MAT1001 Final Examination

Monday, December 20, 2021

Time: 4:00 - 7:00 PM

Notes and Instructions

1. No books, no notes, no dictionaries, and no calculators.

*2. The total score of this examination is **170**.*

*3. There are **15** questions (with parts) in total.*

*4. The symbol [**N**] at the beginning of a question indicates that the question is worth N points.*

*5. Answer all questions on the **answer book**.*

*6. Show your intermediate steps **except Questions 1, 2 and 3** — answers without intermediate steps will receive minimal (or even no) marks.*

7. The symbols arcsin, arctan*, etc., denote the inverse trigonometric functions* sin^{-1}, tan^{-1}, *etc.* .

MAT1001 Final Examination Questions

1. [**9**] Multiple choice (only one answer is correct). No explanation is required.

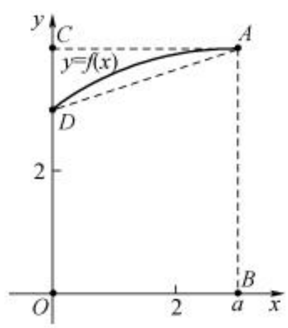
(i) For a concave down curve lying above the x-axis, which of the following approximations for the area under the curve always gives the upper sum?

A) Left-hand rule
B) Right-hand rule
C) Mid-point rule
D) None of the above

(ii) Which one of the following functions grows faster than x^2 as $x \to \infty$?

A) $(1/10)^x$
B) $x^2 + \sqrt{x}$
C) $x^2 e^{-x}$
D) $1.01^{0.1x}$

(iii) Consider the following figure, and suppose that f' is continuous. Which of the following areas is represented by $\int_0^a x f'(x)dx =$?



A) the area of trapezoid $ABOD$
B) the area of the region bounded by AB, BO, OD and arc DA
C) the area of the region bounded by AC, CD and the arc DA
D) the area of triangle ACD

2. [**12**] True or False? No explanation is required.

(i) $1/x^2 = o(\arcsin(1/x))$ as $x \to \infty$.

(ii) $x^{-2} + x^{-4} = O(x^{-4})$ as $x \to \infty$.

(iii) The function defined on $D = [0, 2]$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x \ln x, & \text{if } 0 < x \leq 2 \end{cases}$$

is continuous.

(iv) If f and g are both continuous on $[0, \infty)$, where $\int_0^\infty g(x)\,dx$ is convergent and $f(x) \leq g(x)$ for all $x \geq 0$, then $\int_0^\infty f(x)\,dx$ is also convergent.

(v) If f is integrable on $[1, 2]$ and $F(x) = \int_1^x f(t)\,dt$, then F is differentiable on the interval $(1, 2)$.

(vi) Every autonomous differential equation $dy/dx = f(y)$ having an equilibrium solution must have a stable equilibrium solution.

3. [**21**] Short Questions. No explanation is required.

(i) Let $I_k = \int_0^{k\pi} e^{x^2} \sin x\,dx$. Order I_1, I_2 and I_3 from smallest to biggest.

(ii) The table below shows values of a force function $f(x)$, where x represents distance (in meters) and $f(x)$ is measured in newtons. Use Simpson's rule to estimate the work done by the force in moving an object for 12 m. **You do not need to simplify your expression**.

x	0	3	6	9	12
$f(x)$	9.8	9.1	8.5	8.0	7.7

(iii) Consider the curve C given by $y = x^2 - 4x + 3$. Let L_1 and L_2 be the tangent lines of C at the points $(0, 3)$ and $(3, 0)$, respectively. Find the area of the region bounded by C, L_1 and L_2.

(iv) Let f be continuous on $[a, b]$, and let $F(x) = \int_a^x f(t)(x - t)\,dt$. Then for $x \in [a, b]$, $F''(x) =$ ________.

(v) Let $I = \int_{-1.5}^{-0.5} e^x \sin x \, dx$, and let T and M be the approximated values of I using the trapezoidal rule and the midpoint rule, with 12 subintervals of equal length, respectively. Order the values of I, T and M from smallest to biggest.

(vi) Let $y = \dfrac{x5^x}{\log_3(x)}$. Find dy/dx. (**No need to simplify**.)

(vii) Find the value of

$$\lim_{n\to\infty} \frac{1}{n^{16}} \sum_{k=1}^{n} k^{15}.$$

4. [**24**] Evaluate the following limits, or explain why they do not exist.

(i) $\lim_{x\to\infty} (\sqrt{x^2+x+1} - \sqrt{x^2-x})$

(ii) $\lim_{x\to\infty} (x+e^x)^{2/x}$

(iii) $\lim_{x\to 0} \dfrac{\int_0^x t \arctan(2t)\, dt}{e^{\sin^3 x} - 1}$

(iv) $\lim_{x\to\infty} x^\pi e^{-\sqrt{2}x}$.

5. [**30**] Find the following integrals.

(i) $\int_0^{\ln 9} e^\theta \sqrt{e^\theta - 1}\, d\theta$

(ii) $\int \dfrac{dt}{(3t+1)\sqrt{9t^2+6t}}$

(iii) $\int \tan^5 x \sec^3 x \, dx$

(iv) $\int_0^1 \dfrac{x-4}{x^2-5x+6}\, dx$

(v) $\int \arctan(1/x)\, dx$

6. [**6**] Solve the initial value problem

$$\frac{dy}{dt} = (5-y)t, \quad y(2) = 1.$$

7. [**6**] Let f be differentiable on $D = [0, \infty)$ with $f(0) = 0$, and suppose that it has an inverse function g. If

$$\int_0^{f(x)} g(t)\, dt = x^2 e^x,$$

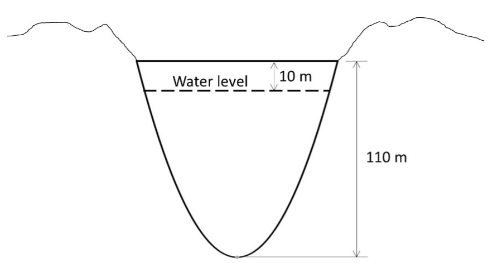
find $f(x)$.

8. [**6**] Find the volume of the solid generated by revolving the region bounded by the curves $y = x^2 - 1$ and $y = -(x-2)^2 + 3$ about the line $y = -2$.

9. [**6**] Find the length of the curve $y = \int_0^x \tan t\, dt$ for $0 \le x \le \pi/3$.

10. [**6**] Find the area of the surface generated by revolving the curve $x = \sqrt{2y-1}$ for $1 \le y \le 2$ about the y-axis.

11. [**3+5**] A dam (水坝) is built across a valley, consisting of a concrete **vertical and flat** wall, 110 m high, which holds the water in a reservoir (an artificial lake). The face of the dam has a parabolic shape shown below (if the bottom is the origin $(0,0)$, then the shape is given by $y = x^2$). The surface of the water reaches a height that is 10 m below the top of the dam. Take the weight-density of water as 10000 N/m^3.



(i) Write an expression for the approximate force that the water imparts upon a narrow horizontal strip of the dam (with thickness Δy), at an elevation y^* from the bottom of the dam.

(ii) What is the total force that the water imparts upon the dam?

12. [**8**] A pizza is removed from the oven after baking thoroughly, and its temperature is 350°F. The temperature of the kitchen is constantly 50°F. Five minutes later, the temperature of the pizza is 320°F. We would like to wait until the temperature of the pizza reaches 290°F before cutting and serving it. Assume that the rate of change of the temperature of the pizza is proportional to its temperature difference from the kitchen temperature. At what time can we start cutting the pizza? (You may take $ln(2) \approx 0.7$, $ln(3) \approx 1.1$ and $ln(10) \approx 2.3$.)

13. [**8**] Let I be an open interval. Use properties of limits to prove **logically** that if a function f is differentiable on I, then it is continuous on I.

14. [**2+6**] Let f be differentiable on its domain I (an interval). Suppose that it has an inverse function g, and let x_0 be a point in the domain of g.

(i) Express $g'(x_0)$ (in terms of information of f, such as f').

(ii) Use (i) to prove that if the cosine function has domain $(0, \pi)$, then

$$\arccos'(x) = \frac{-1}{\sqrt{1-x^2}}.$$

15. [**12**] For each of the following improper integrals, either find its value, or show that it diverges.

(i) $\displaystyle\int_2^\infty \frac{x^2}{(\sqrt{x^5}-1)\ln x}\,dx$

(ii) $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx$