MAT1002 Lecture 1, Thursday, Jan/05/2023

Outline

- Infinite Sequences (10.1)
  - ↳ Definition
  - ↳ Limits and convergence
  - ↳ Properties of limits
  - ↳ Connection with functions of real variables
  - ↳ Sandwich/squeeze theorem
  - ↳ Some common limits
  - ↳ Recursively defined sequences
  - ↳ Bounded sequences

Infinite Sequences

Def: An infinite sequence, or simply sequence, is a list

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

Formally, a sequence is a function f with domain being $\mathbb{Z}_+ := \{1, 2, 3, \ldots\}$, with $f(n)$ written as a_n.

Remarks

- The domain $\{1, 2, 3, \ldots\}$ of the sequence $\{a_1, a_2, a_3, \ldots\}$ is also called the index set of the sequence.
- The index set sometimes can start with a number other than 1, e.g., $\{a_0, a_1, a_2, a_3, \ldots\}$ and $\{b_3, b_4, b_5, \ldots\}$ are also considered sequences.
- We use $\{a_n\}_{n=1}^{\infty}$ to denote $\{a_1, a_2, a_3, \ldots\}$; we may also simply write $\{a_n\}$ if the index set is not important or is clear from the context.

e.g.

- $\{(-1)^{n+1}n^2\}_{n=1}^{\infty} = \{1, -4, 9, -16, 25, -36, \ldots\}$

- A sequence can be defined recursively: the rules

$$F_0 := 0,\quad F_1 := 1,\quad F_i := F_{i-1} + F_{i-2} \ (\text{for } i \geq 2)$$

define the sequence $\{F_n\}_{n=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$

(called the Fibonacci sequence.)

Limits and Convergence

Def: Let $\{a_n\}$ be a sequence.

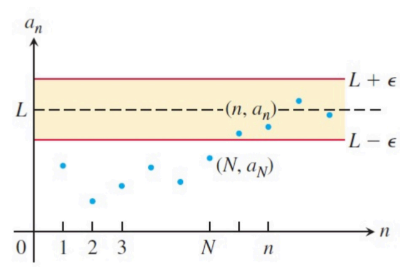
- If $L \in \mathbb{R}$ satisfies the condition

  for which a_n is defined

  $$\forall \varepsilon > 0 \ \exists N := N_\varepsilon \in \mathbb{Z} \text{ such that } \forall n > N,\ |a_n - L| < \varepsilon,$$

  then we say that $\lim_{n\to\infty} a_n = L$, and call L the limit of $\{a_n\}$.
- A sequence having a limit $L \in \mathbb{R}$ is said to be convergent.
- A sequence is said to be divergent if it is not convergent.

Remark



- The choice of N depends on ε, in general.
  (Bigger N may be required for smaller ε.)
- One may replace "$\forall n > N$" with "$\forall n \geq N$" in the definition above (it would be equivalent).

e.g.1 Using definition, show that $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0.

Sol. See example 10.1.1 (a).

e.g.2 Show that $\{a_n\}_{n=1}^{\infty}$ converges using definition, where

$$a_n := \frac{n}{2n+1}.$$

Proof: · It suffices to show that $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$. Guessed from the form of a_n.

· Let $\varepsilon > 0$ be fixed. Let $N := \left\lceil \frac{\frac{1}{\varepsilon}-2}{4} \right\rceil$, where $\lceil x \rceil$ denotes the smallest integer m such that $m \geq x$. Note that $N \geq 0$.

· For all $n \in \mathbb{Z}$, if $n > N$, then a_n is defined, and

$$n > \left\lceil \frac{\frac{1}{\varepsilon}-2}{4} \right\rceil \geq \frac{\frac{1}{\varepsilon}-2}{4}$$

$$\Rightarrow 4n > \frac{1}{\varepsilon} - 2 \Rightarrow 4n+2 > \frac{1}{\varepsilon}$$

$$\Rightarrow \frac{1}{2(2n+1)} < \varepsilon \Rightarrow \left| \frac{n}{2n+1} - \frac{1}{2} \right| < \varepsilon.$$

· By definition, $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$. $\square$

Subsequences

Informally, a subsequence of sequence $\{a_n\}$ is a sequence lying inside $\{a_n\}$ with the terms showing in the same order.

e.g. Given $a_n = \frac{1}{n}$, $\{a_{2k}\}_{k=1}^{\infty} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$, and $\{a_2, a_3, a_5, a_7, a_{11}, \ldots\}$ is another subsequence (where the indices are given by all prime numbers in increasing order).

theorem of subsequential limits

<u>Theorem</u> If $\{a_n\}$ contains two subsequences converge to two different limits, then $\{a_n\}$ diverges.

The proof is omitted and not required for the course. (Big idea: show that if $\{a_n\}$ converges to L, then every subsequence of $\{a_n\}$ must also converge to the same limit L.)

<u>e.g. 3</u> Show that $\{(-1)^n\}_{n=0}^{\infty}$ diverges.

<u>Proof</u>: · Let $a_n := (-1)^n$. Then $\{a_n\}$ contains two subsequences

$$\{b_k\} := \{a_{2k}\} = \{1, 1, 1, \ldots\} \quad \text{and}$$
$$\{c_k\} := \{a_{2k+1}\} = \{-1, -1, -1, \ldots\}$$

· Since the first subsequence converges to 1 while the second converges to -1, $\{a_n\}$ must diverge by theorem.

$\square$

Remark

Adding finitely many terms to a sequence does not change its convergence / divergence, e.g.,

- $\{7, 48, 56, -\pi, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\}$ Converges
- $\{46, 3, e^{-1}, 12, -1, 1, -1, 1, -1, 1, \ldots\}$ diverges.

"For convergence and limit, only the 'tail' matters".

Divergence to Infinity

Definition
Let $\{a_n\}$ be a sequence of real numbers.

- We write $\lim_{n\to\infty} a_n = \infty$, or simply $a_n \to \infty$ (as $n \to \infty$), if for every $M \in \mathbb{R}$, there exists N ~~$\in \mathbb{N}$~~ such that $a_n > M$ for all n satisfying ~~$n \geq N$.~~ $n > N$.
- We write $\lim_{n\to\infty} a_n = -\infty$, or simply $a_n \to -\infty$ (as $n \to \infty$), if for every $M \in \mathbb{R}$, there exists N ~~$\in \mathbb{N}$~~ such that $a_n < M$ for all n satisfying ~~$n \geq N$.~~ $n > N$.

If $\lim_{n\to\infty} a_n = \infty$, then $\{a_n\}$ is said to diverge to ∞.

If $\lim_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ is said to diverge to $-\infty$.

e.g. (without proofs)

- $\{\sqrt{n}\}$ diverges to ∞.
- $\{(-1)^n n\}$ diverges, but not to $\pm\infty$.

Properties of Sequence Limits

The following are some properties of sequence limits similar to those of function limits.

THEOREM ~~1~~ 10.1.1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

e.g.4 $$\lim_{n\to\infty}\frac{n^2+1}{n^3+3} = \lim_{n\to\infty}\frac{\frac{1}{n}+\frac{1}{n^3}}{1+\frac{3}{n^3}} = \frac{\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\frac{1}{n^3}}{\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{3}{n^3}}$$

$$= \frac{\lim_{n\to\infty}\frac{1}{n}+\left(\lim_{n\to\infty}\frac{1}{n}\right)^3}{\lim_{n\to\infty}1+3\left(\lim_{n\to\infty}\frac{1}{n}\right)^3} = \frac{0+0^3}{1+3\cdot 0^3} = 0.$$

Connection with Functions of Real Variables

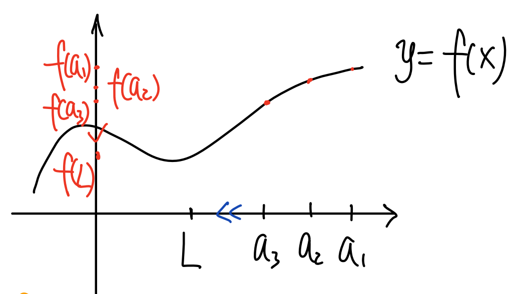
Theorem 10.1.3 Suppose that:

- $f(x)$ is a real variable function that is continuous at L;
- $\{a_n\}$ is a sequence with $\lim_{n\to\infty} a_n = L$.

Then the sequence $\{f(a_n)\}$ converges to $f(L)$ or in other words,

$$\lim_{n\to\infty} f(a_n) = f(L) = f\left(\lim_{n\to\infty} a_n\right)$$

The following proof is optional.



Proof: • Let $\varepsilon > 0$ be arbitrary.

• Since f is continuous at L, there exists $\delta > 0$ such that

$$\text{if } |x - L| < \delta \text{ then } |f(x) - f(L)| < \varepsilon. \quad (1)$$

"x close enough to L" "$f(x)$ close enough to $f(L)$"

• Since $a_n \to L$, there exists N such that for all $n > N$,

$$|a_n - L| < \delta. \quad (2)$$

← This is your "new ε" for sequence limits.

• Combining (1) and (2), we have

$$\forall n > N, \quad |f(a_n) - f(L)| < \varepsilon.$$

• By definition, $f(a_n) \to f(L)$ as $n \to \infty$. $\square$

e.g. 5 Find $\lim_{n\to\infty} 2^{\frac{1}{n}}$.

Sol: The function $f(x) = 2^x$ is continuous. By Thm 10.1.3,

$$\lim_{n\to\infty} 2^{\frac{1}{n}} = 2^{\lim_{n\to\infty}\frac{1}{n}} = 2^0 = 1.$$

THEOREM ~~4~~ 10.1.4 Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\underbrace{\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L.}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x)$$

Seq. limit function limit.

Proof: Suppose that $\lim_{x\to\infty} f(x) = L$. Let $\varepsilon > 0$. Then

$$\exists M \in \mathbb{R} \ \forall x \in \mathbb{R}, \text{ if } x > M \text{ then } |f(x) - L| < \varepsilon.$$

In particular, for all integers n with $n > N := \max\{n_0, \lceil M \rceil\}$,

$$|a_n - L| = |f(n) - L| < \varepsilon.$$

Hence $\lim_{n\to\infty} a_n = L$. $\square$

Theorem 10.1.4 allows one to use techniques for function limits to find sequence limits. (MAT1001!)

<u>e.g.6</u> Find the limit of the sequence $\left\{\frac{\ln n}{n}\right\}$.

<u>Sol</u>. $\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1}$ (L'Hôpital)

$= 0.$

Sandwich Theorem

10.1.2

THEOREM ~~2~~—The Sandwich Theorem for Sequences Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

The proof of this is similar to the function limit version. See Chapter 2 of Thomas'.

<u>e.g.7</u> Since

$$0 < \frac{n!}{n^n} = \frac{n\cdot(n-1)\cdot\ldots\cdot 2\cdot 1}{n\cdot n\cdot\ldots\cdot n\cdot n} \le \frac{1}{n}, \quad \forall n \ge 1,$$

and $\frac{1}{n}\to 0$ as $n\to\infty$, we have $\lim_{n\to\infty}\frac{n!}{n^n} = 0$.

<u>e.g.8</u> If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$, since

$$-|a_n| \le a_n \le |a_n|, \quad \forall n,$$

and $\lim_{n\to\infty}(-|a_n|) = -\lim_{n\to\infty}|a_n| = -0 = 0.$

Some Common Limits

THEOREM ~~5~~ 10.1.5 The following six sequences converge to the limits listed below:

1. $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\displaystyle\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\displaystyle\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\displaystyle\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Proof: 1. ✓

2. $$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x}$$

$$= e^{\lim_{x\to\infty} \frac{\ln x}{x}} = e^0 = 1.$$

6. Since $\displaystyle \frac{-|x|^n}{n!} \le \frac{x^n}{n!} \le \frac{|x|^n}{n!}$ for all x and $\overset{>0}{n}$,

it suffices to show that $\displaystyle\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$.

Choose N s.t. $N \geq |x|$. Then for all $n > N$,

$$0 \le \frac{|x|^n}{n!} = \frac{|x|^N}{N!} \underbrace{\frac{|x|}{N+1} \frac{|x|}{N+2} \cdots}_{<1} \frac{|x|}{n} \le \underbrace{\frac{|x|^N}{N!}}_{\text{Constant}} \cdot \frac{|x|}{n} \to 0$$

as $n \to \infty$.

Hence $\displaystyle\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ by Sandwich Theorem.

3, 4, 5: exercise. $\square$

Recursively Defined Sequences

e.g. 9 (10.1, ex 97) Given that the sequence

$$\left\{2,\ 2+\frac{1}{2},\ 2+\frac{1}{2+\frac{1}{2}},\ 2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}},\ \dots\right\}$$

(Side: the limit is a "continued fraction", commonly appeared in number theory.)

converges, find its limit.

Sol. • The sequence is given by $a_1 := 2$, $a_{n+1} := 2+\frac{1}{a_n}$ for $n \geq 1$.

• Let $L := \lim_{n\to\infty} a_n$. Then $L = \lim_{n\to\infty} a_{n+1}$.

• $a_{n+1} = 2+\frac{1}{a_n} \Rightarrow \lim_{n\to\infty} a_{n+1} = 2+\frac{1}{\lim_{n\to\infty} a_n} \Rightarrow L = 2+\frac{1}{L}$

$$\Rightarrow L^2 = 2L+1 \Rightarrow L^2-2L-1=0$$

$$\Rightarrow L = \frac{2\pm\sqrt{4+4}}{2} = 1\pm\sqrt{2}.$$

• Since $a_n > 0$ for all n, $L \geq 0$. So $L = 1+\sqrt{2}$.

Bounded Sequences

DEFINITIONS A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n. The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

e.g.

- $\{\frac{1}{n}\}$ is bounded above by 2, bounded below by -48.

  Its least upper bound is 1 and greatest lower bound is 0.

- $\{n\} = \{1, 2, 3, 4, \ldots\}$ is unbounded.

It follows that convergent sequences are all bounded.

> **Theorem** If $\{a_n\}$ is convergent, then $\{a_n\}$ is bounded.

Proof: • Let $L := \lim_{n\to\infty} a_n$.

- Then $\exists N$ s.t. $|a_n - L| < 1$ for all $n > N$.
- Let $K := \max\{|a_1 - L|, |a_2 - L|, \ldots, |a_N - L|, 1\}$.
- Then $|a_n - L| \leq K$ for all $n \geq 1$. This means that

$$L - K \leq a_n \leq L + K \quad , \quad \forall n \geq 1,$$

So $\{a_n\}$ is bounded. $\square$

Note that the converse is false : bounded $\not\Rightarrow$ Convergent ; e.g.,

Consider $a_n := (-1)^n$.