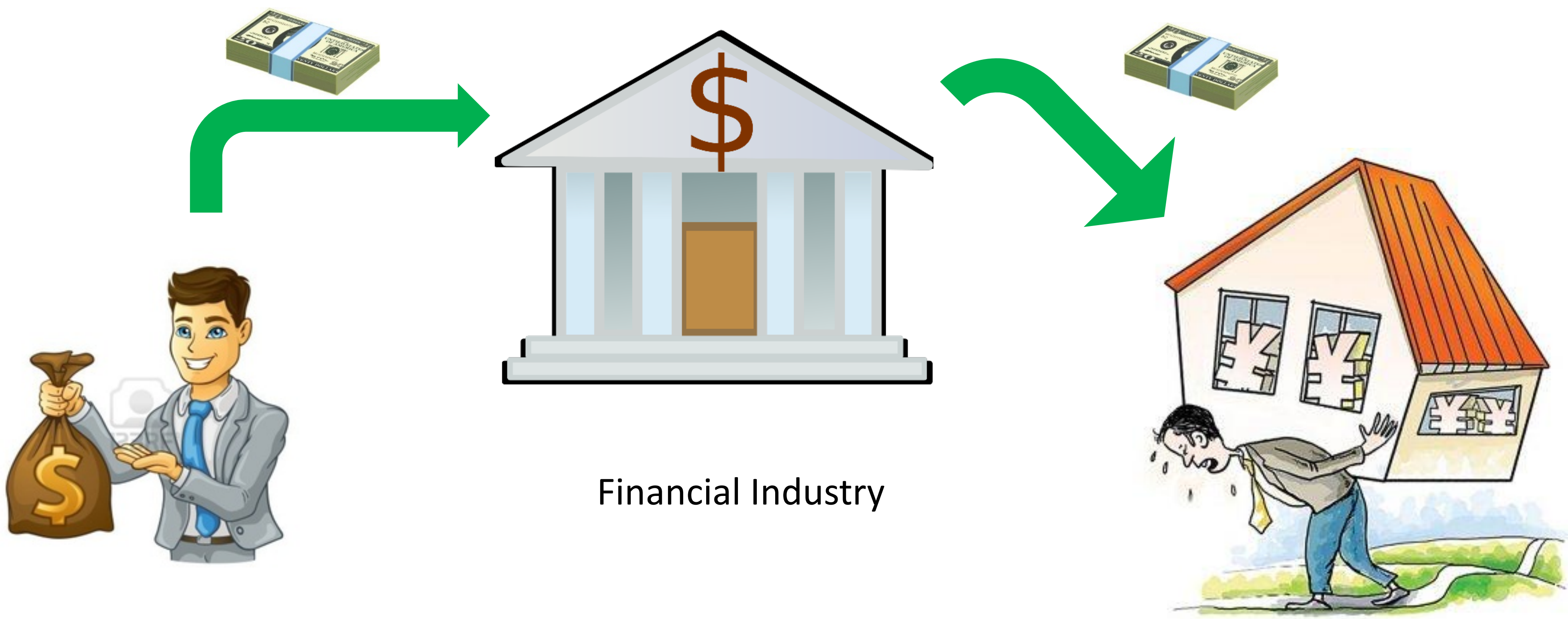
FIN2010 Financial Management
Lecture 2: Time value of money

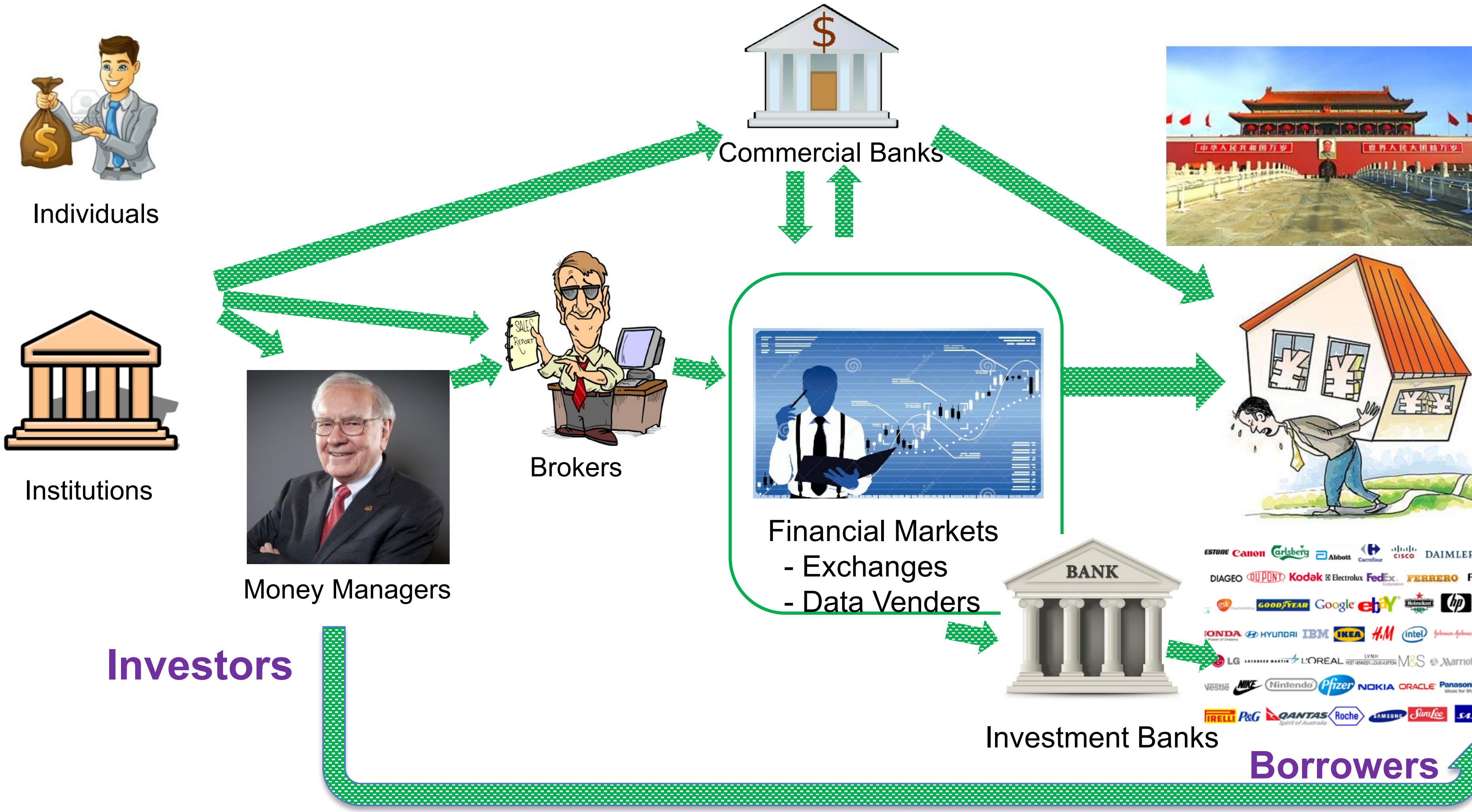
Review—What is Finance?



Investors who have money

Borrowers who need money

Review—Financial Institutions



Agenda

- What is the time value of money (TVM)?
- TVM of a single cash flow
  - APR and EAR
- TVM of cash flow streams

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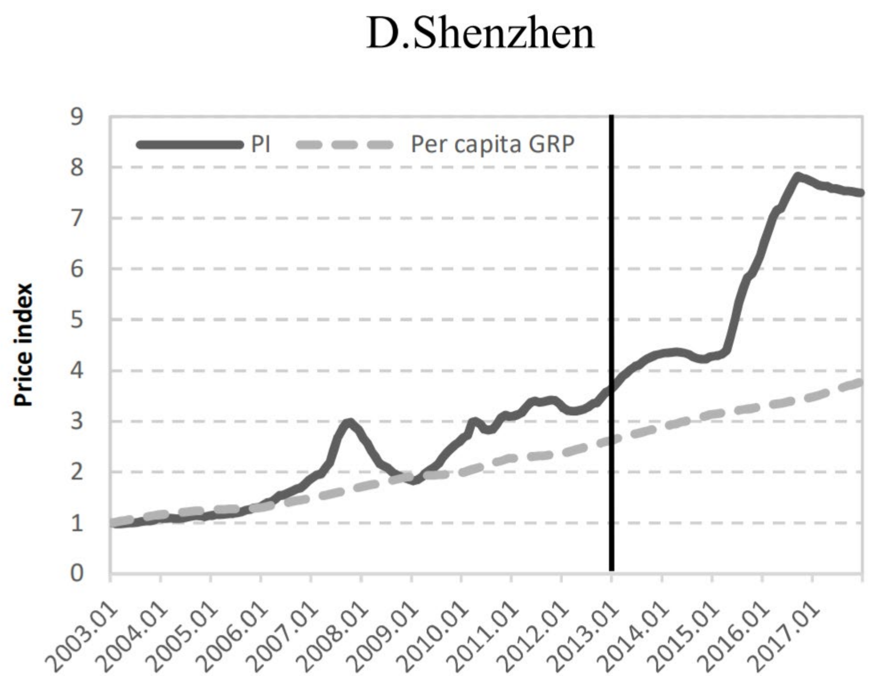
Time Value of Money

- Money has a time value
  - A dollar today is worth more than a dollar tomorrow



[Handbook of China's Financial System]

  - If I had 1 million in 2003 and used it to buy an apartment in Shenzhen, I would have assets worth of ~8 million by now.

Key Concepts

- Present Value – what a cash flow is worth at the beginning of an investment period

- Future Value – what a cash flow is worth in the future

- Interest rate – “exchange rate” between earlier money and later money
  - Discount rate
  - Cost of capital
  - Opportunity cost of capital
  - Required return

Agenda

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Future Values

- Suppose you invest 1,000 for one year at 5% per year. What is the future value in one year?
  - Interest = 1,000*5% = 50
  - Value in one year = initial investment + interest = 1,000 + 50 = 1,050
  - Future value (FV) = 1,000(1 + 5%) = 1,050

Future Values

- Suppose you leave the money in for another year. How much will you have two years from now?
  - **Compound interest**: interest is earned on both
    - the initial investment
    - the interest from previous period
  - FV with compound interest= 1050(1+0.05) = 1,102.50
  - **Simple interest**: interest is earned only on
    - the initial investment
  - FV with simple interest = 1,000 + 50 + 50 = 1,100
  - The extra 2.50 comes from the interest of .05(50) = 2.50 earned on the first interest payment

When do we use simple/compound interest?

In reality,

- Simple interest
  - Is usually charged within a year
  - Chinese banks pay simple interest on timed deposits
    - If you deposit 1000 into 工商银行 for 5 years, you will get $1000(1+2.75\% * 5) = 1137.5$

银行	活期（年利率%）	定期存款（年利率%）					
		3个月	6个月	1 年	2 年	3 年	5 年
工商银行	0.30	1.35	1.55	1.75	2.25	2.75	2.75
建设银行	0.30	1.35	1.55	1.75	2.25	2.75	2.75
交通银行	0.30	1.35	1.55	1.75	2.25	2.75	2.75
农业银行	0.30	1.35	1.55	1.75	2.25	2.75	2.75
中国银行	0.30	1.35	1.55	1.75	2.25	2.75	2.75
广发银行	0.30	1.40	1.65	1.95	2.40	3.10	3.20
光大银行	0.30	1.40	1.65	1.95	2.41	2.75	3.00
华夏银行	0.30	1.40	1.65	1.95	2.40	3.10	3.20
民生银行	0.30	1.40	1.65	1.95	2.45	3.00	3.00
平安银行	0.30	1.40	1.65	1.95	2.50	2.80	2.80

- Compound interest
  - Is charged for debt lasting over multiple years
  - If you owe money, even within a year
    - Your credit card balance compounds every day

Future Values: General Formula

- $FV = PV(1 + r)^t$
  - FV = future value
  - PV = present value
  - r = per period interest rate
  - t = number of periods

- Suppose you deposit 1,000 in a bank today for five years. The banks pays <u>compound</u> interest. How much will you have in five years?
  - $FV = 1{,}000(1+2.75\%)^5 = 1{,}145.27$
- What is the effect of compounding?
  - Simple interest = 1000 + 1000*5*2.75% = 1137.5
  - Compounding added 7.77 to the value of the investment

Interpretation of PV and FV

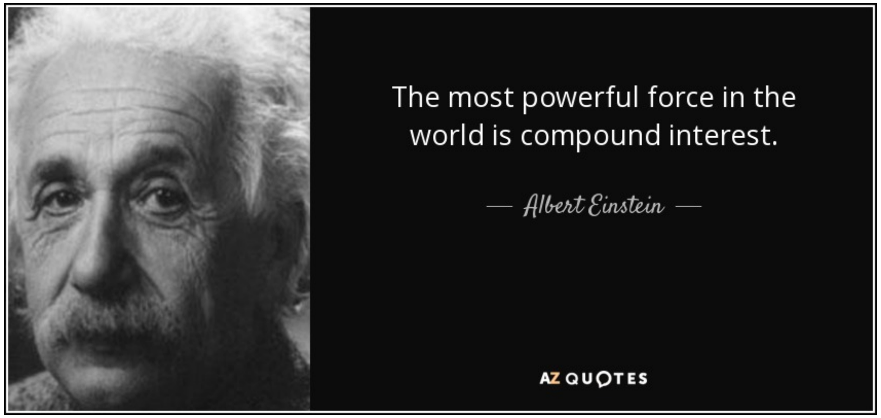
- PV and FV help us establish <u>equivalence</u> between cash now and cash in the future.

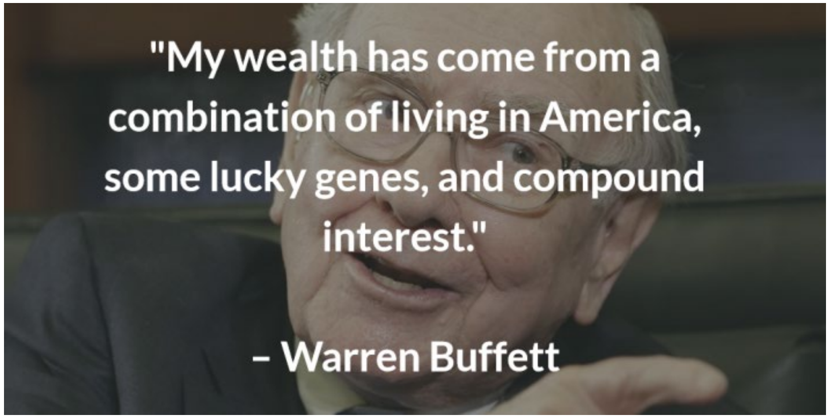
- Therefore, we can use PV and FV to compare cash flow at different points in time.
  - If the interest rate is 3%, will you prefer $1{,}000$ today or $1{,}145.27$ in five years?
  - FV of $1{,}000 = 1{,}000 * (1 + 3\%)^5 = 1{,}159.27$.
  - Today 5 years later

    $1{,}000$ = $1{,}159.27$

    $1{,}145.27$
  - Therefore, $1{,}000$ today is better than $1{,}145.27$ in five years.

The Power of Compounding





- Between 1965 and 2012, Berkshire Hathaway (CEO: Warren Buffett) earned an average return of 19.7% per year.
- Suppose you invested $1000 with Warren Buffet at the beginning of 1965, how much did you have at the end of 2012?
  - $\$1000 * (1 + 19.7\%)^{48} = \$5{,}604{,}272.2$
- When it comes to investment, stay cool and let time be your best friend!

The Power of Compounding

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- Not understanding the compounding interest has cost many people fortune and even lives!

Solve for PV, t, r

- Start with the basic equation: $FV = PV(1+r)^t$
- $PV = FV\ (1+r)^{-t}$
- $R = \left(\frac{FV}{PV}\right)^{1/t} - 1$
- $t = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1+r)}$

Present Values-Example

- Suppose you need $95,000 to pay your tuition next year. If you can earn 7% annually, how much do you need to invest today?
  - $PV = 95{,}000(1.07)^{-1} = 88785.05$

Discount Rate-Example

- Li Ka-shing is famous for "hoarding" land for profit. His company bought 1.64 million m^2 of land (Yangjiashan) in Chongqing for 2.45B in 2007. His company offered to sell 1.03 million m^2 of the land (mostly undeveloped) for 20B in 2018. What is the annual appreciate rate of the land? Assume the land price is the same for the entire area.

  - Original cost of land for 1.03 million m^2 = $2.45\text{B} * \frac{1.03}{1.64} = 1.54\text{B}$
  - $r = \left(\frac{20}{1.54}\right)^{1/11} - 1 = 26.3\%$

Number of Periods – Example

- Suppose you want to buy a new house. You currently have $15,000, and you need $20,000 for the down payment. You can earn 7.5% per year. How long does it take to save enough for the down payment?
  - $t = \ln(20{,}000/15{,}000)/\ln(1.075) = 3.98$ years

Agenda

- What is the time value of money (TVM)?
- TVM of a single cash flow
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- TVM of cash flow streams

Annual Percentage Rate (APR)

- Interests can be charged in any frequency
  - Credit card: daily
  - Home mortgage: monthly
  - Bonds: semiannually

- Interest rate is required by law to be quoted in annual frequency, the so-called annual percentage rate (APR)
  - APR = per period rate * the number of periods per year (m)
  - So, per period rate = APR / number of periods per year (m)
  - $FV = PV * \left(1 + \frac{APR}{m}\right)^{m*n}$

  where n is the number of years

APR vs. Period Rate

- What is the APR if the monthly rate is 0.5%?
  - 0.5% * 12 = 6%

- A 6% bond pays interests semiannually. What is the amount of each interest?
  - 6% / 2 = 3%

- Currently the home mortgage rate is 9%, and interest is charged monthly. What is the interest rate per month?
  - 9% / 12 = 0.75%

Effective Annual Rate (EAR) vs. APR

Annual Percentage Rate (APR) for Purchases	18.99% when you open your account. This APR will vary with the market based on the Prime Rate.
APR for Balance Transfers	**18.99%**. This APR will vary with the market based on the Prime Rate.
APR for Cash Advances	**25.24%**. This APR will vary with the market based on the Prime Rate.
Penalty APR and When it Applies	Up to **29.99%**, based on your creditworthiness. This APR will vary with the market based on the Prime Rate. This APR may be applied to your account if you: 1. Make a late payment; 2. Go over your credit limit; or 3. Make a payment that is returned. **How Long Will the Penalty APR Apply?** If your APRs are increased for any of these reasons, the Penalty APRs may apply indefinitely.

If compounded annually, annual interest for a \$100 debt is \$18.99

If compounded semiannually, interest is \$100*$[(1+0.1899/2)^2 -1]$= \$19.89

If compounded monthly, interest is \$100*$[(1+0.1899/12)^{12} -1]$= \$20.73

If compounded daily, interest is \$100*$[(1+0.1899/365)^{365} -1]$= \$20.91

- **Effective annual rate**: the actual interest rate accounting for compounding that occurs during the year

$$1 + EAR = \left(1 + \frac{APR}{m}\right)^m$$

where m is the compounding frequency

EAR Example—P2P Lending

- $APR = 30\% * 52 = 1560\%$
- $EAR = (1 + 30\%)^{52} - 1 = 841499.4$

EAR-Continuous Compounding

- Sometimes investments or loans are figured based on continuous compounding. In other words, m goes to infinity
- $EAR = \lim_{m \to \infty} \left(1 + \frac{APR}{m}\right)^m - 1 = e^{APR} - 1$

- Example: What is the effective annual rate of 7% compounded continuously?
  - $EAR = e^{7\%} - 1 = 7.25\%$

When should I use EAR vs. APR?

- $FV = PV(1 + r)^t$
  - r should be the effective rate in each period

- If you are given an APR:
  - APR always comes in a pair: {APR, compounding frequency}
  - APR only tells you the interest rate in each compounding period
    Compounding period interest $= APR/m$
  - For a period of any other lengths, do not use APR in this way. Instead, express time in terms of numbers of compounding periods:

$$FV = PV(1 + APR/m)^{tm}$$

- If you are given an EAR:
  - $FV = PV(1 + EAR)^t$, t expressed in terms of years

EAR vs. APR Example

12% APR, monthly compounding ↔ 12.68% EAR

- What is the relationship between the two numbers?
  - $1 + 12.68\% = (1 + 12\%/12)^{12}$
- What is the interest rate for a 2-month period?
  - Using APR: 1-month interest rate $r_{1m} = 1\%$,
    so $r_{2m} = (1 + r_{1m})^2 - 1 = 2.01\%$
  - Using EAR: $r_{2m} = (1 + 12.68\%)^{2/12} - 1 = 2.01\%$
- Is $r_{1m} * 12 = r_{2m} * 6$?
  - No!
- What is the interest rate for a half-month period?
  - Using APR: so $r_{0.5m} = (1 + r_{1m})^{1/2} - 1 = 0.499\%$

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Cash Flows Streams

- Basic principle:
  - The PV (FV) of a stream of cash flows equals to the sum of PV (FV) of individual cash flows.
- A useful tool: timeline
  - A linear representation of the timing and amount of cash flows.
  - Lend out $1,000 today and then receive $200 in the first period, $300 in the second period, $600 in the third period.

Time period	0	1	2	3
Cash flow	-\$1,000	\$200	\$300	\$600

Cash Flows Streams– FV Example

- You think you will be able to deposit $4,000 at the end of each of the next three years in a bank account paying 8 percent interest.
  - You currently have $7,000 in the account.
  - How much will you have in three years?
  - How much will you have in four years?

- Find the value at year 3 of each cash flow and add them together
  - Today (year 0): FV = $7000(1.08)^3$ = 8,817.98
  - Year 1: FV = $4,000(1.08)^2$ = 4,665.60
  - Year 2: FV = 4,000(1.08) = 4,320
  - Year 3: value = 4,000
  - Total value in 3 years = 8,817.98 + 4,665.60 + 4,320 + 4,000 = 21,803.58

  - Value at year 4 = 21,803.58(1.08) = 23,547.87

Cash Flows Streams– PV Example

- You are offered an investment that will pay you $200 in one year, $400 the next year, $600 the following year, and $800 at the end of the fourth year. You can earn 12% on every similar investment. What is the most you should pay for this one?

- Find the PV of each cash flows and add them
  - Year 1 CF: $200 / (1.12)^1 = 178.57$
  - Year 2 CF: $400 / (1.12)^2 = 318.88$
  - Year 3 CF: $600 / (1.12)^3 = 427.07$
  - Year 4 CF: $800 / (1.12)^4 = 508.41$
  - Total PV = 178.57 + 318.88 + 427.07 + 508.41 = 1,432.93

Standardized Cash Flow Streams

- Annuity – finite series of equal payments that occur at regular intervals
  - If the first payment occurs at the end of the period, it is called an *ordinary annuity*
  - If the first payment occurs at the beginning of the period, it is called an *annuity due*
    - Examples: rent payment, tuition payment

- Perpetuity – infinite series of payments that occur at regular intervals
  - Equal payments: level perpetuity
  - Payments grow at a constant rate: growth perpetuity
    - Example: stock dividend

Review – Geometric Series

Sum of geometric series $a + ax + ax^2 + \cdots + ax^{n-1}$:

$$sum = a\frac{1 - x^n}{1 - x}$$

Notes:

- a : the first term of the series
- n: the number of terms in total
- x: common ratio

Infinite geometric series $a + ax + ax^2 + \cdots$:

$$sum = \frac{a}{1 - x}, |x| < 1$$

Standardized Cash Flow Streams-Examples

a.

Time period	0	1	2	3	4	5	6
Cash flow		\$100	\$100	\$100	\$100	\$100	\$100

b.

Time period	0	1	2	3	4	5	6
Cash flow	\$100	\$100	\$100	\$100	\$100	\$100	

c.

Time period	0	1	2	3	4	5	6
Cash flow		\$100	\$200	\$100	\$100	\$100	\$100

d.

Time period	0	1	2	3	4	5	6
Cash flow		\$100		\$100	\$100	\$100	\$100

e.

Time period	0	1	2	3	...	t	...
Cash flow		\$100	\$100	\$100		\$100	...

f.

Time period	0	1	2	3	...
Cash flow		100	200	400	...

a. An ordinary annuity from 0 to 6. An annuity due from 1-7
b. An annuity due from 0-6
c. Complex cash flows from 0-6. An annuity due from 3-7
d. Complex cash flows from 0-6. An ordinary annuity from 2-6
e. Level perpetuity.
f. Growth perpetuity

Annuity

t periods, the amount of each cash flow(PMT), periodic interest rate(r), number of periods per year (m).

Time period	0	1	2	3	...	t-1	t
Cash flow		PMT	PMT	PMT	...	PMT	PMT

$$
\begin{aligned}
FV &= PMT(1+r)^{t-1} + PMT(1+r)^{t-2} + ... + PMT(1+r)^{0} \\
&= PMT[(1+r)^{t-1} + (1+r)^{t-2} + ... + (1+r)^{0}] \quad \text{Inside the bracket: a geometric sequence} \\
&= PMT[\frac{(1+r)^{t}-1}{r}]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{PV} &= \mathrm{PMT} * \frac{1}{(1+r)^1} + \mathrm{PMT} * \frac{1}{(1+r)^2} + \mathrm{PMT} * \frac{1}{(1+r)^3} + ... + \mathrm{PMT} * \frac{1}{(1+r)^t} \\
&= \mathrm{PMT}[\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + ... + \frac{1}{(1+r)^t}]: \quad \text{Geometric sequence} \\
&= \mathrm{PMT} * \frac{1-\frac{1}{(1+r)^t}}{r}
\end{aligned}
$$

Annuities Due

Time period	0	1	2	3	...	t-1	t
Cash flow	PMT	PMT	PMT	PMT	...	PMT	

$$
\begin{aligned}
\mathrm{FV} &= \mathrm{PMT} * (1+r)^t + \mathrm{PMT} * (1+r)^{t-1} + \mathrm{PMT} * (1+r)^{t-2} + \ldots + \mathrm{PMT} * (1+r)^1 \\
&= \mathrm{PMT}[(1+r)^t + (1+r)^{t-1} + (1+r)^{t-2} + \ldots + (1+r)^1] \\
&= \mathrm{PMT} * \frac{(1+r)^t - 1}{r}(1+r) \\
&= \mathrm{FV}_{OA}(1+r)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{PV} &= \mathrm{PMT} * \frac{1}{(1+r)^0} + \mathrm{PMT} * \frac{1}{(1+r)^1} + \mathrm{PMT} * \frac{1}{(1+r)^2} + \ldots + \mathrm{PMT} * \frac{1}{(1+r)^{t-1}} \\
&= \mathrm{PMT}[\frac{1}{(1+r)^0} + \frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^{t-1}}] \\
&= \mathrm{PMT} * \frac{1 - \frac{1}{(1+r)^t}}{r}(1+r) \\
&= \mathrm{PV}_{OA}(1+r)
\end{aligned}
$$

Perpetuity

- **Level perpetuity**

Time period	0	1	2	3	...
Cash flow		PMT	PMT	PMT	...

$$\begin{aligned}
PV &= PMT * \frac{1}{(1+r)^1} + PMT * \frac{1}{(1+r)^2} + PMT * \frac{1}{(1+r)^3} + ... \\
&= PMT[\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + ...] \\
&= PMT * \frac{1}{r}
\end{aligned}$$

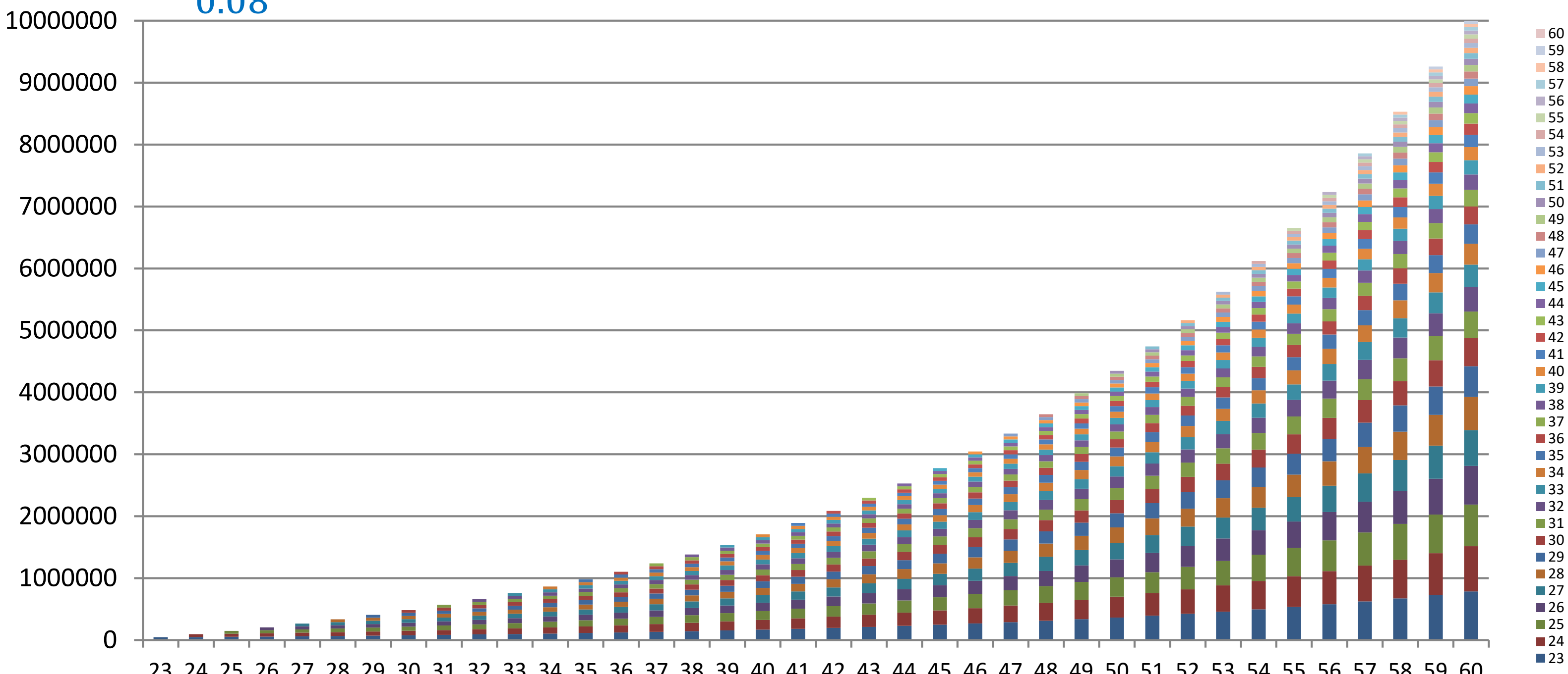
- **Growth perpetuity**

Time period	0	1	2	3	...
Cash flow		PMT_1	$PMT_1(1+g)^1$	$PMT_1(1+g)^2$	

$$\begin{aligned}
PV &= PMT * \frac{1}{(1+r)^1} + PMT(1+g) * \frac{1}{(1+r)^2} + PMT(1+g)^2 * \frac{1}{(1+r)^3} + ... \\
&= PMT[\frac{(1+g)^0}{(1+r)^1} + \frac{(1+g)^1}{(1+r)^2} + \frac{(1+g)^2}{(1+r)^3} + ...] \\
&= PMT * \frac{1}{r-g} \quad \text{(Note that it has to be } g < r\text{)}
\end{aligned}$$

Example—Retirement Planning

- Suppose you want to have 10 million when you retire at the age 60. You plan to save x on every birthday from 23 to 59. Annual interest rate = 8%.
  - $x \cdot 1.08^{37} + x \cdot 1.08^{36} + \cdots + x \cdot 1.08^{1} = 10{,}000{,}000$
  - $x \frac{1.08^{37}-1}{0.08}(1.08) = 10{,}000{,}000 \Rightarrow x = 45{,}596$



Some Takeaways

- Wealth grows exponentially
- Savings of 1.23M in total over the course of 37 years will grow to 10M
- Savings are much more valuable when you are young
  - The first saving will grow 17.25 times after 37 years
  - The first 8 (out of 37) savings accounts for 48% of terminal wealth
- You only reach the half-way mark when you are 52, 29 years into the savings plan

Summary

- For a single cash flow: $FV = PV * \left(1 + \frac{APR}{m}\right)^{m*n}$
  - APR: how interest rate is quoted. Equals to per period interest times m (#periods per year)
  - EAR: actual interest accrued in a year
- For cash flow streams: FV (or PV) is the sum of FV (or PV) of individual cash flows
- Standardized cash flow streams
  - Ordinary annuity: $PV = PMT\frac{1-(1+r)^{-n}}{r}$
  - Annuities due: $PV = PMT\left(1 + \frac{1-(1+r)^{-n+1}}{r}\right) = PMT(1+r)\frac{1-(1+r)^{-n}}{r}$
  - Level perpetuity: $PV = PMT/r$
  - Growth perpetuity: $PV = \frac{PMT}{r-g}$

Next Lecture—Time Value of Money II

- Examples questions of annuities
  - Demonstration of the financial calculator
- Amortized loan
- NPV and IRR