COMS 4236: Introduction to Computational Complexity, Spring 2012

Problem Set 2, due Wednesday February 29, in class.

Problem 1. a. Let $f(n) \geq n$ be any nondecreasing function. Show that NTIME($f(n)$) is closed under union, intersection and concatenation. That is, if L_1, L_2 are two languages in NTIME($f(n)$) then $L_1 \cup L_2, L_1 \cap L_2$ and $L_1 \cdot L_2$ are also in NTIME($f(n)$). Recall that the concatenation of two languages L_1, L_2 is $L_1 \cdot L_2 = \{xy \mid x \in L_1, y \in L_2\}$, i.e. the set of strings that can be written as the concatenation of a string in L_1 and a string in L_2.

b. The *Kleene star* of a language *L* is defined to be the set of all strings that can be written as the concatenation of any number of (0, 1, or more) strings of *L*, i.e. $L^* = \{x_1 \cdots x_k \mid k \geq 0 \text{ and } x_1 \in L, \cdots, x_k \in L\}$. For example, if *L*={a,ab,ba}, then ε (the empty string), aba and ababa are in L^* but abbba is not.
Show that NTIME(n) is closed under Kleene star, i.e. if $L \in$NTIME(n) then also $L^* \in$ NTIME(n).

Problem 2. Use the theorems that we learned on the relations between complexity classes to prove the following.
a. TIME(2^n) $\subset$ TIME(2^{2n})
b. NTIME(n^2) $\subset$ SPACE(n^3)
(*Note*: $\subset$ means that the left hand side is contained but is not equal to the right hand side. Make sure to show both facts.)

Problem 3. Do Problem 7.4.4, parts (a,b,f) in the book.

Problem 4. Show that if P $\neq$ NP then there is a language *L* over the alphabet {0,1} that is in NP - P.
(*Hint:* Encode an arbitrary alphabet into {0,1}. Specifically, if $A = \{a_1, \ldots, a_k\}$ is an arbitrary alphabet, define a mapping $\phi : A \rightarrow \{0,1\}^*$ where $\phi(a_i)$, for *i*=1,..,*k*, is the binary string of length *k* that has 1 in the *i*-th position and 0 in the other positions. Extend ϕ to strings *w* in *A** by applying it on each letter of *w*. If *R* is a language over *A*, then let $\phi(R)$ be the language { $\phi(w)$ | $w \in R$ }. Argue that the languages *R* and $\phi(R)$ have the same time complexity – both with respect to deterministic and nondeterministic time)

Problem 5. Let A be an alphabet, # a symbol not in A, and N the set of natural numbers. A *padding function* is a function pad: $A^* \times N \to (A \cup \{\#\})^*$ defined as $pad(w,l) = w\#^j$, where j=max(0, l-|w|); that is, if the length of a string w is less than l then we pad it with enough #'s so that the resulting string has length l. For any language $L \subseteq A^*$ and function $f : N \to N$ define the language $pad(L,f(n)) = \{\ pad(w,f(|w|))\ \mid\ w \in L\ \}$.

a. Prove that a language L is in TIME(n^2) if and only if $pad(L, n^2)$ is in TIME(n).

b. Argue similarly that L is in NTIME(n^2) if and only if $pad(L, n^2)$ is in NTIME(n).

c. Prove that if TIME(n) = NTIME(n) then TIME(n^2) = NTIME(n^2).

(*Note:* This is an instance of a general translation property: Equality of complexity classes translates upwards to higher resource bounds. You do not have to prove this.)