4}{3}\right)^2}=\sqrt{\frac{20}{9}}=\frac{2\sqrt{5}}{3} \Rightarrow \mathbf{a}(1)=\frac{4}{3}\mathbf{T}+\frac{2\sqrt{5}}{3}\mathbf{N}$

4. $\mathbf{r}=(t\cos t)\mathbf{i}+(t\sin t)\mathbf{j}+t^2\mathbf{k} \Rightarrow \mathbf{v}=(\cos t-t\sin t)\mathbf{i}+(\sin t+t\cos t)\mathbf{j}+2t\mathbf{k}$
$\Rightarrow |\mathbf{v}|=\sqrt{(\cos t-t\sin t)^2+(\sin t+t\cos t)^2+(2t)^2}=\sqrt{5t^2+1} \Rightarrow a_T=\frac{1}{2}\left(5t^2+1\right)^{-1/2}(10t)=\frac{5t}{\sqrt{5t^2+1}}$
$\Rightarrow a_T(0)=0;\ \ \mathbf{a}=(-2\sin t-t\cos t)\mathbf{i}+(2\cos t-t\sin t)\mathbf{j}+2\mathbf{k} \Rightarrow \mathbf{a}(0)=2\mathbf{j}+2\mathbf{k} \Rightarrow |\mathbf{a}(0)|=\sqrt{2^2+2^2}=2\sqrt{2}$
$\Rightarrow a_N=\sqrt{|\mathbf{a}|^2-a_T^2}=\sqrt{\left(2\sqrt{2}\right)^2-0^2}=2\sqrt{2} \Rightarrow \mathbf{a}(0)=(0)\mathbf{T}+2\sqrt{2}\mathbf{N}=2\sqrt{2}\mathbf{N}$

5. $\mathbf{r}=t^2\mathbf{i}+\left(t+\frac{1}{3}t^3\right)\mathbf{j}+\left(t-\frac{1}{3}t^3\right)\mathbf{k} \Rightarrow \mathbf{v}=2t\mathbf{i}+\left(1+t^2\right)\mathbf{j}+\left(1-t^2\right)\mathbf{k} \Rightarrow |\mathbf{v}|=\sqrt{(2t)^2+\left(1+t^2\right)^2+\left(1-t^2\right)^2}$
$=\sqrt{2\left(t^4+2t^2+1\right)}=\sqrt{2}\left(1+t^2\right) \Rightarrow a_T=2t\sqrt{2} \Rightarrow a_T(0)=0;\ \ \mathbf{a}=2\mathbf{i}+2t\mathbf{j}-2t\mathbf{k} \Rightarrow \mathbf{a}(0)=2\mathbf{i} \Rightarrow |\mathbf{a}(0)|=2$
$\Rightarrow a_N=\sqrt{|\mathbf{a}|^2-a_T^2}=\sqrt{2^2-0^2}=2 \Rightarrow \mathbf{a}(0)=(0)\mathbf{T}+2\mathbf{N}=2\mathbf{N}$

6. $\mathbf{r}=\left(e^t\cos t\right)\mathbf{i}+\left(e^t\sin t\right)\mathbf{j}+\sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{v}=\left(e^t\cos t-e^t\sin t\right)\mathbf{i}+\left(e^t\sin t+e^t\cos t\right)\mathbf{j}+\sqrt{2}e^t\mathbf{k}$
$\Rightarrow |\mathbf{v}|=\sqrt{\left(e^t\cos t-e^t\sin t\right)^2+\left(e^t\sin t+e^t\cos t\right)^2+\left(\sqrt{2}e^t\right)^2}=\sqrt{4e^{2t}}=2e^t \Rightarrow a_T=2e^t \Rightarrow a_T(0)=2;$
$\mathbf{a}=\left(e^t\cos t-e^t\sin t-e^t\sin t-e^t\cos t\right)\mathbf{i}+\left(e^t\sin t+e^t\cos t+e^t\cos t-e^t\sin t\right)\mathbf{j}+\sqrt{2}e^t\mathbf{k}$
$=\left(-2e^t\sin t\right)\mathbf{i}+\left(2e^t\cos t\right)\mathbf{j}+\sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{a}(0)=2\mathbf{j}+\sqrt{2}\mathbf{k} \Rightarrow |\mathbf{a}(0)|=\sqrt{2^2+\left(\sqrt{2}\right)^2}=\sqrt{6}$
$\Rightarrow a_N=\sqrt{|\mathbf{a}|^2-a_T^2}=\sqrt{\left(\sqrt{6}\right)^2-2^2}=\sqrt{2} \Rightarrow \mathbf{a}(0)=2\mathbf{T}+\sqrt{2}\mathbf{N}$

7. $\mathbf{r}=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}-\mathbf{k} \Rightarrow \mathbf{v}=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j} \Rightarrow |\mathbf{v}|=\sqrt{(-\sin t)^2+(\cos t)^2}=1$
$\Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j} \Rightarrow \mathbf{T}\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j};\ \ \frac{d\mathbf{T}}{dt}=(-\cos t)\mathbf{i}-(\sin t)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|\sqrt{(-\cos t)^2+(-\sin t)^2}=1 \Rightarrow \mathbf{N}=\frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}=(-\cos t)\mathbf{i}-(\sin t)\mathbf{j} \Rightarrow \mathbf{N}\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\mathbf{i}-\frac{\sqrt{2}}{2}\mathbf{j};$
$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin t & \cos t & 0\\ -\cos t & -\sin t & 0\end{vmatrix}=\mathbf{k} \Rightarrow \mathbf{B}\left(\frac{\pi}{4}\right)=\mathbf{k},$ the normal to the osculating plane; $\mathbf{r}\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j}-\mathbf{k}$
$\Rightarrow P=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},-1\right)$ lies on the osculating plane $\Rightarrow 0\left(x-\frac{\sqrt{2}}{2}\right)+0\left(y-\frac{\sqrt{2}}{2}\right)+(z-(-1))=0 \Rightarrow z=-1$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x-\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(y-\frac{\sqrt{2}}{2}\right)+0(z-(-1))=0$
$\Rightarrow -\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y=0 \Rightarrow -x+y=0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane
$\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x-\frac{\sqrt{2}}{2}\right)+\left(-\frac{\sqrt{2}}{2}\right)\left(y-\frac{\sqrt{2}}{2}\right)+0(z-(-1))=0 \Rightarrow -\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y=-1 \Rightarrow x+y=\sqrt{2}$ is the rectifying plane.

8. $\mathbf{r}=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}+t\mathbf{k} \Rightarrow \mathbf{v}=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}+\mathbf{k} \Rightarrow |\mathbf{v}|=\sqrt{\sin^2 t+\cos^2 t+1}=\sqrt{2}$

$\Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{i}+\left(\frac{1}{\sqrt{2}}\cos t\right)\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt}=\left(-\frac{1}{\sqrt{2}}\cos t\right)\mathbf{i}+\left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{j}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|=\sqrt{\frac{1}{2}\cos^2 t+\frac{1}{2}\sin^2 t}=\frac{1}{\sqrt{2}} \Rightarrow \mathbf{N}=\frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}=(-\cos t)\mathbf{i}-(\sin t)\mathbf{j}$; thus $\mathbf{T}(0)=\frac{1}{\sqrt{2}}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}$ and $\mathbf{N}(0)=-\mathbf{i}$

$\Rightarrow \mathbf{B}(0)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -1 & 0 & 0\end{vmatrix}=-\frac{1}{\sqrt{2}}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}$, the normal to the osculating plane; $\mathbf{r}(0)=\mathbf{i} \Rightarrow P(1,0,0)$ lies on the osculating plane $\Rightarrow 0(x-1)-\frac{1}{\sqrt{2}}(y-0)+\frac{1}{\sqrt{2}}(z-0)=0 \Rightarrow y-z=0$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow 0(x-1)+\frac{1}{\sqrt{2}}(y-0)+\frac{1}{\sqrt{2}}(z-0)=0 \Rightarrow y+z=0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane $\Rightarrow -1(x-1)+0(y-0)+0(z-0)=0 \Rightarrow x=1$ is the rectifying plane.

9. By Exercise 9 in Section 13.4, $\mathbf{T}=\left(\frac{3}{5}\cos t\right)\mathbf{i}+\left(-\frac{3}{5}\sin t\right)\mathbf{j}+\frac{4}{5}\mathbf{k}$ and $\mathbf{N}=(-\sin t)\mathbf{i}-(\cos t)\mathbf{j}$ so that

$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5}\\ -\sin t & -\cos t & 0\end{vmatrix}=\left(\frac{4}{5}\cos t\right)\mathbf{i}-\left(\frac{4}{5}\sin t\right)\mathbf{j}-\frac{3}{5}\mathbf{k}$. Also $\mathbf{v}=(3\cos t)\mathbf{i}+(-3\sin t)\mathbf{j}+4\mathbf{k}$

$\Rightarrow \mathbf{a}=(-3\sin t)\mathbf{i}+(-3\cos t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt}=(-3\cos t)\mathbf{i}+(3\sin t)\mathbf{j}$ and $\mathbf{v}\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3\cos t & -3\sin t & 4\\ -3\sin t & -3\cos t & 0\end{vmatrix}$

$=(12\cos t)\mathbf{i}-(12\sin t)\mathbf{j}-9\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}|^2=(12\cos t)^2+(-12\sin t)^2+(-9)^2=225$. Thus

$$\tau=\frac{\begin{vmatrix}3\cos t & -3\sin t & 4\\ -3\sin t & -3\cos t & 0\\ -3\cos t & 3\sin t & 0\end{vmatrix}}{225}=\frac{4\left(-9\sin^2 t-9\cos^2 t\right)}{225}=\frac{-36}{225}=-\frac{4}{25}$$

10. By Exercise 10 in Section 13.4, $\mathbf{T}=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}$ and $\mathbf{N}=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}$; thus

$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos t & \sin t & 0\\ -\sin t & \cos t & 0\end{vmatrix}=\left(\cos^2 t+\sin^2 t\right)\mathbf{k}=\mathbf{k}$. Also $\mathbf{v}=(t\cos t)\mathbf{i}+(t\sin t)\mathbf{j}$

$\Rightarrow \mathbf{a}=(t(-\sin t)+\cos t)\mathbf{i}+(t\cos t+\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt}=(-t\cos t-\sin t-\sin t)\mathbf{i}+(-t\sin t+\cos t+\cos t)\mathbf{j}$

$=(-t\cos t-2\sin t)\mathbf{i}+(2\cos t-t\sin t)\mathbf{j}$. Thus $\mathbf{v}\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t\cos t & t\sin t & 0\\ -t\sin t+\cos t & t\cos t+\sin t & 0\end{vmatrix}$

$=\left[(t\cos t)(t\cos t+\sin t)-(t\sin t)(-t\sin t+\cos t)\right]\mathbf{k}=t^2\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}|^2=\left(t^2\right)^2=t^4$.

Thus $\tau=\dfrac{\begin{vmatrix}t\cos t & t\sin t & 0\\ \cos t-t\sin t & \sin t+\cos t & 0\\ -2\sin t-t\cos t & 2\cos t-\sin t & 0\end{vmatrix}}{t^4}=\frac{0}{t^4}=0$

11. By Exercise 11 in Section 13.4, $\mathbf{T}=\left(\frac{\cos t-\sin t}{\sqrt{2}}\right)\mathbf{i}+\left(\frac{\sin t+\cos t}{\sqrt{2}}\right)\mathbf{j}$ and $\mathbf{N}=\left(\frac{-\cos t-\sin t}{\sqrt{2}}\right)\mathbf{i}+\left(\frac{-\sin t+\cos t}{\sqrt{2}}\right)\mathbf{j}$; Thus

$$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\cos t-\sin t}{\sqrt{2}} & \frac{\sin t+\cos t}{\sqrt{2}} & 0\\ \frac{-\cos t-\sin t}{\sqrt{2}} & \frac{-\sin t+\cos t}{\sqrt{2}} & 0\end{vmatrix}=\left[\left(\frac{\cos^2 t-2\cos t\sin t+\sin^2 t}{2}\right)+\left(\frac{\sin^2 t+2\sin t\cos t+\cos^2 t}{2}\right)\right]\mathbf{k}$$

$=\left[\left(\frac{1-\sin(2t)}{2}\right)+\left(\frac{1+\sin(2t)}{2}\right)\right]\mathbf{k}=\mathbf{k}$. Also, $\mathbf{v}=\left(e^t\cos t-e^t\sin t\right)\mathbf{i}+\left(e^t\sin t+e^t\cos t\right)\mathbf{j}$

$\Rightarrow \mathbf{a}=\left[e^t(-\sin t-\cos t)+e^t(\cos t-\sin t)\right]\mathbf{i}+\left[e^t(\cos t-\sin t)+e^t(\sin t+\cos t)\right]\mathbf{j}$

$=\left(-2e^t\sin t\right)\mathbf{i}+\left(2e^t\cos t\right)\mathbf{j}\Rightarrow\frac{d\mathbf{a}}{dt}=-2e^t(\cos t+\sin t)\,\mathbf{i}+2e^t(-\sin t+\cos t)\,\mathbf{j}$.

Thus $\mathbf{v}\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ e^t(\cos t-\sin t) & e^t(\sin t+\cos t) & 0\\ -2e^t\sin t & 2e^t\cos t & 0\end{vmatrix}=2e^{2t}\mathbf{k}\Rightarrow|\mathbf{v}\times\mathbf{a}|^2=\left(2e^{2t}\right)^2=4e^{4t}$.

Thus $\tau=\dfrac{\begin{vmatrix}e^t(\cos t-\sin t) & e^t(\sin t+\cos t) & 0\\ -2e^t\sin t & 2e^t\cos t & 0\\ -2e^t(\cos t+\sin t) & 2e^t(-\sin t+\cos t) & 0\end{vmatrix}}{4e^{4t}}=0$

12. By Exercise 12 in Section 13.4, $\mathbf{T}=\left(\frac{12}{13}\cos 2t\right)\mathbf{i}-\left(\frac{12}{13}\sin 2t\right)\mathbf{j}+\frac{5}{13}\mathbf{k}$ and $\mathbf{N}=(-\sin 2t)\mathbf{i}-(\cos 2t)\mathbf{j}$ Thus

$$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{12}{13}\cos 2t & -\frac{12}{13}\sin 2t & \frac{5}{13}\\ -\sin 2t & -\cos 2t & 0\end{vmatrix}=\left(\frac{5}{13}\cos 2t\right)\mathbf{i}-\left(\frac{5}{13}\sin 2t\right)\mathbf{j}-\frac{12}{13}\mathbf{k}.$$

Also, $\mathbf{v}=(12\cos 2t)\mathbf{i}-(12\sin 2t)\mathbf{j}+5\mathbf{k}\Rightarrow\mathbf{a}=(-24\sin 2t)\mathbf{i}-(24\cos 2t)\mathbf{j}$ and $\frac{d\mathbf{a}}{dt}=(-48\cos 2t)\mathbf{i}+(48\sin 2t)\mathbf{j}$

$$\mathbf{v}\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 12\cos 2t & -12\sin 2t & 5\\ -24\sin 2t & -24\cos 2t & 0\end{vmatrix}=(120\cos 2t)\mathbf{i}-(120\sin 2t)\mathbf{j}-288\mathbf{k}$$

$\Rightarrow|\mathbf{v}\times\mathbf{a}|^2=(120\cos 2t)^2+(-120\sin 2t)^2+(-288)^2=120^2\left(\cos^2 2t+\sin^2 2t\right)+288^2=97344$. Thus

$$\tau=\frac{\begin{vmatrix}12\cos 2t & -12\sin 2t & 5\\ -24\sin 2t & -24\cos 2t & 0\\ -48\cos 2t & 48\sin 2t & 0\end{vmatrix}}{97344}=\frac{5\cdot(-24\cdot 48)}{97344}=-\frac{10}{169}$$

13. By Exercise 13 in Section 13.4, $\mathbf{T}=\frac{t}{\sqrt{t^2+1}}\mathbf{i}+\frac{1}{\sqrt{t^2+1}}\mathbf{j}$ and $\mathbf{N}=\frac{1}{\sqrt{t^2+1}}\mathbf{i}-\frac{t}{\sqrt{t^2+1}}\mathbf{j}$ so that

$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{t}{\sqrt{t^2+1}} & \frac{1}{\sqrt{t^2+1}} & 0\\ \frac{1}{\sqrt{t^2+1}} & \frac{-t}{\sqrt{t^2+1}} & 0\end{vmatrix}=-\mathbf{k}$. Also, $\mathbf{v}=t^2\mathbf{i}+t\mathbf{j}\Rightarrow\mathbf{a}=2t\mathbf{i}+\mathbf{j}\Rightarrow\frac{d\mathbf{a}}{dt}=2\mathbf{i}$ so that $\begin{vmatrix}t^2 & t & 0\\ 2t & 1 & 0\\ 2 & 0 & 0\end{vmatrix}=0\Rightarrow\tau=0$

14. By Exercise 14 in Section 13.4, $\mathbf{T}=(-\cos t)\mathbf{i}+(\sin t)\mathbf{j}$ and $\mathbf{N}=(\sin t)\mathbf{i}+(\cos t)\mathbf{j}$ so that

$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\cos t & \sin t & 0\\ \sin t & \cos t & 0\end{vmatrix}=-\mathbf{k}$. Also, $\mathbf{v}=\left(-3\cos^2 t\sin t\right)\mathbf{i}+\left(3\sin^2 t\cos t\right)\mathbf{j}$

$\Rightarrow \mathbf{a}=\frac{d}{dt}\left(-3\cos^2 t\sin t\right)\mathbf{i}+\frac{d}{dt}\left(3\sin^2 t\cos t\right)\mathbf{j}\Rightarrow \frac{d\mathbf{a}}{dt}=\frac{d}{dt}\left[\frac{d}{dt}\left(-3\cos^2 t\sin t\right)\right]\mathbf{i}+\frac{d}{dt}\left[\frac{d}{dt}\left(3\sin^2 t\cos t\right)\right]\mathbf{j}$

$$\Rightarrow\begin{vmatrix}-3\cos^2 t\sin t & 3\sin^2 t\cos t & 0\\ \frac{d}{dt}\left(-3\cos^2 t\sin t\right) & \frac{d}{dt}\left(3\sin^2 t\cos t\right) & 0\\ \frac{d}{dt}\left[\frac{d}{dt}\left(-3\cos^2 t\sin t\right)\right] & \frac{d}{dt}\left[\frac{d}{dt}\left(3\sin^2 t\cos t\right)\right] & 0\end{vmatrix}=0\Rightarrow\tau=0$$

15. By Exercise 15 in Section 13.4, $\mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\left(\operatorname{sech}\frac{t}{a}\right)\mathbf{i}+\left(\tanh\frac{t}{a}\right)\mathbf{j}$ and $\mathbf{N}=\left(-\tanh\frac{t}{a}\right)\mathbf{i}+\left(\operatorname{sech}\frac{t}{a}\right)\mathbf{j}$ so that

$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \operatorname{sech}\left(\frac{t}{a}\right) & \tanh\left(\frac{t}{a}\right) & 0\\ -\tanh\left(\frac{t}{a}\right) & \operatorname{sech}\left(\frac{t}{a}\right) & 0\end{vmatrix}=\mathbf{k}$. Also, $\mathbf{v}=\mathbf{i}+\left(\sinh\frac{t}{a}\right)\mathbf{j}\Rightarrow\mathbf{a}=\left(\frac{1}{a}\cosh\frac{t}{a}\right)\mathbf{j}\Rightarrow\frac{d\mathbf{a}}{dt}=\frac{1}{a^2}\sinh\left(\frac{t}{a}\right)\mathbf{j}$ so

that $\begin{vmatrix}1 & \sinh\left(\frac{t}{a}\right) & 0\\ 0 & \frac{1}{a}\cosh\left(\frac{t}{a}\right) & 0\\ 0 & \frac{1}{a^2}\sinh\left(\frac{t}{a}\right) & 0\end{vmatrix}=0\Rightarrow\tau=0$

16. By Exercise 16 in Section 13.4, $\mathbf{T}=\left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}+\left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k}$ and $\mathbf{N}=(\operatorname{sech} t)\mathbf{i}-(\tanh t)\mathbf{k}$ so that

$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{1}{\sqrt{2}}\tanh t & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\operatorname{sech} t\\ \operatorname{sech} t & 0 & -\tanh t\end{vmatrix}=\left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}+\left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k}$. Also, $\mathbf{v}=(\sinh t)\mathbf{i}-(\cosh t)\mathbf{j}+\mathbf{k}$

$\mathbf{a}=(\cosh t)\mathbf{i}-(\sinh t)\mathbf{j}\Rightarrow\frac{d\mathbf{a}}{dt}=(\sinh t)\mathbf{i}-(\cosh t)\mathbf{j}$ and $\mathbf{v}\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sinh t & -\cosh t & 1\\ \cosh t & -\sinh t & 0\end{vmatrix}$

$=(\sinh t)\mathbf{i}+(\cosh t)\mathbf{j}+\left(\cosh^2 t-\sinh^2 t\right)\mathbf{k}=(\sinh t)\mathbf{i}+(\cosh t)\mathbf{j}+\mathbf{k}\Rightarrow|\mathbf{v}\times\mathbf{a}|^2=\sinh^2 t+\cosh^2 t+1$. Thus

$$\tau=\frac{\begin{vmatrix}\sinh t & -\cosh t & 1\\ \cosh t & -\sinh t & 0\\ \sinh t & -\cosh t & 0\end{vmatrix}}{\sinh^2 t+\cosh^2 t+1}=\frac{-1}{\sinh^2 t+\cosh^2 t+1}=\frac{-1}{2\cosh^2 t}.$$

17. Yes. If the car is moving along a curved path, then $\kappa\neq 0$ and $a_N=k\,|\,\mathbf{v}\,|^2\neq 0\Rightarrow\mathbf{a}=a_T\mathbf{T}+a_N\mathbf{N}\neq\mathbf{0}$.

18. $|\mathbf{v}|$ constant $\Rightarrow a_T=\frac{d}{dt}|\mathbf{v}|=0\Rightarrow\mathbf{a}=a_N\mathbf{N}$ is orthogonal to $\mathbf{T}\Rightarrow$ the acceleration is normal to the path

19. $\mathbf{a}\perp\mathbf{v}\Rightarrow\mathbf{a}\perp\mathbf{T}\Rightarrow a_T=0\Rightarrow\frac{d}{dt}|\mathbf{v}|=0\Rightarrow|\mathbf{v}|$ is constant

20. $\mathbf{a}(t)=a_T\mathbf{T}+a_N\mathbf{N}$, where $a_T=\frac{d}{dt}|\mathbf{v}|=\frac{d}{dt}(10)=0$ and $a_N=\kappa|\mathbf{v}|^2=100\kappa \Rightarrow \mathbf{a}=0\mathbf{T}+100\kappa\,\mathbf{N}$. Now, from Exercise 5(a) Section 13.4, we find for $y=f(x)=x^2$ that $\kappa=\frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}}=\frac{2}{\left[1+(2x)^2\right]^{3/2}}=\frac{2}{\left(1+4x^2\right)^{3/2}}$; also, $\mathbf{r}(t)=t\mathbf{i}+t^2\mathbf{j}$ is the position vector of the moving mass $\Rightarrow \mathbf{v}=\mathbf{i}+2t\mathbf{j}\Rightarrow|\mathbf{v}|=\sqrt{1+4t^2}\Rightarrow \mathbf{T}=\frac{1}{\sqrt{1+4t^2}}(\mathbf{i}+2t\mathbf{j})$.

At $(0,0)$: $\mathbf{T}(0)=\mathbf{i}$, $\mathbf{N}(0)=\mathbf{j}$ and $\kappa(0)=2\Rightarrow \mathbf{F}=m\mathbf{a}=m(100\kappa)\mathbf{N}=200m\,\mathbf{j}$;

At $\left(\sqrt{2},2\right)$: $\mathbf{T}\left(\sqrt{2}\right)=\frac{1}{3}\left(\mathbf{i}+2\sqrt{2}\mathbf{j}\right)=\frac{1}{3}\mathbf{i}+\frac{2\sqrt{2}}{3}\mathbf{j}$, $\mathbf{N}\left(\sqrt{2}\right)=-\frac{2\sqrt{2}}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}$, and $\kappa\left(\sqrt{2}\right)=\frac{2}{27}$

$\Rightarrow \mathbf{F}=m\mathbf{a}=m(100\kappa)\mathbf{N}=\left(\frac{200}{27}m\right)\left(-\frac{2\sqrt{2}}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}\right)=-\frac{400\sqrt{2}}{81}m\mathbf{i}+\frac{200}{81}m\mathbf{j}$

21. $\mathbf{r}=\left(x_0+At\right)\mathbf{i}+\left(y_0+Bt\right)\mathbf{j}+\left(z_0+Ct\right)\mathbf{k}\Rightarrow \mathbf{v}=A\mathbf{i}+B\mathbf{j}+C\mathbf{k}\Rightarrow \mathbf{a}=\mathbf{0}\Rightarrow \mathbf{v}\times\mathbf{a}=\mathbf{0}\Rightarrow\kappa=0$. Since the curve is a plane curve, $\tau=0$.

22. $a_N=0\Rightarrow\kappa|\mathbf{v}|^2=0\Rightarrow\kappa=0$ (since the particle is moving, we cannot have zero speed) $\Rightarrow$ the curvature is zero so the particle is moving along a straight line

23. From Example 1, $|\mathbf{v}|=t$ and $a_N=t$ so that $a_N=\kappa|\mathbf{v}|^2\Rightarrow\kappa=\frac{a_N}{|\mathbf{v}|^2}=\frac{t}{t^2}=\frac{1}{t},\ t\neq 0\Rightarrow\rho=\frac{1}{\kappa}=t$

24. If a plane curve is sufficiently differentiable the torsion is zero as the following argument shows:

$$\mathbf{r}=f(t)\mathbf{i}+g(t)\mathbf{j}\Rightarrow \mathbf{v}=f'(t)\mathbf{i}+g'(t)\mathbf{j}\Rightarrow \mathbf{a}=f''(t)\mathbf{i}+g''(t)\mathbf{j}\Rightarrow \frac{d\mathbf{a}}{dt}=f'''(t)\mathbf{i}+g'''(t)\mathbf{j}\Rightarrow \tau=\frac{\begin{vmatrix} f'(t) & g'(t) & 0\\ f''(t) & g''(t) & 0\\ f'''(t) & g'''(t) & 0\end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2}=0$$

25. $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}\Rightarrow \mathbf{v}=f'(t)\mathbf{i}+g'(t)\mathbf{j}+h'(t)\mathbf{k}$; $\mathbf{v}\cdot\mathbf{k}=0\Rightarrow h'(t)=0\Rightarrow h(t)=C$
$\Rightarrow \mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+C\mathbf{k}$ and $\mathbf{r}(a)=f(a)\mathbf{i}+g(a)\mathbf{j}+C\mathbf{k}=\mathbf{0}\Rightarrow f(a)=0, g(a)=0$ and $C=0\Rightarrow h(t)=0$.

26. From Example 2, $\mathbf{v}=-\left(a\sin t\right)\mathbf{i}+\left(a\cos t\right)\mathbf{j}+b\mathbf{k}\Rightarrow|\mathbf{v}|=\sqrt{a^2+b^2}$

$\Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{\sqrt{a^2+b^2}}\left[-\left(a\sin t\right)\mathbf{i}+\left(a\cos t\right)\mathbf{j}+b\mathbf{k}\right]$; $\frac{d\mathbf{T}}{dt}=\frac{1}{\sqrt{a^2+b^2}}\left[-\left(a\cos t\right)\mathbf{i}-\left(a\sin t\right)\mathbf{j}\right]$

$$\Rightarrow \mathbf{N}=\frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}=-\left(\cos t\right)\mathbf{i}-\left(\sin t\right)\mathbf{j};\quad \mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{-a\sin t}{\sqrt{a^2+b^2}} & \frac{a\cos t}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}}\\ -\cos t & -\sin t & 0\end{vmatrix}=\frac{b\sin t}{\sqrt{a^2+b^2}}\mathbf{i}-\frac{b\cos t}{\sqrt{a^2+b^2}}\mathbf{j}+\frac{a}{\sqrt{a^2+b^2}}\mathbf{k}$$

$$\Rightarrow \frac{d\mathbf{B}}{dt}=\frac{1}{\sqrt{a^2+b^2}}\left[\left(b\cos t\right)\mathbf{i}+\left(b\sin t\right)\mathbf{j}\right]\Rightarrow \frac{d\mathbf{B}}{dt}\cdot\mathbf{N}=-\frac{b}{\sqrt{a^2+b^2}}\Rightarrow\tau=-\frac{1}{|\mathbf{v}|}\left(\frac{d\mathbf{B}}{dt}\cdot\mathbf{N}\right)=\left(-\frac{1}{\sqrt{a^2+b^2}}\right)\left(-\frac{b}{\sqrt{a^2+b^2}}\right)=\frac{b}{a^2+b^2},$$

which is consistent with the result in Example 2.

27-30. Example CAS commands:

Maple:

```
with( LinearAlgebra );
r := < t*cos(t) | t*sin(t) | t >;
t0 := sqrt(3);
rr := eval( r, t=t0 );
v := map( diff, r, t );
vv := eval( v, t=t0 );
a := map( diff, v, t );
aa := eval( a, t=t0 );
s := simplify(Norm( v, 2 )) assuming t::real;
ss := eval( s, t=t0 );
T := v/s;
TT := vv/ss ;
q1 := map( diff, simplify(T), t ):
NN := simplify(eval( q1/Norm(q1,2), t=t0 ));
BB := CrossProduct( TT, NN );
kappa := Norm(CrossProduct(vv,aa),2)/ss^3;
tau := simplify( Determinant(< vv, aa, eval(map(diff,a,t),t=t0) >)/Norm(CrossProduct(vv,aa),2)^3 );
a_t := eval( diff( s, t ), t=t0 );
a_n := evalf[4]( kappa*ss^2 );
```

Mathematica: (assigned functions and value for t0 will vary)

```
Clear[t, v, a, t]
mag[vector_]:=Sqrt[vector.vector]
Print["The position vector is", r[t_]={t Cos[t], t Sin[t], t}]
Print["The velocity vector is", v[t_]=r'[t]]
Print["The acceleration vector is", a[t_]=v'[t]]
Print["The speed is", speed[t_]= mag[v[t]]//Simplify]
Print["The unit tangent vector is", utan[t_]= v[t]/speed[t] //Simplify]
Print["The curvature is", curv[t_]= mag[Cross[v[t],a[t]]] / speed[t]^3 //Simplify]
Print["The torsion is", torsion[t_]= Det[{v[t], a[t], a'[t]}] / mag[Cross[v[t],a[t]]]^2 //Simplify]
Print["The unit normal vector is", unorm[t_]= utan'[t] / mag[utan'[t] //Simplify]
Print["The unit binormal vector is", ubinorm[t_]= Cross[utan[t],unorm[t]] //Simplify]
Print["The tangential component of the acceleration is", at[t_]=a[t].utan[t] //Simplify]
Print["The normal component of the acceleration is", an[t_]=a[t].unorm[t] //Simplify]
```

You can evaluate any of these functions at a specified value of t.

```
t0= Sqrt[3]
{utan[t0], unorm[t0], ubinorm[t0]}
N[{utan[t0], unorm[t0], ubinorm[t0]}]
{curv[t0], torsion[t0]}
```

N[{curv[t0], torsion[t0]}]

{at[t0], an[t0]}

N[{at[t0], an[t0]}]

To verify that the tangential and normal components of the acceleration agree with the formulas in the book:

at[t]== speed'[t] //Simplify

an[t]==curv [t] speed[t]^2 //Simplify

13.6 VELOCITY AND ACCELERATION IN POLAR COORDINATES

1. $\frac{d\theta}{dt} = 3 = \dot{\theta} \Rightarrow \ddot{\theta} = 0,\ \ r = a(1-\cos\theta) \Rightarrow \dot{r} = a\sin\theta \frac{d\theta}{dt} = 3a\sin\theta \Rightarrow \ddot{r} = 3a\cos\theta \frac{d\theta}{dt} = 9a\cos\theta$

$\mathbf{v} = (3a\sin\theta)\mathbf{u}_r + (a(1-\cos\theta))(3)\mathbf{u}_\theta = (3a\sin\theta)\mathbf{u}_r + 3a(1-\cos\theta)\mathbf{u}_\theta$

$\mathbf{a} = \left(9a\cos\theta - a(1-\cos\theta)(3)^2\right)\mathbf{u}_r + \left(a(1-\cos\theta)\cdot 0 + 2(3a\sin\theta)(3)\right)\mathbf{u}_\theta$

$= (9a\cos\theta - 9a + 9a\cos\theta)\mathbf{u}_r + (18a\sin\theta)\mathbf{u}_\theta = 9a(2\cos\theta - 1)\mathbf{u}_r + (18a\sin\theta)\mathbf{u}_\theta$

2. $\frac{d\theta}{dt} 2t = \dot{\theta} \Rightarrow \ddot{\theta} = 2,\ \ r = a\sin 2\theta \Rightarrow \dot{r} = a\cos 2\theta \cdot 2\frac{d\theta}{dt} = 4ta\cos 2\theta \Rightarrow \ddot{r} = 4ta\left(-\sin 2\theta \cdot 2\frac{d\theta}{dt}\right) + 4a\cos 2\theta$

$= -16t^2 a\sin 2\theta + 4a\cos 2\theta;\ \ \mathbf{v} = (4ta\cos 2\theta)\mathbf{u}_r + (a\sin 2\theta)(2t)\mathbf{u}_\theta = (4ta\cos 2\theta)\mathbf{u}_r + (2ta\sin 2\theta)\mathbf{u}_\theta$

$\mathbf{a} = \left[\left(-16t^2 a\sin 2\theta + 4a\cos 2\theta\right) - (a\sin 2\theta)(2t)^2\right]\mathbf{u}_r + \left[(a\sin 2\theta)(2) + 2(4ta\cos 2\theta)(2t)\right]\mathbf{u}_\theta$

$= \left[-16t^2 a\sin 2\theta + 4a\cos 2\theta - 4t^2 a\sin 2\theta\right]\mathbf{u}_r + \left[2a\sin 2\theta + 16t^2 a\cos 2\theta\right]\mathbf{u}_\theta$

$= \left[-20t^2 a\sin 2\theta + 4a\cos 2\theta\right]\mathbf{u}_r + \left[2a\sin 2\theta + 16t^2 a\cos 2\theta\right]\mathbf{u}_\theta$

$= 4a\left(\cos 2\theta - 5t^2\sin 2\theta\right)\mathbf{u}_r + 2a\left(\sin 2\theta + 8t^2\cos 2\theta\right)\mathbf{u}_\theta$

3. $\frac{d\theta}{dt} = 2 = \dot{\theta} \Rightarrow \ddot{\theta} = 0,\ \ r = e^{a\theta} \Rightarrow \dot{r} = e^{a\theta}\cdot a\frac{d\theta}{dt} = 2a\,e^{a\theta} \Rightarrow \ddot{r} = 2a\,e^{a\theta}\cdot a\frac{d\theta}{dt} = 4a^2 e^{a\theta}$

$\mathbf{v} = \left(2a\,e^{a\theta}\right)\mathbf{u}_r + \left(e^{a\theta}\right)(2)\mathbf{u}_\theta = \left(2a\,e^{a\theta}\right)\mathbf{u}_r + \left(2e^{a\theta}\right)\mathbf{u}_\theta$

$\mathbf{a} = \left[\left(4a^2 e^{a\theta}\right) - \left(e^{a\theta}\right)(2)^2\right]\mathbf{u}_r + \left[\left(e^{a\theta}\right)(0) + 2\left(2a\,e^{a\theta}\right)(2)\right]\mathbf{u}_\theta = \left[4a^2 e^{a\theta} - 4e^{a\theta}\right]\mathbf{u}_r + \left[0 + 8a\,e^{a\theta}\right]\mathbf{u}_\theta$

$= 4e^{a\theta}\left(a^2 - 1\right)\mathbf{u}_r + \left(8a\,e^{a\theta}\right)\mathbf{u}_\theta$

4. $\theta = 1 - e^{-t} \Rightarrow \dot{\theta} = e^{-t} \Rightarrow \ddot{\theta} = -e^{-t},\ \ r = a(1+\sin t) \Rightarrow \dot{r} = a\cos t \Rightarrow \ddot{r} = -a\sin t$

$\mathbf{v} = (a\cos t)\mathbf{u}_r + (a(1+\sin t))\left(e^{-t}\right)\mathbf{u}_\theta = (a\cos t)\mathbf{u}_r + a\,e^{-t}(1+\sin t)\mathbf{u}_\theta$

$\mathbf{a} = \left[(-a\sin t) - (a(1+\sin t))\left(e^{-t}\right)^2\right]\mathbf{u}_r + \left[(a(1+\sin t))\left(-e^{-t}\right) + 2(a\cos t)\left(e^{-t}\right)\right]\mathbf{u}_\theta$

$= \left[-a\sin t - a\,e^{-2t}(1+\sin t)\right]\mathbf{u}_r + \left[-a\,e^{-t}(1+\sin t) + 2a\,e^{-t}\cos t\right]\mathbf{u}_\theta$

$= -a\left(\sin t + e^{-2t}(1+\sin t)\right)\mathbf{u}_r + a\,e^{-t}\left(-(1+\sin t) + 2\cos t\right)\mathbf{u}_\theta$

$= -a\left(\sin t + e^{-2t}(1+\sin t)\right)\mathbf{u}_r + a\,e^{-t}(2\cos t - 1 - \sin t)\mathbf{u}_\theta$

5. $\theta = 2t \Rightarrow \dot{\theta} = 2 \Rightarrow \ddot{\theta} = 0,\ \ r = 2\cos 4t \Rightarrow \dot{r} = -8\sin 4t \Rightarrow \ddot{r} = -32\cos 4t$

$\mathbf{v} = (-8\sin 4t)\mathbf{u}_r + (2\cos 4t)(2)\mathbf{u}_\theta = -8(\sin 4t)\mathbf{u}_r + 4(\cos 4t)\mathbf{u}_\theta$

$\mathbf{a} = \left((-32\cos 4t) - (2\cos 4t)(2)^2\right)\mathbf{u}_r + \left((2\cos 4t)\cdot 0 + 2(-8\sin 4t)(2)\right)\mathbf{u}_\theta$

$= (-32\cos 4t - 8\cos 4t)\mathbf{u}_r + (0 - 32\sin 4t)\mathbf{u}_\theta = -40(\cos 4t)\mathbf{u}_r - 32(\sin 4t)\mathbf{u}_\theta$

6. $e = \frac{r_0 v_0^2}{GM} - 1 \Rightarrow v_0^2 = \frac{GM(e+1)}{r_0} \Rightarrow v_0 = \sqrt{\frac{GM(e+1)}{r_0}};$

Circle: $e = 0 \Rightarrow v_0 = \sqrt{\frac{GM}{r_0}}$

Ellipse: $0 < e < 1 \Rightarrow \sqrt{\frac{GM}{r_0}} < v_0 < \sqrt{\frac{2GM}{r_0}}$

Parabola: $e = 1 \Rightarrow v_0 = \sqrt{\frac{2GM}{r_0}}$

Hyperbola: $e > 1 \Rightarrow v_0 > \sqrt{\frac{2GM}{r_0}}$

7. $r = \frac{GM}{v^2} \Rightarrow v^2 = \frac{GM}{r} \Rightarrow v = \sqrt{\frac{GM}{r}}$ which is constant since G, M, and r (the radius of orbit) are constant

8. $\Delta A = \frac{1}{2}\left|\mathbf{r}(t+\Delta t)\times\mathbf{r}(t)\right| \Rightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t)}{\Delta t}\times\mathbf{r}(t)\right| = \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)+\mathbf{r}(t)}{\Delta t}\times\mathbf{r}(t)\right| = \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\times\mathbf{r}(t) + \frac{1}{\Delta t}\mathbf{r}(t)\times\mathbf{r}(t)\right|$

$= \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\times\mathbf{r}(t)\right| \Rightarrow \frac{dA}{dt} = \lim_{\Delta t\to 0}\frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\times\mathbf{r}(t)\right| = \frac{1}{2}\left|\frac{d\mathbf{r}}{dt}\times\mathbf{r}(t)\right| = \frac{1}{2}\left|\mathbf{r}(t)\times\frac{d\mathbf{r}}{dt}\right| = \frac{1}{2}\left|\mathbf{r}\times\dot{\mathbf{r}}\right|$

9. $T = \left(\frac{2\pi a^2}{r_0 v_0}\right)\sqrt{1-e^2} \Rightarrow T^2 = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left(1-e^2\right) = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[1 - \left(\frac{r_0 v_0^2}{GM} - 1\right)^2\right]$ (from Equation 5)

$= \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[-\frac{r_0^2 v_0^4}{G^2M^2} + 2\left(\frac{r_0 v_0^2}{GM}\right)\right] = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[\frac{2GMr_0 v_0^2 - r_0^2 v_0^4}{G^2M^2}\right] = \frac{\left(4\pi^2 a^4\right)\left(2GM - r_0 v_0^2\right)}{r_0 G^2 M^2} = \left(4\pi^2 a^4\right)\left(\frac{2GM - r_0 v_0^2}{2r_0 GM}\right)\left(\frac{2}{GM}\right)$

$= \left(4\pi^2 a^4\right)\left(\frac{1}{2a}\right)\left(\frac{2}{GM}\right)$ (from Equation 10) $\Rightarrow T^2 = \frac{4\pi^2 a^3}{GM} \Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{GM}$

10. For each of the planets listed we form the ratio $\dfrac{T^2/a^3}{(4\pi^2)/(GM)}$. The values we obtain are

Mercury	1.00225
Venus	1.00288
Mars	1.00252
Saturn	1.00019

These values are all close to 1, so they support Kepler's third law.

11. Solve Kepler's third law for a and double this result: $2\cdot\left(\dfrac{(365.256\text{ days})^2}{(4\pi^2)/(GM)}\right)^{1/3} \approx 29.925\times 10^{10}$ m

12. Solve Kepler's third law for a and double this result: $2\cdot\left(\dfrac{(84\text{ years})^2}{(4\pi^2)/(GM)}\right)^{1/3} \approx 573.95\times 10^{10}$ m

13. Assuming Earth has a circular orbit with radius 150×10^6 km, the rate of change of area is

$$\frac{\pi\left(150\times10^6\text{ km}\right)^2}{365.256\text{ days}}\approx 2.24\times10^9\text{ km}^2/\text{s}.$$

14. Solving Kepler's third law for T we find $T=\sqrt{\frac{4\pi^2}{GM}\left(77.8\times10^{10}\text{ m}\right)^3}\approx 11.857$ years.

15. Solving Kepler's third law for the mass M of the body around which Io is orbiting we find

$$M=4\pi^2\frac{a^3}{T^2G}=4\pi^2\frac{\left(0.042\times10^{10}\text{ m}\right)^3}{(1.769\text{ days})^2G}\approx 1.876\times10^{27}\text{ kg}.$$

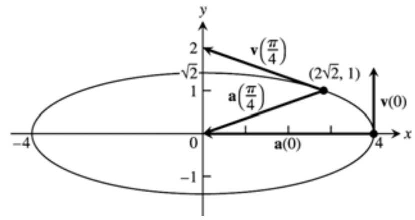
16. To solve this we need a value for the mass of Earth, which is approximately $M=5.972\times10^{24}$ kg. Solving Kepler's third law for the orbital radius we get

$$a=\left(4\pi^2\right)^{-1/3}\left(T^2GM\right)^{1/3}=\left(4\pi^2\right)^{-1/3}\left(\left(2.36055\times10^6\text{ s}\right)G\left(5.972\times10^{24}\text{ kg}\right)\right)^{1/3}\approx 3.831\times10^5\text{ km}.$$

Since Earth's radius is about 6371, the orbit of the moon is about $383{,}143-6371=376.772$ km from the surface, assuming a circular orbit for the moon.

CHAPTER 13 PRACTICE EXERCISES

1. $\mathbf{r}(t) = (4\cos t)\mathbf{i} + \left(\sqrt{2}\sin t\right)\mathbf{j}$



$\Rightarrow x = 4\cos t$ and $y = \sqrt{2}\sin t \Rightarrow \frac{x^2}{16} + \frac{y^2}{2} = 1;$

$\mathbf{v} = (-4\sin t)\mathbf{i} + \left(\sqrt{2}\cos t\right)\mathbf{j}$ and

$\mathbf{a} = (-4\cos t)\mathbf{i} - \left(\sqrt{2}\sin t\right)\mathbf{j};$

$\mathbf{r}(0) = 4\,\mathbf{i},\ \mathbf{v}(0) = \sqrt{2}\mathbf{j},\ \mathbf{a}(0) = -4\mathbf{i};$

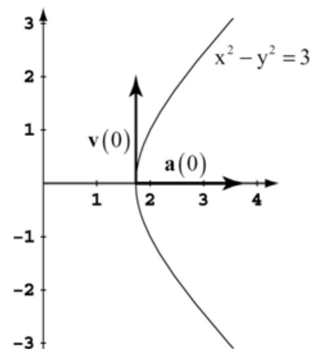
$\mathbf{r}\left(\frac{\pi}{4}\right) = 2\sqrt{2}\mathbf{i} + \mathbf{j},\ \mathbf{v}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\mathbf{i} + \mathbf{j},\ \mathbf{a}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\mathbf{i} - \mathbf{j};\ |\mathbf{v}| = \sqrt{16\sin^2 t + 2\cos^2 t}$

$\Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = \frac{14\sin t\cos t}{\sqrt{16\sin^2 t + 2\cos^2 t}};$ at $t = 0$: $a_T = 0,\ a_N = \sqrt{|\mathbf{a}|^2 - 0} = 4,\ \mathbf{a} = 0\mathbf{T} + 4\mathbf{N} = 4\mathbf{N},$

$\kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{4}{2} = 2;$ at $t = \frac{\pi}{4}$: $a_T = \frac{7}{\sqrt{8+1}} = \frac{7}{3},\ a_N = \sqrt{9 - \frac{49}{9}} = \frac{4\sqrt{2}}{3},\ \ \mathbf{a} = \frac{7}{3}\mathbf{T} + \frac{4\sqrt{2}}{3}\mathbf{N},\ \ \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{4\sqrt{2}}{27}$

2. $\mathbf{r}(t) = \left(\sqrt{3}\sec t\right)\mathbf{i} + \left(\sqrt{3}\tan t\right)\mathbf{j} \Rightarrow x = \sqrt{3}\sec t$ and $y = \sqrt{3}\tan t \Rightarrow \frac{x^2}{3} - \frac{y^2}{3} = \sec^2 t - \tan^2 t = 1 \Rightarrow x^2 - y^2 = 3;$

$\mathbf{v} = \left(\sqrt{3}\sec t\tan t\right)\mathbf{i} + \left(\sqrt{3}\sec^2 t\right)\mathbf{j}$ and



$\mathbf{a} = \left(\sqrt{3}\sec t\tan^2 t + \sqrt{3}\sec^3 t\right)\mathbf{i} - \left(2\sqrt{3}\sec^2 t\tan t\right)\mathbf{j};$

$\mathbf{r}(0) = \sqrt{3}\mathbf{i},\ \mathbf{v}(0) = \sqrt{3}\mathbf{j},\ \mathbf{a}(0) = \sqrt{3}\mathbf{i};$

$|\mathbf{v}| = \sqrt{3\sec^2 t\tan^2 t + 3\sec^4 t}$

$\Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = \frac{6\sec^2 t\tan^3 t + 18\sec^4 t\tan t}{2\sqrt{3\sec^2 t\tan^2 t + 3\sec^4 t}};$

at $t = 0$: $a_T = 0,\ a_N = \sqrt{|\mathbf{a}|^2 - 0} = \sqrt{3},$

$\mathbf{a} = 0\mathbf{T} + \sqrt{3}\mathbf{N} = \sqrt{3}\mathbf{N},\ \ \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$

3. $\mathbf{r} = \frac{1}{\sqrt{1+t^2}}\mathbf{i} + \frac{t}{\sqrt{1+t^2}}\mathbf{j} \Rightarrow \mathbf{v} = -t\left(1+t^2\right)^{-3/2}\mathbf{i} + \left(1+t^2\right)^{-3/2}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\left[-t\left(1+t^2\right)^{-3/2}\right]^2 + \left[\left(1+t^2\right)^{-3/2}\right]^2} = \frac{1}{1+t^2}.$

We want to maximize $|\mathbf{v}|$: $\frac{d|\mathbf{v}|}{dt} = \frac{-2t}{\left(1+t^2\right)^2}$ and $\frac{d|\mathbf{v}|}{dt} = 0 \Rightarrow \frac{-2t}{\left(1+t^2\right)^2} = 0 \Rightarrow t = 0.$ For $t < 0$, $\frac{-2t}{\left(1+t^2\right)^2} > 0;$ for $t > 0$, $\frac{-2t}{\left(1+t^2\right)^2} < 0 \Rightarrow |\mathbf{v}|_{\max}$ occurs when $t = 0 \Rightarrow |\mathbf{v}|_{\max} = 1$

4. $\mathbf{r} = \left(e^t\cos t\right)\mathbf{i} + \left(e^t\sin t\right)\mathbf{j} \Rightarrow \mathbf{v} = \left(e^t\cos t - e^t\sin t\right)\mathbf{i} + \left(e^t\sin t + e^t\cos t\right)\mathbf{j}$

$\Rightarrow \mathbf{a} = \left(e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t\right)\mathbf{i} + \left(e^t\sin t + e^t\cos t + e^t\cos t - e^t\sin t\right)\mathbf{j} = \left(-2e^t\sin t\right)\mathbf{i} + \left(2e^t\cos t\right)\mathbf{j}.$

Let θ be the angle between $\mathbf{r}$ and $\mathbf{a}$. Then $\theta = \cos^{-1}\left(\frac{\mathbf{r}\cdot\mathbf{a}}{|\mathbf{r}||\mathbf{a}|}\right) = \cos^{-1}\left(\frac{-2e^{2t}\sin t\cos t + 2e^{2t}\sin t\cos t}{\sqrt{\left(e^t\cos t\right)^2 + \left(e^t\sin t\right)^2}\sqrt{\left(-2e^t\sin t\right)^2 + \left(2e^t\cos t\right)^2}}\right)$

$= \cos^{-1}\left(\frac{0}{2e^{2t}}\right) = \cos^{-1} 0 = \frac{\pi}{2}$ for all t

5. $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{a} = 5\mathbf{i} + 15\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ 5 & 15 & 0 \end{vmatrix} = 25\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 25;\ |\mathbf{v}| = \sqrt{3^2 + 4^2} = 5 \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{25}{5^3} = \frac{1}{5}$

6. $\kappa = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}} = e^x\left(1+e^{2x}\right)^{-3/2} \Rightarrow \frac{d\kappa}{dx} = e^x\left(1+e^{2x}\right)^{-3/2} + e^x\left[-\frac{3}{2}\left(1+e^{2x}\right)^{-5/2}\left(2e^{2x}\right)\right]$

$= e^x\left(1+e^{2x}\right)^{-3/2} - 3e^{3x}\left(1+e^{2x}\right)^{-5/2} = e^x\left(1+e^{2x}\right)^{-5/2}\left[\left(1+e^{2x}\right) - 3e^{2x}\right] = e^x\left(1+e^{2x}\right)^{-5/2}\left(1-2e^{2x}\right);$

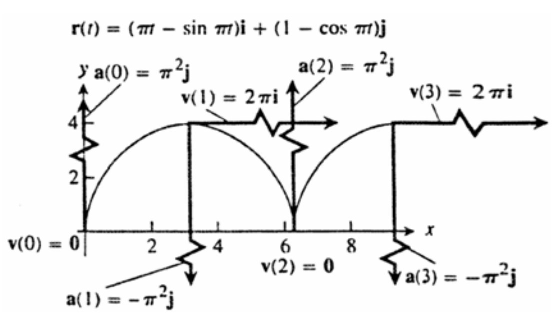
$\frac{d\kappa}{dx} = 0 \Rightarrow \left(1-2e^{2x}\right) = 0 \Rightarrow e^{2x} = \frac{1}{2} \Rightarrow 2x = -\ln 2 \Rightarrow x = -\frac{1}{2}\ln 2 = -\ln\sqrt{2} \Rightarrow y = \frac{1}{\sqrt{2}}$; therefore κ is at a maximum at the point $\left(-\ln\sqrt{2}, \frac{1}{\sqrt{2}}\right)$

7. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ and $\mathbf{v}\cdot\mathbf{i} = y \Rightarrow \frac{dx}{dt} = y$. Since the particle moves around the unit circle $x^2 + y^2 = 1$, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{x}{y}(y) = -x$. Since $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -x$, we have $\mathbf{v} = y\mathbf{i} - x\mathbf{j} \Rightarrow$ at $(1, 0)$, $\mathbf{v} = -\mathbf{j}$ and the motion is clockwise.

8. $9y = x^3 \Rightarrow 9\frac{dy}{dt} = 3x^2\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{3}x^2\frac{dx}{dt}$. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t, then $\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$. Hence $\mathbf{v}\cdot\mathbf{i} = 4 \Rightarrow \frac{dx}{dt} = 4$ and $\mathbf{v}\cdot\mathbf{j} = \frac{dy}{dt} = \frac{1}{3}x^2\frac{dx}{dt} = \frac{1}{3}(3)^2(4) = 12$ at $(3, 3)$. Also, $\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$ and $\frac{d^2y}{dt^2} = \left(\frac{2}{3}x\right)\left(\frac{dx}{dt}\right)^2 + \left(\frac{1}{3}x^2\right)\frac{d^2x}{dt^2}$. Hence $\mathbf{a}\cdot\mathbf{i} = -2 \Rightarrow \frac{d^2x}{dt^2} = -2$ and $\mathbf{a}\cdot\mathbf{j} = \frac{d^2y}{dt^2} = \frac{2}{3}(3)(4)^2 + \frac{1}{3}(3)^2(-2) = 26$ at the point $(x, y) = (3, 3)$.

9. $\frac{d\mathbf{r}}{dt}$ orthogonal to $\mathbf{r} \Rightarrow 0 = \frac{d\mathbf{r}}{dt}\cdot\mathbf{r} = \frac{1}{2}\frac{d\mathbf{r}}{dt}\cdot\mathbf{r} + \frac{1}{2}\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = \frac{1}{2}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) \Rightarrow \mathbf{r}\cdot\mathbf{r} = K$, a constant. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t, then $\mathbf{r}\cdot\mathbf{r} = x^2 + y^2 \Rightarrow x^2 + y^2 = K$, which is the equation of a circle centered at the origin.

10. (a)



(b) $\mathbf{v} = (\pi - \pi\cos\pi t)\mathbf{i} + (\pi\sin\pi t)\mathbf{j}$

$\Rightarrow \mathbf{a} = \left(\pi^2\sin\pi t\right)\mathbf{i} + \left(\pi^2\cos\pi t\right)\mathbf{j};$

$\mathbf{v}(0) = \mathbf{0}$ and $\mathbf{a}(0) = \pi^2\mathbf{j}$;

$\mathbf{v}(1) = 2\pi\mathbf{i}$ and $\mathbf{a}(1) = -\pi^2\mathbf{j}$;

$\mathbf{v}(2) = \mathbf{0}$ and $\mathbf{a}(2) = \pi^2\mathbf{j}$;

$\mathbf{v}(3) = 2\pi\mathbf{i}$; and $\mathbf{a}(3) = -\pi^2\mathbf{j}$

(c) Forward speed at the topmost point is $|\mathbf{v}(1)| = |\mathbf{v}(3)| = 2\pi$ m/s; since the circle makes $\frac{1}{2}$ revolution per second, the center moves π m parallel to the x-axis each second $\Rightarrow$ the forward speed of C is π m/s.

11. $y = y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow y = 2 + (14\text{ m/s})(\sin 45^\circ)(3\text{s}) - \frac{1}{2}\left(9.8\text{ m/s}^2\right)(3\text{s})^2 = 2 + 21\sqrt{2} - 44.1$ $\approx -12.40\text{ m} \Rightarrow$ the shot put is on the ground. Now, $y = 0 \Rightarrow 2 + 7\sqrt{2}t - 4.9t^2 = 0 \Rightarrow t \approx 2.21$ s (the positive root) $\Rightarrow x \approx (14\text{ m/s})(\cos 45^\circ)(2.21\text{s}) \approx 21.88$ m from the stopboard.

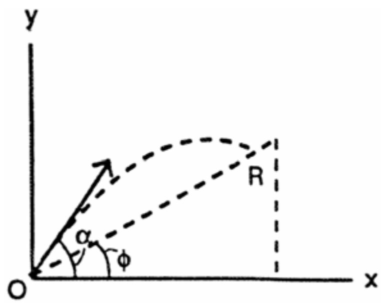
12. $y_{\max} = y_0 + \frac{(v_0 \sin\alpha)^2}{2g} = 2.5\text{ m} + \frac{\left[(24\text{ m/s})(\sin 45^\circ)\right]^2}{(2)(9.8\text{ m/s}^2)} \approx 17.2\text{ m}$

13. $x = (v_0 \cos\alpha)t$ and $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow \tan\phi = \frac{y}{x} = \frac{(v_0 \sin\alpha)t - \frac{1}{2}gt^2}{(v_0\cos\alpha)t} = \frac{(v_0\sin\alpha) - \frac{1}{2}g\,t}{v_0\cos\alpha}$ $\Rightarrow v_0 \cos\alpha \tan\phi = v_0 \sin\alpha - \frac{1}{2}gt \Rightarrow t = \frac{2v_0\sin\alpha - 2v_0\cos\alpha\tan\phi}{g}$, which is the time when the golf ball hits the upward slope. At this time $x = (v_0\cos\alpha)\left(\frac{2v_0\sin\alpha - 2v_0\cos\alpha\tan\phi}{g}\right) = \left(\frac{2}{g}\right)\left(v_0^2\sin\alpha\cos\alpha - v_0^2\cos^2\alpha\tan\phi\right)$. Now

$OR = \frac{x}{\cos\phi} \Rightarrow OR = \left(\frac{2}{g}\right)\left(\frac{v_0^2\sin\alpha\cos\alpha - v_0^2\cos^2\alpha\tan\phi}{\cos\phi}\right)$

$= \left(\frac{2v_0^2\cos\alpha}{g}\right)\left(\frac{\sin\alpha}{\cos\phi} - \frac{\cos\alpha\tan\phi}{\cos\phi}\right) = \left(\frac{2v_0^2\cos\alpha}{g}\right)\left(\frac{\sin\alpha\cos\phi - \cos\alpha\sin\phi}{\cos^2\phi}\right)$

$= \left(\frac{2v_0^2\cos\alpha}{g\cos^2\phi}\right)\left[\sin(\alpha - \phi)\right]$. The distance OR is maximized when x is maximized: $\frac{dx}{d\alpha} = \left(\frac{2v_0^2}{g}\right)(\cos 2\alpha + \sin 2\alpha\tan\phi) = 0$



$\Rightarrow (\cos 2\alpha + \sin 2\alpha \tan\phi) = 0 \Rightarrow \cot 2\alpha + \tan\phi = 0 \Rightarrow \cot 2\alpha = \tan(-\phi) \Rightarrow 2\alpha = \frac{\pi}{2} + \phi \Rightarrow \alpha = \frac{\phi}{2} + \frac{\pi}{4}$

14. (a) $x = v_0\left(\cos 40^\circ\right)t$ and $y = 2 + v_0\left(\sin 40^\circ\right)t - \frac{1}{2}gt^2 = 2 + v_0\left(\sin 40^\circ\right)t - 4.9t^2$; $x = 80$ m and $y = 0\text{ m} \Rightarrow 80 = v_0\left(\cos 40^\circ\right)t$ or $v_0 = \frac{80}{\left(\cos 40^\circ\right)t}$ and $0 = 2 + \left[\frac{80}{\left(\cos 40^\circ\right)t}\right]\left(\sin 40^\circ\right)t - 4.9t^2$

$\Rightarrow t^2 = 14.108 \Rightarrow t \approx 3.756$ s. Therefore, $80 \approx v_0\left(\cos 40^\circ\right)(3.756\text{ s})$

$\Rightarrow v_0 \approx \frac{80}{\left(\cos 40^\circ\right)(3.756\text{ s})} \Rightarrow v_0 \approx 27.8\text{ m/s}$

(b) $y_{\max} = y_0 + \frac{(v_0\sin\alpha)^2}{2g} \approx 2 + \frac{\left((27.8)\left(\sin 40^\circ\right)\right)^2}{(2)(9.8)} \approx 18.3\text{ m}$

15. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (2t)^2}$ $= 2\sqrt{1+t^2} \Rightarrow$ Length $= \int_0^{\pi/4} 2\sqrt{1+t^2}\,dt = \left[t\sqrt{1+t^2} + \ln\left|t + \sqrt{1+t^2}\right|\right]_0^{\pi/4} = \frac{\pi}{4}\sqrt{1+\frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4} + \sqrt{1+\frac{\pi^2}{16}}\right)$

16. $\mathbf{r} = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 2t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 3t^{1/2}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + \left(3t^{1/2}\right)^2}$ $= \sqrt{9+9t} = 3\sqrt{1+t} \Rightarrow$ Length $= \int_0^3 3\sqrt{1+t}\,dt = \left[2(1+t)^{3/2}\right]_0^3 = 14$

17. $\mathbf{r}=\frac{4}{9}(1+t)^{3/2}\,\mathbf{i}+\frac{4}{9}(1-t)^{3/2}\,\mathbf{j}+\frac{1}{3}t\mathbf{k} \Rightarrow \mathbf{v}=\frac{2}{3}(1+t)^{1/2}\,\mathbf{i}-\frac{2}{3}(1-t)^{1/2}\,\mathbf{j}+\frac{1}{3}\mathbf{k}$

$\Rightarrow |\mathbf{v}|=\sqrt{\left[\frac{2}{3}(1+t)^{1/2}\right]^2+\left[-\frac{2}{3}(1-t)^{1/2}\right]^2+\left(\frac{1}{3}\right)^2}=1 \Rightarrow \mathbf{T}=\frac{2}{3}(1+t)^{1/2}\,\mathbf{i}-\frac{2}{3}(1-t)^{1/2}\,\mathbf{j}+\frac{1}{3}\mathbf{k}$

$\Rightarrow \mathbf{T}(0)=\frac{2}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{1}{3}\mathbf{k}$; $\frac{d\mathbf{T}}{dt}=\frac{1}{3}(1+t)^{-1/2}\,\mathbf{i}+\frac{1}{3}(1-t)^{-1/2}\,\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0)=\frac{1}{3}\mathbf{i}+\frac{1}{3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(0)\right|=\frac{\sqrt{2}}{3}$

$\Rightarrow \mathbf{N}(0)=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$; $\mathbf{B}(0)=\mathbf{T}(0)\times\mathbf{N}(0)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{vmatrix}=-\frac{1}{3\sqrt{2}}\mathbf{i}+\frac{1}{3\sqrt{2}}\mathbf{j}+\frac{4}{3\sqrt{2}}\mathbf{k}$;

$\mathbf{a}=\frac{1}{3}(1+t)^{-1/2}\,\mathbf{i}+\frac{1}{3}(1-t)^{-1/2}\,\mathbf{j} \Rightarrow \mathbf{a}(0)=\frac{1}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}$ and $\mathbf{v}(0)=\frac{2}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{1}{3}\mathbf{k} \Rightarrow \mathbf{v}(0)\times\mathbf{a}(0)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & 0\end{vmatrix}$

$=-\frac{1}{9}\mathbf{i}+\frac{1}{9}\mathbf{j}+\frac{4}{9}\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}|=\frac{\sqrt{2}}{3} \Rightarrow \kappa(0)=\frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}=\frac{\left(\frac{\sqrt{2}}{3}\right)}{1^3}=\frac{\sqrt{2}}{3}$;

$\dot{\mathbf{a}}=-\frac{1}{6}(1+t)^{-3/2}\,\mathbf{i}+\frac{1}{6}(1-t)^{-3/2}\,\mathbf{j} \Rightarrow \dot{\mathbf{a}}(0)=-\frac{1}{6}\mathbf{i}+\frac{1}{6}\mathbf{j} \Rightarrow \tau(0)=\frac{\begin{vmatrix}\frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & 0\\ -\frac{1}{6} & \frac{1}{6} & 0\end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2}=\frac{\left(\frac{1}{3}\right)\left(\frac{2}{18}\right)}{\left(\frac{\sqrt{2}}{3}\right)^2}=\frac{1}{6}$

18. $\mathbf{r}=\left(e^t\sin 2t\right)\mathbf{i}+\left(e^t\cos 2t\right)\mathbf{j}+2e^t\mathbf{k} \Rightarrow \mathbf{v}=\left(e^t\sin 2t+2e^t\cos 2t\right)\mathbf{i}+\left(e^t\cos 2t-2e^t\sin 2t\right)\mathbf{j}+2e^t\mathbf{k}$

$\Rightarrow |\mathbf{v}|=\sqrt{\left(e^t\sin 2t+2e^t\cos 2t\right)^2+\left(e^t\cos 2t-2e^t\sin 2t\right)^2+\left(2e^t\right)^2}=3e^t$

$\Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\left(\frac{1}{3}\sin 2t+\frac{2}{3}\cos 2t\right)\mathbf{i}+\left(\frac{1}{3}\cos 2t-\frac{2}{3}\sin 2t\right)\mathbf{j}+\frac{2}{3}\mathbf{k} \Rightarrow \mathbf{T}(0)=\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$;

$\frac{d\mathbf{T}}{dt}=\left(\frac{2}{3}\cos 2t-\frac{4}{3}\sin 2t\right)\mathbf{i}+\left(-\frac{2}{3}\sin 2t-\frac{4}{3}\cos 2t\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0)=\frac{2}{3}\mathbf{i}-\frac{4}{3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(0)\right|=\frac{2}{3}\sqrt{5}$

$\Rightarrow \mathbf{N}(0)=\frac{\left(\frac{2}{3}\mathbf{i}-\frac{4}{3}\mathbf{j}\right)}{\left(\frac{2\sqrt{5}}{3}\right)}=\frac{1}{\sqrt{5}}\mathbf{i}-\frac{2}{\sqrt{5}}\mathbf{j}$; $\mathbf{B}(0)=\mathbf{T}(0)\times\mathbf{N}(0)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3}\\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0\end{vmatrix}=\frac{4}{3\sqrt{5}}\mathbf{i}+\frac{2}{3\sqrt{5}}\mathbf{j}-\frac{5}{3\sqrt{5}}\mathbf{k}$;

$\mathbf{a}=\left(4e^t\cos 2t-3e^t\sin 2t\right)\mathbf{i}+\left(-3e^t\cos 2t-4e^t\sin 2t\right)\mathbf{j}+2e^t\mathbf{k} \Rightarrow \mathbf{a}(0)=4\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ and

$\mathbf{v}(0)=2\mathbf{i}+\mathbf{j}+2\mathbf{k} \Rightarrow \mathbf{v}(0)\times\mathbf{a}(0)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 2\\ 4 & -3 & 2\end{vmatrix}=8\mathbf{i}+4\mathbf{j}-10\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}|=\sqrt{64+16+100}=6\sqrt{5}$ and

$|\mathbf{v}(0)|=3 \Rightarrow \kappa(0)=\frac{6\sqrt{5}}{3^3}=\frac{2\sqrt{5}}{9}$;

$\dot{\mathbf{a}}=\left(4e^t\cos 2t-8e^t\sin 2t-3e^t\sin 2t-6e^t\cos 2t\right)\mathbf{i}+\left(-3e^t\cos 2t+6e^t\sin 2t-4e^t\sin 2t-8e^t\cos 2t\right)\mathbf{j}+2e^t\mathbf{k}$

$=\left(-2e^t\cos 2t-11e^t\sin 2t\right)\mathbf{i}+\left(-11e^t\cos 2t+2e^t\sin 2t\right)\mathbf{j}+2e^t\mathbf{k} \Rightarrow \dot{\mathbf{a}}(0)=-2\mathbf{i}-11\mathbf{j}+2\mathbf{k}$

$\Rightarrow \tau(0)=\frac{\begin{vmatrix}2 & 1 & 2\\ 4 & -3 & 2\\ -2 & -11 & 2\end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2}=\frac{-80}{180}=-\frac{4}{9}$

19. $\mathbf{r} = t\mathbf{i} + \frac{1}{2}e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + e^{2t}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1+e^{4t}} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+e^{4t}}}\mathbf{i} + \frac{e^{2t}}{\sqrt{1+e^{4t}}}\mathbf{j} \Rightarrow \mathbf{T}(\ln 2) = \frac{1}{\sqrt{17}}\mathbf{i} + \frac{4}{\sqrt{17}}\mathbf{j}$;

$\frac{d\mathbf{T}}{dt} = \frac{-2e^{4t}}{\left(1+e^{4t}\right)^{3/2}}\mathbf{i} + \frac{2e^{2t}}{\left(1+e^{4t}\right)^{3/2}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2) = \frac{-32}{17\sqrt{17}}\mathbf{i} + \frac{8}{17\sqrt{17}}\mathbf{j} \Rightarrow \mathbf{N}(\ln 2) = -\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{17}\mathbf{j}$;

$$\mathbf{B}(\ln 2) = \mathbf{T}(\ln 2)\times\mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \end{vmatrix} = \mathbf{k};\ \ \mathbf{a} = 2e^{2t}\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 8\mathbf{j} \text{ and } \mathbf{v}(\ln 2) = \mathbf{i} + 4\mathbf{j}$$

$$\Rightarrow \mathbf{v}(\ln 2)\times\mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 0 & 8 & 0 \end{vmatrix} = 8\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = 8 \text{ and } |\mathbf{v}(\ln 2)| = \sqrt{17} \Rightarrow \kappa(\ln 2) = \frac{8}{17\sqrt{17}};$$

$$\dot{\mathbf{a}} = 4e^{2t}\mathbf{j} \Rightarrow \dot{\mathbf{a}}(\ln 2) = 16\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 16 & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = 0$$

20. $r = (3\cosh 2t)\mathbf{i} + (3\sinh 2t)\mathbf{j} + 6t\mathbf{k} \Rightarrow \mathbf{v} = (6\sinh 2t)\mathbf{i} + (6\cosh 2t)\mathbf{j} + 6\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{36\sinh^2 2t + 36\cosh^2 2t + 36} = 6\sqrt{2}\cosh 2t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}}\tanh 2t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} 2t\right)\mathbf{k}$

$\Rightarrow \mathbf{T}(\ln 2) = \frac{5}{17\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{8}{17\sqrt{2}}\mathbf{k};\quad \frac{d\mathbf{T}}{dt} = \left(\frac{2}{\sqrt{2}}\operatorname{sech}^2 2t\right)\mathbf{i} - \left(\frac{2}{\sqrt{2}}\operatorname{sech} 2t\tanh 2t\right)\mathbf{k}$

$\Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2) = \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)^2\mathbf{i} - \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)\left(\frac{15}{17}\right)\mathbf{k} = \frac{128}{289\sqrt{2}}\mathbf{i} - \frac{240}{289\sqrt{2}}\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(\ln 2)\right| = \sqrt{\left(\frac{128}{289\sqrt{2}}\right)^2 + \left(-\frac{240}{289\sqrt{2}}\right)^2} = \frac{8\sqrt{2}}{17}$

$$\Rightarrow \mathbf{N}(\ln 2) = \frac{8}{17}\mathbf{i} - \frac{15}{17}\mathbf{k};\ \mathbf{B}(\ln 2) = \mathbf{T}(\ln 2)\times\mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{15}{17\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{8}{17\sqrt{2}} \\ \frac{8}{17} & 0 & -\frac{15}{17} \end{vmatrix} = -\frac{15}{17\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - \frac{8}{17\sqrt{2}}\mathbf{k};$$

$\mathbf{a} = (12\cosh 2t)\mathbf{i} + (12\sinh 2t)\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 12\left(\frac{17}{8}\right)\mathbf{i} + 12\left(\frac{15}{8}\right)\mathbf{j} = \frac{51}{2}\mathbf{i} + \frac{45}{2}\mathbf{j}$ and $\mathbf{v}(\ln 2) = 6\left(\frac{15}{8}\right)\mathbf{i} + 6\left(\frac{17}{8}\right)\mathbf{j} + 6\mathbf{k}$

$$= \frac{45}{4}\mathbf{i} + \frac{51}{4}\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(\ln 2)\times\mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \end{vmatrix} = -135\mathbf{i} + 153\mathbf{j} - 72\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = 153\sqrt{2} \text{ and } |\mathbf{v}(\ln 2)| = \frac{51}{4}\sqrt{2}$$

$$\Rightarrow \kappa(\ln 2) = \frac{153\sqrt{2}}{\left(\frac{51}{4}\sqrt{2}\right)^3} = \frac{32}{867};\ \dot{\mathbf{a}} = (24\sinh 2t)\mathbf{i} + (24\cosh 2t)\mathbf{j} \Rightarrow \dot{\mathbf{a}}(\ln 2) = 45\mathbf{i} + 51\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \\ 45 & 51 & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{32}{867}$$

21. $\mathbf{r} = \left(2 + 3t + 3t^2\right)\mathbf{i} + \left(4t + 4t^2\right)\mathbf{j} - (6\cos t)\mathbf{k} \Rightarrow \mathbf{v} = (3 + 6t)\mathbf{i} + (4 + 8t)\mathbf{j} + (6\sin t)\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{(3+6t)^2 + (4+8t)^2 + (6\sin t)^2} = \sqrt{25 + 100t + 100t^2 + 36\sin^2 t}$

$\Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}\left(25 + 100t + 100t^2 + 36\sin^2 t\right)^{-1/2}(100 + 200t + 72\sin t\cos t) \Rightarrow a_T(0) = \frac{d|\mathbf{v}|}{dt}(0) = 10;$

$\mathbf{a} = 6\mathbf{i} + 8\mathbf{j} + (6\cos t)\mathbf{k} \Rightarrow |\mathbf{a}| = \sqrt{6^2 + 8^2 + (6\cos t)^2} = \sqrt{100 + 36\cos^2 t} \Rightarrow |\mathbf{a}(0)| = \sqrt{136}$

$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{136 - 10^2} = \sqrt{36} = 6 \Rightarrow \mathbf{a}(0) = 10\mathbf{T} + 6\mathbf{N}$

22. $\mathbf{r} = (2+t)\mathbf{i} + \left(t + 2t^2\right)\mathbf{j} + \left(1 + t^2\right)\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + (1+4t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (1+4t)^2 + (2t)^2} = \sqrt{2 + 8t + 20t^2}$

$\Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}\left(2 + 8t + 20t^2\right)^{-1/2}(8 + 40t) \Rightarrow a_T = \frac{d|\mathbf{v}|}{dt}(0) = 2\sqrt{2}$; $\mathbf{a} = 4\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{a}| = \sqrt{4^2 + 2^2} = \sqrt{20}$

$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{20 - \left(2\sqrt{2}\right)^2} = \sqrt{12} = 2\sqrt{3} \Rightarrow \mathbf{a}(0) = 2\sqrt{2}\mathbf{T} + 2\sqrt{3}\mathbf{N}$

23. $\mathbf{r} = (\sin t)\mathbf{i} + \left(\sqrt{2}\cos t\right)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow \mathbf{v} = (\cos t)\mathbf{i} - \left(\sqrt{2}\sin t\right)\mathbf{j} + (\cos t)\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t)^2 + \left(-\sqrt{2}\sin t\right)^2 + (\cos t)^2} = \sqrt{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}}\cos t\right)\mathbf{i} - (\sin t)\mathbf{j} + \left(\frac{1}{\sqrt{2}}\cos t\right)\mathbf{k}$;

$\frac{d\mathbf{T}}{dt} = \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}}\sin t\right)\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{1}{\sqrt{2}}\sin t\right)^2 + (-\cos t)^2 + \left(-\frac{1}{\sqrt{2}}\sin t\right)^2} = 1$

$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}}\sin t\right)\mathbf{k}$; $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}}\cos t & -\sin t & \frac{1}{\sqrt{2}}\cos t \\ -\frac{1}{\sqrt{2}}\sin t & -\cos t & -\frac{1}{\sqrt{2}}\sin t \end{vmatrix} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}$;

$\mathbf{a} = (-\sin t)\mathbf{i} - \left(\sqrt{2}\cos t\right)\mathbf{j} - (\sin t)\mathbf{k} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \end{vmatrix} = \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{k}$

$\Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{4} = 2 \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{2}{\left(\sqrt{2}\right)^3} = \frac{1}{\sqrt{2}}$; $\dot{\mathbf{a}} = (-\cos t)\mathbf{i} + \left(\sqrt{2}\sin t\right)\mathbf{j} - (\cos t)\mathbf{k}$

$\Rightarrow \tau = \frac{\begin{vmatrix} \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \\ -\cos t & \sqrt{2}\sin t & -\cos t \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{(\cos t)\left(\sqrt{2}\right) - \left(\sqrt{2}\sin t\right)(0) + (\cos t)\left(-\sqrt{2}\right)}{4} = 0$

24. $\mathbf{r} = \mathbf{i} + (5\cos t)\mathbf{j} + (3\sin t)\mathbf{k} \Rightarrow \mathbf{v} = (-5\sin t)\mathbf{j} + (3\cos t)\mathbf{k} \Rightarrow \mathbf{a} = (-5\cos t)\mathbf{j} - (3\sin t)\mathbf{k}$
$\Rightarrow \mathbf{v}\cdot\mathbf{a} = 25\sin t\cos t - 9\sin t\cos t = 16\sin t\cos t$; $\mathbf{v}\cdot\mathbf{a} = 0 \Rightarrow 16\sin t\cos t = 0 \Rightarrow \sin t = 0$ or $\cos t = 0$
$\Rightarrow t = 0, \frac{\pi}{2}$ or π

25. $\mathbf{r} = 2\mathbf{i} + \left(4\sin\frac{t}{2}\right)\mathbf{j} + \left(3 - \frac{t}{\pi}\right)\mathbf{k} \Rightarrow 0 = \mathbf{r}\cdot(\mathbf{i} - \mathbf{j}) = 2(1) + \left(4\sin\frac{t}{2}\right)(-1) \Rightarrow 0 = 2 - 4\sin\frac{t}{2} \Rightarrow \sin\frac{t}{2} = \frac{1}{2}$

$\Rightarrow \frac{t}{2} = \frac{\pi}{6} \Rightarrow t = \frac{\pi}{3}$ (for the first time)

26. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1 + 4t^2 + 9t^4} \Rightarrow |\mathbf{v}(1)| = \sqrt{14} \Rightarrow \mathbf{T}(1) = \frac{1}{\sqrt{14}}\mathbf{i} + \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}$, which is normal to the normal plane $\Rightarrow \frac{1}{\sqrt{14}}(x-1) + \frac{2}{\sqrt{14}}(y-1) + \frac{3}{\sqrt{14}}(z-1) = 0$ or $x + 2y + 3z = 6$ is an equation of the normal plane. Next we calculate $\mathbf{N}(1)$ which is normal to the rectifying plane. Now, $\mathbf{a} = 2\mathbf{j} + 6t\mathbf{k}$

$\Rightarrow \mathbf{a}(1) = 2\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(1) \times \mathbf{a}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{v}(1) \times \mathbf{a}(1)| = \sqrt{76} \Rightarrow \kappa(1) = \frac{\sqrt{76}}{\left(\sqrt{14}\right)^3} = \frac{\sqrt{19}}{7\sqrt{14}}$;

$\frac{ds}{dt} = |\mathbf{v}(t)| \Rightarrow \left.\frac{d^2s}{dt^2}\right|_{t=1} = \frac{1}{2}\left(1 + 4t^2 + 9t^4\right)^{-1/2}\left.\left(8t + 36t^3\right)\right|_{t=1} = \frac{22}{\sqrt{14}}$, so $\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2 \mathbf{N}$

$\Rightarrow 2\mathbf{j} + 6\mathbf{k} = \frac{22}{\sqrt{14}}\left(\frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{14}}\right) + \frac{\sqrt{19}}{7\sqrt{14}}\left(\sqrt{14}\right)^2 \mathbf{N} \Rightarrow \mathbf{N} = \frac{\sqrt{14}}{2\sqrt{19}}\left(-\frac{11}{7}\mathbf{i} - \frac{8}{7}\mathbf{j} + \frac{9}{7}\mathbf{k}\right)$

$\Rightarrow -\frac{11}{7}(x-1) - \frac{8}{7}(y-1) + \frac{9}{7}(z-1) = 0$ or or $11x + 8y - 9z = 10$ is an equation of the rectifying plane. Finally,

$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \left(\frac{\sqrt{14}}{2\sqrt{19}}\right)\left(\frac{1}{\sqrt{14}}\right)\left(\frac{1}{7}\right)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = \frac{1}{\sqrt{19}}\left(3\mathbf{i} - 3\mathbf{j} + \mathbf{k}\right) \Rightarrow 3(x-1) - 3(y-1) + (z-1) = 0$ or

$3x - 3y + z = 1$ is an equation of the osculating plane.

27. $\mathbf{r} = e^t\mathbf{i} + (\sin t)\mathbf{j} + \ln(1-t)\mathbf{k} \Rightarrow \mathbf{v} = e^t\mathbf{i} + (\cos t)\mathbf{j} - \left(\frac{1}{1-t}\right)\mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}$; $\mathbf{r}(0) = \mathbf{i} \Rightarrow (1, 0, 0)$ is on the line $\Rightarrow x = 1 + t,\ y = t,$ and $z = -t$ are parametric equations of the line

28. $\mathbf{r} = \left(\sqrt{2}\cos t\right)\mathbf{i} + \left(\sqrt{2}\sin t\right)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = \left(-\sqrt{2}\sin t\right)\mathbf{i} + \left(\sqrt{2}\cos t\right)\mathbf{j} + \mathbf{k}$

$\Rightarrow \mathbf{v}\left(\frac{\pi}{4}\right) = \left(-\sqrt{2}\sin\frac{\pi}{4}\right)\mathbf{i} + \left(\sqrt{2}\cos\frac{\pi}{4}\right)\mathbf{j} + \mathbf{k} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ is a vector tangent to the helix when $t = \frac{\pi}{4} \Rightarrow$ the tangent line is parallel to $\mathbf{v}\left(\frac{\pi}{4}\right)$; also $\mathbf{r}\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\cos\frac{\pi}{4}\right)\mathbf{i} + \left(\sqrt{2}\sin\frac{\pi}{4}\right)\mathbf{j} + \frac{\pi}{4}\mathbf{k} \Rightarrow$ the point $\left(1, 1, \frac{\pi}{4}\right)$ is on the line $\Rightarrow x = 1 - t,\ y = 1 + t,$ and $z = \frac{\pi}{4} + t$ are parametric equations of the line

29. $x^2 = \left(v_0^2\cos^2\alpha\right)t^2$ and $\left(y + \frac{1}{2}gt^2\right)^2 = \left(v_0^2\sin^2\alpha\right)t^2 \Rightarrow x^2 + \left(y + \frac{1}{2}gt^2\right)^2 = v_0^2t^2$

30. $\ddot{s} = \frac{d}{dt}\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \Rightarrow \ddot{x}^2 + \ddot{y}^2 - \ddot{s}^2 = \ddot{x}^2 + \ddot{y}^2 - \frac{(\dot{x}\,\ddot{x}+\dot{y}\,\ddot{y})^2}{\dot{x}^2+\dot{y}^2} = \frac{\left(\ddot{x}^2+\ddot{y}^2\right)\left(\dot{x}^2+\dot{y}^2\right) - \left(\dot{x}^2\ddot{x}^2 + 2\dot{x}\,\ddot{x}\,\dot{y}\,\ddot{y} + \dot{y}^2\ddot{y}^2\right)}{\dot{x}^2+\dot{y}^2}$

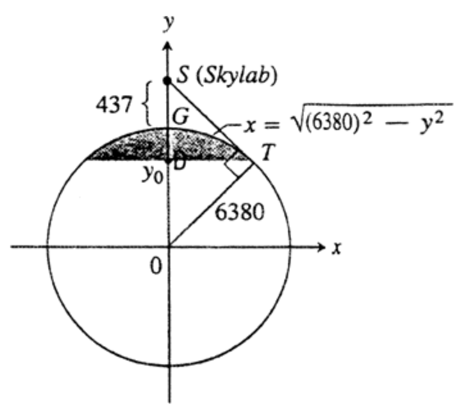
$= \frac{\dot{x}^2\ddot{y}^2 + \dot{y}^2\ddot{x}^2 - 2\dot{x}\,\ddot{x}\,\dot{y}\,\ddot{y}}{\dot{x}^2+\dot{y}^2} = \frac{(\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x})^2}{\dot{x}^2+\dot{y}^2} \Rightarrow \sqrt{\ddot{x}^2 + \ddot{y}^2 - \ddot{s}^2} = \frac{|\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}|}{\sqrt{\dot{x}^2+\dot{y}^2}} \Rightarrow \frac{\dot{x}^2+\dot{y}^2}{\sqrt{\ddot{x}^2+\ddot{y}^2-\ddot{s}^2}} = \frac{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}{|\dot{x}\,\ddot{y} - \dot{y}\ddot{x}|} = \frac{1}{\kappa} = \rho$

31. $s = a\theta \Rightarrow \theta = \frac{s}{a} \Rightarrow \phi = \frac{s}{a} + \frac{\pi}{2} \Rightarrow \frac{d\phi}{ds} = \frac{1}{a} \Rightarrow \kappa = \left|\frac{1}{a}\right| = \frac{1}{a}$ since a > 0

32. (1) $\Delta SOT \approx \Delta TOD \Rightarrow \frac{DO}{OT} = \frac{OT}{SO}$

$\Rightarrow \frac{y_0}{6380} = \frac{6380}{6380+437} \Rightarrow y_0 = \frac{6380^2}{6817}$

$\Rightarrow y_0 \approx 5971$ km;



(2) $VA = \int_{5971}^{6380} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$

$= 2\pi\int_{5971}^{6817}\sqrt{6380^2 - y^2}\left(\frac{6380}{\sqrt{6380^2 - y^2}}\right)dy$

$= 2\pi\int_{5971}^{6817} 6380\,dy = 2\pi\left[6380y\right]_{5971}^{6817}$

$= 16,395,469$ km$^2 \approx 1.639 \times 10^7$ km^2;

(3) percentage visible $\approx \frac{16,395,469 \text{ km}^2}{4\pi(6380 \text{ km})^2} \approx 3.21\%$

CHAPTER 13 ADDITIONAL AND ADVANCED EXERCISES

1. (a) $\mathbf{r}(\theta) = (a\cos\theta)\mathbf{i} + (a\sin\theta)\mathbf{j} + b\theta\mathbf{k} \Rightarrow \frac{d\mathbf{r}}{dt} = \left[(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}\right]\frac{d\theta}{dt}$;
$|\mathbf{v}| = \sqrt{2gz} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{a^2+b^2}\,\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \sqrt{\frac{2gz}{a^2+b^2}} = \sqrt{\frac{2gb\theta}{a^2+b^2}} \Rightarrow \left.\frac{d\theta}{dt}\right|_{\theta=2\pi} = \sqrt{\frac{4\pi gb}{a^2+b^2}} = 2\sqrt{\frac{\pi gb}{a^2+b^2}}$

(b) $\frac{d\theta}{dt} = \sqrt{\frac{2gb\theta}{a^2+b^2}} \Rightarrow \frac{d\theta}{\sqrt{\theta}} = \sqrt{\frac{2gb}{a^2+b^2}}dt \Rightarrow 2\theta^{1/2} = \sqrt{\frac{2gb}{a^2+b^2}}t + C$; $t = 0 \Rightarrow \theta = 0 \Rightarrow C = 0 \Rightarrow 2\theta^{1/2} = \sqrt{\frac{2gb}{a^2+b^2}}t$
$\Rightarrow \theta = \frac{gbt^2}{2\left(a^2+b^2\right)}$; $z = b\theta \Rightarrow z = \frac{gb^2t^2}{2\left(a^2+b^2\right)}$

(c) $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \left[(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}\right]\frac{d\theta}{dt} = \left[(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}\right]\left(\frac{gbt}{a^2+b^2}\right)$, from part (b)
$\Rightarrow \mathbf{v}(t) = \left[\frac{(-a\sin\theta)\mathbf{i}+(a\cos\theta)\mathbf{j}+b\mathbf{k}}{\sqrt{a^2+b^2}}\right]\left(\frac{gbt}{\sqrt{a^2+b^2}}\right) = \frac{gbt}{\sqrt{a^2+b^2}}\mathbf{T}$;
$\frac{d^2\mathbf{r}}{dt^2} = \left[(-a\cos\theta)\mathbf{i} - (a\sin\theta)\mathbf{j}\right]\left(\frac{d\theta}{dt}\right)^2 + \left[(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}\right]\frac{d^2\theta}{dt^2}$
$= \left(\frac{gbt}{a^2+b^2}\right)^2\left[(-a\cos\theta)\mathbf{i} - (a\sin\theta)\mathbf{j}\right] + \left[(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}\right]\left(\frac{gb}{a^2+b^2}\right)$
$= \left[\frac{(-a\sin\theta)\mathbf{i}+(a\cos\theta)\mathbf{j}+b\mathbf{k}}{\sqrt{a^2+b^2}}\right]\left(\frac{gb}{\sqrt{a^2+b^2}}\right) + a\left(\frac{gbt}{a^2+b^2}\right)\left[(-\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j}\right] = \frac{gb}{\sqrt{a^2+b^2}}\mathbf{T} + a\left(\frac{gbt}{a^2+b^2}\right)^2\mathbf{N}$ (there is no component in the direction of $\mathbf{B}$).

2. (a) $\mathbf{r}(\theta) = \left(a\theta\cos\theta\right)\mathbf{i} + \left(a\theta\sin\theta\right)\mathbf{j} + b\theta\mathbf{k} \Rightarrow \frac{d\mathbf{r}}{dt} = \left[\left(a\cos\theta - a\theta\sin\theta\right)\mathbf{i} + \left(a\sin\theta + a\theta\cos\theta\right)\mathbf{j} + b\mathbf{k}\right]\frac{d\theta}{dt}$;
$|\mathbf{v}| = \sqrt{2gz} = \left|\frac{d\mathbf{r}}{dt}\right| = \left(a^2 + a^2\theta^2 + b^2\right)^{1/2}\left(\frac{d\theta}{dt}\right) \Rightarrow \frac{d\theta}{dt} = \frac{\sqrt{2gb\theta}}{\sqrt{a^2+a^2\theta^2+b^2}}$

(b) $s = \int_0^t |\mathbf{v}|\,dt = \int_0^t \left(a^2 + a^2\theta^2 + b^2\right)^{1/2}\frac{d\theta}{dt}dt = \int_0^t \left(a^2 + a^2\theta^2 + b^2\right)^{1/2}d\theta = \int_0^\theta \left(a^2 + a^2u^2 + b^2\right)^{1/2}du$
$= \int_0^\theta a\sqrt{\frac{a^2+b^2}{a^2} + u^2}\,du = a\int_0^\theta a\sqrt{c^2+u^2}\,du$, where $c = \frac{\sqrt{a^2+b^2}}{|a|} \Rightarrow s = a\left[\frac{u}{2}\sqrt{c^2+u^2} + \frac{c^2}{2}\ln\left|u + \sqrt{c^2+u^2}\right|\right]_0^\theta$
$= \frac{a}{2}\left(\theta\sqrt{c^2+\theta^2} + c^2\ln\left|\theta + \sqrt{c^2+\theta^2}\right| - c^2\ln c\right)$

3. $r = \frac{(1+e)r_0}{1+e\cos\theta} \Rightarrow \frac{dr}{d\theta} = \frac{(1+e)r_0(e\sin\theta)}{(1+e\cos\theta)^2}$; $\frac{dr}{d\theta} = 0 \Rightarrow \frac{(1+e)r_0(e\sin\theta)}{(1+e\cos\theta)^2} = 0 \Rightarrow (1+e)r_0(e\sin\theta) = 0 \Rightarrow \sin\theta = 0$
$\Rightarrow \theta = 0$ or π. Note that $\frac{dr}{d\theta} > 0$ when $\sin\theta > 0$ and $\frac{dr}{d\theta} < 0$ when $\sin\theta < 0$. Since $\sin\theta < 0$ on $-\pi < \theta < 0$ and $\sin\theta > 0$ on $0 < \theta < \pi$, r is a minimum when $\theta = 0$ and $r(0) = \frac{(1+e)r_0}{1+e\cos 0} = r_0$

4 (a) $f(x) = x - 1 - \frac{1}{2}\sin x = 0 \Rightarrow f(0) = -1$ and $f(2) = 2 - 1 - \frac{1}{2}\sin 2 \geq \frac{1}{2}$ since $|\sin 2| \leq 1$; since f is continuous on $[0, 2]$, the Intermediate Value Theorem implies there is a root between 0 and 2

(b) Root ≈ 1.4987011335179

5. (a) $\mathbf{v} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j}$ and $\mathbf{v} = \dot{r}\,\mathbf{u}_r + r\dot{\theta}\,\mathbf{u}_\theta = (\dot{r})\left[(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}\right] + \left(r\dot{\theta}\right)\left[(-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}\right] \Rightarrow \mathbf{v}\cdot\mathbf{i} = \dot{x}$ and
$\mathbf{v}\cdot\mathbf{i} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \Rightarrow \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$; $\mathbf{v}\cdot\mathbf{j} = \dot{y}$ and
$\mathbf{v}\cdot\mathbf{j} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \Rightarrow \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

(b) $\mathbf{u}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} \Rightarrow \mathbf{v}\cdot\mathbf{u}_r = \dot{x}\cos\theta + \dot{y}\sin\theta = \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)(\cos\theta) + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)(\sin\theta)$ by part (a), $\Rightarrow \mathbf{v}\cdot\mathbf{u}_r = \dot{r}$; therefore, $\dot{r} = \dot{x}\cos\theta + \dot{y}\sin\theta$; $\mathbf{u}_\theta = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ $\Rightarrow \mathbf{v}\cdot\mathbf{u}_\theta = -\dot{x}\sin\theta + \dot{y}\cos\theta = \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)(-\sin\theta) + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)(\cos\theta)$ by part (a) $\Rightarrow \mathbf{v}\cdot\mathbf{u}_\theta = r\dot{\theta}$; therefore, $r\dot{\theta} = -\dot{x}\sin\theta + \dot{y}\cos\theta$

6. $r = f(\theta) \Rightarrow \frac{dr}{dt} = f'(\theta)\frac{d\theta}{dt} \Rightarrow \frac{d^2r}{dt^2} = f''(\theta)\left(\frac{d\theta}{dt}\right)^2 + f'(\theta)\frac{d^2\theta}{dt^2}$;
$\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta = \left(\cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}\right)\mathbf{i} + \left(\sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt}\right)\mathbf{j}$
$\Rightarrow |\mathbf{v}| = \left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right]^{1/2} = \left[\left(f'\right)^2 + f^2\right]^{1/2}\left(\frac{d\theta}{dt}\right)$; $|\mathbf{v}\times\mathbf{a}| = |\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}|$,
where $x = r\cos\theta$ and $y = r\sin\theta$. Then $\frac{dx}{dt} = (-r\sin\theta)\frac{d\theta}{dt} + (\cos\theta)\frac{dr}{dt}$
$\Rightarrow \frac{d^2x}{dt^2} = (-2\sin\theta)\frac{d\theta}{dt}\frac{dr}{dt} - (r\cos\theta)\left(\frac{d\theta}{dt}\right)^2 - (r\sin\theta)\frac{d^2\theta}{dt^2} + (\cos\theta)\frac{d^2r}{dt^2}$; $\frac{dy}{dt} = (r\cos\theta)\frac{d\theta}{dt} + (\sin\theta)\frac{dr}{dt}$
$\Rightarrow \frac{d^2y}{dt^2} = (2\cos\theta)\frac{d\theta}{dt}\frac{dr}{dt} - (r\sin\theta)\left(\frac{d\theta}{dt}\right)^2 + (r\cos\theta)\frac{d^2\theta}{dt^2} + (\sin\theta)\frac{d^2r}{dt^2}$. Then, after <u>much</u> algebra $|\mathbf{v}\times\mathbf{a}|$
$= r^2\left(\frac{d\theta}{dt}\right)^3 + r\frac{d^2\theta}{dt^2}\frac{dr}{dt} - r\frac{d\theta}{dt}\frac{d^2r}{dt^2} + 2\frac{d\theta}{dt}\left(\frac{dr}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^3\left(f^2 - f\cdot f'' + 2\left(f'\right)^2\right) \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} = \frac{f^2 - f\cdot f'' + 2(f')^2}{\left[(f')^2 + f^2\right]^{3/2}}$

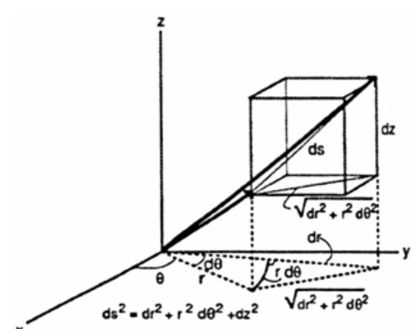
7. (a) Let $r = 2 - t$ and $\theta = 3t \Rightarrow \frac{dr}{dt} = -1$ and $\frac{d\theta}{dt} = 3 \Rightarrow \frac{d^2r}{dt^2} = \frac{d^2\theta}{dt^2} = 0$. The halfway point is $(1, 3) \Rightarrow t = 1$;
$\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta \Rightarrow \mathbf{v}(1) = -\mathbf{u}_r + 3\mathbf{u}_\theta$; $\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta$
$\Rightarrow \mathbf{a}(1) = -9\mathbf{u}_r - 6\mathbf{u}_\theta$

(b) It takes the beetle 2 min to crawl to the origin $\Rightarrow$ the rod has revolved 6 radians
$\Rightarrow L = \int_0^6 \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2}\,d\theta = \int_0^6 \sqrt{\left(2 - \frac{\theta}{3}\right)^2 + \left(-\frac{1}{3}\right)^2}\,d\theta = \int_0^6 \sqrt{4 - \frac{4\theta}{3} + \frac{\theta^2}{9} + \frac{1}{9}}\,d\theta$
$= \int_0^6 \sqrt{\frac{37 - 12\theta + \theta^2}{9}}\,d\theta = \frac{1}{3}\int_0^6 \sqrt{(\theta-6)^2 + 1}\,d\theta = \frac{1}{3}\left[\frac{(\theta-6)}{2}\sqrt{(\theta-6)^2 + 1} + \frac{1}{2}\ln\left|\theta - 6 + \sqrt{(\theta-6)^2 + 1}\right|\right]_0^6$
$= \sqrt{37} - \frac{1}{6}\ln\left(\sqrt{37} - 6\right) \approx 6.5$ cm.

8. (a) $x = r\cos\theta \Rightarrow dx = \cos\theta dr - r\sin\theta\,d\theta$; $y = r\sin\theta \Rightarrow dy = \sin\theta dr + r\cos\theta\,d\theta$; thus
$dx^2 = \cos^2\theta dr^2 - 2r\sin\theta\cos\theta dr\,d\theta + r^2\sin^2\theta\,d\theta^2$ and
$dy^2 = \sin^2\theta\,dr^2 + 2r\sin\theta\cos\theta dr\,d\theta + r^2\cos^2\theta d\theta^2 \Rightarrow ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2d\theta^2 + dz^2$

(c) $r = e^\theta \Rightarrow dr = e^\theta d\theta$
$\Rightarrow L = \int_0^{\ln 8}\sqrt{dr^2 + r^2d\theta^2 + dz^2}$
$= \int_0^{\ln 8}\sqrt{e^{2\theta} + e^{2\theta} + e^{2\theta}}\,d\theta$
$= \int_0^{\ln 8}\sqrt{3}e^\theta d\theta = \left[\sqrt{3}e^\theta\right]_0^{\ln 8}$
$= 8\sqrt{3} - \sqrt{3} = 7\sqrt{3}$

(b)



9. (a) $\mathbf{u}_r \times \mathbf{u}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = \mathbf{k} \Rightarrow a$ right-handed frame of unit vectors

(b) $\frac{d\mathbf{u}_r}{d\theta} = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} = \mathbf{u}_\theta$ and $\frac{d\mathbf{u}_\theta}{d\theta} = (-\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} = -\mathbf{u}_r$

(c) From Eq. (7), $\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k} \Rightarrow \mathbf{a} = \dot{\mathbf{v}} = \left(\ddot{r}\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r\right) + \left(\dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta\right) + \ddot{z}\,\mathbf{k}$

$= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{u}_r + \left(r\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right)\mathbf{u}_\theta + \ddot{z}\,\mathbf{k}$

10. $\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t) \Rightarrow \frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{r}}{dt} \times m\mathbf{v}\right) + \left(\mathbf{r} \times m\frac{d^2\mathbf{r}}{dt^2}\right) \Rightarrow \frac{d\mathbf{L}}{dt} = (\mathbf{v} \times m\mathbf{v}) + (\mathbf{r} \times m\mathbf{a}) = \mathbf{r} \times m\mathbf{a}$;

$\mathbf{F} = m\mathbf{a} \Rightarrow -\frac{c}{|\mathbf{r}|^3}\mathbf{r} = m\mathbf{a} \Rightarrow \frac{d\mathbf{L}}{dt} = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times \left(-\frac{c}{|\mathbf{r}|^3}\mathbf{r}\right) = -\frac{c}{|\mathbf{r}|^3}(\mathbf{r} \times \mathbf{r}) = \mathbf{0} \Rightarrow \mathbf{L} = \text{constant vector}$