On the Number of Classes of Binary Matrices

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Abstract—**Cellular switching theory gives rise to the problems of counting the number of equivalence classes of $m \times n$ matrices of zeros and ones under: 1) row and column permutations; and 2) row and column permutations together with column complementations. A number of techniques are given for the solution of these problems.**

Index Terms—**Binary matrices, cellular logic, counting theory, logical design, switching theory.**

I. Statement of the Problem

IT IS well known that many problems of switching theory can be recast as problems involving 0-1 matrices. In the present paper, we consider a combinatorial problem arising from cellular logic. We wish to find the number of equivalence classes of 0-1 matrices with m rows and n columns under several definitions of equivalence. For instance, we will take equivalence to mean "equivalent under row and column permutations." Another definition will be row permutations together with column permutations and/or complementations. The reader is referred to [5] for the correspondence between the mathematical problems under discussion here and cellular switching theory.

In the remainder of this section, we indicate a formal statement of the problem and some basic definitions. We hope for familiarity with basic counting theory as described by, say, [6] or [3].

Let S_n denote the symmetric group on $\{1, \cdots, n\}$ of order $n!$ Let $M(m, n)$ be the set of all $m \times n$ matrices whose entries are 0 or 1. Let us say that for $A, B \in M(m, n)$, we have

$$A \equiv_1 B \text{ iff } (b_{ij}) = (a_{\sigma^{-1}(i), \tau^{-1}(j)})$$

for some $\sigma \in S_m, \tau \in S_n$. In words, B is equal to A after a row permutation by σ and a column permutation by τ.

Example: Let $m = n = 3$ and $\sigma = (123)$ and $\tau = (12)$. If we compute $(a_{\sigma^{-1}(i), \tau^{-1}(j)})$, we get

$$\begin{bmatrix} a_{32} & a_{31} & a_{33} \\ a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \end{bmatrix}.$$

Our next definition allows column complementations as well. If we have $A, B \in M(m, n)$ we say that

$$A \equiv_2 B \text{ iff } B = (b_1 \cdots b_n)$$

where each column b_k is the kth column of $(a_{\sigma^{-1}(i), \tau^{-1}(j)})$ or its complement.

Example: $m = n = 3$. If we take

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\sigma = (123)$, $\tau = (12)$, and complement columns 2 and 3, we obtain

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can now ask for the number of equivalence classes of matrices in $M(m, n)$ under $\equiv_1$ and under $\equiv_2$. Let s_{mn} be the former quantity and t_{mn} the latter.

II. Several Solutions

We will begin by showing a number of different methods for obtaining the desired results for $\equiv_1$. One of these methods generalizes very naturally for $\equiv_2$.

First, we need some notation and some basic concepts from combinatorial theory [2], [3], [6].

Definition: Let G be a permutation group of finite order g which acts on a finite set S with s elements. Let $f_1, \cdots, f_s$ be s indeterminates and let $g_{(j_1, \cdots, j_s)} = g_{(j)}$ be the number of permutations in G with j_i cycles of length i for $i = 1, \cdots, s$. The *cycle index polynomial* of G acting on S is

$$Z_G(f_1, \cdots, f_s) = (1/g) \sum_{(j)} g_{(j_1, \cdots, j_s)} f_1^{j_1} \cdots f_s^{j_s} \quad (1)$$

where the sum is over all nonnegative integers $j_1, \cdots, j_s$ such that

$$\sum_{i=1}^{s} i j_i = s. \quad (2)$$

As an abbreviation, we may rephrase the previous condition, (2), as a sum over the partitions of s.

Examples: If we let $S = \{0, \cdots, m-1\}$ and let C_m be the cyclic group generated by the cycle $(0, 1, \cdots, m-1)$ then

$$Z_{C_m}(f_1, \cdots, f_m) = (1/m) \sum_{d \mid m} \phi(d) f_d^{m/d}$$

where $\phi(d)$ is the Euler ϕ function.

As a second example, let $S = \{1, \cdots, n\}$ and let S_n be the symmetric group of all $n!$ permutations of S. Then, it is well known that

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$$Z_{S_n}(f_1, \cdots, f_n) = (1/n!) \sum_{(j)} \left[\frac{n!}{\prod_{i=1}^{n} j_i! i^{j_i}} \right] f_1^{j_1} \cdots f_n^{j_n} \quad (3)$$

where the sum is over all partitions of n.

Another of the techniques that we shall need involves the "cross operation" on polynomials. This is defined in the following lemma, which is quoted without proof from [2].

Lemma 1: If α is a permutation on a set X with $|X| = a$ and α has cycle structure denoted by $f_1^{j_1} \cdots f_a^{j_a}$ and β is a permutation on Y with $|Y| = b$ and β has cycle structure $f_1^{k_1} \cdots f_b^{k_b}$, then the permutation (α, β) acting on $X \times Y$ by the rule

$$(\alpha, \beta)(x, y) = (\alpha x, \beta y) \quad (4)$$

has cycle structure denoted by

$$\left(\prod_{p=1}^{a} f_p^{j_p}\right) \times \left(\prod_{q=1}^{b} f_q^{k_q}\right) = \prod_{p=1}^{a} \prod_{q=1}^{b} (f_p^{j_p} \times f_q^{k_q})$$

$$= \prod_{p=1}^{a} \prod_{q=1}^{b} f_{\langle p,q \rangle}^{j_p k_q (p,q)} \quad (5)$$

where $\langle p, q \rangle$ is the least common multiple of p and q while (p, q) is the greatest common divisor of p and q.

Some examples of this operation are

$$f_1^3 \times f_2 = f_2^3$$

$$f_2 \times (f_1 f_4) = (f_2 \times f_1)(f_2 \times f_4) = f_2 f_4^2.$$

One can extend the notation to polynomials in the obvious way. For example,

$$\begin{aligned} Z_{S_2} \times Z_{S_3} &= \tfrac{1}{2}(f_1^2 + f_2) \times \tfrac{1}{6}(f_1^3 + 3f_1 f_2 + 2f_3) \\ &= \tfrac{1}{12}(f_1^2 \times f_1^3 + 3(f_1^2) \times (f_1 f_2) + 2(f_1^2 \times f_3) \\ &\quad + (f_2 \times f_1^3) + 3(f_2 \times (f_1 f_2)) + 2(f_2 \times f_3)) \\ &= \tfrac{1}{12}(f_1^6 + 3f_1^2 f_2^2 + 2f_3^2 + f_2^3 + 3f_2 f_2^2 + 2f_6) \\ &= \tfrac{1}{12}(f_1^6 + 3f_1^2 f_2^2 + 2f_3^2 + 4f_2^3 + 2f_6). \end{aligned} \quad (6)$$

We shall also need a very simple form of Pólya's theorem. Suppose we have a permutation group G acting on a finite set D with $|D| = s$. Let R be a finite set with $|R| = q$. With each $r \in R$, we have an associated weight $w(r) \in N$. If f is any function from D to R, define $w(f) = x^k$ where $k = \Sigma_{d \in D}\, w(f(d))$. Two functions f and g from D to R are said to be *equivalent* if $w(f) = w(g)$. Pólya's theorem will count the number of classes of such functions once we have a generating function on R. So let

$$\psi(x) = \sum_{k \in N} \psi_k x^k \quad (7)$$

where ψ_k is the number of elements of R with weight k. Finally, we need a generating function for the number of functions. We define

$$P(x) = \sum_{k \in N} p_k x^k$$

where p_k is the number of equivalence classes of functions of weight x^k. Pólya's theorem can now be stated. It gives us a relationship between these quantities.

Theorem 1 (Pólya):

$$P(x) = Z_G(\psi(x), \psi(x^2), \cdots, \psi(x^s)).$$

Corollary: The total number of equivalence classes of functions from D to R is

$$P(1) = Z_G(q, \cdots, q).$$

Proof: For each $r \in R$, take $w(r) = 0$. Then $\psi(x) = q$.

We can now use these techniques to solve the counting problem for $\equiv_1$.

Theorem 2: The number of $\equiv_1$-equivalence classes of matrices in $M(m, n)$ is given by

$$s_{mn} = \frac{1}{m!n!} \sum_{(j)} \sum_{(k)} \frac{m!n!}{\left(\prod_{p=1}^{m} j_p! p^{j_p}\right)\left(\prod_{q=1}^{n} k_q! q^{k_q}\right)} \cdot 2^{\sum_{p=1}^{m} \sum_{q=1}^{n} j_p k_q (p,q)} \quad (8)$$

where (p, q) is the greatest common divisor of p and q. The sums over (j) and (k) are over the partitions of m and n, respectively.

Proof: If we have a matrix $A \in M(m, n)$ then $\equiv_1$ allows row and column permutations. That is, we allow permutations $\sigma \in S_m$ and $\tau \in S_n$. Since these permutations may act independently, we are working with a group that is the "direct product" of S_m and S_n and whose cycle index polynomial is $Z_{S_m} \times Z_{S_n}$ (see [2, theorem 4.2]). To use Pólya's theorem, we note that $R = \{0, 1\}$ and take $w(0) = 0$ and $w(1) = 1$. Then $\psi(x) = 1 + x$. By the corollary to Pólya's theorem

$$P(x) = Z_{S_m \times S_n}(2, \cdots, 2).$$

Carrying out the computations with the aid of Lemma 1, we obtain

$$Z_{S_m \times S_n} = \frac{1}{m!n!} \sum_{(j)} \sum_{(k)} \left[\frac{m!}{\prod_{p=1}^{m} j_p! p^{j_p}} \right] \left[\frac{n!}{\prod_{q=1}^{n} k_q! q^{k_q}} \right] \cdot \prod_{p=1}^{m} \prod_{q=1}^{n} f_{\langle p,q \rangle}^{j_p k_q (p,q)}$$

$$s_{mn} = P(1) = \frac{1}{m!n!} \sum_{(j)} \sum_{(k)} \frac{m!n!}{\left[\prod_{p=1}^{m} j_p! p^{j_p}\right]\left[\prod_{q=1}^{n} k_q! q^{k_q}\right]} \cdot 2^{\sum_{p=1}^{m} \sum_{q=1}^{n} j_p k_q (p,q)}. \quad \text{Q.E.D.}$$

In practice, one can compute these results simply without using such formulas, as the following example shows.

Example: Let $m = 2$, $n = 3$. We need to compute $Z_{S_2 \times S_3}$. Since this was done in an earlier example, using (6), we get

$$P(1) = \tfrac{1}{12}(2^6 + 3 \cdot 2^4 + 2 \cdot 2^2 + 4 \cdot 2^3 + 2 \cdot 2)$$
$$= \tfrac{1}{12}(156) = 13.$$

It is interesting to note that the problem that we have just solved admits of a graph-theoretical interpretation. An $m \times n$ matrix of zeros and ones is equivalent to a *bipartite graph*, i.e., a graph whose vertex set can be partitioned into two disjoint sets X and Y such that all edges connect a vertex of X with a vertex of Y [7]. Thus we have the following result.

Corollary: The number of nonisomorphic bipartite graphs with m nodes in one set and n in the other set is given by (8).

The techniques used so far are similar to those used by Harary [1] in counting bicolored graphs. The results are not the same when $m = n$ since a different symmetry group is involved in that case for such graphs. Also see [8] for related work.

We now indicate a totally different way to use Pólya's theorem to solve the same problem for equivalence with respect to $\equiv_1$. This second solution has the advantages of using material available in the switching theory textbooks and, more importantly, it generalizes to the case of equivalence with respect to $\equiv_2$.

The idea of our new approach is to regard S_n as acting on the set of n-tuples of zeros and ones. This is exactly what is done in switching theory for computing the number of classes of functions. Let us call this representation of the symmetric group S_n'. The cycle index of S_n' has been determined [2], [3] and a table of such polynomials is tabulated in [3]. To use these polynomials, we note that this problem is to count the number of classes of functions with domain $\{0, 1\}^n$ and range $R = N$. We choose $w(p) = p$ for $p \in N$ and hence $\psi(x) = 1 + x + x^2 + \cdots = 1/1 - x$ because this allows each n-tuple to appear 0 times, 1 time, $\cdots$. Thus we have the following result.

Theorem 3: The number of $\equiv_1$ equivalence classes of matrices in $M(m, n)$ is given by the coefficient of x^m in

$$\sum_{m_1 \in N} s_{mn} x^m = P(x) = Z_{S_n'}\left(\frac{1}{1-x}, \frac{1}{1-x^2}, \cdots, \frac{1}{1-x^{2^n}}\right).$$

Proof: Using R and ψ as above and taking $G = S_n'$, the symmetric group on $\{0, 1\}^n$, we are enumerating equivalence classes of *sets* of n-tuples under S_n', i.e., elements of $M(m, n)$ under $\equiv_1$. By Pólya's theorem, the result follows.

Example: Let $n = 3$. From [3, Appendix I],

$$Z_{S_3'} = \tfrac{1}{6}(f_1^8 + 3f_1^4 f_2^2 + 2f_1^2 f_3^2)$$
$$P(x) = \tfrac{1}{6}\left[\left(\frac{1}{1-x}\right)^8 + \frac{3}{(1-x)^4(1-x^2)^2} + \frac{2}{(1-x)^2(1-x^3)^2}\right]$$
$$= 1 + 4x + 13x^2 + 36x^3 + 87x^4 + \cdots.$$

This last proof admits an easy generalization to equivalence under $\equiv_2$. We merely replace S_n' by G_n, the group of permutations and/or complementations on $\{0, 1\}^n$. G_n has been extensively studied. Formulas for its cycle index have been computed and a table of them appears in [3]. Thus we have the following result.

TABLE I
THE NUMBER OF 0-1 MATRICES UNDER ROW AND COLUMN PERMUTATIONS

$m \backslash n$	1	2	3	4	5	6
			s_{mn}			
1	2	3	4	5	6	7
2	3	7	13	22	34	50
3	4	13	36	87	190	386
4	5	22	87	317	1053	3250
5	6	34	190	1053	5624	28 576
6	7	50	386	3250	28 576	251 610
7	8	70	734	9343	136 758	2 141 733
8	9	95	1324	25 207	613 894	17 256 831
9	10	125	2284	64 167	2 583 164	130 237 768
10	11	161	3790	155 004	10 208 743	917 558 397

TABLE II
THE NUMBER OF 0-1 MATRICES UNDER ROW PERMUTATIONS WITH COLUMN PERMUTATIONS AND/OR COMPLEMENTATIONS

$m \backslash n$	1	2	3	4	5	6
			t_{mn}			
1	1	1	1	1	1	1
2	2	3	4	5	6	7
3	2	4	7	11	16	23
4	3	8	19	41	81	153
5	3	10	32	101	299	849
6	4	16	68	301	1358	6128
7	4	20	114	757	5567	43 534
8	5	29	210	1981	23 350	319 119
9	5	35	336	4714	91 998	2 255 466
10	6	47	562	11 133	351 058	15 307 395

Theorem 4: The number of $\equiv_2$ equivalence classes of matrices in $M(m, n)$ is given by the coefficient of x^m in

$$\sum_{m \in N} t_{mn} x^m = P(x) = Z_{G_n}\left(\frac{1}{1-x}, \frac{1}{1-x^2}, \cdots, \frac{1}{1-x^{2^n}}\right).$$

Proof: The argument parallels that of the previous theorem and is omitted.

To compute the number of classes for modest values of m and n, the tables in [3, Appendices I and II] were used along with the previous two theorems. G. Mantero programmed the calculations using a higher level computer language for algebraic manipulations. The results are tabulated in Tables I and II.

The tables suggest the following identities which can be established for arbitrary m and n.

Theorem 5:

a) For all $m, n \geqslant 0$, $s_{mn} = s_{nm}$.

b) For all $n \geqslant 0$, $s_{1n} = n + 1$.

c) For all $n \geqslant 0$, $t_{1n} = 1$.

d) For all $n \geqslant 0$, $t_{2n} = n + 1$.

e) For all $m \geqslant 0$, $t_{m1} = 1 + \lfloor m/2 \rfloor$ where $\lfloor x \rfloor$ is the "floor" function, i.e., the largest integer $\leqslant x$.

f) For all m such that $m \equiv 1 \bmod 2$,

$$t_{m2} = \binom{(m+5)/2}{3}.$$

g) For all $m \geqslant 1$,

$$t_{m2} = \frac{1}{8}\left[\binom{m+3}{3} + 2(\lfloor m/2 \rfloor + 1)(\lceil m/2 \rceil + 1) + 3\left(\lfloor m/2 \rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor\right)(\lfloor m/2 \rfloor + 1) + 2\left(\lfloor m/4 \rfloor - \left\lfloor \frac{m-1}{4} \right\rfloor\right)\right].$$

Proof: Clearly a), b), and c) are trivial to verify. To show d), note that each such matrix has $2n$ entries and so may have 0 ones, 1 one, $\cdots$, $2n$ ones. But the matrices with i ones are equivalent (under $\equiv_2$) to those with $2n - i$ ones for $i \leqslant n$. Thus there are exactly $n + 1$ classes.

To prove e), the technique is similar. We note that the classes can be characterized by min $\{k, m - k\}$ where k is the number of ones in the array. We need to find the number of sets of two nonnegative integers i and j such that $i + j = m$. Clearly this is $\lfloor m/2 \rfloor + 1$.

Only g) has an interesting proof, as f) is a special case of g). We know that

$$Z_{G_2} = \frac{1}{8}(f_1^4 + 2f_1^2 f_2 + 3f_2^2 + 2f_4).$$

Thus

$$P(x) = \frac{1}{8}\left[\frac{1}{(1-x)^4} + \frac{2}{(1-x)^2(1-x^2)} + \frac{3}{(1-x^2)^2} + \frac{2}{1-x^4}\right]. \tag{9}$$

By the binomial theorem

$$\frac{1}{(1-x)^4} = \sum_{k \geqslant 0} \binom{k+3}{3} x^k. \tag{10}$$

It is not hard to show that

$$\frac{1}{(1-x)^2(1-x^2)} = \sum_{k \geqslant 0} (\lfloor k/2 \rfloor + 1)(\lceil k/2 \rceil + 1) \tag{11}$$

$$\frac{1}{(1-x^2)^2} = \sum_{m \geqslant 0} (m+1)x^{2m} = \sum_{k \geqslant 0} \left(\lfloor k/2 \rfloor - \left\lfloor \frac{k-1}{2} \right\rfloor\right) \cdot (\lfloor k/2 \rfloor + 1)x^k \tag{12}$$

and

$$\frac{1}{1-x^4} = \sum_{m \geqslant 0} x^{4m} = \sum_{k \geqslant 0} \left(\lfloor k/4 \rfloor - \left\lfloor \frac{k-1}{4} \right\rfloor\right) x^k. \tag{13}$$

Collecting together the coefficients of x^k in (9) after the substitution of (10), (11), (12), and (13) gives the result.

III. Some Open Questions

One often wishes to change the definition of equivalence to rule out degenerate cases. For instance, one might wish to enumerate only nondegenerate equivalence classes of matrices where a matrix is *nondegenerate* if it has no constant or repeated columns.

Example: $m = 3$, $n = 2$. According to Table II, $t_{32} = 4$. Representatives of the four classes are as follows:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

But there is only one class which is nondegenerate, namely,

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Although a number of partial results have been obtained, we have not succeeded in calculating the number of nondegenerate cases for arbitrary m and n.

Although our Theorems 2, 3, and 4 give exact results, they are complicated to compute. It would be desirable to have easily computed approximations. We do know that

$$s_{mn} > \frac{2^{mn}}{m!n!}$$

and

$$t_{mn} > \frac{2^{(m-1)n}}{m!n!}.$$

Unfortunately, these bounds are not the asymptotic limits. It would be desirable to have simply computable asymptotic estimates for s_{mn} and t_{mn}.

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An Improved Algorithm for the Generation of Nonparametric Curves

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***Abstract*—Generation of curves using incremental steps along fixed coordinate axes is important in such diverse areas as computer displays, digital plotters, and numerical control. Direct implementation of a nonparametric representation of a curve, $f(x, y) = 0$, has been shown to be attractive for digital generation. The algorithm in this paper is developed directly from the nonparametric representation of the curve, allows steps to be taken to any point adjacent to the current one, and uses decision variables closely related to an error criterion. Consequently, the algorithm is more general and produces curves closer to the actual curve than do previously reported algorithms.**

***Index Terms*—Computer displays, curve generation, digital plotters, incremental, nonparametric curves.**

I. INTRODUCTION

GENERATION of curves using incremental steps along fixed coordinate axes is important in such diverse areas as computer graphics displays, digital plotters, and numerical control. The classical approach has been to use digital differential analyzers to implement the basically analog technique of introducing an independent parameter, time, and determine a set of differential equations whose solution results in the desired curve. Recently, there has been interest in nonparametric approaches, which are more natural to digital systems. Erdahl [1] developed an algorithm for straight lines, together with associated hardware, which placed a dot in every xy position through which the continuous line passed. Bresenham [2] presented an algorithm for straight lines that basically placed data in those positions which are closest to the original curve. Metzger [3] reiterated Bresenham's algorithm and also produced algorithms for circles and parabolas. All of the above techniques are satisfactory for the generation of approximations to straight lines, but are not general, and Metzger's algorithms are not efficient for other than straight lines. Danielsson [4] considered both the classical DDA approach and proposed an algorithm using a nonparametric technique. He pointed out that the nonparametric approach can produce closed curves without degradation and proceeds along the curve with less velocity variation. While Danielsson's algorithm is general and simple and does produce points close to the actual curve, his basis for selecting the points is not related to any error criteria. Furthermore, incremental steps are limited to either a step in x or a step in y but not both. Consequently, 45° lines are broader and appear fuzzier than horizontal or vertical lines. Finally, curves produced for symmetrical figures (e.g., circles, parabolas) do not retain the symmetry of the original figure. While this may not be important in many applications, in some areas such as numerical control and automatic production of photographic masks it may be a drawback.

The algorithm proposed here retains the generality of Danielsson's approach. However, the ability to step in any of the eight possible directions is retained. Smoother and more well-defined curves are therefore obtained. Furthermore, the variables used to determine each incremental step are related to an error criteria. Consequently, the points generated are closer to the original curve.

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