MAT1002 Calculus II
Final Examination
(Spring 2021)

Solution Key:

1. (30 pts) No partial credits!

   (i) False

   (ii) $\kappa = 1/t$

   (iii) False

   (iv) x

   (v) $\mathbf{i} + 2\mathbf{j}$ which is the negative of the gradient of T at $(0, 0)$.

   (vi) False, need D to be simply connected.

   (vii) True

   (viii) (c) is true

   (ix) (a) is true

   (x) $4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ which is the curl of $\mathbf{V}$ at $(1, 1, 1)$

2. (8 pts) Let $a_n = (-3)^n x^n / \sqrt{n+1}$, then

   ① 求出 $|x| < \frac{1}{3}$,(2分)

   ② 求出 $|x| < \frac{1}{3}$ is absolutely convergent (2分)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-3)x\frac{\sqrt{n}}{\sqrt{n+1}}\right| \to |3x|$$

   ③ $x = -\frac{1}{3}$ 时判断正确 (2分)

   as $n \to \infty$. Hence, when $|x| < 1/3$ the series is absolutely convergent. When $x = -1/3$ it diverges by the p-series results. When $x = 1/3$ it converges (conditionally) by the Alternating Series Test.

   ④ $x = \frac{1}{3}$ 时判断正确 (2分)

3. (6 pts) From

$$\sin x = x - x^3/3! + x^5/5! - x^7/7! + \cdots$$

   and

$$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots,$$

we have

$$2x^2(1-\cos(x^2)) - x^6 = 2x^2(x^4/2! - x^8/4! + x^{12}/6! + \cdots) - x^6 = -x^{10}/12 + O(x^{14})$$

and

$$\sin x^{10} = x^{10} + O(x^{30}).$$

3'

So as $x \to 0$,

$$\frac{(2x^2(1-\cos(x^2)) - x^6)}{\sin x^{10}} = \frac{-x^{10}/12 + O(x^{14})}{x^{10} + O(x^{30})} \to -1/12.$$ 1'

4. (12 pts)

(a) $\vec{BA} =< 1, -1, 5 >, \quad \vec{BC} =< 0, 2, 4 >$. So the cosine of $\angle ABC$ is given by

2' $$\frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}||\vec{BC}|} = \frac{0-2+20}{\sqrt{27(20)}} = \sqrt{\frac{3}{5}}.$$ 1'

(b) The vector projection of $\vec{BA}$ onto $\vec{BC}$ is given by

2' $$\frac{\vec{BA}\cdot\vec{BC}}{|\vec{BC}|^2}\vec{BC} = \frac{18}{20} < 0, 2, 4 >= \frac{9}{5} < 0, 1, 2 > .$$ 1'

(c) The area of the parallelogram is given by $|\vec{BA} \times \vec{BC}|$, which is

1'

the modulus of $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{vmatrix} = | < -14, -4, 2 > | = \sqrt{196 + 16 + 4} = \sqrt{216}.$ 1'

1 ~~2~~'

(d) As $\vec{BA} \times \vec{BC} =< -14, -4, 2 >$ and $B = (1, 0, -1)$, the plane containing the parallelogram is given by

$$0 = (x-1)(-14) + y(-4) + (z+1)2$$ 3'

which is

$$7x + 2y - z = 8.$$

5. (15 pts)

Let $F(x, y, z) = \cos(\pi yz) + 4xz^2$, then S is a level surface of F.

(a)

$$\nabla F =< 4z^2, -\pi z\sin(\pi yz), -\pi y\sin(\pi yz) + 8xz >$$

and

$$\nabla F(1/2, 1, -1) =< 4, 0, -4 > .$$

So the tangent plane at $(1/2, 1, -1)$ is given by

$$0 = (x - 1/2)4 + (z + 1)(-4) = 4x - 4z - 6$$

which is

$$2x - 2z - 3 = 0.$$

(b) For $z = f(x, y)$, we have

$$F_x(1/2, 1, -1) = (4z^2)|_{(1/2,1,-1)} = 4,$$

$$F_y(1/2, 1, -1) = (-\pi z\sin(\pi yz))|_{(1/2,1,-1)} = 0,$$

$$F_z(1/2, 1, -1) = (-\pi y\sin(\pi yz) + 8xz)|_{(1/2,1,-1)} = -4,$$

$$f_x(1/2, 1) = \frac{-F_x(1/2, 1, -1)}{F_z(1/2, 1, -1)} = \frac{-4}{-4} = 1$$

and

$$f_y(1/2, 1) = \frac{-F_y(1/2, 1, -1)}{F_z(1/2, 1, -1)} = 0.$$

So for $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$, the directional derivative is

$$D_{\frac{\mathbf{v}}{|\mathbf{v}|}} f(1/2, 1) = \nabla f(1/2, 1) \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{5}} < 1, 0 > \cdot < 2, -1 >= \frac{2}{\sqrt{5}}.$$ 2

(c) A normal vector to the level curve $f(x, y) = -1$ at $(1/2, 1)$ is $\nabla f(1/2, 1) = \mathbf{i}$. So the tangent line to the level curve at $(1/2, 1)$ is given by

$$1(x - \frac{1}{2}) + 0(y - 1) = 0 \quad \Longrightarrow \quad x = \frac{1}{2},$$ 2

which is a straight line on the xy-plane. Putting the level curve back to the plane $z = -1$, we have the contour curve. So the desired tangent line has parametric equations:

$$x = \frac{1}{2}, \quad y = t, \quad z = -1,$$ 2

for $-\infty < t < \infty$.

6. (9 pts) Given a fixed $(x, y) \in R$, define

2' $$F(t) = f(tx, ty), \quad \text{for} \quad t \in [0, 1].$$

Using Taylor's theorem for functions of a single variable, there is a $c \in (0, 1)$ so that

2' $$F(1) = F(0) + F'(0) + F''(c)/2!.$$

Since

2' $$F'(0) = xf_x(0, 0) + yf_y(0, 0)$$

and

2' $$F''(c) = x^2 f_{xx}(cx, cy) + 2xyf_{xy}(cx, cy) + y^2 f_{yy}(cx, cy),$$

1' $$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2}[x^2 f_{xx}(cx, cy) + 2xyf_{xy}(cx, cy) + y^2 f_{yy}(cx, cy)].$$

7. (8 pts) By Lagrange's method, we first solve

$$1 = 2\lambda x \implies x = \frac{1}{2\lambda},$$

$$2 = 2\lambda y \implies y = \frac{1}{\lambda},$$

$$5 = 2\lambda x \implies z = \frac{5}{2\lambda}.$$ 2'

Subsituting them into the constraint $x^2 + y^2 + z^2 = 1$, we have

$$\lambda^2 = (1/2)^2 + (1)^2 + (5/2)^2 = \frac{1}{4} + 1 + \frac{25}{4} = \frac{15}{2},$$

so

$$\lambda = \pm\sqrt{15/2}$$ 2'

and the corresponding points are

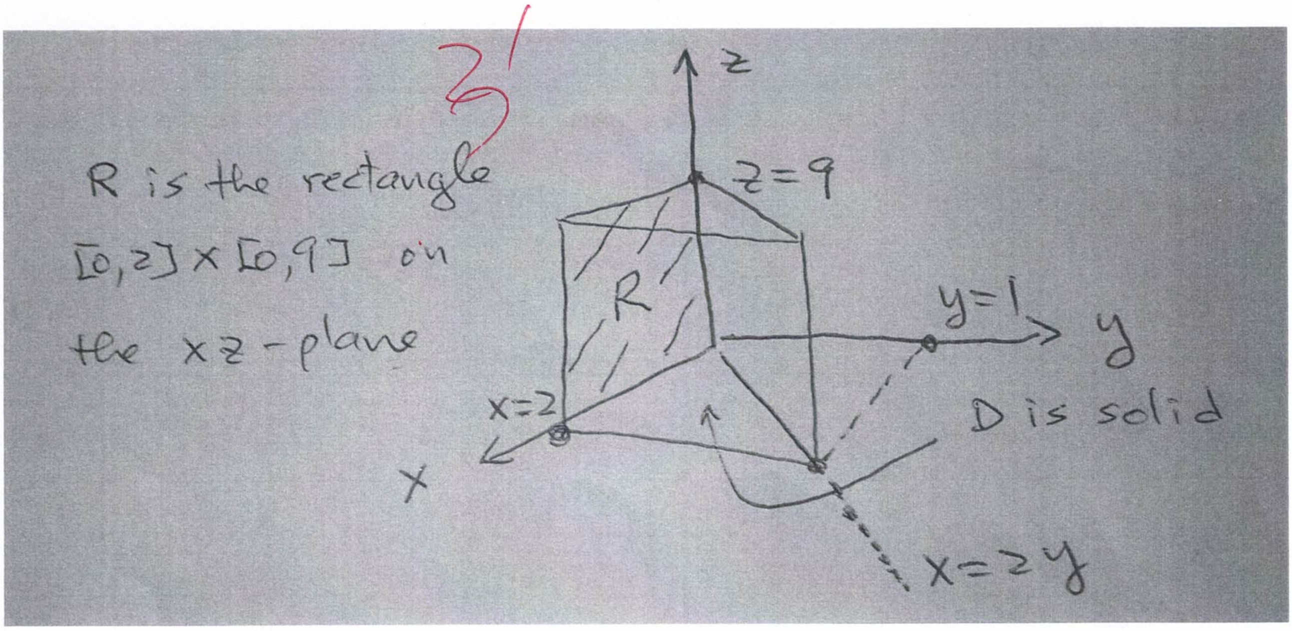
$$(x, y, z) = \pm(\frac{1}{2}\sqrt{\frac{2}{15}}, \sqrt{\frac{2}{15}}, \frac{5}{2}\sqrt{\frac{2}{15}}).$$ 2'

The maximum point is $(\frac{1}{2}\sqrt{\frac{2}{15}}, \sqrt{\frac{2}{15}}, \frac{5}{2}\sqrt{\frac{2}{15}})$ with maximum function value equal to

$$\frac{1}{2}\sqrt{\frac{2}{15}} + 2\sqrt{\frac{2}{15}} + \frac{25}{2}\sqrt{\frac{2}{15}} = 15\sqrt{\frac{2}{15}} = \sqrt{30}.$$ 2'

8. (8 pts)

(a)



(b)

$$\int_0^9\int_0^1\int_{2y}^2 \frac{4\sin x^2}{\sqrt{z}}dxdydz$$
$$=\iiint_D \frac{4\sin x^2}{\sqrt{z}}dV$$
$$=\iint_R(\int_0^{x/2} \frac{4\sin x^2}{\sqrt{z}}dy)dA$$
2' $$=\int_0^2\int_0^9 \frac{2x\sin x^2}{\sqrt{z}}dzdx \quad \text{(or in the order of } dxdz\text{)}$$
$$=\int_0^2 2x\sin x^2[2\sqrt{z}]_0^9dx$$
2' $$=6\int_0^2 2x\sin x^2dx$$
4' $$=6\ [\cancel{\sin}\ -\cos\ x^2]_0^2 = 6\cancel{\sin 4}(-\cos 4+1)$$

9. (6 pts) The cut-out cylinder portion has volume equal to

$$2\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{4-x^2-y^2}} dzdydx,$$ 2'

which, in terms of cylindrical coordinates, is

$$2\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} rdzdrd\theta = 4\pi\int_0^1 r\sqrt{4-r^2}dr = -2\pi\int_4^3 \sqrt{w}dw = \frac{4\pi}{3}(8-3^{3/2}).$$

2' 0

(ANOTHER WAY: The cut-out cylinder portion has volume equal to, with R being the unit disk on the xy-plane,

$$2\iint_R \left(\int_0^{\sqrt{4-x^2-y^2}} dz\right) dA = 2\iint_R \sqrt{4-x^2-y^2}\,dA$$

and then use polar coordinates for the evaluation.)

So, the remaining volume is

$$\frac{4\pi}{3}2^3 - \frac{4\pi}{3}(8-3^{3/2}) = 4\pi\sqrt{3}.$$

2✓

10. (6 pts) For

$$\mathbf{F} = < \tan^{-1}(e^x) + 4y,\ \ln(1+y^2) + x >,$$ 2

$$curl\ \mathbf{F}\cdot\mathbf{k} = \frac{\partial(\ln(1+y^2)+x)}{\partial x} - \frac{\partial(\tan^{-1}(e^x)+4y)}{\partial y} = 1 - 4 = -3.$$ 2

So using Green's Theorem, we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R curl\ \mathbf{F}\cdot\mathbf{k}\ dA = -3\pi.$$ 2✓

Here R is the unit disk bounded by C.

11. (6 pts) Let the boundary curve of S oriented counter-clockwisely be C given by

$$x = \cos t,\ y = \sin t,\ z = 0, \quad 0 \le t < 2\pi.$$

By Stokes' Theorem,

$$\iint_S curl\ \mathbf{F}\cdot\mathbf{n}\ d\sigma = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

On C, $\mathbf{F}\cdot\mathbf{r}'(t) = -\sin^2 t + \cos^2 t$. So

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} -\sin^2 t + \cos^2 t\ dt = \int_0^{2\pi} \cos(2t)\ dt = [\frac{1}{2}\sin(2t)]_0^{2\pi} = 0.$$

(ANOTHER WAY: Let R be the unit disk on the xy-plane oriented by the normal vector $\mathbf{k}$. By Stokes' Theorem,

$$\iint_S curl\ \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint_R curl\ \mathbf{F}\cdot\mathbf{k}\ d\sigma = \iint_R 0\ d\sigma = 0.)$$

1

12. (6 pts) For $\mathbf{F} = < x^2, -2xy, xz >$, $div\ \mathbf{F} = 2x - 2x + x = x$. By Divergence Theorem,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_\Omega div\ \mathbf{F}\ dV.$$ 1

In spherical coordinates, $x = \rho\sin\phi\ \cos\theta$. So the volume integral is equal to

2

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\ \cos\theta)\rho^2\sin\phi\ d\rho d\theta d\phi = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3\sin^2\phi\ \cos\theta\ d\rho d\theta d\phi.$$

which becomes

$$\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{4}\sin^2\phi\ \cos\theta\ d\theta d\phi = \frac{1}{4}\int_0^{\pi/2}\sin^2\phi\ d\phi = \frac{1}{4}\int_0^{\pi/2}\frac{1-\cos 2\phi}{2}\ d\phi$$

and equal to

$$\frac{1}{4}[\frac{\phi}{2}-\frac{\sin 2\phi}{4}]_0^{\pi/2} = \frac{\pi}{16}.$$

2

~~Knowing to use the div thm:~~

If evaluate surface integral directly:

- Set up integral property: 3
- Ans: 3

need multiple faces.