x x x model(Heisenberg model)

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1 Introduction

Hamiltonian of the model

$$H = -\frac{1}{2}J\sum_{j=1}^{N}\sigma_j \cdot \sigma_{j+1} - \frac{1}{2}h\sum_{j=1}^{N}\sigma_j^z \tag{1}$$

We let $h = 0$.
With definition $S^{\pm} = S^x \pm iS^y, S^x = \frac{1}{2}\sigma^x \cdots$. We have

$$[S_j^{\pm}, S_i^{\pm}] = 0 \qquad [S_i^{\pm}, S_j^{\mp}] = \pm 2S_i^z\delta_{ij} \tag{2}$$

$$[S_i^z, S_j^{\pm}] = \pm S_i^{\pm}\delta_{ij} \qquad [S_i^{\pm}, S_j^z] = \mp S_i^{\pm}\delta_{ij} \tag{3}$$

We may let $h = 0$. We can then rewrite (1)

$$H = -\frac{1}{2}J\sum_{j=1}^{N}(S_j^+S_{j+1}^- + S_j^-S_{j+1}^+ + 2S_j^zS_{j+1}^z) - h\sum_{j=1}^{N}S_j^z \tag{4}$$

We have(can be easily verified)

$$S_j^+ \left|\uparrow\right\rangle_j = 0 \quad S_j^- \left|\uparrow\right\rangle_j = \left|\downarrow\right\rangle_j \quad S_j^z \left|\uparrow\right\rangle_j = \frac{1}{2}\left|\uparrow\right\rangle_j \tag{5}$$

Define vaccum state

$$\left|0\right\rangle = \left|\uparrow\right\rangle_1 \otimes \left|\uparrow\right\rangle_2 \otimes \cdots \otimes \left|\uparrow\right\rangle_N \tag{6}$$

We can see $\left|0\right\rangle$ is the eigenstate of (4)

$$H\left|0\right\rangle = (-\frac{J}{4}N - \frac{1}{2}Nh)\left|0\right\rangle \equiv E_0\left|0\right\rangle \tag{7}$$

We will take advantage of symmetries.

- translational invariance → superpositions of plane waves.
- Heisenberg chain possesses full $SU(2)$ rotational invariance.
  One way to understand SU(2) symmetry.
  The generators of $SU(2)$ group are S_x, S_y, S_z or S^+, S^-, S_z which combined with the commutation relations form a algebra. Besides, $[H, \sum_i S_i^{x,y,z}] = 0$ should be satisfied to maintain $SU(2)$ invariance

We define the $z-conponent$ of the total spin $S^z = \sum_{n=1}^{N} S_n^z$. The conservation of S^z can be easily derived by verifying that $[H, S^z] = 0$. Consequently, we can consider separately sectors defined by the quantum number $S^z = \frac{N}{2} - R$, where R is the number of down spins.
We define the state with the nth spin down as:

$$|n\rangle = S_n^- |0\rangle \quad n = 1, \cdots, N \tag{8}$$

However $|n\rangle$ is not the eigenstate of H. We define

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{ikn} |n\rangle \quad k = 2\pi m/N, m = 0, \cdots, N-1 \tag{9}$$

as a linear combination of $|n\rangle$, with translational symmetry(i.e. , the invariance of H with respect to discrete translations) implicitly included by a phase factor.
The vectors $|\psi\rangle$ are eigenstates of translation operator with eigenvalues e^{ik} and also of H with eigenvalues

$$E = E_0 + J(1 - cosk) \tag{10}$$

Question: About the translation operator and its eigenvalues e^{ik}.
Vectors (9) represent *magnon excitations*($\Delta S = 1$ excitations). **The $k = 0$ state is degenerate with** $|0\rangle$. Why is this stressed.

2 Two-body problem

For $R = 2$, we write a generic eigenstates as

$$|\psi\rangle = \sum_{1 \le n_1 \le n_2 \le N} f(n_1, n_2) |n_1, n_2\rangle \tag{11}$$

where $|n_1, n_2\rangle = S_{n_1}^- S_{n_2}^- |F\rangle$. We then consider the eigenvalue equation

$$H|\psi\rangle = E|\psi\rangle \tag{12}$$

which indicates

$$2[E - E_0]f(n_1, n_2) = J(4f(n_1, n_2) - f(n_1 - 1, n_2) - f(n_1 + 1, n_2) - f(n_1, n_2 - 1) - f(n_1, n_2 + 1)) \quad \text{for} \quad n_2 > n_1 + 1 \tag{13}$$

$$2[E - E_0]f(n_1, n_2) = J(2f(n_1, n_2) - f(n_1 - 1, n_2) - f(n_1, n_2 + 1)), \quad \text{for} \quad n_2 = n_1 + 1 \tag{14}$$

Bethe's priliminary ansatz for $f(n_1, n_2)$ is:

$$f(n_1, n_2) = Ae^{i(k_1 n_1 + k_2 n_2)} + A'e^{i(k_1 n_2 + k_2 n_1)} \tag{15}$$

which atomatically satisfy (14) wiht energy

$$E = E_0 + J \sum_{j=1,2} (1 - cosk_j) \tag{16}$$

We have conditions for coefficients,

$$\frac{A}{A'} \equiv = e^{i\theta} = -\frac{e^{i(k_1+k_2)} + 1 - 2e^{ik_1}}{e^{i(k_1+k_2)} + 1 - 2e^{ik_2}} \tag{17}$$

which can be rewritten as

$$2cot\frac{\theta}{2} = cot\frac{k_1}{2} - cot\frac{k_2}{2} \tag{18}$$

The quasi-momenta k_1, k_2 of the Bethe Ansatz is determined by the periodic boundary conditions: $f(n_1, n_2) = f(n_2, n_1 + N)$(We interchange the position of n_1 and n_2) :

$$e^{ik_1 N} = e^{i\theta}, \quad e^{ik_2 N} = e^{-i\theta} \tag{19}$$

Equivalently

$$Nk_1 = 2\pi I_1 + \theta, \quad Nk_2 = 2\pi I_2 - \theta \tag{20}$$

where the $I_j \in \{0, 1, \cdots, N-1\}$ are integer quantum numbers.
We can see

$$k_1 = \frac{2\pi I_1}{N} + \frac{\theta}{N} \quad k_2 = \frac{2\pi I_2}{N} - \frac{\theta}{N} \tag{21}$$

The restrictioin on I_j maintains k in the first Brillouin zone. The total momentum of the state is

$$K = k_1 + k_2 = \frac{2\pi}{N}(I_1 + I_2) \tag{22}$$

Question: Can the phase shift θ be observed. What's the physical significance. The Berry phase is a geometry structure then what about this phase. Is the total momentum conserved. Is (I_1, I_2) describing energy level.

If we restrict I_1, I_2 to $0 \leq I_1 \leq I_2 \leq N-1$, there are $\frac{N(N+1)}{2}$ pairs meet the ordering restriction. But only $\frac{N(N-1)}{2}$ satisfy (18) (20).
We get (23) by combining (18) (20) (22):

$$2\cot\frac{Nk_1}{2} = \cot\frac{k_1}{2} - \cot\frac{K-k_1}{2} \tag{23}$$

Let

$$k_1 \equiv \frac{K}{2} + ik \quad k_2 \equiv \frac{K}{2} - ik \tag{24}$$

which, substituted into (20) yields

$$\theta = \pi(I_2 - I_1) + iNk \tag{25}$$

Equation (18) becomes

$$cos\frac{K}{2}sinh(Nk) = sinh[(N-1)k] + cos[\pi(I_1 - I_2)]sinhk \tag{26}$$

3 The Bethe Solution

We expand the states with generic R overturned spins into the natural(computational) basis

$$|\psi\rangle = \sum_{1\leq n_1\leq\cdots\leq n_R\leq N} f(n_1,\cdots,n_R)\,|n_1,\cdots,n_R\rangle \tag{27}$$

where

$$f(n_1,\cdots,n_R) = \sum_P exp\left(i\sum_{j=1}^{R} k_{P_j}n_j + \frac{i}{2}\sum_{l<j}\theta(k_{P_l},k_{P_j})\right) \tag{28}$$

where P denotes permutations and we introduce the antisymmetric phase shift $\theta(k_l,k_j) = -\theta(k_j,k_l)$. Similarly, we can extract the consistency equations for the coefficients $f(n_1,\cdots,n_R)$ from eigenvalue equation $H|\psi\rangle = E|\psi\rangle$. The energy eigenvalue equation becomes

$$E = E_0 + J\sum_{j=1}^{R}(1-cosk_j) \tag{29}$$

and the eigenstate condition can be written as

$$2f(n_1,\cdots,n_j,n_j+1,\cdots,n_R) = f(n_1,\cdots,n_j,n_j,\cdots,n_R) + f(n_1,\cdots,n_j+1,n_j+1,\cdots,n_R) \tag{30}$$

Similarly we have phase condition:

$$e^{i\theta(k_j,k_l)} = -\frac{e^{i(k_j+k_l)}+1-2e^{ik_j}}{e^{i(k_j+k_l)}+1-2e^{ik_l}} \tag{31}$$

which can be cast in real form as

$$2cot\frac{\theta(k_j,k_l)}{2} = cot\frac{k_j}{2} - cot\frac{k_l}{2}, \quad j,l=1,\cdots,R \tag{32}$$

Similarly the periodicity of the chain for translations by N sites $f(n_1,\cdots,n_R) = f(n_2,\cdots,n_R,n_1+N)$ gives (after taking their logarithm)

$$\sum_{j=1}^{R} k_{P_j}n_j + \frac{1}{2}\sum_{l<j}\theta(k_{P_l},k_{P_j}) = \frac{1}{2}\sum_{l<j}\theta(k_{P'_l},k_{P'_j}) - 2\pi\tilde{I}_{P'R} + \sum_{j=2}^{R}k_{P'_{j-1}}n_j + k_{P'_R}(n_1+N) \tag{33}$$

where $P'(j-1) = Pj, j=2,\cdots,R$; $P'R = P1$. By canceling the same terms on both sides of (33), we derive the relations between the phase shifts and the quasi-momenta(or Bethe equations):

$$Nk_j = 2\pi\tilde{I}_j + \sum_{l\neq j}\theta(k_j,k_l), \quad j=1,\cdots,R \tag{34}$$

We introduce the *rapidities* λ_j to parametrize the quasi-momenta:

$$cot\frac{k_j}{2} = \lambda_j, \quad or \quad k_j = \frac{1}{i}ln\frac{\lambda_j+i}{\lambda_j-i} = \pi - \theta_1(\lambda_j) \quad or \quad e^{ik_j} = \frac{\lambda_j+i}{\lambda_j-i} \tag{35}$$

where

$$\theta_n(\lambda) \equiv 2arctan\frac{\lambda}{n} \tag{36}$$

The (bare) energy and momentum of an individual magnon, characterized by a quasi-momentum k, is

$$p_0(\lambda) = \frac{1}{i}ln\frac{\lambda+i}{\lambda-i} = k \tag{37}$$

$$\epsilon_0(\lambda) = -J\frac{dk}{d\lambda} = \frac{2J}{\lambda^2+1} = J(1-cosk) \tag{38}$$

In terms of these rapidities (31) (35) (36), the scattering phase is

$$\theta(k_j, k_l) = -\theta_2(\lambda_j - \lambda_l) + \pi sgn[R(\lambda_j - \lambda_l)] \tag{39}$$

where $R(x)$ is the real part of x. The Bethe equations (34) become

$$N\theta_1(\lambda_j) = 2\pi I_j + \sum_{l=1}^{R}\theta_2(\lambda_j - \lambda_l), \quad j = 1, \cdots, R \tag{40}$$

Equivalently

$$\left(\frac{\lambda_j - i}{\lambda_j + i}\right)^N = -\prod_{l=1}^{R}\frac{\lambda_j - \lambda_l - 2i}{\lambda_j - \lambda_l + 2i} \tag{41}$$

The state is now defined by these "new" Bethe numbers $\{I_j\}, j = 1, \cdots, R$. Because (40) is translational invariant($[\lambda_j - \lambda_l]$ indicates this), two equal I_j produce the same rapidities and thus a non valid solution.

Using the rapidities to parametrize the eigenstates, their energies and momenta are

$$E = E_0 + \sum_{j=1}^{R}\epsilon_0(\lambda_j) = E_0 + \sum_{j=1}^{R}\frac{2J}{\lambda_j^2+1} \tag{42}$$

$$K = \left(\sum_{j=1}^{R}p_0(\lambda_j)\right) mod\ 2\pi = \left(\pi R - \frac{2\pi}{N}\sum_{j=1}^{R}I_j\right) mod\ 2\pi \tag{43}$$

4 String Solutions

In this section, we take the thermodynamic limit $N \to \infty$ known as *string hypothesis*.

We first look at the $R = 2$ case. The Bethe Equation written in terms of the rapidities are:

$$\left(\frac{\lambda_1 + i}{\lambda_1 - i}\right)^N = \frac{\lambda_1 - \lambda_2 + 2i}{\lambda_1 - \lambda_2 - 2i} \tag{44}$$

$$\left(\frac{\lambda_2 + i}{\lambda_2 - i}\right)^N = \frac{\lambda_2 - \lambda_1 + 2i}{\lambda_2 - \lambda_1 - 2i} \tag{45}$$

In the limit $N \to \infty$, the LHS is strictly zero or ininity supposing $R(\lambda) \neq 0$. We thus have:

$$\lambda_1 - \lambda_2 = \pm 2i, \quad i.e. \quad \lambda \pm i \tag{46}$$

The energy and momenum of this state are real:

$$p_{\frac{1}{2}}(\lambda) = p_0(\lambda + i) + p_0(\lambda - i) = \frac{1}{i} ln \frac{\lambda + 2i}{\lambda - 2i} \tag{47}$$

$$\epsilon_{\frac{1}{2}}(\lambda) = \epsilon_0(\lambda + i) + \epsilon_0(\lambda - i) = \frac{4J}{\lambda^2 + 4} \tag{48}$$

which gives the dispersion relation

$$\epsilon_{\frac{1}{2}}(p) = \frac{J}{2}(1 - \cos p_{\frac{1}{2}}) \tag{49}$$

For $R > 2$, we assume that complex solutions can be organized into *complexes(or strings)* of $2M+1$ rapidities. Here $M = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ and the rapidities have the structure

$$\lambda_m^{(M)} = \lambda_M + 2im, \quad \lambda_M \text{ is real}, \quad m = -M, -M+1, \cdots, M-1, M \tag{50}$$

We denote the number of complexes of length M by ν_M. Then a state with a given magnetization satisfies

$$R = \sum_M (2M+1)\nu_M \tag{51}$$

Since the string containg $2M+1$ spins move together with the same real repidity, we will treat them as a single entity. Note: complex solutions to the Bethe equations always appear in conjugated pairs: $\{\lambda_j\} = \{\lambda_j^*\}$

The energy and momentum of a M-complex are obtained by summing over all rapidities within one string

$$P_M(\lambda_M) = \sum_{m=-M}^{M} p_0(\lambda_M + 2im) = \frac{1}{i} ln \frac{\lambda_M + i(2M+1)}{\lambda_M - i(2M+1)} = \pi - \theta_{2M+1}(\lambda_M), \tag{52}$$

$$\epsilon_M(\lambda_M) = \sum_{m=-M}^{M} \epsilon_0(\lambda_M + 2im) = \frac{2J(2M+1)}{\lambda_M^2 + (2M+1)^2} = \frac{J}{2M+1}(1 - \cos P_M) \tag{53}$$

The scattering phase of a M-complex with a simple magnon(0-complex) is:

$$S_{0,M}(\lambda_0 - \lambda_M) = S_{0,M}(\lambda) = \frac{\lambda + i2M}{\lambda - i2M} \frac{\lambda + i2(M+1)}{\lambda - i2(M+1)} \tag{54}$$

and the scattering of two complexes of length M and M' is

$$S_{M,M'}(\lambda) = \prod_{L=|M-M'|}^{M+M'} S_{0,L}(\lambda) \tag{55}$$

We denote the real part of the j-th M type complex by $\lambda_{M,j}$ where $j = 1, \cdots, \nu_M$. The Bethe equations for the complexes is obtained by grouping all the rapidities $\lambda_j^{(M)}$ belonging to the same complex and performing first the products within each complex so to be left only with cosistency conditions on their real centers $\lambda_{M,j}$:

$$e^{iP_M(\lambda_{M,j})N} = \prod_{M'} \prod_{j',(M',j') \neq (M,j)}^{\nu_{M'}} S_{M,M'}(\lambda_{M,j} - \lambda_{M',j'}), \quad \forall M;\ j = 1, \cdots, , \nu_M \tag{56}$$

Similarly, we take the logarithm of (56) and use the familiar identity

$$\frac{1}{i}ln\frac{\lambda+in}{\lambda-in} = \pi - 2arctan\frac{\lambda}{n} = \pi - \theta_n(\lambda) \tag{57}$$

we get

$$N\theta_{2M+1}(\lambda_{M,j}) = 2\pi I_{M,j} + \sum_{(M',j')\neq(M,j)} \theta_{M,M'}(\lambda_{M,j} - \lambda_{M',j'}) \tag{58}$$

where

$$\theta_{M,M'}(\lambda) \equiv = \sum_{L=|M-M'|}^{M+M'} [\theta_{2L}(\lambda) + \theta_{2L+2}(\lambda)] \tag{59}$$

and the $L = 0$ is omitted to be omitted.
Some constraint on **Bethe numbers** $\{I_{M,j}\}$:

- $I_{M,j} \neq I_{M,j'}$ in order to have a non-vanishing solution.
- Since momenta are constrained within Brillouin zone, the Bethe numbers are bounded. The bounded number is given by taking $\lambda_M^{(\infty)} = \infty$.

  $$I_M^{(\infty)} = -\sum_{M'\neq M}[2min(M,M')+1]\nu_{M'} - \left(2M+\frac{1}{2}\right)(\nu_M - 1) + \frac{N}{2} \tag{60}$$

  The maximum quantum number is given by adding a M-complex shift to $I_M^{(\infty)}$:

  $$I_M^{max} = I_M^{(\infty)} - (2M+\frac{1}{2}) - \frac{1}{2} = \frac{N-1}{2} - \sum_{M'} J(M,M')\nu_{M'} \tag{61}$$

  where

  $$J(M,M') \equiv \begin{cases} 2min(M,M')+1 & M \neq M' \\ 2M+\frac{1}{2} & M = M \end{cases} \tag{62}$$

  The $\frac{1}{2}$ in (61) is from the $\frac{N}{2}$ in (60) or by taking into account that with each rapidity the Bethe numbers shift from integers to half-integers and vice-versa.

  Since all the scattering phases are odd functions, we have that

  $$I_{M,min} = -I_{M,max} \tag{63}$$

  which means that there are

  $$P_M = 2I_M^{max} + 1 = N - 2\sum_{M'} J(M,M')\nu_{M'} \tag{64}$$

  vacancies for a M-complex.

5 The Anti-ferromagnetic Case: J = -1

We consider first single quasi-particle Excitations

$$\nu_0 = \frac{N}{2}: \quad \nu_M = 0, M \geq \frac{1}{2}, \quad \rightarrow \quad R = \frac{N}{2} \tag{65}$$

The number of vacancies for this configuration according to (64) is

$$P_0 = N - 2J(0,0)\nu_0 = N - \frac{N}{2} = \frac{N}{2} \tag{66}$$

which equals the number of particle states. Hence, the quantum numbers occupy all the allowed vacancies:

$$-\frac{N}{4} + \frac{1}{2} \leq I_{0,k} \leq \frac{N}{4} - \frac{1}{2} \tag{67}$$

Excited states over this ground state are constructed by taking away quasi-particles from the single state and moving them into complexes, we will characterize the excited states by κ, with

$$\nu_0 = \frac{N}{2} - \kappa \tag{68}$$

We analyse ground state again.
The linear integral equation for the density $\rho_0(\lambda)$ is

$$\rho_0(\lambda) = \frac{1}{\pi}\frac{1}{1+\lambda^2} - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2}{(\lambda-\mu)^2+4}\rho_0(\mu)d\mu \tag{69}$$

which can be rewritten as

$$\rho_0(\lambda) + \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\lambda-\mu)\rho_0(\mu)d\mu = \frac{1}{2\pi}\theta_1'(\lambda) \tag{70}$$

where we define the kernel

$$K(\lambda) \equiv \frac{d}{d\lambda}\theta_2(\lambda) = \frac{4}{\lambda^2+4} \tag{71}$$

Using Fourier transformation, we get $\rho_0(\lambda)$ as in equation (97)

$$\rho_0(\lambda) = \frac{1}{4cosh(\frac{\pi\lambda}{2})} \tag{72}$$

The momentum and energy of the ground state are then given by

$$K = N\int p_0(\lambda)\rho_0(\lambda)d\lambda = \frac{\pi}{2}Nmod2\pi \equiv K_{AFM} \tag{73}$$

$$E = E_0 + N\int \epsilon_0(\lambda)\rho_0(\lambda)d\lambda = N(\frac{1}{4} - \ln 2) \equiv E_{AFM} \tag{74}$$

We now look at states with $\nu_0 = \frac{N}{2} - 1$ and $\nu_M = 0$ for $M \geq \frac{1}{2}$. They are characterized by two holes.(spinon excitations as in section 6) We let the empty quantum numbers are j_1 and j_2 :

$$I_{0,j} = j + \theta_H(j-j_1) + \theta_H(j-j_2) \tag{75}$$

where $\theta_H(x)$ is Heqviside step-function. The integral equation for the real roots rapidity density $\rho_t(\lambda)$ (where t stands for triplet) is

$$\rho_t(\lambda) + \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\lambda-\mu)\rho_t(\mu)d\mu = \frac{1}{\pi}\frac{1}{1+\lambda^2} - \frac{1}{N}[\delta(\lambda-\lambda_1)+\delta(\lambda-\lambda_2)] \tag{76}$$

where $\lambda_i \to x_i = \frac{j_i}{N}$.Because of the linearity of equation (76). We rewrite $\rho_t(\lambda)$ as:

$$\rho_t(\lambda) = \rho_0(\lambda) + \frac{1}{N}[\tau(\lambda-\lambda_1)+\tau(\lambda-\lambda_2)] \tag{77}$$

where $\tau(\lambda)$ is the solution of equation:

$$\tau(\lambda) + \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\lambda-\mu)\tau(\mu)d\mu = -\delta(\lambda) \tag{78}$$

We give the momentum and energy of the state

$$K = N\int p_0(\lambda)\rho_t(\lambda)d\lambda = K_{AFM} + k(\lambda_1) + k(\lambda_2) \tag{79}$$

$$E = N\int \epsilon_0(\lambda)\rho_t(\lambda)d\lambda = E_{AFM} + \epsilon(\lambda_1) + \epsilon(\lambda_2) \tag{80}$$

The dispersion relation is

$$\epsilon(k) = \frac{\pi}{2}\sin k, \quad -\frac{\pi}{2} \le k \le \frac{\pi}{2} \tag{81}$$

We now look at the state with $\nu_0 = \frac{N}{2} - 2$, $\nu_{\frac{1}{2}} = 1$ and $\nu_M = 0$ for $M \ge 1$. For the density of real roots $\rho_s(\lambda)$ (s is for singlet) we get the integral equation

$$\rho_s(\lambda) + \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\lambda-\mu)\rho_s(\mu)d\mu = \frac{1}{\pi}\frac{1}{1+\lambda^2} - \frac{1}{N}\left(\delta(\lambda-\lambda_1)+\delta(\lambda-\lambda_2)+\frac{1}{\pi}\theta'_{0,\frac{1}{2}}(\lambda-\lambda_{\frac{1}{2}}\right) \tag{82}$$

The sloution of (82) is

$$\rho_s(\lambda) = \rho_0(\lambda) + \frac{1}{N}[\tau(\lambda-\lambda_1)+\tau(\lambda-\lambda_2)+\sigma(\lambda-\lambda_{\frac{1}{2}}] \tag{83}$$

where $\tau(\lambda)$ is given by equation (78) and $\sigma(\lambda)$ is the solution of

$$\sigma(\lambda) + \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\lambda-\mu)\sigma(\mu)d\mu = -\frac{1}{\pi}\theta'_{0,\frac{1}{2}}(\lambda) \tag{84}$$

Given equation (58) and deriving $I_{\frac{1}{2}}^{max} = 0$ from equation (61), we get

$$2arctan\frac{\lambda_{\frac{1}{2}}}{2} = \frac{1}{N}\sum_j \theta_{\frac{1}{2},0}(\lambda_{\frac{1}{2}} - \lambda_{0,j}) = \int_{-\infty}^{\infty}\theta_{\frac{1}{2},0}(\lambda_{\frac{1}{2}}-\lambda)\rho_s(\lambda)d\lambda \tag{85}$$

After some calculations, we get

$$\lambda_{\frac{1}{2}} = \frac{\lambda_1+\lambda_2}{2} \tag{86}$$

6 Ground state and low energy excitations

Theorem 1. *If* $\{\lambda_1, \cdots, \lambda_N\}$ *is a solution of Bethe Ansatz equation* (41), *then* $\forall i, j \quad \lambda_i \neq \lambda_j$

We define

$$Z(\lambda) = \frac{1}{2\pi}[\theta_1(\lambda) - \frac{1}{N}\sum_{l=1}^{R}\theta_2(\lambda - \lambda_l)] \tag{87}$$

$$\sigma(\lambda) = \frac{dZ(\lambda)}{d\lambda} \tag{88}$$

Apparently, $Z(\lambda_j) = I_j/N$.
For any consecutive I_j and $I_{j+1} = I_j + 1$, the following relation holds:

$$\frac{Z(\lambda_{j+1} - Z(\lambda_j)}{\lambda_{j+1} - \lambda_j} = \frac{1}{N\delta\lambda_j} \tag{89}$$

In the limit $N \to \infty$, we obtained

$$\rho(\lambda) = \sigma(\lambda) = a_1(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda - \mu)\rho(\mu)d\mu \tag{90}$$

where

$$a_n(\lambda) = \frac{1}{\pi}\frac{n}{\lambda^2 + n^2} \tag{91}$$

For ground state

$$\rho_g(\lambda) = a_1(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda - \mu)\rho_g(\mu)d\mu \tag{92}$$

To solve (92), we use Fourier transformation:

$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} e^{i\omega\lambda}F(\lambda)d\lambda \tag{93}$$

$$F(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\lambda}\tilde{F}(\omega)d\omega \tag{94}$$

The fourier transformation of $a_n(\lambda)$ is

$$\tilde{a}_n(\omega) = \int_{-\infty}^{\infty} e^{i\omega\lambda}\frac{1}{\pi}\frac{n}{\lambda^2 + n^2}d\lambda = e^{-n|\omega|} \tag{95}$$

Take the fourier transformation of (92), we obtain

$$\tilde{\rho}_g(\omega) = \frac{1}{2cosh\ \omega} \tag{96}$$

Thus the solution of (96) is

$$\rho_g(\lambda) = \frac{1}{4\cosh\frac{\pi\lambda}{2}} \tag{97}$$

The desity of flipped spins relative to the reference state is

$$\frac{R}{N} = \int_{-\infty}^{\infty} \rho_g(\lambda) d\lambda = \frac{1}{2} \tag{98}$$

which means that the magnetization of the ground state is zero. The energy density of the ground state reads

$$e_g = -2\pi \int_{-\infty}^{\infty} a_1(\lambda)\rho_g(\lambda) d\lambda + \frac{1}{2} = \frac{1}{2} - 2\ln 2 \tag{99}$$

6.1 Spinon Excitations

We focus on the even N case. The simplest excitation i.e. $R = \frac{N}{2} - 1$. Thus we have

$$\rho^h(\lambda) = \frac{1}{N}\delta(\lambda - \lambda_1^h) + \frac{1}{N}\delta(\lambda - \lambda_2^h) \tag{100}$$

Why are there two holes

We suppose $\rho(\lambda) = \rho_g(\lambda) + \delta\rho(\lambda)$. Then (90) becomes

$$\delta\rho(\lambda) + \rho^h(\lambda) = -\int_{-\infty}^{\infty} a_2(\lambda - \mu)\delta\rho(\mu) d\mu \tag{101}$$

which can be solved by Fourier transformation. We obtain the total spin S_e of this excitation as

$$S_e = -N \int_{-\infty}^{\infty} \delta\rho(\lambda) d\lambda = 1 \tag{102}$$

with

$$\delta\rho(\lambda) = \frac{1}{4}\left(sech[\frac{\pi}{2}(\lambda + \lambda_1^h)] + sech[\frac{\pi}{2}(\lambda + \lambda_2^h]\right) \tag{103}$$

The excitation energy is

$$\Delta E = 2\pi N J \int_{-\infty}^{\infty} a_1(\lambda)\delta\rho(\lambda) d\lambda = \epsilon(\lambda_1^h) + \epsilon(\lambda_2^h) \tag{104}$$

where $\epsilon(\lambda)$ is the dressed energy with the definition

$$\epsilon(\lambda) = -2\pi J a_1(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda - \mu)\epsilon(\mu) d\mu \tag{105}$$

Solving (105) by Fourier transformation, we obtain(simimar to (92))

$$\epsilon(\lambda) = \frac{-\pi J}{cosh(\pi\lambda)} = -2\pi J \rho_g(\lambda) \tag{106}$$

7 Thermodynamics

$$\epsilon_n^0(\lambda) = -2\pi a_n(\lambda) + nh$$

The number of n-string states in the interval $[\lambda, \lambda + d\lambda]$ is

$$N[\rho_n(\lambda) + \rho_n^h(\lambda)]d\lambda \tag{107}$$

Then the number of the possible physical states in this interval is

$$d\Omega(\lambda) = \prod_{n=1}^{\infty} \frac{[N(\rho_n(\lambda) + \rho_n^h(\lambda)d\lambda]!}{[N\rho_n(\lambda)d\lambda]!\ [N\rho_n^h(\lambda)d\lambda]!} \tag{108}$$

The entropy in the interval

$$dS(\lambda) = lnd\Omega(\lambda) \simeq N\sum_{n=1}^{\infty}\{[\rho_n(\lambda) + \rho_n^h(\lambda)]ln[\rho_n(\lambda) + \rho_n^h(\lambda)] - \rho_n(\lambda)ln\rho_n(\lambda) - \rho_n^h(\lambda)ln\rho_n^h(\lambda)\}d\lambda \tag{109}$$

Density of free energy

$$\begin{aligned} f = &\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\epsilon_n^0(\lambda)\rho_n(\lambda)d\lambda - T\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\{[\rho_n(\lambda) + \rho_n^h(\lambda)]ln[\rho_n(\lambda) + \rho_n^h(\lambda)] \\ &- \rho_n(\lambda)ln\rho_n(\lambda) - \rho_n^h(\lambda)ln\rho_n^h(\lambda)\}d\lambda \end{aligned} \tag{110}$$

To get a thermal equilibrium state

$$\frac{\delta f}{\delta\rho_n(\lambda)} = 0 \tag{111}$$

which leads to

$$\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\{\epsilon_n^0(\lambda)\delta\rho_n(\lambda) - Tln[1 + \eta_n(\lambda)]\delta\rho_n(\lambda) - Tln[1 + \eta_n^{-1}(\lambda)]\delta\rho_n^h(\lambda)\}d\lambda = 0 \tag{112}$$

where

$$\eta_n(\lambda) = \frac{\rho_n^h(\lambda)}{\rho_n(\lambda)} \tag{113}$$

Note that $\rho_n(\lambda)$ and $\rho_n^h(\lambda)$ are related by

$$\delta\rho_n^h(\lambda) = -\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}A_{m,n}(\lambda - \mu)\delta\rho_m(\mu)d\mu \tag{114}$$

Substituting (114) to (112), we get

$$ln[1 + \eta_n(\lambda)] = \frac{\epsilon_n^0(\lambda)}{T} + \sum_{m=1}^{\infty}\int_{-\infty}^{\infty}A_{m,n}(\lambda - \mu)ln[1 + \eta_m^{-1}(\mu)]d\mu \tag{115}$$

We define the integral operator $[n]$ as

$$[n]F(\lambda) = \int_{-\infty}^{\infty}a_n(\lambda - \mu)F(\mu)d\mu \tag{116}$$

Note that under fourier transformation $[n]$ becomes a multiplier $exp(-n|\omega|/2)$.
We have

$$[m][n] = [m+n] \tag{117}$$

It can be easily verified in ω space. Further, we define

$$\begin{aligned}\hat{\mathbf{A}}_{m,n} &= [m+n] + 2[m+n-2] + \cdots + 2[|m-n|+2] + [|m-n|] \\ \hat{\mathbf{G}} &= \frac{[1]}{[0]+[2]}\end{aligned} \tag{118}$$

Where the kernel of the operator $\hat{\mathbf{G}}$ is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda\omega}\frac{e^{-\frac{1}{2}|\omega|}}{1+e^{-|\omega|}}d\omega = \frac{1}{2cosh(\pi\lambda)} = \rho_g(\lambda) \tag{119}$$

The operator identities

$$\hat{\mathbf{G}}[\hat{\mathbf{A}}_{m,n+1} + \hat{\mathbf{A}}_{m,n-1}] = -\delta_{m,n} + \hat{\mathbf{A}}_{m,n}, \quad n > 1 \tag{120}$$

$$\hat{\mathbf{G}}\hat{\mathbf{A}}_{m,2} = -\delta_{1,m} + \hat{\mathbf{A}}_{1,m} \tag{121}$$

We can then rewrite (115) as

$$ln(1+\eta_n(\lambda)) = \frac{\epsilon_n^0(\lambda)}{T} + \sum_{m=1}^{\infty}\hat{\mathbf{A}}_{n,m} ln(1+\eta_m^{-1}(\lambda)) \tag{122}$$

$\hat{\mathbf{G}} \rightarrow$ (122) with $n = n-1$ and $n = n+1$

$$\begin{aligned}&\hat{\mathbf{G}}[ln(1+\eta_{n+1}(\lambda)) + ln(1+\eta_{n-1}(\lambda))] \\ &= \frac{\epsilon_n^0(\lambda)}{T} - ln(1+\eta_n^{-1}(\lambda)) + \sum_{m=1}^{\infty}\hat{\mathbf{A}}_{n,m} ln(1+\eta_m^{-1}(\lambda))\end{aligned} \tag{123}$$

Combining (122) and (123), we get

$$\ln\eta_n(\lambda) = \hat{\mathbf{G}}[ln(1+\eta_{n+1}(\lambda)) + ln(1+\eta_{n-1}(\lambda))] \tag{124}$$

Applying $\hat{\mathbf{G}}$ with $n = 2$, we obtain

$$\ln\eta_1(\lambda) = -\frac{2\pi\rho_g(\lambda)}{T} + \hat{\mathbf{G}}\ln(1+\eta_2(\lambda)) \tag{125}$$

We obtain

$$f = -T\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} a_n(\lambda) ln[1+\eta_n^{-1}(\lambda)]d\lambda \tag{126}$$

From (122), we know that

$$\hat{\mathbf{G}}\ln(1+\eta_1(\lambda)) = \frac{1}{T}\hat{\mathbf{G}}\epsilon_1^0(\lambda) + \sum_{m=1}^{\infty}[m]\ln\big(1+\eta_m^{-1}(\lambda)\big) \tag{127}$$

Let $\lambda = 0$, we get

$$\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} a_m(\lambda) \ln[1 + \eta_m^{-1}(\lambda)] d\lambda$$
$$= \frac{2\ln 2 - \frac{1}{2h}}{T} + \int_{-\infty}^{\infty} \rho_g(\lambda) \ln[1 + \eta_1(\lambda)] d\lambda \tag{128}$$

Substituting (128) into (126), we finally get the expression for the energy

$$\frac{F}{N} = e_g - T \int_{-\infty}^{\infty} \rho_g(\lambda) \ln[1 + \eta_1(\lambda)] d\lambda \tag{129}$$

Below, let us consider the low energy limit $T \to 0$ and $h \to h$. Then $\eta_1 = 0$ and η_n become constants. In this case, (124) (125) are reduced to

$$\eta_n^2 = (1 + \eta_{n+1})(1 + \eta_{n-1}), \quad n > 1 \tag{130}$$

The general solution of the above equations is

$$\eta_n = \left(\frac{bz^n - b^{-1}z^{-n}}{z - z^{-1}} \right)^2 - 1 \tag{131}$$

where b and z are determined by $\eta_1 = 0$ and limitation condition as $n \to \infty$

$$\lim_{n \to \infty} \frac{\ln \eta_n}{n} = \frac{h}{T} \tag{132}$$

In the case of $h = 0$, we have $\eta_n(h = 0) = n^2 - 1$. Let $\eta_n = exp[-\epsilon_n/T]$, then $\epsilon_1 \sim T^0$. Comparing the T^{-1} terms of (122) we have

$$\epsilon_1(\lambda) = -\epsilon_1^0(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda - \mu)\epsilon_1(\mu) d\mu \tag{133}$$

Equivalently

$$\epsilon_1(\lambda) = -2\pi J a_1(\lambda) + H - \int_{-\infty}^{\infty} a_2(\lambda - \mu)\epsilon_1(\mu) d\mu \tag{134}$$

where $a_n = \frac{1}{2\pi} \frac{n}{\lambda^2 + \frac{n^2}{4}}$.(This definition is different from the precedent content of note.)

$$\epsilon_1(\lambda) = -2\pi J a_1(\lambda) + H - \int_{-Q}^{Q} a_2(\lambda - \mu)\epsilon_1(\mu) d\mu \tag{135}$$

$$f_0 = \int_{-Q}^{Q} a_1(\mu)\epsilon_1(\mu) d\mu \tag{136}$$

Where $\epsilon_1(\pm Q) = 0$, by let $\epsilon_1(0) = 0$ we get $H_s = 4J$.
At zero temperature, we consider $H \to H_s$. In that case, Q is small. We expand $\epsilon_1(\lambda)$ in terms of small λ.

$$\epsilon_1(\lambda) = -2\pi J a_1(\lambda) + H - \frac{1}{\pi} \int_{-Q}^{Q} \epsilon_1(\lambda) d\lambda \approx -2\pi J a_1(\lambda) + H - \frac{2Q(H - H_s)}{\pi} \tag{137}$$

Then how could we get $Q = \sqrt{\frac{H_s - H}{16J}}$. Using $\epsilon_1(Q) = 0$?
The zero temperature dressed energy

$$f_0 \approx \int_{-Q}^{Q} a_1(\mu)\epsilon_1(\mu)d\mu \approx \frac{4J}{\pi}\frac{(1-4Q^2)\arctan 2Q - 2Q}{1+4Q^2} + O(Q^4) \tag{138}$$

We can derive some emergent quantities from free energy f

$$M^z = \frac{1}{2} - \frac{\partial f}{\partial H} \quad \chi = \frac{\partial^2 f}{\partial H^2} \quad c_V = -T\frac{\partial^2 f}{\partial T^2} \tag{139}$$

Substituting (138) to (139), we get

$$M^z = \frac{1}{2} - \frac{\partial f_0}{\partial H} = \frac{1}{2} - \frac{2}{\pi}\left(1 - \frac{H}{H_s}\right)^{1/2} \tag{140}$$

$$\chi = \frac{\partial M^z}{\partial H} = \frac{1}{2\pi}\left(\frac{1}{J(H_s - H)}\right)^{1/2} \tag{141}$$

We give the scaling form of (140)

$$1 - \frac{M^z}{M_s} = D\left(1 - \frac{H}{H_s}\right)^{1/\delta} = \frac{4}{\pi}\left(1 - \frac{H}{H_s}\right)^{1/2} \tag{142}$$

7.1 Luttinger Liquid

Luttinger liquid is a collective motion of particle-hole excitations near two Fermi points at low temperature. The condition $|H - H_s|/T >> 1$ is necessary for this excitation. We rewrite the low temperature TBA equation $\epsilon_1^T(\lambda) = \epsilon_1(\lambda) + \eta(\lambda)$,where $\epsilon_1(\lambda)$ is the zero temperature dressed energy (134), η can be regarded as a leading correction to the temperature.
We give first

$$\epsilon_1^T(\lambda) = -2\pi J a_1(\lambda) + H + T\int_{-\infty}^{\infty} a_2(\lambda-\mu)\ln\left(1 + e^{\frac{-|\epsilon_1^T(\mu)|}{T}}\right)d\mu - \int_{-Q}^{Q} a_2(\lambda-\mu)\epsilon_1^T(\mu)d\mu \tag{143}$$

And then

$$\begin{aligned}\epsilon_1^T(\lambda) &= \epsilon_1(\lambda) + \eta(\lambda) \\ &= -2\pi J a_1(\lambda) + H - \int_{-Q}^{Q} a_2(\lambda-\mu)\epsilon_1(\mu)d\mu + \eta(\lambda) \\ &= -2\pi J a_1(\lambda) + H - \int_{-Q}^{Q} a_2(\lambda-\mu)(\epsilon_1^T - \eta(\mu))d\mu + \eta(\lambda)\end{aligned} \tag{144}$$

Thus we have

$$\eta(\lambda) = T\int_{-\infty}^{\infty} a_2(\lambda-\mu)\ln\left(1 + e^{\frac{-|\epsilon_1^T(\mu)|}{T}}\right)d\mu - \int_{-Q}^{Q} a_2(\lambda-\mu)\eta(\mu)d\mu \tag{145}$$

As $T \to 0$, only the zeros of ϵ_1^T contribute to the first integration. We expanding ϵ_1^T at $\lambda = Q$

$$\epsilon_1^T(\lambda) = t(\lambda - Q) + O((\lambda - Q)^2) \tag{146}$$

where $t = \frac{d\epsilon_1^T(\lambda)}{d\lambda}|_{\lambda=Q}$. Then the integration becomes

$$I = \frac{\pi^2 T^2}{6t}[a_2(\lambda+Q) + a_2(\lambda-Q)] \tag{147}$$

In that case

$$\eta(\lambda) = \frac{\pi^2 T^2}{6t}[a_2(\lambda+Q) + a_2(\lambda-Q)] - \int_{-Q}^{Q} a_2(\lambda-\mu)\eta(\mu)d\mu \tag{148}$$

At zero temperature, the free energy $f(T,H)$ is given by

$$f_0(0,H) = \int_{-Q}^{Q} a_1(\lambda)\epsilon_1(\lambda)d\lambda \tag{149}$$

Under such conditions, the free energy is given by

$$f(T,H) = -T\int_{-\infty}^{\infty} a_1(\lambda)\ln\left(1 + e^{\frac{-\epsilon_1^T(\lambda)}{T}}\right)d\lambda \tag{150}$$

It follows that

$$\begin{aligned} f - f_0 &= -T\int_{-\infty}^{\infty} a_1(\lambda)\ln\left(1 + e^{\frac{-\epsilon_1^T(\lambda)}{T}}\right)d\lambda - \int_{-Q}^{Q} a_1(\lambda)\epsilon_1(\lambda)d\lambda \\ &= -T\int_{-\infty}^{\infty} a_1(\lambda)\ln\left(1 + e^{\frac{-|\epsilon_1^T(\lambda)|}{T}}\right)d\lambda + \int_{-Q}^{Q} a_1(\lambda)\eta(\lambda)d\lambda \\ &= -\frac{\pi^2 T^2}{3t}a_1(Q) + \int_{-Q}^{Q} a_1(\lambda)\eta(\lambda)d\lambda \end{aligned} \tag{151}$$

We can express the free energy as

$$f = f_0 - \frac{\pi^2 T^2}{3t}a_1(Q) + \int_{-Q}^{Q} a_1(\lambda)\eta(\lambda)d\lambda \tag{152}$$

To calculate the last term of (152), we first give

$$\rho_0(\lambda) = a_1(\lambda) - \int_{-Q}^{Q} a_2(\lambda-\mu)\rho_0(\mu)d\mu \tag{153}$$

Note that (148) has the same structure as (153) and we have the theorem such that:
If we have

$$f(k) = f_0(k) - \int_{-Q}^{Q} a_2(k-\lambda)f(\lambda)d\lambda \tag{154}$$

$$g(k) = g_0(k) - \int_{-Q}^{Q} a_2(k-\lambda)g(\lambda)d\lambda \tag{155}$$

then

$$\int_{-Q}^{Q} f(\lambda)g_0(\lambda)d\lambda = \int_{-Q}^{Q} g(\lambda)f_0(\lambda)d\lambda \tag{156}$$

Using this theorem we can obtain

$$\int_{-Q}^{Q} \frac{\pi^2 T^2}{6t}[(a_2(\lambda+Q)+a_2(\lambda-Q))]\rho_0(\lambda)d\lambda = \int_{-Q}^{Q} a_1(\lambda)\eta(\lambda)d\lambda \tag{157}$$

Assign $\lambda = \pm Q$ to (153), we can substitute LHS of (157) with

$$\int_{-Q}^{Q} [(a_2(\lambda+Q)+a_2(\lambda-Q))]\rho_0(\lambda)d\lambda = 2a_1(Q) - 2\rho_0(Q) \tag{158}$$

We then obtain

$$\int_{-Q}^{Q} a_1(\lambda)\eta(\lambda)d\lambda = \frac{\pi^2 T^2}{6t}[2a_1(Q) - 2\rho_0(Q)] \tag{159}$$

Finally, we simplify (152) to the following form

$$f = f_0 - \frac{\pi^2 T^2}{3t}\rho_0(Q) \tag{160}$$

We then consider the specific heat at TTL region.

$$c_v = -T\frac{\partial^2 f}{\partial T^2} = \frac{\pi T}{3v_s} \propto T^\alpha \tag{161}$$

where v_s is defined as sound velocity

$$v_s = \frac{1}{2\pi}\frac{d\epsilon_1(\lambda)/d\lambda}{\rho_0(\lambda)}|_{\lambda=Q} = \frac{1}{2\pi}\frac{t}{\rho_0(Q)} \tag{162}$$

We start from the Hamiltonian of Luttinger liquid

$$H = \frac{}{2\pi}\int dx \left(\frac{v_s K_s}{2}(\pi\Pi(x))^2 + \frac{v_s}{K_s}(\delta\phi(x))^2\right) \tag{163}$$