COMS 4236: Introduction to Computational Complexity, Spring 2012

Problem Set 3, due Wednesday March 21, in class

Problem 1.
a. Consider the following problem, called CIRCUIT-SAT-20.
Input: A Boolean circuit C with 20 input variables.
Question: Is C satisfiable, i.e. does there exist an assignment to the input variables of C for which the value of C is 1?

Show that CIRCUIT-SAT-20 is in P.

b. Suppose that A, B, C are three languages such that (1) $A \leq_p B$, (2) $B \leq_p C$, (3) A is NP-complete, and (4) C is NP-complete. Show that B is also NP-complete.

Problem 2.
a. Show that if a language L is complete for a class C under log-space or polynomial time reductions then its complement $\overline{L}$ is complete for the class coC under the same type of reductions.

b. We say that a class C is *closed* under a type of reductions (eg. log-space or polynomial time reductions) if, whenever L reduces to L' and L' is in C, then also L is in C.
Suppose that the class C is closed under a type of reductions and the language M is complete for C. Show that M is in coC if and only if C = coC.

Problem 3.

a. Show that NP, coNP and PSPACE are closed under polynomial time reductions.

b. Show that SPACE(n) is not closed under polynomial time reductions.
Hint: Use a padding function (recall Problem 5 in Homework 2) and the space hierarchy theorem.
(Note: From a and b we can conclude that NP ≠ SPACE(n). We do not know how these two classes compare, i.e. if either one contains the other, but we know they are not equal.)

Problem 4.

a. Consider the following transformation that maps a directed graph G=(V,E) with m nodes to another directed graph G'=(V',E') with m^2 nodes. The set of nodes of G' is $V'=\{ [v,i] \mid v\in V,\ 1\le i\le m \}$, and the set of edges is $E'= \{ ([u,i],[v,i+1]) \mid (u,v)\in E,\ 1\le i\le m-1 \} \cup \{ ([v,i],[v,i+1]) \mid v\in V,\ 1\le i\le m-1 \}$.

Use this transformation to show that the Graph Reachability problem is NL-complete even when the input is restricted to acyclic graphs and the nodes are ordered topologically. Specifically, show that the following *DAG Reachability* problem is NL-complete.

Input: A directed acyclic graph H=(N,A) with set of nodes N={1,…,n} in topological order, i.e. all edges $(i,j) \in A$ satisfy $i < j$; Two nodes s, t.

Question: Is there a path in H from s to t?

b. The *Graph Cyclicity* problem is as follows.

Input: A directed graph G.

Question: Does the graph contain a cycle?

Show that the Graph Cyclicity problem is NL-complete.

c. Is the *Graph Acyclicity* problem (Given a directed graph, is it acyclic?) NL-complete? Justify your answer.

Problem 5.

a. Consider the following set of inequalities: $a \le b$, $a \le c$, $b+c \le a+1$, $0 \le a$.

Show that for every assignment of values 0 or 1 to b and c, there is a unique value of a over the real numbers that satisfies the inequalities, namely the value $a = b \wedge c$, i.e. $a = 1$ if both b, c are 1, and $a = 0$ otherwise.

b. Give a set of inequalities in variables a, b, c that has the analogous property for $a=b\vee c$, i.e., for every assignment of values 0 or 1 to b and c, there is a unique real value of a that satisfies the inequalities, namely the value $a=b\vee c$.

c. Show that the following "*Linear Inequalities*" problem is P-hard under log-space reductions. (The problem is also in P but you do not have to show that.)

Linear Inequalities:

Input*:* A system of linear inequalities in a set of variables.

Question*:* Does the system have a solution over the reals, i.e. is there an assignment of real values to the variables that satisfies all the inequalities?

(*Hint:* Reduce from the Circuit Value problem with fan-in 2. Introduce variables for the gates and the inputs of the circuit and include appropriate inequalities for the gates and the given input assignment to the circuit.)