# STA2001 Probability and Statistical Inference I
# Tutorial 2

1. (1.3-12) You are a member of a class of 30 students. A bowl contains 30 chips: 2 blue and 28 red. Each student is to take 1 chip from the bowl without replacement. The student who draws the blue chip is guaranteed an A for the course.

   (a) If you have a choice of drawing first, tenth, twentieth or last, which position would you choose? Justify your choice on the basis of probability. **(What is the probability to get A when you are in the 10th position?)**

   (b) Suppose the bowl contains 4 blue and 26 red chips. What position would you now choose?

   **Solution:**
   Method 1:

   Let $W_i$ denotes the event that you choose the $i$-th position and draw a blue chip, for $i = 1, \cdots, 30$.

   (a)
   $$P(W_i) = \frac{29!\binom{2}{1}}{30!} = \frac{2}{30} = \frac{1}{15}.$$

   To be specify, the number 29 in the numerator includes 28 red chips and 1 blue chips, and we fix the $i$-th position you choose with a blue chip, and finally we switch the position of these 2 blue chips (this is why there is a $\binom{2}{1}$) and get the desired probability.

   Therefore, according to this results, you have the same probability of getting a blue chip whatever which position you choose.

   (b) Similarly, we have
   $$P(W_i) = \frac{29!\binom{4}{1}}{30!} = \frac{4}{30} = \frac{2}{15}.$$

   Method 2:

   Since it is fair for all positions, the probability of getting $A$ for (a) is 1/15 and for (b) 2/15.
   Take the 10th position in (a) for example:
   Let $A_1$ be the event that "The first 9 position are red" , $A_2$ be the event that "There are 1 blue chip and 8 red chips in the first 9 positions", $A_3$ be the event that "there

are 2 blue chip and 7 red chips in the first 9 positions" and $B$ be event that "I get a blue chip in the 10th position". $A_1, A_2, A_3$ are mutually exclusive and exhaustive.

$$\begin{aligned}
P(B) &= P(B \cap S) = P((B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3)) \\
&= P(B|A_1) \times P(A_1) + P(B|A_2) \times P(A_2) + 0 \\
&= \frac{2}{21} \times \frac{\binom{28}{9}}{\binom{30}{9}} + \frac{1}{21} \times \frac{\binom{28}{8} \times \binom{2}{1}}{\binom{30}{9}} \\
&= \frac{1}{15}
\end{aligned}$$

2. (1.5-7.) A chemist wishes to detect an impurity in a certain compound that she is making. There is a test that detects an impurity with probability 0.90; however, this test indicates that an impurity is there when it is not about 5% of the time. The chemist produces compounds with the impurity about 20% of the time; that is, 80% do not have the impurity. A compound is selected at random from the chemists output. The test indicates that an impurity is present. What is the conditional probability that the compound actually has an impurity?

   **Solution**:

   "There is a test that detects an impurity with probability 0.9" means that the correct rate of testing is 0.9 = P(Test showing impurity | it is truely impurity).

   "This test indicates that an impurity is there when it is not about 5 percent of the time." means that the incorrect rate of testing is 0.05 = P(Test showing impurity | it is not impurity actually).

   Let $A_1$ be the event that compounds include impurity.

   let $A_2$ be the event that compounds don't include impurity.

   let $B$ be the event that the test shows there existing impurity.

   Notice that $A_1$ and $A_2$ are mutually exclusive and exhaustive, then we could get

$$P(A_1) = 0.2 \; P(A_2) = 0.8 \; P(B|A_1) = 0.9 \; P(B|A_2) = 0.05$$

   . In this question, we need to calculate $P(A_1|B)$, therefore we have

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)} = \frac{9}{11}$$

3. Suppose we have 5 fair coins and 10 unfair coins, which look the same and feel the same. For the fair coins, there is a 50% chance of getting heads and of course 50% chance of getting tails. For the unfair coins, there is a 80% probability of getting heads and 20% tails. Now we randomly pick one coin from all 15 coins and flip it for 6 times. Then we get 4 heads. What is the probability that we have pick a fair coin?

   **Solution**:

   According to question stem, we pick up one coin from bag, but we don't know it is fair or unfair coin, and we get 4 heads out of 6 flips which is condition for calculating the probability the question asked.

   Therefore, we need to calculate: P(fair coin | 4 heads out of 6 flips). We define the events as follows:

   Let $B :=$ There are 4 heads out of 6 flips.

   Let $A_1 :=$ We pick a Fair coin.

   Let $A_2 :=$ We pick an Unfair coin.

   Note that $A_1$ and $A_2$ are mutually exclusive and exhaustive, and we have the following results,

$$P(A_1) = \frac{5}{15}$$
$$P(A_2) = \frac{10}{15}$$
$$P(B|A_1) = \binom{6}{4} \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^2, (\binom{6}{4} - \text{because we don't know the position of 4 heads})$$
$$P(B|A_2) = \binom{6}{4} \times (0.8)^4 \times (0.2)^2$$

   Hence, the probability we want to compute is given by

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)} = 0.32287$$

4. Consider two independent fair coin tosses, in which all four possible outcomes are equally likely. Define the following events

$$H_1 = \{\text{1st toss is a head}\},$$
$$H_2 = \{\text{2nd toss is a head}\},$$
$$D = \{\text{the two tosses have different results}\}.$$

Prove or disprove the following four statements:

(i) The events $H_1$ and $H_2$ are dependent.

(ii) Given that the event $D$ has happened, the events $H_1$ and $H_2$ are conditionally independent.

(iii) The events $H_1$, $H_2$ and $D$ are mutually independent.

(iv) The events $H_1$, $H_2$ and $D$ are pairwise independent.

**Solution:**

For (i), note that $P(H_1) = 1/2$, $P(H_2) = 1/2$ and $P(H_1 \cap H_2) = 1/4$, thus we have $P(H_1 \cap H_2) = P(H_1)P(H_2)$, indicating $H_1$ and $H_2$ are independent.

For (ii), note that $P(H_1|D) = 1/2$, $P(H_2|D) = 1/2$ and $P(H_1 \cap H_2|D) = 0$, thus we have $P(H_1 \cap H_2|D) \neq P(H_1|D)P(H_2|D)$, indicating $H_1$ and $H_2$ are conditionally dependent.

For (iii) and (iv), note that $P(H_1) = 1/2$, $P(H_2) = 1/2$, $P(D) = 1/2$, $P(H_1 \cap H_2) = 1/4$, $P(H_1 \cap D) = 1/4$ and $P(H_2 \cap D) = 1/4$ indicating that the events $H_1, H_2$ and $D$ are pairwise independent. However, the events $H_1, H_2$ and $D$ are not mutually independent because $P(H_1 \cap H_2 \cap D) = 0 \neq P(H_1)P(H_2)P(D)$.