CHAPTER 4 APPLICATIONS OF DERIVATIVES

4.1 EXTREME VALUES OF FUNCTIONS

1. An absolute minimum at $x = c_2$, an absolute maximum at $x = b$. Theorem 1 guarantees the existence of such extreme values because h is continuous on $[a, b]$.

2. An absolute minimum at $x = b$, an absolute maximum at $x = c$. Theorem 1 guarantees the existence of such extreme values because f is continuous on $[a, b]$.

3. No absolute minimum. An absolute maximum at $x = c$. Since the function's domain is an open interval, the function does not satisfy the hypotheses of Theorem 1 and need not have absolute extreme values.

4. No absolute extrema. The function is neither continuous nor defined on a closed interval, so it need not fulfill the conclusions of Theorem 1.

5. An absolute minimum at $x = a$ and an absolute maximum at $x = c$. Note that $y = g(x)$ is not continuous but still has extrema. When the hypothesis of Theorem 1 is satisfied then extrema are guaranteed, but when the hypothesis is not satisfied, absolute extrema may or may not occur.

6. Absolute minimum at $x = c$ and an absolute maximum at $x = a$. Note that $y = g(x)$ is not continuous but still has absolute extrema. When the hypothesis of Theorem 1 is satisfied then extrema are guaranteed, but when the hypothesis is not satisfied, absolute extrema may or may not occur.

7. Local minimum at $(-1, 0)$, local maximum at $(1, 0)$.

8. Minima at $(-2, 0)$ and $(2, 0)$, maximum at $(0, 2)$.

9. Maximum at $(0, 5)$. Note that there is no minimum since the endpoint $(2, 0)$ is excluded from the graph.

10. Local maximum at $(-3, 0)$, local minimum at $(2, 0)$, maximum at $(1, 2)$, minimum at $(0, -1)$.

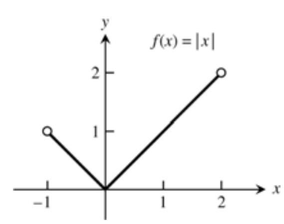
11. Graph (c), since this is the only graph that has positive slope at c.

12. Graph (b), since this is the only graph that represents a differentiable function at a and b and has negative slope at c.

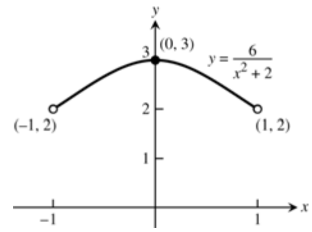
13. Graph (d), since this is the only graph representing a function that is differentiable at b but not at a.

14. Graph (a), since this is the only graph that represents a function that is not differentiable at a or b.

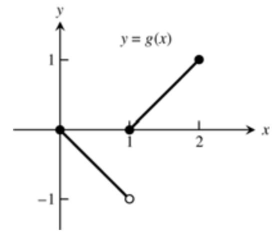
15. f has an absolute min at $x = 0$ but does not have an absolute max. Since the interval on which f is defined, $-1 < x < 2$, is an open interval, we do not meet the conditions of Theorem 1.



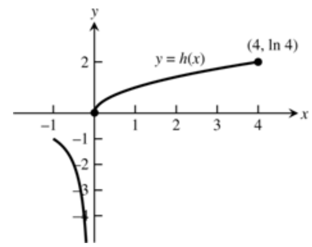
16. f has an absolute max at $x = 0$ but does not have an absolute min. Since the interval on which f is defined, $-1 < x < 1$, is an open interval, we do not meet the conditions of Theorem 1.



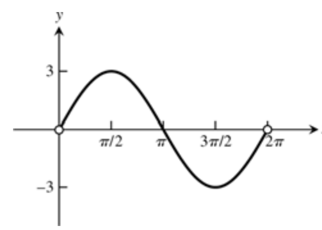
17. f has an absolute max at $x = 2$ but does not have an absolute min. Since the function is not continuous at $x = 1$, we do not meet the conditions of Theorem 1.



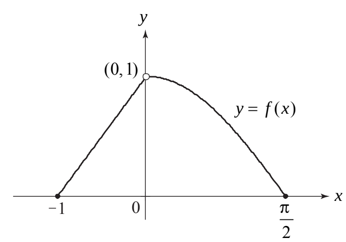
18. f has an absolute max at $x = 4$ but does not have an absolute min. Since the function is not continuous at $x = 0$, we do not meet the conditions of Theorem 1.



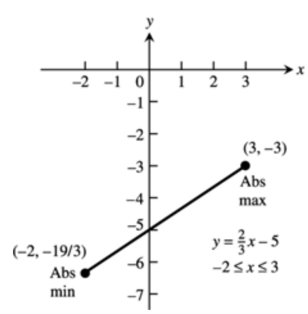
19. f has an absolute max at $x = \frac{\pi}{2}$ and an absolute min at $x = \frac{3\pi}{2}$. Since the interval on which f is defined, $0 < x < 2\pi$, is an open interval we do not meet the conditions of Theorem 1.



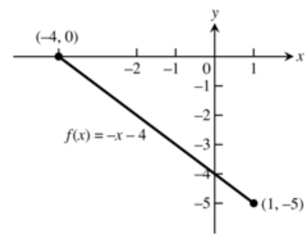
20. f has an absolute max at $x = 0$ and an absolute min at $x = \frac{\pi}{2}$ and $x = -1$ but does not have an absolute maximum. Since f is defined on a union of half-open intervals, we do not meet the conditions of Theorem 1.



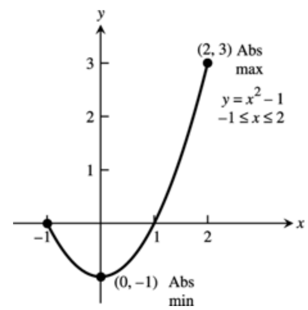
21. $f(x) = \frac{2}{3}x - 5 \Rightarrow f'(x) = \frac{2}{3} \Rightarrow$ no critical points; $f(-2) = -\frac{19}{3}$, $f(3) = -3 \Rightarrow$ the absolute maximum is -3 at $x = 3$ and the absolute minimum is $-\frac{19}{3}$ at $x = -2$



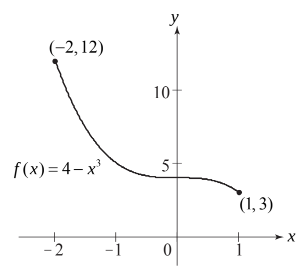
22. $f(x) = -x - 4 \Rightarrow f'(x) = -1 \Rightarrow$ no critical points; $f(-4) = 0,\ f(1) = -5 \Rightarrow$ the absolute maximum is 0 at $x = -4$ and the absolute minimum is -5 at $x = 1$



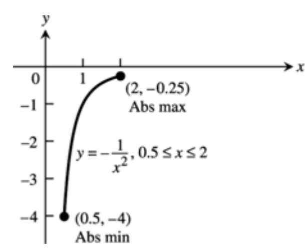
23. $f(x) = x^2 - 1 \Rightarrow f'(x) = 2x \Rightarrow$ a critical point at $x = 0;\ f(-1) = 0,\ f(0) = -1,\ f(2) = 3 \Rightarrow$ the absolute maximum is 3 at $x = 2$ and the absolute minimum is -1 at $x = 0$



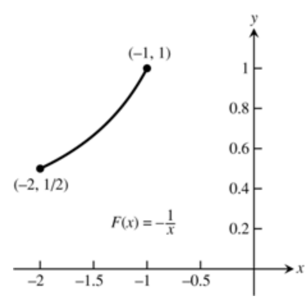
24. $f(x) = 4 - x^3 \Rightarrow f'(x) = -3x^2 \Rightarrow$ a critical point at $x = 0;\ f(-2) = 12,\ f(0) = 4,\ f(1) = 3 \Rightarrow$ the absolute maximum is 12 at $x = -2$ and the absolute minimum is 3 at $x = 1$



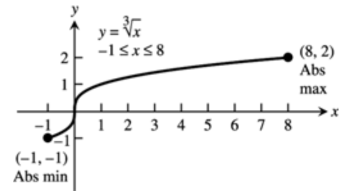
25. $F(x) = -\frac{1}{x^2} = -x^{-2} \Rightarrow F'(x) = 2x^{-3} = \frac{2}{x^3}$, however $x = 0$ is not a critical point since 0 is not in the domain; $F(0.5) = -4,\ F(2) = -0.25 \Rightarrow$ the absolute maximum is -0.25 at $x = 2$ and the absolute minimum is -4 at $x = 0.5$



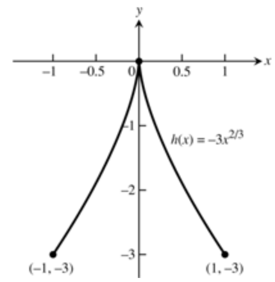
26. $F(x) = -\frac{1}{x} = -x^{-1} \Rightarrow F'(x) = x^{-2} = \frac{1}{x^2}$, however $x = 0$ is not a critical point since 0 is not in the domain; $F(-2) = \frac{1}{2},\ F(-1) = 1 \Rightarrow$ the absolute maximum is 1 at $x = -1$ and the absolute minimum is $\frac{1}{2}$ at $x = -2$



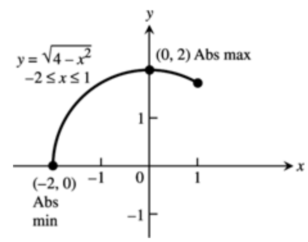
27. $h(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow h'(x) = \frac{1}{3}x^{-2/3} \Rightarrow$ a critical point at $x = 0$; $h(-1) = -1$, $h(0) = 0$, $h(8) = 2 \Rightarrow$ the absolute maximum is 2 at $x = 8$ and the absolute minimum is -1 at $x = -1$



28. $h(x) = -3x^{2/3} \Rightarrow h'(x) = -2x^{-1/3} \Rightarrow$ a critical point at $x = 0$; $h(-1) = -3$, $h(0) = 0$, $h(1) = -3 \Rightarrow$ the absolute maximum is 0 at $x = 0$ and the absolute minimum is -3 at $x = 1$ and $x = -1$

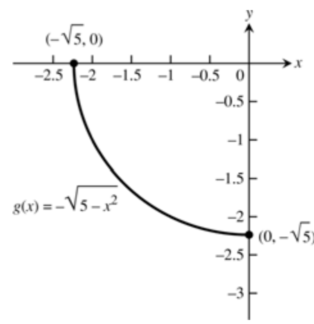


29. $g(x) = \sqrt{4 - x^2} = (4 - x^2)^{1/2}$
$\Rightarrow g'(x) = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}} \Rightarrow$ critical points at $x = -2$ and $x = 0$, but not at $x = 2$ because 2 is not in the domain;
$g(-2) = 0$, $g(0) = 2$, $g(1) = \sqrt{3} \Rightarrow$ the absolute maximum is 2 at $x = 0$ and the absolute minimum is 0 at $x = -2$

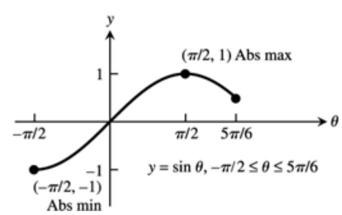


30. $g(x) = -\sqrt{5 - x^2} = -(5 - x^2)^{1/2}$
$\Rightarrow g'(x) = -\left(\frac{1}{2}\right)(5 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{5 - x^2}}$
$\Rightarrow$ critical points at $x = -\sqrt{5}$ and $x = 0$, but not at $x = \sqrt{5}$ because $\sqrt{5}$ is not in the domain;
$f\left(-\sqrt{5}\right) = 0,\ f(0) = -\sqrt{5}$

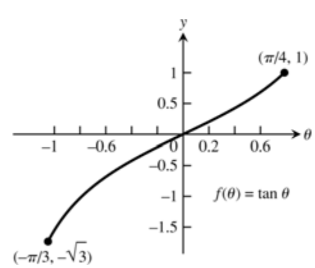
$\Rightarrow$ the absolute maximum is 0 at $x = -\sqrt{5}$ and the absolute minimum is $-\sqrt{5}$ at $x = 0$



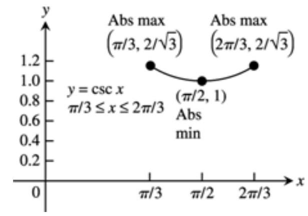
31. $f(\theta) = \sin\theta \Rightarrow f'(\theta) = \cos\theta \Rightarrow \theta = \frac{\pi}{2}$ is a critical point, but $\theta = \frac{-\pi}{2}$ is not a critical point because $\frac{-\pi}{2}$ is not interior to the domain; $f\left(\frac{-\pi}{2}\right) = -1,\ f\left(\frac{\pi}{2}\right) = 1,$ $f\left(\frac{5\pi}{6}\right) = \frac{1}{2} \Rightarrow$ the absolute maximum is 1 at $\theta = \frac{\pi}{2}$ and the absolute minimum is -1 at $\theta = \frac{-\pi}{2}$



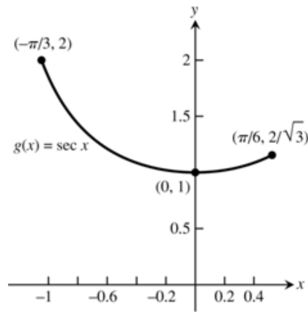
32. $f(\theta) = \tan\theta \Rightarrow f'(\theta) = \sec^2\theta \Rightarrow f$ has no critical points in $\left(\frac{-\pi}{3}, \frac{\pi}{4}\right)$. The extreme values therefore occur at the endpoints: $f\left(\frac{-\pi}{3}\right) = -\sqrt{3}$ and $f\left(\frac{\pi}{4}\right) = 1$
$\Rightarrow$ the absolute maximum is 1 at $\theta = \frac{\pi}{4}$ and the absolute minimum is $-\sqrt{3}$ at $\theta = \frac{-\pi}{3}$



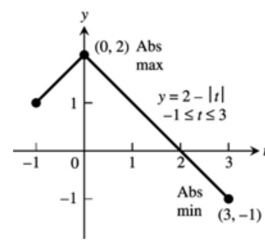
33. $g(x) = \csc x \Rightarrow g'(x) = -(\csc x)(\cot x) \Rightarrow$ a critical point at $x = \frac{\pi}{2}$; $g\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}}$, $g\left(\frac{\pi}{2}\right) = 1$, $g\left(\frac{2\pi}{3}\right) = \frac{2}{\sqrt{3}}$ $\Rightarrow$ the absolute maximum is $\frac{2}{\sqrt{3}}$ at $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$, and the absolute minimum is 1 at $x = \frac{\pi}{2}$



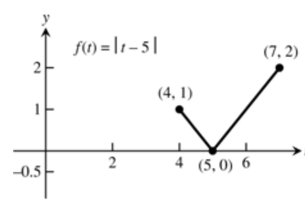
34. $g(x) = \sec x \Rightarrow g'(x) = (\sec x)(\tan x) \Rightarrow$ a critical point at $x = 0$; $g\left(-\frac{\pi}{3}\right) = 2$, $g(0) = 1$, $g\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} \Rightarrow$ the absolute maximum is 2 at $x = -\frac{\pi}{3}$ and the absolute minimum is 1 at $x = 0$



35. $f(t) = 2 - |t| = 2 - \sqrt{t^2} = 2 - (t^2)^{1/2}$ $\Rightarrow f'(t) = -\frac{1}{2}(t^2)^{-1/2}(2t) = -\frac{t}{\sqrt{t^2}} = -\frac{t}{|t|} \Rightarrow$ a critical point at $t = 0$; $f(-1) = 1$, $f(0) = 2$, $f(3) = -1 \Rightarrow$ the absolute maximum is 2 at $t = 0$ and the absolute minimum is -1 at $t = 3$



36. $f(t) = |t-5| = \sqrt{(t-5)^2} = ((t-5)^2)^{1/2}$ $\Rightarrow f'(t) = \frac{1}{2}((t-5)^2)^{-1/2}(2(t-5)) = \frac{t-5}{\sqrt{(t-5)^2}}$ $= \frac{t-5}{|t-5|} \Rightarrow$ a critical point at $t = 5$; $f(4) = 1$, $f(5) = 0$, $f(7) = 2 \Rightarrow$ the absolute maximum is 2 at $t = 7$ and the absolute minimum is 0 at $t = 5$



37. $f(x) = x^{4/3} \Rightarrow f'(x) = \frac{4}{3}x^{1/3} \Rightarrow$ a critical point at $x = 0$; $f(-1) = 1$, $f(0) = 0$, $f(8) = 16 \Rightarrow$ the absolute maximum is 16 at $x = 8$ and the absolute minimum is 0 at $x = 0$

38. $f(x) = x^{5/3} \Rightarrow f'(x) = \frac{5}{3}x^{2/3} \Rightarrow$ a critical point at $x = 0$; $f(-1) = -1$, $f(0) = 0$, $f(8) = 32 \Rightarrow$ the absolute maximum is 32 at $x = 8$ and the absolute minimum is -1 at $x = -1$

39. $g(\theta) = \theta^{3/5} \Rightarrow g'(\theta) = \frac{3}{5}\theta^{-2/5} \Rightarrow$ a critical point at $\theta = 0$; $g(-32) = -8$, $g(0) = 0$, $g(1) = 1 \Rightarrow$ the absolute maximum is 1 at $\theta = 1$ and the absolute minimum is -8 at $\theta = -32$

40. $h(\theta) = 3\theta^{2/3} \Rightarrow h'(\theta) = 2\theta^{-1/3} \Rightarrow$ a critical point at $\theta = 0$; $h(-27) = 27$, $h(0) = 0$, $h(8) = 12 \Rightarrow$ the absolute maximum is 27 at $\theta = -27$ and the absolute minimum is 0 at $\theta = 0$

41. $y = x^2 - 6x + 7 \Rightarrow y' = 2x - 6 \Rightarrow 2x - 6 = 0 \Rightarrow x = 3$. The critical point is $x = 3$.

42. $f(x) = 6x^2 - x^3 \Rightarrow f'(x) = 12x - 3x^2 \Rightarrow 12x - 3x^2 = 0 \Rightarrow 3x(4 - x) = 0 \Rightarrow x = 0$ or $x = 4$. The critical points are $x = 0$ and $x = 4$.

43. $f(x)=x(4-x)^3 \Rightarrow f'(x)=x[3(4-x)^2(-1)]+(4-x)^3=(4-x)^2[-3x+(4-x)]=(4-x)^2(4-4x)$
$=4(4-x)^2(1-x) \Rightarrow 4(4-x)^2(1-x)=0 \Rightarrow x=1$ or $x=4$. The critical points are $x=1$ and $x=4$.

44. $g(x)=(x-1)^2(x-3)^2 \Rightarrow g'(x)=(x-1)^2\cdot 2(x-3)(1)+2(x-1)(1)\cdot(x-3)^2=2(x-3)(x-1)[(x-1)+(x-3)]$
$=4(x-3)(x-1)(x-2) \Rightarrow 4(x-3)(x-1)(x-2)=0 \Rightarrow x=3$ or $x=1$ or $x=2$. The critical points are $x=1$, $x=2$, and $x=3$.

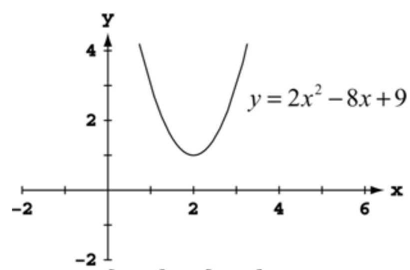
45. $y=x^2+\frac{2}{x} \Rightarrow y'=2x-\frac{2}{x^2}=\frac{2x^3-2}{x^2} \Rightarrow \frac{2x^3-2}{x^2}=0 \Rightarrow 2x^3-2=0 \Rightarrow x=1$; $\frac{2x^3-2}{x^2}=$ undefined $\Rightarrow x^2=0 \Rightarrow x=0$.
The domain of the function is $(-\infty,0)\cup(0,\infty)$, thus $x=0$ is not the domain, so the only critical point is $x=1$.

46. $f(x)=\frac{x^2}{x-2} \Rightarrow f'(x)=\frac{(x-2\cdot)2x-x^2(1)}{(x-2)^2}=\frac{x^2-4x}{(x-2)^2} \Rightarrow \frac{x^2-4x}{(x-2)^2}=0 \Rightarrow x^2-4x=0 \Rightarrow x=0$ or $x=4$; $\frac{x^2-4x}{(x-2)^2}=$ undefined
$\Rightarrow (x-2)^2=0 \Rightarrow x=2$. The domain of the function is $(-\infty,2)\cup(2,\infty)$, thus $x=2$ is not the domain, so the only critical points are $x=0$ and $x=4$

47. $y=x^2-32\sqrt{x} \Rightarrow y'=2x-\frac{16}{\sqrt{x}}=\frac{2x^{3/2}-16}{\sqrt{x}} \Rightarrow \frac{2x^{3/2}-16}{\sqrt{x}}=0 \Rightarrow 2x^{3/2}-16=0 \Rightarrow x=4$; $\frac{2x^{3/2}-16}{\sqrt{x}}$ undefined
$\Rightarrow \sqrt{x}=0 \Rightarrow x=0$. The critical points are $x=4$ and $x=0$.

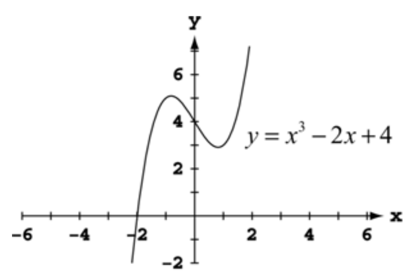
48. $g(x)=\sqrt{2x-x^2} \Rightarrow g'(x)=\frac{1-x}{\sqrt{2x-x^2}} \Rightarrow \frac{1-x}{\sqrt{2x-x^2}}=0 \Rightarrow 1-x=0 \Rightarrow x=1$; $\frac{1-x}{\sqrt{2x-x^2}}=$ undefined $\Rightarrow \sqrt{2x-x^2}=0$
$2x-x^2=0 \Rightarrow x=0$ or $x=2$. The critical points are $x=0$, $x=1$, and $x=2$.

49. Minimum value is 1 at $x=2$.



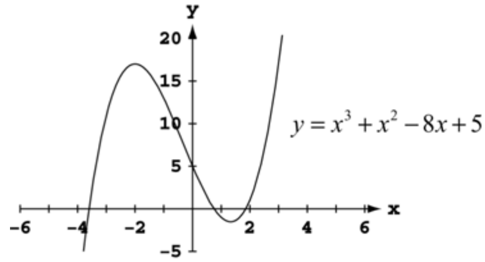
$[-2,6]$ by $[-2,4]$

50. To find the exact values, note that $y'=3x^2-2$, which is zero when $x=\pm\sqrt{\frac{2}{3}}$. Local maximum at $\left(-\sqrt{\frac{2}{3}},\,4+\frac{4\sqrt{6}}{9}\right)\approx(-0.816,\,5.089)$; local minimum at $\left(\sqrt{\frac{2}{3}},\,4-\frac{4\sqrt{6}}{9}\right)\approx(0.816,\,2.911)$



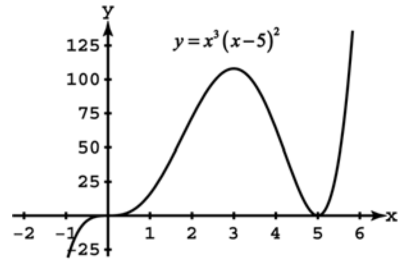
$[-6,6]$ by $[-2,7]$

51. To find the exact values, note that $y' = 3x^2 + 2x - 8$ $= (3x-4)(x+2)$, which is zero when $x = -2$ or $x = \frac{4}{3}$. Local maximum at $(-2, 17)$; local minimum at $\left(\frac{4}{3}, -\frac{41}{27}\right)$



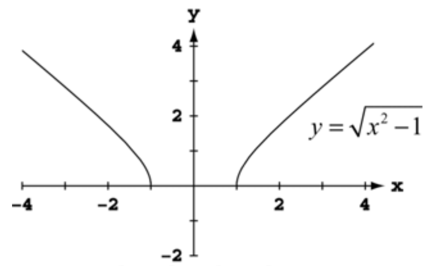
$[-6, 6]$ by $[-5, 20]$

52. Note that $y' = 5x^2(x-5)(x-3)$, which is zero at $x = 0$, $x = 3$, and $x = 5$. Local maximum at $(3, 108)$; local minimum at $(5, 0)$; $(0, 0)$ is neither a maximum nor a minimum.



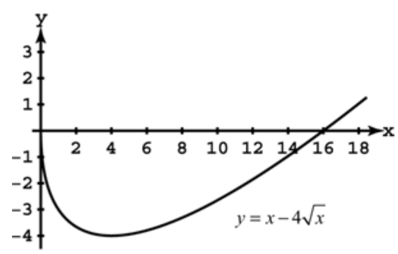
$[-2, 6]$ by $[-25, 125]$

53. Minimum value is 0 when $x = -1$ or $x = 1$.



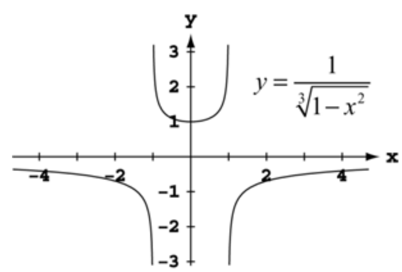
$[-6, 6]$ by $[-2, 4]$

54. Note that $y' = \frac{\sqrt{x}-2}{\sqrt{x}}$, which is zero at $x = 4$ and is undefined when $x = 0$. Local maximum at $(0, 0)$; absolute minimum at $(4, -4)$



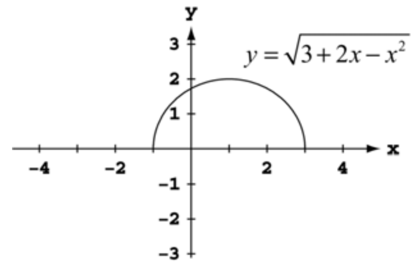
$[-4, 3]$ by $[0, 18]$

55. The actual graph of the function has asymptotes at $x = \pm 1$, so there are no extrema near these values. (This is an example of grapher failure.) There is a local minimum at $(0, 1)$.



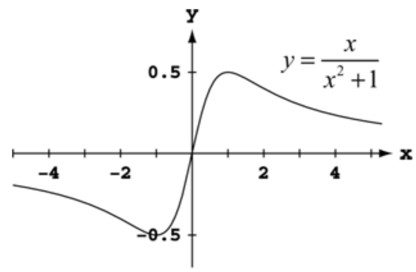
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

56. Maximum value is 2 at $x = 1$;
minimum value is 0 at $x = -1$ and $x = 3$.



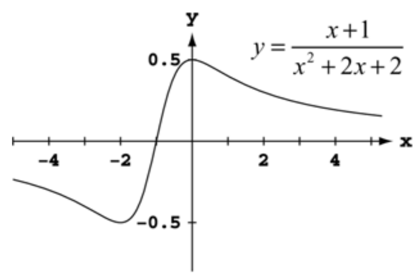
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

57. Maximum value is $\frac{1}{2}$ at $x = 1$;
minimum value is $-\frac{1}{2}$ at $x = -1$.



$[-5, 5]$ by $[-0.7, 0.7]$

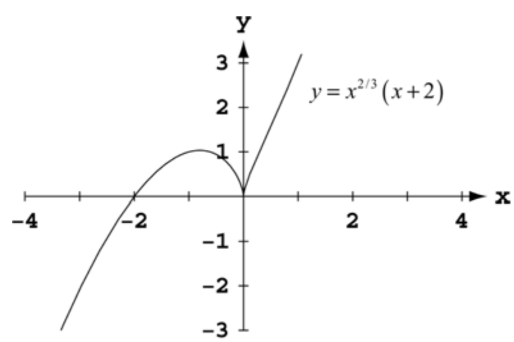
58. Maximum value is $\frac{1}{2}$ at $x = 0$;
minimum value is $-\frac{1}{2}$ as $x = -2$.



$[-5, 5]$ by $[-0.8, 0.6]$

59. $y' = x^{2/3}(1) + \frac{2}{3}x^{-1/3}(x+2) = \frac{5x+4}{3\sqrt[3]{x}}$

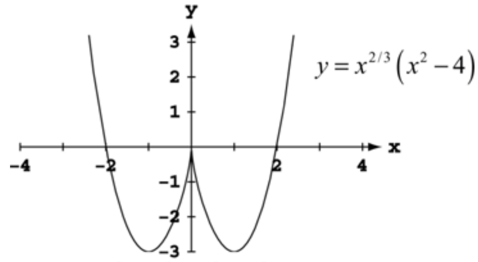
crit. pt.	derivative	extremum	value
$x = -\frac{4}{5}$	0	local max	$\frac{12}{25}10^{1/3} = 1.034$
$x = 0$	undefined	local min	0



$[-4, 4]$ by $[-3, 3]$

60. $y' = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}(x^2 - 4) = \frac{8x^2-8}{3\sqrt[3]{x}}$

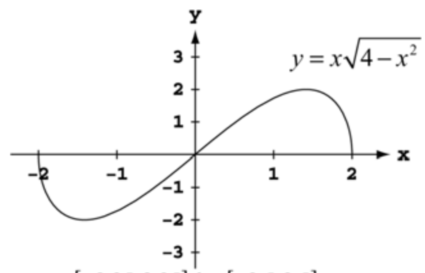
crit. pt.	derivative	extremum	value
$x = -1$	0	minimum	-3
$x = 0$	undefined	local max	0
$x = 1$	0	minimum	3



$[-4, 4]$ by $[-3, 3]$

61. $y' = x\frac{1}{2\sqrt{4-x^2}}(-2x) + (1)\sqrt{4 - x^2} = \frac{-x^2+(4-x^2)}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}}$

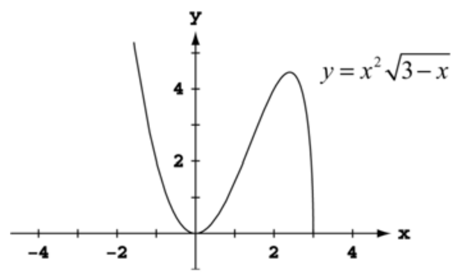
crit. pt.	derivative	extremum	value
$x = -2$	undefined	local max	0
$x = -\sqrt{2}$	0	minimum	-2
$x = \sqrt{2}$	0	maximum	2
$x = 2$	undefined	local min	0



$[-2.35, 2.35]$ by $[-3.5, 3.5]$

62. $y' = x^2\frac{1}{2\sqrt{3-x}}(-1) + 2x\sqrt{3 - x} = \frac{-x^2+(4x)(3-x)}{2\sqrt{3-x}}$
$= \frac{-5x^2+12x}{2\sqrt{3-x}}$

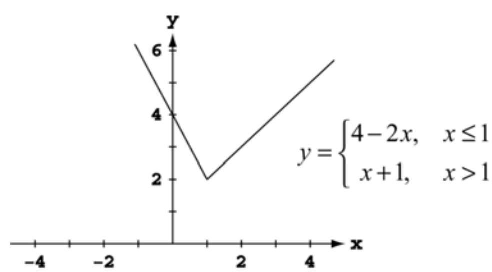
crit. pt.	derivative	extremum	value
$x = 0$	0	minimum	0
$x = \frac{12}{5}$	0	local max	$\frac{144}{125}15^{1/2} \approx 4.462$
$x = 3$	undefined	minimum	0



$[-4.7, 4.7]$ by $[-1, 5]$

63. $y' = \begin{cases} -2, & x < 1 \\ 1, & x > 1 \end{cases}$

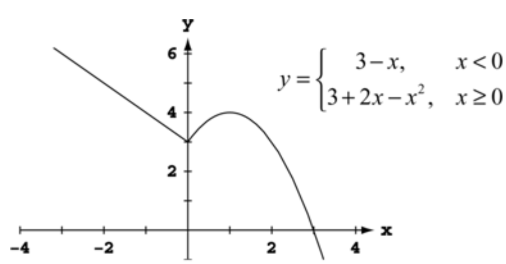
crit. pt.	derivative	extremum	value
$x = 1$	undefined	minimum	2



$[-4.7, 4.7]$ by $[0, 6.2]$

64. $y' = \begin{cases} -1, & x < 0 \\ 2 - 2x, & x > 0 \end{cases}$

crit. pt.	derivative	extremum	value
$x = 0$	undefined	local min	3
$x = 1$	0	local mix	4

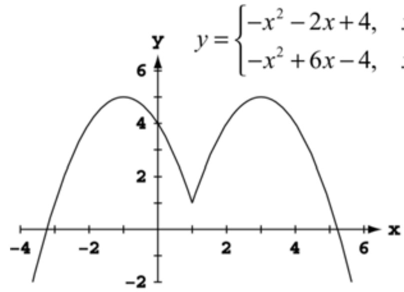


$[-4, 4]$ by $[-1, 6]$

65. $y' = \begin{cases} -2x-2, & x<1 \\ -2x+6, & x>1 \end{cases}$

crit. pt.	derivative	extremum	value
$x=-1$	0	maximum	5
$x=1$	undefined	local min	1
$x=3$	0	maximum	5

$y = \begin{cases} -x^2-2x+4, & x\le 1 \\ -x^2+6x-4, & x>1 \end{cases}$



$[-4,6]$ by $[-2,6]$

66. We begin by determining whether $f'(x)$ is defined at $x=1$, where $f(x) = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

Clearly, $f'(x) = -\frac{1}{2}x - \frac{1}{2}$ if $x<1$, and $\lim\limits_{h\to 0^-} f'(1+h) = -1$. Also, $f'(x) = 3x^2 - 12x + 8$ if $x>1$, and $\lim\limits_{h\to 0^+} f'(1+h) = -1$. Since f is continuous at $x=1$, we have that $f'(1) = -1$.

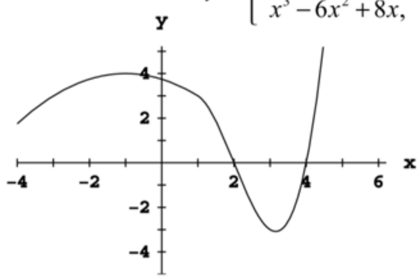
Thus, $f'(x) = \begin{cases} -\frac{1}{2}x - \frac{1}{2}, & x \le 1 \\ 3x^2 - 12x + 8, & x > 1 \end{cases}$

Note that $-\frac{1}{2}x - \frac{1}{2} = 0$ when $x=-1$, and $3x^2 - 12x + 8 = 0$ when $x = \frac{12 \pm \sqrt{12^2 - 4(3)(8)}}{2(3)} = \frac{12 \pm \sqrt{48}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$.

But $2 - \frac{2\sqrt{3}}{3} \approx 0.845 < 1$, so the critical points occur at $x=-1$ and $x = 2 + \frac{2\sqrt{3}}{3} \approx 3.155$.

crit. pt.	derivative	extremum	value
$x=-1$	0	local max	4
$x\approx 3.155$	0	local min	≈ -3.079

$y = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$



$[-4,6]$ by $[-5,5]$

67. (a) No, since $f'(x) = \frac{2}{3}(x-2)^{-1/3}$, which is undefined at $x=2$.
(b) The derivative is defined and nonzero for all $x \ne 2$. Also, $f(2)=0$ and $f(x)>0$ for all $x\ne 2$.
(c) No, $f(x)$ need not have a global maximum because its domain is all real numbers. Any restriction of f to a closed interval of the form $[a, b]$ would have both a maximum value and minimum value on the interval.
(d) The answers are the same as (a) and (b) with 2 replaced by a.

68. Note that $f(x) = \begin{cases} -x^3+9x, & x\le -3 \text{ or } 0\le x<3 \\ x^3-9x, & -3<x<0 \text{ or } x\ge 3 \end{cases}$. Therefore, $f'(x) = \begin{cases} -3x^3+9, & x<-3 \text{ or } 0<x<3 \\ 3x^3-9, & -3<x<0 \text{ or } x>3 \end{cases}$.
(a) No, since the left- and right-hand derivatives at $x=0$, are -9 and 9, respectively.
(b) No, since the left- and right-hand derivatives at $x=3$, are -18 and 18, respectively.
(c) No, since the left- and right-hand derivatives at $x=-3$, are 18 and -18, respectively.

(d) The critical points occur when $f'(x) = 0$ (at $x = \pm\sqrt{3}$) and when $f'(x)$ is undefined (at $x = 0$ and $x = \pm 3$). The minimum value is 0 at $x = -3$, at $x = 0$, and at $x = 3$; local maxima occur at $\left(-\sqrt{3}, 6\sqrt{3}\right)$ and $\left(\sqrt{3}, 6\sqrt{3}\right)$.

69. Yes, since $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = \frac{x}{(x^2)^{1/2}} = \frac{x}{|x|}$ is not defined at $x = 0$. Thus it is not required that f' be zero at a local extreme point since f' may be undefined there.

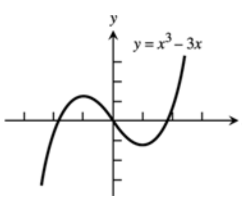
70. If $f(c)$ is a local maximum value of f, then $f(x) \le f(c)$ for all x in some open interval (a, b) containing c. Since f is even, $f(-x) = f(x) \le f(c) = f(-c)$ for all $-x$ in the open interval $(-b, -a)$ containing $-c$. That is, f assumes a local maximum at the point $-c$. This is also clear from the graph of f because the graph of an even function is symmetric about the y-axis.

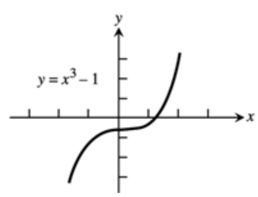
71. If $g(c)$ is a local minimum value of g, then $g(x) \ge g(c)$ for all x in some open interval (a, b) containing c. Since g is odd, $g(-x) = -g(x) \le -g(c) = g(-c)$ for all $-x$ in the open interval $(-b, -a)$ containing $-c$. That is, g assumes a local maximum at the point $-c$. This is also clear from the graph of g because the graph of an odd function is symmetric about the origin.

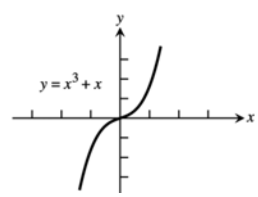
72. If there are no boundary points or critical points the function will have no extreme values in its domain. Such functions do indeed exist, for example $f(x) = x$ for $-\infty < x < \infty$. (Any other linear function $f(x) = mx + b$ with $m \ne 0$ will do as well.)

73. (a) $V(x) = 160x - 52x^2 + 4x^3$
$V'(x) = 160 - 104x + 12x^2 = 4(x-2)(3x-20)$
The only critical point in the interval (0, 5) is at $x = 2$. The maximum value of $V(x)$ is 144 at $x = 2$.
(b) The largest possible volume of the box is 144 cubic units, and it occurs when $x = 2$ units.

74. (a) $f'(x) = 3ax^2 + 2bx + c$ is a quadratic, so it can have 0, 1, or 2 zeros, which would be the critical points of f. The function $f(x) = x^3 - 3x$ has two critical points at $x = -1$ and $x = 1$. The function $f(x) = x^3 - 1$ has one critical point at $x = 0$. The function $f(x) = x^3 + x$ has no critical points.





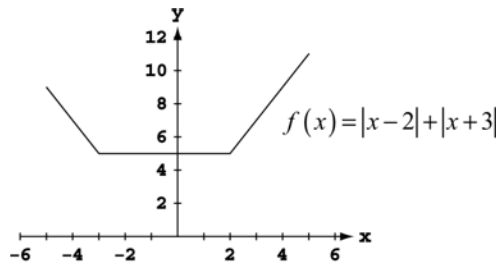


(b) The function can have either two local extreme values or no extreme values. (If there is only one critical point, the cubic function has no extreme values.)

75. $s = -\frac{1}{2}gt^2 + v_0 t + s_0 \Rightarrow \frac{ds}{dt} = -gt + v_0 = 0 \Rightarrow t = \frac{v_0}{g}$. Now $s(t) = s_0 \Leftrightarrow t\left(-\frac{gt}{2} + v_0\right) = 0 \Leftrightarrow t = 0$ or $t = \frac{2v_0}{g}$. Thus $s\left(\frac{v_0}{g}\right) = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\left(\frac{v_0}{g}\right) + s_0 = \frac{v_0^2}{2g} + s_0 > s_0$ is the maximum height over the interval $0 \le t \le \frac{2v_0}{g}$.

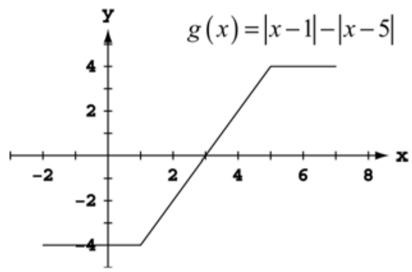
76. $\frac{dI}{dt} = -2\sin t + 2\cos t$, solving $\frac{dI}{dt} = 0 \Rightarrow \tan t = 1 \Rightarrow t = \frac{\pi}{4} + n\pi$ where n is a nonnegative integer (in this exercise t is never negative) $\Rightarrow$ the peak current is $2\sqrt{2}$ amps.

77. Maximum value is 11 at $x = 5$; minimum value is 5 on the interval $[-3, 2]$; local maximum at $(-5, 9)$



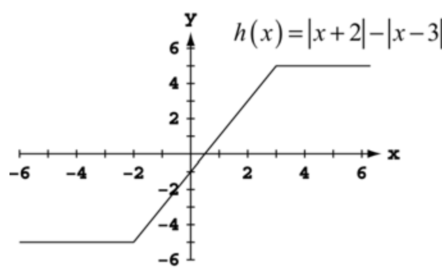
$[-6,6]$ by $[0,12]$

78. Maximum value is 4 on the interval $[5, 7]$; minimum value is -4 on the interval $[-2, 1]$.



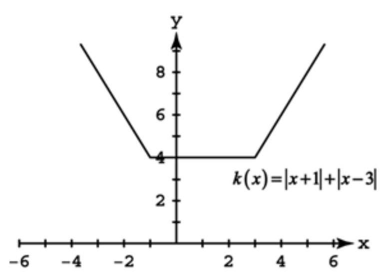
$[-3,8]$ by $[-5,5]$

79. Maximum value is 5 on the interval $[3, \infty)$; minimum value is -5 on the interval $(-\infty, -2]$.



$[-6,6]$ by $[-6,6]$

80. Minimum value is 4 on the interval $[-1, 3]$



$[-6,6]$ by $[0,9]$

81-86. Example CAS commands:

Maple:

```
with(student):
f := x -> x^4-8*x^2+4*x+2;
domain := x=-20/25..64/25;
plot( f(x), domain, color=black, title="Section 4.1 #81(a)" );
Df := D(f);
plot( Df(x), domain, color=black, title="Section 4.1 #81(b)" )
StatPt := fsolve( Df(x)=0, domain )
SingPt := NULL;
EndPt := op(rhs(domain));
```

```
Pts := evalf([EndPt,StatPt,SingPt]);
Values := [seq( f(x), x=Pts )];
```

Maximum value is 2.7608 and occurs at x=2.56 (right endpoint).

Minimum value is -6.2680 and occurs at x=1.86081 (singular point).

Mathematica: (functions may vary):

```
<<Miscellaneous `RealOnly`
Clear[f,x]
a = −1; b = 10/3;
f[x_] =2 + 2x − 3 x^(2/3)
f'[x]
Plot[{f[x], f'[x]}, {x, a, b}]
NSolve[f'[x]==0, x]
{f[a], f[0], f[x]/.%, f[b]}//N
```

In more complicated expressions, NSolve may not yield results. In this case, an approximate solution (say 1.1 here) is observed from the graph and the following command is used:

```
FindRoot[f'[x]==0, {x, 1.1}]
```

4.2 THE MEAN VALUE THEOREM

1. When $f(x) = x^2 + 2x - 1$ for $0 \le x \le 1$, then $\frac{f(1)-f(0)}{1-0} = f'(c) \Rightarrow 3 = 2c + 2 \Rightarrow c = \frac{1}{2}$.

2. When $f(x) = x^{2/3}$ for $0 \le x \le 1$, then $\frac{f(1)-f(0)}{1-0} = f'(c) \Rightarrow 1 = \left(\frac{2}{3}\right)c^{-1/3} \Rightarrow c = \frac{8}{27}$.

3. When $f(x) = x + \frac{1}{x}$ for $\frac{1}{2} \le x \le 2$, then $\frac{f(2)-f(1/2)}{2-1/2} = f'(c) \Rightarrow 0 = 1 - \frac{1}{c^2} \Rightarrow c = 1$.

4. When $f(x) = \sqrt{x-1}$ for $1 \le x \le 3$, then $\frac{f(3)-f(1)}{3-1} = f'(c) \Rightarrow \frac{\sqrt{2}}{2} = \frac{1}{2\sqrt{c-1}} \Rightarrow c = \frac{3}{2}$.

5. When $f(x) = x^3 - x^2$ for $-1 \le x \le 2$, then $\frac{f(2)-f(-1)}{2-(-1)} = f'(c) \Rightarrow 2 = 3c^2 - 2c \Rightarrow c = \frac{1 \pm \sqrt{7}}{3}$.
   $\frac{1+\sqrt{7}}{3} \approx 1.22$ and $\frac{1-\sqrt{7}}{3} \approx -0.549$ are both in the interval $-1 \le x \le 2$.

6. When $g(x) = \begin{cases} x^3 & -2 \le x \le 0 \\ x^2 & 0 < x \le 2 \end{cases}$, then $\frac{g(2)-g(-2)}{2-(-2)} = g'(c) \Rightarrow 3 = g'(c)$. If $-2 \le x < 0$, then $g'(x) = 3x^2 \Rightarrow 3 = g'(c)$
   $\Rightarrow 3c^2 = 3 \Rightarrow c = \pm 1$. Only $c = -1$ is in the interval. If $0 < x \le 2$, then $g'(x) = 2x \Rightarrow 3 = g'(c) \Rightarrow 2c = 3 \Rightarrow c = \frac{3}{2}$.

7. Does not; $f(x)$ is not differentiable at $x = 0$ in $(-1, 8)$.

8. Does; $f(x)$ is continuous for every point of $[0, 1]$ and differentiable for every point in $(0, 1)$.

9. Does; $f(x)$ is continuous for every point of $[0, 1]$ and differentiable for every point in $(0, 1)$.

10. Does not; $f(x)$ is not continuous at $x = 0$ because $\lim_{x \to 0^-} f(x) = 1 \ne 0 = f(0)$.

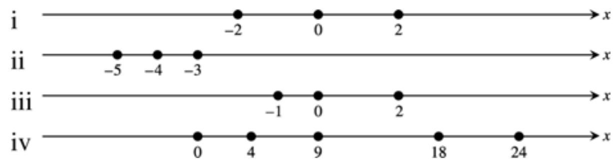
11. Does not; f is not differentiable at $x = -1$ in $(-2, 0)$.

12. Does; $f(x)$ is continuous for every point of $[0, 3]$ and differentiable for every point in $(0, 3)$.

13. Since $f(x)$ is not continuous on $0 \le x \le 1$, Rolle's Theorem does not apply: $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1 \ne 0 = f(1)$.

14. Since $f(x)$ must be continuous at $x = 0$ and $x = 1$ we have $\lim\limits_{x \to 0^+} f(x) = a = f(0) \Rightarrow a = 3$ and $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) \Rightarrow -1 + 3 + a = m + b \Rightarrow 5 = m + b$. Since $f(x)$ must also be differentiable at $x = 1$ we have $\lim\limits_{x \to 1^-} f'(x) = \lim\limits_{x \to 1^+} f'(x) \Rightarrow -2x + 3|_{x=1} = m|_{x=1} \Rightarrow 1 = m$. Therefore, $a = 3$, $m = 1$ and $b = 4$.

15. (a)



(b) Let r_1 and r_2 be zeros of the polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then $P(r_1) = P(r_2) = 0$. Since polynomials are everywhere continuous and differentiable, by Rolle's Theorem $P'(r) = 0$ for some r between r_1 and r_2, where $P'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1$.

16. With f both differentiable and continuous on $[a, b]$ and $f(r_1) = f(r_2) = f(r_3) = 0$ where r_1, r_2 and r_3 are in $[a, b]$, then by Rolle's Theorem there exists a c_1 between r_1 and r_2 such that $f'(c_1) = 0$ and a c_2 between r_2 and r_3 such that $f'(c_2) = 0$. Since f' is both differentiable and continuous on $[a, b]$, Rolle's Theorem again applies and we have a c_3 between c_1 and c_2 such that $f''(c_3) = 0$. To generalize, if f has $n+1$ zeros in $[a, b]$ and $f^{(n)}$ is continuous on $[a, b]$, then $f^{(n)}$ has at least one zero between a and b.

17. Since f'' exists throughout $[a, b]$ the derivative function f' is continuous there. If f' has more than one zero in $[a, b]$, say $f'(r_1) = f'(r_2) = 0$ for $r_1 \ne r_2$, then by Rolle's Theorem there is a c between r_1 and r_2 such that $f''(c) = 0$, contrary to $f'' > 0$ throughout $[a, b]$. Therefore f' has at most one zero in $[a, b]$. The same argument holds if $f'' < 0$ throughout $[a, b]$.

18. If $f(x)$ is a cubic polynomial with four or more zeros, then by Rolle's Theorem $f'(x)$ has three or more zeros, $f''(x)$ has 2 or more zeros and $f'''(x)$ has at least one zero. This is a contradiction since $f'''(x)$ is a non-zero constant when $f(x)$ is a cubic polynomial.

19. With $f(-2) = 11 > 0$ and $f(-1) = -1 < 0$ we conclude from the Intermediate Value Theorem that $f(x) = x^4 + 3x + 1$ has at least one zero between -2 and -1. Then $-2 < x < -1 \Rightarrow -8 < x^3 < -1 \Rightarrow -32 < 4x^3 < -4 \Rightarrow -29 < 4x^3 + 3 < -1 \Rightarrow f'(x) < 0$ for $-2 < x < -1 \Rightarrow f(x)$ is decreasing on $[-2, -1] \Rightarrow f(x) = 0$ has exactly one solution in the interval $(-2, -1)$.

20. $f(x) = x^3 + \frac{4}{x^2} + 7 \Rightarrow f'(x) = 3x^2 - \frac{8}{x^3} > 0$ on $(-\infty, 0) \Rightarrow f(x)$ is increasing on $(-\infty, 0)$. Also, $f(x) < 0$ if $x < -2$ and $f(x) > 0$ if $-2 < x < 0 \Rightarrow f(x)$ has exactly one zero in $(-\infty, 0)$.

21. $g(t) = \sqrt{t} + \sqrt{t+1} - 4 \Rightarrow g'(t) = \frac{1}{2\sqrt{t}} - \frac{1}{2\sqrt{t+1}} > 0 \Rightarrow g(t)$ is increasing for t in $(0, \infty)$; $g(3) = \sqrt{3} - 2 < 0$ and $g(15) = \sqrt{15} > 0 \Rightarrow g(t)$ has exactly one zero in $(0, \infty)$.

22. $g(t) = \frac{1}{1-t} + \sqrt{1+t} - 3.1 \Rightarrow g'(t) = \frac{1}{(1-t)^2} + \frac{1}{2\sqrt{1+t}} > 0 \Rightarrow g(t)$ is increasing for t in $(-1, 1)$; $g(-0.99) = -2.5$ and $g(0.99) = 98.3 \Rightarrow g(t)$ has exactly one zero in $(-1, 1)$.

23. $r(\theta)=\theta+\sin^2\left(\frac{\theta}{3}\right)-8 \Rightarrow r'(\theta)=1+\frac{2}{3}\sin\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)=1+\frac{1}{3}\sin\left(\frac{2\theta}{3}\right)>0$ on $(-\infty,\infty) \Rightarrow r(\theta)$ is increasing on $(-\infty,\infty)$; $r(0)=-8$ and $r(8)=\sin^2\left(\frac{8}{3}\right)>0 \Rightarrow r(\theta)$ has exactly one zero in $(-\infty,\infty)$.

24. $r(\theta)=2\theta-\cos^2\theta+\sqrt{2} \Rightarrow r'(\theta)=2+2\sin\theta\cos\theta=2+\sin 2\theta>0$ on $(-\infty,\infty) \Rightarrow r(\theta)$ is increasing on $(-\infty,\infty)$; $r(-2\pi)=-4\pi-\cos(-2\pi)+\sqrt{2}=-4\pi-1+\sqrt{2}<0$ and $r(2\pi)=4\pi-1+\sqrt{2}>0 \Rightarrow r(\theta)$ has exactly one zero in $(-\infty,\infty)$.

25. $r(\theta)=\sec\theta-\frac{1}{\theta^3}+5 \Rightarrow r'(\theta)=(\sec\theta)(\tan\theta)+\frac{3}{\theta^4}>0$ on $\left(0,\frac{\pi}{2}\right) \Rightarrow r(\theta)$ is increasing on $\left(0,\frac{\pi}{2}\right)$; $r(0.1)\approx-994$ and $r(1.57)\approx 1260.5 \Rightarrow r(\theta)$ has exactly one zero in $\left(0,\frac{\pi}{2}\right)$.

26. $r(\theta)=\tan\theta-\cot\theta-\theta \Rightarrow r'(\theta)=\sec^2\theta+\csc^2\theta-1=\sec^2\theta+\cot^2\theta>0$ on $\left(0,\frac{\pi}{2}\right) \Rightarrow r(\theta)$ is increasing on $\left(0,\frac{\pi}{2}\right)$; $r\left(\frac{\pi}{4}\right)=-\frac{\pi}{4}<0$ and $r(1.57)\approx 1254.2 \Rightarrow r(\theta)$ has exactly one zero in $\left(0,\frac{\pi}{2}\right)$.

27. By Corollary 1, $f'(x)=0$ for all $x \Rightarrow f(x)=C$, where C is a constant. Since $f(-1)=3$ we have $C=3 \Rightarrow f(x)=3$ for all x.

28. $g(x)=2x+5 \Rightarrow g'(x)=2=f'(x)$ for all x. By Corollary 2, $f(x)=g(x)+C$ for some constant C. Then $f(0)=g(0)+C \Rightarrow 5=5+C \Rightarrow C=0 \Rightarrow f(x)=g(x)=2x+5$ for all x.

29. $g(x)=x^2 \Rightarrow g'(x)=2x=f'(x)$ for all x. By Corollary 2, $f(x)=g(x)+C$.
(a) $f(0)=0 \Rightarrow 0=g(0)+C=0+C \Rightarrow C=0 \Rightarrow f(x)=x^2 \Rightarrow f(2)=4$
(b) $f(1)=0 \Rightarrow 0=g(1)+C=1+C \Rightarrow C=-1 \Rightarrow f(x)=x^2-1 \Rightarrow f(2)=3$
(c) $f(-2)=3 \Rightarrow 3=g(-2)+C \Rightarrow 3=4+C \Rightarrow C=-1 \Rightarrow f(x)=x^2-1 \Rightarrow f(2)=3$

30. $g(x)=mx \Rightarrow g'(x)=m$, a constant. If $f'(x)=m$, then by Corollary 2, $f(x)=g(x)+b=mx+b$ where b is a constant. Therefore all functions whose derivatives are constant can be graphed as straight lines $y=mx+b$.

31. (a) $y=\frac{x^2}{2}+C$ (b) $y=\frac{x^3}{3}+C$ (c) $y=\frac{x^4}{4}+C$

32. (a) $y=x^2+C$ (b) $y=x^2-x+C$ (c) $y=x^3+x^2-x+C$

33. (a) $y'=-x^{-2} \Rightarrow y=\frac{1}{x}+C$ (b) $y=x+\frac{1}{x}+C$ (c) $y=5x-\frac{1}{x}+C$

34. (a) $y'=\frac{1}{2}x^{-1/2} \Rightarrow y=x^{1/2}+C \Rightarrow y=\sqrt{x}+C$ (b) $y=2\sqrt{x}+C$
(c) $y=2x^2-2\sqrt{x}+C$

35. (a) $y=-\frac{1}{2}\cos 2t+C$ (b) $y=2\sin\frac{t}{2}+C$
(c) $y=-\frac{1}{2}\cos 2t+2\sin\frac{t}{2}+C$

36. (a) $y=\tan\theta+C$ (b) $y'=\theta^{1/2} \Rightarrow y=\frac{2}{3}\theta^{3/2}+C$ (c) $y=\frac{2}{3}\theta^{3/2}-\tan\theta+C$

37. $f(x)=x^2-x+C; 0=f(0)=0^2-0+C \Rightarrow C=0 \Rightarrow f(x)=x^2-x$

38. $g(x)=-\frac{1}{x}+x^2+C; 1=g(-1)=-\frac{1}{-1}+(-1)^2+C \Rightarrow C=-1 \Rightarrow g(x)=-\frac{1}{x}+x^2-1$

39. $r(\theta) = 8\theta + \cot\theta + C$; $0 = r\left(\frac{\pi}{4}\right) = 8\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right) + C \Rightarrow 0 = 2\pi + 1 + C \Rightarrow C = -2\pi - 1$
$\Rightarrow r(\theta) = 8\theta + \cot\theta - 2\pi - 1$

40. $r(t) = \sec t - t + C$; $0 = r(0) = \sec(0) - 0 + C \Rightarrow C = -1 \Rightarrow r(t) = \sec t - t - 1$

41. $v = \frac{ds}{dt} = 9.8t + 5 \Rightarrow s = 4.9t^2 + 5t + C$; at $s = 10$ and $t = 0$ we have $C = 10 \Rightarrow s = 4.9t^2 + 5t + 10$

42. $v = \frac{ds}{dt} = 32t - 2 \Rightarrow s = 16t^2 - 2t + C$; at $s = 4$ and $t = \frac{1}{2}$ we have $C = 1 \Rightarrow s = 16t^2 - 2t + 1$

43. $v = \frac{ds}{dt} = \sin(\pi t) \Rightarrow s = -\frac{1}{\pi}\cos(\pi t) + C$; at $s = 0$ and $t = 0$ we have $C = \frac{1}{\pi} \Rightarrow s = \frac{1-\cos(\pi t)}{\pi}$

44. $v = \frac{ds}{dt} = \frac{2}{\pi}\cos\left(\frac{2t}{\pi}\right) \Rightarrow s = \sin\left(\frac{2t}{\pi}\right) + C$; at $s = 1$ and $t = \pi^2$ we have $C = 1 \Rightarrow s = \sin\left(\frac{2t}{\pi}\right) + 1$

45. $a = \dfrac{dv}{dt} = 32 \Rightarrow v = 32t + C$; at $v = 20$ and $t = 0$ we have $C = 20 \Rightarrow v = 32t + 20$
$v = \dfrac{ds}{dt} = 32t + 20 \Rightarrow s = 16t^2 + 20t + C$; at $s = 5$ and $t = 0$ we have $C = 5 \Rightarrow s = 16t^2 + 20t + 5$

46. $a = 9.8 \Rightarrow v = 9.8t + C_1$; at $v = -3$ and $t = 0$ we have $C_1 = -3 \Rightarrow v = 9.8t - 3 \Rightarrow s = 4.9t^2 - 3t + C_2$; at $s = 0$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = 4.9t^2 - 3t$

47. $a = -4\sin(2t) \Rightarrow v = 2\cos(2t) + C_1$; at $v = 2$ and $t = 0$ we have $C_1 = 0 \Rightarrow v = 2\cos(2t) \Rightarrow s = \sin(2t) + C_2$; at $s = -3$ and $t = 0$ we have $C_2 = -3 \Rightarrow s = \sin(2t) - 3$

48. $a = \frac{9}{\pi^2}\cos\left(\frac{3t}{\pi}\right) \Rightarrow v = \frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right) + C_1$; at $v = 0$ and $t = 0$ we have $C_1 = 0 \Rightarrow v = \frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right) \Rightarrow s = -\cos\left(\frac{3t}{\pi}\right) + C_2$; at $s = -1$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = -\cos\left(\frac{3t}{\pi}\right)$

49. If $T(t)$ is the temperature of the thermometer at time t, then $T(0) = -19°C$ and $T(14) = -100°C$. From the Mean Value Theorem there exists a $0 < t_0 < 14$ such that $\frac{T(14)-T(0)}{14-0} = 8.5°C/\text{s} = T'(t_0)$, the rate at which the temperature was changing at $t = t_0$ as measured by the rising mercury on the thermometer.

50. Because the trucker's average speed was 115 km/h, by the Mean Value Theorem, the trucker must have been going that speed at least once during the trip.

51. Because its average speed was approximately 7.667 knots, and by the Mean Value Theorem, it must have been going that speed at least once during the trip.

52. The runner's average speed for the marathon was approximately 19.1 km/h. Therefore, by the Mean Value Theorem, the runner must have been going that speed at least once during the marathon. Since the initial speed and final speed are both 0 mph and the runner's speed is continuous, by the Intermediate Value Theorem, the runner's speed must have been 18 km/h at least twice.

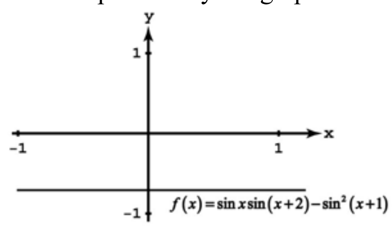
53. Let $d(t)$ represent the distance the automobile traveled in time t. The average speed over $0 \le t \le 2$ is $\frac{d(2)-d(0)}{2-0}$. The Mean Value Theorem says that for some $0 < t_0 < 2$, $d'(t_0) = \frac{d(2)-d(0)}{2-0}$. The value $d'(t_0)$ is the speed of the automobile at time t_0 (which is read on the speedometer).

54. $a(t) = v'(t) = 1.6 \Rightarrow v(t) = 1.6t + C$; at (0, 0) we have $C = 0 \Rightarrow (t) = 1.6t$. When $t = 30$, then $v(30) = 48$ m/s.

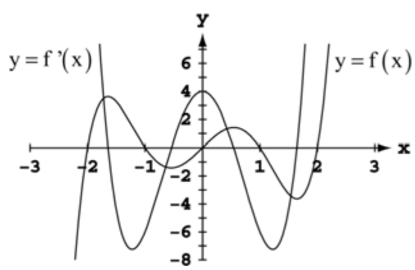
55. The conclusion of the Mean Value Theorem yields $\frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{c^2} \Rightarrow c^2\left(\frac{a-b}{ab}\right) = a - b \Rightarrow c = \sqrt{ab}.$

56. The conclusion of the Mean Value Theorem yields $\frac{b^2-a^2}{b-a} = 2c \Rightarrow c = \frac{a+b}{2}.$

57. $f'(x) = [\cos x \sin(x+2) + \sin x \cos(x+2)] - 2\sin(x+1)\cos(x+1) = \sin(x+x+2) - \sin 2(x+1)$
$= \sin(2x+2) - \sin(2x+2) = 0$. Therefore, the function has the constant value $f(0) = -\sin^2 1 \approx -0.7081$ which explains why the graph is a horizontal line.



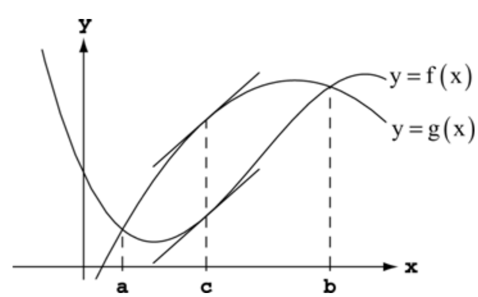
58. (a) $f(x) = (x+2)(x+1)x(x-1)(x-2) = x^5 - 5x^3 + 4x$ is one possibility.
    (b) Graphing $f(x) = x^5 - 5x^3 + 4x$ and $f'(x) = 5x^4 - 15x^2 + 4$ on [−3, 3] by [−7, 7] we see that each x-intercept of $f'(x)$ lies between a pair of x-intercepts of $f(x)$, as expected by Rolle's Theorem.



    (c) Yes, since sin is continuous and differentiable on $(-\infty, \infty)$.

59. $f(x)$ must be zero at least once between a and b by the Intermediate Value Theorem. Now suppose that $f(x)$ is zero twice between a and b. Then by the Mean Value Theorem, $f'(x)$ would have to be zero at least once between the two zeros of $f(x)$, but this can't be true since we are given that $f'(x) \neq 0$ on this interval. Therefore, $f(x)$ is zero once and only once between a and b.

60. Consider the function $k(x) = f(x) - g(x)$. $k(x)$ is continuous and differentiable on $[a, b]$, and since $k(a) = f(a) - g(a)$ and $k(b) = f(b) - g(b)$, by the Mean Value Theorem, there must be a point c in (a, b) where $k'(c) = 0$. But since $k'(c) = f'(c) - g'(c)$, this means that $f'(c) = g'(c)$, and c is a point where the graphs of f and g have tangent lines with the same slope, so these lines are either parallel or are the same line.



61. $f'(x) \leq 1$ for $1 \leq x \leq 4 \Rightarrow f(x)$ is differentiable on $1 \leq x \leq 4 \Rightarrow f$ is continuous on $1 \leq x \leq 4 \Rightarrow f$ satisfies the conditions of the Mean Value Theorem $\Rightarrow \frac{f(4)-f(1)}{4-1} = f'(c)$ for some c in $1 < x < 4 \Rightarrow f'(c) \leq 1 \Rightarrow \frac{f(4)-f(1)}{3} \leq 1$
$\Rightarrow f(4) - f(1) \leq 3$

62. $0 < f'(x) < \frac{1}{2}$ for all $x \Rightarrow f'(x)$ exists for all x, thus f is differentiable on $(-1, 1) \Rightarrow f$ is continuous on $[-1, 1]$ $\Rightarrow f$ satisfies the conditions of the Mean Value Theorem $\Rightarrow \frac{f(1) - f(-1)}{1-(-1)} = f'(c)$ for some c in $[-1, 1]$ $\Rightarrow 0 < \frac{f(1) - f(-1)}{2} < \frac{1}{2} \Rightarrow 0 < f(1) - f(-1) < 1$. Since $f(1) - f(-1) < 1 \Rightarrow f(1) < 1 + f(-1) < 2 + f(-1)$, and since $0 < f(1) - f(-1)$ we have $f(-1) < f(1)$. Together we have $f(-1) < f(1) < 2 + f(-1)$.

63. Let $f(t) = \cos t$ and consider the interval $[0, x]$ where x is a real number. f is continuous on $[0, x]$ and f is differentiable on $(0, x)$ since $f'(t) = -\sin t \Rightarrow f$ satisfies the conditions of the Mean Value Theorem $\Rightarrow \frac{f(x) - f(0)}{x-(0)} = f'(c)$ for some c in $[0, x] \Rightarrow \frac{\cos x - 1}{x} = -\sin c$. Since $-1 \le \sin c \le 1 \Rightarrow -1 \le -\sin c \le 1$ $\Rightarrow -1 \le \frac{\cos x - 1}{x} \le 1$. If $x > 0$, $-1 \le \frac{\cos x - 1}{x} \le 1 \Rightarrow -x \le \cos x - 1 \le x \Rightarrow |\cos x - 1| \le x = |x|$. If $x < 0$, $-1 \le \frac{\cos x - 1}{x} \le 1$ $\Rightarrow -x \ge \cos x - 1 \ge x \Rightarrow x \le \cos x - 1 \le -x \Rightarrow -(-x) \le \cos x - 1 \le -x \Rightarrow |\cos x - 1| \le -x = |x|$. Thus, in both cases, we have $|\cos x - 1| \le |x|$. If $x = 0$, then $|\cos 0 - 1| = |1 - 1| = |0| \le |0|$, thus $|\cos x - 1| \le |x|$ is true for all x.

64. Let $f(x) = \sin x$ for $a \le x \le b$. From the Mean Value Theorem there exists a c between a and b such that $\frac{\sin b - \sin a}{b - a} = \cos c \Rightarrow -1 \le \frac{\sin b - \sin a}{b - a} \le 1 \Rightarrow \frac{\sin b - \sin a}{b - a} \le 1 \Rightarrow |\sin b - \sin a| \le |b - a|$.

65. Yes. By Corollary 2 we have $f(x) = g(x) + c$ since $f'(x) = g'(x)$. If the graphs start at the same point $x = a$, then $f(a) = g(a) \Rightarrow c = 0 \Rightarrow f(x) = g(x)$.

66. Assume f is differentiable and $|f(w) - f(x)| \le |w - x|$ for all values of w and x. Since f is differentiable, $f'(x)$ exists and $f'(x) = \lim\limits_{w \to x} \frac{f(w) - f(x)}{w - x}$ using the alternative formula for the derivative. Let $g(x) = |x|$, which is continuous for all x. By Theorem 10 from Chapter 2, $|f'(x)| = \left|\lim\limits_{w \to x} \frac{f(w) - f(x)}{w - x}\right| = \lim\limits_{w \to x} \left|\frac{f(w) - f(x)}{w - x}\right|$ $= \lim\limits_{w \to x} \frac{|f(w) - f(x)|}{|w - x|}$. Since $|f(w) - f(x)| \le |w - x|$ for all w and $x \Rightarrow \frac{|f(w) - f(x)|}{|w - x|} \le 1$ as long as $w \ne x$. By Theorem 5 from Chapter 2, $|f'(x)| = \lim\limits_{w \to x} \frac{|f(w) - f(x)|}{|w - x|} \le \lim\limits_{w \to x} 1 = 1 \Rightarrow |f'(x)| \le 1 \Rightarrow -1 \le f'(x) \le 1$.

67. By the Mean Value Theorem we have $\frac{f(b) - f(a)}{b - a} = f'(c)$ for some point c between a and b. Since $b - a > 0$ and $f(b) < f(a)$, we have $f(b) - f(a) < 0 \Rightarrow f'(c) < 0$.

68. The condition is that f' should be continuous over $[a, b]$. The Mean Value Theorem then guarantees the existence of a point c in (a, b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$. If f' is continuous, then it has a minimum and maximum value on $[a, b]$, and $\min f' \le f'(c) \le \max f'$, as required.

69. $f'(x) = (1 + x^4 \cos x)^{-1} \Rightarrow f''(x) = -(1 + x^4 \cos x)^{-2}(4x^3 \cos x - x^4 \sin x)$ $= -x^3(1 + x^4 \cos x)^{-2}(4 \cos x - x \sin x) < 0$ for $0 \le x \le 0.1 \Rightarrow f'(x)$ is decreasing when $0 \le x \le 0.1$ $\Rightarrow \min f' \approx 0.9999$ and $\max f' = 1$. Now we have $0.9999 \le \frac{f(0.1) - 1}{0.1} \le 1 \Rightarrow 0.09999 \le f(0.1) - 1 \le 0.1$ $\Rightarrow 1.09999 \le f(0.1) \le 1.1$.

70. $f'(x) = (1 - x^4)^{-1} \Rightarrow f''(x) = -(1 - x^4)^{-2}(-4x^3) = \frac{4x^3}{(1 - x^4)^3} > 0$ for $0 < x \le 0.1 \Rightarrow f'(x)$ is increasing when $0 \le x \le 0.1 \Rightarrow \min f' = 1$ and $\max f' = 1.0001$. Now we have $1 \le \frac{f(0.1) - 2}{0.1} \le 1.0001$ $\Rightarrow 0.1 \le f(0.1) - 2 \le 0.10001 \Rightarrow 2.1 \le f(0.1) \le 2.10001$.

71. (a) Suppose $x < 1$, then by the Mean Value Theorem $\frac{f(x)-f(1)}{x-1} < 0 \Rightarrow f(x) > f(1)$. Suppose $x > 1$, then by the Mean Value Theorem $\frac{f(x)-f(1)}{x-1} > 0 \Rightarrow f(x) > f(1)$. Therefore $f(x) \geq 1$ for all x since $f(1) = 1$.
(b) Yes. From part (a), $\lim\limits_{x\to 1^-} \frac{f(x)-f(1)}{x-1} \leq 0$ and $\lim\limits_{x\to 1^+} \frac{f(x)-f(1)}{x-1} \geq 0$. Since $f'(1)$ exists, these two one-sided limits are equal and have the value $f'(1) \Rightarrow f'(1) \leq 0$ and $f'(1) \geq 0 \Rightarrow f'(1) = 0$.

72. From the Mean Value Theorem we have $\frac{f(b)-f(a)}{b-a} = f'(c)$ where c is between a and b. But $f'(c) = 2pc + q = 0$ has only one solution $c = -\frac{q}{2p}$. (Note: $p \neq 0$ since f is a quadratic function.)

4.3 MONOTONIC FUNCTIONS AND THE FIRST DERIVATIVE TEST

1. (a) $f'(x) = x(x-1) \Rightarrow$ critical points at 0 and 1
(b) $f' = \underset{0}{+++|}\underset{1}{---|}+++ \Rightarrow$ increasing on $(-\infty, 0)$ and $(1, \infty)$, decreasing on $(0, 1)$
(c) Local maximum at $x = 0$ and a local minimum at $x = 1$

2. (a) $f'(x) = (x-1)(x+2) \Rightarrow$ critical points at -2 and 1
(b) $f' = \underset{-2}{+++|}\underset{1}{---|}+++ \Rightarrow$ increasing on $(-\infty, -2)$ and $(1, \infty)$, decreasing on $(-2, 1)$
(c) Local maximum at $x = -2$ and a local minimum at $x = 1$

3. (a) $f'(x) = (x-1)^2(x+2) \Rightarrow$ critical points at -2 and 1
(b) $f' = \underset{-2}{---|}\underset{1}{+++|}+++ \Rightarrow$ increasing on $(-2, 1)$ and $(1, \infty)$, decreasing on $(-\infty, -2)$
(c) No local maximum and a local minimum at $x = -2$

4. (a) $f'(x) = (x-1)^2(x+2)^2 \Rightarrow$ critical points at -2 and 1
(b) $f' = \underset{-2}{+++|}\underset{1}{+++|}+++ \Rightarrow$ increasing on $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$, never decreasing
(c) No local extrema

5. (a) $f'(x) = (x-1)(x+2)(x-3) \Rightarrow \Rightarrow$ critical points at $-2, 1, 3$
(b) $f' = \underset{-2}{---|}\underset{1}{+++|}\underset{3}{---|}+++ \Rightarrow$ increasing on $(-2, 1)$ and $(3, \infty)$, decreasing on $(-\infty, -2)$ and $(1, 3)$
(c) Local maximum at $x = 1$, local minima at $x = -2$ and $x = 3$

6. (a) $f'(x) = (x-7)(x+1)(x+5) \Rightarrow$ critical points at -5, -1 and 7
(b) $f' = \underset{-5}{---|}\underset{-1}{+++|}\underset{7}{---|}+++ \Rightarrow$ increasing on $(-5, -1)$ and $(7, \infty)$, decreasing on $(-\infty, -5)$ and $(-1, 7)$
(c) Local maximum at $x = -1$, local minima at $x = -5$ and $x = 7$

7. (a) $f'(x) = \frac{x^2(x-1)}{(x+2)} \Rightarrow$ critical points at $x = 0$, $x = 1$ and $x = -2$
(b) $f' = \underset{-2}{+++)(}\underset{0}{---|}\underset{1}{---|}+++ \Rightarrow$ increasing on $(-\infty, -2)$ and $(1, \infty)$, decreasing on $(-2, 0)$ and $(0, 1)$
(c) Local minimum at $x = 1$

8. (a) $f'(x) = \frac{(x-2)(x+4)}{(x+1)(x-3)} \Rightarrow$ critical points at $x = 2$, $x = -4$, $x = -1$, and $x = 3$
   (b) $f' = +++ \underset{-4}{|} --- \underset{-1}{)(} +++ \underset{2}{|} --- \underset{3}{)(} +++ \Rightarrow$ increasing on $(-\infty, -4)$, $(-1, 2)$ and $(3, \infty)$, decreasing on $(-4, -1)$ and $(2, 3)$
   (c) Local maximum at $x = -4$ and $x = 2$

9. (a) $f'(x) = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2} \Rightarrow$ critical points at $x = -2$, $x = 2$ and $x = 0$.
   (b) $f' = +++ \underset{-2}{|} --- \underset{0}{)(} --- \underset{2}{|} +++ \Rightarrow$ increasing on $(-\infty, -2)$ and $(2, \infty)$, decreasing on $(-2, 0)$ and $(0, 2)$
   (c) Local maximum at $x = -2$, local minimum at $x = 2$

10. (a) $f'(x) = 3 - \frac{6}{\sqrt{x}} = \frac{3\sqrt{x}-6}{\sqrt{x}} \Rightarrow$ critical points at $x = 4$ and $x = 0$
    (b) $f' = \underset{0}{(} --- \underset{4}{|} +++ \Rightarrow$ increasing on $(4, \infty)$, decreasing on $(0, 4)$
    (c) Local minimum at $x = 4$

11. (a) $f'(x) = x^{-1/3}(x+2) \Rightarrow$ critical points at $x = -2$ and $x = 0$
    (b) $f' = +++ \underset{-2}{|} --- \underset{0}{)(} +++ \Rightarrow$ increasing on $(-\infty, -2)$ and $(0, \infty)$, decreasing on $(-2, 0)$
    (c) Local maximum at $x = -2$, local minimum at $x = 0$

12. (a) $f'(x) = x^{-1/2}(x-3) \Rightarrow$ critical points at $x = 0$ and $x = 3$
    (b) $f' = \underset{0}{(} --- \underset{3}{|} +++ \Rightarrow$ increasing on $(3, \infty)$, decreasing on $(0, 3)$
    (c) No local maximum and a local minimum at $x = 3$

13. (a) $f'(x) = (\sin x - 1)(2\cos x + 1)$, $0 \le x \le 2\pi \Rightarrow$ critical points at $x = \frac{\pi}{2}$, $x = \frac{2\pi}{3}$, and $x = \frac{4\pi}{3}$
    (b) $f' = \underset{0}{[} --- \underset{\frac{\pi}{2}}{|} --- \underset{\frac{2\pi}{3}}{|} +++ \underset{\frac{4\pi}{3}}{|} --- \underset{2\pi}{]} \Rightarrow$ increasing on $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$, decreasing on $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$
    (c) Local maximum at $x = \frac{4\pi}{3}$ and $x = 0$, local minimum at $x = \frac{2\pi}{3}$ and $x = 2\pi$

14. (a) $f'(x) = (\sin x + \cos x)(\sin x - \cos x)$, $0 \le x \le 2\pi \Rightarrow$ critical points at $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, and $x = \frac{7\pi}{4}$
    (b) $f' = \underset{0}{[} --- \underset{\frac{\pi}{4}}{|} +++ \underset{\frac{3\pi}{4}}{|} --- \underset{\frac{5\pi}{4}}{|} +++ \underset{\frac{7\pi}{4}}{|} --- \underset{2\pi}{]} \Rightarrow$ increasing on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$, decreasing on $\left(0, \frac{\pi}{4}\right)$, $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$
    (c) Local maximum at $x = 0$, $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$, local minimum at $x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$ and $x = 2\pi$

15. (a) Increasing on $(-2, 0)$ and $(2, 4)$, decreasing on $(-4, -2)$ and $(0, 2)$
    (b) Absolute maximum at $(-4, 2)$, local maximum at $(0, 1)$ and $(4, -1)$; Absolute minimum at $(2, -3)$, local minimum at $(-2, 0)$

16. (a) Increasing on $(-4, -3.25)$, $(-1.5, 1)$, and $(2, 4)$, decreasing on $(-3.25, -1.5)$ and $(1, 2)$
    (b) Absolute maximum at $(4, 2)$, local maximum at $(-3.25, 1)$ and $(1, 1)$; Absolute minimum at $(-1.5, -1)$, local minimum at $(-4, 0)$ and $(2, 0)$

17. (a) Increasing on $(-4, -1)$, $(0.5, 2)$, and $(2, 4)$, decreasing on $(-1, 0.5)$
    (b) Absolute maximum at $(4, 3)$, local maximum at $(-1, 2)$ and $(2, 1)$; No absolute minimum, local minimum at $(-4, -1)$ and $(0.5, -1)$

18. (a) Increasing on $(-4, -2.5)$, $(-1, 1)$, and $(3, 4)$, decreasing on $(-2.5, -1)$ and $(1, 3)$
 (b) No absolute maximum, local maximum at $(-2.5, 1)$, $(1, 2)$ and $(4, 2)$; No absolute minimum, local minimum at $(-1, 0)$ and $(3, 1)$

19. (a) $g(t) = -t^2 - 3t + 3 \Rightarrow g'(t) = -2t - 3 \Rightarrow$ a critical point at $t = -\frac{3}{2}$; $g' = +++ \underset{-3/2}{|} ---$, increasing on $\left(-\infty, -\frac{3}{2}\right)$, decreasing on $\left(-\frac{3}{2}, \infty\right)$
 (b) local maximum value of $g\left(-\frac{3}{2}\right) = \frac{21}{4}$ at $t = -\frac{3}{2}$, absolute maximum is $\frac{21}{4}$ at $t = -\frac{3}{2}$

20. (a) $g(t) = -3t^2 + 9t + 5 \Rightarrow g'(t) = -6t + 9 \Rightarrow$ a critical point at $t = \frac{3}{2}$; $g' = +++ \underset{3/2}{|} ---$, increasing on $\left(-\infty, \frac{3}{2}\right)$, decreasing on $\left(\frac{3}{2}, \infty\right)$
 (b) local maximum value of $g\left(\frac{3}{2}\right) = \frac{47}{4}$ at $t = \frac{3}{2}$, absolute maximum is $\frac{47}{4}$ at $t = \frac{3}{2}$

21. (a) $h(x) = -x^3 + 2x^2 \Rightarrow h'(x) = -3x^2 + 4x = x(4 - 3x) \Rightarrow$ critical points at $x = 0, \frac{4}{3} \Rightarrow h' = --- \underset{0}{|} +++ \underset{4/3}{|} ---$, increasing on $\left(0, \frac{4}{3}\right)$, decreasing on $(-\infty, 0)$ and $\left(\frac{4}{3}, \infty\right)$
 (b) local maximum value of $h\left(\frac{4}{3}\right) = \frac{32}{27}$ at $x = \frac{4}{3}$; local minimum value of $h(0) = 0$ at $x = 0$, no absolute extrema

22. (a) $h(x) = 2x^3 - 18x \Rightarrow h'(x) = 6x^2 - 18 = 6\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right) \Rightarrow$ critical points at $x = \pm\sqrt{3}$ $\Rightarrow h' = +++ \underset{-\sqrt{3}}{|} --- \underset{\sqrt{3}}{|} +++$, increasing on $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$, decreasing on $\left(-\sqrt{3}, \sqrt{3}\right)$
 (b) a local maximum is $h\left(-\sqrt{3}\right) = 12\sqrt{3}$ at $x = -\sqrt{3}$; local minimum is $h\left(\sqrt{3}\right) = -12\sqrt{3}$ at $x = \sqrt{3}$, no absolute extrema

23. (a) $f(\theta) = 3\theta^2 - 4\theta^3 \Rightarrow f'(\theta) = 6\theta - 12\theta^2 = 6\theta(1 - 2\theta) \Rightarrow$ critical points at $\theta = 0, \frac{1}{2}$ $\Rightarrow f' = --- \underset{0}{|} +++ \underset{1/2}{|} ---$, increasing on $\left(0, \frac{1}{2}\right)$, decreasing on $(-\infty, 0)$ and $\left(\frac{1}{2}, \infty\right)$
 (b) a local maximum is $f\left(\frac{1}{2}\right) = \frac{1}{4}$ at $\theta = \frac{1}{2}$, a local minimum is $f(0) = 0$ at $\theta = 0$, no absolute extrema

24. (a) $f(\theta) = 6\theta - \theta^3 \Rightarrow f'(\theta) = 6 - 3\theta^2 = 3\left(\sqrt{2} - \theta\right)\left(\sqrt{2} + \theta\right) \Rightarrow$ critical points at $\theta = \pm\sqrt{2}$ $\Rightarrow f' = --- \underset{-\sqrt{2}}{|} +++ \underset{\sqrt{2}}{|} ---$, increasing on $\left(-\sqrt{2}, \sqrt{2}\right)$, decreasing on $\left(-\infty, -\sqrt{2}\right)$ and $\left(\sqrt{2}, \infty\right)$
 (b) a local maximum is $f\left(\sqrt{2}\right) = 4\sqrt{2}$ at $\theta = \sqrt{2}$, a local minimum is $f\left(-\sqrt{2}\right) = -4\sqrt{2}$ at $\theta = -\sqrt{2}$, no absolute extrema

25. (a) $f(r) = 3r^3 + 16r \Rightarrow f'(r) = 9r^2 + 16 \Rightarrow$ no critical points $\Rightarrow f' = +++++$, increasing on $(-\infty, \infty)$, never decreasing
 (b) no local extrema, no absolute extrema

26. (a) $h(r) = (r + 7)^3 \Rightarrow h'(r) = 3(r + 7)^2 \Rightarrow$ a critical point at $r = -7 \Rightarrow h' = +++ \underset{-7}{|} +++$, increasing on $(-\infty, -7) \cup (-7, \infty)$, never decreasing
 (b) no local extrema, no absolute extrema

27. (a) $f(x)=x^4-8x^2+16 \Rightarrow f'(x)=4x^3-16x=4x(x+2)(x-2) \Rightarrow$ critical points at $x=0$ and $x=\pm 2$ $\Rightarrow f' = \underset{-2}{---|}\underset{0}{+++|}\underset{2}{---|}+++$, increasing on $(-2, 0)$ and $(2, \infty)$, decreasing on $(-\infty, -2)$ and $(0, 2)$
   (b) a local maximum is $f(0)=16$ at $x=0$, local minima are $f(\pm 2)=0$ at $x=\pm 2$, no absolute maximum; absolute minimum is 0 at $x=\pm 2$

28. (a) $g(x)=x^4-4x^3+4x^2 \Rightarrow g'(x)=4x^3-12x^2+8x=4x(x-2)(x-1) \Rightarrow$ critical points at $x=0, 1, 2$ $\Rightarrow g' = \underset{0}{---|}\underset{1}{+++|}\underset{2}{---|}+++$, increasing on $(0, 1)$ and $(2, \infty)$, decreasing on $(-\infty, 0)$ and $(1, 2)$
   (b) a local maximum is $g(1)=1$ at $x=1$, local minima are $g(0)=0$ at $x=0$ and $g(2)=0$ at $x=2$, no absolute maximum; absolute minimum is 0 at $x=0, 2$

29. (a) $H(t)=\frac{3}{2}t^4-t^6 \Rightarrow H'(t)=6t^3-6t^5=6t^3(1+t)(1-t) \Rightarrow$ critical points at $t=0, \pm 1$ $\Rightarrow H' = \underset{-1}{+++|}\underset{0}{---|}\underset{1}{+++|}---$, increasing on $(-\infty, -1)$ and $(0, 1)$, decreasing on $(-1, 0)$ and $(1, \infty)$
   (b) the local maxima are $H(-1)=\frac{1}{2}$ at $t=-1$ and $H(1)=\frac{1}{2}$ at $t=1$, the local minimum is $H(0)=0$ at $t=0$, absolute maximum is $\frac{1}{2}$ at $t=\pm 1$; no absolute minimum

30. (a) $K(t)=15t^3-t^5 \Rightarrow K'(t)=45t^2-5t^4=5t^2(3+t)(3-t) \Rightarrow$ critical points at $t=0, \pm 3$ $\Rightarrow K' = \underset{-3}{---|}\underset{0}{+++|}\underset{3}{+++|}---$, increasing on $(-3, 0)\cup(0, 3)$, decreasing on $(-\infty, -3)$ and $(3, \infty)$
   (b) a local maximum is $K(3)=162$ at $t=3$, a local minimum is $K(-3)=-162$ at $t=-3$, no absolute extrema

31. (a) $f(x)=x-6\sqrt{x-1} \Rightarrow f'(x)=1-\frac{3}{\sqrt{x-1}}=\frac{\sqrt{x-1}-3}{\sqrt{x-1}} \Rightarrow$ critical points at $x=1$ and $x=10 \Rightarrow f' = \underset{1}{(}\underset{10}{---|}+++$, increasing on $(10, \infty)$, decreasing on $(1, 10)$
   (b) a local minimum is $f(10)=-8$, a local and absolute maximum is $f(1)=1$, absolute minimum of -8 at $x=10$

32. (a) $g(x)=4\sqrt{x}-x^2+3 \Rightarrow g'(x)=\frac{2}{\sqrt{x}}-2x=\frac{2-2x^{3/2}}{\sqrt{x}} \Rightarrow$ critical points at $x=1$ and $x=0 \Rightarrow g' = \underset{0}{(}\underset{1}{+++|}---$, increasing on $(0, 1)$, decreasing on $(1, \infty)$
   (b) a local minimum is $f(0)=3$, a local maximum is $f(1)=6$, absolute maximum of 6 at $x=1$

33. (a) $g(x)=x\sqrt{8-x^2}=x(8-x^2)^{1/2} \Rightarrow g'(x)=(8-x^2)^{1/2}+x\left(\frac{1}{2}\right)(8-x^2)^{-1/2}(-2x) = \frac{2(2-x)(2+x)}{\sqrt{\left(2\sqrt{2}-x\right)\left(2\sqrt{2}+x\right)}}$ $\Rightarrow$ critical points at $x=\pm 2, \pm 2\sqrt{2} \Rightarrow g' = \underset{-2\sqrt{2}}{(}\underset{-2}{---|}\underset{2}{+++|}\underset{2\sqrt{2}}{---)}$, increasing on $(-2, 2)$, decreasing on $\left(-2\sqrt{2}, -2\right)$ and $\left(2, 2\sqrt{2}\right)$
   (b) local maxima are $g(2)=4$ at $x=2$ and $g\left(-2\sqrt{2}\right)=0$ at $x=-2\sqrt{2}$, local minima are $g(-2)=-4$ at $x=-2$ and $g\left(2\sqrt{2}\right)=0$ at $x=2\sqrt{2}$, absolute maximum is 4 at $x=2$; absolute minimum is -4 at $x=-2$

34. (a) $g(x)=x^2\sqrt{5-x}=x^2(5-x)^{1/2} \Rightarrow g'(x)=2x(5-x)^{1/2}+x^2\left(\frac{1}{2}\right)(5-x)^{-1/2}(-1)=\frac{5x(4-x)}{2\sqrt{5-x}} \Rightarrow$ critical points at $x=0, 4$ and $5 \Rightarrow g' = \underset{0}{---|}\underset{4}{+++|}\underset{5}{---)}$, increasing on $(0, 4)$, decreasing on $(-\infty, 0)$ and $(4, 5)$
   (b) a local maximum is $g(4)=16$ at $x=4$, a local minimum is 0 at $x=0$ and $x=5$, no absolute maximum; absolute minimum is 0 at $x=0, 5$

35. (a) $f(x)=\frac{x^2-3}{x-2} \Rightarrow f'(x)=\frac{2x(x-2)-(x^2-3)(1)}{(x-2)^2}=\frac{(x-3)(x-1)}{(x-2)^2} \Rightarrow$ critical points at $x=1, 3$
$\Rightarrow f'=+++\underset{1}{|}---\underset{2}{)(}---\underset{3}{|}+++$, increasing on $(-\infty, 1)$ and $(3, \infty)$, decreasing on $(1, 2)$ and $(2, 3)$, discontinuous at $x=2$
(b) a local maximum is $f(1)=2$ at $x=1$, a local minimum is $f(3)=6$ at $x=3$, no absolute extrema

36. (a) $f(x)=\frac{x^3}{3x^2+1} \Rightarrow f'(x)=\frac{3x^2(3x^2+1)-x^3(6x)}{(3x^2+1)^2}=\frac{3x^2(x^2+1)}{(3x^2+1)^2} \Rightarrow$ a critical point at $x=0 \Rightarrow f'=+++\underset{0}{|}+++$, increasing on $(-\infty, 0)\cup(0, \infty)$, and never decreasing
(b) no local extrema, no absolute extrema

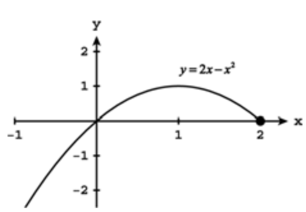
37. (a) $f(x)=x^{1/3}(x+8)=x^{4/3}+8x^{1/3} \Rightarrow f'(x)=\frac{4}{3}x^{1/3}+\frac{8}{3}x^{-2/3}=\frac{4(x+2)}{3x^{2/3}} \Rightarrow$ critical points at $x=0, -2$
$\Rightarrow f'=---\underset{-2}{|}+++\underset{0}{)(}+++$, increasing on $(-2, 0)\cup(0, \infty)$, decreasing on $(-\infty, -2)$
(b) no local maximum, a local minimum is $f(-2)=-6\sqrt[3]{2}\approx -7.56$ at $x=-2$, no absolute maximum; absolute minimum is $-6\sqrt[3]{2}$ at $x=-2$

38. (a) $g(x)=x^{2/3}(x+5)=x^{5/3}+5x^{2/3} \Rightarrow g'(x)=\frac{5}{3}x^{2/3}+\frac{10}{3}x^{-1/3}=\frac{5(x+2)}{3\sqrt[3]{x}} \Rightarrow =\frac{5(x+2)}{3^3\sqrt{x}} \Rightarrow$ critical points at $x=-2$ and $x=0 \Rightarrow g'=+++\underset{-2}{|}---\underset{0}{)(}+++$, increasing on $(-\infty, -2)$ and $(0, \infty)$, decreasing on $(-2, 0)$
(b) local maximum is $g(-2)=3\sqrt[3]{4}\approx 4.762$ at $x=-2$, a local minimum is $g(0)=0$ at $x=0$, no absolute extrema

39. (a) $h(x)=x^{1/3}(x^2-4)=x^{7/3}-4x^{1/3} \Rightarrow h'(x)=\frac{7}{3}x^{4/3}-\frac{4}{3}x^{-2/3}=\frac{\left(\sqrt{7}x+2\right)\left(\sqrt{7}x-2\right)}{3\sqrt[3]{x^2}} \Rightarrow$ critical points at $x=0, \frac{\pm 2}{\sqrt{7}} \Rightarrow h'=+++\underset{-2/\sqrt{7}}{|}---\underset{0}{)(}---\underset{2/\sqrt{7}}{|}+++$, increasing on $\left(-\infty, \frac{-2}{\sqrt{7}}\right)$ and $\left(\frac{2}{\sqrt{7}}, \infty\right)$, decreasing on $\left(\frac{-2}{\sqrt{7}}, 0\right)$ and $\left(0, \frac{2}{\sqrt{7}}\right)$
(b) local maximum is $h\left(\frac{-2}{\sqrt{7}}\right)=\frac{24\sqrt[3]{2}}{7^{7/6}}\approx 3.12$ at $x=\frac{-2}{\sqrt{7}}$, the local minimum is $h\left(\frac{2}{\sqrt{7}}\right)=-\frac{24\sqrt[3]{2}}{7^{7/6}}\approx -3.12$, no absolute extrema

40. (a) $k(x)=x^{2/3}(x^2-4)=x^{8/3}-4x^{2/3} \Rightarrow k'(x)=\frac{8}{3}x^{5/3}-\frac{8}{3}x^{-1/3}=\frac{8(x+1)(x-1)}{3\sqrt[3]{x}} \Rightarrow$ critical points at $x=0, \pm 1$
$\Rightarrow k'=---\underset{-1}{|}+++\underset{0}{)(}---\underset{1}{|}+++$, increasing on $(-1, 0)$ and $(1, \infty)$, decreasing on $(-\infty, -1)$ and $(0, 1)$
(b) local maximum is $k(0)=0$ at $x=0$, local minima are $k(\pm 1)=-3$ at $x=\pm 1$, no absolute maximum; absolute minimum is -3 at $x=\pm 1$

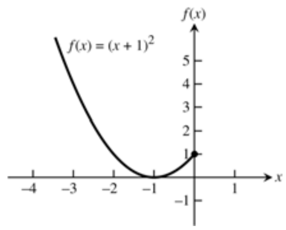
41. (a) $f(x)=2x-x^2 \Rightarrow f'(x)=2-2x \Rightarrow$ a critical point at $x=1 \Rightarrow f'=+++\underset{1}{|}---\underset{2}{]}$ and $f(1)=1$ and $f(2)=0$ a local maximum is 1 at $x=1$, a local minimum is 0 at $x=2$.
(b) There is an absolute maximum of 1 at $x=1$; no absolute minimum.
(c)



42. (a) $f(x)=(x+1)^2 \Rightarrow f'(x)=2(x+1) \Rightarrow$ a critical point at $x=-1 \Rightarrow f' = ---\underset{-1}{|}+++\underset{0}{]}$ and $f(-1)=0,\ f(0)=1 \Rightarrow$ a local maximum is 1 at $x=0$, a local minimum is 0 at $x=-1$

(b) no absolute maximum; absolute minimum is 0 at $x=-1$

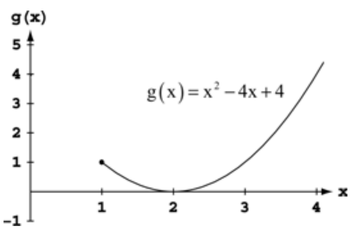
(c)



43. (a) $g(x)=x^2-4x+4 \Rightarrow g'(x)=2x-4=2(x-2) \Rightarrow$ a critical point at $x=2 \Rightarrow g' = \underset{1}{[}---\underset{2}{|}+++$ and $g(1)=1,\ g(2)=0 \Rightarrow$ a local maximum is 1 at $x=1$, a local minimum is $g(2)=0$ at $x=2$

(b) no absolute maximum; absolute minimum is 0 at $x=2$

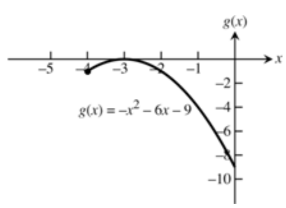
(c)



44. (a) $g(x)=-x^2-6x-9 \Rightarrow g'(x)=-2x-6=-2(x+3) \Rightarrow$ a critical point at $x=-3 \Rightarrow g' = \underset{-4}{[}+++\underset{-3}{|}---$ and $g(-4)=-1,\ g(-3)=0 \Rightarrow$ a local maximum is 0 at $x=-3$, a local minimum is -1 at $x=-4$

(b) absolute maximum is 0 at $x=-3$; no absolute minimum

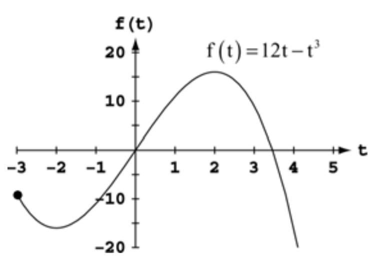
(c)



45. (a) $f(t)=12t-t^3 \Rightarrow f'(t)=12-3t^2=3(2+t)(2-t) \Rightarrow$ critical points at $t=\pm 2 \Rightarrow f' = \underset{-3}{[}---\underset{-2}{|}+++\underset{2}{|}---$ and $f(-3)=-9,\ f(-2)=-16,\ f(2)=16 \Rightarrow$ local maxima are -9 at $t=-3$ and 16 at $t=2$, a local minimum is -16 at $t=-2$

(b) absolute maximum is 16 at $t=2$; no absolute minimum

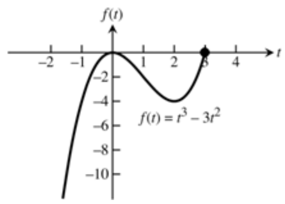
(c)



46. (a) $f(t) = t^3 - 3t^2 \Rightarrow f'(t) = 3t^2 - 6t = 3t(t-2) \Rightarrow$ critical points at $t = 0$ and $t = 2 \Rightarrow f' = +++\underset{0}{|}---\underset{2}{|}+++\underset{3}{]}$ and $f(0) = 0$, $f(2) = -4$, $f(3) = 0 \Rightarrow$ a local maximum is 0 at $t = 0$ and $t = 3$, a local minimum is -4 at $t = 2$

(b) absolute maximum is 0 at $t = 0, 3$; no absolute minimum

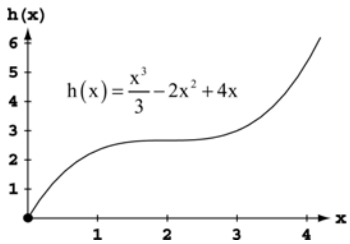
(c)



47. (a) $h(x) = \frac{x^3}{3} - 2x^2 + 4x \Rightarrow h'(x) = x^2 - 4x + 4 = (x-2)^2 \Rightarrow$ a critical point at $x = 2 \Rightarrow h' = \underset{0}{[}+++\underset{2}{|}+++$ and $h(0) = 0 \Rightarrow$ no local maximum, a local minimum is 0 at $x = 0$

(b) no absolute maximum; absolute minimum is 0 at $x = 0$

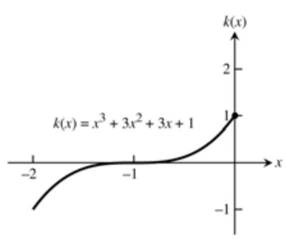
(c)



48. (a) $k(x) = x^3 + 3x^2 + 3x + 1 \Rightarrow k'(x) = 3x^2 + 6x + 3 = 3(x+1)^2 \Rightarrow$ a critical point at $x = -1 \Rightarrow k' = +++\underset{-1}{|}+++\underset{0}{]}$ and $k(-1) = 0$, $k(0) = 1 \Rightarrow$ a local maximum is 1 at $x = 0$, no local minimum

(b) absolute maximum is 1 at $x = 0$; no absolute minimum

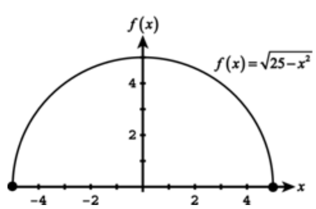
(c)



49. (a) $f(x) = \sqrt{25 - x^2} \Rightarrow f'(x) = \frac{-x}{\sqrt{25-x^2}} \Rightarrow$ critical points at $x = 0$, $x = -5$, and $x = 5 \Rightarrow f' = \underset{-5}{(}+++\underset{0}{|}---\underset{5}{)}$, $f(-5) = 0$, $f(0) = 5$, $f(5) = 0 \Rightarrow$ local maximum is 5 at $x = 0$; local minimum of 0 at $x = -5$ and $x = 5$

(b) absolute maximum is 5 at $x = 0$; absolute minimum of 0 at $x = -5$ and $x = 5$

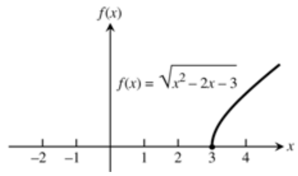
(c)



50. (a) $f(x)=\sqrt{x^2-2x-3}, 3\le x<\infty \Rightarrow f'(x)=\frac{2x-2}{\sqrt{x^2-2x-3}} \Rightarrow$ only critical point in $3\le x<\infty$ is at $x=3$
$\Rightarrow f' = \underset{3}{[}+++,\ f(3)=0 \Rightarrow$ local minimum of 0 at $x=3$, no local maximum

(b) absolute minimum of 0 at $x=3$, no absolute maximum

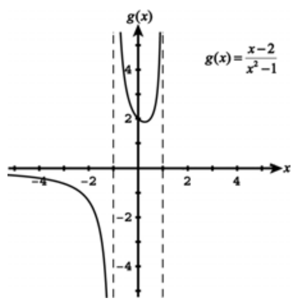
(c)



51. (a) $g(x)=\frac{x-2}{x^2-1}, 0\le x<1 \Rightarrow g'(x)=\frac{-x^2+4x-1}{(x^2-1)^2} \Rightarrow$ only critical point in $0\le x<1$ is $x=2-\sqrt{3}\approx 0.268$
$\Rightarrow g' = \underset{0}{[}---\underset{0.268}{|}+++\underset{1}{)},\ g\left(2-\sqrt{3}\right)=\frac{\sqrt{3}}{4\sqrt{3}-6}\approx 1.866 \Rightarrow$ local minimum of $\frac{\sqrt{3}}{4\sqrt{3}-6}$ at $x=2-\sqrt{3}$, local maximum at $x=0$.

(b) absolute minimum of $\frac{\sqrt{3}}{4\sqrt{3}-6}$ at $x=2-\sqrt{3}$, no absolute maximum

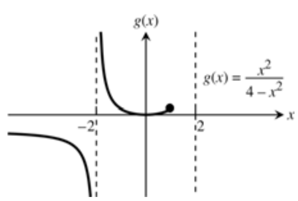
(c)



52. (a) $g(x)=\frac{x^2}{4-x^2}, -2<x\le 1 \Rightarrow g'(x)=\frac{8x}{(4-x^2)^2} \Rightarrow$ only critical point in $-2<x\le 1$ is $x=0$
$\Rightarrow g' = \underset{-2}{(}---\underset{0}{|}+++\underset{1}{]},\ g(0)=0 \Rightarrow$ local minimum of 0 at $x=0$, local maximum of $\frac{1}{3}$ at $x=1$.

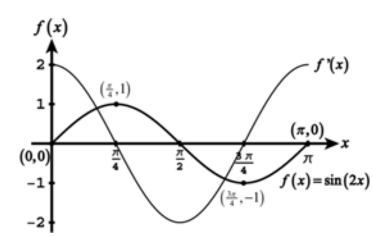
(b) absolute minimum of 0 at $x=0$, no absolute maximum

(c)



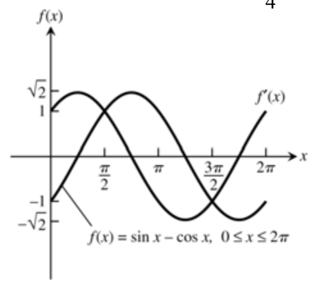
53. (a) $f(x)=\sin 2x,\ 0\le x\le \pi \Rightarrow f'(x)=2\cos 2x,\ f'(x)=0 \Rightarrow \cos 2x=0 \Rightarrow$ critical points are $x=\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$
$\Rightarrow f' = \underset{0}{[}+++\underset{\frac{\pi}{4}}{|}---\underset{\frac{3\pi}{4}}{|}+++\underset{\pi}{]},\ f(0)=0,\ f\left(\frac{\pi}{4}\right)=1,\ f\left(\frac{3\pi}{4}\right)=-1,\ f(\pi)=0 \Rightarrow$ local maxima are 1 at $x=\frac{\pi}{4}$ and 0 at $x=\pi$, and local minima are -1 at $x=\frac{3\pi}{4}$ and 0 at $x=0$.

(b) The graph of f rises when $f'>0$, falls when $f'<0$, and has local extreme values where $f'=0$. The function f has a local minimum value at $x=0$ and $x=\frac{3\pi}{4}$, where the values of f' change from negative to positive. The function f has a local maximum value at $x=\pi$ and $x=\frac{\pi}{4}$, where the values of f' change from positive to negative.



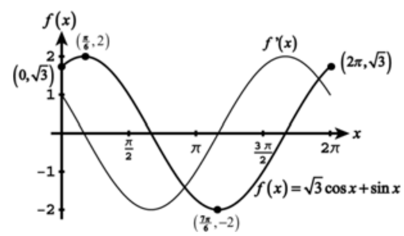
54. (a) $f(x) = \sin x - \cos x,\ 0 \le x \le 2\pi \Rightarrow f'(x) = \cos x + \sin x,\ f'(x) = 0 \Rightarrow \tan x = -1 \Rightarrow$ critical points are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4} \Rightarrow f' = \underset{0}{[}+++\underset{\frac{3\pi}{4}}{|}---\underset{\frac{7\pi}{4}}{|}+++\underset{2\pi}{]}$, $f(0) = -1,\ f\left(\frac{3\pi}{4}\right) = \sqrt{2},\ f\left(\frac{7\pi}{4}\right) = -\sqrt{2},\ f(2\pi) = -1 \Rightarrow$ local maxima are $\sqrt{2}$ at $x = \frac{3\pi}{4}$ and -1 at $x = 2\pi$, and local minima are $-\sqrt{2}$ at $x = \frac{7\pi}{4}$ and -1 at $x = 0$.

(b) The graph of f rises when $f' > 0$, falls when $f' < 0$, and has local extreme values where $f' = 0$. The function f has a local minimum value at $x = 0$ and $x = \frac{7\pi}{4}$, where the values of f' change from negative to positive. The function f has a local maximum value at $x = 2\pi$ and $x = \frac{3\pi}{4}$, where the values of f' change from positive to negative.



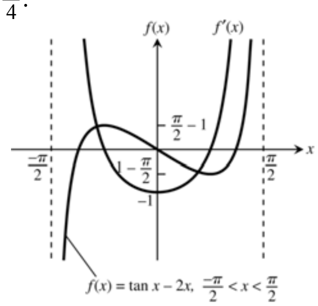
55. (a) $f(x) = \sqrt{3}\cos x + \sin x, 0 \le x \le 2\pi \Rightarrow f'(x) = -\sqrt{3}\sin x + \cos x,\ f'(x) = 0 \Rightarrow \tan x = \frac{1}{\sqrt{3}} \Rightarrow$ critical points $\Rightarrow$ are $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6} \Rightarrow f' = \underset{0}{[}+++\underset{\frac{\pi}{6}}{|}---\underset{\frac{7\pi}{6}}{|}+++\underset{2\pi}{]}$, $f(0) = \sqrt{3},\ f\left(\frac{\pi}{6}\right) = 2,\ f\left(\frac{7\pi}{6}\right) = -2,\ f(2\pi) = \sqrt{3}$ local maxima are 2 at $x = \frac{\pi}{6}$ and $\sqrt{3}$ at $x = 2\pi$, and local minima are -2 at $x = \frac{7\pi}{6}$ and $\sqrt{3}$ at $x = 0$.

(b) The graph of f rises when $f' > 0$, falls when $f' < 0$, and has local extreme values where $f' = 0$. The function f has a local minimum value at $x = 0$ and $x = \frac{7\pi}{6}$, where the values of f' change from negative to positive. The function f has a local maximum value at $x = 2\pi$ and $x = \frac{\pi}{6}$, where the values of f' change from positive to negative.



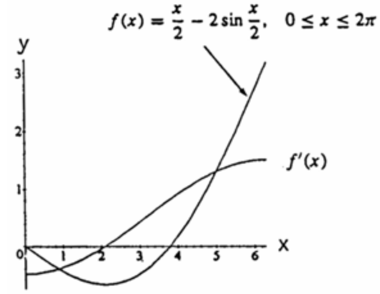
56. (a) $f(x) = -2x + \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow f'(x) = -2 + \sec^2 x,\ f'(x) = 0 \Rightarrow \sec^2 x = 2 \Rightarrow$ critical points are $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4} \Rightarrow f' = \underset{-\frac{\pi}{2}}{(}+++\underset{-\frac{\pi}{4}}{|}---\underset{\frac{\pi}{4}}{|}+++\underset{\frac{\pi}{2}}{)}$, $f\left(-\frac{\pi}{4}\right) = \frac{\pi}{2} - 1,\ f\left(\frac{\pi}{4}\right) = 1 - \frac{\pi}{2} \Rightarrow$ local maximum is $\frac{\pi}{2} - 1$ at $x = -\frac{\pi}{4}$, and local minimum is $1 - \frac{\pi}{2}$ at $x = \frac{\pi}{4}$.

(b) The graph of f rises when $f' > 0$, falls when $f' < 0$, and has local extreme values where $f' = 0$. The function f has a local minimum value at $x = \frac{\pi}{4}$, where the values of f' change from negative to positive. The function f has a local maximum value at $x = -\frac{\pi}{4}$, where the values of f' change from positive to negative.



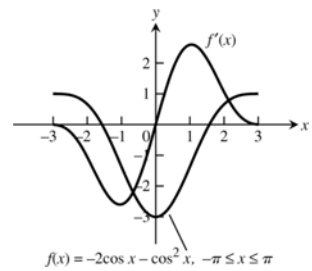
57. (a) $f(x) = \frac{x}{2} - 2\sin\left(\frac{x}{2}\right) \Rightarrow f'(x) = \frac{1}{2} - \cos\left(\frac{x}{2}\right),\ f'(x) = 0 \Rightarrow \cos\left(\frac{x}{2}\right) = \frac{1}{2} \Rightarrow$ a critical point at $x = \frac{2\pi}{3}$ $\Rightarrow f' = \underset{0}{[}---\underset{2\pi/3}{|}+++\underset{2\pi}{]}$ and $f(0) = 0,\ f\left(\frac{2\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3},\ f(2\pi) = \pi \Rightarrow$ local maxima are 0 at $x = 0$ and π at $x = 2\pi$, a local minimum is $\frac{\pi}{3} - \sqrt{3}$ at $x = \frac{2\pi}{3}$

(b) The graph of f rises when $f' > 0$, falls when $f' < 0$, and has a local minimum value at the point where f' changes from negative to positive.



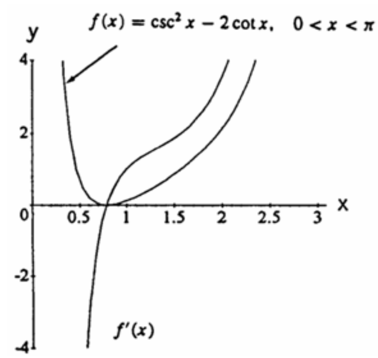
58. (a) $f(x) = -2\cos x - \cos^2 x \Rightarrow f'(x) = 2\sin x + 2\cos x \sin x = 2(\sin x)(1 + \cos x) \Rightarrow$ critical points at $x = -\pi, 0, \pi \Rightarrow f' = \underset{-\pi}{[}\ - - - \underset{0}{|} + + +\ \underset{\pi}{]}$ and $f(-\pi) = 1,\ f(0) = -3,\ f(\pi) = 1 \Rightarrow$ a local maximum is 1 at $x = \pm\pi$, a local minimum is -3 at $x = 0$

(b) The graph of f rises when $f' > 0$, falls when $f' < 0$, and has local extreme values where $f' = 0$. The function f has a local minimum value at $x = 0$, where the values of f' change from negative to positive.



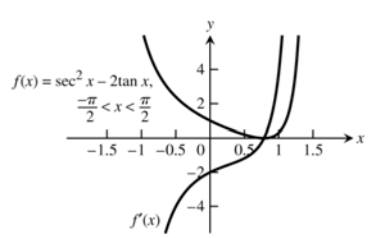
59. (a) $f(x) = \csc^2 x - 2\cot x \Rightarrow f'(x) = 2(\csc x)(-\csc x)(\cot x) - 2(-\csc^2 x) = -2(\csc^2 x)(\cot x - 1) \Rightarrow$ a critical point at $x = \frac{\pi}{4} \Rightarrow f' = \underset{0}{(}- - -\ \underset{\pi/4}{|}\ + + +\underset{\pi}{)}$ and $f\left(\frac{\pi}{4}\right) = 0 \Rightarrow$ no local maximum, a local minimum is 0 at $x = \frac{\pi}{4}$

(b) The graph of f rises when $f' > 0$, falls when $f' < 0$, and has a local minimum value at the point where $f' = 0$ and the values of f' change from negative to positive. The graph of f steepens as $f'(x) \to \pm\infty$.



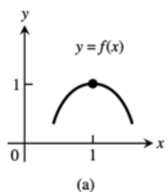
60. (a) $f(x) = \sec^2 x - 2\tan x \Rightarrow f'(x) = 2(\sec x)(\sec x)(\tan x) - 2\sec^2 x = (2\sec^2 x)(\tan x - 1) \Rightarrow$ a critical point at $x = \frac{\pi}{4} \Rightarrow f' = \underset{-\pi/2}{(}\ - - -\ \underset{\pi/4}{|}\ + + +\ \underset{\pi/2}{)}$ and $f\left(\frac{\pi}{4}\right) = 0 \Rightarrow$ no local maximum, a local minimum is 0 at $x = \frac{\pi}{4}$

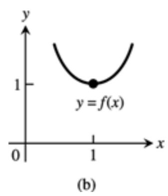
(b) The graph of f rises when $f' > 0$, falls when $f' < 0$, and has a local minimum value where $f' = 0$ and the values of f' change from negative to positive.

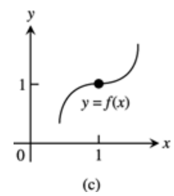


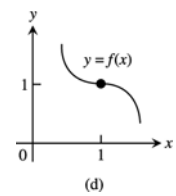
61. $h(\theta) = 3\cos\left(\frac{\theta}{2}\right) \Rightarrow h'(\theta) = -\frac{3}{2}\sin\left(\frac{\theta}{2}\right) \Rightarrow h' = \underset{0}{[}---\underset{2\pi}{]}$, $(0, 3)$ and $(2\pi, -3) \Rightarrow$ a local maximum is 3 at $\theta = 0$, a local minimum is -3 at $\theta = 2\pi$

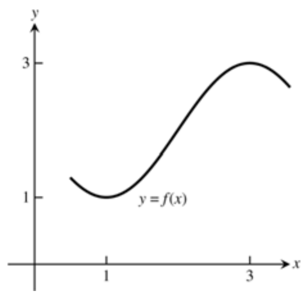
62. $h(\theta) = 5\sin\left(\frac{\theta}{2}\right) \Rightarrow h'(\theta) = \frac{5}{2}\cos\left(\frac{\theta}{2}\right) \Rightarrow h' = \underset{0}{[}+++\underset{\pi}{]}$, $(0, 0)$ and $(\pi, 5) \Rightarrow$ a local maximum is 5 at $\theta = \pi$, a local minimum is 0 at $\theta = 0$

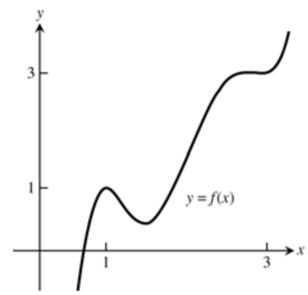
63. (a)


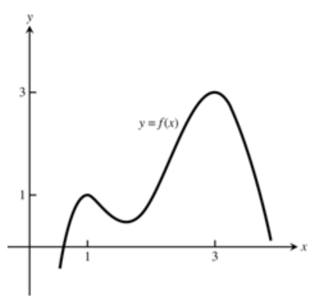
(b)


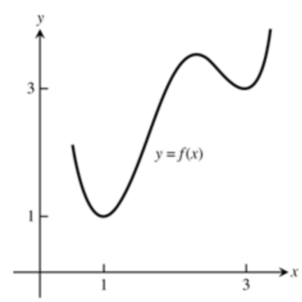
(c)


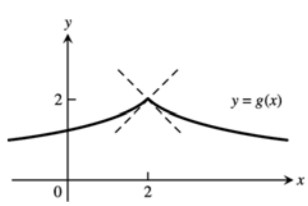
(d)


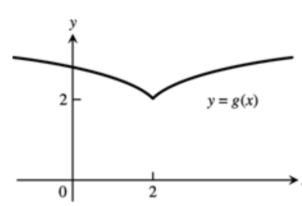
64. (a)


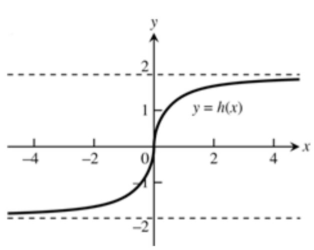
(b)


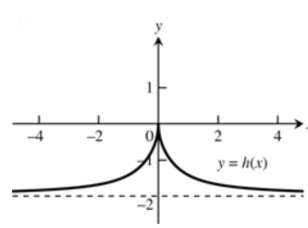
(c)


(d)


65. (a)


(b)


66. (a)


(b)


67. The function $f(x)=x\sin\left(\frac{1}{x}\right)$ has an infinite number of local maxima and minima on its domain, which is $(-\infty,0)\cup(0,\infty)$. The function $\sin x$ has the following properties: a) it is continuous on $(-\infty,\infty)$; b) it is periodic; and c) its range is $[-1,1]$. Also, for $a>0$, the function $\frac{1}{x}$ has a range of $(-\infty,-a]\cup[a,\infty)$ on $\left[-\frac{1}{a},0\right)\cup\left(0,\frac{1}{a}\right]$. In particular, if $a=1$, then $\frac{1}{x}\le -1$ or $\frac{1}{x}\ge 1$ when x is in $[-1,0)\cup(0,1]$. This means $\sin\left(\frac{1}{x}\right)$ takes on the values of 1 and -1 infinitely many times on $[-1,0)\cup(0,1]$, namely at $\frac{1}{x}=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\ldots \Rightarrow x=\pm\frac{2}{\pi},\pm\frac{2}{3\pi},\pm\frac{2}{5\pi},\ldots$. Thus $\sin\left(\frac{1}{x}\right)$ has infinitely many local maxima and minima in $[-1,0)\cup(0,1]$. On the interval $(0,1]$, $-1\le\sin\left(\frac{1}{x}\right)\le 1$ and since $x>0$ we have $-x\le x\sin\left(\frac{1}{x}\right)\le x$. On the interval $[-1,0)$, $-1\le\sin\left(\frac{1}{x}\right)\le 1$ and since $x<0$ we have $-x\ge x\sin\left(\frac{1}{x}\right)\ge x$. Thus $f(x)$ is bounded by the lines $y=x$ and $y=-x$. Since $\sin\left(\frac{1}{x}\right)$ oscillates between 1 and -1 infinitely many times on $[-1,0)\cup(0,1]$ then f will oscillate between $y=x$ and $y=-x$ infinitely many times. Thus f has infinitely many local maxima and minima. We can see from the graph (and verify later in Chapter 7) that $\lim\limits_{x\to\infty} x\sin\left(\frac{1}{x}\right)=1$ and $\lim\limits_{x\to-\infty} x\sin\left(\frac{1}{x}\right)=1$. The graph of f does not have any absolute maxima, but it does have two absolute minima.

68. $f(x)=ax^2+bx+c=a\left(x^2+\frac{b}{a}x\right)+c=a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)-\frac{b^2}{4a}+c=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}$, a parabola whose vertex is at $x=-\frac{b}{2a}$. Thus when $a>0$, f is increasing on $\left(\frac{-b}{2a},\infty\right)$ and decreasing on $\left(-\infty,\frac{-b}{2a}\right)$; when $a<0$, f is increasing on $\left(-\infty,\frac{-b}{2a}\right)$ and decreasing on $\left(\frac{-b}{2a},\infty\right)$. Also note that $f'(x)=2ax+b=2a\left(x+\frac{b}{2a}\right)\Rightarrow$ for $a>0$, $f'=\underset{-b/2a}{---\ |\ +++}$; for $a<0$, $f'=\underset{-b/2a}{+++\ |\ ---}$.

69. $f(x)=ax^2+bx\Rightarrow f'(x)=2ax+b,\ f(1)=2\Rightarrow a+b=2,\ f'(1)=0\Rightarrow 2a+b=0\Rightarrow a=-2, b=4$
$\Rightarrow f(x)=-2x^2+4x$

70. $f(x)=ax^3+bx^2+cx+d\Rightarrow f'(x)=3ax^2+2bx+c,\ f(0)=0\Rightarrow d=0,\ f(1)=-1\Rightarrow a+b+c+d=-1,$
$f'(0)=0\Rightarrow c=0,\ f'(1)=0\Rightarrow 3a+2b+c=0\Rightarrow a=2, b=-3, c=0, d=0\Rightarrow f(x)=2x^3-3x^2$

4.4 CONCAVITY AND CURVE SKETCHING

1. $y=\frac{x^3}{3}-\frac{x^2}{2}-2x+\frac{1}{3}\Rightarrow y'=x^2-x-2=(x-2)(x+1)\Rightarrow y''=2x-1=2\left(x-\frac{1}{2}\right)$. The graph is rising on $(-\infty,-1)$ and $(2,\infty)$, falling on $(-1,2)$, concave up on $\left(\frac{1}{2},\infty\right)$ and concave down on $\left(-\infty,\frac{1}{2}\right)$. Consequently, a local maximum is $\frac{3}{2}$ at $x=-1$, a local minimum is -3 at $x=2$, and $\left(\frac{1}{2},-\frac{3}{4}\right)$ is a point of inflection.

2. $y=\frac{x^4}{4}-2x^2+4\Rightarrow y'=x^3-4x=x(x^2-4)=x(x+2)(x-2)\Rightarrow y''=3x^2-4=\left(\sqrt{3}x+2\right)\left(\sqrt{3}x-2\right)$. The graph is rising on $(-2,0)$ and $(2,\infty)$, falling on $(-\infty,-2)$ and $(0,2)$, concave up on $\left(-\infty,-\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}},\infty\right)$ and concave down on $\left(-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)$. Consequently, a local maximum is 4 at $x=0$, local minima are 0 at $x=\pm 2$, and $\left(-\frac{2}{\sqrt{3}},\frac{16}{9}\right)$ and $\left(\frac{2}{\sqrt{3}},\frac{16}{9}\right)$ are points of inflection.

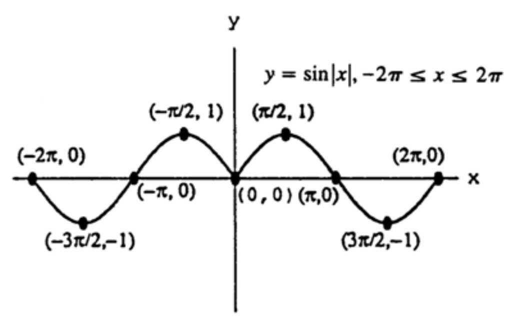
3. $y=\frac{3}{4}(x^2-1)^{2/3} \Rightarrow y'=\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)(x^2-1)^{-1/3}(2x)=x(x^2-1)^{-1/3}$, $y' = ---\underset{-1}{)}\ (+++\underset{0}{|}---\underset{1}{)}(+++ \Rightarrow$ the graph is rising on $(-1, 0)$ and $(1, \infty)$, falling on $(-\infty, -1)$ and $(0, 1) \Rightarrow$ a local maximum is $\frac{3}{4}$ at $x=0$, local minima are 0 at $x=\pm 1$; $y''=(x^2-1)^{-1/3}+(x)\left(-\frac{1}{3}\right)(x^2-1)^{-4/3}(2x)=\frac{x^2-3}{3\sqrt[3]{(x^2-1)^4}}$,
$y'' = +++\underset{-\sqrt{3}}{|}---\underset{-1}{)}\ (---\underset{1}{)}(---\underset{\sqrt{3}}{|}+++ \Rightarrow$ the graph is concave up on $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$, concave down on $\left(-\sqrt{3}, \sqrt{3}\right) \Rightarrow$ points of inflection at $\left(\pm\sqrt{3}, \frac{3\sqrt[3]{4}}{4}\right)$

4. $y=\frac{9}{14}x^{1/3}(x^2-7) \Rightarrow y'=\frac{3}{14}x^{-2/3}(x^2-7)+\frac{9}{14}x^{1/3}(2x)=\frac{3}{2}x^{-2/3}(x^2-1)$, $y' = +++\underset{-1}{|}---\underset{0}{)}(---\underset{1}{|}+++ \Rightarrow$ the graph is rising on $(-\infty, -1)$ and $(1, \infty)$, falling on $(-1, 1) \Rightarrow$ a local maximum is $\frac{27}{7}$ at $x=-1$, a local minimum is $-\frac{27}{7}$ at $x=1$; $y''=-x^{-5/3}(x^2-1)+3x^{1/3}=2x^{1/3}+x^{-5/3}=x^{-5/3}(2x^2+1)$, $y'' = ---\underset{0}{)}(+++ \Rightarrow$ the graph is concave up on $(0, \infty)$, concave down on $(-\infty, 0) \Rightarrow$ a point of inflection at $(0, 0)$.

5. $y=x+\sin 2x \Rightarrow y'=1+2\cos 2x$, $y' = \underset{-2\pi/3}{[}----\underset{-\pi/3}{|}+++\underset{\pi/3}{|}---\underset{2\pi/3}{]} \Rightarrow$ the graph is rising on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, falling on $\left(-\frac{2\pi}{3}, -\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right) \Rightarrow$ local maxima are $-\frac{2\pi}{3}+\frac{\sqrt{3}}{2}$ at $x=-\frac{2\pi}{3}$ and $\frac{\pi}{3}+\frac{\sqrt{3}}{2}$ at $x=\frac{\pi}{3}$, local minima are $-\frac{\pi}{3}-\frac{\sqrt{3}}{2}$ at $x=-\frac{\pi}{3}$ and $\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$ at $x=\frac{2\pi}{3}$; $y''=-4\sin 2x$, $y'' = \underset{-2\pi/3}{[}---\underset{-\pi/2}{|}+++\underset{0}{|}---\underset{\pi/2}{|}+++\underset{2\pi/3}{]} \Rightarrow$ the graph is concave up on $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$, concave down on $\left(-\frac{2\pi}{3}, -\frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right) \Rightarrow$ points of inflection at $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$, $(0, 0)$, and $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

6. $y=\tan x-4x \Rightarrow y'=\sec^2 x-4$, $y' = \underset{-\pi/2}{(}+++\underset{-\pi/3}{|}---\underset{\pi/3}{|}+++\underset{\pi/2}{)} \Rightarrow$ the graph is rising on $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, falling on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \Rightarrow$ a local maximum is $-\sqrt{3}+\frac{4\pi}{3}$ at $x=-\frac{\pi}{3}$, a local minimum is $\sqrt{3}-\frac{4\pi}{3}$ at $x=\frac{\pi}{3}$; $y''=2(\sec x)(\sec x)(\tan x)=2(\sec^2 x)(\tan x)$, $y'' = \underset{-\pi/2}{(}---\underset{0}{|}+++\underset{\pi/2}{)} \Rightarrow$ the graph is concave up on $\left(0, \frac{\pi}{2}\right)$, concave down on $\left(-\frac{\pi}{2}, 0\right) \Rightarrow$ a point of inflection at $(0, 0)$

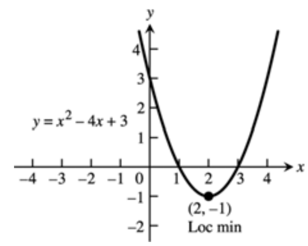
7. If $x \geq 0$, $\sin|x|=\sin x$ and if $x<0$, $\sin|x|=\sin(-x)$ $=-\sin x$. From the sketch the graph is rising on $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$, $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, falling on $\left(-2\pi, -\frac{3\pi}{2}\right)$, $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$; local minima are -1 at $x=\pm\frac{3\pi}{2}$ and 0 at $x=0$; local maxima are 1 at $x=\pm\frac{\pi}{2}$ and 0 at $x=\pm 2\pi$; concave up on $(-2\pi, -\pi)$ and $(\pi, 2\pi)$, and concave down on $(-\pi, 0)$ and $(0, \pi) \Rightarrow$ points of inflection are $(-\pi, 0)$ and $(\pi, 0)$



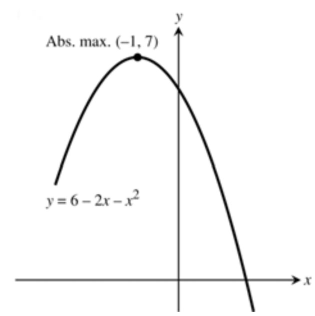
8. $y=2\cos x-\sqrt{2}x \Rightarrow y'=-2\sin x-\sqrt{2}$, $y' = \underset{-\pi}{[}---\underset{-3\pi/4}{|}+++\underset{-\pi/4}{|}---\underset{5\pi/4}{|}+++\underset{3\pi/2}{]} \Rightarrow$ rising on $\left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$, falling on $\left(-\pi, -\frac{3\pi}{4}\right)$ and $\left(-\frac{\pi}{4}, \frac{5\pi}{4}\right) \Rightarrow$ local maxima are $-2+\pi\sqrt{2}$ at $x=-\pi$, $\sqrt{2}+\frac{\pi\sqrt{2}}{4}$ at $x=-\frac{\pi}{4}$ and $-\frac{3\pi\sqrt{2}}{2}$ at $x=\frac{3\pi}{2}$, and local minima are $-\sqrt{2}+\frac{3\pi\sqrt{2}}{4}$ at $x=-\frac{3\pi}{4}$ and $-\sqrt{2}-\frac{5\pi\sqrt{2}}{4}$

at $x = \frac{5\pi}{4}$; $y'' = -2\cos x$, $y'' = \underset{-\pi}{[}\ +++\ \underset{-\pi/2}{|}\ ---\ \underset{\pi/2}{|}\ +++\ \underset{3\pi/2}{]}$ $\Rightarrow$ concave up on $\left(-\pi, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, concave down on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow$ points of inflection at $\left(-\frac{\pi}{2}, \frac{\sqrt{2}\pi}{2}\right)$ and $\left(\frac{\pi}{2}, -\frac{\sqrt{2}\pi}{2}\right)$

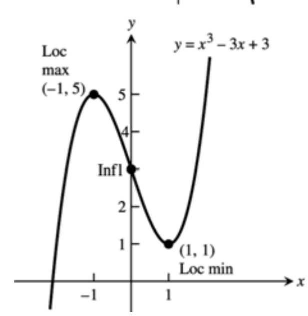
9. When $y = x^2 - 4x + 3$, then $y' = 2x - 4 = 2(x-2)$ and $y'' = 2$. The curve rises on $(2, \infty)$ and falls on $(-\infty, 2)$. At $x = 2$ there is a minimum. Since $y'' > 0$, the curve is concave up for all x.



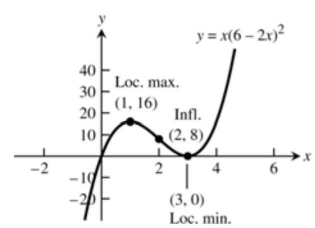
10. When $y = 6 - 2x - x^2$, then $y' = -2 - 2x = -2(1+x)$ and $y'' = -2$. The curve rises on $(-\infty, -1)$ and falls on $(-1, \infty)$. At $x = -1$ there is a maximum. Since $y'' < 0$, the curve is concave down for all x.



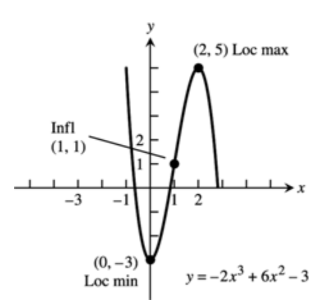
11. When $y = x^3 - 3x + 3$, then $y' = 3x^2 - 3$ $= 3(x-1)(x+1)$ and $y'' = 6x$. The curve rises on $(-\infty, -1) \cup (1, \infty)$ and falls on $(-1, 1)$. At $x = -1$ there is a local maximum and at $x = 1$ a local minimum. The curve is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. There is a point on inflection at $x = 0$.



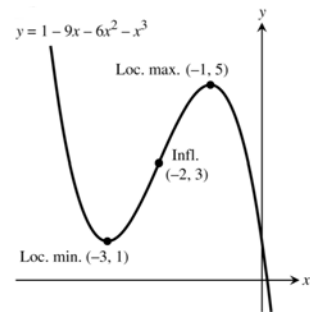
12. When $y = x(6-2x)^2$, then
$y' = -4x(6-2x) + (6-2x)^2 = 12(3-x)(1-x)$ and
$y'' = -12(3-x) - 12(1-x) = 24(x-2)$. The curve rises on $(-\infty, 1) \cup (3, \infty)$ and falls on $(1, 3)$. The curve is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$. At $x = 2$ there is a point of inflection.



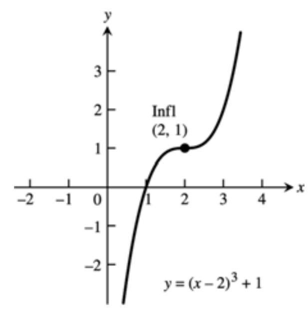
13. When $y = -2x^3 + 6x^2 - 3$, then $y' = -6x^2 + 12x$ $= -6x(x-2)$ and $y'' = -12x + 12 = -12(x-1)$. The curve rises on $(0, 2)$ and falls on $(-\infty, 0)$ and $(2, \infty)$. At $x = 0$ there is a local minimum and at $x = 2$ a local maximum. The curve is concave up on $(-\infty, 1)$ and concave down on $(1, \infty)$. At $x = 1$ there is a point of inflection.



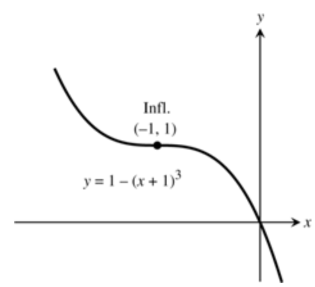
14. When $y = 1 - 9x - 6x^2 - x^3$, then $y' = -9 - 12x - 3x^2$ $= -3(x+3)(x+1)$ and $y'' = -12 - 6x = -6(x+2)$. The curve rises on $(-3, -1)$ and falls on $(-\infty, -3)$ and $(-1, \infty)$. At $x = -1$ there is a local maximum and at $x = -3$ a local minimum. The curve is concave up on $(-\infty, -2)$ and concave down on $(-2, \infty)$. At $x = -2$ there is a point of inflection.



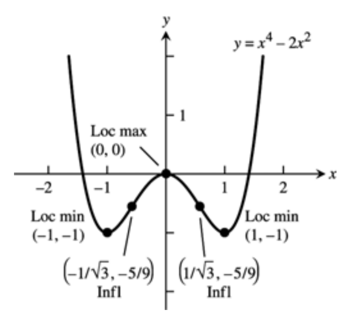
15. When $y = (x-2)^3 + 1$, then $y' = 3(x-2)^2$ and $y'' = 6(x-2)$. The curve never falls and there are no local extrema. The curve is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$. At $x = 2$ there is a point of inflection.



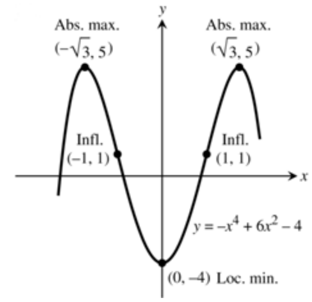
16. When $y = 1 - (x+1)^3$, then $y' = -3(x+1)^2$ and $y'' = -6(x+1)$. The curve never rises and there are no local extrema. The curve is concave up on $(-\infty, -1)$ and concave down on $(-1, \infty)$. At $x = -1$ there is a point of inflection.



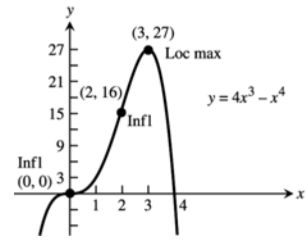
17. When $y = x^4 - 2x^2$, then $y' = 4x^3 - 4x$ $= 4x(x+1)(x-1)$ and $y'' = 12x^2 - 4$ $= 12\left(x + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}\right)$. The curve rises on $(-1, 0)$ and $(1, \infty)$ and falls on $(-\infty, -1)$ and $(0, 1)$. At $x = \pm 1$ there are local minima and at $x = 0$ a local maximum. The curve is concave up on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$ and concave down on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. At $x = \frac{\pm 1}{\sqrt{3}}$ there are points of inflection.



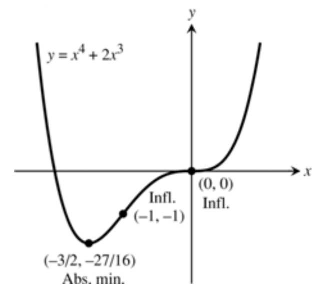
18. When $y = -x^4 + 6x^2 - 4$, then $y' = -4x^3 + 12x$ $= -4x\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right)$ and $y'' = -12x^2 + 12$ $= -12(x+1)(x-1)$. The curve rises on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$, and falls on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$. At $x = \pm\sqrt{3}$ there are local maxima and at $x = 0$ a local minimum. The curve is concave up on $(-1, 1)$ and concave down on $(-\infty, -1)$ and $(1, \infty)$. At $x = \pm 1$ there are points of inflection.



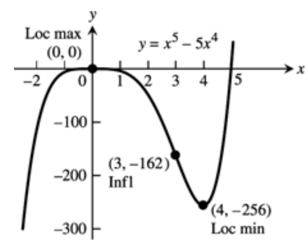
19. When $y = 4x^3 - x^4$, then $y' = 12x^2 - 4x^3 = 4x^2(3-x)$ and $y'' = 24x - 12x^2 = 12x(2-x)$. The curve rises on $(-\infty, 3)$ and falls on $(3, \infty)$. At $x = 3$ there is a local maximum, but there is no local minimum. The graph is concave up on $(0, 2)$ and concave down on $(-\infty, 0)$ and $(2, \infty)$. There are inflection points at $x = 0$ and $x = 2$.



20. When $y = x^4 + 2x^3$, then $y' = 4x^3 + 6x^2 = 2x^2(2x+3)$ and $y'' = 12x^2 + 12x = 12x(x+1)$. The curve rises on $\left(-\frac{3}{2}, \infty\right)$ and falls on $\left(-\infty, -\frac{3}{2}\right)$. There is a local minimum at $x = -\frac{3}{2}$, but no local maximum. The curve is concave up on $(-\infty, -1)$ and $(0, \infty)$, and concave down on $(-1, 0)$. At $x = -1$ and $x = 0$ there are points of inflection.



21. When $y = x^5 - 5x^4$, then $y' = 5x^4 - 20x^3 = 5x^3(x-4)$ and $y'' = 20x^3 - 60x^2 = 20x^2(x-3)$. The curve rises on $(-\infty, 0)$ and $(4, \infty)$, and falls on $(0, 4)$. There is a local maximum at $x = 0$, and a local minimum at $x = 4$. The curve is concave down on $(-\infty, 3)$ and concave up on $(3, \infty)$. At $x = 3$ there is a point of inflection.

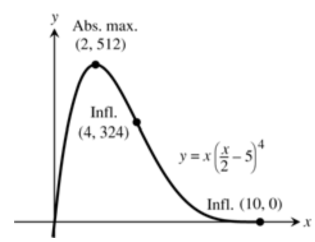


22. When $y = x\left(\frac{x}{2} - 5\right)^4$, then

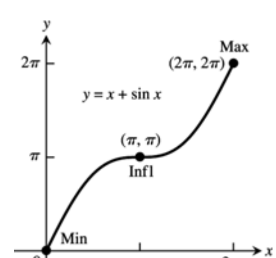
$y' = \left(\frac{x}{2} - 5\right)^4 + x(4)\left(\frac{x}{2} - 5\right)^3\left(\frac{1}{2}\right) = \left(\frac{x}{2} - 5\right)^3\left(\frac{5x}{2} - 5\right),$

and $y'' = 3\left(\frac{x}{2} - 5\right)^2\left(\frac{1}{2}\right)\left(\frac{5x}{2} - 5\right) + \left(\frac{x}{2} - 5\right)^3\left(\frac{5}{2}\right)$

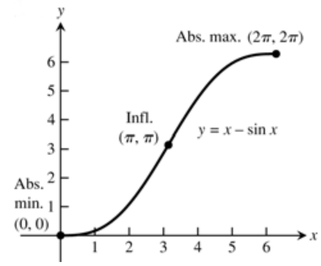
$= 5\left(\frac{x}{2} - 5\right)^2(x-4)$. The curve is rising on $(-\infty, 2)$ and $(10, \infty)$, and falling on $(2, 10)$. There is a local maximum at $x = 2$ and a local minimum at $x = 10$. The curve is concave down on $(-\infty, 4)$ and concave up on $(4, \infty)$. At $x = 4$ there is a point of inflection.



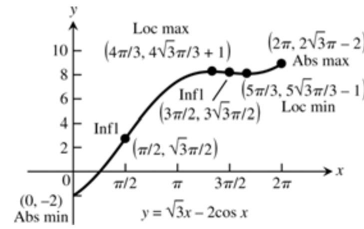
23. When $y = x + \sin x$, then $y' = 1 + \cos x$ and $y'' = -\sin x$. The curve rises on $(0, 2\pi)$. At $x = 0$ there is a local and absolute minimum and at $x = 2\pi$ there is a local and absolute maximum. The curve is concave down on $(0, \pi)$ and concave up on $(\pi, 2\pi)$. At $x = \pi$ there is a point of inflection.



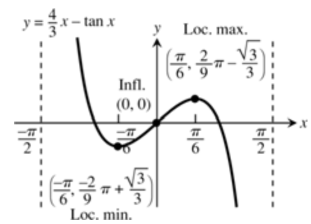
24. When $y = x - \sin x$, then $y' = 1 - \cos x$ and $y'' = \sin x$. The curve rises on $(0, 2\pi)$. At $x = 0$ there is a local and absolute minimum and at $x = 2\pi$ there is a local and absolute maximum. The curve is concave up on $(0, \pi)$ and concave down on $(\pi, 2\pi)$. At $x = \pi$ there is a point of inflection.



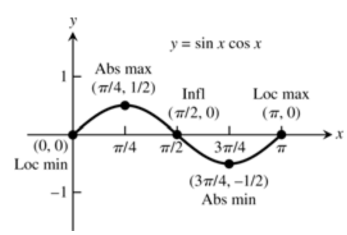
25. When $y = \sqrt{3}x - 2\cos x$, then $y' = \sqrt{3} + 2\sin x$ and $y'' = 2\cos x$. The curve is increasing on $\left(0, \frac{4\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, 2\pi\right)$, and decreasing on $\left(\frac{4\pi}{3}, \frac{5\pi}{3}\right)$. At $x = 0$ there is a local and absolute minimum, at $x = \frac{4\pi}{3}$ there is a local maximum, at $x = \frac{5\pi}{3}$ there is a local minimum, and at $x = 2\pi$ there is a local and absolute maximum. The curve is concave up on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, and is concave down on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. At $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ there are points of inflection.



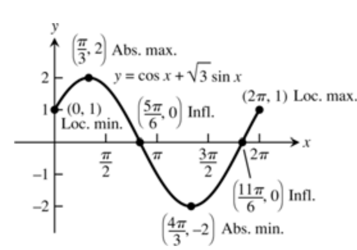
26. When $y = \frac{4}{3}x - \tan x$, then $y' = \frac{4}{3} - \sec^2 x$ and $y'' = -2\sec^2 x \tan x$. The curve is increasing on $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$, and decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{6}\right)$ and $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$. At $x = -\frac{\pi}{6}$ there is a local minimum, at $x = \frac{\pi}{6}$ there is a local maximum, there are no absolute maxima or absolute minima. The curve is concave up on $\left(-\frac{\pi}{2}, 0\right)$, and is concave down on $\left(0, \frac{\pi}{2}\right)$. At $x = 0$ there is a point of inflection.



27. When $y = \sin x \cos x$, then $y' = -\sin^2 x + \cos^2 x = \cos 2x$ and $y'' = -2\sin 2x$. The curve is increasing on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$, and decreasing on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$. At $x = 0$ there is a local minimum, at $x = \frac{\pi}{4}$ there is a local and absolute maximum, at $x = \frac{3\pi}{4}$ there is a local and absolute minimum, and at $x = \pi$ there is a local maximum. The curve is concave down on $\left(0, \frac{\pi}{2}\right)$, and is concave up on $\left(\frac{\pi}{2}, \pi\right)$. At $x = \frac{\pi}{2}$ there is a point of inflection.

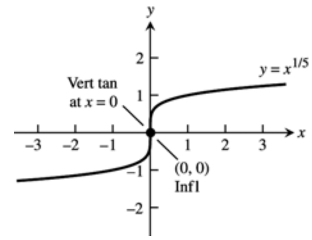


28. When $y = \cos x + \sqrt{3}\sin x$, then $y' = -\sin x + \sqrt{3}\cos x$ and $y'' = -\cos x - \sqrt{3}\sin x$. The curve is increasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$, and decreasing on $\left(\frac{\pi}{3}, \frac{4\pi}{3}\right)$. At $x = 0$ there is a local minimum, at $x = \frac{\pi}{3}$ there is a local and absolute maximum, at $x = \frac{4\pi}{3}$ there is a local and absolute minimum, and at $x = 2\pi$ there is a local maximum. The curve is concave down on

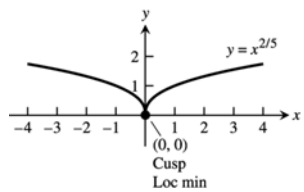


$\left(0, \frac{5\pi}{6}\right)$ and $\left(\frac{11\pi}{6}, 2\pi\right)$, and is concave up on $\left(\frac{5\pi}{6}, \frac{11\pi}{6}\right)$. At $x = \frac{5\pi}{6}$ and $x = \frac{11\pi}{6}$ there are points of inflection.

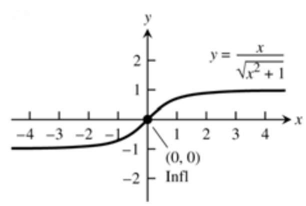
29. When $y = x^{1/5}$, then $y' = \frac{1}{5}x^{-4/5}$ and $y'' = -\frac{4}{25}x^{-9/5}$. The curve rises on $(-\infty, \infty)$ and there are no extrema. The curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. At $x = 0$ there is a point of inflection.



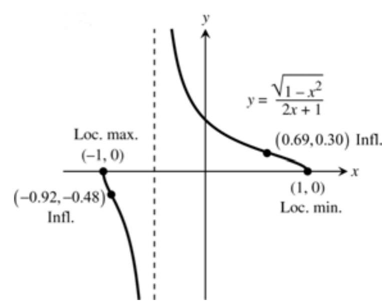
30. When $y = x^{2/5}$, then $y' = \frac{2}{5}x^{-3/5}$ and $y'' = -\frac{6}{25}x^{-8/5}$. The curve is rising on $(0, \infty)$ and falling on $(-\infty, 0)$. At $x = 0$ there is a local and absolute minimum. There is no local or absolute maximum. The curve is concave down on $(-\infty, 0)$ and $(0, \infty)$. There are no points of inflection, but a cusp exists at $x = 0$.



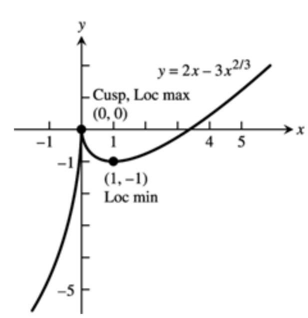
31. When $y = \frac{x}{\sqrt{x^2+1}}$, then $y' = \frac{1}{(x^2+1)^{3/2}}$ and $y'' = \frac{-3x}{(x^2+1)^{5/2}}$. The curve is increasing on $(-\infty, \infty)$. There are no local or absolute extrema. The curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. At $x = 0$ there is a point of inflection.



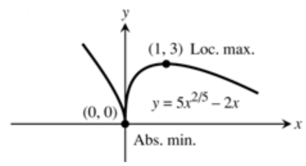
32. When $y = \frac{\sqrt{1-x^2}}{2x+1}$, then $y' = \frac{-(x+2)}{(2x+1)^2\sqrt{1-x^2}}$ and $y'' = \frac{-4x^3-12x^2+7}{(2x+1)^3(1-x^2)^{3/2}}$. The curve is decreasing on $\left(-1, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, 1\right)$. There are no absolute extrema, there is a local maximum at $x = -1$ and a local minimum at $x = 1$. The curve is concave up on $(-1, -0.92)$ and $\left(-\frac{1}{2}, 0.69\right)$, and concave down on $\left(-0.92, -\frac{1}{2}\right)$ and $(0.69, 1)$. At $x \approx -0.92$ and $x \approx 0.69$ there are points of inflection.



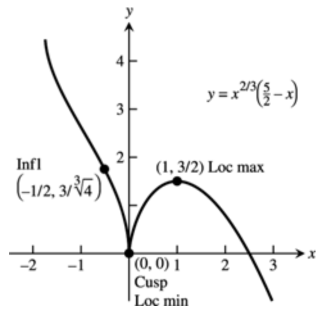
33. When $y = 2x - 3x^{2/3}$, then $y' = 2 - 2x^{-1/3}$ and $y'' = \frac{2}{3}x^{-4/3}$. The curve is rising on $(-\infty, 0)$ and $(1, \infty)$, and falling on $(0, 1)$. There is a local maximum at $x = 0$ and a local minimum at $x = 1$. The curve is concave up on $(-\infty, 0)$ and $(0, \infty)$. There are no points of inflection, but a cusp exists at $x = 0$.



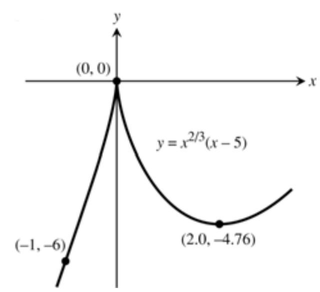
34. When $y = 5x^{2/5} - 2x$, then $y' = 2x^{-3/5} - 2$ $= 2\left(x^{-3/5} - 1\right)$ and $y'' = -\frac{6}{5}x^{-8/5}$. The curve is rising on (0, 1) and falling on $(-\infty, 0)$ and $(1, \infty)$. There is a local minimum at $x = 0$ and a local maximum at $x = 1$. The curve is concave down on $(-\infty, 0)$ and $(0, \infty)$. There are no points of inflection, but a cusp exists at $x = 0$.



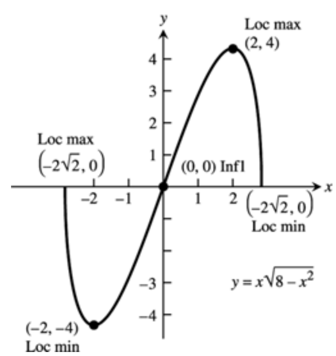
35. When $y = x^{2/3}\left(\frac{5}{2} - x\right) = \frac{5}{2}x^{2/3} - x^{5/3}$, then $y' = \frac{5}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{5}{3}x^{-1/3}(1 - x)$ and $y'' = -\frac{5}{9}x^{-4/3} - \frac{10}{9}x^{-1/3} = -\frac{5}{9}x^{-4/3}(1 + 2x)$. The curve is rising on (0, 1) and falling on $(-\infty, 0)$ and $(1, \infty)$. There is a local minimum at $x = 0$ and a local maximum at $x = 1$. The curve is concave up on $\left(-\infty, -\frac{1}{2}\right)$ and concave down on $\left(-\frac{1}{2}, 0\right)$ and $(0, \infty)$. There is a point of inflection at $x = -\frac{1}{2}$ and a cusp at $x = 0$.



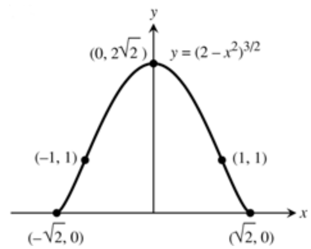
36. When $y = x^{2/3}(x - 5) = x^{5/3} - 5x^{2/3}$, then $y' = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x - 2)$ and $y'' = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x + 1)$. The curve is rising on $(-\infty, 0)$ and $(2, \infty)$, and falling on (0, 2). There is a local minimum at $x = 2$ and a local maximum at $x = 0$. The curve is concave up on $(-1, 0)$ and $(0, \infty)$, and concave down on $(-\infty, -1)$. There is a point of inflection at $x = -1$ and a cusp at $x = 0$.



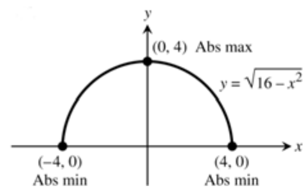
37. When $y = x\sqrt{8 - x^2} = x(8 - x^2)^{1/2}$, then $y' = (8 - x^2)^{1/2} + (x)\left(\frac{1}{2}\right)(8 - x^2)^{-1/2}(-2x)$ $= (8 - x^2)^{-1/2}(8 - 2x^2) = \frac{2(2-x)(2+x)}{\sqrt{\left(2\sqrt{2}+x\right)\left(2\sqrt{2}-x\right)}}$ and $y'' = \left(-\frac{1}{2}\right)(8 - x^2)^{-\frac{3}{2}}(-2x)(8 - 2x^2) + (8 - x^2)^{-\frac{1}{2}}(-4x)$ $= \frac{2x(x^2-12)}{\sqrt{(8-x^2)^3}}$. The curve is rising on $(-2, 2)$, and falling on $\left(-2\sqrt{2}, -2\right)$ and $\left(2, 2\sqrt{2}\right)$. There are local minima $x = -2$ and $x = 2\sqrt{2}$, and local maxima at $x = -2\sqrt{2}$ and $x = 2$. The curve is concave up on $\left(-2\sqrt{2}, 0\right)$ and concave down on $\left(0, 2\sqrt{2}\right)$. There is a point of inflection at $x = 0$.



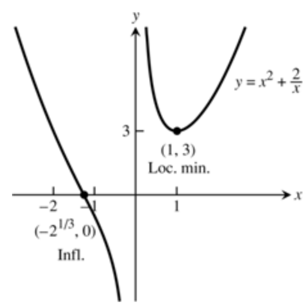
38. When $y=(2-x^2)^{3/2}$, then $y'=\left(\frac{3}{2}\right)(2-x^2)^{1/2}(-2x)$ $=-3x\sqrt{2-x^2}=-3x\sqrt{\left(\sqrt{2}-x\right)\left(\sqrt{2}+x\right)}$ and $y''=(-3)(2-x^2)^{1/2}+(-3x)\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)$ $=\frac{-6(1-x)(1+x)}{\sqrt{\left(\sqrt{2}-x\right)\left(\sqrt{2}+x\right)}}$. The curve is rising on $\left(-\sqrt{2},0\right)$ and falling on $\left(0,\sqrt{2}\right)$. There is a local maximum at $x=0$, and local minima at $x=\pm\sqrt{2}$. The curve is concave down on $(-1,1)$ and concave up on $\left(-\sqrt{2},-1\right)$ and $\left(1,\sqrt{2}\right)$. There are points of inflection at $x=\pm 1$.



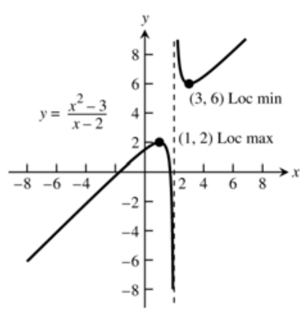
39. When $y=\sqrt{16-x^2}$, then $y'=\frac{-x}{\sqrt{16-x^2}}$ and $y''=\frac{-16}{(16-x^2)^{3/2}}$. The curve is rising on $(-4,0)$ and falling on $(0,4)$. There is a local and absolute maximum at $x=0$ and local and absolute minima at $x=-4$ and $x=4$. The curve is concave down on $(-4,4)$. There are no points of inflection.



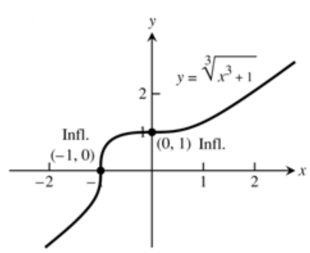
40. When $y=x^2+\frac{2}{x}$, then $y'=2x-\frac{2}{x^2}=\frac{2x^3-2}{x^2}$ and $y''=2+\frac{4}{x^3}=\frac{2x^3+4}{x^3}$. The curve is falling on $(-\infty,0)$ and $(0,1)$, and rising on $(1,\infty)$. There is a local minimum at $x=1$. There are no absolute maxima or absolute minima. The curve is concave up on $\left(-\infty,-\sqrt[3]{2}\right)$ and $(0,\infty)$, and concave down on $\left(-\sqrt[3]{2},0\right)$. There is a point of inflection at $x=-\sqrt[3]{2}$.



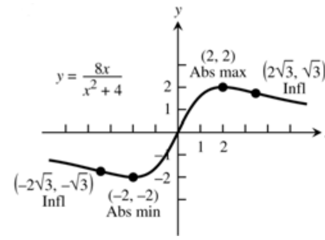
41. When $y=\frac{x^2-3}{x-2}$, then $y'=\frac{2x(x-2)-(x^2-3)(1)}{(x-2)^2}=\frac{(x-3)(x-1)}{(x-2)^2}$ and $y''=\frac{(2x-4)(x-2)^2-(x^2-4x+3)2(x-2)}{(x-2)^4}=\frac{2}{(x-2)^3}$. The curve is rising on $(-\infty,1)$ and $(3,\infty)$, and falling on $(1,2)$ and $(2,3)$. There is a local maximum at $x=1$ and a local minimum at $x=3$. The curve is concave down on $(-\infty,2)$ and concave up on $(2,\infty)$. There are no points of inflection because $x=2$ is not in the domain.



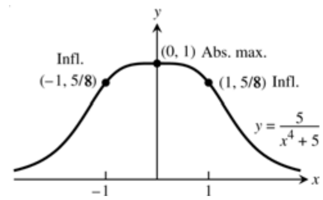
42. When $y=\sqrt[3]{x^3+1}$, then $y'=\frac{x^2}{(x^3+1)^{2/3}}$ and $y''=\frac{2x}{(x^3+1)^{5/3}}$. The curve is rising on $(-\infty,-1)$, $(-1,0)$, and $(0,\infty)$. There are no local or absolute extrema. The curve is concave up on $(-\infty,-1)$ and $(0,\infty)$, and concave down on $(-1,0)$. There are points of inflection at $x=-1$ and $x=0$.



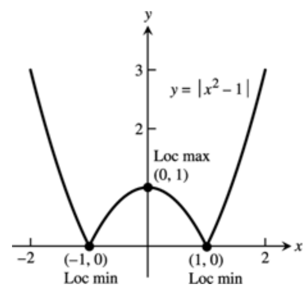
43. When $y = \frac{8x}{x^2+4}$, then $y' = \frac{-8(x^2-4)}{(x^2+4)^2}$ and $y'' = \frac{16x(x^2-12)}{(x^2+4)^3}$. The curve is falling on $(-\infty, -2)$ and $(2, \infty)$, and is rising on $(-2, 2)$. There is a local and absolute minimum at $x = -2$, and a local and absolute maximum at $x = 2$. The curve is concave down on $\left(-\infty, -2\sqrt{3}\right)$ and $\left(0, 2\sqrt{3}\right)$, and concave up on $\left(-2\sqrt{3}, 0\right)$ and $\left(2\sqrt{3}, \infty\right)$. There are points of inflection at $x = -2\sqrt{3}$, $x = 0$, and $x = 2\sqrt{3}$. $y = 0$ is a horizontal asymptote.



44. When $y = \frac{5}{x^4+5}$, then $y' = \frac{-20x^3}{(x^4+5)^2}$ and $y'' = \frac{100x^2(x^4-3)}{(x^4+5)^3}$. The curve is rising on $(-\infty, 0)$, and is falling on $(0, \infty)$. There is a local and absolute maximum at $x = 0$, and there is no local or absolute minimum. The curve is concave up on $\left(-\infty, -\sqrt[4]{3}\right)$ and $\left(\sqrt[4]{3}, \infty\right)$, and concave down on $\left(-\sqrt[4]{3}, 0\right)$ and $\left(0, \sqrt[4]{3}\right)$. There are points of inflection at $x = -\sqrt[4]{3}$ and $x = \sqrt[4]{3}$. There is a horizontal asymptote of $y = 0$.



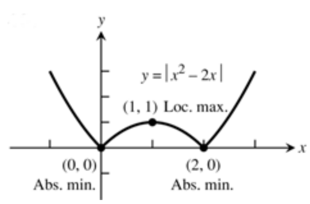
45. When $y = |x^2 - 1| = \begin{cases} x^2 - 1, |x| \ge 1 \\ 1 - x^2, |x| < 1 \end{cases}$, then $y' = \begin{cases} 2x, |x| > 1 \\ -2x, |x| < 1 \end{cases}$ and $y'' = \begin{cases} 2, |x| > 1 \\ -2, |x| < 1 \end{cases}$. The curve rises on $(-1, 0)$ and $(1, \infty)$ and falls on $(-\infty, -1)$ and $(0, 1)$. There is a local maximum at $x = 0$ and local minima at $x = \pm 1$. The curve is concave up on $(-\infty, -1)$ and $(1, \infty)$, and concave down on $(-1, 1)$. There are no points of inflection because y is not differentiable at $x = \pm 1$ (so there is no tangent line at those points).



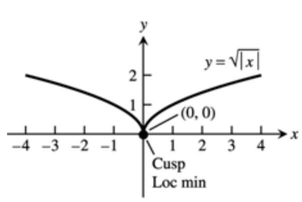
46. When $y = |x^2 - 2x| = \begin{cases} x^2 - 2x, x < 0 \\ 2x - x^2, 0 \le x \le 2, \\ x^2 - 2x, x > 2 \end{cases}$

then $y' = \begin{cases} 2x - 2, x < 0 \\ 2 - 2x, 0 < x < 2 \\ 2x - 2, x > 2 \end{cases}$, and $y'' = \begin{cases} 2, x < 0 \\ -2, 0 < x < 2 \\ 2, x > 2 \end{cases}$.

The curve is rising on $(0, 1)$ and $(2, \infty)$, and falling on $(-\infty, 0)$ and $(1, 2)$. There is a local maximum at $x = 1$ and local minima at $x = 0$ and $x = 2$. The curve is concave up on $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$. There are no points of inflection because y is not differentiable at $x = 0$ and $x = 2$ (so there is no tangent at those points).



47. When $y = \sqrt{|x|} = \begin{cases} \sqrt{x}, & x \ge 0 \\ \sqrt{-x}, & x < 0 \end{cases}$, then $y' = \begin{cases} \frac{1}{2\sqrt{x}}, & x > 0 \\ \frac{-1}{2\sqrt{-x}}, & x < 0 \end{cases}$



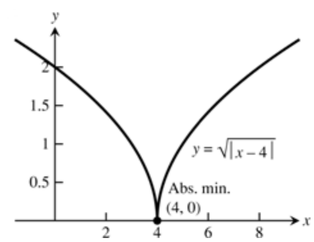
and $y'' = \begin{cases} \frac{-x^{-3/2}}{4}, & x > 0 \\ \frac{-(-x)^{-3/2}}{4}, & x < 0 \end{cases}$.

Since $\lim\limits_{x\to 0^-} y' = -\infty$ and $\lim\limits_{x\to 0^+} y' = \infty$ there is a cusp at $x = 0$. There is a local minimum at $x = 0$, but no local maximum. The curve is concave down on $(-\infty, 0)$ and $(0, \infty)$. There are no points of inflection.

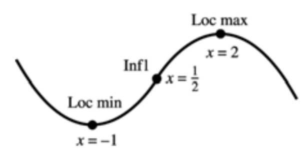
48. When $y = \sqrt{|x-4|} = \begin{cases} \sqrt{x-4}, & x \ge 4 \\ \sqrt{4-x}, & x < 4 \end{cases}$, then

$y' = \begin{cases} \frac{1}{2\sqrt{x-4}}, & x > 4 \\ \frac{-1}{2\sqrt{4-x}}, & x < 4 \end{cases}$ and $y'' = \begin{cases} \frac{-(x-4)^{-3/2}}{4}, & x > 4 \\ \frac{-(4-x)^{-3/2}}{4}, & x < 4 \end{cases}$.

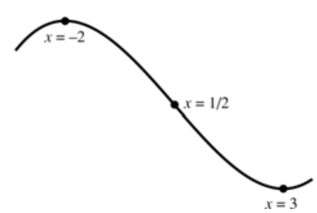
Since $\lim\limits_{x\to 4^-} y' = -\infty$ and $\lim\limits_{x\to 4^+} y' = \infty$ there is a cusp at $x = 4$. There is a local minimum at $x = 4$, but no local maximum. The curve is concave down on $(-\infty, 4)$ and $(4, \infty)$. There are no points of inflection.



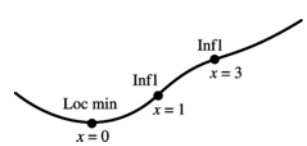
49. $y' = 2 + x - x^2 = (1+x)(2-x)$, $y' = ---\underset{-1}{|}+++\underset{2}{|}---$
$\Rightarrow$ rising on $(-1, 2)$, falling on $(-\infty, -1)$ and $(2, \infty)$
$\Rightarrow$ there is a local maximum at $x = 2$ and a local minimum at $x = -1$; $y'' = 1 - 2x$, $y'' = +++\underset{1/2}{|}---$
$\Rightarrow$ concave up on $\left(-\infty, \frac{1}{2}\right)$, concave down on $\left(\frac{1}{2}, \infty\right)$
$\Rightarrow$ a point of inflection at $x = \frac{1}{2}$



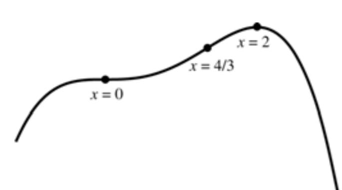
50. $y' = x^2 - x - 6 = (x-3)(x+2)$, $y' = +++\underset{-2}{|}---\underset{3}{|}+++$
$\Rightarrow$ rising on $(-\infty, -2)$ and $(3, \infty)$, falling on $(-2, 3)$
$\Rightarrow$ there is a local maximum at $x = -2$ and a local minimum at $x = 3$; $y'' = 2x - 1$, $y'' = ---\underset{1/2}{|}+++$
$\Rightarrow$ concave up on $\left(\frac{1}{2}, \infty\right)$, concave down on $\left(-\infty, \frac{1}{2}\right)$
$\Rightarrow$ a point of inflection at $x = \frac{1}{2}$



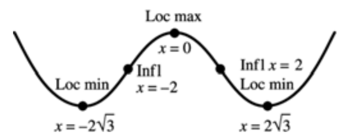
51. $y' = x(x-3)^2$, $y' = ---\underset{0}{|}+++\underset{3}{|}+++ \Rightarrow$ rising on $(0, \infty)$, falling on $(-\infty, 0) \Rightarrow$ no local maximum, but there is a local minimum at $x = 0$; $y'' = (x-3)^2 + x(2)(x-3) = 3(x-3)(x-1)$, $y'' = +++\underset{1}{|}---\underset{3}{|}+++ \Rightarrow$ concave up on $(-\infty, 1)$ and $(3, \infty)$, concave down on $(1, 3) \Rightarrow$ points of inflection at $x = 1$ and $x = 3$



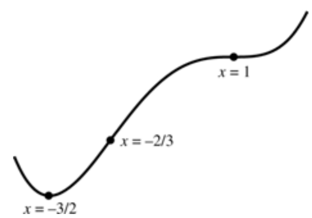
52. $y' = x^2(2-x)$, $y' = +++\underset{0}{|}+++\underset{2}{|}--- \Rightarrow$ rising on $(-\infty, 2)$, falling on $(2, \infty) \Rightarrow$ there is a local maximum at $x = 2$, but no local minimum; $y'' = 2x(2-x) + x^2(-1) = x(4-3x)$, $y'' = ---\underset{0}{|}+++\underset{4/3}{|}--- \Rightarrow$ concave up on $\left(0, \frac{4}{3}\right)$, concave down on $(-\infty, 0)$ and $\left(\frac{4}{3}, \infty\right) \Rightarrow$ points of inflection at $x = 0$ and $x = \frac{4}{3}$



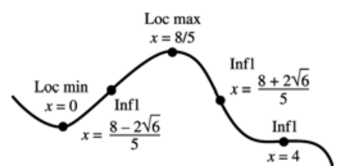
53. $y' = x(x^2-12) = x\left(x-2\sqrt{3}\right)\left(x+2\sqrt{3}\right)$,
$y' = ---\underset{-2\sqrt{3}}{|}+++\underset{0}{|}---\underset{2\sqrt{3}}{|}+++ \Rightarrow$ rising on $\left(-2\sqrt{3}, 0\right)$ and $\left(2\sqrt{3}, \infty\right)$, falling on $\left(-\infty, -2\sqrt{3}\right)$ and $\left(0, 2\sqrt{3}\right) \Rightarrow$ a local maximum at $x = 0$, local minima at $x = \pm 2\sqrt{3}$; $y'' = 1\,(x^2-12) + x(2x) = 3(x-2)(x+2)$,
$y'' = +++\underset{-2}{|}---\underset{2}{|}+++ \Rightarrow$ concave up on $(-\infty, -2)$ and $(2, \infty)$, concave down on $(-2, 2) \Rightarrow$ points of inflection at $x = \pm 2$



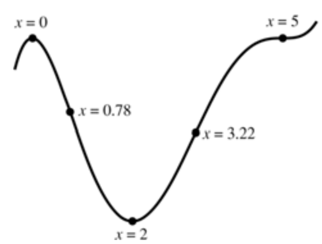
54. $y' = (x-1)^2(2x+3)$, $y' = ---\underset{-3/2}{|}+++\underset{1}{|}+++ \Rightarrow$ rising on $\left(-\frac{3}{2}, \infty\right)$, falling on $\left(-\infty, -\frac{3}{2}\right) \Rightarrow$ no local maximum, a local minimum at $x = -\frac{3}{2}$;
$y'' = 2(x-1)(2x+3) + (x-1)^2(2) = 2(x-1)(3x+2)$,
$y'' = +++\underset{-2/3}{|}---\underset{1}{|}+++ \Rightarrow$ concave up on $\left(-\infty, -\frac{2}{3}\right)$ and $(1, \infty)$, concave down on $\left(-\frac{2}{3}, 1\right) \Rightarrow$ points of inflection at $x = -\frac{2}{3}$ and $x = 1$



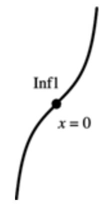
55. $y' = (8x-5x^2)(4-x)^2 = x(8-5x)(4-x)^2$,
$y' = ---\underset{0}{|}+++\underset{8/5}{|}---\underset{4}{|}--- \Rightarrow$ rising on $\left(0, \frac{8}{5}\right)$, falling on $(-\infty, 0)$ and $\left(\frac{8}{5}, \infty\right) \Rightarrow$ a local maximum at $x = \frac{8}{5}$, a local minimum at $x = 0$;
$y'' = (8-10x)(4-x)^2 + (8x-5x^2)(2)(4-x)(-1)$
$= 4(4-x)(5x^2-16x+8)$, $y'' = +++\underset{\frac{8-2\sqrt{6}}{5}}{|}---\underset{\frac{8+2\sqrt{6}}{5}}{|}+++\underset{4}{|}--- \Rightarrow$
concave up on $\left(-\infty, \frac{8-2\sqrt{6}}{5}\right)$ and $\left(\frac{8+2\sqrt{6}}{5}, 4\right)$, concave down on $\left(\frac{8-2\sqrt{6}}{5}, \frac{8+2\sqrt{6}}{5}\right)$ and $(4, \infty) \Rightarrow$ points of inflection at $x = \frac{8\pm 2\sqrt{6}}{5}$ and $x = 4$



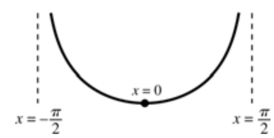
56. $y' = (x^2-2x)(x-5)^2 = x(x-2)(x-5)^2$,
$y' = +++\underset{0}{|}---\underset{2}{|}+++\underset{5}{|}+++ \Rightarrow$ rising on $(-\infty, 0)$ and $(2, \infty)$, falling on $(0, 2) \Rightarrow$ a local maximum at $x = 0$, a local minimum at $x = 2$;
$y'' = (2x-2)(x-5)^2 + 2(x^2-2x)(x-5)$
$= 2(x-5)(2x^2-8x+5)$, $y'' = ---\underset{\frac{4-\sqrt{6}}{2}}{|}+++\underset{\frac{4+\sqrt{6}}{2}}{|}---\underset{5}{|}+++ \Rightarrow$
concave up on $\left(\frac{4-\sqrt{6}}{2}, \frac{4+\sqrt{6}}{2}\right)$ and $(5, \infty)$, concave down on $\left(-\infty, \frac{4-\sqrt{6}}{2}\right)$ and $\left(\frac{4+\sqrt{6}}{2}, 5\right) \Rightarrow$ points of inflection at $x = \frac{4\pm\sqrt{6}}{2}$ and $x = 5$



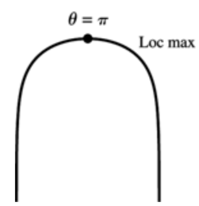
57. $y' = \sec^2 x$, $y' = \underset{-\pi/2}{(}\ +++\ \underset{\pi/2}{)} \Rightarrow$ rising on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, never falling $\Rightarrow$ no local extrema;
$y'' = 2(\sec x)(\sec x)(\tan x) = 2(\sec^2 x)(\tan x)$,
$y'' = \underset{-\pi/2}{(}\ ---\underset{0}{|}+++\ \underset{\pi/2}{)} \Rightarrow$ concave up on $\left(0, \frac{\pi}{2}\right)$, concave down on $\left(-\frac{\pi}{2}, 0\right)$, 0 is a point of inflection.



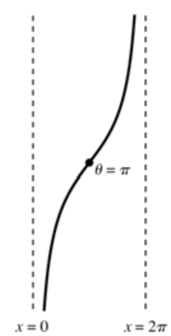
58. $y' = \tan x$, $y' = \underset{-\pi/2}{(}\ ---\underset{0}{|}+++\ \underset{\pi/2}{)} \Rightarrow$ rising on $\left(0, \frac{\pi}{2}\right)$, falling on $\left(-\frac{\pi}{2}, 0\right) \Rightarrow$ no local maximum, a local minimum at $x = 0$; $y'' = \sec^2 x$, $y'' = \underset{-\pi/2}{(}\ +++\ \underset{\pi/2}{)}$
$\Rightarrow$ concave up on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow$ no points of inflection



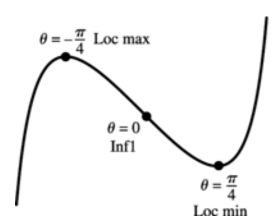
59. $y' = \cot\frac{\theta}{2}$, $y' = \underset{0}{(}+++\underset{\pi}{|}---\ \underset{2\pi}{)} \Rightarrow$ rising on $(0, \pi)$, falling on $(\pi, 2\pi) \Rightarrow$ a local maximum at $\theta = \pi$, no local minimum; $y'' = -\frac{1}{2}\csc^2\frac{\theta}{2}$, $y'' = \underset{0}{(}---\ \underset{2\pi}{)} \Rightarrow$ never concave up, concave down on $(0, 2\pi) \Rightarrow$ no points of inflection



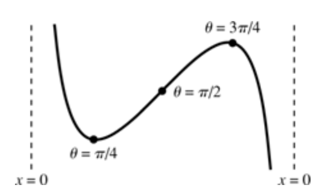
60. $y' = \csc^2\frac{\theta}{2}$, $y' = \underset{0}{(}+++\ \underset{2\pi}{)} \Rightarrow$ rising on $(0, 2\pi)$, never falling $\Rightarrow$ no local extrema;
$y'' = 2\left(\csc\frac{\theta}{2}\right)\left(-\csc\frac{\theta}{2}\right)\left(\cot\frac{\theta}{2}\right)\left(\frac{1}{2}\right)$
$= -\left(\csc^2\frac{\theta}{2}\right)\left(\cot\frac{\theta}{2}\right)$, $y'' = \underset{0}{(}---\underset{\pi}{|}+++\ \underset{2\pi}{)}$
$\Rightarrow$ concave up on $(\pi, 2\pi)$, concave down on $(0, \pi)$
$\Rightarrow$ a point of inflection at $\theta = \pi$



61. $y' = \tan^2\theta - 1 = (\tan\theta - 1)(\tan\theta + 1)$,
$y' = \underset{-\pi/2}{(}\ +++\ \underset{-\pi/4}{|}\ ---\ \underset{\pi/4}{|}\ +++\ \underset{\pi/2}{)} \Rightarrow$ rising on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, falling on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \Rightarrow$ a local maximum at $\theta = -\frac{\pi}{4}$, a local minimum at $\theta = \frac{\pi}{4}$; $y'' = 2\tan\theta\sec^2\theta$,
$y'' = \underset{-\pi/2}{(}\ ---\underset{0}{|}+++\ \underset{\pi/2}{)} \Rightarrow$ concave up on $\left(0, \frac{\pi}{2}\right)$, concave down on $\left(-\frac{\pi}{2}, 0\right) \Rightarrow$ a point of inflection at $\theta = 0$

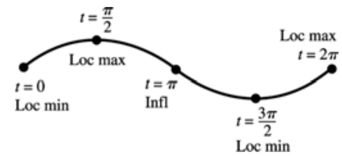


62. $y' = 1 - \cot^2\theta = (1 - \cot\theta)(1 + \cot\theta)$,
$y' = \underset{0}{(}---\ \underset{\pi/4}{|}\ +++\ \underset{3\pi/4}{|}\ ---\underset{\pi}{)} \Rightarrow$ rising on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, falling on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \pi\right) \Rightarrow$ a local maximum at $\theta = \frac{3\pi}{4}$, a local minimum at $\theta = \frac{\pi}{4}$;
$y'' = -2(\cot\theta)(-\csc^2\theta)$, $y'' = \underset{0}{(}+++\ \underset{\pi/2}{|}\ ---\ \underset{\pi}{)}$

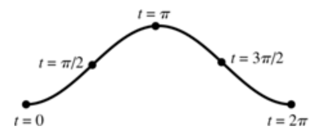


$\Rightarrow$ concave up on $\left(0, \frac{\pi}{2}\right)$, concave down on $\left(\frac{\pi}{2}, \pi\right)$
$\Rightarrow$ a point of infection at $\theta = \frac{\pi}{2}$

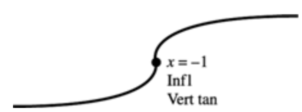
63. $y' = \cos t$, $y' = \underset{0}{[}+++\underset{\pi/2}{|}---\underset{3\pi/2}{|}+++\underset{2\pi}{]} \Rightarrow$ rising on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, falling on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \Rightarrow$ local maxima at $t = \frac{\pi}{2}$ and $t = 2\pi$, local minima at $t = 0$ and $t = \frac{3\pi}{2}$; $y'' = -\sin t$, $y'' = \underset{0}{[}---\underset{\pi}{|}+++\underset{2\pi}{]}$ $\Rightarrow$ concave up on $(\pi, 2\pi)$, concave down on $(0, \pi) \Rightarrow$ a point of inflection at $t = \pi$



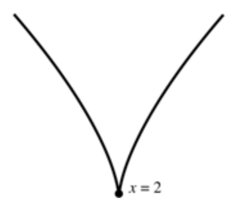
64. $y' = \sin t$, $y' = \underset{0}{[}+++\underset{\pi}{|}---\underset{2\pi}{]} \Rightarrow$ rising on $(0, \pi)$, falling on $(\pi, 2\pi) \Rightarrow$ a local maximum at $t = \pi$, local minima at $t = 0$ and $t = 2\pi$; $y'' = \cos t$, $y'' = \underset{0}{[}+++\underset{\pi/2}{|}---\underset{3\pi/2}{|}+++\underset{2\pi}{]} \Rightarrow$ concave up on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, concave down on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
$\Rightarrow$ points of inflection at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$



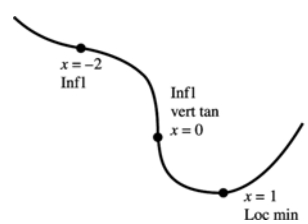
65. $y' = (x+1)^{-2/3}$, $y' = +++\underset{-1}{)}(+++ \Rightarrow$ rising on $(-\infty, \infty)$, never falling $\Rightarrow$ no local extrema; $y'' = -\frac{2}{3}(x+1)^{-5/3}$, $y'' = +++\underset{-1}{)}(---$ $\Rightarrow$ concave up on $(-\infty, -1)$, concave down on $(-1, \infty) \Rightarrow$ a point of inflection and vertical tangent at $x = -1$



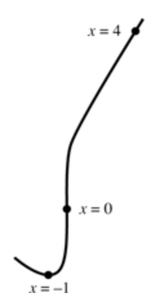
66. $y' = (x-2)^{-1/3}$, $y' = ---\underset{2}{)}(+++ \Rightarrow$ rising on $(2, \infty)$, falling on $(-\infty, 2) \Rightarrow$ no local maximum, but a local minimum at $x = 2$; $y'' = -\frac{1}{3}(x-2)^{-4/3}$, $y'' = ---\underset{2}{)}(--- \Rightarrow$ concave down on $(-\infty, 2)$ and $(2, \infty) \Rightarrow$ no points of inflection, but there is a cusp at $x = 2$



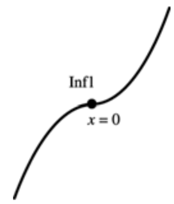
67. $y' = x^{-2/3}(x-1)$, $y' = ---\underset{0}{)}(---\underset{1}{|}+++ \Rightarrow$ rising on $(1, \infty)$, falling on $(-\infty, 1) \Rightarrow$ no local maximum, but a local minimum at $x = 1$; $y'' = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3}$ $= \frac{1}{3}x^{-5/3}(x+2)$, $y'' = +++\underset{-2}{|}---\underset{0}{)}(+++ \Rightarrow$ concave up on $(-\infty, -2)$ and $(0, \infty)$, concave down on $(-2, 0) \Rightarrow$ points of inflection at $x = -2$ and $x = 0$, and a vertical tangent at $x = 0$



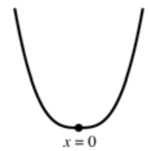
68. $y' = x^{-4/5}(x+1)$, $y' = ---\underset{-1}{|}+++\underset{0}{)}(+++ \Rightarrow$ rising on $(-1, 0)$ and $(0, \infty)$, falling on $(-\infty, -1) \Rightarrow$ no local maximum, but a local minimum at $x = -1$; $y'' = \frac{1}{5}x^{-4/5} - \frac{4}{5}x^{-9/5} = \frac{1}{5}x^{-9/5}(x-4)$, $y'' = +++\underset{0}{)}(---\underset{4}{|}+++ \Rightarrow$ concave up on $(-\infty, 0)$ and $(4, \infty)$, concave down on $(0, 4) \Rightarrow$ points of inflection at $x = 0$ and $x = 4$, and a vertical tangent at $x = 0$



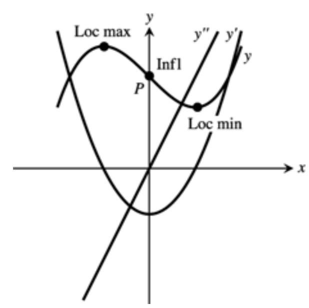
69. $y' = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$, $y' = +++\underset{0}{|}+++ \Rightarrow$ rising on $(-\infty, \infty) \Rightarrow$ no local extrema; $y'' = \begin{cases} -2, & x < 0 \\ 2, & x > 0 \end{cases}$, $y'' = ---\underset{0}{)}(+++ \Rightarrow$ concave up on $(0, \infty)$, concave down on $(-\infty, 0) \Rightarrow$ a point of inflection at $x = 0$



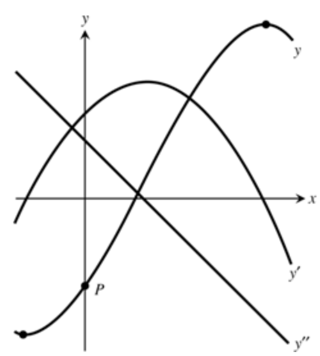
70. $y'' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$, $y' = ---\underset{0}{|}+++ \Rightarrow$ rising on $(0, \infty)$, falling on $(-\infty, 0) \Rightarrow$ no local maximum, but a local minimum at $x = 0$; $y'' = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$, $y'' = +++\underset{0}{|}+++ \Rightarrow$ concave up on $(-\infty, \infty)$ $\Rightarrow$ no point of inflection



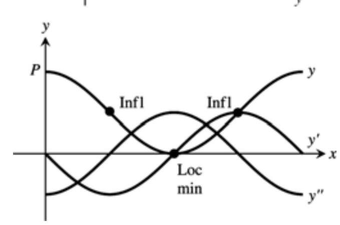
71. The graph of $y = f''(x) \Rightarrow$ the graph of $y = f(x)$ is concave up on $(0, \infty)$, concave down on $(-\infty, 0) \Rightarrow$ a point of inflection at $x = 0$; the graph of $y = f'(x) \Rightarrow y' = +++|---|+++ \Rightarrow$ the graph $y = f(x)$ has both a local maximum and a local minimum



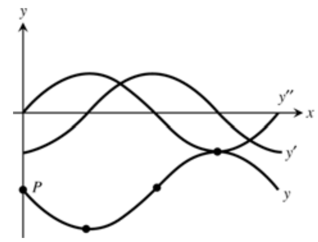
72. The graph of $y = f''(x) \Rightarrow y'' = +++|--- \Rightarrow$ the graph of $y = f(x)$ has a point of inflection, the graph of $y = f'(x) \Rightarrow y' = ---|+++|--- \Rightarrow$ the graph of $y = f(x)$ has both a local maximum and a local minimum



73. The graph of $y = f''(x) \Rightarrow y'' = ---|+++|---$ $\Rightarrow$ the graph of $y = f(x)$ has two points of inflection, the graph of $y = f'(x) \Rightarrow y' = ---|+++ \Rightarrow$ the graph of $y = f(x)$ has a local minimum



74. The graph of $y = f''(x) \Rightarrow y'' = +++|--- \Rightarrow$ the graph of $y = f(x)$ has a point of inflection; the graph of $y = f'(x) \Rightarrow y' = ---|+++|--- \Rightarrow$ the graph of $y = f(x)$ has both a local maximum and a local minimum

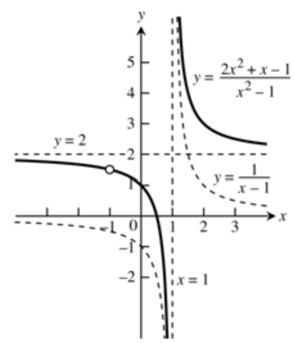


75. $y = \frac{2x^2+x-1}{x^2-1}$

Since -1 and 1 are roots of the denominator, the domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

$$y' = -\frac{1}{(x-1)^2};\; y'' = \frac{2}{(x-1)^3} \quad (x \neq -1)$$

There are no critical points. The function is decreasing on its domain. There are no inflection points. The function is concave down on $(-\infty, -1) \cup (-1, 1)$ and concave up on $(1, \infty)$. The numerator and denominator share a factor of $x+1$. Dividing out this common factor gives $y = \frac{2x-1}{x-1}$ $(x \neq 1)$, which shows that $x = 1$ is a vertical asymptote. Now dividing numerator and denominator by x gives $y = \frac{2-(1/x)}{1-(1/x)}$, which shows that $y = 2$ is a horizontal asymptote. The graph will have a hole at $x = -1$, $y = \frac{2(-1)-1}{1(-1)-1} = \frac{3}{2}$. The x-intercept is $\frac{1}{2}$.

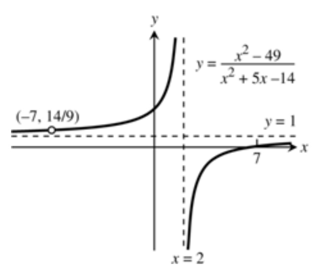


76. $y = \frac{x^2-49}{x^2+5x-14}$

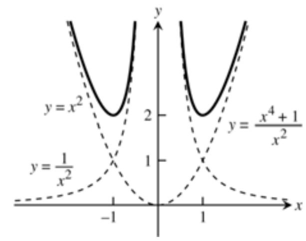
Since -7 and 2 are roots of the denominator, the domain is $(-\infty, -7) \cup (-7, 2) \cup (2, \infty)$.

$$y' = -\frac{5}{(x-2)^2};\; y'' = \frac{-10}{(x-1)^3} \quad (x \neq -7)$$

There are no critical points. The function is increasing on its domain. There are no inflection points. The function is concave up on $(-\infty, -7) \cup (-7, 2)$ and concave down on $(2, \infty)$. The numerator and denominator share a factor of $x+7$. Dividing out this common factor gives $y = \frac{x-7}{x-2}$ $(x \neq -7)$, which shows that $x = 1$ is a vertical asymptote. Now dividing numerator and denominator by x gives $y = \frac{1-(7/x)}{1-(2/x)}$, which shows that $y = 1$ is a horizontal asymptote. The graph will have a hole at $x = -7$, $y = \frac{(-1)-7}{(-7)-2} = \frac{14}{9}$. The x-intercept is $\frac{7}{2}$.



77. $y = \frac{x^4+1}{x^2}$

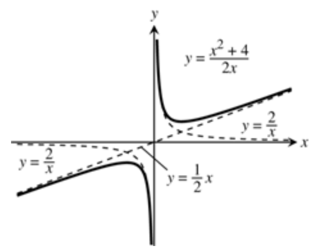


Since 0 is a root of the denominator, the domain is $(-\infty, 0) \cup (0, \infty)$.

$y' = \frac{2x^4 - 2}{x^3}$; $y'' = 2 + \frac{6}{x^4}$

There are critical points at $x = \pm 1$. The function is increasing on $(-1, 0) \cup (1, \infty)$ and decreasing on $(-\infty, -1) \cup (0, 1)$. There are no inflection points. The function is concave up on its domain. The y-axis is a vertical asymptote. Dividing numerator and denominator by x^2 gives $y = \frac{x^2+1/x^2}{1}$, which shows that there are no horizontal asymptotes. For large $|x|$, the graph is close to the graph of $y = x^2$.

78. $y = \frac{x^2+4}{2x}$

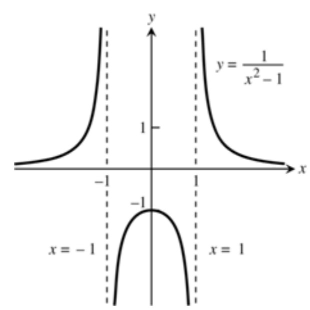


Since 0 is a root of the denominator, the domain is $(-\infty, 0) \cup (0, \infty)$.

$y' = \frac{x^2-4}{2x^2}$; $y'' = \frac{4}{x^3}$

There are no critical points at $x = \pm 2$. The function is increasing on $(-\infty, -2) \cup (2, \infty)$ and decreasing on $(-2, 0) \cup (0, 2)$. There are no inflection points. The function is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. The y-axis is a vertical asymptote. Dividing numerator and denominator by x gives $y = \frac{x+4/x}{2}$, which shows that the line $y = \frac{x}{2}$ is an asymptote.

79. $y = \frac{1}{x^2-1}$

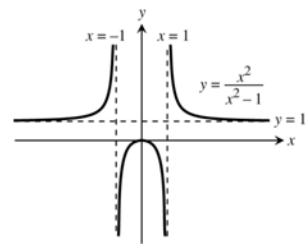


Since 1 and -1 are roots of the denominator, the domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

$y' = -\frac{2x}{(x^2-1)^2}$; $y'' = \frac{6x^2+2}{(x^2-1)^3}$

There is a critical point at $x = 0$, where the function has a local maximum. The function is increasing on $(-\infty, -1) \cup (-1, 0)$ and decreasing on $(0, 1) \cup (1, \infty)$. The function is concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$. The lines $x = 1$ and $x = -1$ are vertical asymptotes. The x-axis is a horizontal asymptote.

80. $y = \frac{x^2}{x^2-1}$

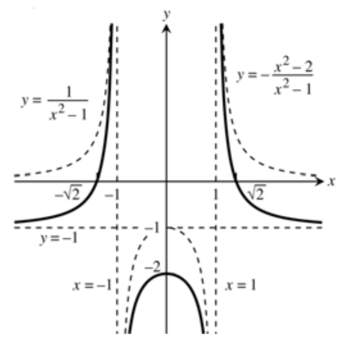


Since 1 and -1 are roots of the denominator, the domain is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

$y' = -\frac{2x}{(x^2-1)^2}$; $y'' = \frac{6x^2+2}{(x^2-1)^3}$

There is a critical point at $x = 0$, where the function has a local maximum. The function is increasing on $(-\infty,-1)\cup(-1,0)$ and decreasing on $(0,1)\cup(1,\infty)$. There are no inflection points. The function is concave up on $(-\infty,-1)\cup(1,\infty)$ and concave down on $(-1,1)$. The lines $x = 1$ and $x = -1$ are vertical asymptotes. Dividing numerator and denominator by x^2 gives $y = \frac{1}{1-(1/x^2)}$ which shows that the line $y = 1$ is a horizontal asymptote. The x-intercept is 0 and the y-intercept is 0.

81. $y = -\frac{x^2-2}{x^2-1}$

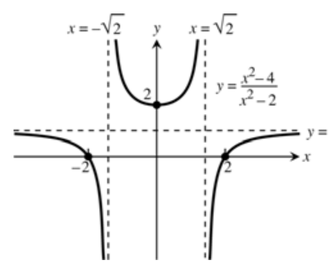


Since 1 and -1 are roots of the denominator, the domain is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

$y' = -\frac{2x}{(x^2-1)^2}$; $y'' = \frac{6x^2+2}{(x^2-1)^3}$

There is a critical point at $x = 0$, where the function has a local maximum. The function is increasing on $(-\infty,-1)\cup(-1,0)$ and decreasing on $(0,1)\cup(1,\infty)$. There are no inflection points. The function is concave up on $(-\infty,-1)\cup(1,\infty)$ and concave down on $(-1,1)$. The lines $x = 1$ and $x = -1$ are vertical asymptotes. Dividing numerator and denominator by x^2 gives $y = -\frac{1-(2/x^2)}{1-(1/x^2)}$ which shows that the line $y = -1$ is a horizontal asymptote. The x-intercepts are $\pm\sqrt{2}$ and the y-intercept is -2.

82. $y = \frac{x^2-4}{x^2-2}$



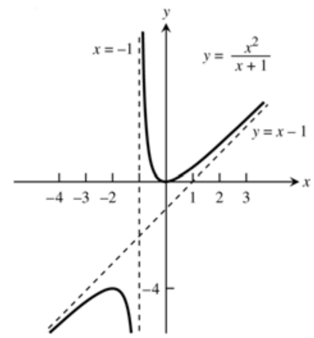
Since $\sqrt{2}$ and $-\sqrt{2}$ are roots of the denominator, the domain is $\left(-\infty,-\sqrt{2}\right)\cup\left(-\sqrt{2},\sqrt{2}\right)\cup\left(\sqrt{2},\infty\right)$.

$y' = \frac{4x}{(x^2-2)^2}$; $y'' = \frac{4(3x^2+2)}{(x^2-2)^3}$

There is a critical point at $x = 0$, where the function has a local minimum. The function is increasing on $\left(0,\sqrt{2}\right)\cup\left(\sqrt{2},\infty\right)$ and decreasing on $\left(-\infty,-\sqrt{2}\right)\cup\left(-\sqrt{2},0\right)$. There are no inflection points. The function is concave up on $\left(-\sqrt{2},\sqrt{2}\right)$ and concave down on $\left(-\infty,-\sqrt{2}\right)\cup\left(\sqrt{2},\infty\right)$. The lines $x = \sqrt{2}$ and $x = -\sqrt{2}$ are vertical asymptotes. Dividing numerator and denominator by

x^2 gives $y = -\frac{1-(4/x^2)}{1-(2/x^2)}$ which shows that the line $y = 1$ is a horizontal asymptote. The x-intercepts are $\pm\sqrt{2}$ and the y-intercept is 2 .

83. $y = \frac{x^2}{x+1}$

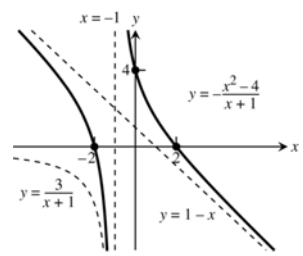


Since -1 is a root of the denominator, the domain is $(-\infty, -1) \cup (-1, \infty)$.

$y' = \frac{x^2+2x}{(x+1)^2}$; $y'' = \frac{2}{(x+1)^3}$

There is a critical point at $x = 0,$ where the function has a local minimum, and a critical point at $x = 2$ where the functions has a local maximum. The function is increasing on $(-\infty, -2) \cup (0, \infty)$ and decreasing on $(-2, -1) \cup (-1, 0)$. There are no inflection points. The function is concave up on $(-1, \infty)$ and concave down on $(-\infty, -1)$. The line $x = -1$ is a vertical asymptote. Dividing numerator by denominator gives $y = x - 1 + \frac{1}{x+1}$, which shows that the line $y = x - 1$ is an oblique asymptote. (See Section 2.6.) The x-intercept is 0 and the y-intercept is 0.

84. $y = -\frac{x^2-4}{x-1}$

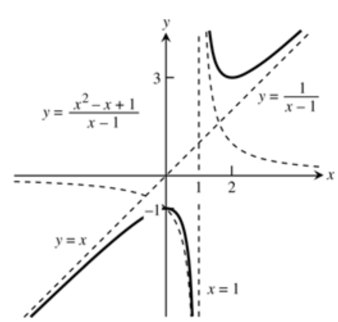


Since -1 is a root of the denominator, the domain is $(-\infty, -1) \cup (-1, \infty)$.

$y' = \frac{x^2+2x+4}{(x+1)^2}$; $y'' = \frac{6}{(x+1)^3}$

There are no critical points. The function is decreasing on its domain. There are no inflection points. The function is concave up on $(-1, \infty)$ and concave down on $(-\infty, -1)$. The line $x = -1$ is a vertical asymptote. Dividing numerator by denominator gives $y = 1 - x + \frac{3}{x+1}$, which shows that the line $y = 1 - x$ is an oblique asymptote. (See Section 2.6.) The x-intercepts are ± 2 and the y-intercept is 4.

85. $y = \frac{x^2-x+1}{x-1}$

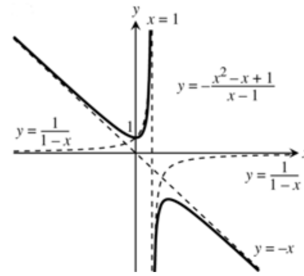


Since 1 is a root of the denominator, the domain is $(-\infty, 1) \cup (1, \infty)$.

$y' = \frac{x^2-2x}{(x-1)^2}$; $y'' = \frac{2}{(x-1)^3}$

There is a critical point at $x = 0,$ where the function has a local maximum, and a critical point at $x = 2$ where the function has a local minimum. The function is increasing on $(-\infty, 0) \cup (2, \infty)$ and decreasing on $(0, 1) \cup (1, 2)$. There are no inflection points. The function is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$. The line $x = 1$ is a vertical asymptote. Dividing numerator by denominator gives $y = x + \frac{1}{x-1}$ which shows that the line $y = x$ is an oblique asymptote. (See Section 2.6.) The y-intercept is -1.

86. $y = -\frac{x^2-x+1}{x-1}$

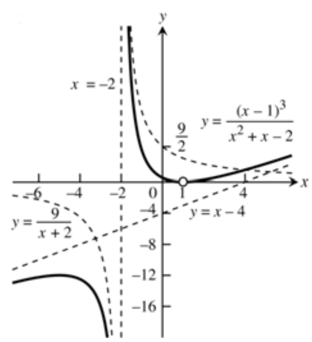


Since 1 is a root of the denominator, the domain is $(-\infty, 1) \cup (1, \infty)$.

$y' = \frac{2x-x^2}{(x-1)^2}$; $y'' = \frac{2}{(x-1)^3}$

There is a critical point at $x = 0$, where the function has a local minimum, and a critical point at $x = 2$ where the function has a local maximum. The function is increasing on $(0, 1) \cup (1, 2)$ and decreasing on $(-\infty, 0) \cup (2, \infty)$. There are no inflection points. The function is concave up on $(-\infty, 1)$ and concave down on $(1, \infty)$. The line $x = 1$ is a vertical asymptote. Dividing numerator by denominator gives $y = -x - \frac{1}{x-1}$ which shows that the line $y = -x$ is an oblique asymptote. (See Section 2.6.) The y-intercept is 1.

87. $y = \frac{x^3-3x^2+3x-1}{x^2+x-2} = \frac{(x-1)^3}{(x-1)(x+2)}$

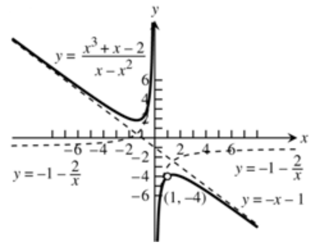


Since 1 and -2 are roots of the denominator, the domain is $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

$y' = \frac{(x-1)(x+5)}{(x+2)^2}, x \neq 1$; $y'' = \frac{18}{(x+2)^3}, x \neq 1$

Since 1 is not in the domain, the only critical point is at $x = -5$, where the function has a local maximum. The function is increasing on $(-\infty, -5) \cup (1, \infty)$ and decreasing on $(-5, -2) \cup (-2, 1)$. There are no inflection points. The function is concave up on $(-2, 1) \cup (1, \infty)$ and concave down on $(-\infty, -2)$. The line $x = -2$ is a vertical asymptote. Dividing numerator by the denominator gives $y = x - 4 + \frac{9}{x+2}$ which shows that the line $y = x - 4$ is an oblique asymptote. (See Section 2.6.) The y-intercept is $\frac{1}{2}$. The graph has a hole at the point $(1, 0)$.

88. $y = \frac{x^3+x-2}{x-x^2} = \frac{(x-1)(x^2+x+2)}{(x-1)(-x)}$

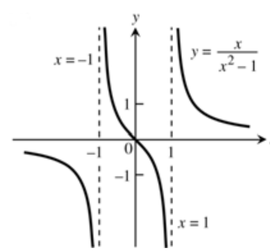


Since 1 and 0 are roots of the denominator, the domain is $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

$y' = -\frac{x^2-2}{x^2}$, $x \neq 1$; $y'' = -\frac{4}{x^2}$, $x \neq 1$

There is a critical point at $x = -\sqrt{2}$ where the function has a local minimum, and a critical point at $x = \sqrt{2}$ where the function has a local maximum. The function is increasing on $\left(-\sqrt{2}, 0\right) \cup \left(0, \sqrt{2}\right)$ and decreasing on $\left(-\infty, -\sqrt{2}\right) \cup \left(\sqrt{2}, \infty\right)$. There are no inflection points. The function is concave up on $(-\infty, 0)$ and concave down on $(0, 1) \cup (1, \infty)$. The y-axis is a vertical asymptote. Dividing numerator by denominator gives $y = -x - 1 - \frac{2}{x}$ which shows that the line $y = -x - 1$ is an oblique asymptote. (See Section 2.6.) The graph has a hole at the point $(1, -4)$.

89. $y = \frac{x}{x^2-1}$

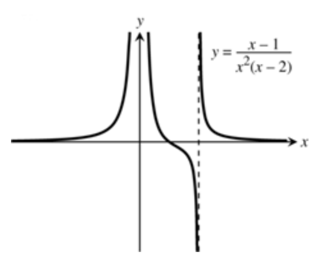


Since 1 and -1 are roots of the denominator, the domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

$y' = -\frac{x^2+1}{(x^2-1)^2}$; $y'' = \frac{2x^3+6x}{(x^2-1)^3}$

There are no critical points. The function is decreasing on its domain. There is an inflection point at $x = 0$. The function is concave up on $(-1, 0) \cup (1, \infty)$ and concave down on $(-\infty, -1) \cup (0, 1)$. The lines $x = 1$ and $x = -1$ are vertical asymptotes. Dividing numerator and denominator by x^2 gives $y = \frac{1/x}{1-(1/x^2)}$ which show that the x-axis is a horizontal asymptote. The x-intercept is 0 and the y-intercept is 0.

90. $y = \frac{x-1}{x^2(x-2)}$



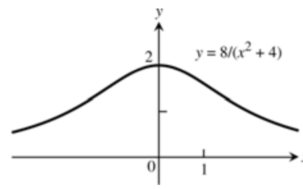
Since 0 and 2 are roots of the denominator, the domain is $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$.

$y' = -\frac{2x^2+5x+4}{x^3(x-2)^2}$; $y'' = \frac{6x^3-24x^2+40x-24}{x^4(x-2)^3}$

There are no critical points. The function is increasing on $(-\infty, 0)$ and decreasing on $(0, 2) \cup (2, \infty)$. There is an inflection point at approximately $x = 1.223$. The function is concave up on $(-\infty, 0) \cup (0, 1.223) \cup (2, \infty)$ and concave down on $(1.223, 2)$. The lines $x = 0$ (the y-axis) and $x = 2$ are vertical asymptotes. Dividing numerator and denominator by x^3 gives $y = \frac{(1/x^2) - (1/x^3)}{1 - (2/x)}$ which shows that the x-axis is a horizontal asymptote. The x-intercept is 1.

91. $y = \frac{8}{x^2+4}$ The domain is $(-\infty, \infty)$.

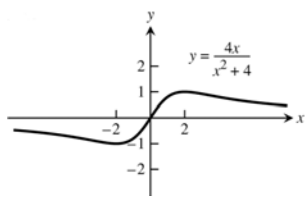
$y' = -\frac{16x}{(x^2+4)^2}$; $y'' = \frac{16(3x^2-4)}{(x^2+4)^3}$



There is a critical point at $x = 0$, where the function has a local maximum. The function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. There are inflection points at $x = -2/\sqrt{3}$ and at $x = 2/\sqrt{3}$. The function is concave up on $\left(-\infty, -2/\sqrt{3}\right) \cup \left(2/\sqrt{3}, \infty\right)$ and concave down on $\left(-2/\sqrt{3}, 2/\sqrt{3}\right)$. Dividing numerator and denominator by x^2 gives $y = \frac{8/x^2}{1+(4/x^2)}$ which shows that the x-axis is a horizontal asymptote. The y-intercept is 2.

92. $y = \frac{4x}{x^2+4}$ The domain is $(-\infty, \infty)$.

$y' = -\frac{4(x^2-4)}{(x^2+4)^2}$; $y'' = \frac{8x(x^2-12)}{(x^2+4)^3}$

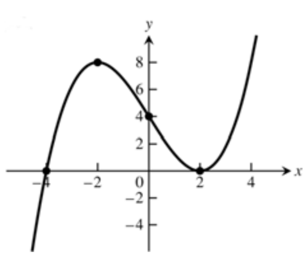


There is a critical point at $x = -2$, where the function has a local minimum, and at $x = 2$, where the function has a local maximum. The function is increasing on $(-2, 2)$ and decreasing on $(-\infty, -2) \cup (2, \infty)$. There are inflection points at $x = -2\sqrt{3}, x = 0$, and $x = 2\sqrt{3}$. The function is concave up on $\left(-2\sqrt{3}, 0\right) \cup \left(2\sqrt{3}, \infty\right)$ and concave down on $\left(-\infty, -2\sqrt{3}\right) \cup \left(0, 2\sqrt{3}\right)$. Dividing numerator and denominator by x^2 gives $y = \frac{4/x}{1+(4/x^2)}$ which shows that the x-axis is a horizontal asymptote. The x-intercept is 0 and the y-intercept is 0.

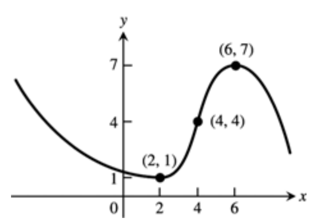
93.

Point	y'	y''
P	–	+
Q	+	0
R	+	–
S	0	–
T	–	–

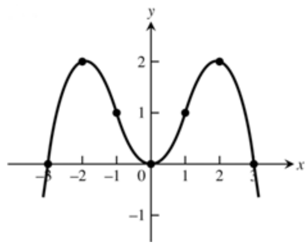
94.



95.



96.



97. Graphs printed in color can shift during a press run, so your values may differ somewhat from those given here.
   (a) The body is moving away from the origin when |displacement| is increasing as t increases, $0 < t < 2$ and $6 < t < 9.5$; the body is moving toward the origin when |displacement| is decreasing as t increases, $2 < t < 6$ and $9.5 < t < 15$.
   (b) The velocity will be zero when the slope of the tangent line for $y = s(t)$ is horizontal. The velocity is zero when t is approximately 2, 6, or 9.5 s.
   (c) The acceleration will be zero at those values of t where the curve $y = s(t)$ has points of inflection. The acceleration is zero when t is approximately 4, 7.5, or 12.5 s.
   (d) The acceleration is positive when the concavity is up, $4 < t < 7.5$ and $12.5 < t < 15$; the acceleration is negative when the concavity is down, $0 < t < 4$ and $7.5 < t < 12.5$.

98. (a) The body is moving away from the origin when |displacement| is increasing as t increases, $1.5 < t < 4$, $10 < t < 12$ and $13.5 < t < 16$; the body is moving toward the origin when |displacement| is decreasing as t increases, $0 < t < 1.5, 4 < t < 10$ and $12 < t < 13.5$.
   (b) The velocity will be zero when the slope of the tangent line for $y = s(t)$ is horizontal. The velocity is zero when t is approximately 0, 4, 12 or 16 s.
   (c) The acceleration will be zero at those values of t where the curve $y = s(t)$ has points of inflection. The acceleration is zero when t is approximately 1.5, 6, 8, 10.5, or 13.5 s.
   (d) The acceleration is positive when the concavity is up, $0 < t < 1.5$, $6 < t < 8$ and $10 < t < 13.5$, the acceleration is negative when the concavity is down, $1.5 < t < 6$, $8 < t < 10$ and $13.5 < t < 16$.

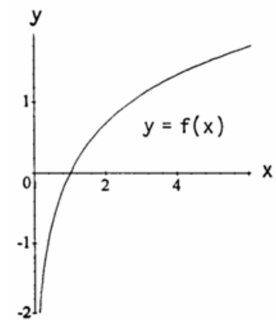
99. The marginal cost is $\frac{dc}{dx}$ which changes from decreasing to increasing when its derivative $\frac{d^2c}{dx^2}$ is zero. This is a point of inflection of the cost curve and occurs when the production level x is approximately 60 thousand units.

100. The marginal revenue is $\frac{dy}{dx}$ and it is increasing when its derivative $\frac{d^2y}{dx^2}$ is positive $\Rightarrow$ the curve is concave up $\Rightarrow 0 < t < 2$ and $5 < t < 9$; marginal revenue is decreasing when $\frac{d^2y}{dx^2} < 0 \Rightarrow$ the curve is concave down $\Rightarrow 2 < t < 5$ and $9 < t < 12$.

101. When $y' = (x-1)^2(x-2)$, then $y'' = 2(x-1)(x-2) + (x-1)^2$. The curve falls on $(-\infty, 2)$ and rises on $(2, \infty)$. At $x = 2$ there is a local minimum. There is no local maximum. The curve is concave upward on $(-\infty, 1)$ and $\left(\frac{5}{3}, \infty\right)$, and concave downward on $\left(1, \frac{5}{3}\right)$. At $x = 1$ or $x = \frac{5}{3}$ there are inflection points.

102. When $y' = (x-1)^2(x-2)(x-4)$, then $y'' = 2(x-1)(x-2)(x-4) + (x-1)^2(x-4) + (x-1)^2(x-2)$ $= (x-1)[2(x^2-6x+8) + (x^2-5x+4) + (x^2-3x+2)] = 2(x-1)(2x^2-10x+11)$. The curve rises on $(-\infty, 2)$ and $(4, \infty)$ and falls on $(2, 4)$. At $x = 2$ there is a local maximum and at $x = 4$ a local minimum. The curve is concave downward on $(-\infty, 1)$ and $\left(\frac{5-\sqrt{3}}{2}, \frac{5+\sqrt{3}}{2}\right)$ and concave upward on $\left(1, \frac{5-\sqrt{3}}{2}\right)$ and $\left(\frac{5+\sqrt{3}}{2}, \infty\right)$. At $x = 1, \frac{5-\sqrt{3}}{2}$ and $\frac{5+\sqrt{3}}{2}$ there are inflection points.

103. The graph must be concave down for $x > 0$ because $f''(x) = -\frac{1}{x^2} < 0$.



104. The second derivative, being continuous and never zero, cannot change sign. Therefore the graph will always be concave up or concave down so it will have no inflection points and no cusps or corners.

105. The curve will have a point of inflection at $x = 1$ if 1 is a solution of $y'' = 0$; $y = x^3 + bx^2 + cx + d$ $\Rightarrow y' = 3x^2 + 2bx + c \Rightarrow y'' = 6x + 2b$ and $6(1) + 2b = 0 \Rightarrow b = -3$.

106. (a) $f(x) = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a}$ a parabola whose vertex is at $x = -\frac{b}{2a} \Rightarrow$ the coordinates of the vertex are $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$

(b) The second derivative, $f''(x) = 2a$, describes concavity $\Rightarrow$ when $a > 0$ the parabola is concave up and when $a < 0$ the parabola is concave down.

107. A quadratic curve never has an inflection point. If $y = ax^2 + bx + c$ where $a \neq 0$, then $y' = 2ax + b$ and $y'' = 2a$. Since $2a$ is a constant, it is not possible for y'' to change signs.

108. A cubic curve always has exactly one inflection point. If $y = ax^3 + bx^2 + cx + d$ where $a \neq 0$, then $y' = 3ax^2 + 2bx + c$ and $y'' = 6ax + 2b$. Since $\frac{-b}{3a}$ is a solution of $y'' = 0$, we have that y'' changes its sign at $x = -\frac{b}{3a}$ and y' exists everywhere (so there is a tangent at $x = -\frac{b}{3a}$). Thus the curve has an inflection point at $x = -\frac{b}{3a}$. There are no other inflection points because y'' changes sign only at this zero.

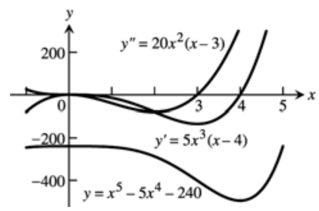
109. $y'' = (x+1)(x-2)$, when $y'' = 0 \Rightarrow x = -1$ or $x = 2$; $y'' = +++ \underset{-1}{|} --- \underset{2}{|} +++ \Rightarrow$ points of inflection at $x = -1$ and $x = 2$

110. $y'' = x^2(x-2)^3(x+3)$, when $y'' = 0 \Rightarrow x = -3$, $x = 0$ or $x = 2$; $y'' = +++ \underset{-3}{|} --- \underset{0}{|} --- \underset{2}{|} +++ \Rightarrow$ points of inflection at $x = -3$ and $x = 2$

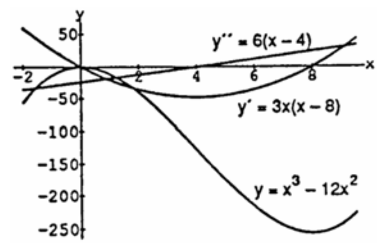
111. $y = ax^3 + bx^2 + cx \Rightarrow y' = 3ax^2 + 2bx + c$ and $y'' = 6ax + 2b$; local maximum at $x = 3$ $\Rightarrow 3a(3)^2 + 2b(3) + c = 0 \Rightarrow 27a + 6b + c = 0$; local minimum at $x = -1 \Rightarrow 3a(-1)^2 + 2b(-1) + c = 0$ $\Rightarrow 3a - 2b + c = 0$; point of inflection at $(1, 11) \Rightarrow a(1)^3 + b(1)^2 + c(1) = 11 \Rightarrow a + b + c = 11$ and $6a(1) + 2b = 0 \Rightarrow 6a + 2b = 0$. Solving $27a + 6b + c = 0$, $3a - 2b + c = 0$, $a + b + c = 11$, and $6a + 2b = 0$ $\Rightarrow a = -1$, $b = 3$, and $c = 9 \Rightarrow y = -x^3 + 3x^2 + 9x$

112. $y = \frac{x^2+a}{bx+c} \Rightarrow y' = \frac{bx^2+2cx-ab}{(bx+c)^2}$; local maximum at $x = 3 \Rightarrow \frac{b(3)^2+2c(3)-ab}{(b(3)+c)^2} = 0 \Rightarrow 9b + 6c - ab = 0$; local minimum at $(-1, -2) \Rightarrow \frac{b(-1)^2+2c(-1)-ab}{(b(-1)+c)^2} = 0 \Rightarrow b - 2c - ab = 0$ and $\frac{(-1)^2+a}{b(-1)+c} = -2 \Rightarrow -a + 2b - 2c = 1$. Solving $9b + 6c - ab = 0$, $b - 2c - ab = 0$, and $-a + 2b - 2c = 1 \Rightarrow a = 3$, $b = 1$, and $c = -1 \Rightarrow y = \frac{x^2+3}{x-1}$.

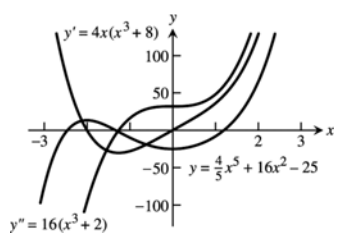
113. If $y = x^5 - 5x^4 - 240$, then $y' = 5x^3(x-4)$ and $y'' = 20x^2(x-3)$. The zeros of y' are extrema, and there is a point of inflection at $x = 3$.



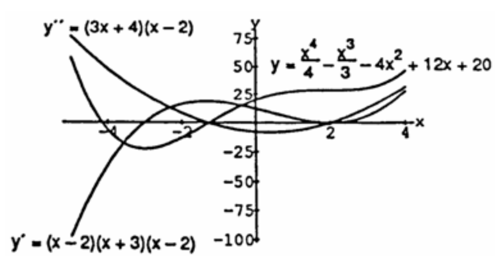
114. If $y = x^3 - 12x^2$ then $y' = 3x(x-8)$ and $y'' = 6(x-4)$. The zeros of y' and y'' are extrema, and points of inflection, respectively.



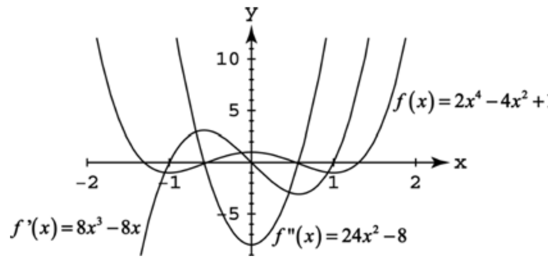
115. If $y = \frac{4}{5}x^5 + 16x^2 - 25$, then $y' = 4x(x^3 + 8)$ and $y'' = 16(x^3 + 2)$. The zeros of y' and y'' are extrema, and points of inflection, respectively.



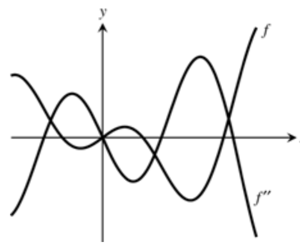
116. If $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$, then $y' = x^3 - x^2 - 8x + 12 = (x+3)(x-2)^2$. So y has a local minimum at $x = -3$ as its only extreme value. Also $y'' = 3x^2 - 2x - 8 = (3x+4)(x-2)$ and there are inflection points at both zeros, $-\frac{4}{3}$ and 2, of y''.



117. The graph of f falls where $f' < 0$, rises where $f' > 0$, and has horizontal tangents where $f' = 0$. It has local minima at points where f' changes from negative to positive and local maxima where f' changes from positive to negative. The graph of f is concave down where $f'' < 0$ and concave up where $f'' > 0$. It has an inflection point each time f'' changes sign, provided a tangent line exists there.



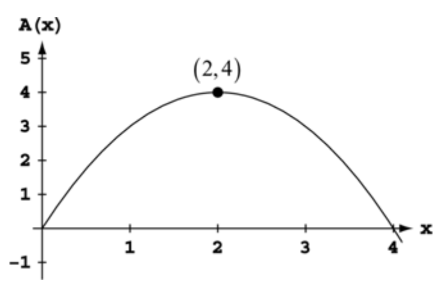
118. The graph f is concave down where $f'' < 0$, and concave up where $f'' > 0$. It has an inflection point each time f'' changes sign, provided a tangent line exists there.



4.5 APPLIED OPTIMIZATION

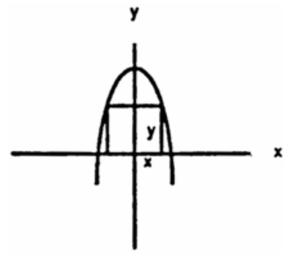
1. Let ℓ and w represent the length and width of the rectangle, respectively. With an area of 16 cm.^2, we have that $(\ell)(w) = 16 \Rightarrow w = 16\ell^{-1} \Rightarrow$ the perimeter is $P = 2\ell + 2w = 2\ell + 32\ell^{-1}$ and $P'(\ell) = 2 - \frac{32}{\ell^2} = \frac{2(\ell^2 - 16)}{\ell^2}$. Solving $P'(\ell) = 0 \Rightarrow \frac{2(\ell+4)(\ell-4)}{\ell^2} = 0 \Rightarrow \ell = -4, 4$. Since $\ell > 0$ for the length of a rectangle, ℓ must be 4 and $w = 4 \Rightarrow$ the perimeter is 16 in., a minimum since $P''(\ell) = \frac{16}{\ell^3} > 0$.
2. Let x represent the length of the rectangle in meters $(0 < x < 4)$. Then the width is $4 - x$ and the area is $A(x) = x(4 - x) = 4x - x^2$. Since $A'(x) = 4 - 2x$, the critical point occurs at $x = 2$. Since, $A'(x) > 0$ for $0 < x < 2$ and $A'(x) < 0$ for $2 < x < 4$, this critical point corresponds to the maximum area. The rectangle with the largest area measures 2 m by $4 - 2 = 2$ m, so it is a square.

   Graphical Support:

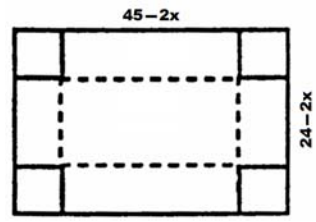


3. (a) The line containing point P also contains the points $(0, 1)$ and $(1, 0) \Rightarrow$ the line containing P is $y = 1 - x \Rightarrow$ a general point on that line is $(x, 1 - x)$.
   (b) The area $A(x) = 2x(1 - x)$, where $0 \le x \le 1$.
   (c) When $A(x) = 2x - 2x^2$, then $A'(x) = 0 \Rightarrow 2 - 4x = 0 \Rightarrow x = \frac{1}{2}$. Since $A(0) = 0$ and $A(1) = 0$, we conclude that $A\left(\frac{1}{2}\right) = \frac{1}{2}$ sq units is the largest area. The dimensions are 1 unit by $\frac{1}{2}$ unit.

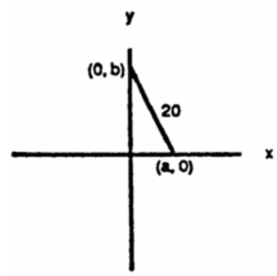
4. The area of the rectangle is $A = 2xy = 2x(12 - x^2)$, where $0 \le x \le \sqrt{12}$. Solving $A'(x) = 0 \Rightarrow 24 - 6x^2 = 0 \Rightarrow x = -2$ or 2. Now -2 is not in the domain, and since $A(0) = 0$ and $A\left(\sqrt{12}\right) = 0$, we conclude that $A(2) = 32$ square units is the maximum area. The dimensions are 4 units by 8 units.



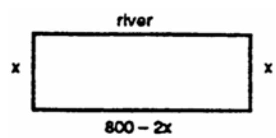
5. The volume of the box is $V(x) = x(45 - 2x)(24 - 2x)$ $= 1080x - 138x^2 + 4x^3$, where $0 \le x \le 12$. Solving $V'(x) = 0 \Rightarrow 1080 - 276x + 12x^2 = 12(x-5)(x-18) = 0$ $\Rightarrow x = 5$ or 18, but 18 is not in the domain. Since $V(0) = V(4) = 0$, $V(5) = 2450 \text{ cm}^3$ must be the maximum volume of the box with dimensions $14 \times 35 \times 5$ cm.



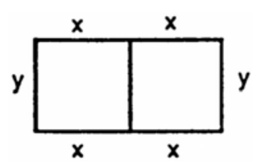
6. The area of the triangle is $A = \frac{1}{2}ba = \frac{b}{2}\sqrt{400 - b^2}$, where $0 \le b \le 20$. Then $\frac{dA}{db} = \frac{1}{2}\sqrt{400 - b^2} - \frac{b^2}{2\sqrt{400 - b^2}}$ $= \frac{200 - b^2}{\sqrt{400 - b^2}} = 0 \Rightarrow$ the interior critical point is $b = 10\sqrt{2}$. When $b = 0$ or 20, the area is zero $\Rightarrow A\left(10\sqrt{2}\right)$ is the maximum area. When $a^2 + b^2 = 400$ and $b = 10\sqrt{2}$, the value of a is also $10\sqrt{2} \Rightarrow$ the maximum area occurs when $a = b$.



7. The area is $A(x) = x(800 - 2x)$, where $0 \le x \le 400$. Solving $A'(x) = 800 - 4x = 0 \Rightarrow x = 200$. With $A(0) = A(400) = 0$, the maximum area is $A(200) = 80{,}000 \text{ m}^2$. The dimensions are 200 m by 400 m.



8. The area is $2xy = 216 \Rightarrow y = \frac{108}{x}$. The amount of fence needed is $P = 4x + 3y = 4x + 324x^{-1}$, where $0 < x$; $\frac{dP}{dx} = 4 - \frac{324}{x^2} = 0 \Rightarrow x^2 - 81 = 0 \Rightarrow$ the critical points are 0 and ± 9, but 0 and -9 are not in the domain. Then $P''(9) > 0 \Rightarrow$ at $x = 9$ there is a minimum $\Rightarrow$ the dimensions of the outer rectangle are 18 m by 12 m $\Rightarrow$ 72 meters of fence will be needed.



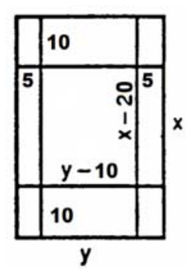
9. (a) We minimize the weight $= tS$ where S is the surface area, and t is the thickness of the steel walls of the tank. The surface area is $S = x^2 + 4xy$ where x is the length of a side of the square base of the tank, and y is its depth. The volume of the tank must be $4 \text{ m}^3 \Rightarrow y = \frac{4}{x^2}$. Therefore, the weight of the tank is $w(x) = t\left(x^2 + \frac{16}{x}\right)$. Treating the thickness as a constant gives $w'(x) = t\left(2x - \frac{16}{x^2}\right)$. The critical value is at $x = 2$. Since $w''(2) = t\left(2 + \frac{32}{2^3}\right) > 0$, there is a minimum at $x = 2$. Therefore, the optimum dimensions of the tank are 20 m on the base edges and 1 m deep.

(b) Minimizing the surface area of the tank minimizes its weight for a given wall thickness. The thickness of the steel walls would likely be determined by other considerations such as structural requirements.

10. (a) The volume of the tank being 20 m^3, we have that $yx^2 = 20 \Rightarrow y = \frac{20}{x^2}$. The cost of building the tank is $c(x) = 5x^2 + 30x\left(\frac{20}{x^2}\right)$, where $0 < x$. Then $c'(x) = 10x - \frac{600}{x^2} = 0 \Rightarrow$ the critical points are 0 and $\sqrt[3]{60}$, but 0 is not in the domain. Thus, $c''\left(\sqrt[3]{60}\right) > 0 \Rightarrow$ at $x = \sqrt[3]{60}$ we have a minimum. The values of $x = \sqrt[3]{60}$ m and $y = \frac{\sqrt[3]{60}}{3}$ m will minimize the cost.

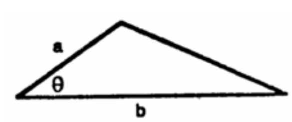
(b) The cost function $c = 5(x^2 + 4xy) + 10xy$, can be separated into two items: (1) the cost of the materials and labor to fabricate the tank, and (2) the cost for the excavation. Since the area of the sides and bottom of the tanks is $(x^2 + 4xy)$, it can be deduced that the unit cost to fabricate the tanks is $\$5/\text{m}^2$. Normally, excavation costs are per unit volume of excavated material. Consequently, the total excavation cost can be taken as $10xy = \left(\frac{10}{x}\right)(x^2 y)$. This suggests that the unit cost of excavation is $\frac{\$10/\text{m}^2}{x}$ where x is the length of a side of the square base of the tank in meters. For the least expensive tank, the unit cost for the excavation is $\frac{\$10/\text{m}^2}{15\text{ m}} = \frac{\$0.67}{\text{m}^3}$. The total cost of the least expensive tank is \$230, which is the sum of \$179 for fabrication and \$51 for the excavation.

11. The area of the printing is $(y-10)(x-20) = 312.5$. Consequently, $y = \left(\frac{312.5}{x-20}\right) + 10$. The area of the paper is $A(x) = x\left(\frac{312.5}{x-20} + 10\right)$, where $20 < x$. Then $A'(x) = \left(\frac{312.5}{x-20} + 10\right) - x\left(\frac{312.5}{(x-20)^2}\right) = \frac{10(x-20)^2 - 6250}{(x-20)^2} = 0$ $\Rightarrow$ the critical points are -5 and 45, but -5 is not in the domain. Thus $A''(45) > 0 \Rightarrow$ at $x = 45$ we have a minimum. Therefore the dimensions 45 by 22.5 cm minimize the amount of paper.

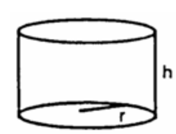


12. The volume of the cone is $V = \frac{1}{3}\pi r^2 h$, where $r = x = \sqrt{9 - y^2}$ and $h = y + 3$ (from the figure in the text). Thus, $V(y) = \frac{\pi}{3}(9 - y^2)(y+3) = \frac{\pi}{3}(27 + 9y - 32y^2 - y^3) \Rightarrow V'(y) = \frac{\pi}{3}(9 - 6y - 3y^2) = \pi(1-y)(3+y)$. The critical points are -3 and 1, but -3 is not in the domain. Thus $V''(1) = \frac{\pi}{3}(-6 - 6(1)) < 0 \Rightarrow$ at $y = 1$ we have a maximum volume of $V(1) = \frac{\pi}{3}(8)(4) = \frac{32\pi}{3}$ cubic units.

13. The area of the triangle is $A(\theta) = \frac{ab\sin\theta}{2}$, where $0 < \theta < \pi$. Solving $A'(\theta) = 0 \Rightarrow \frac{ab\cos\theta}{2} = 0 \Rightarrow \theta = \frac{\pi}{2}$. Since $A''(\theta) = -\frac{ab\sin\theta}{2} \Rightarrow A''\left(\frac{\pi}{2}\right) < 0$, there is a maximum at $\theta = \frac{\pi}{2}$.



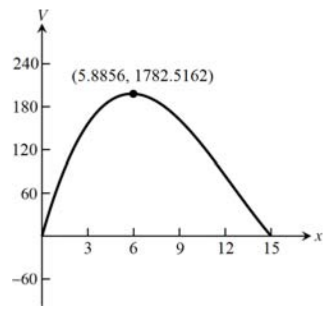
14. A volume $V = \pi r^2 h = 100 \Rightarrow h = \frac{1000}{\pi r^2}$. The amount of material is the surface area given by the sides and bottom of the can $\Rightarrow S = 2\pi r h + \pi r^2 = \frac{2000}{r} + \pi r^2$, $0 < r$. Then $\frac{dS}{dr} = -\frac{2000}{r^2} + 2\pi r = 0 \Rightarrow \frac{\pi r^3 - 1000}{r^2} = 0$. The critical points are 0 and $\frac{10}{\sqrt[3]{\pi}}$, but 0 is not in the domain. Since $\frac{d^2 s}{dr^2} = \frac{4000}{r^3} + 2\pi > 0$, we have a minimum surface area when $r = \frac{10}{\sqrt[3]{\pi}}$ cm and $h = \frac{1000}{\pi r^2} = \frac{10}{\sqrt[3]{\pi}}$ cm. Comparing this result to the result found in Example 2, if we include both ends of the can, then we have a minimum surface area when the can is shorter—specifically, when the height of the can is the same as its diameter.



15. With a volume of 1000 cm^3 and $V = \pi r^2 h$, then $h = \frac{1000}{\pi r^2}$. The amount of aluminum used per can is $A = 8r^2 + 2\pi rh = 8r^2 + \frac{2000}{r}$. Then $A'(r) = 16r - \frac{2000}{r^2} = 0 \Rightarrow \frac{8r^3 - 1000}{r^2} = 0 \Rightarrow$ the critical points are 0 and 5, but $r = 0$ results in no can. Since $A''(r) = 16 + \frac{1000}{r^3} > 0$ we have a minimum at $r = 5 \Rightarrow h = \frac{40}{\pi}$ and $h{:}r = 8{:}\pi$.

16. (a) The base measures $3 - 2x$ cm by $\frac{45-2x}{2}$ cm, so the volume formula is $V(x) = \frac{x(30-2x)(45-2x)}{2}$ $= 2x^3 - 75x^2 + 675x$.

(b) We require $x > 0$, $2x < 30$, and $2x > 45$. Combining these requirements, the domain is the interval $(0, 15)$.

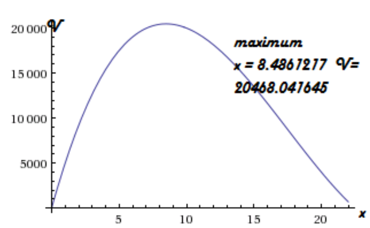


(c) The maximum volume is approximately $1782.5\ \text{cm}^3$ when $x \approx 5.89$ cm.

(d) $V'(x) = 6x^2 - 150x + 675$. The critical point occurs when $V'(x) = 0$, at $x = \frac{150 \pm \sqrt{(-150)^2 - 4(6)(675)}}{2(6)} = \frac{150 \pm \sqrt{6300}}{12} = \frac{25 \pm 5\sqrt{7}}{2}$, that is, $x \approx 5.89$ or $x \approx 19.11$. We discard the larger value because it is not in the domain. Since $V''(x) = 12x - 150$, which is negative when $x \approx 5.89$, the critical point corresponds to the maximum volume. The maximum volume occurs when $x = \frac{25 - 5\sqrt{7}}{2} \approx 5.89$, which confirms the result in (c).

17. (a) The "sides" of the suitcase will measure $60 - 2x$ cm by $45 - 2x$ cm and will be $2x$ cm apart, so the volume formula is $V(x) = 2x(60 - 2x)(45 - 2x) = 8x^3 - 420x^2 + 5400x$.

(b) We require $x > 0$, $2x < 45$, and $2x < 60$. Combining these requirements, the domain is the interval $(0, 9)$.



(c) The maximum volume is approximately $20.68\ \text{cm}^3$ when $x \approx 8.49$ cm.

(d) $V'(x) = 24x^2 - 840x + 5400 = 24(x^2 - 35x + 225)$. The critical point is at $x = \frac{35 \pm \sqrt{(-35)^2 - 4(1)(225)}}{2(1)} = \frac{35 \pm \sqrt{325}}{2} = \frac{35 \pm 5\sqrt{13}}{2}$, that is, $x \approx 8.49$ or $x \approx 26.51$. We discard the larger value because it is not in the domain. Since $V''(x) = 24(2x - 35)$ which is negative when $x \approx 8.49$, the critical point corresponds to the maximum volume. The maximum value occurs at $x = \frac{35 - 5\sqrt{13}}{2} \approx 8.49$, which confirms the results in (c).

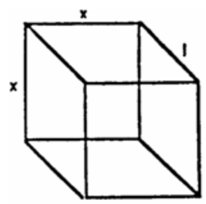
(e) $8x^3 - 468x^2 + 5400x = 17500 \Rightarrow$ $4(2x^3 - 105x^2 + 1350x - 4375) = 0 \Rightarrow$ $4(2x - 25)(x - 5)(x - 35) = 0$. Since 35 is not in the domain, the possible values of x are $x = 12.5$ cm or $x = 5$ cm.

(f) The dimensions of the resulting box are $2x$ cm, $(60-2x)$ cm, and $(45-2x)$ cm. Each of these measurements must be positive, so that gives the domain of $(0, 22.5)$.

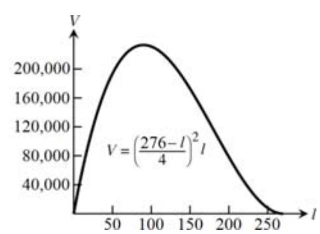
18. If the upper right vertex of the rectangle is located at $(x, 4\cos 0.5x)$ for $0 < x < \pi$, then the rectangle has width $2x$ and height $4\cos 0.5x$, so the area is $A(x) = 8x\cos 0.5x$.. Solving $A'(x) = 0$ graphically for $0 < x < \pi$, we find that $x \approx 2.214$. Evaluating $2x$ and $4\cos 0.5x$ for $x \approx 2.214$, the dimensions of the rectangle are approximately 4.43 (width) by 1.79 (height), and the maximum area is approximately 7.923.

19. Let the radius of the cylinder be r cm, $0 < r < 10$. Then the height is $2\sqrt{100-r^2}$ and the volume is $V(r) = 2\pi r^2\sqrt{100-r^2}\,\text{cm}^3$. Then, $V'(r) = 2\pi r^2\left(\frac{1}{2\sqrt{100-r^2}}\right)(-2r) + \left(2\pi\sqrt{100-r^2}\right)(2r) = \frac{-2\pi r^3+4\pi r(100-r^2)}{\sqrt{100-r^2}}$ $= \frac{2\pi r(200-3r^2)}{\sqrt{100-r^2}}$. The critical point for $0 < r < 10$ occurs at $r = \sqrt{\frac{200}{3}} = 10\sqrt{\frac{2}{3}}$. Since $V'(r) > 0$ for $0 < r < 10\sqrt{\frac{2}{3}}$ and $V'(r) < 0$ for $10\sqrt{\frac{2}{3}} < r < 10$, the critical point corresponds to the maximum volume. The dimensions are $r = 10\sqrt{\frac{2}{3}} \approx 8.16$ cm and $h = \frac{20}{\sqrt{3}} \approx 11.55$ cm, and the volume is $\frac{4000\pi}{3\sqrt{3}} \approx 2418.40\ \text{cm}^3$.

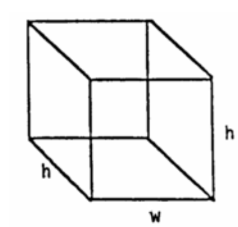
20. (a) From the diagram we have $4x + \ell = 276$ and $V = x^2\ell$. The volume of the box is $V(x) = x^2(276-4x)$, where $0 \le x \le 69$. Then $V'(x) = 552x - 12x^2 = 12x(46-x) = 0 \Rightarrow$ the critical points are 0 and 46, but $x = 0$ results in no box. Since $V''(x) = 552 - 24x < 0$ at $x = 46$ we have a maximum. The dimensions of the box are $46 \times 46 \times 92$ cm.



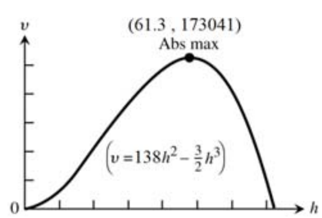
(b) In terms of length, $V(\ell) = x^2\ell = \left(\frac{276-\ell}{4}\right)^2\ell$. The graph indicates that the maximum volume occurs near $\ell = 92$, which is consistent with the result of part (a).



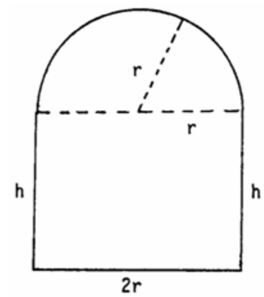
21. (a) From the diagram we have $3h + 2w = 276$ and $V = h^2w \Rightarrow V(h) = h^2\left(138 - \frac{3}{2}h\right) = 138h^2 - \frac{3}{2}h^3$ Then $V'(h) = 276h - \frac{9}{2}h^2 = \frac{9}{2}h\left(\frac{184}{3} - h\right) = 0$ $\Rightarrow h = 0$ or $h = \frac{184}{3}$, but $h = 0$ results in no box. Since $V''(h) = 276 - 9h < 0$ at $h = \frac{184}{3}$, we have a maximum volume at $h = \frac{184}{3}$ and $w = 138 - \frac{3}{2}h = 46$.



(b)



22. From the diagram the perimeter is $P = 2r + 2h + \pi r$, where r is the radius of the semicircle and h is the height of the rectangle. The amount of light transmitted proportional to $A = 2rh + \frac{1}{4}\pi r^2$ $= r(P - 2r - \pi r) + \frac{1}{4}\pi r^2 = rP - 2r^2 - \frac{3}{4}\pi r^2$. Then $\frac{dA}{dr} = P - 4r - \frac{3}{2}\pi r = 0 \Rightarrow r = \frac{2P}{8+3\pi}$ $\Rightarrow 2h = P - \frac{4P}{8+3\pi} - \frac{2\pi P}{8+3\pi} = \frac{(4+\pi)P}{8+3\pi}$. Therefore, $\frac{2r}{h} = \frac{8}{4+\pi}$ gives the proportions that admit the <u>most</u> light since $\frac{d^2A}{dr^2} = -4 - \frac{3}{2}\pi < 0$.



23. The fixed volume is $V = \pi r^2 h + \frac{2}{3}\pi r^3 \Rightarrow h = \frac{V}{\pi r^2} - \frac{2r}{3}$, where h is the height of the cylinder and r is the radius of the hemisphere. To minimize the cost we must minimize surface area of the cylinder added to twice the surface area of the hemisphere. Thus, we minimize $C = 2\pi rh + 4\pi r^2 = 2\pi r\left(\frac{V}{\pi r^2} - \frac{2r}{3}\right) + 4\pi r^2 = \frac{2V}{r} + \frac{8}{3}\pi r^2$. Then $\frac{dC}{dr} = -\frac{2V}{r^2} + \frac{16}{3}\pi r = 0 \Rightarrow V = \frac{8}{3}\pi r^3 \Rightarrow r = \left(\frac{3V}{8\pi}\right)^{1/3}$. From the volume equation, $h = \frac{V}{\pi r^2} - \frac{2r}{3}$ $= \frac{4V^{1/3}}{\pi^{1/3}\cdot 3^{2/3}} - \frac{2\cdot 3^{1/3}\cdot V^{1/3}}{3\cdot 2\cdot \pi^{1/3}} = \frac{3^{1/3}\cdot 2\cdot 4\cdot V^{1/3} - 2\cdot 3^{1/3}\cdot V^{1/3}}{3\cdot 2\cdot \pi^{1/3}} = \left(\frac{3V}{\pi}\right)^{1/3}$. Since $\frac{d^2C}{dr^2} = \frac{4V}{r^3} + \frac{16}{3}\pi > 0$, these dimensions do minimize the cost.

24. The volume of the trough is maximized when the area of the cross section is maximized. From the diagram the area of the cross section is $A(\theta) = \cos\theta + \sin\theta\cos\theta$, $0 < \theta < \frac{\pi}{2}$. Then $A'(\theta) = -\sin\theta + \cos^2\theta - \sin^2\theta$ $= -(2\sin^2\theta + \sin\theta - 1) = -(2\sin\theta - 1)(\sin\theta + 1)$ so $A'(\theta) = 0 \Rightarrow \sin\theta = \frac{1}{2}$ or $\sin\theta = -1 \Rightarrow \theta = \frac{\pi}{6}$ because $\sin\theta \neq -1$ when $0 < \theta < \frac{\pi}{2}$. Also, $A'(\theta) > 0$ for $0 < \theta < \frac{\pi}{6}$ and $A'(\theta) < 0$ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$. Therefore, at $\theta < \frac{\pi}{6}$ there is a maximum.

25. (a) From the diagram we have: $\overline{AP} = x$, $\overline{RA} = \sqrt{L - x^2}$, $\overline{PB} = 21.6 - x$,

$\overline{CH} = \overline{DR} = 28 - \overline{RA} = 28 - \sqrt{L - x^2}$, $\overline{QB} = \sqrt{x^2 - (21.6 - x)^2}$,

$\overline{HQ} = 28 - \overline{CH} - \overline{QB} = 28 - \left[28 - \sqrt{L - x^2} + \sqrt{x^2 - (21.6 - x)^2}\right]$

$= \sqrt{L - x^2} - \sqrt{x^2 - (21.6 - x)^2}$, $\overline{RQ}^2 = \overline{RH}^2 + \overline{HQ}^2$

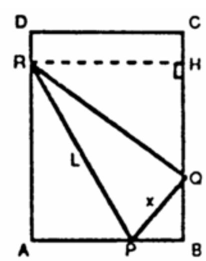
$= (21.6)^2 + \left(\sqrt{L - x^2} - \sqrt{x^2 - (21.6 - x^2)}\right)^2$. It follows that $\overline{RP}^2 = \overline{PQ}^2 + \overline{RQ}^2$

$\Rightarrow L^2 = x^2 + \left(\sqrt{L^2 - x^2} - \sqrt{x^2 - (x - 21.6)^2}\right)^2 + (21.6)^2$

$\Rightarrow L^2 = x^2 + L^2 - x^2 - 2\sqrt{L^2 - x^2} - \sqrt{43.2x - (21.6)^2} + 43.2x - (21.6)^2 + (21.6)^2$

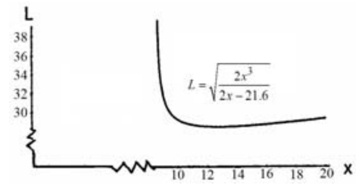
$\Rightarrow 43.2^2 x^2 = 4(L^2 - x^2)(43.2x - (21.6)^2) \Rightarrow L^2 = x^2 + \frac{43.2^2 x^2}{4[43.2x - (21.6)^2]}$

$= \frac{43.2x^3}{43.2x - (21.6)^2} = \frac{43.2x^3}{43.2x - \left(\frac{43.2}{2}\right)^2} = \frac{4x^3}{4x - 43.2} = \frac{2x^3}{2x - 21.6}$.



(b) If $f(x) = \frac{4x^3}{4x - 43.2}$ is minimized, then L^2 is minimized. Now $f'(x) = \frac{4x^2(8x - 129.6)}{(4x - 43.2)^2} \Rightarrow f'(x) < 0$ when $x < \frac{129.6}{8}$ and $f'(x) > 0$ when $x > \frac{129.6}{8}$. Thus L^2 is minimized when $x = \frac{129.6}{8}$.

(c) When $x > 16.2$, then $L \approx 28$ cm.



26. (a) From the figure in the text we have $P = 2x + 2y \Rightarrow y = \frac{P}{2} - x.$ If $P = 36$, then $y = 18 - x.$ When the cylinder is formed, $x = 2\pi r \Rightarrow r = \frac{x}{2\pi}$ and $h = y \Rightarrow h = 18 - x.$ The volume of the cylinder is $V = \pi r^2 h \Rightarrow V(x) = \frac{18x^2 - x^3}{4\pi}$. Solving $V'(x) = \frac{3x(12-x)}{4\pi} = 0 \Rightarrow x = 0$ or 12; but when $x = 0$ there is no cylinder. Then $V''(x) = \frac{3}{\pi}\left(3 - \frac{x}{2}\right) \Rightarrow V''(12) < 0 \Rightarrow$ there is a maximum at $x = 12.$ The values of $x = 12$ cm and $y = 6$ cm give the largest volume.

(b) In this case $V(x) = \pi x^2(18 - x).$ Solving $V'(x) = 3\pi x(12 - x) = 0 \Rightarrow x = 0$ or 12; but $x = 0$ would result in no cylinder. Then $V''(x) = 6\pi(6 - x) \Rightarrow V''(12) < 0 \Rightarrow$ there is a maximum at $x = 12.$ The values of $x = 12$ cm and $y = 6$ cm give the largest volume.

27. Note that $h^2 + r^2 = 3$ and so $r = \sqrt{3 - h^2}$. Then the volume is given by $V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(3 - h^2)h = \pi h - \frac{\pi}{3} h^3$ for $0 < h < \sqrt{3}$, and so $\frac{dV}{dh} = \pi - \pi r^2 = \pi(1 - r^2).$ The critical point (for $h > 0$) occurs at $h = 1.$ Since $\frac{dV}{dh} > 0$ for $0 < h < 1$, and $\frac{dV}{dh} < 0$ for $1 < h < \sqrt{3}$, the critical point corresponds to the maximum volume. The cone of greatest volume has radius $\sqrt{2}$ m, height 1 m, and volume $\frac{2\pi}{3}\,\text{m}^3.$

28. Let $d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$ and $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow y = -\frac{b}{a}x + b.$ We can minimize d by minimizing $D = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + \left(-\frac{b}{a}x + b\right)^2 \Rightarrow D' = 2x + 2\left(-\frac{b}{a}x + b\right)\left(-\frac{b}{a}\right) = 2x + \frac{2b^2}{a^2}x - \frac{2b^2}{a}.$ $D' = 0$ $\Rightarrow 2\left(x + \frac{b^2}{a^2}x - \frac{b^2}{a}\right) = 0 \Rightarrow x = \frac{ab^2}{a^2+b^2}$ is the critical point $\Rightarrow y = -\frac{b}{a}\left(\frac{ab^2}{a^2+b^2}\right) + b = \frac{a^2 b}{a^2+b^2}. \Rightarrow D'' = 2 + \frac{2b^2}{a^2}$ $\Rightarrow D''\left(\frac{ab^2}{a^2+b^2}\right) = 2 + \frac{2b^2}{a^2} > 0 \Rightarrow$ the critical point is a local minimum $\Rightarrow \left(\frac{ab^2}{a^2+b^2}, \frac{a^2 b}{a^2+b^2}\right)$ is the point on the line $\frac{x}{a} + \frac{y}{b} = 1$ that is closest to the origin.

29. Let $S(x) = x + \frac{1}{x}$, $x > 0 \Rightarrow S'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}.$ $S'(x) = 0 \Rightarrow \frac{x^2-1}{x^2} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1.$ Since $x > 0$, we only consider $x = 1.$ $S''(x) = \frac{2}{x^3} \Rightarrow S''(1) = \frac{2}{1^3} > 0 \Rightarrow$ local minimum when $x = 1$

30. Let $S(x) = \frac{1}{x} + 4x^2$, $x > 0 \Rightarrow S'(x) = -\frac{1}{x^2} + 8x = \frac{8x^3-1}{x^2}.$ $S'(x) = 0 \Rightarrow \frac{8x^3-1}{x^2} = 0 \Rightarrow 8x^3 - 1 = 0 \Rightarrow x = \frac{1}{2}.$ $S''(x) = \frac{2}{x^3} + 8 \Rightarrow S''\left(\frac{1}{2}\right) = \frac{2}{(1/2)^3} + 8 > 0 \Rightarrow$ local minimum when $x = \frac{1}{2}.$

31. The length of the wire b = perimeter of the triangle + circumference of the circle. Let x = length of a side of the equilateral triangle $\Rightarrow P = 3x$, and let r = radius of the circle $\Rightarrow C = 2\pi r.$ Thus $b = 3x + 2\pi r \Rightarrow r = \frac{b-3x}{2\pi}.$ The area of the circle is πr^2 and the area of an equilateral triangle whose sides are x is $\frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2.$

Thus, the total area is given by $A = \frac{\sqrt{3}}{4}x^2 + \pi r^2 = \frac{\sqrt{3}}{4}x^2 + \pi\left(\frac{b-3x}{2\pi}\right)^2 = \frac{\sqrt{3}}{4}x^2 + \frac{(b-3x)^2}{4\pi}$

$\Rightarrow A' = \frac{\sqrt{3}}{2}x - \frac{3}{2\pi}(b - 3x) = \frac{\sqrt{3}}{2}x - \frac{3b}{2\pi} + \frac{9}{2\pi}x.$ $A' = 0 \Rightarrow \frac{\sqrt{3}}{2}x - \frac{3b}{2\pi} + \frac{9}{2\pi}x = 0 \Rightarrow x = \frac{3b}{\sqrt{3}\pi + 9}.$

$A'' = \frac{\sqrt{3}}{2} + \frac{9}{2\pi} > 0 \Rightarrow$ local minimum at the critical point. $P = 3\left(\frac{3b}{\sqrt{3}\pi+9}\right) = \frac{9b}{\sqrt{3}\pi+9}$ m is the length of the triangular segment and $C = 2\pi\left(\frac{b-3x}{2\pi}\right) = b - 3x = b - \frac{9b}{\sqrt{3}\pi+9} = \frac{\sqrt{3}\pi b}{\sqrt{3}\pi+9}$ m is the length of the circular segment.

32. The length of the wire b = perimeter of the triangle + circumference of the circle. Let x = length of a side of the square $\Rightarrow P = 4x$, and let r = radius of the circle $\Rightarrow C = 2\pi r$. Thus $b = 4x + 2\pi\, r \Rightarrow r = \frac{b-4x}{2\pi}$. The area of the circle is πr^2 and the area of a square whose sides are x is x^2. Thus, the total area is given by $A = x^2 + \pi r^2$ $= x^2 + \pi\left(\frac{b-4x}{2\pi}\right)^2 = x^2 + \frac{(b-4x)^2}{4\pi} \Rightarrow A' = 2x - \frac{4}{2\pi}(b-4x) = 2x - \frac{2b}{\pi} + \frac{8}{\pi}x$, $A' = 0 \Rightarrow 2x - \frac{2b}{\pi} + \frac{8}{\pi}x = 0$ $\Rightarrow x = \frac{b}{4+\pi}$. $A'' = 2 + \frac{8}{\pi} > 0 \Rightarrow$ local minimum at the critical point. $P = 4\left(\frac{b}{4+\pi}\right) = \frac{4b}{4+\pi}$ m is the length of the square segment and $C = 2\pi\left(\frac{b-4x}{2\pi}\right) = b - 4x = b - \frac{4b}{4+\pi} = \frac{b\pi}{4+\pi}$ m is the length of the circular segment.

33. Let $(x, y) = \left(x, \frac{4}{3}x\right)$ be the coordinates of the corner that intersects the line. Then base $= 3 - x$ and height $= y = \frac{4}{3}x$, thus the area of the rectangle is given by $A = (3-x)\left(\frac{4}{3}x\right) = 4x - \frac{4}{3}x^2$, $0 \le x \le 3$. $A' = 4 - \frac{8}{3}x$, $A' = 0$ $\Rightarrow x = \frac{3}{2}$. $A'' = -\frac{4}{3} \Rightarrow A''\left(\frac{3}{2}\right) < 0 \Rightarrow$ local maximum at the critical point. The base $= 3 - \frac{3}{2} = \frac{3}{2}$ and the height $= \frac{4}{3}\left(\frac{3}{2}\right) = 2$.

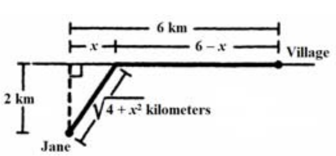
34. Let $(x, y) = \left(x, \sqrt{9-x^2}\right)$ be the coordinates of the corner that intersects the semicircle. Then base $= 2x$ and height $= y = \sqrt{9-x^2}$, thus the area of the inscribed rectangle is given by $A = (2x)\sqrt{9-x^2}$, $0 \le x \le 3$. Then $A' = 2\sqrt{9-x^2} + (2x)\frac{-x}{\sqrt{9-x^2}} = \frac{2(9-x^2)-2x^2}{\sqrt{9-x^2}} = \frac{18-4x^2}{\sqrt{4-x^2}}$, $A' = 0 \Rightarrow 18 - 4x^2 = 0 \Rightarrow x = \pm\frac{3\sqrt{2}}{2}$, only $x = \frac{3\sqrt{2}}{2}$ lies in $0 \le x \le 3$. A is continuous on the closed interval $0 \le x \le 3 \Rightarrow A$ has an absolute maxima and absolute minima. $A(0) = 0$, $A(3) = 0$, and $A\left(\frac{3\sqrt{2}}{2}\right) = \left(3\sqrt{2}\right)\left(\frac{3\sqrt{2}}{2}\right) = 9 \Rightarrow$ absolute maxima. Base of rectangle is $3\sqrt{2}$ and height is $\frac{3\sqrt{2}}{2}$.

35. (a) $f(x) = x^2 + \frac{a}{x} \Rightarrow f'(x) = x^{-2}\,(2x^3 - a)$, so that $f'(x) = 0$ when $x = 2$ implies $a = 16$
(b) $f(x) = x^2 + \frac{a}{x} \Rightarrow f''(x) = 2x^{-3}\,(x^3 + a)$, so that $f''(x) = 0$ when $x = 1$ implies $a = -1$

36. If $f(x) = x^3 + ax^2 + bx$, then $f'(x) = 3x^2 + 2ax + b$ and $f''(x) = 6x + 2a$.
(a) A local maximum at $x = -1$ and local minimum at $x = 3 \Rightarrow f'(-1) = 0$ and $f'(3) = 0 \Rightarrow 3 - 2a + b = 0$ and $27 + 6a + b = 0 \Rightarrow a = -3$ and $b = -9$.
(b) A local minimum at $x = 4$ and a point inflection at $x = 1 \Rightarrow f'(4) = 0$ and $f''(1) = 0 \Rightarrow 48 + 8a + b = 0$ and $6 + 2a = 0 \Rightarrow a = -3$ and $b = -24$.

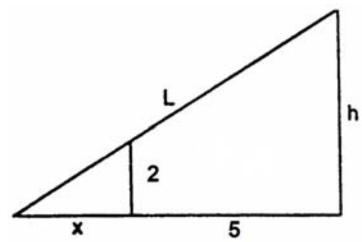
37. (a) $s(t) = -4.9t^2 + 29.4t + 34.3 \Rightarrow v(t) = s'(t) = -9.8t + 29.4$. At $t = 0$, the velocity is $v(0) = 29.4$ m/s.
(b) The maximum height occurs when $v(t) = 0$, when $t = 3$. The maximum height is $s(3) = 78.4$ m and it occurs at $t = 3$ s.
(c) Note that $s(t) = -4.9t^2 + 29.4t + 34.3 = -4.9(t+1)(t-7)$, so $s = 0$ at $t = -1$ or $t = 7$. Choosing the positive value of t, the velocity when $s = 0$ is $v(7) = -39.2$ m/s.

38.



Let x be the distance from the point on the shoreline nearest Jane's boat to the point where she lands her boat. Then she needs to row $\sqrt{4+x^2}$ km at 2 km/h and walk $6-x$ km at 5 km/h. The total amount of time to reach the village is $f(x)=\frac{\sqrt{4+x^2}}{2}+\frac{6-x}{5}$ hours $(0\le x\le 6)$. Then $f'(x)=\frac{1}{2}\frac{1}{2\sqrt{4+x^2}}(2x)-\frac{1}{5}=\frac{x}{2\sqrt{4+x^2}}-\frac{1}{5}$. Solving $f'(x)=0$. we have: $\frac{x}{2\sqrt{4+x^2}}=\frac{1}{5}\Rightarrow 5x=2\sqrt{4+x^2}\Rightarrow 25x^2=4\left(4+x^2\right)\Rightarrow 21x^2=16\Rightarrow x=\pm\frac{4}{\sqrt{21}}$. We discard the negative value of x because it is not in the domain. Checking the endpoints and critical point, we have $f(0)=2.2,\ f\left(\frac{4}{\sqrt{21}}\right)\approx 2.12,$ and $f(6)\approx 3.16$. Jane should land her boat $\frac{4}{\sqrt{21}}\approx 0.87$ kilometers down the shoreline from the point nearest her boat.

39. $\frac{2}{x}=\frac{h}{x+5}\Rightarrow h=2+\frac{10}{x}$ and $L(x)=\sqrt{h^2+(x+5)^2}$ $=\sqrt{\left(2+\frac{10}{x}\right)^2+(x+5)^2}$ when $x\ge 0$. Note that $L(x)$ is minimized when $f(x)=\left(2+\frac{10}{x}\right)^2+\ (x+5)^2$ is minimized. If $f'(x)=0$, then
$2\left(2+\frac{10}{x}\right)\left(-\frac{10}{x^2}\right)+2(x+5)=0\ \Rightarrow(x+5)\left(1-\frac{20}{x^3}\right)=0$
$\Rightarrow x=-5$ (not acceptable since distance is never negative) or $x=2.71$. Then
$L(2.71)=\sqrt{91.82}\approx 9.58$ m.



40. (a) $s_1=s_2\Rightarrow\sin t=\sin\left(t+\frac{\pi}{3}\right)\Rightarrow\sin t=\sin t\cos\frac{\pi}{3}+\sin\frac{\pi}{3}\cos t\Rightarrow\sin t=\frac{1}{2}\sin t+\frac{\sqrt{3}}{2}\cos t\Rightarrow\tan t=\sqrt{3}$ $\Rightarrow t=\frac{\pi}{3}$ or $\frac{4\pi}{3}$

(b) The distance between the particles is $s(t)=|s_1-s_2|=\left|\sin t-\sin\left(t+\frac{\pi}{3}\right)\right|=\frac{1}{2}\left|\sin t-\sqrt{3}\cos t\right|$ $\Rightarrow s'(t)=\frac{\left(\sin t-\sqrt{3}\cos t\right)\left(\cos t+\sqrt{3}\sin t\right)}{2\left|\sin t-\sqrt{3}\cos t\right|}$ since $\frac{d}{dx}|x|=\frac{x}{|x|}\Rightarrow$ critical times and endpoints are $0,\frac{\pi}{3},\frac{5\pi}{6},\frac{4\pi}{3},\frac{11\pi}{6},2\pi$; then $s(0)=\frac{\sqrt{3}}{2},\ s\left(\frac{\pi}{3}\right)=0,\ s\left(\frac{5\pi}{6}\right)=1,\ s\left(\frac{4\pi}{3}\right)=0,\ s\left(\frac{11\pi}{6}\right)=1,\ s(2\pi)=\frac{\sqrt{3}}{2}\Rightarrow$ the greatest distance between the particles is 1.

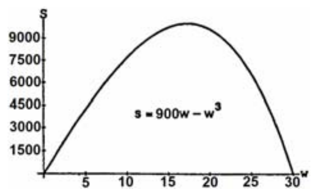
(c) Since $s'(t)=\frac{\left(\sin t-\sqrt{3}\cos t\right)\left(\cos t+\sqrt{3}\sin t\right)}{2\left|\sin t-\sqrt{3}\cos t\right|}$ we can conclude that at $t=\frac{\pi}{3}$ and $\frac{4\pi}{3}$, $s'(t)$ has cusps and the distance between the particles is changing the fastest near these points.

41. $I=\frac{k}{d^2}$, let x = distance the point is from the stronger light source $\Rightarrow 6-x$ = distance the point is from the other light source. The intensity of illumination at the point from the stronger light is $I_1=\frac{k_1}{x^2}$, and intensity of illumination at the point from the weaker light is $I_2=\frac{k_2}{(6-x)^2}$. Since the intensity of the first light is eight times the intensity of the second light $\Rightarrow k_1=8k_2.\Rightarrow I_1=\frac{8k_2}{x^2}$. The total intensity is given by $I=I_1+I_2=\frac{8k_2}{x^2}+\frac{k_2}{(6-x)^2}$ $\Rightarrow I'=-\frac{16k_2}{x^3}+\frac{2k_2}{(6-x)^3}=\frac{-16(6-x)^3k_2+2x^3k_2}{x^3(6-x)^3}$ and $I'=0\Rightarrow\frac{-16(6-x)^3k_2+2x^3k_2}{x^3(6-x)^3}=0\Rightarrow -16(6-x)^3k_2+2x^3k_2=0$ $\Rightarrow x=4$ m. $I''=\frac{48k_2}{x^4}+\frac{6k_2}{(6-x)^4}\Rightarrow I''(4)=\frac{48k_2}{4^4}+\frac{6k_2}{(6-4)^4}>0\Rightarrow$ local minimum. The point should be 4 m from the stronger light source.

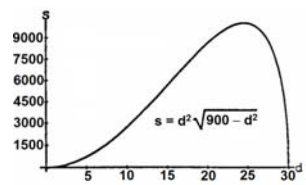
42. $R = \frac{v_0^2}{g}\sin 2\alpha \Rightarrow \frac{dR}{d\alpha} = \frac{2v_0^2}{g}\cos 2\alpha$ and $\frac{dR}{d\alpha} = 0 \Rightarrow \frac{2v_0^2}{g}\cos 2\alpha = 0 \Rightarrow \alpha = \frac{\pi}{4}$. $\frac{d^2R}{d\alpha^2} = -\frac{4v_0^2}{g}\sin 2\alpha$
$\Rightarrow \left.\frac{d^2R}{d\alpha^2}\right|_{\alpha=\frac{\pi}{4}} = -\frac{4v_0^2}{g}\sin 2\left(\frac{\pi}{4}\right) = -\frac{4v_0^2}{g} < 0 \Rightarrow$ local maximum. Thus, the firing angle of $\alpha = \frac{\pi}{4} = 45°$ will maximize the range R.

43. (a) From the diagram we have $d^2 = 4r^2 - w^2$. The strength of the beam is $S = kwd^2 = kw(4r^2 - w^2)$. When $r = 15$, then $S = 900kw - kw^3$. Also, $S'(w) = 900k - 3kw^2 = 3k(300 - w^2)$ so $S'(w) = 0 \Rightarrow w = \pm 10\sqrt{3}$; $S''\left(10\sqrt{3}\right) < 0$ and $-10\sqrt{3}$ is not acceptable. Therefore $S\left(10\sqrt{3}\right)$ is the maximum strength. The dimensions of the strongest beam are $10\sqrt{3}$ by $10\sqrt{6}$ centimeters.

(b)



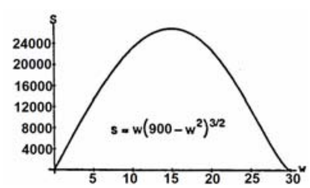
(c)



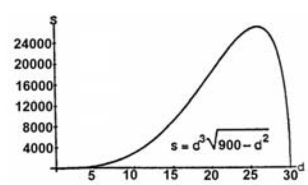
Both graphs indicate the same maximum value and are consistent with each other. Changing k does not change the dimensions that give the strongest beam (i.e., do not change the values of w and d that produce the strongest beam).

44. (a) From the situation we have $w^2 = 900 - d^2$. The stiffness of the beam is $S = kwd^3 = kd^3(900 - d^2)^{1/2}$, where $0 \le d \le 30$. Also, $S'(d) = \frac{4kd^2(675 - d^2)}{\sqrt{900 - d^2}} \Rightarrow$ critical points at 0, 30, and $15\sqrt{3}$. Both $d = 0$ and $d = 30$ cause $S = 0$. The maximum occurs at $d = 15\sqrt{3}$. The dimensions are 15 by $15\sqrt{3}$ centimeters.

(b)



(c)



Both graphs indicate the same maximum value and are consistent with each other. The changing of k has no effect.

45. (a) $s = 10\cos(\pi t) \Rightarrow v = -10\pi\sin(\pi t) \Rightarrow$ speed $= |10\pi\sin(\pi t)| = 10\pi\,|\sin(\pi t)| \Rightarrow$ the maximum speed is $10\pi \approx 31.42$ cm/s since the maximum value of $|\sin(\pi t)|$ is 1; the cart is moving the fastest at $t = 0.5$ s, 1.5 s, 2.5 s and 3.5 s when $|\sin(\pi t)|$ is 1. At these times the distance is $s = 10\cos\left(\frac{\pi}{2}\right) = 0$ cm and $a = -10\pi^2\cos(\pi t) \Rightarrow |a| = 10\pi^2\,|\cos(\pi t)| \Rightarrow |a| = 0$ cm/s^2

(b) $|a| = 10\pi^2\,|\cos(\pi t)|$ is greatest at $t = 0.0$ s, 1.0 s, 2.0 s, 3.0 s, and 4.0 s, and at these times the magnitude of the cart's position is $|s| = 10$ cm from the rest position and the speed is 0 cm/s.

46. (a) $2\sin t = \sin 2t \Rightarrow 2\sin t - 2\sin t\cos t = 0 \Rightarrow (2\sin t)(1 - \cos t) = 0 \Rightarrow t = k\pi$ where k is a positive integer

(b) The vertical distance between the masses is $s(t) = |s_1 - s_2| = \left((s_1 - s_2)^2\right)^{1/2} = ((\sin 2t - 2\sin t)^2)^{1/2}$
$\Rightarrow s'(t) = \left(\frac{1}{2}\right)((\sin 2t - 2\sin t)^2)^{-1/2}(2)(\sin 2t - 2\sin t)(2\cos 2t - 2\cos t) = \frac{2(\cos 2t - 2\cos t)(\sin 2t - 2\sin t)}{|\sin 2t - 2\sin t|}$
$= \frac{4(2\cos t + 1)(\cos t - 1)(\sin t)(\cos t - 1)}{|\sin 2t - 2\sin t|} \Rightarrow$ critical times at 0, $\frac{2\pi}{3}$, π, $\frac{4\pi}{3}$, 2π; then $s(0) = 0$,

$s\left(\frac{2\pi}{3}\right) = \left|\sin\left(\frac{4\pi}{3}\right) - 2\sin\left(\frac{2\pi}{3}\right)\right| = \frac{3\sqrt{3}}{2}$, $s(\pi) = 0$, $s\left(\frac{4\pi}{3}\right) = \left|\sin\left(\frac{8\pi}{3}\right) - 2\sin\left(\frac{4\pi}{3}\right)\right| = \frac{3\sqrt{3}}{2}$, $s(2\pi) = 0$

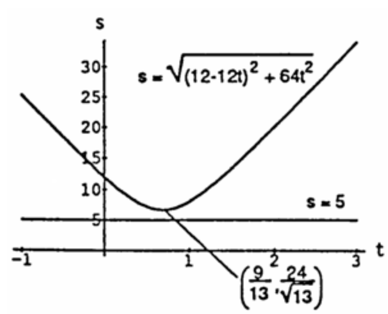
$\Rightarrow$ the greatest distance is $\frac{3\sqrt{3}}{2}$ at $t = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$

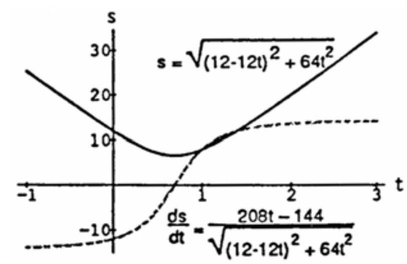
47. (a) $s = \sqrt{(12-12t)^2 + (8t)^2} = ((12-12t)^2 + 64t^2)^{1/2}$

(b) $\frac{ds}{dt} = \frac{1}{2}((12-12t)^2 + 64t^2)^{-1/2}[2(12-12t)(-12) + 128t] = \frac{208t-144}{\sqrt{(12-12t^2)+64t^2}} \Rightarrow \left.\frac{ds}{dt}\right|_{t=0} = -12$ knots and $\left.\frac{ds}{dt}\right|_{t=1} = 8$ knots

(c) The graph indicates that the ships did not see each other because $s(t) > 5$ for all values of t.

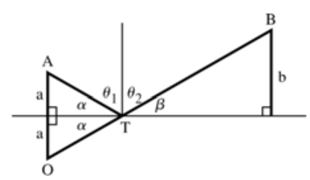
(d) The graph supports the conclusions in parts (b) and (c).





(e) $\lim\limits_{t\to\infty} \frac{ds}{dt} = \sqrt{\lim\limits_{t\to\infty} \frac{(208t-144)^2}{144(1-t)^2+64t^2}} = \sqrt{\lim\limits_{t\to\infty} \frac{\left(208 - \frac{144}{t}\right)^2}{144\left(\frac{1}{t}-1\right)^2 + 64}} = \sqrt{\frac{208^2}{144+64}} = \sqrt{208} = 4\sqrt{13}$ which equals the square root of the sums of the squares of the individual speeds.

48. The distance $OT + TB$ is minimized when $\overline{OB}$ is a straight line. Hence $\angle\alpha = \angle\beta \Rightarrow \theta_1 = \theta_2$.



49. If $v = kax - kx^2$, then $v' = ka - 2kx$ and $v'' = -2k$, so $v' = 0 \Rightarrow x = \frac{a}{2}$. At $x = \frac{a}{2}$ there is a maximum since $v''\left(\frac{a}{2}\right) = -2k < 0$. The maximum value of v is $\frac{ka^2}{4}$.

50. (a) According to the graph, $y'(0) = 0$.

(b) According to the graph, $y'(-L) = 0$.

(c) $y(0) = 0$, so $d = 0$. Now $y'(x) = 3ax^2 + 2bx + c$, so $y'(0) = 0$ implies that $c = 0$. Therefore, $y'(x) = ax^3 + bx^2$ and $y'(x) = 3ax^2 + 2bx$. then $y(-L) = -aL^3 + bL^2 = H$ and $y'(-L) = 3aL^2 + 2bL = 0$, so we have two linear equations in two unknowns a and b. The second equation gives $b = \frac{3aL}{2}$. Substituting into the first equation, we have $-aL^3 + \frac{3aL^3}{2} = H$, or $\frac{aL^3}{2} = H$, so $a = 2\frac{H}{L^3}$. Therefore, $b = 3\frac{H}{L^2}$ and the equation for y is $y(x) = 2\frac{H}{L^3}x^3 + 3\frac{H}{L^2}x^2$, or $y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right]$.

51. The profit is $p = nx - nc = n(x-c) = [a(x-c)^{-1} + b(100-x)](x-c) = a + b(100-x)(x-c)$ $= a + (bc + 100b)x - 100bc - bx^2$. Then $p'(x) = bc + 100b - 2bx$ and $p''(x) = -2b$. Solving $p'(x) = 0 \Rightarrow x = \frac{c}{2} + 50$. At $x = \frac{c}{2} + 50$ there is a maximum profit since $p''(x) = -2b < 0$ for all x.

52. Let x represent the number of people over 50. The profit is $p(x)=(50+x)(200-2x)-32(50+x)-6000$ $=-2x^2+68x+2400.$ Then $p'(x)=-4x+68$ and $p''=-4.$ Solving $p'(x)=0 \Rightarrow x=17.$ At $x=17$ there is a maximum since $p''(17)<0.$ It would take 67 people to maximize the profit.

53. (a) $A(q)=kmq^{-1}+cm+\frac{h}{2}q$, where $q>0 \Rightarrow A'(q)=-kmq^{-2}+\frac{h}{2}=\frac{hq^2-2km}{2q^2}$ and $A''(q)=2kmq^{-3}$. The critical points are $-\sqrt{\frac{2km}{h}}$, 0, and $\sqrt{\frac{2km}{h}}$, but only $\sqrt{\frac{2km}{h}}$ is in the domain. Then $A''\left(\sqrt{\frac{2km}{h}}\right)>0 \Rightarrow$ at $q=\sqrt{\frac{2km}{h}}$ there is a minimum average weekly cost.

(b) $A(q)=\frac{(k+bq)m}{q}+cm+\frac{h}{2}q=kmq^{-1}+bm+cm+\frac{h}{2}q$, where $q>0 \Rightarrow A'(q)=0$ at $q=\sqrt{\frac{2km}{h}}$ as in (a). Also $A''(q)=2kmq^{-3}>0$ so the most economical quantity to order is still $q=\sqrt{\frac{2km}{h}}$ which minimizes the average weekly cost.

54. We start with $c(x)=$ the cost of producing x items, $x>0$, and $\frac{c(x)}{x}=$ the average cost of producing x items, assumed to be differentiable. If the average cost can be minimized, it will be at a production level at which $\frac{d}{dx}\left(\frac{c(x)}{x}\right)=0 \Rightarrow \frac{xc'(x)-c(x)}{x^2}=0$ (by the quotient rule) $\Rightarrow xc'(x)-c(x)=0$ (multiply both sides by x^2) $\Rightarrow c'(x)=\frac{c(x)}{x}$ where $c'(x)$ is the marginal cost. This concludes the proof. (Note: The theorem does not assure a production level that will give a minimum cost, but rather, it indicates where to look to see if there is one. Find the production levels where the average cost equals the marginal cost, then check to see if any of them give a minimum.)

55. The profit $p(x)=r(x)-c(x)=6x-(x^3-6x^2+15x)=-x^3+6x^2-9x$, where $x\ge 0.$ Then $p'(x)=-3x^2+12x-9$ $=-3(x-3)(x-1)$ and $p''(x)=-6x+12.$ The critical points are 1 and 3. Thus $p''(1)=6>0 \Rightarrow$ at $x=1$ there is a local minimum, and $p''(3)=-6<0 \Rightarrow$ at $x=3$ there is a local maximum. But $p(3)=0 \Rightarrow$ the best you can do is break even.

56. The average cost of producing x items is $\bar{c}(x)=\frac{c(x)}{x}=x^2-20x+20{,}000 \Rightarrow \bar{c}'(x)=2x-20=0 \Rightarrow x=10$, the only critical value. The average cost is $\bar{c}(10)=\$19{,}900$ per item is a minimum cost because $\bar{c}''(10)=2>0.$

57. Let $x=$ the length of a side of the square base of the box and $h=$ the height of the box. $V=x^2h=6 \Rightarrow h=\frac{6}{x^2}.$ The total cost is given by

$C=60\cdot x^2+40(4\cdot xh)=60x^2+160x\left(\frac{6}{x^2}\right)=60x^2+\frac{960}{x},\ x>0 \Rightarrow C'=120x-\frac{960}{x^2}=\frac{120x^3-960}{x^2}$

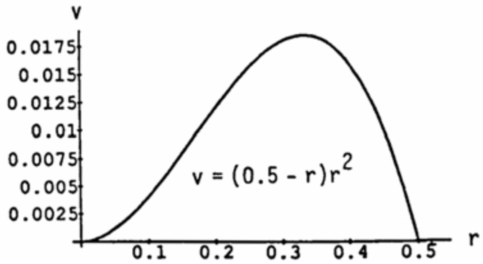
$C'=0 \Rightarrow \frac{12x^3-96}{x^2}=0 \Rightarrow 12x^3-96=0 \Rightarrow x=4;\ C''=120+\frac{1920}{x^2} \Rightarrow C''(4)=120+\frac{1920}{4^2}>0 \Rightarrow$ local minimum. $x=2 \Rightarrow h=\frac{6}{2^2}=3/2$ and $C(2)=60(2)^2+\frac{960}{2}=720 \Rightarrow$ the box is $2\text{ m}\times 2\text{ m}\times 3/2\text{ m},$ with a minimum cost of \$720.

58. Let $x=$ the number of \$10 increases in the charge per room, then price per room $=50+10x$, and the number of rooms filled each night $=800-40x \Rightarrow$ the total revenue is $R(x)=(50+10x)(800-40x)$ $=-400x^2+6000x+40000,\ 0\le x\le 20 \Rightarrow R'(x)=-800x+6000;\ R'(x)=0 \Rightarrow -800x+6000=0$ $\Rightarrow x=\frac{15}{2};\ R''(x)=-800 \Rightarrow R''\left(\frac{15}{2}\right)=-800<0 \Rightarrow$ local maximum. The price per room is $50+10\left(\frac{15}{2}\right)=\$125.$

59. We have $\frac{dR}{dM}=CM-M^2.$ Solving $\frac{d^2R}{dM^2}=C-2M=0 \Rightarrow M=\frac{C}{2}.$ Also, $\frac{d^3R}{dM^3}=-2<0 \Rightarrow$ at $M=\frac{C}{2}$ there is a maximum.

60. (a) If $v = cr_0r^2 - cr^3$, then $v' = 2cr_0r - 3cr^2 = cr(2r_0 - 3r)$ and $v'' = 2cr_0 - 6cr = 2c(r_0 - 3r)$. The solution of $v' = 0$ is $r = 0$ or $\frac{2r_0}{3}$, but 0 is not in the domain. Also, $v' > 0$ for $r < \frac{2r_0}{3}$ and $v' < 0$ for $r > \frac{2r_0}{3} \Rightarrow$ at $r = \frac{2r_0}{3}$ there is a maximum.

(b) The graph confirms the findings in (a).



61. If $x > 0$, then $(x-1)^2 \geq 0 \Rightarrow x^2 + 1 \geq 2x \Rightarrow \frac{x^2+1}{x} \geq 2$. In particular if a, b, c and d are positive integers, then $\left(\frac{a^2+1}{a}\right)\left(\frac{b^2+1}{b}\right)\left(\frac{c^2+1}{c}\right)\left(\frac{d^2+1}{d}\right) \geq 16$.

62. (a) $f(x) = \frac{x}{\sqrt{a^2+x^2}} \Rightarrow f'(x) = \frac{(a^2+x^2)^{1/2} - x^2(a^2+x^2)^{-1/2}}{(a^2+x^2)} = \frac{a^2+x^2-x^2}{(a^2+x^2)^{3/2}} = \frac{a^2}{(a^2+x^2)^{3/2}} > 0 \Rightarrow f(x)$ is an increasing function of x

(b) $g(x) = \frac{d-x}{\sqrt{b^2+(d-x)^2}} \Rightarrow g'(x) = \frac{-\left(b^2+(d-x)^2\right)^{1/2} + (d-x)^2\left(b^2+(d-x)^2\right)^{-1/2}}{b^2+(d-x)^2} = \frac{-\left(b^2+(d-x)^2\right)+(d-x)^2}{\left(b^2+(d-x)^2\right)^{3/2}}$

$= \frac{-b^2}{\left(b^2+(d-x)^2\right)^{3/2}} < 0 \Rightarrow g(x)$ is a decreasing function of x

(c) Since $c_1, c_2 > 0$, the derivative $\frac{dt}{dx}$ is an increasing function of x (from part (a)) minus a decreasing function of x (from part (b)): $\frac{dt}{dx} = \frac{1}{c_1}f(x) - \frac{1}{c_2}g(x) \Rightarrow \frac{d^2t}{dx^2} = \frac{1}{c_1}f'(x) - \frac{1}{c_2}g'(x) > 0$ since $f'(x) > 0$ and $g'(x) < 0 \Rightarrow \frac{dt}{dx}$ is an increasing function of x.

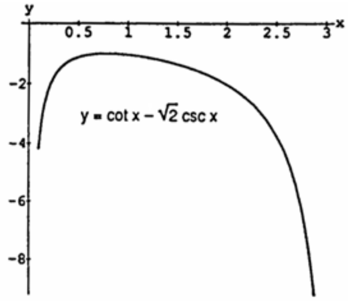
63. At $x = c$, the tangents to the curves are parallel. Justification: The vertical distance between the curves is $D(x) = f(x) - g(x)$, so $D'(x) = f'(x) - g'(x)$. The maximum value of D will occur at a point c where $D' = 0$. At such a point, $f'(c) - g'(c) = 0$, or $f'(c) = g'(c)$.

64. (a) $f(x) = 3 + 4\cos x + \cos 2x$ is a periodic function with period 2π

(b) No, $f(x) = 3 + 4\cos x + \cos 2x = 3 + 4\cos x + (2\cos^2 x - 1) = 2(1 + 2\cos x + \cos^2 x) = 2(1+\cos x)^2 \geq 0$ $\Rightarrow f(x)$ is never negative.

65. (a) If $y = \cot x - \sqrt{2}\csc x$ where $0 < x < \pi$, then $y' = (\csc x)\left(\sqrt{2}\cot x - \csc x\right)$. Solving $y' = 0 \Rightarrow \cos x = \frac{1}{\sqrt{2}}$ $\Rightarrow x = \frac{\pi}{4}$. For $0 < x < \frac{\pi}{4}$ we have $y' > 0$ and $y' < 0$ when $\frac{\pi}{4} < x < \pi$. Therefore, at $x = \frac{\pi}{4}$ there is a maximum value of $y = -1$.

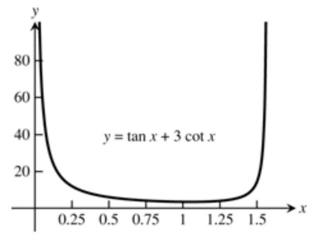
(b)



The graph confirms the findings in (a).

66. (a) If $y = \tan x + 3\cot x$ where $0 < x < \frac{x}{2}$, then $y' = \sec^2 x - 3\csc^2 x$. Solving $y' = 0 \Rightarrow \tan x = \pm\sqrt{3} \Rightarrow x = \pm\frac{\pi}{3}$, but $-\frac{\pi}{3}$ is not in the domain. Also, $y'' = 2\sec^2 x\tan x + 6\csc^2 x\cot x > 0$ for all $0 < x < \frac{\pi}{2}$. Therefore at $x = \frac{\pi}{3}$ there is a minimum value of $y = 2\sqrt{3}$.

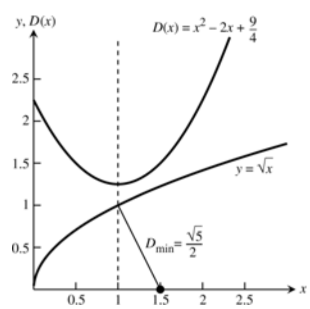
(b)



The graph confirms the findings in (a).

67. (a) The square of the distance is $D(x) = \left(x - \frac{3}{2}\right)^2 + \left(\sqrt{x} + 0\right)^2 = x^2 - 2x + \frac{9}{4}$, so $D'(x) = 2x - 2$ and the critical point occurs at $x = 1$. Since $D'(x) < 0$ for $x < 1$ and $D'(x) > 0$ for $x > 1$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(1)} = \frac{\sqrt{5}}{2}$.

(b)



The minimum distance is from the point $\left(\frac{3}{2}, 0\right)$ to the point $(1, 1)$ on the graph of $y = \sqrt{x}$, and this occurs at the value $x = 1$ where $D(x)$, the distance squared, has its minimum value.

68. (a) Calculus Method:

The square of the distance from the point $\left(1, \sqrt{3}\right)$ to $\left(x, \sqrt{16 - x^2}\right)$ is given by

$$D(x) = (x-1)^2 + \left(\sqrt{16 - x^2} - \sqrt{3}\right)^2 = x^2 - 2x + 1 + 16 - x^2 - 2\sqrt{48 - 3x^2} + 3 = -2x + 20 - 2\sqrt{48 - 3x^2}.$$

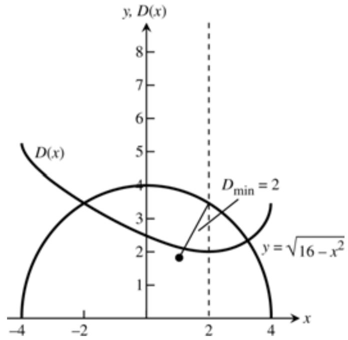
Then $D'(x) = -2 - \frac{1}{2}\cdot\frac{2}{\sqrt{48-3x^2}}(-6x) = -2 + \frac{6x}{\sqrt{48-3x^2}}$. Solving $D'(x) = 0$ we have:

$6x = 2\sqrt{48 - 3x^2} \Rightarrow 36x^2 = 4(48 - 3x^2) \Rightarrow 9x^2 = 48 - 3x^2 \Rightarrow 12x^2 = 48 \Rightarrow x = \pm 2$ We discard $x = -2$ as an extraneous solution, leaving $x = 2$. Since $D'(x) < 0$ for $-4 < x < 2$ and $D'(x) > 0$ for $2 < x < 4$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(2)} = 2$.

Geometry Method:

The semicircle is centered at the origin and has radius 4. The distance from the origin to $\left(1, \sqrt{3}\right)$ is $\sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$. The shortest distance from the point to the semicircle is the distance along the radius containing the point $\left(1, \sqrt{3}\right)$. That distance is $4 - 2 = 2$.

(b)



The minimum distance is from the point $\left(1, \sqrt{3}\right)$ to the point $\left(2, 2\sqrt{3}\right)$ on the graph of $y = \sqrt{16 - x^2}$, and this occurs at the value $x = 2$ where $D(x)$, the distance squared, has its minimum value.

4.6 NEWTON'S METHOD

1. $y = x^2 + x - 1 \Rightarrow y' = 2x + 1 \Rightarrow x_{n+1} = x_n - \frac{x_n^2 + x_n - 1}{2x_n + 1}$; $x_0 = 1 \Rightarrow x_1 = 1 - \frac{1+1-1}{2+1} = \frac{2}{3} \Rightarrow x_2 = \frac{2}{3} - \frac{\frac{4}{9} + \frac{2}{3} - 1}{\frac{4}{3} + 1}$
$\Rightarrow x_2 = \frac{2}{3} - \frac{4+6-9}{12+9} = \frac{2}{3} - \frac{1}{21} = \frac{13}{21} \approx .61905$; $x_0 = -1 \Rightarrow x_1 = -1 - \frac{1-1-1}{-2+1} = -2 \Rightarrow x_2 = -2 - \frac{4-2-1}{-4+1} = -\frac{5}{3} \approx -1.66667$

2. $y = x^3 + 3x + 1 \Rightarrow y' = 3x^2 + 3 \Rightarrow x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}$; $x_0 = 0 \Rightarrow x_1 = 0 - \frac{1}{3} = -\frac{1}{3} \Rightarrow x_2 = -\frac{1}{3} - \frac{-\frac{1}{27} - 1 + 1}{\frac{1}{3} + 3}$
$= -\frac{1}{3} + \frac{1}{90} = -\frac{29}{90} \approx -0.32222$

3. $y = x^4 + x - 3 \Rightarrow y' = 4x^3 + 1 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 + x_n - 3}{4x_n^3 + 1}$; $x_0 = 1 \Rightarrow x_1 = 1 - \frac{1+1-3}{4+1} = \frac{6}{5} \Rightarrow x_2 = \frac{6}{5} - \frac{\frac{1296}{625} + \frac{6}{5} - 3}{\frac{864}{125} + 1}$
$= \frac{6}{5} - \frac{1296+750-1875}{4320+625} = \frac{6}{5} - \frac{171}{4945} = \frac{5763}{4945} \approx 1.16542$; $x_0 = -1 \Rightarrow x_1 = -1 - \frac{1-1-3}{-4+1} = -2 \Rightarrow x_2 = -2 - \frac{16-2-3}{-32+1}$
$= -2 + \frac{11}{31} = -\frac{51}{31} \approx -1.64516$

4. $y = 2x - x^2 + 1 \Rightarrow y' = 2 - 2x \Rightarrow x_{n+1} = x_n - \frac{2x_n - x_n^2 + 1}{2 - 2x_n}$; $x_0 = 0 \Rightarrow x_1 = 0 - \frac{0-0+1}{2-0} = -\frac{1}{2} \Rightarrow x_2 = -\frac{1}{2} - \frac{-1-\frac{1}{4}+1}{2+1}$
$= -\frac{1}{2} + \frac{1}{12} = -\frac{5}{12} \approx -.41667$; $x_0 = 2 \Rightarrow x_1 = 2 - \frac{4-4+1}{2-4} = \frac{5}{2} \Rightarrow x_2 = \frac{5}{2} - \frac{5 - \frac{25}{4} + 1}{2-5} = \frac{5}{2} - \frac{20-25+4}{-12} = \frac{5}{2} - \frac{1}{12}$
$= \frac{29}{12} \approx 2.41667$

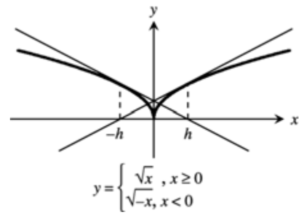
5. $y = x^4 - 2 \Rightarrow y' = 4x^3 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 - 2}{4x_n^3}$; $x_0 = 1 \Rightarrow x_1 = 1 - \frac{1-2}{4} = \frac{5}{4} \Rightarrow x_2 = \frac{5}{4} - \frac{\frac{625}{256} - 2}{\frac{125}{16}} = \frac{5}{4} - \frac{625-512}{2000}$
$= \frac{5}{4} - \frac{113}{2000} = \frac{2500-113}{2000} = \frac{2387}{2000} \approx 1.1935$

6. From Exercise 5, $x_{n+1} = x_n - \frac{x_n^4 - 2}{4x_n^3}$; $x_0 = -1 \Rightarrow x_1 = -1 - \frac{1-2}{-4} = -1 - \frac{1}{4} = -\frac{5}{4} \Rightarrow x_2 = -\frac{5}{4} - \frac{\frac{625}{256} - 2}{-\frac{125}{16}}$
$= -\frac{5}{4} - \frac{625-512}{-2000} = -\frac{5}{4} + \frac{113}{2000} \approx -1.1935$

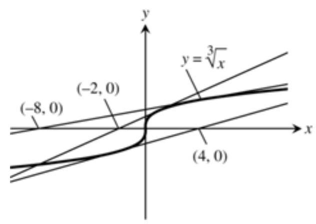
7. $f(x_0) = 0$ and $f'(x_0) \neq 0 \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ gives $x_1 = x_0 \Rightarrow x_2 = x_0 \Rightarrow x_n = x_0$ for all $n \geq 0$. That is all, of the approximations in Newton's method will be the root of $f(x) = 0$.

8. It does matter. If you start too far away from $x = \frac{\pi}{2}$, the calculated values may approach some other root. Starting with $x_0 = -0.5$, for instance, leads to $x = -\frac{\pi}{2}$ as the root, not $x = \frac{\pi}{2}$.

9. If $x_0 = h > 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = h - \frac{f(h)}{f'(h)}$
$= h - \frac{\sqrt{h}}{\left(\frac{1}{2\sqrt{h}}\right)} = h - \left(\sqrt{h}\right)\left(2\sqrt{h}\right) = -h;$
if $x_0 = -h < 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -h - \frac{f(-h)}{f'(-h)}$
$= -h - \frac{\sqrt{h}}{\left(\frac{-1}{2\sqrt{h}}\right)} = -h + \left(\sqrt{h}\right)\left(2\sqrt{h}\right) = h.$



10. $f(x) = x^{1/3} \Rightarrow f'(x) = \left(\frac{1}{3}\right)x^{-2/3}$
$\Rightarrow x_{n+1} = x_n - \frac{x_n^{1/3}}{\left(\frac{1}{3}\right)x_n^{-2/3}} = -2x_n; \ \ x_0 = 1$
$\Rightarrow x_1 = -2,\ x_2 = 4,\ x_3 = -8,$ and $x_4 = 16$ and so forth. Since $|x_n| = 2|x_{n-1}|$ we may conclude that $n \to \infty \Rightarrow |x_n| \to \infty.$



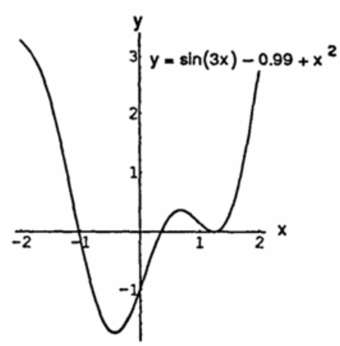
11. i) is equivalent to solving $x^3 - 3x - 1 = 0$.
ii) is equivalent to solving $x^3 - 3x - 1 = 0$.
iii) is equivalent to solving $x^3 - 3x - 1 = 0$.
iv) is equivalent to solving $x^3 - 3x - 1 = 0$.
All four equations are equivalent.

12. $f(x) = x - 1 - 0.5\sin x \Rightarrow f'(x) = 1 - 0.5\cos x \Rightarrow x_{n+1} = x_n - \frac{x_n - 1 - 0.5\sin x_n}{1 - 0.5\cos x_n}$; if $x_0 = 1.5$, then $x_1 = 1.49870$

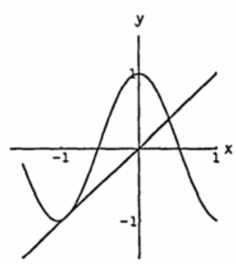
13. $f(x) = \tan x - 2x \Rightarrow f'(x) = \sec^2 x - 2 \Rightarrow x_{n+1} = x_n - \frac{\tan(x_n) - 2x_n}{\sec^2(x_n)}; \ \ x_0 = 1 \Rightarrow x_1 = 1.2920445$
$\Rightarrow x_2 = 1.155327774 \ \ \Rightarrow x_{16} = x_{17} = 1.165561185$

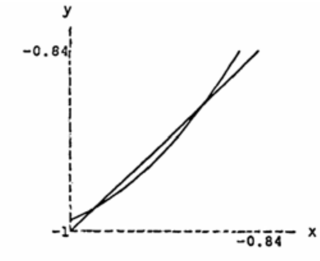
14. $f(x) = x^4 - 2x^3 - x^2 - 2x + 2 \Rightarrow f'(x) = 4x^3 - 6x^2 - 2x - 2 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 - 2x_n^3 - x_n^2 - 2x_n + 2}{4x_n^3 - 6x_n^2 - 2x_n - 2}$; if $x_0 = 0.5$, then $x_4 = 0.630115396$; if $x_0 = 2.5$, then $x_4 = 2.57327196$

15. (a) The graph of $f(x) = \sin 3x - 0.99 + x^2$ in the window $-2 \le x \le 2, -2 \le y \le 3$ suggests three roots. However, when you zoom in on the x-axis near $x = 1.2$, you can see that the graph lies above the axis there. There are only two roots, one near $x = -1$, the other near $x = 0.4$.
(b) $f(x) = \sin 3x - 0.99 + x^2$
$\Rightarrow f'(x) = 3\cos 3x + 2x$
$\Rightarrow x_{n+1} = x_n - \frac{\sin(3x_n) - 0.99 + x_n^2}{3\cos(3x_n) + 2x_n}$
and the solutions are approximately 0.35003501505249 and –1.0261731615301



16. (a) Yes, three times as indicted by the graphs
(b) $f(x)=\cos 3x - x \Rightarrow f'(x) = -3\sin 3x - 1$
$\Rightarrow x_{n+1} = x_n - \frac{\cos(3x_n) - x_n}{-3\sin(3x_n) - 1}$; at approximately
-0.979367, -0.887726, and 0.39004 we have
$\cos 3x = x$

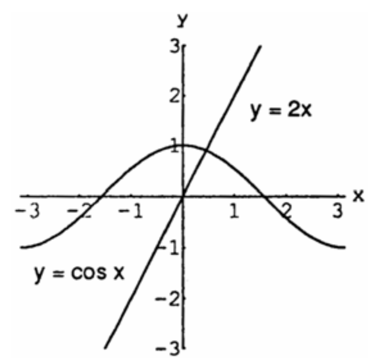




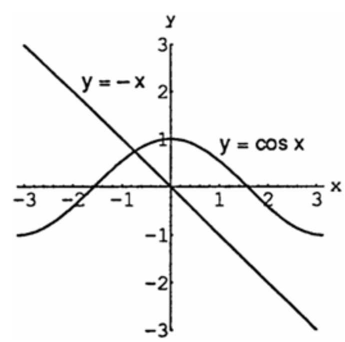
17. $f(x) = 2x^4 - 4x^2 + 1 \Rightarrow f'(x) = 8x^3 - 8x \Rightarrow x_{n+1} = x_n - \frac{2x_n^4 - 4x_n^2 + 1}{8x_n^3 - 8x_n}$; if $x_0 = -2$, then $x_6 = -1.30656296$; if $x_0 = -0.5$, then $x_3 = -0.5411961$; the roots are approximately ± 0.5411961 and ± 1.30656296 because $f(x)$ is an even function.

18. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow x_{n+1} = x_n - \frac{\tan(x_n)}{\sec^2(x_n)}$; $x_0 = 3 \Rightarrow x_1 = 3.13971 \Rightarrow x_2 = 3.14159$ and we approximate π to be 3.14159.

19. From the graph we let $x_0 = 0.5$ and $f(x) = \cos x - 2x$
$\Rightarrow x_{n+1} = x_n - \frac{\cos(x_n) - 2x_n}{-\sin(x_n) - 2} \Rightarrow x_1 = .45063$
$\Rightarrow x_2 = .45018 \Rightarrow$ at $x \approx 0.45$ we have $\cos x = 2x$.



20. From the graph we let $x_0 = -0.7$ and
$f(x) = \cos x + x \Rightarrow x_{n+1} = x_n - \frac{x_n + \cos(x_n)}{1 - \sin(x_n)}$
$\Rightarrow x_1 = -.73944 \Rightarrow x_2 = -.73908 \Rightarrow$ at $x \approx -0.74$
we have $\cos x = -x$.



21. The x-coordinate of the point of intersection of $y = x^2(x+1)$ and $y = \frac{1}{x}$ is the solution of $x^2(x+1) = \frac{1}{x}$
$\Rightarrow x^3 + x^2 - \frac{1}{x} = 0 \Rightarrow$ The x-coordinate is the root of $f(x) = x^3 + x^2 - \frac{1}{x} \Rightarrow f'(x) = 3x^2 + 2x + \frac{1}{x^2}$. Let $x_0 = 1$
$\Rightarrow x_{n+1} = x_n - \frac{x_n^3 + x_n^2 - \frac{1}{x_n}}{3x_n^2 + 2x_n + \frac{1}{x_n^2}} \Rightarrow x_1 = 0.83333 \Rightarrow x_2 = 0.81924 \Rightarrow x_3 = 0.81917 \Rightarrow x_7 = 0.81917 \Rightarrow r \approx 0.8192$

22. The x-coordinate of the point of intersection of $y=\sqrt{x}$ and $y=3-x^2$ is the solution of $\sqrt{x}=3-x^2$ $\Rightarrow \sqrt{x}-3+x^2=0 \Rightarrow$ The x-coordinate is the root of $f(x)=\sqrt{x}-3+x^2 \Rightarrow f'(x)=\frac{1}{2\sqrt{x}}+2x$. Let $x_0=1$ $\Rightarrow x_{n+1}=x_n-\frac{\sqrt{x_n}-3+x_n{}^2}{\frac{1}{2\sqrt{x_n}}+2x_n} \Rightarrow x_1=1.4 \Rightarrow x_2=1.35556 \Rightarrow x_3=1.35498 \Rightarrow x_7=1.35498 \Rightarrow r\approx 1.3550$

23. If $f(x)=x^3+2x-4$, then $f(1)=-1<0$ and $f(2)=8>0 \Rightarrow$ by the Intermediate Value Theorem the equation $x^3+2x-4=0$ has a solution between 1 and 2. Consequently, $f'(x)=3x^2+2$ and $x_{n+1}=x_n-\frac{x_n^3+2x_n-4}{3x_n^2+2}$. Then $x_0=1 \Rightarrow x_1=1.2 \Rightarrow x_2=1.17975 \Rightarrow x_3=1.179509 \Rightarrow x_4=1.1795090 \Rightarrow$ the root is approximately 1.17951.

24. We wish to solve $8x^4-14x^3-9x^2+11x-1=0$. Let $f(x)=8x^4-14x^3-9x^2+11x-1$, then $f'(x)=32x^3-42x^2-18x+11 \Rightarrow x_{n+1}=x_n-\frac{8x_n^4-14x_n^3-9x_n^2+11x_n-1}{32x_n^3-42x_n^2-18x_n+11}$.

x_0	approximation of corresponding root
–1.0	–0.976823589
0.1	0.100363332
0.6	0.642746671
2.0	1.983713587

25. $f(x)=4x^4-4x^2 \Rightarrow f'(x)=16x^3-8x \Rightarrow x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}=x_i-\frac{x_i^3-x_i}{4x_i^2-2}$. Iterations are performed using the procedure in problem 13 in this section.

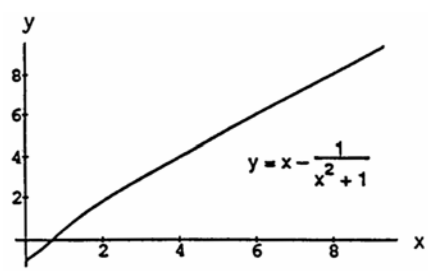
(a) For $x_0=-2$ or $x_0=-0.8$, $x_i\to -1$ as i gets large.

(b) For $x_0=-0.5$ or $x_0=0.25$, $x_i\to 0$ as i gets large.

(c) For $x_0=0.8$ or $x_0=2$, $x_i\to 1$ as i gets large.

(d) (If your calculator has a CAS, put it in exact mode, otherwise approximate the radicals with a decimal value.) For $x_0=-\frac{\sqrt{21}}{7}$ or $x_0=\frac{\sqrt{21}}{7}$, Newton's method does not converge. The values of x_i alternate between $x_0=-\frac{\sqrt{21}}{7}$ or $x_0=\frac{\sqrt{21}}{7}$ as i increases.

26. (a) The distance can be represented by $D(x)=\sqrt{(x-2)^2+\left(x^2+\frac{1}{2}\right)^2}$, where $x\ge 0$. The distance $D(x)$ is minimized when $f(x)=(x-2)^2+\left(x^2+\frac{1}{2}\right)^2$ is minimized. If $f(x)=(x-2)^2+\left(x^2+\frac{1}{2}\right)^2$, then $f'(x)=4(x^3+x-1)$ and $f''(x)=4(3x^2+1)>0$. Now $f'(x)=0 \Rightarrow x^3+x-1=0 \Rightarrow x(x^2+1)=1 \Rightarrow x=\frac{1}{x^2+1}$.



(b) Let $g(x)=\frac{1}{x^2+1}-x=(x^2+1)^{-1}-x \Rightarrow g'(x)=-(x^2+1)^{-2}(2x)-1=\frac{-2x}{(x^2+1)^2}-1 \Rightarrow x_{n+1}=x_n-\frac{\left(\frac{1}{x_n^2+1}-x_n\right)}{\left(\frac{-2x_n}{\left(x_n^2+1\right)^2}-1\right)}$;

$x_0=1 \Rightarrow x_4=0.68233$ to five decimal places.

27. $f(x)=(x-1)^{40} \Rightarrow f'(x)=40(x-1)^{39} \Rightarrow x_{n+1}=x_n-\frac{(x_n-1)^{40}}{40(x_n-1)^{39}}=\frac{39x_n+1}{40}$. With $x_0=2$, our computer gave $x_{87}=x_{88}=x_{89}=\cdots=x_{200}=1.11051$, coming within 0.11051 of the root $x=1$.

28. Since $s=r\theta \Rightarrow 3=r\theta \Rightarrow \theta=\frac{3}{r}$. Bisect the angle θ to obtain a right triangle with hypotenuse r and opposite side of length 1. Then $\sin\frac{\theta}{2}=\frac{1}{r} \Rightarrow \sin\frac{\left(\frac{3}{r}\right)}{2}=\frac{1}{r} \Rightarrow \sin\left(\frac{3}{2r}\right)=\frac{1}{r} \Rightarrow \sin\frac{3}{2r}-\frac{1}{r}=0$. Thus the solution r is a root of $f(r)=\sin\left(\frac{3}{2r}\right)-\frac{1}{r} \Rightarrow f'(r)=-\frac{3}{2r^2}\cos\left(\frac{3}{2r}\right)+\frac{1}{r^2}$; $r_0=1 \Rightarrow r_{n+1}=r_n-\frac{\sin\left(\frac{3}{2r_n}\right)-\frac{1}{r_n}}{-\frac{3}{2r_n^2}\cos\left(\frac{3}{2r_n}\right)+\frac{1}{r_n^2}} \Rightarrow r_1=1.00280$ $\Rightarrow r_2=1.00282 \Rightarrow r_3=1.00282 \Rightarrow r\approx 1.0028 \Rightarrow \theta=\frac{3}{1.00282}\approx 2.9916$

4.7 ANTIDERIVATIVES

1. (a) x^2 (b) $\frac{x^3}{3}$ (c) $\frac{x^3}{3}-x^2+x$

2. (a) $3x^2$ (b) $\frac{x^8}{8}$ (c) $\frac{x^8}{8}-3x^2+8x$

3. (a) x^{-3} (b) $-\frac{x^{-3}}{3}$ (c) $-\frac{x^{-3}}{3}+x^2+3x$

4. (a) $-x^{-2}$ (b) $-\frac{x^{-2}}{4}+\frac{x^3}{3}$ (c) $\frac{x^{-2}}{2}+\frac{x^2}{2}-x$

5. (a) $\frac{-1}{x}$ (b) $\frac{-5}{x}$ (c) $2x+\frac{5}{x}$

6. (a) $\frac{1}{x^2}$ (b) $\frac{-1}{4x^2}$ (c) $\frac{x^4}{4}+\frac{1}{2x^2}$

7. (a) $\sqrt{x^3}$ (b) $\sqrt{x}$ (c) $\frac{2}{3}\sqrt{x^3}+2\sqrt{x}$

8. (a) $x^{4/3}$ (b) $\frac{1}{2}x^{2/3}$ (c) $\frac{3}{4}x^{4/3}+\frac{3}{2}x^{2/3}$

9. (a) $x^{2/3}$ (b) $x^{1/3}$ (c) $x^{-1/3}$

10. (a) $\frac{1}{\sqrt{3}+1}x^{\sqrt{3}+1}$ (b) $\frac{1}{\pi+1}x^{\pi+1}$ (c) $\frac{1}{\sqrt{2}}x^{\sqrt{2}}$

11. (a) $\cos(\pi x)$ (b) $-3\cos x$ (c) $\frac{-\cos(\pi x)}{\pi}+\cos(3x)$

12. (a) $\sin(\pi x)$ (b) $\sin\left(\frac{\pi x}{2}\right)$ (c) $\left(\frac{2}{\pi}\right)\sin\left(\frac{\pi x}{2}\right)+\pi\sin x$

13. (a) $\frac{1}{2}\tan x$ (b) $2\tan\left(\frac{x}{3}\right)$ (c) $-\frac{2}{3}\tan\left(\frac{3x}{2}\right)$

14. (a) $-\cot x$ (b) $\cot\left(\frac{3x}{2}\right)$ (c) $x+4\cot(2x)$

15. (a) $-\csc x$ (b) $\frac{1}{5}\csc(5x)$ (c) $2\csc\left(\frac{\pi x}{2}\right)$

16. (a) $\sec x$ (b) $\frac{4}{3}\sec(3x)$ (c) $\frac{2}{\pi}\sec\left(\frac{\pi x}{2}\right)$

17. $\int(x+1)\,dx = \frac{x^2}{2}+x+C$

18. $\int(5-6x)\,dx = 5x-3x^2+C$

19. $\int\left(3t^2+\frac{t}{2}\right)dt = t^3+\frac{t^2}{4}+C$

20. $\int\left(\frac{t^2}{2}+4t^3\right)dt = \frac{t^3}{6}+t^4+C$

21. $\int(2x^3-5x+7)dx = \frac{1}{2}x^4-\frac{5}{2}x^2+7x+C$

22. $\int(1-x^2-3x^5)dx = x-\frac{1}{3}x^3-\frac{1}{2}x^6+C$

23. $\int\left(\frac{1}{x^2}-x^2-\frac{1}{3}\right)dx = \int\left(x^{-2}-x^2-\frac{1}{3}\right)dx = \frac{x^{-1}}{-1}-\frac{x^3}{3}-\frac{1}{3}x+C = -\frac{1}{x}-\frac{x^3}{3}-\frac{x}{3}+C$

24. $\int\left(\frac{1}{5}-\frac{2}{x^3}+2x\right)dx = \int\left(\frac{1}{5}-2x^{-3}+2x\right)dx = \frac{1}{5}x-\left(\frac{2x^{-2}}{-2}\right)+\frac{2x^2}{2}+C = \frac{x}{5}+\frac{1}{x^2}+x^2+C$

25. $\int x^{-1/3}dx = \frac{x^{2/3}}{\frac{2}{3}}+C = \frac{3}{2}x^{2/3}+C$

26. $\int x^{-5/4}dx = \frac{x^{-1/4}}{-\frac{1}{4}}+C = \frac{-4}{\sqrt[4]{x}}+C$

27. $\int\left(\sqrt{x}+\sqrt[3]{x}\right)dx = \int\left(x^{1/2}+x^{1/3}\right)dx = \frac{x^{3/2}}{\frac{3}{2}}+\frac{x^{4/3}}{\frac{4}{3}}+C = \frac{2}{3}x^{3/2}+\frac{3}{4}x^{4/3}+C$

28. $\int\left(\frac{\sqrt{x}}{2}+\frac{2}{\sqrt{x}}\right)dx = \int\left(\frac{1}{2}x^{1/2}+2x^{-1/2}\right)dx = \frac{1}{2}\left(\frac{x^{3/2}}{\frac{3}{2}}\right)+2\left(\frac{x^{1/2}}{\frac{1}{2}}\right)+C = \frac{1}{3}x^{3/2}+4x^{1/2}+C$

29. $\int\left(8y-\frac{2}{y^{1/4}}\right)dy = \int\left(8y-2y^{-1/4}\right)dy = \frac{8y^2}{2}-2\left(\frac{y^{3/4}}{\frac{3}{4}}\right)+C = 4y^2-\frac{8}{3}y^{3/4}+C$

30. $\int\left(\frac{1}{7}-\frac{1}{y^{5/4}}\right)dy = \int\left(\frac{1}{7}-y^{-5/4}\right)dy = \frac{1}{7}y-\left(\frac{y^{-1/4}}{-\frac{1}{4}}\right)+C = \frac{y}{7}+\frac{4}{y^{1/4}}+C$

31. $\int 2x\left(1-x^{-3}\right)dx = \int\left(2x-2x^{-2}\right)dx = \frac{2x^2}{2}-2\left(\frac{x^{-1}}{-1}\right)+C = x^2+\frac{2}{x}+C$

32. $\int x^{-3}(x+1)\,dx = \int\left(x^{-2}+x^{-3}\right)dx = \frac{x^{-1}}{-1}+\left(\frac{x^{-2}}{-2}\right)+C = -\frac{1}{x}-\frac{1}{2x^2}+C$

33. $\int\frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt = \int\left(\frac{t^{3/2}}{t^2}+\frac{t^{1/2}}{t^2}\right)dt = \int\left(t^{-1/2}+t^{-3/2}\right)dt = \frac{t^{1/2}}{\frac{1}{2}}+\left(\frac{t^{-1/2}}{-\frac{1}{2}}\right)+C = 2\sqrt{t}-\frac{2}{\sqrt{t}}+C$

34. $\int\frac{4+\sqrt{t}}{t^3}\,dt = \int\left(\frac{4}{t^3}+\frac{t^{1/2}}{t^3}\right)dt = \int\left(4t^{-3}+t^{-5/2}\right)dt = 4\left(\frac{t^{-2}}{-2}\right)+\left(\frac{t^{-3/2}}{-\frac{3}{2}}\right)+C = -\frac{2}{t^2}-\frac{2}{3t^{3/2}}+C$

35. $\int -2\cos t\,dt = -2\sin t+C$

36. $\int -5\sin t\,dt = 5\cos t+C$

37. $\int 7\sin\frac{\theta}{3}\,d\theta = -21\cos\frac{\theta}{3}+C$

38. $\int 3\cos 5\theta\,d\theta = \frac{3}{5}\sin 5\theta+C$

39. $\int -3\csc^2 x\,dx = 3\cot x + C$

40. $\int -\frac{\sec^2 x}{3}\,dx = -\frac{\tan x}{3} + C$

41. $\int \frac{\csc\theta\cot\theta}{2}\,d\theta = -\frac{1}{2}\csc\theta + C$

42. $\int \frac{2}{5}\sec\theta\tan\theta\,d\theta = \frac{2}{5}\sec\theta + C$

43. $\int (4\sec x\tan x - 2\sec^2 x)\,dx = 4\sec x - 2\tan x + C$

44. $\int \frac{1}{2}(\csc^2 x - \csc x\cot x)\,dx = -\frac{1}{2}\cot x + \frac{1}{2}\csc x + C$

45. $\int (\sin 2x - \csc^2 x)\,dx = -\frac{1}{2}\cos 2x + \cot x + C$

46. $\int (2\cos 2x - 3\sin 3x)\,dx = \sin 2x + \cos 3x + C$

47. $\int \frac{1+\cos 4t}{2}\,dt = \int \left(\frac{1}{2} + \frac{1}{2}\cos 4t\right)dt = \frac{1}{2}t + \frac{1}{2}\left(\frac{\sin 4t}{4}\right) + C = \frac{t}{2} + \frac{\sin 4t}{8} + C$

48. $\int \frac{1-\cos 6t}{2}\,dt = \int \left(\frac{1}{2} - \frac{1}{2}\cos 6t\right)dt = \frac{1}{2}t - \frac{1}{2}\left(\frac{\sin 6t}{6}\right) + C = \frac{t}{2} - \frac{\sin 6t}{12} + C$

49. $\int 3x^{\sqrt{3}}\,dx = \frac{3x^{(\sqrt{3}+1)}}{\sqrt{3}+1} + C$

50. $\int x^{(\sqrt{2}-1)}\,dx = \frac{x^{\sqrt{2}}}{\sqrt{2}} + C$

51. $\int (1+\tan^2\theta)\,d\theta = \int \sec^2\theta\,d\theta = \tan\theta + C$

52. $\int (2+\tan^2\theta)\,d\theta = \int (1+1+\tan^2\theta)\,d\theta = \int (1+\sec^2\theta)\,d\theta = \theta + \tan\theta + C$

53. $\int \cot^2 x\,dx = \int (\csc^2 x - 1)\,dx = -\cot x - x + C$

54. $\int (1-\cot^2 x)\,dx = \int (1-(\csc^2 x - 1))\,dx = \int (2-\csc^2 x)\,dx = 2x + \cot x + C$

55. $\int \cos\theta(\tan\theta + \sec\theta)\,d\theta = \int (\sin\theta + 1)\,d\theta = -\cos\theta + \theta + C$

56. $\int \frac{\csc\theta}{\csc\theta - \sin\theta}\,d\theta = \int \left(\frac{\csc\theta}{\csc\theta-\sin\theta}\right)\left(\frac{\sin\theta}{\sin\theta}\right)\,d\theta = \int \frac{1}{1-\sin^2\theta}\,d\theta = \int \frac{1}{\cos^2\theta}\,d\theta = \int \sec^2\theta\,d\theta = \tan\theta + C$

57. $\frac{d}{dx}\left(\frac{(7x-2)^4}{28} + C\right) = \frac{4(7x-2)^3(7)}{28} = (7x-2)^3$

58. $\frac{d}{dx}\left(-\frac{(3x+5)^{-1}}{3} + C\right) = -\left(-\frac{(3x+5)^{-2}(3)}{3}\right) = (3x+5)^{-2}$

59. $\frac{d}{dx}\left(\frac{1}{5}\tan(5x-1) + C\right) = \frac{1}{5}(\sec^2(5x-1))(5) = \sec^2(5x-1)$

60. $\frac{d}{dx}\left(-3\cot\left(\frac{x-1}{3}\right) + C\right) = -3\left(-\csc^2\left(\frac{x-1}{3}\right)\right)\left(\frac{1}{3}\right) = \csc^2\left(\frac{x-1}{3}\right)$

61. $\frac{d}{dx}\left(\frac{-1}{x+1} + C\right) = (-1)(-1)(x+1)^{-2} = \frac{1}{(x+1)^2}$

62. $\frac{d}{dx}\left(\frac{x}{x+1} + C\right) = \frac{(x+1)(1)-x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$

63. (a) Wrong: $\frac{d}{dx}\left(\frac{x^2}{2}\sin x + C\right) = \frac{2x}{2}\sin x + \frac{x^2}{2}\cos x = x\sin x + \frac{x^2}{2}\cos x \neq x\sin x$

(b) Wrong: $\frac{d}{dx}(-x\cos x + C) = -\cos x + x\sin x \neq x\sin x$

(c) Right: $\frac{d}{dx}(-x\cos x + \sin x + C) = -\cos x + x\sin x + \cos x = x\sin x$

64. (a) Wrong: $\frac{d}{d\theta}\left(\frac{\sec^3\theta}{3} + C\right) = \frac{3\sec^2\theta}{3}(\sec\theta\tan\theta) = \sec^3\theta\tan\theta \neq \tan\theta\sec^2\theta$

(b) Right: $\frac{d}{d\theta}\left(\frac{1}{2}\tan^2\theta + C\right) = \frac{1}{2}(2\tan\theta)\sec^2\theta = \tan\theta\sec^2\theta$

(c) Right: $\frac{d}{d\theta}\left(\frac{1}{2}\sec^2\theta + C\right) = \frac{1}{2}(2\sec\theta)\sec\theta\tan\theta = \tan\theta\sec^2\theta$

65. (a) Wrong: $\frac{d}{dx}\left(\frac{(2x+1)^3}{3} + C\right) = \frac{3(2x+1)^2(2)}{3} = 2(2x+1)^2 \neq (2x+1)^2$

(b) Wrong: $\frac{d}{dx}((2x+1)^3 + C) = 3(2x+1)^2(2) = 6(2x+1)^2 \neq 3(2x+1)^2$

(c) Right: $\frac{d}{dx}((2x+1)^3 + C) = 6(2x+1)^2$

66. (a) Wrong: $\frac{d}{dx}(x^2+x+C)^{1/2} = \frac{1}{2}(x^2+x+C)^{-1/2}(2x+1) = \frac{2x+1}{2\sqrt{x^2+x+C}} \neq \sqrt{2x+1}$

(b) Wrong: $\frac{d}{dx}\left((x^2+x)^{1/2} + C\right) = \frac{1}{2}(x^2+x)^{-1/2}(2x+1) = \frac{2x+1}{2\sqrt{x^2+x}} \neq \sqrt{2x+1}$

(c) Right: $\frac{d}{dx}\left(\frac{1}{3}\left(\sqrt{2x+1}\right)^3 + C\right) = \frac{d}{dx}\left(\frac{1}{3}(2x+1)^{3/2} + C\right) = \frac{3}{6}(2x+1)^{1/2}(2) = \sqrt{2x+1}$

67. Right: $\frac{d}{dx}\left(\left(\frac{x+3}{x-2}\right)^3 + C\right) = 3\left(\frac{x+3}{x-2}\right)^2 \frac{(x-2)\cdot 1-(x+3)\cdot 1}{(x-2)^2} = 3\frac{(x+3)^2}{(x-2)^2}\frac{-5}{(x-2)^2} = \frac{-15(x+3)^2}{(x-2)^4}$

68. Wrong: $\frac{d}{dx}\left(\frac{\sin(x^2)}{x} + C\right) = \frac{x\cdot\cos(x^2)(2x)-\sin(x^2)\cdot 1}{x^2} = \frac{2x^2\cos(x^2)-\sin(x^2)}{x^2} \neq \frac{x\cos(x^2)-\sin(x^2)}{x^2}$

69. Graph (b), because $\frac{dy}{dx} = 2x \Rightarrow y = x^2 + C$. Then $y(1) = 4 \Rightarrow C = 3$.

70. Graph (b), because $\frac{dy}{dx} = -x \Rightarrow y = -\frac{1}{2}x^2 + C$. Then $y(-1) = 1 \Rightarrow C = \frac{3}{2}$.

71. $\frac{dy}{dx} = 2x - 7 \Rightarrow y = x^2 - 7x + C$; at $x = 2$ and $y = 0$ we have $0 = 2^2 - 7(2) + C \Rightarrow C = 10 \Rightarrow y = x^2 - 7x + 10$

72. $\frac{dy}{dx} = 10 - x \Rightarrow y = 10x - \frac{x^2}{2} + C$; at $x = 0$ and $y = -1$ we have $-1 = 10(0) - \frac{0^2}{2} + C \Rightarrow C = -1 \Rightarrow y = 10x - \frac{x^2}{2} - 1$

73. $\frac{dy}{dx} = \frac{1}{x^2} + x = x^{-2} + x \Rightarrow y = -x^{-1} + \frac{x^2}{2} + C$; at $x = 2$ and $y = 1$ we have $1 = -2^{-1} + \frac{2^2}{2} + C \Rightarrow C = -\frac{1}{2}$
$\Rightarrow y = -x^{-1} + \frac{x^2}{2} - \frac{1}{2}$ or $y = -\frac{1}{x} + \frac{x^2}{2} - \frac{1}{2}$

74. $\frac{dy}{dx} = 9x^2 - 4x + 5 \Rightarrow y = 3x^3 - 2x^2 + 5x + C$; at $x = -1$ and $y = 0$ we have $0 = 3(-1)^3 - 2(-1)^2 + 5(-1) + C$
$\Rightarrow C = 10 \Rightarrow y = 3x^3 - 2x^2 + 5x + 10$

75. $\frac{dy}{dx} = 3x^{-2/3} \Rightarrow y = \frac{3x^{1/3}}{\frac{1}{3}} + C = 9 \Rightarrow y = 9x^{1/3} + C$; at $x = -1$ and $y = -5$ we have $-5 = 9(-1)^{1/3} + C \Rightarrow C = 4$
$\Rightarrow y = 9x^{1/3} + 4$

76. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2} \Rightarrow y = x^{1/2} + C$; at $x = 4$ and $y = 0$ we have $0 = 4^{1/2} + C \Rightarrow C = -2 \Rightarrow y = x^{1/2} - 2$

77. $\frac{ds}{dt}=1+\cos t \Rightarrow s=t+\sin t+C$; at $t=0$ and $s=4$ we have $4=0+\sin 0+C \Rightarrow C=4 \Rightarrow s=t+\sin t+4$

78. $\frac{ds}{dt}=\cos t+\sin t \Rightarrow s=\sin t-\cos t+C$; at $t=\pi$ and $s=1$ we have $1=\sin\pi-\cos\pi+C \Rightarrow C=0$
$\Rightarrow s=\sin t-\cos t$

79. $\frac{dr}{d\theta}=-\pi\sin\pi\theta \Rightarrow r=\cos(\pi\theta)+C$; at $r=0$ and $\theta=0$ we have $0=\cos(\pi 0)+C \Rightarrow C=-1 \Rightarrow r=\cos(\pi\theta)-1$

80. $\frac{dr}{d\theta}=\cos\pi\theta \Rightarrow r=\frac{1}{\pi}\sin(\pi\theta)+C$; at $r=1$ and $\theta=0$ we have $1=\frac{1}{\pi}\sin(\pi 0)+C \Rightarrow C=1 \Rightarrow r=\frac{1}{\pi}\sin(\pi\theta)+1$

81. $\frac{dv}{dt}=\frac{1}{2}\sec t\tan t \Rightarrow v=\frac{1}{2}\sec t+C$; at $v=1$ and $t=0$ we have $1=\frac{1}{2}\sec(0)+C \Rightarrow C=\frac{1}{2} \Rightarrow v=\frac{1}{2}\sec t+\frac{1}{2}$

82. $\frac{dv}{dt}=8t+\csc^2 t \Rightarrow v=4t^2-\cot t+C$; at $v=-7$ and $t=\frac{\pi}{2}$ we have $-7=4\left(\frac{\pi}{2}\right)^2-\cot\left(\frac{\pi}{2}\right)+C \Rightarrow C=-7-\pi^2$
$\Rightarrow v=4t^2-\cot t-7-\pi^2$

83. $\frac{d^2y}{dx^2}=2-6x \Rightarrow \frac{dy}{dx}=2x-3x^2+C_1$; at $\frac{dy}{dx}=4$ and $x=0$ we have $4=2(0)-3(0)^2+C_1 \Rightarrow C_1=4$
$\Rightarrow \frac{dy}{dx}=2x-3x^2+4 \Rightarrow y=x^2-x^3+4x+C_2$; at $y=1$ and $x=0$ we have $1=0^2-0^3+4(0)+C_2 \Rightarrow C_2=1$
$\Rightarrow y=x^2-x^3+4x+1$

84. $\frac{d^2y}{dx^2}=0 \Rightarrow \frac{dy}{dx}=C_1$; at $\frac{dy}{dx}=2$ and $x=0$ we have $C_1=2 \Rightarrow \frac{dy}{dx}=2 \Rightarrow y=2x+C_2$; at $y=0$ and $x=0$ we have
$0=2(0)+C_2 \Rightarrow C_2=0 \Rightarrow y=2x$

85. $\frac{d^2r}{dt^2}=\frac{2}{t^3}=2t^{-3} \Rightarrow \frac{dr}{dt}=-t^{-2}+C_1$; at $\frac{dr}{dt}=1$ and $t=1$ we have $1=-(1)^{-2}+C_1 \Rightarrow C_1=2 \Rightarrow \frac{dr}{dt}=-t^{-2}+2$
$\Rightarrow r=t^{-1}+2t+C_2$; at $r=1$ and $t=1$ we have $1=1^{-1}+2(1)+C_2 \Rightarrow C_2=-2 \Rightarrow r=t^{-1}+2t-2$ or $r=\frac{1}{t}+2t-2$

86. $\frac{d^2s}{dt^2}=\frac{3t}{8} \Rightarrow \frac{ds}{dt}=\frac{3t^2}{16}+C_1$; at $\frac{ds}{dt}=3$ and $t=4$ we have $3=\frac{3(4)^2}{16}+C_1 \Rightarrow C_1=0 \Rightarrow \frac{ds}{dt}=\frac{3t^2}{16} \Rightarrow s=\frac{t^3}{16}+C_2$; at
$s=4$ and $t=4$ we have $4=\frac{4^3}{16}+C_2 \Rightarrow C_2=0 \Rightarrow s=\frac{t^3}{16}$

87. $\frac{d^3y}{dx^3}=6 \Rightarrow \frac{d^2y}{dx^2}=6x+C_1$; at $\frac{d^2y}{dx^2}=-8$ and $x=0$ we have $-8=6(0)+C_1 \Rightarrow C_1=-8 \Rightarrow \frac{d^2y}{dx^2}=6x-8$
$\Rightarrow \frac{dy}{dx}=3x^2-8x+C_2$; at $\frac{dy}{dx}=0$ and $x=0$ we have $0=3(0)^2-8(0)+C_2 \Rightarrow C_2=0 \Rightarrow \frac{dy}{dx}=3x^2-8x$
$\Rightarrow y=x^3-4x^2+C_3$; at $y=5$ and $x=0$ we have $5=0^3-4(0)^2+C_3 \Rightarrow C_3=5 \Rightarrow y=x^3-4x^2+5$

88. $\frac{d^3\theta}{dt^3}=0 \Rightarrow \frac{d^2\theta}{dt^2}=C_1$; at $\frac{d^2\theta}{dt^2}=-2$ and $t=0$ we have $\frac{d^2\theta}{dt^2}=-2 \Rightarrow \frac{d\theta}{dt}=-2t+C_2$; at $\frac{d\theta}{dt}=-\frac{1}{2}$ and $t=0$ we have
$-\frac{1}{2}=-2(0)+C_2 \Rightarrow C_2=-\frac{1}{2} \Rightarrow \frac{d\theta}{dt}=-2t-\frac{1}{2} \Rightarrow \theta=-t^2-\frac{1}{2}t+C_3$; at $\theta=\sqrt{2}$ and $t=0$ we have
$\sqrt{2}=-0^2-\frac{1}{2}(0)+C_3 \Rightarrow C_3=\sqrt{2} \Rightarrow \theta=-t^2-\frac{1}{2}t+\sqrt{2}$

89. $y^{(4)}=-\sin t+\cos t \Rightarrow y'''=\cos t+\sin t+C_1$; at $y'''=7$ and $t=0$ we have $7=\cos(0)+\sin(0)+C_1 \Rightarrow C_1=6$
$\Rightarrow y'''=\cos t+\sin t+6 \Rightarrow y''=\sin t-\cos t+6t+C_2$; at $y''=-1$ and $t=0$ we have
$-1=\sin(0)-\cos(0)+6(0)+C_2 \Rightarrow C_2=0 \Rightarrow y''=\sin t-\cos t+6t \Rightarrow y'=-\cos t-\sin t+3t^2+C_3$; at

$y' = -1$ and $t = 0$ we have $-1 = -\cos(0) - \sin(0) + 3(0)^2 + C_3 \Rightarrow C_3 = 0 \Rightarrow y' = -\cos t - \sin t + 3t^2$
$\Rightarrow y = -\sin t + \cos t + t^3 + C_4$; at $y = 0$ and $t = 0$ we have $0 = -\sin(0) + \cos(0) + 0^3 + C_4$
$\Rightarrow C_4 = -1 \Rightarrow y = -\sin t + \cos t + t^3 - 1$

90. $y^{(4)} = -\cos x + 8\sin(2x) \Rightarrow y''' = -\sin x - 4\cos(2x) + C_1$; at $y''' = 0$ and $x = 0$ we have $0 = -\sin(0) - 4\cos(2(0)) + C_1 \Rightarrow C_1 = 4 \Rightarrow y''' = -\sin x - 4\cos(2x) + 4 \Rightarrow y'' = \cos x - 2\sin(2x) + 4x + C_2$; at $y'' = 1$ and $x = 0$ we have $1 = \cos(0) - 2\sin(2(0)) + 4(0) + C_2 \Rightarrow C_2 = 0 \Rightarrow y'' = \cos x - 2\sin(2x) + 4x$
$\Rightarrow y' = \sin x + \cos(2x) + 2x^2 + C_3$; at $y' = 1$ and $x = 0$ we have $1 = \sin(0) + \cos(2(0)) + 2(0)^2 + C_3 \Rightarrow C_3 = 0$
$\Rightarrow y' = \sin x + \cos(2x) + 2x^2 \Rightarrow y = -\cos x + \frac{1}{2}\sin(2x) + \frac{2}{3}x^3 + C_4$; at $y = 3$ and $x = 0$ we have
$3 = -\cos(0) + \frac{1}{2}\sin(2(0)) + \frac{2}{3}(0)^3 + C_4 \Rightarrow C_4 = 4 \Rightarrow y = -\cos x + \frac{1}{2}\sin(2x) + \frac{2}{3}x^3 + 4$

91. $m = y' = 3\sqrt{x} = 3x^{1/2} \Rightarrow y = 2x^{3/2} + C$; at $(9, 4)$ we have $4 = 2(9)^{3/2} + C \Rightarrow C = -50 \Rightarrow y = 2x^{3/2} - 50$

92. (a) $\frac{d^2y}{dx^2} = 6x \Rightarrow \frac{dy}{dx} = 3x^2 + C_1$; at $y' = 0$ and $x = 0$ we have $0 = 3(0)^2 + C_1 \Rightarrow C_1 = 0 \Rightarrow \frac{dy}{dx} = 3x^2$
$\Rightarrow y = x^3 + C_2$; at $y = 1$ and $x = 0$ we have $C_2 = 1 \Rightarrow y = x^3 + 1$

(b) One, because any other possible function would differ from $x^3 + 1$ by a constant that must be zero because of the initial conditions

93. $\frac{dy}{dx} = 1 - \frac{4}{3}x^{1/3} \Rightarrow y = \int\left(1 - \frac{4}{3}x^{1/3}\right)dx = x - x^{4/3} + C$; at $(1, 0.5)$ on the curve we have
$0.5 = 1 - 1^{4/3} + C \Rightarrow C = 0.5 \Rightarrow x - x^{4/3} + \frac{1}{2}$

94. $\frac{dy}{dx} = x - 1 \Rightarrow y = \int(x-1)dx = \frac{x^2}{2} - x + C$; at $(-1, 1)$ on the curve we have
$1 = \frac{(-1)^2}{2} - (-1) + C \Rightarrow C = -\frac{1}{2} \Rightarrow y = \frac{x^2}{2} - x - \frac{1}{2}$

95. $\frac{dy}{dx} = \sin x - \cos x \Rightarrow y = \int(\sin x - \cos x)dx = -\cos x - \sin x + C$; at $(-\pi, -1)$ on the curve we have
$-1 = -\cos(-\pi) - \sin(-\pi) + C \Rightarrow C = -2 \Rightarrow y = -\cos x - \sin x - 2$

96. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \pi\sin\pi x = \frac{1}{2}x^{-1/2} + \pi\sin\pi x \Rightarrow y = \int\left(\frac{1}{2}x^{-1/2} + \sin\pi x\right)dx = x^{1/2} - \cos\pi x + C$; at $(1, 2)$ on the curve we have $2 = 1^{1/2} - \cos\pi(1) + C \Rightarrow C = 0 \Rightarrow y = \sqrt{x} - \cos\pi x$

97. (a) $\frac{ds}{dt} = 9.8t - 3 \Rightarrow s = 4.9t^2 - 3t + C$; (i) at $s = 5$ and $t = 0$ we have $C = 5 \Rightarrow s = 4.9t^2 - 3t + 5$; displacement $= s(3) - s(1) = ((4.9)(9) - 9 + 5) - (4.9 - 3 + 5) = 33.2$ units; (ii) at $s = -2$ and $t = 0$ we have $C = -2 \Rightarrow s = 4.9t^2 - 3t - 2$; displacement $= s(3) - s(1) = ((4.9)(9) - 9 - 2) - (4.9 - 3 - 2) = 33.2$ units; (iii) at $s = s_0$ and $t = 0$ we have $C = s_0 \Rightarrow s = 4.9t^2 - 3t + s_0$;
displacement $= s(3) - s(1) = ((4.9)(9) - 9 + s_0) - (4.9 - 3 + s_0) = 33.2$ units

(b) True. Given an antiderivative $f(t)$ of the velocity function, we know that the body's position function is $s = f(t) + C$ for some constant C. Therefore, the displacement from $t = a$ to $t = b$ is $(f(b) + C) - (f(a) + C) = f(b) - f(a)$. Thus we can find the displacement from any antiderivative f as the numerical difference $f(b) - f(a)$ without knowing the exact values of C and s.

98. $a(t)=v'(t)=20 \Rightarrow v(t)=20t+C$; at (0, 0) we have $C=0 \Rightarrow v(t)=20t$. When $t=60$, then $v(60)=20(60)=1200$ m/s.

99. Step 1: $\frac{d^2s}{dt^2}=-k \Rightarrow \frac{ds}{dt}=-kt+C_1$; at $\frac{ds}{dt}=30$ and $t=0$ we have

$C_1=30 \Rightarrow \frac{ds}{dt}=-kt+30 \Rightarrow s=-k\left(\frac{t^2}{2}\right)+30t+C_2$; at $s=0$ and $t=0$ we have $C_2=0 \Rightarrow s=-\frac{kt^2}{2}+30t$

Step 2: $\frac{ds}{dt}=0 \Rightarrow 0=-kt+30 \Rightarrow t=\frac{30}{k}$

Step 3: $75=\frac{-k\left(\frac{30}{k}\right)^2}{2}+30\left(\frac{30}{k}\right) \Rightarrow 75=-\frac{(30)^2}{2k}+\frac{(30)^2}{k} \Rightarrow 75=\frac{(30)^2}{2k} \Rightarrow k=6$

100. $\frac{d^2s}{dt^2}=-k \Rightarrow \frac{ds}{dt}=\int -k\,dt=-kt+C$; at $\frac{ds}{dt}=13.3$ when $t=0$ we have $13.3=-k(0)+C \Rightarrow C=13.3 \Rightarrow \frac{ds}{dt}=-kt+13.3 \Rightarrow s=-\frac{kt^2}{2}+13.3t+C_1$; at $s=0$ when $t=0$ we have $0=-\frac{k(0)^2}{2}+13.3(0)+C_1 \Rightarrow C_1=0 \Rightarrow s=-\frac{kt^2}{2}+13.3t$. Then $\frac{ds}{dt}=0 \Rightarrow -kt+13.3=0 \Rightarrow t=\frac{13.3}{k}$ and $s\left(\frac{13.3}{k}\right)=-\frac{k\left(\frac{13.3}{k}\right)^2}{2}+13.3\left(\frac{13.3}{k}\right)=13.7 \Rightarrow -\frac{88.445}{k}+\frac{176.89}{k}=13.7 \Rightarrow k=\frac{88.445}{13.7}\approx 6.46\ \frac{\text{m}}{\text{s}^2}$.

101. (a) $v=\int a\,dt=\int(15t^{1/2}-3t^{-1/2})dt=10t^{3/2}-6t^{1/2}+C$;

$\frac{ds}{dt}(1)=4 \Rightarrow 4=10(1)^{3/2}-6(1)^{1/2}+C \Rightarrow C=0 \Rightarrow v=10t^{3/2}-6t^{1/2}$

(b) $s=\int v\,dt=\int(10t^{3/2}-6t^{1/2})dt=4t^{5/2}-4t^{3/2}+C$;

$s(1)=0 \Rightarrow 0=4(1)^{5/2}-4(1)^{3/2}+C \Rightarrow C=0 \Rightarrow s=4t^{5/2}-4t^{3/2}$

102 . $\frac{d^2s}{dt^2}=-1.6 \Rightarrow \frac{ds}{dt}=-1.6t+C_1$; at $\frac{ds}{dt}=0$ and $t=0$ we have $C_1=0 \Rightarrow \frac{ds}{dt}=-1.6t \Rightarrow s=-0.8t^2+C_2$; at $s=1.2$ and $t=0$ we have $C_2=1.2 \Rightarrow s=-0.8t^2+1.2$. Then $s=0 \Rightarrow 0=-0.8t^2+1.2 \Rightarrow t=\sqrt{\frac{1.2}{0.8}}\approx 1.22$ s, since $t>0$

103. $\frac{d^2s}{dt^2}=a \Rightarrow \frac{ds}{dt}=\int a\,dt=at+C$; $\frac{ds}{dt}=v_0$ when $t=0 \Rightarrow C=v_0 \Rightarrow \frac{ds}{dt}=at+v_0 \Rightarrow s=\frac{at^2}{2}+v_0t+C_1$; $s=s_0$ when $t=0 \Rightarrow s_0=\frac{a(0)^2}{2}+v_0(0)+C_1 \Rightarrow C_1=s_0 \Rightarrow s=\frac{at^2}{2}+v_0t+s_0$

104. The appropriate initial value problem is: Differential Equation: $\frac{d^2s}{dt^2}=-g$ with Initial Conditions: $\frac{ds}{dt}=v_0$ and $s=s_0$ when $t=0$. Thus $\frac{ds}{dt}=\int -g\,dt=-gt+C_1$; $\frac{ds}{dt}(0)=v_0 \Rightarrow v_0=(-g)(0)+C_1 \Rightarrow C_1=v_0 \Rightarrow \frac{ds}{dt}=-gt+v_0$.
Thus $s=\int(-gt+v_0)dt=-\frac{1}{2}gt^2+v_0t+C_2$; $s(0)=s_0=-\frac{1}{2}(g)(0)^2+v_0(0)+C_2 \Rightarrow C_2=s_0$
Thus $s=-\frac{1}{2}gt^2+v_0t+s_0$.

105 (a) $\int f(x)\,dx=1-\sqrt{x}+C_1=-\sqrt{x}+C$ (b) $\int g(x)\,dx=x+2+C_1=x+C$

(c) $\int -f(x)\,dx=-\left(1-\sqrt{x}\right)+C_1=\sqrt{x}+C$ (d) $\int -g(x)\,dx=-(x+2)+C_1=-x+C$

(e) $\int[f(x)+g(x)]dx=\left(1-\sqrt{x}\right)+(x+2)+C_1=x-\sqrt{x}+C$

(f) $\int[f(x)-g(x)]dx=\left(1-\sqrt{x}\right)-(x+2)+C_1=-x-\sqrt{x}+C$

106. Yes. If $F(x)$ and $G(x)$ both solve the initial value problem on an interval I then they both have the same first derivative. Therefore, by Corollary 2 of the Mean Value Theorem there is a constant C such that $F(x) = G(x) + C$ for all x. In particular, $F(x_0) = G(x_0) + C$, so $C = F(x_0) - G(x_0) = 0$. Hence $F(x) = G(x)$ for all x.

107−110. Example CAS commands:

Maple:

```
with(student):
f := x -> cos(x)^2 + sin(x);
ic := [x=Pi,y=1];
F := unapply( int( f(x), x ) + C, x );
eq := eval( y=F(x), ic );
solnC := solve( eq, {C} );
Y := unapply( eval( F(x), solnC ), x );
DEplot( diff(y(x),x) = f(x), y(x), x=0..2*Pi, [[y(Pi)=1]],
    color=black, linecolor=black, stepsize=0.05, title="Section 4.7 #107" );
```

Mathematica: (functions and values may vary)

The following commands use the definite integral and the Fundamental Theorem of calculus to construct the solution of the initial value problems for Exercises 107-110.

```
Clear[x, y, yprime]
yprime[x_] = Cos[x]^2 + Sin[x];
initxvalue = π; inityvalue = 1;
y[x_] = Integrate[yprime[t], {t, initxvalue, x}] + inityvalue
```

If the solution satisfies the differential equation and initial condition, the following yield True

```
yprime[x]==D[y[x], x]//Simplify
y[initxvalue]==inityvalue
```

Since exercise 110 is a second order differential equation, two integrations will be required.

```
Clear[x, y, yprime]
y2prime[x_] = 3 Exp[x/2] + 1;
initxval = 0; inityval = 4; inityprimeval = −1;
yprime[x_] = Integrate[y2prime[t], {t, initxval, x}] + inityprimeval
  y[x_] = Integrate[yprime[t], {t, initxval, x}] + inityval
```

Verify that y[x] solves the differential equation and initial condition and plot the solution (red) and its derivative (blue).

```
y2prime[x]==D[y[x], {x, 2}]//Simplify
y[initxval]==inityval
yprime[initxval]==inityprimeval
Plot[{y[x], yprime[x]}, {x, initxval − 3, initxval + 3}, PlotStyle → {RGBColor[1,0,0], RGBColor[0,0,1]}]
```

CHAPTER 4 PRACTICE EXERCISES

1. No, since $f(x) = x^3 + 2x + \tan x \Rightarrow f'(x) = 3x^2 + 2 + \sec^2 x > 0 \Rightarrow f(x)$ is always increasing on its domain

2. No, since $g(x) = \csc x + 2\cot x \Rightarrow g'(x) = -\csc x \cot x - 2\csc^2 x = -\frac{\cos x}{\sin^2 x} - \frac{2}{\sin^2 x} = -\frac{1}{\sin^2 x}(\cos x + 2) < 0$ $\Rightarrow g(x)$ is always decreasing on its domain

3. No absolute minimum because $\lim_{x\to\infty} (7+x)(11-3x)^{1/3} = -\infty$. Next $f'(x) = (11-3x)^{1/3} - (7+x)(11-3x)^{-2/3}$ $= \frac{(11-3x)-(7+x)}{(11-3x)^{2/3}} = \frac{4(1-x)}{(11-3x)^{2/3}} \Rightarrow x = 1$ and $x = \frac{11}{3}$ are critical points. Since $f' > 0$ if $x < 1$ and $f' < 0$ if $x > 1$, $f(1) = 16$ is the absolute maximum.

4. $f(x) = \frac{ax+b}{x^2-1} \Rightarrow f'(x) = \frac{a(x^2-1)-2x(ax+b)}{(x^2-1)^2} = \frac{-(ax^2+2bx+a)}{(x^2-1)^2}$; $f'(3) = 0 \Rightarrow -\frac{1}{64}(9a + 6b + a) = 0 \Rightarrow 5a + 3b = 0$. We require also that $f(3) = 1$. Thus $1 = \frac{3a+b}{8} \Rightarrow 3a + b = 8$. Solving both equations yields $a = 6$ and $b = -10$. Now, $f'(x) = \frac{-2(3x-1)(x-3)}{(x^2-1)^2}$ so that $f' = \underset{-1}{---\,|}\underset{1/3}{---\,|}\underset{1}{+++\,|}\underset{3}{+++\,|}---$. Thus f' changes sign at $x = 3$ from positive to negative so there is a local maximum at $x = 3$ which has a value $f(3) = 1$.

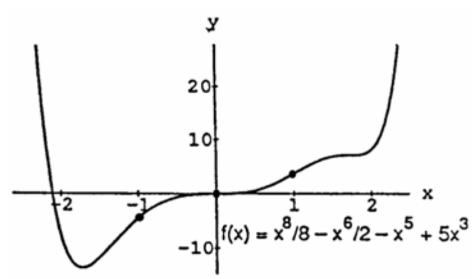
5. Yes, because at each point of $[0, 1)$ except $x = 0$, the function's value is a local minimum value as well as a local maximum value. At $x = 0$ the function's value, 0, is not a local minimum value because each open interval around $x = 0$ on the x-axis contains points to the left of 0 where f equals -1.

6. (a) The first derivative of the function $f(x) = x^3$ is zero at $x = 0$ even though f has no local extreme value at $x = 0$.
   (b) Theorem 2 says only that if f is differentiable and f has a local extreme at $x = c$ then $f'(c) = 0$. It does not assert the (false) reverse implication $f'(c) = 0 \Rightarrow f$ has a local extreme at $x = c$.

7. No, because the interval $0 < x < 1$ fails to be closed. The Extreme Value Theorem says that if the function is continuous throughout a finite closed interval $a \le x \le b$ then the existence of absolute extrema is guaranteed on that interval.

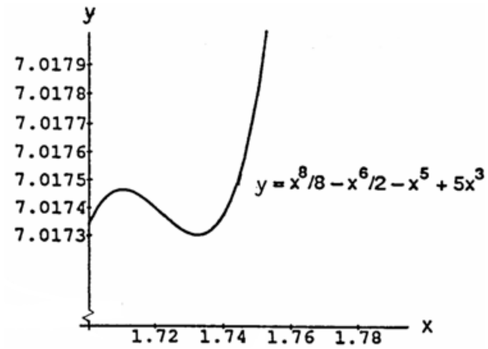
8. The absolute maximum is $|-1| = 1$ and the absolute minimum is $|0| = 0$. This is not inconsistent with the Extreme Value Theorem for continuous functions, which says a continuous function on a closed interval attains its extreme values on that interval. The theorem says nothing about the behavior of a continuous function on an interval which is half open and half closed, such as $[-1, 1)$, so there is nothing to contradict.

9. (a) There appear to be local minima at $x = -1.75$ and 1.8. Points of inflection are indicated at approximately $x = 0$ and $x = \pm 1$.

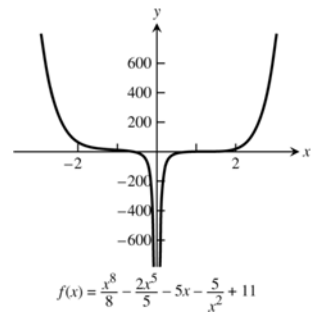


   (b) $f'(x) = x^7 - 3x^5 - 5x^4 + 15x^2 = x^2(x^2-3)(x^3-5)$. The pattern $y' = \underset{-\sqrt{3}}{---\,|}\underset{0}{+++\,|}\underset{\sqrt[3]{5}}{+++\,|}\underset{\sqrt{3}}{---\,|}+++$ indicates a local maximum at $x = \sqrt[3]{5}$ and local minima at $x = \pm\sqrt{3}$.

(c)

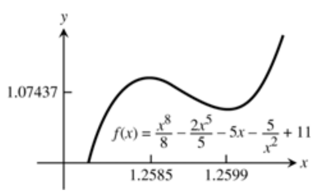


10. (a) The graph does not indicate any local extremum. Points of inflection are indicated at approximately $x = -\frac{3}{4}$ and $x = 1$.



(b) $f'(x) = x^7 - 2x^4 - 5 + \frac{10}{x^3} = x^{-3}(x^3 - 2)(x^7 - 5)$. The pattern $f' = ---\underset{0}{)}(+++\underset{\sqrt[7]{5}}{|}---\underset{\sqrt[3]{2}}{|}+++$ indicates a local maximum at $x = \sqrt[7]{5}$ and a local minimum at $x = \sqrt[3]{2}$.

(c)



11. (a) $g(t) = \sin^2 t - 3t \Rightarrow g'(t) = 2\sin t \cos t - 3 = \sin(2t) - 3 \Rightarrow g' < 0 \Rightarrow g(t)$ is always falling and hence must decrease on every interval in its domain.

(b) One, since $\sin^2 t - 3t - 5 = 0$ and $\sin^2 t - 3t = 5$ have the same solutions: $f(t) = \sin^2 t - 3t - 5$ has the same derivative as $g(t)$ in part (a) and is always decreasing with $f(-3) > 0$ and $f(0) < 0$. The Intermediate Value Theorem guarantees the continuous function f has a root in $[-3, 0]$.

12. (a) $y = \tan\theta \Rightarrow \frac{dy}{d\theta} = \sec^2\theta > 0 \Rightarrow y = \tan\theta$ is always rising on its domain $\Rightarrow y = \tan\theta$ increases on every interval in its domain

(b) The interval $\left[\frac{\pi}{4}, \pi\right]$ is not in the tangent's domain because $\tan\theta$ is undefined at $\theta = \frac{\pi}{2}$. Thus the tangent need not increase on this interval.

13. (a) $f(x) = x^4 + 2x^2 - 2 \Rightarrow f'(x) = 4x^3 + 4x$. Since $f(0) = -2 < 0$, $f(1) = 1 > 0$ and $f'(x) \geq 0$ for $0 \leq x \leq 1$, we may conclude from the Intermediate Value Theorem that $f(x)$ has exactly one solution when $0 \leq x \leq 1$.

(b) $x^2 = \frac{-2 \pm \sqrt{4+8}}{2} > 0 \Rightarrow x^2 = \sqrt{3} - 1$ and $x \geq 0 \Rightarrow x \approx \sqrt{.7320508076} \approx .8555996772$

14. (a) $y = \frac{x}{x+1} \Rightarrow y' = \frac{1}{(x+1)^2} > 0$, for all x in the domain of $\frac{x}{x+1} \Rightarrow y = \frac{x}{x+1}$ is increasing in every interval in its domain.

(b) $y = x^3 + 2x \Rightarrow y' = 3x^2 + 2 > 0$ for all $x \Rightarrow$ the graph of $y = x^3 + 2x$ is always increasing and can never have a local maximum or minimum

15. Let $V(t)$ represent the volume of the water in the reservoir at time t, in minutes, let $V(0) = a_0$ be the initial amount and $V(1440) = a_0 + 1{,}000{,}000{,}000$ liters be the amount of water contained in the reservoir after the rain, where 24 h $= 1440$ min. Assume that $V(t)$ is continuous on $[0, 1440]$ and differentiable on $(0, 1440)$. The Mean Value Theorem says that for some t_0 in $(0, 1440)$ we have $V'(t_0) = \frac{V(1400) - V(0)}{1440 - 0} = \frac{a_0 + 1{,}000{,}000{,}000 - a_0}{1440}$ $= \frac{1{,}000{,}000{,}000 \text{ L}}{1440 \text{ min}} = 694{,}444$ L/min. Therefore at t_0 the reservoir's volume was increasing at a rate in excess of 500,000 L/min.

16. Yes, all differentiable functions $g(x)$ having 3 as a derivative differ by only a constant. Consequently, the difference $3x - g(x)$ is a constant K because $g'(x) = 3 = \frac{d}{dx}(3x)$. Thus $g(x) = 3x + K$, the same form as $F(x)$.

17. No, $\frac{x}{x+1} = 1 + \frac{-1}{x+1} \Rightarrow \frac{x}{x+1}$ differs from $\frac{-1}{x+1}$ by the constant 1. Both functions have the same derivative $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2} = \frac{d}{dx}\left(\frac{-1}{x+1}\right)$.

18. $f'(x) = g'(x) = \frac{2x}{(x^2+1)^2} \Rightarrow f(x) - g(x) = C$ for some constant $C \Rightarrow$ the graphs differ by a vertical shift.

19. The global minimum value of $\frac{1}{2}$ occurs at $x = 2$.

20. (a) The function is increasing on the intervals $[-3, -2]$ and $[1, 2]$.

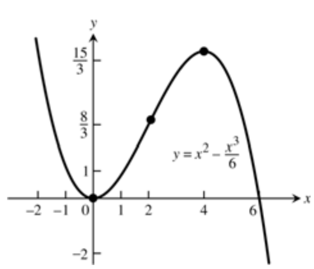
(b) The function is decreasing on the intervals $[-2, 0)$ and $(0, 1]$.

(c) The local maximum values occur only at $x = -2$, and at $x = 2$; local minimum values occur at $x = -3$ and at $x = 1$ provided f is continuous at $x = 0$.

21. (a) $t = 0, 6, 12$ (b) $t = 3, 9$ (c) $6 < t < 12$ (d) $0 < t < 6, 12 < t < 14$

22. (a) $t = 4$ (b) at no time (c) $0 < t < 4$ (d) $4 < t < 8$

23.



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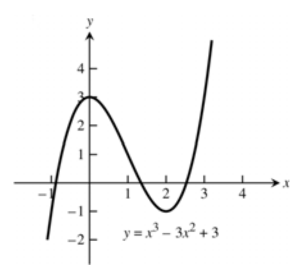

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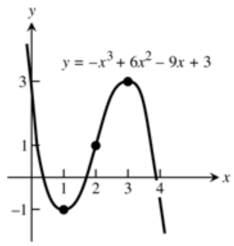

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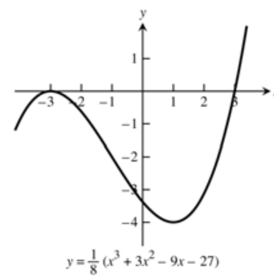

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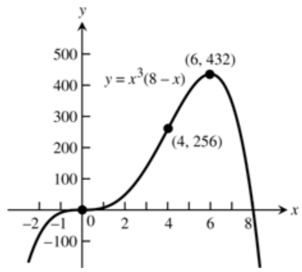

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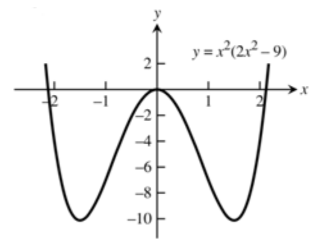

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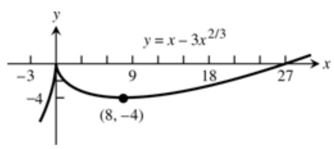

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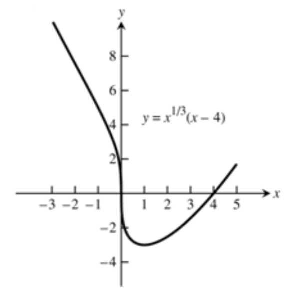

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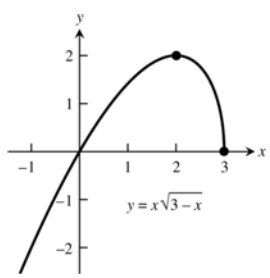
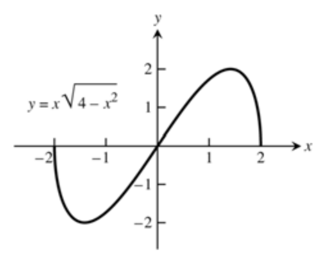
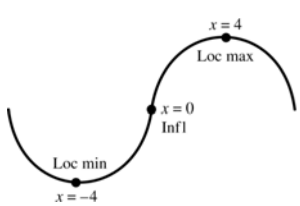


32.



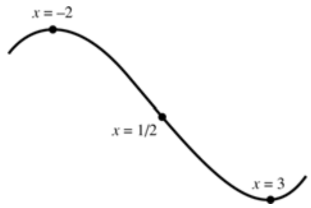
33. (a) $y' = 16 - x^2 \Rightarrow y' = ---\underset{-4}{|}+++\underset{4}{|}--- \Rightarrow$ the curve is rising on $(-4, 4)$, falling on $(-\infty, -4)$ and $(4, \infty)$ $\Rightarrow$ a local maximum at $x = 4$ and a local minimum at $x = -4$; $y'' = -2x \Rightarrow y'' = +++\underset{0}{|}--- \Rightarrow$ the curve is concave up on $(-\infty, 0)$, concave down on $(0, \infty) \Rightarrow$ a point of inflection at $x = 0$

(b)



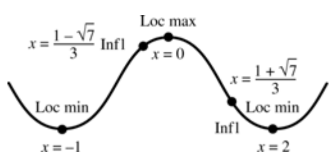
34. (a) $y' = x^2 - x - 6 = (x-3)(x+2) \Rightarrow y' = +++\ \underset{-2}{|}\ ---\ \underset{3}{|}\ +++ \Rightarrow$ the curve is rising on $(-\infty, -2)$ and $(3, \infty)$, falling on $(-2, 3) \Rightarrow$ local maximum at $x = -2$ and a local minimum at $x = 3$; $y'' = 2x - 1 \Rightarrow y'' = ---\ \underset{1/2}{|}\ +++$ $\Rightarrow$ concave up on $\left(\frac{1}{2}, \infty\right)$, concave down on $\left(-\infty, \frac{1}{2}\right) \Rightarrow$ a point of inflection at $x = \frac{1}{2}$

(b)



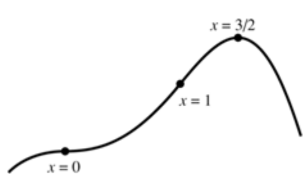
35. (a) $y' = 6x(x+1)(x-2) = 6x^3 - 6x^2 - 12x \Rightarrow y' = ---\ \underset{-1}{|}\ +++\ \underset{0}{|}\ ---\ \underset{2}{|}\ +++ \Rightarrow$ the graph is rising on $(-1, 0)$ and $(2, \infty)$, falling on $(-\infty, -1)$ and $(0, 2) \Rightarrow$ a local maximum at $x = 0$, local minima at $x = -1$ and $x = 2$; $y'' = 18x^2 - 12x - 12 = 6(3x^2 - 2x - 2) = 6\left(x - \frac{1-\sqrt{7}}{3}\right)\left(x - \frac{1+\sqrt{7}}{3}\right) \Rightarrow y'' = +++\ \underset{\frac{1-\sqrt{7}}{3}}{|}\ ---\ \underset{\frac{1+\sqrt{7}}{3}}{|}\ +++$ $\Rightarrow$ the curve is concave up on $\left(-\infty, \frac{1-\sqrt{7}}{3}\right)$ and $\left(\frac{1+\sqrt{7}}{3}, \infty\right)$, concave down on $\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right) \Rightarrow$ points of inflection at $x = \frac{1 \pm \sqrt{7}}{3}$

(b)



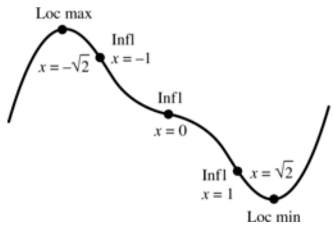
36. (a) $y' = x^2(6-4x) = 6x^2 - 4x^3 \Rightarrow y' = +++\ \underset{0}{|}\ +++\ \underset{3/2}{|}\ --- \Rightarrow$ the curve is rising on $\left(-\infty, \frac{3}{2}\right)$, falling on $\left(\frac{3}{2}, \infty\right) \Rightarrow$ a local maximum at $x = \frac{3}{2}$; $y'' = 12x - 12x^2 = 12x(1-x) \Rightarrow y'' = ---\ \underset{0}{|}\ +++\ \underset{1}{|}\ --- \Rightarrow$ concave up on $(0, 1)$, concave down on $(-\infty, 0)$ and $(1, \infty) \Rightarrow$ points of inflection at $x = 0$ and $x = 1$

(b)



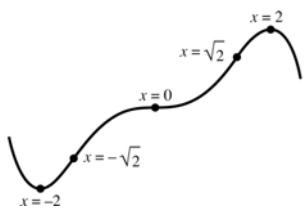
37. (a) $y' = x^4 - 2x^2 = x^2(x^2 - 2) \Rightarrow y' = +++\ \underset{-\sqrt{2}}{|}\ ---\ \underset{0}{|}\ ---\ \underset{\sqrt{2}}{|}\ +++ \Rightarrow$ the curve is rising on $\left(-\infty, -\sqrt{2}\right)$ and $\left(\sqrt{2}, \infty\right)$, falling on $\left(-\sqrt{2}, \sqrt{2}\right) \Rightarrow$ a local maximum at $x = -\sqrt{2}$ and a local minimum at $x = \sqrt{2}$; $y'' = 4x^3 - 4x = 4x(x-1)(x+1) \Rightarrow y'' = ---\ \underset{-1}{|}\ +++\ \underset{0}{|}\ ---\ \underset{1}{|}\ +++ \Rightarrow$ concave up on $(-1, 0)$ and $(1, \infty)$, concave down on $(-\infty, -1)$ and $(0, 1) \Rightarrow$ points of inflection at $x = 0$ and $x = \pm 1$

(b)

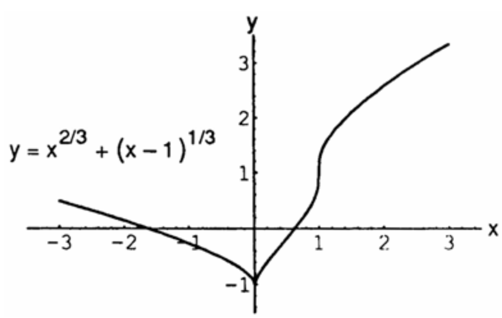


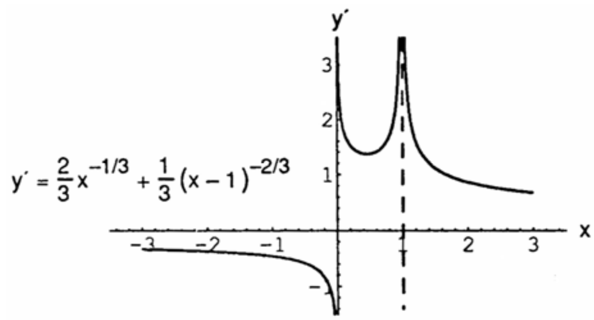
38. (a) $y' = 4x^2 - x^4 = x^2(4-x^2) \Rightarrow y' = ---\underset{-2}{|}+++\underset{0}{|}+++\underset{2}{|}--- \Rightarrow$ the curve is rising on $(-2, 0)$ and $(0, 2)$, falling on $(-\infty, -2)$ and $(2, \infty) \Rightarrow$ a local maximum at $x = 2$, a local minimum at $x = -2$; $y'' = 8x - 4x^3$ $= 4x\ (2-x^2) \Rightarrow y'' = +++\underset{-\sqrt{2}}{|}---\underset{0}{|}+++\underset{\sqrt{2}}{|}--- \Rightarrow$ concave up on $\left(-\infty, -\sqrt{2}\right)$ and $\left(0, \sqrt{2}\right)$, concave down on $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, \infty\right) \Rightarrow$ points of inflection at $x = 0$ and $x = \pm\sqrt{2}$

(b)

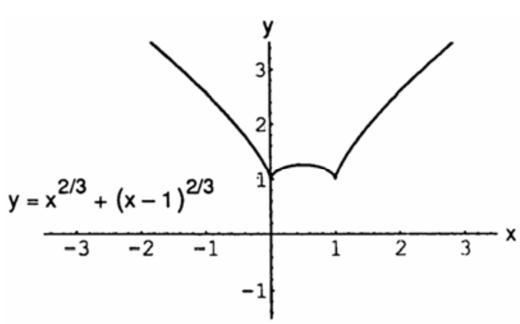


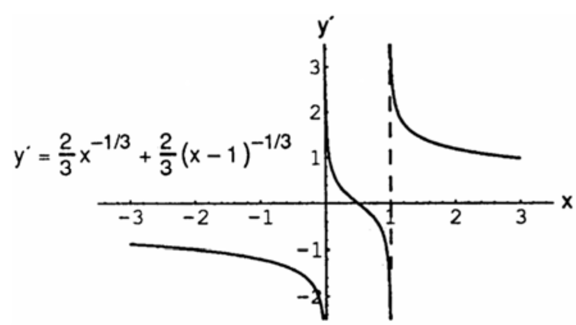
39. The values of the first derivative indicate that the curve is rising on $(0, \infty)$ and falling on $(-\infty, 0)$. The slope of the curve approaches $-\infty$ as $x \to 0^-$, and approaches ∞ as $x \to 0^+$ and $x \to 1$. The curve should therefore have a cusp and local minimum at $x = 0$, and a vertical tangent at $x = 1$.



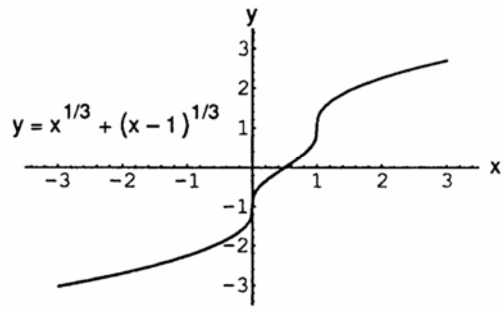


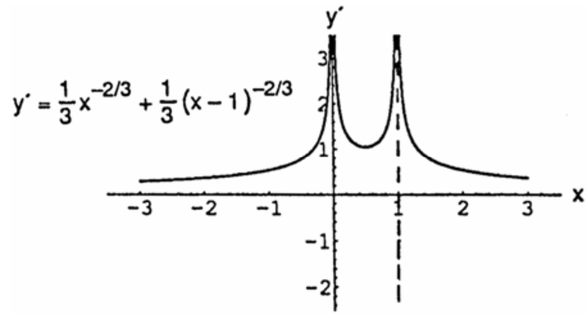
40. The values of the first derivative indicate that the curve is rising on $\left(0, \frac{1}{2}\right)$ and $(1, \infty)$, and falling on $(-\infty, 0)$ and $\left(\frac{1}{2}, 1\right)$. The derivative changes from positive to negative at $x = \frac{1}{2}$, indicating a local maximum there. The slope of the curve approaches $-\infty$ as $x \to 0^-$ and $x \to 1^-$, and approaches ∞ as $x \to 0^+$ and as $x \to 1^+$, indicating cusps and local minima at both $x = 0$ and $x = 1$.



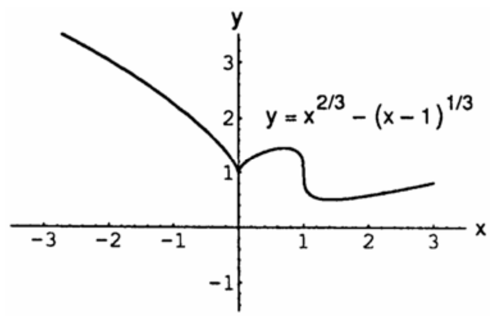


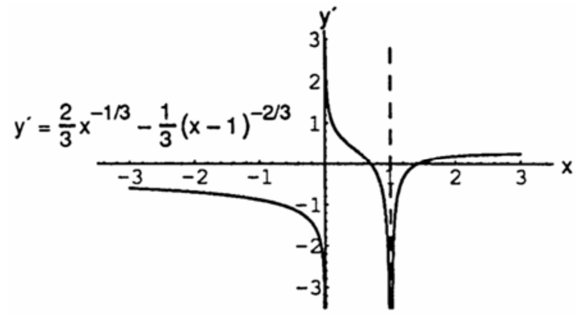
41. The values of the first derivative indicate that the curve is always rising. The slope of the curve approaches ∞ as $x \to 0$ and as $x \to 1$, indicating vertical tangents at both $x = 0$ and $x = 1$.



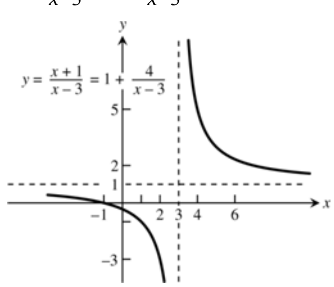


42. The graph of the first derivative indicates that the curve is rising on $\left(0, \frac{17-\sqrt{33}}{16}\right)$ and $\left(\frac{17+\sqrt{33}}{16}, \infty\right)$, falling on $(-\infty, 0)$ and $\left(\frac{17-\sqrt{33}}{16}, \frac{17+\sqrt{33}}{16}\right) \Rightarrow$ a local maximum at $x = \frac{17-\sqrt{33}}{16}$, a local minimum at $x = \frac{17+\sqrt{33}}{16}$. The derivative approaches $-\infty$ as $x \to 0^-$ and $x \to 1$, and approaches ∞ as $x \to 0^+$, indicating a cusp and local minimum at $x = 0$ and a vertical tangent at $x = 1$.

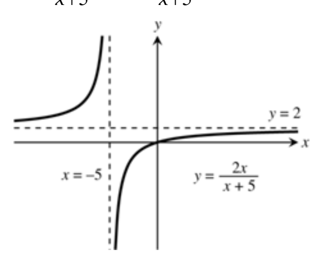




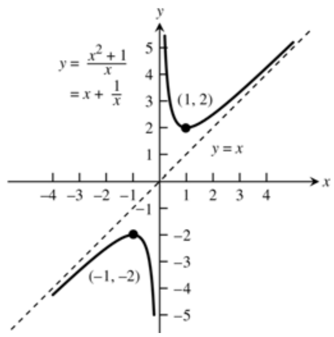
43. $y = \frac{x+1}{x-3} = 1 + \frac{4}{x-3}$



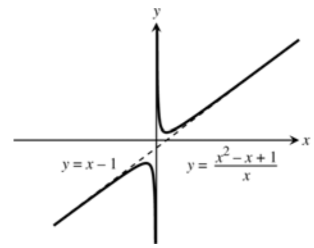
44. $y = \frac{2x}{x+5} = 2 - \frac{10}{x+5}$



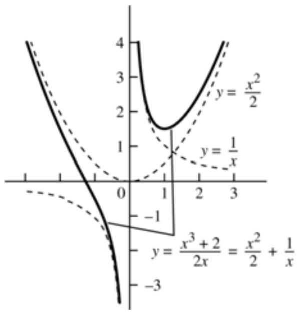
45. $y = \frac{x^2+1}{x} = x + \frac{1}{x}$



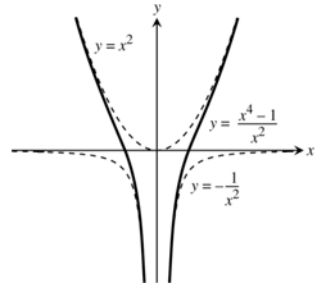
46. $y = \frac{x^2-x+1}{x} = x - 1 + \frac{1}{x}$



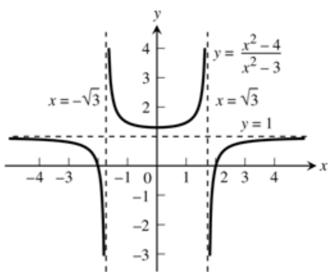
47. $y = \frac{x^3+2}{2x} = \frac{x^2}{2} + \frac{1}{x}$



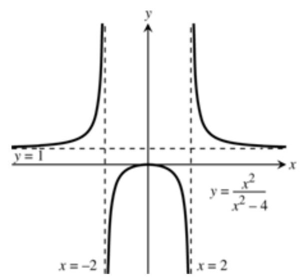
48. $y = \frac{x^4-1}{x^2} = x^2 - \frac{1}{x^2}$



49. $y = \frac{x^2-4}{x^2-3} = 1 - \frac{1}{x^2-3}$



50. $y = \frac{x^2}{x^2-4} = 1 + \frac{4}{x^2-4}$



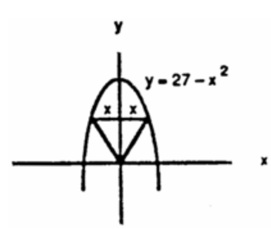
51. (a) Maximize $f(x) = \sqrt{x} - \sqrt{36-x} = x^{1/2} - (36-x)^{1/2}$ where $0 \le x \le 36$ $\Rightarrow f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}(36-x)^{-1/2}(-1) = \frac{\sqrt{36-x}+\sqrt{x}}{2\sqrt{x}\sqrt{36-x}} \Rightarrow$ derivative fails to exist at 0 and 36; $f(0) = -6$, and $f(36) = 6 \Rightarrow$ the numbers are 0 and 36

(b) Maximize $g(x) = \sqrt{x} + \sqrt{36-x} = x^{1/2} + (36-x)^{1/2}$ where $0 \le x \le 36 \Rightarrow g'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{2}(36-x)^{-1/2}(-1)$ $= \frac{\sqrt{36-x}-\sqrt{x}}{2\sqrt{x}\sqrt{36-x}} \Rightarrow$ critical points at 0, 18 and 36; $g(0) = 6$, $g(18) = 2\sqrt{18} = 6\sqrt{2}$ and $g(36) = 6 \Rightarrow$ the numbers are 18 and 18

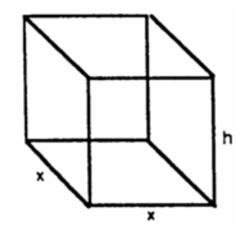
52. (a) Maximize $f(x) = \sqrt{x}(20-x) = 20x^{1/2} - x^{3/2}$ where $0 \le x \le 20 \Rightarrow f'(x) = 10x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{20-3x}{2\sqrt{x}} = 0$ $\Rightarrow x = 0$ and $x = \frac{20}{3}$ are critical points; $f(0) = f(20) = 0$ and $f\left(\frac{20}{3}\right) = \sqrt{\frac{20}{3}}\left(20 - \frac{20}{3}\right) = \frac{40\sqrt{20}}{3\sqrt{3}} \Rightarrow$ the numbers are $\frac{20}{3}$ and $\frac{40}{3}$.

(b) Maximize $g(x) = x + \sqrt{20-x} = x + (20-x)^{1/2}$ where $0 \le x \le 20 \Rightarrow g'(x) = \frac{2\sqrt{20-x}-1}{2\sqrt{20-x}} = 0 \Rightarrow \sqrt{20-x} = \frac{1}{2}$ $\Rightarrow x = \frac{79}{4}$. The critical points are $x = \frac{79}{4}$ and $x = 20$. Since $g\left(\frac{79}{4}\right) = \frac{81}{4}$ and $g(20) = 20$, the numbers must be $\frac{79}{4}$ and $\frac{1}{4}$.

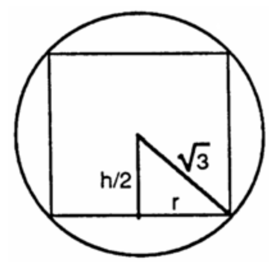
53. $A(x) = \frac{1}{2}(2x)(27 - x^2)$ for $0 \le x \le \sqrt{27}$ $\Rightarrow A'(x) = 3(3+x)(3-x)$ and $A''(x) = -6x$. The critical points are -3 and 3, but -3 is not in the domain. Since $A''(3) = -18 < 0$ and $A\left(\sqrt{27}\right) = 0$, the maximum occurs at $x = 3 \Rightarrow$ the largest area is $A(3) = 54$ sq units.



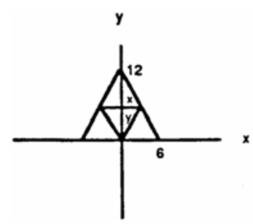
54. The volume is $V = x^2h = 1 \Rightarrow h = \frac{1}{x^2}$. The surface area is $S(x) = x^2 + 4x\left(\frac{1}{x^2}\right) = x^2 + \frac{4}{x}$, where $x > 0$ $\Rightarrow S'(x) = \frac{2(x-2)2(x^3-2)}{x^2} \Rightarrow$ the critical points are 0 and $\sqrt[3]{2}$, but 0 is not in the domain. Now $S''(\sqrt[3]{2}) = 2 + \frac{8}{2^3} > 0 \Rightarrow$ at $x = \sqrt[3]{2}$ there is a minimum. The dimensions $\sqrt[3]{2}$ m by $\sqrt[3]{2}$ m by $\sqrt[3]{2}/2$ m minimize the surface area.



55. From the diagram we have $\left(\frac{h}{2}\right)^2 + r^2 = \left(\sqrt{3}\right)^2$ $\Rightarrow r^2 = \frac{12-h^2}{4}$. The volume of the cylinder is $V = \pi r^2 h = \pi\left(\frac{12-h^2}{4}\right)h = \frac{\pi}{4}(12h - h^3)$, where $0 \le h \le 2\sqrt{3}$. Then $V'(h) = \frac{3\pi}{4}(2+h)(2-h) \Rightarrow$ the critical points are -2 and 2, but -2 is not in the domain. At $h = 2$ there is a maximum since $V''(2) = -3\pi < 0$. The dimensions of the largest cylinder are radius $= \sqrt{2}$ and height $= 2$.

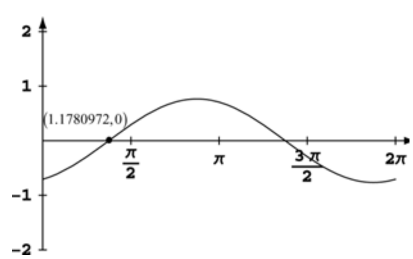


56. From the diagram we have $x =$ radius and $y =$ height $= 12 - 2x$ and $V(x) = \frac{1}{3}\pi x^2(12-2x)$, where $0 \le x \le 6$ $\Rightarrow V'(x) = 2\pi x(4-x)$ and $V''(4) = -8\pi$. The critical points are 0 and 4; $V(0) = V(6) = 0 \Rightarrow x = 4$ gives the maximum. Thus the values of $r = 4$ and $h = 4$ yield the largest volume for the smaller cone.



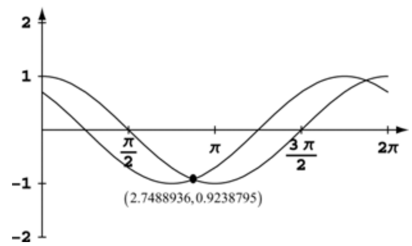
57. The profit $P = 2px + py = 2px + p\left(\frac{40-10x}{5-x}\right)$, where p is the profit on grade B tires and $0 \le x \le 4$. Thus $P'(x) = \frac{2p}{(5-x)^2}(x^2 - 10x + 20) \Rightarrow$ the critical points are $\left(5-\sqrt{5}\right)$, 5, and $\left(5+\sqrt{5}\right)$, but only $\left(5-\sqrt{5}\right)$ is in the domain. Now $P'(x) > 0$ for $0 < x < \left(5-\sqrt{5}\right)$ and $P'(x) < 0$ for $\left(5-\sqrt{5}\right) < x < 4 \Rightarrow$ at $x = \left(5-\sqrt{5}\right)$ there is a local maximum. Also $P(0) = 8p$, $P\left(5-\sqrt{5}\right) = 4p\left(5-\sqrt{5}\right) \approx 11p$, and $P(4) = 8p \Rightarrow$ at $x = \left(5-\sqrt{5}\right)$ there is an absolute maximum. The maximum occurs when $x = \left(5-\sqrt{5}\right)$ and $y = 2\left(5-\sqrt{5}\right)$, the units are hundreds of tires, i.e., $x \approx 276$ tires and $y \approx 553$ tires.

58. (a) The distance between the particles is $|f(t)|$ where $f(t) = -\cos t + \cos\left(t + \frac{\pi}{4}\right)$. Then, $f'(t) = \sin t - \sin\left(t + \frac{\pi}{4}\right)$. Solving $f'(t) = 0$ graphically, we obtain $t \approx 1.178$, $t \approx 4.320$, and so on.



Alternatively, $f'(t) = 0$ may be solved analytically as follows. $f'(t) = \sin\left[\left(t + \frac{\pi}{8}\right) - \frac{\pi}{8}\right] - \sin\left[\left(t + \frac{\pi}{8}\right) + \frac{\pi}{8}\right]$ $= \left[\sin\left(t + \frac{\pi}{8}\right)\cos\frac{\pi}{8} - \cos\left(t + \frac{\pi}{8}\right)\sin\frac{\pi}{8}\right] - \left[\sin\left(t + \frac{\pi}{8}\right)\cos\frac{\pi}{8} + \cos\left(t + \frac{\pi}{8}\right)\sin\frac{\pi}{8}\right] = -2\sin\frac{\pi}{8}\cos\left(t + \frac{\pi}{8}\right)$ so the critical points occur when $\cos\left(t + \frac{\pi}{8}\right) = 0$, or $t = \frac{3\pi}{8} + k\pi$. At each of these values, $f(t) = \pm\cos\frac{3\pi}{8} \approx \pm 0.765$ units, so the maximum distance between the particles is 0.765 units.

(b) Solving $\cos t = \cos\left(t + \frac{\pi}{4}\right)$ graphically, we obtain $t \approx 2.749, t \approx 5.890$, and so on.



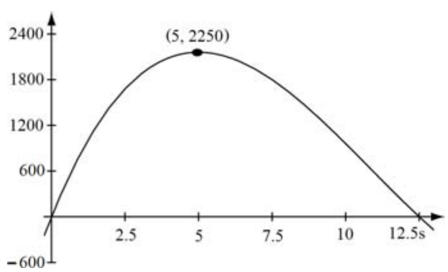
Alternatively, this problem can be solved analytically as follows.

$$\cos t = \cos\left(t + \frac{\pi}{4}\right)$$
$$\cos\left[\left(t + \frac{\pi}{8}\right) - \frac{\pi}{8}\right] = \cos\left[\left(t + \frac{\pi}{8}\right) + \frac{\pi}{8}\right]$$
$$\cos\left(t + \frac{\pi}{8}\right)\cos\frac{\pi}{8} + \sin\left(t + \frac{\pi}{8}\right)\sin\frac{\pi}{8} = \cos\left(t + \frac{\pi}{8}\right)\cos\frac{\pi}{8} - \sin\left(t + \frac{\pi}{8}\right)\sin\frac{\pi}{8}$$
$$2\sin\left(t + \frac{\pi}{8}\right)\sin\frac{\pi}{8} = 0$$
$$\sin\left(t + \frac{\pi}{8}\right) = 0;\ t = \frac{7\pi}{8} + k\pi$$

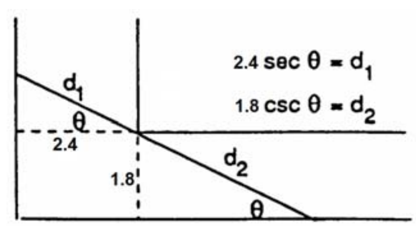
The particles collide when $t = \frac{7\pi}{8} \approx 2.749$. (Plus multiples of π if they keep going.)

59. The dimensions will be x cm by $25 - 2x$ cm by $40 - 2x$ cm, so $V(x) = x(25 - 2x)(40 - 2x)$ $= 4x^3 - 130x^2 + 1000x$ for $0 < x < 12$. Then $V'(x) = 12x^2 - 260x + 1000 = 4(x - 5)(3x - 50)$, so the critical point in the correct domain is $x = 5$. This critical point corresponds to the maximum possible volume because $V'(x) > 0$ for $0 < x < 5$ and $V'(x) < 0$ for $5 < x < 12.5$. The box of largest volume has a height of 5 cm and a base measuring 15 cm by 30 cm, and its volume is 2250 cm^3.

Graphical support:



60. The length of the ladder is $d_1 + d_2 = 2.4\sec\theta + 1.8\csc\theta$. We wish to maximize $I(\theta) = 2.4\sec\theta + 1.8\csc\theta$ $\Rightarrow I'(\theta) = 2.4\sec\theta\tan\theta - 1.8\csc\theta\cot\theta$. Then $I'(\theta) = 0$ $\Rightarrow 2.4\sin^3\theta - 1.8\cos^3\theta = 0 \Rightarrow \tan\theta = \frac{\sqrt[3]{6}}{2}$ $\Rightarrow d_1 = 3.243$ and $d_2 = 2.677 \Rightarrow$ the length of the ladder is about 5.92 m.



61. $g(x) = 3x - x^3 + 4 \Rightarrow g(2) = 2 > 0$ and $g(3) = -14 < 0 \Rightarrow g(x) = 0$ in the interval $[2, 3]$ by the Intermediate Value Theorem. Then $g'(x) = 3 - 3x^2 \Rightarrow x_{n+1} = x_n - \frac{3x_n - x_n^3 + 4}{3 - 3x_n^2}$; $x_0 = 2 \Rightarrow x_1 = 2.\overline{22} \Rightarrow x_2 = 2.196215$, and so forth to $x_5 = 2.195823345$.

62. $g(x) = x^4 - x^3 - 75 \Rightarrow g(3) = -21 < 0$ and $g(4) = 117 > 0 \Rightarrow g(x) = 0$ in the interval $[3, 4]$ by the Intermediate Value Theorem. Then $g'(x) = 4x^3 - 3x^2 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 - x_n^3 - 75}{4x_n^3 - 3x_n^2}$; $x_0 = 3 \Rightarrow x_1 = 3.259259 \Rightarrow x_2 = 3.229050$, and so forth to $x_5 = 3.22857729$.

63. $\int (x^5 + 4x^2 + 9)\,dx = \int x^5 dx + 4\int x^2 dx + 9\int 1 dx = \frac{x^6}{6} + \frac{4x^3}{3} + 9x + C$

64. $\int \left(6t^5 - \frac{t^3}{4} - t\right) dt = \int 6t^5 dt - \int \frac{t^3}{4} - \int t dt = 6\frac{t^6}{6} - \frac{1}{4}\frac{t^4}{4} - \frac{t^2}{2} + C = t^6 - \frac{t^4}{16} - \frac{t^2}{2} + C$

65. $\int \left(t\sqrt{t} + \frac{5}{t^3}\right) dt = \int t\sqrt{t}dt + \int \frac{5}{t^3} dt = \int t^{3/2} dt + \int 5t^{-3} dt = \frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 5\frac{t^{-3+1}}{-3+1} + C = \frac{t^{5/2}}{\frac{5}{2}} - \frac{5}{2t^2} + C = \frac{2}{5}t^{5/2} - \frac{5}{2t^2} + C$

66. $\int \left(\frac{5}{t\sqrt{t}} - \frac{4}{t^3}\right) dt = \int 5t^{-3/2} dt - \int 4t^{-3} dt = 5\frac{t^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} - 4\frac{t^{-3+1}}{-3+1} + C = -10t^{-1/2} + \frac{2}{t^2} + C$

67. $\int \frac{dr}{(r-3)^3} = \int (r-3)^{-3} dr = \frac{(r-3)^{-3+1}}{-3+1} + C = \frac{(r-3)^{-2}}{-2} + C = \frac{-1}{2(r-3)^2} + C$

68. $\int \frac{5\,dr}{\left(r+\sqrt{3}\right)^2} = 5\int \left(r+\sqrt{3}\right)^{-2} dr = 5\frac{\left(r+\sqrt{3}\right)^{-2+1}}{-2+1}(1) + C = \frac{-5}{\left(r+\sqrt{3}\right)} + C$

69. $\int 9\theta^2\sqrt{\theta^3 - 1}\,d\theta$ - - - - - - - (1)

Put $\theta^3 - 1 = t \Rightarrow 3\theta^2 d\theta = dt \Rightarrow \theta^2 d\theta = \frac{dt}{3}$

$\therefore$ (1) becomes $\int 9\theta^2\sqrt{\theta^3 - 1}\,d\theta = \int \frac{9}{3}\sqrt{t}\,dt = 3\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = 3\cdot\frac{2}{3}t^{\frac{3}{2}} + C = 2\left(\theta^3 - 1\right)^{3/2} + C$

70. $\int \frac{x}{\sqrt{5-x^2}} dx$ - - - - - - - (1)

Put $5 - x^2 = t \Rightarrow -2xdx = dt \Rightarrow xdx = -\frac{dt}{2}$

$\therefore$ (1) becomes $\int \frac{x}{\sqrt{5-x^2}} dx = \int \frac{-dt/2}{\sqrt{t}} = -\frac{1}{2}\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = -\frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = -\sqrt{5 - x^2} + C.$

71. $\int x^4\left(1-x^5\right)^{-1/5} dx$ - - - - - - - (1)

Put $1-x^5 = t \Rightarrow -5x^4 dx = dt \Rightarrow x^4 dx = -\frac{dt}{5}$

$\therefore$ (1) becomes $\int (t)^{-1/5}\left(\frac{dt}{5}\right) = -\frac{1}{5}\frac{t^{-\frac{1}{5}+1}}{-\frac{1}{5}+1} = -\frac{1}{5}\frac{t^{\frac{4}{5}}}{\frac{4}{5}} + C = -\frac{1}{5}\cdot\frac{5}{4}t^{\frac{4}{5}} + C = -\frac{1}{4}\left(5-x^2\right)^{4/5} + C.$

72. $\int (5-x)^{5/7} dx$ - - - - - - - (1)

Put $5-x = t \Rightarrow -dx = dt$

$\therefore$ (1) becomes $\int (5-x)^{5/7} dx = \int t^{5/7}(-dt) = -\frac{t^{\frac{5}{7}+1}}{\frac{5}{7}+1} + C = -\frac{7}{12}(5-x)^{12/7} + C.$

73. Our trial solution based on the chain rule is $10\tan\frac{s}{10} + C$. Differentiate the solution to check:

$\frac{d}{ds}\left[10\tan\frac{s}{10} + C\right] = \sec^2\frac{s}{10}$. Thus $\int \sec^2\frac{s}{10} ds = 10\tan\frac{s}{10} + C.$

74. $\int \csc^2 es\, ds$ - - - - - - - (1)

Put $es = t \Rightarrow eds = dt \Rightarrow ds = \frac{dt}{e}$

$\therefore$ (1) becomes $\int \csc^2 es\, ds = \frac{1}{e}(-\cot es) + C = -\frac{\cot es}{e} + C.$

75. $\int \csc\sqrt{3}\theta \cot\sqrt{3}\theta\, d\theta$ - - - - - - - (1)

Put $\sqrt{3}\theta = t \Rightarrow \sqrt{3}d\theta = dt \Rightarrow d\theta = \frac{dt}{\sqrt{3}}$

$\therefore$ (1) becomes $\int \csc\sqrt{3}\theta\cot\sqrt{3}\theta\, d\theta = \frac{1}{\sqrt{3}}\csc t\cot t\ dt = \frac{1}{\sqrt{3}}(-\csc t) + C = -\frac{1}{\sqrt{3}}\csc t\sqrt{3}\theta + C.$

76. $\int \sec\frac{t}{5}\tan\frac{t}{5} dt$ - - - - - - - (1)

Put $\frac{t}{5} = p \Rightarrow dt = 5dp$

$\therefore$ (1) becomes $\int \sec\frac{t}{5}\tan\frac{t}{5} dt = \int \sec p\tan p\, dp = 5\sec p + C = 5\sec\frac{t}{5} + C.$

77. Using the hint $\cos^2\frac{x}{6} = \frac{1+\cos^2\left(\frac{x}{6}\right)}{2} = \frac{1+\cos\left(\frac{x}{3}\right)}{2}$

$\therefore \int \cos^2\frac{x}{6} dx = \frac{1}{2}\int 1+\cos\left(\frac{x}{3}\right) dx = \frac{1}{2}x + 3\sin\left(\frac{x}{3}\right) + C.$

78. Put $\frac{x}{2} = t \Rightarrow dx = 2dt$

$\therefore$ (1) becomes $\int \sec^2\frac{x}{2} dx = \int \sec^2 t\, dt = 2\tan t + C.$

79. $y = \int \frac{x^2+1}{x^2} dx = \int (1+x^{-2})\ dx = x - x^{-1} + C = x - \frac{1}{x} + C$; $y = -1$ when $x = 1 \Rightarrow 1 - \frac{1}{1} + C = -1 \Rightarrow C = -1$
$\Rightarrow y = x - \frac{1}{x} - 1$

80. $y = \int \left(x + \frac{1}{x}\right)^2 dx = \int (x^2 + 2 + \frac{1}{x^2})\, dx = \int (x^2 + 2 + x^{-2})\, dx = \frac{x^3}{3} + 2x - x^{-1} + C = \frac{x^3}{3} + 2x - \frac{1}{x} + C;$
$y = 1$ when $x = 1 \Rightarrow \frac{1}{3} + 2 - \frac{1}{1} + C = 1 \Rightarrow C = -\frac{1}{3} \Rightarrow y = \frac{x^3}{3} + 2x - \frac{1}{x} - \frac{1}{3}$

81. $\frac{dr}{dt} = \int\left(15\sqrt{t} + \frac{3}{\sqrt{t}}\right) dt = \int (15t^{1/2} + 3t^{-1/2})\, dt = 10t^{3/2} + 6t^{1/2} + C;\ \frac{dr}{dt} = 8$ when $t = 1 \Rightarrow 10(1)^{3/2} + 6(1)^{1/2} + C = 8$
$\Rightarrow C = -8$. Thus $\frac{dr}{dt} = 10t^{3/2} + 6t^{1/2} - 8 \Rightarrow r = \int (10t^{3/2} + 6t^{1/2} - 8)\, dt = 4t^{5/2} + 4t^{3/2} - 8t + C;\ r = 0$ when $t = 1$
$\Rightarrow 4(1)^{5/2} + 4(1)^{3/2} - 8(1) + C_1 = 0 \Rightarrow C_1 = 0$. Therefore, $r = 4t^{5/2} + 4t^{3/2} - 8t$

82. $\frac{d^2r}{dt^2} = \int -\cos t\ dt = -\sin t + C;\ r'' = 0$ when $t = 0 \Rightarrow -\sin 0 + C = 0 \Rightarrow C = 0$. Thus, $\frac{d^2r}{dt^2} = -\sin t$
$\Rightarrow \frac{dr}{dt} = \int -\sin t\ dt = \cos t + C_1;\ r' = 0$ when $t = 0 \Rightarrow 1 + C_1 = 0 \Rightarrow C_1 = -1$. Then
$\frac{dr}{dt} = \cos t - 1 \Rightarrow r = \int (\cos t - 1)\ dt = \sin t - t + C_2;\ r = -1$ when $t = 0 \Rightarrow 0 - 0 + C_2 = -1 \Rightarrow C_2 = -1$. Therefore,
$r = \sin t - t - 1$

CHAPTER 4 ADDITIONAL AND ADVANCED EXERCISES

1. If M and m are the maximum and minimum values, respectively, then $m \le f(x) \le M$ for all $x \in I$. If $m = M$ then f is constant on I.

2. No, the function $f(x) = \begin{cases} 3x + 6, & -2 \le x < 0 \\ 9 - x^2, & 0 \le x \le 2 \end{cases}$ has an absolute minimum value of 0 at $x = -2$ and an absolute maximum value of 9 at $x = 0$, but it is discontinuous at $x = 0$.

3. On an open interval the extreme values of a continuous function (if any) must occur at an interior critical point. On a half-open interval the extreme values of a continuous function may be at a critical point or at the closed endpoint. Extreme values occur only where $f' = 0$, f' does not exist, or at the endpoints of the interval. Thus the extreme points will not be at the ends of an open interval.

4. The pattern $f' = +++\underset{1}{|}----\underset{2}{|}----\underset{3}{|}++++\underset{4}{|}+++$ indicates a local maximum at $x = 1$ and a local minimum at $x = 3$.

5. (a) If $y' = 6(x+1)(x-2)^2$, then $y' < 0$ for $x < -1$ and $y' > 0$ for $x > -1$. The sign pattern is $f' = ---\underset{-1}{|}+++\underset{2}{|}+++ \Rightarrow f$ has a local minimum at $x = -1$. Also $y'' = 6(x-2)^2 + 12(x+1)(x-2)$ $= 6(x-2)(3x) \Rightarrow y'' > 0$ for $x < 0$ or $x > 2$, while $y'' < 0$ for $0 < x < 2$. Therefore f has points of inflection at $x = 0$ and $x = 2$. There is no local maximum.
   (b) If $y' = 6x(x+1)(x-2)$, then $y' < 0$ for $x < -1$ and $0 < x < 2$; $y' > 0$ for $-1 < x < 0$ and $x > 2$. The sign pattern is $y' = ---\underset{-1}{|}+++\underset{0}{|}---\underset{2}{|}+++$. Therefore f has a local maximum at $x = 0$ and local minima at $x = -1$ and $x = 2$. Also, $y'' = 18\left[x - \left(\frac{1-\sqrt{7}}{3}\right)\right]\left[x - \left(\frac{1+\sqrt{7}}{3}\right)\right]$, so $y'' < 0$ for $\frac{1-\sqrt{7}}{3} < x < \frac{1+\sqrt{7}}{3}$ and $y'' > 0$ for all other $x \Rightarrow f$ has points of inflection at $x = \frac{1\pm\sqrt{7}}{3}$.

6. The Mean Value Theorem indicates that $\frac{f(6)-f(0)}{6-0} = f'(c) \le 2$ for some c in $(0, 6)$. Then $f(6) - f(0) \le 12$ indicates the most that f can increase is 12.

7. If f is continuous on $[a, c)$ and $f'(x) \le 0$ on $[a, c)$, then by the Mean Value Theorem for all $x \in [a, c)$ we have $\frac{f(c)-f(x)}{c-x} \le 0 \Rightarrow f(c) - f(x) \le 0 \Rightarrow f(x) \ge f(c)$. Also if f is continuous on $(c, b]$ and $f'(x) \ge 0$ on $(c, b]$, then for all $x \in (c, b]$ we have $\frac{f(x)-f(c)}{x-c} \ge 0 \Rightarrow f(x) - f(c) \ge 0 \Rightarrow f(x) \ge f(c)$. Therefore $f(x) \ge f(c)$ for all $x \in [a, b]$.

8. (a) For all $x, -(x+1)^2 \le 0 \le (x-1)^2 \Rightarrow -(1+x^2) \le 2x \le (1+x^2) \Rightarrow -\frac{1}{2} \le \frac{x}{1+x^2} \le \frac{1}{2}$.
   (b) There exists $c \in (a, b)$ such that $\frac{c}{1+c^2} = \frac{f(b)-f(a)}{b-a} \Rightarrow \left|\frac{f(b)-f(a)}{b-a}\right| = \left|\frac{c}{1+c^2}\right| \le \frac{1}{2}$, from part (a) $\Rightarrow |f(b)-f(a)| \le \frac{1}{2}|b-a|$.

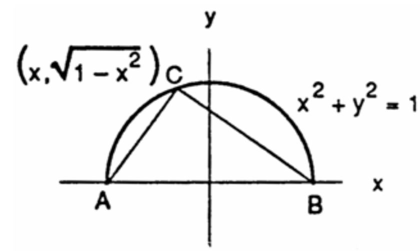
9. No. Corollary 1 requires that $f'(x) = 0$ for <u>all</u> x in some interval I, not $f'(x) = 0$ at a single point in I.

10. (a) $h(x) = f(x)g(x) \Rightarrow h'(x) = f'(x)g(x) + f(x)g'(x)$ which changes signs at $x = a$ since $f'(x), g'(x) > 0$ when $x < a$, $f'(x), g'(x) < 0$ when $x > a$ and $f(x), g(x) > 0$ for all x. Therefore $h(x)$ does have a local maximum at $x = a$.
    (b) No, let $f(x) = g(x) = x^3$ which have points of inflection at $x = 0$, but $h(x) = x^6$ has no point of inflection (it has a local minimum at $x = 0$).

11. From (ii), $f(-1) = \frac{-1+a}{b-c+2} = 0 \Rightarrow a = 1$; from (iii), either $1 = \lim_{x\to\infty} f(x)$ or $1 = \lim_{x\to-\infty} f(x)$. In either case, $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x+1}{bx^2+cx+2} = \lim_{x\to\pm\infty} \frac{1+\frac{1}{x}}{bx+c+\frac{2}{x}} = 1 \Rightarrow b = 0$ and $c = 1$. For if $b = 1$, then $\lim_{x\to\pm\infty} \frac{1+\frac{1}{x}}{x+c+\frac{2}{x}} = 0$ and if $c = 0$, then $\lim_{x\to\pm\infty} \frac{1+\frac{1}{x}}{bx+\frac{2}{x}} = \lim_{x\to\pm\infty} \frac{1+\frac{1}{x}}{\frac{2}{x}} = \pm\infty$. Thus $a = 1$, $b = 0$, and $c = 1$.

12. $\frac{dy}{dx} = 3x^2 + 2kx + 3 = 0 \Rightarrow x = \frac{-2k\pm\sqrt{4k^2-36}}{6} \Rightarrow x$ has only one value when $4k^2 - 36 = 0 \Rightarrow k^2 = 9$ or $k = \pm 3$.

13. The area of the ΔABC is $A(x) = \frac{1}{2}(2)\sqrt{1-x^2}$ $= (1-x^2)^{1/2}$, where $0 \le x \le 1$. Thus $A'(x) = \frac{-x}{\sqrt{1-x^2}}$ $\Rightarrow 0$ and ± 1 are critical points. Also $A(\pm 1) = 0$ so $A(0) = 1$ is the maximum. When $x = 0$ the ΔABC is isosceles since $AC = BC = \sqrt{2}$.

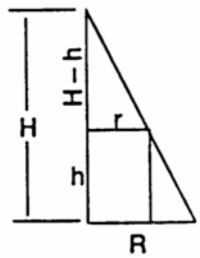


14. $\lim_{h\to 0} \frac{f'(c+h)-f'(c)}{h} = f''(c) \Rightarrow$ for $\varepsilon = \frac{1}{2}|f''(c)| > 0$ there exists a $\delta > 0$ such that $0 < |h| < \delta$ $\Rightarrow \left|\frac{f'(c+h)-f'(c)}{h} - f''(c)\right| < \frac{1}{2}|f''(c)|$. Then $f'(c) = 0 \Rightarrow -\frac{1}{2}|f''(c)| < \frac{f'(c+h)}{h} - f''(c) < \frac{1}{2}|f''(c)|$ $\Rightarrow f''(c) - \frac{1}{2}|f''(c)| < \frac{f'(c+h)}{h} < f''(c) + \frac{1}{2}|f''(c)|$. If $f''(c) < 0$, then $|f''(c)| = -f''(c)$ $\Rightarrow \frac{3}{2}f''(c) < \frac{f'(c+h)}{h} < \frac{1}{2}f''(c) < 0$; likewise if $f''(c) > 0$, then $0 < \frac{1}{2}f''(c) < \frac{f'(c+h)}{h} < \frac{3}{2}f''(c)$.
    (a) If $f''(c) < 0$, then $-\delta < h < 0 \Rightarrow f'(c+h) > 0$ and $0 < h < \delta \Rightarrow f'(c+h) < 0$. Therefore, $f(c)$ is a local maximum.
    (b) If $f''(c) < 0$, then $-\delta < h < 0 \Rightarrow f'(c+h) < 0$ and $0 < h < \delta \Rightarrow f'(c+h) < 0$. Therefore, $f(c)$ is a local minimum.

15. The time it would take the water to hit the ground from height y is $\sqrt{\frac{2y}{g}}$, where g is the acceleration of gravity. The product of time and exit velocity (rate) yields the distance the water travels: $D(y) = \sqrt{\frac{2y}{g}}\sqrt{64(h-y)} = 8\sqrt{\frac{2}{g}}(hy-y^2)^{1/2}$, $0 \le y \le h \Rightarrow D'(y) = -4\sqrt{\frac{2}{g}}(hy-y^2)^{-1/2}(h-2y) \Rightarrow 0$, $\frac{h}{2}$ and h are critical points. Now $D(0) = 0$, $D\left(\frac{h}{2}\right) = 8\sqrt{\frac{2}{g}}\left(h\left(\frac{h}{2}\right)-\left(\frac{h}{2}\right)^2\right)^{1/2} = 4h\sqrt{\frac{2}{g}}$ and $D(h) = 0 \Rightarrow$ the best place to drill the hole is at $y = \frac{h}{2}$.

16. From the figure in the text, $\tan(\beta+\theta)=\frac{b+a}{n}$; $\tan(\beta+\theta)=\frac{\tan\beta+\tan\theta}{1-\tan\beta\tan\theta}$; and $\tan\theta=\frac{a}{h}$. These equations give $\frac{b+a}{h}=\frac{\tan\beta+\frac{a}{h}}{1-\frac{a}{h}\tan\beta}=\frac{h\tan\beta+a}{h-a\tan\beta}$. Solving for $\tan\beta$ gives $\tan\beta=\frac{bh}{h^2+a(b+a)}$ or $(h^2-a(b+a))\tan\beta=bh$. Differentiating both sides with respect to h gives $2h\tan\beta+(h^2+a(b+a))\sec^2\beta\frac{d\beta}{dh}=b$. Then $\frac{d\beta}{dh}=0\Rightarrow 2h\tan\beta=b\Rightarrow 2h\left(\frac{bh}{h^2+a(b+a)}\right)=b\Rightarrow 2bh^2=bh^2+ab(b+a)\Rightarrow h^2=a(b+a)\Rightarrow h=\sqrt{a(a+b)}$.

17. The surface area of the cylinder is $S=2\pi r^2+2\pi rh$.
From the diagram we have $\frac{r}{R}=\frac{H-h}{H}\Rightarrow h=\frac{RH-rH}{R}$
and $S(r)=2\pi r(r+h)=2\pi r\left(r+H-r\frac{H}{R}\right)$
$=2\pi\left(1-\frac{H}{R}\right)r^2+2\pi Hr$, where $0\le r\le R$.



Case 1: $H<R\Rightarrow S(r)$ is a quadratic equation containing the origin and concave upward $\Rightarrow S(r)$ is maximum at $r=R$.
Case 2: $H=R\Rightarrow S(r)$ is a linear equation containing the origin with a positive slope $\Rightarrow S(r)$ is maximum at $r=R$.
Case 3: $H>R\Rightarrow S(r)$ is a quadratic equation containing the origin and concave downward.
Then $\frac{dS}{dr}=4\pi\left(1-\frac{H}{R}\right)r+2\pi H$ and $\frac{dS}{dr}=0\Rightarrow 4\pi\left(1-\frac{H}{R}\right)r+2\pi H=0\Rightarrow r=\frac{RH}{2(H-R)}$. For simplification we let $r^*=\frac{RH}{2(H-R)}$.

(a) If $R<H<2R$, then $0>H-2R\Rightarrow H>2(H-R)\Rightarrow r^*=\frac{RH}{2(H-R)}>R$. Therefore, the maximum occurs at the right endpoint R of the interval $0\le r\le R$ because $S(r)$ is an increasing function of r.

(b) If $H=2R$, then $r^*=\frac{2R^2}{2R}=R\Rightarrow S(r)$ is maximum at $r=R$.

(c) If $H>2R$, then $2R+H<2H\Rightarrow H<2(H-R)\Rightarrow\frac{H}{2(H-R)}<1\Rightarrow\frac{RH}{2(H-R)}<R\Rightarrow r^*<R$. Therefore, $S(r)$ is a maximum at $r=r^*=\frac{RH}{2(H-R)}$.

Conclusion: If $H\in(0,\ 2R]$, then the maximum surface area is at $r=R$. If $H\in(2R,\ \infty)$, then the maximum is at $r=r^*=\frac{RH}{2(H-R)}$.

18. $f(x)=mx-1+\frac{1}{x}\Rightarrow f'(x)=m-\frac{1}{x^2}$ and $f''(x)=\frac{2}{x^3}>0$ when $x>0$. Then $f'(x)=0\Rightarrow x=\frac{1}{\sqrt{m}}$ yields a minimum. If $f\left(\frac{1}{\sqrt{m}}\right)\ge 0$, then $\sqrt{m}-1+\sqrt{m}=2\sqrt{m}-1\ge 0\Rightarrow m\ge\frac{1}{4}$. Thus the smallest acceptable value for m is $\frac{1}{4}$.

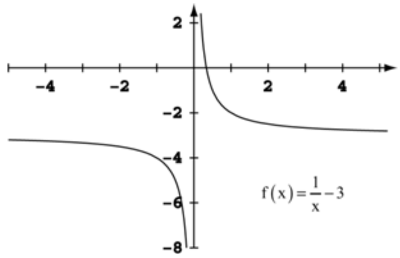
19. (a) The profit function is $P(x)=(c-ex)x-(a+bx)=-ex^2+(c-b)x-a$. $P'(x)=-2ex+c-b=0$ $\Rightarrow x=\frac{c-b}{2e}$. $P''(x)=-2e<0$ if $e>0$ so that the profit function is maximized at $x=\frac{c-b}{2e}$.

(b) The price therefore that corresponds to a production level yielding a maximum profit is $p\Big|_{x=\frac{c-b}{2e}}=c-e\left(\frac{c-b}{2e}\right)=\frac{c+b}{2}$ dollars.

(c) The weekly profit at this production level is $P(x)=-e\left(\frac{c-b}{2e}\right)^2+(c-b)\left(\frac{c-b}{2e}\right)-a=\frac{(c-b)^2}{4e}-a$.

(d) The tax increases cost to the new profit function is $F(x)=(c-ex)x-(a+bx+tx)=-ex^2+(c-b-t)x-a$. Now $F'(x)=-2ex+c-b-t=0$ when $x=\frac{t+b-c}{-2e}=\frac{c-b-t}{2e}$. Since $F''(x)=-2e<0$ if $e>0$, F is maximized when $x=\frac{c-b-t}{2e}$ units per week. Thus the price per unit is $p=c-e\left(\frac{c-b-t}{2e}\right)=\frac{c+b+t}{2}$ dollars. Thus, such a tax increases the cost per unit by $\frac{c+b+t}{2}-\frac{c+b}{2}=\frac{t}{2}$ dollars if units are priced to maximize profit.

20. (a)



The x-intercept occurs when $\frac{1}{x} - 3 = 0 \Rightarrow \frac{1}{x} = 3 \Rightarrow x = \frac{1}{3}$.

(b) By Newton's method, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Here $f'(x_n) = -x_n^{-2} = \frac{-1}{x_n^2}$. So $x_{n+1} = x_n - \frac{\frac{1}{x_n} - 3}{\frac{-1}{x_n^2}} = x_n + \left(\frac{1}{x_n} - 3\right)x_n^2$

$= x_n + x_n - 3x_n^2 = 2x_n - 3x_n^2 = x_n(2 - 3x_n)$.

21. $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^q - a}{qx_0^{q-1}} = \frac{qx_0^q - x_0^q + a}{qx_0^{q-1}} = \frac{x_0^q(q-1) + a}{qx_0^{q-1}} = x_0\left(\frac{q-1}{q}\right) + \frac{a}{x_0^{q-1}}\left(\frac{1}{q}\right)$ so that x_1 is a weighted average of x_0 and $\frac{a}{x_0^{q-1}}$ with weights $m_0 = \frac{q-1}{q}$ and $m_1 = \frac{1}{q}$.

In the case where $x_0 = \frac{a}{x_0^{q-1}}$ we have $x_0^q = a$ and $x_1 = \frac{a}{x_0^{q-1}}\left(\frac{q-1}{q}\right) + \frac{a}{x_0^{q-1}}\left(\frac{1}{q}\right) = \frac{a}{x_0^{q-1}}\left(\frac{q-1}{q} + \frac{1}{q}\right) = \frac{a}{x_0^{q-1}}$.

22. We have that $(x-h)^2 + (y-h)^2 = r^2$ and so $2(x-h) + 2(y-h)\frac{dy}{dx} = 0$ and $2 + 2\frac{dy}{dx} + 2(y-h)\frac{d^2y}{dx^2} = 0$ hold. Thus $2x + 2y\frac{dy}{dx} = 2h + 2h\frac{dy}{dx}$, by the former. Solving for h, we obtain $h = \frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}$. Substituting this into the second equation yields $2 + 2\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} - 2\left(\frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right) = 0$. Dividing by 2 results in $1 + \frac{dy}{dx} + y\frac{d^2y}{dx^2} - \left(\frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right) = 0$.

23. (a) $a(t) = s''(t) = -k\ (k > 0) \Rightarrow s'(t) = -kt + C_1$, where $s'(0) = 30 \Rightarrow C_1 = 30 \Rightarrow s'(t) = -kt + 30$. So $s(t) = \frac{-kt^2}{2} + 30t + C_2$ where $s(0) = 0 \Rightarrow C_2 = 0$ so $s(t) = \frac{-kt^2}{2} + 30t$. Now $s(t) = 30$ when $\frac{-kt^2}{2} + 30t = 30$. Solving for t we obtain $t = \frac{30 \pm \sqrt{30^2 - 60k}}{k}$. At such t we want $s'(t) = 0$, thus $-k\left(\frac{30 + \sqrt{30^2 - 60k}}{k}\right) + 30 = 0$ or $-k\left(\frac{30 - \sqrt{30^2 - 60k}}{k}\right) + 30 = 0$. In either case we obtain $30^2 - 60k = 0$ so that $k = \frac{30^2}{60} \approx 15\ \text{m/s}^2$.

(b) The initial condition that $s'(0) = 15$ m/s implies that $s'(t) = -kt + 15$ and $s(t) = \frac{-kt^2}{2} + 15t$ where k is as above. The car is stopped at a time t such that $s'(t) = -kt + 15 = 0 \Rightarrow t = \frac{15}{k}$. At this time the car has traveled a distance $s\left(\frac{15}{k}\right) = \frac{-k}{2}\left(\frac{15}{k}\right)^2 + 15\left(\frac{15}{k}\right) = \frac{15^2}{2k} = \frac{112.5}{k} = 112.5\left(\frac{60}{30^2}\right) = 7.5$ m. Thus halving the initial velocity quarters stopping distance.

24. $h(x) = f^2(x) + g^2(x) \Rightarrow h'(x) = 2f(x)f'(x) + 2g(x)g'(x) = 2[f(x)f'(x) + g(x)g'(x)]$
$= 2[f(x)g(x) + g(x)(-f(x))] = 2 \cdot 0 = 0$. Thus $h(x) = c$, a constant. Since $h(0) = 5$, $h(x) = 5$ for all x in the domain of h. Thus $h(10) = 5$.

25. Yes. The curve $y = x$ satisfies all three conditions since $\frac{dy}{dx} = 1$ everywhere, when $x = 0$, $y = 0$, and $\frac{d^2y}{dx^2} = 0$ everywhere.

26. $y' = 3x^2 + 2$ for all $x \Rightarrow y = x^3 + 2x + C$ where $-1 = 1^3 + 2 \cdot 1 + C \Rightarrow C = -4 \Rightarrow y = x^3 + 2x - 4$.

27. $s''(t) = a = -t^2 \Rightarrow v = s'(t) = \frac{-t^3}{3} + C$. We seek $v_0 = s'(0) = C$. We know that $s(t^*) = b$ for some t^* and s is at a maximum for this t^*. Since $s(t) = \frac{-t^4}{12} + Ct + k$ and $s(0) = 0$ we have that $s(t) = \frac{-t^4}{12} + Ct$ and also $s'(t^*) = 0$ so that $t^* = (3C)^{1/3}$. So $\frac{[-(3C)^{1/3}]^4}{12} + C(3C)^{1/3} = b \Rightarrow (3C)^{1/3}(C - \frac{3C}{12}) = b \Rightarrow (3C)^{1/3}\left(\frac{3C}{4}\right) = b \Rightarrow 3^{1/3}C^{4/3} = \frac{4b}{3}$ $\Rightarrow C = \frac{(4b)^{3/4}}{3}$. Thus $v_0 = s'(0) = \frac{(4b)^{3/4}}{3} = \frac{2\sqrt{2}}{3}b^{3/4}$.

28. (a) $s''(t) = t^{1/2} - t^{-1/2} \Rightarrow v(t) = s'(t) = \frac{2}{3}t^{3/2} - 2t^{1/2} + k$ where $v(0) = k = \frac{4}{3} \Rightarrow v(t) = \frac{2}{3}t^{3/2} - 2t^{1/2} + \frac{4}{3}$.
(b) $s(t) = \frac{4}{15}t^{5/2} - \frac{4}{3}t^{3/2} + \frac{4}{3}t + k_2$ where $s(0) = k_2 = -\frac{4}{15}$. Thus $s(t) = \frac{4}{15}t^{5/2} - \frac{4}{3}t^{3/2} + \frac{4}{3}t - \frac{4}{15}$.

29. The graph of $f(x) = ax^2 + bx + c$ with $a > 0$ is a parabola opening upwards. Thus $f(x) \ge 0$ for all x if $f(x) = 0$ for at most one real value of x. The solutions to $f(x) = 0$ are, by the quadratic equation $\frac{-2b \pm \sqrt{(2b)^2 - 4ac}}{2a}$. Thus we require $(2b)^2 - 4ac \le 0 \Rightarrow b^2 - ac \le 0$.

30. (a) Clearly $f(x) = (a_1x + b_1)^2 + \ldots + (a_nx + b_n)^2 \ge 0$ for all x. Expanding we see

$$f(x) = \left(a_1^2x^2 + 2a_1b_1x + b_1^2\right) + \ldots + \left(a_n^2x^2 + 2a_nb_nx + b_n^2\right)$$

$$= \left(a_1^2 + a_2^2 + \ldots + a_n^2\right)x^2 + 2\left(a_1b_1 + a_2b_2 + \ldots + a_nb_n\right)x + \left(b_1^2 + b_2^2 + \ldots + b_n^2\right) \ge 0.$$ Thus

$$\left(a_1b_1 + a_2b_2 + \ldots + a_nb_n\right)^2 - \left(a_1^2 + a_2^2 + \ldots + a_n^2\right)\left(b_1^2 + b_2^2 + \ldots + b_n^2\right) \le 0$$ by Exercise 29. Thus

$$\left(a_1b_1 + a_2b_2 + \ldots + a_nb_n\right)^2 \le \left(a_1^2 + a_2^2 + \ldots + a_n^2\right)\left(b_1^2 + b_2^2 + \ldots + b_n^2\right).$$

(b) Referring to Exercise 29: It is clear that $f(x) = 0$ for some real $x \Leftrightarrow b^2 - 4ac = 0$, by quadratic formula. Now notice that this implies that $f(x) = (a_1x + b_1)^2 + \ldots + (a_nx + b_n)^2$

$$= \left(a_1^2 + a_2^2 + \ldots + a_n^2\right)x^2 + 2\left(a_1b_1 + a_2b_2 + \ldots + a_nb_n\right)x + \left(b_1^2 + b_2^2 + \ldots + b_n^2\right) = 0$$

$$\Leftrightarrow \left(a_1b_1 + a_2b_2 + \ldots + a_nb_n\right)^2 - \left(a_1^2 + a_2^2 + \ldots + a_n^2\right)\left(b_1^2 + b_2^2 + \ldots + b_n^2\right) = 0$$

$$\Leftrightarrow \left(a_1b_1 + a_2b_2 + \ldots + a_nb_n\right)^2 = \left(a_1^2 + a_2^2 + \ldots + a_n^2\right)\left(b_1^2 + b_2^2 + \ldots + b_n^2\right)$$ But now $f(x) = 0 \Leftrightarrow a_ix + b_i = 0$ for all $i = 1, 2, \ldots, n \Leftrightarrow a_ix = -b_i = 0$ for all $i = 1, 2, \ldots, n$.