STA2001 Tutorial 9

1. 4.3-10. Let $f_X(x) = 1/10, x = 0, 1, 2, \cdots, 9$, and $h(y|x) = 1/(10-x), y = x, x+1, \cdots, 9$. Find

   (a) $f(x, y)$.

   (b) $f_Y(y)$.

   (c) $E(Y|x)$.

   Solution:

   (a) Using Bayes' theorem for continuous random variables, we have

   $$f(x,y) = f_X(x) \cdot h(y|x) = \frac{1}{10(10-x)}$$

   where $x = 0, 1, \cdots, 9$ and $y = x, x+1, \cdots, 9$.

   (b) Since we have obtained the joint pmf from (a) and notice that $0 \leq x \leq y \leq 9$, we could compute the marginal pmf as follows

   $$f_Y(y) = \sum_x f(x,y) = \sum_{x=0}^{y} f(x,y) = \sum_{x=0}^{y} \frac{1}{10(10-x)}$$

   where $y = 0, 1, 2, \cdots, 9$.

   (c) Using the condition probability density $h(y|x)$, we have

   $$\begin{aligned} E(Y|x) &= \sum_{y=x}^{9} y h(y|x) \\ &= \sum_{y=x}^{9} y \frac{1}{10-x} \\ &= \frac{1}{10-x} \cdot \sum_{y=x}^{9} y \\ &= \frac{1}{10-x} \frac{(9+x)(9-x+1)}{2} \\ &= \frac{9+x}{2} \end{aligned}$$

   where $x = 0, 1, \cdots, 9$.

2. 4.4-11. Let X and Y have the joint pdf $f(x, y) = cx(1-y)$, $0 < y < 1$, and $0 < x < 1 - y$.

   (a) Determine the value of c.

   (b) Compute $P(Y < X | X \leq 1/4)$.

   Solution:

   (a) Since $f(x, y)$ is a joint pdf with support on $0 < y < 1$ and $0 < x < 1 - y$, then we have

$$\begin{aligned}
&\int_0^1 \int_0^{1-y} cx(1-y)dx\, dy = 1 \\
&\Rightarrow c \cdot \int_0^1 (1-y) \cdot \frac{x^2}{2}\Big|_0^{1-y} dy = 1 \\
&\Rightarrow c \cdot \left(-\frac{1}{8}(1-y)^4\right)\Big|_0^1 = 1 \\
&\Rightarrow c = 8
\end{aligned}$$

   Hence,

$$f(x, y) = 8x(1-y), \qquad 0 < y < 1,\ 0 < x < 1 - y$$

   (b) Before answering the question, it's better to sketch the region such the join pdf has non-zero density, which may help you to determine the integration region later.

$$\begin{aligned}
P(Y < X | X \leq 1/4) &= \frac{P(Y < X, X \leq 1/4)}{P(X \leq 1/4)} \\
&= \frac{\int_0^{1/4}\int_0^x 8x(1-y)dy\, dx}{\int_0^{1/4}\int_0^{1-x} 8x(1-y)dy\, dx} \\
&= \frac{29/6144}{31/2048} \\
&= \frac{29}{93}
\end{aligned}$$

3. 4.4-20. Let X have a uniform distribution on the interval $(0, 1)$. Given that $X = x$, let Y have a uniform distribution on the interval $(0, x + 1)$.

   (a) Find the joint pdf of X and Y. Sketch the region where $f(x, y) > 0$.

   (b) Find $E(Y|x)$, the conditional mean of Y, given that $X = x$. Draw this line on the region sketched in part (a).

   (c) Find $f_Y(y)$, the marginal pdf of Y. Be sure to include the domain.

Solution:

(a) Since we are given two uniform distributions,

$$f_X(x) = 1, \qquad 0 < x < 1$$
$$h(y|x) = \frac{1}{x+1}, \qquad 0 < y < x + 1 \text{ when } 0 < x < 1$$

then by Bayes' theorem we have

$$f(x, y) = f_X(x) \cdot h(y|x) = \frac{1}{x+1}$$

where $0 < x < 1$, $0 < y < x + 1$.

The sketched region will be given later.

(b) By definition and use the conditional distribution, we have

$$\begin{aligned} E(Y|x) &= \int_0^{x+1} y h(y|x) dy \\ &= \int_0^{x+1} y \frac{1}{x+1} dy \\ &= \frac{1}{x+1} \cdot \frac{y^2}{2}\Big|_0^{x+1} \\ &= \frac{x+1}{2}, \qquad 0 < x < 1 \end{aligned}$$

The red line in the following graph is the desired line, and the region in purple is the required region in question (a).

(c) The marginal pdf of Y is the integral of the joint pdf over all possible values of X. Recall that the joint pdf obtained in question (a) is

$$f(x, y) = f_X(x) \cdot h(y|x) = \frac{1}{x+1}$$

where $0 < x < 1$, $0 < y < x + 1$.

Let's take a look at the domain for y. For $0 < y \le 1$, there is no constraint for X, which means X may take any values within $(0, 1)$. However, for $1 < y < x + 1$,

we must have $0 < y-1 < x$, which leads to a constraint on X, and X may take values within $(y-1, 1)$. Hence, we have
If $0 < y \leq 1$,

$$f_Y(y) = \int_0^1 f(x,y)dx \, = \int_0^1 \frac{1}{x+1}dx \, = \ln(x+1)\Big|_0^1 = \ln 2$$

If $1 < y < 2$,

$$f_Y(y) = \int_{y-1}^1 f(x,y)dx \, = \int_{y-1}^1 \frac{1}{x+1}dx \, = \ln(x+1)\Big|_{y-1}^1 = \ln 2 - \ln y$$

To summary,

$$f_Y(y) = \begin{cases} \ln 2 & 0 < y \leq 1 \\ \ln 2 - \ln y & 1 < y < 2 \end{cases}$$