MAT1001 Midterm Examination

Saturday, October 30, 2021

Time: 9:30 - 11:30 AM

Notes and Instructions

1. No books, no notes, no dictionaries, and no calculators.

2. The total score of this examination is 140.

*3. There are **11** questions (with parts) in total.*

*4. The symbol [**N**] at the beginning of a question indicates that the question is worth N points.*

*5. Answer all questions on the **answer book**.*

*6. Show your intermediate steps **except Questions 1, 2 and 3** — answers without intermediate steps will receive minimal (or even no) marks.*

MAT1001 Midterm Questions

1. [**15**] Multiple Choice. No explanation is required.

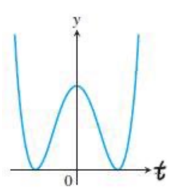
(i) $\lim_{x\to 1} 2^{\frac{3}{x-1}} =$ ________. D

A) 0
B) 1
C) ∞
D) None of the above

(i) $\lim_{x\to\infty} \dfrac{4x^2 - 3x + 2}{x^4 - 2x^2 + x - 5} =$ ________. A

A) 0
B) 4
C) ∞
D) $-\infty$

(iii) Given the graph of the velocity of a particle moving along a horizontal line, which of the following could be the position function $s = f(t)$ for the particle? D



A) $s = \sin|t|$
B) $s = \frac{t^4}{4} - 2t^2 + 4$
C) $s = \frac{3}{4}(t^2 - 1)^{\frac{2}{3}}$
D) $s = \frac{1}{5}t^5 - \frac{8}{3}t^3 + 16t$

(iv) For $x \neq 0$, we have $\dfrac{d}{dx}\left(\sqrt{|x|}\right) =$ ________. C

A) $\frac{1}{2\sqrt{|x|}}$
B) $\frac{x}{2\sqrt{|x|}}$
C) $\frac{\sqrt{|x|}}{2x}$
D) $\frac{\sqrt{|x|}}{-2x}$

(v) What is the normal line of $y = \sqrt{1-x}$ through the point $(-3, 2)$? D

A) $x + 4y - 5 = 0$
B) $4x - y - 14 = 0$
C) $x + 4y + 5 = 0$
D) $4x - y + 14 = 0$

2. [**10**] True or False (in general)? No explanation is required.

(i) Let f, g be functions defined for all real numbers and both not continuous at $x = 0$. Then $f + g$ must be non-differentiable at $x = 0$. F

(iii)

(ii) Let $y = f(x)$ be defined for all real numbers such that the left-hand derivative and the right-hand derivative at $x = a$ both exist. Then $y = f(x)$ is differentiable at $x = a$. F

eg. $f(x) = \frac{1}{x}$, $g(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{x} + 1, & x < 0 \end{cases}$

(iii) Let f and g be functions having the same domain D. If $f'(x) = g'(x)$ for all $x \in D$, then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in D$. F

(iv) If $\lim_{x \to a} \big(f(x) - g(x)\big)$ exists and $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} g(x)$ exists. T

(v) Let f be differentiable on the interval $(0, 6)$. Then $\lim_{x \to 3} f'(x)$ must exist. F

3. [**21**] Short questions: no explanation is required.

(i) Find the values of a and b that make f continuous on $\mathbb{R}$, where

$$f(x) = \begin{cases} x + 2, & \text{if } x < 2 \\ ax^2 - bx + 3, & \text{if } 2 \le x < 3 \\ 2x - a + b, & \text{if } x \ge 3 \end{cases}.$$

$\frac{1}{2}$, $\frac{1}{2}$

(ii) If $\lim_{x \to 2} \dfrac{f(x) - 5}{x - 2} = 4$, find $\lim_{x \to 2} f(x)$. 5

(iii) Let $f(x) = x^3 + 2x - 4$. Starting with $x_0 = 2$, find x_1 using Newton's method. $\frac{10}{7}$

(iv) Estimate the area under the curve $y = x^2 + (1/x)$ for x from 1 to 5 using the midpoint sum S with two subintervals.

$\frac{83}{2}$

(v) In (iv) above, is S greater than, smaller than, or equal to the exact area under the curve? Smaller

(vi) Find the function $y = f(x)$ that satisfies $y' = \sqrt{x} + \dfrac{2}{x^4} - \sin(\pi x)$ and $y(1) = \pi$. $\cos(\pi x) + \pi + \frac{1}{\pi}$

(vii) Let f and g be differentiable functions such that

$$f(3) = 3, \quad g(3) = -4, \quad f'(3) = 2\pi \quad \text{and} \quad g'(3) = 5.$$ $\frac{6}{5}\pi - 4$

Find the derivative of $\sqrt{(f(x))^2 + (g(x))^2} + 5^\pi$ at $x = 3$.

4. [**24**] Evaluate the following limits, or explain why they do not exist.

(i) $\displaystyle\lim_{x \to -6} \frac{2x + 12}{|x + 6|}$ DNE

(ii) $\displaystyle\lim_{x \to \infty} \sqrt{4x^2 + 3x} + 2x$ $+\infty$

(iii) $\displaystyle\lim_{x \to 0} \frac{|2x - 1| - |2x + 1|}{x}$ -4

(iv) $\displaystyle\lim_{x \to 0} \frac{\sin(1 - \cos x)}{x}$ 0

5. [**6**] A curve is given by $y^3 - 4\sin(xy) = 8$. Find the tangent line to the curve at $x = 0$.

$y - 2 = \frac{2}{3}x$

6. [**2+6+6=14**] Consider the function rule $f(x) = 3x^2 + \dfrac{x^2 - 1}{(x - 1)(x - \sin x)}$.

(i) Find its natural domain D (that is, the biggest domain in $\mathbb{R}$). $(-\infty, 0) \cup (0, 1) \cup (1, +\infty)$

(ii) Extend the function to have domain $\mathbb{R}$ by giving the function some values at the points missing in D (in part (i)). At which of these points can the function be extended continuously? $x = 1$

(iii) Find all asymptotes (horizontal/vertical/oblique) for $y = f(x)$.

DNE $x=0$ DNE

7. [**6+6+4+4=20**] Consider the function $f(x) = x^{2/3}(6-x)^{1/3}$ defined on $\mathbb{R}$.

(i) Find all intervals on which the function is increasing/decreasing. $(0,4)$ $(-\infty,0)\cup(4,+\infty)$

(ii) Find all intervals on which the function is concave up/concave down. $(6,+\infty)$ $(-\infty,0)\cup(0,6)$

(iii) Find all inflection points. (State their x-coordinates.) $x=6$.

(iv) Find all local extrema and global extrema by stating their x-coordinates, or explain why they do not exist.

gmin DNE gmax=lmax : $x=4$

lmin : $x=0$

8. [**6**] A projectile is fired from a canon over horizontal ground and lands a distance s away from the canon, where s is given by the equation

$$s = \frac{v_0^2}{9.8}\sin 2\alpha,$$

where v_0 is the initial velocity of the projectile when it is fired, and α is the angle to the horizontal at which it is fired. At what angle should the canon be fired to maximize the distance travelled by the projectile?

$\alpha = \frac{\pi}{4}$

9. [**8**] Let

$$f(x) = \begin{cases} x\cos\frac{\pi}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

Show that $f(x)$ is continuous but not differentiable at $x = 0$.

10. [**8**] Prove that $\sin x < x$ for all $x \in (0, 2\pi]$.

11. [**8**] A Ferris wheel with radius 10 m is rotating at a rate of one cycle every 2 minutes. During the rising process, how fast is the rider rising when her seat is 16 m above the ground? You may assume that the base of the wheel is just touching the ground level.

$\frac{2}{15}\pi$ m/s