

MAT1002 Lecture 7, Thursday, Feb/16/2023

Outline

- Binomial Series (10.10)
- Approximation of nonelementary integrals (10.10)
- Arctangent (10.10)
- Euler's formula/identity (10.10)
- Parametric plane curves (11.1)
- Calculus with parametric curves (11.2)
 - ↳ Derivatives

Binomial Series

Consider the binomial theorem, which implies that

$$(1+x)^m = \sum_{n=0}^m \binom{m}{n} x^n,$$

where $\binom{m}{n} = C_m^n$ is the binomial coefficient "m choose n".

Here $m \in \mathbb{N}$.

Now consider $f(x) := (1+x)^\alpha$, where $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} f'(x) &= \alpha(1+x)^{\alpha-1} &\Rightarrow f'(0) &= \alpha \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} &\Rightarrow f''(0) &= \alpha(\alpha-1) \\ &&\vdots & \\ f^{(n)}(x) &= \alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n} &\Rightarrow f^{(n)}(0) &= \alpha(\alpha-1)\dots(\alpha-n+1) \end{aligned}$$

MacLaurin series of $f(x) = (1+x)^\alpha$ is $\sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n$.

Def:

- For any $\alpha \in \mathbb{R}$ and $n \in \mathbb{N} := \{0, 1, 2, \dots\}$,

$$\text{"}\alpha\text{ choose } n\text{"} \rightarrow \binom{\alpha}{n} := \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$$

is a **binomial coefficient**. By default, $\binom{\alpha}{0} = 1$.

- The series $\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ is a **binomial series**.

For $\alpha \in \mathbb{N}$,
this is
the
"usual"
binomial
coefficient
with counting
meaning.

Convergence

If $\alpha = m \in \mathbb{N}$, then the binomial series is a finite sum

$$\sum_{n=0}^{\infty} \binom{m}{n} x^n = \sum_{n=0}^m \binom{m}{n} x^n, \quad \begin{pmatrix} \text{Since } \binom{m}{n} = 0 \\ \text{for } n > m. \end{pmatrix}$$

So it converges for all $x \in \mathbb{R}$.

If $\alpha \notin \mathbb{N}$, then the binomial series is infinite. Since

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\alpha(\alpha-1)\dots(\alpha-n)}{(n+1)!} \cdot \frac{n!}{\alpha(\alpha-1)\dots(\alpha-n+1)} \right| |x| = \frac{|\alpha-n|}{n+1} |x|$$

$$= \frac{\left| \frac{\alpha}{n} - 1 \right|}{1 + \frac{1}{n}} |x| \rightarrow |x| \text{ as } n \rightarrow \infty.$$

By Ratio Test, radius of convergence is $R=1$.

Fact: $\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^\alpha \quad \text{for } x \in (-1, 1).$

Proof: Ex. 58 in 10.10. (Assignment 3.)

e.g. For $\alpha = \frac{1}{2}$, $\binom{\frac{1}{2}}{0} = 1$, $\binom{\frac{1}{2}}{1} = \frac{1}{2}$, $\binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} = -\frac{1}{8}$,

and for $n \geq 2$,

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!} = \frac{\frac{1}{2}(-\frac{1}{2})\dots(-\frac{2n-3}{2})}{n!}$$

$$= (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \dots (2n-3)}{2^n \cdot n!}$$

These give the coefficients for the binomial series for

$$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}.$$

Approximation of Nonelementary Integrals

The Maclaurin series of some elementary functions are alternating series, e.g., $\sin x$, $\cos x$, $\ln(1+x)$, etc..

We may use alternating series approximation to estimate the values of some nonelementary functions involving these.

e.g. Estimate $\int_0^1 \sin(x^2) dx$ with an error < 0.001 .

Sol. $\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}$, $y \in \mathbb{R}$

$$\Rightarrow \sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}, \quad \forall x \in \mathbb{R}$$

$$\begin{aligned} \int_0^1 \sin(x^2) dx &= \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} dx = \sum_{n=0}^{\infty} \int_0^1 (-1)^n \frac{x^{4n+2}}{(2n+1)!} dx \\ &= \sum_{n=0}^{\infty} \left. \left(\frac{(-1)^n}{(2n+1)! \cdot (4n+3)} x^{4n+3} \right) \right|_{x=0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! (4n+3)} \end{aligned}$$

$U_n = \frac{1}{(2n+1)! (4n+3)}$ is decreasing and $\rightarrow 0$ as $n \rightarrow \infty$.

When $n=2$, $U_n = U_2 = \frac{1}{5! \cdot 11} \approx 0.00076 < 0.001$, so

$$\int_0^1 \sin(x^3) dx \approx \frac{1}{3} - \frac{1}{3! \cdot 7} = \frac{1}{3} - \frac{1}{42} \approx 0.310.$$

Arctangent

In Lecture 5, we showed that

$$(*) \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}, \quad \forall x \in (-1, 1).$$

Does this hold for the endpoints $x = \pm 1$? Consider

$$Q_n(x) := \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{2k+1} = P_{2n+1}(x),$$

and let $r_n(x) := \arctan x - Q_n(x)$. We show that $\lim_{n \rightarrow \infty} r_n(x) = 0$ for $x = \pm 1$. This would imply that $(*)$ holds for $x = \pm 1$. Recall

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt.$$

Since

$$1+y+y^2+\dots+y^n = \frac{1-y^{n+1}}{1-y} = \frac{1}{1-y} - \frac{y^{n+1}}{1-y}, \quad \forall y \in \mathbb{R},$$

We have

$$\frac{1}{1-y} = \left(\sum_{k=0}^n y^k \right) + \frac{y^{n+1}}{1-y}. \quad (**)$$

Now

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt \stackrel{(**) \text{ and } y = -t^2}{=} \int_0^x \left(\sum_{k=0}^n (-1)^k t^{2k} \right) dt + \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt$$

$$= \underbrace{\sum_{k=0}^n \frac{(-1)^k}{2k+1} x^{2k+1}}_{Q_n(x)} + \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt ,$$

$$\text{So } r_n(x) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt .$$

For $x=1$,

$$|r_n(1)| = \left| \int_0^1 \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt \right| \leq \int_0^1 \frac{t^{2n+2}}{1+t^2} dt < \int_0^1 t^{2n+2} dt \\ = \frac{t^{2n+3}}{2n+3} \Big|_{t=0}^1 = \frac{1}{2n+3} \rightarrow 0 \text{ as } n \rightarrow \infty .$$

For $x=-1$,

$$|r_n(-1)| = \left| \int_0^{-1} \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt \right| = \left| \int_{-1}^0 \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt \right| \leq \int_{-1}^0 \frac{t^{2n+2}}{1+t^2} dt \\ < \int_{-1}^0 t^{2n+2} dt = \frac{t^{2n+3}}{2n+3} \Big|_{t=-1}^0 = \frac{1}{2n+3} \rightarrow 0 \text{ as } n \rightarrow \infty .$$

Hence, $\lim_{n \rightarrow \infty} r_n(1) = \lim_{n \rightarrow \infty} r_n(-1) = 0$, as desired.

Frequently Used MacLaurin Series that Converge to Their Functions

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \leq 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1$$

Euler's Formula set : \mathbb{C}

A complex number is a number of the form $x+yi$, where $x, y \in \mathbb{R}$, and i satisfies $i^2 = -1$. Multiplications are done as usual : $(x+yi)(a+bi) = ax + xbi + ya + ybi^2$
 $= (ax - by) + (bx + ay)i$.

With multiplications, one could define \mathbb{Z}^n naturally $\forall z \in \mathbb{C}, \forall n \in \mathbb{N}$.

With Taylor series, one could extend the definition of many elementary functions to cover complex variables.

The complex exponential function $\exp : \mathbb{C} \rightarrow \mathbb{C}$ is defined by

$$\text{Also written as } e^z \rightarrow \exp(z) := 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

One could define the complex sine and cosine by using their MacLaurin series similarly.

Since $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = -i^2 = 1 = i^0,$

for $\theta \in \mathbb{R}$, we have

$$\begin{aligned}\exp(i\theta) &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right) \\ &= \cos \theta + i \sin \theta\end{aligned}$$

Euler's formula: $e^{i\theta} = \cos \theta + i \sin \theta$.

← The book takes it as a definition, and call it Euler's identity.

One consequence of Euler's formula is the following identity:

If $\theta = \pi$, we have $e^{i\pi} = -1 + 0$, or

$e^{i\pi} + 1 = 0$,

Connecting the five constants $e, i, \pi, 1$ and 0 in one equation.

Parametric Plane Curves

In MAT1001, we studied the computation of quantities, such as lengths and areas, associated to a curve if the curve is described as the graph of a function $y = f(x)$. In this chapter we look at curves that have a more general description.

Definition

If a plane curve has the form

$$\{(x, y) : x = f(t), y = g(t), t \in I\},$$

where I is an interval, then the curve is called a **parametric curve**, and the equations

$$x = f(t), y = g(t), \quad t \in I$$

t is called the parameter

are called **parametric equations** of the curve.

e.g. (a) If a curve is the graph of a function $y = f(x)$, where $x \in I$, then it is (trivially) a parametric curve, with parametric equations

$$x = t, y = f(t), t \in I.$$

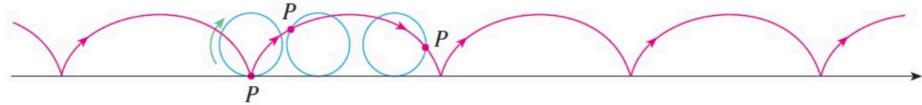
e.g. (b) Sketch the curve

$$x = \sin t, \quad y = \sin^2 t, \quad t \in \mathbb{R}$$

Remark A curve is a point set, not the "movement": a curve has many parametric representations.

Give two different sets of parametric equations to the curve above.

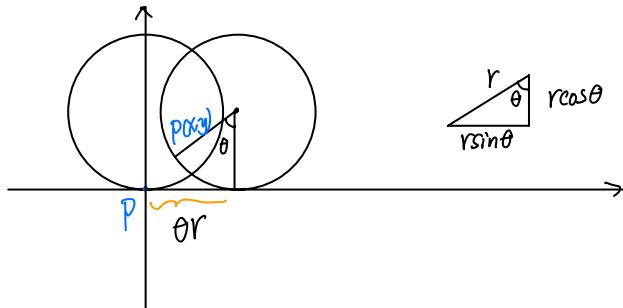
e.g. (c) The curve traced out by a point P on a disk as the disk rolls along a straight line is called a **cycloid** (see the figure below). Assume that the disk has radius r and rolls along the x -axis. If one position of P is the origin, find parametric equations for the cycloid.



For $\theta \in [0, \frac{\pi}{2}]$:

$$x = \theta r - r \sin \theta = r(\theta - \sin \theta)$$

$$y = r - r \cos \theta = r(1 - \cos \theta)$$



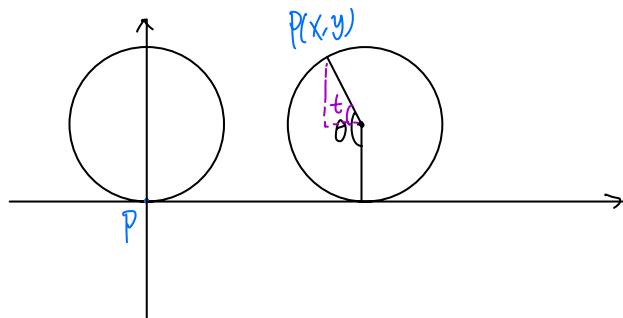
For $\theta \in [\frac{\pi}{2}, \pi]$:

$$x = \theta r - r \cos(\theta - \frac{\pi}{2})$$

$$= \theta r - r \sin \theta = r(\theta - \sin \theta)$$

$$y = r + r \sin(\theta - \frac{\pi}{2})$$

$$= r + r(-\cos \theta) = r(1 - \cos \theta)$$



Using similar arguments for $\theta \in [\pi, \frac{3\pi}{2}]$ and $\theta \in [\frac{3\pi}{2}, 2\pi]$, one can show that one arc of the cycloid can be parametrized by

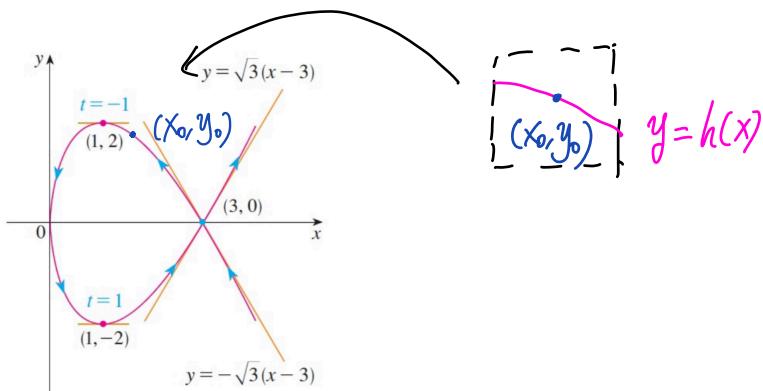
$$x = r(\theta - \sin \theta), \quad y = r(1 - \cos \theta), \quad 0 \leq \theta \leq 2\pi.$$

Since changing θ by $2k\pi$ changes the x -value by $2kr$ and the y -value should remain the same by curve nature, the full cycloid is parametrized by

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta), \quad \theta \in \mathbb{R}.$$

Tangents

Consider the curve given by $x = t^2$, $y = t^3 - 3t$, $t \in \mathbb{R}$.



How can one find the slope of the tangent at the point (x_0, y_0) ? Note that near the point, the curve is the graph of $y = h(x)$. More generally, if $x = f(t)$, $y = g(t)$, and y is a differentiable function of x (near a point), then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad (\text{if } \frac{dx}{dt} \neq 0).$$

expressed in terms of t

For the example above, $\left. \frac{dy}{dx} \right|_{(x_0, y_0)} = \left. \frac{3t^2 - 3}{2t} \right|_{t=t_0}$, where $(x_0, y_0) = (f(t_0), g(t_0))$.

Second Derivatives

Similarly, if $x = f(t)$, $y = g(t)$, and y is a twice differentiable function of x near a point, then at the point,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} y' = \frac{dy'/dt}{dx/dt}, \text{ where } y' := \frac{dy}{dx}$$

For the example above, at $(x_0, y_0) = (f(t_0), g(t_0))$.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3}{2}(t - \frac{1}{t})\right)}{2t} = \frac{\frac{3}{2}(1 + \frac{1}{t^2})}{2t} \\ &= \frac{3(t^2+1)}{4t^3} \quad \begin{cases} > 0, & \text{if } t > 0 \\ < 0, & \text{if } t < 0 \end{cases} \quad \begin{array}{l} \text{Curve is} \\ \text{Concave up} \end{array} \\ &\quad \begin{array}{l} \text{Curve is} \\ \text{Concave down} \end{array}\end{aligned}$$