7.3

Exponential Functions

Generalization of Power to an Irrational Number

For a rational number $r = p/q$, where p and q are integers and $q > 0$, then

$$e^r = \sqrt[q]{e^p}$$

What if r is irrational? Let's take the logarithm of e^r,

$$\ln e^r = r \ln e = r \times 1 = r\,, \qquad \text{(since } \ln e = 1\text{)}$$

If we apply the inverse function of ln (.) to the above, denote it by exp (.), then for r rational,

$$e^r = \ln^{-1} (\ln e^r) = \exp (\ln e^r) = \exp (r)\,.$$

Since the inverse logarithmic function can be applied to any real number, rational or irrational, exp (r) allows us to generalize the power to an irrational number.

Likewise, since $a = e^{\ln a}$, we can think of a^x as $(e^{\ln a})^x = e^{x \ln a}$.

The Inverse of ln x and the Number e

The function ln x, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1} x$ is the graph of ln x reflected across the line $y = x$. As you can see in Figure 7.10,

$$\lim_{x \to \infty} \ln^{-1} x = \infty \qquad \text{and} \qquad \lim_{x \to -\infty} \ln^{-1} x = 0.$$

The function $\ln^{-1} x$ is usually denoted as $\exp x$.

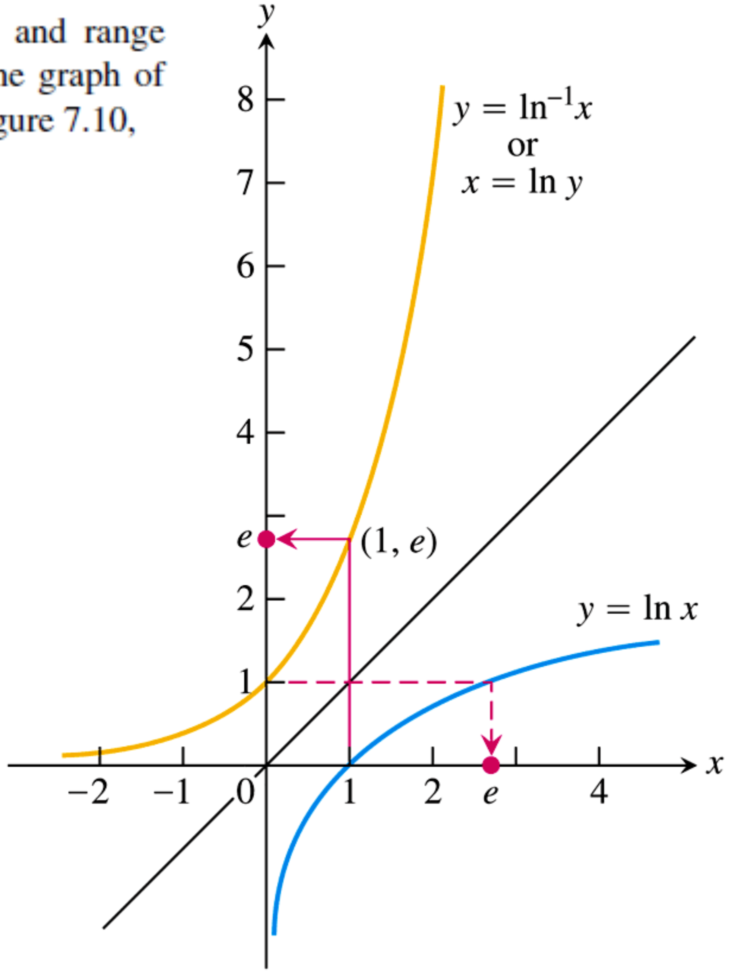

FIGURE 7.10 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

DEFINITION For every real number x, we define the **natural exponential function** to be $e^x = \exp x$.

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0)$$
$$\ln(e^x) = x \qquad (\text{all } x)$$

Since

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1$$

we have $e = \ln^{-1}(1) = \exp(1) = 2.72$.

Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

EXAMPLE 2 A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m?

Solution Suppose the point of tangency occurs at the unknown point $x = a > 0$. Then we know that the point $(a, \ln a)$ lies on the graph and that the tangent line at that point has slope $m = 1/a$ (Figure 7.11). Since the tangent line passes through the origin, its slope is

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

Setting these two formulas for m equal to each other, we have

$$\begin{aligned} \frac{\ln a}{a} &= \frac{1}{a} \\ \ln a &= 1 \\ e^{\ln a} &= e^1 \\ a &= e \\ m &= \frac{1}{e}. \end{aligned}$$

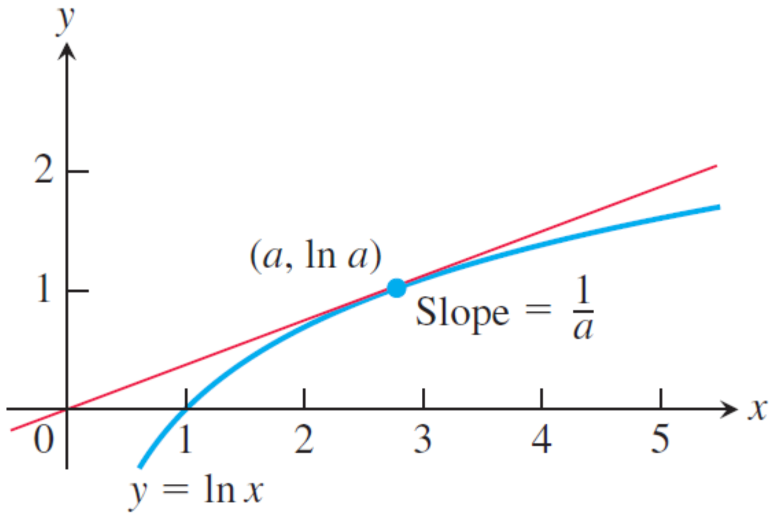

FIGURE 7.11 The tangent line intersects the curve at some point $(a, \ln a)$, where the slope of the curve is $1/a$ (Example 2).

The Derivative and Integral of e^x

According to Theorem 1, the natural exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero. We calculate its derivative using the inverse relationship and Chain Rule:

$$\ln(e^x) = x \qquad \text{Inverse relationship}$$

$$\frac{d}{dx}\ln(e^x) = 1 \qquad \text{Differentiate both sides.}$$

$$\frac{1}{e^x}\cdot\frac{d}{dx}(e^x) = 1 \qquad \text{Eq. (2), Section 7.2, with } u = e^x$$

$$\frac{d}{dx}e^x = e^x. \qquad \text{Solve for the derivative.}$$

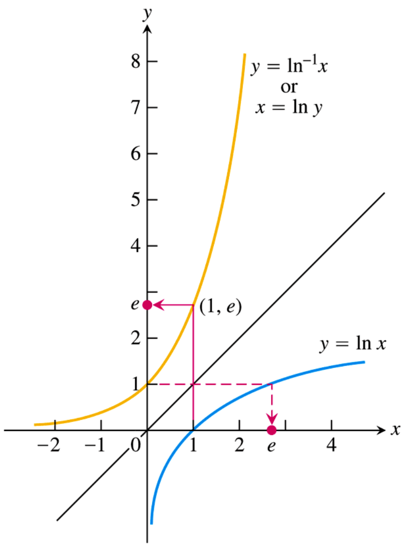

FIGURE 7.10 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

That is, for $y = e^x$, we find that $dy/dx = e^x$ so the natural exponential function e^x is its own derivative. Moreover, if $f(x) = e^x$, then $f'(0) = e^0 = 1$. This means that the natural exponential function e^x has slope 1 as it crosses the y-axis at $x = 0$.

The Chain Rule extends the derivative result for the natural exponential function to a more general form involving a function $u(x)$:

If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}. \qquad (2)$$

EXAMPLE 3 We find derivatives of the exponential using Equation (2).

(a) $\frac{d}{dx}(5e^x) = 5\frac{d}{dx}e^x = 5e^x$

(b) $\frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (2) with $u = -x$

(c) $\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$ Eq. (2) with $u = \sin x$

(d) $\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}} \cdot \frac{d}{dx}\left(\sqrt{3x+1}\right)$ Eq. (2) with $u = \sqrt{3x+1}$

$$= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x+1)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{3x+1}}e^{\sqrt{3x+1}}$$

Since e^x is its own derivative, it is also its own antiderivative. So the integral equivalent of Equation (2) is the following.

The general antiderivative of the exponential function

$$\int e^u \, du = e^u + C$$

EXAMPLE 4

(a) $$\int_0^{\ln 2} e^{3x}\,dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}du$$

$u = 3x,\quad \frac{1}{3}du = dx,\quad u(0) = 0,$
$u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$

$$= \frac{1}{3}\int_0^{\ln 8} e^u\,du$$

$$= \frac{1}{3}e^u\Big]_0^{\ln 8}$$

$$= \frac{1}{3}(8 - 1) = \frac{7}{3}$$

(b) $$\int_0^{\pi/2} e^{\sin x}\cos x\,dx = e^{\sin x}\Big]_0^{\pi/2}$$

Antiderivative from Example 3c

$$= e^1 - e^0 = e - 1$$

THEOREM 3 For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^r = e^{rx_1}$, if r is rational

Proof of Law 1 Let $y_1 = e^{x_1}$ and $y_2 = e^{x_2}$. Then

$$
\begin{aligned}
x_1 &= \ln y_1 \quad \text{and} \quad x_2 = \ln y_2 && \text{Inverse equations} \\
x_1 + x_2 &= \ln y_1 + \ln y_2 \\
&= \ln y_1 y_2 && \text{Product Rule for logarithms} \\
e^{x_1+x_2} &= e^{\ln y_1 y_2} && \text{Exponentiate.} \\
&= y_1 y_2 && e^{\ln u} = u \\
&= e^{x_1} e^{x_2}. && \blacksquare
\end{aligned}
$$

Proof of Law 4 Let $y = (e^{x_1})^r$. Then

$$
\begin{aligned}
\ln y &= \ln (e^{x_1})^r \\
&= r \ln (e^{x_1}) && \text{Power Rule for logarithms, rational } r \\
&= rx_1 && \ln e^u = u \text{ with } u = x_1
\end{aligned}
$$

Thus, exponentiating each side,

$$
y = e^{rx_1}. \qquad e^{\ln y} = y \qquad \blacksquare
$$

Laws 2 and 3 follow from Law 1. Like the Power Rule for logarithms, Law 4 holds for all real numbers r.

The General Exponential Function a^x

Since $a = e^{\ln a}$ for any positive number a, we can think of a^x as $(e^{\ln a})^x = e^{x \ln a}$. We therefore use the function e^x to define the other exponential functions, which allow us to raise *any* positive number to an irrational exponent.

DEFINITION For any numbers $a > 0$ and x, the **exponential function with base a** is

$$a^x = e^{x \ln a}.$$

Theorem 3 is also valid for a^x, the exponential function with base a.

DEFINITION For any $x > 0$ and for any real number n,

$$x^n = e^{n \ln x}.$$

General Power Rule for Derivatives

For $x > 0$ and any real number n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $x \leq 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

Proof Differentiating x^n with respect to x gives

$$\begin{aligned}
\frac{d}{dx}x^n &= \frac{d}{dx}e^{n \ln x} && \text{Definition of } x^n,\ x > 0 \\
&= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) && \text{Chain Rule for } e^u\text{, Eq. (2)} \\
&= x^n \cdot \frac{n}{x} && \text{Definition and derivative of } \ln x \\
&= nx^{n-1}. && x^n \cdot x^{-1} = x^{n-1}
\end{aligned}$$

In short, whenever $x > 0$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

For $x < 0$, if $y = x^n$, y', and x^{n-1} all exist, then

$$\ln|y| = \ln|x|^n = n \ln|x|.$$

Using implicit differentiation (which *assumes* the existence of the derivative y') and Equation (3) in Section 7.2, we have

$$\frac{y'}{y} = \frac{n}{x}.$$

Solving for the derivative,

$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}.$$

It can be shown directly from the definition of the derivative that the derivative equals 0 when $x = 0$ and $n \geq 1$. This completes the proof of the general version of the Power Rule for all values of x. ■

EXAMPLE 5 Differentiate $f(x) = x^x$, $x > 0$.

Solution We cannot apply the power rule here because the exponent is the *variable* x rather than being a constant value n (rational or irrational). However, from the definition of the general exponential function we note that $f(x) = x^x = e^{x \ln x}$, and differentiation gives

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(e^{x \ln x}\right) \\
&= e^{x \ln x} \frac{d}{dx}(x \ln x) && \text{Eq. (2) with } u = x \ln x \\
&= e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right) \\
&= x^x (\ln x + 1). && x > 0
\end{aligned}
$$

■

THEOREM 4—The Number e as a Limit The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Proof If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \to 0} \ln(1+x)^{1/x} = \ln\left[\lim_{x \to 0} (1+x)^{1/x}\right]. \end{aligned}$$

$\ln 1 = 0$

ln is continuous; use Theorem 10 in Chapter 2.

Because $f'(1) = 1$, we have

$$\ln\left[\lim_{x \to 0} (1+x)^{1/x}\right] = 1$$

Therefore, exponentiating both sides we get (see Figure 7.12)

$$\lim_{x \to 0} (1+x)^{1/x} = e.$$

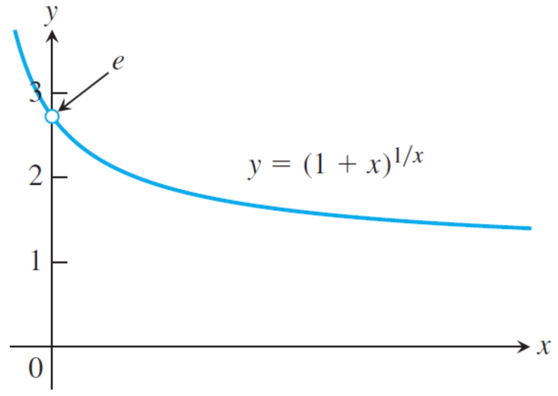

FIGURE 7.12 The number e is the limit of the function graphed here as $x \to 0$.

Approximating the limit in Theorem 4 by taking x very small gives approximations to $e \approx 2.718281828459045$

Letting $n = 1/x$, then $n \to +\infty$ as $x \to 0+$, and $n \to -\infty$ as $x \to 0-$.

Since

$$e = \lim_{x \to 0+} (1 + x)^{1/x} = \lim_{x \to 0-} (1 + x)^{1/x}$$

Therefore

$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \to -\infty} \left(1 + \frac{1}{n}\right)^n$$

The Derivative of a^u

To find this derivative, we start with the defining equation $a^x = e^{x \ln a}$. Then we have

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \qquad \frac{d}{dx}e^u = e^u \frac{du}{dx}$$

$$= a^x \ln a.$$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}. \tag{3}$$

The integral equivalent of this last result gives the general antiderivative

$$\int a^u \, du = \frac{a^u}{\ln a} + C. \tag{4}$$

From Equation (3) with $u = x$, we see that the derivative of a^x is positive if $\ln a > 0$, or $a > 1$, and negative if $\ln a < 0$, or $0 < a < 1$. Thus, a^x is an increasing function of x if $a > 1$ and a decreasing function of x if $0 < a < 1$. In each case, a^x is one-to-one. The second derivative

$$\frac{d^2}{dx^2}(a^x) = \frac{d}{dx}(a^x \ln a) = (\ln a)^2 a^x$$

is positive for all x, so the graph of a^x is concave up on every interval of the real line. Figure 7.13 displays the graphs of several exponential functions.

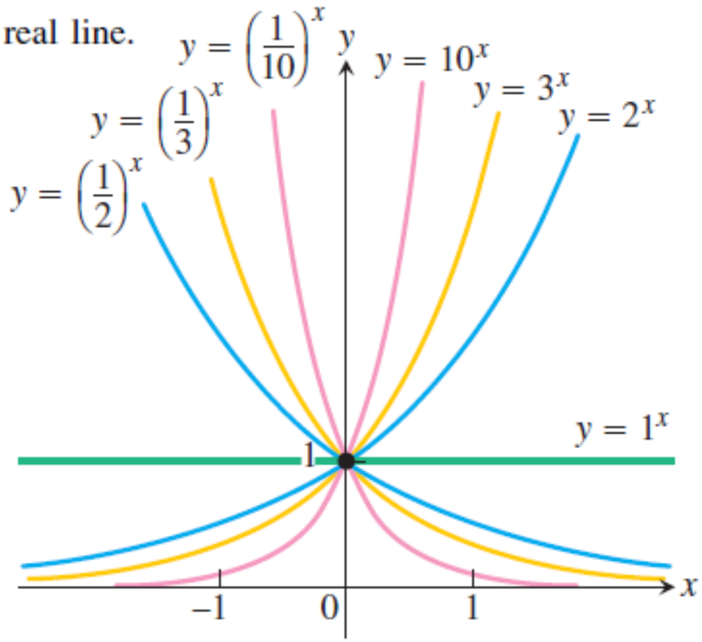

FIGURE 7.13 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

EXAMPLE 6 We find derivatives and integrals using Equations (3) and (4).

(a) $\frac{d}{dx}3^x = 3^x \ln 3$ — Eq. (3) with $a = 3, u = x$

(b) $\frac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x} \ln 3$ — Eq. (3) with $a = 3, u = -x$

(c) $\frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$ — $\ldots, u = \sin x$

(d) $\int 2^x \, dx = \frac{2^x}{\ln 2} + C$ — Eq. (4) with $a = 2, u = x$

(e) $\int 2^{\sin x} \cos x \, dx = \int 2^u \, du = \frac{2^u}{\ln 2} + C$ — $u = \sin x, du = \cos x \, dx$, and Eq. (4)

$= \frac{2^{\sin x}}{\ln 2} + C$ — u replaced by $\sin x$

Logarithms with Base a

If a is any positive number other than 1, the function a^x is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the **logarithm of x with base a** and denote it by $\log_a x$.

DEFINITION For any positive number $a \neq 1$,

$$\log_a x \text{ is the inverse function of } a^x.$$

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the 45° line $y = x$ (Figure 7.14).

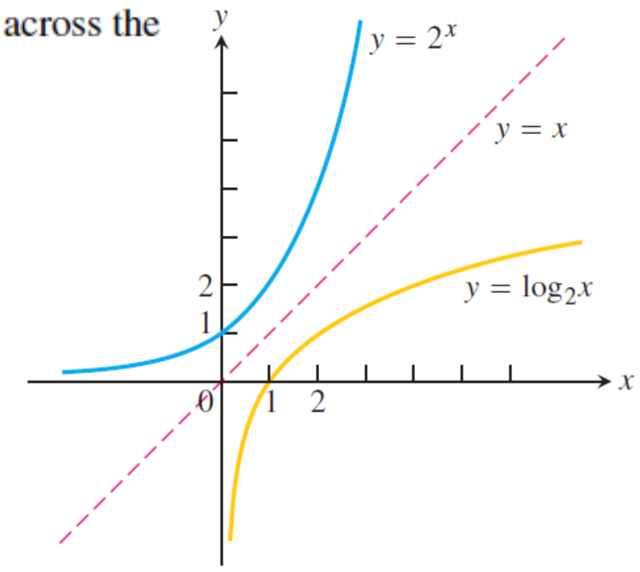

FIGURE 7.14 The graph of 2^x and its inverse, $\log_2 x$.

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*
   $$\log_a xy = \log_a x + \log_a y$$
2. *Quotient Rule:*
   $$\log_a \frac{x}{y} = \log_a x - \log_a y$$
3. *Reciprocal Rule:*
   $$\log_a \frac{1}{y} = -\log_a y$$
4. *Power Rule:*
   $$\log_a x^y = y \log_a x$$

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0)$$

$$\log_a (a^x) = x \qquad (\text{all } x)$$

The function $\log_a x$ is actually just a numerical multiple of $\ln x$. To see this, we let $y = \log_a x$ and then take the natural logarithm of both sides of the equivalent equation $a^y = x$ to obtain $y \ln a = \ln x$. Solving for y gives

$$\log_a x = \frac{\ln x}{\ln a}. \qquad (5)$$

Derivatives and Integrals Involving $\log_a x$

To find derivatives or integrals involving base a logarithms, we convert them to natural logarithms. If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}.$$

$$\boxed{\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}}$$

EXAMPLE 7

(a) $\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$

(b) $\int \frac{\log_2 x}{x}dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}dx$ $\qquad \log_2 x = \frac{\ln x}{\ln 2}$

$= \frac{1}{\ln 2}\int u\,du$ $\qquad u = \ln x, \quad du = \frac{1}{x}dx$

$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$ ■

7.5

Indeterminate Forms and L′Hôpital's Rule

Sometimes we encounter limits of the form 0/0 or $\dfrac{\infty}{\infty}$.

In such cases, we can apply *L'Hospital's Rule* (or *L'Hopital's Rule*)

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

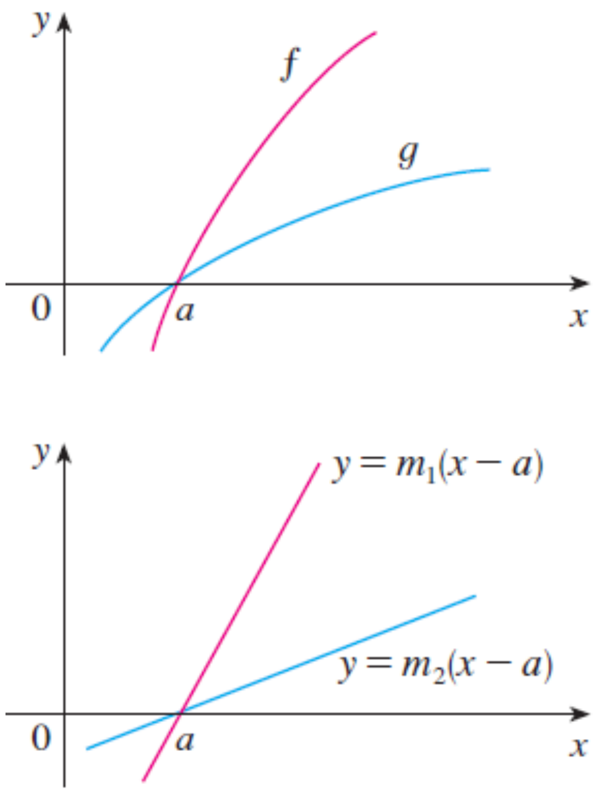

FIGURE 1

Figure 1 suggests visually why l'Hospital's Rule might be true. The first graph shows two differentiable functions f and g, each of which approaches 0 as $x \to a$. If we were to zoom in toward the point $(a, 0)$, the graphs would start to look almost linear. But if the functions actually *were* linear, as in the second graph, then their ratio would be

$$\frac{m_1(x - a)}{m_2(x - a)} = \frac{m_1}{m_2}$$

which is the ratio of their derivatives. This suggests that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

NOTE 1 L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied.

NOTE 2 L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

NOTE 3 For the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$, it is easy to see why l'Hospital's Rule is true. In fact, using the alternative form of the definition of a derivative, we have

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\displaystyle\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)} \qquad [\text{since } f(a) = g(a) = 0]$$

EXAMPLE 1 The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $$\lim_{x\to 0} \frac{3x - \sin x}{x} = \lim_{x\to 0} \frac{3 - \cos x}{1} = \left.\frac{3 - \cos x}{1}\right|_{x=0} = 2$$

(b) $$\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$$

(c) $$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$$ $\frac{0}{0}$; apply l'Hôpital's Rule.

$$= \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$= \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$$ Not $\frac{0}{0}$; limit is found.

(d) $$\lim_{x\to 0} \frac{x - \sin x}{x^3}$$ $\frac{0}{0}$; apply l'Hôpital's Rule.

$$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$= \lim_{x\to 0} \frac{\sin x}{6x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$ Not $\frac{0}{0}$; limit is found.

EXAMPLE 2 Be careful to apply l'Hôpital's Rule correctly:

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{1 + 2x} \qquad \text{Not } \frac{0}{0}$$

It is tempting to try to apply l'Hôpital's Rule again, which would result in

$$\lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2},$$

but this is not the correct limit. L'Hôpital's Rule can be applied only to limits that give indeterminate forms, and $\lim_{x\to 0}(\sin x)/(1 + 2x)$ does not give an indeterminate form. Instead, this limit is $0/1 = 0$, and the correct answer for the original limit is 0. ■

EXAMPLE 3 In this example the one-sided limits are different.

(a) $$\lim_{x\to 0^+} \frac{\sin x}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\cos x}{2x} = \infty \qquad \text{Positive for } x > 0$$

(b) $$\lim_{x\to 0^-} \frac{\sin x}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty \qquad \text{Negative for } x < 0$$ ■

EXAMPLE 4 Find the limits of these ∞/∞ forms:

(a) $\lim_{x \to \pi/2} \dfrac{\sec x}{1 + \tan x}$ **(b)** $\lim_{x \to \infty} \dfrac{\ln x}{2\sqrt{x}}$ **(c)** $\lim_{x \to \infty} \dfrac{e^x}{x^2}$.

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x \to (\pi/2)^-} \frac{\sec x}{1 + \tan x} \qquad \frac{\infty}{\infty} \text{ from the left so we apply l'Hôpital's Rule.}$$

$$= \lim_{x \to (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to (\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $$\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \to \infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0 \qquad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

(c) $$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$$

EXAMPLE 5 Find the limits of these $\infty \cdot 0$ forms:

(a) $\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right)$ **(b)** $\lim_{x \to 0^+} \sqrt{x} \ln x$

Solution

(a) $$\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right) = \lim_{h \to 0^+} \left(\frac{1}{h} \sin h \right) = \lim_{h \to 0^+} \frac{\sin h}{h} = 1$$ $\infty \cdot 0$; let $h = 1/x$.

(b) $$\lim_{x \to 0^+} \sqrt{x} \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/\sqrt{x}}$$ $\infty \cdot 0$ converted to ∞/∞

$$= \lim_{x \to 0^+} \frac{1/x}{-1/2x^{3/2}}$$ l'Hôpital's Rule applied

$$= \lim_{x \to 0^+} \left(-2\sqrt{x} \right) = 0$$

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}. \qquad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\begin{aligned}
\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x \to 0} \frac{x - \sin x}{x \sin x} && \frac{0}{0} \\
&= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} && \text{Still } \frac{0}{0} \\
&= \lim_{x \to 0} \frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2} = 0.
\end{aligned}$$

■

Indeterminate Powers

Limits that lead to the indeterminate forms 1^∞, 0^0, and ∞^0 can sometimes be handled by first taking the logarithm of the function. We use l'Hôpital's Rule to find the limit of the logarithm expression and then exponentiate the result to find the original function limit.

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x\to 0^+}(1+x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞. We let $f(x) = (1+x)^{1/x}$ and find $\lim_{x\to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln (1+x)^{1/x} = \frac{1}{x}\ln (1+x),$$

l'Hôpital's Rule now applies to give

$$\begin{aligned}
\lim_{x\to 0^+} \ln f(x) &= \lim_{x\to 0^+} \frac{\ln (1+x)}{x} && \frac{0}{0} \\
&= \lim_{x\to 0^+} \frac{\dfrac{1}{1+x}}{1} && \text{l'Hôpital's Rule applied} \\
&= \frac{1}{1} = 1.
\end{aligned}$$

Therefore, $\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e.$ ■

EXAMPLE 8 Find $\lim_{x\to\infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

$$\begin{aligned}
\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty} \frac{\ln x}{x} && \frac{\infty}{\infty} \\
&= \lim_{x\to\infty} \frac{1/x}{1} && \text{l'Hôpital's Rule applied} \\
&= \frac{0}{1} = 0.
\end{aligned}$$

Therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$ ■

The proof of l'Hôpital's Rule is based on Cauchy's Mean Value Theorem, an extension of the Mean Value Theorem that involves two functions instead of one. We prove Cauchy's Theorem first and then show how it leads to l'Hôpital's Rule.

THEOREM 6—Cauchy's Mean Value Theorem Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

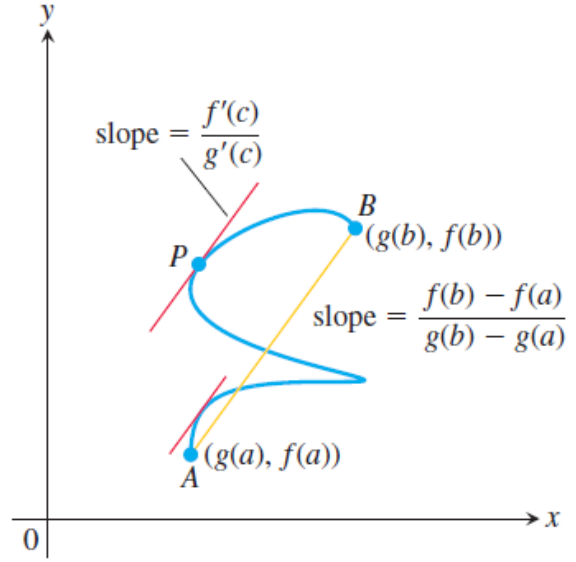

FIGURE 7.20 There is at least one point P on the curve C for which the slope of the tangent to the curve at P is the same as the slope of the secant line joining the points $A(g(a), f(a))$ and $B(g(b), f(b))$.

Proof We apply the Mean Value Theorem of Section 4.2 twice. First we use it to show that $g(a) \neq g(b)$. For if $g(b)$ did equal $g(a)$, then the Mean Value Theorem would give

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0$$

for some c between a and b, which cannot happen because $g'(x) \neq 0$ in (a, b).

We next apply the Mean Value Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)].$$

This function is continuous and differentiable where f and g are, and $F(b) = F(a) = 0$. Therefore, there is a number c between a and b for which $F'(c) = 0$. When expressed in terms of f and g, this equation becomes

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}[g'(c)] = 0$$

so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad \blacksquare$$

Proof of l'Hôpital's Rule We first establish the limit equation for the case $x \to a^+$. The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.

Suppose that x lies to the right of a. Then $g'(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from a to x. This step produces a number c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

As x approaches a, c approaches a because it always lies between a and x. Therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)},$$

which establishes l'Hôpital's Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy's Mean Value Theorem to the closed interval $[x, a]$, $x < a$. ■

7.8

Relative Rates of Growth

Growth Rates of Functions

It is often important in mathematics, computer science, and engineering to compare the rates at which functions of x grow as x becomes large. Exponential functions are important in these comparisons because of their very fast growth, and logarithmic functions because of their very slow growth. In this section we introduce the *little-oh* and *big-oh* notation used to describe the results of these comparisons. We restrict our attention to functions whose values eventually become and remain positive as $x \to \infty$.

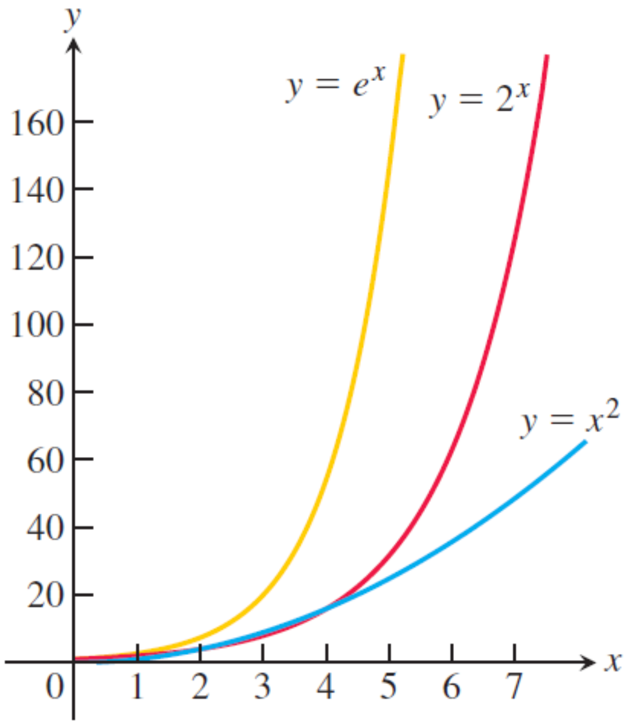

FIGURE 7.34 The graphs of e^x, 2^x, and x^2.

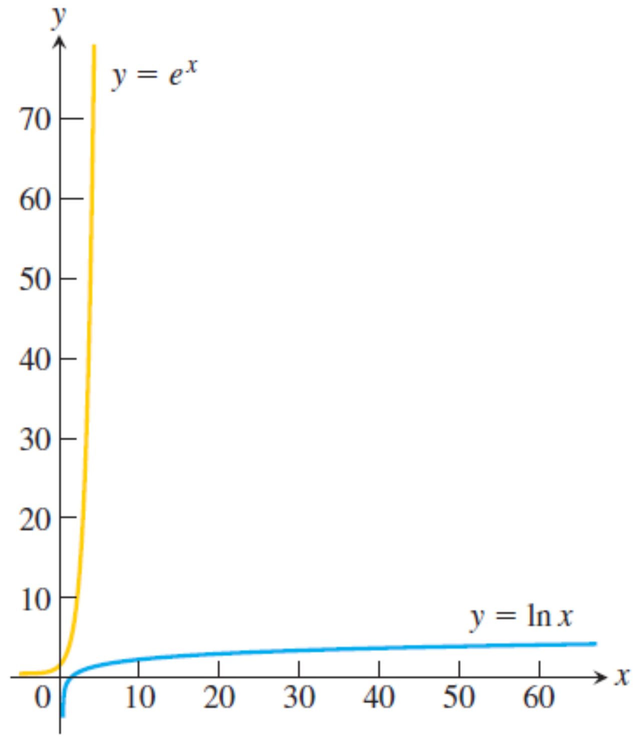

FIGURE 7.35 Scale drawings of the graphs of e^x and $\ln x$.

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

or, equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

2. f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where L is finite and positive.

EXAMPLE 1 Let's compare the growth rates of several common functions.

(a) e^x grows faster than x^2 as $x \to \infty$ because

$$\underbrace{\lim_{x\to\infty} \frac{e^x}{x^2}}_{\infty/\infty} = \underbrace{\lim_{x\to\infty} \frac{e^x}{2x}}_{\infty/\infty} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Using l'Hôpital's Rule twice

(b) 3^x grows faster than 2^x as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{3^x}{2^x} = \lim_{x\to\infty} \left(\frac{3}{2}\right)^x = \infty.$$

(c) x^2 grows faster than $\ln x$ as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{x^2}{\ln x} = \lim_{x\to\infty} \frac{2x}{1/x} = \lim_{x\to\infty} 2x^2 = \infty.$$

l'Hôpital's Rule

(d) $\ln x$ grows slower than $x^{1/n}$ as $x \to \infty$ for any positive integer n because

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/n}} = \lim_{x\to\infty} \frac{1/x}{(1/n)x^{(1/n)-1}}$$

l'Hôpital's Rule

$$= \lim_{x\to\infty} \frac{n}{x^{1/n}} = 0.$$

n is constant.

(e) As part (b) suggests, exponential functions with different bases never grow at the same rate as $x \to \infty$. If $a > b > 0$, then a^x grows faster than b^x. Since $(a/b) > 1$,

$$\lim_{x\to\infty} \frac{a^x}{b^x} = \lim_{x\to\infty} \left(\frac{a}{b}\right)^x = \infty.$$

(f) In contrast to exponential functions, logarithmic functions with different bases $a > 1$ and $b > 1$ always grow at the same rate as $x \to \infty$:

$$\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\ln x/\ln a}{\ln x/\ln b} = \frac{\ln b}{\ln a}.$$

The limiting ratio is always finite and never zero. ■

EXAMPLE 2 Show that $\sqrt{x^2 + 5}$ and $(2\sqrt{x} - 1)^2$ grow at the same rate as $x \to \infty$.

Solution We show that the functions grow at the same rate by showing that they both grow at the same rate as the function $g(x) = x$:

$$\lim_{x\to\infty} \frac{\sqrt{x^2 + 5}}{x} = \lim_{x\to\infty} \sqrt{1 + \frac{5}{x^2}} = 1,$$

$$\lim_{x\to\infty} \frac{(2\sqrt{x} - 1)^2}{x} = \lim_{x\to\infty} \left(\frac{2\sqrt{x} - 1}{\sqrt{x}}\right)^2 = \lim_{x\to\infty} \left(2 - \frac{1}{\sqrt{x}}\right)^2 = 4. \quad ■$$

DEFINITION A function f is **of smaller order than** g as $x \to \infty$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $\boldsymbol{f = o(g)}$ ("f is little-oh of g").

EXAMPLE 3 Here we use little-oh notation.

(a) $\ln x = o(x)$ as $x \to \infty$ because $\lim_{x \to \infty} \frac{\ln x}{x} = 0$

(b) $x^2 = o(x^3 + 1)$ as $x \to \infty$ because $\lim_{x \to \infty} \frac{x^2}{x^3 + 1} = 0$ ■

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \leq M,$$

for x sufficiently large. We indicate this by writing $\boldsymbol{f = O(g)}$ ("f is big-oh of g").

EXAMPLE 4 Here we use big-oh notation.

(a) $x + \sin x = O(x)$ as $x \to \infty$ because $\dfrac{x + \sin x}{x} \leq 2$ for x sufficiently large.

(b) $e^x + x^2 = O(e^x)$ as $x \to \infty$ because $\dfrac{e^x + x^2}{e^x} \to 1$ as $x \to \infty$.

(c) $x = O(e^x)$ as $x \to \infty$ because $\dfrac{x}{e^x} \to 0$ as $x \to \infty$. ■

If you look at the definitions again, you will see that $f = o(g)$ implies $f = O(g)$ for functions that are positive for x sufficiently large. Also, if f and g grow at the same rate, then

$$f = O(g) \text{ and } g = O(f)$$

Week 10

Assignment 10

7.3:
#18,24,28,29,43,48,50,59,66,74,92,100,114,116,136,142(a),143

7.5:
#5,15,20,26,27,30,32,38,40,46,54,62,65,66,68,72,80,82,85

These need to be submitted. Deadline: 10 PM, Friday, Nov 24

Required Reading (Textbook)

- Sections 7.3, 7.5, 7.8