MAT1002 Final Examination

Monday, May 16, 2022

Time: 4:00 - 7:00 PM

Notes and Instructions

1. This exam is closed-book. No book, note, dictionary, or calculator is allowed.

*2. The total score of this examination is **110**.*

*3. There are **twelve** questions (with parts) in total.*

*4. The symbol [**N**] at the beginning of a question indicates that the question is worth N points.*

5. State your answers in exact form, e.g., write $\sqrt{2}$ instead of 1.414.

*6. Show your intermediate steps **except Questions 1 and 2** — answers without intermediate steps will receive minimal (or even no) marks.*

7. Onsite examinees should answer all questions in the answer book.

MAT1002 Final Examination Questions

1. [**6**] True (T) or False (F)? No explanation is required.

   (i) Consider $f(x, y, z) = xy^2z^6 - \sin(e^{yz}) - \ln(x^2)$ defined on

   $$D = \{(x, y, z) : 4 \le x \le 8,\ 3 \le y \le 4,\ -1 \le z \le 1\}.$$

   Then there must exist (x_1, y_1, z_1) and (x_2, y_2, z_2) in D such that $f(x_1, y_1, z_1) \ge f(x, y, z) \ge f(x_2, y_2, z_2)$ for all $(x, y, z) \in D$.

   (ii) If a function $f(x, y)$ is differentiable at $(0, 0)$, then the partial derivatives f_x and f_y must both be continuous at $(0, 0)$.

   (iii) If the series $\sum_{n=1}^{\infty} u_n$ converges, then the series $\sum_{n=1}^{\infty}(u_{2n-1} - u_{2n})$ must also converge.

2. [**27**] Short questions. No explanation or intermediate steps are required.

   (i) Let $f(x, y, z) = x^2 + 2xy + yz$. Then $\operatorname{div}(\nabla f) = (\qquad)$.

   (ii) Let $f(x, y, z) = \frac{1}{\sqrt{4-x^2-y^2-z^2}}$. Describe all the level surfaces of f.

   (iii) The temperature H is described by a differentiable function $H = f(x, y, z, t)$, where (x, y, z) is the position in the space and t is time. A space curve C is described by a smooth parametrization $x = x(t)$, $y = y(t)$, and $z = z(t)$, where t is time (same as above). What is the rate of change of temperature with respect to time along C at $t = 0$? Describe with an algebraic expression.

   (iv) Find the curvature of the curve $y = \sin x$ at the point $P(\pi/2, 1)$.

   (v) Find the principle unit normal for $y = -x^2$ at the point $P(1/2, -1/4)$.

   (vi) State the first three **nonzero** terms in the Maclaurin series of $\dfrac{x^2}{\sqrt{2 + x}}$.

   (vii) Let $H = 5x^2 - 3xy + xyz$. At the point $P(3, 4, 5)$, in which direction does the value of H decrease the fastest? (You do not have to normalize your direction.)

(viii) Let E be the solid that is outside the sphere $x^2+y^2+z^2=z$ and inside the sphere $x^2+y^2+z^2=2z$. Write the following triple integral in the spherical coordinates (you do not need to compute the value):

$$\iiint_E z\,dV = (\qquad).$$

(ix) Let C be the circle that is the intersection of the sphere $x^2+y^2+z^2=1$ and the plane $x+y+z=0$. Find the line integral $\int_C x^2\,ds$.

3. [**6+4=10**] Consider the surface S given by $xz^2-yz+\cos(xy)=1$.

(i) Find the tangent plane M and normal line ℓ to the surface S at the point $P(0,0,1)$.

(ii) Show that the tangent line to the curve

$$\mathbf{r}(t)=(\ln t)\,\mathbf{i}+(t\ln t)\,\mathbf{j}+t\,\mathbf{k}$$

at $P(0,0,1)$ is lying on M.

4. [**6**] Let

$$f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2y^2+(x-y)^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

Determine whether f is continuous at the origin. Justify.

5. [**6**] Find the global extrema of the function $f(x,y,z)=x+y+z$ subject to the constraints

$$x^2+z^2=2 \quad\text{and}\quad x+y=1.$$

6. [**5+5**] Evaluate the following integrals:

(i) $\displaystyle\int_0^1\int_{3y}^3 e^{(x^2)}\,dx\,dy.$

(ii) $\displaystyle\iint_R v(u+v^2)^4\,dA$, where R is the rectangle $0\le u\le 1$, $0\le v\le 1$.

7. [**5**] **Use a double integral** to find the area of the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.

8. [**5**] Find the volume of the solid enclosed by the surface $(x^2+y^2)^2+z^2=1$.

9. [**5+5**] Suppose that the vector field

$$\mathbf{F} = (e^{kx} \ln y)\,\mathbf{i} + \left(\frac{e^{kx}}{y} + \sin z\right)\mathbf{j} + (my \cos z)\,\mathbf{k}$$

is conservative on $\{(x, y, z) : y > 0\}$, where k and m are two constants.

(i) Find the values of k and m.

(ii) Find one potential function of $\mathbf{F}$.

10. [**5+3+5**]

(i) Suppose that C is a piecewise smooth, simple closed curve that is counterclockwise. Show that the area $A(R)$ of the region R enclosed by C is given by

$$A(R) = \oint_C x\,dy.$$

(ii) Now consider the simple closed curve C in the xy-plane given by the polar equation $r = \sqrt{\sin\theta}$. State a parametrization of C.

(iii) **Use the formula in part (i)** to find the area of the region enclosed by the curve C in part (ii).

11. [**6**] Let S be the cone $x = \sqrt{y^2 + z^2}$, $x \leq 2$. Given the vector field

$$\mathbf{F} = (\sin(x^3 y^2 z))\,\mathbf{i} + \left(x^2 y\right)\mathbf{j} + (x^2 z^2)\,\mathbf{k},$$

find the flux done by the curl of $\mathbf{F}$ across S,

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n}\,d\sigma,$$

with unit normals pointing to the positive x-direction.

12. [**6**] Let E be the solid that lies above the cone $z = \sqrt{\frac{x^2+y^2}{3}}$ and below the sphere $x^2 + y^2 + z^2 = 1$, and let S be the boundary surface of E. Given the vector field

$$\mathbf{F} = (\cos(z^2))\,\mathbf{i} + \left(z^3(y + x)\right)\mathbf{j} + (e^{x+y} x)\,\mathbf{k},$$

find the value of the outward flux across S:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,d\sigma.$$