introduce paper's orthogonal poly

Proof. Suppose $w(x)$ is an arbitrary weight function and $A = \{u_0, u_1, \dots, u_n\}$ be the associated orthogonal polynomial set, that spans all polynomials of the associated orthogonal polynomial with degree $\leq n$. Let u_{n+1} be the associated orthogonal polynomial with degree $n+1$. Now, consider a monomial, or single term expression, x^γ where $\gamma \leq n$. Now, since orthogonal polynomials form a basis, we can write x^γ in terms of A s.t.

$$x^\gamma = \sum_{i=0}^{\gamma} \gamma_i u_i(x) \quad \text{where } \gamma_i \text{ is some scalar,}$$

Then, we can rewrite $\langle p, q \rangle = \int_a^b p(x) q(x) w(x)\, dx$ as

$$\int_a^b x^\gamma u_{n+1}(x) w(x)\, dx = \int_a^b \left(\sum_{i=0}^{\gamma} \gamma_i u_i(x) \right) u_{n+1}(x) w(x)\, dx$$

which can be rewritten as our defined inner product with $p(x) = u_i(x)$ and $q(x) = u_{n+1}(x)$ such that

$$\Rightarrow \sum_{i=0}^{\gamma} \gamma_i \langle u_i, u_{n+1} \rangle$$

However, u_{n+1} is orthogonal to the set A, and $\forall u_i \in A$. Thus,

$$\Rightarrow \sum_{i=0}^{\gamma} \gamma_i \langle u_i, u_{n+1} \rangle = 0$$

Now, let the points of our quadrature be $x_0, x_1, \dots, x_n$ be our roots for the polynomial $u_{n+1}(x)$ and define our weights as

$$\omega_k = \int_a^b P_k(x) w(x)\, dx \quad \text{where } P_k(x) \text{ is the } k^{th} \text{ basis polynomial}$$

Then, we can write an arbitrary polynomial (of degree $\leq 2n+1$) as the divisible decomposition

$$f(x) = p(x) u_{n+1}(x) + r(x)$$

where p and r are polynomials of at most degree n. The important term to note is the remainder term $r(x)$.

Thus, we can write an integral of $f(x)$ such that

$$I[f] = \int_a^b f(x)\omega(x)\,dx = \int_a^b \left(p(x)\,u_{n+1}(x) + r(x)\right)\omega(x)\,dx$$

$$= \int_a^b (p(x)\,u_{n+1}(x)\,\omega(x)\,dx + \int_a^b r(x)\omega(x)\,dx$$

$$= \int_a^b r(x)\,\omega(x)\,dx$$

where the first term can be written as an inner product and since $p(x)$ is a polynomial of at most degree n, then it is orthogonal to $u_{n+1}(x)$.

Applying our quadratic algorithm, we find

$$Q[f] = \sum_{K=0}^{n} \omega_K f(x_K) = \sum_{K=0}^{n} \omega_K \left(p(x_K)u_{n+1}(x_K) + r(x_K)\right)$$

$$= \sum_{K=0}^{n} \omega_K r(x_K)$$

These p terms vanish because we let x_K to be the roots of u_{n+1}, so $u_{n+1}(x_K) = 0 \Rightarrow p(x_K)u_{n+1}(x_K) = 0$.

Since $Q[f]$ approximates $\int_a^b f(x)\omega(x)\,dx$, but given our $f(x)$ we find that

$$Q[f] = \sum_{K=0}^{n} \omega_K r(x_K) = \int_a^b r(x)\,\omega(x)\,dx = I[f]$$

Therefore, this is an exact integration for an $f \in \mathbb{P}_{2n+1}$.

This proof is complete once you let $\omega(x) = 1$ for Legendre polynomials ($u_K = P_K$). The PDF notes though that you can do a similar method to derive other quadrature polynomial estimates.

Therefore, the optimal points for a gaussian quadrature are the zeros of P_N because it integrates an exact integration. ☒