SIZES OF FLAT MAXIMAL ANTICHAINS OF SUBSETS

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Abstract. This is the second of two papers investigating for which positive integers m there exists a maximal antichain of size m in the Boolean lattice B_n (the power set of $[n] := \{1, 2, \dots, n\}$, ordered by inclusion). In the first part, the sizes of maximal antichains have been characterized. Here we provide an alternative construction with the benefit of showing that almost all sizes of maximal antichains can be obtained using antichains containing only l-sets and $(l+1)$-sets for some l.

1. Introduction

In the Boolean lattice B_n of all subsets of $[n] := \{1, 2, \dots, n\}$, ordered by inclusion, an *antichain* is a family of subsets such that no one contains any other one. By a classical theorem of Sperner [12], the maximum size of an antichain in B_n is $\binom{n}{\lfloor n/2 \rfloor}$, and it is obtained only by the collections $\binom{[n]}{\lfloor n/2 \rfloor}$ and $\binom{[n]}{\lceil n/2 \rceil}$, where we write $\binom{[n]}{l}$ for the l-th *level* of B_n, i.e., for the collection of all l-element subsets of $[n]$. An antichain is called *maximal* if no $B \subseteq [n]$ with $B \notin \mathcal{A}$ can be added to $\mathcal{A}$ without destroying the antichain property. Let $S(n) = \{|\mathcal{A}| : \mathcal{A} \text{ is a maximal antichain in } B_n\}$. In [6], we have proved the following characterization for the sets $S(n)$.

Theorem 1 ([6])**.** *Let $k = \lceil n/2 \rceil$, and let m be an integer with $1 \leqslant m \leqslant \binom{n}{k}$. Then $m \in S(n)$ if and only if $m \leqslant \binom{n}{k} - k\lceil (k+1)/2 \rceil$ or m has the form*

$$m = \binom{n}{k} - tl + \binom{a}{2} + \binom{b}{2} + c$$

for some integers $l \in \{k, n-k\}$, $t \in \{0, \dots, k\}$, $a \geqslant b \geqslant c \geqslant 0$, $a + b \leqslant t$. Moreover, if $m \in S(n)$ satisfies $m \geqslant \binom{n}{k} - k^2$ then there exists a maximal antichain $\mathcal{A}$ with $|\mathcal{A}| = m$ and $\mathcal{A} \subseteq \binom{[n]}{k} \cup \binom{[n]}{k+1}$ or $\mathcal{A} \subseteq \binom{[n]}{k-1} \cup \binom{[n]}{k}$.

The last statement in Theorem 1 says that all sizes close to the maximum possible size $\binom{n}{k}$ are obtained by *flat* antichains, which means that there exists an integer l such that $|A| \in \{l, l+1\}$ for every $A \in \mathcal{A}$. In the present paper, we add this as a constraint and investigate for which $m \in S(n)$ there exists a *flat* maximal antichain of size m. Our main result is the following answer to this question.

Theorem 2. *Let $k = \lceil n/2 \rceil$, and let m be an integer with*

$$\binom{n}{2} - \left\lfloor \frac{(n+1)^2}{8} \right\rfloor \leqslant m \leqslant \binom{n}{k} - k\left\lceil \frac{k+1}{2} \right\rceil.$$

Then there exist an $l \in \{2, 3, \dots, \lfloor (n-2)/2 \rfloor\}$ and a maximal antichain $\mathcal{A} \subseteq \binom{[n]}{l} \cup \binom{[n]}{l+1}$ with $|\mathcal{A}| = m$.

The lower bound on m in Theorem 2 is best possible: In [7] it was shown that $\binom{n}{2} - \left\lfloor \frac{(n+1)^2}{8} \right\rfloor$ is the minimum size of a maximal antichain consisting of 2-sets and 3-sets, and with singletons and 2-sets we only

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get the sizes $\binom{t}{2}+n-t$ for $t\in\{0,1,\dots,n\}$. Therefore, at least one of the two numbers $\binom{n}{2}-\left\lfloor\frac{(n+1)^2}{8}\right\rfloor-1$ and $\binom{n}{2}-\left\lfloor\frac{(n+1)^2}{8}\right\rfloor-2$ is not the size of a flat maximal antichain.[1] Combining Theorems 1 and 2, we obtain the following characterization of the sizes of flat maximal antichains.

Corollary 1. *Let n and m be integers with $1\leqslant m\leqslant\binom{n}{k}$ where $k=\lceil n/2\rceil$. There exists a flat maximal antichain of size m in B_n if and only if at least one of the following three conditions is satisfied:*

(i) $\binom{n}{2}-\left\lfloor\frac{(n+1)^2}{8}\right\rfloor\leqslant m\leqslant\binom{n}{k}-k\left\lceil\frac{k+1}{2}\right\rceil$

(ii) $m=\binom{n}{k}-tl+\binom{a}{2}+\binom{b}{2}+c$ *for some integers* $l\in\{k,n-k\}$, $t\in\{0,\dots,k\}$, $a\geqslant b\geqslant c\geqslant 0$, $a+b\leqslant t$.

(iii) $m=\binom{t}{2}+n-t$ *for some* $t\in\{0,1,\dots,n\}$.

In particular, almost all sizes of maximal antichains are obtained by flat maximal antichains, the exceptions being the elements of the set $\left\{1,2,\dots,\binom{n}{2}-\left\lfloor\frac{(n+1)^2}{8}\right\rfloor-1\right\}\setminus\left\{\binom{t}{2}+n-t\,:\,t\in\{0,\dots,n\}\right\}$.

Related work. In [7] the problem of finding the smallest size of a maximal antichain in $\binom{n}{l}\cup\binom{n}{l+1}$ was raised and a complete answer was provided for $l=2$. A stability version of this result together with asymptotic bounds for general l have been presented in [3]. The more general question of minimizing the size of an antichain on two arbitrary levels, not necessarily consecutive, has been studied in [8]. Recently, there has been an increasing interest in poset saturation problems, asking for the possible sizes of posets which are maximal with respect to the property of not containing a fixed poset (see [10] for an overview).

Another related line of research is the investigation of the domination number of the bipartite graph with vertex set $\binom{[n]}{l}\cup\binom{[n]}{l+1}$ (an more generally $\binom{[n]}{l}\cup\binom{[n]}{l+t}$ for $t\in\{1,\dots,n-l\}$) where adjacency is given by set inclusion [1, 2]. A dominating set in a graph is a set W of vertices with the property that every vertex outside W has a neighbor in W, and the *domination number* of a graph is the minimum size of a dominating set. Clearly, a maximal antichain $\mathcal{A}$ is a minimal dominating set which is also an independent set (that is, there are no edges between elements of $\mathcal{A}$). The problem of finding the minimum size of a flat maximal antichain can be viewed as asking for the *independent domination number* of the graph: the minimum size of an independent dominating set. For $l=2$, the domination number is asymptotically equal to the independent domination number [2], and it is conjectured that the exact values are equal.

2. High level overview

Before going into the technical details we provide a sketch of the main ideas behind the construction. This will be informal and without proofs. It turns out that this informal description is more straightforward in the upper half of B_n, that is, using sets of size at least $\lfloor n/2\rfloor$. The statement of Theorem 2 and the formal proofs below take place in the lower half. Clearly, these two settings are equivalent via the observation that the family obtained by taking complements of the members of a maximal antichain is again a maximal antichain.

For $\mathcal{F}\subseteq\binom{[n]}{l+1}$, the *shadow* $\Delta\mathcal{F}$ is the collection of all l-sets contained in some member of $\mathcal{F}$, that is, $\Delta\mathcal{F}=\{X\subseteq[n]\,:\,|X|=l\text{ and }X\subseteq A\text{ for some }A\in\mathcal{F}\}$. This concept is relevant because a flat maximal antichain $\mathcal{A}\subseteq\binom{[n]}{l+1}\cup\binom{[n]}{l}$ is determined by its collection of $(l+1)$-sets: $\mathcal{A}=\mathcal{F}\cup\binom{[n]}{l}\setminus\Delta\mathcal{F}$. Such an antichain is maximal if and only if $\Delta A\not\subseteq\Delta\mathcal{F}$ for every $A\in\binom{[n]}{l+1}\setminus\mathcal{F}$. Similarly, the *shade* (or *upper shadow*) $\nabla\mathcal{G}$ of $\mathcal{G}\subseteq\binom{[n]}{l}$ (with respect to the fixed ground set $[n]$) is the family of all $(l+1)$-subsets of $[n]$ which are a superset of some member of $\mathcal{G}$, i.e., $\nabla\mathcal{G}=\{Y\,:\,Y\subseteq[n],\ |Y|=l+1,\ B\subseteq Y\text{ for some }B\in\mathcal{G}\}$. In verifying maximality of certain flat antichains constructed below it is convenient to notice that an antichain $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2$ with $\mathcal{A}_1\subseteq\binom{[n]}{l}$ and $\mathcal{A}_2\subseteq\binom{[n]}{l+1}$, is maximal if and only if $\mathcal{A}_1=\binom{[n]}{l}\setminus\Delta\mathcal{A}_2$ and $\mathcal{A}_2=\binom{[n]}{l+1}\setminus\nabla\mathcal{A}_1$.

Given a positive integer t, the Kruskal-Katona Theorem [9, 11] says that among the families $\mathcal{F}\subseteq\binom{[n]}{l+1}$ with $|\mathcal{F}|=t$, the shadow is minimized by taking the first t sets in *squashed order* (also known as *colexicographic*

[1] Usually $\binom{n}{2}-\left\lfloor\frac{(n+1)^2}{8}\right\rfloor-1$ is the largest missing value, but if $s=4(n-1)(n-2)-8\left\lfloor\frac{(n+1)^2}{8}\right\rfloor-7$ turns out to be a square then $\binom{n}{2}-\left\lfloor\frac{(n+1)^2}{8}\right\rfloor-1=\binom{t}{2}+n-t=$ with $t=\frac{3}{2}+\frac{1}{2}\sqrt{s}$ is the size of a flat maximal antichain consisting of 1-sets and 2-sets. For $n\leqslant 1\,000\,000$ this happens only for the following 23 values of n: 5, 6, 7, 8, 9, 33, 48, 63, 78, 93, 429, 638, 847, 1056, 1265, 5945, 8856, 11 767, 14 678, 17 589, 82 773, 123 318, 163 863, 204 408, 244 953.

order), where A precedes B in this order if $\max((A\cup B)\setminus(A\cap B))\in B$. The theorem also specifies the size of the shadow: Writing t in the cascade representation $t=\binom{a_{l+1}}{l+1}+\binom{a_l}{l}+\dots+\binom{a_r}{r}$ with $a_{l+1}>a_l>\dots>a_r\geqslant r\geqslant 1$, the shadow of the first t sets of size $l+1$ in squashed order has size $\binom{a_{l+1}}{l}+\binom{a_l}{l-1}+\dots+\binom{a_r}{r-1}$.

We call a flat antichain $\mathcal{A}$ *squashed* if it has the form $\mathcal{A}=\mathcal{F}\cup\binom{[n]}{l}\setminus\Delta\mathcal{F}$, where $\mathcal{F}$ is the collection of the first $|\mathcal{F}|$ elements of $\binom{[n]}{l+1}$ in squashed order. Note that such a squashed flat antichain is a maximal antichain if and only if $r\geqslant 2$ (see [5, Proposition 3.1] for a formal proof). The sizes of maximal squashed flat antichains have been studied in [5], and they form the backbone of our construction.

The main idea in the proof of Theorem 2 is to begin with the families of squashed flat maximal antichains on levels $l+1$ and l, and consider the gaps in the corresponding collection of sizes. The ultimate aim is to show that each gap value is attained by a flat maximal antichain on levels $i+1$ and i for some i. The difference of any two consecutive sizes of squashed flat maximal antichains on levels $l+1$ and l is at most l, and the idea is to fill the gaps between them by modifying the last few sets in an initial segment $\mathcal{F}$ of $\binom{[n]}{l+1}$. More precisely, if $\mathcal{A}=\mathcal{F}\cup\binom{[n]}{l}\setminus\Delta\mathcal{F}$ is a squashed flat maximal antichain of size m, and $\delta\leqslant l$, we change the last few sets of $\mathcal{F}$ to obtain a family $\mathcal{F}'$ such that $\mathcal{A}'=\mathcal{F}'\cup\binom{[n]}{l}\setminus\Delta\mathcal{F}'$ is a maximal antichain of size $m-\delta$. We show that this is possible by an induction argument: The base case is $l=n-3$ for $l\leqslant n-4$ recursive constructions are used.

The base case $l=n-3$. We start by constructing, for every $i\in\{0,1,\dots,\lambda\}$, a family $\mathcal{F}_i\subseteq\binom{[n]}{l+1}$ with $|\mathcal{F}_i|=\left\lceil\frac{l+1}{2}\right\rceil$ and $|\Delta\mathcal{F}_i|=(l+1)\left\lceil\frac{l+1}{2}\right\rceil-i$ with the property that $n\in A$ for every A in every $\mathcal{F}_i$. Here λ is a parameter depending on n: $\lambda=f(n-2)$ in Proposition 2 below, in particular $\lambda=l$ for $n\geqslant 13$. For $t=0,1,\dots,l$, we set $\mathcal{F}_{i,t}=\mathcal{F}_i\cup\{A_1,\dots,A_t\}$ where A_j is the j-th $(l+1)$-set in squashed order (note that $n\notin A_j$ for all j), and $\mathcal{A}_{i,t}=\mathcal{F}_{i,t}\cup\binom{[n]}{l}\setminus\Delta\mathcal{F}_{i,t}$. The families $\mathcal{F}_i$ are constructed in such a way that, for all i and t, adding A_t to $\mathcal{A}_{i,t-1}$ increases the shadow by $l+1-t$ for $t\leqslant\lceil(n-2)/2\rceil$ and by $l+2-t$ if $t>\lceil(n-2)/2\rceil$. As a consequence, the sizes of the maximal antichains $\mathcal{A}_{i,t}$ cover a certain interval ending at $|\mathcal{A}_{\lambda,0}|=\binom{n}{3}-l\left\lceil\frac{l+1}{2}\right\rceil+\lambda$. By an induction argument we also have an interval of sizes starting at $\binom{n}{l+1}-5$ and ending above the starting point of the previously mentioned interval. Combining these two intervals, we have an interval of sizes from $\binom{n}{2}-5$ to $|\mathcal{A}_{\lambda,0}|$ (see Lemma 3.2). For example, for $(n,l)=(13,10)$, the families $\mathcal{F}_0,\dots,\mathcal{F}_{10}$ are illustrated in Figures 1 to 4. Here we use that we can think of a set of size $n-2$ as the complement of an edge of a graph, and a family of sets of size $n-2$ can be represented by a graph. In this sense, the members of $\mathcal{F}_i$ are the complements of the edges of the corresponding graph, for instance

$$\mathcal{F}_0=\{[13]\setminus\{1,12\},\ [13]\setminus\{2,11\},\ [13]\setminus\{3,10\},\ [13]\setminus\{4,9\},\ [13]\setminus\{5,8\},\ [13]\setminus\{6,7\}\}.$$

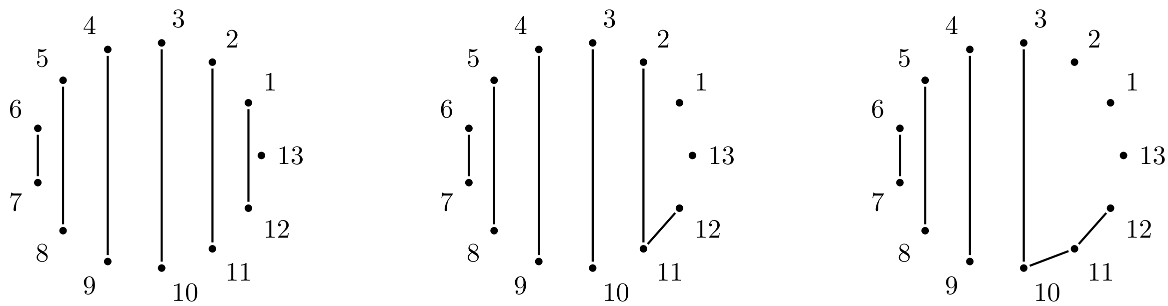


FIGURE 1. The edges of the graphs are the complements of the members of $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$.

Adding the first t 11-sets in squashed order corresponds to adding the edges $\{13,12\}$, $\{13,11\},\dots,\{13,13-t\}$ to these graphs, and as a result we obtain the following maximal antichains:

- $\mathcal{A}_{i,0}$, $i=0,1,\dots,10$, each with 6 11-sets, cover the sizes $\binom{13}{10}-60=226,\dots,\binom{13}{10}-50=236$,
- $\mathcal{A}_{i,1}$, $i=0,1,\dots,10$, each with 7 11-sets, cover the sizes $\binom{13}{10}-69=217,\dots,\binom{13}{10}-59=227$,
- $\mathcal{A}_{i,2}$, $i=0,1,\dots,10$, each with 8 11-sets, cover the sizes $\binom{13}{10}-77=209,\dots,\binom{13}{10}-67=219$,

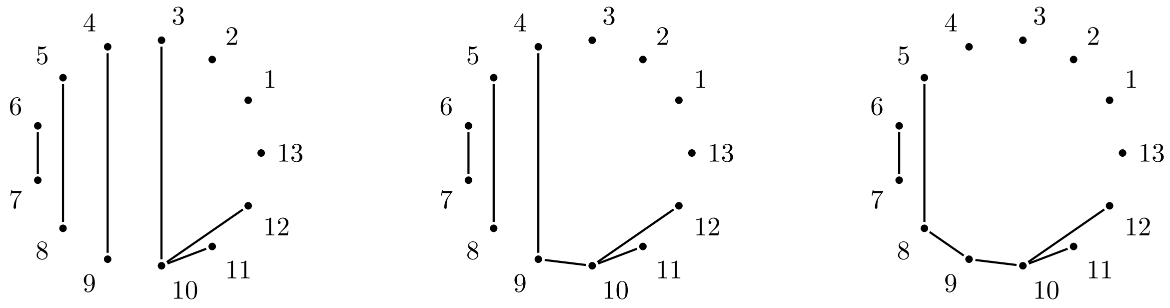


FIGURE 2. The edges of the graphs are the complements of the members of $\mathcal{F}_3$, $\mathcal{F}_4$, $\mathcal{F}_5$.

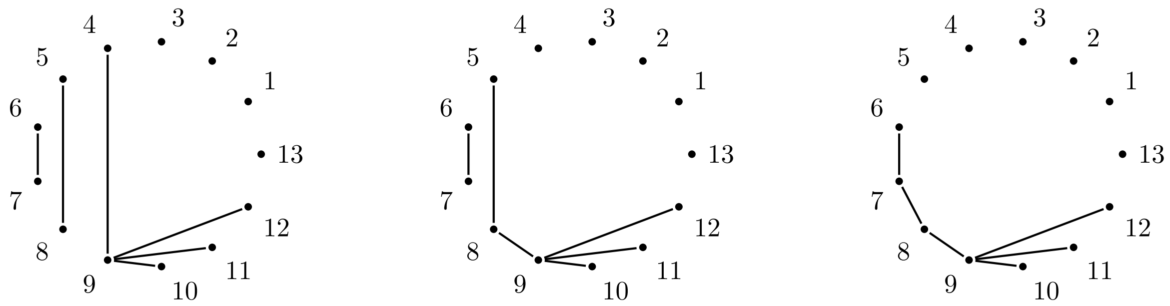


FIGURE 3. The edges of the graphs are the complements of the members of $\mathcal{F}_6$, $\mathcal{F}_7$, $\mathcal{F}_8$.

FIGURE 4. The edges of the graphs are the complements of the members of $\mathcal{F}_9$ and $\mathcal{F}_{10}$.

- $\mathcal{A}_{i,3}$, $i = 0, 1, \dots, 10$, each with 9 11-sets, cover the sizes $\binom{13}{10} - 84 = 202, \dots, \binom{13}{10} - 74 = 212$,
- $\mathcal{A}_{i,4}$, $i = 0, 1, \dots, 10$, each with 10 11-sets, cover the sizes $\binom{13}{10} - 90 = 196, \dots, \binom{13}{10} - 80 = 206$,
- $\mathcal{A}_{i,5}$, $i = 0, 1, \dots, 10$, each with 11 11-sets, cover the sizes $\binom{13}{10} - 95 = 191, \dots, \binom{13}{10} - 85 = 201$,
- $\mathcal{A}_{i,6}$, $i = 0, 1, \dots, 10$, each with 12 11-sets, cover the sizes $\binom{13}{10} - 99 = 187, \dots, \binom{13}{10} - 89 = 197$,
- $\mathcal{A}_{i,7}$, $i = 0, 1, \dots, 10$, each with 13 11-sets, cover the sizes $\binom{13}{10} - 103 = 183, \dots, \binom{13}{10} - 93 = 193$,
- $\mathcal{A}_{i,8}$, $i = 0, 1, \dots, 10$, each with 14 11-sets, cover the sizes $\binom{13}{10} - 106 = 180, \dots, \binom{13}{10} - 96 = 190$,
- $\mathcal{A}_{i,9}$, $i = 0, 1, \dots, 10$, each with 15 11-sets, cover the sizes $\binom{13}{10} - 108 = 178, \dots, \binom{13}{10} - 98 = 188$,
- $\mathcal{A}_{i,10}$, $i = 0, 1, \dots, 10$, each with 16 11-sets, cover the sizes $\binom{13}{10} - 109 = 177, \dots, \binom{13}{10} - 99 = 187$.

Overall we get maximal antichains of all sizes in the interval $[\binom{13}{10} - 109, \binom{13}{10} - 50] = [177, 236]$ (illustrated by the plot on the left in Figure 5). By induction, we also have maximal antichains in $\binom{[12]}{9} \cup \binom{[12]}{10}$ of all sizes between $\binom{12}{2} - 5 = 61$ and $\binom{12}{3} - 38 = 182$. For each such antichain $\mathcal{A}$ we obtain a maximal antichain $\mathcal{A}' \subseteq \binom{[13]}{10} \cup \binom{[13]}{11}$ with $|\mathcal{A}'| = |\mathcal{A}| + 12$ by (1) adding the element 13 to every $A \in \mathcal{A}$ and (2) adding the 11-sets $[12] \setminus \{i\}$ for $i = 1, 2, \dots, 12$ to the antichain (Lemma 3.1(i)). This yields the sizes from $\binom{13}{2} - 5 = 73$

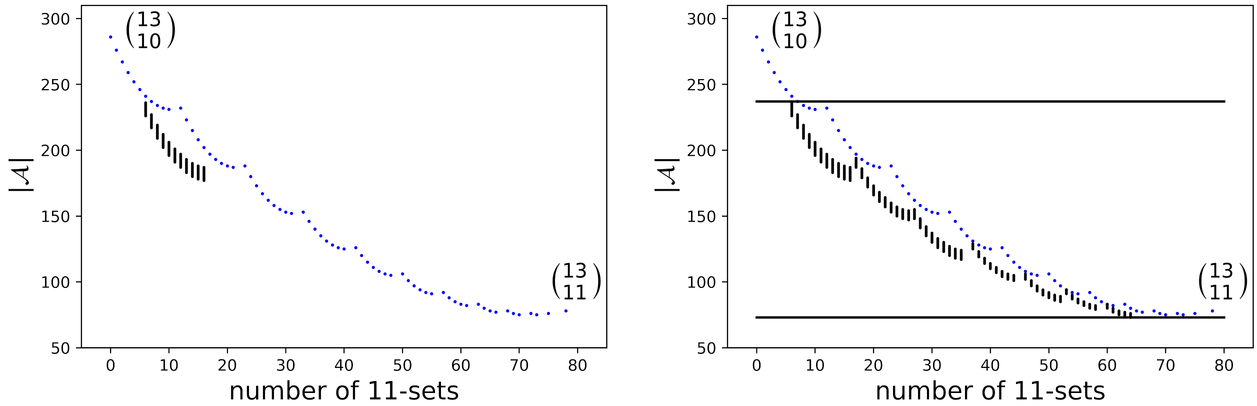


FIGURE 5. The dots correspond to the squashed maximal flat antichains $\mathcal{A} \subseteq \binom{[13]}{10} \cup \binom{[13]}{11}$. In the left plot, the segments correspond to the maximal antichains $\mathcal{A}_{i,t}$ ($i = 0, 1, \dots, 10$, $t = 0, 1, \dots, 10$). In the right plot, we have also added the maximal flat antichains obtained by recursion from $n = 12$. The interval $[73, 237]$ of sizes of maximal antichains contained in $\binom{[13]}{11} \cup \binom{[13]}{10}$ is indicated by horizontal lines.

to $182 + 12 > 177$. Thus, we have maximal antichains of all sizes between $\binom{13}{2} - 5$ and $\binom{13}{3} - 50$ (see the right hand side plot in Figure 5).

The case $l \leqslant n - 4$**.** For $t = 0, 1, \dots, \binom{n}{l+1}$, let $\mathcal{I}_t$ be the collection of the first t elements of $\binom{[n]}{l+1}$ in squashed order, and define the function $f : \{0, 1, \dots, \binom{n}{l+1}\} \to \mathbb{N}$ by $f(t) = t + \binom{n}{l} - |\Delta\mathcal{I}_t|$. In other words, $f(t)$ is the size of the squashed full flat antichain with t $(l+1)$-sets. For t with cascade representation $t = \binom{a_{l+1}}{l+1} + \binom{a_l}{l} + \dots + \binom{a_r}{r}$ with $a_r \geqslant r + 2$, we can use the base case to obtain antichains of all sizes between $f(t + \binom{r+1}{r-1}) - 5$ and a value just below $f(t)$. The base construction provides, for every $\delta \in \{\delta_0, \delta_0 + 1, \dots, \binom{r+1}{r-2} - \binom{r+1}{r-1} + 5\}$ a family $\mathcal{F}' \subseteq \binom{[r+1]}{r-1}$ with $|\Delta\mathcal{F}'| - |\mathcal{F}'| = \delta$, where $\delta_0 < r^2/2$ (see Lemma 3.2 for the precise value). The maximal antichain $\mathcal{A} = \mathcal{F} \cup \binom{[n]}{l} \setminus \Delta\mathcal{F}$ with $\mathcal{F} = \mathcal{I}_t \cup \{A \cup \{a_j + 1 : r \leqslant j \leqslant l + 1\} : A \in \mathcal{F}'\}$ has size $|\mathcal{A}| = f(t) - \delta$ since we have the partition $\Delta\mathcal{F} = \Delta\mathcal{I}_t \cup \{A \cup \{a_j + 1 : r \leqslant j \leqslant l + 1\} : A \in \Delta\mathcal{F}'\}$, hence

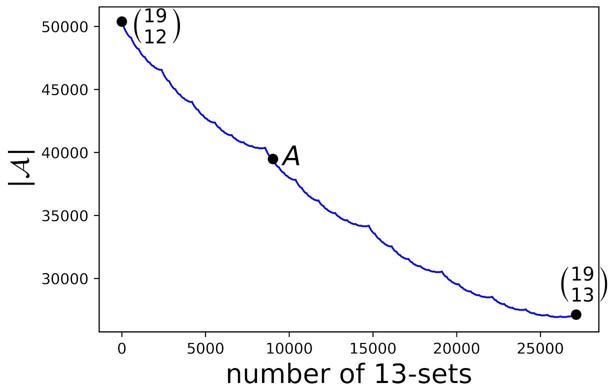
$$|\mathcal{A}| = |\mathcal{F}| + \binom{n}{l} - |\Delta\mathcal{F}| = t + |\mathcal{F}'| + \binom{n}{l} - |\Delta\mathcal{I}_t| - |\Delta\mathcal{F}'| = f(t) - \delta.$$

For example, for $n = 19$, $l = 12$ and $t = \binom{18}{13} + \binom{15}{12}$, we use the base case for $\binom{[13]}{10} \cup \binom{[13]}{11}$ to obtain maximal antichains $\mathcal{A} \subseteq \binom{[19]}{12} \cup \binom{[19]}{13}$ of all sizes between $f(t + \binom{13}{11}) - 5$ and $f(t) - 49$. The base case provides, for every $\delta \in \{49, \dots, \binom{13}{10} - \binom{13}{11} + 5\}$ a family $\mathcal{F}' \subseteq \binom{[13]}{11}$ with $|\Delta\mathcal{F}'| - |\mathcal{F}'| = \delta$, and then the maximal antichain $\mathcal{A} = \mathcal{F} \cup \binom{[19]}{12} \setminus \Delta\mathcal{F}$ for

$$\mathcal{F} = \binom{[18]}{13} \cup \left\{A \cup \{19\} : A \in \binom{[15]}{12}\right\} \cup \{A \cup \{16, 19\} : A \in \mathcal{F}'\}$$

has size $|\mathcal{A}| = f(t) - \delta$ (see Figure 6 for an illustration).

According to the description above, we obtain maximal antichains $\mathcal{A} \subseteq \binom{[n]}{l} \cup \binom{[n]}{l+1}$ whose sizes form many short intervals inside the large interval that goes roughly from $\binom{n}{l+1}$ to $\binom{n}{l}$, but there are many gaps between those intervals. In order to avoid these gaps, we proceed a bit more carefully when the base case is transferred: Instead of jumping directly from $l = n - 3$ to an arbitrary l, we move recursively from level to level (using the recursions given in Lemma 3.1). Keeping careful track of the constructed intervals, it turns out that we can push the lower end of the interval in each level of the recursion a bit further down, and, as a consequence, the recursively constructed sizes form a single interval. This is the content of Proposition 2 which is proved in Section 3. Finally, the gaps between these intervals for varying l are closed in Section 4.



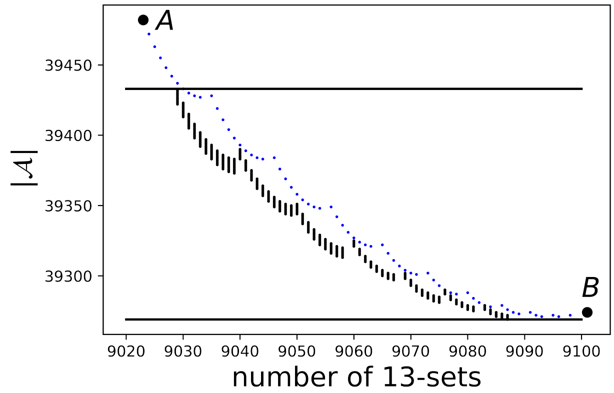


FIGURE 6. Constructing an interval of sizes of maximal antichains in $\binom{[19]}{12} \cup \binom{19}{13}$. The labels on the dots are the sizes of the corresponding maximal flat antichains. The part of the graph shown in the plot on the right has the same structure as the graph for the sizes of the maximal flat antichains in $\binom{[13]}{10} \cup \binom{[13]}{11}$ (see Figure 5). The point A corresponds to a maximal antichain of size $\binom{19}{12} + \binom{18}{13} - \binom{18}{12} + \binom{15}{12} - \binom{15}{11} = 39\,482$, and the point B represents size $A + \binom{13}{11} - \binom{13}{10} = 39\,274$. The interval $[39\,269, 39\,433]$ of sizes of maximal antichains contained in $\binom{[19]}{13} \cup \binom{[19]}{12}$ is indicated by horizontal lines.

For the rest of the paper we work in the lower half of B_n, that is, with subsets of size at most $\lceil n/2 \rceil$. The main reason is that a major part of the argument (Section 3.2) is about the base case which deals with maximal antichains in $\binom{[n]}{n-2} \cup \binom{[n]}{n-3}$ or in $\binom{[n]}{2} \cup \binom{[n]}{3}$. Working with sets of size $n-2$ and $n-3$ was convenient for the informal overview above, because the $(n-2)$-sets are added in squashed order, and this ordering is very familiar for many people working in the field. For our formal proof it is easier to work with sets of size 2 and 3, because this allows us to interpret the 2-sets in the antichain directly as the edges of a graph (without the need to take complements), and using concepts from graph theory turns out to be useful for our argument. Note that the plots in Figures 5 and 6 look essentially the same if we work in the lower half: We just have to change the labels of the x-axes to "number of 2-sets" and "number of 6-sets", respectively, and replace $\binom{19}{12}$ with $\binom{19}{7}$ etc.

For notational convenience, we reserve the letter k for $\lceil n/2 \rceil$, so that $n \in \{2k-1, 2k\}$.

Outline of the proof. Let $S(n,l)$ be the subset of $S(n)$ containing the sizes of flat antichains on levels l and $l+1$:

$$S(n,l) = \left\{ |\mathcal{A}| \,:\, \mathcal{A} \subseteq \binom{[n]}{l} \cup \binom{[n]}{l+1} \text{ is a maximal antichain in } B_n \right\}.$$

The following proposition, to be proved in Sections 3 and 4, states that $S(n,l)$ covers an interval from slightly below $\binom{n}{l}$ to slightly below $\binom{n}{l+1}$. For convenience, we denote by C_l the sum of the first l Catalan numbers, that is, $C_l = \sum_{i=1}^{l} \frac{1}{i+1}\binom{2i}{i}$.

Proposition 1.

(i) $$\left[\binom{n}{2} - \left\lfloor \frac{(n+1)^2}{8} \right\rfloor,\ \binom{n}{3} - (n-3) \left\lceil \frac{n-2}{2} \right\rceil \right] \subseteq S(n,2)$$

(ii) $$\left[\binom{n}{l} - \gamma,\ \binom{n}{l+1} - (n-l-1) \left\lceil \frac{n-l}{2} \right\rceil \right] \subseteq S(n,l) \text{ for } 3 \leqslant l \leqslant \frac{n-2}{2},$$

$$\text{where } \gamma = \max\left\{ 3 + C_l,\ (1+(l-1)(n-k)) \left(\left\lceil \frac{1}{k}\binom{k}{l} \right\rceil - l + 2 \right) \right\}$$

It has been shown in [5] that the minimum size of a squashed maximal flat antichain on levels l and $l+1$ is $\binom{n}{l}-C_l$. The first options for γ in the second part of Proposition 1 says that we can make it smaller by 3 if we drop the squashed condition.[2] The right end of the interval corresponds to an antichain obtained by taking $\lceil\frac{n-l}{2}\rceil$ l-sets such that no two of them have a common superset of size $l+1$, together with all $(l+1)$-sets which are not a superset of any of the chosen l-sets.

For $n\geqslant 15$, Proposition 1 is already sufficient to deduce the statement of Theorem 2. For instance, for $n=15$ the intervals obtained from Proposition 1 are

- $\left[\binom{15}{2}-32,\ \binom{15}{3}-84\right]=[73,371]\subseteq S(15,2)$,
- $\left[\binom{15}{3}-90,\ \binom{15}{4}-66\right]=[365,1299]\subseteq S(15,3)$,
- $\left[\binom{15}{4}-154,\ \binom{15}{5}-60\right]=[1211,2943]\subseteq S(15,4)$,
- $\left[\binom{15}{5}-116,\ \binom{15}{6}-45\right]=[2887,4960]\subseteq S(15,5)$,
- $\left[\binom{15}{6}-199,\ \binom{15}{7}-40\right]=[4860,6395]\subseteq S(15,6)$.

The union of these intervals is $\left[\binom{15}{2}-32,\binom{15}{7}-40\right]=[73,6395]$ which is the one required by Theorem 2.

The proof of Proposition 1 comes in two parts corresponding to the two possible values of γ. In Section 3 we construct maximal antichains $\mathcal{A}\subseteq\binom{[n]}{l}\cup\binom{[n]}{l+1}$ for all sizes m with

$$\binom{n}{l}-3-\sum_{i=1}^{l}\frac{1}{i+1}\binom{2i}{i}\leqslant m\leqslant\binom{n}{l+1}-(n-l-1)\left\lceil\frac{n-l}{2}\right\rceil,$$

and in Section 4.1 we use a result from coding theory to obtain the sizes m with

$$\binom{n}{l}-(1+(l-1)(n-k))\left(\left\lceil\frac{1}{k}\binom{k}{l}\right\rceil-l+2\right)\leqslant m\leqslant\binom{n}{l}-(l-1)^2-1.$$

Combining these two results, we obtain Proposition 1. The proof of Theorem 2 is completed in Section 4.2 by verifying that the intervals for $l=2,3,\dots,\lfloor\frac{n-2}{2}\rfloor$ in Proposition 1, together with some extra constructions for small n, cover the required interval.

3. Large maximal flat antichains

In this section we prove the following half of Proposition 1 (one of the two possible values of γ).

Proposition 2. *For every integer l with $2\leqslant l\leqslant(n-2)/2$,*

$$\left[\binom{n}{l}-3-C_l,\ \binom{n}{l+1}-(n-l-1)\left\lceil\frac{n-l}{2}\right\rceil+f(n-l)\right]\subseteq S(n,l),$$

where the function $f:\mathbb{N}\to\mathbb{N}$ is given by

$$f(t)=\begin{cases}0 & \text{for } t\leqslant 2,\\ 1 & \text{for } t\in\{3,4\},\\ 3 & \text{for } t\in\{5,6\},\\ 4 & \text{for } t\in\{7,8\},\\ 7 & \text{for } t\in\{9,10\},\\ t-1 & \text{for } t\geqslant 11.\end{cases}$$

The interval in Proposition 2 goes a bit further to the right than needed for Proposition 1. The proof is by induction using the recursions provided in Section 3.1, and the purpose of the function f is to facilitate the induction step by ensuring that certain shorter intervals of sizes overlap. Another point where the slight extension of the interval to the right turns out to be useful is the proof of Lemma 4.4 where it reduces the number of small values of (n,l) that have to be treated by an alternative method (see Table 1).

Section 3.2 contains the base case $l=2$. For this, we exploit an observation from [7] about a correspondence between maximal antichains $\mathcal{A}\subseteq\binom{[n]}{2}\cup\binom{[n]}{3}$ and graphs on n vertices with the property that every edge is

[2] The flat maximal antichain of size $\binom{n}{l}-C_l-3$ which comes out of our proof of Proposition 1 has the following $(l+1)$-sets: the first $2+\sum_{i=4}^{l+1}\binom{2i-2}{i}$ sets in squashed order, and the two sets $\{3,5,6\}\cup\{7,9,\dots,2l+1\}$ and $\{4,5,6\}\cup\{7,9,\dots,2l+1\}$.

contained in at least one triangle. Section 3.3 contains the induction step for the induction on l and n to conclude the proof of Proposition 2.

3.1. **Recurrences.** In this subsection we prove Lemma 3.1 which contains three recurrence relations between the sets $S(n,l)$. The ideas for these recursions is that we obtain maximal antichains in $\binom{[n]}{l}\cup\binom{[n]}{l+1}$ by any of the following three operations:

(1) Start with a maximal antichain in $\binom{[n-1]}{[l]}\cup\binom{[n-1]}{[l+1]}$ and add all l-subsets of $[n]$ which contain the element n.
(2) Start with a maximal antichain in $\binom{[n-1]}{[l-1]}\cup\binom{[n-1]}{[l]}$, add the element n to each of its members, and add all $(l+1)$-subsets of $[n-1]$ to the resulting family.
(3) Start with two maximal antichain $\mathcal{A}'$ and $\mathcal{A}''$ in $\binom{[n-2]}{[l-1]}\cup\binom{[n-2]}{[l]}$. Add the element $n-1$ to the members of $\mathcal{A}'$, and the element n to the members of $\mathcal{A}''$. Take the union of the two resulting families and add all $(l+1)-$subsets of $[n-2]$ together with all the l-subsets of $[n]$ which contain $\{n-1,n\}$ as a subset.

For convenience, for a set of integers X and an integer y, we write $X+y$ to denote the set $\{x+y\,:\,x\in X\}$, and for two sets X and Y, we write $X+Y$ for $\{x+y\,:\,x\in X,\,y\in Y\}$.

Lemma 3.1.

$$S(n,l)\supseteq S(n-1,l)+\binom{n-1}{l-1}\qquad \textit{for } 2\leqslant l\leqslant n-1, \tag{1}$$

$$S(n,l)\supseteq S(n-1,l-1)+\binom{n-1}{l+1}\qquad \textit{for } 3\leqslant l\leqslant n-2, \tag{2}$$

$$S(n,l)\supseteq S(n-2,l-1)+S(n-2,l-1)+\binom{n-2}{l+1}+\binom{n-2}{l-2}\qquad \textit{for } 3\leqslant l\leqslant n-3. \tag{3}$$

Proof. For (1), let $\mathcal{A}'\subseteq\binom{[n-1]}{[l]}\cup\binom{[n-1]}{[l+1]}$ be a maximal antichain in B_{n-1}. Then

$$\mathcal{A}=\mathcal{A}'\cup\left\{A\cup\{n\}\,:\,A\in\binom{[n-1]}{l-1}\right\}\subseteq\binom{[n]}{l}\cup\binom{[n]}{l+1}$$

is a maximal antichain in B_n. For (2), let $\mathcal{A}'\subseteq\binom{[n-1]}{[l-1]}\cup\binom{[n-1]}{[l]}$ be a maximal antichain in B_{n-1}. Then

$$\mathcal{A}=\{A\cup\{n\}\,:\,A\in\mathcal{A}'\}\cup\binom{[n-1]}{l+1}\subseteq\binom{[n]}{l}\cup\binom{[n]}{l+1}$$

is a maximal antichain in B_n. For (3), let $\mathcal{A}',\mathcal{A}''\subseteq\binom{[n-2]}{l-1}\cup\binom{[n-2]}{l}$ be maximal antichains in B_{n-2}. Then

$$\mathcal{A}=\{A\cup\{n-1\}\,:\,A\in\mathcal{A}'\}\cup\{A\cup\{n\}\,:\,A\in\mathcal{A}''\}\cup\left\{A\cup\{n-1,n\}\,:\,A\in\binom{[n-2]}{l-2}\right\}\cup\binom{[n-2]}{l+1}$$

is a maximal antichain in B_n. To see this, write $\mathcal{A}'=\mathcal{A}_1'\cup\mathcal{A}_2'$, $\mathcal{A}''=\mathcal{A}_1''\cup\mathcal{A}_2''$, $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2$ with $\mathcal{A}_1',\mathcal{A}_1''\subseteq\binom{[n-2]}{l-1}$, $\mathcal{A}_2',\mathcal{A}_2''\subseteq\binom{[n-2]}{l}$, $\mathcal{A}_1\subseteq\binom{[n]}{l}$, and $\mathcal{A}_2\subseteq\binom{[n]}{l+1}$. Then

$$\mathcal{A}_1=\{A\cup\{n-1\}\,:\,A\in\mathcal{A}_1'\}\cup\{A\cup\{n\}\,:\,A\in\mathcal{A}_1''\}\cup\left\{A\cup\{n-1,n\}\,:\,A\in\binom{[n-2]}{l-2}\right\},$$

$$\mathcal{A}_2=\{A\cup\{n-1\}\,:\,A\in\mathcal{A}_2'\}\cup\{A\cup\{n\}\,:\,A\in\mathcal{A}_2''\}\cup\binom{[n-2]}{l+1}.$$

This implies

$$\Delta\mathcal{A}_2=\{A\cup\{n-1\}\,:\,A\in\Delta\mathcal{A}_2'\}\cup\{A\cup\{n\}\,:\,A\in\Delta\mathcal{A}_2''\}\cup\binom{[n-2]}{l},$$

$$\nabla\mathcal{A}_1=\{A\cup\{n-1\}\,:\,A\in\nabla'\mathcal{A}_1'\}\cup\{A\cup\{n\}\,:\,A\in\nabla'\mathcal{A}_1''\}\cup\left\{A\cup\{n-1,n\}\,:\,A\in\binom{[n-2]}{l-1}\right\},$$

where ∇' denotes the shade in B_{n-2}, that is, $\nabla'(A)=\{A\cup\{i\}\,:\,i\in[n-2]\setminus A\}$. Now the required equations $\mathcal{A}_1=\binom{[n]}{l}\setminus\Delta\mathcal{A}_2$ and $\mathcal{A}_2=\binom{[n]}{l+1}\setminus\nabla\mathcal{A}_1$ follow from

$$\mathcal{A}_1'=\binom{[n-2]}{l-1}\setminus\Delta\mathcal{A}_2',\quad \mathcal{A}_2'=\binom{[n-2]}{l}\setminus\nabla'\mathcal{A}_1',\quad \mathcal{A}_1''=\binom{[n-2]}{l-1}\setminus\Delta\mathcal{A}_2'',\quad \mathcal{A}_2''=\binom{[n-2]}{l}\setminus\nabla'\mathcal{A}_1''. \square$$

3.2. **The base case.** In this subsection we prove the statement of Proposition 2 for $l=2$.

Lemma 3.2. *For every $n\geqslant 6$, $\left[\binom{n}{2}-6,\ \binom{n}{3}-(n-3)\left\lceil\frac{n-2}{2}\right\rceil+f(n-2)\right]\subseteq S(n,2)$.*

For the lower end of the interval, a maximal antichain of size $\binom{n}{2}-6$ is $\mathcal{A}=\mathcal{F}\cup\binom{[n]}{2}\setminus\Delta\mathcal{F}$ where $\mathcal{F}=\{123,124,356,456\}$. For the upper end of the interval, a maximal antichain of size $\binom{n}{3}-(n-3)\left\lceil\frac{n-2}{2}\right\rceil+f(n-2)$ is obtained by taking as 2-sets the edges of a tree on n vertices with $\left\lceil\frac{n-2}{2}\right\rceil$ edges with exactly $f(n-2)$ pairs of adjacent edges (the existence of these trees is the contents of Lemma 3.6). For example, the last tree in Figure 4 corresponds to a maximal antichain $\mathcal{A}\subseteq\binom{[13]}{2}\cup\binom{[13]}{3}$ with six 2-sets and size $|\mathcal{A}|=\binom{13}{3}-10\times 6+f(11)=\binom{13}{3}-50$. Lemma 3.2 states that all integers in between are also obtained as sizes of flat maximal antichains.

In [7] it was observed that the sets $S(n,2)$ can be characterized in terms of graphs with the property that every edge is contained in a triangle. We call such graphs *T-graphs*. For a graph G, let $e(G)$ be the number of edges, and let $t(G)$ be the number of triangles. Moreover, let $\overline{G}$ denote the complement, that is, the graph with the same vertex set as G, where two vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.

Lemma 3.3. *$S(n,2)=\left\{t(G)+e\left(\overline{G}\right)\,:\,G$ is a T-graph on n vertices.$\right\}$.*

Proof. Let G be a T-graph. We can take the triangles of G as the 3-sets of the antichain and the non-edges as the 2-sets to obtain a maximal antichain of size $t(G)+e\left(\overline{G}\right)$. Conversely, from a maximal antichain $\mathcal{A}\subseteq\binom{[n]}{2}\cup\binom{[n]}{3}$, we get the required T-graph by taking $\binom{[n]}{2}\setminus\mathcal{A}$ as the edge set. $\square$

In the following lemma, we express the size of the antichain corresponding to a T-graph G in terms of G and the line graph $L\left(\overline{G}\right)$, that is, the graph whose vertices are the edges of $\overline{G}$ and two of these edges are adjacent in $L\left(\overline{G}\right)$ if they have a vertex in common.

Lemma 3.4. *Let G be a T-graph. The size of the corresponding maximal flat antichain $\mathcal{A}$ is*

$$|\mathcal{A}|=\binom{n}{3}-(n-3)e\left(\overline{G}\right)+e\left(L\left(\overline{G}\right)\right)-t\left(\overline{G}\right).$$

Proof. This follows by the principle of inclusion and exclusion:

$$|\mathcal{A}|=t(G)+e\left(\overline{G}\right)=\binom{n}{3}-e\left(\overline{G}\right)(n-2)+e\left(L\left(\overline{G}\right)\right)-t\left(\overline{G}\right)+e\left(\overline{G}\right). \qquad \square$$

Lemmas 3.3 and 3.4 provide a correspondence between the elements of $S(n,2)$ and certain parameters of T-graphs. As a consequence, the proof of Lemma 3.2 can be done in terms of T-graphs. Our strategy for proving Lemma 3.2 is summarized as follows.

- We start with a collection of T-graphs G_j, $j=0,1,\dots,f(n-2)$ on vertex set $[n]$, which satisfy $e\left(\overline{G_j}\right)=\lceil(n-2)/2\rceil$, $t\left(\overline{G_j}\right)=0$ and $e\left(L\left(\overline{G_j}\right)\right)=j$. By Lemma 3.4, the corresponding maximal antichains have sizes in the interval
$$\left[\binom{n}{3}-(n-3)\left\lceil\frac{n-2}{2}\right\rceil,\ \binom{n}{3}-(n-3)\left\lceil\frac{n-2}{2}\right\rceil+f(n-2)\right].$$
- In every G_j, vertex n has degree $n-1$, and we delete edges $\{n,n-1\}$, $\{n,n-2\}$,…,$\{n,3\}$ one by one. These edge deletions preserve the property that every edge is contained in a triangle, and we can express the effect on the size of the corresponding maximal antichain: we add one 2-set (the deleted edge) and we remove all 3-sets containing the deleted edge. This gives a collection of overlapping short intervals in $S(n,2)$ whose union is a longer interval.
- We conclude the proof of Lemma 3.2 using induction on n and the recursion (1).

For the rest of the subsection we argue in terms of T-graphs without explicitly stating the translation into sizes of maximal antichains which is provided by Lemmas 3.3 and 3.4. We start by defining the almost complete graphs G_j which will serve as the starting points of the construction.

Definition 3.1. An *n-starter* is a graph G on n vertices whose complement $\overline{G}$ is a forest with $\lceil \frac{n-2}{2} \rceil$ edges.

An n-starter G is a T-graph with triangle-free complement, and by Lemma 3.4, the corresponding maximal antichain has size $\binom{n}{3} - (n-3)\left\lceil \frac{n-2}{2} \right\rceil + e\left(L\left(\overline{G}\right)\right)$. By varying $e\left(L\left(\overline{G}\right)\right)$ we obtain the final part of the interval in Proposition 2.

Example 3.1. The graph G which is obtained by removing the matching

$$E\left(\overline{G}\right) = \left\{\{n-1,1\},\{n-2,2\},\ldots,\left\{\left\lceil\frac{n+1}{2}\right\rceil,\left\lceil\frac{n-2}{2}\right\rceil\right\}\right\}$$

from K_n is an n-starter with $e\left(L\left(\overline{G}\right)\right) = 0$, and the corresponding antichain has size $\binom{n}{3} - (n-3)\left\lceil \frac{n-2}{2} \right\rceil$.

Example 3.2. Figure 7 shows the complement $\overline{G}$ of a starter for $n = 14$ and its line graph. The corresponding antichain has size $\binom{14}{3} - 11 \times 6 + 9 = \binom{14}{3} - 57$.

FIGURE 7. The complement $\overline{G}$ of a starter G for $n = 14$ and the line graph $L(\overline{G})$.

By Example 3.1 there is always a starter G with $e\left(L\left(\overline{G}\right)\right) = 0$. In view of the comment after Definition 3.1 we would like to have a family of starters such that $e\left(L\left(\overline{G}\right)\right)$ varies over an interval starting at 0. We now define the maximal length of such an interval.

Definition 3.2. Let $i^*(n)$ be the largest integer such that for every $i \in \{0, 1, \ldots, i^*(n)\}$ there exists an n-starter G with $e\left(L\left(\overline{G}\right)\right) = i$.

For our purpose (the proof of Lemma 3.2) we only need a crude lower bound on $i^*(n)$, which is provided in the next two lemmas.

Lemma 3.5. *(i) For every integer $n \geqslant 3$, $i^*(n+1) \geqslant i^*(n)$ with equality if n is odd.*
(ii) For integers $n = 2k-1 \geqslant 5$, $t \leqslant k-1$, and an $(n-2t)$-starter G, there is an n-starter G' with $e\left(L\left(\overline{G'}\right)\right) = e\left(L\left(\overline{G}\right)\right) + \binom{t+1}{2}$.

Proof. Part (i) is obvious. For part (ii), let G be a $(2k-2t-1)$-starter. We obtain the required $(2k-1)$-starter G' by adding $2t$ new vertices and joining a leaf of G to t of the new vertices. □

Lemma 3.6.

$$i^*(n) \geqslant \begin{cases} 0 & \text{for } n \in \{3,4\}, \\ 1 & \text{for } n \in \{5,6\}, \\ 3 & \text{for } n \in \{7,8\}, \\ 4 & \text{for } n \in \{9,10\}, \\ 7 & \text{for } n \in \{11,12\}, \\ n-2 & \text{for odd } n \geqslant 13, \\ n-3 & \text{for even } n \geqslant 14. \end{cases}$$

In particular, $i^*(n) \geqslant f(n-2)$ for all $n \geqslant 3$.

Proof. By Lemma 3.5(i), we can focus on the case that n is odd. For $n=3$, the starter G with $E\left(\overline{G}\right)=\{\{1,2\}\}$ witnesses $i^*(3)\geqslant 0$.

- **$n=5$:** Lemma 3.5(ii) with $t=1$ gives a 5-starter with $e\left(L\left(\overline{G}\right)\right)=1$.
- **$n=7$:** Lemma 3.5(ii) with $t=1$ gives a starter with $e\left(L\left(\overline{G}\right)\right)=2$, and the complement of a 3-star (plus 3 isolated vertices) is a 7-starter with $e\left(\overline{G}\right)=3$.
- **$n=9$:** Lemma 3.5(ii) with $t=1$ gives a 9-starter with $e\left(L\left(\overline{G}\right)\right)=4$.
- **$n=11$:** Lemma 3.5(ii) with $t=2$ gives 11-starters G_i with $e\left(L\left(\overline{G_i}\right)\right)=i$ for all $i\in\{3,4,5,6\}$, and together with the 11-starter whose complement is a tree with degrees $4,2,1,1,1,1$ (plus 5 isolated vertices) we conclude $i^*(11)\geqslant 7$.
- **$n=13$:** Lemma 3.5(ii) with $t=3$ gives 13-starters G_i with $e\left(L\left(\overline{G_i}\right)\right)=i$ for $i\in\{6,7,8,9\}$, and with $t=4$ we cover $i\in\{10,11\}$.
- **$n=15$:** Lemma 3.5(ii) with $t=4$ gives 15-starters G_i with $e\left(L\left(\overline{G_i}\right)\right)=i$ for $i\in\{10,11,12,13\}$
- **$n=17$:** Lemma 3.5(ii) with $t=4$ gives 17-starters G_i with $e\left(L\left(\overline{G_i}\right)\right)=i$ for $i\in\{10,\dots,14\}$, and with $t=5$ we cover $i\in\{15,16,17,18\}$.
- **$n\geqslant 19$:** Lemma 3.5(ii) with $t=3$ gives n-starters G_i with $e\left(L\left(\overline{G_i}\right)\right)=i$ for $i\in\{6,\dots,i^*(n-6)+6\}$, and this concludes the proof because $i^*(n-6)\geqslant n-8$ by induction on n. □

As a consequence of Lemma 3.6, to prove Lemma 3.2 it suffices to verify

$$\left[\binom{n}{2}-6,\ \binom{n}{3}-(n-3)\left\lceil\frac{n-2}{2}\right\rceil+i^*(n)\right]\subseteq S(n,2)$$

So far we have established the existence of starters corresponding to maximal antichains $\mathcal{A}\subseteq\binom{[n]}{2}\cup\binom{[n]}{3}$ with all sizes of the form

$$\binom{n}{3}-(n-3)\left\lceil\frac{n-2}{2}\right\rceil+i\qquad\text{for } i=0,1,2,\dots,i^*(n).$$

We can always label the vertices of an n-starter G in such a way that

(i) n is an isolated vertex in $\overline{G}$ (that is, $\{i,n\}\in E(G)$ for all $i\in[n-1]$),
(ii) for $i=1,2,\dots,\left\lceil\frac{n-2}{2}\right\rceil$, the vertex $n-i$ has degree 1 in $\overline{G}-\{n,n-1,\dots,n-i+1\}$.

Let us call an n-starter satisfying (i) and (ii) *properly labeled.* The second condition (ii) says that the edge set of $\overline{G}$ has the form $E\left(\overline{G}\right)=\left\{\{n-l,b_l\}\ :\ l=1,\dots,\left\lceil\frac{n-2}{2}\right\rceil\right\}$ with $b_l<n-l$ for every l. In this setup, we will now delete the edges $\{n,n-1\},\dots\{n,3\}$ from G one by one, and this shifts the sizes of the corresponding maximal antichains, as illustrated in the following example.

Example 3.3. For $n=14$, the starters for $i=1,2,\dots,i^*(14)=11$ correspond to maximal antichains with sizes in the interval $\left[\binom{14}{3}-66,\binom{14}{3}-55\right]$. In Figure 8, it is illustrated how deleting the edges $\{14,13\}$, $\{14,12\}$,…,$\{14,3\}$ shifts the size of the corresponding antichain. Doing this for all starters, we obtain

$$\begin{aligned}S(14,2)\supseteq\binom{14}{3}-\big([55,66]\cup[65,76]\cup[74,85]\cup[82,93]\cup[89,100]\cup[95,106]\cup[100,111]\cup[105,116]\\ \cup[109,120]\cup[112,123]\cup[114,125]\cup[115,126]\big)=\binom{14}{3}-[55,126].\end{aligned}$$

This interval is the first part of the interval in the statement of Lemma 3.2 for $n=14$. The missing sizes between $\binom{14}{2}-5=86$ and $\binom{14}{3}-126=238$ will come from induction on n.

In the next lemma we state precisely how deleting the edges incident with vertex n from a starter affects the size of the corresponding maximal antichain. For a graph G on n vertices, let

$$\varphi(G)=\binom{n}{3}-(n-3)e\left(\overline{G}\right)+e\left(L\left(\overline{G}\right)\right)-t\left(\overline{G}\right).$$

By Lemma 3.4, if G is a T-graph then $\varphi(G)$ is the size of the corresponding maximal antichain $\mathcal{A}$.

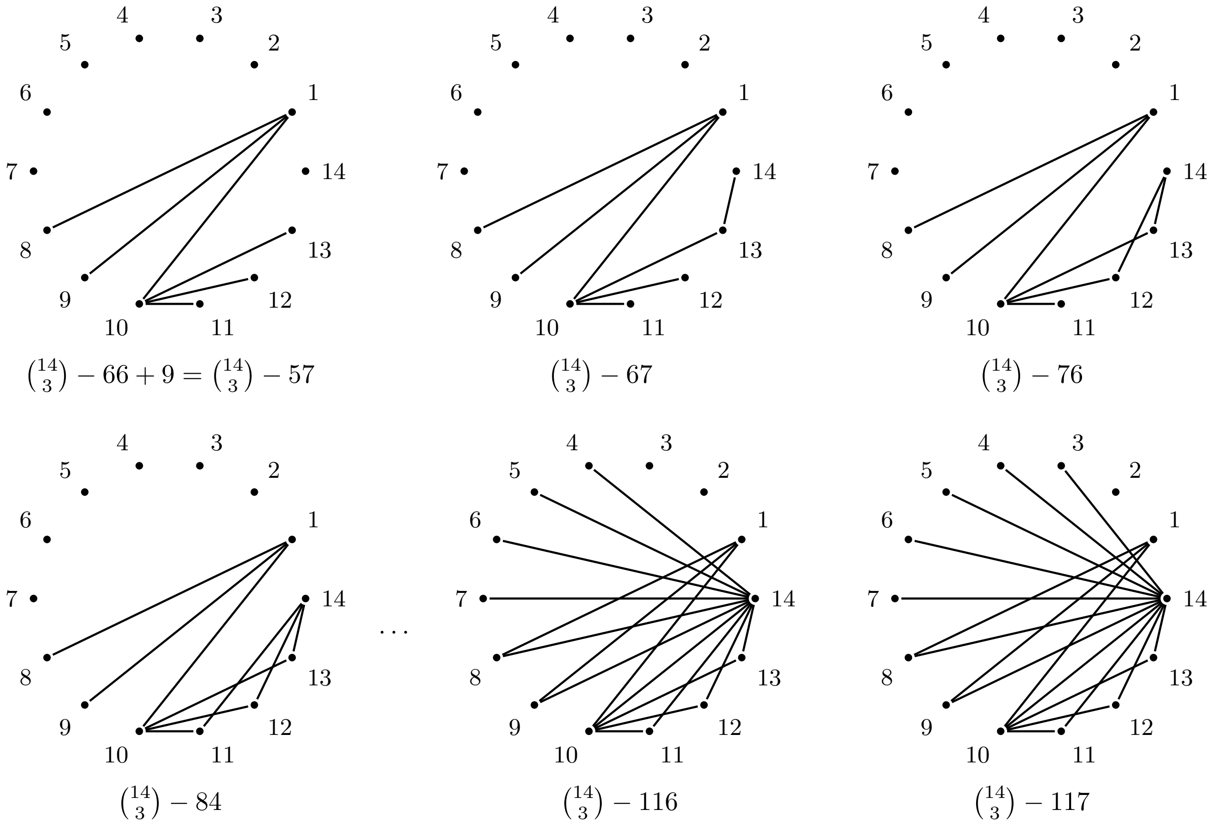


FIGURE 8. Deleting the edges $\{n, n-1\}, \{n, n-2\}, \ldots, \{n, 3\}$ from a starter leads to graphs corresponding to maximal antichains. The number under each graph is the size of the corresponding maximal antichain.

Lemma 3.7. *Let $n \geqslant 6$, and let G_0 be a properly labeled n-starter. Furthermore, for $l = 1, 2, \ldots, n-3$, let G_l be the graph obtained from G_0 by deleting the edges $\{n, n-1\}$, $\{n, n-2\}, \ldots,$ $\{n, n-l\}$. Then, for every $l \in [n-1]$, $\varphi(G_l) = \varphi(G_{l-1}) - \alpha_l$, where*

$$\alpha_l = \begin{cases} n-l-3 & \text{for } l = 1, 2, \ldots, \left\lceil \frac{n-2}{2} \right\rceil, \\ n-l-2 & \text{for } l = \left\lceil \frac{n-2}{2} \right\rceil + 1, \ldots, n-1. \end{cases}$$

Moreover, the graphs $G_1, \ldots, G_{n-3}$ are T-graphs.

Proof. In each step one edge is deleted, hence

$$e\left(\overline{G_l}\right) = e\left(\overline{G_{l-1}}\right) + 1. \tag{4}$$

The new edge $\{n, n-l\}$ becomes a new vertex in the line graph. This new vertex is adjacent to

- all edges $\{n, n-l'\}$, $l' = 1, \ldots, l-1$,
- all $\{n-l, n-j\} \in E\left(\overline{G_0}\right)$ with $j \in \{1, \ldots, l-1\}$, and
- the edge $\{n-l, b_l\}$ if $l \leqslant \lceil (n-2)/2 \rceil$.

The second group of additional edges in the line digraph is compensated by additional triangles in $\overline{G_l}$. More precisely,

$$e\left(L\left(\overline{G_l}\right)\right) = e\left(L\left(\overline{G_{l-1}}\right)\right) + (l-1) + \left|\left\{j\,:\,\{n-l,n-j\}\in E\left(\overline{G_0}\right),\,1\leqslant j\leqslant l-1\right\}\right| + \begin{cases}1 & \text{if } l\leqslant\lceil(n-2)/2\rceil\\ 0 & \text{if } l>\lceil(n-2)/2\rceil\end{cases} \tag{5}$$

$$t\left(\overline{G_l}\right) = t\left(\overline{G_{l-1}}\right) + \left|\left\{j\,:\,\{n-l,n-j\}\in E\left(\overline{G_0}\right),\,1\leqslant j\leqslant l-1\right\}\right|. \tag{6}$$

Combining (4), (5) and (6), we get

$$\varphi\left(G_l\right) = \varphi\left(G_{l-1}\right) - (n-3) + (l-1) + \begin{cases}1 & \text{if } l\leqslant\lceil(n-2)/2\rceil,\\ 0 & \text{if } l>\lceil(n-2)/2\rceil.\end{cases}$$

To see that the graphs G_l, $l=1,\dots,n-3$, are T-graphs we check that every edge $\{p,q\}$ is contained in a triangle. For $p,q<n$, the edges of $\overline{G_0}$ remove at most $\lceil(n-2)/2\rceil$ of the $n-2$ triangles $\{p,q,r\}$, and deleting the edges $\{n,l\}$ can remove only one additional triangle, namely the triangle $\{p,q,n\}$. From $\lceil(n-2)/2\rceil+1<n-2$ it follows that there is always a triangle $\{p,q,r\}$ left. For edges $\{p,n\}$ with $2\leqslant p\leqslant n/2$, there is the triangle $\{1,p,n\}$, for the edge $\{1,n\}$ there is the triangle $\{1,2,n\}$, and for an edge $\{p,n\}$ with $p=n-i$, $i\in\{1,\dots,\lceil(n-2)/2\rceil\}$, we can take any triangle $\{j,p,n\}$ with $j\in\{1,\dots,p-1\}\setminus\{b_i\}$. ☐

In the next lemma, we state the interval of maximal antichain sizes obtained by the above construction.

Lemma 3.8. *For every integer $n\geqslant 6$,*

$$S(n,2)\supseteq\left[\binom{n}{3}-\binom{n-2}{2}-(n-4)\left\lceil\frac{n-2}{2}\right\rceil,\ \binom{n}{3}-(n-3)\left\lceil\frac{n-2}{2}\right\rceil+i^*(n)\right].$$

Proof. For $n=6$, we obtain the sizes in the interval $[10,15]$ from the T-graphs whose complements are shown in Figure 9. For $n\geqslant 7$, let m be any number in the interval on the right hand side of the claimed inclusion,

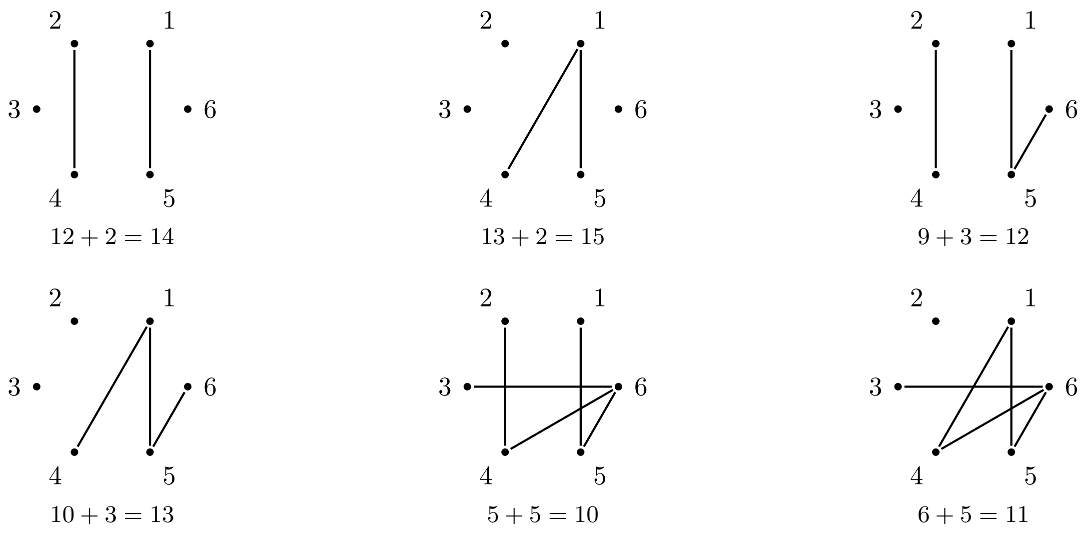


FIGURE 9. Complements of the graphs corresponding to maximal antichains with sizes in $[10,15]$. The line below each graph has the format "#3-sets+#2-sets = size of the antichain".

and let m_0 be the smallest size of an antichain coming from a starter, that is,

$$m_0=\binom{n}{3}-(n-3)\left\lceil\frac{n-2}{2}\right\rceil.$$

By definition of $i^*(n)$, there is nothing to do if $m\in[m_0,m_0+i^*(n)]$. We define the sequence $\alpha_1,\dots,\alpha_{n-1}$ as in the proof of Lemma 3.7. Using Lemma 3.7, we can obtain a maximal antichain of size

$$m_0-(\alpha_1+\alpha_2+\cdots+\alpha_{n-3})=m_0-\binom{n-2}{2}+\left\lceil\frac{n-2}{2}\right\rceil=\binom{n}{3}-\binom{n-2}{2}-(n-4)\left\lceil\frac{n-2}{2}\right\rceil.$$

Let $t\in\{0,1,\dots,n-3\}$ be the smallest index with $m_0-(\alpha_1+\alpha_2+\cdots+\alpha_t)\leqslant m$, and set $i=m-m_0+(\alpha_1+\alpha_2+\cdots+\alpha_t)$. Then $i\leqslant\alpha_1-1=n-5$ with equality only if $t=1$ and $m=m_0-1$. By Lemma 3.6, this leaves two cases.

Case 1: $i\leqslant i^*(n)$. Let G be a starter with $t(G)+e\left(\overline{G}\right)=m_0+i$. By Lemma 3.7, the graph G' with

$$E\left(\overline{G'}\right)=E\left(\overline{G}\right)\cup\{\{n,n-1\},\,\{n,n-2\},\dots,\,\{n,n-t\}\},$$

corresponds to a maximal antichain of size m.

Case 2: $n=10$ and $m=m_0-1=91$. In this case, we conclude by pointing to the T-graph G with

$$E\left(\overline{G}\right)=\{\{10,9\},\,\{1,2\},\,\{1,3\},\,\{1,4\},\,\{1,5\}\},$$

which satisfies $t(G)+e\left(\overline{G}\right)=86+5=91$. □

Finally, we combine Lemma 3.8 with the recursion (1) to prove Lemma 3.2.

Proof of Lemma 3.2. For the base case $n=6$, Lemma 3.8 gives the interval $[10,15]$, and size 9 is obtained by the flat maximal antichain with 3-sets $\{1,2,5\}$, $\{1,2,6\}$, $\{3,4,5\}$, $\{3,4,6\}$. Now assume $n\geqslant 7$ and

$$\left[\binom{n-1}{2}-6,\ \binom{n-1}{3}-(n-4)\left\lceil\frac{n-3}{2}\right\rceil+i^*(n-1)\right]\subseteq S(n-1,2).$$

Then (1) implies

$$\begin{aligned}S(n,2)&\supseteq\left[\binom{n-1}{2}-6,\ \binom{n-1}{3}-(n-4)\left\lceil\frac{n-3}{2}\right\rceil+i^*(n-1)\right]+(n-1)\\&=\left[\binom{n}{2}-6,\ \binom{n-1}{3}-(n-4)\left\lceil\frac{n-3}{2}\right\rceil+i^*(n-1)+(n-1)\right],\end{aligned}$$

while Lemma 3.8 ensures that

$$S(n,2)\supseteq\left[\binom{n}{3}-\binom{n-2}{2}-(n-4)\left\lceil\frac{n-2}{2}\right\rceil,\ \binom{n}{3}-(n-3)\left\lceil\frac{n-2}{2}\right\rceil+i^*(n)\right].$$

To conclude the proof it is sufficient to verify

$$\binom{n-1}{3}-(n-4)\left\lceil\frac{n-3}{2}\right\rceil+i^*(n-1)+(n-1)\geqslant\binom{n}{3}-\binom{n-2}{2}-(n-4)\left\lceil\frac{n-2}{2}\right\rceil-1.$$

This simplifies to

$$i^*(n-1)\geqslant(n-4)\left(\left\lceil\frac{n-3}{2}\right\rceil-\left\lceil\frac{n-2}{2}\right\rceil\right)-2,$$

which is obvious because the right hand side is negative. □

3.3. **Intervals in $S(n,l)$.** In this subsection we prove Proposition 2 by induction on l and n. The structure of the argument can be summarized as follows.

- The base case for the induction on l, that is, the case $l=2$, is Lemma 3.2 and has been proved in Section 3.2.
- For $l\geqslant 3$, the base case for the induction on n is $n=2l+2$. By induction on l, we have intervals $I_1\subseteq S(2l+1,l-1)$ and $I_2\subseteq S(2l,l-1)$. By (2), $I_1+\binom{2l+1}{l+1}\subseteq S(n,l)$, and by (2) and (1), $I_2+\binom{2l}{l+1}+\binom{2l+1}{l-1}\subseteq S(n,l)$. In Lemma 3.9 we verify that the union of these two intervals is the required subset of $S(n,l)$.
- For $l\geqslant 3$ and $n\geqslant 2l+3$, we have three ingredients from the induction, and apply (1), (2) and (3):
  - induction on n gives us an interval $I_1\subseteq S(n-1,l)$ and then $I_1+\binom{n-1}{l-1}\subseteq S(n,l)$ by (1),
  - induction on l gives us an interval $I_2\subseteq S(n-1,l-1)$ and then $I_2+\binom{n-1}{l+1}\subseteq S(n,l)$ by (2),

– induction on l gives us an interval $I_3 \subseteq S(n-2, l-1)$ and then $I_3 + I_3 + \binom{n-2}{l+1} + \binom{n-2}{l-2} \subseteq S(n, l)$ by (3).

We conclude the proof of Proposition 2 by verifying that the union of these three intervals is the required subset of $S(n, l)$.

Recall that C_l denotes the sum of the first l Catalan numbers: $C_l = \sum_{i=1}^{l} \frac{1}{i+1}\binom{2i}{i}$.

Lemma 3.9. *For every $l \geqslant 2$,*

$$S(2l+2, l) \supseteq \left[\binom{2l+2}{l} - 3 - C_l,\ \binom{2l+2}{l+1} - (l+1)\left\lceil\frac{l+2}{2}\right\rceil + f(l+2)\right].$$

Proof. By induction on l (using Lemma 3.2 for the base case $l = 2$),

$$S(2l+1, l-1) \supseteq \left[\binom{2l+1}{l-1} - 3 - C_{l-1},\ \binom{2l+1}{l} - (l+1)\left\lceil\frac{l+2}{2}\right\rceil + f(l+2)\right],$$

and with (2),

$$S(2l+2, l) \supseteq \left[\binom{2l+2}{l} - 3 - C_{l-1},\ \binom{2l+2}{l+1} - (l+1)\left\lceil\frac{l+2}{2}\right\rceil + f(l+2)\right].$$

Similarly,

$$S(2l, l-1) \supseteq \left[\binom{2l}{l-1} - 3 - C_{l-1},\ \binom{2l}{l} - l\left\lceil\frac{l+1}{2}\right\rceil + f(l+1)\right],$$

and then, using (2) and (1),

$$\begin{aligned} S(2l+2, l) \supseteq \left[\binom{2l}{l-1} - 3 - C_{l-1},\ \binom{2l}{l} - l\left\lceil\frac{l+1}{2}\right\rceil + f(l+1)\right] + \binom{2l}{l+1} + \binom{2l+1}{l-1} \\ = \left[\binom{2l+2}{l} - 3 - C_l,\ \binom{2l+2}{l} - l\left\lceil\frac{l+1}{2}\right\rceil + f(l+1)\right]. \end{aligned}$$

Here the left end of the interval comes from

$$\binom{2l}{l-1} + \binom{2l}{l+1} + \binom{2l+1}{l-1} = \binom{2l+2}{l} - \frac{1}{l+1}\binom{2l}{l}.$$

We conclude the proof by observing that $l\left\lceil\frac{l+1}{2}\right\rceil - f(l+1) \leqslant 4 + C_{l-1}$. □

Proof of Proposition 2. For $n = 2l+2$ the statement follows from Lemma 3.9. For $n \geqslant 2l+3$, we have by induction that $I_1 \subseteq S(n-1, l)$, $I_2 \subseteq S(n-1, l-1)$ and $I_3 \subseteq S(n-2, l-1)$, where

$$\begin{aligned} I_1 &= \left[\binom{n-1}{l} - 3 - C_l,\ \binom{n-1}{l+1} - (n-l-2)\left\lceil\frac{n-l-1}{2}\right\rceil + f(n-l-1)\right], \\ I_2 &= \left[\binom{n-1}{l-1} - 3 - C_{l-1},\ \binom{n-1}{l} - (n-l-1)\left\lceil\frac{n-l}{2}\right\rceil + f(n-l)\right], \\ I_3 &= \left[\binom{n-2}{l-1} - 3 - C_{l-1},\ \binom{n-2}{l} - (n-l-2)\left\lceil\frac{n-l-1}{2}\right\rceil + f(n-l-1)\right]. \end{aligned}$$

With (1) and (2) we obtain from $I_1 \subseteq S(n-1, l)$ and $I_2 \subseteq S(n-1, l-1)$ that

$$\begin{aligned} S(n, l) &\supseteq \left[\binom{n}{l} - 3 - C_l,\ \binom{n-1}{l+1} + \binom{n-1}{l-1} - (n-l-2)\left\lceil\frac{n-l-1}{2}\right\rceil + f(n-l-1)\right], \\ S(n, l) &\supseteq \left[\binom{n-1}{l+1} + \binom{n-1}{l-1} - 3 - C_{l-1},\ \binom{n}{l+1} - (n-l-1)\left\lceil\frac{n-l}{2}\right\rceil + f(n-l)\right]. \end{aligned}$$

From (3) and $I_3 \subseteq S(n-2, l-1)$, we have

$$\begin{aligned}
S(n,l) \supseteq \Bigg[&\binom{n-2}{l-1} - 3 - C_{l-1},\ \binom{n-2}{l} - (n-l-2)\left\lceil\frac{n-l-1}{2}\right\rceil + f(n-l-1)\Bigg] \\
&+\binom{n-2}{l} - (n-l-2)\left\lceil\frac{n-l-1}{2}\right\rceil + f(n-l-1) + \binom{n-2}{l+1} + \binom{n-2}{l-2} \\
&\supseteq \Bigg[\binom{n-1}{l+1} + \binom{n-1}{l-1} - (n-l-2)\left\lceil\frac{n-l-1}{2}\right\rceil + f(n-l-1), \\
&\qquad\qquad \binom{n-1}{l+1} + \binom{n-2}{l} + \binom{n-2}{l-2} - (n-l-2)(n-l) + 2f(n-l-1)\Bigg],
\end{aligned}$$

so the result follows once we verify that

$$\binom{n-1}{l+1} + \binom{n-2}{l} + \binom{n-2}{l-2} - (n-l-2)(n-l) + 2f(n-l-1) \geqslant \binom{n-1}{l+1} + \binom{n-1}{l-1} - 4 - C_{l-1}.$$

This inequality simplifies to

$$\binom{n-2}{l} - \binom{n-2}{l-1} - (n-l-2)(n-l) + 2f(n-l-1) + 4 + C_{l-1} \geqslant 0.$$

For $(l,n) = (3,9)$ and $(l,n) = (3,10)$, the left-hand side is equal to 3 and 6, respectively, and for all other pairs (l,n) with $l \geqslant 3$ and $n \geqslant 2l+3$, the inequality is a consequence of the following lemma. □

Lemma 3.10. *For integers l and n with $l = 3$ and $n \geqslant 11$, or $l \geqslant 4$ and $n \geqslant 2l+3$,*

$$\binom{n-2}{l} - \binom{n-2}{l-1} \geqslant (n-l-2)(n-l).$$

Proof. The left-hand side is

$$\binom{n-2}{l} - \binom{n-2}{l-1} = \binom{n-2}{l} - \frac{l}{n-l-1}\binom{n-2}{l} = \frac{n-2l-1}{n-l-1}\binom{n-2}{l},$$

and after cancelling common factors the claimed inequality becomes

$$\frac{n-2l-1}{n-l-2}\prod_{k=n-l+1}^{n-2} k \geqslant l!.$$

For fixed l, the left-hand side is increasing in n, so it is sufficient to consider the smallest relevant n, that is, $n = 11$ if $l = 3$ and $n = 2l+3$ if $l \geqslant 4$. For $l = 3$, $\frac{4}{6} \times 9 = 6 = 3!$. For $l \geqslant 4$ the claim is $\frac{2}{l+1}\prod_{k=l+4}^{2l+1} k \geqslant l!$, and we proceed by induction on l. The base case $l = 4$: $\frac{2}{5} \times 8 \times 9 = \frac{144}{5} > 4!$, and for $l \geqslant 5$, the induction step is

$$\frac{2}{l+1}\prod_{k=l+4}^{2l+1} k = \frac{l(2l)(2l+1)}{(l+1)(l+3)}\left(\frac{2}{l}\prod_{k=l+3}^{2l-1} k\right) \geqslant \frac{l(2l)(2l+1)}{(l+1)(l+3)}(l-1)! > l!. \qquad \square$$

4. Proofs of Proposition 1 and Theorem 2

4.1. **Proof of Proposition 1.** In this subsection we construct small maximal antichains in $\binom{[n]}{l} \cup \binom{[n]}{l+1}$ in order to extend the interval of sizes given by Proposition 2 to the left. This generalizes the construction for the minimum of $S(n,2)$ from [7]. The corresponding maximal antichain of minimum size $\binom{n}{2} - \left\lfloor\frac{(n+1)^2}{8}\right\rfloor$ is obtained by partitioning the ground set into two parts A and B of almost equal size with $|A|$ even, say $|A| = \left\lfloor\frac{n}{2}\right\rfloor$ if $n \equiv 0$ or $1 \pmod 4$ and $|A| = \left\lfloor\frac{n+2}{2}\right\rfloor$ if $n \equiv 2$ or $3 \pmod 4$. With M being a perfect matching on A the minimum size of a maximal antichain in $\binom{[n]}{2} \cup \binom{[n]}{3}$ is obtained by taking the 3-sets xyz with $xy \in M$ and $z \in B$. We observe that, essentially by dropping the 3-sets one by one, we obtain flat maximal antichains of all sizes between the minimum $\binom{n}{2} - \left\lfloor\frac{(n+1)^2}{8}\right\rfloor$ and $\binom{n}{2} - 2$ (coming from the flat maximal antichain which consists of the 3-set $\{1,2,3\}$ and all 2-sets except the subsets of $\{1,2,3\}$). Indeed, fixing an edge e in M, the size of the corresponding maximal antichain increases by 1 as we drop one 3-set containing e, as long as this

3-set is not the last one containing e. Dropping this last 3-set D increases the size by 2. If e is not the last remaining element of M, the missing size can be obtained by leaving D in the family and dropping a 3-set containing another edge $e' \in M$.

For arbitrary l, we generalize this construction as follows. Fix an integer t and a family $\mathcal{F} \subseteq \binom{[t]}{l}$ of pairwise shadow-disjoint l-sets, that is, such that $|A \cap B| \leqslant l-2$ for any distinct members $A, B \in \mathcal{F}$. Additionally, for each member $A \in \mathcal{F}$ we specify a non-empty set $X(A) \subseteq \{t+1, \dots, n\}$, and we set

$$\mathcal{A}_{l+1} = \{A \cup \{i\} \,:\, A \in \mathcal{F},\, i \in X(A)\}, \qquad \mathcal{A}_l = \binom{[n]}{l} \setminus \Delta\mathcal{A}_{l+1}. \tag{7}$$

Then $\mathcal{A} = \mathcal{A}_{l+1} \cup \mathcal{A}_l$ is an antichain of size

$$|\mathcal{A}| = \sum_{A \in \mathcal{F}} |X(A)| + \binom{n}{l} - |\mathcal{F}| - \sum_{A \in \mathcal{F}} l\,|X(A)| = \binom{n}{l} - |\mathcal{F}| - (l-1) \sum_{A \in \mathcal{F}} |X(A)|.$$

Moreover, $\mathcal{A}$ is maximal. Clearly, one cannot add an l-set to $\mathcal{A}$ without destroying the antichain property. Assume there was an $S \in \binom{[n]}{l+1} \setminus \mathcal{A}$ with $\Delta S \subseteq \Delta\mathcal{A}_{l+1}$. By

$$\Delta\mathcal{A}_{l+1} = \mathcal{F} \cup \{A' \cup \{i\} : A' \in \Delta A,\ A \in \mathcal{F},\ i \in X(A)\},$$

we have $|S \cap \{t+1, \dots, n\}| \leq 1$. If S and $\{t+1, \dots, n\}$ are disjoint, then $\Delta S \subseteq \mathcal{F}$, a contradiction to the choice of $\mathcal{F}$. Consequently, $S = T \cup \{x\}$ with $T \subseteq [t]$ and $x \in [n] \setminus [t]$. By $T \in \Delta S$ and $S \notin \mathcal{A}_{l+1}$, we obtain $T \in \mathcal{F}$ and $x \notin X(T)$. Let $y \in T$. As $T \setminus \{y\} \cup \{x\} \in \Delta S$, it follows that $T \setminus \{y\} \in \Delta A$ for some $A \in \mathcal{F}$ with $x \in X(A)$. Now T and A are distinct elements of $\mathcal{F}$ which are not shadow-disjoint. Again, this contradicts the choice of $\mathcal{F}$.

Example 4.1. For $n = 9$, $l = 3$ and $t = 6$, we can take for $\mathcal{F}$ any subfamily of $\{123, 145, 246, 356\}$. For $\mathcal{F} = \{123\}$ we obtain the maximal antichains with collections of 4-sets

- $\mathcal{A}_4 = \{1237\}$ (size $\binom{9}{3} - 3 = 81$)
- $\mathcal{A}_4 = \{1237, 1238\}$ (size $\binom{9}{3} - 5 = 79$)
- $\mathcal{A}_4 = \{1237, 1238, 1239\}$ (size $\binom{9}{3} - 7 = 77$)

For $\mathcal{F} = \{123, 145\}$ we obtain the maximal antichains with collections of 4-sets

- $\mathcal{A}_4 = \{1237, 1457\}$ (size $\binom{9}{3} - 6 = 78$)
- $\mathcal{A}_4 = \{1237, 1457, 1238\}$ (size $\binom{9}{3} - 8 = 76$)
- $\mathcal{A}_4 = \{1237, 1457, 1238, 1458\}$ (size $\binom{9}{3} - 10 = 74$)
- $\mathcal{A}_4 = \{1237, 1457, 1238, 1458, 1239\}$ (size $\binom{9}{3} - 12 = 72$)
- $\mathcal{A}_4 = \{1237, 1457, 1238, 1458, 1239, 1459\}$ (size $\binom{9}{3} - 14 = 70$)

For $\mathcal{F} = \{123, 145, 246\}$ we obtain the maximal antichains with collections of 4-sets

- $\mathcal{A}_4 = \{1237, 1457, 2467\}$ (size $\binom{9}{3} - 9 = 75$)
- $\mathcal{A}_4 = \{1237, 1457, 2467, 1238\}$ (size $\binom{9}{3} - 11 = 73$)
- ...
- $\mathcal{A}_4 = \{1237, 1457, 2467, 1238, 1458, 2468, 1239, 1459, 2469\}$ (size $\binom{9}{3} - 21 = 63$)

For $\mathcal{F} = \{123, 145, 246, 357\}$ we obtain the maximal antichains with collections of 4-sets

- $\mathcal{A}_4 = \{1237, 1457, 2467, 3567\}$ (size $\binom{9}{3} - 12 = 72$)
- $\mathcal{A}_4 = \{1237, 1457, 2467, 3567, 1238\}$ (size $\binom{9}{3} - 14 = 70$)
- ...
- $\mathcal{A}_4 = \{1237, 1457, 2467, 3567, 1238, 1458, 2468, 3568, 1239, 1459, 2469, 3569\}$ (size $\binom{9}{3} - 28 = 56$)

In particular, we obtain flat maximal antichains of all sizes in the interval $[63, 79] = \left[\binom{9}{3} - 21, \binom{9}{3} - 5\right]$ (see Lemma 4.2 below for the general version).

To give a precise statement about the sizes of flat maximal antichains that can be obtained by the above construction we introduce the following notation. For integers $l < t$, let $\varphi(t, l)$ be the maximum size of a pairwise shadow-disjoint family $\mathcal{F} \subseteq \binom{[t]}{l}$. In particular $\varphi(t, 2) = \lfloor t/2 \rfloor$, and in general $\varphi(t, l) \geqslant \left\lceil \frac{1}{t}\binom{t}{l} \right\rceil$ by [4, Theorem 1]. The following lemma is then an immediate consequence of the above construction, and will be sufficient for constructing flat maximal antichains of the required sizes in Lemma 4.2.

Lemma 4.1. *If n, t, l, s and α are non-negative integers with $2 \leqslant l < t < n$, $s \leqslant \varphi(t,l)$ and $s \leqslant \alpha \leqslant (n-t)s$, then $\binom{n}{l} - s - \alpha(l-1) \in S(n,l)$.*

We use Lemma 4.1 to find an interval in $S(n,l)$ as follows. For fixed $|\mathcal{F}|$, varying $\sum_{A\in\mathcal{F}} |X(A)|$ yields maximal flat antichains whose sizes form an arithmetic progression with common difference $l-1$ (see Example 4.1 for an illustration). The union of these progressions for varying $|\mathcal{F}|$ contains an interval whose boundaries are stated in the following lemma. These sizes are what will needed for the final step in the proof of Theorem 2 which is provided in Lemma 4.4.

Lemma 4.2. *For $l \geqslant 2$ and $l+3 \leqslant t \leqslant n-l$,*
$$\left[\binom{n}{l} - (1+(l-1)(n-t))\left(\varphi(t,l)-l+2\right),\ \binom{n}{l} - (l-1)^2 - 1\right] \subseteq S(n,l).$$

Proof. Let $m = \binom{n}{l} - x$ be any integer in the interval on the left-hand side of the claimed inclusion, that is, $(l-1)^2+1 \leqslant x \leqslant (1+(l-1)(n-t))(\varphi(t,l)-l+2)$. Let s be the largest integer satisfying $s \equiv x \pmod{l-1}$ and $s \leqslant \min\{\lfloor x/l\rfloor, \varphi(t,l)\}$, and define $\alpha = (x-s)/(l-1)$. It follows from $t \geqslant l+3$ and $\varphi(t,l) \geqslant \frac{1}{t}\binom{t}{l}$ that $\varphi(t,l) \geqslant l-2$, and therefore, $s \geqslant 0$. By Lemma 4.1 (and noting that $s \leqslant \alpha$ follows from $s \leqslant \frac{x}{l}$), it is sufficient to verify $\alpha \leqslant (n-t)s$, or equivalently,
$$x \leqslant (1+(l-1)(n-t))\, s. \tag{8}$$
By construction, $s = \min\{\lfloor x/l\rfloor, \varphi(t,l)\} - j$ for some $j \in \{0,1,\dots,l-2\}$. If $s = \varphi(t,l)-j$ then (8) is immediate from the assumed upper bound on x. For $s = \lfloor x/l\rfloor - j$ we will verify $x \leqslant (1+l(l-1))s$ which implies (8) because $n-t \geqslant l$ by assumption. We write $x = q_1 l(l-1) + q_2(l-1) + r$ with integers $q_1 \geqslant 0$, $0 \leqslant q_2 \leqslant l-1$ and $0 \leqslant r \leqslant l-2$. Then s is the unique integer with $s \equiv r \pmod{l-1}$ and $sl \leqslant x < (s+l-1)l$, hence
$$s = \begin{cases} q_1(l-1)+r & \text{if } r \leqslant q_2,\\ (q_1-1)(l-1)+r & \text{if } r \geqslant q_2+1.\end{cases}$$
If $r \leqslant q_2$ then
$$\begin{aligned}(1+l(l-1))s &= q_1(l-1) + r + q_1 l(l-1)^2 + rl(l-1)\\ &= q_1 l(l-1) + \left[q_1 l(l-1)(l-2) + q_1 + rl\right](l-1) + r \geqslant q_1 l(l-1) + l(l-1) + r > x,\end{aligned}$$
where the term in square brackets is at least l because $(q_1,r) \neq (0,0)$ which follows from $x > (l-1)^2$. If $r \geqslant q_2+1$, then $q_2 \leqslant l-3$ and $q_1 \geqslant 1$. The value of $(1+l(l-1))s$ is $(1+l(l-1))(l-1)$ less than the one above, that is,
$$\begin{aligned}(1+l(l-1))s &= q_1 l(l-1) + \left[q_1 l(l-1)(l-2) + q_1 + rl - 1 - l(l-1)\right](l-1) + r\\ & q_1 l(l-1) + rl(l-1) + r \geqslant q_1 l(l-1) + l(l-1) + r > x,\end{aligned}$$
which concludes the proof. $\square$

We have now all the ingredients to prove Proposition 1. For $l=2$ we apply Lemma 4.2 with
$$t = \begin{cases} \lfloor n/2\rfloor & \text{if } n \equiv 0 \text{ or } 1 \pmod 4,\\ \lfloor (n+2)/2\rfloor & \text{if } n \equiv 2 \text{ or } 3 \pmod 4.\end{cases}$$
Then $\varphi(t,2) = t/2$ and with $(n-t+1)\frac{t}{2} = \left\lfloor \frac{(n+1)^2}{8}\right\rfloor$, we obtain
$$\left[\binom{n}{2} - \left\lfloor\frac{(n+1)^2}{8}\right\rfloor,\ \binom{n}{2} - 2\right] \subseteq S(n,2),$$
and the result follows with Proposition 2. For $3 \leqslant l \leqslant (n-2)/2$ we distinguish three cases.

Case 1: $l = k-1$. From $\left\lceil \frac{1}{k}\binom{k}{l}\right\rceil - l + 2 = 3 - l \leqslant 0$ it follows that $\gamma = 3 + \sum_{i=1}^{l} \frac{1}{i+1}\binom{2i}{i}$, and we are done by Proposition 2.

Case 2: $l=k-2$. If $l\geqslant 5$, then $\left\lceil\frac{1}{k}\binom{k}{l}\right\rceil-l+2\leqslant 0$, hence $\gamma=3+\sum_{i=1}^{l}\frac{1}{i+1}\binom{2i}{i}$, and the same can be checked directly for $l\in\{3,4\}$. Hence, Proposition 2 does the job.

Case 3: $l\leqslant k-3$. We apply Lemma 4.2 with $t=k$, and use $\varphi(k,l)\geqslant\left\lceil\frac{1}{k}\binom{k}{l}\right\rceil$, to deduce

$$\left[\binom{n}{l}-(1+(l-1)(n-k))\left(\left\lceil\frac{1}{k}\binom{k}{l}\right\rceil-l+2\right),\ \binom{n}{l}-(l-1)^2-1\right]\subseteq S(n,l).$$

The result follows with Proposition 2 and $(l-1)^2+1\leqslant 4+\sum_{i=1}^{l}\frac{1}{i+1}\binom{2i}{i}$.

4.2. **Proof of Theorem 2.** Let $I_{n,l}$ be the interval which, according to Proposition 1, is contained in $S(n,l)$ $(2\leqslant l\leqslant\lfloor\frac{n-2}{2}\rfloor)$. Observing that the left end of $I_{n,2}$ is $\binom{n}{2}-\left\lfloor\frac{(n+1)^2}{8}\right\rfloor$, and the right end of $I_{n,\lfloor(n-2)/2\rfloor}$ is $\binom{n}{k}-k\left\lceil\frac{k+1}{2}\right\rceil$, we would be done if we could show that, for $3\leqslant l\leqslant\lfloor\frac{n-2}{2}\rfloor$ the left end of $I_{n,l}$ is at most one more than the right end of $I_{n,l-1}$. If $l\geqslant 5$ or $n\geqslant 15$, this is indeed true (as shown in the next lemma). For $l\in\{3,4\}$ and $n\leqslant 14$, we will then check that the small gaps between the intervals $I_{n,l}$ can be closed using Proposition 2 and Lemma 4.1.

Lemma 4.3. *Let l and n be integers with $l\geqslant 3$ and $n\geqslant 2l+3$. Assume further that $l\geqslant 5$ or $n\geqslant 15$. Then*

$$(n-l)\left\lceil\frac{n-l+1}{2}\right\rceil-1\leqslant\max\left\{3+\sum_{i=1}^{l}\frac{1}{i+1}\binom{2i}{i},\ (1+(l-1)(n-k))\left(\left\lceil\frac{1}{k}\binom{k}{l}\right\rceil-l+2\right)\right\}.$$

Proof. For $n\leqslant 2l+6$, we can assume $l\geqslant 5$. Bounding the left-hand side from above and the right-hand side from below, it is sufficient to verify

$$(l+6)\left\lceil\frac{l+7}{2}\right\rceil\leqslant 3+\sum_{i=1}^{l}\frac{1}{i+1}\binom{2i}{i}.$$

For $l=5$, this is $66\leqslant 67$, and for $l\geqslant 6$ we proceed by induction on l:

$$(l+6)\left\lceil\frac{l+7}{2}\right\rceil\leqslant(l+5)\left\lceil\frac{l+6}{2}\right\rceil+\frac{3}{2}l+9\leqslant 3+\sum_{i=1}^{l-1}\frac{1}{i+1}\binom{2i}{i}+\frac{3}{2}l+9\leqslant 3+\sum_{i=1}^{l}\frac{1}{i+1}\binom{2i}{i}.$$

For $n\geqslant 2l+7$, we complete the proof by showing

$$(n-l)\left\lceil\frac{n-l+1}{2}\right\rceil-1\leqslant(1+(l-1)(n-k))\left(\left\lceil\frac{1}{k}\binom{k}{l}\right\rceil-l+2\right).$$

For $n\leqslant 26$, we just check all pairs (l,n) by computer. For $n\geqslant 27$, the following crude bounds are good enough. For the left-hand side,

$$(n-l)\left\lceil\frac{n-l+1}{2}\right\rceil-1\leqslant(2k-3)(k-1)-1=2k^2-5k+2,$$

and for the right-hand side,

$$(1+(l-1)(n-k))\left(\left\lceil\frac{1}{k}\binom{k}{l}\right\rceil-l+2\right)\geqslant(2k-1)\left(\frac{1}{k}\binom{k}{3}-k\right)=\frac{2k^3-19k^2+13k-2}{6}.$$

The claim follows from $2k^3-31k^2+43k-14\geqslant 0$, and this inequality is a consequence of $2k^3-31k^2+43k-14=(2k-1)(k-1)(k-14)$ and $k\geqslant 14$. □

Finally, we deal with the cases $l\in\{3,4\}$ and $n\leqslant 14$, and prove that the gap between the intervals $I_{n,l-1}\subseteq S(n,l-1)$ and $I_{n,l}\subseteq S(n,l)$ that are obtained from Proposition 1 is contained in $S(n,l-1)\cup S(n,l)$.

Lemma 4.4. *Let $l\in\{3,4\}$ and $2l+2\leqslant n\leqslant 14$. The integers between the interval in $S(n,l-1)$ and the interval in $S(n,l)$ from Proposition 1 are contained in $S(n,l-1)\cup S(n,l)$.*

Proof. The intervals from Proposition 2, shown in Table 1, are a bit longer, and this is already enough to close the gap in most cases. The only remaining gaps occur between the interval in $S(n,l-1)$ and the interval in $S(n,l)$ for $(l,n)\in\{(3,9),(3,10),(4,12)\}$, and those are closed using Lemma 4.2.

TABLE 1. Intervals from Proposition 2

n	interval in $S(n,2)$	interval in $S(n,3)$	interval in $S(n,4)$
8	$[18,44]$	$[45,61]$	–
9	$[24,64]$	$[73,114]$	–
10	$[30,96]$	$[109,190]$	$[185,240]$
11	$[37,132]$	$[132,306]$	$[305,442]$
12	$[45,182]$	$[181,462]$	$[470,768]$
13	$[54,236]$	$[234,677]$	$[658,1254]$
14	$[63,309]$	$[304,951]$	$[935,1964]$

- For $(l,n)=(3,9)$, Lemma 4.2 with $t=6$ implies $[63,79]\subseteq S(9,3)$.
- For $(l,n)=(3,10)$, Lemma 4.2 with $t=6$ implies $[93,115]\subseteq S(10,3)$.
- For $(l,n)=(4,12)$, Lemma 4.2 with $t=7$ implies $[447,485]\subseteq S(12,4)$. □

Combining Proposition 1 with Lemmas 4.3 and 4.4, we obtain that

$$\left[\binom{n}{2}-\left\lfloor\frac{(n+1)^2}{8}\right\rfloor,\ \binom{n}{k}-k\left\lceil\frac{k+1}{2}\right\rceil\right]\subseteq\bigcup_{l=2}^{\lfloor(n-2)/2\rfloor}S(n,l),$$

and this concludes the proof of Theorem 2.

5. Open problems

In this paper we proved that almost all sizes of maximal antichains are attained by flat antichains (the exception being sizes below $\binom{n}{2}-\left\lfloor\frac{(n+1)^2}{2}\right\rfloor$). We obtained this by establishing the existence of sufficiently long intervals in the flat maximal antichain size spectra

$$S(n,l)=\left\{|\mathcal{A}|\ :\ \mathcal{A}\subseteq\binom{[n]}{l}\cup\binom{[n]}{l+1}\text{ is a maximal antichain in }B_n\right\}$$

for $2\leqslant l\leqslant\lceil n/2\rceil$. We think it would be interesting to determine the sets $S(n,l)$ exactly, and construct the corresponding flat maximal antichains.

Problem 1. Determine the sets $S(n,l)$ and describe the corresponding constructions for flat maximal antichains.

We already know that $S(n,l)$ contains the long interval from Proposition 1, and it remains to be determined what happens above $\binom{n}{l+1}-(n-l-1)\left\lceil\frac{n-l}{2}\right\rceil$ and for $l\geqslant 3$ also what happens below $\binom{n}{l}-\gamma$. For $l\geqslant 4$, it follows from [6, Theorem 2] that for m with $\binom{n}{l+1}-(n-l-1)\left\lceil\frac{n-l}{2}\right\rceil<m<\binom{n}{l+1}$, there is a maximal antichain $\mathcal{A}\subseteq\binom{[n]}{l}\cup\binom{[n]}{l+1}$ of size $|\mathcal{A}|=m$ if and only if

$$m=\binom{n}{l+1}-t(n-l-1)+\binom{a}{2}+\binom{b}{2}+c \tag{9}$$

for some $t\in\{1,2,\dots,n-l+1\}$, $a\geqslant b\geqslant c\geqslant 0$, $1\leqslant a+b\leqslant t$. For $l\in\{2,3\}$ this condition on m is still necessary, but its sufficiency does not follow immediately from the results in [6].

Problem 2. Assume that $l\in\{2,3\}$ and $\binom{n}{l+1}-(n-l-1)\left\lceil\frac{n-l}{2}\right\rceil<m<\binom{n}{l+1}$, where m has the form (9). Prove that $m\in S(n,l)$.

For $l=2$, this would provide a complete description of the set $S(n,2)$: it's the interval from $\min S(n,2)=\binom{n}{2}-\lfloor\frac{(n+1)^2}{8}\rfloor$ to $\binom{n}{3}$ with some gaps above $\binom{n}{3}-(n-3)\lceil\frac{n-2}{2}\rceil$. It seems plausible that $S(n,l)$ has this form also for $l\geqslant 3$.

Problem 3. For $l\geqslant 3$, prove that every integer m between $\min S(n,l)$ and $\binom{n}{l+1}-(n-l-1)\lceil\frac{n-l}{2}\rceil$ is an element of $S(n,l)$.

For a complete description of $S(n,l)$ it would then be sufficient to solve the following problem.

Problem 4. Determine $\min S(n,l)$, at least asymptotically for $n \to \infty$ and fixed $l \geqslant 3$.

This is also interesting with a view towards the recent work on the domination number of the graph induced by two consecutive levels of the n-cube [1, 2], as $\min S(n,l)$ is the independent domination number of this graph. Problem 4 appears to be challenging. For $l \geqslant 3$, the best bounds we are aware of are from [3]:

$$\left(1 - \frac{l-1}{l}t_l - o(1)\right)\binom{n}{l} \leqslant \min S(n,l) \leqslant \left(1 - \frac{1}{2}\left(\frac{l-1}{l}\right)^{l-1} + o(1)\right)\binom{n}{l}, \tag{10}$$

where the *Turán number* t_l is the limit, for $n \to \infty$, of the maximal density of an l-uniform hypergraph on n vertices which does not contain a complete l-uniform hypergraph on $l+1$ vertices. As a more modest aim, it would be nice to solve the following problem, the second part of which has been stated already in [3].

Problem 5. (a) Find constructions for maximal flat antichains in $\binom{[n]}{l} \cup \binom{[n]}{l+1}$ which are smaller than the upper bound from [3].
(b) Find lower bounds for $\min S(n,l)$ which do not depend on Turán numbers.

The lower bound for $\min S(n,l)$ in (10) is still valid for the domination number without the requirement that the dominating set is independent [2]. For $l=3$, the authors of [2] provide a construction of a dominating set whose size is asymptotically equal to the lower bound (assuming the conjectured value of $t_3 = \frac{5}{9}$), and they conjecture that the lower bound is asymptotically sharp in general. Hence part (b) of Problem 5 is a step towards separating the domination number from the independent domination number for $l \geqslant 3$.

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