CHAPTER 7 TRANSCENDENTAL FUNCTIONS

7.1 INVERSE FUNCTIONS AND THEIR DERIVATIVES

1. Yes one-to-one, the graph passes the horizontal line test.

2. Not one-to-one, the graph fails the horizontal line test.

3. Not one-to-one since (for example) the horizontal line $y = 2$ intersects the graph twice.

4. Not one-to-one, the graph fails the horizontal line test.

5. Yes one-to-one, the graph passes the horizontal line test.

6. Yes one-to-one, the graph passes the horizontal line test.

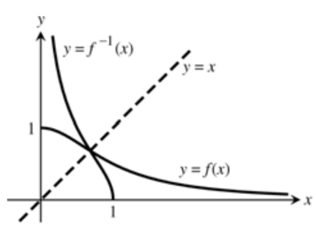
7. Not one-to one since the horizontal line $y = 3$ intersects the graph an infinite number of times.

8. Yes one-to-one, the graph passes the horizontal line test.

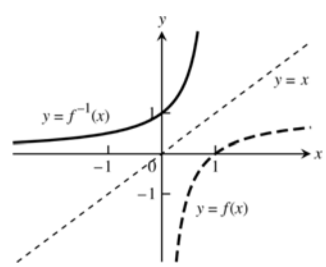
9. Yes one-to-one, the graph passes the horizontal line test.

10. Not one-to one since (for example) the horizontal line $y = 1$ intersects the graph twice.

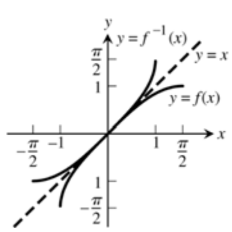
11. Domain: $0 < x \le 1$, Range: $0 \le y$



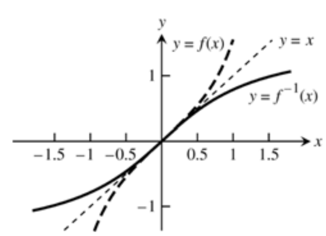
12. Domain: $x < 1$, Range: $y > 0$



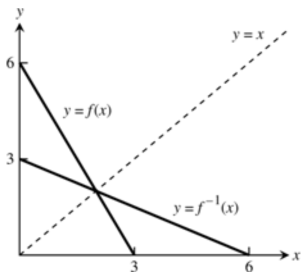
13. Domain: $-1 \le x \le 1$, Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



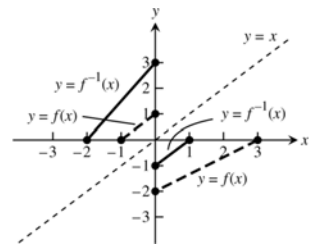
14. Domain: $-\infty < x < \infty$, Range: $-\frac{\pi}{2} < y \le \frac{\pi}{2}$



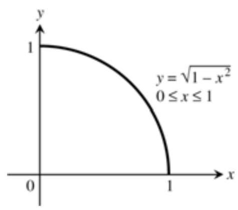
15. Domain: $0 \le x \le 6$, Range: $0 \le y \le 3$



16. Domain: $-2 \le x \le 1$, Range: $-1 \le y < 3$

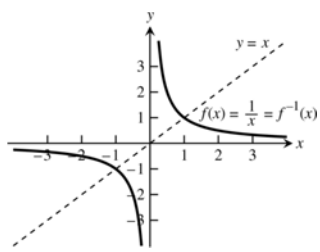


17. The graph is symmetric about $y = x$.



(b) $y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x^2 = 1-y^2 \Rightarrow x = \sqrt{1-y^2} \Rightarrow y = \sqrt{1-x^2} = f^{-1}(x)$

18. The graph is symmetric about $y = x$.



$y = \frac{1}{x} \Rightarrow x = \frac{1}{y} \Rightarrow y = \frac{1}{x} = f^{-1}(x)$

19. Step 1: $y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y-1}$
    Step 2: $y = \sqrt{x-1} = f^{-1}(x)$

20. Step 1: $y = x^2 \Rightarrow x = -\sqrt{y}$, since $x \le 0$.
    Step 2: $y = -\sqrt{x} = f^{-1}(x)$

21. Step 1: $y = x^3 - 1 \Rightarrow x^3 = y + 1 \Rightarrow x = (y+1)^{1/3}$
    Step 2: $y = \sqrt[3]{x+1} = f^{-1}(x)$

22. Step 1: $y = x^2 - 2x + 1 \Rightarrow y = (x-1)^2 \Rightarrow \sqrt{y} = x - 1$, since $x \ge 1 \Rightarrow x = 1 + \sqrt{y}$
    Step 2: $y = 1 + \sqrt{x} = f^{-1}(x)$

23. Step 1: $y = (x+1)^2 \Rightarrow \sqrt{y} = x + 1$, since $x \ge -1 \Rightarrow x = \sqrt{y} - 1$
    Step 2: $y = \sqrt{x} - 1 = f^{-1}(x)$

24. Step 1: $y = x^{2/3} \Rightarrow x = y^{3/2}$

Step 2: $y = x^{3/2} = f^{-1}(x)$

25. Step 1: $y = x^5 \Rightarrow x = y^{1/5}$

Step 2: $y = \sqrt[5]{x} = f^{-1}(x)$;

Domain and Range of f^{-1}: all reals; $f\left(f^{-1}(x)\right) = \left(x^{1/5}\right)^5 = x$ and $f^{-1}\left(f(x)\right) = \left(x^5\right)^{1/5} = x$

26. Step 1: $y = x^4 \Rightarrow x = y^{1/4}$

Step 2: $y = \sqrt[4]{x} = f^{-1}(x)$;

Domain of $f^{-1}: x \geq 0$, Range of $f^{-1}: y \geq 0$; $f\left(f^{-1}(x)\right) = \left(x^{1/4}\right)^4 = x$ and $f^{-1}\left(f(x)\right) = \left(x^4\right)^{1/4} = x$

27. Step 1: $y = x^3 + 1 \Rightarrow x^3 = y - 1 \Rightarrow x = (y-1)^{1/3}$

Step 2: $y = \sqrt[3]{x-1} = f^{-1}(x)$;

Domain and Range of f^{-1}: all reals;

$f\left(f^{-1}(x)\right) = \left((x-1)^{1/3}\right)^3 + 1 = (x-1) + 1 = x$ and $f^{-1}\left(f(x)\right) = \left(\left(x^3+1\right)-1\right)^{1/3} = \left(x^3\right)^{1/3} = x$

28. Step 1: $y = \frac{1}{2}x - \frac{7}{2} \Rightarrow \frac{1}{2}x = y + \frac{7}{2} \Rightarrow x = 2y + 7$

Step 2: $y = 2x + 7 = f^{-1}(x)$;

Domain and Range of f^{-1}: all reals;

$f\left(f^{-1}(x)\right) = \frac{1}{2}(2x+7) - \frac{7}{2} = \left(x + \frac{7}{2}\right) - \frac{7}{2} = x$ and $f^{-1}\left(f(x)\right) = 2\left(\frac{1}{2}x - \frac{7}{2}\right) + 7 = (x-7) + 7 = x$

29. Step 1: $y = \frac{1}{x^2} \Rightarrow x^2 = \frac{1}{y} \Rightarrow x = \frac{1}{\sqrt{y}}$

Step 2: $y = \frac{1}{\sqrt{x}} = f^{-1}(x)$

Domain of $f^{-1}: x > 0$, Range of $f^{-1}: y > 0$; $f\left(f^{-1}(x)\right) = \frac{1}{\left(\frac{1}{\sqrt{x}}\right)^2} = \frac{1}{\left(\frac{1}{x}\right)} = x$ and $f^{-1}\left(f(x)\right) = \frac{1}{\sqrt{\frac{1}{x^2}}} = \frac{1}{\left(\frac{1}{x}\right)} = x$

since $x > 0$

30. Step 1: $y = \frac{1}{x^3} \Rightarrow x^3 = \frac{1}{y} \Rightarrow x = \frac{1}{y^{1/3}}$

Step 2: $y = \frac{1}{x^{1/3}} = \sqrt[3]{\frac{1}{x}} = f^{-1}(x)$;

Domain of $f^{-1}: x \neq 0$, Range of $f^{-1}: y \neq 0$; $f\left(f^{-1}(x)\right) = \frac{1}{\left(x^{-1/3}\right)^3} = \frac{1}{x^{-1}} = x$ and

$f^{-1}\left(f(x)\right) = \left(\frac{1}{x^3}\right)^{-1/3} = \left(\frac{1}{x}\right)^{-1} = x$

31. Step 1: $y = \frac{x+3}{x-2} \Rightarrow y(x-2) = x + 3 \Rightarrow xy - 2y = x + 3 \Rightarrow xy - x = 2y + 3 \Rightarrow x = \frac{2y+3}{y-1}$

Step 2: $y = \frac{2x+3}{x-1} = f^{-1}(x)$;

Domain of $f^{-1}: x \neq 1$, Range of $f^{-1}: y \neq 2$; $f\left(f^{-1}(x)\right) = \frac{\left(\frac{2x+3}{x-1}\right)+3}{\left(\frac{2x+3}{x-1}\right)-2} = \frac{(2x+3)+3(x-1)}{(2x+3)-2(x-1)} = \frac{5x}{5} = x$ and

$f^{-1}\left(f(x)\right) = \frac{2\left(\frac{x+3}{x-2}\right)+3}{\left(\frac{x+3}{x-2}\right)-1} = \frac{2(x+3)+3(x-2)}{(x+3)-(x-2)} = \frac{5x}{5} = x$

32. Step 1: $y = \frac{\sqrt{x}}{\sqrt{x}-3} \Rightarrow y\left(\sqrt{x}-3\right) = \sqrt{x} \Rightarrow y\sqrt{x} - 3y = \sqrt{x} \Rightarrow y\sqrt{x} - \sqrt{x} = 3y \Rightarrow x = \left(\frac{3y}{y-1}\right)^2$

Step 2: $y = \left(\frac{3x}{x-1}\right)^2 = f^{-1}(x)$;

Domain of $f^{-1}: (-\infty, 0] \cup (1, \infty)$, Range of $f^{-1}: [0, 9) \cup (9, \infty)$; $f\left(f^{-1}(x)\right) = \frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2}-3}$; If $x > 1$ or

$x \leq 0 \Rightarrow \frac{3x}{x-1} \geq 0 \Rightarrow \frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2}-3} = \frac{\frac{3x}{x-1}}{\frac{3x}{x-1}-3} = \frac{3x}{3x-3(x-1)} = \frac{3x}{3} = x$ and $f^{-1}\left(f(x)\right) = \left(\frac{3\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)}{\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)-1}\right)^2 = \frac{9x}{\left(\sqrt{x}-\left(\sqrt{x}-3\right)\right)^2}$

$= \frac{9x}{9} = x$

33. Step 1: $y = x^2 - 2x, x \leq 1 \Rightarrow y+1 = (x-1)^2, x \leq 1 \Rightarrow -\sqrt{y+1} = x-1, x \leq 1 \Rightarrow x = 1 - \sqrt{y+1}$

Step 2: $y = 1 - \sqrt{x+1} = f^{-1}(x)$;

Domain of $f^{-1}: [-1, \infty)$, Range of $f^{-1}: (-\infty, 1]$;

$f\left(f^{-1}(x)\right) = \left(1-\sqrt{x+1}\right)^2 - 2\left(1-\sqrt{x+1}\right) = 1 - 2\sqrt{x+1} + x + 1 - 2 + 2\sqrt{x+1} = x$ and

$f^{-1}\left(f(x)\right) = 1 - \sqrt{\left(x^2-2x\right)+1}, x \leq 1 = 1 - \sqrt{(x-1)^2}, x \leq 1 = 1 - |x-1| = 1 - (1-x) = x$

34. Step 1: $y = \left(2x^3+1\right)^{1/5} \Rightarrow y^5 = 2x^3 + 1 \Rightarrow y^5 - 1 \Rightarrow 2x^3 \Rightarrow \frac{y^5-1}{2} = x^3 \Rightarrow x = \sqrt[3]{\frac{y^5-1}{2}}$

Step 2: $y = \sqrt[3]{\frac{x^5-1}{2}} = f^{-1}(x)$;

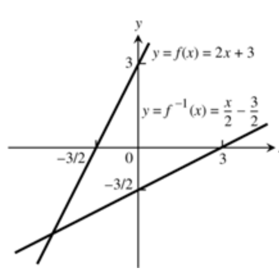
Domain of $f^{-1}: (-\infty, \infty)$, Range of $f^{-1}: (-\infty, \infty)$; $f\left(f^{-1}(x)\right) = \left(2\left(\sqrt[3]{\frac{x^5-1}{2}}\right)^3 + 1\right)^{1/5} = \left(2\left(\frac{x^5-1}{2}\right)+1\right)^{1/5}$

$= \left(\left(x^5-1\right)+1\right)^{1/5} = \left(x^5\right)^{1/5} = x$ and $f^{-1}\left(f(x)\right) = \sqrt[3]{\frac{\left[\left(2x^3+1\right)^{1/5}\right]^5-1}{2}} = \sqrt[3]{\frac{(2x^3+1)-1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = x$

35. (a) $y = 2x + 3 \Rightarrow 2x = y - 3$

$\Rightarrow x = \frac{y}{2} - \frac{3}{2} \Rightarrow f^{-1}(x) = \frac{x}{2} - \frac{3}{2}$

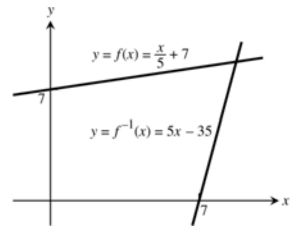
(b)



(c) $\left.\frac{df}{dx}\right|_{x=-1} = 2$, $\left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{2}$

36. (a) $y = \frac{1}{5}x + 7 \Rightarrow \frac{1}{5}x = y - 7$
$\Rightarrow x = 5y - 35 \Rightarrow f^{-1}(x) = 5x - 35$

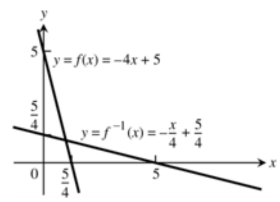
(b)



(c) $\left.\frac{df}{dx}\right|_{x=-1} = \frac{1}{5}, \left.\frac{df^{-1}}{dx}\right|_{x=34/5} = 5$

37. (a) $y = 5 - 4x \Rightarrow 4x = 5 - y$
$\Rightarrow x = \frac{5}{4} - \frac{y}{4} \Rightarrow f^{-1}(x) = \frac{5}{4} - \frac{x}{4}$

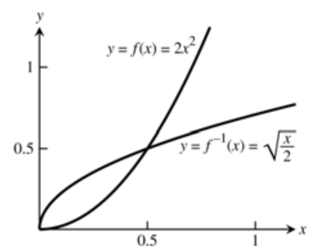
(b)



(c) $\left.\frac{df}{dx}\right|_{x=1/2} = -4, \left.\frac{df^{-1}}{dx}\right|_{x=3} = -\frac{1}{4}$

38. (a) $y = 2x^2 \Rightarrow x^2 = \frac{1}{2}y$
$\Rightarrow x = \frac{1}{\sqrt{2}}\sqrt{y} \Rightarrow f^{-1}(x) = \sqrt{\frac{x}{2}}$

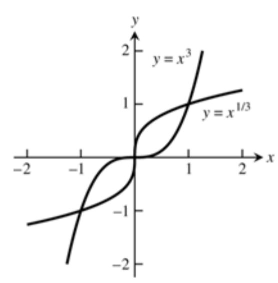
(b)



(c) $\left.\frac{df}{dx}\right|_{x=5} = 4x\big|_{x=5} = 20,$
$\left.\frac{df^{-1}}{dx}\right|_{x=50} = \left.\frac{1}{2\sqrt{2}}x^{-1/2}\right|_{x=50} = \frac{1}{20}$

39. (a) $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x, g(f(x)) = \sqrt[3]{x^3} = x$

(b)

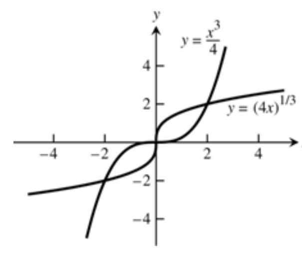


(c) $f'(x) = 3x^2 \Rightarrow f'(1) = 3, f'(-1) = 3;$
$g'(x) = \frac{1}{3}x^{-2/3} \Rightarrow g'(1) = \frac{1}{3}, g'(-1) = \frac{1}{3}$

(d) The line $y = 0$ is tangent to $f(x) = x^3$ at $(0, 0)$; the line $x = 0$ is tangent to $g(x) = \sqrt[3]{x}$ at $(0, 0)$

40. (a) $h(k(x)) = \frac{1}{4}\left((4x)^{1/3}\right)^3 = x,$
$k(h(x)) = \left(4 \cdot \frac{x^3}{4}\right)^{1/3} = x$

(b)



(c) $h'(x) = \frac{3x^2}{4} \Rightarrow h'(2) = 3, h'(-2) = 3;$
$k'(x) = \frac{4}{3}(4x)^{-2/3} \Rightarrow k'(2) = \frac{1}{3}, k'(-2) = \frac{1}{3}$

(d) The line $y = 0$ is tangent to $h(x) = \frac{x^3}{4}$ at $(0, 0)$; the line $x = 0$ is tangent to $k(x) = (4x)^{1/3}$ at $(0, 0)$

41. $\frac{df}{dx} = 3x^2 - 6x \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=f(3)} = \left.\frac{1}{\frac{df}{dx}}\right|_{x=3} = \frac{1}{9}$

42. $\frac{df}{dx} = 2x - 4 \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=f(5)} = \left.\frac{1}{\frac{df}{dx}}\right|_{x=5} = \frac{1}{6}$

43. $\left.\frac{df^{-1}}{dx}\right|_{x=4} = \left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \left.\frac{1}{\frac{df}{dx}}\right|_{x=2} = \frac{1}{\left(\frac{1}{3}\right)} = 3$

44. $\left.\frac{dg^{-1}}{dx}\right|_{x=0} = \left.\frac{dg^{-1}}{dx}\right|_{x=f(0)} = \left.\frac{1}{\frac{dg}{dx}}\right|_{x=0} = \frac{1}{2}$

45. (a) $y = mx \Rightarrow x = \frac{1}{m}y \Rightarrow f^{-1}(x) = \frac{1}{m}x$

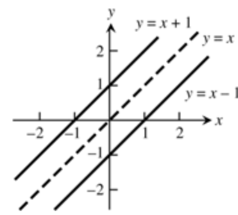
(b) The graph of $y = f^{-1}(x)$ is a line through the origin with slope $\frac{1}{m}$.

46. $y = mx + b \Rightarrow x = \frac{y}{m} - \frac{b}{m} \Rightarrow f^{-1}(x) = \frac{1}{m}x - \frac{b}{m}$; the graph of $f^{-1}(x)$ is a line with slope $\frac{1}{m}$ and y-intercept $-\frac{b}{m}$.

47. (a) $y = x + 1 \Rightarrow x = y - 1 \Rightarrow f^{-1}(x) = x - 1$

(b) $y = x + b \Rightarrow x = y - b \Rightarrow f^{-1}(x) = x - b$

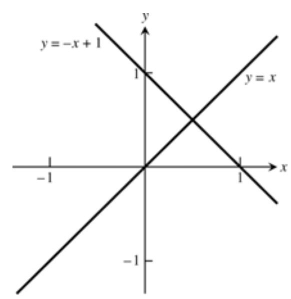
(c) Their graphs will be parallel to one another and lie on opposite sides of the line $y = x$ equidistant from that line.



48. (a) $y = -x + 1 \Rightarrow x = -y + 1 \Rightarrow f^{-1}(x) = 1 - x$; the lines intersect at a right angle

(b) $y = -x + b \Rightarrow x = -y + b \Rightarrow f^{-1}(x) = b - x$; the lines intersect at a right angle

(c) Such a function is its own inverse.



49. Let $x_1 \neq x_2$ be two numbers in the domain of an increasing function *f*. Then, either $x_1 < x_2$ or $x_1 > x_2$ which implies $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, since $f(x)$ is increasing. In either case, $f(x_1) \neq f(x_2)$ and *f* is one-to-one. Similar arguments hold if *f* is decreasing.

50. $f(x)$ is increasing since $x_2 > x_1 \Rightarrow \frac{1}{3}x_2 + \frac{5}{6} > \frac{1}{3}x_1 + \frac{5}{6}$; $\frac{df}{dx} = \frac{1}{3} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\left(\frac{1}{3}\right)} = 3$

51. $f(x)$ is increasing since $x_2 > x_1 \Rightarrow 27x_2^3 > 27x_1^3$; $y = 27x^3 \Rightarrow x = \frac{1}{3}y^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{3}x^{1/3}$;

$\frac{df}{dx} = 81x^2 \Rightarrow \frac{df^{-1}}{dx} = \left.\frac{1}{81x^2}\right|_{\frac{1}{3}x^{1/3}} = \frac{1}{9x^{2/3}} = \frac{1}{9}x^{-2/3}$

52. $f(x)$ is decreasing since $x_2 > x_1 \Rightarrow 1 - 8x_2^3 < 1 - 8x_1^3$; $y = 1 - 8x^3 \Rightarrow x = \frac{1}{2}(1-y)^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{2}(1-x)^{1/3}$;

$\frac{df}{dx} = -24x^2 \Rightarrow \frac{df^{-1}}{dx} = \left.\frac{1}{-24x^2}\right|_{\frac{1}{2}(1-x)^{1/3}} = \frac{-1}{6(1-x)^{2/3}} = -\frac{1}{6}(1-x)^{-2/3}$

53. $f(x)$ is decreasing since $x_2 > x_1 \Rightarrow \left(1 - x_2\right)^3 < \left(1 - x_1\right)^3$; $y = (1-x)^3 \Rightarrow x = 1 - y^{1/3} \Rightarrow f^{-1}(x) = 1 - x^{1/3}$;

$\frac{df}{dx} = -3(1-x)^2 \Rightarrow \frac{df^{-1}}{dx} = \left.\frac{1}{-3(1-x)^2}\right|_{1-x^{1/3}} = \frac{-1}{3x^{2/3}} = -\frac{1}{3}x^{-2/3}$

54. $f(x)$ is increasing since $x_2 > x_1 \Rightarrow x_2^{5/3} > x_1^{5/3}$; $y = x^{5/3} \Rightarrow x = y^{3/5} \Rightarrow f^{-1}(x) = x^{3/5}$;

$$\frac{df}{dx} = \frac{5}{3}x^{2/3} \Rightarrow \frac{df^{-1}}{dx} = \left.\frac{1}{\frac{5}{3}x^{2/3}}\right|_{x^{3/5}} = \frac{3}{5x^{2/5}} = \frac{3}{5}x^{-2/5}$$

55. The function $g(x)$ is also one-to-one. The reasoning: $f(x)$ is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $-f(x_1) \neq -f(x_2)$ and therefore $g(x_1) \neq g(x_2)$. Therefore $g(x)$ is one-to-one as well.

56. The function $h(x)$ is also one-to-one. The reasoning: $f(x)$ is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $\frac{1}{f(x_1)} \neq \frac{1}{f(x_2)}$, and therefore $h(x_1) \neq h(x_2)$.

57. The composite is one-to-one also. The reasoning: If $x_1 \neq x_2$ then $g(x_1) \neq g(x_2)$ because *g* is one-to-one. Since $g(x_1) \neq g(x_2)$, we also have $f(g(x_1)) \neq f(g(x_2))$ because *f* is one-to-one. Thus, $f \circ g$ is one-to-one because $x_1 \neq x_2 \Rightarrow f(g(x_1)) \neq f(g(x_2))$.

58. Yes, *g* must be one-to-one. If *g* were not one-to-one, there would exist numbers $x_1 \neq x_2$ in the domain of *g* with $g(x_1) = g(x_2)$. For these numbers we would also have $f(g(x_1)) = f(g(x_2))$, contradicting the assumption that $f \circ g$ is one-to-one.

59. $(g \circ f)(x) = x \Rightarrow g(f(x)) = x \Rightarrow g'(f(x))f'(x) = 1$

60. $W(a) = \int_{f(a)}^{f(a)} \pi\left[\left(f^{-1}(y)\right)^2 - a^2\right] dy = 0 = \int_a^a 2\pi x[f(a) - f(x)]\, dx = S(a)$;

$W'(t) = \pi\left[\left(f^{-1}(f(t))\right)^2 - a^2\right] f'(t) = \pi\left(t^2 - a^2\right) f'(t)$; also

$S(t) = 2\pi f(t)\int_a^t x\, dx - 2\pi\int_a^t x f(x)\, dx = \left[\pi f(t)t^2 - \pi f(t)a^2\right] - 2\pi\int_a^t x f(x)\, dx$

$\Rightarrow S'(t) = \pi t^2 f'(t) + 2\pi t f(t) - \pi a^2 f'(t) - 2\pi t f(t) = \pi\left(t^2 - a^2\right) f'(t) \Rightarrow W'(t) = S'(t)$. Therefore, $W(t) = S(t)$ for all $t \in [a, b]$.

61-68. Example CAS commands:

<u>Maple</u>:

```
with(plots); #61
f := x -> sqrt(3*x-2);
domain := 2/3..4;
x0 := 3;
Df := D(f);                                    #(a)
plot( [f(x), Df(x)], x=domain, color=[red,blue], linestyle=[1,3], legend=["y=f(x)","y=f '(x)"],
   title="#61(a) (Section 7.1)");
q1 := solve( y=f(x), x );                      # (b)
g := unapply( q1, y );
m1 := Df(x0);                                  # (c)
```

```
t1 := f(x0)+m1*(x -x0);
y=t1;
m2 := 1/Df(x0);                          # (d)
t2 := g(f(x0)) + m2 * (x-f(x0));
y=t2;
domaing := map(f, domain);               # (e)
p1 :=plot( [f(x), x], x=domain, color= [pink, green], linestyle=[1, 9], thickness=[3, 0] ):
p2 := plot( g(x), x=domaing, color=cyan, linestyle=3, thickness=4 ):
p3 :=plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 ) :
p4 :=plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1) :
p5 :=plot([[x0, f(x0)], [f(x0),x0]], color=green ) :
display( [p1, p2, p3, p4, p5], scaling=constrained, title="#61(e) (Section 7.1)" );
```

Mathematica: (assigned function and values for a, b, and x0 may vary)
If a function requires the odd root of a negative number, begin by loading the RealOnly package that allows Mathematica to do this.

```
<<Miscellaneous `RealOnly`
Clear[x, y]
{a,b} = {–2, 1}; x0 = 1/2;
f[x_] = (3x + 2) / (2x – 11)
Plot[{f[x], f'[x]}, {x, a, b}]
solx = Solve[y == f[x], x]
g[y_] = x /. solx[[1]]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x-x0)
gtan[y_] = x0 + 1 / f'[x0] (y – y0)
Plot[{f[x], ftan[x], g[x], gtan[x], Identity[x]},{x, a, b},
Epilog → Line[{{x0, y0},{y0, x0}}], PlotRange → {{a,b},{a,b}}, AspectRatio → Automatic]
```

69-70. Example CAS commands:
Maple:

```
with(plots);
eq := cos(y) = x^(1/5);
domain:= 0..1;
x0:= 1/2;
f := unapply( solve( eq, y ), x);        # (a)
Df := D(f);
plot( [f(x), Df(x), x=domain, color=[red, blue], linestyle=[1, 3], legend=["y=f(x)","y=f '(x)"],
   title="#70(a) (Section 7.1)" );
q1 := solve( eq, x );                    # (b)
g := unapply( q1, y );
```

```
m1 := Df(x0);                          # (c)
t1 := f(x0)+m1*(x-x0);
y=t1;
m2 := 1/Df(x0);                        # (d)
t2 := g(f(x0)) + m2 * (x-f(x0));
y=t2;
domaing := map(f, domain);             # (e)
p1 := plot( [f(x), x], x=domain, color=[pink, green], linestyle=[1, 9], thickness=[3, 0] ):
p2 := plot( g(x), x=domaing, color=cyan, linestyle=3, thickness=4 ) :
p3 := plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 ) :
p4 := plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1 ) :
p5 := plot( [[x0, f(x0)], [f(x0), x0]], color=green ) :
display( [p1, p2, p3, p4, p5], scaling=constrained, title="#70(e) (Section 7.1)" );
```

Mathematica: (assigned function and values for a, b, and x0 may vary)
For problems 69 and 70, the code is just slightly altered. At times, different "parts" of solutions need to be used, as in the definitions of f[x] and g[y]

```
Clear[x, y]
{a,b} = {0, 1}; x0 = 1 / 2 ;
eqn = Cos[y] == x^(1/5)
soly = Solve[eqn, y]
f[x_] = y /. soly[[2]]
Plot[{f[x], f '[x]}, {x, a, b}]
solx = Solve[eqn, x]
g[y_] = x /. sol x[[1]]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x − x0)
gtan[y_] = x0 + 1 / f'[x0] (y − y0)
Plot [{f[x], ftan[x], g[x], gtan[x], Idenity[x]}, {x, a, b},
Epilog → Line[{{x0, y0}, {y0, x0}}], PlotRange → {{a, b}, {a, b}}, AspectRatio → Automatic]
```

7.2 NATURAL LOGARITHMS

1. (a) $\ln 0.75 = \ln \frac{3}{4} = \ln 3 - \ln 4 = \ln 3 - \ln 2^2 = \ln 3 - 2\ln 2$

   (b) $\ln \frac{4}{9} = \ln 4 - \ln 9 = \ln 2^2 - \ln 3^2 = 2\ln 2 - 2\ln 3$

   (c) $\ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$

   (d) $\ln \sqrt[3]{9} = \frac{1}{3}\ln 9 = \frac{1}{3}\ln 3^2 = \frac{2}{3}\ln 3$

   (e) $\ln 3\sqrt{2} = \ln 3 + \ln 2^{1/2} = \ln 3 + \frac{1}{2}\ln 2$

   (f) $\ln \sqrt{13.5} = \frac{1}{2}\ln 13.5 = \frac{1}{2}\ln \frac{27}{2} = \frac{1}{2}\left(\ln 3^3 - \ln 2\right) = \frac{1}{2}(3\ln 3 - \ln 2)$

2. (a) $\ln \frac{1}{125} = \ln 1 - 3\ln 5 = -3\ln 5$

   (b) $\ln 9.8 = \ln \frac{49}{5} = \ln 7^2 - \ln 5 = 2\ln 7 - \ln 5$

(c) $\ln 7\sqrt{7} = \ln 7^{3/2} = \frac{3}{2}\ln 7$

(d) $\ln 1225 = \ln 35^2 = 2\ln 35 = 2\ln 5 + 2\ln 7$

(e) $\ln 0.056 = \ln \frac{7}{125} = \ln 7 - \ln 5^3 = \ln 7 - 3\ln 5$

(f) $\frac{\ln 35 + \ln \frac{1}{7}}{\ln 25} = \frac{\ln 5 + \ln 7 - \ln 7}{2\ln 5} = \frac{1}{2}$

3. (a) $\ln \sin\theta - \ln\left(\frac{\sin\theta}{5}\right) = \ln\left(\frac{\sin\theta}{\left(\frac{\sin\theta}{5}\right)}\right) = \ln 5$

(b) $\ln\left(3x^2 - 9x\right) + \ln\left(\frac{1}{3x}\right) = \ln\left(\frac{3x^2-9x}{3x}\right) = \ln(x-3)$

(c) $\frac{1}{2}\ln\left(4t^4\right) - \ln 2 = \ln\sqrt{4t^4} - \ln 2 = \ln 2t^2 - \ln 2 = \ln\left(\frac{2t^2}{2}\right) = \ln\left(t^2\right)$

4. (a) $\ln \sec\theta + \ln\cos\theta = \ln\left[(\sec\theta)(\cos\theta)\right] = \ln 1 = 0$

(b) $\ln(8x+4) - \ln 2^2 = \ln(8x+4) - \ln 4 = \ln\left(\frac{8x+4}{4}\right) = \ln(2x+1)$

(c) $3\ln\sqrt[3]{t^2-1} - \ln(t+1) = 3\ln\left(t^2-1\right)^{1/3} - \ln(t+1) = 3\left(\frac{1}{3}\right)\ln\left(t^2-1\right) - \ln(t+1) = \ln\left(\frac{(t+1)(t-1)}{(t+1)}\right) = \ln(t-1)$

5. $y = \ln 3x \Rightarrow y' = \left(\frac{1}{3x}\right)(3) = \frac{1}{x}$

6. $y = \ln kx \Rightarrow y' = \left(\frac{1}{kx}\right)(k) = \frac{1}{x}$

7. $y = \ln\left(t^2\right) \Rightarrow \frac{dy}{dt} = \left(\frac{1}{t^2}\right)(2t) = \frac{2}{t}$

8. $y = \ln\left(t^{3/2}\right) \Rightarrow \frac{dy}{dt} = \left(\frac{1}{t^{3/2}}\right)\left(\frac{3}{2}t^{1/2}\right) = \frac{3}{2t}$

9. $y = \ln\frac{3}{x} = \ln 3x^{-1} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{3x^{-1}}\right)\left(-3x^{-2}\right) = -\frac{1}{x}$

10. $y = \ln\frac{10}{x} = \ln 10x^{-1} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{10x^{-1}}\right)(-10x^{-2}) = -\frac{1}{x}$

11. $y = \ln(\theta+1) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\theta+1}\right)(1) = \frac{1}{\theta+1}$

12. $y = \ln(2\theta+2) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{2\theta+2}\right)(2) = \frac{1}{\theta+1}$

13. $y = \ln x^3 \Rightarrow \frac{dy}{dx} = \left(\frac{1}{x^3}\right)\left(3x^2\right) = \frac{3}{x}$

14. $y = (\ln x)^3 \Rightarrow \frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{d}{dx}(\ln x) = \frac{3(\ln x)^2}{x}$

15. $y = t(\ln t)^2 \Rightarrow \frac{dy}{dx} = (\ln t)^2 + 2t(\ln t)\cdot\frac{d}{dt}(\ln t) = (\ln t)^2 + \frac{2t\ln t}{t} = (\ln t)^2 + 2\ln t$

16. $y = t\sqrt{\ln t} = t(\ln t)^{1/2} \Rightarrow \frac{dy}{dt} = (\ln t)^{1/2} + \frac{1}{2}t(\ln t)^{-1/2}\cdot\frac{d}{dt}(\ln t) = (\ln t)^{1/2} + \frac{t(\ln t)^{-1/2}}{2t} = (\ln t)^{1/2} + \frac{1}{2(\ln t)^{1/2}}$

17. $y = \frac{x^4}{4}\ln x - \frac{x^4}{16} \Rightarrow \frac{dy}{dx} = x^3\ln x + \frac{x^4}{4}\cdot\frac{1}{x} - \frac{4x^3}{16} = x^3\ln x$

18. $y = \left(x^2\ln x\right)^4 \Rightarrow \frac{dy}{dx} = 4\left(x^2\ln x\right)^3\left(x^2\cdot\frac{1}{x} + 2x\ln x\right) = 4x^6(\ln x)^3(x + 2x\ln x) = 4x^7(\ln x)^3 + 8x^7(\ln x)^4$

19. $y = \frac{\ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (\ln t)(1)}{t^2} = \frac{1-\ln t}{t^2}$

20. $y=\frac{1+\ln t}{t}\Rightarrow\frac{dy}{dt}=\frac{t\left(\frac{1}{t}\right)-(1+\ln t)(1)}{t^2}=\frac{1-1-\ln t}{t^2}=-\frac{\ln t}{t^2}$

21. $y=\frac{\ln x}{1+\ln x}\Rightarrow y'=\frac{(1+\ln x)\left(\frac{1}{x}\right)-(\ln x)\left(\frac{1}{x}\right)}{(1+\ln x)^2}=\frac{\frac{1}{x}+\frac{\ln x}{x}-\frac{\ln x}{x}}{(1+\ln x)^2}=\frac{1}{x(1+\ln x)^2}$

22. $y=\frac{x\ln x}{1+\ln x}\Rightarrow y'=\frac{(1+\ln x)\left(\ln x+x\cdot\frac{1}{x}\right)-(x\ln x)\left(\frac{1}{x}\right)}{(1+\ln x)^2}=\frac{(1+\ln x)^2-\ln x}{(1+\ln x)^2}=1-\frac{\ln x}{(1+\ln x)^2}$

23. $y=\ln(\ln x)\Rightarrow y'=\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)=\frac{1}{x\ln x}$

24. $y=\ln\left(\ln(\ln x)\right)\Rightarrow y'=\frac{1}{\ln(\ln x)}\cdot\frac{d}{dx}\left(\ln(\ln x)\right)=\frac{1}{\ln(\ln x)}\cdot\frac{1}{\ln x}\cdot\frac{d}{dx}(\ln x)=\frac{1}{x(\ln x)\ln(\ln x)}$

25. $y=\theta[\sin(\ln\theta)+\cos(\ln\theta)]\Rightarrow\frac{dy}{d\theta}=[\sin(\ln\theta)+\cos(\ln\theta)]+\theta\left[\cos(\ln\theta)\cdot\frac{1}{\theta}-\sin(\ln\theta)\cdot\frac{1}{\theta}\right]$
$=\sin(\ln\theta)+\cos(\ln\theta)+\cos(\ln\theta)-\sin(\ln\theta)=2\cos(\ln\theta)$

26. $y=\ln(\sec\theta+\tan\theta)\Rightarrow\frac{dy}{d\theta}=\frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}=\frac{\sec\theta(\tan\theta+\sec\theta)}{\tan\theta+\sec\theta}=\sec\theta$

27. $y=\ln\frac{1}{x\sqrt{x+1}}=-\ln x-\frac{1}{2}\ln(x+1)\Rightarrow y'=-\frac{1}{x}-\frac{1}{2}\left(\frac{1}{x+1}\right)=-\frac{2(x+1)+x}{2x(x+1)}=-\frac{3x+2}{2x(x+1)}$

28. $y=\frac{1}{2}\ln\frac{1+x}{1-x}=\frac{1}{2}[\ln(1+x)-\ln(1-x)]\Rightarrow y'=\frac{1}{2}\left[\frac{1}{1+x}-\left(\frac{1}{1-x}\right)(-1)\right]=\frac{1}{2}\left[\frac{1-x+1+x}{(1+x)(1-x)}\right]=\frac{1}{1-x^2}$

29. $y=\frac{1+\ln t}{1-\ln t}\Rightarrow\frac{dy}{dt}=\frac{(1-\ln t)\left(\frac{1}{t}\right)-(1+\ln t)\left(\frac{-1}{t}\right)}{(1-\ln t)^2}=\frac{\frac{1}{t}-\frac{\ln t}{t}+\frac{1}{t}+\frac{\ln t}{t}}{(1-\ln t)^2}=\frac{2}{t(1-\ln t)^2}$

30. $y=\sqrt{\ln\sqrt{t}}=\left(\ln t^{1/2}\right)^{1/2}\Rightarrow\frac{dy}{dt}=\frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2}\cdot\frac{d}{dt}\left(\ln t^{1/2}\right)=\frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2}\cdot\frac{1}{t^{1/2}}\cdot\frac{d}{dt}\left(t^{1/2}\right)$
$=\frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2}\cdot\frac{1}{t^{1/2}}\cdot\frac{1}{2}t^{-1/2}=\frac{1}{4t\sqrt{\ln\sqrt{t}}}$

31. $y=\ln\left(\sec(\ln\theta)\right)\Rightarrow\frac{dy}{d\theta}=\frac{1}{\sec(\ln\theta)}\cdot\frac{d}{d\theta}\left(\sec(\ln\theta)\right)=\frac{\sec(\ln\theta)\tan(\ln\theta)}{\sec(\ln\theta)}\cdot\frac{d}{d\theta}(\ln\theta)=\frac{\tan(\ln\theta)}{\theta}$

32. $y=\ln\frac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta}=\frac{1}{2}(\ln\sin\theta+\ln\cos\theta)-\ln(1+2\ln\theta)\Rightarrow\frac{dy}{d\theta}=\frac{1}{2}\left(\frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}\right)-\frac{\frac{2}{\theta}}{1+2\ln\theta}$
$=\frac{1}{2}\left[\cot\theta-\tan\theta-\frac{4}{\theta(1+2\ln\theta)}\right]$

33. $y=\ln\left(\frac{\left(x^2+1\right)^5}{\sqrt{1-x}}\right)=5\ln\left(x^2+1\right)-\frac{1}{2}\ln(1-x)\Rightarrow y'=\frac{5\cdot 2x}{x^2+1}-\frac{1}{2}\left(\frac{1}{1-x}\right)(-1)=\frac{10x}{x^2+1}+\frac{1}{2(1-x)}$

34. $y=\ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}=\frac{1}{2}[5\ln(x+1)-20\ln(x+2)]\Rightarrow y'=\frac{1}{2}\left(\frac{5}{x+1}-\frac{20}{x+2}\right)=\frac{5}{2}\left[\frac{(x+2)-4(x+1)}{(x+1)(x+2)}\right]=-\frac{5}{2}\left[\frac{3x+2}{(x+1)(x+2)}\right]$

35. $y=\int_{x^2/2}^{x^2} \ln\sqrt{t}\,dt \Rightarrow \frac{dy}{dx}=\left(\ln\sqrt{x^2}\right)\cdot\frac{d}{dx}\left(x^2\right)-\left(\ln\sqrt{\frac{x^2}{2}}\right)\cdot\frac{d}{dx}\left(\frac{x^2}{2}\right)=2x\ln|x|-x\ln\frac{|x|}{\sqrt{2}}$

36. $y=\int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\,dt \Rightarrow \frac{dy}{dx}=\left(\ln\sqrt[3]{x}\right)\cdot\frac{d}{dx}\left(\sqrt[3]{x}\right)-\left(\ln\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)=\left(\ln\sqrt[3]{x}\right)\left(\frac{1}{3}x^{-2/3}\right)-\left(\ln\sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right)$
$=\frac{\ln\sqrt[3]{x}}{3\sqrt[3]{x^2}}-\frac{\ln\sqrt{x}}{2\sqrt{x}}$

37. $\int_{-3}^{-2}\frac{1}{x}\,dx=\left[\ln|x|\right]_{-3}^{-2}=\ln 2-\ln 3=\ln\frac{2}{3}$

38. $\int_{-1}^{0}\frac{3}{3x-2}\,dx=\left[\ln|3x-2|\right]_{-1}^{0}=\ln 2-\ln 5=\ln\frac{2}{5}$

39. $\int\frac{2y}{y^2-25}\,dy=\ln\left|y^2-25\right|+C$

40. $\int\frac{8r}{4r^2-5}\,dr=\ln\left|4r^2-5\right|+C$

41. $\int_0^{\pi}\frac{\sin t}{2-\cos t}\,dt=\left[\ln|2-\cos t|\right]_0^{\pi}=\ln 3-\ln 1=\ln 3$; or let $u=2-\cos t\Rightarrow du=\sin t\,dt$ with $t=0\Rightarrow u=1$ and $t=\pi\Rightarrow u=3\Rightarrow\int_0^{\pi}\frac{\sin t}{2-\cos t}\,dt=\int_1^3\frac{1}{u}\,du=\left[\ln|u|\right]_1^3=\ln 3-\ln 1=\ln 3$

42. $\int_0^{\pi/3}\frac{4\sin\theta}{1-4\cos\theta}\,d\theta=[\ln|1-4\cos\theta|]_0^{\pi/3}=\ln|1-2|=-\ln 3=\ln\frac{1}{3}$; or let $u=1-4\cos\theta\Rightarrow du=4\sin\theta\,d\theta$ with $\theta=0\Rightarrow u=-3$ and $\theta=\frac{\pi}{3}\Rightarrow u=-1\Rightarrow\int_0^{\pi/3}\frac{4\sin\theta}{1-4\cos\theta}\,d\theta=\int_{-3}^{-1}\frac{1}{u}\,du=[\ln|u|]_{-3}^{-1}=-\ln 3=\ln\frac{1}{3}$

43. Let $u=\ln x\Rightarrow du=\frac{1}{x}\,dx$; $x=1\Rightarrow u=0$ and $x=2\Rightarrow u=\ln 2$; $\int_1^2\frac{2\ln x}{x}\,dx=\int_0^{\ln 2}2u\,du=[u^2]_0^{\ln 2}=(\ln 2)^2$

44. Let $u=\ln x\Rightarrow du=\frac{1}{x}\,dx$; $x=2\Rightarrow u=\ln 2$ and $x=4\Rightarrow u=\ln 4$;
$\int_2^4\frac{dx}{x\ln x}=\int_{\ln 2}^{\ln 4}\frac{1}{u}\,du=\left[\ln u\right]_{\ln 2}^{\ln 4}=\ln(\ln 4)-\ln(\ln 2)=\ln\left(\frac{\ln 4}{\ln 2}\right)=\ln\left(\frac{\ln 2^2}{\ln 2}\right)=\ln\left(\frac{2\ln 2}{\ln 2}\right)=\ln 2$

45. Let $u=\ln x\Rightarrow du=\frac{1}{x}\,dx$; $x=2\Rightarrow u=\ln 2$ and $x=4\Rightarrow u=\ln 4$;
$\int_2^4\frac{dx}{x(\ln x)^2}=\int_{\ln 2}^{\ln 4}u^{-2}\,du=\left[-\frac{1}{u}\right]_{\ln 2}^{\ln 4}=-\frac{1}{\ln 4}+\frac{1}{\ln 2}=-\frac{1}{\ln 2^2}+\frac{1}{\ln 2}=-\frac{1}{2\ln 2}+\frac{1}{\ln 2}=\frac{1}{2\ln 2}=\frac{1}{\ln 4}$

46. Let $u=\ln x\Rightarrow du=\frac{1}{x}\,dx$; $x=2\Rightarrow u=\ln 2$ and $x=16\Rightarrow u=\ln 16$;
$\int_2^{16}\frac{dx}{2x\sqrt{\ln x}}=\frac{1}{2}\int_{\ln 2}^{\ln 16}u^{-1/2}\,du=\left[u^{1/2}\right]_{\ln 2}^{\ln 16}=\sqrt{\ln 16}-\sqrt{\ln 2}=\sqrt{4\ln 2}-\sqrt{\ln 2}=2\sqrt{\ln 2}-\sqrt{\ln 2}=\sqrt{\ln 2}$

47. Let $u=6+3\tan t\Rightarrow du=3\sec^2 t\,dt$; $\int\frac{3\sec^2 t}{6+3\tan t}\,dt=\int\frac{du}{u}=\ln|u|+C=\ln|6+3\tan t|+C$

48. Let $u=2+\sec y\Rightarrow du=\sec y\tan y\,dy$; $\int\frac{\sec y\tan y}{2+\sec y}\,dy=\int\frac{du}{u}=\ln|u|+C=\ln|2+\sec y|+C$

49. Let $u=\cos\frac{x}{2}\Rightarrow du=-\frac{1}{2}\sin\frac{x}{2}\,dx\Rightarrow -2\,du=\sin\frac{x}{2}\,dx$; $x=0\Rightarrow u=1$ and $x=\frac{\pi}{2}\Rightarrow u=\frac{1}{\sqrt{2}}$;
$\int_0^{\pi/2}\tan\frac{x}{2}\,dx=\int_0^{\pi/2}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\,dx=-2\int_1^{1/\sqrt{2}}\frac{du}{u}=\left[-2\ln|u|\right]_1^{1/\sqrt{2}}=-2\ln\frac{1}{\sqrt{2}}=2\ln\sqrt{2}=\ln 2$

50. Let $u = \sin t \Rightarrow du = \cos t\, dt$; $t = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$ and $t = \frac{\pi}{2} \Rightarrow u = 1$;

$$\int_{\pi/4}^{\pi/2} \cot t\, dt = \int_{\pi/4}^{\pi/2} \frac{\cos t}{\sin t}\, dt = \int_{1/\sqrt{2}}^{1} \frac{du}{u} = \left[\ln|u|\right]_{1/\sqrt{2}}^{1} = -\ln \frac{1}{\sqrt{2}} = \ln \sqrt{2}$$

51. Let $u = \sin \frac{\theta}{3} \Rightarrow du = \frac{1}{3} \cos \frac{\theta}{3}\, d\theta \Rightarrow 6\, du = 2 \cos \frac{\theta}{3}\, d\theta$; $\theta = \frac{\pi}{2} \Rightarrow u = \frac{1}{2}$ and $\theta = \pi \Rightarrow u = \frac{\sqrt{3}}{2}$;

$$\int_{\pi/2}^{\pi} 2 \cot \frac{\theta}{3}\, d\theta = \int_{\pi/2}^{\pi} \frac{2\cos\frac{\theta}{3}}{\sin\frac{\theta}{3}}\, d\theta = 6\int_{1/2}^{\sqrt{3}/2} \frac{du}{u} = 6\left[\ln|u|\right]_{1/2}^{\sqrt{3}/2} = 6\left(\ln \frac{\sqrt{3}}{2} - \ln \frac{1}{2}\right) = 6 \ln \sqrt{3} = \ln 27$$

52. Let $u = \cos 3x \Rightarrow du = -3 \sin 3x\, dx \Rightarrow -2du = 6 \sin 3x\, dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{12} \Rightarrow u = \frac{1}{\sqrt{2}}$;

$$\int_0^{\pi/12} 6 \tan 3x\, dx = \int_0^{\pi/12} \frac{6 \sin 3x}{\cos 3x}\, dx = -2\int_1^{1/\sqrt{2}} \frac{du}{u} = -2\left[\ln|u|\right]_1^{1/\sqrt{2}} = -2 \ln \frac{1}{\sqrt{2}} - \ln 1 = 2 \ln \sqrt{2} = \ln 2$$

53. $\int \frac{dx}{2\sqrt{x}+2x} = \int \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)}$; let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\, dx$; $\int \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)} = \int \frac{du}{u} = \ln|u| + C$

$= \ln\left|1+\sqrt{x}\right| + C = \ln\left(1+\sqrt{x}\right) + C$

54. Let $u = \sec x + \tan x \Rightarrow du = \left(\sec x \tan x + \sec^2 x\right) dx = (\sec x)(\tan x + \sec x)\, dx \Rightarrow \sec x\, dx = \frac{du}{u}$;

$$\int \frac{\sec x\, dx}{\sqrt{\ln(\sec x + \tan x)}} = \int \frac{du}{u\sqrt{\ln u}} = \int (\ln u)^{-1/2} \cdot \frac{1}{u}\, du = 2(\ln u)^{1/2} + C = 2\sqrt{\ln(\sec x + \tan x)} + C$$

55. $y = \sqrt{x(x+1)} = \left(x(x+1)\right)^{1/2} \Rightarrow \ln y = \frac{1}{2} \ln\left(x(x+1)\right) \Rightarrow 2 \ln y = \ln(x) + \ln(x+1) \Rightarrow \frac{2y'}{y} = \frac{1}{x} + \frac{1}{x+1}$

$\Rightarrow y' = \left(\frac{1}{2}\right) \sqrt{x(x+1)} \left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{\sqrt{x(x+1)}(2x+1)}{2x(x+1)} = \frac{2x+1}{2\sqrt{x(x+1)}}$

56. $y = \sqrt{\left(x^2+1\right)(x-1)^2} \Rightarrow \ln y = \frac{1}{2}\left[\ln\left(x^2+1\right) + 2 \ln(x-1)\right] \Rightarrow \frac{y'}{y} = \frac{1}{2}\left(\frac{2x}{x^2+1} + \frac{2}{x-1}\right)$

$\Rightarrow y' = \sqrt{\left(x^2+1\right)(x-1)^2} \left(\frac{x}{x^2+1} + \frac{1}{x-1}\right) = \sqrt{\left(x^2+1\right)(x-1)^2} \left[\frac{x^2-x+x^2+1}{\left(x^2+1\right)(x-1)}\right] = \frac{\left(2x^2-x+1\right)|x-1|}{\sqrt{x^2+1}\,(x-1)}$

57. $y = \sqrt{\frac{t}{t+1}} = \left(\frac{t}{t+1}\right)^{1/2} \Rightarrow \ln y = \frac{1}{2}\left[\ln t - \ln(t+1)\right] \Rightarrow \frac{1}{y}\frac{dy}{dt} = \frac{1}{2}\left(\frac{1}{t} - \frac{1}{t+1}\right)$

$\Rightarrow \frac{dy}{dt} = \frac{1}{2}\sqrt{\frac{t}{t+1}}\left(\frac{1}{t} - \frac{1}{t+1}\right) = \frac{1}{2}\sqrt{\frac{t}{t+1}}\left[\frac{1}{t(t+1)}\right] = \frac{1}{2\sqrt{t}\,(t+1)^{3/2}}$

58. $y = \sqrt{\frac{1}{t(t+1)}} = \left[t(t+1)\right]^{-1/2} \Rightarrow \ln y = -\frac{1}{2}\left[\ln t + \ln(t+1)\right] \Rightarrow \frac{1}{y}\frac{dy}{dt} = -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)$

$\Rightarrow \frac{dy}{dt} = -\frac{1}{2}\sqrt{\frac{1}{t(t+1)}}\left[\frac{2t+1}{t(t+1)}\right] = -\frac{2t+1}{2\left(t^2+t\right)^{3/2}}$

59. $y = \sqrt{\theta+3}\,(\sin\theta) = (\theta+3)^{1/2} \sin\theta \Rightarrow \ln y = \frac{1}{2}\ln(\theta+3) + \ln(\sin\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{1}{2(\theta+3)} + \frac{\cos\theta}{\sin\theta}$

$\Rightarrow \frac{dy}{d\theta} = \sqrt{\theta+3}\,(\sin\theta)\left[\frac{1}{2(\theta+3)} + \cot\theta\right]$

60. $y=(\tan\theta)\sqrt{2\theta+1}=(\tan\theta)(2\theta+1)^{1/2}\Rightarrow \ln\ y=\ln\ (\tan\theta)+\frac{1}{2}\ln\ (2\theta+1)\Rightarrow \frac{1}{y}\frac{dy}{d\theta}=\frac{\sec^2\theta}{\tan\theta}+\left(\frac{1}{2}\right)\left(\frac{2}{2\theta+1}\right)$
$\Rightarrow \frac{dy}{d\theta}=(\tan\theta)\sqrt{2\theta+1}\left(\frac{\sec^2\theta}{\tan\theta}+\frac{1}{2\theta+1}\right)=(\sec^2\theta)\sqrt{2\theta+1}+\frac{\tan\theta}{\sqrt{2\theta+1}}$

61. $y=t(t+1)(t+2)\Rightarrow \ln y=\ln t+\ln\ (t+1)+\ln\ (t+2)\Rightarrow \frac{1}{y}\frac{dy}{dt}=\frac{1}{t}+\frac{1}{t+1}+\frac{1}{t+2}\Rightarrow \frac{dy}{dt}=t(t+1)(t+2)\left(\frac{1}{t}+\frac{1}{t+1}+\frac{1}{t+2}\right)$
$=t(t+1)(t+2)\left[\frac{(t+1)(t+2)+t(t+2)+t(t+1)}{t(t+1)(t+2)}\right]=3t^2+6t+2$

62. $y=\frac{1}{t(t+1)(t+2)}\Rightarrow \ln y=\ln 1-\ln t-\ln\ (t+1)-\ln\ (t+2)\Rightarrow \frac{1}{y}\frac{dy}{dt}=-\frac{1}{t}-\frac{1}{t+1}-\frac{1}{t+2}$
$\Rightarrow \frac{dy}{dt}=\frac{1}{t(t+1)(t+2)}\left[-\frac{1}{t}-\frac{1}{t+1}-\frac{1}{t+2}\right]=\frac{-1}{t(t+1)(t+2)}\left[\frac{(t+1)(t+2)+t(t+2)+t(t+1)}{t(t+1)(t+2)}\right]=-\frac{3t^2+6t+2}{\left(t^3+3t^2+2t\right)^2}$

63. $y=\frac{\theta+5}{\theta\cos\theta}\Rightarrow \ln\ y=\ln\ (\theta+5)-\ln\theta-\ln\ (\cos\theta)\Rightarrow \frac{1}{y}\frac{dy}{d\theta}=\frac{1}{\theta+5}-\frac{1}{\theta}+\frac{\sin\theta}{\cos\theta}\Rightarrow \frac{dy}{d\theta}=\left(\frac{\theta+5}{\theta\cos\theta}\right)\left(\frac{1}{\theta+5}-\frac{1}{\theta}+\tan\theta\right)$

64. $y=\frac{\theta\sin\theta}{\sqrt{\sec\theta}}\Rightarrow \ln y=\ln\theta+\ln\ (\sin\theta)-\frac{1}{2}\ln\ (\sec\theta)\Rightarrow \frac{1}{y}\frac{dy}{d\theta}=\left[\frac{1}{\theta}+\frac{\cos\theta}{\sin\theta}-\frac{(\sec\theta)(\tan\theta)}{2\sec\theta}\right]$
$\Rightarrow \frac{dy}{d\theta}=\frac{\theta\sin\theta}{\sqrt{\sec\theta}}\left(\frac{1}{\theta}+\cot\theta-\frac{1}{2}\tan\theta\right)$

65. $y=\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\Rightarrow \ln y=\ln x+\frac{1}{2}\ln\left(x^2+1\right)-\frac{2}{3}\ln\ (x+1)\Rightarrow \frac{y'}{y}=\frac{1}{x}+\frac{x}{x^2+1}-\frac{2}{3(x+1)}$
$\Rightarrow y'=\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left[\frac{1}{x}+\frac{x}{x^2+1}-\frac{2}{3(x+1)}\right]$

66. $y=\sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}\Rightarrow \ln y=\frac{1}{2}\left[10\ln\ (x+1)-5\ln\ (2x+1)\right]\Rightarrow \frac{y'}{y}=\frac{5}{x+1}-\frac{5}{2x+1}\Rightarrow y'=\sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}\left(\frac{5}{x+1}-\frac{5}{2x+1}\right)$

67. $y=\sqrt[3]{\frac{x(x-2)}{x^2+1}}\Rightarrow \ln y=\frac{1}{3}\left[\ln x+\ln(x-2)-\ln\left(x^2+1\right)\right]\Rightarrow \frac{y'}{y}=\frac{1}{3}\left(\frac{1}{x}+\frac{1}{x-2}-\frac{2x}{x^2+1}\right)$
$\Rightarrow y'=\frac{1}{3}\sqrt[3]{\frac{x(x-2)}{x^2+1}}\left(\frac{1}{x}+\frac{1}{x-2}-\frac{2x}{x^2+1}\right)$

68. $y=\sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\Rightarrow \ln\ y=\frac{1}{3}\left[\ln x+\ln\ (x+1)+\ln\ (x-2)-\ln\left(x^2+1\right)-\ln\ (2x+3)\right]$
$\Rightarrow y'=\frac{1}{3}\sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\left(\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x-2}-\frac{2x}{x^2+1}-\frac{2}{2x+3}\right)$

69. (a) $f(x)=\ln\ (\cos x)\Rightarrow f'(x)=-\frac{\sin x}{\cos x}=-\tan x=0\Rightarrow x=0$; $f'(x)>0$ for $-\frac{\pi}{4}\le x<0$ and $f'(x)<0$ for $0<x\le\frac{\pi}{3}\Rightarrow$ there is a relative maximum at $x=0$ with $f(0)=\ln\ (\cos 0)=\ln 1=0$; $f\left(-\frac{\pi}{4}\right)$ $=\ln\left(\cos\left(-\frac{\pi}{4}\right)\right)=\ln\left(\frac{1}{\sqrt{2}}\right)=-\frac{1}{2}\ln 2$ and $f\left(\frac{\pi}{3}\right)=\ln\left(\cos\left(\frac{\pi}{3}\right)\right)=\ln\frac{1}{2}=-\ln 2$. Therefore, the absolute minimum occurs at $x=\frac{\pi}{3}$ with $f\left(\frac{\pi}{3}\right)=-\ln 2$ and the absolute maximum occurs at $x=0$ with $f(0)=0$.

(b) $f(x)=\cos(\ln x) \Rightarrow f'(x)=\frac{-\sin(\ln x)}{x}=0 \Rightarrow x=1$; $f'(x)>0$ for $\frac{1}{2}\le x<1$ and $f'(x)<0$ for $1<x\le 2$ $\Rightarrow$ there is a relative maximum at $x=1$ with $f(1)=\cos(\ln 1)=\cos 0=1$; $f\left(\frac{1}{2}\right)=\cos\left(\ln\left(\frac{1}{2}\right)\right)$ $=\cos(-\ln 2)=\cos(\ln 2)$ and $f(2)=\cos(\ln 2)$. Therefore, the absolute minimum occurs at $x=\frac{1}{2}$ and $x=2$ with $f\left(\frac{1}{2}\right)=f(2)=\cos(\ln 2)$, and the absolute maximum occurs at $x=1$ with $f(1)=1$.

70. (a) $f(x)=x-\ln x \Rightarrow f'(x)=1-\frac{1}{x}$; if $x>1$, then $f'(x)>0$ which means that $f(x)$ is increasing
(b) $f(1)=1-\ln 1=1 \Rightarrow f(x)=x-\ln x>0$, if $x>1$ by part (a) $\Rightarrow x>\ln x$ if $x>1$

71. $\int_1^5 (\ln 2x-\ln x)\,dx=\int_1^5(-\ln x+\ln 2+\ln x)\,dx=(\ln 2)\int_1^5 dx=(\ln 2)(5-1)=\ln 2^4=\ln 16$

72. $A=\int_{-\pi/4}^{0}(-\tan x)\,dx+\int_0^{\pi/3}\tan x\,dx=\int_{-\pi/4}^{0}\frac{-\sin x}{\cos x}\,dx-\int_0^{\pi/3}\frac{-\sin x}{\cos x}\,dx=[\ln|\cos x|]_{-\pi/4}^{0}-[\ln|\cos x|]_0^{\pi/3}$
$=\left(\ln 1-\ln\frac{1}{\sqrt{2}}\right)-\left(\ln\frac{1}{2}-\ln 1\right)=\ln\sqrt{2}+\ln 2=\frac{3}{2}\ln 2$

73. $V=\pi\int_0^3\left(\frac{2}{\sqrt{y+1}}\right)^2 dy=4\pi\int_0^3\frac{1}{y+1}\,dy=4\pi\,[\ln|y+1|]_0^3=4\pi(\ln 4-\ln 1)=4\pi\ln 4$

74. $V=\pi\int_{\pi/6}^{\pi/2}\cot x\,dx=\pi\int_{\pi/6}^{\pi/2}\frac{\cos x}{\sin x}\,dx=\pi\,[\ln(\sin x)]_{\pi/6}^{\pi/2}=\pi\left(\ln 1-\ln\frac{1}{2}\right)=\pi\ln 2$

75. $V=2\pi\int_{1/2}^{2}x\left(\frac{1}{x^2}\right)dx=2\pi\int_{1/2}^{2}\frac{1}{x}\,dx=2\pi\,[\ln|x|]_{1/2}^{2}=2\pi\left(\ln 2-\ln\frac{1}{2}\right)=2\pi(2\ln 2)=\pi\ln 2^4=\pi\ln 16$

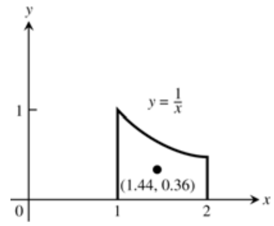
76. $V=\pi\int_0^3\left(\frac{9x}{\sqrt{x^3+9}}\right)^2 dx=27\pi\int_0^3\frac{3x^2}{x^3+9}\,dx=27\pi\left[\ln\left(x^3+9\right)\right]_0^3=27\pi(\ln 36-\ln 9)=27\pi(\ln 4+\ln 9-\ln 9)$
$=27\pi\ln 4=54\pi\ln 2$

77. (a) $y=\frac{x^2}{8}-\ln x \Rightarrow 1+(y')^2=1+\left(\frac{x}{4}-\frac{1}{x}\right)^2=1+\left(\frac{x^2-4}{4x}\right)^2=\left(\frac{x^2+4}{4x}\right)^2 \Rightarrow L=\int_4^8\sqrt{1+(y')^2}\,dx$
$=\int_4^8\frac{x^2+4}{4x}\,dx=\int_4^8\left(\frac{x}{4}+\frac{1}{x}\right)dx=\left[\frac{x^2}{8}+\ln|x|\right]_4^8=(8+\ln 8)-(2+\ln 4)=6+\ln 2$
(b) $x=\left(\frac{y}{4}\right)^2-2\ln\left(\frac{y}{4}\right) \Rightarrow \frac{dx}{dy}=\frac{y}{8}-\frac{2}{y} \Rightarrow 1+\left(\frac{dx}{dy}\right)^2=1+\left(\frac{y}{8}-\frac{2}{y}\right)^2=1+\left(\frac{y^2-16}{8y}\right)^2=\left(\frac{y^2+16}{8y}\right)^2$
$\Rightarrow L=\int_4^{12}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_4^{12}\frac{y^2+16}{8y}\,dy=\int_4^{12}\left(\frac{y}{8}+\frac{2}{y}\right)dy=\left[\frac{y^2}{16}+2\ln y\right]_4^{12}=(9+2\ln 12)-(1+2\ln 4)$
$=8+2\ln 3=8+\ln 9$

78. $L=\int_1^2\sqrt{1+\frac{1}{x^2}}\,dx \Rightarrow \frac{dy}{dx}=\frac{1}{x} \Rightarrow y=\ln|x|+C=\ln x+C$ since $x>0 \Rightarrow 0=\ln 1+C \Rightarrow C=0 \Rightarrow y=\ln x$

79. (a) $M_y = \int_1^2 x\left(\frac{1}{x}\right) dx = 1,\ M_x = \int_1^2 \left(\frac{1}{2x}\right)\left(\frac{1}{x}\right) dx$
$= \frac{1}{2}\int_1^2 \frac{1}{x^2}\, dx = \left[-\frac{1}{2x}\right]_1^2 = \frac{1}{4},\ M = \int_1^2 \frac{1}{x}\, dx$
$= \left[\ln|x|\right]_1^2 = \ln 2 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{1}{\ln 2} \approx 1.44$ and
$\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{1}{4}\right)}{\ln 2} \approx 0.36$

(b)



80. (a) $M_y = \int_1^{16} x\left(\frac{1}{\sqrt{x}}\right) dx = \int_1^{16} x^{1/2}\, dx = \frac{2}{3}\left[x^{3/2}\right]_1^{16} = 42;\ M_x = \int_1^{16}\left(\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}}\right) dx = \frac{1}{2}\int_1^{16}\frac{1}{x}\, dx$
$= \frac{1}{2}\left[\ln|x|\right]_1^{16} = \ln 4;\ M = \int_1^{16}\frac{1}{\sqrt{x}}\, dx = \left[2x^{1/2}\right]_1^{16} = 6 \Rightarrow\ \bar{x} = \frac{M_y}{M} = 7$ and $\bar{y} = \frac{M_x}{M} = \frac{\ln 4}{6}$

(b) $M_y = \int_1^{16} x\left(\frac{1}{\sqrt{x}}\right)\left(\frac{4}{\sqrt{x}}\right) dx = 4\int_1^{16} dx = 60,\ M_x = \int_1^{16}\left(\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}}\right)\left(\frac{4}{\sqrt{x}}\right) dx = 2\int_1^{16} x^{-3/2}\ dx$
$= -4\left[x^{-1/2}\right]_1^{16} = 3;\ M = \int_1^{16}\left(\frac{1}{\sqrt{x}}\right)\left(\frac{4}{\sqrt{x}}\right) dx = 4\int_1^{16}\frac{1}{x}\, dx = \left[4\ln|x|\right]_1^{16} = 4\ln 16 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{15}{\ln 16}$ and
$\bar{y} = \frac{M_x}{M} = \frac{3}{4\ln 16}$

81. $f(x) = \ln\left(x^3 - 1\right)$, domain of f: $(1, \infty) \Rightarrow f'(x) = \frac{3x^2}{x^3-1}; f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$, not in the domain: $f'(x) =$ undefined $\Rightarrow x^3 - 1 = 0 \Rightarrow x = 1$, not a domain. On $(1, \infty)$, $f'(x) > 0 \Rightarrow f$ is increasing on $(1, \infty) \Rightarrow f$ is one-to-one

82. $g(x) = \sqrt{x^2 + \ln x}$, domain of g: $x > 0.652919 \Rightarrow g'(x) = \frac{2x + \frac{1}{x}}{2\sqrt{x^2 + \ln x}} = \frac{2x^2+1}{2x\sqrt{x^2+\ln x}};\ g'(x) = 0 \Rightarrow 2x^2 + 1 = 0$ $\Rightarrow$ no real solutions; $g'(x) =$ undefined $\Rightarrow 2x\sqrt{x^2 + \ln x} = 0 \Rightarrow x = 0$ or $x \approx 0.652919$, neither in domain. On $x > 0.652919$, $g'(x) > 0 \Rightarrow g$ is increasing for $x > 0.652919 \Rightarrow$ g is one-to-one

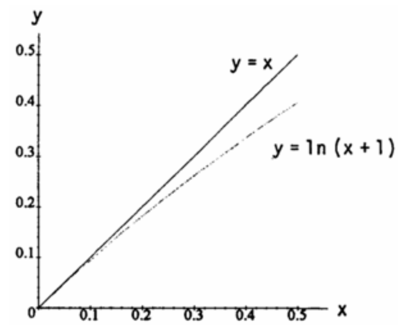
83. $\frac{dy}{dx} = 1 + \frac{1}{x}$ at $(1, 3) \Rightarrow y = x + \ln|x| + C;\ y = 3$ at $x = 1 \Rightarrow C = 2 \Rightarrow y = x + \ln|x| + 2$

84. $\frac{d^2y}{dx^2} = \sec^2 x \Rightarrow \frac{dy}{dx} = \tan x + C$ and $1 = \tan 0 + C \Rightarrow \frac{dy}{dx} = \tan x + 1 \Rightarrow y = \int(\tan x + 1)\, dx = \ln|\sec x| + x + C_1$ and $0 = \ln|\sec 0| + 0 + C_1\ \Rightarrow C_1 = 0 \Rightarrow y = \ln|\sec x| + x$

85. (a) $L(x) = f(0) + f'(0)\cdot x$, and $f(x) = \ln(1 + x) \Rightarrow f'(x)\big|_{x=0} = \frac{1}{1+x}\Big|_{x=0} = 1 \Rightarrow L(x) = \ln 1 + 1\cdot x \Rightarrow L(x) = x$

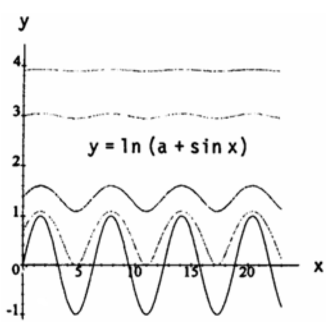
(b) Let $f(x) = \ln(x + 1)$. Since $f''(x) = -\frac{1}{(x+1)^2} < 0$ on $[0, 0.1]$, the graph of f is concave down on this interval and the largest error in the linear approximation will occur when $x = 0.1$. This error is $0.1 - \ln(1.1) \approx 0.00469$ to five decimal places.

(c) The approximation $y = x$ for $\ln(1+x)$ is best for smaller positive values of x; in particular for $0 \le x \le 0.1$ in the graph. As x increases, so does the error $x - \ln(1+x)$. From the graph an upper bound for the error is $0.5 - \ln(1+0.5) \approx 0.095$; i.e., $|E(x)| \le 0.095$ for $0 \le x \le 0.5$. Note from the graph that $0.1 - \ln(1+0.1) \approx 0.00469$ estimates the error in replacing $\ln(1+x)$ by x over $0 \le x \le 0.1$. This is consistent with the estimate given in part (b) above.



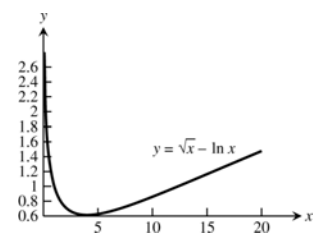
86. For all positive values of x, $\frac{d}{dx}\left[\ln \frac{a}{x}\right] = \frac{1}{\frac{a}{x}} \cdot -\frac{a}{x^2} = -\frac{1}{x}$ and $\frac{d}{dx}\left[\ln a - \ln x\right] = 0 - \frac{1}{x} = -\frac{1}{x}$. Since in $\frac{a}{x}$ and $\ln a - \ln x$ have the same derivative, then $\ln \frac{a}{x} = \ln a - \ln x + C$ for some constant C. Since this equation holds for all positive values of x, it must be true for $x = 1 \Rightarrow \ln \frac{a}{1} = \ln a - \ln 1 + C = \ln a - 0 + C \Rightarrow \ln \frac{a}{1} = \ln a + C$. Thus $\ln a = \ln a + C \Rightarrow C = 0 \Rightarrow \ln \frac{a}{x} = \ln a - \ln x$.

87. (a)



(b) $y' = \frac{\cos x}{a + \sin x}$. Since $|\sin x|$ and $|\cos x|$ are less than or equal to 1, we have for $a > 1$
$\frac{-1}{a-1} \le y' \le \frac{1}{a-1}$ for all x.
Thus, $\lim\limits_{a \to +\infty} y' = 0$ for all $x \Rightarrow$ the graph of y looks more and more horizontal as $a \to +\infty$.

88. (a) The graph of $y = \sqrt{x} - \ln x$ appears to be concave upward for all $x > 0$.



(b) $y = \sqrt{x} - \ln x \Rightarrow y' = \frac{1}{2\sqrt{x}} - \frac{1}{x} \Rightarrow y'' = -\frac{1}{4x^{3/2}} + \frac{1}{x^2} = \frac{1}{x^2}\left(-\frac{\sqrt{x}}{4} + 1\right) = 0 \Rightarrow \sqrt{x} = 4 \Rightarrow x = 16$. Thus $y'' > 0$ if $0 < x < 16$ and $y'' < 0$ if $x > 16$ so a point of inflection exists at $x = 16$. The graph of $y = \sqrt{x} - \ln x$ closely resembles a straight line $x \ge 10$ and it is impossible to discuss the point of inflection visually from the graph.

7.3 EXPONENTIAL FUNCTIONS

1. (a) $e^{-0.3t} = 27 \Rightarrow \ln e^{-0.3t} = \ln 3^3 \Rightarrow (-0.3t)\ln e = 3\ln 3 \Rightarrow -0.3t = 3\ln 3 \Rightarrow t = -10\ln 3$

(b) $e^{kt} = \frac{1}{2} \Rightarrow \ln e^{kt} = \ln 2^{-1} = kt \ln e = -\ln 2 \Rightarrow t = -\frac{\ln 2}{k}$

(c) $e^{(\ln 0.2)t} = 0.4 \Rightarrow \left(e^{\ln 0.2}\right)^t = 0.4 \Rightarrow 0.2^t = 0.4 \Rightarrow \ln 0.2^t = \ln 0.4 \Rightarrow t \ln 0.2 = \ln 0.4 \Rightarrow t = \frac{\ln 0.4}{\ln 0.2}$

2. (a) $e^{-0.01t} = 1000 \Rightarrow \ln e^{-0.01t} = \ln 1000 \Rightarrow (-0.01t)\ln e = \ln 1000 \Rightarrow -0.01t = \ln 1000 \Rightarrow t = -100 \ln 1000$

(b) $e^{kt} = \frac{1}{10} \Rightarrow \ln e^{kt} = \ln 10^{-1} \Rightarrow kt \ln e = -\ln 10 \Rightarrow kt = -\ln 10 \Rightarrow t = -\frac{\ln 10}{k}$

(c) $e^{(\ln 2)t} = \frac{1}{2} \Rightarrow \left(e^{\ln 2}\right)^t = 2^{-1} \Rightarrow 2^t = 2^{-1} \Rightarrow t = -1$

3. $e^{\sqrt{t}} = x^2 \Rightarrow \ln e^{\sqrt{t}} = \ln x^2 \Rightarrow \sqrt{t} = 2\ln x \Rightarrow t = 4(\ln x)^2$

4. $e^{x^2} e^{2x+1} = e^t \Rightarrow e^{x^2+2x+1} = e^t \Rightarrow \ln e^{x^2+2x+1} = \ln e^t \Rightarrow t = x^2 + 2x + 1$

5. $y = e^{-5x} \Rightarrow y' = e^{-5x} \frac{d}{dx}(-5x) \Rightarrow y' = -5e^{-5x}$

6. $y = e^{2x/3} \Rightarrow y' = e^{2x/3} \frac{d}{dx}\left(\frac{2x}{3}\right) \Rightarrow y' = \frac{2}{3} e^{2x/3}$

7. $y = e^{5-7x} \Rightarrow y' = e^{5-7x} \frac{d}{dx}(5-7x)\ y' = -7e^{5-7x}$

8. $y = e^{\left(4\sqrt{x}+x^2\right)} \Rightarrow y' = e^{\left(4\sqrt{x}+x^2\right)} \frac{d}{dx}\left(4\sqrt{x}+x^2\right) \Rightarrow y' = \left(\frac{2}{\sqrt{x}} + 2x\right) e^{\left(4\sqrt{x}+x^2\right)}$

9. $y = xe^x - e^x \Rightarrow y' = \left(e^x + xe^x\right) - e^x = xe^x$

10. $y = (1+2x)e^{-2x} \Rightarrow y' = 2e^{-2x} + (1+2x)e^{-2x} \frac{d}{dx}(-2x) \Rightarrow y' = 2e^{-2x} - 2(1+2x)e^{-2x} = -4xe^{-2x}$

11. $y = \left(x^2 - 2x + 2\right)e^x \Rightarrow y' = (2x-2)e^x + \left(x^2 - 2x + 2\right)e^x = x^2 e^x$

12. $y = \left(9x^2 - 6x + 2\right)e^{3x} \Rightarrow y' = (18x-6)e^{3x} + \left(9x^2 - 6x + 2\right)e^{3x} \frac{d}{dx}(3x)$
$\Rightarrow y' = (18x-6)e^{3x} + 3\left(9x^2 - 6x + 2\right)e^{3x} = 27x^2 e^{3x}$

13. $y = e^{\theta}(\sin\theta + \cos\theta) \Rightarrow y' = e^{\theta}(\sin\theta + \cos\theta) + e^{\theta}(\cos\theta - \sin\theta) = 2e^{\theta}\cos\theta$

14. $y = \ln\left(3\theta e^{-\theta}\right) = \ln 3 + \ln\theta + \ln e^{-\theta} = \ln 3 + \ln\theta - \theta \Rightarrow \frac{dy}{d\theta} = \frac{1}{\theta} - 1$

15. $y = \cos\left(e^{-\theta^2}\right) \Rightarrow \frac{dy}{d\theta} = -\sin\left(e^{-\theta^2}\right) \frac{d}{d\theta}\left(e^{-\theta^2}\right) = \left(-\sin\left(e^{-\theta^2}\right)\right)\left(e^{-\theta^2}\right) \frac{d}{d\theta}(-\theta^2) = 2\theta e^{-\theta^2} \sin\left(e^{-\theta^2}\right)$

16. $y = \theta^3 e^{-2\theta} \cos 5\theta \Rightarrow \frac{dy}{d\theta} = \left(3\theta^2\right)\left(e^{-2\theta} \cos 5\theta\right) + \left(\theta^3 \cos 5\theta\right) e^{-2\theta} \frac{d}{d\theta}(-2\theta) - 5(\sin 5\theta)\left(\theta^3 e^{-2\theta}\right)$
$= \theta^2 e^{-2\theta}\left(3\cos 5\theta - 2\theta\cos 5\theta - 5\theta\sin 5\theta\right)$

17. $y = \ln\left(3te^{-t}\right) = \ln 3 + \ln t + \ln e^{-t} = \ln 3 + \ln t - t \Rightarrow \frac{dy}{dt} = \frac{1}{t} - 1 = \frac{1-t}{t}$

18. $y=\ln\left(2e^{-t}\sin t\right)=\ln 2+\ln e^{-t}+\ln \sin t=\ln 2-t+\ln\sin t \Rightarrow \frac{dy}{dt}=-1+\left(\frac{1}{\sin t}\right)\frac{d}{dt}(\sin t)=-1+\frac{\cos t}{\sin t}=\frac{\cos t-\sin t}{\sin t}$

19. $y=\ln\frac{e^{\theta}}{1+e^{\theta}}=\ln e^{\theta}-\ln\left(1+e^{\theta}\right)=\theta-\ln\left(1+e^{\theta}\right)\Rightarrow \frac{dy}{d\theta}=1-\left(\frac{1}{1+e^{\theta}}\right)\frac{d}{d\theta}\left(1+e^{\theta}\right)=1-\frac{e^{\theta}}{1+e^{\theta}}=\frac{1}{1+e^{\theta}}$

$\frac{1}{2}\ln\theta \Rightarrow \frac{1}{2\theta}$ $\quad \frac{1}{\sqrt{\theta}}\left(\theta^{\frac{1}{2}}\right)'=\frac{1}{\sqrt{\theta}}\cdot\frac{1}{2}\theta^{-\frac{1}{2}}=\frac{1}{2\theta}$

20. $y=\ln\frac{\sqrt{\theta}}{1+\sqrt{\theta}}=\ln\sqrt{\theta}-\ln\left(1+\sqrt{\theta}\right)\Rightarrow \frac{dy}{d\theta}=\left(\frac{1}{\sqrt{\theta}}\right)\frac{d}{d\theta}\left(\sqrt{\theta}\right)-\left(\frac{1}{1+\sqrt{\theta}}\right)=\frac{d}{d\theta}\left(1+\sqrt{\theta}\right)$

$=\left(\frac{1}{\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right)-\left(\frac{1}{1+\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right)=\frac{\left(1+\sqrt{\theta}\right)-\sqrt{\theta}}{2\theta\left(1+\sqrt{\theta}\right)}=\frac{1}{2\theta\left(1+\sqrt{\theta}\right)}=\frac{1}{2\theta\left(1+\theta^{1/2}\right)}$

21. $y=e^{(\cos t+\ln t)}=e^{\cos t}e^{\ln t}=te^{\cos t}\Rightarrow \frac{dy}{dt}=e^{\cos t}+te^{\cos t}\frac{d}{dt}(\cos t)=(1-t\sin t)e^{\cos t}$

22. $y=e^{\sin t}\left(\ln t^2+1\right)\Rightarrow \frac{dy}{dt}=e^{\sin t}(\cos t)\left(\ln t^2+1\right)+\frac{2}{t}e^{\sin t}=e^{\sin t}\left[\left(\ln t^2+1\right)(\cos t)+\frac{2}{t}\right]$

23. $\int_0^{\ln x}\sin e^t\,dt \Rightarrow y'=\left(\sin e^{\ln x}\right)\cdot\frac{d}{dx}(\ln x)=\frac{\sin x}{x}$

24. $y=\int_{e^{4\sqrt{x}}}^{e^{2x}}\ln t\,dt \Rightarrow y'=\left(\ln e^{2x}\right)\cdot\frac{d}{dx}\left(e^{2x}\right)-\left(\ln e^{4\sqrt{x}}\right)\cdot\frac{d}{dx}\left(e^{4\sqrt{x}}\right)=(2x)\left(2e^{2x}\right)-\left(4\sqrt{x}\right)\left(e^{4\sqrt{x}}\right)\cdot\frac{d}{dx}\left(4\sqrt{x}\right)$

$=4xe^{2x}-4\sqrt{x}e^{4\sqrt{x}}\left(\frac{2}{\sqrt{x}}\right)=4xe^{2x}-8e^{4\sqrt{x}}$

25. $\ln y=e^{y}\sin x\Rightarrow\left(\frac{1}{y}\right)y'=\left(y'e^{y}\right)(\sin x)+e^{y}\cos x\Rightarrow y'\left(\frac{1}{y}-e^{y}\sin x\right)=e^{y}\cos x$

$\Rightarrow y'\left(\frac{1-ye^{y}\sin x}{y}\right)=e^{y}\cos x\Rightarrow y'=\frac{ye^{y}\cos x}{1-ye^{y}\sin x}$

26. $\ln xy=e^{x+y}\Rightarrow \ln x+\ln y=e^{x+y}\Rightarrow \frac{1}{x}+\left(\frac{1}{y}\right)y'=\left(1+y'\right)e^{x+y}\Rightarrow y'\left(\frac{1}{y}-e^{x+y}\right)=e^{x+y}-\frac{1}{x}$

$\Rightarrow y'\left(\frac{1-ye^{x+y}}{y}\right)=\frac{xe^{x+y}-1}{x}\Rightarrow y'=\frac{y\left(xe^{x+y}-1\right)}{x\left(1-ye^{x+y}\right)}$

27. $e^{2x}=\sin(x+3y)\Rightarrow 2e^{2x}=\left(1+3y'\right)\cos(x+3y)\Rightarrow 1+3y'=\frac{2e^{2x}}{\cos(x+3y)}\Rightarrow 3y'=\frac{2e^{2x}}{\cos(x+3y)}-1$

$\Rightarrow y'=\frac{2e^{2x}-\cos(x+3y)}{3\cos(x+3y)}$

28. $\tan y=e^{x}+\ln x\Rightarrow\left(\sec^2 y\right)y'=e^{x}+\frac{1}{x}\Rightarrow y'=\frac{\left(xe^{x}+1\right)\cos^2 y}{x}$

29. $\int\left(e^{3x}+5e^{-x}\right)dx=\frac{e^{3x}}{3}-5e^{-x}+C$

30. $\int\left(2e^{x}-3e^{-2x}\right)dx=2e^{x}+\frac{3}{2}e^{-2x}+C$

31. $\int_{\ln 2}^{\ln 3}e^{x}\,dx=\left[e^{x}\right]_{\ln 2}^{\ln 3}=e^{\ln 3}-e^{\ln 2}=3-2=1$

32. $\int_{-\ln 2}^{0}e^{-x}\,dx=\left[-e^{-x}\right]_{-\ln 2}^{0}=-e^{0}+e^{\ln 2}=-1+2=1$

33. $\int 8e^{(x+1)}\ dx = 8e^{(x+1)} + C$

34. $\int 2e^{(2x-1)}\ dx = e^{(2x-1)} + C$

35. $\int_{\ln 4}^{\ln 9} e^{x/2}\ dx = \left[2e^{x/2}\right]_{\ln 4}^{\ln 9} = 2\left[e^{(\ln 9)/2} - e^{(\ln 4)/2}\right] = 2\left(e^{\ln 3} - e^{\ln 2}\right) = 2(3-2) = 2$

36. $\int_0^{\ln 16} e^{x/4}\ dx = \left[4e^{x/4}\right]_0^{\ln 16} = 4\left(e^{(\ln 16)/4} - e^0\right) = 4\left(e^{\ln 2} - 1\right) = 4(2-1) = 4$

37. Let $u = r^{1/2} \Rightarrow du = \frac{1}{2}r^{-1/2}\ dr \Rightarrow 2\ du = r^{-1/2}\ dr$;
$\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\ dr = \int e^{r^{1/2}} \cdot r^{-1/2}\ dr = 2\int e^u\ du = 2e^u + C = 2e^{r^{1/2}} + C = 2e^{\sqrt{r}} + C$

38. Let $u = -r^{1/2} \Rightarrow du = -\frac{1}{2}r^{-1/2}\ dr \Rightarrow -2\ du = r^{-1/2}\ dr$;
$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\ dr = \int e^{-r^{1/2}} \cdot r^{-1/2}\ dr = -2\int e^u\ du = -2e^{-r^{1/2}} + C = -2e^{-\sqrt{r}} + C$

39. Let $u = -t^2 \Rightarrow du = -2t\ dt \Rightarrow -du = 2t\ dt$; $\int 2te^{-t^2}\ dt = -\int e^u du = -e^u + C = -e^{-t^2} + C$

40. Let $u = t^4 \Rightarrow du = 4t^3\ dt \Rightarrow \frac{1}{4}\ du = t^3\ dt$; $\int t^3 e^{t^4}\ dt = \frac{1}{4}\int e^u du = \frac{1}{4}e^{t^4} + C$

41. Let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}dx \Rightarrow -du = \frac{1}{x^2}\ dx$; $\int \frac{e^{1/x}}{x^2}\ dx = \int -e^u\ du = -e^u + C = -e^{1/x} + C$

42. Let $u = -x^{-2} \Rightarrow du = 2x^{-3}\ dx \Rightarrow \frac{1}{2}\ du = x^{-3}\ dx$;
$\int \frac{e^{-1/x^2}}{x^3}\ dx = \int e^{-x^{-2}} \cdot x^{-3}\ dx = \frac{1}{2}\int e^u\ du = \frac{1}{2}e^u + C = \frac{1}{2}e^{-x^{-2}} + C = \frac{1}{2}e^{-1/x^2} + C$

43. Let $u = \tan\theta \Rightarrow du = \sec^2\theta\ d\theta; \theta = 0 \Rightarrow u = 0, \theta = \frac{\pi}{4} \Rightarrow u = 1$;
$\int_0^{\pi/4}\left(1 + e^{\tan\theta}\right)\sec^2\theta\ d\theta = \int_0^{\pi/4}\sec^2\theta\ d\theta + \int_0^1 e^u du = \left[\tan\theta\right]_0^{\pi/4} + \left[e^u\right]_0^1 = \left[\tan\left(\frac{\pi}{4}\right) - \tan(0)\right] + \left(e^1 - e^0\right)$
$= (1-0) + (e-1) = e$

44. Let $u = \cot\theta \Rightarrow du = -\csc^2\theta\ d\theta; \theta = \frac{\pi}{4} \Rightarrow u = 1, \theta = \frac{\pi}{2} \Rightarrow u = 0$;
$\int_{\pi/4}^{\pi/2}\left(1 + e^{\cot\theta}\right)\csc^2\theta\ d\theta = \int_{\pi/4}^{\pi/2}\csc^2\theta\ d\theta - \int_1^0 e^u\ du = \left[-\cot\theta\right]_{\pi/4}^{\pi/2} - \left[e^u\right]_1^0 = \left[-\cot\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{4}\right)\right] - \left(e^0 - e^1\right)$
$= (0+1) - (1-e) = e$

45. Let $u = \sec\pi t \Rightarrow du = \pi\sec\pi t\tan\pi t\ dt \Rightarrow \frac{du}{\pi} = \sec\pi t\tan\pi t\ dt$;
$\int e^{\sec(\pi t)}\sec(\pi t)\tan(\pi t)\ dt = \frac{1}{\pi}\int e^u\ du = \frac{e^u}{\pi} + C = \frac{e^{\sec(\pi t)}}{\pi} + C$

46. Let $u = \csc(\pi + t) \Rightarrow du = -\csc(\pi + t)\cot(\pi + t)\ dt$;
$\int e^{\csc(\pi+t)}\csc(\pi + t)\cot(\pi + t)\ dt = -\int e^u\ du = -e^u + C = -e^{\csc(\pi+t)} + C$

47. Let $u = e^v \Rightarrow du = e^v dv \Rightarrow 2\,du = 2e^v\,dv$; $v = \ln\frac{\pi}{6} \Rightarrow u = \frac{\pi}{6}$, $v = \ln\frac{\pi}{2} \Rightarrow u = \frac{\pi}{2}$;
$\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v\,dv = 2\int_{\pi/6}^{\pi/2} \cos u\,du = [2\sin u]_{\pi/6}^{\pi/2} = 2\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right] = 2\left(1-\frac{1}{2}\right) = 1$

48. Let $u = e^{x^2} \Rightarrow du = 2xe^{x^2}dx$; $x = 0 \Rightarrow u = 1$, $x = \sqrt{\ln\pi} \Rightarrow u = e^{\ln\pi} = \pi$;
$\int_0^{\sqrt{\ln\pi}} 2xe^{x^2}\cos\left(e^{x^2}\right)dx = \int_1^{\pi} \cos u\,du = [\sin u]_1^{\pi} = \sin(\pi) - \sin(1) = -\sin(1) \approx -0.84147$

49. Let $u = 1 + e^r \Rightarrow du = e^r\,dr$; $\int \frac{e^r}{1+e^r}dr = \int \frac{1}{u}du = \ln|u| + C = \ln\left(1+e^r\right) + C$

50. $\int \frac{1}{1+e^x}dx = \int \frac{e^{-x}}{e^{-x}+1}dx$; let $u = e^{-x} + 1 \Rightarrow du = -e^{-x}\,dx \Rightarrow -du = e^{-x}\,dx$;
$\int \frac{e^{-x}}{e^{-x}+1}dx = -\int \frac{1}{u}du = -\ln|u| + C = -\ln\left(e^{-x}+1\right) + C$

51. $\frac{dy}{dt} = e^t \sin\left(e^t - 2\right) \Rightarrow y = \int e^t \sin\left(e^t-2\right)dt$;
let $u = e^t - 2 \Rightarrow du = e^t\,dt \Rightarrow y = \int \sin u\,du = -\cos u + C = -\cos\left(e^t-2\right) + C$; $y(\ln 2) = 0$
$\Rightarrow -\cos\left(e^{\ln 2} - 2\right) + C = 0 \Rightarrow -\cos(2-2) + C = 0 \Rightarrow C = \cos 0 = 1$; thus, $y = 1 - \cos\left(e^t - 2\right)$

52. $\frac{dy}{dt} = e^{-t}\sec^2\left(\pi e^{-t}\right) \Rightarrow y = \int e^{-t}\sec^2\left(\pi e^{-t}\right)dt$;
let $u = \pi e^{-t} \Rightarrow du = -\pi e^{-t}\,dt \Rightarrow -\frac{1}{\pi}du = e^{-t}\,dt \Rightarrow y = -\frac{1}{\pi}\int \sec^2 u\,du = -\frac{1}{\pi}\tan u + C$
$= -\frac{1}{\pi}\tan\left(\pi e^{-t}\right) + C$; $y(\ln 4) = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan\left(\pi e^{-\ln 4}\right) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan\left(\pi\cdot\frac{1}{\pi}\right) + C = \frac{2}{\pi}$
$\Rightarrow -\frac{1}{\pi}(1) + C = \frac{2}{\pi} \Rightarrow C = \frac{3}{\pi}$; thus, $y = \frac{3}{\pi} - \frac{1}{\pi}\tan\left(\pi e^{-t}\right)$

53. $\frac{d^2y}{dx^2} = 2e^{-x} \Rightarrow \frac{dy}{dx} = -2e^{-x} + C$; $x = 0$ and $\frac{dy}{dx} = 0 \Rightarrow 0 = -2e^0 + C \Rightarrow C = 2$; thus $\frac{dy}{dx} = -2e^{-x} + 2$
$\Rightarrow y = 2e^{-x} + 2x + C_1$; $x = 0$ and $y = 1 \Rightarrow 1 = 2e^0 + C_1 \Rightarrow C_1 = -1 \Rightarrow y = 2e^{-x} + 2x - 1 = 2\left(e^{-x} + x\right) - 1$

54. $\frac{d^2y}{dt^2} = 1 - e^{2t} \Rightarrow \frac{dy}{dt} = t - \frac{1}{2}e^{2t} + C$; $t = 1$ and $\frac{dy}{dt} = 0 \Rightarrow 0 = 1 - \frac{1}{2}e^2 + C \Rightarrow C = \frac{1}{2}e^2 - 1$; thus
$\frac{dy}{dt} = t - \frac{1}{2}e^{2t} + \frac{1}{2}e^2 - 1 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t + C_1$; $t = 1$ and $y = -1 \Rightarrow -1 = \frac{1}{2} - \frac{1}{4}e^2 + \frac{1}{2}e^2 - 1 + C_1$
$\Rightarrow C_1 = -\frac{1}{2} - \frac{1}{4}e^2 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t - \left(\frac{1}{2} + \frac{1}{4}e^2\right)$

55. $y = 2^x \Rightarrow y' = 2^x \ln 2$

56. $y = 3^{-x} \Rightarrow y' = 3^{-x}(\ln 3)(-1) = -3^{-x}\ln 3$

57. $y = 5^{\sqrt{s}} \Rightarrow \frac{dy}{ds} = 5^{\sqrt{s}}(\ln 5)\left(\frac{1}{2}s^{-1/2}\right) = \left(\frac{\ln 5}{2\sqrt{s}}\right)5^{\sqrt{s}}$

58. $y = 2^{s^2} \Rightarrow \frac{dy}{ds} = 2^{s^2}(\ln 2)2s = \left(\ln 2^2\right)\left(s2^{s^2}\right) = (\ln 4)s2^{s^2}$

59. $y = x^{\pi} \Rightarrow y' = \pi x^{(\pi-1)}$

60. $y = t^{1-e} \Rightarrow \frac{dy}{dt} = (1-e)t^{-e}$

61. $y = (\cos\theta)^{\sqrt{2}} \Rightarrow \frac{dy}{d\theta} = -\sqrt{2}(\cos\theta)^{\left(\sqrt{2}-1\right)}(\sin\theta)$

62. $y = (\ln\theta)^{\pi} \Rightarrow \frac{dy}{d\theta} = \pi(\ln\theta)^{(\pi-1)}\left(\frac{1}{\theta}\right) = \frac{\pi(\ln\theta)^{(\pi-1)}}{\theta}$

63. $y = 7^{\sec\theta}\ln 7 \Rightarrow \frac{dy}{d\theta} = \left(7^{\sec\theta}\ln 7\right)(\ln 7)(\sec\theta\tan\theta) = 7^{\sec\theta}(\ln 7)^2(\sec\theta\tan\theta)$

64. $y = 3^{\tan\theta}\ln 3 \Rightarrow \frac{dy}{d\theta} = \left(3^{\tan\theta}\ln 3\right)(\ln 3)\sec^2\theta = 3^{\tan\theta}(\ln 3)^2\sec^2\theta$

65. $y = 2^{\sin 3t} \Rightarrow \frac{dy}{dt} = \left(2^{\sin 3t}\ln 2\right)(\cos 3t)(3) = (3\cos 3t)\left(2^{\sin 3t}\right)(\ln 2)$

66. $y = 5^{-\cos 2t} \Rightarrow \frac{dy}{dt} = \left(5^{-\cos 2t}\ln 5\right)(\sin 2t)(2) = (2\sin 2t)\left(5^{-\cos 2t}\right)(\ln 5)$

67. $y = \log_2 5\theta = \frac{\ln 5\theta}{\ln 2} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 2}\right)\left(\frac{1}{5\theta}\right)(5) = \frac{1}{\theta\ln 2}$

68. $y = \log_3(1+\theta\ln 3) = \frac{\ln(1+\theta\ln 3)}{\ln 3} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 3}\right)\left(\frac{1}{1+\theta\ln 3}\right)(\ln 3) = \frac{1}{1+\theta\ln 3}$

69. $y = \frac{\ln x}{\ln 4} + \frac{\ln x^2}{\ln 4} = \frac{\ln x}{\ln 4} + 2\frac{\ln x}{\ln 4} = 3\frac{\ln x}{\ln 4} \Rightarrow y' = \frac{3}{x\ln 4}$

70. $y = \frac{x\ln e}{\ln 25} - \frac{\ln x}{2\ln 5} = \frac{x}{2\ln 5} - \frac{\ln x}{2\ln 5} = \left(\frac{1}{2\ln 5}\right)(x - \ln x) \Rightarrow y' = \left(\frac{1}{2\ln 5}\right)\left(1-\frac{1}{x}\right) = \frac{x-1}{2x\ln 5}$

71. $y = x^3\log_{10} x = x^3\left(\frac{\ln x}{\ln 10}\right) = \frac{1}{\ln 10}x^3\ln x \Rightarrow y' = \frac{1}{\ln 10}\left(x^3\cdot\frac{1}{x} + 3x^2\ln x\right) = \frac{1}{\ln 10}x^2 + 3x^2\frac{\ln x}{\ln 10}$
$= \frac{1}{\ln 10}x^2 + 3x^2\log_{10} x$

72. $y = \log_3 r\cdot\log_9 r = \left(\frac{\ln r}{\ln 3}\right)\left(\frac{\ln r}{\ln 9}\right) = \frac{\ln^2 r}{(\ln 3)(\ln 9)} \Rightarrow \frac{dy}{dr} = \left[\frac{1}{(\ln 3)(\ln 9)}\right](2\ln r)\left(\frac{1}{r}\right) = \frac{2\ln r}{r(\ln 3)(\ln 9)}$

73. $y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)^{\ln 3}}{\ln 3} = \frac{(\ln 3)\ln\left(\frac{x+1}{x-1}\right)}{\ln 3} = \ln\left(\frac{x+1}{x-1}\right) = \ln(x+1) - \ln(x-1) \Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$

74. $y = \log_5\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}} = \log_5\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2} = \frac{\ln\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2}}{\ln 5} = \left(\frac{\ln 5}{2}\right)\left[\frac{\ln\left(\frac{7x}{3x+2}\right)}{\ln 5}\right] = \frac{1}{2}\ln\left(\frac{7x}{3x+2}\right)$
$= \frac{1}{2}\ln 7x - \frac{1}{2}\ln(3x+2) \Rightarrow \frac{dy}{dx} = \frac{7}{2\cdot 7x} - \frac{3}{2\cdot(3x+2)} = \frac{(3x+2)-3x}{2x(3x+2)} = \frac{1}{x(3x+2)}$

75. $y = \theta\sin\left(\log_7\theta\right) = \theta\sin\left(\frac{\ln\theta}{\ln 7}\right) \Rightarrow \frac{dy}{d\theta} = \sin\left(\frac{\ln\theta}{\ln 7}\right) + \theta\left[\cos\left(\frac{\ln\theta}{\ln 7}\right)\right]\left(\frac{1}{\theta\ln 7}\right) = \sin\left(\log_7\theta\right) + \frac{1}{\ln 7}\cos\left(\log_7\theta\right)$

76. $y = \log_7\left(\frac{\sin\theta\cos\theta}{e^\theta 2^\theta}\right) = \frac{\ln(\sin\theta)+\ln(\cos\theta)-\ln e^\theta-\ln 2^\theta}{\ln 7} = \frac{\ln(\sin\theta)+\ln(\cos\theta)-\theta-\theta\ln 2}{\ln 7}$

$\Rightarrow \frac{dy}{d\theta} = \frac{\cos\theta}{(\sin\theta)(\ln 7)} - \frac{\sin\theta}{(\cos\theta)(\ln 7)} - \frac{1}{\ln 7} - \frac{\ln 2}{\ln 7} = \left(\frac{1}{\ln 7}\right)(\cot\theta - \tan\theta - 1 - \ln 2)$

77. $y = \log_{10} e^x = \frac{\ln e^x}{\ln 10} = \frac{x}{\ln 10} \Rightarrow y' = \frac{1}{\ln 10}$

78. $y = \frac{\theta\cdot 5^\theta}{2-\log_5\theta} = \frac{\theta\cdot 5^\theta}{2-\frac{\ln\theta}{\ln 5}} \Rightarrow y' = \frac{\left(2-\frac{\ln\theta}{\ln 5}\right)\left(\theta\cdot 5^\theta\ln 5+5^\theta(1)\right)-\left(\theta\cdot 5^\theta\right)\left(-\frac{1}{\theta\ln 5}\right)}{\left(2-\frac{\ln\theta}{\ln 5}\right)^2} = \frac{5^\theta\ln 5(2-\log_5\theta)(\theta\ln 5+1)+5^\theta}{\ln 5(2-\log_5\theta)^2}$

79. $y = 3^{\log_2 t} = 3^{(\ln t)/(\ln 2)} \Rightarrow \frac{dy}{dt} = \left[3^{(\ln t)/(\ln 2)}(\ln 3)\right]\left(\frac{1}{t\ln 2}\right) = \frac{1}{t}\left(\log_2 3\right)3^{\log_2 t}$

80. $y = 3\log_8\left(\log_2 t\right) = \frac{3\ln(\log_2 t)}{\ln 8} = \frac{3\ln\left(\frac{\ln t}{\ln 2}\right)}{\ln 8} \Rightarrow \frac{dy}{dt} = \left(\frac{3}{\ln 8}\right)\left[\frac{1}{(\ln t)/(\ln 2)}\right]\left(\frac{1}{t\ln 2}\right) = \frac{3}{t(\ln t)(\ln 8)} = \frac{1}{t(\ln t)(\ln 2)}$

81. $y = \log_2\left(8t^{\ln 2}\right) = \frac{\ln 8+\ln\left(t^{\ln 2}\right)}{\ln 2} = \frac{3\ln 2+(\ln 2)(\ln t)}{\ln 2} = 3+\ln t \Rightarrow \frac{dy}{dt} = \frac{1}{t}$

82. $y = \frac{t\ln\left(\left(e^{\ln 3}\right)^{\sin t}\right)}{\ln 3} = \frac{t\ln\left(3^{\sin t}\right)}{\ln 3} = \frac{t(\sin t)(\ln 3)}{\ln 3} = t\sin t \Rightarrow \frac{dy}{dt} = \sin t + t\cos t$

83. $\int 5^x\,dx = \frac{5^x}{\ln 5} + C$

84. Let $u = 3-3^x \Rightarrow du = -3^x\ln 3\,dx \Rightarrow -\frac{1}{\ln 3}du = 3^x\,dx$; $\int\frac{3^x}{3-3^x}dx = -\frac{1}{\ln 3}\int\frac{1}{u}du = -\frac{1}{\ln 3}\ln|u| + C = -\frac{\ln\left|3-3^x\right|}{\ln 3} + C$

85. $\int_0^1 2^{-\theta}d\theta = \int_0^1\left(\frac{1}{2}\right)^\theta d\theta = \left[\frac{\left(\frac{1}{2}\right)^\theta}{\ln\left(\frac{1}{2}\right)}\right]_0^1 = \frac{\frac{1}{2}}{\ln\left(\frac{1}{2}\right)} - \frac{1}{\ln\left(\frac{1}{2}\right)} = -\frac{\frac{1}{2}}{\ln\left(\frac{1}{2}\right)} = \frac{-1}{2(\ln 1-\ln 2)} = \frac{1}{2\ln 2}$

86. $\int_{-2}^0 5^{-\theta}d\theta = \int_{-2}^0\left(\frac{1}{5}\right)^\theta d\theta = \left[\frac{\left(\frac{1}{5}\right)^\theta}{\ln\left(\frac{1}{5}\right)}\right]_{-2}^0 = \frac{1}{\ln\left(\frac{1}{5}\right)} - \frac{\left(\frac{1}{5}\right)^{-2}}{\ln\left(\frac{1}{5}\right)} = \frac{1}{\ln\left(\frac{1}{5}\right)}(1-25) = \frac{-24}{\ln 1-\ln 5} = \frac{24}{\ln 5}$

87. Let $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$; $x = 1 \Rightarrow u = 1$, $x = \sqrt{2} \Rightarrow u = 2$;

$\int_1^{\sqrt{2}} x2^{\left(x^2\right)}\,dx = \int_1^2\left(\frac{1}{2}\right)2^u\,du = \frac{1}{2}\left[\frac{2^u}{\ln 2}\right]_1^2 = \left(\frac{1}{2\ln 2}\right)\left(2^2-2^1\right) = \frac{1}{\ln 2}$

88. Let $u = x^{1/2} \Rightarrow du = \frac{1}{2}x^{-1/2}\,dx \Rightarrow 2\,du = \frac{dx}{\sqrt{x}}$; $x = 1 \Rightarrow u = 1$, $x = 4 \Rightarrow u = 2$;

$\int_1^4\frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^4 2^{x^{1/2}}\cdot x^{-1/2}\,dx = 2\int_1^2 2^u\,du = \left[\frac{2^{(u+1)}}{\ln 2}\right]_1^2 = \left(\frac{1}{\ln 2}\right)\left(2^3-2^2\right) = \frac{4}{\ln 2}$

89. Let $u = \cos t \Rightarrow du = -\sin t\, dt \Rightarrow -du = \sin t\, dt$; $t = 0 \Rightarrow u = 1$, $t = \frac{\pi}{2} \Rightarrow u = 0$;

$$\int_0^{\pi/2} 7^{\cos t} \sin t\, dt = -\int_1^0 7^u\, du = \left[-\frac{7^u}{\ln 7}\right]_1^0 = \left(\frac{-1}{\ln 7}\right)\left(7^0 - 7\right) = \frac{6}{\ln 7}$$

90. Let $u = \tan t \Rightarrow du = \sec^2 t\, dt$; $t = 0 \Rightarrow u = 0$, $t = \frac{\pi}{4} \Rightarrow u = 1$;

$$\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t\, dt = \int_0^1 \left(\frac{1}{3}\right)^u du = \left[\frac{\left(\frac{1}{3}\right)^u}{\ln\left(\frac{1}{3}\right)}\right]_0^1 = \left(-\frac{1}{\ln 3}\right)\left[\left(\frac{1}{3}\right)^1 - \left(\frac{1}{3}\right)^0\right] = \frac{2}{3 \ln 3}$$

91. Let $u = x^{2x} \Rightarrow \ln u = 2x \ln x \Rightarrow \frac{1}{u}\frac{du}{dx} = 2\ln x + (2x)\left(\frac{1}{x}\right) \Rightarrow \frac{du}{dx} = 2u(\ln x + 1) \Rightarrow \frac{1}{2}\, du = x^{2x}(1 + \ln x)\, dx$;

$x = 2 \Rightarrow u = 2^4 = 16$, $x = 4 \Rightarrow u = 4^8 = 65,536$;

$$\int_2^4 x^{2x}(1 + \ln x)\, dx = \frac{1}{2}\int_{16}^{65,536} du = \frac{1}{2}\left[u\right]_{16}^{65,536} = \frac{1}{2}(65,536 - 16) = \frac{65,520}{2} = 32,760$$

92. Let $u = 1 + 2^{x^2} \Rightarrow du = 2x^2(2x)\ln 2 dx \Rightarrow \frac{1}{2\ln 2}\, du = 2^{x^2}\, x\, dx$

$$\int \frac{x 2^{x^2}}{1 + 2^{x^2}}\, dx = \frac{1}{2\ln 2}\int \frac{1}{u}\, du = \frac{1}{2\ln 2}\ln|u| + C = \frac{\ln\left(1 + 2^{x^2}\right)}{2\ln 2} + C$$

93. $\int 3x^{\sqrt{3}}\, dx = \frac{3x^{\left(\sqrt{3}+1\right)}}{\sqrt{3}+1} + C$

94. $\int x^{\left(\sqrt{2}-1\right)}\, dx = \frac{x^{\sqrt{2}}}{\sqrt{2}} + C$

95. $\int_0^3 \left(\sqrt{2} + 1\right) x^{\sqrt{2}}\, dx = \left[x^{\left(\sqrt{2}+1\right)}\right]_0^3 = 3^{\left(\sqrt{2}+1\right)}$

96. $\int_1^e x^{(\ln 2) - 1}\, dx = \left[\frac{x^{\ln 2}}{\ln 2}\right]_1^e = \frac{e^{\ln 2} - 1^{\ln 2}}{\ln 2} = \frac{2-1}{\ln 2} = \frac{1}{\ln 2}$

97. $\int \frac{\log_{10} x}{x}\, dx = \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx$; $\left[u = \ln x \Rightarrow du = \frac{1}{x}\, dx\right]$

$\to \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx = \frac{1}{\ln 10}\int u\, du = \left(\frac{1}{\ln 10}\right)\left(\frac{1}{2}u^2\right) + C = \frac{(\ln x)^2}{2\ln 10} + C$

98. $\int_1^4 \frac{\log_2 x}{x}\, dx = \int_1^4 \left(\frac{\ln x}{\ln 2}\right)\left(\frac{1}{x}\right) dx$; $\left[u = \ln x \Rightarrow du = \frac{1}{x}\, dx;\ x = 1 \Rightarrow u = 0,\ x = 4 \Rightarrow u = \ln 4\right]$

$\to \int_1^4 \left(\frac{\ln x}{\ln 2}\right)\left(\frac{1}{x}\right) dx = \int_0^{\ln 4}\left(\frac{1}{\ln 2}\right) u\, du = \left(\frac{1}{\ln 2}\right)\left[\frac{1}{2}u^2\right]_0^{\ln 4} = \left(\frac{1}{\ln 2}\right)\left[\frac{1}{2}(\ln 4)^2\right] = \frac{(\ln 4)^2}{2\ln 2} = \frac{(\ln 4)^2}{\ln 4} = \ln 4$

99. $\int_1^4 \frac{\ln 2 \log_2 x}{x}\, dx = \int_1^4 \left(\frac{\ln 2}{x}\right)\left(\frac{\ln x}{\ln 2}\right) dx = \int_1^4 \frac{\ln x}{x}\, dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}[(\ln 4)^2 - (\ln 1)^2] = \frac{1}{2}(\ln 4)^2$

$= \frac{1}{2}(2\ln 2)^2 = 2(\ln 2)^2$

100. $\int_1^e \frac{2\ln 10\left(\log_{10} x\right)}{x}\, dx = \int_1^e \frac{(\ln 10)(2\ln x)}{(\ln 10)}\left(\frac{1}{x}\right) dx = \left[(\ln x)^2\right]_1^e = (\ln e)^2 - (\ln 1)^2 = 1$

101. $\int_0^2 \frac{\log_2(x+2)}{x+2}\,dx = \frac{1}{\ln 2}\int_0^2 [\ln(x+2)]\left(\frac{1}{x+2}\right)dx = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln(x+2))^2}{2}\right]_0^2 = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln 4)^2}{2} - \frac{(\ln 2)^2}{2}\right]$

$= \left(\frac{1}{\ln 2}\right)\left[\frac{4(\ln 2)^2}{2} - \frac{(\ln 2)^2}{2}\right] = \frac{3}{2}\ln 2$

102. $\int_{1/10}^{10} \frac{\log_{10}(10x)}{x}\,dx = \frac{10}{\ln 10}\int_{1/10}^{10} [\ln(10x)]\left(\frac{1}{10x}\right)dx = \left(\frac{10}{\ln 10}\right)\left[\frac{(\ln(10x))^2}{20}\right]_{1/10}^{10} = \left(\frac{10}{\ln 10}\right)\left[\frac{(\ln 100)^2}{20} - \frac{(\ln 1)^2}{2}\right]$

$= \left(\frac{10}{\ln 10}\right)\left[\frac{4(\ln 10)^2}{20}\right] = 2\ln 10$

103. $\int_0^9 \frac{2\log_{10}(x+1)}{x+1}\,dx = \frac{2}{\ln 10}\int_0^9 \ln(x+1)\left(\frac{1}{x+1}\right)dx = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln(x+1))^2}{2}\right]_0^9 = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln 10)^2}{2} - \frac{(\ln 1)^2}{2}\right] = \ln 10$

104. $\int_2^3 \frac{2\log_2(x-1)}{x-1}\,dx = \frac{2}{\ln 2}\int_2^3 \ln(x-1)\left(\frac{1}{x-1}\right)dx = \left(\frac{2}{\ln 2}\right)\left[\frac{(\ln(x-1))^2}{2}\right]_2^3 = \left(\frac{2}{\ln 2}\right)\left[\frac{(\ln 2)^2}{2} - \frac{(\ln 1)^2}{2}\right] = \ln 2$

105. $\int \frac{dx}{x\log_{10}x} = \int\left(\frac{\ln 10}{\ln x}\right)\left(\frac{1}{x}\right)dx = (\ln 10)\int\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)dx;\ \left[u = \ln x \Rightarrow du = \frac{1}{x}\,dx\right]$

$\to (\ln 10)\int\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)dx = (\ln 10)\int \frac{1}{u}\,du = (\ln 10)\ln|u| + C = (\ln 10)\ln|\ln x| + C$

106. $\int \frac{dx}{x(\log_8 x)^2} = \int \frac{dx}{x\left(\frac{\ln x}{\ln 8}\right)^2} = (\ln 8)^2\int \frac{(\ln x)^{-2}}{x}\,dx = (\ln 8)^2\,\frac{(\ln x)^{-1}}{-1} + C = -\frac{(\ln 8)^2}{\ln x} + C$

107. $\int_1^{\ln x} \frac{1}{t}\,dt = [\ln|t|]_1^{\ln x} = \ln|\ln x| - \ln 1 = \ln(\ln x),\ x > 1$

108. $\int_1^{e^x} \frac{1}{t}\,dt = [\ln|t|]_1^{e^x} = \ln e^x - \ln 1 = x\ln e = x$

109. $\int_1^{1/x} \frac{1}{t}\,dt = [\ln|t|]_1^{1/x} = \ln\left|\frac{1}{x}\right| - \ln 1 = (\ln 1 - \ln|x|) - \ln 1 = -\ln x,\ x > 0$

110. $\frac{1}{\ln a}\int_1^x \frac{1}{t}\,dt = \left[\frac{1}{\ln a}\ln|t|\right]_1^x = \frac{\ln x}{\ln a} - \frac{\ln 1}{\ln a} = \log_a x,\ x > 0$

111. $y = (x+1)^x \Rightarrow \ln y = \ln(x+1)^x = x\ln(x+1) \Rightarrow \frac{y'}{y} = \ln(x+1) + x\cdot\frac{1}{(x+1)} \Rightarrow y' = (x+1)^x\left[\frac{x}{x+1} + \ln(x+1)\right]$

112. $y = x^2 + x^{2x} \Rightarrow y - x^2 = x^{2x} \Rightarrow \ln\left(y - x^2\right) = \ln x^{2x} = 2x\ln x \Rightarrow \frac{1}{y-x^2}\left(y' - 2x\right) = 2x\cdot\frac{1}{x} + 2\cdot\ln x = 2 + 2\ln x$

$\Rightarrow y' - 2x = \left(y - x^2\right)(2 + 2\ln x) \Rightarrow y' = \left(\left(x^2 + x^{2x}\right) - x^2\right)(2 + 2\ln x) + 2x = 2\left(x + x^{2x} + x^{2x}\ln x\right)$

113. $y = \left(\sqrt{t}\right)^t = \left(t^{1/2}\right)^t = t^{1/2} \Rightarrow \ln y = \ln t^{1/2} = \left(\frac{t}{2}\right)\ln t \Rightarrow \frac{1}{y}\frac{dy}{dt} = \left(\frac{1}{2}\right)(\ln t) + \left(\frac{t}{2}\right)\left(\frac{1}{t}\right) = \frac{\ln t}{2} + \frac{1}{2} \Rightarrow \frac{dy}{dt} = \left(\sqrt{t}\right)^t\left(\frac{\ln t}{2} + \frac{1}{2}\right)$

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114. $y = t^{\sqrt{t}} = t^{\left(t^{1/2}\right)} \Rightarrow \ln y = \ln t^{\left(t^{1/2}\right)} = \left(t^{1/2}\right)(\ln t) \Rightarrow \frac{1}{y}\frac{dy}{dt} = \left(\frac{1}{2}t^{-1/2}\right)(\ln t) + t^{1/2}\left(\frac{1}{t}\right) = \frac{\ln t + 2}{2\sqrt{t}} \Rightarrow \frac{dy}{dt} = \left(\frac{\ln t+2}{2\sqrt{t}}\right)t^{\sqrt{t}}$

115. $y = (\sin x)^x \Rightarrow \ln y = \ln(\sin x)^x = x\ln(\sin x) \Rightarrow \frac{y'}{y} = \ln(\sin x) + x\left(\frac{\cos x}{\sin x}\right) \Rightarrow y' = (\sin x)^x\left[\ln(\sin x) + x\cot x\right]$

116. $y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = (\sin x)(\ln x) \Rightarrow \frac{y'}{y} = (\cos x)(\ln x) + (\sin x)\left(\frac{1}{x}\right) = \frac{\sin x + x(\ln x)(\cos x)}{x}$
$\Rightarrow y' = x^{\sin x}\left[\frac{\sin x + x(\ln x)(\cos x)}{x}\right]$

117. $y = \sin x^x \Rightarrow y' = \cos x^x \frac{d}{dx}\left(x^x\right)$; if $u = x^x \Rightarrow \ln u = \ln x^x = x\ln x \Rightarrow \frac{u'}{u} = x\cdot\frac{1}{x} + 1\cdot\ln x = 1 + \ln x$
$\Rightarrow u' = x^x(1+\ln x) \Rightarrow y' = \cos x^x \cdot x^x(1+\ln x) = x^x\cos x^x(1+\ln x)$

118. $y = (\ln x)^{\ln x} \Rightarrow \ln y = (\ln x)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{x}\right)\ln(\ln x) + (\ln x)\left(\frac{1}{\ln x}\right)\frac{d}{dx}(\ln x) = \frac{\ln(\ln x)}{x} + \frac{1}{x}$
$\Rightarrow y' = \left(\frac{\ln(\ln x)+1}{x}\right)(\ln x)^{\ln x}$

119. $f(x) = e^x - 2x \Rightarrow f'(x) = e^x - 2$; $f'(x) = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2$; $f(0) = 1$, the absolute maximum; $f(\ln 2) = 2 - 2\ln 2 \approx 0.613706$, the absolute minimum; $f(1) = e - 2 \approx 0.71828$, a relative or local maximum since $f''(x) = e^x$ is always positive.

120. The function $f(x) = 2e^{\sin(x/2)}$ has a maximum whenever $\sin\frac{x}{2} = 1$ and a minimum whenever $\sin\frac{x}{2} = -1$. Therefore the maximums occur at $x = \pi + 2k(2\pi)$ and the minimums occur at $x = 3\pi + 2k(2\pi)$, where k is any integer. The maximum is $2e \approx 5.43656$ and the minimum is $\frac{2}{e} \approx 0.73576$.

121. $f(x) = xe^{-x} \Rightarrow f'(x) = xe^{-x}(-1) + e^{-x} = e^{-x} - xe^{-x} \Rightarrow f''(x) = -e^{-x} - \left(xe^{-x}(-1) + e^{-x}\right) = xe^{-x} - 2e^{-x}$

(a) $f'(x) = 0 \Rightarrow e^{-x} - xe^{-x} = e^{-x}(1-x) = 0 \Rightarrow e^{-x} = 0$ or $1 - x = 0 \Rightarrow x = 1$, $f(1) = (1)e^{-1} = \frac{1}{e}$; using second derivative test, $f''(1) = (1)e^{-1} - 2e^{-1} = -\frac{1}{e} < 0 \Rightarrow$ absolute maximum at $\left(1, \frac{1}{e}\right)$

(b) $f''(x) = 0 \Rightarrow xe^{-x} - 2e^{-x} = e^{-x}(x-2) = 0 \Rightarrow e^{-x} = 0$ or $x - 2 = 0 \Rightarrow x = 2$, $f(2) = (2)e^{-2} = \frac{2}{e^2}$; since $f''(1) < 0$ and $f''(3) = e^{-3}(3-2) = \frac{1}{e^3} > 0 \Rightarrow$ point of inflection at $\left(2, \frac{2}{e^2}\right)$

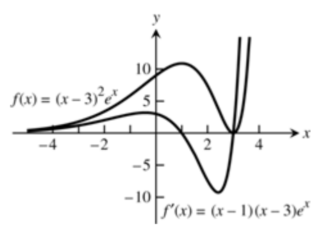
122. $f(x) = \frac{e^x}{1+e^{2x}} \Rightarrow f'(x) = \frac{\left(1+e^{2x}\right)e^x - e^x\left(2e^{2x}\right)}{\left(1+e^{2x}\right)^2} = \frac{e^x - e^{3x}}{\left(1+e^{2x}\right)^2} \Rightarrow f''(x) = \frac{\left(1+e^{2x}\right)^2\left(e^x - 3e^{3x}\right) - \left(e^x - e^{3x}\right)2\left(1+e^{2x}\right)\left(2e^{2x}\right)}{\left[\left(1+e^{2x}\right)^2\right]^2}$
$= \frac{e^x\left(1 - 6e^{2x} + e^{4x}\right)}{\left(1+e^{2x}\right)^3}$

(a) $f'(x)=0 \Rightarrow e^x - e^{3x} = 0 \Rightarrow e^x\left(1-e^{2x}\right)=0 \Rightarrow e^{2x}=1 \Rightarrow x=0;$ $f(0)=\frac{e^0}{1+e^{2(0)}}=\frac{1}{2};$ $f'(x)=$ undefined $\Rightarrow \left(1+e^{2x}\right)^2=0 \Rightarrow e^{2x}=-1 \Rightarrow$ no real solutions. Using the second derivative test, $f''(0)=\frac{e^0\left(1-6e^{2(0)}+e^{4(0)}\right)}{\left(1+e^{2(0)}\right)^3}=\frac{-4}{8}<0 \Rightarrow$ absolute maximum at $\left(0,\frac{1}{2}\right)$

(b) $f''(x)=0 \Rightarrow e^x\left(1-6e^{2x}+e^{4x}\right) \Rightarrow e^x=0$ or $1-6e^{2x}+e^{4x}=0 \Rightarrow e^{2x}=\frac{-(-6)\pm\sqrt{36-4}}{2}=3\pm2\sqrt{2},$ $\Rightarrow x=\frac{\ln\left(3+2\sqrt{2}\right)}{2}$ or $x=\frac{\ln\left(3-2\sqrt{2}\right)}{2}\cdot f\left(\frac{\ln\left(3+2\sqrt{2}\right)}{2}\right)=\frac{\sqrt{3+2\sqrt{2}}}{4+2\sqrt{2}}$ and $f\left(\frac{\ln\left(3-2\sqrt{2}\right)}{2}\right)=\frac{\sqrt{3-2\sqrt{2}}}{4-2\sqrt{2}};$ since $f''(-1)>0,\ f''(0)<0,$ and $f''(1)>0 \Rightarrow$ points of inflection at $\left(\frac{\ln\left(3+2\sqrt{2}\right)}{2},\frac{\sqrt{3+2\sqrt{2}}}{4+2\sqrt{2}}\right)$ and $\left(\frac{\ln\left(3-2\sqrt{2}\right)}{2},\frac{\sqrt{3-2\sqrt{2}}}{4-2\sqrt{2}}\right).$

123. $f(x)=x^2\ln\frac{1}{x} \Rightarrow f'(x)=2x\ln\frac{1}{x}+x^2\left(\frac{1}{\frac{1}{x}}\right)\left(-x^{-2}\right)=2x\ln\frac{1}{x}-x=-x(2\ln x+1);$ $f'(x)=0 \Rightarrow x=0$ or $\ln x=-\frac{1}{2}.$ Since $x=0$ is not in the domain of f, $x=e^{-1/2}=\frac{1}{\sqrt{e}}.$ Also, $f'(x)>0$ for $0<x<\frac{1}{\sqrt{e}}$ and $f'(x)<0$ for $x>\frac{1}{\sqrt{e}}.$ Therefore, $f\left(\frac{1}{\sqrt{e}}\right)=\frac{1}{e}\ln\sqrt{e}=\frac{1}{e}\ln e^{1/2}=\frac{1}{2e}\ln e=\frac{1}{2e}$ is the absolute maximum value of f assumed at $x=\frac{1}{\sqrt{e}}.$

124. $f(x)=(x-3)^2e^x \Rightarrow f'(x)=2(x-3)e^x+(x-3)^2e^x$ $=(x-3)e^x(2+x-3)=(x-1)(x-3)e^x;$ thus $f'(x)>0$ for $x<1$ or $x>3,$ and $f'(x)<0$ for $1<x<3 \Rightarrow f(1)=4e\approx10.87$ is a local maximum and $f(3)=0$ is a local minimum. Since $f(x)\ge0$ for all x, $f(3)=0$ is also an absolute minimum.



125. $\int_0^{\ln 3}\left(e^{2x}-e^x\right)dx=\left[\frac{e^{2x}}{2}-e^x\right]_0^{\ln 3}=\left(\frac{e^{2\ln 3}}{2}-e^{\ln 3}\right)-\left(\frac{e^0}{2}-e^0\right)=\left(\frac{9}{2}-3\right)-\left(\frac{1}{2}-1\right)=\frac{8}{2}-2=2$

126. $\int_0^{2\ln 2}\left(e^{x/2}-e^{-x/2}\right)dx=\left[2e^{x/2}+2e^{-x/2}\right]_0^{2\ln 2}=\left(2e^{\ln 2}+2e^{-\ln 2}\right)-\left(2e^0+2e^0\right)=(4+1)-(2+2)=5-4=1$

127. $L=\int_0^1\sqrt{1+\frac{e^x}{4}}\,dx \Rightarrow \frac{dy}{dx}=\frac{e^{x/2}}{2} \Rightarrow y=e^{x/2}+C;\ y(0)=0 \Rightarrow 0=e^0+C \Rightarrow C=-1 \Rightarrow y=e^{x/2}-1$

128. $S=2\pi\int_0^{\ln 2}\left(\frac{e^y+e^{-y}}{2}\right)\sqrt{1+\left(\frac{e^y-e^{-y}}{2}\right)^2}\,dy=2\pi\int_0^{\ln 2}\left(\frac{e^y+e^{-y}}{2}\right)\sqrt{1+\frac{1}{4}\left(e^{2y}-2+e^{-2y}\right)}\,dy$

$=2\pi\int_0^{\ln 2}\left(\frac{e^y+e^{-y}}{2}\right)\sqrt{\left(\frac{e^y+e^{-y}}{2}\right)^2}\,dy=2\pi\int_0^{\ln 2}\left(\frac{e^y+e^{-y}}{2}\right)^2dy=\frac{\pi}{2}\int_0^{\ln 2}\left(e^{2y}+2+e^{-2y}\right)dy$

$= \frac{\pi}{2}\left[\frac{1}{2}e^{2y} + 2y - \frac{1}{2}e^{-2y}\right]_0^{\ln 2} = \frac{\pi}{2}\left[\left(\frac{1}{2}e^{2\ln 2} + 2\ln 2 - \frac{1}{2}e^{-2\ln 2}\right) - \left(\frac{1}{2} + 0 - \frac{1}{2}\right)\right]$

$= \frac{\pi}{2}\left(\frac{1}{2}\cdot 4 + 2\ln 2 - \frac{1}{2}\cdot\frac{1}{4}\right) = \frac{\pi}{2}\left(2 - \frac{1}{8} + 2\ln 2\right) = \pi\left(\frac{15}{16} + \ln 2\right)$

129. $y = \frac{1}{2}\left(e^x + e^{-x}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(e^x - e^{-x}\right)$; $L = \int_0^1 \sqrt{1 + \left(\frac{1}{2}\left(e^x - e^{-x}\right)\right)^2}\, dx = \int_0^1 \sqrt{1 + \frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}}\, dx$

$= \int_0^1 \sqrt{\frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4}}\, dx = \int_0^1 \sqrt{\left(\frac{1}{2}\left(e^x + e^{-x}\right)\right)^2}\, dx = \int_0^1 \frac{1}{2}\left(e^x + e^{-x}\right) dx = \frac{1}{2}\left[e^x - e^{-x}\right]_0^1 = \frac{1}{2}\left(e - \frac{1}{e}\right) - 0 = \frac{e^2 - 1}{2e}$

130. $y = \ln\left(e^x - 1\right) - \ln(e^x + 1) \Rightarrow \frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = \frac{2e^x}{e^{2x} - 1}$; $L = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{2e^x}{e^{2x} - 1}\right)^2}\, dx = \int_{\ln 2}^{\ln 3} \sqrt{1 + \frac{4e^{2x}}{\left(e^{2x} - 1\right)^2}}\, dx$

$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{\left(e^{2x} - 1\right)^2}}\, dx = \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{\left(e^{2x} - 1\right)^2}}\, dx = \int_{\ln 2}^{\ln 3} \sqrt{\frac{\left(e^{2x} + 1\right)^2}{\left(e^{2x} - 1\right)^2}}\, dx = \int_{\ln 2}^{\ln 3} \frac{e^{2x} + 1}{e^{2x} - 1}\, dx = \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x} + 1}{e^x}}{\frac{e^{2x} - 1}{e^x}}\, dx$

$= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx$; $\left[\text{let } u = e^x - e^{-x} \Rightarrow du = \left(e^x + e^{-x}\right)dx,\ x = \ln 2 \Rightarrow u = e^{\ln 2} - e^{-\ln 2} = 2 - \frac{1}{2} = \frac{3}{2},\right.$

$\left. x = \ln 3 \Rightarrow u = e^{\ln 3} - e^{-\ln 3} = 3 - \frac{1}{3} = \frac{8}{3}\right] \rightarrow \int_{3/2}^{8/3} \frac{1}{u}\, du = \left[\ln|u|\right]_{3/2}^{8/3} = \ln\left(\frac{8}{3}\right) - \ln\left(\frac{3}{2}\right) = \ln\left(\frac{16}{9}\right)$

131. $y = \ln\cos x \Rightarrow \frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$; $L = \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$

$= \int_0^{\pi/4} \sec x\, dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/4} = \left(\ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right|\right) - (0) = \ln\left(\sqrt{2} + 1\right)$

132. $y = \ln\csc x \Rightarrow \frac{dy}{dx} = \frac{-\cos x \cot x}{\csc x} = -\cot x$; $L = \int_{\pi/6}^{\pi/4} \sqrt{1 + (-\cot x)^2}\, dx = \int_{\pi/6}^{\pi/4} \sqrt{1 + \cot^2 x}\, dx = \int_{\pi/6}^{\pi/4} \sqrt{\csc^2 x}\, dx$

$= \int_{\pi/6}^{\pi/4} \csc x\, dx = \left[-\ln|\csc x + \cot x|\right]_{\pi/6}^{\pi/4} = \left(-\ln\left|\csc\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right)\right|\right) + \left(\ln\left|\csc\left(\frac{\pi}{6}\right) + \cot\left(\frac{\pi}{6}\right)\right|\right)$

$= -\ln\left(\sqrt{2} + 1\right) + \ln\left(2 + \sqrt{3}\right) = \ln\left(\frac{2 + \sqrt{3}}{\sqrt{2} + 1}\right)$

133. (a) $\frac{d}{dx}(x\ln x - x + C) = x\cdot\frac{1}{x} + \ln x - 1 + 0 = \ln x$

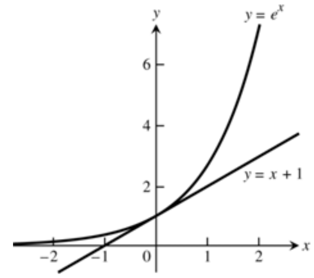
(b) average value $= \frac{1}{e-1}\int_1^e \ln x\, dx = \frac{1}{e-1}\left[x\ln x - x\right]_1^e = \frac{1}{e-1}\left[(e\ln e - e) - (1\ln 1 - 1)\right] = \frac{1}{e-1}(e - e + 1) = \frac{1}{e-1}$

134. average value $= \frac{1}{2-1}\int_1^2 \frac{1}{x}\, dx = \left[\ln|x|\right]_1^2 = \ln 2 - \ln 1 = \ln 2$

135. (a) $f(x) = e^x \Rightarrow f'(x) = e^x$; $L(x) = f(0) + f'(0)(x - 0) \Rightarrow L(x) = 1 + x$

(b) $f(0) = 1$ and $L(0) = 1 \Rightarrow$ error $= 0$; $f(0.2) = e^{0.2} \approx 1.22140$ and $L(0.2) = 1.2 \Rightarrow$ error ≈ 0.02140

(c) Since $y'' = e^x > 0$, the tangent line approximation always lies below the curve $y = e^x$. Thus $L(x) = x+1$ never overestimates e^x.



136. (a) $y = e^x \Rightarrow y'' = e^x > 0$ for all $x \Rightarrow$ the graph of $y = e^x$ is always concave upward

(b) area of the trapezoid *ABCD* $< \int_{\ln a}^{\ln b} e^x \, dx <$ area of the trapezoid *AEFD*

$\Rightarrow \frac{1}{2}(AB+CD)(\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x \, dx < \left(\frac{e^{\ln a}+e^{\ln b}}{2}\right)(\ln b - \ln a)$. Now $\frac{1}{2}(AB+CD)$ is the height of the midpoint $M = e^{(\ln a + \ln b)/2}$ since the curve containing the points *B* and *C* is linear

$\Rightarrow e^{(\ln a+\ln b)/2}(\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x dx < \left(\frac{e^{\ln a}+e^{\ln b}}{2}\right)(\ln b - \ln a)$

(c) $\int_{\ln a}^{\ln b} e^x \, dx = \left[e^x\right]_{\ln a}^{\ln b} = e^{\ln b} - e^{\ln a} = b - a$, so part (b) implies that

$e^{(\ln a+\ln b)/2}(\ln b - \ln a) < b - a < \left(\frac{e^{\ln a}+e^{\ln b}}{2}\right)(\ln b - \ln a) \Rightarrow e^{(\ln a+\ln b)/2} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$

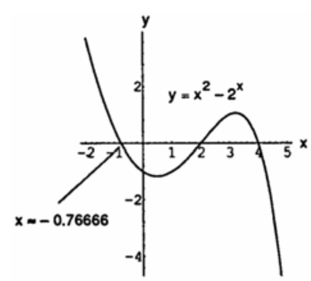
$\Rightarrow e^{\ln a/2} \cdot e^{\ln b/2} < \frac{b-a}{\ln b-\ln a} < \frac{a+b}{2} \Rightarrow \sqrt{e^{\ln a}}\sqrt{e^{\ln b}} < \frac{b-a}{\ln b-\ln a} < \frac{a+b}{2} \Rightarrow \sqrt{ab} < \frac{b-a}{\ln b-\ln a} < \frac{a+b}{2}$

137. $A = \int_{-2}^{2} \frac{2x}{1+x^2} \, dx = 2\int_0^2 \frac{2x}{1+x^2} \, dx; [u = 1+x^2 \Rightarrow du = 2x \, dx; x = 0 \Rightarrow u = 1, x = 2 \Rightarrow u = 5]$

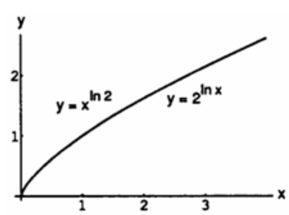
$\to A = 2\int_1^5 \frac{1}{u} du = 2\left[\ln|u|\right]_1^5 = 2(\ln 5 - \ln 1) = 2\ln 5$

138. $A = \int_{-1}^{1} 2^{(1-x)} \, dx = 2\int_{-1}^{1} \left(\frac{1}{2}\right)^x dx = 2\left[\frac{\left(\frac{1}{2}\right)^x}{\ln\left(\frac{1}{2}\right)}\right]_{-1}^{1} = -\frac{2}{\ln 2}\left(\frac{1}{2} - 2\right) = \left(-\frac{2}{\ln 2}\right)\left(-\frac{3}{2}\right) = \frac{3}{\ln 2}$

139. From zooming in on the graph at the right, we estimate the third root to be $x \approx -0.76666$

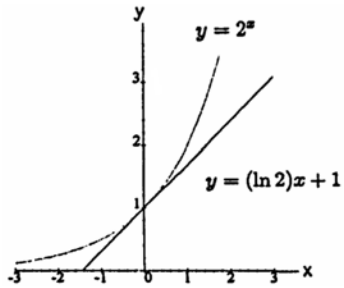


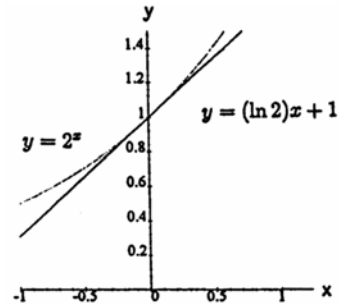
140. The functions $f(x) = x^{\ln 2}$ and $g(x) = 2^{\ln x}$ appear to have identical graphs for $x > 0$. This is no accident, because $x^{\ln 2} = e^{\ln 2 \cdot \ln x} = \left(e^{\ln 2}\right)^{\ln x} = 2^{\ln x}$.



141. (a) $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2; L(x) = \left(2^0 \ln 2\right)x + 2^0 = x\ln 2 + 1 \approx 0.69x + 1$

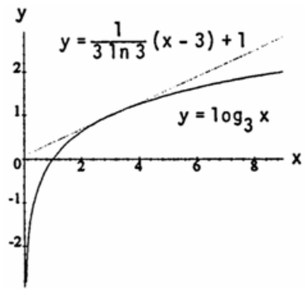
(b)

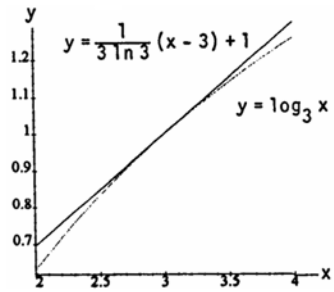




142. (a) $f(x)=\log_3 x \Rightarrow f'(x)=\frac{1}{x\ln 3}$, and $f(3)=\frac{\ln 3}{\ln 3} \Rightarrow L(x)=\frac{1}{3\ln 3}(x-3)+\frac{\ln 3}{\ln 3}=\frac{x}{3\ln 3}-\frac{1}{\ln 3}+1 \approx 0.30x+0.09$

(b)





143. (a) The point of tangency is $(p, \ln p)$ and $m_{\text{tangent}}=\frac{1}{p}$ since $\frac{dy}{dx}=\frac{1}{x}$. The tangent line passes through $(0, 0)$ $\Rightarrow$ the equation of the tangent line is $y=\frac{1}{p}x$. The tangent line also passes through $(p, \ln p)$ $\Rightarrow \ln p=\frac{1}{p}p=1 \Rightarrow p=e$, and the tangent line equation is $y=\frac{1}{e}x$.

(b) $\frac{d^2y}{dx^2}=-\frac{1}{x^2}$ for $x \neq 0 \Rightarrow y=\ln x$ is concave downward over its domain. Therefore, $y=\ln x$ lies below the graph of $y=\frac{1}{e}x$ for all $x>0, x\neq e$, and $\ln x<\frac{x}{e}$ for $x>0, x\neq e$.

(c) Multiplying by e, $e\ln x<x$ or $\ln x^e<x$.

(d) Exponentiating both sides of $\ln x^e<x$, we have $e^{\ln x^e}<e^x$, or $x^e<e^x$ for all positive $x\neq e$.

(e) Let $x=\pi$ to see that $\pi^e<e^\pi$. Therefore, e^π is bigger.

144. Using Newton's Method: $f(x)=\ln(x)-1 \Rightarrow f'(x)=\frac{1}{x} \Rightarrow x_{n+1}=x_n-\frac{\ln(x_n)-1}{\frac{1}{xn}} \Rightarrow x_{n+1}=x_n\left[2-\ln(x_n)\right]$. Then, $x_1=2, x_2=2.61370564, x_3=2.71624393$, and $x_5=2.71828183$. Many other methods may be used. For example, graph $y=\ln x-1$ and determine the zero of y.

7.4 EXPONENTIAL CHANGE AND SEPARABLE DIFFERENTIAL EQUATIONS

1. (a) $y=e^{-x} \Rightarrow y'=-e^{-x} \Rightarrow 2y'+3y=2\left(-e^{-x}\right)+3e^{-x}=e^{-x}$

(b) $y=e^{-x}+e^{-3x/2} \Rightarrow y'=-e^{-x}-\frac{3}{2}e^{-3x/2} \Rightarrow 2y'+3y=2\left(-e^{-x}-\frac{3}{2}e^{-3x/2}\right)+3\left(e^{-x}+e^{-3x/2}\right)=e^{-x}$

(c) $y=e^{-x}+Ce^{-3x/2} \Rightarrow y'=-e^{-x}-\frac{3}{2}Ce^{-3x/2} \Rightarrow 2y'+3y=2\left(-e^{-x}-\frac{3}{2}Ce^{-3x/2}\right)+3\left(e^{-x}+Ce^{-3x/2}\right)=e^{-x}$

2. (a) $y = -\frac{1}{x} \Rightarrow y' = \frac{1}{x^2} = \left(-\frac{1}{x}\right)^2 = y^2$

(b) $y = -\frac{1}{x+3} \Rightarrow y' = \frac{1}{(x+3)^2} = \left[-\frac{1}{(x+3)}\right]^2 = y^2$

(c) $y = \frac{1}{x+C} \Rightarrow y' = \frac{1}{(x+C)^2} = \left[-\frac{1}{(x+C)}\right]^2 = y^2$

3. $y = \frac{1}{x}\int_1^x \frac{e^t}{t}\,dt \Rightarrow y' = -\frac{1}{x^2}\int_1^x \frac{e^t}{t}\,dt + \left(\frac{1}{x}\right)\left(\frac{e^x}{x}\right) \Rightarrow x^2 y' = -\int_1^x \frac{e^t}{t}\,dt + e^x = -x\left(\frac{1}{x}\int_1^x \frac{e^t}{t}\,dt\right) + e^x = -xy + e^x$
$\Rightarrow x^2 y' + xy = e^x$

4. $y = \frac{1}{\sqrt{1+x^4}}\int_1^x \sqrt{1+t^4}\,dt \Rightarrow y' = -\frac{1}{2}\left[\frac{4x^3}{\left(\sqrt{1+x^4}\right)^3}\right]\int_1^x \sqrt{1+t^4}\,dt + \frac{1}{\sqrt{1+x^4}}\left(\sqrt{1+x^4}\right)$
$\Rightarrow y' = \left(\frac{-2x^3}{1+x^4}\right)\left(\frac{1}{\sqrt{1+x^4}}\int_1^x \sqrt{1+t^4}\,dt\right) + 1 \Rightarrow y' = \left(\frac{-2x^3}{1+x^4}\right)y + 1 \Rightarrow y' + \frac{2x^3}{1+x^4}\cdot y = 1$

5. $y = e^{-x}\tan^{-1}\left(2e^x\right) \Rightarrow y' = -e^{-x}\tan^{-1}\left(2e^x\right) + e^{-x}\left[\frac{1}{1+\left(2e^x\right)^2}\right]\left(2e^x\right) = -e^{-x}\tan^{-1}\left(2e^x\right) + \frac{2}{1+4e^{2x}}$
$\Rightarrow y' = -y + \frac{2}{1+4e^{2x}} \Rightarrow y' + y = \frac{2}{1+4e^{2x}}$; $y(-\ln 2) = e^{-(-\ln 2)}\tan^{-1}\left(2e^{-\ln 2}\right) = 2\tan^{-1} 1 = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

6. $y = (x-2)e^{-x^2} \Rightarrow y' = e^{-x^2} + \left(-2xe^{-x^2}\right)(x-2) \Rightarrow y' = e^{-x^2} - 2xy$; $y(2) = (2-2)e^{-2^2} = 0$

7. $y = \frac{\cos x}{x} \Rightarrow y' = \frac{-x\sin x - \cos x}{x^2} \Rightarrow y' = -\frac{\sin x}{x} - \frac{1}{x}\left(\frac{\cos x}{x}\right) \Rightarrow y' = -\frac{\sin x}{x} - \frac{y}{x} \Rightarrow xy' = -\sin x - y$
$\Rightarrow xy' + y = -\sin x$; $y\left(\frac{\pi}{2}\right) = \frac{\cos(\pi/2)}{(\pi/2)} = 0$

8. $y = \frac{x}{\ln x} \Rightarrow y' = \frac{\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} \Rightarrow y' = \frac{1}{\ln x} - \frac{1}{(\ln x)^2} \Rightarrow x^2 y' = \frac{x^2}{\ln x} - \frac{x^2}{(\ln x)^2} \Rightarrow x^2 y' = xy - y^2$; $y(e) = \frac{e}{\ln e} = e.$

9. $2\sqrt{xy}\,\frac{dy}{dx} = 1 \Rightarrow 2x^{1/2}y^{1/2}\,dy = dx \Rightarrow 2y^{1/2}\,dy = x^{-1/2}\,dx \Rightarrow \int 2y^{1/2}\,dy = \int x^{-1/2}\,dx$
$\Rightarrow 2\left(\frac{2}{3}y^{3/2}\right) = 2x^{1/2} + C_1 \Rightarrow \frac{2}{3}y^{3/2} - x^{1/2} = C$, where $C = \frac{1}{2}C_1$

10. $\frac{dy}{dx} = x^2\sqrt{y} \Rightarrow dy = x^2 y^{1/2}dx \Rightarrow y^{-1/2}dy = x^2 dx \Rightarrow \int y^{-1/2}\,dy = \int x^2\,dx \Rightarrow 2y^{1/2} = \frac{x^3}{3} + C \Rightarrow 2y^{1/2} - \frac{1}{3}x^3 = C$

11. $\frac{dy}{dx} = e^{x-y} \Rightarrow dy = e^x e^{-y}\,dx \Rightarrow e^y\,dy = e^x\,dx \Rightarrow \int e^y\,dy = \int e^x\,dx \Rightarrow e^y = e^x + C \Rightarrow e^y - e^x = C$

12. $\frac{dy}{dx} = 3x^2 e^{-y} \Rightarrow dy = 3x^2 e^{-y}dx \Rightarrow e^y\,dy = 3x^2 dx \Rightarrow \int e^y\,dy = \int 3x^2 dx \Rightarrow e^y = x^3 + C \Rightarrow e^y - x^3 = C$

13. $\frac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y} \Rightarrow dy = \left(\sqrt{y}\cos^2\sqrt{y}\right)dx \Rightarrow \frac{\sec^2\sqrt{y}}{\sqrt{y}}dy = dx \Rightarrow \int \frac{\sec^2\sqrt{y}}{\sqrt{y}}dy = \int dx.$ In the integral on the left-hand side, substitute $u = \sqrt{y} \Rightarrow du = \frac{1}{2\sqrt{y}}dy \Rightarrow 2\ du = \frac{1}{\sqrt{y}}dy,$ and we have

$\int \sec^2 u\ du = \int dx \Rightarrow 2\tan u = x + C \Rightarrow -x + 2\tan\sqrt{y} = C$

14. $\sqrt{2xy}\frac{dy}{dx} = 1 \Rightarrow dy = \frac{1}{\sqrt{2xy}}dx \Rightarrow \sqrt{2}\sqrt{y}dy = \frac{1}{\sqrt{x}}dx \Rightarrow \sqrt{2}y^{1/2}\,dy = x^{-1/2}\,dx \Rightarrow \sqrt{2}\int y^{1/2}\,dy = \int x^{-1/2}dx$

$\Rightarrow \sqrt{2}\frac{y^{3/2}}{\frac{3}{2}}\,dy = \frac{x^{1/2}}{\frac{1}{2}} + C_1 \Rightarrow \sqrt{2}y^{3/2} = 3\sqrt{x} + \frac{3}{2}C_1 \Rightarrow \sqrt{2}\left(\sqrt{y}\right)^3 - 3\sqrt{x} = C,$ where $C = \frac{3}{2}C_1$

15. $\sqrt{x}\frac{dy}{dx} = e^{y+\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}} \Rightarrow dy = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}}dx \Rightarrow e^{-y}\ dy = \frac{e^{\sqrt{x}}}{\sqrt{x}}dx \Rightarrow \int e^{-y}dy = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx.$ In the integral on the right-hand side, substitute $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2\ du = \frac{1}{\sqrt{x}}dx,$ and we have

$\int e^{-y}dy = 2\int e^u du \Rightarrow -e^{-y} = 2e^u + C_1 \Rightarrow -e^{-y} = 2e^{\sqrt{x}} + C,$ where $C = -C_1$

16. $(\sec x)\frac{dy}{dx} = e^{y+\sin x} \Rightarrow \frac{dy}{dx} = e^{y+\sin x}\cos x \Rightarrow dy = \left(e^y e^{\sin x}\cos x\right)dx \Rightarrow e^{-y}dy = e^{\sin x}\cos x\ dx$

$\Rightarrow \int e^{-y}dy = \int e^{\sin x}\cos x\ dx \Rightarrow -e^{-y} = e^{\sin x} + C_1 \Rightarrow e^{-y} + e^{\sin x} = C,$ where $C = -C_1$

17. $\frac{dy}{dx} = 2x\sqrt{1-y^2} \Rightarrow dy = 2x\sqrt{1-y^2}\ dx \Rightarrow \frac{dy}{\sqrt{1-y^2}} = 2x\ dx \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int 2x\ dx \Rightarrow \sin^{-1} y = x^2 + C$ since

$|y| < 1 \Rightarrow y = \sin\left(x^2 + C\right)$

18. $\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}} \Rightarrow dy = \frac{e^{2x-y}}{e^{x+y}}dx \Rightarrow dy = \frac{e^{2x}e^{-y}}{e^x e^y}dx = \frac{e^x}{e^{2y}}dx \Rightarrow e^{2y}\ dy = e^x\ dx \Rightarrow \int e^{2y}\ dy = \int e^x\ dx \Rightarrow \frac{e^{2y}}{2} = e^x + C_1$

$\Rightarrow e^{2y} - 2e^x = C$ where $C = 2C_1$

19. $y^2\frac{dy}{dx} = 3x^2y^3 - 6x^2 \Rightarrow y^2dy = 3x^2\left(y^3 - 2\right)dx \Rightarrow \frac{y^2}{y^3-2}\ dy = 3x^2dx \Rightarrow \int \frac{y^2}{y^3-2}dy = \int 3x^2dx$

$\Rightarrow \frac{1}{3}\ln\left|y^3 - 2\right| = x^3 + C$

20. $\frac{dy}{dx} = xy + 3x - 2y - 6 = (y+3)(x-2) \Rightarrow \frac{1}{y+3}dy = (x-2)dx \Rightarrow \int \frac{1}{y+3}dy = \int (x-2)dx$

$\Rightarrow \ln|y+3| = \frac{1}{2}x^2 - 2x + C$

21. $\frac{1}{x}\frac{dy}{dx} = ye^{x^2} + 2\sqrt{y}e^{x^2} = e^{x^2}\left(y + 2\sqrt{y}\right) \Rightarrow \frac{1}{y+2\sqrt{y}}dy = xe^{x^2}dx \Rightarrow \int \frac{1}{y+2\sqrt{y}}dy = \int xe^{x^2}dx$

$\Rightarrow \int \frac{1}{\sqrt{y}\left(\sqrt{y}+2\right)}dy = \int xe^{x^2}dx \Rightarrow 2\ln\left|\sqrt{y}+2\right| = \frac{1}{2}e^{x^2} + C \Rightarrow 4\ln\left|\sqrt{y}+2\right| = e^{x^2} + C \Rightarrow 4\ln\left(\sqrt{y}+2\right) = e^{x^2} + C$

22. $\frac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1 = \left(e^{-y}+1\right)\left(e^x+1\right) \Rightarrow \frac{1}{e^{-y}+1}dy = \left(e^x+1\right)dx \Rightarrow \int \frac{1}{e^{-y}+1}\ dy = \int\left(e^x+1\right)dx$

$\Rightarrow \int \frac{e^y}{1+e^y}dy = \int\left(e^x+1\right)dx \Rightarrow \ln\left|1+e^y\right| = e^x + x + C \Rightarrow \ln\left(1+e^y\right) = e^x + x + C$

23. (a) $y = y_0 e^{kt} \Rightarrow 0.99 y_0 = y_0 e^{1000k} \Rightarrow k = \frac{\ln 0.99}{1000} \approx -0.00001$

(b) $0.9 = e^{(-0.00001)t} \Rightarrow (-0.00001)t = \ln(0.9) \Rightarrow t = \frac{\ln(0.9)}{-0.00001} \approx 10{,}536$ years

(c) $y = y_0 e^{(20{,}000)k} \approx y_0 e^{-0.2} = y_0(0.82) \Rightarrow 82\%$

24. (a) $\frac{dp}{dh} = kp \Rightarrow p = p_0 e^{kh}$ where $p_0 = 1013$; $90 = 1013e^{20k} \Rightarrow k = \frac{\ln(90) - \ln(1013)}{20} \approx -0.121$

(b) $p = 1013e^{-6.05} \approx 2.389$ hectopascals

(c) $900 = 1013e^{(-0.121)h} \Rightarrow -0.121h = \ln\left(\frac{900}{1013}\right) \Rightarrow h = \frac{\ln(1013) - \ln(900)}{0.121} \approx 0.9777$ km

25. $\frac{dy}{dt} = -0.6y \Rightarrow y = y_0 e^{-0.6t}$; $y_0 = 100 \Rightarrow y = 100e^{-0.6t} \Rightarrow y = 100e^{-0.6} \approx 54.88$ grams when $t = 1$ h

26. $A = A_0 e^{kt} \Rightarrow 800 = 1000e^{10k} \Rightarrow k = \frac{\ln(0.8)}{10} \Rightarrow A = 1000e^{(\ln(0.8)/10)t}$, where A represents the amount of sugar that remains after time t. Thus after another 14 hours, $A = 1000e^{(\ln(0.8)/10)24} \approx 585.35$ kg

27. $L(x) = L_0 e^{-kx} \Rightarrow \frac{L_0}{2} = L_0 e^{-6k} \Rightarrow \ln\frac{1}{2} = -6k \Rightarrow k = \frac{\ln 2}{6} \approx 0.1155 \Rightarrow L(x) = L_0 e^{-0.1155x}$; when the intensity is one-tenth of the surface value, $\frac{L_0}{10} = L_0 e^{-0.1155x} \Rightarrow \ln 10 = 0.1155x \Rightarrow x \approx 19.9$ m

28. $V(t) = V_0 e^{-t/40} \Rightarrow 0.1V_0 = V_0 e^{-t/40}$ when the voltage is 10% of its original value $\Rightarrow t = -40\ln(0.1) \approx 92.1$ s

29. $y = y_0 e^{kt}$ and $y_0 = 1 \Rightarrow y = e^{kt} \Rightarrow$ at $y = 2$ and $t = 0.5$ we have $2 = e^{0.5k} \Rightarrow \ln 2 = 0.5k \Rightarrow k = \frac{\ln 2}{0.5} = \ln 4$. Therefore, $y = e^{(\ln 4)t} \Rightarrow y = e^{24\ln 4} = 4^{24} = 2.81474978 \times 10^{14}$ at the end of 24 hours

30. $y = y_0 e^{kt}$ and $y(3) = 10{,}000 \Rightarrow 10{,}000 = y_0 e^{3k}$; also $y(5) = 40{,}000 = y_0 e^{5k}$. Therefore $y_0 e^{5k} = 4y_0 e^{3k} \Rightarrow e^{5k} = 4e^{3k} \Rightarrow e^{2k} = 4 \Rightarrow k = \ln 2$. Thus, $y = y_0 e^{(\ln 2)t} \Rightarrow 10{,}000 = y_0 e^{3\ln 2} = y_0 e^{\ln 8}$ $\Rightarrow 10{,}000 = 8y_0 \Rightarrow y_0 = \frac{10{,}000}{8} = 1250$

31. (a) $10{,}000e^{k(1)} = 7500 \Rightarrow e^k = 0.75 \Rightarrow k = \ln 0.75$ and $y = 10{,}000e^{(\ln 0.75)t}$. Now $1000 = 10{,}000e^{(\ln 0.75)t}$ $\Rightarrow \ln 0.1 = (\ln 0.75)t \Rightarrow t = \frac{\ln 0.1}{\ln 0.75} \approx 8.00$ years (to the nearest hundredth of a year)

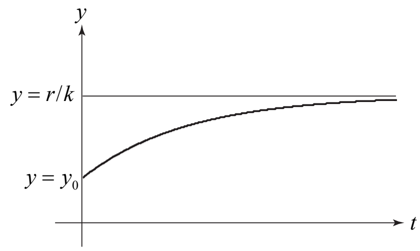
(b) $1 = 10{,}000e^{(\ln 0.75)t} \Rightarrow \ln 0.0001 = (\ln 0.75)t \Rightarrow t = \frac{\ln 0.0001}{\ln 0.75} \approx 32.02$ years (to the nearest hundredth of a year)

32. Let $z = r - ky$. Then $\frac{dz}{dt} = -k\frac{dy}{dt} = -k(r - ky) = -kz$. The equation $dz/dt = -kz$ has solution $z = ce^{-kt}$, so $r - ky = ce^{-kt}$ and $y = \frac{1}{k}\left(r - ce^{-kt}\right)$.

(a) Since $y(0) = y_0$, we have $y_0 = \frac{1}{k}(r - c)$ and thus $c = r - ky_0$. So

$$y = \frac{1}{k}\left(r - [r - ky_0]e^{-kt}\right) = \left(y_0 - \frac{r}{k}\right)e^{-kt} + \frac{r}{k}.$$

(b) Since $k > 0$, $\lim_{t\to\infty}\left[\left(y_0 - \frac{r}{k}\right)e^{-kt} + \frac{r}{k}\right] = \frac{r}{k}$.



33. Let $y(t)$ be the population at time t, so $t(0) = 1147$ and we are interested in $t(20)$. If the population continues to decline at 39% per year, the population in 20 years would be $1147\cdot(0.61)^{20} \approx 0.06 < 1$, so the species would be extinct.

34. (a) We will ignore leap years. There are $(60)(60)(24)(365) = 31{,}536{,}000$ seconds in a year. Thus, assuming exponential growth, $P = 314{,}419{,}198e^{kt}$, with t in years, and
$314{,}419{,}199 = 314{,}419{,}198e^{12k/31{,}536{,}000} \Rightarrow \; k = \frac{31{,}536{,}000}{12}\ln\left(\frac{314{,}419{,}199}{314{,}419{,}198}\right) \approx 0.0083583.$
(You don't really need to compute that logarithm: it will be very nearly equal to 1 over the denominator of the fraction.)

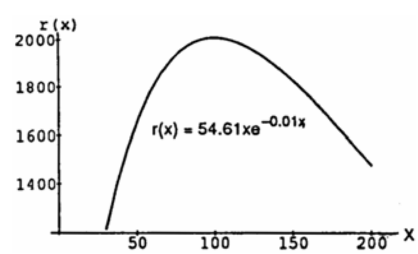
(b) In seven years, $P = 314{,}419{,}198e^{(0.0083583)(7)} \approx 333{,}664{,}000$. (We certainly can't estimate this population to better than six significant digits.)

35. $0.9P_0 = P_0e^k \Rightarrow k = \ln 0.9$; when the well's output falls to one-fifth of its present value $P = 0.2P_0$ $\Rightarrow 0.2P_0 = P_0e^{(\ln 0.9)t} \Rightarrow 0.2 = e^{(\ln 0.9)t} \Rightarrow \ln(0.2) = (\ln 0.9)t \Rightarrow t = \frac{\ln 0.2}{\ln 0.9} \approx 15.28$ years

36. (a) $\frac{dp}{dx} = -\frac{1}{100}p \Rightarrow \frac{dp}{p} = -\frac{1}{100}dx \Rightarrow \ln p = -\frac{1}{100}x + C \Rightarrow p = e^{(-0.01x+C)} = e^Ce^{-0.01x} = C_1e^{-0.01x}$;
$p(100) = 20.09 \Rightarrow 20.09 = C_1e^{(-0.01)(100)} \Rightarrow C_1 = 20.09e \approx 54.61 \Rightarrow p(x) = 54.61e^{-0.01x}$ (in dollars)

(b) $p(10) = 54.61e^{(-0.01)(10)} = \$49.41$, and $p(90) = 54.61e^{(-0.01)(90)} = \$22.20$

(c) $r(x) = xp(x) \Rightarrow r'(x) = p(x) + xp'(x)$;
$p'(x) = -.5461e^{-0.01x}$
$\Rightarrow r'(x) = (54.61 - .5461x)e^{-0.01x}$. Thus, $r'(x) = 0 \Rightarrow 54.61 = .5461x \Rightarrow x = 100$. Since $r' > 0$ for any $x < 100$ and $r' < 0$ for $x > 100$, then $r(x)$ must be a maximum at $x = 100$.



37. $A = A_0e^{kt}$ and $A_0 = 10 \Rightarrow A = 10e^{kt}$, $5 = 10e^{k(24360)} \Rightarrow k = \frac{\ln(0.5)}{24360} \approx -0.000028454 \Rightarrow A = 10e^{-0.000028454t}$, then $0.2(10) = 10e^{-0.000028454t} \Rightarrow t = \frac{\ln 0.2}{-0.000028454} \approx 56563$ years

38. $A = A_0e^{kt}$ and $\frac{1}{2}A_0 = A_0e^{139k} \Rightarrow \frac{1}{2} = e^{139k} \Rightarrow k = \frac{\ln(0.5)}{139} \approx -0.00499$; then $0.05A_0 = A_0e^{-0.00499t} \Rightarrow t = \frac{\ln 0.05}{-0.00499} \approx 600$ days

39. $y = y_0 e^{-kt} = y_0 e^{-(k)(3/k)} = y_0 e^{-3} = \frac{y_0}{e^3} < \frac{y_0}{20} = (0.05)(y_0) \Rightarrow$ after three mean lifetimes less than 5% remains

40. (a) $A = A_0 e^{-kt} \Rightarrow \frac{1}{2} = e^{-2.645k} \Rightarrow k = \frac{\ln 2}{2.645} \approx 0.262$

(b) $\frac{1}{k} \approx 3.816$ years

(c) $(0.05)A = A\exp\left(-\frac{\ln 2}{2.645}t\right) \Rightarrow -\ln 20 = \left(-\frac{\ln 2}{2.645}\right)t \Rightarrow t = \frac{2.645 \ln 20}{\ln 2} \approx 11.431$ years

41. $T - T_s = (T_0 - T_s)e^{-kt}$, $T_0 = 90°\text{C}, T_s = 20°\text{C}, T = 60°\text{C} \Rightarrow 60 - 20 = 70e^{-10k} \Rightarrow \frac{4}{7} = e^{-10k}$

$\Rightarrow k = \frac{\ln\left(\frac{7}{4}\right)}{10} \approx 0.05596$

(a) $35 - 20 = 70e^{-0.05596t} \Rightarrow t \approx 27.5$ min is the total time $\Rightarrow$ it will take $27.5 - 10 = 17.5$ minutes longer to reach $35°\text{C}$

(b) $T - T_s = (T_0 - T_s)e^{-kt}$, $T_0 = 90°\text{C}, T_s = -15°\text{C} \Rightarrow 35 + 15 = 105e^{-0.05596t} \Rightarrow t \approx 13.26$ min

42. $T - 18° = (T_0 - 18°)e^{-kt} \Rightarrow 2° - 18° = (T_0 - 18°)e^{-10k}$ and $10° - 18° = (T_0 - 18°)e^{-20k}$. Solving $-16° = (T_0 - 18°)e^{-10k}$ and $-8° = (T_0 - 18°)e^{-20k}$ simultaneously $\Rightarrow (T_0 - 18°)e^{-10k} = 2(T_0 - 18°)e^{-20k}$

$\Rightarrow e^{10k} = 2 \Rightarrow k = \frac{\ln 2}{10}$ and $-16° = \frac{T_0 - 18°}{e^{10k}} \Rightarrow -16°\left[e^{10\left(\frac{\ln 2}{10}\right)}\right] = T_0 - 18° \Rightarrow T_0 = 18° - 16°\left(e^{\ln 2}\right)$

$= 18° - 32° = -14°$

43. $T - T_s = (T_o - T_s)e^{-kt} \Rightarrow 39 - T_s = (46 - T_s)e^{-10k}$ and $33 - T_s = (46 - T_s)e^{-20k} \Rightarrow \frac{39 - T_s}{46 - T_s} = e^{-10k}$ and

$\frac{33 - T_s}{46 - T_s} = e^{-20k} = \left(e^{-10k}\right)^2 \Rightarrow \frac{33 - T_s}{46 - T_s} = \left(\frac{39 - T_s}{46 - T_s}\right)^2 \Rightarrow (33 - T_s)(46 - T_s) = (39 - T_s)^2$

$\Rightarrow 1518 - 79T_s + T_s^2 = 1521 - 78T_s + T_s^2 \Rightarrow -T_s = 3 \Rightarrow T_s = -3°\text{C}$

44. Let *x* represent how far above room temperature the silver will be 15 min from now, *y* how far above room temperature the silver will be 120 min from now, and t_0 the time the silver will be 10°C above room temperature. We then have the following time-temperature table:

time in min.	0	20 (Now)	35	140	t_0
temperature	$T_s + 70°$	$T_s + 60°$	$T_s + x$	$T_s + y$	$T_s + 10°$

$T - T_s = (T_0 - T_s)e^{-kt} \Rightarrow (60 + T_s) - T_s = [(70 + T_s) - T_s]e^{-20k} \Rightarrow 60 = 70e^{-20k} \Rightarrow k = \left(-\frac{1}{20}\right)\ln\left(\frac{6}{7}\right) \approx 0.00771$

(a) $T - T_s = (T_0 - T_s)e^{-0.00771t} \Rightarrow (T_s + x) - T_s = [(70 + T_s) - T_s]e^{-(0.00771)(35)} \Rightarrow x = 70e^{-0.26985} \approx 53.44°C$

(b) $T - T_s = (T_0 - T_s)e^{-0.00771t} \Rightarrow (T_s + y) - T_s = [(70 + T_s) - T_s]e^{-(0.00771)(140)}$

$\Rightarrow y = 70e^{-1.0794} \approx 23.79°\text{C}$

(c) $T - T_s = (T_0 - T_s)e^{-0.00771t} \Rightarrow (T_s + 10) - T_s = [(70 + T_s) - T_s]e^{-(0.00771)t_0} \Rightarrow 10 = 70e^{-0.00771t_0}$

$\Rightarrow \ln\left(\frac{1}{7}\right) = -0.00771t_0 \Rightarrow t_0 = \left(-\frac{1}{0.00771}\right)\ln\left(\frac{1}{7}\right) = 252.39 \Rightarrow 252.39 - 20 \approx 232$ minutes from now the silver will be 10°C above room temperature

45. From Example 4, the half-life of carbon-14 is 5700 yr $\Rightarrow \frac{1}{2}c_0 = c_0 e^{-k(5700)} \Rightarrow k = \frac{\ln 2}{5700} \approx 0.0001216$
$\Rightarrow c = c_0 e^{-0.0001216t} \Rightarrow (0.445)c_0 = c_0 e^{-0.0001216t} \Rightarrow t = \frac{\ln(0.445)}{-0.0001216} \approx 6659$ years

46. From Exercise 45, $k \approx 0.0001216$ for carbon-14.
(a) $c = c_0 e^{-0.0001216t} \Rightarrow (0.17)c_0 = c_0 e^{-0.0001216t} \Rightarrow t \approx 14{,}571.44$ years $\Rightarrow 12{,}571$ BC
(b) $(0.18)c_0 = c_0 e^{-0.0001216t} \Rightarrow t \approx 14{,}101.41$ years $\Rightarrow 12{,}101$ BC
(c) $(0.16)c_0 = c_0 e^{-0.0001216t} \Rightarrow t \approx 15{,}069.98$ years $\Rightarrow 13{,}070$ BC

47. From Exercise 45, $k \approx 0.0001216$ for carbon-14 $\Rightarrow y = y_0 e^{-0.0001216t}$. When $t = 5000$
$\Rightarrow y = y_0 e^{-0.0001216(5000)} \approx 0.5444 y_0 \Rightarrow \frac{y}{y_0} \approx 0.5444 \Rightarrow$ approximately 54.44% remains

48. From Exercise 45, $k \approx 0.0001216$ for carbon-14. Thus, $c = c_0 e^{-0.0001216t} \Rightarrow (0.995)c_0 = c_0 e^{-0.0001216t}$
$\Rightarrow t = \frac{\ln(0.995)}{-0.0001216} \approx 41$ years old

49. $e^{-(\ln 2/5730)t} = 0.15 \Rightarrow -\dfrac{\ln 2}{5730}t = \ln(0.15) \Rightarrow t = -\dfrac{5730\ln(0.15)}{\ln 2} \approx 15{,}683$ years

50. (a) $e^{-(\ln 2/5730)(500)} \approx 0.94131$, or about 94%.
(b) We'll assume that the error could be 1% of the original amount. If the percentage of carbon-14 remaining were 0.93131, the Ice Maiden's actual age would be $-\dfrac{5730\ln(0.93131)}{\ln 2} \approx 588$ years.

7.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

1. l'Hôpital: $\lim\limits_{x\to 2} \frac{x-2}{x^2-4} = \frac{1}{2x}\Big|_{x=2} = \frac{1}{4}$ or $\lim\limits_{x\to 2} \frac{x-2}{x^2-4} = \lim\limits_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim\limits_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

2. l'Hôpital: $\lim\limits_{x\to 0} \frac{\sin 5x}{x} = \frac{5\cos 5x}{1}\Big|_{x=0} = 5$ or $\lim\limits_{x\to 0} \frac{\sin 5x}{x} = 5\left[\lim\limits_{5x\to 0} \frac{\sin 5x}{5x}\right] = 5\cdot 1 = 5$

3. l'Hôpital: $\lim\limits_{x\to\infty} \frac{5x^2-3x}{7x^2+1} = \lim\limits_{x\to\infty} \frac{10x-3}{14x} = \lim\limits_{x\to\infty} \frac{10}{14} = \frac{5}{7}$ or $\lim\limits_{x\to\infty} \frac{5x^2-3x}{7x^2+1} = \lim\limits_{x\to\infty} \frac{5-\frac{3}{x}}{7+\frac{1}{x^2}} = \frac{5}{7}$

4. l'Hôpital: $\lim\limits_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim\limits_{x\to 1} \frac{3x^2}{12x^2-1} = \frac{3}{11}$ or $\lim\limits_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim\limits_{x\to 1} \frac{(x-1)\left(x^2+x+1\right)}{(x-1)\left(4x^2+4x+3\right)} = \lim\limits_{x\to 1} \frac{x^2+x+1}{4x^2+4x+3} = \frac{3}{11}$

5. l'Hôpital: $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \frac{\sin x}{2x} = \lim\limits_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$ or $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x\to 0}\left[\frac{(1-\cos x)}{x^2}\left(\frac{1+\cos x}{1+\cos x}\right)\right]$
$= \lim\limits_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim\limits_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x}\right)\left(\frac{1}{1+\cos x}\right)\right] = \frac{1}{2}$

6. l'Hôpital: $\lim\limits_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim\limits_{x\to\infty} \frac{4x+3}{3x^2+1} = \lim\limits_{x\to\infty} \frac{4}{6x} = 0$ or $\lim\limits_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim\limits_{x\to\infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{0}{1} = 0$

7. $\lim\limits_{x\to 2} \frac{x-2}{x^2-4} = \lim\limits_{x\to 2} \frac{1}{2x} = \frac{1}{4}$

8. $\lim\limits_{x\to -5} \frac{x^2-25}{x+5} = \lim\limits_{x\to -5} \frac{2x}{1} = -10$

9. $\lim\limits_{t\to -3} \frac{t^3-4t+15}{t^2-t-12} = \lim\limits_{t\to -3} \frac{3t^2-4}{2t-1} = \frac{3(-3)^2-4}{2(-3)-1} = -\frac{23}{7}$

10. $\lim\limits_{t\to -1} \frac{3t^3+3}{4t^3-t+3} = \lim\limits_{t\to -1} \frac{9t^2}{12t^2-1} = \frac{9}{11}$

11. $\lim\limits_{x\to\infty} \frac{5x^3-2x}{7x^3+3} = \lim\limits_{x\to\infty} \frac{15x^2-2}{21x^2} = \lim\limits_{x\to\infty} \frac{30x}{42x} = \lim\limits_{x\to\infty} \frac{30}{42} = \frac{5}{7}$

12. $\lim\limits_{x\to\infty} \frac{x-8x^2}{12x^2+5x} = \lim\limits_{x\to\infty} \frac{1-16x}{24x+5} = \lim\limits_{x\to\infty} \frac{-16}{24} = -\frac{2}{3}$

13. $\lim\limits_{t\to 0} \frac{\sin t^2}{t} = \lim\limits_{t\to 0} \frac{\left(\cos t^2\right)(2t)}{1} = 0$

14. $\lim\limits_{t\to 0} \frac{\sin 5t}{2t} = \lim\limits_{t\to 0} \frac{5\cos 5t}{2} = \frac{5}{2}$

15. $\lim\limits_{x\to 0} \frac{8x^2}{\cos x-1} = \lim\limits_{x\to 0} \frac{16x}{-\sin x} = \lim\limits_{x\to 0} \frac{16}{-\cos x} = \frac{16}{-1} = -16$

16. $\lim\limits_{x\to 0} \frac{\sin x-x}{x^3} = \lim\limits_{x\to 0} \frac{\cos x-1}{3x^2} = \lim\limits_{x\to 0} \frac{-\sin x}{6x} = \lim\limits_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$

17. $\lim\limits_{\theta\to\frac{\pi}{2}} \frac{2\theta-\pi}{\cos(2\pi-\theta)} = \lim\limits_{\theta\to\frac{\pi}{2}} \frac{2}{\sin(2\pi-\theta)} = \frac{2}{\sin\left(\frac{3\pi}{2}\right)} = -2$

18. $\lim\limits_{\theta\to -\frac{\pi}{3}} \frac{3\theta+\pi}{\sin\left(\theta+\frac{\pi}{3}\right)} = \lim\limits_{\theta\to -\frac{\pi}{3}} \frac{3}{\cos\left(\theta+\frac{\pi}{3}\right)} = 3$

19. $\lim\limits_{\theta\to\frac{\pi}{2}} \frac{1-\sin\theta}{1+\cos 2\theta} = \lim\limits_{\theta\to\frac{\pi}{2}} \frac{-\cos\theta}{-2\sin 2\theta} = \lim\limits_{\theta\to\frac{\pi}{2}} \frac{\sin\theta}{-4\cos 2\theta} = \frac{1}{(-4)(-1)} = \frac{1}{4}$

20. $\lim\limits_{x\to 1} \frac{x-1}{\ln x-\sin(\pi x)} = \lim\limits_{x\to 1} \frac{1}{\frac{1}{x}-\pi\cos(\pi x)} = \frac{1}{1+\pi}$

21. $\lim\limits_{x\to 0} \frac{x^2}{\ln(\sec x)} = \lim\limits_{x\to 0} \frac{2x}{\left(\frac{\sec x\tan x}{\sec x}\right)} = \lim\limits_{x\to 0} \frac{2x}{\tan x} = \lim\limits_{x\to 0} \frac{2}{\sec^2 x} = \frac{2}{1^2} = 2$

22. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\ln(\csc x)}{\left(x-\left(\frac{\pi}{2}\right)\right)^2} = \lim\limits_{x\to\frac{\pi}{2}} \frac{-\left(\frac{\csc x\cot x}{\csc x}\right)}{2\left(x-\left(\frac{\pi}{2}\right)\right)} = \lim\limits_{x\to\frac{\pi}{2}} \frac{-\cot x}{2\left(x-\left(\frac{\pi}{2}\right)\right)} = \lim\limits_{x\to\frac{\pi}{2}} \frac{\csc^2 x}{2} = \frac{1^2}{2} = \frac{1}{2}$

23. $\lim\limits_{t\to 0} \frac{t(1-\cos t)}{t-\sin t} = \lim\limits_{t\to 0} \frac{(1-\cos t)+t(\sin t)}{1-\cos t} = \lim\limits_{t\to 0} \frac{\sin t+(\sin t+t\cos t)}{\sin t} = \lim\limits_{t\to 0} \frac{\cos t+\cos t+\cos t-t\sin t}{\cos t} = \frac{1+1+1-0}{1} = 3$

24. $\lim\limits_{t\to 0} \frac{t\sin t}{1-\cos t} = \lim\limits_{t\to 0} \frac{\sin t+t\cos t}{\sin t} = \lim\limits_{t\to 0} \frac{\cos t+(\cos t-t\sin t)}{\cos t} = \frac{1+(1-0)}{1} = 2$

25. $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\left(x-\frac{\pi}{2}\right)\sec x=\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\left(x-\frac{\pi}{2}\right)}{\cos x}=\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\left(\frac{1}{-\sin x}\right)=\frac{1}{-1}=-1$

26. $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\left(\frac{\pi}{2}-x\right)\tan x=\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\left(\frac{\pi}{2}-x\right)}{\cot x}=\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\left(\frac{-1}{-\csc^2 x}\right)=\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\sin^2 x=1$

27. $\lim\limits_{\theta\to 0}\frac{3^{\sin\theta}-1}{\theta}=\lim\limits_{\theta\to 0}\frac{3^{\sin\theta}(\ln 3)(\cos\theta)}{1}=\frac{\left(3^0\right)(\ln 3)(1)}{1}=\ln 3$

28. $\lim\limits_{\theta\to 0}\frac{\left(\frac{1}{2}\right)^{\theta}-1}{\theta}=\lim\limits_{\theta\to 0}\frac{\left(\ln\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)^{\theta}}{1}=\ln\left(\frac{1}{2}\right)=\ln 1-\ln 2=-\ln 2$

29. $\lim\limits_{x\to 0}\frac{x\,2^x}{2^x-1}=\lim\limits_{x\to 0}\frac{(1)\left(2^x\right)+(x)(\ln 2)\left(2^x\right)}{(\ln 2)\left(2^x\right)}=\frac{1\cdot 2^0+0}{(\ln 2)\cdot 2^0}=\frac{1}{\ln 2}$

30. $\lim\limits_{x\to 0}\frac{3^x-1}{2^x-1}=\lim\limits_{x\to 0}\frac{3^x\ln 3}{2^x\ln 2}=\frac{3^0\cdot\ln 3}{2^0\cdot\ln 2}=\frac{\ln 3}{\ln 2}$

31. $\lim\limits_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}=\lim\limits_{x\to\infty}\frac{\ln(x+1)}{\left(\frac{\ln x}{\ln 2}\right)}=(\ln 2)\lim\limits_{x\to\infty}\frac{\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x}\right)}=(\ln 2)\lim\limits_{x\to\infty}\frac{x}{x+1}=(\ln 2)\lim\limits_{x\to\infty}\frac{1}{1}=\ln 2$

32. $\lim\limits_{x\to\infty}\frac{\log_2 x}{\log_3(x+3)}=\lim\limits_{x\to\infty}\frac{\left(\frac{\ln x}{\ln 2}\right)}{\left(\frac{\ln(x+3)}{\ln 3}\right)}=\left(\frac{\ln 3}{\ln 2}\right)\lim\limits_{x\to\infty}\frac{\ln x}{\ln(x+3)}=\left(\frac{\ln 3}{\ln 2}\right)\lim\limits_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x+3}\right)}=\left(\frac{\ln 3}{\ln 2}\right)\lim\limits_{x\to\infty}\frac{x+3}{x}=\left(\frac{\ln 3}{\ln 2}\right)\lim\limits_{x\to\infty}\frac{1}{1}=\frac{\ln 3}{\ln 2}$

33. $\lim\limits_{x\to 0^+}\frac{\ln\left(x^2+2x\right)}{\ln x}=\lim\limits_{x\to 0^+}\frac{\left(\frac{2x+2}{x^2+2x}\right)}{\left(\frac{1}{x}\right)}=\lim\limits_{x\to 0^+}\frac{2x^2+2x}{x^2+2x}=\lim\limits_{x\to 0^+}\frac{4x+2}{2x+2}=\lim\limits_{x\to 0^+}\frac{2}{2}=1$

34. $\lim\limits_{x\to 0^+}\frac{\ln\left(e^x-1\right)}{\ln x}=\lim\limits_{x\to 0^+}\frac{\left(\frac{e^x}{e^x-1}\right)}{\left(\frac{1}{x}\right)}=\lim\limits_{x\to 0^+}\frac{xe^x}{e^x-1}=\lim\limits_{x\to 0^+}\frac{e^x+xe^x}{e^x}=\frac{1+0}{1}=1$

35. $\lim\limits_{y\to 0}\frac{\sqrt{5y+25}-5}{y}=\lim\limits_{y\to 0}\frac{(5y+25)^{1/2}-5}{y}=\lim\limits_{y\to 0}\frac{\left(\frac{1}{2}\right)(5y+25)^{-1/2}(5)}{1}=\lim\limits_{y\to 0}\frac{5}{2\sqrt{5y+25}}=\frac{1}{2}$

36. $\lim\limits_{y\to 0}\frac{\sqrt{ay+a^2}-a}{y}=\lim\limits_{y\to 0}\frac{\left(ay+a^2\right)^{1/2}-a}{y}=\lim\limits_{y\to 0}\frac{\left(\frac{1}{2}\right)\left(ay+a^2\right)^{-1/2}(a)}{1}=\lim\limits_{y\to 0}\frac{a}{2\sqrt{ay+a^2}}=\frac{1}{2},\,a>0$

37. $\lim\limits_{x\to\infty}\left[\ln 2x-\ln(x+1)\right]=\lim\limits_{x\to\infty}\ln\left(\frac{2x}{x+1}\right)=\ln\left(\lim\limits_{x\to\infty}\frac{2x}{x+1}\right)=\ln\left(\lim\limits_{x\to\infty}\frac{2}{1}\right)=\ln 2$

38. $\lim\limits_{x\to 0^+}(\ln x-\ln\sin x)=\lim\limits_{x\to 0^+}\ln\left(\frac{x}{\sin x}\right)=\ln\left(\lim\limits_{x\to 0^+}\frac{x}{\sin x}\right)=\ln\left(\lim\limits_{x\to 0^+}\frac{1}{\cos x}\right)=\ln 1=0$

39. $\lim_{x\to 0^+} \frac{(\ln x)^2}{\ln(\sin x)} = \lim_{x\to 0^+} \frac{2(\ln x)\left(\frac{1}{x}\right)}{\frac{\cos x}{\sin x}} = \lim_{x\to 0^+} \frac{2(\ln x)(\sin x)}{x\cos x} = \lim_{x\to 0^+} \left[\frac{2(\ln x)}{\cos x}\cdot\frac{\sin x}{x}\right] = -\infty\cdot 1 = -\infty$

40. $\lim_{x\to 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \left(\frac{(3x+1)(\sin x)-x}{x\sin x}\right) = \lim_{x\to 0^+} \frac{3\sin x+(3x+1)(\cos x)-1}{\sin x + x\cos x} = \lim_{x\to 0^+} \left(\frac{3\cos x+3\cos x+(3x+1)(-\sin x)}{\cos x+\cos x - x\sin x}\right)$
$= \frac{3+3+(1)(0)}{1+1-0} = \frac{6}{2} = 3$

41. $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1^+} \left(\frac{\ln x-(x-1)}{(x-1)(\ln x)}\right) = \lim_{x\to 1^+} \left(\frac{\frac{1}{x}-1}{(\ln x)+(x-1)\left(\frac{1}{x}\right)}\right) = \lim_{x\to 1^+} \left(\frac{1-x}{(x\ln x)+x-1}\right)$
$= \lim_{x\to 1^+} \left(\frac{-1}{(\ln x+1)+1}\right) = \frac{-1}{(0+1)+1} = -\frac{1}{2}$

42. $\lim_{x\to 0^+} (\csc x - \cot x + \cos x) = \lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} + \cos x\right) = \lim_{x\to 0^+} \left(\frac{(1-\cos x)+(\sin x)(\cos x)}{\sin x}\right)$
$= \lim_{x\to 0^+} \left(\frac{\sin x+\cos^2 x-\sin^2 x}{\cos x}\right) = \frac{0+1-0}{1} = 1$

43. $\lim_{\theta\to 0} \frac{\cos\theta-1}{e^\theta-\theta-1} = \lim_{\theta\to 0} \frac{-\sin\theta}{e^\theta-1} = \lim_{\theta\to 0} \frac{-\cos\theta}{e^\theta} = -1$

44. $\lim_{h\to 0} \frac{e^h-(1+h)}{h^2} = \lim_{h\to 0} \frac{e^h-1}{2h} = \lim_{h\to 0} \frac{e^h}{2} = \frac{1}{2}$

45. $\lim_{t\to\infty} \frac{e^t+t^2}{e^t-1} = \lim_{t\to\infty} \frac{e^t+2t}{e^t} = \lim_{t\to\infty} \frac{e^t+2}{e^t} = \lim_{t\to\infty} \frac{e^t}{e^t} = 1$

46. $\lim_{x\to\infty} x^2e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

47. $\lim_{x\to 0} \frac{x-\sin x}{x\tan x} = \lim_{x\to 0} \frac{1-\cos x}{x\sec^2 x+\tan x} = \lim_{x\to 0} \frac{\sin x}{2x\sec^2 x\tan x+2\sec^2 x} = \frac{0}{2} = 0$

48. $\lim_{x\to 0} \frac{\left(e^x-1\right)^2}{x\sin x} = \lim_{x\to 0} \frac{2\left(e^x-1\right)e^x}{x\cos x+\sin x} = \lim_{x\to 0} \frac{2e^{2x}-2e^x}{x\cos x+\sin x} = \lim_{x\to 0} \frac{4e^{2x}-2e^x}{-x\sin x+2\cos x} = \frac{2}{2} = 1$

49. $\lim_{\theta\to 0} \frac{\theta-\sin\theta\cos\theta}{\tan\theta-\theta} = \lim_{\theta\to 0} \frac{1+\sin^2\theta-\cos^2\theta}{\sec^2\theta-1} = \lim_{\theta\to 0} \frac{2\sin^2\theta}{\tan^2\theta} = \lim_{\theta\to 0} 2\cos^2\theta = 2$

50. $\lim_{x\to 0} \frac{\sin 3x-3x+x^2}{\sin x\sin 2x} = \lim_{x\to 0} \frac{3\cos 3x-3+2x}{2\sin x\cos 2x+\cos x\sin 2x} = \lim_{x\to 0} \frac{3\cos 3x-3+2x}{\sin x\cos 2x+\sin 3x} = \lim_{x\to 0} \frac{-9\sin 3x+2}{-2\sin x\sin 2x+\cos x\cos 2x+3\cos 3x}$
$= \frac{2}{4} = \frac{1}{2}$

51. The limit leads to the indeterminate form 1^∞. Let $f(x) = x^{1/(1-x)} \Rightarrow \ln f(x) = \ln\left(x^{1/(1-x)}\right) = \frac{\ln x}{1-x}$. Now
$\lim_{x\to 1^+} \ln f(x) = \lim_{x\to 1^+} \frac{\ln x}{1-x} = \lim_{x\to 1^+} \frac{\left(\frac{1}{x}\right)}{-1} = -1$. Therefore $\lim_{x\to 1^+} x^{1/(1-x)} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$

52. The limit leads to the indeterminate form 1^{∞}. Let $f(x) = x^{1/(x-1)} \Rightarrow \ln f(x) = \ln\left(x^{1/(x-1)}\right) = \frac{\ln x}{x-1}$. Now

$\lim\limits_{x\to1^+} \ln f(x) = \lim\limits_{x\to1^+} \frac{\ln x}{x-1} = \lim\limits_{x\to1^+} \frac{\left(\frac{1}{x}\right)}{1} = 1$. Therefore $\lim\limits_{x\to1^+} x^{1/(x-1)} = \lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} = e^{\ln f(x)} = e^1 = e$

53. The limit leads to the indeterminate form ∞^0. Let $f(x) = (\ln x)^{1/x} \Rightarrow \ln f(x) = \ln(\ln x)^{1/x} = \frac{\ln(\ln x)}{x}$. Now

$\lim\limits_{x\to\infty} \ln f(x) = \lim\limits_{x\to\infty} \frac{\ln(\ln x)}{x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x\ln x}\right)}{1} = 0$. Therefore $\lim\limits_{x\to\infty} (\ln x)^{1/x} = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^0 = 1$

54. The limit leads to the indeterminate form 1^{∞}. Let $f(x) = (\ln x)^{1/(x-e)} \Rightarrow \ln f(x) = \frac{\ln(\ln x)}{x-e} = \lim\limits_{x\to e^+} \ln f(x)$

$= \lim\limits_{x\to e^+} \frac{\ln(\ln x)}{x-e} = \lim\limits_{x\to e^+} \frac{\left(\frac{1}{x\ln x}\right)}{1} = \frac{1}{e}$. Therefore $(\ln x)^{1/(x-e)} = \lim\limits_{x\to e^+} f(x) = \lim\limits_{x\to e^+} e^{\ln f(x)} = e^{1/e}$

55. The limit leads to the indeterminate form 0^0. Let $f(x) = x^{-1/\ln x} \Rightarrow \ln f(x) = -\frac{\ln x}{\ln x} = -1$. Therefore $\lim\limits_{x\to0^+} x^{-1/\ln x} = \lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$

56. The limit leads to the indeterminate form ∞^0. Let $f(x) = x^{1/\ln x} \Rightarrow \ln f(x) = \frac{\ln x}{\ln x} = 1$. Therefore $\lim\limits_{x\to\infty} x^{1/\ln x} = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^1 = e$

57. The limit leads to the indeterminate form ∞^0. Let $f(x) = (1+2x)^{1/(2\ln x)} \Rightarrow \ln f(x) = \frac{\ln(1+2x)}{2\ln x}$

$\Rightarrow \lim\limits_{x\to\infty} \ln f(x) = \lim\limits_{x\to\infty} \frac{\ln(1+2x)}{2\ln x} = \lim\limits_{x\to\infty} \frac{x}{1+2x} = \lim\limits_{x\to\infty} \frac{1}{2} = \frac{1}{2}$. Therefore $\lim\limits_{x\to\infty} (1+2x)^{1/(2\ln x)} = \lim\limits_{x\to\infty} f(x)$

$= \lim\limits_{x\to\infty} e^{\ln f(x)} = e^{1/2}$

58. The limit leads to the indeterminate form 1^{∞}. Let $f(x) = \left(e^x + x\right)^{1/x} \Rightarrow \ln f(x) = \frac{\ln\left(e^x+x\right)}{x}$

$\Rightarrow \lim\limits_{x\to0} \ln f(x) = \lim\limits_{x\to0} \frac{\ln\left(e^x+x\right)}{x} = \lim\limits_{x\to0} \frac{e^x+1}{e^x+x} = 2$. Therefore $\lim\limits_{x\to0} \left(e^x + x\right)^{1/x} = \lim\limits_{x\to0} f(x) = \lim\limits_{x\to0} e^{\ln f(x)} = e^2$

59. The limit leads to the indeterminate form 0^0. Let $f(x) = x^x \Rightarrow \ln f(x) = x\ln x \Rightarrow \ln f(x) = \frac{\ln x}{\left(\frac{1}{x}\right)}$

$= \lim\limits_{x\to0^+} \ln f(x) = \lim\limits_{x\to0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim\limits_{x\to0^+} (-x) = 0$. Therefore $\lim\limits_{x\to0^+} x^x = \lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} e^{\ln f(x)}$

$= e^0 = 1$

$f(x) = (1+x^{-1})^x$

$\ln f(x) = x \ln(1+x^{-1})$

60. The limit leads to the indeterminate form ∞^0. Let $f(x) = \left(1+\frac{1}{x}\right)^x \Rightarrow \ln f(x) = \frac{\ln\left(1+x^{-1}\right)}{x^{-1}} \Rightarrow \lim\limits_{x\to0^+} \ln f(x)$

$= \lim\limits_{x\to0^+} \frac{\left(\frac{-x^{-2}}{1+x^{-1}}\right)}{-x^{-2}} = \lim\limits_{x\to0^+} \frac{1}{1+x^{-1}} = \lim\limits_{x\to0^+} \frac{x}{x+1} = 0$. Therefore $\lim\limits_{x\to0^+} \left(1+\frac{1}{x}\right)^x = \lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} e^{\ln f(x)} = e^0 = 1$

61. The limit leads to the indeterminate form 1^{∞}. Let $f(x)=\left(\frac{x+2}{x-1}\right)^x \Rightarrow \ln f(x)=\ln\left(\frac{x+2}{x-1}\right)^x = x\ln\left(\frac{x+2}{x+1}\right)$

$\Rightarrow \lim\limits_{x\to\infty} \ln f(x) = \lim\limits_{x\to\infty} x\ln\left(\frac{x+2}{x-1}\right) = \lim\limits_{x\to\infty}\left(\frac{\ln\left(\frac{x+2}{x-1}\right)}{\frac{1}{x}}\right) = \lim\limits_{x\to\infty}\left(\frac{\ln(x+2)-\ln(x-1)}{\frac{1}{x}}\right) = \lim\limits_{x\to\infty}\left(\frac{\frac{1}{x+2}-\frac{1}{x-1}}{-\frac{1}{x^2}}\right) = \lim\limits_{x\to\infty}\left(\frac{\frac{-3}{(x+2)(x-1)}}{-\frac{1}{x^2}}\right)$

$= \lim\limits_{x\to\infty}\left(\frac{3x^2}{(x+2)(x-1)}\right) = \lim\limits_{x\to\infty}\left(\frac{6x}{2x+1}\right) = \lim\limits_{x\to\infty}\left(\frac{6}{2}\right) = 3$. Therefore, $\lim\limits_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^3$

62. The limit leads to the indeterminate form ∞^0. Let $f(x)=\left(\frac{x^2+1}{x+2}\right)^{1/x} \Rightarrow \ln f(x)=\ln\left(\frac{x^2+1}{x+2}\right)^{1/x} = \frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right)$

$\Rightarrow \lim\limits_{x\to\infty} \ln f(x) = \lim\limits_{x\to\infty} \frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right) = \lim\limits_{x\to\infty} \frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x} = \lim\limits_{x\to\infty} \frac{\ln\left(x^2+1\right)-\ln(x+2)}{x} = \lim\limits_{x\to\infty} \frac{\frac{2x}{x^2+1}-\frac{1}{x+2}}{1} = \lim\limits_{x\to\infty} \frac{x^2+4x-1}{(x^2+1)(x+2)}$

$= \lim\limits_{x\to\infty} \frac{x^2+4x-1}{x^3+2x^2+x+2} = \lim\limits_{x\to\infty} \frac{2x+4}{3x^2+4x+1} = \lim\limits_{x\to\infty} \frac{2}{6x+4} = 0$. Therefore, $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x+2}\right)^{1/x} = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} e^{\ln f(x)}$

$= e^0 = 1$

63. $\lim\limits_{x\to 0^+} x^2\ln x = \lim\limits_{x\to 0^+}\left(\frac{\ln x}{\frac{1}{x^2}}\right) = \lim\limits_{x\to 0^+}\left(\frac{\frac{1}{x}}{-\frac{2}{x^3}}\right) = \lim\limits_{x\to 0^+}\left(-\frac{x^3}{2x}\right) = \lim\limits_{x\to 0^+}\left(-\frac{3x^2}{2}\right) = 0$

64. $\lim\limits_{x\to 0^+} x(\ln x)^2 = \lim\limits_{x\to 0^+}\left(\frac{(\ln x)^2}{\frac{1}{x}}\right) = \lim\limits_{x\to 0^+}\left(\frac{2(\ln x)\frac{1}{x}}{-\frac{1}{x^2}}\right) = \lim\limits_{x\to 0^+}\left(\frac{2\ln x}{-\frac{1}{x}}\right) = \lim\limits_{x\to 0^+}\left(\frac{\frac{2}{x}}{\frac{1}{x^2}}\right) = \lim\limits_{x\to 0^+}\left(\frac{2x^2}{x}\right) = \lim\limits_{x\to 0^+}(2x) = 0$

65. $\lim\limits_{x\to 0^+} x\tan\left(\frac{\pi}{2}-x\right) = \lim\limits_{x\to 0^+}\left(\frac{x}{\cot\left(\frac{\pi}{2}-x\right)}\right) = \lim\limits_{x\to 0^+}\left(\frac{1}{\csc^2\left(\frac{\pi}{2}-x\right)}\right) = \frac{1}{1} = 1$

66. $\lim\limits_{x\to 0^+} \sin x\cdot\ln x = \lim\limits_{x\to 0^+}\left(\frac{\ln x}{\csc x}\right) = \lim\limits_{x\to 0^+}\left(\frac{\frac{1}{x}}{-\csc x\cot x}\right) = \lim\limits_{x\to 0^+}\left(-\frac{\sin x\tan x}{x}\right) = \lim\limits_{x\to 0^+}\left(-\frac{\sin x\sec^2 x+\cos x\tan x}{1}\right) = \frac{0}{1} = 0$

67. $\lim\limits_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} = \sqrt{\lim\limits_{x\to\infty} \frac{9x+1}{x+1}} = \sqrt{\lim\limits_{x\to\infty} \frac{9}{1}} = \sqrt{9} = 3$

68. $\lim\limits_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = \sqrt{\frac{1}{\lim\limits_{x\to 0^+} \frac{\sin x}{x}}} = \sqrt{\frac{1}{1}} = 1$

69. $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\sec x}{\tan x} = \lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right) = \lim\limits_{x\to\left(\frac{\pi}{2}\right)^-} \frac{1}{\sin x} = 1$

70. $\lim\limits_{x\to 0^+} \frac{\cot x}{\csc x} = \lim\limits_{x\to 0^+} \frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)} = \lim\limits_{x\to 0^+} \cos x = 1$

71. $\lim\limits_{x\to\infty} \frac{2^x-3^x}{3^x+4^x} = \lim\limits_{x\to\infty} \frac{\left(\frac{2}{3}\right)^x-1}{1+\left(\frac{4}{3}\right)^x} = 0$

72. $\lim\limits_{x\to-\infty}\frac{2^x+4^x}{5^x-2^x} = \lim\limits_{x\to-\infty}\frac{1+\left(\frac{4}{2}\right)^x}{\left(\frac{5}{2}\right)^x-1} = \lim\limits_{x\to-\infty}\frac{1+2^x}{\left(\frac{5}{2}\right)^x-1} = \frac{1+0}{0-1} = -1$

73. $\lim\limits_{x\to\infty}\frac{e^{x^2}}{xe^x} = \lim\limits_{x\to\infty}\frac{e^{x^2-x}}{x} = \lim\limits_{x\to\infty}\frac{e^{x(x-1)}}{x} = \lim\limits_{x\to\infty}\frac{e^{x(x-1)}(2x-1)}{1} = \infty$

74. $\lim\limits_{x\to0^+}\frac{x}{e^{-1/x}} = \lim\limits_{x\to0^+}\frac{e^{1/x}}{\frac{1}{x}} = \lim\limits_{x\to0^+}\frac{e^{1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim\limits_{x\to0^+} e^{1/x} = \infty$

75. Part (b) is correct because part (a) is neither in the $\frac{0}{0}$ nor $\frac{\infty}{\infty}$ form and so l'Hôpital's rule may not be used.

76. Part (b) is correct; the step $\lim\limits_{x\to0}\frac{2x-2}{2x-\cos x} = \lim\limits_{x\to0}\frac{2}{2+\sin x}$ in part (a) is false because $\lim\limits_{x\to0}\frac{2x-2}{2x-\cos x}$ is not an indeterminate quotient form.

77. Part (d) is correct, the other parts are indeterminate forms and cannot be calculated by the incorrect arithmetic

78. (a) We seek c in $(-2,0)$ so that $\frac{f'(c)}{g'(c)} = \frac{f(0)-f(-2)}{g(0)-g(-2)} = \frac{0+2}{0-4} = -\frac{1}{2}$. Since $f'(c)=1$ and $g'(c)=2c$ we have that $\frac{1}{2c} = -\frac{1}{2} \Rightarrow c = -1$.

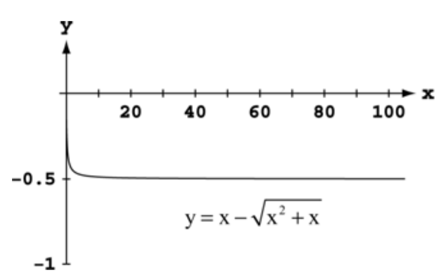
(b) We seek c in (a,b) so that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{b-a}{b^2-a^2} = \frac{1}{b+a}$. Since $f'(c)=1$ and $g'(c)=2c$ we have that $\frac{1}{2c} = \frac{1}{b+a} \Rightarrow c = \frac{b+a}{2}$.

(c) We seek c in $(0,3)$ so that $\frac{f'(c)}{g'(c)} = \frac{f(3)-f(0)}{g(3)-g(0)} = -\frac{-3-0}{9-0} = -\frac{1}{3}$. Since $f'(c)=c^2-4$ and $g'(c)=2c$ we have that $\frac{c^2-4}{2c} = -\frac{1}{3} \Rightarrow c = \frac{-1\pm\sqrt{37}}{3} \Rightarrow c = \frac{-1+\sqrt{37}}{3}$.

79. If $f(x)$ is to be continuous at $x=0$, then $\lim\limits_{x\to0} f(x) = f(0) \Rightarrow c = f(0) = \lim\limits_{x\to0}\frac{9x-3\sin 3x}{5x^3} = \lim\limits_{x\to0}\frac{9-9\cos 3x}{15x^2}$
$= \lim\limits_{x\to0}\frac{27\sin 3x}{30x} = \lim\limits_{x\to0}\frac{81\cos 3x}{30} = \frac{27}{10}$.

80. $\lim\limits_{x\to0}\left(\frac{\tan 2x}{x^3}+\frac{a}{x^2}+\frac{\sin bx}{x}\right) = \lim\limits_{x\to0}\left(\frac{\tan 2x+ax+x^2\sin bx}{x^3}\right) = \lim\limits_{x\to0}\left(\frac{2\sec^2 2x+a+bx^2\cos bx+2x\sin bx}{3x^2}\right)$ will be in $\frac{0}{0}$ form if $\lim\limits_{x\to0}(2\sec^2 2x+a+bx^2\cos bx+2x\sin bx) = a+2 = 0 \Rightarrow a = -2$; $\lim\limits_{x\to0}\left(\frac{2\sec^2 2x-2+bx^2\cos bx+2x\sin bx}{3x^2}\right)$
$= \lim\limits_{x\to0}\left(\frac{8\sec^2 2x\tan 2x-b^2x^2\sin bx+4bx\cos bx+2\sin bx}{6x}\right) = \lim\limits_{x\to0}\left(\frac{32\sec^2 2x\tan^2 2x+16\sec^4 2x-b^3x^2\cos bx-6b^2x\sin bx+6b\cos bx}{6}\right)$
$= \frac{16+6b}{6} = 0 \Rightarrow 16+6b = 0 \Rightarrow b = -\frac{8}{3}$

81. (a)



(b) The limit leads to the indeterminate form $\infty-\infty$:

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)=\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)\left(\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}\right)=\lim_{x\to\infty}\left(\frac{x^2-\left(x^2+x\right)}{x+\sqrt{x^2+x}}\right)=\lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}}$$

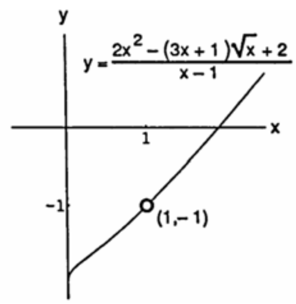
$$=\lim_{x\to\infty}\frac{-1}{1+\sqrt{1+\frac{1}{x}}}=\frac{-1}{1+\sqrt{1+0}}=-\frac{1}{2}$$

82. $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x}\right)=\lim_{x\to\infty}x\left(\frac{\sqrt{x^2+1}}{x}-\frac{\sqrt{x}}{x}\right)=\lim_{x\to\infty}x\left(\sqrt{\frac{x^2+1}{x^2}}-\sqrt{\frac{x}{x^2}}\right)=\lim_{x\to\infty}x\left(\sqrt{1+\frac{1}{x^2}}-\sqrt{\frac{1}{x}}\right)=\infty$

83. The graph indicates a limit near -1. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 1}\frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$

$$=\lim_{x\to 1}\frac{2x^2-3x^{3/2}-x^{1/2}+2}{x-1}=\lim_{x\to 1}\frac{4x-\frac{9}{2}x^{1/2}-\frac{1}{2}x^{-1/2}}{1}$$

$$=\frac{4-\frac{9}{2}-\frac{1}{2}}{1}=\frac{4-5}{1}=-1$$



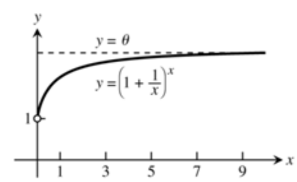
84. (a) The limit leads to the indeterminate form 1^∞. Let $f(x)=\left(1+\frac{1}{x}\right)^x \Rightarrow \ln f(x)=x\ln\left(1+\frac{1}{x}\right)\Rightarrow \lim_{x\to\infty}\ln f(x)$

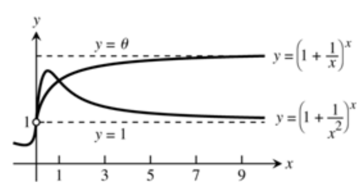
$$=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{\ln\left(1+x^{-1}\right)}{x^{-1}}=\lim_{x\to\infty}\frac{\left(\frac{-x^{-2}}{1+x^{-1}}\right)}{-x^{-2}}=\lim_{x\to\infty}\frac{1}{1+\left(\frac{1}{x}\right)}=\frac{1}{1+0}=1\Rightarrow\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to\infty}f(x)$$

$$=\lim_{x\to\infty}e^{\ln f(x)}=e^1=e$$

(b)

x	$\left(1+\frac{1}{x}\right)x$
10	2.5937424601
100	2.70481382942
1000	2.71692393224
10,000	2.71814592683
100,000	2.71826823717





Both functions have limits as x approaches infinity. The function f has a maximum but no minimum while g has no extrema. The limit of $f(x)$ leads to the indeterminate form 1^∞.

(c) Let $f(x)=\left(1+\frac{1}{x^2}\right)^x \Rightarrow \ln f(x)=x\ln\left(1+x^{-2}\right)$

$$\Rightarrow \lim_{x\to\infty}\ln f(x)=\lim_{x\to\infty}\frac{\ln\left(1+x^{-2}\right)}{x^{-1}}=\lim_{x\to\infty}\frac{\left(\frac{-2x^{-3}}{1+x^{-2}}\right)}{-x^{-2}}=\lim_{x\to\infty}\frac{2x^2}{x^3+x}=\lim_{x\to\infty}\frac{4x}{3x^2+1}=\lim_{x\to\infty}\frac{4}{6x}=0.$$

Therefore $\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x=\lim_{x\to\infty}f(x)=\lim_{x\to\infty}e^{\ln f(x)}=e^0=1$

85. Let $f(k)=\left(1+\frac{r}{k}\right)^k \Rightarrow \ln f(k)=\frac{\ln\left(1+rk^{-1}\right)}{k^{-1}} \Rightarrow \lim\limits_{k\to\infty}\frac{\ln\left(1+rk^{-1}\right)}{k^{-1}}=\lim\limits_{k\to\infty}\frac{\left(\frac{-rk^{-2}}{1+rk^{-1}}\right)}{-k^{-2}}=\lim\limits_{k\to\infty}\frac{r}{1+rk^{-1}}=\lim\limits_{k\to\infty}\frac{rk}{k+r}$
$=\lim\limits_{k\to\infty}\frac{r}{1}=r$. Therefore $\lim\limits_{k\to\infty}\left(1+\frac{r}{k}\right)^k=\lim\limits_{k\to\infty}f(k)=\lim\limits_{k\to\infty}e^{\ln f(k)}=e^r$.

86. (a) $y=x^{1/x} \Rightarrow \ln y=\frac{\ln x}{x} \Rightarrow \frac{y'}{y}=\frac{\left(\frac{1}{x}\right)(x)-\ln x}{x^2} \Rightarrow y'=\left(\frac{1-\ln x}{x^2}\right)\left(x^{1/x}\right)$. The sign pattern is $y'=\underset{0}{|}+++++\underset{e}{|}----$ which indicates a maximum value of $y=e^{1/e}$ when $x=e$

(b) $y=x^{1/x^2} \Rightarrow \ln y=\frac{\ln x}{x^2} \Rightarrow \frac{y'}{y}=\frac{\left(\frac{1}{x}\right)\left(x^2\right)-2x\ln x}{x^4} \Rightarrow y'=\left(\frac{1-2\ln x}{x^3}\right)\left(x^{1/x^2}\right)$. The sign pattern is $y'=\underset{0}{|}+++\underset{\sqrt{e}}{|}----$ which indicates a maximum of $y=e^{1/(2e)}$ when $x=\sqrt{e}$

(c) $y=x^{1/x^n} \Rightarrow \ln y=\frac{\ln x}{x^n}=\frac{\left(\frac{1}{x}\right)\left(x^n\right)-(\ln x)\left(nx^{n-1}\right)}{x^{2n}} \Rightarrow y'=\frac{x^{n-1}(1-n\ln x)}{x^{2n}}\cdot x^{1/x^n}$. The sign pattern is $y'=\underset{0}{|}+++\underset{\sqrt[n]{e}}{|}----$ which indicates a maximum of $y=e^{1/(ne)}$ when $x=\sqrt[n]{e}$

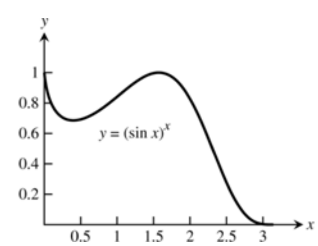
(d) $\lim\limits_{x\to\infty}x^{1/x^n}=\lim\limits_{x\to\infty}\left(e^{\ln x}\right)^{1/x^n}=\lim\limits_{x\to\infty}e^{(\ln x)x^n}=\exp\left(\lim\limits_{x\to\infty}\frac{\ln x}{x^n}\right)=\exp\left(\lim\limits_{x\to\infty}\left(\frac{1}{nx^n}\right)\right)=e^0=1$

87. (a) $y=x\tan\left(\frac{1}{x}\right)$, $\lim\limits_{x\to\infty}\left(x\tan\left(\frac{1}{x}\right)\right)=\lim\limits_{x\to\infty}\left(\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}\right)=\lim\limits_{x\to\infty}\left(\frac{\sec^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right)=\lim\limits_{x\to\infty}\sec^2\left(\frac{1}{x}\right)=1$; $\lim\limits_{x\to-\infty}\left(x\tan\left(\frac{1}{x}\right)\right)$
$=\lim\limits_{x\to-\infty}\left(\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}\right)=\lim\limits_{x\to-\infty}\left(\frac{\sec^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right)=\lim\limits_{x\to-\infty}\sec^2\left(\frac{1}{x}\right)=1 \Rightarrow$ the horizontal asymptote is $y=1$ as $x\to\infty$ and as $x\to-\infty$.

(b) $y=\frac{3x+e^{2x}}{2x+e^{3x}}$, $\lim\limits_{x\to\infty}\left(\frac{3x+e^{2x}}{2x+e^{3x}}\right)=\lim\limits_{x\to\infty}\left(\frac{3+2e^{2x}}{2+3e^{3x}}\right)=\lim\limits_{x\to\infty}\left(\frac{4e^{2x}}{9e^{3x}}\right)=\lim\limits_{x\to\infty}\left(\frac{4}{9e^x}\right)=0$; $\lim\limits_{x\to-\infty}\left(\frac{3x+e^{2x}}{2x+e^{3x}}\right)=\lim\limits_{x\to-\infty}\left(\frac{3+e^{2x}}{2+3^{ex}}\right)$
$=\frac{3}{2}\Rightarrow$ the horizontal asymptotes are $y=0$ as $x\to\infty$ and $y=\frac{3}{2}$ as $x\to-\infty$.

88. $f'(0)=\lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\frac{e^{-1/h^2}-0}{h}=\lim\limits_{h\to 0}\frac{e^{-1/h^2}}{h}=\lim\limits_{h\to 0}\left(\frac{\frac{1}{h}}{e^{1/h^2}}\right)=\lim\limits_{h\to 0}\left(\frac{-\frac{1}{h^2}}{e^{1/h^2}\left(-\frac{2}{h^3}\right)}\right)=\lim\limits_{h\to 0}\left(\frac{h}{2e^{1/h^2}}\right)$
$=\lim\limits_{h\to 0}\left(\frac{h}{2}e^{-1/h^2}\right)=0$

89. (a) We should assign the value 1 to $f(x)=(\sin x)^x$ to make it continuous at $x=0$.

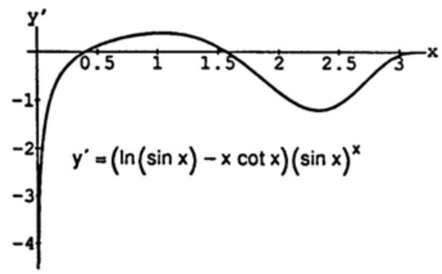


(b) $\ln f(x) = x\ln(\sin x) = \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} \Rightarrow \lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} = \lim_{x\to 0^+} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0} \frac{-x^2}{\tan x}$

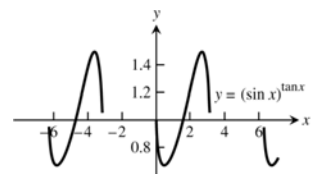
$= \lim_{x\to 0} \frac{-2x}{\sec^2 x} = 0 \Rightarrow \lim_{x\to 0} f(x) = e^0 = 1$

(c) The maximum value of $f(x)$ is close to 1 near the point $x \approx 1.55$ (see the graph in part (a)).

(d) The root in question is near 1.57.



90. (a) When sin $x < 0$ there are gaps in the sketch. The width of each gap is π.



(b) Let $f(x) = (\sin x)^{\tan x}$

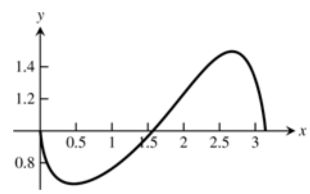
$\Rightarrow \ln f(x) = (\tan x)\ln(\sin x) \Rightarrow \lim_{x\to\left(\frac{\pi}{2}\right)^-} \ln f(x)$

$= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\ln(\sin x)}{\cot x} = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{-\csc^2 x}$

$= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\cos x}{(-\csc x)} = 0 \Rightarrow \lim_{x\to\left(\frac{\pi}{2}\right)^-} f(x) = e^0 = 1.$

Similarly, $\lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x) = e^0 = 1$. Therefore,

$\lim_{x\to\frac{\pi}{2}} f(x) = 1.$



(c) From the graph in part (b) we have a minimum of about 0.665 at $x \approx 0.47$ and the maximum is about 1.491 at $x \approx 2.66$.

7.6 INVERSE TRIGONOMETRIC FUNCTIONS

1. (a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$

2. (a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $-\frac{\pi}{6}$

3. (a) $-\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{3}$

4. (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$ (c) $\frac{\pi}{3}$

5. (a) $\frac{\pi}{3}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{6}$

6. (a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$

7. (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$

8. (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$

9. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

10. $\sec\left(\cos^{-1}\frac{1}{2}\right)=\sec\left(\frac{\pi}{3}\right)=2$

11. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)=\tan\left(-\frac{\pi}{6}\right)=-\frac{1}{\sqrt{3}}$

12. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)=\cot\left(-\frac{\pi}{3}\right)=-\frac{1}{\sqrt{3}}$

13. $\lim\limits_{x\to 1^-}\sin^{-1}x=\frac{\pi}{2}$

14. $\lim\limits_{x\to -1^+}\cos^{-1}x=\pi$

15. $\lim\limits_{x\to\infty}\tan^{-1}x=\frac{\pi}{2}$

16. $\lim\limits_{x\to-\infty}\tan^{-1}x=-\frac{\pi}{2}$

17. $\lim\limits_{x\to\infty}\sec^{-1}x=\frac{\pi}{2}$

18. $\lim\limits_{x\to-\infty}\sec^{-1}x=\lim\limits_{x\to-\infty}\cos^{-1}\left(\frac{1}{x}\right)=\frac{\pi}{2}$

19. $\lim\limits_{x\to\infty}\csc^{-1}x=\lim\limits_{x\to\infty}\sin^{-1}\left(\frac{1}{x}\right)=0$

20. $\lim\limits_{x\to-\infty}\csc^{-1}x=\lim\limits_{x\to-\infty}\sin^{-1}\left(\frac{1}{x}\right)=0$

21. $y=\cos^{-1}\left(x^2\right)\Rightarrow\frac{dy}{dx}=-\frac{2x}{\sqrt{1-\left(x^2\right)^2}}=\frac{-2x}{\sqrt{1-x^4}}$

22. $y=\cos^{-1}\left(\frac{1}{x}\right)=\sec^{-1}x\Rightarrow\frac{dy}{dx}=\frac{1}{|x|\sqrt{x^2-1}}$

23. $y=\sin^{-1}\sqrt{2}t\Rightarrow\frac{dy}{dt}=\frac{\sqrt{2}}{\sqrt{1-\left(\sqrt{2}t\right)^2}}=\frac{\sqrt{2}}{\sqrt{1-2t^2}}$

24. $y=\sin^{-1}(1-t)\Rightarrow\frac{dy}{dt}=\frac{-1}{\sqrt{1-(1-t)^2}}=\frac{-1}{\sqrt{2t-t^2}}$

25. $y=\sec^{-1}(2s+1)\Rightarrow\frac{dy}{ds}=\frac{2}{|2s+1|\sqrt{(2s+1)^2-1}}=\frac{2}{|2s+1|\sqrt{4s^2+4s}}=\frac{1}{|2s+1|\sqrt{s^2+s}}$

26. $y=\sec^{-1}5s\Rightarrow\frac{dy}{ds}=\frac{5}{|5s|\sqrt{(5s)^2-1}}=\frac{1}{|s|\sqrt{25s^2-1}}$

27. $y=\csc^{-1}\left(x^2+1\right)\Rightarrow\frac{dy}{dx}=-\frac{2x}{\left|x^2+1\right|\sqrt{\left(x^2+1\right)^2-1}}=\frac{-2x}{\left(x^2+1\right)\sqrt{x^4+2x^2}}$

28. $y=\csc^{-1}\left(\frac{x}{2}\right)\Rightarrow\frac{dy}{dx}=-\frac{\left(\frac{1}{2}\right)}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}}=\frac{-1}{|x|\sqrt{\frac{x^2-4}{4}}}=\frac{-2}{|x|\sqrt{x^2-4}}$

29. $y=\sec^{-1}\left(\frac{1}{t}\right)=\cos^{-1}t\Rightarrow\frac{dy}{dt}=\frac{-1}{\sqrt{1-t^2}}$

30. $y=\sin^{-1}\left(\frac{3}{t^2}\right)=\csc^{-1}\left(\frac{t^2}{3}\right)\Rightarrow\frac{dy}{dt}=-\frac{\left(\frac{2t}{3}\right)}{\left|\frac{t^2}{3}\right|\sqrt{\left(\frac{t^2}{3}\right)^2-1}}=\frac{-2t}{t^2\sqrt{\frac{t^4-9}{9}}}=\frac{-6}{t\sqrt{t^4-9}}$

31. $y=\cot^{-1}\sqrt{t}=\cot^{-1}\left(t^{1/2}\right)\Rightarrow\frac{dy}{dt}=-\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1+\left(t^{1/2}\right)^2}=\frac{-1}{2\sqrt{t}(1+t)}$

32. $y=\cot^{-1}\sqrt{t-1}=\cot^{-1}(t-1)^{1/2} \Rightarrow \frac{dy}{dt}=-\frac{\left(\frac{1}{2}\right)(t-1)^{-1/2}}{1+\left[(t-1)^{1/2}\right]^2}=\frac{-1}{2\sqrt{t-1}(1+t-1)}=\frac{-1}{2t\sqrt{t-1}}$

33. $y=\ln\left(\tan^{-1}x\right) \Rightarrow \frac{dy}{dx}=\frac{\left(\frac{1}{1+x^2}\right)}{\tan^{-1}x}=\frac{1}{\left(\tan^{-1}x\right)\left(1+x^2\right)}$

34. $y=\tan^{-1}(\ln x) \Rightarrow \frac{dy}{dx}=\frac{\left(\frac{1}{x}\right)}{1+(\ln x)^2}=\frac{1}{x\left[1+(\ln x)^2\right]}$

35. $y=\csc^{-1}\left(e^t\right) \Rightarrow \frac{dy}{dt}=-\frac{e^t}{|e^t|\sqrt{\left(e^t\right)^2-1}}=\frac{-1}{\sqrt{e^{2t}-1}}$

$\cos^{-1}\left(\frac{1}{e^t}\right)=\sec^{-1}\left(e^t\right)$

$\frac{\cancel{e^t}\,1}{\cancel{e^t}\sqrt{e^{2t}-1}}$

36. $y=\cos^{-1}\left(e^{-t}\right) \Rightarrow \frac{dy}{dt}=-\frac{-e^{-t}}{\sqrt{1-\left(e^{-t}\right)^2}}=\frac{e^{-t}}{\sqrt{1-e^{-2t}}}$ $=\frac{\frac{1}{e^t}}{\sqrt{1-\left(\frac{1}{e^t}\right)^2}}$

$=\frac{1}{\sqrt{e^{2t}-1}}$ ✓

37. $y=s\sqrt{1-s^2}+\cos^{-1}s=s\left(1-s^2\right)^{1/2}+\cos^{-1}s \Rightarrow \frac{dy}{ds}=\left(1-s^2\right)^{1/2}+s\left(\frac{1}{2}\right)\left(1-s^2\right)^{-1/2}(-2s)-\frac{1}{\sqrt{1-s^2}}$

$=\sqrt{1-s^2}-\frac{s^2}{\sqrt{1-s^2}}-\frac{1}{\sqrt{1-s^2}}=\sqrt{1-s^2}-\frac{s^2+1}{\sqrt{1-s^2}}=\frac{1-s^2-s^2-1}{\sqrt{1-s^2}}=\frac{-2s^2}{\sqrt{1-s^2}}$

38. $y=\sqrt{s^2-1}-\sec^{-1}s=\left(s^2-1\right)^{1/2}-\sec^{-1}s \Rightarrow \frac{dy}{dx}=\left(\frac{1}{2}\right)\left(s^2-1\right)^{-1/2}(2s)-\frac{1}{|s|\sqrt{s^2-1}}=\frac{s}{\sqrt{s^2-1}}-\frac{1}{|s|\sqrt{s^2-1}}=\frac{s|s|-1}{|s|\sqrt{s^2-1}}$

39. $y=\tan^{-1}\sqrt{x^2-1}+\csc^{-1}x=\tan^{-1}\left(x^2-1\right)^{1/2}+\csc^{-1}x \Rightarrow \frac{dy}{dx}=\frac{\left(\frac{1}{2}\right)\left(x^2-1\right)^{-1/2}(2x)}{1+\left[\left(x^2-1\right)^{1/2}\right]^2}-\frac{1}{|x|\sqrt{x^2-1}}=\frac{1}{x\sqrt{x^2-1}}-\frac{1}{|x|\sqrt{x^2-1}}=0,$

for $x>1$

40. $y=\cot^{-1}\left(\frac{1}{x}\right)-\tan^{-1}x=\frac{\pi}{2}-\tan^{-1}\left(x^{-1}\right)-\tan^{-1}x \Rightarrow \frac{dy}{dx}=0-\frac{-x^{-2}}{1+\left(x^{-1}\right)^2}-\frac{1}{1+x^2}=\frac{1}{x^2+1}-\frac{1}{1+x^2}=0$

41. $y=x\sin^{-1}x+\sqrt{1-x^2}=x\sin^{-1}x+\left(1-x^2\right)^{1/2} \Rightarrow \frac{dy}{dx}=\sin^{-1}x+x\left(\frac{1}{\sqrt{1-x^2}}\right)+\left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)$

$=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}=\sin^{-1}x$

42. $y=\ln\left(x^2+4\right)-x\tan^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx}=\frac{2x}{x^2+4}-\tan^{-1}\left(\frac{x}{2}\right)-x\left[\frac{\left(\frac{1}{2}\right)}{1+\left(\frac{x}{2}\right)^2}\right]=\frac{2x}{x^2+4}-\tan^{-1}\left(\frac{x}{2}\right)-\frac{2x}{4+x^2}=-\tan^{-1}\left(\frac{x}{2}\right)$

43. $\int\frac{1}{\sqrt{9-x^2}}\,dx=\sin^{-1}\left(\frac{x}{3}\right)+C$

44. $\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int \frac{2}{\sqrt{1-(2x)^2}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = 2x$ and $du = 2dx$

$= \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1}(2x) + C$

45. $\int \frac{1}{17+x^2}\,dx = \int \frac{1}{\left(\sqrt{17}\right)^2 + x^2}\,dx = \frac{1}{\sqrt{17}}\tan^{-1}\frac{x}{\sqrt{17}} + C$

46. $\int \frac{1}{9+3x^2}\,dx = \frac{1}{3}\int \frac{1}{\left(\sqrt{3}\right)^2 + x^2}dx = \frac{1}{3\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C = \frac{\sqrt{3}}{9}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

47. $\int \frac{dx}{x\sqrt{25x^2-2}} = \int \frac{du}{u\sqrt{u^2-2}}$, where $u = 5x$ and $du = 5\,dx$

$= \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{u}{\sqrt{2}}\right| + C = \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{5x}{\sqrt{2}}\right| + C$

48. $\int \frac{dx}{x\sqrt{5x^2-4}} = \int \frac{du}{u\sqrt{u^2-4}}$, where $u = \sqrt{5}x$ and $du = \sqrt{5}\,dx$

$= \frac{1}{2}\sec^{-1}\left|\frac{u}{2}\right| + C = \frac{1}{2}\sec^{-1}\left|\frac{\sqrt{5}x}{2}\right| + C$

49. $\int_0^1 \frac{4ds}{\sqrt{4-s^2}} = \left[4\sin^{-1}\frac{s}{2}\right]_0^1 = 4\left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right) = 4\left(\frac{\pi}{6} - 0\right) = \frac{2\pi}{3}$

50. $\int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9-4s^2}} = \frac{1}{2}\int_0^{3\sqrt{2}/4} \frac{du}{\sqrt{9-u^2}}$, where $u = 2s$ and $du = 2ds$; $s = 0 \Rightarrow u = 0, s = \frac{3\sqrt{2}}{4} \Rightarrow u = \frac{3\sqrt{2}}{2}$

$= \left[\frac{1}{2}\sin^{-1}\frac{u}{3}\right]_0^{3\sqrt{2}/2} = \frac{1}{2}\left(\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1} 0\right) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$

51. $\int_0^2 \frac{dt}{8+2t^2} = \frac{1}{\sqrt{2}}\int_0^{2\sqrt{2}} \frac{du}{8+u^2}$, where $u = \sqrt{2}t$ and $du = \sqrt{2}dt$; $t = 0 \Rightarrow u = 0, t = 2 \Rightarrow u = 2\sqrt{2}$

$= \left[\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{8}}\tan^{-1}\frac{u}{\sqrt{8}}\right]_0^{2\sqrt{2}} = \frac{1}{4}\left(\tan^{-1}\frac{2\sqrt{2}}{\sqrt{8}} - \tan^{-1} 0\right) = \frac{1}{4}\left(\tan^{-1} 1 - \tan^{-1} 0\right) = \frac{1}{4}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{16}$

52. $\int_{-2}^2 \frac{dt}{4+3t^2} = \frac{1}{\sqrt{3}}\int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{du}{4+u^2}$, where $u = \sqrt{3}t$ and $du = \sqrt{3}dt$; $t = -2 \Rightarrow u = -2\sqrt{3}, t = 2 \Rightarrow u = 2\sqrt{3}$

$= \left[\frac{1}{\sqrt{3}}\cdot\frac{1}{2}\tan^{-1}\frac{u}{2}\right]_{-2\sqrt{3}}^{2\sqrt{3}} = \frac{1}{2\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\left(-\sqrt{3}\right)\right] = \frac{1}{2\sqrt{3}}\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \frac{\pi}{3\sqrt{3}}$

53. $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2-1}} = \int_{-2}^{-\sqrt{2}} \frac{du}{u\sqrt{u^2-1}}$, where $u = 2y$ and $du = 2dy$; $y = -1 \Rightarrow u = -2, y = -\frac{\sqrt{2}}{2} \Rightarrow u = -\sqrt{2}$

$= \left[\sec^{-1}|u|\right]_{-2}^{-\sqrt{2}} = \sec^{-1}\left|-\sqrt{2}\right| - \sec^{-1}|-2| = \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$

54. $\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}} = \int_{-2}^{-\sqrt{2}} \frac{du}{u\sqrt{u^2-1}}$, where $u = 3y$ and $du = 3dy$; $y = -\frac{2}{3} \Rightarrow u = -2, y = -\frac{\sqrt{2}}{3} \Rightarrow u = -\sqrt{2}$

$= \left[\sec^{-1}|u|\right]_{-2}^{-\sqrt{2}} = \sec^{-1}\left|-\sqrt{2}\right| - \sec^{-1}|-2| = \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$

55. $\int \frac{3dr}{\sqrt{1-4(r-1)^2}} = \frac{3}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = 2(r-1)$ and $du = 2dr$

$= \frac{3}{2}\sin^{-1} u + C = \frac{3}{2}\sin^{-1} 2(r-1) + C$

56. $\int \frac{6dr}{\sqrt{4-(r+1)^2}} = 6\int \frac{du}{\sqrt{4-u^2}}$, where $u = r+1$ and $du = dr$

$= 6\sin^{-1} \frac{u}{2} + C = 6\sin^{-1}\left(\frac{r+1}{2}\right) + C$

57. $\int \frac{dx}{2+(x-1)^2} = \int \frac{du}{2+u^2}$, where $u = x-1$ and $du = dx$

$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C$

58. $\int \frac{dx}{1+(3x+1)^2} = \frac{1}{3}\int \frac{du}{1+u^2}$, where $u = 3x+1$ and $du = 3dx$

$= \frac{1}{3}\tan^{-1} u + C = \frac{1}{3}\tan^{-1}(3x+1) + C$

59. $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}} = \frac{1}{2}\int \frac{du}{u\sqrt{u^2-4}}$, where $u = 2x-1$ and $du = 2dx$

$= \frac{1}{2}\cdot\frac{1}{2}\sec^{-1}\left|\frac{u}{2}\right| + C = \frac{1}{4}\sec^{-1}\left|\frac{2x-1}{2}\right| + C$

60. $\int \frac{dx}{(x+3)\sqrt{(x+3)^2-25}} = \int \frac{du}{u\sqrt{u^2-25}}$, where $u = x+3$ and $du = dx$

$= \frac{1}{5}\sec^{-1}\left|\frac{u}{5}\right| + C = \frac{1}{5}\sec^{-1}\left|\frac{x+3}{5}\right| + C$

61. $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{1+(\sin\theta)^2} = 2\int_{-1}^{1} \frac{du}{1+u^2}$, where $u = \sin\theta$ and $du = \cos\theta\, d\theta$; $\theta = -\frac{\pi}{2} \Rightarrow u = -1, \theta = \frac{\pi}{2} \Rightarrow u = 1$

$= \left[2\tan^{-1} u\right]_{-1}^{1} = 2\left(\tan^{-1} 1 - \tan^{-1}(-1)\right) = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

62. $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\, dx}{1+(\cot x)^2} = -\int_{\sqrt{3}}^{1} \frac{du}{1+u^2}$, where $u = \cot x$ and $du = -\csc^2 x\, dx$; $x = \frac{\pi}{6} \Rightarrow u = \sqrt{3}, x = \frac{\pi}{4} \Rightarrow u = 1$

$= \left[-\tan^{-1} u\right]_{\sqrt{3}}^{1} = -\tan^{-1} 1 + \tan^{-1}\sqrt{3} = -\frac{\pi}{4} + \frac{\pi}{3} = \frac{\pi}{12}$

63. $\int_{0}^{\ln\sqrt{3}} \frac{e^x dx}{1+e^{2x}} = \int_{1}^{\sqrt{3}} \frac{du}{1+u^2}$, where $u = e^x$ and $du = e^x dx$; $x = 0 \Rightarrow u = 1, x = \ln\sqrt{3} \Rightarrow u = \sqrt{3}$

$= \left[\tan^{-1} u\right]_{1}^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

64. $\int_{1}^{e^{\pi/4}} \frac{4dt}{t\left(1+\ln^2 t\right)} = 4\int_{0}^{\pi/4} \frac{du}{1+u^2}$, where $u = \ln t$ and $du = \frac{1}{t}dt$; $t = 1 \Rightarrow u = 0, t = e^{\pi/4} \Rightarrow u = \frac{\pi}{4}$

$= \left[4\tan^{-1} u\right]_{0}^{\pi/4} = 4\left(\tan^{-1}\frac{\pi}{4} - \tan^{-1} 0\right) = 4\tan^{-1}\frac{\pi}{4}$

65. $\int \frac{y\,dy}{\sqrt{1-y^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = y^2$ and $du = 2y\,dy$

$= \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} y^2 + C$

66. $\int \frac{\sec^2 y\, dy}{\sqrt{1-\tan^2 y}} = \int \frac{du}{\sqrt{1-u^2}}$, where $u = \tan y$ and $du = \sec^2 y\, dy$
$= \sin^{-1} u + C = \sin^{-1}(\tan y) + C$

67. $\int \frac{dx}{\sqrt{-x^2+4x-3}} = \int \frac{dx}{\sqrt{1-\left(x^2-4x+4\right)}} = \int \frac{dx}{\sqrt{1-(x-2)^2}} = \sin^{-1}(x-2) + C$

68. $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-\left(x^2-2x+1\right)}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1) + C$

69. $\int_{-1}^{0} \frac{6\,dt}{\sqrt{3-2t-t^2}} = 6\int_{-1}^{0} \frac{dt}{\sqrt{4-\left(t^2+2t+1\right)}} = 6\int_{-1}^{0} \frac{dt}{\sqrt{2^2-(t+1)^2}} = 6\left[\sin^{-1}\left(\frac{t+1}{2}\right)\right]_{-1}^{0} = 6\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0\right] = 6\left(\frac{\pi}{6} - 0\right) = \pi$

70. $\int_{1/2}^{1} \frac{6\,dt}{\sqrt{3+4t-4t^2}} = 3\int_{1/2}^{1} \frac{2\,dt}{\sqrt{4-\left(4t^2-4t+1\right)}} = 3\int_{1/2}^{1} \frac{2\,dt}{\sqrt{2^2-(2t-1)^2}} = 3\left[\sin^{-1}\left(\frac{2t-1}{2}\right)\right]_{1/2}^{1} = 3\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0\right]$
$= 3\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{2}$

71. $\int \frac{dy}{y^2-2y+5} = \int \frac{dy}{4+y^2-2y+1} = \int \frac{dy}{2^2+(y-1)^2} = \frac{1}{2}\tan^{-1}\left(\frac{y-1}{2}\right) + C$

72. $\int \frac{dy}{y^2+6y+10} = \int \frac{dy}{1+\left(y^2+6y+9\right)} = \int \frac{dy}{1+(y+3)^2} = \tan^{-1}(y+3) + C$

73. $\int_{1}^{2} \frac{8\,dx}{x^2-2x+2} = 8\int_{1}^{2} \frac{dx}{1+\left(x^2-2x+1\right)} = 8\int_{1}^{2} \frac{dx}{1+(x-1)^2} = 8\left[\tan^{-1}(x-1)\right]_{1}^{2} = 8\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 8\left(\frac{\pi}{4} - 0\right) = 2\pi$

74. $\int_{2}^{4} \frac{2\,dx}{x^2-6x+10} = 2\int_{2}^{4} \frac{dx}{1+\left(x^2-6x+9\right)} = 2\int_{2}^{4} \frac{dx}{1+(x-3)^2} = 2\left[\tan^{-1}(x-3)\right]_{2}^{4} = 2\left[\tan^{-1} 1 - \tan^{-1}(-1)\right] = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

75. $\int \frac{x+4}{x^2+4}dx = \int \frac{x}{x^2+4}dx + \int \frac{4}{x^2+4}dx;\ \int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{1}{u}du$ where $u = x^2+4 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$
$\Rightarrow \int \frac{x+4}{x^2+4}\,dx = \frac{1}{2}\ln\left(x^2+4\right) + 2\tan^{-1}\left(\frac{x}{2}\right) + C$

76. $\int \frac{t-2}{t^2-6t+10}dt = \int \frac{t-2}{(t-3)^2+1}dt\left[\text{Let } w = t-3 \Rightarrow w+3 = t \Rightarrow dw = dt\right] \to \int \frac{w+1}{w^2+1}dw = \int \frac{w}{w^2+1}dw + \int \frac{1}{w^2+1}dw;$
$\int \frac{w}{w^2+1}dw = \frac{1}{2}\int \frac{1}{u}du$ where $u = w^2+1 \Rightarrow du = 2w\,dw \Rightarrow \frac{1}{2}du = w\,dw \Rightarrow \int \frac{w}{w^2+1}dw + \int \frac{1}{w^2+1}dw$
$= \frac{1}{2}\ln\left(w^2+1\right) + \tan^{-1}(w) + C = \frac{1}{2}\ln\left((t-3)^2+1\right) + \tan^{-1}(t-3) + C = \frac{1}{2}\ln\left(t^2-6t+10\right) + \tan^{-1}(t-3) + C$

77. $\int \frac{x^2+2x-1}{x^2+9}dx = \int\left(1 + \frac{2x-10}{x^2+9}\right)dx = \int dx + \int \frac{2x}{x^2+9}dx - 10\int \frac{1}{x^2+9}dx;\ \int \frac{2x}{x^2+9}dx = \int \frac{1}{u}du$ where
$u = x^2+9 \Rightarrow du = 2x\,dx \Rightarrow \int dx + \int \frac{2x}{x^2+9}dx - 10\int \frac{1}{x^2+9}dx = x + \ln\left(x^2+9\right) - \frac{10}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

78. $\int \frac{t^3-2t^2+3t-4}{t^2+1}dt = \int\left(t-2+\frac{2t-2}{t^2+1}\right)dt = \int(t-2)\,dt + \int\frac{2t}{t^2+1}dt - 2\int\frac{1}{t^2+1}dt;\ \int\frac{2t}{t^2+1}dt = \int\frac{1}{u}\,du$ where $u = t^2+1 \Rightarrow du = 2t\,dt \Rightarrow \int(t-2)dt + \int\frac{2t}{t^2+1}dt - 2\int\frac{1}{t^2+1}dt = \frac{1}{2}t^2 - 2t + \ln\left(t^2+1\right) - 2\tan^{-1}(t) + C$

79. $\int\frac{dx}{(x+1)\sqrt{x^2+2x}} = \int\frac{dx}{(x+1)\sqrt{x^2+2x+1-1}} = \int\frac{dx}{(x-1)\sqrt{(x+1)^2-1}} = \int\frac{du}{u\sqrt{u^2-1}}$, where $u = x+1$ and $du = dx$
$= \sec^{-1}|u| + C = \sec^{-1}|x+1| + C$

80. $\int\frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int\frac{dx}{(x-2)\sqrt{x^2-4x+4-1}} = \int\frac{dx}{(x-2)\sqrt{(x-2)^2-1}} = \int\frac{1}{u\sqrt{u^2-1}}du$, where $u = x-2$ and $du = dx$
$= \sec^{-1}|u| + C = \sec^{-1}|x-2| + C$

81. $\int\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx = \int e^u\,du$, where $u = \sin^{-1}x$ and $du = \frac{dx}{\sqrt{1-x^2}}$
$= e^u + C = e^{\sin^{-1}x} + C$

82. $\int\frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}dx = -\int e^u\,du$, where $u = \cos^{-1}x$ and $du = \frac{-dx}{\sqrt{1-x^2}}$
$= -e^u + C = -e^{\cos^{-1}x} + C$

83. $\int\frac{\left(\sin^{-1}x\right)^2}{\sqrt{1-x^2}}dx = \int u^2\,du$, where $u = \sin^{-1}x$ and $du = \frac{dx}{\sqrt{1-x^2}}$
$= \frac{u^3}{3} + C = \frac{\left(\sin^{-1}x\right)^3}{3} + C$

84. $\int\frac{\sqrt{\tan^{-1}x}}{1+x^2}dx = \int u^{1/2}\,du$, where $u = \tan^{-1}x$ and $du = \frac{dx}{1+x^2}$
$= \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(\tan^{-1}x\right)^{3/2} + C = \frac{2}{3}\sqrt{\left(\tan^{-1}x\right)^3} + C$

85. $\int\frac{1}{\left(\tan^{-1}y\right)\left(1+y^2\right)}dy = \int\frac{\left(\frac{1}{1+y^2}\right)}{\tan^{-1}y}dy = \int\frac{1}{u}\,du$, where $u = \tan^{-1}y$ and $du = \frac{dy}{1+y^2}$
$= \ln|u| + C = \ln\left|\tan^{-1}y\right| + C$

86. $\int\frac{1}{\left(\sin^{-1}y\right)\sqrt{1-y^2}}dy = \int\frac{\left(\frac{1}{\sqrt{1-y^2}}\right)}{\sin^{-1}y}dy = \int\frac{1}{u}\,du$, where $u = \sin^{-1}y$ and $du = \frac{dy}{\sqrt{1-y^2}}$
$= \ln|u| + C = \ln\left|\sin^{-1}y\right| + C$

87. $\int_{\sqrt{2}}^{2}\frac{\sec^2\left(\sec^{-1}x\right)}{x\sqrt{x^2-1}}dx = \int_{\pi/4}^{\pi/3}\sec^2u\,du$, where $u = \sec^{-1}x$ and $du = \frac{dx}{x\sqrt{x^2-1}}$; $x = \sqrt{2} \Rightarrow u = \frac{\pi}{4}, x = 2 \Rightarrow u = \frac{\pi}{3}$
$= \left[\tan u\right]_{\pi/4}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{4} = \sqrt{3} - 1$

88. $\int_{2/\sqrt{3}}^{2} \frac{\cos\left(\sec^{-1} x\right)}{x\sqrt{x^2-1}}\,dx = \int_{\pi/6}^{\pi/3} \cos u\,du$, where $u = \sec^{-1} x$ and $du = \frac{dx}{x\sqrt{x^2-1}}$; $x = \frac{2}{\sqrt{3}} \Rightarrow u = \frac{\pi}{6}, x = 2 \Rightarrow u = \frac{\pi}{3}$

$$= \left[\sin u\right]_{\pi/6}^{\pi/3} = \sin\frac{\pi}{3} - \sin\frac{\pi}{6} = \frac{\sqrt{3}-1}{2}$$

89. $\int \frac{1}{\sqrt{x}(x+1)\left[\left(\tan^{-1}\sqrt{x}\right)^2+9\right]}\,dx = 2\int \frac{1}{u^2+9}\,du$ where $u = \tan^{-1}\sqrt{x} \Rightarrow du = \frac{1}{1+\left(\sqrt{x}\right)^2}\frac{1}{2\sqrt{x}}\,dx \Rightarrow 2du = \frac{1}{(1+x)\sqrt{x}}\,dx$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan^{-1}\sqrt{x}}{3}\right) + C$$

90. $\int \frac{e^x \sin^{-1} e^x}{\sqrt{1-e^{2x}}}\,dx = \int u\,du$ where $u = \sin^{-1} e^x \Rightarrow du = \frac{1}{\sqrt{1-e^{2x}}} e^x\,dx$

$$= \frac{1}{2}\left(\sin^{-1} e^x\right)^2 + C$$

91. $\lim\limits_{x\to 0} \frac{\sin^{-1} 5x}{x} = \lim\limits_{x\to 0} \frac{\left(\frac{5}{\sqrt{1-25x^2}}\right)}{1} = 5$

92. $\lim\limits_{x\to 1^+} \frac{\sqrt{x^2-1}}{\sec^{-1} x} = \lim\limits_{x\to 1^+} \frac{\left(x^2-1\right)^{1/2}}{\sec^{-1} x} = \lim\limits_{x\to 1^+} \frac{\left(\frac{1}{2}\right)\left(x^2-1\right)^{-1/2}(2x)}{\left(\frac{1}{|x|\sqrt{x^2-1}}\right)} = \lim\limits_{x\to 1^+} x\,|x| = 1$

93. $\lim\limits_{x\to\infty} x\tan^{-1}\left(\frac{2}{x}\right) = \lim\limits_{x\to\infty} \frac{\tan^{-1}\left(2x^{-1}\right)}{x^{-1}} = \lim\limits_{x\to\infty} \frac{\left(\frac{-2x^{-2}}{1+4x^{-2}}\right)}{-x^{-2}} = \lim\limits_{x\to\infty} \frac{2}{1+4x^{-2}} = 2$

94. $\lim\limits_{x\to 0} \frac{2\tan^{-1} 3x^2}{7x^2} = \lim\limits_{x\to 0} \frac{\left(\frac{12x}{1+9x^4}\right)}{14x} = \lim\limits_{x\to 0} \frac{6}{7\left(1+9x^4\right)} = \frac{6}{7}$

95. $\lim\limits_{x\to 0} \frac{\tan^{-1} x^2}{x\sin^{-1} x} = \lim\limits_{x\to 0}\left(\frac{\frac{2x}{1+x^4}}{x\cdot\frac{1}{\sqrt{1-x^2}}+\sin^{-1} x}\right) = \lim\limits_{x\to 0}\left(\frac{\frac{-2\left(3x^4-1\right)}{\left(1+x^4\right)^2}}{\frac{-x^2+2}{\left(1-x^2\right)^{3/2}}}\right) = \frac{\frac{-2(0-1)}{1^2}}{\frac{-0+2}{(1-0)^{3/2}}} = \frac{2}{2} = 1$

96. $\lim\limits_{x\to\infty} \frac{e^x\tan^{-1} e^x}{e^{2x}+x} = \lim\limits_{x\to\infty} \frac{e^x\tan^{-1} e^x + \frac{e^{2x}}{e^{2x}+1}}{2e^{2x}+1} = \lim\limits_{x\to\infty} \frac{e^x\tan^{-1} e^x + \frac{e^{2x}}{e^{2x}+1} + \frac{2e^{2x}}{\left(e^{2x}+1\right)^2}}{4e^{2x}} = \lim\limits_{x\to\infty} \frac{e^x\tan^{-1} e^x + \frac{e^{2x}\left(e^{2x}+3\right)}{\left(e^{2x}+1\right)^2}}{4e^{2x}}$

$$= \lim_{x\to\infty}\left[\frac{\tan^{-1} e^x}{4e^x} + \frac{\left(e^{2x}+3\right)}{4\left(e^{2x}+1\right)^2}\right] = \lim_{x\to\infty}\left[\frac{\tan^{-1} e^x}{4e^x} + \frac{\left(1+3e^{-2x}\right)}{4\left(e^x+e^{-x}\right)^2}\right] = 0 + 0 = 0$$

97. $\lim\limits_{x\to 0^+} \frac{\left[\tan^{-1}\left(\sqrt{x}\right)\right]^2}{x\sqrt{x+1}} = \lim\limits_{x\to 0^+} \frac{\tan^{-1}\left(\sqrt{x}\right)\frac{1}{\sqrt{x}(1+x)}}{\frac{x}{2\sqrt{x+1}}+\sqrt{x+1}} = \lim\limits_{x\to 0^+} \frac{\frac{\tan^{-1}\left(\sqrt{x}\right)}{\sqrt{x}(1+x)}}{\frac{3x+2}{2\sqrt{x+1}}} = \lim\limits_{x\to 0^+}\left(\frac{2\tan^{-1}\left(\sqrt{x}\right)}{(3x+2)\sqrt{x}\sqrt{x+1}}\right) = \lim\limits_{x\to 0^+}\left(\frac{\frac{1}{\sqrt{x}(1+x)}}{\frac{12x^2+13x+2}{2\sqrt{x}\sqrt{x+1}}}\right)$

$= \lim\limits_{x\to 0^+}\left(\frac{2}{\left(12x^2+13x+2\right)\sqrt{x+1}}\right) = \frac{2}{2} = 1$

98. $\lim\limits_{x\to 0^+} \frac{\sin^{-1}\left(x^2\right)}{\left(\sin^{-1}x\right)^2} = \lim\limits_{x\to 0^+}\left(\frac{\frac{2x}{\sqrt{1-x^4}}}{2\left(\sin^{-1}x\right)\frac{1}{\sqrt{1-x^2}}}\right) = \lim\limits_{x\to 0^+}\left(\frac{x}{\sin^{-1}x\sqrt{1+x^2}}\right) = \lim\limits_{x\to 0^+}\left(\frac{1}{\sin^{-1}x\cdot\frac{x}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1-x^2}}\sqrt{1+x^2}}\right)$

$= \lim\limits_{x\to 0^+}\left(\frac{\sqrt{1+x^2}\sqrt{1-x^2}}{1+x^2+x\sqrt{1-x^2}\,\sin^{-1}x}\right) = \frac{1}{1} = 1$

99. If $y = \ln x - \frac{1}{2}\ln\left(1+x^2\right) - \frac{\tan^{-1}x}{x} + C$, then $dy = \left[\frac{1}{x} - \frac{x}{1+x^2} - \frac{\left(\frac{x}{1+x^2}\right)-\tan^{-1}x}{x^2}\right]dx$

$= \left(\frac{1}{x} - \frac{x}{1+x^2} - \frac{1}{x\left(1+x^2\right)} + \frac{\tan^{-1}x}{x^2}\right)dx = \frac{x\left(1+x^2\right)-x^3-x+\left(\tan^{-1}x\right)\left(1+x^2\right)}{x^2\left(1+x^2\right)}dx = \frac{\tan^{-1}x}{x^2}dx$, which verifies the formula

100. If $y = \frac{x^4}{4}\cos^{-1}5x + \frac{5}{4}\int\frac{x^4}{\sqrt{1-25x^2}}dx$, then

$dy = \left[x^3\cos^{-1}5x + \left(\frac{x^4}{4}\right)\left(\frac{-5}{\sqrt{1-25x^2}}\right) + \frac{5}{4}\left(\frac{x^4}{\sqrt{1-25x^2}}\right)\right]dx = \left(x^3\cos^{-1}5x\right)dx$, which verifies the formula

101. If $y = x\left(\sin^{-1}x\right)^2 - 2x + 2\sqrt{1-x^2}\,\sin^{-1}x + C$, then

$dy = \left[\left(\sin^{-1}x\right)^2 + \frac{2x\left(\sin^{-1}x\right)}{\sqrt{1-x^2}} - 2 + \frac{-2x}{\sqrt{1-x^2}}\sin^{-1}x + 2\sqrt{1-x^2}\left(\frac{1}{\sqrt{1-x^2}}\right)\right]dx = \left(\sin^{-1}x\right)^2dx$, which verifies the formula

102. If $y = x\ln\left(a^2+x^2\right) - 2x + 2a\tan^{-1}\left(\frac{x}{a}\right) + C$, then $dy = \left[\ln\left(a^2+x^2\right) + \frac{2x^2}{a^2+x^2} - 2 + \frac{2}{1+\left(\frac{x^2}{a^2}\right)}\right]dx$

$= \left[\ln\left(a^2+x^2\right) + 2\left(\frac{a^2+x^2}{a^2+x^2}\right) - 2\right]dx = \ln\left(a^2+x^2\right)dx$, which verifies the formula

103. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow dy = \frac{dx}{\sqrt{1-x^2}} \Rightarrow y = \sin^{-1}x + C; x = 0$ and $y = 0 \Rightarrow 0 = \sin^{-1}0 + C \Rightarrow C = 0 \Rightarrow y = \sin^{-1}x$

104. $\frac{dy}{dx} = \frac{1}{x^2+1} - 1 \Rightarrow dy = \left(\frac{1}{1+x^2} - 1\right)dx \Rightarrow y = \tan^{-1}(x) - x + C; x = 0$ and $y = 1 \Rightarrow 1 = \tan^{-1}0 - 0 + C \Rightarrow C = 1$

$\Rightarrow y = \tan^{-1}(x) - x + 1$

105. $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} \Rightarrow dy = \frac{dx}{x\sqrt{x^2-1}} \Rightarrow y = \sec^{-1}|x| + C; x = 2$ and $y = \pi \Rightarrow \pi = \sec^{-1}2 + C \Rightarrow C = \pi - \sec^{-1}2$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3} \Rightarrow y = \sec^{-1}(x) + \frac{2\pi}{3},\ x > 1$

106. $\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} \Rightarrow dy = \left(\frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}}\right)dx \Rightarrow y = \tan^{-1} x - 2\sin^{-1} x + C; x = 0$ and $y = 2$
$\Rightarrow 2 = \tan^{-1} 0 - 2\sin^{-1} 0 + C \Rightarrow C = 2 \Rightarrow y = \tan^{-1} x - 2\sin^{-1} x + 2$

107. (a) The angle α is the large angle between the wall and the right end of the blackboard minus the small angle between the left end of the blackboard and the wall $\Rightarrow \alpha = \cot^{-1}\left(\frac{x}{5}\right) - \cot^{-1}(x)$.

(b) $\frac{d\alpha}{dt} = -\frac{\frac{1}{5}}{1+\left(\frac{x}{5}\right)^2} + \frac{1}{1+(x)^2} = -\frac{5}{25+x^2} + \frac{1}{1+x^2} = \frac{20-4x^2}{(25+x^2)(1+x^2)}$; $\frac{d\alpha}{dt} = 0 \Rightarrow 20 - 4x^2 = 0 \Rightarrow x = \pm\sqrt{5}$. Since $x > 0$, consider only $x = \sqrt{5} \Rightarrow \alpha(\sqrt{5}) = \cot^{-1}\left(\frac{\sqrt{5}}{5}\right) - \cot^{-1}(\sqrt{5}) \approx 0.729728 \approx 41.8103°$. Using the first derivative test, $\left.\frac{d\alpha}{dt}\right|_{x=1} = \frac{16}{52} > 0$ and $\left.\frac{d\alpha}{dt}\right|_{x=10} = -\frac{380}{1375} < 0 \Rightarrow$ local maximum of 41.8103° when $x = \sqrt{5} \approx 2.24$ m.

108. $V = \pi\int_0^{\pi/3}\left[2^2 - (\sec y)^2\right]dy = \pi\left[4y - \tan y\right]_0^{\pi/3} = \pi\left(\frac{4\pi}{3} - \sqrt{3}\right)$

109. $V = \left(\frac{1}{3}\right)\pi r^2 h = \left(\frac{1}{3}\right)\pi(3\sin\theta)^2(3\cos\theta) = 9\pi\left(\cos\theta - \cos^3\theta\right)$, where $0 \le \theta \le \frac{\pi}{2}$
$\Rightarrow \frac{dV}{d\theta} = -9\pi(\sin\theta)\left(1 - 3\cos^2\theta\right) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = \pm\frac{1}{\sqrt{3}} \Rightarrow$ the critical points are: 0, $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, and $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$; but $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is not in the domain. When $\theta = 0$, we have a minimum and when $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7°$, we have a maximum volume.

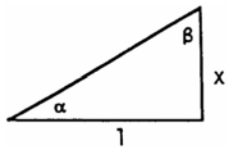
110. $65° + (90° - \beta) + (90° - \alpha) = 180° \Rightarrow \alpha = 65° - \beta = 65° - \tan^{-1}\left(\frac{21}{50}\right) \approx 65° - 22.78° \approx 42.22°$

111. Take each square as a unit square. From the diagram we have the following: the smallest angle α has a tangent of $1 \Rightarrow \alpha = \tan^{-1} 1$; the middle angle β has a tangent of $2 \Rightarrow \beta = \tan^{-1} 2$; and the largest angle γ has a tangent of $3 \Rightarrow \gamma = \tan^{-1} 3$. The sum of these three angles is $\pi \Rightarrow \alpha + \beta + \gamma = \pi \Rightarrow \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$.

112. (a) From the symmetry of the diagram, we see that $\pi - \sec^{-1} x$ is the vertical distance from the graph of $y = \sec^{-1} x$ to the line $y = \pi$ and this distance is the same as the height of $y = \sec^{-1} x$ above the x-axis at $-x$; i.e., $\pi - \sec^{-1} x = \sec^{-1}(-x)$.

(b) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, where $-1 \le x \le 1 \Rightarrow \cos^{-1}\left(-\frac{1}{x}\right) = \pi - \cos^{-1}\left(\frac{1}{x}\right)$, where $x \ge 1$ or $x \le -1$
$\Rightarrow \sec^{-1}(-x) = \pi - \sec^{-1} x$

113. $\sin^{-1}(1) + \cos^{-1}(1) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$; $\sin^{-1}(0) + \cos^{-1}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2}$; and $\sin^{-1}(-1) + \cos^{-1}(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$. If $x \in (-1, 0)$ and $x = -a$, then $\sin^{-1}(x) + \cos^{-1}(x) = \sin^{-1}(-a) + \cos^{-1}(-a) = -\sin^{-1} a + \left(\pi - \cos^{-1} a\right)$
$= \pi - \left(\sin^{-1} a + \cos^{-1} a\right) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$ from Equations (3) and (4) in the text.

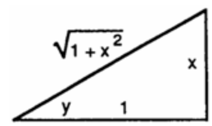
114. 

$x \Rightarrow \tan\alpha = x$ and $\tan\beta = \frac{1}{x} \Rightarrow \frac{\pi}{2} = \alpha + \beta = \tan^{-1} x + \tan^{-1} \frac{1}{x}$.

115. $\csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u \Rightarrow \frac{d}{dx}\left(\csc^{-1} u\right) = \frac{d}{dx}\left(\frac{\pi}{2} - \sec^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}} = -\frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}}, |u| > 1$

116. $y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$

$\Rightarrow \left(\sec^2 y\right)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\left(\sqrt{1+x^2}\right)^2} = \frac{1}{1+x^2}$, as indicated by the triangle



117. $f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} = \frac{1}{\sec\left(\sec^{-1} b\right)\tan\left(\sec^{-1} b\right)} = \frac{1}{b\left(\pm\sqrt{b^2-1}\right)}$. Since the slope of $\sec^{-1} x$ is always positive, we obtain the right sign by writing $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$.

118. $\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u \Rightarrow \frac{d}{dx}\left(\cot^{-1} u\right) = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{1+u^2} = -\frac{\frac{du}{dx}}{1+u^2}$

119. The function f and g have the same derivative (for $x \ge 0$), namely $\frac{1}{\sqrt{x}(x+1)}$. The functions therefore differ by a constant. To identify the constant we can set x equal to 0 in the equation $f(x) = g(x) + C$, obtaining $\sin^{-1}(-1) = 2\tan^{-1}(0) + C \Rightarrow -\frac{\pi}{2} = 0 + C \Rightarrow C = -\frac{\pi}{2}$. For $x \ge 0$, we have $\sin^{-1}\left(\frac{x-1}{x+1}\right) = 2\tan^{-1}\sqrt{x} - \frac{\pi}{2}$.

120. The functions f and g have the same derivative for $x > 0$, namely $\frac{-1}{1+x^2}$. The functions therefore differ by a constant for $x > 0$. To identify the constant we can set x equal to 1 in the equation $f(x) = g(x) + C$, obtaining $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \tan^{-1} 1 + C \Rightarrow \frac{\pi}{4} = \frac{\pi}{4} + C \Rightarrow C = 0$. For $x > 0$, we have $\sin^{-1}\frac{1}{\sqrt{x^2+1}} = \tan^{-1}\frac{1}{x}$.

121. $V = \pi\int_{-\sqrt{3}/3}^{\sqrt{3}}\left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx = \pi\int_{-\sqrt{3}/3}^{\sqrt{3}}\frac{1}{1+x^2}dx = \pi\left[\tan^{-1} x\right]_{-\sqrt{3}/3}^{\sqrt{3}} = \pi\left[\tan^{-1}\sqrt{3} - \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right]$

$= \pi\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi^2}{2}$

122. Consider $y = \sqrt{r^2 - x^2} \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$; Since $\frac{dy}{dx}$ is undefined at $x = r$ and $x = -r$, we will find the length from $x = 0$ to $x = \frac{r}{\sqrt{2}}$ (in other words, the length of $\frac{1}{8}$ of a circle) $\Rightarrow L = \int_0^{r/\sqrt{2}}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx$

$= \int_0^{r/\sqrt{2}}\sqrt{1+\frac{x^2}{r^2-x^2}}dx = \int_0^{r/\sqrt{2}}\sqrt{\frac{r^2}{r^2-x^2}}dx = \int_0^{r/\sqrt{2}}\frac{r}{\sqrt{r^2-x^2}}dx = \left[r\sin^{-1}\left(\frac{x}{r}\right)\right]_0^{r/\sqrt{2}} = r\sin^{-1}\left(\frac{r/\sqrt{2}}{r}\right) - r\sin^{-1}(0)$

$= r\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 0 = r\left(\frac{\pi}{4}\right) = \frac{\pi r}{4}$. The total circumference of the circle is $C = 8L = 8\left(\frac{\pi r}{4}\right) = 2\pi r$.

123. (a) $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}}\right)\right]^2 = \frac{\pi}{1+x^2} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-1}^{1} \frac{\pi\,dx}{1+x^2} = \pi\left[\tan^{-1} x\right]_{-1}^{1}$
$= (\pi)(2)\left(\frac{\pi}{4}\right) = \frac{\pi^2}{2}$

(b) $A(x) = (\text{edge})^2 = \left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}}\right)\right]^2 = \frac{4}{1+x^2} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-1}^{1} \frac{4dx}{1+x^2} = 4\left[\tan^{-1} x\right]_{-1}^{1}$
$= 4[\tan^{-1}(1) - \tan^{-1}(-1)] = 4\left[\frac{\pi}{4} - (-\frac{\pi}{4})\right] = 2\pi$

124. (a) $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\frac{2}{\sqrt[4]{1-x^2}} - 0\right)^2 = \frac{\pi}{4}\left(\frac{4}{\sqrt{1-x^2}}\right) = \frac{\pi}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{\pi}{\sqrt{1-x^2}}\,dx$
$= \pi\left[\sin^{-1} x\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = \pi\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right] = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}$

(b) $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{1}{2}\left(\frac{2}{\sqrt[4]{1-x^2}} - 0\right)^2 = \frac{2}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{2}{\sqrt{1-x^2}}\,dx = 2\left[\sin^{-1} x\right]_{-\sqrt{2}/2}^{\sqrt{2}/2}$
$= 2\left(\frac{\pi}{4} \cdot 2\right) = \pi$

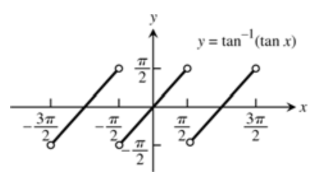
125. (a) $\sec^{-1} 1.5 = \cos^{-1} \frac{1}{1.5} \approx 0.84107$ (b) $\csc^{-1}(-1.5) = \sin^{-1}\left(-\frac{1}{1.5}\right) \approx -0.72973$

(c) $\cot^{-1} 2 = \frac{\pi}{2} - \tan^{-1} 2 \approx 0.46365$

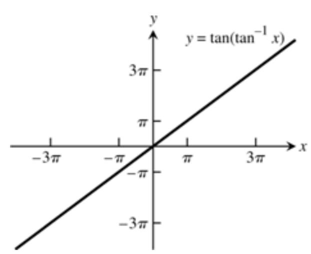
126. (a) $\sec^{-1}(-3) = \cos^{-1}\left(-\frac{1}{3}\right) \approx 1.91063$ (b) $\csc^{-1} 1.7 = \sin^{-1}\left(\frac{1}{1.7}\right) \approx 0.62887$

(c) $\cot^{-1}(-2) = \frac{\pi}{2} - \tan^{-1}(-2) \approx 2.67795$

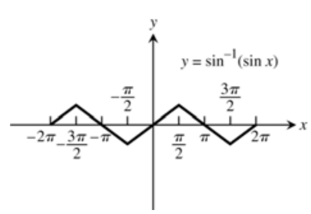
127. (a) Domain: all real numbers except those having the form $\frac{\pi}{2} + k\pi$ where k is an integer.
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



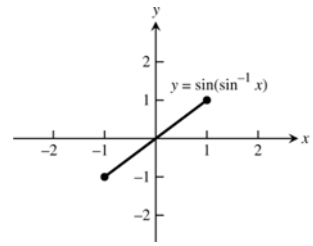
(b) Domain: $-\infty < x < \infty$; Range: $-\infty < y < \infty$
The graph of $y = \tan^{-1}(\tan x)$ is periodic, the graph of $y = \tan(\tan^{-1} x) = x$ for $-\infty \le x < \infty$.



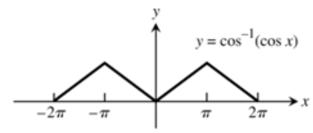
128. (a) Domain: $-\infty < x < \infty$; Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



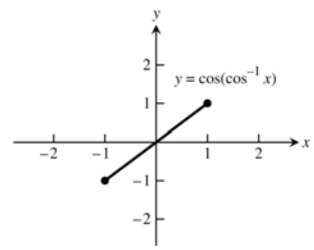
(b) Domain: $-1 \le x \le 1$; Range: $-1 \le y \le 1$
The graph of $y = \sin^{-1}(\sin x)$ is periodic; the graph of $y = \sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$.



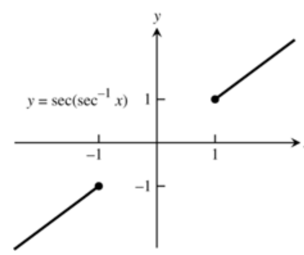
129. (a) Domain: $-\infty < x < \infty$; Range: $0 \le y \le \pi$



(b) Domain: $-1 \le x \le 1$; Range: $-1 \le y \le 1$
The graph of $y = \cos^{-1}(\cos x)$ is periodic; the graph of $y = \cos(\cos^{-1} x) = x$ for $-1 \le x \le 1$.



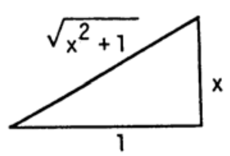
130. Since the domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$, we have $\sec(\sec^{-1} x) = x$ for $|x| \ge 1$. The graph of $y = \sec(\sec^{-1} x)$ is the line $y = x$ with the open line segment from $(-1, -1)$ to $(1, 1)$ removed.

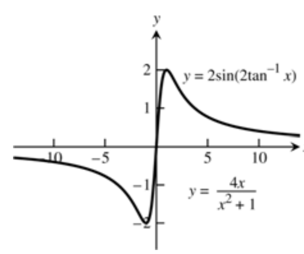


131. The graphs are identical for $y = 2\sin\left(2\tan^{-1} x\right)$

$= 4\left[\sin\left(\tan^{-1} x\right)\right]\left[\cos\left(\tan^{-1} x\right)\right]$

$= 4\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right) = \frac{4x}{x^2+1}$ from the triangle

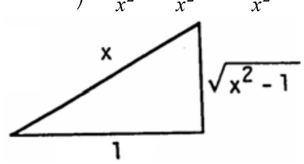


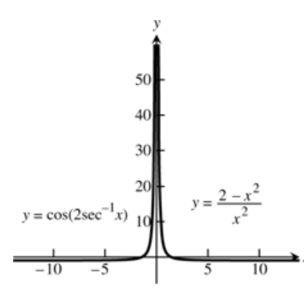


132. The graphs are identical for $y = \cos\left(2\sec^{-1} x\right)$

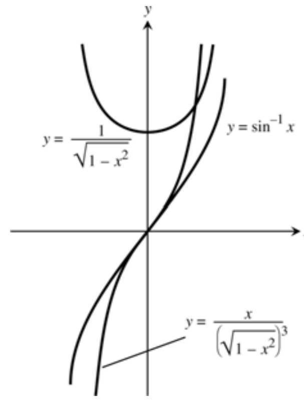
$\cos^2\left(\sec^{-1} x\right) - \sin^2\left(\sec^{-1} x\right) = \frac{1}{x^2} - \frac{x^2-1}{x^2} = \frac{2-x^2}{x^2}$

from the triangle

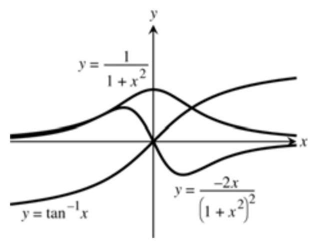




133. The values of f increase over the interval $[-1, 1]$ because $f' > 0$, and the graph of f steepens as the values of f' increase toward the ends of the interval. The graph of f is concave down to the left of the origin where $f'' < 0$, and concave up to the right of the origin where $f'' > 0$, There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local minimum value.



134. The values of f increase throughout the interval $(-\infty, \infty)$ because $f' > 0$, and they increase most rapidly near the origin where the values of f' are relatively large. The graph of f is concave up to the left of the origin where $f'' > 0$, and concave down to the right of the origin where $f'' < 0$. There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local maximum value.



7.7 HYPERBOLIC FUNCTIONS

1. $\sinh x = -\frac{3}{4} \Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\left(-\frac{3}{4}\right)^2} = \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(-\frac{3}{4}\right)}{\left(\frac{5}{4}\right)} = -\frac{3}{5}$,

   $\coth x = \frac{1}{\tanh x} = -\frac{5}{3}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{4}{5}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = -\frac{4}{3}$

2. $\sinh x = \frac{4}{3} \Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{5}{3}\right)} = \frac{4}{5}$, $\coth x = \frac{1}{\tanh x} = \frac{5}{4}$,

   $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{3}{5}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{3}{4}$

3. $\cosh x = \frac{17}{15}, x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \sqrt{\frac{289}{225} - 1} = \sqrt{\frac{64}{225}} = \frac{8}{15}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{8}{15}\right)}{\left(\frac{17}{15}\right)} = \frac{8}{17}$,

   $\coth x = \frac{1}{\tanh x} = \frac{17}{8}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{15}{17}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{15}{8}$

4. $\cosh x = \frac{13}{5}, x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\frac{169}{25} - 1} = \sqrt{\frac{144}{25}} = \frac{12}{5}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{12}{5}\right)}{\left(\frac{13}{5}\right)} = \frac{12}{13}$,

   $\coth x = \frac{1}{\tanh x} = \frac{13}{12}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{5}{13}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{5}{12}$

5. $2\cosh(\ln x) = 2\left(\frac{e^{\ln x} + e^{-\ln x}}{2}\right) = e^{\ln x} + \frac{1}{e^{\ln x}} = x + \frac{1}{x}$

6. $\sinh(2\ln x) = \frac{e^{2\ln x} - e^{-2\ln x}}{2} = \frac{e^{\ln x^2} - e^{\ln x^{-2}}}{2} = \frac{\left(x^2 - \frac{1}{x^2}\right)}{2} = \frac{x^4 - 1}{2x^2}$

7. $\cosh 5x + \sinh 5x = \frac{e^{5x} + e^{-5x}}{2} + \frac{e^{5x} - e^{-5x}}{2} = e^{5x}$

8. $\cosh 3x - \sinh 3x = \frac{e^{3x} + e^{-3x}}{2} - \frac{e^{3x} - e^{-3x}}{2} = e^{-3x}$

9. $(\sinh x + \cosh x)^4 = \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}\right)^4 = \left(e^x\right)^4 = e^{4x}$

10. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) = \ln\left(\cosh^2 x - \sinh^2 x\right) = \ln 1 = 0$

11. (a) $\sinh 2x = \sinh(x + x) = \sinh x \cosh x + \cosh x \sinh x = 2\sinh x \cosh x$

(b) $\cosh 2x = \cosh(x + x) = \cosh x \cosh x + \sinh x \sin x = \cosh^2 x + \sinh^2 x$

12. $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left[\left(e^x + e^{-x}\right) + \left(e^x - e^{-x}\right)\right]\left[\left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right)\right]$
$= \frac{1}{4}\left(2e^x\right)\left(2e^{-x}\right) = \frac{1}{4}\left(4e^0\right) = \frac{1}{4}(4) = 1$

13. $y = 6\sinh\frac{x}{3} \Rightarrow \frac{dy}{dx} = 6\left(\cosh\frac{x}{3}\right)\left(\frac{1}{3}\right) = 2\cosh\frac{x}{3}$

14. $y = \frac{1}{2}\sinh(2x + 1) \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\cosh(2x + 1)\right](2) = \cosh(2x + 1)$

15. $y = 2\sqrt{t}\tanh\sqrt{t} = 2t^{1/2}\tanh t^{1/2} \Rightarrow \frac{dy}{dt} = \left[\operatorname{sech}^2\left(t^{1/2}\right)\left(\frac{1}{2}t^{-1/2}\right)\right]\left(2t^{1/2}\right) + \left(\tanh t^{1/2}\right)\left(t^{-1/2}\right) = \operatorname{sech}^2\sqrt{t} + \frac{\tanh\sqrt{t}}{\sqrt{t}}$

16. $y = t^2\tanh\frac{1}{t} = t^2\tanh t^{-1} \Rightarrow \frac{dy}{dt} = \left[\operatorname{sech}^2\left(t^{-1}\right)\left(-t^{-2}\right)\right]\left(t^2\right) + (2t)\left(\tanh t^{-1}\right) = -\operatorname{sech}^2\frac{1}{t} + 2t\tanh\frac{1}{t}$

17. $y = \ln(\sinh z) \Rightarrow \frac{dy}{dz} = \frac{\cosh z}{\sinh z} = \coth z$

18. $y = \ln(\cosh z) \Rightarrow \frac{dy}{dz} = \frac{\sinh z}{\cosh z} = \tanh z$

19. $y = (\operatorname{sech}\theta)(1 - \ln\operatorname{sech}\theta) \Rightarrow \frac{dy}{d\theta} = \left(-\frac{-\operatorname{sech}\theta\tanh\theta}{\operatorname{sech}\theta}\right)(\operatorname{sech}\theta) + (-\operatorname{sech}\theta\tanh\theta)(1 - \ln\operatorname{sech}\theta)$
$= \operatorname{sech}\theta\tanh\theta - (\operatorname{sech}\theta\tanh\theta)(1 - \ln\operatorname{sech}\theta) = (\operatorname{sech}\theta\tanh\theta)\left[1 - (1 - \ln\operatorname{sech}\theta)\right]$
$= (\operatorname{sech}\theta\tanh\theta)(\ln\operatorname{sech}\theta)$

20. $y = (\operatorname{csch}\theta)(1 - \ln\operatorname{csch}\theta) \Rightarrow \frac{dy}{d\theta} = (\operatorname{csch}\theta)\left(-\frac{-\operatorname{csch}\theta\coth\theta}{\operatorname{csch}\theta}\right) + (1 - \ln\operatorname{csch}\theta)(-\operatorname{csch}\theta\coth\theta)$
$= \operatorname{csch}\theta\coth\theta - (1 - \ln\operatorname{csch}\theta)(\operatorname{csch}\theta\coth\theta) = (\operatorname{csch}\theta\coth\theta)(1 - 1 + \ln\operatorname{csch}\theta)$
$= (\operatorname{csch}\theta\coth\theta)(\ln\operatorname{csch}\theta)$

21. $y = \ln\cosh v - \frac{1}{2}\tanh^2 v \Rightarrow \frac{dy}{dv} = \frac{\sinh v}{\cosh v} - \left(\frac{1}{2}\right)(2\tanh v)\left(\operatorname{sech}^2 v\right) = \tanh v - (\tanh v)\left(\operatorname{sech}^2 v\right)$
$= (\tanh v)\left(1 - \operatorname{sech}^2 v\right) = (\tanh v)\left(\tanh^2 v\right) = \tanh^3 v$

22. $y = \ln\sinh v - \frac{1}{2}\coth^2 v \Rightarrow \frac{dy}{dv} = \frac{\cosh v}{\sinh v} - \left(\frac{1}{2}\right)(2\coth v)\left(-\operatorname{csch}^2 v\right) = \coth v + (\coth v)\left(\operatorname{csch}^2 v\right)$
$= (\coth v)\left(1 + \operatorname{csch}^2 v\right) = (\coth v)\left(\coth^2 v\right) = \coth^3 v$

23. $y = \left(x^2+1\right)\operatorname{sech}(\ln x) = \left(x^2+1\right)\left(\frac{2}{e^{\ln x} + e^{-\ln x}}\right) = \left(x^2+1\right)\left(\frac{2}{x + x^{-1}}\right) = \left(x^2+1\right)\left(\frac{2x}{x^2+1}\right) = 2x \Rightarrow \frac{dy}{dx} = 2$

24. $y = \left(4x^2-1\right)\operatorname{csch}(\ln 2x) = \left(4x^2-1\right)\left(\frac{2}{e^{\ln 2x} - e^{-\ln 2x}}\right) = \left(4x^2-1\right)\left(\frac{2}{2x - (2x)^{-1}}\right) = \left(4x^2-1\right)\left(\frac{4x}{4x^2-1}\right) = 4x \Rightarrow \frac{dy}{dx} = 4$

25. $y = \sinh^{-1}\sqrt{x} = \sinh^{-1}\left(x^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)x^{-1/2}}{\sqrt{1 + \left(x^{1/2}\right)^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}} = \frac{1}{2\sqrt{x(1+x)}}$

26. $y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1}\left(2(x+1)^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{(2)\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{\left[2(x+1)^{1/2}\right]^2 - 1}} = \frac{1}{\sqrt{x+1}\sqrt{4x+3}} = \frac{1}{\sqrt{4x^2+7x+3}}$

27. $y = (1-\theta)\tanh^{-1}\theta \Rightarrow \frac{dy}{d\theta} = (1-\theta)\left(\frac{1}{1-\theta^2}\right) + (-1)\tanh^{-1}\theta = \frac{1}{1+\theta} - \tanh^{-1}\theta$

28. $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta+1) \Rightarrow \frac{dy}{d\theta} = \left(\theta^2 + 2\theta\right)\left[\frac{1}{1-(\theta+1)^2}\right] + (2\theta+2)\tanh^{-1}(\theta+1)$
$= \frac{\theta^2+2\theta}{-\theta^2-2\theta} + (2\theta+2)\tanh^{-1}(\theta+1) = (2\theta+2)\tanh^{-1}(\theta+1) - 1$

29. $y = (1-t)\coth^{-1}\sqrt{t} = (1-t)\coth^{-1}\left(t^{1/2}\right) \Rightarrow \frac{dy}{dt} = (1-t)\left[\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1 - \left(t^{1/2}\right)^2}\right] + (-1)\coth^{-1}\left(t^{1/2}\right) = \frac{1}{2\sqrt{t}} - \coth^{-1}\sqrt{t}$

30. $y = \left(1-t^2\right)\coth^{-1} t \Rightarrow \frac{dy}{dt} = \left(1-t^2\right)\left(\frac{1}{1-t^2}\right) + (-2t)\coth^{-1} t = 1 - 2t\coth^{-1} t$

31. $y = \cos^{-1} x - x\operatorname{sech}^{-1} x \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} - \left[x\left(\frac{-1}{x\sqrt{1-x^2}}\right) + (1)\operatorname{sech}^{-1} x\right] = \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} - \operatorname{sech}^{-1} x = -\operatorname{sech}^{-1} x$

32. $y = \ln x + \sqrt{1-x^2}\operatorname{sech}^{-1} x = \ln x + \left(1-x^2\right)^{1/2}\operatorname{sech}^{-1} x$
$\Rightarrow \frac{dy}{dx} = \frac{1}{x} + \left(1-x^2\right)^{1/2}\left(\frac{-1}{x\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)\operatorname{sech}^{-1} x = \frac{1}{x} - \frac{1}{x} - \frac{x}{\sqrt{1-x^2}}\operatorname{sech}^{-1} x = \frac{-x}{\sqrt{1-x^2}}\operatorname{sech}^{-1} x$

33. $y=\operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta} \Rightarrow \frac{dy}{d\theta}=-\frac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^{\theta}}{\left(\frac{1}{2}\right)^{\theta}\sqrt{1+\left[\left(\frac{1}{2}\right)^{\theta}\right]^{2}}}=-\frac{\ln(1)-\ln(2)}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}=\frac{\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$

34. $y=\operatorname{csch}^{-1} 2^{\theta} \Rightarrow \frac{dy}{d\theta}=-\frac{(\ln 2)2^{\theta}}{2^{\theta}\sqrt{1+\left(2^{\theta}\right)^{2}}}=\frac{-\ln 2}{\sqrt{1+2^{2\theta}}}$

35. $y=\sinh^{-1}(\tan x) \Rightarrow \frac{dy}{dx}=\frac{\sec^2 x}{\sqrt{1+(\tan x)^2}}=\frac{\sec^2 x}{\sqrt{\sec^2 x}}=\frac{\sec^2 x}{|\sec x|}=\frac{|\sec x||\sec x|}{|\sec x|}=|\sec x|$

36. $y=\cosh^{-1}(\sec x) \Rightarrow \frac{dy}{dx}=\frac{(\sec x)(\tan x)}{\sqrt{\sec^2 x-1}}=\frac{(\sec x)(\tan x)}{\sqrt{\tan^2 x}}=\frac{(\sec x)(\tan x)}{|\tan x|}=\sec x,\quad 0<x<\frac{\pi}{2}$

37. (a) If $y=\tan^{-1}(\sinh x)+C$, then $\frac{dy}{dx}=\frac{\cosh x}{1+\sinh^2 x}=\frac{\cosh x}{\cosh^2 x}=\operatorname{sech} x$, which verifies the formula

(b) If $y=\sin^{-1}(\tanh x)+C$, then $\frac{dy}{dx}=\frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}}=\frac{\operatorname{sech}^2 x}{\operatorname{sech} x}=\operatorname{sech} x$, which verifies the formula

38. If $y=\frac{x^2}{2}\operatorname{sech}^{-1}x-\frac{1}{2}\sqrt{1-x^2}+C$, then $\frac{dy}{dx}=x\operatorname{sech}^{-1}x+\frac{x^2}{2}\left(\frac{-1}{x\sqrt{1-x^2}}\right)+\frac{2x}{4\sqrt{1-x^2}}=x\operatorname{sech}^{-1}x$, which verifies the formula

39. If $y=\frac{x^2-1}{2}\coth^{-1}x+\frac{x}{2}+C$, then $\frac{dy}{dx}=x\coth^{-1}x+\left(\frac{x^2-1}{2}\right)\left(\frac{1}{1-x^2}\right)+\frac{1}{2}=x\coth^{-1}x$, which verifies the formula

40. If $y=x\tanh^{-1}x+\frac{1}{2}\ln\left(1-x^2\right)+C$, then $\frac{dy}{dx}=\tanh^{-1}x+x\left(\frac{1}{1-x^2}\right)+\frac{1}{2}\left(\frac{-2x}{1-x^2}\right)=\tanh^{-1}x$, which verifies the formula

41. $\int \sinh 2x\,dx=\frac{1}{2}\int \sinh u\,du$, where $u=2x$ and $du=2\,dx$

$=\frac{\cosh u}{2}+C=\frac{\cosh 2x}{2}+C$

42. $\int \sinh \frac{x}{5}\,dx=5\int \sinh u\,\mathrm{du}$, where $u=\frac{x}{5}$ and $du=\frac{1}{5}dx$

$=5\cosh u+C=5\cosh\frac{x}{5}+C$

43. $\int 6\cosh\left(\frac{x}{2}-\ln 3\right)dx=12\int \cosh u\,du$, where $u=\frac{x}{2}-\ln 3$ and $du=\frac{1}{2}dx$

$=12\sinh u+C=12\sinh\left(\frac{x}{2}-\ln 3\right)+C$

44. $\int 4\cosh(3x-\ln 2)\,dx=\frac{4}{3}\int \cosh u\,du$, where $u=3x-\ln 2$ and $du=3\,dx$

$=\frac{4}{3}\sinh u+C=\frac{4}{3}\sinh(3x-\ln 2)+C$

45. $\int \tanh \frac{x}{7}\, dx = 7\int \frac{\sinh u}{\cosh u}\, du$, where $u = \frac{x}{7}$ and $du = \frac{1}{7}dx$

$$= 7\ln|\cosh u| + C_1 = 7\ln\left|\cosh \frac{x}{7}\right| + C_1 = 7\ln\left|\frac{e^{x/7}+e^{-x/7}}{2}\right| + C_1 = 7\ln\left|e^{x/7}+e^{-x/7}\right| - 7\ln 2 + C_1$$

$$= 7\ln\left|e^{x/7}+e^{-x/7}\right| + C$$

46. $\int \coth \frac{\theta}{\sqrt{3}}\, d\theta = \sqrt{3}\int \frac{\cosh u}{\sinh u}\, du$, where $u = \frac{\theta}{\sqrt{3}}$ and $du = \frac{d\theta}{\sqrt{3}}$

$$= \sqrt{3}\ln|\sinh u| + C_1 = \sqrt{3}\ln\left|\sinh \frac{\theta}{\sqrt{3}}\right| + C_1 = \sqrt{3}\ln\left|\frac{e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}}{2}\right| + C_1$$

$$= \sqrt{3}\ln\left|e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}\right| - \sqrt{3}\ln 2 + C_1 = \sqrt{3}\ln\left|e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}\right| + C$$

47. $\int \operatorname{sech}^2\left(x - \frac{1}{2}\right)dx = \int \operatorname{sech}^2 u\, du$, where $u = \left(x - \frac{1}{2}\right)$ and $du = dx$

$$= \tanh u + C = \tanh\left(x - \frac{1}{2}\right) + C$$

48. $\int \operatorname{csch}^2(5-x)dx = -\int \operatorname{csch}^2 u\, du$, where $u = (5-x)$ and $du = -dx$

$$= -(-\coth u) + C = \coth u + C = \coth(5-x) + C$$

49. $\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}}{\sqrt{t}}\, dt = 2\int \operatorname{sech} u \tanh u\, du$, where $u = \sqrt{t} = t^{1/2}$ and $du = \frac{dt}{2\sqrt{t}}$

$$= 2(-\operatorname{sech} u) + C = -2\operatorname{sech}\sqrt{t} + C$$

50. $\int \frac{\operatorname{csch}(\ln t)\coth(\ln t)}{t}\, dt = \int \operatorname{csch} u \coth u\, du$, where $u = \ln t$ and $du = \frac{dt}{t}$

$$= -\operatorname{csch} u + C = -\operatorname{csch}(\ln t) + C$$

51. $\int_{\ln 2}^{\ln 4} \coth x\, dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\, dx = \int_{3/4}^{15/8} \frac{1}{u}\, du$ where $u = \sinh x, du = \cosh x\, dx$;

$x = \ln 2 \Rightarrow u = \sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \left(\frac{1}{2}\right)}{2} = \frac{3}{4}, x = \ln 4 \Rightarrow u = \sinh(\ln 4) = \frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4 - \left(\frac{1}{4}\right)}{2} = \frac{15}{8}$

$$= \left[\ln|u|\right]_{3/4}^{15/8} = \ln\left|\frac{15}{8}\right| - \ln\left|\frac{3}{4}\right| = \ln\left|\frac{15}{8}.\frac{4}{3}\right| = \ln\frac{5}{2}$$

52. $\int_0^{\ln 2} \tanh 2x\, dx = \int_0^{\ln 2} \frac{\sinh 2x}{\cosh 2x}\, dx = \frac{1}{2}\int_1^{17/8} \frac{1}{u}\, du$ where $u = \cosh 2x, du = 2\sinh(2x)\, dx$,

$x = 0 \Rightarrow u = \cosh 0 = 1, x = \ln 2 \Rightarrow u = \cosh(2\ln 2) = \cosh(\ln 4) = \frac{e^{\ln 4} + e^{-\ln 4}}{2} = \frac{4 + \left(\frac{1}{4}\right)}{2} = \frac{17}{8}$

$$= \frac{1}{2}\left[\ln|u|\right]_1^{17/8} = \frac{1}{2}\left[\ln\left(\frac{17}{8}\right) - \ln 1\right] = \frac{1}{2}\ln\frac{17}{8}$$

53. $\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\, d\theta = \int_{-\ln 4}^{-\ln 2} 2e^{\theta}\left(\frac{e^{\theta}+e^{-\theta}}{2}\right)d\theta = \int_{-\ln 4}^{-\ln 2}\left(e^{2\theta}+1\right)d\theta = \left[\frac{e^{2\theta}}{2} + \theta\right]_{-\ln 4}^{-\ln 2}$

$$= \left(\frac{e^{-2\ln 2}}{2} - \ln 2\right) - \left(\frac{e^{-2\ln 4}}{2} - \ln 4\right) = \left(\frac{1}{8} - \ln 2\right) - \left(\frac{1}{32} - \ln 4\right) = \frac{3}{32} - \ln 2 + 2\ln 2 = \frac{3}{32} + \ln 2$$

54. $\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\, d\theta = \int_0^{\ln 2} 4e^{-\theta}\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)d\theta = 2\int_0^{\ln 2}\left(1-e^{-2\theta}\right)d\theta = 2\left[\theta+\frac{e^{-2\theta}}{2}\right]_0^{\ln 2}$
$= 2\left[\left(\ln 2+\frac{e^{-2\ln 2}}{2}\right)-\left(0+\frac{e^0}{2}\right)\right] = 2\left(\ln 2+\frac{1}{8}-\frac{1}{2}\right) = 2\ln 2+\frac{1}{4}-1 = \ln 4-\frac{3}{4}$

55. $\int_{-\pi/4}^{\pi/4}\cosh(\tan\theta)\sec^2\theta\, d\theta = \int_{-1}^{1}\cosh u\, du$ where $u=\tan\theta$, $du=\sec^2\theta\, d\theta$, $x=-\frac{\pi}{4}\Rightarrow u=-1, x=\frac{\pi}{4}\Rightarrow u=1,$
$= \left[\sinh u\right]_{-1}^{1} = \sinh(1)-\sinh(-1) = \left(\frac{e^1-e^{-1}}{2}\right)-\left(\frac{e^{-1}-e^1}{2}\right) = \frac{e-e^{-1}-e^{-1}+e}{2} = e-e^{-1}$

56. $\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\, d\theta = 2\int_0^1 \sinh u\, du$ where $u=\sin\theta, du=\cos\theta\, d\theta,\ x=0\Rightarrow u=0, x=\frac{\pi}{2}\Rightarrow u=1$
$= 2\left[\cosh u\right]_0^1 = 2(\cosh 1-\cosh 0) = 2\left(\frac{e+e^{-1}}{2}-1\right) = e+e^{-1}-2$

57. $\int_1^2 \frac{\cosh(\ln t)}{t}\, dt = \int_0^{\ln 2}\cosh u\, du$ where $u=\ln t, du=\frac{1}{t}\, dt,\ x=1\Rightarrow u=0, x=2\Rightarrow u=\ln 2$
$= \left[\sinh u\right]_0^{\ln 2} = \sinh(\ln 2)-\sinh(0) = \frac{e^{\ln 2}-e^{-\ln 2}}{2}-0 = \frac{2-\frac{1}{2}}{2} = \frac{3}{4}$

58. $\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}}\, dx = 16\int_1^2 \cosh u\, du$ where $u=\sqrt{x}=x^{1/2}, du=\frac{1}{2}x^{-1/2}dx=\frac{dx}{2\sqrt{x}},\ x=1\Rightarrow u=1, x=4\Rightarrow u=2$
$= 16\left[\sinh u\right]_1^2 = 16(\sinh 2-\sinh 1) = 16\left[\left(\frac{e^2-e^{-2}}{2}\right)-\left(\frac{e-e^{-1}}{2}\right)\right] = 8\left(e^2-e^{-2}-e+e^{-1}\right)$

59. $\int_{-\ln 2}^{0}\cosh^2\left(\frac{x}{2}\right)dx = \int_{-\ln 2}^{0}\frac{\cosh x+1}{2}\, dx = \frac{1}{2}\int_{-\ln 2}^{0}(\cosh x+1)dx = \frac{1}{2}\left[\sinh x+x\right]_{-\ln 2}^{0}$
$= \frac{1}{2}\left[(\sinh 0+0)-(\sinh(-\ln 2)-\ln 2)\right] = \frac{1}{2}\left[(0+0)-\left(\frac{e^{-\ln 2}-e^{\ln 2}}{2}-\ln 2\right)\right] = \frac{1}{2}\left[-\frac{\left(\frac{1}{2}\right)-2}{2}+\ln 2\right] = \frac{1}{2}\left(1-\frac{1}{4}+\ln 2\right)$
$= \frac{3}{8}+\frac{1}{2}\ln 2 = \frac{3}{8}+\ln\sqrt{2}$

60. $\int_0^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right)dx = \int_0^{\ln 10} 4\left(\frac{\cosh x-1}{2}\right)dx = 2\int_0^{\ln 10}(\cosh x-1)dx = 2\left[\sinh x-x\right]_0^{\ln 10}$
$= 2\left[(\sinh(\ln 10)-\ln 10)-(\sinh 0-0)\right] = e^{\ln 10}-e^{-\ln 10}-2\ln 10 = 10-\frac{1}{10}-2\ln 10 = 9.9-2\ln 10$

61. $\sinh^{-1}\left(\frac{-5}{12}\right) = \ln\left(-\frac{5}{12}+\sqrt{\frac{25}{144}+1}\right) = \ln\left(\frac{2}{3}\right)$

62. $\cosh^{-1}\left(\frac{5}{3}\right) = \ln\left(\frac{5}{3}+\sqrt{\frac{25}{9}-1}\right) = \ln 3$

63. $\tanh^{-1}\left(-\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1-(1/2)}{1+(1/2)}\right) = -\frac{\ln 3}{3}$

64. $\coth^{-1}\left(\frac{5}{4}\right) = \frac{1}{2}\ln\left(\frac{(9/4)}{(1/4)}\right) = \frac{1}{2}\ln 9 = \ln 3$

65. $\operatorname{sech}^{-1}\left(\frac{3}{5}\right) = \ln\left(\frac{1+\sqrt{1-(9/25)}}{(3/5)}\right) = \ln 3$

66. $\operatorname{csch}^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \ln\left(-\sqrt{3}+\frac{\sqrt{4/3}}{\left(1/\sqrt{3}\right)}\right) = \ln\left(-\sqrt{3}+2\right)$

67. (a) $\int_0^{2\sqrt{3}}\frac{dx}{\sqrt{4+x^2}} = \left[\sinh^{-1}\frac{x}{2}\right]_0^{2\sqrt{3}} = \sinh^{-1}\sqrt{3}-\sinh 0 = \sinh^{-1}\sqrt{3}$
(b) $\sinh^{-1}\sqrt{3} = \ln\left(\sqrt{3}+\sqrt{3+1}\right) = \ln\left(\sqrt{3}+2\right)$

68. (a) $\int_0^{1/3} \frac{6dx}{\sqrt{1+9x^2}} = 2\int_0^1 \frac{dx}{\sqrt{a^2+u^2}}$, where $u = 3x, du = 3\ dx,\ a = 1$

$$= \left[2\sinh^{-1} u\right]_0^1 = 2\left(\sinh^{-1} 1 - \sinh^{-1} 0\right) = 2\sinh^{-1} 1$$

(b) $2\sinh^{-1} 1 = 2\ln\left(1+\sqrt{1^2+1}\right) = 2\ln\left(1+\sqrt{2}\right)$

69. (a) $\int_{5/4}^{2} \frac{1}{1-x^2}\,dx = \left[\coth^{-1} x\right]_{5/4}^{2} = \coth^{-1} 2 - \coth^{-1} \frac{5}{4}$

(b) $\coth^{-1} 2 - \coth^{-1} \frac{5}{4} = \frac{1}{2}\left[\ln 3 - \ln\left(\frac{9/4}{1/4}\right)\right] = \frac{1}{2}\ln\frac{1}{3}$

70. (a) $\int_0^{1/2} \frac{1}{1-x^2}\,dx = \left[\tanh^{-1} x\right]_0^{1/2} = \tanh^{-1}\frac{1}{2} - \tanh^{-1} 0 = \tanh^{-1}\frac{1}{2}$

(b) $\tanh^{-1}\frac{1}{2} = \frac{1}{2}\ln\left(\frac{1+(1/2)}{1-(1/2)}\right) = \frac{1}{2}\ln 3$

71. (a) $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = \int_{4/5}^{12/13} \frac{du}{u\sqrt{a^2-u^2}}$, $u = 4x, du = 4\ dx, a = 1$

$$= \left[-\operatorname{sech}^{-1} u\right]_{4/5}^{12/13} = -\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5}$$

(b) $-\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5} = -\ln\left(\frac{1+\sqrt{1-(12/13)^2}}{(12/13)}\right) + \ln\left(\frac{1+\sqrt{1-(4/5)^2}}{(4/5)}\right) = -\ln\left(\frac{13+\sqrt{169-144}}{12}\right) + \ln\left(\frac{5+\sqrt{25-16}}{4}\right)$

$= \ln\left(\frac{5+3}{4}\right) - \ln\left(\frac{13+5}{12}\right) = \ln 2 - \ln\frac{3}{2} = \ln\left(2\cdot\frac{2}{3}\right) = \ln\frac{4}{3}$

72. (a) $\int_1^2 \frac{dx}{x\sqrt{4+x^2}} = \left[-\frac{1}{2}\operatorname{csch}^{-1}\left|\frac{x}{2}\right|\right]_1^2 = -\frac{1}{2}\left(\operatorname{csch}^{-1} 1 - \operatorname{csch}^{-1}\frac{1}{2}\right) = \frac{1}{2}\left(\operatorname{csch}^{-1}\frac{1}{2} - \operatorname{csch}^{-1} 1\right)$

(b) $\frac{1}{2}\left(\operatorname{csch}^{-1}\frac{1}{2} - \operatorname{csch}^{-1} 1\right) = \frac{1}{2}\left[\ln\left(2+\frac{\sqrt{5/4}}{(1/2)}\right) - \ln\left(1+\sqrt{2}\right)\right] = \frac{1}{2}\ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)$

73. (a) $\int_0^{\pi} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^0 \frac{1}{\sqrt{1+u^2}}\,du$ where $u = \sin x, du = \cos x\ dx$;

$$= \left[\sinh^{-1} u\right]_0^0 = \sinh^{-1} 0 - \sinh^{-1} 0 = 0$$

(b) $\sinh^{-1} 0 - \sinh^{-1} 0 = \ln\left(0+\sqrt{0+1}\right) - \ln\left(0+\sqrt{0+1}\right) = 0$

74. (a) $\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}} = \int_0^1 \frac{du}{\sqrt{a^2+u^2}}$, where $u = \ln x, du = \frac{1}{x}dx,\ a = 1$

$$= \left[\sinh^{-1} u\right]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0 = \sinh^{-1} 1$$

(b) $\sinh^{-1} 1 - \sinh^{-1} 0 = \ln\left(1+\sqrt{1^2+1}\right) - \ln\left(0+\sqrt{0^2+1}\right) = \ln\left(1+\sqrt{2}\right)$

75. Let $E(x) = \frac{f(x)+f(-x)}{2}$ and $O(x) = \frac{f(x)+f(-x)}{2}$. Then $E(x)+O(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = \frac{2f(x)}{2} = f(x)$.
Also, $E(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(x)+f(-x)}{2} = E(x) \Rightarrow E(x)$ is even, and $O(-x) = \frac{f(-x)-f(-(-x))}{2}$

$= -\frac{f(x)-f(-x)}{2} = -O(x) \Rightarrow O(x)$ is odd. Consequently, $f(x)$ can be written as a sum of an even and an odd function. $f(x) = \frac{f(x)+f(-x)}{2}$ because $\frac{f(x)-f(-x)}{2} = 0$ if *f* is even, and $f(x) = \frac{f(x)-f(-x)}{2}$ because $\frac{f(x)+f(-x)}{2} = 0$ if *f* is odd. Thus, if *f* is even $f(x) = \frac{2f(x)}{2} + 0$ and if *f* is odd, $f(x) = 0 + \frac{2f(x)}{2}$

76. $y = \sinh^{-1} x \Rightarrow x = \sinh y \Rightarrow x = \frac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - \frac{1}{e^y} \Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y - 1 = 0$
$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} \Rightarrow e^y = x + \sqrt{x^2+1} \Rightarrow \sinh^{-1} x = y = \ln\left(x + \sqrt{x^2+1}\right)$. Since $e^y > 0$, we cannot choose $e^y = x - \sqrt{x^2+1}$ because $x - \sqrt{x^2+1} < 0$.

77. (a) $v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}}\, t\right) \Rightarrow \frac{dv}{dt} = \sqrt{\frac{mg}{k}}\left[\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\, t\right)\right]\left(\sqrt{\frac{gk}{m}}\right) = g \operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\, t\right)$. Thus
$m\frac{dv}{dt} = mg \operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\, t\right) = mg\left(1 - \tanh^2\left(\sqrt{\frac{gk}{m}}\, t\right)\right) = mg - kv^2$. Also, since $\tanh x = 0$ when $x = 0$, $v = 0$ when $t = 0$.

(b) $\lim\limits_{t\to\infty} v = \lim\limits_{t\to\infty} \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}} t\right) = \sqrt{\frac{mg}{k}} \lim\limits_{t\to\infty} \tanh\left(\sqrt{\frac{kg}{m}} t\right) = \sqrt{\frac{mg}{k}}(1) = \sqrt{\frac{mg}{k}}$

(c) $\sqrt{\frac{735}{0.235}} = \sqrt{\frac{735{,}000}{235}} \approx 55.93$ s

78. (a) $s(t) = a\cos kt + b\sin kt \Rightarrow \frac{ds}{dt} = -ak\sin kt + bk\cos kt \Rightarrow \frac{d^2s}{dt^2} = -ak^2\cos kt - bk^2\sin kt$
$= -k^2(a\cos kt + b\sin kt) = -k^2 s(t) \Rightarrow$ acceleration is proportional to *s*. The negative constant $-k^2$ implies that the acceleration is directed toward the origin.

(b) $s(t) = a\cosh kt + b\sinh kt \Rightarrow \frac{ds}{dt} = ak\sinh kt + bk\cosh kt \Rightarrow \frac{d^2s}{dt^2} = ak^2\cosh kt + bk^2\sinh kt$
$= k^2(a\cosh kt + b\sinh kt) = k^2 s(t) \Rightarrow$ acceleration is proportional to *s*. The positive constant k^2 implies that the acceleration is directed away from the origin.

79. $V = \pi\int_0^2 \left(\cosh^2 x - \sinh^2 x\right)dx = \pi\int_0^2 1\, dx = 2\pi$

80. $V = 2\pi\int_0^{\ln\sqrt{3}} \operatorname{sech}^2 x\, dx = 2\pi\left[\tanh x\right]_0^{\ln\sqrt{3}} = 2\pi\left[\frac{\sqrt{3}-(1/\sqrt{3})}{\sqrt{3}+(1/\sqrt{3})}\right] = \pi$

81. $y = \frac{1}{2}\cosh 2x \Rightarrow y' = \sinh 2x \Rightarrow L = \int_0^{\ln\sqrt{5}} \sqrt{1+(\sinh 2x)^2}\, dx = \int_0^{\ln\sqrt{5}} \cosh 2x\, dx = \left[\frac{1}{2}\sinh 2x\right]_0^{\ln\sqrt{5}}$
$= \left[\frac{1}{2}\left(\frac{e^{2x}-e^{-2x}}{2}\right)\right]_0^{\ln\sqrt{5}} = \frac{1}{4}\left(5 - \frac{1}{5}\right) = \frac{6}{5}$

82. (a) $\lim\limits_{x\to\infty} \tanh x = \lim\limits_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x\to\infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{\left(e^x - \frac{1}{e^x}\right)}{\left(e^x \frac{1}{e^x}\right)} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{1 - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}} = \frac{1-0}{1+0} = 1$

(b) $\lim\limits_{x\to-\infty} \tanh x = \lim\limits_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x\to-\infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim\limits_{x\to-\infty} \frac{\left(e^x - \frac{1}{e^x}\right)}{\left(e^x + \frac{1}{e^x}\right)} \cdot \frac{e^x}{e^x} = \lim\limits_{x\to-\infty} \frac{e^{2x}-1}{e^{2x}+1} = \frac{0-1}{0+1} = -1$

(c) $\lim\limits_{x\to\infty} \sinh x = \lim\limits_{x\to\infty} \frac{e^x - e^{-x}}{2} = \lim\limits_{x\to\infty} \frac{e^x - \frac{1}{ex}}{2} = \lim\limits_{x\to\infty} \left(\frac{e^x}{2} - \frac{1}{2ex}\right) = \infty - 0 = \infty$

(d) $\lim\limits_{x\to-\infty} \sinh x = \lim\limits_{x\to-\infty} \frac{e^x - e^{-x}}{2} = \lim\limits_{x\to-\infty} \left(\frac{e^x}{2} - \frac{e^{-x}}{2}\right) = 0 - \infty = -\infty$

(e) $\lim\limits_{x\to\infty} \operatorname{sech} x = \lim\limits_{x\to\infty} \frac{2}{e^x + e^{-x}} = \lim\limits_{x\to\infty} \frac{2}{e^x + \frac{1}{e^x}} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{\frac{2}{e^x}}{1 + \frac{1}{e^{2x}}} = \frac{0}{1+0} = 0$

(f) $\lim\limits_{x\to\infty} \coth x = \lim\limits_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x\to\infty} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{\left(e^x + \frac{1}{e^x}\right)}{\left(e^x - \frac{1}{e^x}\right)} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{1 + \frac{1}{e^{2x}}}{1 - \frac{1}{e^{2x}}} = \frac{1+0}{1-0} = 1$

(g) $\lim\limits_{x\to 0^+} \coth x = \lim\limits_{x\to 0^+} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x\to 0^+} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim\limits_{x\to 0^+} \frac{e^{2x}+1}{e^{2x}-1} = +\infty$

(h) $\lim\limits_{x\to 0^-} \coth x = \lim\limits_{x\to 0^-} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x\to 0^-} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim\limits_{x\to 0^-} \frac{e^{2x}+1}{e^{2x}-1} = -\infty$

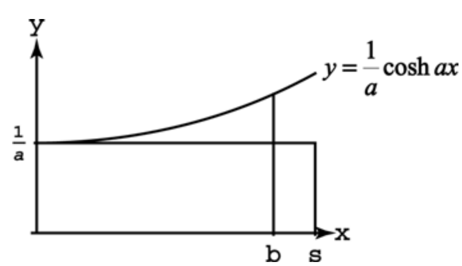
(i) $\lim\limits_{x\to-\infty} \operatorname{csch} x = \lim\limits_{x\to-\infty} \frac{2}{e^x - e^{-x}} = \lim\limits_{x\to-\infty} \frac{2}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim\limits_{x\to-\infty} \frac{2e^x}{e^{2x}-1} = \frac{0}{0-1} = 0$

83. (a) $y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right) \Rightarrow \tan\phi = \frac{dy}{dx} = \left(\frac{H}{w}\right)\left[\frac{w}{H}\sinh\left(\frac{w}{H}x\right)\right] = \sinh\left(\frac{w}{H}x\right)$

(b) The tension at P is given by $T\cos\phi = H \Rightarrow T = H\sec\phi = H\sqrt{1+\tan^2\phi} = H\sqrt{1+\left(\sinh\frac{w}{H}x\right)^2}$
$= H\cosh\left(\frac{w}{H}x\right) = w\left(\frac{H}{w}\right)\cosh\left(\frac{w}{H}x\right) = wy$

84. $s = \frac{1}{a}\sinh ax \Rightarrow \sinh ax = as \Rightarrow ax = \sinh^{-1} as \Rightarrow x = \frac{1}{a}\sinh^{-1} as$; $y = \frac{1}{a}\cosh ax = \frac{1}{a}\sqrt{\cosh^2 ax}$
$= \frac{1}{a}\sqrt{\sinh^2 ax + 1} = \frac{1}{a}\sqrt{a^2 s^2 + 1} = \sqrt{s^2 + \frac{1}{a^2}}$

85. To find the length of the curve: $y = \frac{1}{a}\cosh ax \Rightarrow y' = \sinh ax \Rightarrow L = \int_0^b \sqrt{1+(\sinh ax)^2}\,dx \Rightarrow L = \int_0^b \cosh ax\,dx$
$= \left[\frac{1}{a}\sinh ax\right]_0^b = \frac{1}{a}\sinh ab$. The area under the curve is $A = \int_0^b \frac{1}{a}\cosh ax\,dx = \left[\frac{1}{a^2}\sinh ax\right]_0^b = \frac{1}{a^2}\sinh ab$
$= \left(\frac{1}{a}\right)\left(\frac{1}{a}\sinh ab\right)$ which is the area of the rectangle of height $\frac{1}{a}$ and length L as claimed, and which is illustrated below.



86. (a) Let the point located at $(\cosh u, 0)$ be called T. Then $A(u) =$ area of the triangle ΔOTP minus the area under the curve $y=\sqrt{x^2-1}$ from A to $T \Rightarrow A(u)=\frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u}\sqrt{x^2-1}\,dx$.

(b) $A(u)=\frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u}\sqrt{x^2-1}\,dx \Rightarrow A'(u)=\frac{1}{2}\left(\cosh^2 u+\sinh^2 u\right)-\left(\sqrt{\cosh^2 u-1}\right)(\sinh u)$
$=\frac{1}{2}\cosh^2 u+\frac{1}{2}\sinh^2 u-\sinh^2 u=\frac{1}{2}\left(\cosh^2 u-\sinh^2 u\right)=\left(\frac{1}{2}\right)(1)=\frac{1}{2}$

(c) $A'(u)=\frac{1}{2} \Rightarrow A(u)=\frac{u}{2}+C$, and from part (a) we have $A(0)=0 \Rightarrow C=0 \Rightarrow A(u)=\frac{u}{2} \Rightarrow u=2A$

7.8 RELATIVE RATES OF GROWTH

1. (a) slower, $\lim\limits_{x\to\infty}\frac{x+3}{e^x}=\lim\limits_{x\to\infty}\frac{1}{e^x}=0$

(b) slower, $\lim\limits_{x\to\infty}\frac{x^3+\sin^2 x}{e^x}=\lim\limits_{x\to\infty}\frac{3x^2+2\sin x\cos x}{e^x}=\lim\limits_{x\to\infty}\frac{6x+2\cos 2x}{e^x}=\lim\limits_{x\to\infty}\frac{6-4\sin 2x}{e^x}=0$ by the Sandwich Theorem because $\frac{2}{e^x}\le\frac{6-4\sin 2x}{e^x}\le\frac{10}{e^x}$ for all reals, and $\lim\limits_{x\to\infty}\frac{2}{e^x}=0=\lim\limits_{x\to\infty}\frac{10}{e^x}$

(c) slower, $\lim\limits_{x\to\infty}\frac{\sqrt{x}}{e^x}=\lim\limits_{x\to\infty}\frac{x^{1/2}}{e^x}=\lim\limits_{x\to\infty}\frac{\left(\frac{1}{2}\right)x^{-1/2}}{e^x}=\lim\limits_{x\to\infty}\frac{1}{2\sqrt{x}e^x}=0$

(d) faster, $\lim\limits_{x\to\infty}\frac{4^x}{e^x}=\lim\limits_{x\to\infty}\left(\frac{4}{e}\right)^x=\infty$ since $\frac{4}{e}>1$

(e) slower, $\lim\limits_{x\to\infty}\frac{\left(\frac{3}{2}\right)^x}{e^x}=\lim\limits_{x\to\infty}\left(\frac{3}{2e}\right)^x=0$ since $\frac{3}{2e}<1$

(f) slower, $\lim\limits_{x\to\infty}\frac{e^{x/2}}{e^x}=\lim\limits_{x\to\infty}\frac{1}{e^{x/2}}=0$

(g) same, $\lim\limits_{x\to\infty}\frac{\left(\frac{e^x}{2}\right)}{e^x}=\lim\limits_{x\to\infty}\frac{1}{2}=\frac{1}{2}$

(h) slower, $\lim\limits_{x\to\infty}\frac{\log_{10}x}{e^x}=\lim\limits_{x\to\infty}\frac{\ln x}{(\ln 10)e^x}=\lim\limits_{x\to\infty}\frac{\frac{1}{x}}{(\ln 10)e^x}=\lim\limits_{x\to\infty}\frac{1}{(\ln 10)xe^x}=0$

2. (a) slower, $\lim\limits_{x\to\infty}\frac{10x^4+30x+1}{e^x}=\lim\limits_{x\to\infty}\frac{40x^3+30}{e^x}=\lim\limits_{x\to\infty}\frac{120x^2}{e^x}=\lim\limits_{x\to\infty}\frac{240x}{e^x}=\lim\limits_{x\to\infty}\frac{240}{e^x}=0$

(b) slower, $\lim\limits_{x\to\infty}\frac{x\ln x-x}{e^x}=\lim\limits_{x\to\infty}\frac{x(\ln x-1)}{e^x}=\lim\limits_{x\to\infty}\frac{\ln x-1+x\left(\frac{1}{x}\right)}{e^x}=\lim\limits_{x\to\infty}\frac{\ln x-1+1}{e^x}=\lim\limits_{x\to\infty}\frac{\ln x}{e^x}=\lim\limits_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{e^x}=\lim\limits_{x\to\infty}\frac{1}{xe^x}=0$

(c) slower, $\lim\limits_{x\to\infty}\frac{\sqrt{1+x^4}}{e^x}=\sqrt{\lim\limits_{x\to\infty}\frac{1+x^4}{e^{2x}}}=\sqrt{\lim\limits_{x\to\infty}\frac{4x^3}{2e^{2x}}}=\sqrt{\lim\limits_{x\to\infty}\frac{12x^2}{4e^{2x}}}=\sqrt{\lim\limits_{x\to\infty}\frac{24x}{8e^{2x}}}=\sqrt{\lim\limits_{x\to\infty}\frac{24}{16^{2x}}}=\sqrt{0}=0$

(d) slower, $\lim\limits_{x\to\infty}\frac{\left(\frac{5}{2}\right)^x}{e^x}=\lim\limits_{x\to\infty}\left(\frac{5}{2e}\right)^x=0$ since $\frac{5}{2e}<1$

(e) slower, $\lim\limits_{x\to\infty}\frac{e^{-x}}{e^x}=\lim\limits_{x\to\infty}\frac{1}{e^{2x}}=0$

(f) faster, $\lim\limits_{x\to\infty}\frac{xe^x}{e^x}=\lim\limits_{x\to\infty}x=\infty$

(g) slower, since for all reals we have $-1\le\cos x\le 1 \Rightarrow e^{-1}\le e^{\cos x}\le e^1 \Rightarrow \frac{e^{-1}}{e^x}\le\frac{e^{\cos x}}{e^x}\le\frac{e^1}{e^x}$ and also $\lim\limits_{x\to\infty}\frac{e^{-1}}{e^x}=0=\lim\limits_{x\to\infty}\frac{e^1}{e^x}$, so by the Sandwich Theorem we conclude that $\lim\limits_{x\to\infty}\frac{e^{\cos x}}{e^x}=0$

(h) same, $\lim_{x\to\infty} \frac{e^{x-1}}{e^x} = \lim_{x\to\infty} \frac{1}{e^{(x-x+1)}} = \lim_{x\to\infty} \frac{1}{e} = \frac{1}{e}$

3. (a) same, $\lim_{x\to\infty} \frac{x^2+4x}{x^2} = \lim_{x\to\infty} \frac{2x+4}{2x} = \lim_{x\to\infty} \frac{2}{2} = 1$

(b) slower, $\lim_{x\to\infty} \frac{x^5-x^2}{x^2} = \lim_{x\to\infty} \left(x^3-1\right) = \infty$

(c) same, $\lim_{x\to\infty} \frac{\sqrt{x^4+x^3}}{x^2} = \sqrt{\lim_{x\to\infty} \frac{x^4+x^3}{x^4}} = \sqrt{\lim_{x\to\infty} \left(1+\frac{1}{x}\right)} = \sqrt{1} = 1$

(d) same, $\lim_{x\to\infty} \frac{(x+3)^2}{x^2} = \lim_{x\to\infty} \frac{2(x+3)}{2x} = \lim_{x\to\infty} \frac{2}{2} = 1$

(e) slower, $\lim_{x\to\infty} \frac{x\ln x}{x^2} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$

(f) slower, $\lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{(\ln 2)2^x}{2x} = \lim_{x\to\infty} \frac{(\ln 2)^2 2^x}{2} = \infty$

(g) slower, $\lim_{x\to\infty} \frac{x^3e^{-x}}{x^2} = \lim_{x\to\infty} \frac{x}{e^x} \lim_{x\to\infty} \frac{1}{e^x} = 0$

(h) same, $\lim_{x\to\infty} \frac{8x^2}{x^2} = \lim_{x\to\infty} 8 = 8$

4. (a) same, $\lim_{x\to\infty} \frac{x^2+\sqrt{x}}{x^2} = \lim_{x\to\infty} \left(1+\frac{1}{x^{3/2}}\right) = 1$

(b) same, $\lim_{x\to\infty} \frac{10x^2}{x^2} = \lim_{x\to\infty} 10 = 10$

(c) slower, $\lim_{x\to\infty} \frac{x^2e^{-x}}{x^2} = \lim_{x\to\infty} \frac{1}{e^x} = 0$

(d) slower, $\lim_{x\to\infty} \frac{\log_{10} x^2}{x^2} = \lim_{x\to\infty} \frac{\left(\frac{\ln x^2}{\ln 10}\right)}{x^2} = \frac{1}{\ln 10} \lim_{x\to\infty} \frac{2\ln x}{x^2} = \frac{2}{\ln 10} \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{2x} = \frac{1}{\ln 10} \lim_{x\to\infty} \frac{1}{x^2} = 0$

(e) faster, $\lim_{x\to\infty} \frac{x^3-x^2}{x^2} = \lim_{x\to\infty} (x-1) = \infty$

(f) slower, $\lim_{x\to\infty} \frac{\left(\frac{1}{10}\right)^x}{x^2} = \lim_{x\to\infty} \frac{1}{10^x x^2} = 0$

(g) faster, $\lim_{x\to\infty} \frac{(1.1)^x}{x^2} = \lim_{x\to\infty} \frac{(\ln 1.1)(1.1)^x}{2x} = \lim_{x\to\infty} \frac{(\ln 1.1)^2(1.1)^x}{2} = \infty$

(h) same, $\lim_{x\to\infty} \frac{x^2+100x}{x^2} = \lim_{x\to\infty} \left(1-\frac{100}{x}\right) = 1$

5. (a) same, $\lim_{x\to\infty} \frac{\log_3 x}{\ln x} = \lim_{x\to\infty} \frac{\left(\frac{\ln x}{\ln 3}\right)}{\ln x} = \lim_{x\to\infty} \frac{1}{\ln 3} = \frac{1}{\ln 3}$

(b) same, $\lim_{x\to\infty} \frac{\ln 2x}{\ln x} = \lim_{x\to\infty} \frac{\left(\frac{2}{2x}\right)}{\left(\frac{1}{x}\right)} = 1$

(c) same, $\lim_{x\to\infty} \frac{\ln\sqrt{x}}{\ln x} = \lim_{x\to\infty} \frac{\left(\frac{1}{2}\right)\ln x}{\ln x} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$

(d) faster, $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x} = \lim_{x\to\infty} \frac{x^{1/2}}{\ln x} = \lim_{x\to\infty} \frac{\left(\frac{1}{2}\right)x^{-1/2}}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{2} = \infty$

(e) faster, $\lim\limits_{x\to\infty} \frac{x}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} x = \infty$

(f) same, $\lim\limits_{x\to\infty} \frac{5\ln x}{\ln x} = \lim\limits_{x\to\infty} 5 = 5$

(g) slower, $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{x\ln x} = 0$

(h) faster, $\lim\limits_{x\to\infty} \frac{e^x}{\ln x} = \lim\limits_{x\to\infty} \frac{e^x}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} xe^x = \infty$

6. (a) same, $\lim\limits_{x\to\infty} \frac{\log_2 x^2}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{\ln x^2}{\ln 2}\right)}{\ln x} = \frac{1}{\ln 2} \lim\limits_{x\to\infty} \frac{\ln x^2}{\ln x} = \frac{1}{\ln 2} \lim\limits_{x\to\infty} \frac{2\ln x}{\ln x} = \frac{1}{\ln 2} \lim\limits_{x\to\infty} 2 = \frac{2}{\ln 2}$

(b) same, $\lim\limits_{x\to\infty} \frac{\log_{10} 10x}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{\ln 10x}{\ln 10}\right)}{\ln x} = \frac{1}{\ln 10} \lim\limits_{x\to\infty} \frac{\ln 10x}{\ln x} = \frac{1}{\ln 10} \lim\limits_{x\to\infty} \frac{\left(\frac{10}{10x}\right)}{\left(\frac{1}{x}\right)} = \frac{1}{\ln 10} \lim\limits_{x\to\infty} 1 = \frac{1}{\ln 10}$

(c) slower, $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x}}\right)}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{\left(\sqrt{x}\right)(\ln x)} = 0$

(d) slower, $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{x^2}\right)}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{x^2 \ln x} = 0$

(e) faster, $\lim\limits_{x\to\infty} \frac{x-2\ln x}{\ln x} = \lim\limits_{x\to\infty} \left(\frac{x}{\ln x} - 2\right) = \left(\lim\limits_{x\to\infty} \frac{x}{\ln x}\right) - 2 = \left(\lim\limits_{x\to\infty} \frac{1}{\left(\frac{1}{x}\right)}\right) - 2 = \left(\lim\limits_{x\to\infty} x\right) - 2 = \infty$

(f) slower, $\lim\limits_{x\to\infty} \frac{e^{-x}}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{e^x \ln x} = 0$

(g) slower, $\lim\limits_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1/x}{\ln x}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{1}{\ln x} = 0$

(h) same, $\lim\limits_{x\to\infty} \frac{\ln(2x+5)}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{2}{2x+5}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{2x}{2x+5} = \lim\limits_{x\to\infty} \frac{2}{2} = \lim\limits_{x\to\infty} 1 = 1$

7. $\lim\limits_{x\to\infty} \frac{e^x}{e^{x/2}} = \lim\limits_{x\to\infty} e^{x/2} = \infty \Rightarrow e^x$ grows faster then $e^{x/2}$; since for $x > e^e$ we have $\ln x > e$ and $\lim\limits_{x\to\infty} \frac{(\ln x)^x}{e^x} = \lim\limits_{x\to\infty} \left(\frac{\ln x}{e}\right)^x = \infty \Rightarrow (\ln x)^x$ grows faster then e^x; since $x > \ln x$ for all $x > 0$ and $\lim\limits_{x\to\infty} \frac{x^x}{(\ln x)^x} = \lim\limits_{x\to\infty} \left(\frac{x}{\ln x}\right)^x = \infty \Rightarrow x^x$ grows faster then $(\ln x)^x$. Therefore, slowest to fastest are: $e^{x/2}, e^x, (\ln x)^x, x^x$ so the order is *d*, *a*, *c*, *b*

8. $\lim\limits_{x\to\infty} \frac{(\ln 2)^x}{x^2} = \lim\limits_{x\to\infty} \frac{(\ln(\ln 2))(\ln 2)^x}{2x} = \lim\limits_{x\to\infty} \frac{(\ln(\ln 2))^2 (\ln 2)^x}{2} = \frac{(\ln(\ln 2))^2}{2} \lim\limits_{x\to\infty} (\ln 2)^x = 0 \Rightarrow (\ln 2)^x$ grows slower than x^2; $\lim\limits_{x\to\infty} \frac{x^2}{2^x} = \lim\limits_{x\to\infty} \frac{2x}{(\ln 2)2^x} = \lim\limits_{x\to\infty} \frac{2}{(\ln 2)^2 2^x} = 0 \Rightarrow x^2$ grows slower than 2^x; $\lim\limits_{x\to\infty} \frac{2x}{e^x} = \lim\limits_{x\to\infty} \left(\frac{2}{e}\right)^x = 0 \Rightarrow 2^x$ grows slower than e^x. Therefore, the slowest to the fastest is: $(\ln 2)^x, x^2, 2^x$ and e^x so the order is *c*, *b*, *a*, *d*

9. (a) false; $\lim\limits_{x\to\infty} \frac{x}{x} = 1$

(b) false; $\lim\limits_{x\to\infty} \frac{x}{x+5} = \frac{1}{1} = 1$

(c) true; $x < x+5 \Rightarrow \frac{x}{x+5} < 1$ if $x > 1$ (or sufficiently large)

(d) true; $x < 2x \Rightarrow \frac{x}{2x} < 1$ if $x > 1$ (or sufficiently large)

(e) true; $\lim\limits_{x\to\infty} \frac{e^x}{e^{2x}} = \lim\limits_{x\to\infty} \frac{1}{e^x} = 0$

(f) true; $\frac{x+\ln x}{x} = 1 + \frac{\ln x}{x} < 1 + \frac{\sqrt{x}}{x} = 1 + \frac{1}{\sqrt{x}} < 2$ if $x > 1$ (or sufficiently large)

(g) false; $\lim\limits_{x\to\infty} \frac{\ln x}{\ln 2x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{2}{2x}\right)} = \lim\limits_{x\to\infty} 1 = 1$

(h) true; $\frac{\sqrt{x^2+5}}{x} < \frac{\sqrt{(x+5)^2}}{x} < \frac{x+5}{x} = 1 + \frac{5}{x} < 6$ if $x > 1$ (or sufficiently large)

10. (a) true; $\frac{\left(\frac{1}{x+3}\right)}{\left(\frac{1}{x}\right)} = \frac{x}{x+3} < 1$ if $x > 1$ (or sufficiently large)

(b) true; $\frac{\left(\frac{1}{x}+\frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} = 1 + \frac{1}{x} < 2$ if $x > 1$ (or sufficiently large)

(c) false; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}-\frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \left(1 - \frac{1}{x}\right) = 1$

(d) true; $2 + \cos x \le 3 \Rightarrow \frac{2+\cos x}{2} \le \frac{3}{2}$ if x is sufficiently large

(e) true; $\frac{e^x+x}{e^x} = 1 + \frac{x}{e^x}$ and $\frac{x}{e^x} \to 0$ as $x \to \infty \Rightarrow 1 + \frac{x}{e^x} < 2$ if x is sufficiently large

(f) true; $\lim\limits_{x\to\infty} \frac{x\ln x}{x^2} = \lim\limits_{x\to\infty} \frac{\ln x}{x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$

(g) true; $\frac{\ln(\ln x)}{\ln x} < \frac{\ln x}{\ln x} = 1$ if x is sufficiently large

(h) false; $\lim\limits_{x\to\infty} \frac{\ln x}{\ln(x^2+1)} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{2x}{x^2+1}\right)} = \lim\limits_{x\to\infty} \frac{x^2+1}{2x^2} = \lim\limits_{x\to\infty} \left(\frac{1}{2} + \frac{1}{2x^2}\right) = \frac{1}{2}$

11. If $f(x)$ and $g(x)$ grow at the same rate, then $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L \ne 0 \Rightarrow \lim\limits_{x\to\infty} \frac{g(x)}{f(x)} = \frac{1}{L} \ne 0$. Then $\left|\frac{f(x)}{g(x)} - L\right| < 1$ if x is sufficiently large $\Rightarrow L - 1 < \frac{f(x)}{g(x)} < L + 1 \Rightarrow \frac{f(x)}{g(x)} \le |L| + 1$ if x is sufficiently large $\Rightarrow f = O(g)$. Similarly, $\frac{g(x)}{f(x)} \le \left|\frac{1}{L}\right| + 1 \Rightarrow g = O(f)$.

12. When the degree of f is less than the degree of g since in that case $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

13. When the degree of f is less than or equal to the degree of g since $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$ when the degree of f is smaller than the degree of g, and $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \frac{a}{b}$ (the ratio of the leading coefficients) when the degrees are the same.

14. Polynomials of a greater degree grow at a greater rate than polynomials of a lesser degree. Polynomials of the same degree grow at the same rate.

15. $\lim\limits_{x\to\infty} \frac{\ln(x+1)}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{x}{x+1} = \lim\limits_{x\to\infty} \frac{1}{1} = 1$ and $\lim\limits_{x\to\infty} \frac{\ln(x+999)}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x+999}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{x}{x+999} = 1$

16. $\lim\limits_{x\to\infty} \frac{\ln(x+a)}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x+a}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{x}{x+a} = \lim\limits_{x\to\infty} \frac{1}{1} = 1.$ Therefore, the relative rates are the same.

17. $\lim\limits_{x\to\infty} \frac{\sqrt{10x+1}}{\sqrt{x}} = \sqrt{\lim\limits_{x\to\infty} \frac{10x+1}{x}} = \sqrt{10}$ and $\lim\limits_{x\to\infty} \frac{\sqrt{x+1}}{\sqrt{x}} = \sqrt{\lim\limits_{x\to\infty} \frac{x+1}{x}} = \sqrt{1} = 1.$ Since the growth rate is transitive, we conclude that $\sqrt{10x+1}$ and $\sqrt{x+1}$ have the same growth rate (that of $\sqrt{x}$).

18. $\lim\limits_{x\to\infty} \frac{\sqrt{x^4+x}}{x^2} = \sqrt{\lim\limits_{x\to\infty} \frac{x^4+x}{x^4}} = 1$ and $\lim\limits_{x\to\infty} \frac{\sqrt{x^4-x^3}}{x^2} = \sqrt{\lim\limits_{x\to\infty} \frac{x^4-x^3}{x^4}} = 1.$ Since the growth rate is transitive, we conclude that $\sqrt{x^4+x}$ and $\sqrt{x^4-x^3}$ have the same growth rate (that of x^2).

19. $\lim\limits_{x\to\infty} \frac{x^n}{e^x} = \lim\limits_{x\to\infty} \frac{nx^{n-1}}{e^x} = \cdots = \lim\limits_{x\to\infty} \frac{n!}{e^x} = 0 \Rightarrow x^n = o\left(e^x\right)$ for any non-negative integer n

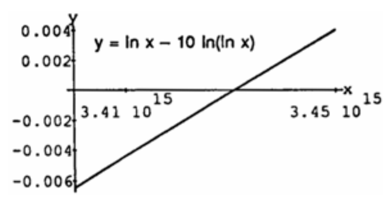
20. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $\lim\limits_{x\to\infty} \frac{p(x)}{e^x} = a_n \lim\limits_{x\to\infty} \frac{x^n}{e^x} + a_{n-1} \lim\limits_{x\to\infty} \frac{x^{n-1}}{ex} + \cdots + a_1 \lim\limits_{x\to\infty} \frac{x}{e^x} + a_0 \lim\limits_{x\to\infty} \frac{1}{e^x}$ where each limit is zero (from Exercise 19). Therefore, $\lim\limits_{x\to\infty} \frac{p(x)}{e^x} = 0 \Rightarrow e^x$ grows faster than any polynomial.

21. (a) $\lim\limits_{x\to\infty} \frac{x^{1/n}}{\ln x} = \lim\limits_{x\to\infty} \frac{x(1-n)/n}{n\left(\frac{1}{n}\right)} = \left(\frac{1}{n}\right) \lim\limits_{x\to\infty} x^{1/n} = \infty \Rightarrow \ln x = o\left(x^{1/n}\right)$ for any positive integer n

(b) $\ln\left(e^{17,000,000}\right) = 17,000,000 < \left(e^{17\times 10^6}\right)^{1/10^6} = e^{17} \approx 24,154,952.75$

(c) $x \approx 3.430631121 \times 10^{15}$

(d) In the interval $\left[3.41\times 10^{15}, 3.45\times 10^{15}\right]$ we have $\ln x = 10\ln(\ln x)$. The graphs cross at about 3.4306311×10^{15}.



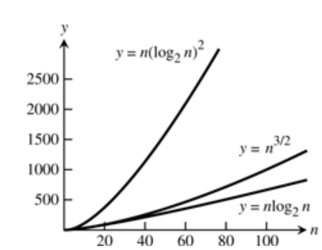
22. $\lim\limits_{x\to\infty} \frac{\ln x}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} = \frac{\lim\limits_{x\to\infty}\left(\frac{\ln x}{x^n}\right)}{\lim\limits_{x\to\infty}\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)} = \frac{\lim\limits_{x\to\infty}\left[\frac{1/x}{nx^{n-1}}\right]}{a_n} = \lim\limits_{x\to\infty} \frac{1}{(a_n)\left(nx^n\right)} = 0 \Rightarrow \ln x$ grows slower than any non-constant polynomial ($n \ge 1$)

23. (a) $\lim\limits_{n\to\infty} \frac{n\log_2 n}{n(\log_2 n)^2} = \lim\limits_{n\to\infty} \frac{1}{\log_2 n} = 0 \Rightarrow n\log_2 n$ grows slower then $n(\log_2 n)^2$;

$\lim\limits_{n\to\infty} \frac{n\log_2 n}{n^{3/2}} = \lim\limits_{n\to\infty} \frac{\left(\frac{\ln n}{\ln 2}\right)}{n^{1/2}} = \frac{1}{\ln 2} \lim\limits_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{2}\right)n^{-1/2}}$

$= \frac{2}{\ln 2} \lim\limits_{n\to\infty} \frac{1}{n^{1/2}} = 0 \Rightarrow n\log_2 n$ grows slower than $n^{3/2}$. Therefore, $n\log_2 n$ grows at the slowest rate $\Rightarrow$ the algorithm that takes $O(n\log_2 n)$ steps is the most efficient in the long run.

(b)



24. (a) $\lim_{n\to\infty} \frac{(\log_2 n)^2}{n} = \lim_{n\to\infty} \frac{\left(\frac{\ln n}{\ln 2}\right)^2}{n} = \lim_{n\to\infty} \frac{(\ln n)^2}{n(\ln 2)^2} = \lim_{n\to\infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{(\ln 2)^2} = \frac{2}{(\ln 2)^2} \lim_{n\to\infty} \frac{\ln n}{n} = \frac{2}{(\ln 2)^2} \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0$

$\Rightarrow (\log_2 n)^2$ grows slower then n; $\lim_{n\to\infty} \frac{(\log_2 n)^2}{\sqrt{n}\log_2 n} = \lim_{n\to\infty} \frac{\log_2 n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\left(\frac{\ln n}{\ln 2}\right)}{n^{1/2}} = \frac{1}{\ln 2} \lim_{n\to\infty} \frac{\ln n}{n^{1/2}}$

$= \frac{1}{\ln 2} \lim_{x\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{2}\right)n^{-1/2}} = \frac{2}{\ln 2} \lim_{n\to\infty} \frac{1}{n^{1/2}} = 0$

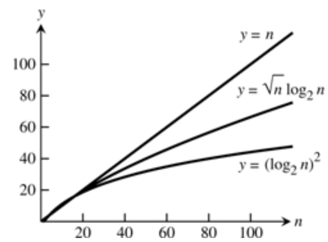
$\Rightarrow (\log_2 n)^2$ grows slower than $\sqrt{n}\log_2 n$.

Therefore $(\log_2 n)^2$ grows at the slowest rate

$\Rightarrow$ the algorithm that takes $O\left((\log_2 n)^2\right)$ steps

is the most efficient in the long run.

(b)



25. It could take one million steps for a sequential search, but at most 20 steps for a binary search because $2^{19} = 524,288 < 1,000,000 < 1,048,576 = 2^{20}$.

26. It could take 450,000 steps for a sequential search, but at most 19 steps for a binary search because $2^{18} = 262,144 < 450,000 < 524,288 = 2^{19}$.

CHAPTER 7 PRACTICE EXERCISES

1. $y = 10e^{-x/5} \Rightarrow \frac{dy}{dx} = (10)\left(-\frac{1}{5}\right)e^{-x/5} = -2e^{-x/5}$

2. $y = \sqrt{2}e^{\sqrt{2}x} \Rightarrow \frac{dy}{dx} = \left(\sqrt{2}\right)\left(\sqrt{2}\right)e^{\sqrt{2}x} = 2e^{\sqrt{2}x}$

3. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} \Rightarrow \frac{dy}{dx} = \frac{1}{4}\left[x\left(4e^{4x}\right) + e^{4x}(1)\right] - \frac{1}{16}\left(4e^{4x}\right) = xe^{4x} + \frac{1}{4}e^{4x} - \frac{1}{4}e^{4x} = xe^{4x}$

4. $y = x^2e^{-2/x} = x^2e^{-2x^{-1}} \Rightarrow \frac{dy}{dx} = x^2\left[\left(2x^{-2}\right)e^{-2x^{-1}}\right] + e^{-2x^{-1}}(2x) = (2+2x)e^{-2x^{-1}} = 2e^{-2/x}(1+x)$

5. $y = \ln\left(\sin^2\theta\right) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sin\theta)(\cos\theta)}{\sin^2\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta$

6. $y = \ln\left(\sec^2\theta\right) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sec\theta)(\sec\theta\tan\theta)}{\sec^2\theta} = 2\tan\theta$

7. $y = \log_2\left(\frac{x^2}{2}\right) = \frac{\ln\left(\frac{x^2}{2}\right)}{\ln 2} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 2}\left(\frac{x}{\left(\frac{x^2}{2}\right)}\right) = \frac{2}{(\ln 2)x}$

8. $y = \log_5(3x-7) = \frac{\ln(3x-7)}{\ln 5} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\ln 5}\right)\left(\frac{3}{3x-7}\right) = \frac{3}{(\ln 5)(3x-7)}$

9. $y = 8^{-t} \Rightarrow \frac{dy}{dt} = 8^{-t}(\ln 8)(-1) = -8^{-t}(\ln 8)$

10. $y=9^{2t} \Rightarrow \frac{dy}{dt}=9^{2t}(\ln 9)(2)=9^{2t}(2\ln 9)$

11. $y=5x^{3.6} \Rightarrow \frac{dy}{dx}=5(3.6)x^{2.6}=18x^{2.6}$

12. $y=\sqrt{2}x^{-\sqrt{2}} \Rightarrow \frac{dy}{dx}=\left(\sqrt{2}\right)\left(-\sqrt{2}\right)x^{\left(-\sqrt{2}-1\right)}=-2x^{\left(-\sqrt{2}-1\right)}$

13. $y=(x+2)^{x+2} \Rightarrow \ln y=\ln(x+2)^{x+2}=(x+2)\ln(x+2) \Rightarrow \frac{y'}{y}=(x+2)\left(\frac{1}{x+2}\right)+(1)\ln(x+2)$
$\Rightarrow \frac{dy}{dx}=(x+2)^{x+2}\left[\ln(x+2)+1\right]$

14. $y=2(\ln x)^{x/2} \Rightarrow \ln y=\ln\left[2(\ln x)^{x/2}\right]=\ln(2)+\left(\frac{x}{2}\right)\ln(\ln x) \Rightarrow \frac{y'}{y}=0+\left(\frac{x}{2}\right)\left[\frac{\left(\frac{1}{x}\right)}{\ln x}\right]+\left(\ln(\ln x)\right)\left(\frac{1}{2}\right)$
$\Rightarrow y'=\left[\frac{1}{2\ln x}+\left(\frac{1}{2}\right)\ln(\ln x)\right]2(\ln x)^{x/2}=(\ln x)^{x/2}\left[\ln(\ln x)+\frac{1}{\ln x}\right]$

15. $y=\sin^{-1}\sqrt{1-u^2}=\sin^{-1}\left(1-u^2\right)^{1/2} \Rightarrow \frac{dy}{du}=\frac{\frac{1}{2}\left(1-u^2\right)^{-1/2}(-2u)}{\sqrt{1-\left[\left(1-u^2\right)^{1/2}\right]^2}}=\frac{-u}{\sqrt{1-u^2}\sqrt{1-\left(1-u^2\right)}}=\frac{-u}{|u|\sqrt{1-u^2}}$
$=\frac{-u}{u\sqrt{1-u^2}}=\frac{-1}{\sqrt{1-u^2}}, 0<u<1$

16. $y=\sin^{-1}\left(\frac{1}{\sqrt{v}}\right)=\sin^{-1}\left(v^{-1/2}\right) \Rightarrow \frac{dy}{dv}=\frac{-\frac{1}{2}v^{-3/2}}{\sqrt{1-\left(v^{-1/2}\right)^2}}=\frac{-1}{2v^{3/2}\sqrt{1-v^{-1}}}=\frac{-1}{2v^{3/2}\sqrt{\frac{v-1}{v}}}=\frac{-\sqrt{v}}{2v^{3/2}\sqrt{v-1}}=\frac{-1}{2v\sqrt{v-1}}$

17. $y=\ln\left(\cos^{-1}x\right) \Rightarrow y'=\frac{\left(\frac{-1}{\sqrt{1-x^2}}\right)}{\cos^{-1}x}=\frac{-1}{\sqrt{1-x^2}\cos^{-1}x}$

18. $y=z\cos^{-1}z-\sqrt{1-z^2}=z\cos^{-1}z-\left(1-z^2\right)^{1/2} \Rightarrow \frac{dy}{dz}=\cos^{-1}z-\frac{z}{\sqrt{1-z^2}}-\left(\frac{1}{2}\right)\left(1-z^2\right)^{-1/2}(-2z)$
$=\cos^{-1}z-\frac{z}{\sqrt{1-z^2}}+\frac{z}{\sqrt{1-z^2}}=\cos^{-1}z$

19. $y=t\tan^{-1}t-\left(\frac{1}{2}\right)\ln t \Rightarrow \frac{dy}{dt}=\tan^{-1}t+t\left(\frac{1}{1+t^2}\right)-\left(\frac{1}{2}\right)\left(\frac{1}{t}\right)=\tan^{-1}t+\frac{t}{1+t^2}-\frac{1}{2t}$

20. $y=\left(1+t^2\right)\cot^{-1}2t \Rightarrow \frac{dy}{dx}=2t\cot^{-1}2t+\left(1+t^2\right)\left(\frac{-2}{1+4t^2}\right)$

21. $y=z\sec^{-1}z-\sqrt{z^2-1}=z\sec^{-1}z-\left(z^2-1\right)^{1/2} \Rightarrow \frac{dy}{dz}=z\left(\frac{1}{|z|\sqrt{z^2-1}}\right)+\left(\sec^{-1}z\right)(1)-\frac{1}{2}\left(z^2-1\right)^{-1/2}(2z)$
$=\frac{z}{|z|\sqrt{z^2-1}}-\frac{z}{\sqrt{z^2-1}}+\sec^{-1}z=\frac{1-z}{\sqrt{z^2-1}}+\sec^{-1}z, z>1$

22. $y = 2\sqrt{x-1}\sec^{-1}\sqrt{x} = 2(x-1)^{1/2}\sec^{-1}\left(x^{1/2}\right) \Rightarrow \frac{dy}{dx} = 2\left[\left(\frac{1}{2}\right)(x-1)^{-1/2}\sec^{-1}\left(x^{1/2}\right) + (x-1)^{1/2}\left(\frac{\left(\frac{1}{2}x^{-1/2}\right)}{\sqrt{x}\sqrt{x-1}}\right)\right]$

$= 2\left(\frac{\sec^{-1}\sqrt{x}}{2\sqrt{x-1}} + \frac{1}{2x}\right) = \frac{\sec^{-1}\sqrt{x}}{\sqrt{x-1}} + \frac{1}{x}$

23. $y = \csc^{-1}(\sec\theta) \Rightarrow \frac{dy}{d\theta} = \frac{-\sec\theta\tan\theta}{|\sec\theta|\sqrt{\sec^2\theta - 1}} = -\frac{\tan\theta}{|\tan\theta|} = -1, 0 < \theta < \frac{\pi}{2}$

24. $y = \left(1+x^2\right)e^{\tan^{-1}x} \Rightarrow y' = 2xe^{\tan^{-1}x} + \left(1+x^2\right)\left(\frac{e^{\tan^{-1}x}}{1+x^2}\right) = 2xe^{\tan^{-1}x} + e^{\tan^{-1}x}$

25. $y = \frac{2\left(x^2+1\right)}{\sqrt{\cos 2x}} \Rightarrow \ln y = \ln\left(\frac{2\left(x^2+1\right)}{\sqrt{\cos 2x}}\right) = \ln(2) + \ln\left(x^2+1\right) - \frac{1}{2}\ln(\cos 2x) \Rightarrow \frac{y'}{y} = 0 + \frac{2x}{x^2+1} - \left(\frac{1}{2}\right)\frac{(-2\sin 2x)}{\cos 2x}$

$\Rightarrow y' = \left(\frac{2x}{x^2+1} + \tan 2x\right)y = \frac{2\left(x^2+1\right)}{\sqrt{\cos 2x}}\left(\frac{2x}{x^2+1} + \tan 2x\right)$

26. $y = \sqrt[10]{\frac{3x+4}{2x-4}} \Rightarrow \ln y = \ln\sqrt[10]{\frac{3x+4}{2x-4}} = \frac{1}{10}\left[\ln(3x+4) - \ln(2x-4)\right] \Rightarrow \frac{y'}{y} = \frac{1}{10}\left(\frac{3}{3x+4} - \frac{2}{2x-4}\right)$

$\Rightarrow y' = \frac{1}{10}\left(\frac{3}{3x+4} - \frac{1}{x-2}\right)y = \sqrt[10]{\frac{3x+4}{2x-4}}\left(\frac{1}{10}\right)\left(\frac{3}{3x+4} - \frac{1}{x-2}\right)$

27. $y = \left[\frac{(t+1)(t-1)}{(t-2)(t+3)}\right]^5 \Rightarrow \ln y = 5\left[\ln(t+1) + \ln(t-1) - \ln(t-2) - \ln(t+3)\right] \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{dt}\right) = 5\left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3}\right)$

$\Rightarrow \frac{dy}{dt} = 5\left[\frac{(t+1)(t-1)}{(t-2)(t+3)}\right]^5\left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-1} - \frac{1}{t+3}\right)$

28. $y = \frac{2u2^u}{\sqrt{u^2+1}} \Rightarrow \ln y = \ln 2 + \ln u + u\ln 2 - \frac{1}{2}\ln\left(u^2+1\right) \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{du}\right) = \frac{1}{u} + \ln 2 - \frac{1}{2}\left(\frac{2u}{u^2+1}\right)$

$\Rightarrow \frac{dy}{du} = \frac{2u2^u}{\sqrt{u^2+1}}\left(\frac{1}{u} + \ln 2 - \frac{u}{u^2+1}\right)$

29. $y = (\sin\theta)^{\sqrt{\theta}} \Rightarrow \ln y = \sqrt{\theta}\ln y(\sin\theta) \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{d\theta}\right) = \sqrt{\theta}\left(\frac{\cos\theta}{\sin\theta}\right) + \frac{1}{2}\theta^{-1/2}\ln(\sin\theta)$

$\Rightarrow \frac{dy}{d\theta} = (\sin\theta)^{\sqrt{\theta}}\left(\sqrt{\theta}\cot\theta + \frac{\ln(\sin\theta)}{2\sqrt{\theta}}\right)$

30. $y = (\ln x)^{1/\ln x} \Rightarrow \ln y = \left(\frac{1}{\ln x}\right)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{\ln x}\right)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) + \ln(\ln x)\left[\frac{-1}{(\ln x)^2}\right]\left(\frac{1}{x}\right) \Rightarrow y' = (\ln x)^{1/\ln x}\left[\frac{1-\ln(\ln x)}{x(\ln x)^2}\right]$

31. $\int e^x\cos\left(e^x\right)dx = \int \cos t\, dt$, where $e^x = t$ and $e^x\, dx = dt$

$= \sin t + C = \sin\left(e^x\right) + C$

32. $\int e^x\sin\left(5e^x - 7\right)dx = \frac{1}{5}\int \sin t\, dt$, where $5e^x - 7 = t$ and $e^x dx = \frac{dt}{5}$

$= -\cos t + C = -\frac{1}{5}\cos\left(5e^x - 7\right) + C$

33. $\int e^x \sec^2\left(e^x - 7\right) dx = \int \sec^2 u\, du$, where $u = e^x - 7$ and $du = e^x dx$

$= \tan u + C = \tan\left(e^x - 7\right) + C$

34. $\int e^y \csc\left(e^y + 1\right)\cot\left(e^y + 1\right) dy = \int \csc u \cot u\, du$, where $u = e^y + 1$ and $du = e^y dy$

$= -\csc u + C = -\csc\left(e^y + 1\right) + C$

35. $\int \left(\sec^2 x\right) e^{\tan x}\, dx = \int e^u du$, where $u = \tan x$ and $du = \sec^2 x\, dx$

$= e^u + C = e^{\tan x} + C$

36. $\int \left(\csc^2 x\right) e^{\cot x} dx = -\int e^u du$, where $u = \cot x$ and $du = -\csc^2 x\, dx$

$= -e^u + C = -e^{\cot x} + C$

37. $\int_{-1}^{1} \frac{1}{3x-4}\, dx = \frac{1}{3}\int_{-7}^{-1} \frac{1}{u}\, du$, where $u = 3x - 4, du = 3\, dx$; $x = -1 \Rightarrow u = -7, x = 1 \Rightarrow u = -1$

$= \frac{1}{3}\left[\ln[u]\right]_{-7}^{-1} = \frac{1}{3}\left[\ln|-1| - \ln|-7|\right] = \frac{1}{3}\left[0 - \ln 7\right] = -\frac{\ln 7}{3}$

38. $\int_1^e \frac{\sqrt{\ln x}}{x}\, dx = \int_0^1 u^{1/2} du$, where $u = \ln x, du = \frac{1}{x}\, dx$; $x = 1 \Rightarrow u = 0$, $x = e \Rightarrow u = 1$

$= \left[\frac{2}{3} u^{3/2}\right]_0^1 = \left[\frac{2}{3} 1^{3/2} - \frac{2}{3} 0^{3/2}\right] = \frac{2}{3}$

39. $\int_0^{\pi} \tan\left(\frac{x}{3}\right) dx = \int_0^{\pi} \frac{\sin\left(\frac{x}{3}\right)}{\cos\left(\frac{x}{3}\right)} dx = -3\int_1^{1/2} \frac{1}{u}\, du$, where $u = \cos\left(\frac{x}{3}\right)$, $du = -\frac{1}{3}\sin\left(\frac{x}{3}\right) dx$; $x = 0 \Rightarrow u = 1, x = \pi \Rightarrow u = \frac{1}{2}$

$= -3\left[\ln|u|\right]_1^{1/2} = -3\left[\ln\left|\frac{1}{2}\right| - \ln|1|\right] = -3\ln\frac{1}{2} = \ln 2^3 = \ln 8$

40. $\int_{1/6}^{1/4} 2\cot \pi x\, dx = 2\int_{1/6}^{1/4} \frac{\cos \pi x}{\sin \pi x}\, dx = \frac{2}{\pi}\int_{1/2}^{1/\sqrt{2}} \frac{1}{u}\, du$, where $u = \sin \pi x, du = \pi \cos \pi x\, dx$;

$x = \frac{1}{6} \Rightarrow u = \frac{1}{2}, x = \frac{1}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$

$= \frac{2}{x}\left[\ln|u|\right]_{1/2}^{1/\sqrt{2}} = \frac{2}{\pi}\left[\ln\left|\frac{1}{\sqrt{2}}\right| - \ln\left|\frac{1}{2}\right|\right] = \frac{2}{\pi}\left[\ln 1 - \frac{1}{2}\ln 2 - \ln 1 + \ln 2\right] = \frac{2}{\pi}\left[\frac{1}{2}\ln 2\right] = \frac{\ln 2}{\pi}$

41. $\int_0^4 \frac{2t}{t^2 - 25}\, dt = \int_{-25}^{-9} \frac{1}{u}\, du$, where $u = t^2 - 25, du = 2t\, dt$; $t = 0 \Rightarrow u = -25, t = 4 \Rightarrow u = -9$

$= \left[\ln|u|\right]_{-25}^{-9} = \ln|-9| - \ln|-25| = \ln 9 - \ln 25 = \ln\frac{9}{25}$

42. $\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1 - \sin t}\, dt = -\int_2^{1/2} \frac{1}{u} du$, where $u = 1 - \sin t$, $du = -\cos t\, dt$; $t = -\frac{\pi}{2} \Rightarrow u = 2, t = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}$

$= -\left[\ln|u|\right]_2^{1/2} = -\left[\ln\left|\frac{1}{2}\right| - \ln|2|\right] = -\ln 1 + \ln 2 + \ln 2 = 2\ln 2 = \ln 4$

43. $\int \frac{\tan(\ln v)}{v} dv = \int \tan u \, du = \int \frac{\sin u}{\cos u} du, \quad u = \ln v$ and $du = \frac{1}{v} dv$
$= -\ln|\cos u| + C = -\ln|\cos(\ln v)| + C$

44. $\int \frac{1}{v \ln v} dv = \int \frac{1}{u} du$, where $u = \ln v$ and $du = \frac{1}{v} dv$
$= \ln|u| + C = \ln|\ln v| + C$

45. $\int \frac{(\ln x)^{-3}}{x} dx = \int u^{-3} du$, where $u = \ln x$ and $du = \frac{1}{x} dx$
$= \frac{u^{-2}}{-2} + C = -\frac{1}{2}(\ln x)^{-2} + C$

46. $\int \frac{\ln(x+9)}{x+9} dx = \int p \, dp$, where $\ln(x+9) = p$ and $\frac{1}{x+9} dx = dp$
$= \frac{p^2}{2} + C = \frac{1}{2}[\ln(x+9)]^2 + C$

47. $\int \frac{1}{r} \csc^2(1 + \ln r) dr = \int \csc^2 u \, du$, where $u = 1 + \ln r$ and $du = \frac{1}{r} dr$
$= -\cot u + C = -\cot(1 + \ln r) + C$

48. $\int \frac{\sin(2+\ln x)}{x} dx = \int \sin t \, dt$, where $2 + \ln x = t$ and $\frac{1}{x} dx = dt$
$= -\cos t + C = -\cos(2 + \ln x) + C$

49. $\int x 3^{x^2} dx = \frac{1}{2} \int 3^u \, du$, where $u = x^2$ and $du = 2x \, dx$
$= \frac{1}{2 \ln 3}\left(3^u\right) + C = \frac{1}{2 \ln 3}\left(3^{x^2}\right) + C$

50. $\int 2^{\tan x} \sec^2 x \, dx = \int 2^u du$, where $u = \tan x$ and $du = \sec^2 x \, dx$
$= \frac{1}{\ln 2}\left(2^u\right) + C = \frac{2^{\tan x}}{\ln 2} + C$

51. $\int_1^9 \frac{5}{x} dx = [5 \ln x]_1^9 = 5(\ln 9 - \ln 1) = 5 \ln 9$

52. $\int_1^{81} \frac{1}{4x} dx = \frac{1}{4}[\ln x]_1^{81} = \frac{1}{4}[\ln 81 - \ln 1] = \frac{1}{4}\left(\ln 3^4 - 0\right) = \frac{1}{4} 4 \ln 3 = \ln 3$

53. $\int_1^4 \left(\frac{x}{8} + \frac{1}{2x}\right) dx = \frac{1}{2} \int_1^4 \left(\frac{1}{4} x + \frac{1}{x}\right) dx = \frac{1}{2}\left[\frac{1}{8} x^2 + \ln|x|\right]_1^4 = \frac{1}{2}\left[\left(\frac{16}{8} + \ln 4\right) - \left(\frac{1}{8} + \ln 1\right)\right] = \frac{15}{16} + \frac{1}{2} \ln 4$
$= \frac{15}{16} + \ln \sqrt{4} = \frac{15}{16} + \ln 2$

54. $\int_1^8 \left(\frac{2}{3x} - \frac{8}{x^2}\right) dx = \frac{2}{3} \int_1^8 \left(\frac{1}{x} - 12x^{-2}\right) dx = \frac{2}{3}\left[\ln|x| + 12x^{-1}\right]_1^8 = \frac{2}{3}\left[\left(\ln 8 + \frac{12}{8}\right) - (\ln 1 + 12)\right]$
$= \frac{2}{3}\left(\ln 8 + \frac{3}{2} - 12\right) = \frac{2}{3}\left(\ln 8 - \frac{21}{2}\right) = \frac{2}{3}(\ln 8) - 7 = \ln\left(8^{2/3}\right) - 7 = \ln 4 - 7$

55. $\int_{-2}^{-1} e^{-(x+1)}dx = -\int_1^0 e^u du$, where $u = -(x+1), du = -dx$; $x = -2 \Rightarrow u = 1, x = -1 \Rightarrow u = 0$

$$= -\left[e^u\right]_1^0 = -\left(e^0 - e^1\right) = e - 1$$

56. $\int_{-\ln 2}^{0} e^{2w}dw = \frac{1}{2}\int_{\ln(1/4)}^{0} e^u\ du$, where $u = 2w, du = 2dw$; $w = -\ln 2 \Rightarrow u = \ln\frac{1}{4}, w = 0 \Rightarrow u = 0$

$$= \frac{1}{2}\left[e^u\right]_{\ln(1/4)}^0 = \frac{1}{2}\left[e^0 - e^{\ln(1/4)}\right] = \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}$$

57. $\int_1^{\ln 5} e^r\left(3e^r + 1\right)^{-3/2} dr = \frac{1}{3}\int_4^{16} u^{-3/2}du$, where $u = 3e^r + 1, du = 3e^r dr$; $r = 0 \Rightarrow u = 4, r = \ln 5 \Rightarrow u = 16$

$$= -\frac{2}{3}\left[u^{-1/2}\right]_4^{16} = -\frac{2}{3}\left(16^{-1/2} - 4^{-1/2}\right) = \left(-\frac{2}{3}\right)\left(\frac{1}{4} - \frac{1}{2}\right) = \left(-\frac{2}{3}\right)\left(-\frac{1}{4}\right) = \frac{1}{6}$$

58. $\int_0^{\ln 9} e^\theta\left(e^\theta - 1\right)^{1/2} d\theta = \int_0^8 u^{1/2}du$, where $u = e^\theta - 1, du = e^\theta d\theta$; $\theta = 0 \Rightarrow u = 0, \theta = \ln 9 \Rightarrow u = 8$

$$= \frac{2}{3}\left[u^{3/2}\right]_0^8 = \frac{2}{3}\left(8^{3/2} - 0^{3/2}\right) = \frac{2}{3}\left(2^{9/2} - 0\right) = \frac{2^{11/2}}{3} = \frac{32\sqrt{2}}{3}$$

59. $\int_1^e \frac{1}{x}(1 + 7\ln x)^{-1/3} dx = \frac{1}{7}\int_1^8 u^{-1/3}du$, where $u = 1 + 7\ln x, du = \frac{7}{x}dx$; $x = 1 \Rightarrow u = 1, x = e \Rightarrow u = 8$

$$= \frac{3}{14}\left[u^{2/3}\right]_1^8 = \frac{3}{14}\left(8^{2/3} - 1^{2/3}\right) = \left(\frac{3}{14}\right)(4 - 1) = \frac{9}{14}$$

60. $\int \frac{5}{x\sqrt{\ln x}} dx = \int \frac{5}{\sqrt{t}} dt = 5\frac{t^{1/2}}{1/2} = 10\sqrt{t} + C = 10\sqrt{\ln x} + C$, where $\ln x = t, \frac{1}{x}dx = dt$;

$$\int_{e^2}^{e^3} \frac{5}{x\sqrt{\ln x}} dx = \left[10\sqrt{\ln x}\right]_{e^2}^{e^3} = 10\sqrt{\ln e^3} - 10\sqrt{\ln e^2} = 10\left(\sqrt{3} - \sqrt{2}\right)$$

61. $\int_1^3 \frac{\left[\ln(v+1)\right]^2}{v+1} dv = \int_1^3 \left[\ln(v+1)\right]^2 \frac{1}{v+1} dv = \int_{\ln 2}^{\ln 4} u^2 du$, where $u = \ln(v+1), du = \frac{1}{v+1}dv$;

$v = 1 \Rightarrow u = \ln 2, v = 3 \Rightarrow u = \ln 4$

$$= \frac{1}{3}\left[u^3\right]_{\ln 2}^{\ln 4} = \frac{1}{3}\left[(\ln 4)^3 - (\ln 2)^3\right] = \frac{1}{3}\left[(2\ln 2)^3 - (\ln 2)^3\right] = \frac{(\ln 2)^3}{3}(8 - 1) = \frac{7}{3}(\ln 2)^3$$

62. $\int (1 + \ln t)(t\ln t)dt = \int p\,dp = \frac{p^2}{2} + C = \frac{(t\ln t)^2}{2} + C$ where $t\ln t = p, \left((t)\left(\frac{1}{t}\right) + (\ln t)\right)dt = dp, \left(1 + \ln t\right)dt = dp$;

$$\int_3^9 (1 + \ln t)(t\ln t)dt = \left[\frac{(t\ln t)^2}{2}\right]_3^9 = \frac{(9\ln 9)^2}{2} - \frac{(3\ln 3)^2}{2} = \frac{1}{2}\left[(18\ln 3)^2 - (3\ln 3)^2\right] = \frac{1}{2}315(\ln 3)^2$$

63. $\int_1^8 \frac{\log_4\theta}{\theta} d\theta = \frac{1}{\ln 4}\int_1^8 (\ln\theta)\left(\frac{1}{\theta}\right)d\theta = \frac{1}{\ln 4}\int_0^{\ln 8} u\ du$, where $u = \ln\theta, du = \frac{1}{\theta}d\theta$; $\theta = 1 \Rightarrow u = 0, \theta = 8 \Rightarrow u = \ln 8$

$$= \frac{1}{2\ln 4}\left[u^2\right]_0^{\ln 8} = \frac{1}{\ln 16}\left[(\ln 8)^2 - 0^2\right] = \frac{(3\ln 2)^2}{4\ln 2} = \frac{9\ln 2}{4}$$

64. $\int_1^e \frac{8(\ln 3)\left(\log_3\theta\right)}{\theta} d\theta = \int_1^e \frac{8(\ln 3)(\ln\theta)}{\theta(\ln 3)} d\theta = 8\int_1^e (\ln\theta)\left(\frac{1}{\theta}\right)d\theta = 8\int_0^1 u\ du$, where $u = \ln\theta, du = \frac{1}{\theta}d\theta$

$\theta = 1 \Rightarrow u = 0, \theta = e \Rightarrow u = 1$

$$= 4\left[u^2\right]_0^1 = 4\left(1^2 - 0^2\right) = 4$$

65. $\int_{-3/4}^{3/4} \frac{6}{\sqrt{9-4x^2}}\,dx = 3\int_{-3/4}^{3/4} \frac{2}{\sqrt{3^2-(2x)^2}}\,dx = 3\int_{-3/2}^{3/2} \frac{1}{\sqrt{3^2-u^2}}\,du$, where $u = 2x, du = 2\,dx$;

$x = -\frac{3}{4} \Rightarrow u = -\frac{3}{2}, x = \frac{3}{4} \Rightarrow u = \frac{3}{2}$

$$= 3\left[\sin^{-1}\left(\frac{u}{3}\right)\right]_{-3/2}^{3/2} = 3\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right] = 3\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = 3\left(\frac{\pi}{3}\right) = \pi$$

66. $\int_{-1/5}^{1/5} \frac{6}{\sqrt{4-25x^2}}\,dx = \frac{6}{5}\int_{-1/5}^{1/5} \frac{5}{\sqrt{2^2-(5x)^2}}dx = \frac{6}{5}\int_{-1}^{1} \frac{1}{\sqrt{2^2-u^2}}\,du$, where $u = 5x, du = 5dx$;

$x = -\frac{1}{5} \Rightarrow u = -1, x = \frac{1}{5} \Rightarrow u = 1$

$$= \frac{6}{5}\left[\sin^{-1}\left(\frac{u}{2}\right)\right]_{-1}^{1} = \frac{6}{5}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \frac{6}{5}\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = \frac{6}{5}\left(\frac{\pi}{3}\right) = \frac{2\pi}{5}$$

67. $\int_{-2}^{2} \frac{3}{4+3t^2}\,dt = \sqrt{3}\int_{-2}^{2} \frac{\sqrt{3}}{2^2+\left(\sqrt{3}\,t\right)^2}dt = \sqrt{3}\int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{1}{2^2+u^2}\,du$, where $u = \sqrt{3}\,t, du = \sqrt{3}dt$;

$t = -2 \Rightarrow u = -2\sqrt{3}, t = 2 \Rightarrow u = 2\sqrt{3}$

$$= \sqrt{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right]_{-2\sqrt{3}}^{2\sqrt{3}} = \frac{\sqrt{3}}{2}\left[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}\left(-\sqrt{3}\right)\right] = \frac{\sqrt{3}}{2}\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \frac{\pi}{\sqrt{3}}$$

68. $\int_{\sqrt{3}}^{3} \frac{1}{3+t^2}dt = \int_{\sqrt{3}}^{3} \frac{1}{\left(\sqrt{3}\right)^2+t^2}\,dt = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_{\sqrt{3}}^{3} = \frac{1}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}1\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{3}\pi}{36}$

69. $\int \frac{1}{y\sqrt{4y^2-1}}\,dy = \int \frac{2}{(2y)\sqrt{(2y)^2-1}}\,dy = \int \frac{1}{u\sqrt{u^2-1}}\,du$ where $u = 2y$ and $du = 2\,dy$

$$= \sec^{-1}|u| + C = \sec^{-1}\left|2y\right| + C$$

70. $\int \frac{24}{y\sqrt{y^2-16}}\,dy = 24\int \frac{1}{y\sqrt{y^2-4^2}}dy = 24\left(\frac{1}{2}\sec^{-1}\left|\frac{y}{4}\right|\right) + C = 6\sec^{-1}\left|\frac{y}{4}\right| + C$

71. $\int_{\sqrt{2}/3}^{2/3} \frac{1}{|y|\sqrt{9y^2-1}}\,dy = \int_{\sqrt{2}/3}^{2/3} \frac{3}{|3y|\sqrt{(3y)^2-1}}\,dy = \int_{\sqrt{2}}^{2} \frac{1}{|u|\sqrt{u^2-1}}\,du$, where $u = 3y, du = 3\,dy$;

$y = \frac{\sqrt{2}}{3} \Rightarrow u = \sqrt{2},\ y = \frac{2}{3} \Rightarrow u = 2$

$$= \left[\sec^{-1}u\right]_{\sqrt{2}}^{2} = \left[\sec^{-1}2 - \sec^{-1}\sqrt{2}\right] = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

72. $\int_{-2\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{1}{|y|\sqrt{5y^2-3}}\,dy = \int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{\sqrt{5}}{-\sqrt{5}\sqrt{\left(\sqrt{5}y\right)^2-\left(\sqrt{3}\right)^2}}\,dy = \int_{-2}^{-\sqrt{6}} \frac{1}{-u\sqrt{u^2-\left(\sqrt{3}\right)^2}}\,du$, where $u = \sqrt{5}y, du = \sqrt{5}dy$;

$y = -\frac{2}{\sqrt{5}} \Rightarrow u = -2, y = -\frac{\sqrt{6}}{\sqrt{5}} \Rightarrow u = -\sqrt{6}$

$$= \left[-\frac{1}{\sqrt{3}} \sec^{-1} \left| \frac{u}{\sqrt{3}} \right| \right]_{-2}^{-\sqrt{6}} = \frac{-1}{\sqrt{3}} \left[\sec^{-1} \sqrt{2} - \sec^{-1} \frac{2}{\sqrt{3}} \right] = \frac{-1}{\sqrt{3}} \left(\frac{\pi}{4} - \frac{\pi}{6} \right) = \frac{-1}{\sqrt{3}} \left[\frac{3\pi}{12} - \frac{2\pi}{12} \right] = \frac{-\pi}{12\sqrt{3}} = \frac{-\sqrt{3}\pi}{36}$$

73. $\int \frac{1}{\sqrt{-2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-\left(x^2+2x+1\right)}}\,dx = \int \frac{1}{\sqrt{1-(x+1)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du$, where $u = x+1$ and $du = dx$

$= \sin^{-1} u + C = \sin^{-1}(x+1) + C$

74. $\int \frac{1}{\sqrt{-x^2+4x-1}}\,dx = \int \frac{1}{\sqrt{3-\left(x^2-4x+4\right)}}\,dx = \int \frac{1}{\sqrt{\left(\sqrt{3}\right)^2-(x-2)^2}}\,dx = \int \frac{1}{\sqrt{\left(\sqrt{3}\right)^2-u^2}}\,du$ where $u = x-2$ and $du = dx$

$= \sin^{-1}\left(\frac{u}{\sqrt{3}}\right) + C = \sin^{-1}\left(\frac{x-2}{\sqrt{3}}\right) + C$

75. $\int_{-2}^{-1} \frac{2}{v^2+4v+5}\,dv = 2\int_{-2}^{-1} \frac{1}{1+\left(v^2+4v+4\right)}\,dv = 2\int_{-2}^{-1} \frac{1}{1+(v+2)^2}\,dv = 2\int_0^1 \frac{1}{1+u^2}\,du$, where $u = v+2$, $du = dv$;

$v = -2 \Rightarrow u = 0, v = -1 \Rightarrow u = 1$

$= 2\left[\tan^{-1} u\right]_0^1 = 2\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 2\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{2}$

76. $\int_{-1}^{1} \frac{3}{4v^2+4v+4}\,dv = \frac{3}{4}\int_{-1}^{1} \frac{1}{\frac{3}{4}+\left(v^2+v+\frac{1}{4}\right)}\,dv = \frac{3}{4}\int_{-1}^{1} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+\left(v+\frac{1}{2}\right)^2}\,dv = \frac{3}{4}\int_{-1/2}^{3/2} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+u^2}\,du$ where $u = v+\frac{1}{2}$, $du = dv$;

$v = -1 \Rightarrow u = -\frac{1}{2}$, $v = 1 \Rightarrow u = \frac{3}{2}$

$= \frac{3}{4}\left[\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\right]_{-1/2}^{3/2} = \frac{\sqrt{3}}{2}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \frac{\sqrt{3}}{2}\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \frac{\sqrt{3}}{2}\left(\frac{2\pi}{6} + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \cdot \frac{\pi}{2} = \frac{\sqrt{3}\pi}{4}$

77. $\int \frac{1}{(t+1)\sqrt{t^2+2t-8}}\,dt = \int \frac{1}{(t+1)\sqrt{\left(t^2+2t+1\right)-9}}\,dt = \int \frac{1}{(t+1)\sqrt{(t+1)^2-3^2}}\,dt = \int \frac{1}{u\sqrt{u^2-3^2}}\,du$, where $u = t+1$ and $du = dt$

$= \frac{1}{3}\sec^{-1}\left|\frac{u}{3}\right| + C = \frac{1}{3}\sec^{-1}\left|\frac{t+1}{3}\right| + C$

78. $\int \frac{1}{(3t+1)\sqrt{9t^2+6t}}\,dt = \int \frac{1}{(3t+1)\sqrt{\left(9t^2+6t+1\right)-1}}\,dt = \int \frac{1}{(3t+1)\sqrt{(3t+1)^2-1^2}}\,dt = \frac{1}{3}\int \frac{1}{u\sqrt{u^2-1}}\,du$, where $u = 3t+1$ and $du = 3dt$

$= \frac{1}{3}\sec^{-1}|u| + C = \frac{1}{3}\sec^{-1}|3t+1| + C$

79. $3^y = 2^{y+1} \Rightarrow \ln 3^y = \ln 2^{y+1} \Rightarrow y(\ln 3) = (y+1)\ln 2 \Rightarrow (\ln 3 - \ln 2)y = \ln 2 \Rightarrow \left(\ln \frac{3}{2}\right)y = \ln 2 \Rightarrow y = \frac{\ln 2}{\ln\left(\frac{3}{2}\right)}$

80. $4^{-y} = 3^{y+2} \Rightarrow \ln 4^{-y} = \ln 3^{y+2} \Rightarrow -y\ln 4 = (y+2)\ln 3 \Rightarrow -2\ln 3 = (\ln 3 + \ln 4)y$

$\Rightarrow (\ln 12)y = -2\ln 3 \Rightarrow y = -\frac{\ln 9}{\ln 12}$

81. $9e^{2y} = x^2 \Rightarrow e^{2y} = \frac{x^2}{9} \Rightarrow \ln e^{2y} = \ln\left(\frac{x^2}{9}\right) \Rightarrow 2y(\ln e) = \ln\left(\frac{x^2}{9}\right) \Rightarrow y = \frac{1}{2}\ln\left(\frac{x^2}{9}\right) = \ln\sqrt{\frac{x^2}{9}} = \ln\left|\frac{x}{3}\right| = \ln|x| - \ln 3$

82. $3^y = 3\ln x \Rightarrow \ln 3^y = \ln(3\ln x) \Rightarrow y\ln 3 = \ln(3\ln x) \Rightarrow y = \frac{\ln(3\ln x)}{\ln 3} = \frac{\ln 3 + \ln(\ln x)}{\ln 3}$

83. $\ln(y-1)=x+\ln y \Rightarrow e^{\ln(y-1)}=e^{(x+\ln y)}=e^x e^{\ln y} \Rightarrow y-1=ye^x \Rightarrow y-ye^x=1 \Rightarrow y\left(1-e^x\right)=1 \Rightarrow y=\frac{1}{1-e^x}$

84. $\ln(10\ln y)=\ln 5x \Rightarrow e^{\ln(10\ln y)}=e^{\ln 5x} \Rightarrow 10\ln y=5x \Rightarrow \ln y=\frac{x}{2} \Rightarrow e^{\ln y}=e^{x/2} \Rightarrow y=e^{x/2}$

85. $\lim\limits_{x\to 1}\frac{x^2+3x-4}{x-1}=\lim\limits_{x\to 1}\frac{2x+3}{1}=5$

86. $\lim\limits_{x\to 1}\frac{x^a-1}{x^b-1}=\lim\limits_{x\to 1}\frac{ax^{a-1}}{bx^{b-1}}=\frac{a}{b}$

87. $\lim\limits_{x\to\pi}\frac{\tan x}{x}=\frac{\tan\pi}{\pi}=0$

88. $\lim\limits_{x\to 0}\frac{\tan x}{x+\sin x}=\lim\limits_{x\to 0}\frac{\sec^2 x}{1+\cos x}=\frac{1}{1+1}=\frac{1}{2}$

89. $\lim\limits_{x\to 0}\frac{\sin^2 x}{\tan\left(x^2\right)}=\lim\limits_{x\to 0}\frac{2\sin x\cdot\cos x}{2x\sec^2\left(x^2\right)}=\lim\limits_{x\to 0}\frac{\sin(2x)}{2x\sec^2\left(x^2\right)}=\lim\limits_{x\to 0}\frac{2\cos(2x)}{2x\left(2\sec^2\left(x^2\right)\tan\left(x^2\right)\cdot 2x\right)+2\sec^2\left(x^2\right)}=\frac{2}{0+2\cdot 1}=1$

90. $\lim\limits_{x\to 0}\frac{\sin(mx)}{\sin(nx)}=\lim\limits_{x\to 0}\frac{m\cos(mx)}{n\cos(nx)}=\frac{m}{n}$

91. $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\sec(7x)\cos(3x)=\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\cos(3x)}{\cos(7x)}=\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{-3\sin(3x)}{-7\sin(7x)}=\frac{3}{7}$

92. $\lim\limits_{x\to 0^+}\sqrt{x}\sec x=\lim\limits_{x\to 0^+}\frac{\sqrt{x}}{\cos x}=\frac{0}{1}=0$

93. $\lim\limits_{x\to 0}(\csc x-\cot x)=\lim\limits_{x\to 0}\frac{1-\cos x}{\sin x}=\lim\limits_{x\to 0}\frac{\sin x}{\cos x}=\frac{0}{1}=0$

94. $\lim\limits_{x\to 0}\left(\frac{1}{x^4}-\frac{1}{x^2}\right)=\lim\limits_{x\to 0}\left(\frac{1-x^2}{x^4}\right)=\lim\limits_{x\to 0}\left(1-x^2\right)\cdot\frac{1}{x^4}=\lim\limits_{x\to 0}\left(1-x^2\right)\cdot\lim\limits_{x\to 0}\frac{1}{x^4}=1\cdot\infty=\infty$

95. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)=\lim\limits_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)\cdot\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}=\lim\limits_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$

Notice that $x=\sqrt{x^2}$ for $x>0$ so this is equivalent to

$=\lim\limits_{x\to\infty}\frac{\frac{2x+1}{x}}{\sqrt{\frac{x^2+x+1}{x^2}}+\sqrt{\frac{x^2-x}{x^2}}}=\lim\limits_{x\to\infty}\frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}}=\frac{2}{\sqrt{1}+\sqrt{1}}=1$

96. $\lim\limits_{x\to\infty}\left(\frac{x^3}{x^2-1}-\frac{x^3}{x^2+1}\right)=\lim\limits_{x\to\infty}\frac{x^3\left(x^2+1\right)-x^3\left(x^2-1\right)}{\left(x^2-1\right)\left(x^2+1\right)}=\lim\limits_{x\to\infty}\frac{2x^3}{x^4-1}=\lim\limits_{x\to\infty}\frac{6x^2}{4x^3}=\lim\limits_{x\to\infty}\frac{12x}{12x^2}=\lim\limits_{x\to\infty}\frac{12}{24x}=\lim\limits_{x\to\infty}\frac{1}{2x}=0$

97. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x\to 0}\frac{10^x-1}{x}=\lim\limits_{x\to 0}\frac{(\ln 10)10^x}{1}=\ln 10$

98. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{\theta\to 0}\frac{3^\theta-1}{\theta}=\lim\limits_{\theta\to 0}\frac{(\ln 3)3^\theta}{1}=\ln 3$

99. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x\to 0}\frac{2^{\sin x}-1}{e^x-1}=\lim\limits_{x\to 0}\frac{2^{\sin x}(\ln 2)(\cos x)}{e^x}=\ln 2$

100. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x\to 0}\frac{2^{-\sin x}-1}{e^x-1}=\lim\limits_{x\to 0}\frac{2^{-\sin x}(\ln 2)(-\cos x)}{e^x}=-\ln 2$

101. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 0} \frac{5-5\cos x}{e^x-x-1} = \lim_{x\to 0} \frac{5\sin x}{e^x-1} = \lim_{x\to 0} \frac{5\cos x}{e^x} = 5$

102. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 0} \frac{x\sin x^2}{\tan^3 x} = \lim_{x\to 0} \frac{2x^2\cos x^2+\sin x^2}{3\tan^2 x\sec^2 x} = \lim_{x\to 0} \frac{2x^2\cos x^2+\sin x^2}{3\tan^4 x+3\tan^2 x}$

$= \lim_{x\to 0} \frac{6x\cos x^2-4x^2\sin x^2}{12\tan^3 x\sec^2 x+6\tan x\sec^2 x} = \lim_{x\to 0} \frac{6x\cos x^2-4x^3\sin x^2}{12\tan^5 x+18\tan^3 x+6\tan x} = \lim_{x\to 0} \frac{\left(6-8x^4\right)\cos x^2-24x^2\sin x^2}{60\tan^4 x\sec^2 x+54\tan^2 x\sec^2 x+6\sec^2 x} = \frac{6}{6} = 1$

103. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{t\to 0^+} \frac{t-\ln(1+2t)}{t^2} = \lim_{t\to 0^+} \frac{\left(1-\frac{2}{1+2t}\right)}{2t} = -\infty$

104. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 4} \frac{\sin^2(\pi x)}{e^{x-4}+3-x} = \lim_{x\to 4} \frac{2\pi(\sin\pi x)(\cos\pi x)}{e^{x-4}-1} = \lim_{x\to 4} \frac{\pi\sin(2\pi x)}{e^{x-4}-1}$

$= \lim_{x\to 4} \frac{2\pi^2\cos(2\pi x)}{e^{x-4}} = 2\pi^2$

105. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{t\to 0^+} \left(\frac{e^t}{t}-\frac{1}{t}\right) = \lim_{t\to 0^+} \left(\frac{e^t-1}{t}\right) = \lim_{t\to 0^+} \frac{e^t}{1} = 1$

106. The limit leads to the indeterminate form $\frac{\infty}{\infty}$: $\lim_{y\to 0^+} e^{-1/y}\ln y = \lim_{y\to 0^+} \frac{\ln y}{e^{y^{-1}}} = \lim_{y\to 0^+} \frac{y^{-1}}{-e^{y^{-1}}\left(y^{-2}\right)} = \lim_{y\to 0^+} \left(-\frac{y}{e^{y^{-1}}}\right) = 0$

107. Let $f(x) = \left(\frac{e^x+1}{e^x-1}\right)^{\ln x} \Rightarrow \ln f(x) = \ln x \ln\left(\frac{e^x+1}{e^x-1}\right) \Rightarrow \lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \ln x \ln\left(\frac{e^x+1}{e^x-1}\right)$; this limit is currently of the form $0\cdot\infty$. Before we put in one of the indeterminate forms, we rewrite $\frac{e^x+1}{e^x-1} = \frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} = \coth\left(\frac{x}{2}\right)$; the limit is $\lim_{x\to\infty} \ln x \ln\coth\left(\frac{x}{2}\right) = \lim_{x\to\infty} \frac{\ln\coth\left(\frac{x}{2}\right)}{\frac{1}{\ln x}}$; the limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to\infty} \frac{\ln\coth\left(\frac{x}{2}\right)}{\frac{1}{\ln x}}$

$= \lim_{x\to\infty} \left(\frac{\frac{\operatorname{csch}^2\left(\frac{x}{2}\right)}{\coth\left(\frac{x}{2}\right)}\left(-\frac{1}{2}\right)}{-\frac{1}{(\ln x)^2}\left(\frac{1}{x}\right)}\right) = \lim_{x\to\infty} \left(\frac{x(\ln x)^2}{2\sinh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)}\right) = \lim_{x\to\infty} \left(\frac{x(\ln x)^2}{\sinh x}\right) = \lim_{x\to\infty} \left(\frac{2x(\ln x)\left(\frac{1}{x}\right)+(\ln x)^2}{\cosh x}\right) = \lim_{x\to\infty} \left(\frac{2\ln x+(\ln x)^2}{\cosh x}\right)$

$= \lim_{x\to\infty} \left(\frac{2\left(\frac{1}{x}\right)+2(\ln x)\left(\frac{1}{x}\right)}{\sinh x}\right) = \lim_{x\to\infty} \left(\frac{2+2\ln x}{x\sinh x}\right) = \lim_{x\to\infty} \left(\frac{\frac{2}{x}}{x\cosh x+\sinh x}\right) = \lim_{x\to\infty} \left(\frac{2}{x^2\cosh x+x\sinh x}\right) = 0 \Rightarrow \lim_{x\to\infty} \left(\frac{e^x+1}{e^x-1}\right)^{\ln x}$
$= \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$

108. Let $f(x) = \left(1+\frac{3}{x}\right)^x \Rightarrow \ln f(x) = x\ln\left(1+\frac{3}{x}\right) \Rightarrow \lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \frac{\ln\left(1+3x^{-1}\right)}{x^{-1}}$; the limit leads to the indeterminate form $\frac{\infty}{\infty}$: $\lim_{x\to 0^+} \frac{\left(\frac{-3x^{-2}}{1+3x^{-1}}\right)}{-x^{-2}} = \lim_{x\to 0^+} \frac{3x}{x+3} = 0 \Rightarrow \lim_{x\to 0^+} \left(1+\frac{3}{x}\right)^x = \lim_{x\to 0^+} e^{\ln f(x)} = e^0 = 1$

109. (a) $\lim_{x\to\infty} \frac{\log_2 x}{\log_3 x} = \lim_{x\to\infty} \frac{\left(\frac{\ln x}{\ln 2}\right)}{\left(\frac{\ln x}{\ln 3}\right)} = \lim_{x\to\infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2} \Rightarrow$ same rate

(b) $\lim\limits_{x\to\infty} \frac{x}{x+\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{x^2}{x^2+1} = \lim\limits_{x\to\infty} \frac{2x}{2x} = \lim\limits_{x\to\infty} 1 = 1 \Rightarrow$ same rate

(c) $\lim\limits_{x\to\infty} \frac{\left(\frac{x}{100}\right)}{xe^{-x}} = \lim\limits_{x\to\infty} \frac{xe^x}{100x} = \lim\limits_{x\to\infty} \frac{e^x}{100} = \infty \Rightarrow$ faster

(d) $\lim\limits_{x\to\infty} \frac{x}{\tan^{-1} x} = \infty \Rightarrow$ faster

(e) $\lim\limits_{x\to\infty} \frac{\csc^{-1} x}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{\sin^{-1}\left(x^{-1}\right)}{x^{-1}} = \lim\limits_{x\to\infty} \frac{\left(\frac{-x^{-2}}{\sqrt{1-\left(x^{-1}\right)^2}}\right)}{-x^{-2}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1-\left(\frac{1}{x^2}\right)}} = 1 \Rightarrow$ same rate

(f) $\lim\limits_{x\to\infty} \frac{\sinh x}{e^x} = \lim\limits_{x\to\infty} \frac{\left(e^x - e^{-x}\right)}{2e^x} = \lim\limits_{x\to\infty} \frac{1-e^{-2x}}{2} = \frac{1}{2} \Rightarrow$ same rate

110. (a) $\lim\limits_{x\to\infty} \frac{3^{-x}}{2^{-x}} = \lim\limits_{x\to\infty} \left(\frac{2}{3}\right)^x = 0 \Rightarrow$ slower

(b) $\lim\limits_{x\to\infty} \frac{\ln 2x}{\ln x^2} = \lim\limits_{x\to\infty} \frac{\ln 2 + \ln x}{2(\ln x)} = \lim\limits_{x\to\infty} \left(\frac{\ln 2}{2\ln x} + \frac{1}{2}\right) = \frac{1}{2} \Rightarrow$ same rate

(c) $\lim\limits_{x\to\infty} \frac{10x^3+2x^2}{e^x} = \lim\limits_{x\to\infty} \frac{30x^2+4x}{e^x} = \lim\limits_{x\to\infty} \frac{60x+4}{e^x} = \lim\limits_{x\to\infty} \frac{60}{e^x} = 0 \Rightarrow$ slower

(d) $\lim\limits_{x\to\infty} \frac{\tan^{-1}\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{\tan^{-1}\left(x^{-1}\right)}{x^{-1}} = \lim\limits_{x\to\infty} \frac{\left(\frac{-x^{-2}}{1+x^{-2}}\right)}{-x^{-2}} = \lim\limits_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = 1 \Rightarrow$ same rate

(e) $\lim\limits_{x\to\infty} \frac{\sin^{-1}\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2}\right)} = \lim\limits_{x\to\infty} \frac{\sin^{-1}\left(x^{-1}\right)}{x^{-2}} = \lim\limits_{x\to\infty} \frac{\left(\frac{-x^{-2}}{\sqrt{1-\left(x^{-1}\right)^2}}\right)}{-2x^{-3}} = \lim\limits_{x\to\infty} \frac{x}{2\sqrt{1-\frac{1}{x^2}}} = \infty \Rightarrow$ faster

(f) $\lim\limits_{x\to\infty} \frac{\operatorname{sech} x}{e^{-x}} = \lim\limits_{x\to\infty} \frac{\left(\frac{2}{e^x+e^{-x}}\right)}{e^{-x}} = \lim\limits_{x\to\infty} \frac{2}{e^{-x}\left(e^x+e^{-x}\right)} = \lim\limits_{x\to\infty} \left(\frac{2}{1+e^{-2x}}\right) = 2 \Rightarrow$ same rate

111. (a) $\frac{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)}{\left(\frac{1}{x^2}\right)} = 1 + \frac{1}{x^2} \le 2$ for x sufficiently large $\Rightarrow$ true

(b) $\frac{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)} = x^2 + 1 > M$ for any positive integer M whenever $x > \sqrt{M} \Rightarrow$ false

(c) $\lim\limits_{x\to\infty} \frac{x}{x+\ln x} = \lim\limits_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1 \Rightarrow$ the same growth rate $\Rightarrow$ false

(d) $\lim\limits_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim\limits_{x\to\infty} \frac{\left[\frac{(1/x)}{\ln x}\right]}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{1}{\ln x} = 0 \Rightarrow$ grows slower $\Rightarrow$ true

(e) $\frac{\tan^{-1} x}{1} \le \frac{\pi}{2}$ for all $x \Rightarrow$ true

(f) $\frac{\cosh x}{e^x} = \frac{1}{2}\left(1 + e^{-2x}\right) \le \frac{1}{2}(1+1) = 1$ if $x > 0 \Rightarrow$ true

112. (a) $\frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)} = \frac{1}{x^2+1} \le 1$ if $x > 0 \Rightarrow$ true

(b) $\lim\limits_{x\to\infty} \dfrac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)} = \lim\limits_{x\to\infty} \left(\dfrac{1}{x^2+1}\right) = 0 \Rightarrow \text{true}$

(c) $\lim\limits_{x\to\infty} \frac{\ln x}{x+1} = \lim\limits_{x\to\infty} \dfrac{\left(\frac{1}{x}\right)}{1} = 0 \Rightarrow \text{true}$

(d) $\frac{\ln 2x}{\ln x} = \frac{\ln 2}{\ln x} + 1 \le 1+1 = 2$ if $x \ge 2 \Rightarrow$ true

(e) $\frac{\sec^{-1} x}{1} = \dfrac{\cos^{-1}\left(\frac{1}{x}\right)}{1} \le \dfrac{\left(\frac{\pi}{2}\right)}{1} = \frac{\pi}{2}$ if $x > 1 \Rightarrow$ true

(f) $\frac{\sinh x}{e^x} = \frac{1}{2}\left(1-e^{-2x}\right) \le \frac{1}{2}$ if $x > 0 \Rightarrow$ true

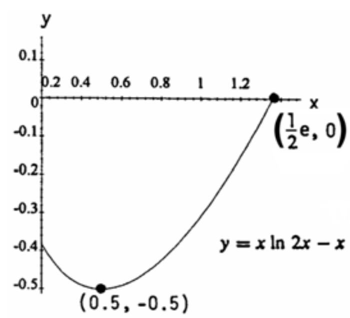
113. $\frac{df}{dx} = e^x + 1 \Rightarrow \left(\frac{df^{-1}}{dx}\right)_{x=f(\ln 2)} = \dfrac{1}{\left(\frac{df}{dx}\right)_{x=\ln 2}} \Rightarrow \left(\frac{df^{-1}}{dx}\right)_{x=f(\ln 2)} = \dfrac{1}{\left(e^x+1\right)_{x=\ln 2}} = \frac{1}{2+1} = \frac{1}{3}$

114. $y = f(x) \Rightarrow y = 1 + \frac{1}{x} \Rightarrow \frac{1}{x} = y - 1 \Rightarrow x = \frac{1}{y-1} \Rightarrow f^{-1}(x) = \frac{1}{x-1}$; $f^{-1}\left(f(x)\right) = \dfrac{1}{\left(1+\frac{1}{x}\right)-1} = \dfrac{1}{\left(\frac{1}{x}\right)} = x$ and

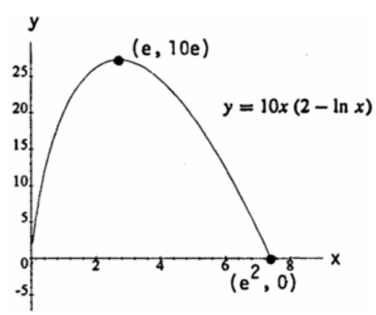
$f\left(f^{-1}(x)\right) = 1 + \dfrac{1}{\left(\frac{1}{x-1}\right)} = 1 + (x-1) = x$; $\left.\frac{df^{-1}}{dx}\right|_{f(x)} = \left.\frac{-1}{(x-1)^2}\right|_{f(x)} = \dfrac{-1}{\left[\left(1+\frac{1}{x}\right)-1\right]^2} = -x^2$;

$f'(x) = -\frac{1}{x^2} \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{f(x)} = \frac{1}{f'(x)}$

115. $y = x\ln 2x - x \Rightarrow y' = x\left(\frac{2}{2x}\right) + \ln(2x) - 1 = \ln 2x$; solving $y' = 0 \Rightarrow x = \frac{1}{2}$; $y' > 0$ for $x > \frac{1}{2}$ and $y' < 0$ for $x < \frac{1}{2} \Rightarrow$ relative minimum of $-\frac{1}{2}$ at $x = \frac{1}{2}$; $f\left(\frac{1}{2e}\right) = -\frac{1}{e}$ and $f\left(\frac{e}{2}\right) = 0 \Rightarrow$ absolute minimum is $-\frac{1}{2}$ at $x = \frac{1}{2}$ and the absolute maximum is 0 at $x = \frac{e}{2}$



116. $y = 10x(2-\ln x) \Rightarrow y' = 10(2-\ln x) - 10x\left(\frac{1}{x}\right) = 20 - 10\ln x - 10 = 10(1-\ln x)$; solving $y' = 0 \Rightarrow x = e$; $y' < 0$ for $x > e$ and $y' > 0$ for $x < e \Rightarrow$ relative maximum at $x = e$ of $10e$; $y \ge 0$ on $(0, e^2]$ and $y\left(e^2\right) = 10e^2(2 - 2\ln e) = 0 \Rightarrow$ absolute minimum is 0 at $x = e^2$ and the absolute maximum is 10e at $x = e$



117. $A = \int_1^e \frac{2\ln x}{x}\,dx = \int_0^1 2u\,du = \left[u^2\right]_0^1 = 1$, where $u = \ln x$ and $du = \frac{1}{x}dx$; $x = 1 \Rightarrow u = 0, x = e \Rightarrow u = 1$

118. (a) $A_1 = \int_{10}^{20} \frac{1}{x}\,dx = \left[\ln|x|\right]_{10}^{20} = \ln 20 - \ln 10 = \ln\frac{20}{10} = \ln 2$, and $A_2 = \int_1^2 \frac{1}{x}\,dx = \left[\ln|x|\right]_1^2 = \ln 2 - \ln 1 = \ln 2$

(b) $A_1 = \int_{ka}^{kb} \frac{1}{x}\,dx = \left[\ln|x|\right]_{ka}^{kb} = \ln kb - \ln ka = \ln \frac{kb}{ka} = \ln \frac{b}{a} = \ln b - \ln a,$ and $A_2 = \int_a^b \frac{1}{x}\,dx = \left[\ln|x|\right]_a^b = \ln b - \ln a$

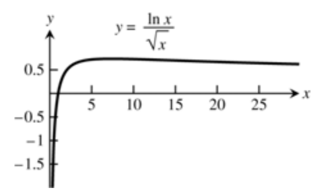
119. $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$; $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \left(\frac{1}{x}\right)\sqrt{x} = \frac{1}{\sqrt{x}} \Rightarrow \left.\frac{dy}{dt}\right|_{e^2} = \frac{1}{e}$ m/s

120. $y = 3e^{-x/3} \Rightarrow \frac{dy}{dx} = -e^{-x/3}$; $\frac{dx}{dt} = \frac{(dy/dt)}{(dy/dx)} \Rightarrow \frac{dx}{dt} = \frac{\left(-\frac{1}{4}\right)\sqrt{3-y}}{-e^{-x/3}}$; $x = 3 \Rightarrow y = 3e^{-1} \Rightarrow \left.\frac{dx}{dt}\right|_{x=3} = \frac{\left(-\frac{1}{4}\right)\sqrt{3-\frac{3}{e}}}{\left(-\frac{1}{e}\right)}$
$= \frac{1}{4}\sqrt{e}\sqrt{e-1} \approx 0.54$ m/s

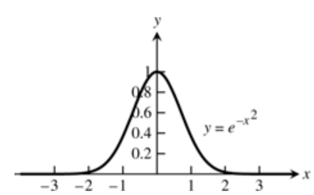
121. $A = xy = xe^{-x^2} \Rightarrow \frac{dA}{dx} = e^{-x^2} + (x)(-2x)e^{-x^2} = e^{-x^2}\left(1-2x^2\right)$. Solving $\frac{dA}{dx} = 0 \Rightarrow 1-2x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$;
$\frac{dA}{dx} < 0$ for $x > \frac{1}{\sqrt{2}}$ and $\frac{dA}{dx} > 0$ for $0 < x < \frac{1}{\sqrt{2}} \Rightarrow$ absolute maximum of $\frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2e}}$ at $x = \frac{1}{\sqrt{2}}$ units long by $y = e^{-1/2} = \frac{1}{\sqrt{e}}$ units high.

122. $A = xy = x\left(\frac{\ln x}{x^2}\right) = \frac{\ln x}{x} \Rightarrow \frac{dA}{dx} = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1-\ln x}{x^2}$. Solving $\frac{dA}{dx} = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow x = e$; $\frac{dA}{dx} < 0$ for $x > e$ and $\frac{dA}{dx} > 0$ for $x < e \Rightarrow$ absolute maximum of $\frac{\ln e}{e} = \frac{1}{e}$ at $x = e$ units long and $y = \frac{1}{e^2}$ units high.

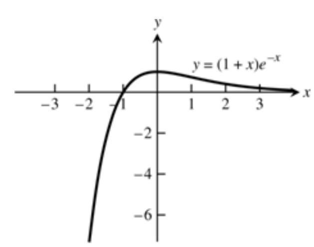
123. (a) $y = \frac{\ln x}{\sqrt{x}} \Rightarrow y' = \frac{1}{x\sqrt{x}} - \frac{\ln x}{2x^{3/2}} = \frac{2-\ln x}{2x\sqrt{x}}$
$\Rightarrow y'' = -\frac{3}{4}x^{-5/2}(2-\ln x) - \frac{1}{2}x^{-5/2} = x^{-5/2}\left(\frac{3}{4}\ln x - 2\right)$;
solving $y' = 0 \Rightarrow \ln x = 2 \Rightarrow x = e^2$; $y' < 0$ for $x > e^2$ and $y' > 0$ for $x < e^2 \Rightarrow$ a maximum of $\frac{2}{e}$; $y'' = 0 \Rightarrow \ln x = \frac{8}{3} \Rightarrow x = e^{8/3}$;
the curve is concave down on $\left(0, e^{8/3}\right)$ and concave up on $\left(e^{8/3}, \infty\right)$; so there is an inflection point at $\left(e^{8/3}, \frac{8}{3e^{4/3}}\right)$.



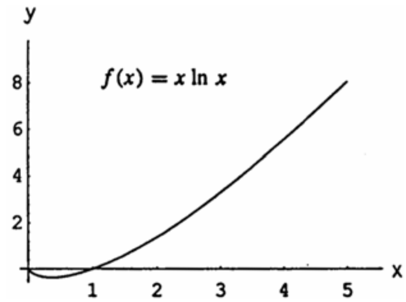
(b) $y = e^{-x^2} \Rightarrow y' = -2xe^{-x^2} \Rightarrow y'' = -2e^{-x^2} + 4x^2e^{-x^2}$
$= \left(4x^2 - 2\right)e^{-x^2}$; solving $y' = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$ for $x < 0 \Rightarrow$ a maximum at $x = 0$ of $e^0 = 1$; there are points of inflection at $x = \pm\frac{1}{\sqrt{2}}$; the curve is concave down for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and concave up otherwise.



(c) $y = (1+x)e^{-x} \Rightarrow y' = e^{-x} - (1+x)e^{-x} = -xe^{-x}$
$\Rightarrow y'' = -e^{-x} + xe^{-x} = (x-1)e^{-x}$; solving
$y' = 0 \Rightarrow -xe^{-x} = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$ for $x < 0 \Rightarrow$ a maximum at $x = 0$ of $(1+0)e^0 = 1$; there is a point of inflection at $x = 1$ and the curve is concave up for $x > 1$ and concave down for $x < 1$.



124. $y = x\ln x \Rightarrow y' = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$; solving $y' = 0 \Rightarrow \ln x + 1 = 0$
$\Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$; $y' > 0$ for $x > e^{-1}$ and $y' < 0$ for $x < e^{-1} \Rightarrow$ a minimum of $e^{-1}\ln e^{-1} = -\frac{1}{e}$ at $x = e^{-1}$. This minimum is an absolute minimum since $y'' = \frac{1}{x}$ is positive for all $x > 0$.



125. $\frac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y} \Rightarrow \frac{dy}{\sqrt{y}\cos^2\sqrt{y}} = dx \Rightarrow 2\tan\sqrt{y} = x + C \Rightarrow y = \left(\tan^{-1}\left(\frac{x+C}{2}\right)\right)^2$

126. $y' = \frac{3y(x+1)^2}{y-1} \Rightarrow \frac{(y-1)}{y}dy = 3(x+1)^2\,dx \Rightarrow y - \ln y = (x+1)^3 + C$

127. $yy' = \sec\left(y^2\right)\sec^2 x \Rightarrow \frac{ydy}{\sec\left(y^2\right)} = \sec^2 x\,dx \Rightarrow \frac{\sin\left(y^2\right)}{2} = \tan x + C \Rightarrow \sin\left(y^2\right) = 2\tan x + C_1$

128. $y\cos^2(x)\,dy + \sin x\,dx = 0 \Rightarrow y\,dy = -\frac{\sin x}{\cos^2(x)}dx \Rightarrow \frac{y^2}{2} = -\frac{1}{\cos(x)} + C \Rightarrow y = \pm\sqrt{\frac{-2}{\cos(x)} + C_1}$

129. $\frac{dy}{dx} = e^{-x-y-2} \Rightarrow e^y dy = e^{-(x+2)}dx \Rightarrow e^y = -e^{-(x+2)} + C$. We have $y(0) = -2$, so $e^{-2} = -e^{-2} + C \Rightarrow C = 2e^{-2}$ and $e^y = -e^{-(x+2)} + 2e^{-2} \Rightarrow y = \ln\left(-e^{-(x+2)} + 2e^{-2}\right)$

130. $\frac{dy}{dx} = \frac{y\ln y}{1+x^2} \Rightarrow \frac{dy}{y\ln y} = \frac{dx}{1+x^2} \Rightarrow \ln(\ln y) = \tan^{-1}(x) + C \Rightarrow y = e^{e^{\tan^{-1}(x)+C}}$. We have $y(0) = e^2 \Rightarrow e^2 = e^{e^{\tan^{-1}(0)+C}}$
$\Rightarrow e^{\tan^{-1}(0)+C} = 2 \Rightarrow \tan^{-1}(0) + C = \ln 2 \Rightarrow 0 + C = \ln 2 \Rightarrow C = \ln 2 \Rightarrow y = e^{e^{\tan^{-1}(x)+\ln 2}}$

131. $x\,dy - \left(y + \sqrt{y}\right)dx = 0 \Rightarrow \frac{dy}{\left(y+\sqrt{y}\right)} = \frac{dx}{x} \Rightarrow 2\ln\left(\sqrt{y}+1\right) = \ln x + C$. We have $y(1) = 1 \Rightarrow 2\ln\left(\sqrt{1}+1\right) = \ln 1 + C$
$\Rightarrow 2\ln 2 = C = \ln 2^2 = \ln 4$. So $2\ln\left(\sqrt{y}+1\right) = \ln x + \ln 4 = \ln(4x) \Rightarrow \ln\left(\sqrt{y}+1\right) = \frac{1}{2}\ln(4x) = \ln(4x)^{1/2}$
$\Rightarrow e^{\ln\left(\sqrt{y}+1\right)} = e^{\ln(4x)^{1/2}} \Rightarrow \sqrt{y} + 1 = 2\sqrt{x} \Rightarrow y = \left(2\sqrt{x} - 1\right)^2$

132. $y^{-2}\frac{dx}{dy} = \frac{e^x}{e^{2x}+1} \Rightarrow \frac{e^{2x}+1}{e^x}dx = \frac{dy}{y^{-2}} \Rightarrow \frac{y^3}{3} = e^x - e^{-x} + C$. We have $y(0) = 1 \Rightarrow \frac{(1)^3}{3} = e^0 - e^0 + C \Rightarrow C = \frac{1}{3}$. So $\frac{y^3}{3} = e^x - e^{-x} + \frac{1}{3} \Rightarrow y^3 = 3\left(e^x - e^{-x}\right) + 1 \Rightarrow y = \left[3\left(e^x - e^{-x}\right) + 1\right]^{1/3}$

133. Since the half life is 5700 years and $A(t) = A_0e^{kt}$ we have $\frac{A_0}{2} = A_0e^{5700k} \Rightarrow \frac{1}{2} = e^{5700k} \Rightarrow \ln(0.5) = 5700k$
$\Rightarrow k = \frac{\ln(0.5)}{5700}$. With 10% of the original carbon-14 remaining we have $0.1A_0 = A_0e^{\frac{\ln(0.5)}{5700}t} \Rightarrow 0.1 = e^{\frac{\ln(0.5)}{5700}t}$
$\Rightarrow \ln(0.1) = \frac{\ln(0.5)}{5700}t \Rightarrow t = \frac{(5700)\ln(0.1)}{\ln(0.5)} \approx 18{,}935$ years (rounded to the nearest year).

134. $T - T_s = (T_o - T_s)e^{-kt} \Rightarrow 82 - 5 = (104 - 5)e^{-k/4}$, time in hours, $\Rightarrow k = -4\ln\left(\frac{7}{9}\right) = 4\ln\left(\frac{9}{7}\right)$
$\Rightarrow 21 - 5 = (104 - 5)e^{-4\ln(9/7)t} \Rightarrow t = \frac{\ln 6}{4\ln\left(\frac{9}{7}\right)} \approx 1.78\,\text{h} \approx 107$ min, the total time $\Rightarrow$ the time it took to cool from $82°$ C to $21°$ C was $107 - 15 = 92\,\text{min}$

135. $\theta = \pi - \cot^{-1}\left(\frac{x}{60}\right) - \cot^{-1}\left(\frac{5}{3} - \frac{x}{30}\right), 0 < x < 50 \Rightarrow \frac{d\theta}{dx} = \frac{\left(\frac{1}{60}\right)}{1+\left(\frac{x}{60}\right)^2} + \frac{\left(-\frac{1}{30}\right)}{1+\left(\frac{50-x}{30}\right)^2} = 30\left[\frac{2}{60^2+x^2} - \frac{1}{30^2+(50-x)^2}\right]$; solving $\frac{d\theta}{dx} = 0 \Rightarrow x^2 - 200x + 3200 = 0 \Rightarrow x = 100 \pm 20\sqrt{17}$, but $100 + 20\sqrt{17}$ is not in the domain; $\frac{d\theta}{dx} > 0$ for $x < 20\left(5 - \sqrt{17}\right)$ and $\frac{d\theta}{dx} < 0$ for $20\left(5 - \sqrt{17}\right) < x < 50 \Rightarrow x = 20\left(5 - \sqrt{17}\right) \approx 17.54$ m maximizes θ

136. $v = x^2 \ln\left(\frac{1}{x}\right) = x^2(\ln 1 - \ln x) = -x^2 \ln x \Rightarrow \frac{dv}{dx} = -2x\ln x - x^2\left(\frac{1}{x}\right) = -x(2\ln x + 1)$; solving $\frac{dv}{dx} = 0 \Rightarrow 2\ln x + 1 = 0 \Rightarrow \ln x = -\frac{1}{2} \Rightarrow x = e^{-1/2}$; $\frac{dv}{dx} < 0$ for $x > e^{-1/2}$ and $\frac{dv}{dx} > 0$ for $x < e^{-1/2} \Rightarrow$ a relative maximum at $x = e^{-1/2}$; $\frac{r}{h} = x$ and $r = 1 \Rightarrow h = e^{1/2} = \sqrt{e} \approx 1.65$ cm

CHAPTER 7 ADDITIONAL AND ADVANCED EXERCISES

1. $\lim\limits_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim\limits_{b\to 1^-}\left[\sin^{-1} x\right]_0^b = \lim\limits_{b\to 1^-}\left(\sin^{-1} b - \sin^{-1} 0\right) = \lim\limits_{b\to 1^-}\left(\sin^{-1} b - 0\right) = \lim\limits_{b\to 1^-} \sin^{-1} b = \frac{\pi}{2}$

2. $\lim\limits_{x\to\infty} \frac{1}{x}\int_0^x \tan^{-1} t\,dt = \lim\limits_{x\to\infty} \frac{\int_0^x \tan^{-1} t\,dt}{x}$, $\frac{\infty}{\infty}$ form
$= \lim\limits_{x\to\infty} \frac{\tan^{-1} x}{1} = \frac{\pi}{2}$

3. $y = \left(\cos\sqrt{x}\right)^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln\left(\cos\sqrt{x}\right)$ and $\lim\limits_{x\to 0^+} \frac{\ln\left(\cos\sqrt{x}\right)}{x} = \lim\limits_{x\to 0^+} \frac{-\sin\sqrt{x}}{2\sqrt{x}\cos\sqrt{x}} = \frac{-1}{2}\lim\limits_{x\to 0^+} \frac{\tan\sqrt{x}}{\sqrt{x}}$
$= -\frac{1}{2}\lim\limits_{x\to 0^+} \frac{\frac{1}{2}x^{-1/2}\sec^2\sqrt{x}}{\frac{1}{2}x^{-1/2}} = -\frac{1}{2} \Rightarrow \lim\limits_{x\to 0^+}\left(\cos\sqrt{x}\right)^{1/x} = e^{-1/2} = \frac{1}{\sqrt{e}}$

4. $y = \left(x + e^x\right)^{2/x} \Rightarrow \ln y = \frac{2\ln\left(x+e^x\right)}{x} \Rightarrow \lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \frac{2\left(1+e^x\right)}{x+e^x} = \lim\limits_{x\to\infty} \frac{2e^x}{1+e^x} = \lim\limits_{x\to\infty} \frac{2e^x}{e^x} = 2$
$\Rightarrow \lim\limits_{x\to\infty}\left(x + e^x\right)^{2/x} = \lim\limits_{x\to\infty} e^y = e^2$

5. $\lim\limits_{x\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right) = \lim\limits_{x\to\infty}\left(\left(\frac{1}{n}\right)\left[\frac{1}{1+\left(\frac{1}{n}\right)}\right] + \left(\frac{1}{n}\right)\left[\frac{1}{1+2\left(\frac{1}{n}\right)}\right] + \ldots + \left(\frac{1}{n}\right)\left[\frac{1}{1+n\left(\frac{1}{n}\right)}\right]\right)$ which can be interpreted as a Riemann sum with partitioning $\Delta x = \frac{1}{n} \Rightarrow \lim\limits_{x\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right) = \int_0^1 \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_0^1 = \ln 2$

6. $\lim\limits_{x\to\infty} \frac{1}{n}[e^{1/n}+e^{2/n}+\ldots+e] = \lim\limits_{x\to\infty}\left[\left(\frac{1}{n}\right)e^{(1/n)}+\left(\frac{1}{n}\right)e^{2(1/n)}+\ldots+\left(\frac{1}{n}\right)e^{n(1/n)}\right]$ which can be interpreted as a Riemann sum with partitioning $\Delta x=\frac{1}{n} \Rightarrow \lim\limits_{x\to\infty}\frac{1}{n}[e^{1/n}+e^{2/n}+\ldots+e]=\int_0^1 e^x\,dx=\left[e^x\right]_0^1=e-1$

7. $A(t)=\int_0^t e^{-x}dx=\left[-e^{-x}\right]_0^t=1-e^{-t}, V(t)=\pi\int_0^t e^{-2x}dx=\left[-\frac{\pi}{2}e^{-2x}\right]_0^t=\frac{\pi}{2}\left(1-e^{-2t}\right)$

(a) $\lim\limits_{t\to\infty} A(t)=\lim\limits_{t\to\infty}\left(1-e^{-t}\right)=1$

(b) $\lim\limits_{t\to\infty}\frac{V(t)}{A(t)}=\lim\limits_{t\to\infty}\frac{\frac{\pi}{2}\left(1-e^{-2t}\right)}{1-e^{-t}}=\frac{\pi}{2}$

(c) $\lim\limits_{t\to0^+}\frac{V(t)}{A(t)}=\lim\limits_{t\to0^+}\frac{\frac{\pi}{2}\left(1-e^{-2t}\right)}{1-e^{-t}}=\lim\limits_{t\to0^+}\frac{\frac{\pi}{2}\left(1-e^{-t}\right)\left(1+e^{-t}\right)}{\left(1-e^{-t}\right)}=\lim\limits_{t\to0^+}\frac{\pi}{2}\left(1+e^{-t}\right)=\pi$

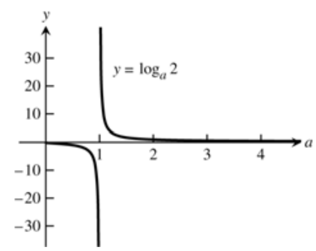
8. (a) $\lim\limits_{a\to0^+}\log_a 2=\lim\limits_{a\to0^+}\frac{\ln 2}{\ln a}=0;$

$\lim\limits_{a\to1^-}\log_a 2=\lim\limits_{a\to1^-}\frac{\ln 2}{\ln a}=-\infty;$

$\lim\limits_{a\to1^+}\log_a 2=\lim\limits_{a\to1^+}\frac{\ln 2}{\ln a}=\infty;$

$\lim\limits_{a\to\infty}\log_a 2=\lim\limits_{a\to\infty}\frac{\ln 2}{\ln a}=0$

(b)

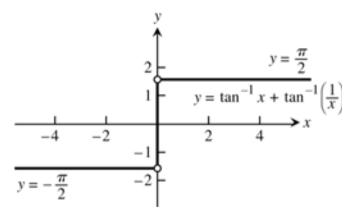


9. $A_1=\int_1^e\frac{2\log_2 x}{x}dx=\frac{2}{\ln 2}\int_1^e\frac{\ln x}{x}dx=\left[\frac{(\ln x)^2}{\ln 2}\right]_1^e=\frac{1}{\ln 2};\;\; A_2=\int_1^e\frac{2\log_4 x}{4}dx=\frac{2}{\ln 4}\int_1^e\frac{\ln x}{x}dx=\left[\frac{(\ln x)^2}{2\ln 2}\right]_1^e=\frac{1}{2\ln 2}$

$\Rightarrow A_1:A_2=2:1$

10. $y=\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)\Rightarrow y'=\frac{1}{1+x^2}+\frac{\left(-\frac{1}{x^2}\right)}{\left(1+\frac{1}{x^2}\right)}$

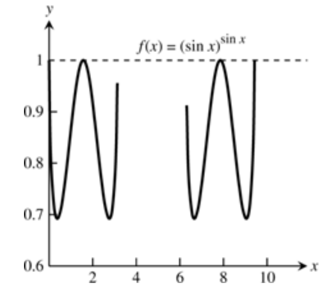
$=\frac{1}{1+x^2}-\frac{1}{1+x^2}=0\Rightarrow\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)$ is a constant and the constant is $\frac{\pi}{2}$ for $x>0$; it is $-\frac{\pi}{2}$ for $x<0$ since $\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)$ is odd. Next the



$\lim\limits_{x\to0^+}\left[\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)\right]=0+\frac{\pi}{2}=\frac{\pi}{2}$ and $\lim\limits_{x\to0^-}\left(\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)\right)=0+\left(-\frac{\pi}{2}\right)=-\frac{\pi}{2}$

11. $\ln x^{(x^x)}=x^x\ln x$ and $\ln(x^x)^x=x\ln x^x=x^2\ln x$; then, $x^x\ln x=x^2\ln x\Rightarrow\left(x^x-x^2\right)\ln x=0\Rightarrow x^x=x^2$ or $\ln x=0$. $\ln x=0\Rightarrow x=1; x^x=x^2\Rightarrow x\ln x=2\ln x\Rightarrow x=2$. Therefore, $x^{(x^x)}=(x^x)^x$ when $x=2$ or $x=1$.

12. In the interval $\pi < x < 2\pi$ the function $\sin x < 0 \Rightarrow (\sin x)^{\sin x}$ is not defined for all values in that interval or its translation by 2π.



13. $f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)}g'(x)$, where $g'(x) = \frac{x}{1+x^4} \Rightarrow f'(2) = e^0\left(\frac{2}{1+16}\right) = \frac{2}{17}$

14. (a) $\frac{df}{dx} = \frac{2\ln e^x}{e^x} \cdot e^x = 2x$ (b) $f(0) = \int_1^1 \frac{2\ln t}{t}\,dt = 0$

(c) $\frac{df}{dx} = 2x \Rightarrow f(x) = x^2 + C; f(0) = 0 \Rightarrow C = 0 \Rightarrow f(x) = x^2 \Rightarrow$ the graph of $f(x)$ is a parabola

15. (a) $g(x) + h(x) = 0 \Rightarrow g(x) = -h(x)$; also $g(x) + h(x) = 0 \Rightarrow g(-x) + h(-x) = 0 \Rightarrow g(x) - h(x) = 0$ $\Rightarrow g(x) = h(x)$; therefore $-h(x) = h(x) \Rightarrow h(x) = 0 \Rightarrow g(x) = 0$

(b) $\frac{f(x)+f(-x)}{2} = \frac{[f_E(x)+f_O(x)]+[f_E(-x)+f_O(-x)]}{2} = \frac{f_E(x)+f_O(x)+f_E(x)-f_O(x)}{2} = f_E(x)$;

$\frac{f(x)-f(-x)}{2} = \frac{[f_E(x)+f_O(x)]-[f_E(-x)+f_O(-x)]}{2} = \frac{f_E(x)+f_O(x)-f_E(x)+f_O(x)}{2} = f_O(x)$

(c) Part $b \Rightarrow$ such a decomposition is unique.

16. (a) $g(0+0) = \frac{g(0)+g(0)}{1-g(0)g(0)} \Rightarrow \left[1-g^2(0)\right]g(0) = 2g(0) \Rightarrow g(0) - g^3(0) = 2g(0) \Rightarrow g^3(0) + g(0) = 0$ $\Rightarrow g(0)\left[g^2(0)+1\right] = 0 \Rightarrow g(0) = 0$

(b) $g'(x) = \lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim\limits_{h\to 0} \frac{\left[\frac{g(x)+g(h)}{1-g(x)g(h)}\right]-g(x)}{h} = \lim\limits_{h\to 0} \frac{g(x)+g(h)-g(x)+g^2(x)g(h)}{h[1-g(x)g(h)]}$

$= \lim\limits_{h\to 0}\left[\frac{g(h)}{h}\right]\left[\frac{1+g^2(x)}{1-g(x)g(h)}\right] = 1\cdot\left[1+g^2(x)\right] = 1 + g^2(x) = 1 + [g(x)]^2$

(c) $\frac{dy}{dx} = 1 + y^2 \Rightarrow \frac{dy}{1+y^2} = dx \Rightarrow \tan^{-1} y = x + C \Rightarrow \tan^{-1}(g(x)) = x + C; g(0) = 0 \Rightarrow \tan^{-1} 0 = 0 + C$

$\Rightarrow C = 0 \Rightarrow \tan^{-1}(g(x)) = x \Rightarrow g(x) = \tan x$

17. $M = \int_0^1 \frac{2}{1+x^2}\,dx = 2\left[\tan^{-1} x\right]_0^1 = \frac{\pi}{2}$ and $M_y = \int_0^1 \frac{2x}{1+x^2}\,dx = \left[\ln\left(1+x^2\right)\right]_0^1 = \ln 2 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{\ln 2}{\left(\frac{\pi}{2}\right)} = \frac{\ln 4}{\pi}$; $\bar{y} = 0$ by symmetry

18. (a) $V=\pi\int_{1/4}^{4}\left(\frac{1}{2\sqrt{x}}\right)^2 dx=\frac{\pi}{4}\int_{1/4}^{4}\frac{1}{x}\,dx=\frac{\pi}{4}\left[\ln|x|\right]_{1/4}^{4}=\frac{\pi}{4}\left(\ln 4-\ln\frac{1}{4}\right)=\frac{\pi}{4}\ln 16=\frac{\pi}{4}\ln\left(2^4\right)=\pi\ln 2$

(b) $M_y=\int_{1/4}^{4}x\left(\frac{1}{2\sqrt{x}}\right)dx=\frac{1}{2}\int_{1/4}^{4}x^{1/2}dx=\left[\frac{1}{3}x^{3/2}\right]_{1/4}^{4}=\left(\frac{8}{3}-\frac{1}{24}\right)=\frac{64-1}{24}=\frac{63}{24}$;

$M_x=\int_{1/4}^{4}\frac{1}{2}\left(\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{2\sqrt{x}}\right)dx=\frac{1}{8}\int_{1/4}^{4}\frac{1}{x}\,dx=\left[\frac{1}{8}\ln|x|\right]_{1/4}^{4}=\frac{1}{8}\ln 16=\frac{1}{2}\ln 2$;

$M=\int_{1/4}^{4}\frac{1}{2\sqrt{x}}\,dx=\int_{1/4}^{4}\frac{1}{2}x^{-1/2}dx=\left[x^{1/2}\right]_{1/4}^{4}=2-\frac{1}{2}=\frac{3}{2}$; therefore, $\bar{x}=\frac{M_y}{M}=\left(\frac{63}{24}\right)\left(\frac{2}{3}\right)=\frac{21}{12}=\frac{7}{4}$ and

$\bar{y}=\frac{M_x}{M}=\left(\frac{1}{2}\ln 2\right)\left(\frac{2}{3}\right)=\frac{\ln 2}{3}$

19. (a) $L=k\left(\frac{a-b\cot\theta}{R^4}+\frac{b\csc\theta}{r^4}\right)\Rightarrow\frac{dL}{d\theta}=k\left(\frac{b\csc^2\theta}{R^4}-\frac{b\csc\theta\cot\theta}{r^4}\right)$; solving $\frac{dL}{d\theta}=0$

$\Rightarrow r^4 b\csc^2\theta-bR^4\csc\theta\cot\theta=0\Rightarrow(b\csc\theta)\left(r^4\csc\theta-R^4\cot\theta\right)=0$; but $b\csc\theta\neq 0$ since

$\theta\neq\frac{\pi}{2}\Rightarrow r^4\csc\theta-R^4\cot\theta=0\Rightarrow\cos\theta=\frac{r^4}{R^4}\Rightarrow\theta=\cos^{-1}\left(\frac{r^4}{R^4}\right)$, the critical value of θ

(b) $\theta=\cos^{-1}\left(\frac{5}{6}\right)^4\approx\cos^{-1}(0.48225)\approx 61°$

20. In order to maximize the amount of sunlight, we need to maximize the angle θ formed by extending the two red line segments to their vertex. The angle between the two lines is given by $\theta=\pi-\left(\theta_1+(\pi-\theta_2)\right)$. From trig we have $\tan\theta_1=\frac{105}{135-x}\Rightarrow\theta_1=\tan^{-1}\left(\frac{105}{135-x}\right)$ and $\tan\left(\pi-\theta_2\right)=\frac{60}{x}\Rightarrow\left(\pi-\theta_2\right)=\tan^{-1}\left(\frac{60}{x}\right)$

$\Rightarrow\theta=\pi-(\theta_1+(\pi-\theta_2))=\pi-\tan^{-1}\left(\frac{105}{135-x}\right)-\tan^{-1}\left(\frac{60}{x}\right)$

$\Rightarrow\frac{d\theta}{dx}=-\frac{1}{1+\left(\frac{105}{135-x}\right)^2}\cdot\frac{105}{(135-x)^2}-\frac{1}{1+\left(\frac{60}{x}\right)^2}\cdot\left(-\frac{60}{x^2}\right)=\frac{-105}{(135-x)^2+11{,}025}+\frac{60}{x^2+3600}$

$\frac{d\theta}{dx}=0\Rightarrow\frac{-105}{(135-x)^2+11{,}025}+\frac{60}{x^2+3600}=0\Rightarrow 60\left((135-x)^2+11{,}025\right)=105(x^2+3600)$

$\Rightarrow x^2+360x-30{,}600=0\Rightarrow x=-180\pm 30\sqrt{70}$. Since $x>0$, consider only $x=-180+30\sqrt{70}$. Using the first derivative test, $\left.\frac{d\theta}{dx}\right|_{x=30}=\frac{9}{1050}>0$ and $\left.\frac{d\theta}{dx}\right|_{x=120}=\frac{-9}{1500}<0\Rightarrow$ local max when $x=-180+30\sqrt{70}\approx 71$ m.