MAT1002 Lecture 4, Tuesday, Jan/17/2023

Outline

- Alternating series test (10.6)
- Alternating series approximation (10.6)
- Conditional convergence and rearrangement

Alternating Series

Definition
An alternating series is a series of the form $\sum(-1)^{n+1}u_n$, where $u_n > 0$ for all n.

Remark
The starting index does not have to be 1 (so the first term can be negative). For example, both

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \quad \text{and} \quad -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$$

are alternating series.

10.6.15

THEOREM ~~15~~—The Alternating Series Test The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

e.g. The alternating p-series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$ converges for all $p>0$, since:

- $\left\{\frac{1}{n^p}\right\}$ positive and $\searrow$;
- $\lim_{n\to\infty}\frac{1}{n^p} = 0$.

In particular, the alternating harmonic series $\sum(-1)^{n+1}\frac{1}{n}$ converges.

Remark:

If we have an alternating series that starts with a negative term, may multiply the series by -1, then use the test.

e.g.,

$$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots=-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right).$$

That is, we do not need to worry whether the series starts with a "+" or "−" term.

e.g. For $\sum (-1)^{n+1}\frac{n^2}{n^3+1}$:

- Let $f(x) := \frac{x^2}{x^3+1}$.
- Since $f'(x)=\frac{(x^3+1)2x-x^2(3x^2)}{(x^3+1)^2}=\frac{2x-x^4}{(x^3+1)^2}=\frac{x(2-x^3)}{(x^3+1)^2}$

  <0 on $(\sqrt[3]{2},\infty)$,

  $a_n=\frac{n^2}{n^3+1}$ is $\searrow$ starting at $n=2$.

- Series converges by alternating series test, since

$$\lim_{n\to\infty}\frac{n^2}{n^3+1}=0.$$

Alternating Series Test : Proof

<u>Fact</u>: If $\{a_1, a_3, a_5, \ldots\}$ and $\{a_2, a_4, a_6, \ldots\}$ both converge to the limit L, then so does $\{a_1, a_2, a_3, a_4, \ldots\}$, i.e., $\lim_{n\to\infty} a_n = L$. <u>Proof</u>: Omitted. (10.1, ex 133.)

<u>Proof</u> (of the Alternating series test): (For notational convenience, assume $N = 1$.)

- Let S_n be the partial sum. Consider $b_k := S_{2k}$.
- $\forall k \geq 1$, $b_k = S_{2k} = (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2k-1} - u_{2k})$,

  So $\{b_k\}$ is nondecreasing, since $u_{2i-1} - u_{2i} \geq 0$, $\forall i$.
- $\{b_k\}$ is bounded, since

$$
\begin{aligned}
0 &\leq (u_1 - u_2) + \ldots + (u_{2k-1} - u_{2k}) \\
&= u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots - (u_{2k-2} - u_{2k-1}) - u_{2k} \\
&\leq u_1 .
\end{aligned}
$$

- By Monotonic Sequence Theorem, $\{b_k\} = \{S_{2k}\}$ converges, say $S_{2k} \to L$.
- Consider the sequence $\{S_{2k-1}\}_{k=1}^{\infty}$. Then

$$S_{2k} = S_{2k-1} - u_{2k} \quad \Rightarrow \quad S_{2k-1} = S_{2k} + u_{2k}$$

$$\Rightarrow \lim_{k\to\infty} S_{2k-1} = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} u_{2k} = L + 0 = L.$$

- By Fact before proof, $\lim_{n\to\infty} S_n = L$, so series converges. $\square$

Alternating Series Approximation

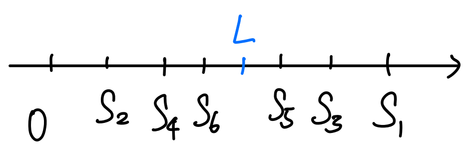
Suppose for a given series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$, we have

- $u_n > 0$, $\forall n \geq 1$; • $\{u_n\}$ is nonincreasing;
- $u_n \to 0$ as $n \to \infty$.

Then $\sum(-1)^{n+1}u_n = L$ for some $L \in \mathbb{R}$. If we want to approximate L using the sum of the first K terms (S_K), then

- L is between S_K and S_{K+1};
- $\underbrace{|L - S_K|}_{\text{error}} < \underbrace{u_{K+1}}_{\text{first unused term, in absolute value}}$;
- $L - S_K$ has same sign as first unused term.

Intuition:



- $|L - S_5| = S_5 - L < S_5 - S_6 = u_6$
- First unused term: $-u_6$. (<0)

e.g. Find an approximated value of $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}$ with error less than 0.001.

Sol: • $u_n = \frac{1}{(n-1)!}$ is positive, nonincreasing.

• For $n \geq 2$, $0 < \frac{1}{(n-1)!} \leq \frac{1}{n-1} \to 0$, so $\lim_{n\to\infty} u_n = 0$.

• Hence series converges (by alternating series test).

- When $K=8$, $\frac{1}{(K-1)!} = \frac{1}{7!} = \frac{1}{5040} < \frac{1}{5000} = 0.0002$.
- Take $S_7 = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{1}{6!} = 0.36805\dots$ as an approximated value.

Fact: exact value is $0.367879\dots = e^{-1}$.

Conditional Convergence

Def: A series $\sum a_n$ is said to be convergent conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

e.g. $\sum_n (-1)^{n+1} \frac{1}{n^p}$ $\begin{cases} \text{converges absolutely}, & \text{if } p>1; \\ \text{converges conditionally}, & \text{if } 0<p\le 1; \\ \text{diverges}, & \text{if } p\le 0 \end{cases}$

Alternating p-series

Q: What is the key difference between absolute convergence and conditional convergence?

Re-arrangements (Optional)

Consider the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$, which converges (conditionally). Let

$$L := \sum_{n=3}^{\infty} (-1)^{n+1} \frac{1}{n} = \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Since $\frac{1}{3} - \frac{1}{4}, \frac{1}{5} - \frac{1}{6}, \ldots, \frac{1}{2k-1} - \frac{1}{2k}, \ldots$ are positive, $L > 0$.

Consider rearranging the terms in $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{1}{n}$ as follows.

$$\frac{1}{3} - \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{8} - \frac{1}{10} + \frac{1}{7} - \frac{1}{12} - \frac{1}{14} + \cdots$$

(One odd term followed by two even terms)

Note that $\frac{1}{3} - \frac{1}{4} - \frac{1}{6} = \frac{4-3-2}{12} < 0$, $\frac{1}{5} - \frac{1}{8} - \frac{1}{10} = \frac{8-5-4}{40} < 0$,

$\ldots, \frac{1}{2k+1} - \frac{1}{4k} - \frac{1}{4k+2} = \frac{-1}{(2k+1)(4k)} < 0$, so

the rearranged series cannot converge to L, since $L > 0$.

Fact: Changing the order of the terms of a conditionally convergent series may change its value. In fact, a conditionally convergent series can converge to ANY value by rearranging its terms.

In contrast, an absolutely convergent series can only converge to one value, no matter how you order its terms.

Fact: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+$ is any bijection (i.e., permutation of positive integers), then

= injective + surjective

= one-to-one + onto

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n .$$

We omit the proofs of the facts.