COMS 6253 Problem 1

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Problem 1. The parity function on k 0/1-valued variables is

$$PAR(x_1, ..., x_k) = x_1 + ... + x_k \text{ mod } 2,$$

a) Show that the parity function on $\log s$ variables can be computed by a decision tree of size s.

b) Show that any PTF for the parity function on k variables must have degree at least k.

Answer:

a) Since decision tree is an universal representation scheme, any Boolean function with k variables can be computed by a complete decision tree with depth k and 2^k leaves that exhaustively queries all k variables on every path.

For the parity function with $\log s$ variables, its complete decision tree has depth $\log s$ with $2^{\log s} = s$ leaves. So the parity function on $\log s$ variables can be computed by a decision tree of size s.

b) We know from class that polynomial threshold function is an universal representation scheme. So any Boolean function with k variables can be computed by a PTF with degree at most k. As a result, the parity function can also be represented by a PTF with degree at most k.

Observed from $PAR(x_1, ..., x_k) = x_1 + ... + x_k$ mod 2, we can see that one important property of parity function is that the value of a parity function depends only on the sum of all k variables. In other words, $PAR(x_1, ..., x_k)$ is invariant under permutation of its input. That is, $PAR(x_1, ..., x_k)$ is a symmetric polynomial.

One property of symmetric polynomial is that *for any symmetric polynomial* p *of degree* d*, there exists a polynomial* q *of degree* d *such that*

$$p(x_1, ..., x_n) = q(x_1 + ... + x_n)$$

Suppose $PAR(x_1, ..., x_k)$ can be computed by a symmetric polynomial threshold function $p(x_1, ..., x_k)$ of degree d. Then $PAR(x_1, ..., x_k)$ can also be computed by an univariate polynomial threshold function $q(x_1 + ... + x_k)$ of degree d.

Note that for all $x_1, ..., x_k \in \{0, 1\}^k$, $x_1 + ... + x_k \in \{0, 1, ..., k\}$. Since $q(x_1 + ... + x_k)$ compute the parity, we have

$$\begin{aligned} q(0) &< 0 \\ q(1) &> 0 \\ q(2) &< 0 \\ q(3) &> 0 \\ &\vdots \end{aligned}$$

So, in order to fit the $k + 1$ input values, function q must have at least k roots. As a result, we get $d \geq k$.

To sum up, any PTF for the parity function on k variables must have degree at least k.