# MAT1002 Lecture 5, Thursday, Feb/09/2023

## Outline

- Power series (10.7)
  - ↳ Overview
  - ↳ Convergence
  - ↳ Operations
- Taylor series (10.8)

## Power Series

We now shift our attention to series of functions. Consider the

$$f(x) := 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n .$$

By geometric series,

$$f(x) = \begin{cases} \frac{1}{1-x}, & \text{if } |x| < 1; \\ \underbrace{\text{D.N.E}}_{\text{(divergent)}}, & \text{if } |x| \geq 1. \end{cases}$$

This means that the series is not always convergent (not for all $x \in \mathbb{R}$), and the domain of the function is $(-1, 1)$.

**Definition**

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} \underline{\underline{c_n}}(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots .$$

$\{c_n\}$ sequence of real numbers

The number $a$ above is called the center of the power series. In particular, a power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots .$$

**Remark**: A power series can be thought of as a polynomial of infinite degree, but it may not converge for all $x \in \mathbb{R}$.

Graphical intuition

- For the power series $\sum_{n=0}^{\infty} x^n$, its partial sums are

$$S_0(x) = 1, \quad S_1(x) = 1 + x, \quad \ldots, \quad S_k(x) = \sum_{n=0}^{k} x^n$$

Graph them using Desmos and observe the pattern.

- Try the same for $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n$. Note that

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2}x\right)^n = \frac{1}{1 - 1 + \frac{1}{2}x} = \frac{2}{x},$$

given that $\left|1 - \frac{1}{2}x\right| < 1$, i.e., $\underbrace{0 < x < 4}_{(0,4) \text{ is an interval centered at } 2}$.

**Remark**:

- In power series notation, $0^0 := 1$, e.g. if $f(x) := \sum_{n=0}^{\infty} x^n$, then $f(0) = \sum_{n=0}^{\infty} 0^n = 0^0 + 0^1 + 0^2 + \ldots = 1$.
  Indeed, $f(x) = \frac{1}{1-x}$ for $|x| < 1$, so $f(0) = 1$.
- Any power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ always converges at its center $a$, since

$$\sum_{n=0}^{\infty} c_n (a-a)^n = c_0.$$

Q: For which values of $x$ does a given power series converge?

A: Try Ratio Test / Root Test.

**Example**

For what values of $x$ do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$. Ans: Converges $\Leftrightarrow$ $x \in (-1, 1]$.

(b) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$. Ans: Converges $\Leftrightarrow$ $x \in (-\infty, \infty)$.

(c) $\sum_{n=0}^{\infty} n! x^n$. Ans: Converges $\Leftrightarrow$ $x = 0$.

Note that all three series converges for $x=0$. Fix $x$ with $x \neq 0$.

(a) $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n} = |x| \cdot \frac{n}{n+1} \to |x| \quad \text{as } n \to \infty$$

By Ratio Test, if $|x|<1$, series converges (absolutely); if $|x|>1$, series diverge. If $x=1$, series converges (alt. harmonic); if $x=-1$, series diverges (negative harmonic).

(b) $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1} \longrightarrow 0 \quad \text{as } n \to 0.$$

By Ratio Test, series always converges (i.e., converges for all $x$).

(c) $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\,|x|^{n+1}}{n!\,|x|^n} = (n+1)|x| \to \infty \quad (\text{since } x \neq 0).$$

By Ratio Test, series diverges for every $x \neq 0$.

## Convergence

From the three examples above, we saw that they all converge on intervals. What is the general pattern?

**Theorem** (10.7.18) If $\sum_{n=0}^{\infty} C_n (x-a)^n$ converges at $x = x_0$, then it converges absolutely for all $x$ with $|x-a| < |x_0 - a|$. If it diverges at $x = x_0$, then it diverges for all $x$ with $|x-a| > |x_0 - a|$.

*Proof*: Suppose series converges at $x = x_0$. May assume $x_0 \neq a$.

So $|x_0 - a| > 0$. Since $\lim_{n\to\infty} C_n (x_0 - a)^n = 0$, $\exists N$ s.t. $\forall n \geq N$,

$$|C_n (x_0 - a)^n| < 1 \Rightarrow |C_n| < \frac{1}{|x_0 - a|^n}.$$

For any $x_1$ with $|x_1 - a| < |x_0 - a|$, we have

$$0 \leq |C_n||x_1 - a|^n < \frac{|x_1 - a|^n}{|x_0 - a|^n} = \left|\frac{x_1 - a}{x_0 - a}\right|^n.$$

$<1$

Direct Comparison

Since $\sum_{n=N}^{\infty} \left|\frac{x_1 - a}{x_0 - a}\right|^n$ converges, $\sum_{n=N}^{\infty} |C_n||x_1 - a|^n$ converges, so $\sum C_n (x-a)^n$ converges absolutely for all $x$ with $|x-a| < |x_0 - a|$.

For the second part, suppose $\sum C_n (x_0 - a)^n$ diverges. If $\exists x_1$ such that $|x_1 - a| > |x_0 - a|$ but $\sum C_n (x_1 - a)^n$ converges, then by the first part, $\sum C_n (x_0 - a)^n$ would converge (absolutely). This is a contradiction, so no such $x_1$ can exist, i.e., if $|x_1 - a| > |x_0 - a|$ then $\sum C_n (x_1 - a)^n$ must diverge. $\square$

# Radius of Convergence

**Theorem (Existence of the Radius of Convergence)**

*For any power series $\sum c_n(x-a)^n$, one of the following three statements holds:*

(i) *There exists a positive real number $R$ such that the series converges absolutely for all $x$ with $|x-a| < R$ but diverges for all $x$ with $|x-a| > R$. The series may or may not converge for $x$ with $|x-a| = R$.*

(ii) *The series converges absolutely for all $x$ ($R = \infty$).*

(iii) *The series converges at $x = a$ and diverges elsewhere ($R = 0$).*

This $R$ is called the radius of convergence

## Remark

- The radius of convergence may be found by the Ratio Test (or the Root test).
- If $R$ is the radius of convergence for $\sum C_n(x-a)^n$, where $R \neq 0$ and $R \neq \infty$, then $\sum C_n(x-a)^n$ may or may not converge at $x = a \pm R$.

  e.g. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ cvgs for $x \in (-1, 1]$.

  e.g. $\sum x^n$ cvgs for $x \in (-1, 1)$.

  One needs to test $x = a \pm R$ separately.

Proof: See proof of "corollary of theorem 18" in Chap 10.7 in the book.

# Operations on Power Series

## Addition/Subtraction

This is done in the same way as series of numbers.

## Multiplication

Let $\sum a_n(x-c)^n$ and $\sum b_n(x-c)^n$ be power series with radii of convergence $R_a$ and $R_b$, respectively. Let $R := \min\{R_a, R_b\}$. Define

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k} \quad \text{for } n \geq 0.$$

Then

$$\left(\sum_{n=0}^{\infty} a_n(x-c)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-c)^n\right) = \sum_{n=0}^{\infty} c_n(x-c)^n$$

for all $x$ with $|x-c|<R$.

## Substitution

**THEOREM** ~~20~~ 10.7.20 If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function $f$ on $|f(x)| < R$.

e.g., Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for all $x \in (-1,1)$, we have

$\sum_{n=0}^{\infty} (3x)^n = \frac{1}{1-3x}$ valid for $|3x|<1$, i.e., $x \in (\frac{-1}{3}, \frac{1}{3})$.

# Differentiation and Integration

**Theorem (Term-by-Term Differentiation and Integration)**

*Suppose that* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has a radius of convergence* $R$, *with* $R>0$. *Define* $f$ *on* $(a-R, a+R)$ *by*

$$f(x) := \sum_{n=0}^{\infty} c_n(x-a)^n.$$

*Then on* $(a-R, a+R)$, *the function* $f$ *is differentiable and has an antiderivative, with*

(i) $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

(ii) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.

The theorem above may be useful in converting a function between its elementary form and power series form.

e.g. $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_n (-1)^n x^n$ for $|x|<1$

$\Rightarrow \int \frac{1}{1+x}dx = C + \sum_{n=0}^{\infty} \frac{1}{n+1}(-1)^n x^{n+1}$ for $|x|<1$

$\Rightarrow \ln(1+x) = C + \sum_{n=0}^{\infty} \frac{1}{n+1}(-1)^n x^{n+1}$ for $|x|<1$

$(= \ln|1+x|)$

For $x=0$, $0 = \ln 1 = C + 0 \Rightarrow C = 0$

$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n+1}(-1)^n x^{n+1} = \ln(1+x)$ for $x \in (-1,1)$.

For an approach using definite integrals, see e.g. 10.7.6.

<u>e.g.</u> Let $f(x) := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

$(x \neq 0)$

- By Ratio Test, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+3}}{2n+3} \cdot \frac{2n+1}{|x|^{2n+1}} = \left(\frac{2n+1}{2n+3}\right)|x|^2 \to |x|^2$,

  so series converges when $|x|^2 < 1 \Rightarrow |x| < 1$
  and diverges when $|x|^2 > 1 \Rightarrow |x| > 1$

- $f$ is defined on $(-1,1)$. <u>What is an elementary function expression?</u>

- $f'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1+x^2}$, for $|x| < 1$

$\Rightarrow f(x) = \int f'(x)dx = \int \frac{1}{1+x^2} dx = \arctan x + C$

- $0 = f(0) = \arctan 0 + C = C \Rightarrow C = 0$

Series expression

$\Rightarrow$ <u>$f(x) = \arctan x$</u>, $x \in (-1,1)$.

elementary expression for $f$ on $(-1,1)$.

<u>Remark</u>: After term-by-term differentiation/integration, convergence at endpoints $a \pm R$ is <u>NOT</u> guaranteed; e.g. $\sum \frac{x^n}{n^2}$.

- $\frac{|x|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|x|^n} = \left(\frac{n}{n+1}\right)^2 |x| \to |x| \Rightarrow R = 1$.

- At $x=1$, $\sum_{n=0}^{\infty}\frac{x^n}{n^2}$ converges.
- But $\sum_{n=1}^{\infty} n\frac{x^{n-1}}{n^2} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n}$ does not converge at $x=1$.

## Taylor Series

Suppose $\sum_n c_n(x-a)^n$ converges to $f(x)$ on some open interval $I$. i.e., $f(x)=\sum_n c_n(x-a)^n$ $\forall x\in I$. By the previous theorem,

$$f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1},\quad f''(x)=\sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},\ \ldots$$

i.e., if $f$ has a power series representation on $I$, it must be infinitely differentiable on $I$ ( $f^{(n)}$ exists for all $n\geq 0$).

Conversely, given an infinitely differentiable function $f$ on $I$:

Q1: Can $f$ be represented as a power series on $I$?

Q2: If so, what is the form of this power series?

We look at Q2 first. Suppose $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$ on some open interval $I$ containing $a$. What are the values of $c_n$?

- $f(a)=\sum_{n=0}^{\infty}c_n(a-a)^n = c_0$.
- $f'(a)=\sum_{n=1}^{\infty}nc_n(a-a)^{n-1} \implies f'(a)=c_1$.

$\cdot\ f''(a) = \sum_{n=2}^{\infty} n(n-1)C_n(a-a)^{n-2} \Rightarrow f''(a) = 2\cdot 1\cdot C_2$

$\Rightarrow C_2 = \frac{f''(a)}{2!}$

$\cdot\ f^{(k)}(a) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\,C_n(x-a)^{n-k}\Big|_{x=a} = k(k-1)\cdots 1\cdot C_k$

$\Rightarrow C_k = \frac{f^{(k)}(a)}{k!}$

Hence, if $f(x)$ has a power series representation $\sum_{n=0}^{\infty} C_n(x-a)^n$ on $I$, then it must be

$C_n$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

**Definition**

Let $f$ be a function such that for all $n \in \mathbb{N}$, $f^{(n)}$ exists on some open interval containing $a$. The Taylor series of $f$ centered at $a$ is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

or "Taylor series generated by $f$"

The Maclaurin series of $f$ is the Taylor series of $f$ centered at 0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

**Example**

(a) The Maclaurin series of $f(x) := e^x$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(b) The Maclaurin series of $f(x) := \cos x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

Partial sums of a Taylor series are Taylor polynomials. More generally:

**Definition**

Let $f$ be a function such that for all $n \in \{0, 1, \ldots, N\}$, $f^{(n)}$ exists on some open interval containing $a$. The Taylor polynomial of $f$ (of order $n$) centered at $a$ is the polynomial

$$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$