MAT1002 Lecture 9, Thursday, Feb/23/2023

Outline

- The 3D space (12.1)
- Vectors (12.2)

<u>Extra</u>: Let A, B and C be sets. The Cartesian product of A and B is

$$A \times B := \{(a,b) : a \in A,\ b \in B\}.$$

Similarly, $A \times B \times C := \{(a,b,c) : a \in A,\ b \in B,\ c \in C\}$. The Cartesian product for more than 3 sets are defined similarly. For $n \in \mathbb{Z}_+$, we define

$$A^n := \underbrace{A \times A \times \dots \times A}_{\text{Cartesian product of } n \text{ copies of } A}.$$

In particular, $\mathbb{R}^n$ is the set of all elements with n <u>real</u> coordinates: $(x_1, x_2, \dots, x_n)$, so $\mathbb{R}^n$ can be thought of as the n-dimensional space.
($x_i \in \mathbb{R}$ for each i)

The 3D Space

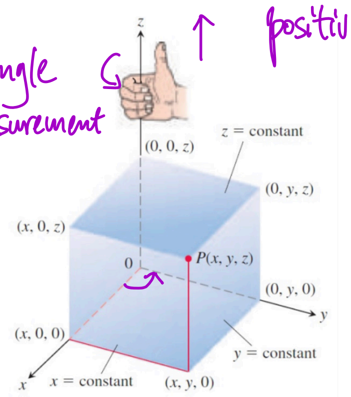
Coordinate Axes and Their Placement

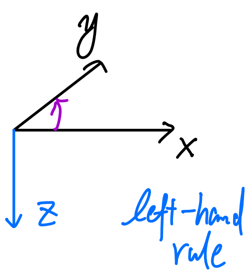
We will represent the (three-dimensional) space using $\mathbb{R}^3$, where

$$\mathbb{R}^3 := \{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}.$$

The numbers x, y and z represent the position of the point $P(x, y, z)$, with respect to the coordinate axes (x-axis, y-axis and z-axis). The coordinate axes are usually arranged according to the right-hand rule, as shown in the following figure.

angle measurement ↑ positive z direction





Coordinate planes.

$x = 0$: yz-plane
$y = 0$: xz-plane
$z = 0$: xy-plane

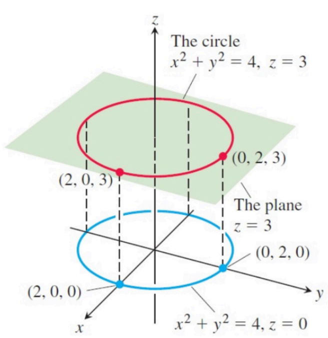
First octant : $\{(x,y,z) : x>0, y>0, z>0\}$.

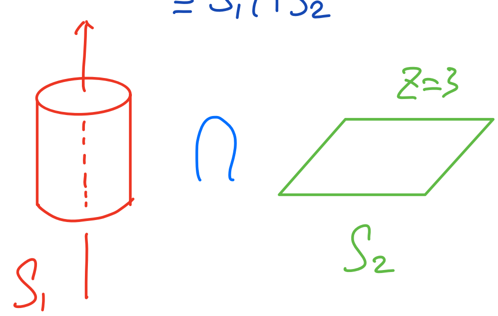
The set of all points (x, y, z) which satisfy

$$\underbrace{x^2 + y^2 = 4}_{S_1}, \quad \underbrace{z = 3}_{S_2}$$

$\{(x,y,z): x^2+y^2=4 \ \& \ z=3\}$
$= \{(x,y,z): x^2+y^2=4\}$
$\cap \{(x,y,z) : z=3\}$
$= S_1 \cap S_2$

can be represented geometrically by the pink disk in the following figure.





Distance & Spheres

Definition

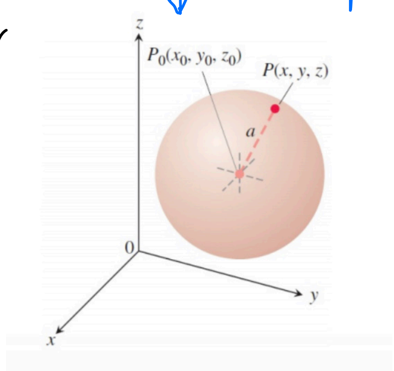
The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is defined by

$$d(P_1, P_2) := |P_1P_2| := \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

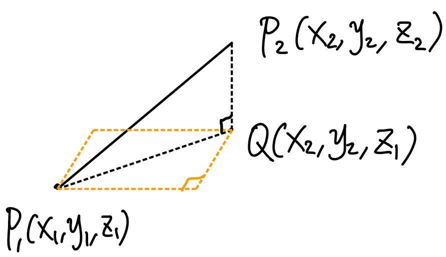
The sphere with radius a centered at the point (x_0, y_0, z_0) is the set of all points (x, y, z) that satisfy

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

when $a=0$, it is a "degenerate sphere" which is a point.



The definition of $|P_1P_2|$ is motivated by our intuition in the following figure.



$$|P_1P_2| = \sqrt{|P_1Q|^2 + |P_2Q|^2} = \sqrt{\left((x_2-x_1)^2 + (y_2-y_1)^2\right) + (z_2-z_1)^2}$$

Generalizing from 1D, 2D and 3D cases, the distance between $P_1 := (x_1, x_2, \ldots, x_n)$ and $P_2 := (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is defined by

$$|P_1P_2| := \sqrt{(y_1-x_1)^2 + \ldots + (y_n-x_n)^2} = \left(\sum_{i=1}^{n} (y_i - x_i)^2\right)^{\frac{1}{2}}.$$

Remark Consider the equation

$$x^2+y^2+z^2+\alpha x+\beta y+\gamma z+\delta=0. \quad (*)$$

Since

$$(*) \iff x^2+\alpha x+\left(\frac{\alpha}{2}\right)^2+y^2+\beta y+\left(\frac{\beta}{2}\right)^2+z^2+\gamma z+\left(\frac{\gamma}{2}\right)^2 - \left(\frac{\alpha}{2}\right)^2-\left(\frac{\beta}{2}\right)^2-\left(\frac{\gamma}{2}\right)^2+\delta=0$$

$$\iff \left(x+\frac{\alpha}{2}\right)^2+\left(y+\frac{\beta}{2}\right)^2+\left(z+\frac{\gamma}{2}\right)^2=\left(\frac{\alpha}{2}\right)^2+\left(\frac{\beta}{2}\right)^2+\left(\frac{\gamma}{2}\right)^2-\delta,$$

if $K:=\left(\frac{\alpha}{2}\right)^2+\left(\frac{\beta}{2}\right)^2+\left(\frac{\gamma}{2}\right)^2-\delta \geqslant 0$, then $(*)$ describes a sphere of radius $\sqrt{K}$ centered at $\left(-\frac{\alpha}{2}, -\frac{\beta}{2}, -\frac{\gamma}{2}\right)$.

Example

Show that $x^2+y^2+z^2+4x-6y+2z+6=0$ $(*)$ is the equation of a sphere, and find its center and radius.

Sol. $(*) \iff (x^2+4x+4)+(y^2-6y+9)+(z^2+2z+1)-14+6=0$

$$\iff (x+2)^2+(y-3)^2+(z+1)^2=(\sqrt{8})^2.$$

Center: $(-2,3,-1)$; radius: $2\sqrt{2}$.

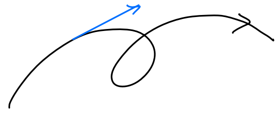
Remark With center (x_0,y_0,z_0) and radius a:

$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2$: Sphere.

$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\leq a^2$: Closed ball. (Interior + sphere.)

$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2<a^2$: Open ball. (Interior)

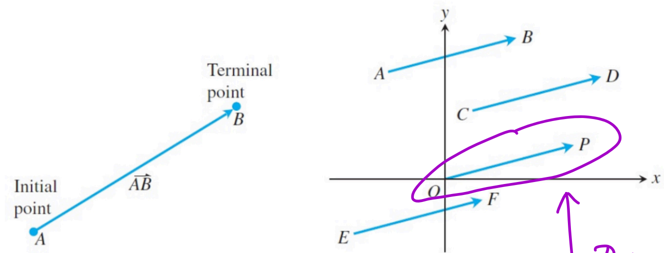
Vectors



Roughly speaking, a vector is a directed line segment that goes from a point A (the initial point) to a point B (the terminal point). Such a vector is denoted by $\overrightarrow{AB}$.

A & B in the same space $\mathbb{R}^n$.

- The two points A and B can be the same. $\overrightarrow{AA}$: zero vector.
- There are two defining properties of a vector: length and direction.
- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are considered exactly the same if they have the same length and direction, even if $A \neq C$.



$\overrightarrow{AB}$ is also written as $\vec{AB}$.

Reprensentative

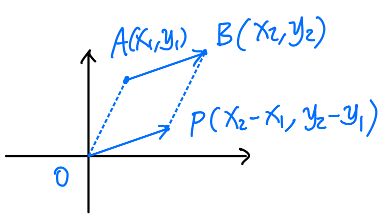
All of the same vectors above can be represented by $\overrightarrow{OP}$, the one that starts at the origin O.

Let $\overrightarrow{AB}$ be a vector in the xy-plane, where $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Consider translating A to the origin O. It is not hard to see that $\overrightarrow{OP} = \overrightarrow{AB}$ if and only if

$$P = (x_2 - x_1, y_2 - y_1).$$

Therefore $\overrightarrow{AB}$ can be recorded by the position of P alone, and we write

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle \quad \text{or} \quad \overrightarrow{AB} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}.$$



Remark

If $A=(x_1, y_1, z_1)$ and $B=(x_2, y_2, z_2)$ are points in the 3D space, then $\overrightarrow{AB} = \overrightarrow{OP}$, where $P=(x_2-x_1, y_2-y_1, z_2-z_1)$.

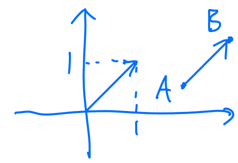
In this case, we write

$$\overrightarrow{AB} = < x_2-x_1, y_2-y_1, z_2-z_1 > \quad \text{or} \quad \overrightarrow{AB} = \begin{bmatrix} x_2-x_1 \\ y_2-y_1 \\ z_2-z_1 \end{bmatrix}.$$

Component Forms and Length

- In general, every vector in the xy-plane can be represented as a vector $\vec{v} = \langle v_1, v_2 \rangle$. The numbers v_1 and v_2 are called the components of $\vec{v}$.
- A common alternative notation for $\vec{v}$ is $\mathbf{v}$. or $\vec{\mathrm{v}}$ ← Usually used for printing.
- The length (or norm or magnitude) of the vector $\vec{v} = \langle v_1, v_2 \rangle$ is defined to be

$$\sqrt{v_1^2 + v_2^2}.$$



$\vec{AB} = \langle 1, 1 \rangle$.

  The length of $\vec{v}$ is commonly denoted by $|\vec{v}|$ or $\|\vec{v}\|$.
- Similarly, every vector in the space can be written as $\vec{v} = \langle v_1, v_2, v_3 \rangle$, with

$$|\vec{v}| := \|\vec{v}\| := \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

More generally, we can talk about vectors in the n-dimensional space: each such vector $\vec{v}$ can be represented by

$$\langle v_1, v_2, \ldots, v_n \rangle,$$

$v_1, \ldots, v_n$ components of $\vec{v}$.

where $|\vec{v}| := \|\vec{v}\| := \sqrt{v_1^2 + \cdots + v_n^2}$, although they are hard to draw for $d \geq 4$.

Algebraic Operations of Vectors

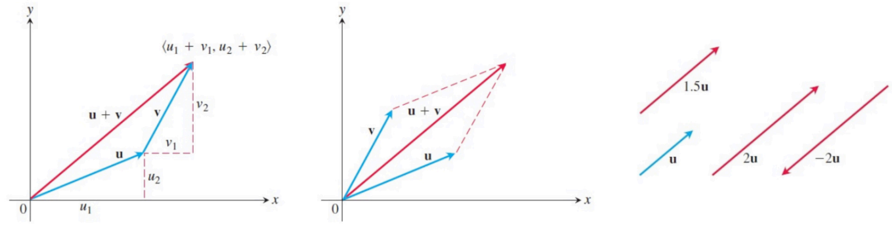
Definition
Given two vectors $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$, $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ and a scalar $k \in \mathbb{R}$, define additions and scalar multiplications as follows:

- $\vec{u} + \vec{v} := \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n \rangle$.
- $k\vec{u} := \langle ku_1, ku_2, \ldots, ku_n \rangle$.

Similar to real numbers, we write $\vec{u} - \vec{v}$ to mean $\vec{u} + (-1)\vec{v}$.

In short: these two operations are componentwise.

The ~~geometry~~ geometric effect of these operations on vectors in $\mathbb{R}^2$ is displayed in the following figure.



- $(-1)\vec{v}$ is pointing to the "opposite" direction as $\vec{v}$.
- For $\vec{v}$ in $\mathbb{R}^n$,

$$|k\vec{v}| = \sqrt{(kv_1)^2+\dots+(kv_n)^2} = |k|\sqrt{v_1^2+\dots+v_n^2} = |k||\vec{v}|,$$

So length of $k\vec{v}$ is $|k|$ times that of $\vec{v}$.

Properties of Algebraic Operations

Definition

The zero vector, denoted by $\vec{0}$ or $\mathbf{0}$, is the vector whose components are all zeros.

Properties of Vector Operations

Let **u**, **v**, **w** be vectors and a, b be scalars.

in the same space $\mathbb{R}^n$

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Proof is straightforward (direct verification).

Unit Vectors and Standard Unit Vectors

Definition

- A unit vector is a vector whose length is 1.
- The standard unit vectors in $\mathbb{R}^3$ are the unit vectors

$$\vec{i} := \mathbf{i} := \langle 1, 0, 0 \rangle, \quad \vec{j} := \mathbf{j} := \langle 0, 1, 0 \rangle, \quad \vec{k} := \mathbf{k} := \langle 0, 0, 1 \rangle.$$

std. unit vectors in $\mathbb{R}^2$: $\vec{i} = \langle 1, 0 \rangle$ & $\vec{j} = \langle 0, 1 \rangle$.

Remark

- Any vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ can be written as a linear combination of $\vec{i}$, $\vec{j}$ and $\vec{k}$:

$$\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}.$$

- Whenever $|\vec{v}| \neq 0$, the vector $\vec{v}/|\vec{v}|$ is a unit vector, which is called the direction of $\vec{v}$.

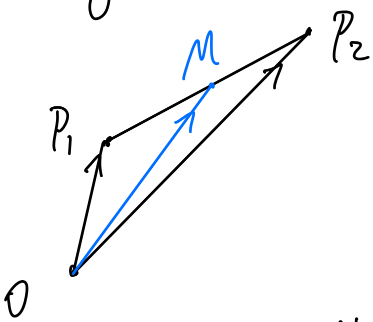
$$\left| \frac{1}{|\vec{v}|} \vec{v} \right| = \frac{1}{|\vec{v}|} |\vec{v}| = 1$$

For any nonzero vector $\vec{v}$, $\vec{v} = \underbrace{|\vec{v}|}_{\text{length}} \underbrace{\left(\frac{\vec{v}}{|\vec{v}|} \right)}_{\text{direction}}$

e.g. If $\vec{v} = \langle 3, -4 \rangle$ is the velocity vector, then $|\vec{v}| = 5$ is the speed, and $\langle \frac{3}{5}, \frac{-4}{5} \rangle$ is the direction of the motion.

Midpoint Formula

If midpoint of the line segment P_1P_2 joining $P_1(x_1, y_1, z_1)$ & $P_2(x_2, y_2, z_2)$ is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$. This can be derived using vectors:

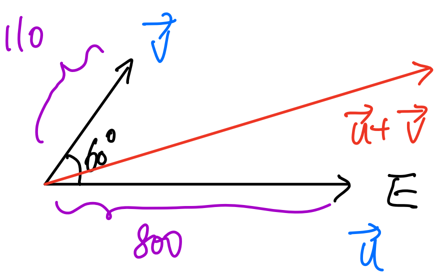


$$\vec{OM} = \vec{OP_1} + \frac{1}{2}\vec{P_1P_2}$$
$$= \left\langle x_1 + \frac{x_2-x_1}{2}, y_1 + \frac{y_2-y_1}{2}, z_1 + \frac{z_2-z_1}{2} \right\rangle$$
$$= \left\langle \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right\rangle.$$

Hence $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

Applications of Vectors

EXAMPLE 8 A jet airliner, flying due east at 800 km/h in still air, encounters a 110 km/h tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?



Sol: $\vec{u} = \langle 800, 0 \rangle$.

$v_1 = 110 \cdot \cos\frac{\pi}{3} = 55$

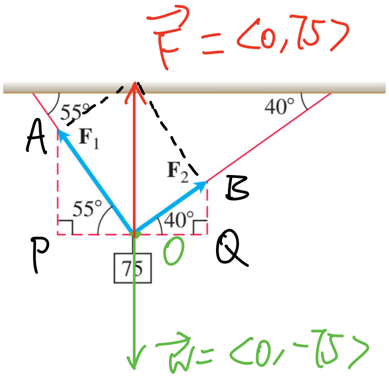
$v_2 = 110 \cdot \sin\frac{\pi}{3} = 55\sqrt{3}$

$\vec{u}+\vec{v} = \langle 855, 55\sqrt{3} \rangle$

Speed $= |\vec{u}+\vec{v}| = \sqrt{855^2 + 55^2 \cdot 3}$ (≈ 860.3) km/h

Direction $= \dfrac{\vec{u}+\vec{v}}{|\vec{u}+\vec{v}|}$

EXAMPLE 9 A 75-N weight is suspended by two wires, as shown in Figure 12.18a. Find the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting in both wires.



<u>Sol</u>: • $\vec{F}_1 + \vec{F}_2 = \vec{F} = \langle 0, 75 \rangle$.

• $\vec{F}_1 = \langle -|\vec{F}_1| \cos 55°, |\vec{F}_1| \sin 55° \rangle$
$\vec{F}_2 = \langle |\vec{F}_2| \cos 40°, |\vec{F}_2| \sin 40° \rangle$ (*)

• $\begin{cases} 0 = |\vec{F}_2| \cos 40° - |\vec{F}_1| \cos 55° \\ 75 = |\vec{F}_2| \sin 40° + |\vec{F}_1| \sin 55° \end{cases}$

$\Rightarrow$ Solve for $|\vec{F}_1|$ and $|\vec{F}_2|$: $|\vec{F}_1| \approx 57.67$ (N) and $|\vec{F}_2| \approx 43.18$ (N).

Plug back to (*) to get $\vec{F}_1$ and $\vec{F}_2$.