Distributive law:

- $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
- $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

De Morgan's law:

- $(A\cup B)'=A'\cap B'$
- $(A\cap B)'=A'\cup B'$

permutation: ${}_nP_r=n(n-1)\cdots(n-r+1)=\frac{n!}{(n-r)!}$

combination: ${}_nC_r=\binom{n}{r}=\frac{n!}{r!(n-r)!}={}_nC_{n-r}=\frac{{}_nP_r}{r!}$

Conditional Probability: $P(A|B)=\frac{P(A\cap B)}{P(B)}$

$P(A\cap B\cap C)=P(A\cap B)P(C|A\cap B)=P(A)P(B|A)P(C|A\cap B)$

Independent Events: $P(A\cap B)=P(A)P(B)$

$\Rightarrow P(A|B)=P(A),\ P(B|A)=P(B)$

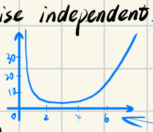
$\Rightarrow$ A and B', A' and B, A' and B' ✓

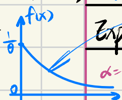
Events $A.B.C$ are mutually independent, i.e.,

1. $A.B.C$ are pairwise independent

$$\begin{cases}P(A\cap B)=P(A)P(B)\\ P(A\cap C)=P(A)P(C)\\ P(B\cap C)=P(B)P(C)\end{cases}$$

2. $P(A\cap B\cap C)=P(A)P(B)P(C)$





Bayes' Theorem:

total probability

(a) For any event A, $P(A)=\sum_{i=1}^{m}P(A\cap B_i)=\sum_{i=1}^{m}P(B_i)P(A|B_i)$

(b) If $P(A)>0$, then $P(B_k|A)=\frac{P(B_k\cap A)}{P(A)}$, $k=1,\cdots,m$

$P(B_k|A)=\frac{P(B_k)P(A|B_k)}{P(A)=\sum_{i=1}^{m}P(B_i)P(A|B_i)}$ $\longrightarrow$ Bayes Theorem

$P(B_k)$: prior / $P(B_k|A)$: posterior probability 先/后验概率

$P(A|B_k)$: likelihood of B_k, A is a data

$f(x):\bar S\to(0,1]$

Probability Mass Function (pmf):

$$\begin{cases}1.\ f(x)>0,\ x\in\bar S\\ 2.\ \sum_{x\in\bar S}f(x)=1\\ 3.\ P(X\in A)=\sum_{x\in A}f(x),\ A\subseteq\bar S\end{cases}$$

We often extend the domain of $f(x)$ from $\bar S$ to R and let $f(x)=0$, $x\notin\bar S$. In this case, $\bar S$ is called the support of $f(x)$.

$f(x)=c$ for $x\in\bar S\Rightarrow$ uniform distribution

Cumulative Distribution Function (cdf):

$F(x):R\to[0,1]$: $F(x)=P(X\le x)$

$$\begin{cases}1.\ \text{nondecreasing},\ P(X\le x)=\sum_{x'\le x,\,x'\in\bar S}f(x')\\ 2.\ P(a<X\le b)=F(b)-F(a)\end{cases}$$

Mathematical Expectation: $E[g(x)]=\sum_{x\in\bar S}g(x)f(x)$

(1) $E(c)=c$ (2) $E[cg(x)]=cE[g(x)]$

(3) $E[c_1g_1(x)+c_2g_2(x)]=c_1E[g_1(x)]+c_2E[g_2(x)]$ (linear operator)

Mean of a RV $[g(x)=X]$:

the average value of X $E(X)=\sum_{x\in\bar S}xf(x)\ \overset{\bar S=\{x_1,\cdots,x_k\}}{=}\ \sum_{i=1}^{k}x_if(x_i)$

$=\int_{\bar S}xf(x)dx$

Variance of a RV $[g(x)=(X-E[X])^2]$:

$Var(X)=E[(X-E[X])^2]=\sum_{x\in\bar S}(x-E[X])^2f(x)=E(X^2)-(E[X])^2$ (second, first moment)

$=\int_{\bar S}(x-E[x])^2f(x)dx$

Standard deviation of a RV: $\sqrt{Var(X)}$

(1) $Var(c)=0$ (2) $Var(cX)=c^2Var(X)$

The rth Moment $[g(x)=X^r$ with r a positive integer$]$:

$E(X^r)=\sum_{x\in\bar S}x^rf(x)\Longrightarrow$ the rth moment of X $=\int_{\bar S}x^rf(x)dx$

$E[(X-b)^r]=\sum_{x\in\bar S}(x-b)^rf(x)\Longrightarrow$ the rth moment of X about b

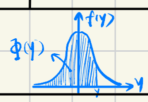
$E[(X)_r]=E[X(X-1)\cdots(X-r+1)]\Longrightarrow$ the rth factorial moment

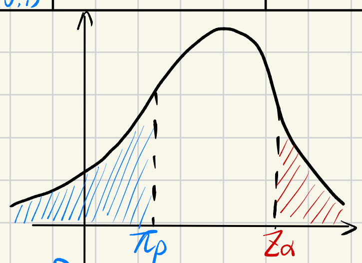
Moment Generating Function (mgf): (for $-h<t<h$)

$M(t)=E[e^{tX}]=\sum_{x\in\bar S}e^{tx}f(x)=\int_{\bar S}e^{tx}f(x)dx$

1. $M(0)=1$
2. two RVs, same mgf, same probability distribution, same pmf
3. $M'(t)=\sum_{x\in\bar S}xe^{tx}f(x)$ $M''(t)=\sum_{x\in\bar S}x^2e^{tx}f(x)$ $M^{(r)}(t)=\sum_{x\in\bar S}x^re^{tx}f(x)$

$M'(0)=E(X)$ $M''(0)=E(X^2)$ $M^{(r)}(0)=E[X^r]$

	pmf/pdf	mgf $M(t)$	mean	variance	cdf	解释
Bernoulli	$f(x):\bar S\to[0,1]$; $f(x)=p^x(1-p)^{1-x},\ x\in\bar S$	$p\cdot e^t+(1-p)$, $t\in(-\infty,\infty)$	p	$(1-p)p$		RV: $X: S\to\bar S$, $S=\{\text{success},\text{failure}\}$, $X(✓)=1$ $X(✗)=0$ $\bar S=[0,1]$
Binomial	$f(x)=\binom{n}{x}p^x(1-p)^{n-x}$, $x=0,1,\cdots,n$; $X\sim b(n,p)$	$[(1-p)+pe^t]^n$, $t\in(-\infty,\infty)$	np	$np(1-p)$	$F(x)=P(X\le x)=\sum_{y\in\{X\le x\}}f(y)=\sum_{y=0}^{[x]}\binom{n}{y}p^y(1-p)^{n-y}$, $x\in(-\infty,\infty)$, $[x]:\le x$	n Bernoulli trials 中成功总次数
Negative Binomial	$f(x)=\binom{x-1}{r-1}p^r(1-p)^{x-r}$, $x=r,r+1,\cdots$; $(1-w)^{-r}=\sum_{x=r}^{\infty}\binom{x-1}{r-1}w^{x-r}$	$\frac{(pe^t)^r}{[1-(1-p)e^t]^r}$ for $(1-p)e^t<1$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$		对于给定的r，第r次Benoulli成功的次数
Geometric	$f(x)=p(1-p)^{x-1}$ ($r=1$)	$P(X>k)=\sum_{x=k+1}^{\infty}p(1-p)^{x-1}=(1-p)^k$; $P(X\le k)=\sum_{x=1}^{k}p(1-p)^{x-1}=1-P(X>k)=1-(1-p)^k$				
Poisson	$f(x)=\frac{\lambda^xe^{-\lambda}}{x!}$, $x=0,1,\cdots$; $X\sim\text{Poisson}(\lambda)$	$e^{\lambda(e^t-1)}$	λ	λ		特定事件在给定时间间隔内发生的次数
Uniform $X\sim U(a,b)$	$f(x)=\begin{cases}\frac{1}{b-a}, & a\le x\le b\\ 0, & \text{otherwise}\end{cases}$	略 $\begin{cases}\frac{e^{tb}-e^{ta}}{t(b-a)}, & t\ne0\\ 1, & t=0\end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$F(x)=\begin{cases}0, & x<a\\ \frac{x-a}{b-a}, & a\le x\le b\\ 1, & x>b\end{cases}$	
Exponential	$f(x)=\frac{1}{\theta}e^{-\frac{x}{\theta}}$, $x\ge0,\theta>0$	$\frac{1}{1-t\theta}$, $t<\frac{1}{\theta}$	θ	θ^2		
Gamma ($\alpha=1$; $w\Leftrightarrow x$, $\theta\Leftrightarrow\frac{1}{\lambda}$; $\theta=2,\alpha=\frac{r}{2}$)	$f(x)=\frac{1}{\Gamma(\alpha)\theta^\alpha}x^{\alpha-1}e^{-\frac{x}{\theta}}$; $\Gamma(n)=(n-1)!$ $x\ge0,\alpha>0,\theta>0$	$\frac{1}{(1-\theta t)^\alpha}$, $t<\frac{1}{\theta}$	$\alpha\theta$	$\alpha\theta^2$		某特定事件第α次发生APP等待时间
Chi-square $X\sim\chi^2(r)$	$f(x)=\frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}}x^{\frac{r}{2}-1}e^{-\frac{x}{2}}$, $x>0$	$(1-2t)^{-\frac{r}{2}}$, $t<\frac{1}{2}$	r	$2r$		
Normal $X\sim N(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{1}{2}\cdot\frac{(x-\mu)^2}{\sigma^2})$, $-\infty<x<\infty$	$\exp(\mu t+\frac{1}{2}\sigma^2t^2)$	μ	σ^2		(Gaussian)
Standard Normal $Y\sim N(0,1)$	$f(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$	$\Phi(-y)=1-\Phi(y)$ 			$\Phi(y)=P(Y\le y)=\int_{-\infty}^{y}f(z)dz=\int_{-\infty}^{y}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$	



$P(X\le\pi_p)=p$

(100p)th percentile

$P(X\ge z_\alpha)=\alpha$

The upper 100α percent point

the 100(1-α)th percentile

25th / first

$p=\int_{-\infty}^{\pi_p}f(x)dx=F(\pi_p)$

75th / third

50th / median, 2nd quantile

Normal Distribution 与其他 Distribution 的转化

1. If Y is $N(\mu,\sigma^2)$, then $X=\frac{Y-\mu}{\sigma}$ is $N(0,1)$

X has the same cdf as $N(0,1)$

证: $P(X\le x)=P(\frac{Y-\mu}{\sigma}\le x)=P(Y\le\sigma x+\mu)$

$=\int_{-\infty}^{\sigma x+\mu}f(y)dy=\int_{-\infty}^{\sigma x+\mu}\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{1}{2}(\frac{(y-\mu)^2}{\sigma^2}))dy$

$\overset{w=\frac{y-\mu}{\sigma}}{=}\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}w^2)dw\Rightarrow$ cdf of $N(0,1)$

$\Rightarrow P(a\le X\le b)=P(\frac{a-\mu}{\sigma}\le\frac{X-\mu}{\sigma}\le\frac{b-\mu}{\sigma})$

$=\Phi(\frac{b-\mu}{\sigma})-\Phi(\frac{a-\mu}{\sigma})$

2. If X is $N(\mu,\sigma^2)$ with $\sigma^2>0$, then $\frac{(X-\mu)^2}{\sigma^2}\sim\chi^2(1)$

证: Let $V=\frac{(X-\mu)^2}{\sigma^2}$. Then consider the cdf of V,

$G(v)=P(V\le v)=P(-\sqrt{v}\le Z\le\sqrt{v})$,

where $Z=\frac{X-\mu}{\sigma}$, with $v\ge0$

$G(v)=\int_{-\sqrt{v}}^{\sqrt{v}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz=2\int_0^{\sqrt{v}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$

$z=\sqrt{y},\ \frac{dz}{dy}=\frac{1}{2\sqrt{y}}$: $2\int_0^{v}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y}\frac{1}{2\sqrt{y}}dy=\int_0^{v}\frac{1}{\sqrt{2\pi y}}e^{-\frac{y}{2}}dy$, $v\ge0$

$g(v)=G'(v)=\frac{1}{\sqrt{2\pi}}v^{\frac{1}{2}-1}e^{-\frac{1}{2}v}$, $v>0$

Now recall the pdf of $\chi^2(1)$:

$f(x)=\frac{1}{\Gamma(\frac{1}{2})2^{\frac{1}{2}}}x^{\frac{1}{2}-1}e^{-\frac{1}{2}x}$, $x\ge0$

Since $g(v)$ is a pdf, then $\int_0^\infty g(v)dv=1$

$1=\int_0^\infty\frac{1}{\sqrt{2\pi}}v^{\frac{1}{2}-1}e^{-\frac{1}{2}v}dv\ \overset{x=\frac{1}{2}v}{=}\ \frac{1}{\sqrt{\pi}}\int_0^\infty x^{\frac{1}{2}-1}e^{-x}dx$

$=\frac{1}{\sqrt{\pi}}\Gamma(\frac{1}{2})\Rightarrow\Gamma(\frac{1}{2})=\sqrt{\pi}$

$\Rightarrow g(v)=\frac{1}{\Gamma(\frac{1}{2})2^{\frac{1}{2}}}v^{\frac{1}{2}-1}e^{-\frac{v}{2}}$, $v>0$

Normal Distribution 的详细内容

$M(t)=E[e^{tX}]=\int_{-\infty}^{\infty}e^{tx}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}dx$

$e^{tx}e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}=\exp\{-\frac{1}{2\sigma^2}[x^2-2(\mu+\sigma^2t)x+\mu^2]\}$

Consider $x^2-2(\mu+\sigma^2t)x+\mu^2=[x-(\mu+\sigma^2t)]^2-2\mu\sigma^2t-\sigma^4t^2$

$\therefore M(t)=\exp(\frac{-2\mu\sigma^2t-\sigma^4t^2}{-2\sigma^2})\cdot\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{1}{2\sigma^2}[x-(\mu+\sigma^2t)]^2)dx$

Recall that $I=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\exp[-\frac{(x-\mu)^2}{2\sigma^2}]dx=1$, independent of μ

Therefore, $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{1}{2\sigma^2}[x-(\mu+\sigma^2t)]^2)dx=1$

$\therefore M(t)=\cdots$ ✓ $M(0)=1$

$M'(t)=(\mu+\sigma^2t)\exp(\mu t+\frac{1}{2}\sigma^2t^2)\Rightarrow M'(0)=\mu$

$M''(t)=\sigma^2\exp(\mu t+\frac{1}{2}\sigma^2t^2)+(\mu+\sigma^2t)^2\exp(\mu t+\frac{1}{2}\sigma^2t^2)$

$\Rightarrow M''(0)=\mu^2+\sigma^2$

Recall that $E[X]=M'(0)=\mu$,

$Var[X]=E[X^2]-(E[X])^2=M''(0)-M'(0)^2=\sigma^2$

For $X\sim N(\mu,\sigma^2)$, $E[X]=\mu$, $Var[X]=\sigma^2$

$\therefore$ The two parameters μ, σ^2 are the mean and variance, respectively.

证 Gamma Distribution $\int$pdf = 1

Let $t=\frac{x}{\theta}\Rightarrow x=\theta t,\ dx=\theta dt$

(原式) $=\int_0^\infty\frac{1}{\Gamma(\alpha)\theta^\alpha}(\theta t)^{\alpha-1}e^{-t}\theta dt$

$=\frac{1}{\Gamma(\alpha)\theta^\alpha}\cdot\theta^{\alpha-1}\cdot\theta\int_0^\infty t^{\alpha-1}e^{-t}dt$

$=\frac{1}{\Gamma(\alpha)}\cdot\Gamma(\alpha)=1$

Gamma Function (generalized factorial):

$\Gamma(t)=\int_0^\infty y^{t-1}e^{-y}dy=-y^{t-1}e^{-y}\Big|_0^\infty+\int_0^\infty(t-1)y^{t-2}e^{-y}dy$

$=(t-1)\Gamma(t-1)$