MAT1002 Lecture 12, Tuesday, Mar/07/2023

Outline

- Integrals of vector functions (13.2)
- Ideal projectile (13.2)
- Arc length and the arc length parameter (13.3)
- Tangent, normal and curvature (13.4)

Integrals of Vector Functions

Definition

Let $\vec{r} : I \to \mathbb{R}^n$ be a vector function, with $\vec{r} = \langle r_1, r_2, \ldots, r_n \rangle$.

- A function $\vec{R}$ is called an antiderivative of $\vec{r}$ if $\vec{R}'(t) = \vec{r}(t)$ for all $t \in I$.
- The indefinite integral of $\vec{r}$, denoted by $\int \vec{r}(t)\,dt$, is defined to be the set of all antiderivatives of $\vec{r}$. We write
$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C},$$
where $\vec{C}$ represents any constant vector.
- The definite integral, denoted by $\int_a^b \vec{r}(t)\,dt$, is defined by
$$\int_a^b \vec{r}(t)\,dt := \left\langle \int_a^b r_1(t)\,dt, \int_a^b r_2(t)\,dt, \ldots, \int_a^b r_n(t)\,dt \right\rangle.$$

Since integrals are defined componentwise, the FTC also works for vector functions: if $\vec{R}$ is an antiderivative of $\vec{r}$ on $[a,b]$ and $\vec{r}$ is continuous, then
$$\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a).$$

e.g. If $\vec{r}(t)$ is the position vector at time t, then
$$\vec{r}\,'(t) = \underbrace{\vec{v}(t)}_{\text{velocity}}, \qquad \vec{v}\,'(t) = \underbrace{\vec{a}(t)}_{\text{acceleration}},$$

$$\vec{v}(b) - \vec{v}(a) = \int_a^b \vec{a}(t)\,dt, \qquad \underbrace{\vec{r}(b) - \vec{r}(a)}_{\text{displacement}} = \int_a^b \vec{v}(t)\,dt.$$

13.2.3

EXAMPLE ~~3~~ Suppose we do not know the path of a hang glider, but only its acceleration vector $\mathbf{a}(t) = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}$. We also know that initially (at time $t = 0$) the glider departed from the point $(4, 0, 0)$ with velocity $\mathbf{v}(0) = 3\mathbf{j}$. Find the glider's position as a function of t.

Sol: · Want: $\vec{r}(t)$. Need to find $\vec{v}(t)$.

· $\int_0^t \vec{a}(u)\,du = \vec{v}(t) - \vec{v}(0)$

$$\Rightarrow \vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du = 3\vec{j} + \left(-3\sin u\,\vec{i} + 3\cos u\,\vec{j} + 2u\,\vec{k}\right)\Big|_{u=0}^{t}$$

$$= -3\sin t\,\vec{i} + 3\cos t\,\vec{j} + 2t\,\vec{k}$$

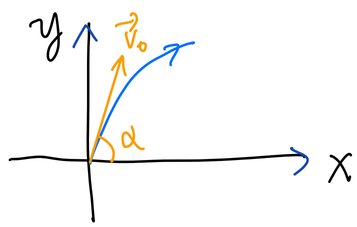
· $\int_0^t \vec{v}(u)\,du = \vec{r}(t) - \vec{r}(0)$

$$\Rightarrow \vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\,du = 4\vec{i} + \left(3\cos u\,\vec{i} + 3\sin u\,\vec{j} + u^2\vec{k}\right)\Big|_{u=0}^{t}$$

$$= (3\cos t + 1)\,\vec{i} + 3\sin t\,\vec{j} + t^2\,\vec{k}.$$

Ideal Projectile

$(0<\alpha<\frac{\pi}{2})$



- Object is fired at origin at angle α.
- Assume only constant force $\vec{F}$ of gravity is applied to the object after launching.

Q: What is an algebraic formula for the position of the object, expressed in v_0 and α?

given by $\vec{r}$

- Initial velocity $\vec{v}_0$ is obtained by

$v_0 := |\vec{v}_0| =$ initial speed

$$\vec{v}_0 = v_0 \cos\alpha\, \vec{i} + v_0 \sin\alpha\, \vec{j}. \qquad ①$$

- Acceleration due to gravity is $\vec{a} = g(-\vec{j}) = -g\vec{j}$. ②

- Since $\vec{v}\,'(t) = a(t)$, by FTC2,

$$\vec{v}(t) - \vec{v}_0 = \vec{v}(t) - \vec{v}(0) = \int_0^t \vec{a}(u)\,du \overset{②}{=} \int_0^t -g\vec{j}\,du = -gt\vec{j}$$

$$\Rightarrow \quad \vec{v}(t) = \vec{v}_0 - gt\vec{j}. \qquad ③$$

- Since $\vec{r}\,'(t) = \vec{v}(t)$, by FTC2,

$$\vec{r}(t) = \vec{r}(t) - \underbrace{\vec{r}(0)}_{\vec{0}} = \int_0^t \vec{v}(u)\,du \overset{③}{=} \int_0^t (\vec{v}_0 - gu\vec{j})\,du$$

$$= \left(\vec{v}_0 u - \tfrac{1}{2}gu^2\vec{j}\right)\Big|_{u=0}^{t} = t\vec{v}_0 - \tfrac{1}{2}gt^2\vec{j} \qquad ④$$

- By ① and ④, we have:

$$\boxed{\vec{r}(t) = v_0(\cos\alpha)t\,\vec{i} + \left(v_0(\sin\alpha)t - \frac{1}{2}gt^2\right)\vec{j}.}$$

In parametric form, we have

$$\boxed{x = \underbrace{v_0(\cos\alpha)}_{x\text{-component of } \vec{v_0}}t, \quad y = \underbrace{v_0(\sin\alpha)}_{y\text{-component of } \vec{v_0}}t - \frac{1}{2}gt^2, \quad t \geq 0.}$$

By substituting $t = \frac{x}{v_0\cos\alpha}$ into $y = v_0(\sin\alpha)t - \frac{1}{2}gt^2$, we can write y as an explicit function of x:

$$\boxed{y = \tan\alpha\, x - \frac{1}{2}g\frac{1}{v_0^2\cos^2\alpha}x^2.}$$

This shows that any ideal projectile moves along a parabola.

Q: What is the maximum height? Flight time (until hitting the ground)? Range?

Max Height

$y = v_0(\sin\alpha)t - \frac{1}{2}gt^2 \implies y'(t) = v_0\sin\alpha - gt$

So $y' = 0$ if and only if $t = \frac{v_0\sin\alpha}{g}$. Substituting $t = \frac{v_0\sin\alpha}{g}$

into $y = v_0(\sin\alpha)t - \frac{1}{2}gt^2$ yields

$$y_{max} = \frac{v_0^2 \sin^2\alpha}{g} - \frac{1}{2} g \frac{v_0^2 \sin^2\alpha}{g^2} = \frac{1}{2g} v_0^2 \sin^2\alpha .$$

(The extremum above must be a maximum by physical nature ; can also be checked by using first/second derivative test.)

Flight Time

When object hits the ground again, $y=0$ but $t \neq 0$. Since

$$v_0(\sin\alpha)t - \frac{1}{2}gt^2 = 0 \overset{t\neq 0}{\iff} v_0 \sin\alpha - \frac{1}{2}gt = 0 \iff t = \frac{2v_0 \sin\alpha}{g},$$

the flight time is $\frac{2v_0\sin\alpha}{g}$.

Range

The total horizontal distance travelled by the object (when hitting the ground) is given by

$$v_0(\cos\alpha)\cdot(\text{flight time}) = v_0 \cos\alpha \frac{2v_0 \sin\alpha}{g} = \frac{v_0^2}{g} 2\cos\alpha \sin\alpha = \frac{v_0^2}{g} \sin 2\alpha .$$

Height, Flight Time, and Range for Ideal Projectile Motion

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α:

Maximum height: $y_{max} = \dfrac{(v_0 \sin\alpha)^2}{2g}$

Flight time: $t = \dfrac{2v_0 \sin\alpha}{g}$

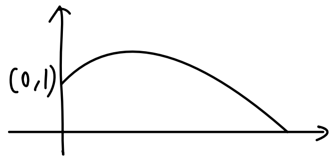
Range: $R = \dfrac{v_0{}^2}{g} \sin 2\alpha.$

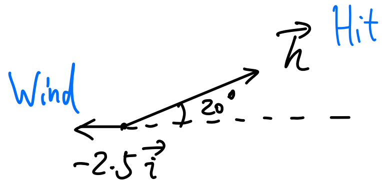
EXAMPLE ~~5~~ 13.2.5 A baseball is hit when it is 1 m above the ground. It leaves the bat with initial speed of 50 m/s, making an angle of 20° with the horizontal. At the instant the ball is hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of $-2.5\mathbf{i}$ (m/s) to the ball's initial velocity (2.5 m/s = 9 km/h).

(a) Find a vector equation (position vector) for the path of the baseball.

(b) How high does the baseball go, and when does it reach maximum height?

(c) Assuming that the ball is not caught, find its range and flight time.





Read the book for details.

Arc Length

Definition

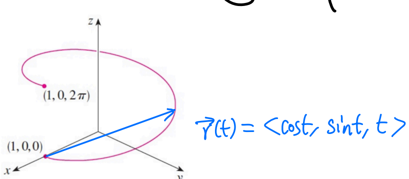
Let $\vec{r}: [a,b] \to \mathbb{R}^n$ be a continuously differentiable vector function (that is, $\vec{r}\,'$ is continuous). The arc length (or length) of a curve parametrized by $\vec{r}$ is defined to be the number L, where

$$L := \int_a^b |\vec{r}\,'(t)|\, dt,$$

assuming that the curve is traced exactly once (i.e. $\vec{r}$ is one-to-one). In particular, for a space curve ($n = 3$) parametrized by $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, its length is given by

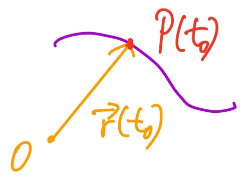
$$\int_a^b \sqrt{\underbrace{(f'(t))^2}_{dx/dt} + \underbrace{(g'(t))^2}_{dy/dt} + \underbrace{(h'(t))^2}_{dz/dt}}\, dt.$$

e.g. The length of a part of the helix below is given by



$$L = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = 2\pi\sqrt{2}.$$

The Arc Length Function



Consider a curve parametrized by a one-to-one continuously differentiable function $\vec{r}(t)$ defined for $t \in \cancel{[a,b]}\ I$. Fix a value $t_0 \in \cancel{[a,b]}\ I$, and define $s : \cancel{[a,b]}\ I \to \mathbb{R}$ by

$$s(t) := \int_{t_0}^{t} |\vec{r}\,'(u)|\, du.$$

" Signed distance from $P(t_0)$ to $P(t)$ along the curve "

- $s(t) > 0$ if $t > t_0$;
- $s(t) < 0$ if $t < t_0$;
- By FTC, $s'(t) = \frac{ds}{dt} = |\vec{r}\,'(t)|$, the speed of the motion. If t is time.
- If $\vec{r}\,'(t) \neq \vec{0}$ $\forall t$, then $s'(t) > 0$ $\forall t \in I$.
- $s \uparrow$ as $t \uparrow$.

Such a curve can be re-parametrized using the arc length parameter s (i.e., each point of the curve can be given in terms of its signed distance from a fixed base point P_0.)

Example

Re-parametrize the helix $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ with respect to arc length measured from $(1,0,0)$ in the direction of increasing t.

Sol: • Base point $P_0 = (1,0,0)$.

- $\vec{r}(t) = \langle 1, 0, 0 \rangle \iff t = 0$.
- Signed distance from P_0 to a point $P(t)$ along helix is

$$s = s(t) = \int_0^t |\vec{r}\,'(u)|\, du = \int_0^t \sqrt{\sin^2 u + \cos^2 u + 1^2}\, du = \sqrt{2}\, t.$$

- $t = \frac{s}{\sqrt{2}}$.

- $\vec{r}\left(\frac{s}{\sqrt{2}}\right) = \left\langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle =: \vec{g}(s)$, $s \in (-\infty, \infty)$

  is a new parametrization (in terms of the arc length parameter s).

Sometimes it is helpful to consider the curve in terms of the arc length parameter, since arc length is a geometric concept and does not depend on motion, and considering it may reveal geometric properties of the curve (e.g. curvature) instead of motion properties (e.g. velocity).

Tangent, Normal and Curvature

in $\mathbb{R}^2$

Let $\vec{r}$ be the position vector of a plane curve, and suppose it has a smooth parametrization $\vec{r}(t)$. Consider representing $\vec{r}$ using the arc length parameter s (with a base point fixed). What is $d\vec{r}/ds$?

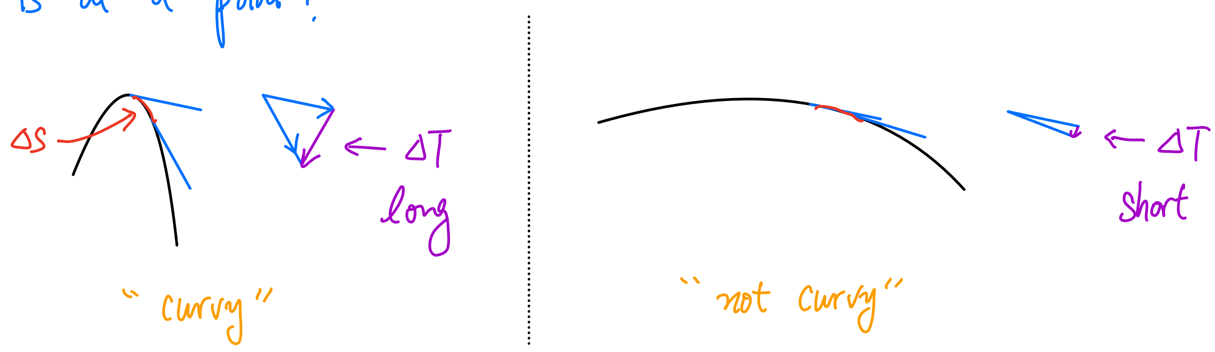
- Note that $\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} |\vec{r}\,'(u)|\,du = |\vec{r}\,'(t)| \neq 0 \quad \forall t$ ($\vec{r}$ smooth).

- By the chain rule, $\frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \underbrace{\vec{T}(t)}_{\text{unit tangent vector}}$.

$$\boxed{\frac{d\vec{r}}{ds} = \vec{T}.}$$

Curvature

Q: What would be a reasonable way to capture how "curvy" a curve is at a point?



Def: The curvature of a (smooth) curve is the scalar κ defined by

$$\kappa := \left| \frac{d\vec{T}}{ds} \right| .$$

Remarks

- The bigger κ is, the sharper turn the curve would make at a point.
- If position vector $\vec{r}$ is given by a smooth parametrization in t, then

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}/dt}{ds/dt} \right| = \frac{1}{|\vec{r}\,'(t)|} \left| \frac{d\vec{T}}{dt} \right| .$$

This gives us a way to compute κ without computing s.

e.g. Consider three curves:

C_1: $\vec{r}_1(t) = \langle t, t \rangle$, $t \in \mathbb{R}$.

C_2: $\vec{r}_2(t) = \langle a\cos t, a\sin t \rangle$, $t \in [0, 2\pi)$. (Here $a>0$.)

C_3: $\vec{r}_3(t) = \langle t, t^2 \rangle$, $t \in \mathbb{R}$.

C_1: $\vec{r}_1'(t) = \langle 1, 1 \rangle$, $\vec{T} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$, $\left|\frac{d\vec{T}}{dt}\right| = 0$, $\kappa = 0$

at all points.

C_2: $\vec{r}_2'(t) = \langle -a\sin t, a\cos t \rangle$, $\vec{T} = \langle -\sin t, \cos t \rangle$,

$\frac{d\vec{T}}{dt} = \langle -\cos t, -\sin t \rangle$, $\left|\frac{d\vec{T}}{dt}\right| = 1$

$\therefore \kappa = \frac{1}{|\vec{r}_2'(t)|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{a}$ at all points.

C_3: $\vec{r}_3'(t) = \langle 1, 2t \rangle$, $\vec{T} = \frac{1}{\sqrt{1+4t^2}}\langle 1, 2t \rangle$,

$$\frac{d\vec{T}}{dt} = \left\langle -\frac{1}{2}(1+4t^2)^{-\frac{3}{2}} \cdot 8t, \frac{2\sqrt{1+4t^2} - 2t\frac{1}{2}(1+4t^2)^{-\frac{1}{2}} \cdot 8t}{1+4t^2} \right\rangle$$

$$= \left\langle -4t(1+4t^2)^{-\frac{3}{2}}, \left(2(1+4t^2) - 8t^2\right)(1+4t^2)^{-\frac{3}{2}} \right\rangle$$

$$= (1+4t^2)^{-\frac{3}{2}} \langle -4t, 2 \rangle$$

$$\Rightarrow \kappa(t) = \frac{1}{|\vec{r}_3'(t)|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{\sqrt{1+4t^2}} \frac{1}{(1+4t^2)^{3/2}} \sqrt{16t^2+4} = \frac{2\sqrt{4t^2+1}}{(1+4t^2)^2} = \frac{2}{(1+4t^2)^{3/2}}.$$

Note that κ is maximized to 2 when $t=0$, so the curve turns the most sharply at $(0,0)$.

(Try to plot $\kappa(t)$ using desmos; also try re-doing it using the formula in Q5 of Exercises 13.4.)

Principle Unit Normal

- Consider parametring a curve (in $\mathbb{R}^2$) using the arc length parameter s.
- Consider the vector $\frac{d\vec{T}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \vec{T}}{\Delta s}$.
- It points to a direction in which $\vec{T}$ turns as the curve bends. (if $\frac{d\vec{T}}{ds} \neq \vec{0}$)
- Since $\vec{T}$ has length 1 for all points on curve, $\vec{T}$ can be viewed as a vector function of s with constant length. By a result last time, $\frac{d\vec{T}}{ds}$ is orthogonal to $\vec{T}$ at every point.

> **Def:** At a point (on a curve) where $\kappa \neq 0$, the **principal unit normal vector** is
> $$\vec{N} := \frac{d\vec{T}}{ds} \Big/ \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{\kappa}\frac{d\vec{T}}{ds}.$$

- If position vector $\vec{r}$ is given by a smooth parametrization in t, then

$$\vec{N} = \frac{d\vec{T}/ds}{|d\vec{T}/ds|} = \frac{\frac{d\vec{T}}{dt}\Big/\frac{ds}{dt}}{\left|\frac{d\vec{T}}{dt}\Big/\frac{ds}{dt}\right|} = \frac{\frac{d\vec{T}}{dt}\Big/\frac{ds}{dt}}{\left|\frac{d\vec{T}}{dt}\right|\Big/\frac{ds}{dt}} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}.$$

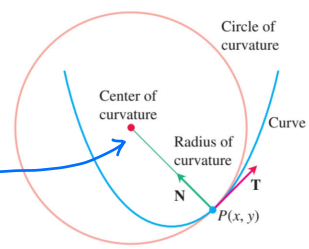
This gives us a way to compute $\vec{N}$ without computing s and κ.

Def: Given a plane curve C and a point P on C at which $\kappa \neq 0$, the osculating circle at P is the circle that

- shares the same tangent line at P as C, and;
- has the same curvature as C at P, and;
- has center that lies on the side to which $\vec{N}$ points.

It is also called the circle of curvature, and its radius called the radius of curvature. By previous example, we saw that a circle of radius $a>0$ has constant curvature $\kappa = \frac{1}{a}$. Hence, at a point (of a curve) with curvature $\kappa \neq 0$, its radius of curvature is $\rho = 1/\kappa$.

$\vec{r} + \frac{1}{\kappa}\vec{N}$



e.g. Find the osculating circle of the graph of $y = x^2$ at $(0,0)$. (C)

Sol: • By previous exercise, we know the curvature of C at $(0,0)$ is 2. Hence radius of curvature is $\rho = \frac{1}{2}$.

• We know $\vec{N} = \vec{j}$ geometrically. (This may also be found by considering parametrization $\vec{r}(t) = \langle t, t^2 \rangle$ of C and then use the formula $\vec{N} = \frac{d\vec{T}}{dt} / \left|\frac{d\vec{T}}{dt}\right|$.)

- Hence centre of curvature is given by

$$\langle 0,0\rangle + \frac{1}{2}\langle 0,1\rangle = \langle 0, \frac{1}{2}\rangle .$$

- Circle is $(x-0)^2 + (y-\frac{1}{2})^2 = \frac{1}{4}$.

The concept of κ and $\vec{N}$ can be defined for space curves, and can be computed using the same formulae.

$(a \geq 0,\ ab \neq 0)$

e.g. For the helix given by $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$,

$\vec{r}'(t) = \langle -a\sin t, a\cos t, b\rangle \Rightarrow \vec{T}(t) = \frac{1}{\sqrt{a^2+b^2}}\langle -a\sin t, a\cos t, b\rangle$

$\Rightarrow \vec{T}'(t) = \frac{1}{\sqrt{a^2+b^2}}\langle -a\cos t, -a\sin t, 0\rangle$

$\Rightarrow \kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{a}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}$

- $b \uparrow \Rightarrow \kappa \downarrow$
- $a=0 \Rightarrow \kappa = 0$. (line)
- $b=0 \Rightarrow \kappa = \frac{1}{a}$. (circle)