Review of Chapters 4-5

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Outline

1. Bivariate Random Variable
2. Multivariate Random Variable

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1. Bivariate Random Variable
2. Multivariate Random Variable

Joint pmf and joint pdf

Motivation: the outcome of the random experiment is a pair of two scalars.

Given a pair of discrete or continuous RVs (X,Y) taking values in $\overline{S}$, we define accordingly a pmf or pdf:

1. pmf for discrete RV: $\quad f(x,y) : \overline{S} \to (0,1]$
   (1) $f(x,y) > 0, \quad (x,y) \in \overline{S}$
   (2) $\sum_{(x,y)\in\overline{S}} f(x) = 1$
   (3) $P((X,Y) \in A) = \sum_{(x,y)\in A} f(x,y), \quad A \subseteq \overline{S}$
2. pdf for continuous RV: $\quad f(x,y) : \overline{S} \to (0,\infty)$
   (1) $f(x,y) > 0, \quad (x,y) \in \overline{S}$
   (2) $\iint_{\overline{S}} f(x,y)dxdy = 1$
   (3) $P((X,Y) \in A) = \iint_{A} f(x,y)dxdy, \quad A \subseteq \overline{S}$

Mathematical Expectation

$$E[u(X,Y)] = \begin{cases} \sum_{(x,y)\in\overline{S}} u(x,y)f(x,y), & \text{discrete RV} \\ \iint_{\overline{S}} u(x,y)f(x,y)dxdy, & \text{continuous RV} \end{cases}$$

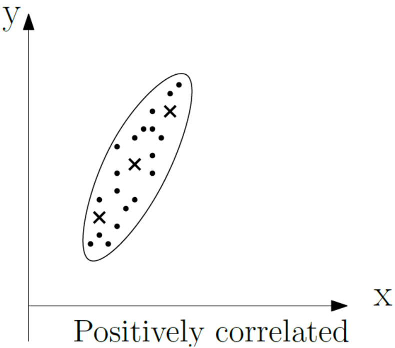
- Covariance and Correlation Coefficient

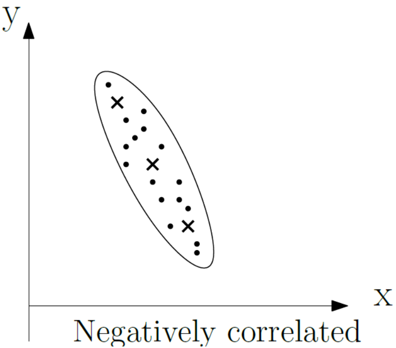
$$Cov(X,Y) = E[(X-E[X])(Y-E[Y])]$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(x)}\sqrt{Var(Y)}}, \quad |\rho| \le 1$$

Covariance and Correlation Coefficient

1. $Cov(X, Y) > 0, (X - E[X])$ and $(Y - E[Y])$ tend to have the same sign; $\rho = 1 \Rightarrow X - E[X] = c(Y - E[Y])$ with $c > 0$
2. $Cov(X, Y) < 0, (X - E[X])$ and $(Y - E[Y])$ tend to have the opposite sign; $\rho = -1 \Rightarrow X - E[X] = c(Y - E[Y])$ with $c < 0$





Independent Random Variables

X and Y are said to be independent if

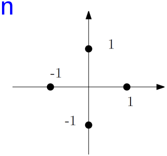
$$f(x, y) = f_X(x) f_Y(y)$$

A necessary condition for X and Y to be independent

$$\overline{S} = \overline{S_X} \times \overline{S_Y}$$

$$\boxed{\text{Independence} \Rightarrow \text{Uncorrelation}}$$

1. The Converse is not true in general.
2. The Converse is however true for multivariate normal (Gaussian) distribution



Marginal distributions

Given two discrete or continuous RVs (X,Y) taking values in $\overline{S}$ and their joint pmf or joint pdf $f(x,y)$, define accordingly the marginal pmf or marginal pdf to assign the probability of events for RV X:

$$\overline{S_X} = \{\text{all possible values of } X\}, \overline{S_Y}(x) = \left\{y|(x,y) \in \overline{S}\right\}, \text{for } x \in \overline{S_X}$$
$$\overline{S_Y} = \{\text{all possible values of } Y\}, \overline{S_X}(y) = \left\{x|(x,y) \in \overline{S}\right\}, \text{for } y \in \overline{S_Y}$$

1. marginal pmf for discrete RV: $f_X(x) : \overline{S_X} \to (0,1]$

$$f_X(x) = \sum_{y \in \overline{S_Y}(x)} f(x,y)$$

2. marginal pdf for continuous RV: $f_X(x) : \overline{S_X} \to (0,\infty)$

$$f_X(x) = \int_{\overline{S_Y}(x)} f(x,y)dy$$

Conditional Distributions

Conditional pmf and Conditional pdf

Given two discrete or continuous RVs (X,Y) taking values in $\overline{S}$ and their joint pmf or joint pdf $f(x, y)$, and marginal pmf or marginal pdf $f_X(x)$, define accordingly the conditional pmf or conditional pdf to assign the probability of events for RV Y given that $X = x$:

$$h(y|x) = \frac{f(x,y)}{f_X(x)}, \quad f_X(x) > 0, y \in \overline{S_Y}(x)$$

In particular, for $A \subseteq \overline{S_Y}(x)$,

$$P(Y \in A|X = x) = \sum_{y \in A} h(y|x), \quad \text{or} \quad \int_{y \in A} h(y|x)dy$$

Conditional mathematical expectation

$$E[u(Y)|X = x] = \begin{cases} \sum_{y \in \overline{S_Y}(x)} u(y)h(y|x), & \text{discrete RV} \\ \int_{\overline{S_Y}(x)} u(y)h(y|x)dy, & \text{continuous RV} \end{cases}$$

Bivariate Normal Distribution

Definition

Let X and Y be 2 continuous RVs and have the joint pdf

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp[-\frac{1}{2}q(x,y)], x \in R, y \in R$$

where $|\rho| < 1$ and

$$q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] \geq 0$$

Then X and Y are said to be bivariate normally distributed.

Key components: Scaled exponential function with a quadratic and negative function as its exponent.

Bivariate Normal Distribution

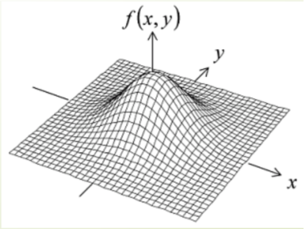
1. Marginal distribution is normal

$$X \sim N(\mu_X, \sigma_X^2) \text{ and } Y \sim N(\mu_Y, \sigma_Y^2)$$

2. Conditional distribution is normal
   given $X = x$, Y is normally distribution with

$$E[Y|X=x] = \mu_Y + \rho\sigma_Y \frac{x-\mu_X}{\sigma_X},$$
$$Var(Y|X=x) = (1-\rho^2)\sigma_Y^2.$$

3. Uncorrelation is equivalent to independence



Outline

1. Bivariate Random Variable
2. Multivariate Random Variable

Function of One Random Variable

Assume $Y = u(X)$ has an inverse function $X = v(Y)$ and for continuous case, further assume $Y = u(X)$ is strictly increasing or decreasing and $v'(y)$ exists. Then the pmf or pdf of Y is

$$g(y) = \begin{cases} f(v(y)), \quad y \in u(\overline{S}) & \text{discrete RV} \\ f(v(y))|v'(y)|, y \in u(\overline{S}) & \text{continuous RV} \end{cases}$$

For continuous RV:

1. cdf of Y: $G(y) = P(Y \le y), \qquad y \in u(\overline{S})$
2. pdf of Y: $g(y) = G'(y), \qquad y \in u(\overline{S})$

Multivariate Random Variable

Motivation: the outcome of the random experiment is a tuple of several scalars.

Joint pmf and joint pdf:

1. pmf for discrete RV: $f(x_1, \cdots, x_n) : \overline{S} \to (0, 1]$
   (1) $f(x_1, \cdots, x_n) > 0, \quad (x_1, \cdots, x_n) \in \overline{S}$
   (2) $\sum_{(x_1, \cdots, x_n) \in \overline{S}} f(x_1, \cdots, x_n) = 1$
   (3) $P((X_1, \cdots, X_n) \in A) = \sum_{(x_1, \cdots, x_n) \in A} f(x_1, \cdots, x_n), \quad A \subset \overline{S}$
2. pdf for continuous RV: $f(x_1, \cdots, x_n) : \overline{S} \to (0, \infty)$
   (1) $f(x_1, \cdots, x_n) > 0, \quad (x_1, \cdots, x_n) \in \overline{S}$
   (2) $\int_{\overline{S}} f(x_1, \cdots, x_n) dx_1, \cdots, dx_n = 1$
   (3) $P((X_1, \cdots, X_n) \in A) = \int_A f(x_1, \cdots, x_n) dx_1, \cdots, dx_n, \quad A \subset \overline{S}$

Independent Random Variables

RVs $X_1, X_2, \cdots, X_n$ are said to be independent if

$$f(x_1, x_2, \cdots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

A necessary condition for $X_1, X_2, \cdots, X_n$ to be independent

$$\overline{S} = \overline{S}_{X_1} \times \cdots \overline{S}_{X_n}$$

Random sample of size n from a common distribution: n independent and identically distributed, i.i.d., RVs $X_1, X_2, \cdots, X_n$.

Theorem 5.3-1

Assume $X_1, X_2, \cdots, X_n$ are independent random variables and $Y = u_1(X_1)u_2(X_2)\cdots u_n(X_n)$. If $E[u_i(X_i)]$, $i = 1, 2, \cdots, n$, exist, then

$$E(Y) = E[u_1(X_1)\cdots u_n(X_n)] = E[u_1(X_1)]\cdots E[u_n(X_n)]$$

Independent Random Variables

Theorem 5.3-2

If $X_1, X_2, \cdots, X_n$ are independent random variables with respective means $\mu_1, \mu_2, \cdots \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2$, then the mean and the variance of $Y = \sum_{i=1}^n a_i X_i$, where $a_1, a_2, \cdots, a_n$ are constants, are, respectively,

$$E(Y) = \sum_{i=1}^n a_i \mu_i \quad \text{and} \quad Var(Y) = \sum_{i=1}^n a_i^2 \sigma_i^2.$$

Moment Generating Function Technique

Mgf, if exists, uniquely determines the distribution of the RV. So distribution of a random variable can be found through its mgf, e.g., normal or Gaussian $N(\mu, \sigma^2)$: $M(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$

Theorem 5.4-1

If $X_1, X_2, \cdots, X_n$ are independent random variables with respective moment generating functions $M_{X_i}(t), i = 1, 2, 3, \cdots, n$, where $-h_i < t < h_i, i = 1, 2, \cdots, n$, for positive numbers $h_i, i = 1, 2, \cdots, n$, then the moment-generating function of $Y = \sum_{i=1}^{n} a_i X_i$ is

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t), \text{where} -h_i < a_i t < h_i, i = 1, 2, \cdots, n.$$

Moment Generating Function Technique

Theorem 5.4-2

Let $X_1, X_2, \cdots, X_n$ be independent chi-square random variables with $r_1, r_2, \cdots, r_n$ degrees of freedom, respectively. Then $Y = X_1 + X_2 + \cdots + X_n$ is $\chi^2(r_1 + r_2 + \cdots + r_n)$.

Corollary 5.4-2

Let $Z_1, Z_2, \cdots, Z_n$ have standard normal distributions, $N(0, 1)$. If these random variables are independent, then $W = Z_1^2 + Z_2^2 + \cdots + Z_n^2$ has a distribution that is $\chi^2(n)$.

Functions of Multivariate Normal RVs(1/2)

Theorem 5.5-1

If $X_1, X_2, \cdots, X_n$ are n independent normal variables with means $\mu_1, \mu_2, \cdots, \mu_n$ and variances σ_1^2, σ_2^2, $\cdots$, σ_n^2 , respectively, then $Y = \sum_{i=1}^{n} a_i X_i$ has the normal distribution

$$Y \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

Functions of Multivariate Normal RVs(1/2)

Theorem 5.5-2

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the normal distribution $N(\mu, \sigma^2)$. Then the sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and the sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ [$E(S^2) = \sigma^2$] are independent, and

$$\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i - \overline{X}}{\sigma}\right)^2 \sim \chi^2(n-1)$$

$$\sum_{i=1}^{n}\left(\frac{X_i - \overline{X}}{\sigma}\right)^2 \sim \chi^2(n-1)$$

$$\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2(n)$$

Functions of Multivariate Normal RVs(2/2)

Theorem 5.5-3 (Student's t distribution)

Let

$$T = \frac{Z}{\sqrt{U/r}}$$

where $Z \sim N(0,1), U \sim \chi^2(r)$, and Z and U are independent. Then T has a student's t distribution

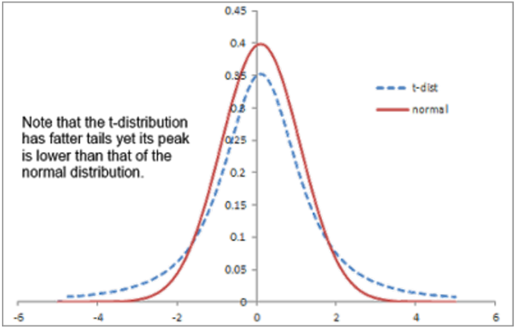
$$f(t) = \frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\Gamma(\frac{r}{2})} \frac{1}{(1+\frac{t^2}{r})^{\frac{r+1}{2}}}, \quad -\infty < t < \infty$$

Student's t distribution is a heavy tailed distribution in contrast with the normal distribution.

Functions of Multivariate Normal RVs(2/2)

$$T = \frac{\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}} = \frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$$

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$



Heavy-tailed distribution

Central Limit Theorem

Definition

A sequence of random variables Z_1, Z_2, ... is said to converge in distribution to a random variable Z, denoted by $Z_n \xrightarrow{d} Z$, if

$$\lim_{n\to\infty} F_n(z) = F(z),$$

for every number $z \in R$ at which $F(z)$ is continuous, where $F_n(z)$ and $F(z)$ are the cdfs of random variables Z_n and Z, respectively.

CLT

Let $\overline{X}$ be the sample mean of the random sample of size n, $X_1,X_2 ,\cdots,X_n$ from a distribution with a finite mean μ and a finite nonzero variance σ^2, then as $n \to \infty$, the random variable $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ converges in distribution to $N(0,1)$.

Practical use of CLT

For large n, the probabilities of events of $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$, $\overline{X}$ and $\sum_{i=1}^{n} X_i$ can be calculated approximately by treating them as if they are $N(0,1)$, $N(\mu, \frac{\sigma^2}{n})$, and $N(n\mu, n\sigma^2)$, respectively, and by looking up tables of normal distributions.

Recall that if $Y \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(a \leq Y \leq b) &= P(\frac{a-\mu}{\sigma} \leq \frac{Y-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}) \\
&= \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})
\end{aligned}
$$

where $\Phi(\cdot)$ is the cdf of $N(0,1)$

Approximation for discrete distribution

To find a continuous distribution whose pdf is “close” to the histogram of the discrete distribution.

Let $Y = \sum_{i=1}^{n} X_i$, where $X_1, \cdots, X_n$ are i.i.d. random sample drawn from discrete distributions with mean μ and variance σ^2, then

$P(Y = k)$	$\approx$	$P(k - \frac{1}{2} < Y < k + \frac{1}{2})$
discrete RV		approximate by continuous RV
pmf $f(y)$		by CLT for large n, Y can be approximated by $N(n\mu, n\sigma^2)$ in the sense that the pdf of the normal distribution is close to the histogram of Y
hard to calculate		easy to calculate

Chebyshev's inequality

Chebyshev's Inequality

If the random variable X has a finite mean μ and finite nonzero variance σ^2, then for every $k \geq 1$,

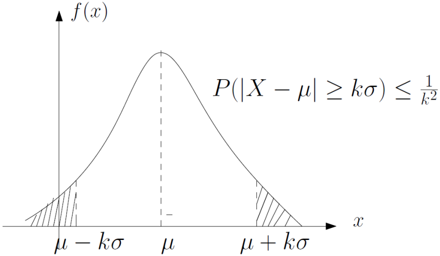
$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

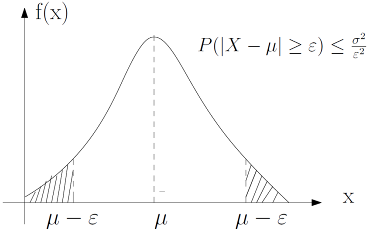
Or equivalently, if $\epsilon = k\sigma$, then

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

Chebyshev's inequality

Graphical interpretation:





This links to the interpretation of σ^2: a measure of dispersion of the values that X can take with respect to its mean μ.

Convergence in Probability and Law of large numbers

Convergence in Probability

A sequence of RVs $Z_1, Z_2, \cdots,$ is said to converge in probability to a RV Z, denoted by, $Z_n \xrightarrow{p} Z$, if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|Z_n - Z| \geq \varepsilon) = 0.$$

Law of Large Numbers

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n drawn from a distribution with finite mean μ and finite nonzero variance, and let $\overline{X}$ be the sample mean. Then $\overline{X}$ converges in probability to μ, i.e., for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\left|\overline{X} - \mu\right| \geq \varepsilon\right) = 0$$

Limiting mgf technique

limiting mgf technique

Let $\{M_n(t)\}_{n=1}^{\infty}$ be a sequence of mgfs for t in an open interval around $t = 0$. If $\lim_{n\to\infty} M_n(t) = M(t)$, for t in the open interval around $t = 0$. Then the sequence of RVs (or random distributions) associated with $M_n(t)$ converges in distribution to the one associated with $M(t)$.

Convergence of $b(n, p)$

1. as $n \to \infty$ with $\lambda = np$ being a constant, and let $Z_n \sim b(n, p)$ and $Z \sim \text{Poisson}(\lambda)$, then $Z_n \xrightarrow{d} Z$.
2. as $n \to \infty$ with p being a constant, and let $Z_n \sim b(n, p)$ and $Z \sim N(0, 1)$, then
$$\frac{Z_n - np}{\sqrt{np(1-p)}} \xrightarrow{d} Z$$

Final Exam

- The exam will cover Chapters 1-5, excluding Sections 3.4, 5.2
- The final exam contains 10 regular questions, which are in the similar format of the ones in the assignments. Among the 10 questions, 5 of them are taken from the assignments and created by merging a couple of questions.
- You are allowed to bring ONE sheet of A4 paper, on which you can write/draw anything you want, but all notes on the paper must be hand-written by yourself (you can print out on a blank A4 paper your own notes hand-written on ipad or the similar, but it is NOT allowed to print out other's notes).
- Tables of probabilities for distributions will be provided.
- You can bring a non-electronic dictionary, but no mathematical formula, notations, etc should be found in your dictionary.
- Please bring your student ID card to the exam for verification of identity and attendance taking.
- Those who arrive more than 30 minutes late shall NOT be permitted to take the exam.
- Calculators are allowed in the exam.

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