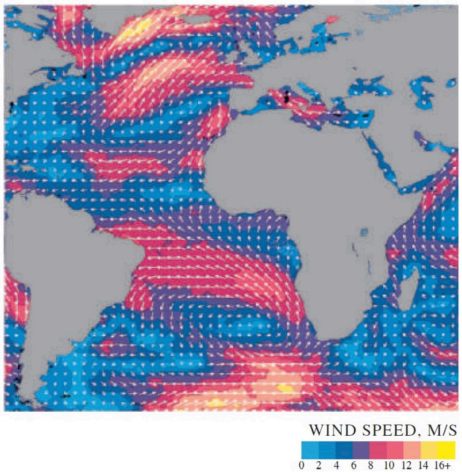
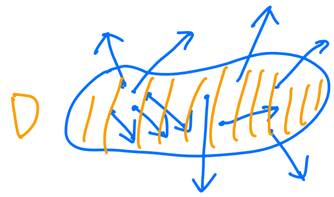
MAT1002 Lecture 22, Tuesday, Apr/11/2023

Outline

- Vector fields (16.2)
  - ↳ Definition and examples
  - ↳ Line integrals of vector fields
  - ↳ Flow, circulation and flux
- Conservative fields (16.3)

Vector Fields





Def: Let $D \subseteq \mathbb{R}^n$ be a set of points. A vector field (in $\mathbb{R}^n$) is a function $\vec{F}: D \to \mathbb{R}^n$ that assigns each point in D a vector in $\mathbb{R}^n$.

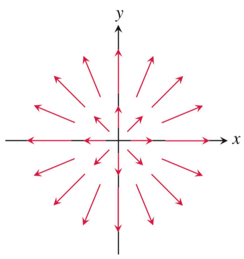


FIGURE 16.11 The radial field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ of position vectors of points in the plane. Notice the convention that an arrow is drawn with its tail, not its head, at the point where $\mathbf{F}$ is evaluated.

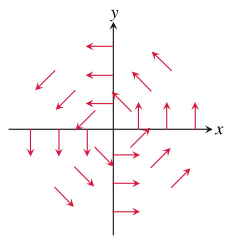


FIGURE 16.12 A "spin" field of rotating unit vectors

$$\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$$

in the plane. The field is not defined at the origin.

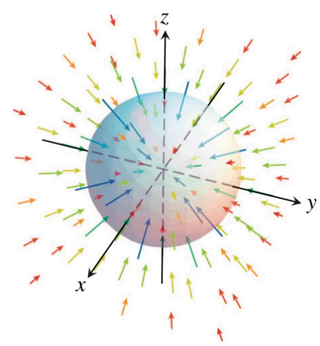
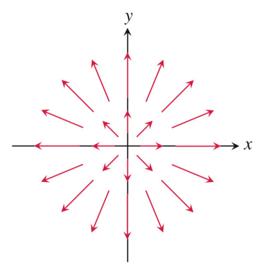


FIGURE 16.8 Vectors in a gravitational field point toward the center of mass that gives the source of the field.

e.g. For $f(x,y) = x^2+y^2$, $\nabla f(x,y) = \langle 2x, 2y \rangle$ $\rightarrow$

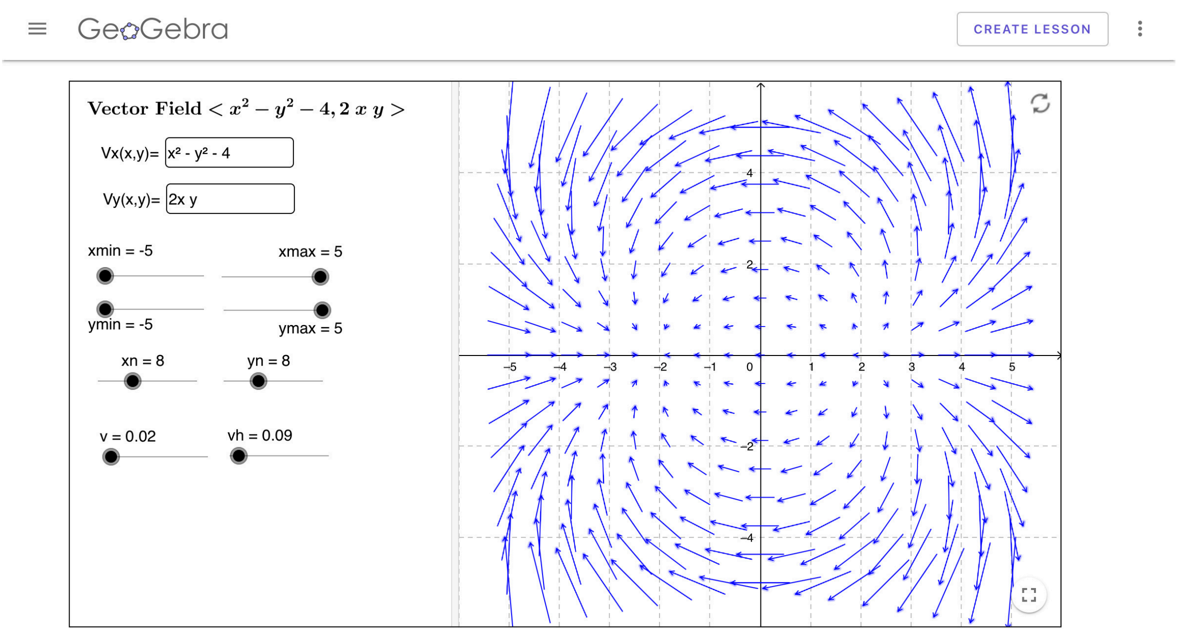


Direction of fastest increase

Every scalar function $f: \underset{(\subseteq \mathbb{R}^n)}{D} \to \mathbb{R}$, $f(x_1, x_2, \ldots, x_n)$, gives a gradient field ∇f, which is a vector field in $\mathbb{R}^n$.

Examples of vector fields:

- Velocity fields
- Force fields
- Gradient fields



We will focus on vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$, which we will write as

$$\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$$

and

$$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle.$$

Line Integrals of Vector Fields

A Motivation : Work

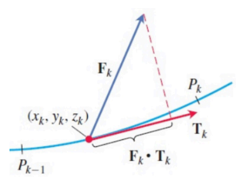
A basic formula in physics states that

$$\text{Work} = \text{Force} \cdot \text{Distance},$$

if the force is a constant. Suppose that a force field $\mathbf{F}$ in the space ~~drags~~ acts on a particle moving along a smooth curve C. To approximate the total work done by $\mathbf{F}$ in moving the particle, we may do the following.

- Partition C into $\{P_0, P_1, \ldots, P_n\}$.
- The work done by $\mathbf{F}$ in moving the particle from P_{k-1} to P_k is approximately $\mathbf{F}(x_k, y_k, z_k) \cdot \mathbf{T}(x_k, y_k, z_k)\, \Delta s_k$, as shown in the following figure.
- We expect the total work done to be

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} \mathbf{F}(x_k, y_k, z_k) \cdot \mathbf{T}(x_k, y_k, z_k)\, \Delta s_k.$$



We define the work done by a continuous force field $\mathbf{F}$ in moving a particle along C to be

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

This is an example of a line integral of a vector field.

Note that $f(x,y,z) := \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)$ is a real-valued function, so $\int_C \vec{F} \cdot \vec{T}\, ds = \int_C f\, ds$.

talked about this in 16.1.

Definition
Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C. The line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

Remark
If C is parametrized by $\mathbf{r}(t)$, for $t \in [a, b]$, then

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

For this reason, the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is often also written as

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Alternatively

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot \frac{d\vec{r}}{ds}\, ds = \int_C \vec{F} \cdot d\vec{r}$$

Computation

For a space curve C given by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $(a \le t \le b)$

$$d\vec{r} = \langle dx, dy, dz \rangle.$$

If $\vec{F} = \langle M, N, P \rangle$, then ← $M(x,y,z)$

Different notations for the same integral.

$$\boxed{\begin{aligned} \int_C \vec{F}\cdot\vec{T}\,ds &= \int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz \\ &= \int_C M\,dx + \int_C N\,dy + \int_C P\,dz. \end{aligned}}$$

For computation, we may change all variables into the parameter t;

e.g., $\int_C M\,dx = \int_a^b M(x(t), y(t), z(t))\, x'(t)\,dt$, etc.

Note that $\int_C M\,dx + N\,dy + P\,dz = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$.

Example

Find the work done by the force field $\mathbf{F} = \langle x, y, z \rangle$ in moving a particle along the curve C parametrized by

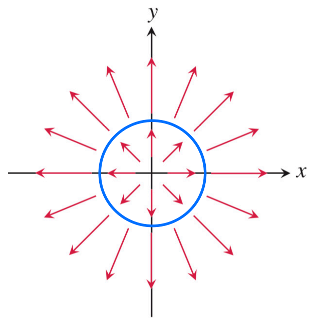
$$\mathbf{r}(t) = \langle \cos(\pi t), t^2, \sin(\pi t) \rangle, \quad t \in [0,1].$$

Ans: $1/2$ (energy unit).

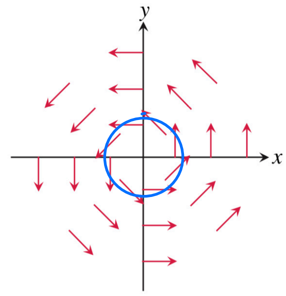
(e.g. 16.2.3)

e.g. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \langle -y, z, 2x \rangle$ and C is the Helix $\langle x,y,z \rangle = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$.

Ans: π.



$\vec{F}_1 = \langle x, y \rangle$



$\vec{F}_2 = \dfrac{1}{\sqrt{x^2+y^2}} \langle -y, x \rangle$

Imagine the two fields above being the velocity fields of some flowing fluid ("waterflow").

- $\vec{F}_1$ is not causing any rotation (0 "circulation") but is causing an expansion to the blue curve (positive "flux").
- $\vec{F}_2$ is not causing any expansion (0 "flux") but is causing a (counterclockwise) rotation to the blue curve (positive "circulation").

We will define the mathematical abstractions for these two concepts.

Flows and Circulations

- The line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ of $\vec{F}$ along C is also called a flow integral.
- A curve given by $\vec{r}(t)$, $a \le t \le b$ is said to be closed if

$\vec{r}(a) = \vec{r}(b)$. If additionally, $\vec{r}$ is one-to-one on $[a,b)$, then it is a simple closed curve.

does not cross itself, except possibly at endpoints.

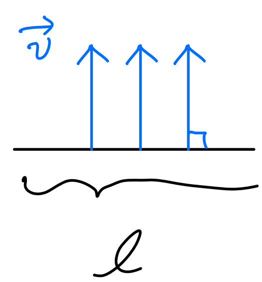
- When C is closed, we may write $\oint_C \vec{F}\cdot d\vec{s}$ instead of $\int_C \vec{F}\cdot d\vec{s}$. This integral is often called a circulation.

- Note that the orientation (direction) of C matters for flows and circulations, since reversing C changes the direction of $\vec{T}$.

We may write $\oint_C$ for $\oint$ to emphasize the orientation of C

Flux (for Vector Fields in $\mathbb{R}^2$)

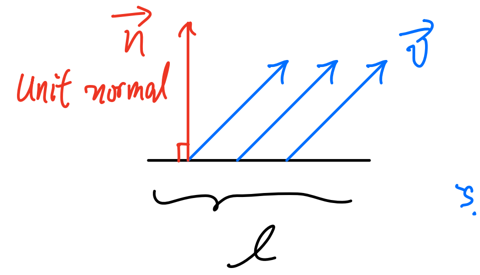
- Consider water flowing with a constant velocity $\vec{v}$ (m/s) directly perpendicular to a line segment L with length ℓ. (m) Then $|\vec{v}|\ell$ (m²/s) measures the amount of water flowing through L per unit time.



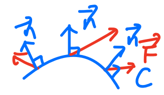
- If $\vec{v}$ is not directly perpendicular to L, then dot product would be required:

$(\vec{v}\cdot\vec{n})\,l$, m^2/s

water through l per unit time



not essential;



Def: If C is a simple closed curve and $\vec{F} = \langle M(x,y), N(x,y) \rangle$ is a vector field in $\mathbb{R}^2$, and $\vec{n}$ is the outward normal C, then the (outward) flux of $\vec{F}$ across C is the integral

$$\int_C \vec{F}\cdot\vec{n}\,ds.$$

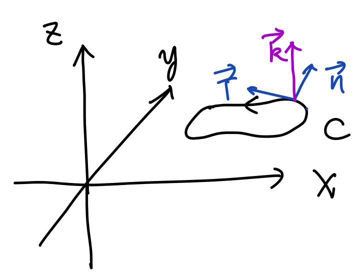
If $\vec{F}$ is the velocity field of some flowing fluid, then the (outward) flux measures the total amount (e.g., m^2) of fluid flowing out of C per unit time (e.g., sec).

Computation

Using the Right-hand rule, one can see that

if C is counterclockwise, then $\vec{n} = \vec{T}\times\vec{k}$, and

if C is clockwise, then $\vec{n} = \vec{k}\times\vec{T}$.



Assuming that C is oriented counterclockwise,

$$\vec{n} = \vec{T} \times \vec{k} = \frac{d\vec{r}}{ds} \times \vec{k} = \left\langle \frac{dx}{ds}, \frac{dy}{ds}, 0 \right\rangle \times \langle 0, 0, 1 \rangle$$

$$= \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j}$$

Hence, $\vec{n}\,ds = dy\,\vec{i} - dx\,\vec{j}$, and

$$\boxed{\int_C \vec{F}\cdot\vec{n}\,ds = \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C (M\,dy - N\,dx).}$$

<u>e.g.</u> Find the flux of $\vec{F} = \langle -y, x \rangle$ across the circle C $x^2+y^2=1$. Then find the (counterclockwise) circulation of $\vec{F}$ along C.

<u>Sol</u>: • C: $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$, Counterclockwise, does not repeat except $\vec{r}(0) = \vec{r}(2\pi)$.

• $dy = \cos t\,dt$, $dx = -\sin t\,dt$

• flux $= \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx$

$$= \int_0^{2\pi} -\sin t \cos t\,dt - \cos t(-\sin t)\,dt = 0$$

• Circulation $= \oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy$

$$= \int_0^{2\pi} -\sin t(-\sin t\,dt) + \cos t(\cos t\,dt) = 2\pi.$$

Also see e.g. 16.2.7 and 16.2.8.

Conservative Fields

A conservative force is a force with the property that the total work done in moving a particle between two points is independent of the path taken. Gravitational force is an example of a conservative force.

Definition

Let $\mathbf{F}$ be a vector field defined on an open region D. Then $\mathbf{F}$ is said to be conservative on D if the following condition holds: for any two points A and B in D, if C_1 and C_2 are piecewise smooth curves in D from A to B, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Any such line integral, which depends only on the initial and terminal points (but not the path itself), is said to be path independent.

Consider the vector field $\mathbf{F}(x, y) := \langle -y, x \rangle$. One can check that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

if C_1 is the line segment from $(0, 0)$ to $(1, 1)$ and C_2 is the curve given by

$$\langle t, t^2 \rangle, \quad t \in [0, 1].$$

Although both C_1 and C_2 are smooth curves from $(0, 0)$ to $(1, 1)$, the line integrals of $\mathbf{F}$ along them are different.

Hence, the field $\vec{F} := -y\vec{i} + x\vec{j}$ is not conservative.

Q: Which fields are conservative? What properties do they have?