A continuous analogue of Erdős' k-Sperner theorem

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Abstract

A chain in the unit n-cube is a set $C \subset [0,1]^n$ such that for every $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$ in C we either have $x_i \leq y_i$ for all $i \in [n]$, or $x_i \geq y_i$ for all $i \in [n]$. We show that the 1-dimensional Hausdorff measure of a chain in the unit n-cube is at most n, and that the bound is sharp. Given this result, we consider the problem of maximising the n-dimensional Lebesgue measure of a measurable set $A \subset [0,1]^n$ subject to the constraint that it satisfies $\mathcal{H}^1(A \cap C) \leq \kappa$ for all chains $C \subset [0,1]^n$, where κ is a fixed real number from the interval (0,n]. We show that the measure of A is not larger than the measure of the following optimal set:

$$A_{\kappa}^* = \left\{ (x_1, \dots, x_n) \in [0, 1]^n : \frac{n - \kappa}{2} \le \sum_{i=1}^n x_i \le \frac{n + \kappa}{2} \right\}.$$

Our result may be seen as a continuous counterpart to a theorem of Erdős, regarding k-Sperner families of finite sets.

Keywords: chains; *k*-Sperner families; Hausdorff measure; Lebesgue measure *MSC* (2010): 05D05; 28A78; 05C35

1 Prologue, related work and main results

Let [n] denote the set of positive integers $\{1,\ldots,n\}$, and $2^{[n]}$ denote the power-set of [n]. A family $\mathcal{C} \subset 2^{[n]}$ is called a *chain* if for every distinct $C_1, C_2 \in \mathcal{C}$ we either have $C_1 \subset C_2$ or $C_2 \subset C_1$. We assume that the chains under consideration do not contain the empty set. Here and later, the cardinality of a finite set F is denoted |F|. Let $k \in [n]$

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be a positive integer. A family $\mathcal{F} \subset 2^{[n]}$ is called k-Sperner if there is no chain $\mathcal{C} \subset \mathcal{F}$ such that $|\mathcal{C}| = k+1$. In other words, a k-Sperner family is a collection $\mathcal{F} \subset 2^{[n]}$ such that $|\mathcal{F} \cap \mathcal{C}| \leq k$, for all chains $\mathcal{C} \subset 2^{[n]}$. Given two points $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in \mathbb{R}^n , we write $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$, for all $i \in [n]$.

Let us begin with a well-known result of Erdős, that provides a sharp upper bound on the size of k-Sperner families.

Theorem 1.1 (Erdős [7]). Let \mathcal{F} be a k-Sperner family of $2^{[n]}$. Then the cardinality of \mathcal{F} is not greater than the sum of the k largest binomial coefficients.

For k=1, Theorem 1.1 is due to Sperner (see [16]). The notion of k-Sperner families is fundamental in extremal set theory and has inspired a vast amount of research. We refer the reader to [1, 5] for legible textbooks on the topic. In this article we shall be interested in a continuous analogue of Erdős' result. It has been almost half a century (see [2, 12, 13, 15]) since the idea was conceived that several results from extremal combinatorics have continuous counterparts. This idea has inspired several continuous analogues of results from extremal combinatorics both in a "measure-theoretic setting" (see, for example, [2, 3, 4, 6, 12, 13]) and in a "vector space setting" (see, for example, [9, 14]). In this article we investigate a continuous analogue of Theorem 1.1. Let us proceed by stating a result due to Konrad Engel [4] that is similar to our main result. Here and later, $\mathcal{L}^n(\cdot)$ denotes n-dimensional Lebesgue measure.

Theorem 1.2 (Engel [4]). Let $\kappa > 0$ be a real number and let A be a Lebesgue measurable subset of $[0,1]^n$ that does not contain two elements $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$ such that $\mathbf{x} \leq \mathbf{y}$ and $\sum_{i=1}^n (y_i - x_i) \geq \kappa$. Then the n-dimensional Lebesgue measure of A is not greater than the measure of the following optimal set:

$$A_{\kappa} := \left\{ (x_1, \dots, x_n) \in [0, 1]^n : \frac{n - \kappa}{2} \le \sum_{i=1}^n x_i < \frac{n + \kappa}{2} \right\}.$$

Moreover, if we set $v_n(\kappa) := 1 - \frac{2}{n!} \sum_{j=0}^{\lfloor \frac{n-\kappa}{2} \rfloor} (-1)^j \binom{n}{j} \left(\frac{n-\kappa}{2} - j \right)^n$, where $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x, then we have $\mathcal{L}^n(A_\kappa) = v_n(\kappa)$.

Notice that the measure of set A_{κ} , in Theorem 1.2, depends continuously on κ and therefore $v_n(\kappa)$ is a continuous function of κ .

Before stating our main results, let us proceed with some remarks. Notice that one can associate a binary vector $\mathbf{x} \in \{0,1\}^n$ to each subset F of [n]: simply put 1 in the i-th coordinate if $i \in F$, and 0 otherwise. Notice that this correspondence is bijective and one may choose to not distinguish between subsets of [n] and binary vectors of length n. Hence, another way to think of chains in $2^{[n]}$ is to consider subsets $C \subset \{0,1\}^n$ such that for every distinct $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in C we either have $\mathbf{x} \leq \mathbf{y}$, or $\mathbf{y} \leq \mathbf{x}$. Clearly, the maximum size of a chain, which does not contain the empty set, is at most n. Given the aforementioned observations, Theorem 1.1 can be equivalently expressed as follows.

Theorem 1.3 (Theorem 1.1 restated). Fix a positive integer $k \in [n]$. Let $A \subset \{0,1\}^n$ be such that

$$|A \cap C| \le k$$
, for all chains $C \subset \{0,1\}^n$.

Then
$$|A| \leq \sum_{i=1}^k {n \choose \lfloor \frac{n-k}{2} \rfloor + i}$$
.

It seems natural to ask what happens if one replaces the binary n-cube $\{0,1\}^n$ with the unit n-cube $[0,1]^n$ in Theorem 1.3. Bearing this in mind, we proceed with the following.

Definition 1 (Chains). A chain is a set $C \subset \mathbb{R}^n$ such that for every distinct $\mathbf{x}, \mathbf{y} \in C$ we either have $\mathbf{x} \leq \mathbf{y}$, or $\mathbf{y} \leq \mathbf{x}$.

An example of a chain in the unit n-cube is the set

$$C = \{(f_1(x), \dots, f_n(x)) : x \in [0, 1]\},\$$

where, for $i \in [n]$, $f_i : [0,1] \rightarrow [0,1]$ is a non-decreasing function.

What is the maximum "size" of a chain in the unit n-cube? Since we are dealing with subsets of the unit n-cube we have to choose a suitable notion of "size". A first choice could be the n-dimensional Lebesgue measure. However, it is not difficult to see, using Lebesgue's density theorem, that the Lebesgue measure of a chain in the unit n-cube equals zero. Given this observation, it is then natural to ask for sharp upper bounds on the Hausdorff dimension and the corresponding Hausdorff measure of chains in the unit n-cube. Our first result provides best possible bounds on both quantities. Throughout the text, given $s \in [0,\infty)$, $\mathcal{H}^s(\cdot)$ denotes s-dimensional Hausdorff outer measure (see [8, p. 81 and [9, 1]).

Theorem 1.4. Let $C \subset [0,1]^n$ be a chain. Then $\mathcal{H}^1(C) \leq n$.

The bound provided by Theorem 1.4 is best possible, as can be seen from the chain

$$C = \bigcup_{i=1}^{n} \{ (x_1, \dots, x_n) \in [0, 1]^n : x_1 = \dots = x_{i-1} = 1, \ x_{i+1} = \dots = x_n = 0 \}.$$

A more "exotic" example of a chain in the unit n-cube whose 1-dimensional Hausdorff measure equals n can be found in the proof of [3, Theorem 1.5]. Now, given Theorem 1.4 and Theorem 1.3, it seems natural to ask for upper bounds on the maximum "size" of a subset of the unit n-cube whose intersection with every chain has \mathcal{H}^1 -measure which is not larger than a given number from the interval (0,n]. This leads to the following continuous analogue of Erdős' theorem. Throughout the text, the term *measurable set* refers to a set that is Lebesgue measurable.

Theorem 1.5. Fix a real number $\kappa \in (0, n]$. Let $A \subset [0, 1]^n$ be a measurable set that satisfies

$$\mathcal{H}^1(A \cap C) \leq \kappa$$
, for all chains $C \subset [0,1]^n$.

Then the n-dimensional Lebesgue measure of A is not greater than the measure of the following optimal set:

$$A_{\kappa}^* := \left\{ (x_1, \dots, x_n) \in [0, 1]^n : \frac{n - \kappa}{2} \le \sum_{i=1}^n x_i \le \frac{n + \kappa}{2} \right\}.$$

Moreover, we have $\mathcal{L}^n(A_{\kappa}^*) = v_n(\kappa)$, where $v_n(\kappa)$ is as in Theorem 1.2.

1.1 Organisation

In Section 2 we prove Theorem 1.4 by showing that the \mathcal{H}^1 -measure of A is less than or equal to the sum of the \mathcal{H}^1 -measures of its n "anti-diagonal" projections onto the n coordinate axes. Sections 3 and 4 are devoted to the proof of Theorem 1.5. The proof is based on, and is inspired from, the proof of Theorem 1.2 (see [4]) and proceeds by discretising the problem and by employing well know results from the theory of (finite) partially ordered sets. Finally, in Section 5 we collect some remarks and an open problem.

2 Proof of Theorem 1.4

Given a chain $C \subset [0,1]^n$ and $i \in [n]$, let $C^{(i)}$ denote the set

$$C^{(i)} = C \cap \left\{ (x_1, \dots, x_n) \in [0, 1]^n : i - 1 \le \sum_{i=1}^n x_i \le i \right\}.$$

Moreover, given $\mathbf{x} \in C^{(i)}$, let $S_i(\mathbf{x}) = (\sum_{j=1}^n x_j) - (i-1)$. For each $i \in [n]$ consider the "anti-diagonal" projections $\vartheta_i : C^{(i)} \to [0,1]^n$ defined by

$$(x_1,\ldots,x_n)\mapsto (0,\ldots,0,S_i(\mathbf{x}),0,\ldots,0),$$

where $S_i(\mathbf{x})$ is on the *i*-th coordinate. Notice that, for each $i \in [n]$, ϑ_i restricted on $C^{(i)}$ is injective and therefore is a bijection from $C^{(i)}$ onto its image $\vartheta_i(C^{(i)})$. Let $\mathbf{a}, \mathbf{b} \in \vartheta_i(C^{(i)})$ be distinct and suppose that $\vartheta_i^{-1}(\mathbf{a}) = \mathbf{x}$ and $\vartheta_i^{-1}(\mathbf{b}) = \mathbf{y}$, for some $\mathbf{x}, \mathbf{y} \in C^{(i)}$. Suppose, without loss of generality, that $S_i(\mathbf{x}) \geq S_i(\mathbf{y})$. Now notice that

$$\|\vartheta_i^{-1}(\mathbf{a}) - \vartheta_i^{-1}(\mathbf{b})\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \le \sum_{i=1}^n (x_i - y_i) = S_i(\mathbf{x}) - S_i(\mathbf{y}) = \|\mathbf{a} - \mathbf{b}\|.$$

This implies that, for each $i \in [n]$, the function $\vartheta_i^{-1} : \vartheta_i(C^{(i)}) \to C^{(i)}$ is Lipschitz with constant 1 and therefore (see [8, Theorem 2.8]) we have

$$\mathcal{H}^1(C^{(i)}) = \mathcal{H}^1(\vartheta_i^{-1}(\vartheta_i(C^{(i)}))) \leq \mathcal{H}^1(\vartheta_i(C^{(i)}))\,.$$

Hence

$$\mathcal{H}^1(C) \le \sum_{i=1}^n \mathcal{H}^1(\vartheta_i(C^{(i)}))$$

and, since we clearly have $\mathcal{H}^1(\vartheta_i(C^{(i)})) \leq 1$, the result follows.

3 Proof of Theorem 1.5

In this section we prove Theorem 1.5. The proof requires some extra piece of notation. Throughout this section, $[m-1]_0$ denotes the set of integers $\{0,1,\ldots,m-1\}$. Given positive integers j and $m \geq 2$ such that j < m, we denote by $I_{j,m}$ the intervals

$$I_{j,m} = \begin{cases} \left[\frac{j}{m}, \frac{j+1}{m} \right), & \text{if } j \in [m-2]_0, \\ \left[\frac{j}{m}, \frac{j+1}{m} \right], & \text{if } j = m-1. \end{cases}$$

The approach we embark on is based on, and is inspired from, the approach in [4]. In particular, we make use of the following result from [4, Lemma 2].

Lemma 3.1 ([4]). Let $V_{n,m}(\kappa)$ be the sum of the $\lceil \kappa m + n \rceil$ largest coefficients in the polynomial $p(x) = (1 + x + \dots + x^{m-1})^n$. Then we have $\lim_{m \to \infty} \frac{V_{n,m}(\kappa)}{m^n} = v_n(\kappa)$, where $v_n(\kappa)$ is defined in Theorem 1.2.

The sum of the k largest coefficients in the polynomial $p(x) = (1 + x + \cdots + x^{m-1})^n$ are also referred to as the k largest Whitney numbers of $[m-1]_0^n$ (see [10, p. 25]).

We now proceed with the proof of Theorem 1.5. Let $A \subset [0,1]^n$ be a measurable set that satisfies $\mathcal{H}^1(A \cap C) \leq \kappa$, for all chains $C \subset [0,1]^n$. Notice that Theorem 1.2 implies that it is enough to show

$$\mathcal{L}^n(A) \le v_n(\kappa) \,, \tag{1}$$

where $v_n(\kappa)$ is as in Theorem 1.2. Moreover, the inner regularity of Lebesgue measure implies that it is enough to assume that A is *compact*.

If $\kappa = n$, then Theorem 1.4 implies that the unit n-cube has maximum \mathcal{L}^n -measure. We may therefore assume that $\kappa < n$.

Fix $\varepsilon < \frac{1}{2n+2}$ which is additionally assumed to be sufficiently small so that it satisfies

$$\kappa < n(1 - (2n+2)\varepsilon) \cdot (1 - \varepsilon)^n. \tag{2}$$

Write the unit n-cube $[0,1]^n$ as a union of cubes of the form

$$Q_{\mathbf{d}} := I_{d_1,m} \times I_{d_2,m} \times \cdots \times I_{d_n,m},$$

where $\mathbf{d} = (d_1, \dots, d_n) \in [m-1]_0^n$. Notice that each cube $Q_{\mathbf{d}}$ can be uniquely identified by the vector $\mathbf{d} \in [m-1]_0^n$. Given $F \subset [m-1]_0^n$, we denote

$$Q_F := \bigcup_{\mathbf{d} \in F} Q_{\mathbf{d}} \,.$$

Consider the set of *n*-tuples

$$D_A = \{ \mathbf{d} \in [m-1]_0^n : Q_{\mathbf{d}} \cap A \neq \emptyset \}.$$

Notice that $A \subset Q_{D_A}$. Moreover, since A is compact, we may assume that m is large enough so that it holds

$$\mathcal{L}^n(Q_{D_A} \setminus A) < \varepsilon^{2n+1}. \tag{3}$$

Now consider the set

$$D_{\varepsilon} = \{ \mathbf{d} \in D_A : \mathcal{L}^n(Q_{\mathbf{d}} \cap A) > (1 - 2^{-n} \varepsilon^{2n}) \cdot \mathcal{L}^n(Q_{\mathbf{d}}) \}.$$

The next lemma provides an upper bound on the maximum size of a chain in D_{ε} .

Lemma 3.2. Let t be the maximum size of a chain in D_{ε} . Then $t \leq \lceil m\kappa' + n \rceil$, where $\kappa' = \frac{\kappa}{(1-(2n+2)\varepsilon)\cdot(1-\varepsilon)^n}$.

The proof of Lemma 3.2 is rather technical and is deferred to Section 4. For the remaining part of this section, let us assume that Lemma 3.2 holds true. Notice that (2) guarantees that $\kappa' < n$.

Now Lemma 3.2 implies that D_{ε} is a partially ordered set that does not contain a chain of length $\lceil m\kappa' + n \rceil$ and therefore (see [5, Theorem 5.1.4 and Example 5.1.1]) it follows that $|D_{\varepsilon}|$ is not larger than the sum of the $\lceil m\kappa' + n \rceil$ largest Whitney numbers of $\lceil m - 1 \rceil_0^n$, i.e., we have

$$|D_{\varepsilon}| \le V_{n,m}(\kappa'), \tag{4}$$

where $V_{n,m}(\kappa')$ is defined in Lemma 3.1.

Claim: We have $\mathcal{L}^n(A) \leq \mathcal{L}^n(Q_{D_{\varepsilon}}) + \frac{1-2^{-n}\varepsilon^{2n}}{2^{-n}} \cdot \varepsilon$.

Proof of Claim. Notice that (3) implies $\mathcal{L}^n(Q_{D_A}) < \mathcal{L}^n(A) + \varepsilon^{2n+1}$. Moreover, the definition of D_{ε} implies $\mathcal{L}^n(Q_{D_A \setminus D_{\varepsilon}} \cap A) \leq (1 - 2^{-n} \varepsilon^{2n}) \cdot \mathcal{L}^n(Q_{D_A \setminus D_{\varepsilon}})$. Therefore, we have

$$\mathcal{L}^{n}(A) = \mathcal{L}^{n}(A \cap Q_{D_{\varepsilon}}) + \mathcal{L}^{n}(A \cap Q_{D_{A} \setminus D_{\varepsilon}})$$

$$\leq \mathcal{L}^{n}(Q_{D_{\varepsilon}}) + (1 - 2^{-n}\varepsilon^{2n}) \cdot (\mathcal{L}^{n}(Q_{D_{A}}) - \mathcal{L}^{n}(Q_{D_{\varepsilon}}))$$

$$\leq \mathcal{L}^{n}(Q_{D_{\varepsilon}}) + (1 - 2^{-n}\varepsilon^{2n}) \cdot (\mathcal{L}^{n}(A) + \varepsilon^{2n+1} - \mathcal{L}^{n}(Q_{D_{\varepsilon}}))$$

which in turn implies

$$\mathcal{L}^{n}(A) \leq \mathcal{L}^{n}(Q_{D_{\varepsilon}}) + \frac{(1 - 2^{-n} \varepsilon^{2n}) \cdot \varepsilon^{2n+1}}{2^{-n} \varepsilon^{2n}},$$

as desired. \Box

Set $\delta(\varepsilon) := \frac{1-2^{-n}\varepsilon^{2n}}{2^{-n}} \cdot \varepsilon$. To finish the proof of Theorem 1.5, notice that the Claim and (4) imply

$$\mathcal{L}^n(A) \leq \mathcal{L}^n(Q_{D_{\varepsilon}}) + \delta(\varepsilon) = \frac{|D_{\varepsilon}|}{m^n} + \delta(\varepsilon) \leq \frac{V_{n,m}(\kappa')}{m^n} + \delta(\varepsilon)$$

and therefore, using Lemma 3.1, we conclude

$$\mathcal{L}^n(A) \le v_n(\kappa') + \delta(\varepsilon)$$
.

The continuity of $v_n(\cdot)$ implies (1), upon letting $\varepsilon \to 0$, and thus Theorem 1.5 follows.

4 Proof of Lemma 3.2

This section is devoted to the proof of Lemma 3.2. We begin by introducing some additional piece of notation.

We denote by $\mathbf{1}_n=(1,\dots,1)$ the point in \mathbb{R}^n all of whose n coordinates are equal to 1, and we occasionally drop the index when the underlying dimension is clear from the context. Given a positive integer $k\in[n]$, we denote by $\binom{[n]}{k}$ the family consisting of all subsets of [n] whose cardinality equals k and, given $F=\{i_1<\dots< i_k\}\in\binom{[n]}{k}$, we let $\pi_F(\cdot)$ denote the function $\pi_F:\mathbb{R}^n\to\mathbb{R}^k$ which maps every point (x_1,\dots,x_n) to the point (x_{i_1},\dots,x_{i_k}) . That is, $\pi_F(\cdot)$ is the projection onto the coordinates corresponding to F. Moreover, given $v\in\mathbb{R}^n$ and $F\subset[n]$, we denote by v_F the vector in \mathbb{R}^n whose i-th coordinate equals v_i if $i\in F$, and 0 otherwise. For $i\in[n]$, we denote by e^i the i-th basis vector, i.e., the vector $(0,\dots,0,1,0,\dots,0)$ whose i-th coordinate equals 1. If $A,B\subset\mathbb{R}^n$, we write $A\leq B$ if for every $x\in A$ and every $y\in B$ we have $x\leq y$. Given $a,b\in\mathbb{R}^n$ such that $a\leq b$, we denote by $R_{a,b}\subset\mathbb{R}^n$ the rectangle

$$R_{\boldsymbol{a},\boldsymbol{b}} := \{ \boldsymbol{x} \in \mathbb{R}^n; \ \boldsymbol{a} \le \boldsymbol{x} \le \boldsymbol{b} \}.$$

If $b = a + 1_n$, we simply write R_a instead of $R_{a,b}$. Thus R_0 is another way to denote the set $[0,1]^n$. Finally, suppose we are given k < n and $M \in {[n] \choose k}$, a point $t \in \mathbb{R}^{n-k}$ and a set $A \subset \mathbb{R}^n$. Then we define

$$A(t, M) := A \cap \pi_{[n] \setminus M}^{-1}(\{t\}).$$

We begin with a simple consequence of Fubini's theorem.

Lemma 4.1. Let $n \in \mathbb{N}$ and $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ be fixed. Suppose that $\varepsilon > 0$ is such that for every $j \in [n]$ we have $\frac{1}{v_j+1} \ge \varepsilon$ and let $A \subset [0,1]^n$ be a compact set that satisfies

$$\mathcal{L}^n(A) > 1 - \varepsilon^{n+1}. \tag{5}$$

Then, for every $i \in [n]$, there exists a chain $V \subset [0,1]^n$ such that $\mathcal{H}^1(A \cap V) \geq 1 - (v_i + 1)\varepsilon$ and

$$\{\varepsilon \mathbf{v}\} \le V \le \{\varepsilon(\mathbf{v} + \mathbf{1}_n)_{[n]\setminus\{i\}} + \mathbf{e}^i\}.$$
 (6)

Proof. Fix $i \in n$ and denote by $v^{(i)}$ the vector $\pi_{[n]\setminus\{i\}}(v)$. Consider the rectangle $R:=\varepsilon R_{v^{(i)}}$. Since $\frac{1}{v_j+1}\geq \varepsilon$ for every $j\in [n]$ we have $R\subset [0,1]^{n-1}$. We now show that there exists $t_0\in R$ such that $\mathcal{H}^1(A(t_0,\{i\}))\geq 1-\varepsilon$. Assume, towards a contradiction, that there does not exist such a $t_0\in R$. Then for every $t\in R$ we have

$$\mathcal{H}^1([0,1]^n \setminus A(t,\{i\})) = \mathcal{H}^1(([0,1]^n \setminus A)(t,\{i\})) > \varepsilon. \tag{7}$$

By (5), (7), Fubini's theorem and the fact that $R \subset [0,1]^{n-1}$ and $\mathcal{L}^{n-1}(R) = \varepsilon^{n-1}$, we conclude

$$\varepsilon^{n+1} > \mathcal{L}^n([0,1]^n \setminus A) \ge \mathcal{L}^n\left(\bigcup_{t \in R} ([0,1]^n \setminus A)(t,\{i\})\right) \ge \varepsilon \cdot \mathcal{L}^{n-1}(R) = \varepsilon^{n+1},$$

which is a contradiction. Hence there exists $t_0 \in R$ such that $\mathcal{H}^1(A(t_0, \{i\})) \ge 1 - \varepsilon$. Now consider the set

$$V := A(t_0, \{i\}) \cap R_{\varepsilon n, \mathbf{1}_n}.$$

Since $t_0 \in R$ and $V \subset R_{\varepsilon v, \mathbf{1}_n}$ we readily obtain (6). Clearly V is chain and, since $V = A \cap V$, we conclude

$$\mathcal{H}^1(A \cap V) = \mathcal{H}^1(V) \ge \mathcal{H}^1(A(t_0, \{i\})) - \varepsilon v_i \ge 1 - (v_i + 1)\varepsilon,$$

as desired. \Box

Given $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$, let $\mathbf{O}(\mathbf{a}, \mathbf{b}) = \{l \in [n] : a_l \neq b_l\}$ be the set of indices for which the corresponding coordinates are different.

Lemma 4.2. Let $n, k \in \mathbb{N}$ be fixed. Let $v = (v_1, \dots, v_n) \in \mathbb{R}^n_+$ be given and assume that $d^1 \leq \dots \leq d^k \in \mathbb{N}^n$ is a chain of vectors with non-negative integer coordinates. Set $\alpha := \max\{2n, v_1, \dots, v_n\}$ and $O := O(d^1, d^k)$, and fix $\varepsilon \in (0, \frac{1}{\alpha+2})$. Let $A \subset \mathbb{R}^n$ be a compact set and assume further that there exists $W \subset [k]$ such that for every $i \in W$ we have

$$\mathcal{L}^n(A \cap R_{\boldsymbol{d}^i}) > 1 - \varepsilon^{2n} 2^{-n} .$$

Then there exists a chain $L \subset \mathbb{R}^n$ such that

$$\mathcal{H}^1(A \cap L) \ge (1 - (\alpha + 2)\varepsilon)(1 - \varepsilon)^n(|W| - 1) \tag{8}$$

and

$$\{d^1 + \varepsilon v\} \le L \le \{d^k + \varepsilon(2|O|\mathbf{1}_O + (v+1)_{[n]\setminus O})\}.$$
(9)

Proof. If $|W| \le 1$, then we may choose $L = \emptyset$ and the result follows. So we may sssume that $k \ge |W| > 1$. In particular, we have $O \ne \emptyset$. Notice that the assumptions imply that ε satisfies $\varepsilon(v+2\cdot 1_n) \le 1_n$, a fact that will be used several times in the proof.

We proceed by induction on the dimension, n. The case n = 1 is trivial; we may choose L to be the interval $[d^1 + v\varepsilon, d^k]$. Now, assuming that the lemma holds true for every integer less than or equal to n - 1, we prove it for n. We distinguish two cases.

First assume that |O| < n. Clearly, we have |O| > 0 and

$$\pi_{[n]\setminus \mathbf{O}}(\mathbf{d}^l) = \pi_{[n]\setminus \mathbf{O}}(\mathbf{d}^1), \text{ for all } l \in [k].$$
 (10)

To simplify notation, set $\boldsymbol{a} := \pi_{[n] \setminus \boldsymbol{O}}(\boldsymbol{d}^1 + \varepsilon \boldsymbol{v})$, $\boldsymbol{b} := \pi_{[n] \setminus \boldsymbol{O}}(\boldsymbol{d}^1 + \varepsilon (\boldsymbol{v} + \boldsymbol{1}))$ and $C := \bigcup_{j \in W} R_{\boldsymbol{d}^j}$. Notice that $\mathcal{L}^n(C) = |W|$. By (10) we have

$$\mathcal{L}^{n}\left(\bigcup_{t\in R_{\boldsymbol{a},\boldsymbol{b}}} C(t,\boldsymbol{O})\right) = \mathcal{L}^{n-|\boldsymbol{O}|}(R_{\boldsymbol{a},\boldsymbol{b}}) \cdot \mathcal{L}^{n}(C) = \varepsilon^{n-|\boldsymbol{O}|} \mathcal{L}^{n}(C).$$
(11)

Since $\mathcal{L}^n(A\cap R_{\boldsymbol{d}^i})>1-\varepsilon^{2n}2^{-n}$ for every $i\in W$, we have

$$\mathcal{L}^n(C \setminus A) < \mathcal{L}^n(C)\varepsilon^{2n}2^{-n}$$

which, combined with (11), yields

$$\mathcal{L}^{n}\left(\bigcup_{t\in R_{\boldsymbol{a},\boldsymbol{b}}}C(t,\boldsymbol{O})\setminus A\right) \leq \mathcal{L}^{n}(C\setminus A) < \mathcal{L}^{n}(C)\varepsilon^{2n}2^{-n}$$

$$= \mathcal{L}^{n}\left(\bigcup_{t\in R_{\boldsymbol{a},\boldsymbol{b}}}C(t,\boldsymbol{O})\right)\varepsilon^{n+|\boldsymbol{O}|}2^{-n}$$

$$\leq \mathcal{L}^{n}\left(\bigcup_{t\in R_{\boldsymbol{a},\boldsymbol{b}}}C(t,\boldsymbol{O})\right)\varepsilon^{2|\boldsymbol{O}|+1}2^{-|\boldsymbol{O}|-1}.$$

Observe that $\pi_{O}: \pi_{[n]\backslash O}^{-1}(\{t\}) \to \mathbb{R}^{|O|}$ preserves $\mathcal{H}^{|O|}$ -measure. Moreover, notice that $\mathcal{L}^{|O|}(\pi_{O}(C(t,O))) = |W|$. Using Fubini Theorem we find $t \in R_{a,b}$ such that

$$\mathcal{L}^{|O|}(\pi_{O}(C(t, O) \setminus A)) \le \mathcal{L}^{|O|}(\pi_{O}(C(t, O)))\varepsilon^{2|O|+1}2^{-|O|-1}.$$
(12)

Now, we find $\tilde{W} \subset W$ such that for every $l \in \tilde{W}$ we have

$$\mathcal{L}^{|\boldsymbol{O}|}(\pi_{\boldsymbol{O}}(R_{\boldsymbol{d}^{l}}(t, \boldsymbol{O}) \cap A)) \ge 1 - \varepsilon^{2|\boldsymbol{O}|} 2^{-|\boldsymbol{O}|}. \tag{13}$$

We show that $|\tilde{W}| - 1 \ge (1 - \varepsilon)(|W| - 1)$. Assume, for the sake of contradiction, that

$$|W \setminus \tilde{W}| > \varepsilon(|W| - 1) \ge \varepsilon \frac{1}{2}|W|.$$

Then for every $l \in W \setminus \tilde{W}$ we have

$$\mathcal{L}^{|O|}(\pi_O(R_{d^l}(t, O) \setminus A)) > \varepsilon^{2|O|} 2^{-|O|}.$$

Thus

$$\begin{split} \mathcal{L}^{|O|}(\pi_{O}(C(t, O) \setminus A)) &\geq \mathcal{L}^{|O|}\left(\pi_{O}\left(\bigcup_{l \in W \setminus \tilde{W}} R_{d^{l}}(t, O) \setminus A\right)\right) \\ &> |W \setminus \tilde{W}|\varepsilon^{2|O|}2^{-|O|} > \varepsilon \frac{1}{2}|W|\varepsilon^{2|O|}2^{-|O|} \\ &= \varepsilon^{2|O|+1}2^{-|O|-1}\mathcal{L}^{|O|}(\pi_{O}(C(t, O))), \end{split}$$

which contradicts (12). Since |O| < n we may apply the induction hypothesis for $\pi_O(v)$, |O|, k, ε , $\pi_O(d^1)$, ..., $\pi_O(d^k)$, \tilde{W} , $\pi_O(A(t,O))$ instead of v, n, k, ε , d^1 , ..., d^k , W, A and obtain a chain $\tilde{L} \subset \mathbb{R}^{|O|}$ such that

$$\mathcal{H}^{1}(\tilde{L} \cap \pi_{O}(A(t, O))) \geq (1 - (\tilde{\alpha} + 2)\varepsilon)(1 - \varepsilon)^{|O|}(|\tilde{W}| - 1)$$

$$\geq (1 - (\alpha + 2)\varepsilon)(1 - \varepsilon)^{|O|+1}(|W| - 1)$$

$$\geq (1 - (\alpha + 2)\varepsilon)(1 - \varepsilon)^{n}(|W| - 1),$$

where $\tilde{\alpha} = \max\{2|\boldsymbol{O}|, v_1, \dots, v_n\}$ and

$$\{\pi_{O}(d^{1}+\varepsilon v)\} \leq \tilde{L} \leq \{\pi_{O}(d^{k}+2|O|\varepsilon 1)\}.$$

Now define the set $L:=\pi_{\boldsymbol{O}}^{-1}(\tilde{L})\cap\pi_{[n]\backslash\boldsymbol{O}}^{-1}(t)$. Clearly, L is a chain. Since $t\in R_{\boldsymbol{a},\boldsymbol{b}}$ we obtain (9). Finally, the fact that $\pi_{\boldsymbol{O}}$ is a linear isometry on $A(t,\boldsymbol{O})$ implies (8). The proof of the first case is thus completed.

Now assume that |O| = n. First we show that we may additionally suppose that W = [k]. Let $W = \{i_1, \dots, i_p\}$, where p > 1 and $i_1 < \dots < i_p$. Consider the sets

$$ilde{O}:=m{O}(m{d}^{i_1},m{d}^{i_p}),\, ilde{O}_1:=m{O}(m{d}^1,m{d}^{i_1})\setminus ilde{O} \,\, ext{and}\,\,\, ilde{O}_2:=m{O}(m{d}^{i_p},m{d}^{i_k})\setminus (ilde{O}\cup ilde{O}_1)\,.$$

Clearly, $\tilde{O} \cup \tilde{O}_1 \cup \tilde{O}_2 = O = [n]$. Now we can replace v, d_1, \ldots, d_k with $v_{\tilde{O} \cup \tilde{O}_2}$, d^{i_1}, \ldots, d^{i_p} in the assumptions of the lemma, and obtain the desired L for the latter. If we have the desired L for $v_{\tilde{O} \cup \tilde{O}_2}$, d^{i_1}, \ldots, d^{i_p} , then (8) will readily follow, and we may deduce (9) upon observing that

$$egin{aligned} d^1 + arepsilon v & \leq d^{i_1} + v_{ ilde{O} \cup ilde{O}_2}, \ d^{i_p} + arepsilon (2| ilde{O}|\mathbf{1}_{ ilde{O}} + (v_{ ilde{O} \cup ilde{O}_2} + \mathbf{1})_{[n] \setminus ilde{O}}) & = d^{i_p} + arepsilon (2| ilde{O}|\mathbf{1}_{ ilde{O}} + v_{ ilde{O}_2} + \mathbf{1}_{[n] \setminus ilde{O}}) \ & \leq d^k + arepsilon (2| ilde{O}|\mathbf{1}_{ ilde{O}} + \mathbf{1}_{[n] \setminus ilde{O}}) \ & \leq d^k + arepsilon 2n\mathbf{1}. \end{aligned}$$

Hence we may assume that |O|=n and |W|=k>1. We proceed by finding $s\in\mathbb{N}$, a sequence of non-negative integers $n_1,\ldots,n_s\in\mathbb{N}$, vectors $v^1,\ldots,v^s\in\mathbb{R}^n$, and sets $O_1,\ldots,O_{s-1}\subset[n]$ and $L_1,\ldots,L_{s-1}\subset\mathbb{R}^n$ that satisfy, and are defined via, the following seven conditions:

- (i) Set $n_1 = 1$ and, for $i \in [s-2]$, let $n_{i+1} = \min\{l \in [k]; O(d^{n_i}, d^l) = [n]\}$. We clearly have $1 = n_1 < n_2 < \cdots < n_s = k+1$.
- (ii) Define $v^1 = v$ and $v^i = 2n\mathbf{1}$, for $i \in \{2, \dots, s\}$.
- (iii) For $i \in [s-1]$, let $O_i := O(d^{n_i}, d^{n_{i+1}-1})$.
- (iv) For every $i \in [s-2]$ the set L_i is a chain and $\{d^{n_i} + \varepsilon v^i\} \le L_i \le \{d^{n_{i+1}} + \varepsilon v^{i+1}\}$.
- (v) The set L_{s-1} is a chain and $\{d^{n_{s-1}} + \varepsilon v^{s-1}\} \le L_{s-1} \le \{d^k + \varepsilon v^s\}$.
- (vi) For every $i \in [s-2]$ we have $\mathcal{H}^1(A \cap L_i) \ge (1-(\alpha+2)\varepsilon)(1-\varepsilon)^n(n_{i+1}-n_i)$.
- (vii) $\mathcal{H}^1(A \cap L_{s-1}) \ge (1 (\alpha + 2)\varepsilon)(1 \varepsilon)^n (k n_{s-1}).$

Notice that, since k > 1 and O = [n], we have $s \ge 3$. It remains to show how to find the sets L_i , for $i \in [s-1]$, such that (iv)-(vii) are satisfied. To this end, we first construct sets $T_i, V_i \subset \mathbb{R}^n$ that satisfy the following:

- (a) The sets T_i , V_i are chains, for all $i \in [s-1]$.
- (b) For $i \in [s-1]$ we have $\{d^{n_i} + \varepsilon v^i\} \le T_i \le \{d^{n_{i+1}-1} + \varepsilon(2|O_i|1_{O_i} + (v^i+1)_{[n]\setminus O_i})\}.$
- (c) For $i \in [s-2]$ we have $\{d^{n_{i+1}-1} + \varepsilon(2|O_i|\mathbf{1}_{O_i} + (v^i+1)_{[n]\setminus O_i})\} \le V_i \le \{d^{n_{i+1}} + \varepsilon v^{i+1}\}.$
- (d) $\mathcal{H}^1(T_i \cap A) \ge (1 (v+2)\varepsilon)(1-\varepsilon)^{2n+1}(n_{i+1} n_i 1)$ for $i \in [s-1]$.
- (e) $\mathcal{H}^1(V_i \cap A) \ge (1 (v+2)\varepsilon)(1-\varepsilon)^{2n+1}$, for $i \in [s-2]$.

We begin with the sets T_i . Consider $d^{n_i}, \ldots, d^{n_{i+1}-1}$ and v^i instead of d^1, \ldots, d^k and v and observe that we are in the same situation as in the first part of the proof (i.e., the case |O| < n). So, we are able to find T_i satisfying (a), (b), (d).

Now, we proceed with the sets V_i . Put $V_{s-1} := \emptyset$. For $i \in [s-2]$ choose some $l \in O(d^{n_{i+1}-1}, d^{n_{i+1}})$ and use Lemma 4.1 for $2|O_i|1_{O_i} + (v^i+1)_{[n]\setminus O_i}$, ε and l. Since $\frac{1}{n+2} > \varepsilon$, $\frac{1}{v_{i+2}} > \varepsilon$ and $n_{i+1} - 1 \in W$ we can find V_i satisfying (a), (e) and

$$\begin{aligned} \{\boldsymbol{d}^{n_{i+1}-1} + \varepsilon(2|\boldsymbol{O}_i|\mathbf{1}_{\boldsymbol{O}_i} + (\boldsymbol{v}^i + \mathbf{1})_{[n]\setminus\boldsymbol{O}_i})\} \\ &\leq V_i \leq \{\boldsymbol{d}^{n_{i+1}-1} + \boldsymbol{e}^l + \varepsilon((2|\boldsymbol{O}_i| + 1)\mathbf{1}_{\boldsymbol{O}_i} + (\boldsymbol{v}^i + 2 \cdot \mathbf{1})_{[n]\setminus(\boldsymbol{O}_i \cup \{l\})})\} \\ &\leq \boldsymbol{d}^{n_{i+1}} + \varepsilon(2|\boldsymbol{O}_i| + 1)\mathbf{1}_{\boldsymbol{O}_i} \leq \{\boldsymbol{d}^{n_{i+1}} + \varepsilon\boldsymbol{v}^{i+1}\}. \end{aligned}$$

In the last but one inequality, we used (ii). Thus V_i also satisfy (c).

Now let $L_i := T_i \cup V_i$. Since $V_{s-1} = \emptyset$ we immediately deduce (iv) and (v) from (a), (b) and (c). We can also derive (vi) and (vii) from (d) and (e). We have thus completed the construction of the sets L_i .

Finally, we put $L := \bigcup_{i=1}^{s-1} L_i$. By (iv), (v), (vi) and (vii), it follows that L is a chain satisfying (8) and (9).

Recall, from Section 3, the definition of cubes Q_d , where $d \in [m-1]_0^n$. Given a set of cubes $Q = \{Q_{d_1}, \dots, Q_{d_k}\}$, we say that Q is a *chain of m-cubes* if $d_1 \leq \dots \leq d_k$ is a chain in $[m-1]_0^n$.

Theorem 4.3. Let $n \in \mathbb{N}$, $\frac{1}{2n+2} > \varepsilon > 0$, $A \subset \mathbb{R}^n$ be measurable and Q be a chain of m-cubes such that for every $Q \in Q$ we have $\lambda_n(A \cap Q) > 1 - \varepsilon^{2n} 2^{-n}$. Then there exists chain $L \subset \mathbb{R}^n$ such that $\mathcal{H}^1(A \cap L) \geq (1 - (2n+2)\varepsilon)(1-\varepsilon)^n \frac{|Q|-1}{m}$.

Proof. Since Lebesque measure is inner regular, we may assume that A is compact. The result follows immediately from Lemma 4.2 upon setting v=0 and rescaling to m-cubes.

The proof of Lemma 3.2 is almost complete.

Proof of Lemma 3.2. Let $d_1 \leq \cdots \leq d_t$ be a chain in D_{ε} and assume contrariwise that $t \geq m\kappa' + n + 1$, where $\kappa' = \frac{\kappa}{(1 - (2n + 2)\varepsilon) \cdot (1 - \varepsilon)^n}$. Then $\mathcal{Q} = \{Q_{d_1}, \dots, Q_{d_t}\}$ is a chain of m-cubes that satisfies the hypothesis of Theorem 4.3 and so there exists chain a $L \subset \mathbb{R}^n$ such that

$$\mathcal{H}^1(A \cap L) > \kappa$$
,

contrary to the assumption that $\mathcal{H}^1(A \cap C) \leq \kappa$, for all chains $C \subset [0,1]^n$. The result follows.

5 Concluding remarks

Notice that in Theorem 1.5 we assume that $\kappa > 0$. However, it seems reasonable to ask what happens when $\kappa = 0$. In this case we are dealing with a measurable set $A \subset [0,1]^n$ which satisfies

$$\mathcal{H}^1(A \cap C) = 0$$
, for all chains $C \subset [0,1]^n$. (14)

What is the maximum "size" of a measurable $A \subset [0,1]^n$ that satisfies (14)? It is shown in [6] that if $A \subset [0,1]^n$ is such that

$$\mathcal{H}^0(A \cap C) \le 1$$
, for all chains $C \subset [0,1]^n$, (15)

then the Hausdorff dimension of A is less than or equal to n-1, and that the bound is sharp. Let us remark that a set A satisfying (15) is referred to as an *antichain*. An example of an antichain in the unit n-cube is the set

$$A_t = \left\{ (x_1, \dots, x_n) \in [0, 1]^n : \sum_{i=1}^n x_i = t \right\}, \text{ where } t \in [0, n].$$
 (16)

Since an antichain, by definition, satisfies (15) it readily follows that it satisfies (14). This implies that there exist subsets of the unit n-cube that satisfy (14) and whose Hausdorff

dimension is equal to n-1. In fact, we can say a bit more. Recall that $\dim_H(\cdot)$ denotes Hausdorff dimension (see [8, p. 86]).

Theorem 5.1. Fix $s \in [0,1]$ and $\beta \geq 0$. Then there exists $A \subset [0,1]^n$ that satisfies $\dim_H(A) = n-1+s$ and

$$\mathcal{H}^s(A \cap C) \leq \beta$$
, for all chains $C \subset [0,1]^n$.

Proof. Let $A_t, t \in [0, n]$, be as in (16). Let $K \subset [0, n]$ be a set such that $\dim_H(K) = s$ and $\mathcal{H}^s(K) = \beta$, and define the set

$$A = \bigcup_{t \in K} A_t \,.$$

It follows from [11, Theorem 1.2] that $\dim_H(A) = n - 1 + s$, and it remains to show the second statement.

Let $C \subset [0,1]^n$ be a chain and consider the function $\theta : [0,1]^n \to [0,n]$ defined via

$$\theta(x_1,\ldots,x_n)=\sum_{i=1}^n x_i.$$

Notice that θ restricted on C is injective and therefore θ is a bijection from C onto its image $\theta(C)$. Using a similar argument as in the proof of Theorem 1.4, we conclude that $\theta^{-1}:\theta(C)\to C$ is Lipschitz with constant 1. Since $\theta(A\cap C)\subset K$ and θ is 1-Lipschitz, we have

$$\mathcal{H}^s(A \cap C) \le \mathcal{H}^s(\theta^{-1}(K)) \le \mathcal{H}^s(K) = \beta,$$

as desired. \Box

Given Theorem 5.1, the following problem arises naturally.

Problem 5.2. Fix $s \in [0,1]$ and $\beta \geq 0$. Let $A \subset [0,1]^n$ be a measurable set such that $\dim_H(A) = n - 1 + s$ and $\mathcal{H}^s(A \cap C) \leq \beta$, for all chains $C \subset [0,1]^n$. What is a sharp upper bound on $\mathcal{H}^{n-1+s}(A)$?

In this article we considered the case $s=1, \beta\in [0,n]$ in Problem 5.2 . The case $s=0,\ \beta=1$ in Problem 5.2 has been considered in [6] where it is shown that a set $A\subset [0,1]^n$ for which it holds $\mathcal{H}^0(A\cap C)\leq 1$, for all chains $C\subset [0,1]^n$, satisfies $\mathcal{H}^{n-1}(A)\leq n$, and that the bound is best possible.

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