Online Clustering with Nearly Optimal Consistency

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  - **Competitive ratio:** the ratio between the algorithm's k-Means cost and the optimal k-Means cost (with full information).
  - **Lower bound**:No bounded competitive ratio [LSS16].

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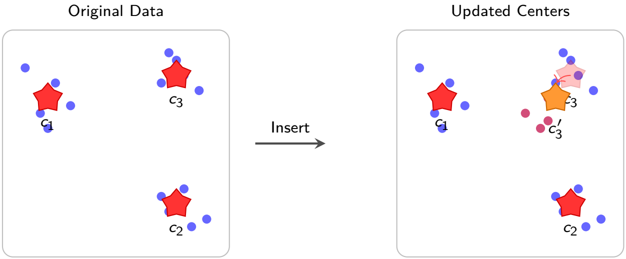

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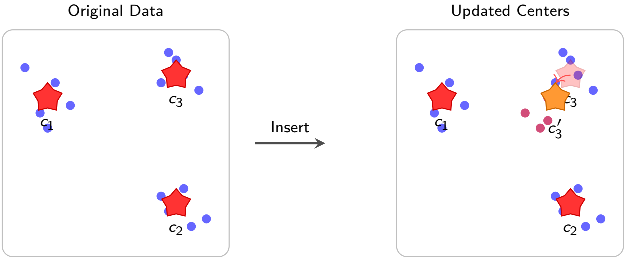

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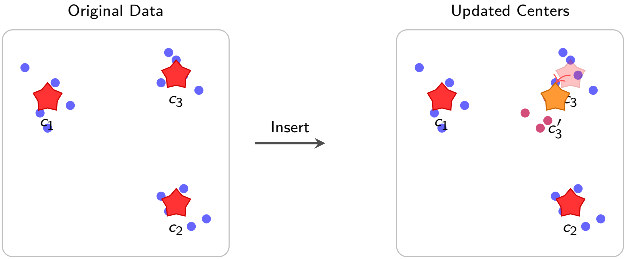

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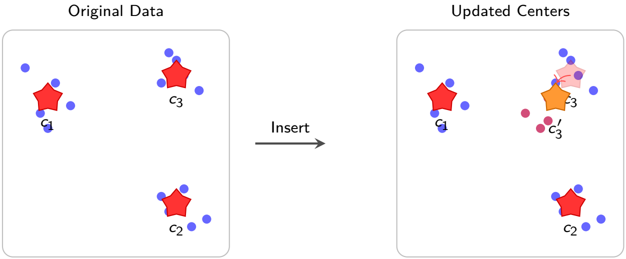

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- **Goal:** Minimizing the consistency while maintaining a bounded competitive ratio.
- **Lowerbound:** any $O(1)$-competitive algorithms for k-Means must be $\Omega(k \log(n))$-consistent [LV17].

Previous Work and Limitations

- Relative works:

Algs	Ratio	Consistency	Problems
[LV17]	$O(1)$	$O(k^2 \operatorname{polylog}(n))$	k-Means and k-Median
[FLNS21]	$O(1)$	$O(k \operatorname{polylog}(n))$	k-Median

- **Gaps:**
  - If k-Means also admits $O(1)$-competitive ratio with $O(k \operatorname{polylog}(n))$-consistency?
  - Moreover, $(1+\epsilon)$-competitive ratio (might require an exponential running time)?
  - A framework, turning any offline clustering algorithm into an online algorithm with good consistency?
- **Question:** Can we close these gaps?

Main Contributions

Theorem

Given an offline α-approximate algorithm for k-Means *that runs in $T(n)$ time, there exists an $\tilde{O}_\epsilon(k)$[a]-consistent $(1+\epsilon)\alpha^2$-competitive algorithm for online* k-Means, *and the running time is $\tilde{O}_\epsilon(nk + k^3 \cdot T(\tilde{O}_\epsilon(k)))$.*

[a] $\tilde{O}_\epsilon(k)$ hides poly$(\epsilon^{-1} d \log(n))$ factor.

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Algorithm	**Ratio**	**Consistency**
[LV17](k-Median, k-Means)	$O(1)$	$\tilde{O}(k^2)$
[FLNS21](k-Median)	$O(1)$	$\tilde{O}(k)$
Our work (k-Median, k-Means**)**	$1+\epsilon$	$\tilde{O}_\epsilon(k)$

Key Ideas

- Consistent Coreset Construction: reduce the number of points to be considered: from n points to $\tilde{O}_\epsilon(k)$ weighted points.

coreset

A weighted set $S \subseteq \mathbb{R}^d$ such that $\forall C \subseteq \mathbb{R}^d, |C| = k$,
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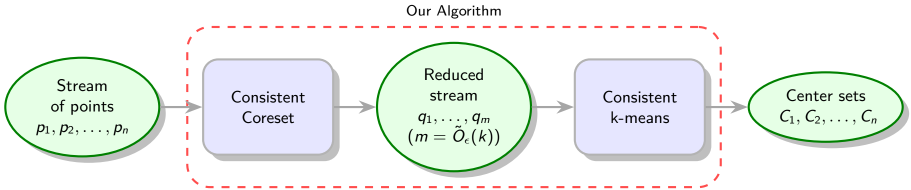

Experiment

Datasets

Dataset	Dimension	Size
SKIN [BD09]	3	245057
SHUTTLE [Cat]	7	58000
COVERTYPE [Bla98]	54	581012

Algorithms:

- Baseline1 (LV17): Algorithm in [LV17].
- Baseline2 (naive): Directly running k-Means++ on the consistent coreset instead of running any additional algorithm.
- Our algorithm (ours-faithful): plugging k-Means++ to our framework.
- A heuristic implementation of our algorithm (ours-heuristic): when a new point is added, it replaces only one existing center if doing so maximizes cost reduction.

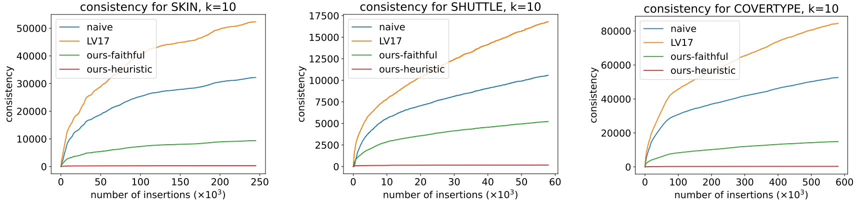

Figure: The consistency curve over the insertions of points, for all datasets and $k = 10$.

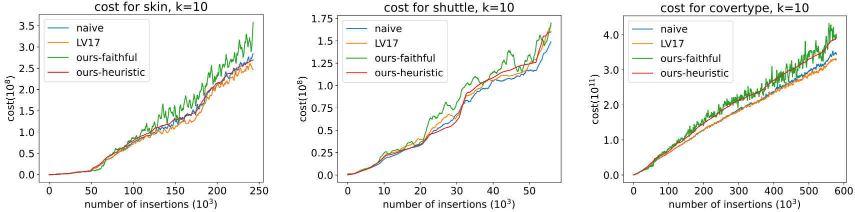

Figure: The cost curve over the insertions of points, for all datasets and $k = 10$. We plot the curve after applying a moving average with a window size equal to 1% of the dataset size.

Thank you!

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