WEEK 6

12.6

7. b, cylinder 8. j, hyperboloid 9. k, hyperbolic paraboloid

10. f, paraboloid 11. h, cone 12. c, ellipsoid

13.1

3. $x=e^t,\ y=\frac{2}{9}e^{2t} \Rightarrow y=\frac{2}{9}x^2$; $\vec{v}=\frac{d\vec{r}}{dt}=e^t\vec{i}+\frac{4}{9}e^{2t}\vec{j} \Rightarrow \vec{a}=e^t\vec{i}+\frac{8}{9}e^{2t}\vec{j} \Rightarrow$

$\vec{v}=3\vec{i}+4\vec{j}$, $\vec{a}=3\vec{i}+8\vec{j}$ at $t=\ln 3$

13. $\vec{r}=2\ln(t+1)\vec{i}+t^2\vec{j}+\frac{t^2}{2}\vec{k} \Rightarrow \vec{v}=\frac{d\vec{r}}{dt}=\frac{2}{t+1}\vec{i}+2t\vec{j}+t\vec{k} \Rightarrow \vec{a}=-\frac{2}{(t+1)^2}\vec{i}+2\vec{j}+\vec{k}$

$|\vec{v}(1)|=\sqrt{6}$, $\vec{v}(1)=\sqrt{6}\left(\frac{1}{\sqrt{6}}\vec{i}+\frac{2}{\sqrt{6}}\vec{j}+\frac{1}{\sqrt{6}}\vec{k}\right)$

17. $\vec{v}=\frac{2t}{t^2+1}\vec{i}+\frac{1}{t^2+1}\vec{j}+t(t^2+1)^{-\frac{1}{2}}\vec{k}$, $\vec{a}=\frac{-2t^2+2}{(t^2+1)^2}\vec{i}-\frac{2t}{(t^2+1)^2}\vec{j}+\frac{1}{(t^2+1)^{\frac{3}{2}}}\vec{k} \Rightarrow$

$\vec{v}(0)=\vec{j}$, $\vec{a}(0)=2\vec{i}+\vec{k} \Rightarrow |\vec{v}(0)|=1$, $|\vec{a}(0)|=\sqrt{5}$; $\vec{v}(0)\cdot\vec{a}(0)=0 \Rightarrow \theta=\frac{\pi}{2}$

21. $\vec{r}(t)=(\ln t)\vec{i}+\frac{t-1}{t+2}\vec{j}+(t\ln t)\vec{k} \Rightarrow \vec{v}(t)=\frac{1}{t}\vec{i}+\frac{3}{(t+2)^2}\vec{j}+(\ln t+1)\vec{k}$; $t_0=1 \Rightarrow \vec{v}(1)=\vec{i}+\frac{1}{3}\vec{j}+\vec{k}$,

$\vec{r}(t_0)=P_0=(0,0,0) \Rightarrow x=0+t=t$, $y=0+\frac{1}{3}t=\frac{1}{3}t$, $z=0+t=t$

23. (b) $\vec{v}(t)=-(2\sin 2t)\vec{i}+(2\cos 2t)\vec{j} \Rightarrow \vec{a}(t)=-(4\cos 2t)\vec{i}-(4\sin 2t)\vec{j}$

(i) $|\vec{v}(t)|=2 \Rightarrow$ constant speed

(ii) $\vec{v}\cdot\vec{a}=0 \Rightarrow$ yes, orthogonal

(iii) counterclockwise movement

(iv) yes, $\vec{r}(0)=\vec{i}+0\vec{j}$

(c) $\vec{v}(t)=-\sin(t-\frac{\pi}{2})\vec{i}+\cos(t-\frac{\pi}{2})\vec{j} \Rightarrow \vec{a}(t)=-\cos(t-\frac{\pi}{2})\vec{i}-\sin(t-\frac{\pi}{2})\vec{j}$

(i) $|\vec{v}(t)|=1 \Rightarrow$ constant speed

(ii) $\vec{v}\cdot\vec{a}=0 \Rightarrow$ yes, orthogonal

(iii) ~~clockwise~~ counterclockwise movement

~~(iv) yes, $\vec{r}(0)=\vec{i}+0\vec{j}$~~

(iv) no, $\vec{r}(0)=0\vec{i}-\vec{j}$ instead of $\vec{i}+0\vec{j}$

(d) $\vec{v}(t)=-\sin t\,\vec{i}-\cos t\,\vec{j} \Rightarrow \vec{a}(t)=-\cos t\,\vec{i}+\sin t\,\vec{j}$

(i) $|\vec{v}(t)|=1 \Rightarrow$ ~~constand~~ constant speed

(ii) $\vec{v}\cdot\vec{a}=0 \Rightarrow$ yes, orthogonal

(iii) clockwise movement

(iv) yes, $\vec{r}(0)=\vec{i}-0\vec{j}$

27. $\frac{d}{dt}(\vec{r}\cdot\vec{r})=\vec{r}\cdot\frac{d\vec{r}}{dt}+\frac{d\vec{r}}{dt}\cdot\vec{r}=2\vec{r}\cdot\frac{d\vec{r}}{dt}=0 \Rightarrow |\vec{r}|$ is a constant

13.2

7. $\int_0^1 (te^{t^2}\vec{i} + e^{-t}\vec{j} + \vec{k})\,dt = [\frac{1}{2}e^{t^2}]_0^1\vec{i} - [e^{-t}]_0^1\vec{j} + (t)_0^1\vec{k} = \frac{e-1}{2}\vec{i} + \frac{e-1}{e}\vec{j} + \vec{k}$

13. $\vec{r} = \int [(\frac{3}{2}(t+1)^{\frac{1}{2}})\vec{i} + e^{-t}\vec{j} + \frac{1}{t+1}\vec{k}]\,dt = (t+1)^{\frac{3}{2}}\vec{i} - e^{-t}\vec{j} + \ln(t+1)\vec{k} + \vec{C}$

$\vec{r}(0) = \vec{k} \Rightarrow \vec{C} = -\vec{i} + \vec{j} + \vec{k} \Rightarrow \vec{r} = [(t+1)^{\frac{3}{2}} - 1]\vec{i} + (1-e^{-t})\vec{j} + [1+\ln(t+1)]\vec{k}$

21. (a) $t = \frac{2v_0\sin\alpha}{g} \approx 72.2s$; $R = \frac{v_0^2}{g}\sin 2\alpha \approx 25510.2m$

(b) $x = (v_0\cos\alpha)t \Rightarrow t \approx 14.14s$, $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow y \approx 4020m$

(c) $y_{max} = \frac{(v_0\sin\alpha)^2}{2g} \approx 6378m$

24. $v_0 = 5\times10^6 m/s$, $x = 0.4m$, $t = 8\times10^{-8}s$

$\Rightarrow y = y_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2 \cancel{=3} \Rightarrow \cancel{y=3.316\times10^{-14}m}\ y = -3.136\times10^{-14}m$

30. $x^2 = \frac{v_0^4\sin^2\alpha\cos^2\alpha}{g^2} = \frac{v_0^2}{g}(2y) - (2y)^2 \Rightarrow x^2 + 4y^2 - \frac{2v_0^2}{g}y = 0$

$\Rightarrow x^2 + 4(y - \frac{v_0^2}{4g})^2 = \frac{v_0^4}{4g^2}$, where $x \geq 0$.

31. (a) $\tan\beta = \frac{\frac{1}{2}gt - v_0\sin(\alpha-\beta)}{v_0\cos(\alpha-\beta)} \Rightarrow t = \frac{2v_0\sin(\alpha-\beta) + 2v_0\cos(\alpha-\beta)\tan\beta}{g} \Rightarrow x = \frac{2v_0^2}{g}[\cos^2(\alpha-\beta)\tan\beta + \sin(\alpha-\beta)\cos(\alpha-\beta)]$

$\Rightarrow -\sin 2(\alpha-\beta)\tan\beta + \cos(\alpha-\beta) = 0 \Rightarrow \alpha - \beta = \frac{1}{2}(90° - \beta) \Rightarrow \alpha = \frac{1}{2}$ of $\angle AOR$

(b) $\tan\beta = \frac{[v_0\sin(\alpha+\beta)]t - \frac{1}{2}gt^2}{[v_0\cos(\alpha+\beta)]t} = \frac{v_0\sin(\alpha+\beta) - \frac{1}{2}gt}{v_0\cos(\alpha+\beta)} \Rightarrow t = \frac{2v_0\sin(\alpha+\beta) - 2v_0\cos(\alpha+\beta)\tan\beta}{g}$

$\Rightarrow \cot 2(\alpha+\beta) + \tan\beta = 0 \Rightarrow \alpha = \frac{1}{2}(90° - \beta) = \frac{1}{2}$ of $\angle AOR$

Therefore v_0 would bisect $\angle AOR$ for maximum range uphill.

32. $v_0 = 35.5m/s$. $\alpha = 45°$. $x = (v_0\cos\alpha)t \Rightarrow t = 4.\cancel{15}s \Rightarrow \cancel{y \approx 14.5m}$ $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2 \approx 14.5m$

44. $y_{max} = \frac{(v_0\sin\alpha)^2}{2g}$, $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow 3(v_0\sin\alpha)^2 = (8gv_0\sin\alpha)t - 4g^2t^2$

$\Rightarrow t = \frac{3v_0\sin\alpha}{2g}$ or $t = \frac{v_0\sin\alpha}{2g} \Rightarrow t = \frac{v_0\sin\alpha}{2g}$

13.3

10. $\vec{v} = 12\cos t\,\vec{i} + 12\sin t\,\vec{j} \Rightarrow t_0 = -\pi \Rightarrow P(\pi) = (0, 12, -5\pi)$ $\cancel{(12\sin(-\pi), \ldots)}$

12. $\vec{v} = (t\cos t)\vec{i} + (t\sin t)\vec{j} \Rightarrow |\vec{v}| = t \Rightarrow s(t) = \int_0^t \tau\,d\tau = \frac{t^2}{2} \Rightarrow \text{Length} = s(\pi) - s(\frac{\pi}{2}) = \frac{3\pi^2}{8}$

14. $\vec{r} = (1+2t)\vec{i} + (1+3t)\vec{j} + (6-6t)\vec{k} \Rightarrow \vec{v} = 2\vec{i} + 3\vec{j} - 6\vec{k} \Rightarrow |\vec{v}| = 7 \Rightarrow s(t) = 7t \Rightarrow \text{Length} = 7$

19. $x = OB + BC = OB + DP = \cos t + t\sin t$, $y = PC = QB - QD = \sin t - t\cos t$

21. $\vec{v} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k} = \vec{u} \Rightarrow s(t) = \int_0^t |\vec{v}|\,dt = t$

13.4

3. $\vec{r} = (2t+3)\vec{i} + (5-t^2)\vec{j} \Rightarrow \vec{v} = 2\vec{i} - 2t\vec{j} \Rightarrow |\vec{v}| = 2\sqrt{1+t^2} \Rightarrow \frac{d\vec{T}}{dt} = \frac{-t}{(\sqrt{1+t^2})^3}\vec{i} - \frac{1}{(\sqrt{1+t^2})^3}\vec{j}$

$\Rightarrow \vec{N} = \frac{-t}{\sqrt{t^2+1}}\vec{i} - \frac{1}{\sqrt{t^2+1}}\vec{j}$; $\kappa = \frac{1}{2(1+t^2)^{\frac{3}{2}}}$

5. (a) $\frac{d\vec{T}}{dt}(x) = \frac{-f'(x)f''(x)}{(1+[f'(x)]^2)^{\frac{3}{2}}}\vec{i} + \frac{f''(x)}{(1+[f'(x)]^2)^{\frac{3}{2}}}\vec{j} \Rightarrow \kappa(x) = \frac{|f''(x)|}{(1+[f'(x)]^2)^{\frac{3}{2}}}$

(b) $y = \ln(\cos x) \Rightarrow \frac{d^2y}{dx^2} = -\sec^2 x \Rightarrow \kappa = \cos x$

(c) Note that $f''(x) = 0$ at an inflection point

12. $\vec{v} = (12\cos 2t)\vec{i} - (12\sin 2t)\vec{j} + 5\vec{k} \Rightarrow \vec{T} = \left(\frac{12}{13}\cos 2t\right)\vec{i} - \left(\frac{12}{13}\sin 2t\right)\vec{j} + \frac{5}{13}\vec{k} \Rightarrow$

$\frac{d\vec{T}}{dt} = \left(-\frac{24}{13}\sin 2t\right)\vec{i} - \left(\frac{24}{13}\cos 2t\right)\vec{j} \Rightarrow \vec{N} = -\sin 2t\,\vec{i} - \cos 2t\,\vec{j}$; $k = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{24}{169}$

17. $y = ax^2 \Rightarrow y' = 2ax \Rightarrow y'' = 2a \Rightarrow k(x) = \frac{|2a|}{(1+4a^2x^2)^{\frac{3}{2}}} \Rightarrow k'(x) = -\frac{3}{2}|2a|(1+4a^2x^2)^{-\frac{5}{2}}$

$\Rightarrow k(x)$ has a maximum at $x=0$. no minimum value

18. $\vec{r} = (a\cos t)\vec{i} + (b\sin t)\vec{j} \Rightarrow \vec{v} = -(a\sin t)\vec{i} + (b\cos t)\vec{j} \Rightarrow \vec{a} = -(a\cos t)\vec{i} - (b\sin t)\vec{j}$

$\Rightarrow \vec{v}\times\vec{a} = ab\vec{k}$. $k(t) = ab(a^2\sin^2 t + b^2\cos^2 t)^{-\frac{3}{2}}$; $k'(t) = -\frac{3}{2}(ab)(a^2-b^2)(\sin t)(a^2\sin^2 t + b^2\cos^2 t)^{-\frac{5}{2}}$

maximum : $t=0$ and $t=\pi$. minimum: $t=\frac{\pi}{2}$ and $t=\frac{3\pi}{2}$

20. (a) $|\vec{v}| = a^2+b^2 \Rightarrow k = \frac{3}{3^2+1^2} = \frac{3}{10}$ and $|\vec{v}| = \sqrt{10} \Rightarrow K = \int_0^{4\pi} \frac{3\sqrt{10}}{10}\,dt = \frac{12\pi}{\sqrt{10}}$

(b) $y = x^2 \Rightarrow x = t$ and $y = t^2 \Rightarrow \vec{v} = \vec{i} + 2t\vec{j} \Rightarrow |\vec{v}| = \sqrt{1+t^2}$

$k = \frac{2}{(\sqrt{1+4t^2})^3}$, $K = \int_{-\infty}^{\infty} \frac{2}{(\sqrt{1+4t^2})^3}(\sqrt{1+4t^2})\,dt = \pi$

27. $|\vec{v}(t)| = \sqrt{1+4t^2}$, $\frac{d\vec{T}}{dt} = 2(1+4t^2)^{-\frac{3}{2}}(-2t\vec{i}+\vec{j})$

~~At~~ At $t=a$. $k = \frac{1}{|\vec{v}(a)|}\left|\frac{d\vec{T}(a)}{dt}\right| = \frac{2}{(1+4a^2)^{\frac{3}{2}}} \Rightarrow r = \frac{1}{2}(1+4a^2)^{\frac{3}{2}}$

$\Rightarrow d = \sqrt{(-4a^3-a)^2 + (3a^2+\frac{1}{2}-a^2)^2} = \frac{1}{2}\sqrt{(1+4a^2)^3} \Rightarrow \frac{3a^2+\frac{1}{2}-a^2}{-4a^3-a} = -\frac{1}{2a}$. correct