STA2001 Probability and Statistics (I)

Lecture 4

Tianshi Chen

The Chinese University of Hong Kong, Shenzhen

Review

- Conditional probability of an event A, given that event B has occurred, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) > 0$. Note: conditional probability is a probability function.

- Events A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

The occurrence of one of them does not change the probability of the occurrence of the other.

Properties of Independent Events

Theorem 1.4-1

A and B are independent, if and only if any pair of the following events are independent

(a) A and B'

(b) A' and B

(c) A' and B'

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Proof:

$$\begin{aligned} P(A) &= P(A \cap (B \cup B')) = P((A \cap B) \cup (A \cap B')) \\ &= P(A \cap B) + P(A \cap B') = P(A)P(B) + P(A \cap B') \end{aligned}$$

$$P(A \cap B') = P(A)(1 - P(B)) = P(A)P(B')$$

Independent Events

Definition

Events A, B and C are mutually independent if

1. A, B, C are pairwise independent, i.e.,

$$\begin{cases} P(A \cap B) & = P(A)P(B) \\ P(A \cap C) & = P(A)P(C) \\ P(B \cap C) & = P(B)P(C) \end{cases}$$

2. $P(A \cap B \cap C) = P(A)P(B)P(C)$

    - multiplication rule for three independent events.

Example 3, page 39

An urn contains four balls number 1,2,3,4 and we draw one ball randomly from the urn.

$$A = \{1, 2\}, \quad B = \{1, 3\}, \quad C = \{1, 4\}$$

Then are A, B, C mutually independent?

Example 3, page 39

$$
\begin{aligned}
P(A) =& P(B) = P(C) = \frac{1}{2} \\
P(A \cap B) &= P(\{1\}) = \frac{1}{4} = P(A)P(B) \\
P(A \cap C) &= P(A)P(C) = \frac{1}{4} \\
P(B \cap C) &= P(B)P(C) = \frac{1}{4}
\end{aligned}
$$

$$P(A \cap B \cap C) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}$$

So A, B, C are pairwise independent but not mutually independent.

Properties of Independent Events (Continued)

- Mutual independence can be extended to four or more events: Each pair, triple, quartet of the events are independent and moreover

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n)$$

- If A, B, C are mutually independent, then
  1. A and $(B \cap C)$ independent,
  2. A' and $(B \cap C')$ independent,
  3. A and $(B \cup C)$ independent,
  4. A', B', C' independent

Properties of Independent Events (Continued)

①A and $(B \cap C)$ independent

$$P(A \cap (B \cap C)) = P(A)P(B)P(C) = P(A)P(B \cap C)$$

②A' and $(B \cap C')$ independent,

By Theorem 1.4-1, ② $\Leftrightarrow$ A and $B \cap C'$ independent

$$\begin{aligned} P(A \cap B \cap C') &= P(A \cap B) - P(A \cap B \cap C) = P(A \cap B)P(C') \\ &= P(A)P(B)P(C') = P(A)P(B \cap C') \end{aligned}$$

Properties of Independent Events (Continued)

③A and $(B \cup C)$ independent

$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)) = P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) = P(A)(P(B) + P(C) - P(B)P(C)) = P(A)P(B \cup C)$

④A', B', C' independent

The pairwise independence is obvious and then from ③ $\Leftrightarrow$ A' and $B' \cap C'$ independent

$$P(A' \cap (B' \cap C')) = P(A')P(B' \cap C') = P(A')P(B')P(C')$$

Properties of Independent Events (Continued)

- Many experiments consist of a sequence of n trials. If the outcomes of ith trial, in fact, does not have anything to do with the others, then events such that each is associated with a different trial should be independent in the probability sense. That is, if the event A_i is associated with the ith trial, $i = 1, 2, \cdots, n$, then $A_1, A_2, \cdots, A_n$ are mutually independent and in particular

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n)$$

Example 4, page 40

Question

A fair 6-sided die is rolled six independent times. Let $A_i = \{\text{a match on the ith roll, i.e., the side } i \text{ is observed on the ith roll}\}$, $i = 1, 2, \cdots, 6$. Let $B = \{\text{at least one match occur}\}$, what is $P(B)$?

Example 4, page 40

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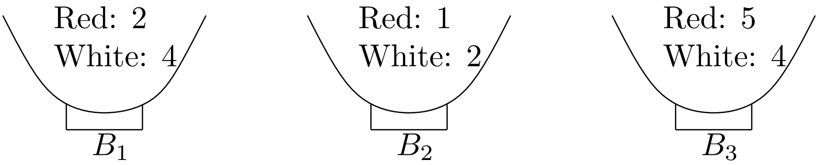
$$P(B) = 1 - P(B') \quad \text{where } B' = \{\text{no matches occur in 6 rolls}\}$$

$$= 1 - P(A_1' \cap A_2' \cdots \cap A_6') \quad \text{since } A_1' \cdots A_6' \text{ are independent}$$

$$= 1 - P(A_1')P(A_2') \cdots P(A_6') = 1 - \left(\frac{5}{6}\right)^6$$

Section 1.5 Bayes's Theorem

A Motivation Example



Experiment: Select a bowl first, and then draw a chip from the selected bowl.

Assumption: All chips are "equally likely" and moreover,

$$P(B_1) = \frac{1}{3}, \quad P(B_2) = \frac{1}{6}, \quad P(B_3) = \frac{1}{2}.$$

$P(B_i)$: the probability to select the ith bowl.

A Motivation Example

Question 1

Let R=$\{$draw a red chip$\}$. What is $P(R)$?

A Motivation Example

Question 1

Let R={draw a red chip}. What is $P(R)$?

$$P(R) = P(S \cap R), \text{ where } S=\{\text{all chips}\}$$

$$= P((B_1 \cup B_2 \cup B_3) \cap R) = P((B_1 \cap R) \cup (B_2 \cap R) \cup (B_3 \cap R))$$

$$= P(B_1 \cap R) + P(B_2 \cap R) + P(B_3 \cap R)$$

$$= P(B_1)P(R|B_1) + P(B_2)P(R|B_2) + P(B_3)P(R|B_3)$$

$$= \frac{1}{3} \cdot \frac{2}{6} + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{5}{9} = \frac{4}{9}$$

A Motivation Example

Question 2

Suppose now that the outcome of the experiment is a red chip but we don't know from which bowl the chip was drawn. We are interested in

$$P(B_1|R), \quad P(B_2|R), \quad P(B_3|R)$$

A Motivation Example

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From the definition of conditional probability, e.g., Consider

$$P(B_i|R) = \frac{P(B_i \cap R)}{P(R)} = \frac{P(B_i)P(R|B_i)}{P(R)}, \quad i = 1, 2, 3.$$

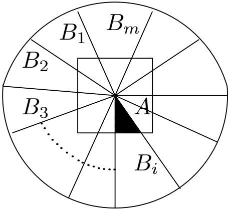
$$P(B_1|R) = \frac{1}{4}, \quad P(B_2|R) = \frac{1}{8}, \quad P(B_3|R) = \frac{5}{8}$$

Bayes' Theorem

Assume that

1. S is a sample space, and $B_1, B_2, \cdots, B_m$ are mutually exclusive and exhaustive w.r.t the sample space S.
2. the prior probabilities of B_i is positive, i.e., $P(B_i) > 0, i = 1, \cdots, m$. Then we have

Bayes' Theorem



(a) For any event A,

$$P(A) = \sum_{i=1}^{m} P(A \cap B_i) = \sum_{i=1}^{m} P(B_i)P(A|B_i)$$

$\rightarrow$ total probability

(b) If $P(A) > 0$, then

$$P(B_k|A) = \frac{P(B_k \cap A)}{P(A)}, \quad k = 1, \cdots, m$$

$$P(B_k|A) = \frac{P(B_k)P(A|B_k)}{P(A) = \sum_{i=1}^{m} P(B_i)P(A|B_i)} \rightarrow \text{Bayes Theorem}$$

Bayes' Theorem

$$P(B_k) \to \text{ prior probability}$$

$$P(B_k|A) \to \text{ posterior probability}$$

$$P(A|B_k) \to \text{ likelihood of } B_k,\ A \text{ is called a data}$$

Thomas Bayes

Thomas Bayes is known for formulating a specific case of the theorem that bears his name: Bayes' theorem.

Figure: Thomas Bayes (1701 – 1761) was an English statistician, philosopher and Presbyterian minister.

Pierre-Simon Laplace

However, it was Pierre-Simon Laplace (1749–1827) who introduced what is now called Bayes' theorem, and the Bayesian was in fact pioneered and popularised by Pierre-Simon Laplace.

Figure: Pierre-Simon Laplace (1749–1827) was a French scholar and polymath whose work was important to the development of engineering, mathematics, statistics, physics, astronomy, and philosophy.