STA2001 Assignment 11

1. (5.7-2). Suppose that among gifted seventh-graders who score very high on a mathematics exam, approximately 20% are left-handed or ambidextrous. Let X equal the number of left-handed or ambidextrous students among a random sample of $n = 25$ gifted seventh-graders. Find $P(2 < X < 9)$.

   (a) Using Table II in Appendix B.
   (b) Approximately, using the central limit theorem

   Solution:

   (a) Note $n = 25$ and $p = 0.2$, and $X \sim b(25, 0.2)$. According to Table II in Appendix B (or you could directly compute by hand using the cdf of binomial distribution),

$$P(2 < X < 9) = P(X \leq 8) - P(X \leq 2) = 0.9532 - 0.0982 = 0.8550$$

   (b) Since $X \sim b(25, 0.2)$, the mean and variance of X can be easily obtained by

$$\begin{aligned} E(X) &= np = 25 \times 0.2 = 5 \\ \mathrm{Var}(X) &= npq = 25 \times 0.2 \times 0.8 = 4 \end{aligned}$$

   where $q = 1 - p$.

   Using central limit theorem,

$$\begin{aligned} P(2 < X < 9) &= P(3 \leq X \leq 8) \\ &\approx P\left(\frac{3 - 1/2 - 5}{2} \leq \frac{X - 5}{2} \leq \frac{8 + 1/2 - 5}{2}\right) \\ &\approx P(-1.25 \leq Z \leq 1.75) \\ &= 0.8543 \end{aligned}$$

   Note that we have applied half-unit correction for continuity here as X is a discrete random variable.

2. (5.7-12). If X is $b(100, 0.1)$, find the approximate value of $P(12 \leq X \leq 14)$, using

   (a) The normal approximation.
   (b) The Poisson approximation.
   (c) The binomial.

   Solution:

   (a) Since $X \sim b(100, 0.1)$, the mean and variance of X can be easily obtained by

$$\begin{aligned} E(X) &= np = 100 \times 0.1 = 10 \\ \mathrm{Var}(X) &= npq = 100 \times 0.1 \times 0.9 = 9 \end{aligned}$$

   By normal approximation (i.e. using CLT),

$$\begin{aligned} P(12 \leq X \leq 14) &= P\left(\frac{12 - 1/2 - 10}{3} \leq \frac{X - 10}{3} \leq \frac{14 + 1/2 - 10}{3}\right) \\ &\approx P(0.5 \leq Z \leq 1.5) \\ &= \Phi(1.5) - \Phi(0.5) \\ &= 0.9332 - 0.6915 \\ &= 0.2417 \end{aligned}$$

Note that we have applied half-unit correction for continuity here as X is a discrete random variable.

(b) Let $\lambda \approx np = 100 \times 0.1 = 10$. The Poisson approximation is given by (directly compute it using the Poisson cdf, or using Table III in Appendix B),

$$P(12 \le X \le 14) = P(X \le 14) - P(X \le 11) \approx 0.917 - 0.697 = 0.220$$

(c) We compute it directly

$$P(12 \le X \le 14) = \sum_{x=12}^{14} \binom{100}{x} (0.1)^x (0.9)^{100-x} = 0.2244$$

3. (5.7-18). Assume that the background noise X of a digital signal has a normal distribution with $\mu = 0$ volts and $\sigma = 0.5$ volt. If we observe $n = 100$ independent measurements of this noise, what is the probability that at least 7 of them exceed 0.98 in absolute value?

   (a) Use the Poisson distribution to approximate this probability.
   (b) Use the normal distribution to approximate this probability.
   (c) Use the binomial distribution to approximate this probability.

   Solution:
   Note that $X \sim N\left(0, 0.5^2\right)$. The probability that one single item exceeds 0.98 in absolute value is

$$\begin{aligned} P(|X| > 0.98) &= 1 - P(-0.98 \le X \le 0.98) \\ &= 1 - P\left(\frac{-0.98 - 0}{0.5} \le \frac{X - 0}{0.5} \le \frac{0.98 - 0}{0.5}\right) \\ &= 1 - P(-1.96 \le Z \le 1.96) \\ &= 1 - 0.95 \\ &= 0.05 \end{aligned}$$

   Let Y be the number of items that that exceed 0.98 in absolute value, so apparently $Y \sim b(100, 0.05)$.
   (a) let $\lambda \approx np = 5$. The Poisson approximation is given by (directly compute it using the Poisson cdf, or using Table III in Appendix B),

$$P(7 \le Y) = 1 - P(Y \le 6) \approx 1 - 0.762 = 0.238$$

   (b) Since $Y \sim b(100, 0.05)$, we have

$$\begin{aligned} E(Y) &= np = 5 \\ \mathrm{Var}(Y) &= npq = 4.75 \end{aligned}$$

   where $q = 1 - p$.
   So,

$$\begin{aligned} P(7 \le Y) &\approx P\left(\frac{7 - 1/2 - 5}{\sqrt{4.75}} \le \frac{Y - 5}{\sqrt{4.75}}\right) \\ &\approx P(0.688 \le Z) \\ &= 1 - P(Z < 0.688) \\ &= 0.2447 \end{aligned}$$

   Note that we have applied half-unit correction for continuity here as Y is a discrete random variable.
   (c) We compute it directly

$$P(7 \le Y) = 1 - P(Y \le 6) = 1 - \sum_{y=0}^{6} \binom{100}{y} (0.05)^y (0.95)^{100-y} = 1 - 0.7660 = 0.2340$$

4. (5.8-3). Let X denote the outcome when a fair die is rolled. Then $\mu = 7/2$ and $\sigma^2 = 35/12$. Note that the maximum deviation of X from μ equals $5/2$. Express this deviation in terms of the number of standard deviations; that is, find k, where $k\sigma = 5/2$. Determine a lower bound for $P(|X - 3.5| < 2.5)$.

   Solution:
   Solve the equation
   $$\frac{5}{2} = k\sigma$$
   we have $k = \frac{5}{2} \times \sqrt{\frac{12}{35}} = \sqrt{\frac{15}{7}}$.
   Apply Chebyshev's inequality,
   $$P\left(|X - 3.5| < 2.5\right) \geq 1 - \frac{1}{k^2} = 1 - \frac{7}{15} = \frac{8}{15}$$

5. (5.8-4). If the distribution of Y is $b(n, 0.5)$, give a lower bound for $P(|Y/n - 0.5| < 0.08)$ when
   (a) $n = 100$.
   (b) $n = 500$.
   (c) $n = 1000$.

   Solution:
   Since $Y \sim b(n, 0.5)$, $E(Y) = np = 0.5n$, $\text{Var}(Y) = npq = 0.25n$.
   Apply Chebyshev's inequality, we can write
   $$\begin{aligned} P(|Y/n - 0.5| < 0.08) &= P\left(|Y/n - 0.5|n < 0.08n\right) \\ &= P\left(|Y - 0.5n| < 0.08n\right) \\ &= P\left(|Y - \mu| < 0.08n\right) \\ &= P\left(|Y - \mu| < \frac{4\sqrt{n}}{25} \cdot 0.5\sqrt{n}\right) \\ &\geq 1 - \frac{625}{16n} \end{aligned}$$
   (a) If $n = 100$,
   $$P(|Y/100 - 0.5| < 0.08) \geq 1 - \frac{625}{16 \cdot 100} = 0.609$$
   (b) If $n = 500$,
   $$P(|Y/500 - 0.5| < 0.08) \geq 1 - \frac{625}{16 \cdot 500} = 0.922$$
   (c) If $n = 1000$,
   $$P(|Y/1000 - 0.5| < 0.08) \geq 1 - \frac{625}{16 \cdot 1000} = 0.961$$

6. (5.8-6). Let $\bar{X}$ be the mean of a random sample of size $n = 15$ from a distribution with mean $\mu = 80$ and variance $\sigma^2 = 60$. Use Chebyshev's inequality to find a lower bound for $P(75 < \bar{X} < 85)$.

   Solution:
   Since $\bar{X}$ is the mean of a random sample of size $n = 15$ from a distribution with mean $\mu = 80$ and

variance $\sigma^2 = 60$, we have

$$E(\bar{X}) = \mu = 80$$
$$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n} = 4$$

Apply Chebyshev's inequality,

$$\begin{aligned}
P(75 < \bar{X} < 85) &= P(75 - 80 < \bar{X} - 80 < 85 - 80) \\
&= P(-5 < \bar{X} - 80 < 5) \\
&= P\left(\left|\bar{X} - 80\right| < 5\right) \\
&= P\left(\left|\bar{X} - 80\right| < \frac{5}{2} \cdot 2\right) \\
&\geq 1 - \frac{4}{25} \\
&= 0.84
\end{aligned}$$

Note that $k = \frac{5}{2}$ and $\sigma_{\bar{X}} = 2$.

7. (5.9-1). Let Y be the number of defectives in a box of 50 articles taken from the output of a machine. Each article is defective with probability 0.01. Find the probability that $Y = 0, 1, 2$, or 3.

   (a) By using the binomial distribution.
   (b) By using the Poisson approximation.

   Solution:
   (a) Note that $Y \sim b(50, 0.01)$. We compute the probability directly,

$$P(Y \leq 3) = \sum_{y=0}^{3} \binom{50}{y} (0.01)^y (0.99)^{50-y} = 0.9984$$

   (b) Let $\lambda \approx np = 50 \times 0.01 = 0.5$, then we compute the probability using the Poisson cdf,

$$P(Y \leq 3) \approx \sum_{k=0}^{3} \frac{0.5^k e^{-0.5}}{k!} = 0.9982$$

8. (5.9-4). Let Y be $\chi^2(n)$. Use the central limit theorem to demonstrate that $W = (Y - n)/\sqrt{2n}$ has a limiting cdf that is $N(0, 1)$. Hint: Think of Y as being the sum of a random sample from a certain distribution.

   Solution:
   Recall the construction of the random variable $\chi^2(n)$, we may let $Y = \sum_{i=1}^{n} X_i$, where $X_1, X_2, \ldots, X_n$ is a random sample of size n from the same distribution $\chi^2(1)$.
   Therefore, $X_1, X_2, \ldots, X_n$ are i.i.d and

$$\mu_Y = n \cdot E(X_i) = n \cdot 1 = n$$
$$\sigma_Y^2 = n \cdot \sigma_{X_i}^2 = n \cdot 2 = 2n$$

The conditions of central limit theorem are satisfied (Y is a sum of independent random variables, and we minus μ_Y and then divide σ_Y to construct W), so the random variable

$$W = \frac{Y - n}{\sqrt{2n}} = \frac{Y - \mu_Y}{\sigma_Y}$$

has a limiting distribution $N(0, 1)$.

9. (5.9-5). Let Y have a Poisson distribution with mean $3n$. Use the central limit theorem to show that the limiting distribution of $W = (Y - 3n)/\sqrt{3n}$ is $N(0, 1)$.

Solution:

First, we state a fact that is useful to solve this question

If $X_1 \sim \text{Possion}(\lambda_1)$, $X_2 \sim \text{Possion}(\lambda_2)$ and they are independent, then $X_1 + X_2 \sim \text{Possion}(\lambda_1 + \lambda_2)$.

This can be proved by computing the moment-generating function of $(X_1 + X_2)$, and apparently it can be generalized to a finite sum of independent Poisson random variables.

Now, using this fact we may let $Y = \sum_{i=1}^{n} X_i$, where $X_1, X_2, \ldots, X_n$ are i.i.d random variables which follow a Poisson distribution with mean 3 (or equivalently, a random sample of size n from the Poisson distribution with mean 3).

Therefore,

$$\mu_Y = n \cdot E(X_i) = n \cdot 3 = 3n$$
$$\sigma_Y^2 = n \cdot \sigma_{X_i}^2 = n \cdot 3 = 3n$$

The conditions for central limit theorem are satisfied, so the random variable

$$W = \frac{Y - 3n}{\sqrt{3n}} = \frac{Y - \mu_Y}{\sigma_Y}$$

has a limiting distribution $N(0, 1)$.

10. Let Z be a random variable, $X \sim N(1, 1)$, and $Z_n = Z + \frac{1}{n}X$ with $n = 1, 2, \cdots$. Prove that the sequence of random variables $Z_1, Z_2, \cdots$, converges in probability to Z, i.e., $Z_n \xrightarrow{p} Z$.

Solution:

It is to prove that for any $\epsilon > 0$,

$$\lim_{n \to \infty} P(|Z_n - Z| \geq \epsilon) = 0.$$

First, let $Y_n = \frac{1}{n}X$ and then $\text{E}(Y_n) = \frac{1}{n}, \text{Var}(Y_n) = \frac{1}{n^2}$. Then note that

$$|Y_n| = |Y_n - \text{E}(Y_n) + \text{E}(Y_n)| \leq |Y_n - \text{E}(Y_n)| + |\text{E}(Y_n)| = |Y_n - \frac{1}{n}| + \frac{1}{n}.$$

Now for any $\epsilon > 0$, we have

$$\begin{aligned} 0 \leq P(|Z_n - Z| \geq \epsilon) = P(|Y_n| \geq \epsilon) &\leq P(|Y_n - \frac{1}{n}| + \frac{1}{n} \geq \epsilon) \\ &= P(|Y_n - \frac{1}{n}| \geq \epsilon - \frac{1}{n}) \end{aligned}$$

Then by using Chebyshev inequality, we have

$$0 \leq P(|Z_n - Z| \geq \epsilon) \leq \frac{1}{(n\epsilon - 1)^2},$$

where the right hand side goes to 0 as n goes to ∞. Therefore, we conclude

$$\lim_{n \to \infty} P(|Z_n - Z| \geq \epsilon) = 0.$$