MAT1002 Lecture 2, Tuesday, Jan/10/2023

Outline

- Infinite Sequences (10.1)
  - ↳ Bounded monotonic sequences
- Infinite Series (10.2)
  - ↳ Definitions
  - ↳ Algebraic properties
  - ↳ The n^{th}-term test
  - ↳ Integral test (10.3)
    - ↳ Approximation

Bounded Monotonic Sequences

or weakly decreasing

or weakly increasing

DEFINITIONS A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all n. That is, $a_1 \leq a_2 \leq a_3 \leq \ldots$. The sequence is **nonincreasing** if $a_n \geq a_{n+1}$ for all n. The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

or monotone

e.g. • Any constant sequence is both nondecreasing and nonincreasing.

• $\{(-1)^n\}$ is not monotonic.

Another name: Monotone Convergence Theorem

THEOREM ~~6~~ 10.1.6—The Monotonic Sequence Theorem If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

The proof is optional.

Proof: • Suppose $\{a_n\}$ is nondecreasing. (The other case: exercise.)

• Since $\{a_n\}$ is bounded, it is bounded above. Let M be the least upper bound for $\{a_n\}$. Will show that $a_n \to M$ as $n \to \infty$.

• Let $\varepsilon > 0$ be arbitrary. Then $M - \varepsilon$ is not an upper bound for $\{a_n\}$ (since M is the least), so $\exists N$ s.t. $a_N > M - \varepsilon$.

• Since $\{a_n\}$ is nondecreasing, $a_n \geq a_N > M - \varepsilon$ $\forall n > N$.

• Also, $a_n \leq M$, $\forall n$.

• Therefore, $\forall n > N$, $M - \varepsilon < a_n \leq M$, so $|a_n - M| < \varepsilon$.

• By def, $\lim_{n \to \infty} a_n = M$. $\square$

Infinite Series

Consider "adding" infinitely many numbers of the form $1/2^n$:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}+\ldots.$$

How do we formalize these?

One might argue that, intuitively, this "sum" in equal to 2.

On the other hand, "adding" infinitely many numbers does not always give a number: intuitively, the "sum" $1+1+1+1+\ldots$ tends to infinity.

Such "infinite sums" are formally understood as series.

Def Given a sequence $\{a_n\}_{n=1}^{\infty}$, define $S_k := \sum_{n=1}^{k} a_n$ for $k \geq 1$, and

$$\sum_{n=1}^{\infty} a_n := \lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} S_k.$$

The symbol $\sum_{n=1}^{\infty} a_n$ is called an infinite series (or series) of $\{a_n\}$. The number a_n is called the n^{th} term of the series, and S_k is called the k^{th} partial sum. The series is said to be convergent if the limit exists (as a real number), and is said to be divergent otherwise.

Remark

(or $\sum_n a_n$)

It is common to use $\sum_n a_n$ or $\sum a_n$ to denote a series for short. Also, sometimes the initial index is an integer other than 1.

↳ e.g. $\sum_{n=89}^{\infty} a_n$ is also a series e.g. (a) For $a_n \equiv a$ (Constant sequence)

$$\sum_{n=1}^{\infty} a = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \underbrace{a+\cdots+a}_{k \text{ times}} = \lim_{k\to\infty} ka = \begin{cases} 0, & \text{if } a=0 \\ \infty, & \text{if } a>0 \\ -\infty, & \text{if } a<0 \end{cases}$$

Serie is divergent

In particular, $\sum_{n=1}^{\infty} 1 = 1+1+1+\ldots$ diverges.

(b) Given any constant a and r (with $a \neq 0$), the series

$$\sum_{n=1}^{\infty} ar^{n-1} = a+ar+ar^2+ar^3+\cdots$$

is called a geometric series. If $r \neq 1$, then

$$S_k = a(1+r+\cdots+r^{k-1}) = a\frac{1-r^k}{1-r},$$

So $\lim_{k\to\infty} S_k = \begin{cases} \frac{a}{1-r}, & \text{if } |r|<1; \\ \text{D.N.E.}, & \text{if } |r|>1. \end{cases}$

(In particular, $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} = 2.$)

If $r=1$, then $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} a$ diverges.

Geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ $\begin{cases} \text{converges} & \text{if } \|r\|<1; \\ \text{diverges} & \text{if } \|r\|\geq 1. \end{cases}$

(c) $0.\overline{9} = 0.999\ldots = \sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n = \frac{1}{10}\sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^{n-1} = \frac{1}{10}\frac{9}{1-\frac{1}{10}} = 1.$

$\therefore 0.\overline{9} = 1.$

(d) For series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$, Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$,

$$S_k = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1-\frac{1}{k+1},$$

So $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \lim_{k\to\infty}\left(1-\frac{1}{k+1}\right) = 1$. This is an example of a telescoping series, which is a series of the form $\sum_n (a_n - a_{n+1})$.

Remark

- A series can be written in many ways by index shifting:

$$a + ar + ar^2 + ar^3 + \ldots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

- Removing finitely many terms from a series does not affect its convergence.

Algebraic Properties

THEOREM ~~8~~ 5.2.8 If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (any number k).

Proof:

1. $$\sum_{n=1}^{\infty}(a_n+b_n) = \lim_{k\to\infty}\sum_{n=1}^{k}(a_n+b_n) = \lim_{k\to\infty}\left(\sum_{n=1}^{k}a_n+\sum_{n=1}^{k}b_n\right)$$
$$= \left(\lim_{k\to\infty}\sum_{n=1}^{k}a_n\right)+\left(\lim_{k\to\infty}\sum_{n=1}^{k}b_n\right) = \sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n.$$

2 & 3 are similar. $\square$

Remark:

- If $k\neq 0$ is a constant, then
$$\sum a_n \text{ diverges} \Leftrightarrow \sum ka_n \text{ diverges}.$$
- If $\sum a_n$ converges but $\sum b_n$ diverges, then $\sum(a_n+b_n)$ diverges.

Convergence Tests

Unlike geometric series, it is generally hard to find the exact value of a series. We will discuss various tests for testing whether a series converges or not without computing its value.

The n^{th} Term Test

Suppose that $\sum a_n$ converges to S. Consider the partial sums. Then
$$\lim_{k\to\infty} S_k = S = \lim_{k\to\infty} S_{k-1}.$$
Since $a_n = S_n - S_{n-1}$ for all n, we have
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

Theorem (The n-th Term Test)

If a series $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$; in other words, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Example

(a) The series $\displaystyle\sum_{n=1}^{\infty} \frac{e^n - 2}{4e^n + 5}$ diverges since $\displaystyle\lim_{n\to\infty} \frac{e^n - 2}{4e^n + 5} = \frac{1}{4} \neq 0$.

(b) The series $\displaystyle\sum_{n=1}^{\infty} \sin\frac{n\pi}{2}$ diverges since $\displaystyle\lim_{n\to\infty} \sin\frac{n\pi}{2}$ does not exist.

Note that the converse of the n-th term test is false. That is, just having $a_n \to 0$ as $n \to \infty$ does not imply that $\sum a_n$ converges.

(c) The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

is called the harmonic series. Although $1/n \to 0$ as $n \to \infty$, this series is divergent.

Q: Why is the harmonic series divergent?

Series with Nonnegative Terms

Consider $\sum a_n$ with $a_n \geq 0$ $\forall n$. Then its partial sum sequence $\{S_k\}$ is nondecreasing, which converges if and only if it is bounded. Furthermore, since $S_k \geq 0$ $\forall k$, $\{S_k\}$ is bounded below automatically.

<u>Theorem</u> Let $\sum a_n$ be a series of nonnegative terms a_n. Then $\sum a_n$ converges if and only if its partial sum sequence is bounded above.

Integral Test

Theorem (Integral Test) $(N \geq 1)$

Suppose that $a_n = f(n) \geq 0$ for all n satisfying $n \geq N$ (with N being a fixed integer), where f is a nonincreasing continuous function on $[N, \infty)$. Then $\sum a_n$ converges if and only if the improper integral $\int_N^\infty f(x)\,dx$ converges.

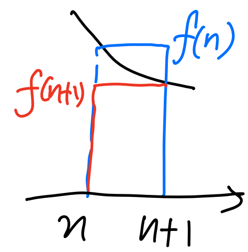
integrable is enough.

<u>Remark</u> Note that the conditions above imply that $f(x) \geq 0$, $\forall x \in [N, \infty)$.

Proof: • Note that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\underbrace{\sum_{n=n_0}^{\infty} a_n}_{\text{a "tail"}}$ converges.

• For any $n \geq N$,

$$a_{n+1} = f(n+1) \leq \int_n^{n+1} f(x)\,dx \leq f(n) = a_n$$



• Hence

$$\sum_{n=N+1}^{N+k} a_n \leq \underbrace{\int_N^{N+k} f(x)\,dx}_{= \int_N^{N+1} + \int_{N+1}^{N+2} + \cdots + \int_{N+k-1}^{N+k}} \leq \sum_{n=N}^{N+k-1} a_n .$$

• If $\int_N^{\infty} f(x)\,dx$ converges, since $\sum_{n=N+1}^{N+k} a_n \leq \int_N^{N+k} f(x)\,dx$, the partial sums of $\sum a_n$ are bounded. Hence $\sum a_n$ converges.

(above by $\int_N^{\infty} f(x)\,dx$)

• If $\int_N^{\infty} f(x)\,dx$ diverges, then it must diverge to ∞. Since

$$\underbrace{\int_N^{N+k} f(x)\,dx}_{\text{arbitrarily big}} \leq \sum_{n=N}^{N+k-1} a_n ,$$

By theorem.

the partial sums of $\sum a_n$ is unbounded. Hence $\sum a_n$ diverges.

$\square$

e.g. a) The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is fixed, is called a p-series.

Case 1. If $p \leq 0$, then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so series diverges.

Case 2. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is positive and decreasing on $[1, \infty)$.

Since $\int_1^{\infty} \frac{1}{x^p} dx$ converges for $p > 1$ and diverges for $0 < p \leq 1$

(done in MAT1001),

by integral test on Case 2, together with Case 1, we have

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges}, & \text{if } p > 1; \\ \text{diverges}, & \text{if } p \leq 1. \end{cases}}$$

In particular, the harmonic series $(p = 1)$ diverges.

b) Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is convergent.

Integral Test : Approximation

Suppose that we have a convergent series $\sum a_n$ satisfying the conditions in the integral test, and suppose that $\sum a_n = S$. Consider the error term $R_N := S - s_N$. Then

i.e., $\int_N^{\infty} f(x) dx$ also converges.

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \ldots = \sum_{n=N+1}^{\infty} a_n.$$

By the argument in the proof of the integral test, it follows that

$$\int_{N+1}^{\infty} f(x)\, dx \leq R_N \leq \int_N^{\infty} f(x)\, dx.$$

If we add s_N to the inequality above, we get

$$s_N + \int_{N+1}^{\infty} f(x)\,dx \le S \le s_N + \int_{N}^{\infty} f(x)\,dx. \qquad (1)$$

This gives an interval I in which S lies, where $S = \sum a_n$. Hence, if we take the midpoint of I as an approximation of $\sum a_n$, the error is at most half the length of I, which is

$$\frac{1}{2}\left(\int_{N}^{\infty} f(x)\,dx - \int_{N+1}^{\infty} f(x)\,dx\right).$$

Note that this error is at most $a_N/2$. (Why?)

Example

Approximate the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ using the method above ~~on the previous slide,~~ with $N = 10$. Find an upper bound for the error.

Solution

- An easy computation shows that $\int_N^\infty (1/x^2)\,dx = 1/N$.
- By Inequality (1) on the previous slide, we have

$$s_{10} + \frac{1}{11} \le S \le s_{10} + \frac{1}{10}.$$

- $s_{10} = 1 + 1/4 + 1/9 + \ldots + 1/100 \approx 1.54977$, so

$$1.64068 \le S \le 1.64977.$$

  Then we take the midpoint of the interval, which is 1.645225. The error of this is at most $0.5 \times (1.64977 - 1.64068)$, which is 0.004545.

Fact: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.644934\ldots$. Proof of this is not elementary (no need to know for this course).