Half-time Review of STA2001 (I)

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Outline

Outline

What is probability theory?

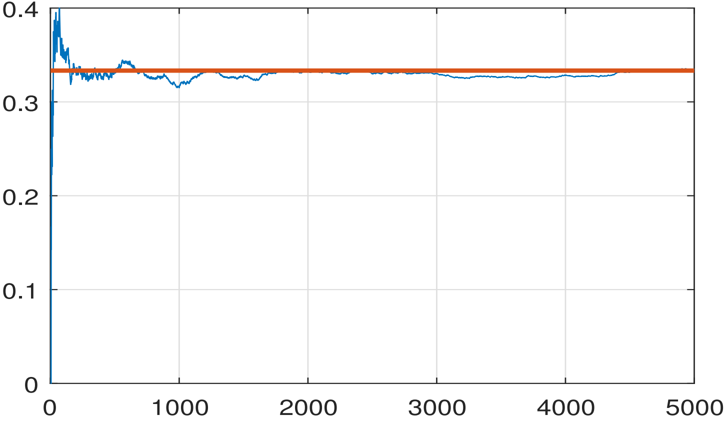
Probability theory is the branch of mathematics concerned with probability, the analysis of random phenomena - wikipedia

1. Random experiment
2. Sample space
3. Event
4. **Event A has happened**

A first definition of probability

The limit of relative frequency:

$$P(A) = \lim_{n\to\infty} \frac{\mathcal{N}(A)}{n}$$



A more formal definition of probability

Probability function is a function that assigns $P(A)$ to an event A, $A \subseteq S$ such that the following three conditions are satisfied

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $A_1, A_2, \cdots$ mutually exclusive ~~and exhaustive~~

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

Properties for probability function

- $P(A) = 1 - P(A')$
- $P(\emptyset) = 0$
- $P(A) \leq 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

How to calculate probability of an event

For random experiments that satisfy

Assumption 1: S contains m possible outcomes

$$e_k, \quad k = 1, 2, \cdots, m, \quad i.e., \quad S = \{e_1, e_2, \cdots, e_m\}.$$

Assumption 2: The m outcomes are "equally likely"

$$P(\{e_k\}) = \frac{1}{m}, \quad k = 1, \cdots, m.$$

$$P(A) = \frac{\mathcal{N}(A)}{\mathcal{N}(S)},$$

where $\mathcal{N}(X)$ is the number of outcomes in $X \subseteq S$.

Counting techniques — tools to calculate probability

Multiplication principle:

$$\to \boxed{\text{Experiment } E_1} \to n_1 \text{ outcomes}$$

$$\to \boxed{\text{Experiment } E_2} \to n_2 \text{ outcomes}$$

$$\to \underbrace{\boxed{\text{Experiment } E_1} \to \boxed{\text{Experiment } E_2}}_{\text{sequential implementation}} \to n_1 n_2 \text{ possible outcomes}$$

- permutation
- combination
- distinguishable permutation

Conditional Probability

Conditional probability of an event A, given that event B has occurred, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) > 0$.

- The sample space shrinks to B.
- Conditional probability is a probability function.
- $P(A \cap B) = P(A)P(B|A)$.

Independent Events

Events A and B are independent if

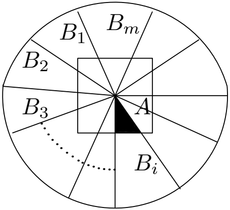
$$P(A \cap B) = P(A)P(B).$$

The occurrence of one of them does NOT change the probability of the occurrence of the other.

- A and B are independent, if and only if any pair of the following events are independent
  (a) A and B'
  (b) A' and B
  (c) A' and B'

- A, B, C are independent if
  1. pairwise independent
  2. $P(A \cap B \cap C) = P(A)P(B)P(C)$

Bayes' Theorem



Assume

1. $S = B_1 \cup B_2 \cup \cdots B_m, \quad B_i \cap B_j = \Phi$
2. $P(B_i) > 0$

Then

$$P(A) = \sum_{k=1}^{m} P(A \cap B_i) = \sum_{k=1}^{m} P(B_i) P(A|B_i)$$

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(A)}, \text{ provided } P(A) > 0$$

Outline

1. Basic Concepts of Probability Theory
2. Univariate Random Variable
3. Typical Univariate Random Distributions

Random variable

Definition[Random Variable]

Given a random experiment with sample space S, a function $X : S \to \overline{S} \subseteq R$ that assign one real number $X(s) = x$ to each $s \in S$ is called Random Variable (RV).

Note: repeat a random experiment $\Leftrightarrow$ generate a number from $\overline{S}$

Depending on the property of $\overline{S}$

1. Discrete RV, if $\overline{S}$ is finite or countably infinite.
2. Continuous RV, if $\overline{S}$ is union of intervals.

Discrete RV Vs. Continuous RV

RV X is a function $X : S \to \overline{S} \subseteq R$

Discrete RV:

pmf $f(x) : \overline{S} \to (0,\ 1]$

1. $f(x) > 0$
2. $\sum_{x \in \overline{S}} f(x) = 1$
3. $P(X \in A) = \sum_{x \in A} f(x)$

Continuous RV:

pdf $f(x) : \overline{S} \to (0, \infty)$

1. $f(x) > 0$
2. $\int_{\overline{S}} f(x)dx = 1$
3. $P(X \in A) = \int_A f(x)dx$

Very often, we extend the definition domain of $f(x)$ from $\overline{S}$ to R by letting $f(x) = 0$ for $x \notin \overline{S}$.

Discrete RV Vs. Continuous RV

RV X is a function $X : S \to \overline{S} \subseteq R$

Discrete RV:

pmf $f(x) : R \to [0,\ 1]$

1. $f(x) \geq 0$
2. $\sum_{x \in \overline{S}} f(x) = 1$
3. $P(X \in A) = \sum_{x \in A} f(x)$

Continuous RV:

pdf $f(x) : R \to [0, \infty)$

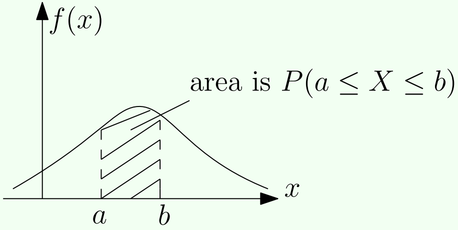
1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x) dx = 1$
3. $P(X \in A) = \int_A f(x) dx$

with $f(x) = 0$ for $x \notin \overline{S}$; $\overline{S}$ is called the support set of $f(x)$.

Interpretation of pdf

Interpretation

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$



Cdf and its properties

Definition

cdf $F(x) : R \to [0, 1]$

$$F(x) = P(X \leq x), \quad \text{nondecreasing function!}$$

1. relation between the probability function and the cdf

$$P(a \leq X \leq b) = F(b) - F(a)$$

2. for continuous RV,

$$f(x) = F'(x)$$

for those values of x at which $F(x)$ is differentiable.

Mathematical Expectation

Definition[Mathematical Expectation]

Let X be a RV with range $\overline{S}$ and $f(x)$ be its pmf or pdf. The mathematical expectation of $g(X)$, if exists, is denoted by

$$E[g(X)] = \begin{cases} \sum_{x \in \overline{S}} g(x) f(x), & \text{discrete RV} \\ \int_{x \in \overline{S}} g(x) f(x) dx, & \text{continuous RV} \end{cases}$$

Property[Mathematical Expectation]

Mathematical expectation is a linear operator, i.e.,

$$E[c_1 g_1(X) + c_2 g_2(X)] = c_1 E[g_1(X)] + c_2 E[g_2(X)]$$

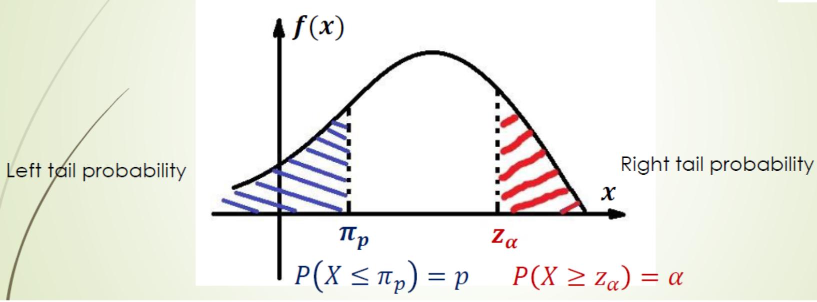
Special mathematical expectation

Definition[Special mathematical expectation]

$$E[g(X)] = \begin{cases} \sum_{x \in \overline{S}} g(x)f(x), & \text{discrete RV} \\ \int_{x \in \overline{S}} g(x)f(x)dx, & \text{continuous RV} \end{cases}$$

$$g(X) = \begin{cases} X \rightarrow \text{ Mean} \\ (X - EX)^2 \rightarrow \text{ Variance, } Var(X) = EX^2 - (EX)^2 \\ X^r \rightarrow \text{ Moment} \\ e^{tX}, \text{ for } |t| < h, \rightarrow \text{ Mgf:} M^{(r)}(0) = EX^r \end{cases}$$

(100p)th percentile, the upper 100α percent point



Definition [(100p)th percentile]

The number π_p such that $P(X \leq \pi_p) = p$. $\pi_{0.5}, \pi_{025}$ and $\pi_{0.75}$ are called the median (the second quantile), the first and third quantiles, respectively.

Definition [the upper (100α) percent point]

The number z_α such that $P(X \geq z_\alpha) = \alpha$.

Outline

Table of distributions

	pmf/pdf	mgf	mean	variance
Binomial $(n = 1, \text{Bernoulli})$	$\binom{n}{x} p^x (1-p)^{n-x}$ $x = 0,1,2,\dots,n$	$[(1-p)+pe^t]^n$ $-\infty < t < \infty$	np	$np(1-p)$
Negative Binomial	$\binom{x-1}{r-1} p^r (1-p)^{x-r}$ $x = r, r+1, r+2, \dots$	$\frac{(pe^t)^r}{[1-(1-p)e^t]^r}$ $(1-p)e^t < 1$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Poisson	$\frac{\lambda^x e^{-\lambda}}{x!}, x = 0,1,2,\dots$	$e^{\lambda(e^t-1)}$	λ	λ
Gamma $(\alpha = 1, \text{Expnential})$ $(\theta = 2, \alpha = r/2, \chi^2)$	$\frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}}$ $0 \le x < \infty$	$\frac{1}{(1-\theta t)^\alpha}$ $t < \frac{1}{\theta}$	$\alpha\theta$	$\alpha\theta^2$
Normal $(\mu = 0, \sigma^2$ $= 1, \text{standard normal})$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2)$ $-\infty < x < \infty$	$\exp(\mu t + \frac{1}{2}\sigma^2 t^2)$ $-\infty < t < \infty$	μ	σ^2

Descriptions of distributions

	Random Phenomena
Binomial $(n = 1, \text{Bernoulli})$	The total number of successes in n Bernoulli trials (the order does not matter)
Negative Binomial	For a given natural number r, the number of Bernoulli trials on which the rth success is observed
Poisson	The number of occurrences that a particular event happens in a given time interval or for a given physical object that can be described by APP
Gamma $(\alpha = 1, \text{Expnential})$ $(\theta = 2, \alpha = r/2, \chi^2)$	The waiting time until the αth occurrences of a particular event for an APP
Normal $(\mu = 0, \sigma^2 = 1, \text{standard normal})$	When a large number of outcomes are observed, the outcomes have a “bell-shaped” relative frequency distribution.

Univariate normal distribution

Definition

A continuous RV X is said to be normal or Gaussian if it has a pdf of the form.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2} \cdot \frac{(x-\mu)^2}{\sigma^2}), \quad -\infty < x < \infty$$

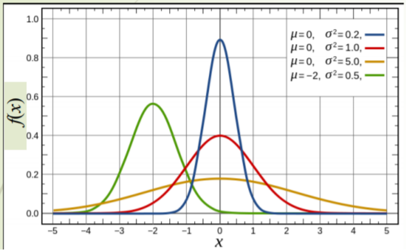
where μ and σ^2 are two parameters characterizing the normal distribution. Briefly, $X \sim N(\mu, \sigma^2)$

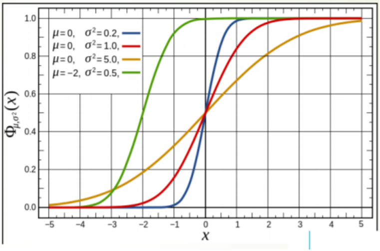
$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right) dx = 1$$

Univariate normal distribution

1. Mgf:
$$M(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right), \quad t \in R$$
2. Y is said to be a standard normal distribution if
$$Y \sim N(0,1) \Leftrightarrow f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$
3. for $Y \sim N(0,1)$, $P(a \leq Y \leq b) = \Phi(b) - \Phi(a)$
4. if $X \sim N(\mu, \sigma^2)$, $(X - \mu)/\sigma \sim N(0,1)$ and
$$\begin{aligned} P(a \leq X \leq b) &= P(\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \end{aligned}$$
5. if $Y \sim N(0,1)$, $Y^2 \sim \chi^2(1)$

Univariate normal distribution





$$P(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$\Phi(-z) = 1 - \Phi(z)$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026

The end

- Arrnagement of the midterm exam
- **Any questions?**

Motivations for the double integral

To prove

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(s-\mu)^2}{\sigma^2}\right) ds = 1$$

we need to handle double integrals!

Proof of $\int_{-\infty}^{\infty} f(s)ds = 1$

Let

$$I = \int_{-\infty}^{\infty} f(s)ds = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(s-\mu)^2}{\sigma^2}\right) ds \stackrel{?}{=} 1$$

Take a coordinate change $x = \frac{s-\mu}{\sigma}$

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \stackrel{?}{=} 1.$$

Since $I > 0$, then if $I^2 = 1$, then $I = 1$.

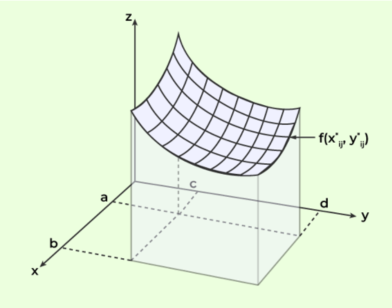
$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy$$

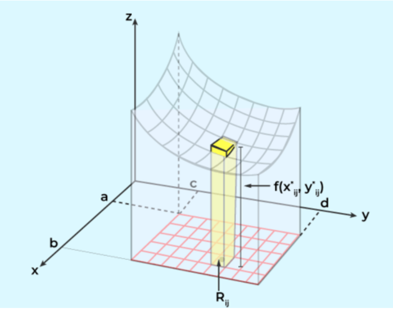
Geometric interpretation of double integral

For $f(x, y) \geq 0$

$$\text{Volume} = \iint_A f(x, y)dxdy$$

The double integral calculates the volume under the surface $f(x, y)$ over the definition domain of $f(x, y)$ or any region of interest A.





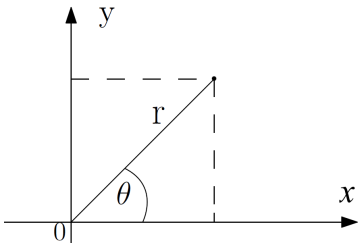
Geometric interpretation of double integral in polar coordinate system

Take a coordinate change

$$x = r\cos\theta$$

$$y = r\sin\theta$$

(polar coordinate)



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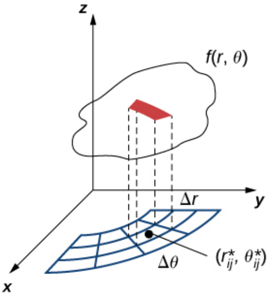

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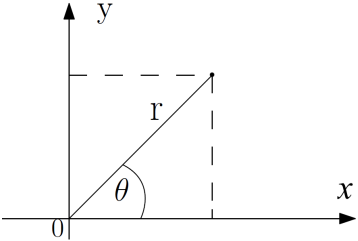
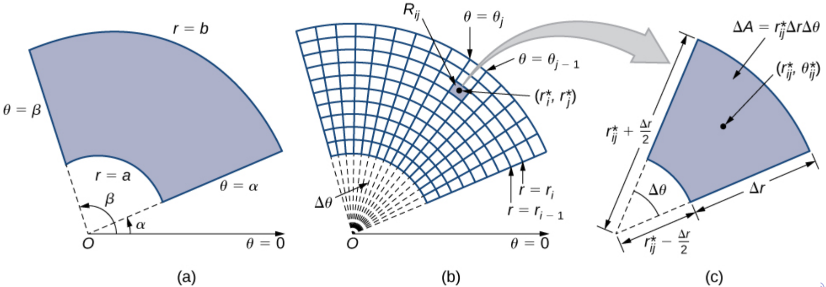


The problem lies in to how to calculate the area of small trapezoid.



Proof of $\int_{-\infty}^{\infty} f(s)ds = 1$

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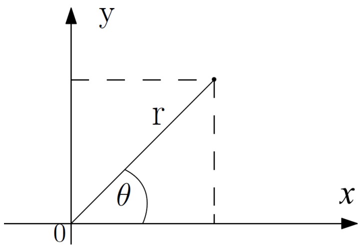

$$I^2 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} r dr d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{\infty} e^{-\frac{r^2}{2}} d\frac{r^2}{2}$$
$$= \frac{1}{2\pi}\cdot 2\pi \cdot (-1)\cdot e^{-\frac{r^2}{2}}\Big|_0^{\infty} = 1$$