

MAT 1001 Midterm Exam, 10:00am -Noon, March 27, 2021

Your Name and Student ID:

Your Lecture Class(e.g, L1) and **your tutorial class** (e.g, T01):

Instruction: (i) This is a closed-book and closed-notes exam; no calculators, no dictionaries and no cell phones; (ii) Show your work unless otherwise instructed—a correct answer without showing your work when required shall be given no credits; (iii) Write down ALL your work and your answers(including the answers for short questions) in the Answer Book.

1. (27 points) **Short Questions** (for these questions, NO need to show your work, just write down your answers in the Answer Book; NO partial credits for each question)

(i). Which of the following statements is false?

- (a) Let f be a continuous function defined for all real numbers. If $a_1 = a$ and $a_{n+1} = f(a_n)$ (for all $n \geq 1$) define a convergent sequence $\{a_n\}$, then f has a fixed point (i.e. $f(x_0) = x_0$ for some x_0).
- (b) If the sequence $\{a_1, a_3, a_5, \dots\}$ converges to L_1 and $\{a_2, a_4, a_6, \dots\}$ converges to L_2 with $L_1 \neq L_2$, then $\{a_n\}_{n \geq 1}$ cannot converge.
- (c) If $\sum |a_n|$ diverges, then $\sum a_n$ cannot converge.

(ii). Which of the following statements is false?

- (a) Given an alternating series, if it does not satisfy the conditions in the alternating series test, then it must diverge.
- (b) Rearranging the terms (which means changing the order of the terms) in $\sum_{n=1}^{\infty} (-1)^n (1/n^\pi)$ will never change the value of the series.
- (c) If $\sum a_n$ is convergent but $\sum b_n$ is divergent, then $\sum (a_n + b_n)$ must be divergent.

(iii). Which of the following is the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{6(5+\sin x)^n}$?

- (a) $\{x \mid x \text{ is not an integer multiple of } 2\pi\}$.
- (b) $\{x \mid x \text{ is not an integer multiple of } \pi\}$.
- (c) $-\infty < x < \infty$.
- (d) diverges for all x .

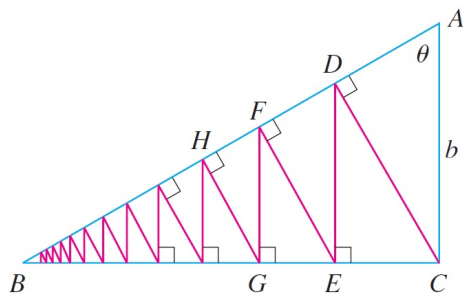
(iv). Which of the following is the Maclaurin series for $\cos 2x$?

- (a) $\sum_{n=0}^{\infty} \frac{(-1)^{2n} 2^{2n} x^{2n}}{n!}$.
- (b) $\sum_{n=0}^{\infty} \frac{(-1)^{2n} 2^{2n} x^{2n}}{(2n)!}$.
- (c) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{n!}$.
- (d) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$.
- (v). Which of the following is the Taylor series generated by $f(x) = \frac{1}{x^2}$ at $x = 8$?
- (a) $\sum_{n=0}^{\infty} \frac{(-1)^n n(x-8)^n}{8^{n+2}}$.
- (b) $\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(x-8)^n}{8^{n+1}}$.
- (c) $\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(x-8)^n}{8^{n+2}}$.
- (d) $\sum_{n=0}^{\infty} \frac{(-1)^n n(x-8)^n}{8^{n+1}}$.
- (vi). $\frac{2}{3} - \frac{2^3}{3^3 \cdot 3!} + \frac{2^5}{3^5 \cdot 5!} - \frac{2^7}{3^7 \cdot 7!} + \dots =$
- (a) $\sin \frac{3}{2}$.
- (b) $\cos \frac{2}{3}$.
- (c) $\ln \frac{2}{3}$.
- (d) $\sin \frac{2}{3}$.
- (vii). Let \vec{u} , \vec{v} and \vec{w} be nonzero vectors in space. Which of the following statements is false?
- (a) $\vec{u} \times \vec{v} = \vec{0}$ is equivalent to $\vec{u} = k\vec{v}$ for some scalar k .
- (b) The vector $\vec{u} \times (\vec{v} \times \vec{w})$ is parallel to the plane spanned by \vec{v} and \vec{w} .
- (c) $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|$ if \vec{u} and \vec{v} are parallel.
- (viii). Which of the following statements is false?
- (a) If we want to expand a function $f(x)$ as a power series about $x = 0$ in the interval $(-1, 1)$, then necessarily f has to be infinitely differentiable on the interval (i.e., derivative of all orders of f exist on the interval).
- (b) If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $x \in (-1, 1)$, then $f(x) = \sum_{n=0}^{2021} c_n x^n + O(x^{2022})$ as $x \rightarrow 0$.
- (c) If $f(0) = 0 = f^{(n)}(0)$ for all positive integers n , then f has to be identically equal to 0, at least in a small open interval containing 0.
- (ix). Write down the parametric equations of the line tangent to the curve $\vec{r}(t) = \cos(e^t)\vec{i} + (3 - t^2)\vec{j} + t\vec{k}$ at $t = 0$:
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2. (5 points) In the triangle ACB below, the angle at corner C is a right angle, and line segments CD , EF , GH , etc., are parallel, while AC , DE , FG , etc., are parallel. The drawing $CDEFGHI\dots$ continues indefinitely. Is the length

$$|CD| + |DE| + |EF| + |FG| + |GH| + \dots$$

finite? If so, find it.



3. (24 points) For each of the following series, determine whether it is convergent absolutely, convergent conditionally, or divergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{n + n \cos^2 n}$. cos² n < 1

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$. ln n > 1 n > 3

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \dots \cdot (3n-1)}$. 比值

(d) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$, where $0 < p < 1$. 积分

4. (8 points) Determine the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{2^n x^n}{4^n + 1}$. (-2, 2)

5. (8 points) Find

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{(1 + x^3)^\pi - 1}.$$

6. (10 points) Let $f(x)$ be continuously differentiable on the finite interval $[a, b]$ (i.e., f' exists and is continuous on $[a, b]$), and suppose f'' exists on the open interval (a, b) . Prove the following special case of Taylor's Theorem: there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)}{2!}(b-a)^2.$$

7. (24 points) Consider the cycloid parametrized by

$$x = t - \sin t, \quad y = 1 - \cos t, \quad t \in [0, 2\pi].$$

- (a) Find its arclength.

- (b) Find the area of the surface generated by revolving the cycloid about the x -axis.
- (c) Find the curvature of the cycloid at $t = \pi$.
- (d) Let the y -axis **point downward** (and the x -axis be horizontal, pointing to the right). Consider a particle sliding frictionlessly on the cycloid under the influence of gravity, from the origin $O = (0, 0)$ to the bottom $B = (\pi, 2)$ of the cycloid, with 0 initial speed. Find the time T it takes for the particle to reach B from O . Length is measured in meters, and time in seconds.
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8. (15 points) Consider the polar curve $r = \sin(2\theta)$, $\theta \in [0, \pi]$.
- (a) Sketch the curve on the xy -plane.
- (b) Compute the slope of the curve at $\theta = \pi/4$.
- (c) Find the area of the region bounded by the polar curve for $0 \leq \theta \leq \pi/2$.
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9. (16 points) Let $P = (2, 4, 5)$, $Q = (1, 5, 7)$ and $R = (-1, 6, 8)$.
- (a) Find the area of the triangle ΔPQR ;
- (b) Find an equation of the plane containing the triangle mentioned above;
- (c) Find parametric equations of the line which is perpendicular to the plane mentioned above, and passes through the origin $(0, 0, 0)$.
- (d) Find the distance from the origin to the plane mentioned above.
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10. (16 points) Let $\vec{r}(t)$, $-\infty < t < \infty$, be the position vector function of a particle moving, with positive speed, along a smooth curve C in space. Suppose $\vec{r}(t)$ is twice differentiable (which means the second order derivative exists) for all t .
- (a) Prove that the speed of the particle is constant if and only if the velocity vector $\vec{v}(t)$ is always orthogonal to the acceleration vector $\vec{a}(t)$.
- (b) Prove that if the curvature of curve C is 0 everywhere, then the curve has to be a straight line.
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