Phenomenological theory of unconventional superconductivity

[Manfred Sigrist and kazuo Ueda] Rev.Mod.Phys.63,239- April,1991

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References I

[1] James F Annett. "Symmetry of the order parameter for high-temperature superconductivity". In: *Advances in Physics* 39.2 (1990), pp. 83–126.

[2] Manfred Sigrist and Kazuo Ueda. "Phenomenological theory of unconventional superconductivity". In: *Reviews of Modern physics* 63.2 (1991), p. 239.

[3] GE Volovik and LP Gorkov. "A symmetry classification of superfluid 3He phases". In: *Soviet Phys. JETP* 61 (1985), p. 843.

1. BCS theory

2. symmetry and order parameter

3. Coexistence of antiferromagnetism and superconductivity

Hamiltonian

The effective Hamiltonian:

$$H = \sum_{k,s} \epsilon(k) a^{\dagger}_{k,s} a_{k,s} + \frac{1}{2} \sum_{k,k',s_1,s_2,s_3,s_4} V_{s_1,s_2,s_3,s_4}(k,k') a^{\dagger}_{-k,s_1} a^{\dagger}_{k,s_2} a_{k',s_3} a_{-k',s_4} \tag{1}$$

where

$$V_{s_1,s_2,s_3,s_4} = \left\langle -k, s_1 \kappa s_2 \right| \hat{\mathbf{V}} \left| -k', s_4 k', s_3 \right\rangle \tag{2}$$

gap function:

$$\Delta_{s,s'}(k) = -\sum_{k',s_3,s_4} V_{s',s,s_3,s_4}(k,k') < a_{k',s_3} a_{-k',s_4} > \tag{3}$$

$$\Delta^{*}_{s,s'}(-k) = \sum_{k',s_1,s_2} V_{s_1,s_2,s',s}(k',k) < a^{\dagger}_{-k',s_1} a^{\dagger}_{k',s_2} > \tag{4}$$

Here we have use the symmetry property of V

$$V_{s_1 s_2 s_3 s_4}(k,k') = -V_{s_2 s_1 s_3 s_4}(-k,k') = -V_{s_1 s_2 s_4 s_3}(k,-k') = V_{s_4 s_3 s_2 s_1}(k',k) \tag{5}$$

Meanfield approximation

$$H = \sum_{k,s} \epsilon(k) a^\dagger_{ks} a_{ks} + \frac{1}{2} \sum_{k,s_1,s_2} [\Delta_{s_1,s_2}(k) a^\dagger_{ks_1} a^\dagger_{-ks_2} - \Delta^*_{s_1 s_2}(-k) a_{-ks_1} a_{ks_2}] \quad (6)$$

Bogoliubov(or canonical) transformation:

$$a_k = U_k \alpha_k \quad (7)$$

where:

$$a_k = [a_{k,\uparrow}, a_{k,\downarrow}, a^\dagger_{-k,\uparrow}, a^\dagger_{-k,\downarrow}] \quad (8)$$

$$\alpha_k = [\alpha_{k,\uparrow}, \alpha_{k,\downarrow}, \alpha^\dagger_{-k,\uparrow}, \alpha^\dagger_{-k,\downarrow}] \quad (9)$$

We have the diagonalization equation:

$$E_k = U^\dagger_k \mathscr{E}_k U_k \quad (10)$$

where:

$$U_k = \begin{bmatrix} \mu_k & \nu_k \\ \nu^*_{-k} & \mu^*_{-k} \end{bmatrix} \quad and \quad U_k U_k^\dagger = 1$$

$$E_k = \begin{bmatrix} E_{k+} & 0 & 0 & 0 \\ 0 & E_{k-} & 0 & 0 \\ 0 & 0 & -E_{-k+} & 0 \\ 0 & 0 & 0 & -E_{-k-} \end{bmatrix} \tag{11}$$

$$\mathscr{E}_k = \begin{bmatrix} \epsilon(k)\sigma_0 & \Delta(k) \\ -\Delta^*(-k) & -\epsilon(k)\sigma_0 \end{bmatrix} \tag{12}$$

We can separate $\Delta(k)$ according to its parity.

$$\Delta(k) = i\sigma_y \Psi(k) = \begin{bmatrix} 0 & \Psi(k) \\ -\Psi(k) & 0 \end{bmatrix} \tag{13}$$

and

$$\Delta(k) = i(d(k) \cdot \sigma)\sigma_y = \begin{bmatrix} -d_x(k) + id_y(k) & d_z(k) \\ d_z(k) & d_x(k) + id_y(k) \end{bmatrix} \tag{14}$$

by solving (10), we can get U_k.

symmetry and eigenvalue problem

We rewrite (3) as

$$\Delta_{s,s'}(k) = -\sum_{k',s_3,s_4} V_{s'ss_3s_4}(k,k')\mathscr{F}_{s_3,s_4}(k',\beta) \tag{15}$$

where in the case of unitary pairing state

$$\mathscr{F}(k,\beta) = \frac{\Delta(k)}{2E_k} tanh(\frac{\beta E_k}{2}) \tag{16}$$

We give here the self-consistency(or gap) equation;

$$v\Delta_{s_1s_2}(k) = -\sum_{s_3s_4} < V_{s_2s_1s_3s_4}(k,k')\Delta_{s_3s_4}(k') >_{k'} \tag{17}$$

where

$$\frac{1}{\nu} = N(0)\int_0^{\epsilon_c} d\epsilon \frac{tanh(\frac{\beta_c\epsilon(k)}{2})}{\epsilon(k)} = ln(1.14\beta_c\epsilon_c) \tag{18}$$

We shall use the property of eigenvalue problems that the eigenfunction spaces of the singlet eigenvalue forms a basis of a irreducible representation of the symmetry group of the equation.

$$\Delta(k) = \sum_m \eta(\Gamma, m)\Delta(\Gamma, m; k) \tag{19}$$

symmetry and order parameter

- broken symmetry and phase
- broken time reversal symmetry and magnetism[Landau]
- classification

Coexistence of antiferromagnetism and superconductivity

The paper start from the microscopic derivation of the coupling term between the two order parameter by weak-coupling theory.In the presense of antiferromagnetic ordering, the original symmetry group was lowered from g to g'.
The magnetic free energy in second order can be written as(Ueda and Konno, 1988; Konno and Ueda, 1989)

$$F_M^{(2)} = \frac{1}{2}\sum_{i,j} M_i[(\chi^{-1})_{ij} - I\delta_{ij}]M_j \tag{20}$$

We have

$$\delta(\chi^{-1})_{ij} = -\sum_{i',j'}(\chi^{-1})_{i,i'}\delta\chi_{i',j'}(\chi^{-1})_{j',j} \tag{21}$$

to prove it just note that $\chi\chi^{-1} = I$.
define:

$$M_i = \sum \chi_{ij} B_j \tag{22}$$

the coupling term has the form:

$$F_{\Delta M} = -\frac{1}{2}\sum_{i,j} B_i \delta\chi_{ij} B_j \tag{23}$$

To calculate χ :

$$\chi_{ij} = \int_0^\beta d\tau < s_i(Q,\tau)s_j(-Q,0) > \tag{24}$$

with

$$s_j(Q) = \sum_k c^\dagger_{k+Q,\alpha}(\sigma_j)_{\alpha\beta}c_{k\beta} \tag{25}$$

$$< s_i(Q,\tau)s_j(-Q,0) > \tag{26}$$

$$=< \sum_k C^\dagger_{k+Q,\alpha}(\sigma_i)_{\alpha\beta}C_{k,\beta}\sum_{k'} C^\dagger_{k'-Q,\alpha'}(0)(\sigma_j)_{\alpha'\beta'}C_{k',\beta'}(0) > \tag{27}$$

Wick's theorem fails, anomalous green's function is not zero due to copper pairing. Sum over matsubara frequencies $i\omega_n$ via Fourier transformation.

The Mausubara frequency summation:

$$G_{ss'}(k,k';\tau) = k_B T \sum_n G_{ss'}(k,k',i\omega_n)e^{-i\omega_n\tau} \tag{28}$$

$$F_{ss'}(k,k',\tau) = k_B T \sum_n F_{ss'}(k,k',i\omega_n)e^{-i\omega_n\tau} \tag{29}$$

$$F^\dagger_{ss'}(k,k';\tau) = k_B T \sum_n F^\dagger_{ss'}(k,k'\pi\omega_n)e^{-i\omega_n\tau} \tag{30}$$

Here I only show the first term of (28):

$$= \int_0^\beta -(\sigma_i)_{\alpha\beta}(\sigma_j)_{\alpha'\beta'}G_{\beta'\alpha}(k+Q,\tau)G_{\beta\alpha'}(k,-\tau) \tag{31}$$

$$= \int_0^\beta -(\sigma_i)_{\alpha\beta}(\sigma_j)_{\alpha'\beta'}(k_B T)^2 \sum_{n,n'} e^{-i\omega_n\tau}G_{\beta'\alpha}(k+Q;i\omega_n)e^{i\omega_{n'}\tau}G_{\beta\alpha'}(k,i\omega_{n'}) \tag{32}$$

$$= -(\sigma_i)_{\alpha\beta}(\sigma_j)_{\alpha'\beta'}(k_B T)^2 \int_0^\beta e^{-i(\omega_n-\omega_{n'})\tau} \sum G_{\beta'\alpha}(k+Q;i\omega_n)G_{\beta\alpha'}(k,i\omega_{n'})$$

$$\chi_{ij} = k_B T \sum_{i\omega_n} \sum_{\alpha\alpha'\beta\beta'} \sum_{k} [- (\sigma_i)_{\alpha\beta}(\sigma_j)_{\alpha'\beta'} G_{\beta'\alpha}(k+Q, i\omega_n) G_{\beta\alpha'}(k, i\omega_n) \tag{35}$$

$$+ (\sigma_i)_{\alpha\beta}(\sigma_j)_{\alpha'\beta'} F^{\dagger}_{\alpha'\alpha}(k+Q, i\omega) F_{\beta\beta'}(k, i\omega_n)] \tag{36}$$

using the following expansion for green's functions:

$$G_{\alpha\beta}(k, i\omega_n) = \frac{\delta_{\alpha\beta}}{i\omega_n - \epsilon(k)} + \frac{\sum_\gamma \Delta_{\alpha\gamma}\Delta^{\dagger}_{\gamma\beta}}{[i\omega_n - \epsilon(k)][\omega_n^2 + \epsilon(k)^2]} \tag{37}$$

$$F_{\alpha\beta}(k, i\omega_n) = \frac{\Delta_{\alpha\beta}(k)}{\omega_n^2 + \epsilon(k)^2} \tag{38}$$

$$F^{\dagger}_{\alpha\beta}(k, i\omega_n Z) = -\frac{\Delta^{\dagger}_{\alpha\beta}(k)}{\omega_n^2 + \epsilon(k)^2} \tag{39}$$

Green's function formualtion

We start from the Hamiltonian:

$$H = H_0 + H_{pair} \tag{40}$$

$$= \sum_{k,k',s,s'} \langle ks| H_0 \left|k's'\right\rangle a^\dagger_{k_1 s_1} a_{k_2 s_2} \tag{41}$$

$$+ \sum_{k,k',q,s_{1,2,3,4}} V_{s_1,s_2,s_3,s_4}(k,k') a^\dagger_{(\frac{q}{2})-k,s_1} a^\dagger_{(\frac{q}{2})+k,s_2} a_{(\frac{q}{2})+k',s_3} a_{\frac{q}{2}-k',s_4} \tag{42}$$

green's function:

$$G_{ss'}(k,k';\tau) = - < T(a_{ks}(\tau) a^\dagger_{k's'}(0)) > \tag{43}$$

$$F_{ss'}(k,k';\tau) = < T(a_{ks}(\tau) a_{k's'}(0)) > \tag{44}$$

$$F^\dagger_{ss'}(k,k';\tau) = < T(a^\dagger_{k's'}(\tau) a^\dagger_{ks}(0)) > \tag{45}$$

Use now the equation of motion for the Green's functions:

$$\frac{\partial a_{ks}}{\partial \tau} = [H, a_{ks}] \tag{46}$$

$$\sum_{\mathbf{k}'',s''} \left\{ \langle \mathbf{k}s | i\omega_n - \mathcal{H}_0 | \mathbf{k}''s'' \rangle G_{s''s'}(\mathbf{k}'', \mathbf{k}'; i\omega_n) \right. \tag{47}$$

$$- \sum_{\mathbf{q}} \Delta_{ss''}(\mathbf{k}'', \mathbf{q}'') F^{\dagger}_{s''s'} \left[\frac{\mathbf{q}''}{2} - \mathbf{k}'', \mathbf{k}'; i\omega_n \right] \delta_{\mathbf{q}''/2+\mathbf{k}'',\mathbf{k}} = \delta_{\mathbf{k},\mathbf{k}'} \delta_{s,s'}, \qquad t(48)$$

$$\Delta_{ss'}(k,q) = - \sum_{k',s_1,s_2} V_{s'ss_1s_2}(k,k') < a_{\frac{q}{2}+k',s_1} a_{\frac{q}{2}-k',s_2} > \tag{49}$$

$$= -k_B T \sum_n \sum_{k',s_1,s_2} V_{s'ss_1s_2}(k,k') F_{s_1s_2}(\frac{q}{2}+k', \frac{q}{2}-k'; i\omega_n) \tag{50}$$

$$[i\omega_n - \varepsilon(\mathbf{k})]\, G_{ss'}(\mathbf{k}, i\omega_n) - \sum_{\mathbf{q},s''} \Delta_{ss''}(\mathbf{k},\mathbf{q}) F^{\dagger}_{s''s'}\left(\frac{\mathbf{q}}{2} - \mathbf{k}, i\omega_n\right) = \delta_{ss'}, \quad (51)$$

$$[i\omega_n + \varepsilon(\mathbf{k})]\, F^{\dagger}_{ss'}(\mathbf{k}, i\omega_n) - \sum_{\mathbf{q},s''} \Delta^{\dagger}_{s''s'}(\mathbf{k},\mathbf{q}) G_{s''s'}\left(\frac{\mathbf{q}}{2} + \mathbf{k}, i\omega_n\right) = 0, \quad (52)$$

$$[i\omega_n - \varepsilon(\mathbf{k})]\, F_{ss'}(\mathbf{k}, i\omega_n) - \sum_{\mathbf{q},s''} \Delta_{ss''}(\mathbf{k},\mathbf{q}) G^{\dagger}_{s''s'}\left(\mathbf{k}, \frac{\mathbf{q}}{2} - \mathbf{k}, -i\omega_n\right) = 0. \quad (53)$$

$$\Delta_{ss'}(k,q) = -\sum_{k',s_1,s_2} V_{s'ss_1s_2}(k,k') < a_{\frac{q}{2}+k',s_1} a_{\frac{q}{2}-k',s_2} > \tag{54}$$

$$= -k_B T \sum_n \sum_{k',s_1,s_2} V_{s'ss_1s_2}(k,k') F_{s_1s_2}(\frac{q}{2}+k', \frac{q}{2}-k'; i\omega_n) \tag{55}$$