MAT 1001 Final Exam, December 20, 2020

Your Name and Student ID:

Your Lecture Class(e.g, L1):

Instruction: (i) This is a closed-book and closed-notes exam; no calculators, no dictionaries and no cell phones; (ii) Show your work unless otherwise instructed–a correct answer without showing your work when required shall be given no credits; (iii) Write down ALL your work and your answers(including the answers for short questions) in the Exam Book.

1. (30 points) Short Questions (for these questions, NO need to show your work, just write down your answers in the Exam Book; NO partial credits for each question)

   (i). If $f(x) > 1$ for all x and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$.

   True False

   (ii). Suppose we want to approximate the positive root of $x^4 + x^2 - 1 = 0$ by using Newton's method, and we choose the first approximation x_1 between 0.25 and 0.5, then $x_1 <$ the positive root and $x_2 > x_3 > \cdots >$ the positive root.

   True False

   (iii). Assume that $f(x)$ is continuous on $[0, 1]$, and $g(x)$ is equal to $f(x)$ at all points in $[0, 1]$, except at $x = 0, 1/n$ where $n = 1, \cdots, 100$. Which of the following statement is correct? (Only one correct answer from below.)

   (a) g is discontinuous at $x = 0, 1/n$ where $n = 1, \cdots, 100$; g is not integrable on $[0, 1]$.

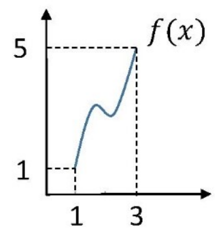
   (b) g is discontinuous at $x = 0, 1/n$ where $n = 1, \cdots, 100$; g is integrable on $[0, 1]$ but the integrals of f and g on $[0, 1]$ may not be equal.

   (c) $\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx$.

   (d) There exists a function $H(x)$ such that $H'(x) = g(x)$ for all $x \in [0, 1]$.

   (iv). Let $f(x)$ be a function that is differentiable on $[1, 3]$, as shown in the figure below. What's the average value of $f'(x)$ on $[1, 3]$? ____2____.

$$\frac{\int_1^3 f'(x)\,dx}{2}$$



(v). The improper integral $\int_{-\infty}^{\infty} \sin x dx$ is convergent and equals 0 because the integrand is an odd function.

True False

(vi). Let $f(x)$ be infinitely differentiable on the interval $[0,1]$ (namely, derivatives of all orders of f exist on $[0,1]$). Which of the following statement is incorrect?

(a) If $f(x)$ is a polynomial of order less than 4, then Simpson's rule gives the exact value of $\int_0^1 f(x)dx$.

(b) If $f(x)$ is concave down on $[0,1]$, then the midpoint rule gives an over-estimate of $\int_0^1 f(x)dx$; the trapezoidal rule an under-estimate.

(c) In general, as estimates of $\int_0^1 f(x)dx$ the midpoint rule and the left sum are equally good.

(d) In general, as estimates of $\int_0^1 f(x)dx$ Simpson's rule is better than the midpoint rule.

(e) The trapezoidal rule is a Riemann sum, and is the average of the left sum and the right sum.

(vii). $1-\cos(1/x) = O(x^{-2})$ as $x \to \infty$.

True False

(viii). Let $p(x)$ be a polynomial whose order is bigger than 0; let $a > 1$ be a constant. Then as $x \to \infty$ we have $p(x) = o(a^x)$ and $\ln x = o(p(x))$.

True False

(ix). The arc length of the graph of $y = x^{3/2}$, $x \in [0,1]$, is $\frac{13\sqrt{13}}{27} - \frac{8}{27}$.

(x). A particle moves on the unit circle $x^2+y^2=1$. Assume at the moment when the particle is at point $(1/\sqrt{3}, \sqrt{2/3})$, its horizontal velocity is $10m/s$. Then at that moment its vertical velocity is $10\sqrt{2}$ m/s.

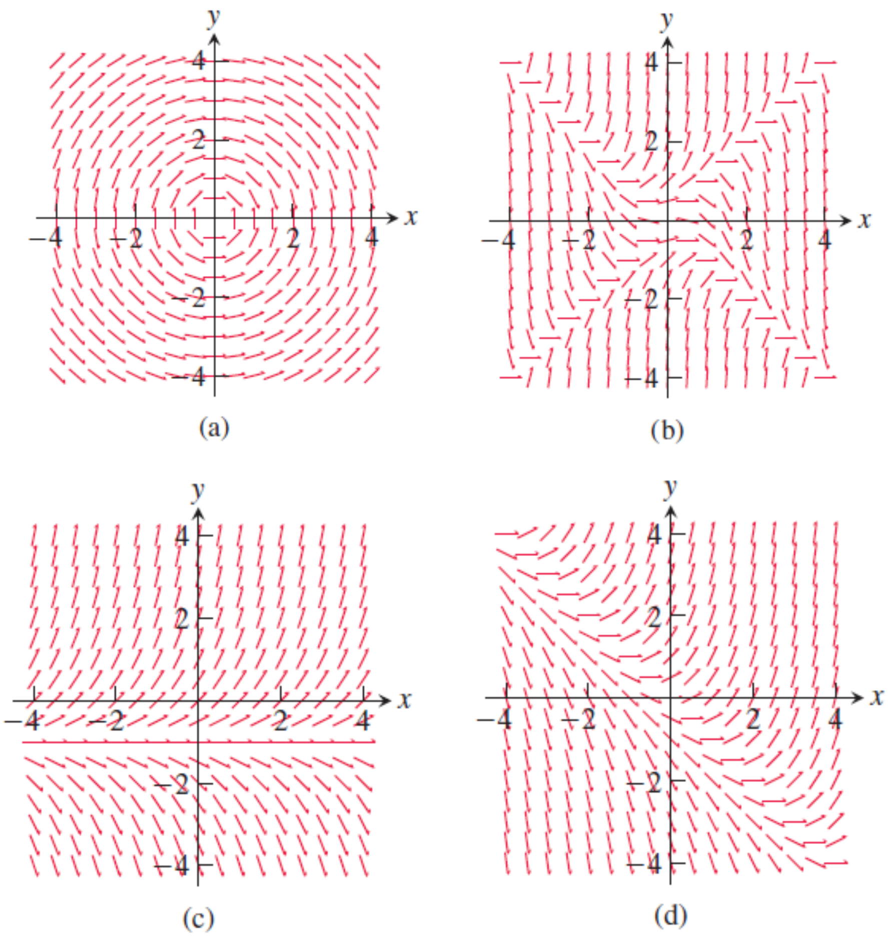
2. (4 points) Match the differential equations with their slope fields.

(i) $y' = x + y$ matches (d);

(ii) $y' = y + 1$ matches (c);

(iii) $y' = -\frac{x}{y}$ matches (a);

(iv) $y' = y^2 - x^2$ matches (b).



3. (24 points) Find each of the following limits or explain why the limit does not exist (the limit is allowed to be ∞ or $-\infty$).

   (a) $\lim\limits_{z\to 0} \dfrac{\sin(z^2)}{z}$. $\lim\limits_{z\to 0} \dfrac{\sin z}{z} \cdot \lim\limits_{z\to 0} \sin z = 0$

   (b) $\lim\limits_{t\to\infty} \left(\dfrac{t+a}{t-a}\right)^t$. e^{2a}

   (c) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{1-\cos^2(4x)}$. 0

   (d) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(1+\frac{i}{n})^{2020}$. $\int_0^1 (1+x)^{2020}$ $\frac{1}{2021}(1+x)^{2021}$

   $\dfrac{(2^{2021}-1)}{2021}$

4. (18 points) Find the derivatives.

(a) $\frac{d}{dx}(\sin x)^x$ $\quad e^{x\ln\sin x}\cdot\left(\ln\sin x + x\frac{\cos x}{\sin x}\right)$

(b) $\frac{dy}{dx}$, where

$$y = \log_5\left(\frac{\sin x \cos x}{2^x}\right) \qquad \frac{1}{\ln 5}\cdot\frac{(2\cos 2x - \sin 2x\ln 2)}{\sin 2x}$$

(c) Suppose that x and y are related by the equation

$$y = \int_1^x \frac{dt}{\sqrt{10+3t^2}}.$$

Find d^2x/dy^2. $\quad \frac{dy}{dx} = \frac{1}{\sqrt{10+3x^2}} \qquad \frac{dx}{dy} = \sqrt{10+3x^2}\ \ 6x\cdot\frac{1}{2\sqrt{10+3x^2}}$

5. (16 points)

(a) Is the derivative of the following function continuous at x=0? → 振荡

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \qquad f(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad \text{F}$$

(b) How about the derivative of $g(x) = xf(x)$? Given reasons for your answers.

6. (8 points) Prove that

$$e^x - x - 1 \geq 0 \qquad \text{T}$$

for all real numbers x. *Hint:* Find the global minimum of the function.

7. (12 points) Suppose that a differentiable function f satisfies $f(2) = -4$ and $f'(x) = \sqrt{x^2+5}$ for all x.

$f(1.95) = -4 - 3\times 0.05 = -4.15$

(a) Use a standard linear approximation to estimate $f(1.95)$ and $f(2.05)$.

(b) Are your estimates in part (a) too large or too small? Explain. $\quad f(2.05) = -3.85$

$f''(x)>0$. small

8. (16 points) Consider the function $f(x) = xe^{-x}$ defined on $(-\infty, \infty)$.

(a) Find the intervals on which f is increasing or decreasing. $\quad (-\infty, 1)\uparrow \quad (1, +\infty)\downarrow$

(b) Find the intervals on which f is concave up or concave down. $\quad (-\infty, 2)\downarrow \quad (2, +\infty)\uparrow$

(c) Find all asymptotes (horizontal, vertical and oblique) of the graph $y = f(x)$.

(d) Sketch the graph of f. $\quad \frac{x}{e^x} \quad x \to \infty.$

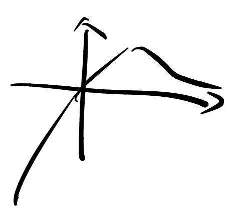
9. (12 points) State and prove Part I of Fundamental Theorem of Calculus.

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad x \to -\infty$$

↓

$$= \lim_{x\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{x\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} f(c). = f(x)$$



10. (42 points) Compute the following integrals

(i)

$-\frac{1}{x}+\ln|x|+2\sin x-3e^{x}+C$

$$\int (\frac{1+x}{x^2}+2\cos x-3e^x)dx.$$

(ii)

$$\int_{-\pi/4}^{\pi/2} |\sin x| dx.$$

$2-\frac{\sqrt{2}}{2}$

(iii)

$$\int_0^{\pi} \cos(nx)\cos(mx)dx$$

$\frac{1}{2}\left[\cos(n-m)x+\cos(n+m)x\right]$

$=\frac{1}{2}\frac{\sin(n-m)\pi}{n-m}$

$+\frac{1}{2}\frac{\sin(n+m)\pi}{n+m}$

where m and n are positive integers.

(iv) $2e^{\sqrt{x}}(\sqrt{x}-1)+C$

$$\int e^{\sqrt{x}}\, dx$$

(v)

$$\int \frac{dx}{(x^2+1)^{\frac{3}{2}}}.$$

$\frac{x}{\sqrt{x^2+1}}+C$

$\frac{1}{2}(\tan x\sec x+\ln|\sec x+\tan x|)$

(vi)

$+C$

$$\int \sec^3 x\, dx.$$

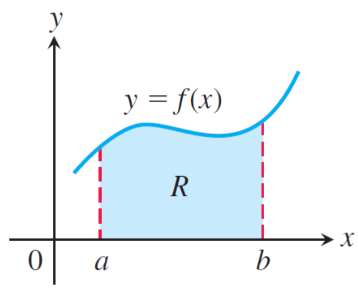
$\frac{1}{3}\ln(x+1)-\frac{1}{6}\ln(x^2-x+1)$

(vii)

$$\int \frac{dx}{x^3-1}.$$

$+\frac{1}{\sqrt{3}}\arctan(\frac{2x}{\sqrt{3}}-\frac{1}{\sqrt{3}})+C$

11. (8 points) Consider the region R bounded by the graphs of $y=f(x)>0$, $x=a>0$, $x=b>a$, and $y=0$. If the volume of the solid formed by revolving R around the x-axis is 10π, and the volume of the solid formed by revolving R around the line $y=-2$ is 20π, find the area of R.



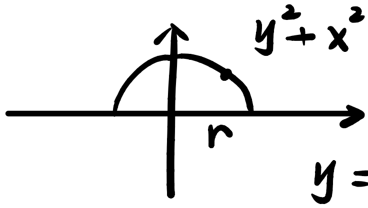
$\pi\left(\int_a^b f(x)\,dx\right)^2=10\pi$

$\pi\left(\int_a^b f(x)\,dx+2\right)^2$

$-\pi\cdot 2^2=20\pi.$

$10\pi+4\pi\int_a^b f(x)\,dx=20\pi.$

$R=\frac{5}{2}$



$y^2+x^2=r^2$

$y=\sqrt{r^2-x^2}$

surface $= \int_{-r}^{r} 2\pi\, y\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$

$= \int_{-r}^{r} 2\pi\cdot r\,dx = 2\pi r(r+r)$

$= 4\pi r^2$

12. (8 points) Use Calculus to prove that the surface area of a ball is $4\pi r^2$, where r is the radius of the ball.

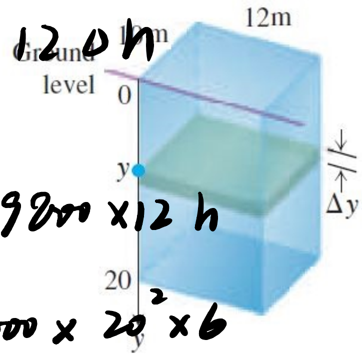
13. (16 points) The rectangular tank shown below, with its top at ground level, is used to catch runoff water.

(b) $= \int_0^{20} 9800(20-h)\,12\,dh$

$= \int_0^{20} 9800\times20\times12 - \int_0^{20} 9800\times12\,h$

$= 9800\times20^2\times12 - 9800\times20^2\times6$

$= 2.352\times10^7\,J$



(a)

$W = Fh$

$= \int_0^{20} 120\,\Delta y\cdot y\times9.8\times10^3$

$= 2.352\times10^8\,J$

Assume that the water weighs 9800 N/m^3.

(a) How much work does it take to empty the tank by pumping the water back to ground level once the tank is full?

(b) Compute the fluid force on the side of the tank that faces you (i.e., the side with height $20m$ and width $12m$).

14. (12 points) Determine the convergence of each of the following improper integrals; if the integral is convergent, find its value.

(i)

divergent

$$\int_0^{\pi/2} \frac{1+e^x}{\cos x}dx.$$

$\int_0^{\frac{\pi}{2}} \frac{1+e^x}{\cos x}dx$

$> \int_0^{\frac{\pi}{2}} \sec x\,dx$ divergent

(ii)

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx.$$

$\int_1^{\infty} e^{-u}\cdot 2\,du = +2\frac{1}{e}.$

15. (8 points) An object is moving on a straight line with positive velocity $v(t)$. We do not have a formula for v but we have the following measurements:

$S = \frac{1}{2}(1+4+4+2+3)$

$= 7$

$$v(0)=1,\quad v(1)=2,\quad v(2)=2,\quad v(3)=1,\quad v(4)=3.$$

Use trapezoidal rule to estimate the total distance travelled from $t=0$ to $t=4$.

16. (12 pts) Solve the following differential equations

(a) $\frac{dy}{dx} = 4x^3e^{-y}$; $y = \ln(x^4+C)$

(b) $xy' + 2y = x^3 - 1$, $y(1) = 2$.

$\frac{x^3}{5} - \frac{1}{2} + \frac{23}{10x^2}$

17. (10 points) A tank initially contains 50 liters of saltwater with concentration of salt being 0.1 $kgs/liter$. At time $t = 0$, saltwater of concentration 0.2 $kgs/liter$ of salt is pumped into the tank at the rate of 5 $liters/minute$. Well-mixed saltwater flows out of the tank at the same rate. **Derive** the Differential Equation for the amount $S(t)$ of salt inside the tank at time t, and specify the initial condition $S(0)$. **DO NOT SOLVE the DIFFERENTIAL EQUATION**.

$S(0) = 5kg$

$$\frac{ds}{dt} = 1 - \frac{S(t)}{50} \cdot 5$$

18. (15 points) Compare the following two population models

$$\frac{dP}{dt} = P(100 - P), \quad \text{(Logistic Model)}$$

and

$$\frac{dP}{dt} = P(P - 2)(100 - P). \quad \text{(Cubic Model)}$$

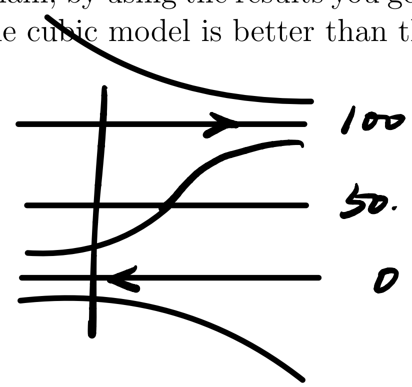
$(-3P^2 + 204P - 200)$

(a) Perform phase-line analysis on both models (that is, on the P-line you draw arrows to indicate the direction of motion of solutions of these two differential equations);

(b) For each case, find $\lim_{t\to\infty} P(t)$ if $P(0) = 200$.

(c) Explain, by using the results you get in (a), the difference between these two models. Why the cubic model is better than the logistic model?

(a)

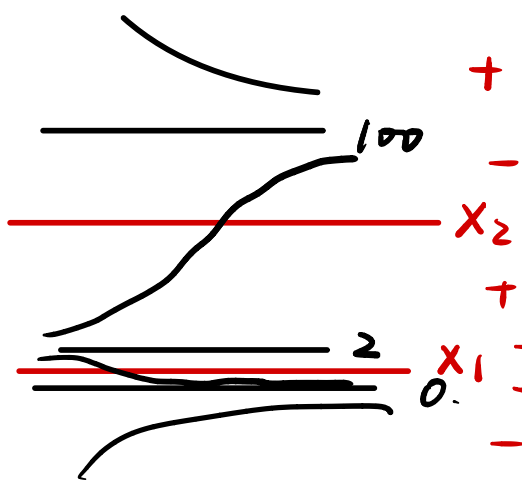


(b) $\to 100$.

(c) Cubic better

(lower than 2

go extinct)



$$x_2 = \frac{1}{3}\sqrt{102 + \sqrt{100^2 + 2^2 - 200}}$$

$$x_1 = \frac{1}{3}\sqrt{102 - \sqrt{100^2 + 2^2 - 200}}$$