COMS 6253 problems

April 14, 2012

Here is the final list of COMS 6253 problems. If you haven't done so yet, please turn in a detailed writeup of *one* of the problems below to Rocco by May 3.

Problem 1. The parity function on k 0/1-valued variables is

$$PAR(x_1, ..., x_k) = x_1 + ... + x_k \bmod 2,$$

i.e. the output is 0 if the number of bits that are set to 1 is even and is 1 if the number of bits that are set to 1 is odd.

(a) Show that the parity function on $\log s$ variables can be computed by a decision tree of size s.

(b) Show that any PTF for the parity function on k variables must have degree at least k. Hint: Convert this into a problem about *univariate* polynomials; then the desired result will follow from a standard fact about univariate polynomials.

Problem 2.

(a) Show that every LTF over $\{0, 1\}^n$ can be expressed using integer weights.

(b) Show that there are LTFs over $\{0, 1\}^n$ such that any representation using integer weights must have some weight at least as large as $2^{\Omega(n)}$.

Problem 3.

(a) Give a precise definition of $T_d(x)$, the degree-d Chebychev polynomial of the first kind. (There are a number of different possible definitions that can be shown equivalent; you can use whichever you please.)

(b) Use your definition for $T_d(x)$ to prove the following properties of $T_d(x)$ which we used in class:

(i) $|T_d(x)| \leq 1$ for $x \in [-1, 1]$;
(ii) $T_d(1) = 1$; and
(iii) $T'_d(x) \geq d^2$ for $x >= 1$.

Problem 4. Suppose that $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is a Boolean function which has Fourier degree k (i.e. all of its Fourier coefficients $\hat{f}(S)$ are 0 for sets $|S| > k$). Show that f can be written as a multilinear polynomial of degree at most k where each coefficient is of the form $C/2^k$, where C is an integer which is at most 2^k in absolute value.

Problem 5. Show that the Linial/Mansour/Nisan Fourier concentration bound for size-M depth-d circuits is not far from optimal, by displaying a size-M depth-d circuit whose Fourier concentration is not much better than guaranteed by the theorem. (Hint: Design such a circuit to compute the parity function over some number of variables – as many as you can in this size and depth.)

Problem 6. Prove the following lemma which we stated in class: Let $f : \{-1,1\}^n \to \{-1,1\}$ be any Boolean function. Let (I, z) be a ρ-random restriction. Then for any t such that $t\rho \geq 8$, we have

$$\sum_{|S|>t} \hat{f}(S)^2 \leq 2 \Pr_{(I,z)}[f_{\bar{I} \leftarrow z} \text{ has decision tree depth } > \rho t/2].$$

(Hint: Mimic the proof of the key lemma we used for Mansour's DNF Fourier concentration bound.)

Problem 7. In class we proved the following statement, which bounds the noise sensitivity of any LTF for "special" noise rates of the form 1/integer:

(*) Let $f : \{-1,1\}^n \to \{-1,1\}$ be an n-variable LTF. Let $\epsilon \in (0, 1/2)$ be of the form $\epsilon = 1/t$ where t is an integer. Then $NS_\epsilon(f) \leq \sqrt{\epsilon}$.

Prove the following bound on noise sensitivity for LTFs which holds for all values of ϵ:

Let $f : \{-1,1\}^n \to \{-1,1\}$ be an n-variable LTF. Let $\epsilon \in (0, 1/2)$. Then $NS_\epsilon(f) \leq 2\sqrt{\epsilon}$.

You may use anything we proved in class, including (*).

Problem 8. Prove the following: Let $f : \{0,1\}^n \to \{0,1\}$. Then

(i) f has a unique representation as a multilinear polynomial p_R over the reals. All the coefficients of this polynomial p_R are integers.

(ii) f has a unique representation as a multilinear $GF(2)$ polynomial $p_{GF(2)}$.

(iii) If $f, g : \{0,1\}^n \to \{0,1\}$ are two degree-d $GF(2)$ polynomials then $\Pr_{x \in 0,1^n}[f \neq g] \geq 2^{-d}$.

Problem 9. The quantity $\sum_{S \subseteq [n]} |\hat{f}(S)|$ is sometimes written $L_1(f)$ and is referred to as the "L_1 norm of f."

(i) Show that functions with small L_1 norm have almost all of their Fourier weight on few coefficients. More precisely, let $f : \{-1,1\}^n \to \{-1,1\}$ be such that

$$\sum_{S \subseteq [n]} |\hat{f}(S)| \leq L.$$

Let $\mathcal{S}_\epsilon = \{S \subseteq [n] : |\hat{f}(S)| \geq \frac{\epsilon}{L}\}$. Show that $|\mathcal{S}_\epsilon| \leq L^2/\epsilon$ and $\sum_{S \in \mathcal{S}_\epsilon} \hat{f}(S)^2 \geq 1 - \epsilon$.

(ii) Let f be a Boolean function that is computed by an s-leaf decision tree. Show that $L_1(f) \leq s$.