MAT1002 Lecture 26, Tuesday, Apr/25/2023

Outline

- Stokes' theorem (16.7)
  - ↳ Curl of a vector field
  - ↳ Surface independent property
  - ↳ 3D component test
- Divergence (16.8)

Curl of a Vector Field

Definition

The curl of a vector field $\mathbf{F} = \langle M, N, P \rangle$ is defined to be

$$\operatorname{curl} \mathbf{F} := \underbrace{\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)}_{\text{Circulation density projected on } yz\text{-plane}} \mathbf{i} + \underbrace{\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)}_{\text{" } zx\text{-plane}} \mathbf{j} + \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{" } xy\text{-plane}} \mathbf{k}.$$

Remark

If we think of ∇ as the symbolic vector $\langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$, then the curl can be remembered as the symbolic expression

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}.$$

e.g. Investigate curl $\vec{F}$, where:

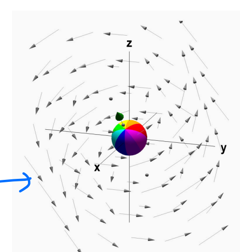
(i) $\vec{F}_1 = \langle -y, x, 0 \rangle$. $\left(\operatorname{Curl} \vec{F}_1 = \langle 0, 0, 2 \rangle\right)$

(ii) $\vec{F}_2 = \langle 0, xyz, yz \rangle$. $\left(\operatorname{Curl} \vec{F}_2 = \langle z - xy, 0, yz \rangle\right)$

Remark

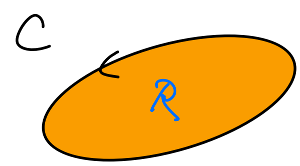
- Curl $\vec{F}$ measures rotation behaviours caused by $\vec{F}$. At a point P_0:
  - ↳ Direction of curl $\vec{F}$ indicates positive axis (w.r.t. right-hand rule) of rotation of a "very very small" sphere centered at P_0. (More precisely, its direction is the limit of the rotation axis direction of such a sphere as the radius approaches 0.)
  - ↳ Length measures "how fast" the small sphere rotates.

See "Math Insight" for more details. →



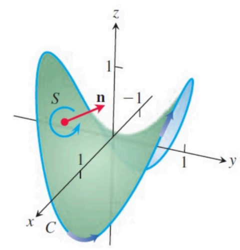
Stokes' Theorem

Recall the circulation version of Green's theorem:

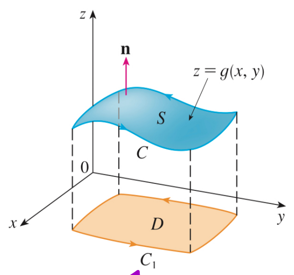


$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dA.$$

Q: What if C is a space curve? $\oint_C \vec{F}\cdot d\vec{r} = ?$



Let us investigate a special case below first.



- Assume $\vec{F} = \langle M, N, P\rangle$, M, N, P have cts partials.
- Suppose g has cts second partials.
- $\oint_C \vec{F}\cdot d\vec{r} = ?$

$C_1: \begin{cases} x = x(t) \\ y = y(t) \end{cases}, \quad a \le t \le b$ $\qquad$ $C: \begin{cases} x = x(t) \\ y = y(t) \\ z = g(x(t), y(t)) \end{cases}, \quad a \le t \le b$

$$\begin{aligned}
\oint_C \vec{F}\cdot d\vec{r} &= \oint_C M\,dx + N\,dy + P\,dz \\
&= \int_a^b M x'(t)\,dt + N y'(t)\,dt + P z'(t)\,dt \\
&= \int_a^b M x'(t)\,dt + N y'(t)\,dt + P\left(g_x x'(t) + g_y y'(t)\right) dt \\
&= \int_a^b (M + P g_x)\, x'(t)\,dt + (N + P g_y)\, y'(t)\,dt \\
&= \oint_{C_1} (\bar{M} + \bar{P} g_x)\,dx + (\bar{N} + \bar{P} g_y)\,dy
\end{aligned}$$

← Here $\bar{M}(x,y) := M(x,y,g(x,y))$; Same for $\bar{N}$ & $\bar{P}$.

$$\overset{\text{Green's}}{=} \iint_D \left(\frac{\partial(\bar{N} + \bar{P} g_y)}{\partial x} - \frac{\partial(\bar{M} + \bar{P} g_x)}{\partial y}\right) dA$$

Product rule
$$= \iint_D \left(\frac{\partial \bar N}{\partial x} + \bar P g_{yx} + \frac{\partial \bar P}{\partial x} g_y - \frac{\partial \bar M}{\partial y} - \bar P g_{xy} - \frac{\partial \bar P}{\partial y} g_x \right) dA$$

$$= \iint_D \left(\underline{\frac{\partial \bar N}{\partial x}} + \underline{\frac{\partial \bar P}{\partial x}} g_y - \underline{\frac{\partial \bar M}{\partial y}} - \underline{\frac{\partial \bar P}{\partial y}} g_x \right) dA$$

Chain rule
$$= \iint_D \left(\underline{\left(\frac{\partial N}{\partial x} + \frac{\partial N}{\partial z} g_x\right)} + \underline{\left(\frac{\partial P}{\partial x} + \frac{\partial P}{\partial z} g_x\right)} g_y - \underline{\left(\frac{\partial M}{\partial y} + \frac{\partial M}{\partial z} g_y\right)} - \underline{\left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} g_y\right)} g_x \right) dA$$

$$= \iint_D \left[-\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) g_x - \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) g_y + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \right] dA$$

$$= \iint_D \underline{\operatorname{curl} \vec F \cdot \langle -g_x, -g_y, 1 \rangle} \, dA$$

curl is evaluated at $(x, y, g(x,y))$, $(x,y) \in D$

$$= \iint_S (\operatorname{curl} \vec F \cdot \vec n) \, d\sigma.$$

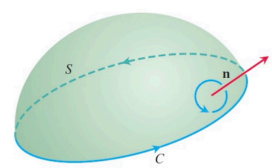
This turns out to be true in a more general setting.

Theorem (Stokes' Theorem) orientable, with a side given by $\vec n$

Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C, and let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region containing S. If C is positively oriented (counterclockwise) with respect to the surface's unit normal $\mathbf{n}$, then

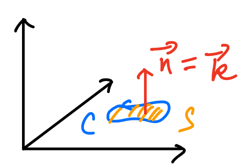
if $\vec n$ is "up", then C is counterclockwise when viewed from above.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$



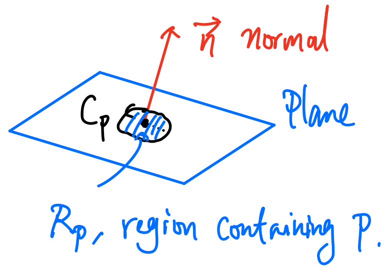
Remarks

- Green's thm is a special case of Stokes' theorem, where C and S are entirely on the xy-plane and $\vec n = \vec k = \langle 0, 0, 1 \rangle$.



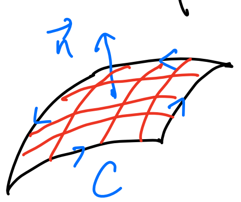
· $(\text{Curl}(\vec F\cdot\vec n)(P)$ gives the circulation density on a tiny plane containing P with unit normal $\vec n$.

Counterclockwise w.r.t. upward normal $\vec n$

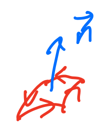


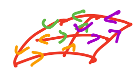
$$(\text{Curl}(\vec F\cdot\vec n)(P)=\lim_{A(R_p)\to 0}\frac{\oint_{C_p}\vec F\cdot d\vec r}{A(R_p)}$$

· Intuition of Stokes' theorem :



Total circulation around C

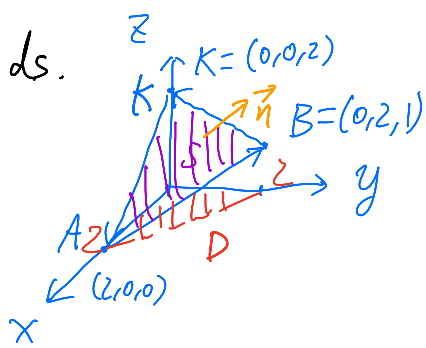
$= \sum$ circulation around small closed curve 

(due to cancellation  as in Green's thm)

$\approx \sum$ (circulation density) $\cdot$ (Surface area enclosed by small closed curve)

$= \sum (\text{Curl}(\vec F\cdot\vec n)\,\Delta\sigma$

e.g. $\vec F=\langle x+2y,\ x+3z,\ 2x+y\rangle$, find $\oint_C \vec F\cdot\vec T\,ds$.



Sol.

· C is counterclockwise when $\vec n$ is "upward" (z-component >0).

· $\vec{AB}\times\vec{AK}=\begin{vmatrix}\vec i & \vec j & \vec k\\ -2 & 2 & 1\\ -2 & 0 & 2\end{vmatrix}=\langle 4,2,4\rangle$, upward normal.

$\cdot$ S is given by $\quad 4(x-2)+2y+4z=0$

$$\iff x-2+\tfrac{1}{2}y+z=0 \iff z=-x-\tfrac{1}{2}y+2 =: f(x,y)$$

$$D=\{(x,y):\ 0\le x\le 2,\ 0\le y\le 2-x\}$$

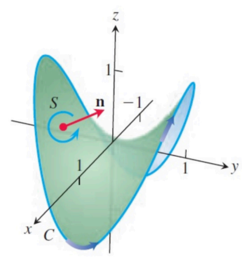
$\cdot$ $\mathrm{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y & x+3z & 2x+y \end{vmatrix} = \langle 1-3,\ 0-2,\ 1-2\rangle = \langle -2,-2,-1\rangle$

$\cdot$ $\displaystyle\oint_C \vec{F}\cdot d\vec{r} \overset{\text{Stokes'}}{=} \iint_S (\mathrm{curl}(\vec{F}\cdot\vec{n}))\,d\sigma = \iint_D \left(\mathrm{curl}\,\vec{F}\cdot\langle -f_x, -f_y, 1\rangle\right)\underset{dxdy}{dA}$

$$= \iint_D (2f_x+2f_y-1)\,dA = \iint_D \left(2(-1)+2(-\tfrac{1}{2})-1\right)dA$$

$$= \iint_D (-4)\,dA = -4\,\mathrm{Area}(D) = -8.$$

e.g. Let S be the surface $z=y^2-x^2$ inside the cylinder $x^2+y^2\le 1$. Find upward flux of $\mathrm{curl}\,\vec{F}$ across S, where $\vec{F}=\langle y,-x,x^2\rangle$.



Sol: $\cdot$ $\displaystyle\text{Flux} = \iint_S (\mathrm{curl}(\vec{F}))\cdot\vec{n}\,d\sigma \overset{\text{Stokes'}}{=} \oint_C \vec{F}\cdot d\vec{r}$,

where $C=\{(x,y,y^2-x^2):\ x^2+y^2=1\}$

$$=\{(\cos t, \sin t, \sin^2 t-\cos^2 t):\ 0\le t\le 2\pi\}.$$

$\cdot$ $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx+N\,dy+P\,dz$

$$=\int_0^{2\pi} \sin t\cdot(-\sin t)dt + (-\cos t)\cos t\,dt + \cos^2 t\,(2\sin t\cos t+2\cos t\sin t)\,dt$$

$$=\int_0^{2\pi} (-1+4\cos^3 t\sin t)\,dt = \ldots\ldots = -2\pi.$$

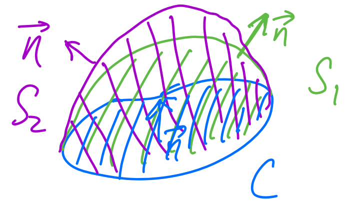
Surface Independence

← having $\vec{n}$ pointing the same side

If S_1 and S_2 are two oriented surfaces having the same boundary curve, which is counterclockwise, then Stokes' theorem implies that

that is a simple closed curve

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$



In other words, any curl field is "surface independent" (the flux by a curl field across a surface with boundary depends only on the boundary but not the surface).

This is a reason why curl fields are important: they are like gradient fields!

e.g. Find circulation of $\vec{F} = \langle x^2+z,\ y^2+2x,\ z^2-y \rangle$ along C, where C is intersection of $S_1: x^2+y^2+z^2=1$ ① and $S_2: z=\sqrt{x^2+y^2}$ ②, counterclockwise when viewed from above. (e.g. 16.7.7)

Sol: • ①, ② $\Rightarrow$ $x^2+y^2+(x^2+y^2)=1 \Rightarrow x^2+y^2=\frac{1}{2}$

$\Rightarrow$ C is circle $x^2+y^2=\frac{1}{2}$, in plane $z=\frac{1}{\sqrt{2}}$.

• Can take surface S lying on the plane; i.e.,

$$S=\Big\{(x,y,\tfrac{1}{\sqrt{2}}) : \underbrace{x^2+y^2\le\tfrac{1}{2}}_{D}\Big\}.$$

• Take upward unit normal $\vec{n}=\vec{k}$. By Stokes' theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\vec{k}\, d\sigma = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_D (2-0)\, dA = 2\,\mathrm{Area}(D) = 2\pi\tfrac{1}{2} = \pi.$$

3D Component Test

If $\vec{F} = \langle M, N, P\rangle$, M, N, P have cts partials on $D \subseteq \mathbb{R}^3$ (D open, simply connected), then

$$\underbrace{\vec{F} \text{ is conservative on } D}_{①} \iff \underbrace{P_y = N_z,\ M_z = P_x,\ N_x = M_y \text{ on } D}_{②}$$

Note that ② is the same as $\operatorname{curl}\vec{F} \equiv \vec{0}$ on D. "irrotational on D"

Proof:

① ⇒ ② : Assume ①. Then $\vec{F} = \langle f_x, f_y, f_z\rangle$ for some f on D.
Now $P_y = f_{zy} \underset{\text{Mixed Derivative Thm}}{=} f_{yz} = N_z$. Similarly, $M_z = P_x$, $N_x = M_y$.

② ⇒ ① : Let C be any closed curve in D. May assume C is simple. Take any piecewise smooth surface with C being its boundary curve, say S. By Stokes' theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\, d\sigma.$$

But $\operatorname{curl}\vec{F} \equiv \vec{0}$ on D by ②, so $RHS = 0$; i.e. $\oint_C \vec{F}\cdot d\vec{r} = 0$.

D satisfies the loop property, so it is conservative on D. $\square$

If $\mathrm{curl}\,\vec{F} \equiv \vec{0}$ for all points in D, then $\vec{F}$ is called irrotational on D. The 3D component test states that conservative fields and irrotational fields are the same thing (on an open simply connected domain).

Divergence

Def: Given a vector field $\vec{F}$, the divergence of $\vec{F}$ is

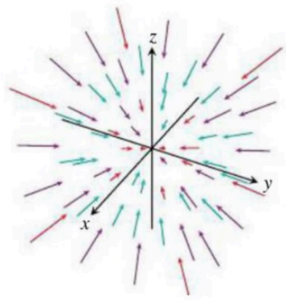
$$\mathrm{div}(\vec{F}) := \nabla \cdot \vec{F}.$$

- If $\vec{F} = \langle M, N, P \rangle$, then $\mathrm{div}\,\vec{F} = \frac{\partial}{\partial x}M + \frac{\partial}{\partial y}N + \frac{\partial}{\partial z}P$.
- If $\vec{F} = \langle M, N \rangle$, then $\mathrm{div}\,\vec{F} = \frac{\partial}{\partial x}M + \frac{\partial}{\partial y}N$.

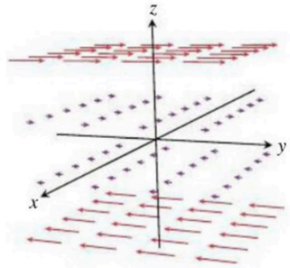
Intuitively, $\mathrm{div}(\vec{F})$ at P describes the outward flux density at P, as discussed in the section of Green's thm (except now it is outward flux across a small sphere centered at P divided by the volume of of the sphere).

$$\mathrm{div}(\vec{F})(P) \begin{cases} > 0 \Rightarrow \text{expanding (diverging) at } P \\ < 0 \Rightarrow \text{compressing (shrinking) at } P \\ = 0 \Rightarrow \text{neither.} \end{cases}$$

e.g.

(a) 

$\mathbf{F}(x, y, z) = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

(b) 

$\mathbf{F}(x, y, z) = z\mathbf{j}$

(a) $div(\vec{F}) = M_x + N_y + P_z = -1-1-1 = -3.$ (Compressing)

(b) $div(\vec{F}) = M_x + N_y + P_z = 0+0+0 = 0.$

We saw that any curl field is "surface independent". Another property of a curl field is the following.

Theorem

If **F** *is a vector field in* $\mathbb{R}^3$ *whose components have continuous second partial derivatives, then*

$$div\,(curl\,\mathbf{F}) \equiv 0.$$

This follows from direct computation with the mixed derivative theorem.

Proof:

$$\begin{aligned} \text{div}\,(\text{curl}\,\mathbf{F}) &= \nabla \cdot (\nabla \times \mathbf{F}) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} \\ &= 0. \end{aligned}$$

Example

Show that $\mathbf{F}(x, y, z) := \langle xy, xyz, -y^2 \rangle$ is not the curl field of any field.