18794 Pattern Recognition Theory
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Contents

1 Introduction

Back to Top

2 Decision Theory

2.1 Terms

- Feature Space:
  - **Feature**: a distinctive characteristic or quality of the object
  - **Feature vector**: combine more than one feature as a vector
  - **Feature space**: The space defined by the feature vectors
- Classifiers:
  - **Decision regions**: a classifier partitions the feature space into class-corresponding decision regions.
  - **Decision boundaries**: the borders between the decision regions.

2.2 Bayes Rule

$$P(w_i|x) = \frac{P(x, w_i)}{P(x)} = \frac{P(x|w_i)P(w_i)}{\sum_{k=1}^{C} P(x|w_k)P(w_k)} \tag{1}$$

- **Posterior Probability** $P(w_i|x)$: the conditional probability of correct class being w_i given that feature value x has been observed.
- **Evidence** $P(x)$: the total probability of observing the feature value of x.
- **Likelihood** $P(x|w_i)$: the conditional probability of observing a feature value of x given that the correct class is w_i.
- **Prior Probability** $P(w_i)$: the probability of class w_i, $\sum_{k=1}^{C} P(w_k) = 1$.
- **Bayes Classifiers** decide on the class that has the **largest posterior probability** ($\max_{w_i} P(w_i|x)$). They are statistically the best classifiers i.e. they are minimum error classifiers (optimal).

2.3 Minimum Probability of Error

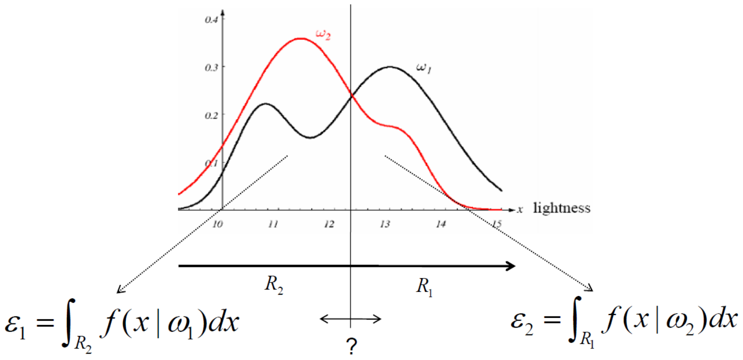
- $\epsilon = P(error|class)$: probability of assigning x to the wrong class w.
- $P_e = \sum_{k=1}^{C} P(w_k)\epsilon_k$: total probability of error.

For the two class case shown above, we want to minimize P_e as below:

$$\begin{array}{rcl} P_e & = & P(w_1)\epsilon_1 + P(w_2)\epsilon_2 \\ & = & P(w_1)\int_{R_2} f(x|w_1)dx + P(w_2)\int_{R_1} f(x|w_2)dx \\ & = & P(w_1)(1 - \int_{R_1} f(x|w_1)dx) + P(w_2)\int_{R_1} f(x|w_2)dx \\ & = & P(w_1) + \int_{R_1} (P(w_2)f(x|w_2) - P(w_1)f(x|w_1))dx \end{array} \tag{2}$$

To minimize P_e, we want $P(w_2)f(x|w_2) - P(w_1)f(x|w_1)$ to be always negative(< 0) in the region R_1:

$$\begin{array}{lcl} P(w_1)f(x|w_1) - P(w_2)f(x|w_2) > 0 & \Rightarrow & w_1 \\ P(w_1)f(x|w_1) - P(w_2)f(x|w_2) < 0 & \Rightarrow & w_2 \end{array} \tag{3}$$



2.4 Likelihood Ratio

- **Likelihood ratio**: $l(x) = \frac{f(x|w_1)}{f(x|w_2)}$
- **Log likelihood ratio**: $\ln(l(x)) = \ln(\frac{f(x|w_1)}{f(x|w_2)}) = \ln(f(x|w_1)) - \ln(f(x|w_2))$
- **Ratio of a priori probabilities**: $T = \frac{P(w_2)}{P(w_1)}$
- **Log ratio of a priori probabilities**: $\ln(T) = \ln(\frac{P(w_2)}{P(w_1)}) = \ln(P(w_2)) - \ln(P(w_1))$

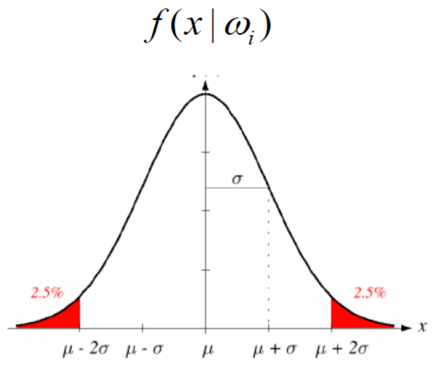
$$\begin{array}{ll} \ln(l(x)) = \ln(\frac{f(x|w_1)}{f(x|w_2)}) > \ln(\frac{P(w_2)}{P(w_1)}) = \ln(T) & \Rightarrow \quad w_1 \\ \ln(l(x)) = \ln(\frac{f(x|w_1)}{f(x|w_2)}) < \ln(\frac{P(w_2)}{P(w_1)}) = \ln(T) & \Rightarrow \quad w_2 \end{array} \tag{4}$$

2.4.1 Likelihood as Gaussian distribution

Assume likelihood $f(x|w_i)$ are Gaussian distributions with mean μ_i and variance σ_i^2.

$$f(x|w_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(x-\mu_i)^2}{2\sigma_i^2}\right) \tag{5}$$

The log likelihood ratio:



$$\begin{aligned} \ln(l(x)) &= \ln\left(\frac{\frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right)}{\frac{1}{\sqrt{2\pi\sigma_2^2}}\exp\left(-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right)}\right) \\ &= \ln(\frac{\sigma_2}{\sigma_1}) + \frac{(x-\mu_2)^2}{2\sigma_2^2} - \frac{(x-\mu_1)^2}{2\sigma_1^2} \end{aligned} \tag{6}$$

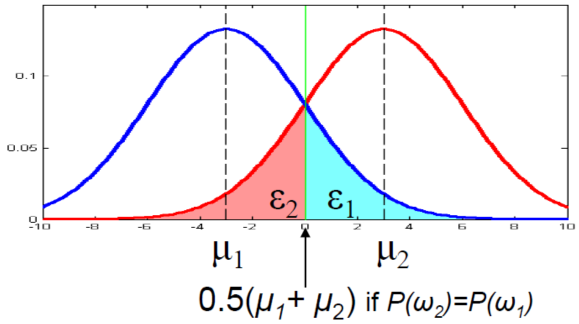
Case: $\sigma_1 = \sigma_2 = \sigma$

$$\ln(l(x)) \quad = \quad \tfrac{2x(\mu_1-\mu_2)-(\mu_1^2-\mu_2^2)}{2\sigma^2} \tag{7}$$

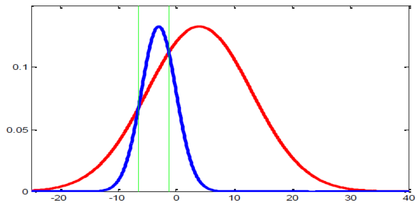
$$\begin{array}{l} x(\mu_1-\mu_2) - \frac{\mu_1^2-\mu_2^2}{2} > \sigma^2 \ln(\frac{P(w_2)}{P(w_1)}) \quad \Rightarrow \quad w_1 \\ x(\mu_1-\mu_2) - \frac{\mu_1^2-\mu_2^2}{2} < \sigma^2 \ln(\frac{P(w_2)}{P(w_1)}) \quad \Rightarrow \quad w_2 \end{array} \tag{8}$$

If $P(w_1) = P(w_2)$

$$x = \frac{\mu_1 + \mu_2}{2} \tag{9}$$



Case: $\sigma_1 \neq \sigma_2$



$$\ln\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2 - \frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2 \overset{\omega_1}{\underset{\omega_2}{\gtrless}} \ln\left(\frac{P(\omega_2)}{P(\omega_1)}\right)$$

$$ax^2 + bx + c \gtrless 0$$

$$(x - x_1)(x - x_2) \gtrless 0$$

Back to Top

3 Parametric & Non-Parametric Density Estimation

Learning & Classification

- Supervised:
  - Bayes Density:
    - Parametric: assumes a particular form of a PDF (e.g. Gaussian) is known → determine the parameters (e.g. mean and variance)

        - · Maximum Likelihood ($f(x|w_i)$) Estimation (MLE)
        - · Maximum A Posteriori (Bayesian $p(w_i|x)$) Estimation (MAPE)
      - ∗ Non-Parametric: no assumption about the density
        - · Parzen Windows
        - · Kernel Density Estimation
        - · K-Nearest Neighbor Rule
  - – Discriminant Analysis:
    - ∗ Linear
    - ∗ Non-Linear

- Unsupervised:
  - – Clustering

3.1 ML Estimation (MLE)

Steps:

- Assume $P(x|w)$ has a known parametric form uniquely determined by the parameter vector θ
- The parameters are assumed to be fixed (i.e. non random) but unknown
- Suppose we have a dataset D with the samples in D having been drawn **independently** according to the probability law $P(x|w)$
- The MLE is the value of θ that best explains the data and once we know this value, we know $P(x|w)$

$$\hat{\theta} = \arg\max_{\theta}\{P(D|\theta)\} = \arg\max_{\theta}\{\prod_{k=1}^{N} P(x_k|\theta)\} = \arg\max_{\theta}\{\sum_{k=1}^{N} \log(P(x_k|\theta))\} \tag{10}$$

Choose the value of θ that is the most likely to give rise to the data we observe.

Method:

- Assume $\theta = [\theta_1, \theta_2, \ldots, \theta_p]^T$
- Assume gradient operator $\nabla_\theta = [\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \ldots, \frac{\partial}{\partial\theta_p}]^T$
- Assume the log-likelihood of the objective function $l(\theta) = \sum_{k=1}^{N} \log(P(x_k|\theta))$
- Therefore: $\nabla_\theta l(\theta) = \sum_{k=1}^{N} \nabla_\theta \log(P(x_k|\theta)) = 0$

3.1.1 E.g. Univariate Gaussian

$$P(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2}) \tag{11}$$

$$\theta = \left[\begin{array}{c} \theta_1 = \mu \\ \theta_2 = \sigma^2 \end{array}\right] \tag{12}$$

$$\begin{aligned} \log(P(x_k|\theta)) &= \log(\tfrac{1}{\sqrt{2\pi\theta_2}} \exp(-\tfrac{(x_k-\theta_1)^2}{2\theta_2})) \\ &= -\tfrac{1}{2}\log(2\pi\theta_2) - \tfrac{(x_k-\theta_1)^2}{2\theta_2} \end{aligned} \tag{13}$$

$$\begin{array}{rcl} \nabla_\theta \log(P(x_k|\theta)) & = & \left[\begin{array}{c} \frac{x_k-\theta_1}{\theta_2} \\ -\frac{1}{2\theta_2} + \frac{(x_k-\theta_1)^2}{2\theta_2} \end{array} \right] \\ \Rightarrow \left\{ \begin{array}{rcl} \sum_{k=1}^{n} \frac{1}{\theta_2}(x_k-\theta_1) & = & 0 \\ -\sum_{k=1}^{n} \frac{1}{\theta_2} + \sum_{k=1}^{n} \frac{(x_k-\theta_1)^2}{\theta_2^2} & = & 0 \end{array} \right. & \Rightarrow & \left\{ \begin{array}{rcl} \hat{\mu} & = & \frac{1}{n}\sum_{k=1}^{n} x_k \\ \hat{\sigma}^2 & = & \frac{1}{n}\sum_{k=1}^{n} (x_k-\hat{\mu})^2 \end{array} \right. \end{array} \tag{14}$$

3.1.2 E.g. Multivariate Gaussian

$$P(x|\mu,\Sigma) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)) \tag{15}$$

Case: only the mean μ is unknown

$$\log(P(x_k|\mu)) = -\frac{1}{2}\log((2\pi)^d|\Sigma|) - \frac{1}{2}(x_k-\mu)^T\Sigma^{-1}(x_k-\mu) \tag{16}$$

$$\begin{array}{rcl} \nabla_\theta \log(P(x_k|\mu)) & = & \Sigma^{-1}(x_k-\mu) \\ \Rightarrow \sum_{k=1}^{n} \Sigma^{-1}(x_k-\mu) = 0 & \Rightarrow & \hat{\mu} = \frac{1}{n}\sum_{k=1}^{n} x_k \end{array} \tag{17}$$

Case: Neither the mean μ nor the covariance matrix Σ are known

$$\theta = \left[\begin{array}{ccc} \theta_1 & = & \mu \\ \theta_2 & = & \Sigma \end{array} \right] \tag{18}$$

Convert the covariance matrix Σ

$$\begin{array}{rcl} P(D|\Sigma) & = & \prod_{k=1}^{n} P(x_k|\Sigma) \\ & = & \prod_{k=1}^{n} \frac{1}{\sqrt{(2\pi)^d|\Sigma|}} \exp(-\frac{1}{2}(x_k-\mu)^T\Sigma^{-1}(x_k-\mu)) \\ & = & \frac{1}{((2\pi)^d|\Sigma|)^{n/2}} \exp(-\frac{1}{2}\sum_{n}^{k=1}(x_k-\mu)^T\Sigma^{-1}(x_k-\mu)) \end{array} \tag{19}$$

The property of matrix trace

- scalar $b^TBb = trace(b^TBb) = trace(Bbb^T)$
- $trace(A+B) = trace(A) + trace(B)$
- $trace(C(A+B)) = trace(CA) + trace(CB)$
- $trace(A)$ is the sum of A's eigenvalues

$$\begin{array}{rcl} \sum_{n}^{k=1}(x_k-\mu)^T\Sigma^{-1}(x_k-\mu) & = & \sum_{n}^{k=1} trace((x_k-\mu)^T\Sigma^{-1}(x_k-\mu)) \\ & = & \sum_{n}^{k=1} trace(\Sigma^{-1}(x_k-\mu)(x_k-\mu)^T) \\ & = & trace(\Sigma^{-1}\sum_{n}^{k=1}(x_k-\mu)(x_k-\mu)^T) \end{array} \tag{20}$$

Assume

$$A = \frac{1}{n}\sum_{k=1}^{n}(x_k-\mu)(x_k-\mu)^T \tag{21}$$

and A is fixed.

Then

$$\sum_{n}^{k=1} (x_k - \mu)^T \Sigma^{-1} (x_k - \mu) = n \; trace(\Sigma^{-1} A) \tag{22}$$

Therefore

$$P(D|\Sigma) = \frac{1}{((2\pi)^d |\Sigma|)^{n/2}} \exp(-\frac{n}{2} trace(\Sigma^{-1} A)) \tag{23}$$

Assume

$$B = \Sigma^{-1} A \tag{24}$$

and B's eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_d$

The property of matrix determinant

- $det(AB) = det(A)det(B)$
- $det(A^{-1}) = det(A)^{-1}$
- $det(A)$ is the product of A's eigenvalues

Therefore

$$\begin{aligned} P(D|\Sigma) &= \tfrac{1}{((2\pi)^d |\Sigma|)^{n/2}} \exp(-\tfrac{n}{2} trace(\Sigma^{-1} A)) \\ &= \tfrac{1}{(2\pi)^{dn/2}} |\Sigma|^{-n/2} \exp(-\tfrac{n}{2} trace(B)) \\ &= \tfrac{1}{(2\pi)^{dn/2}} (\tfrac{|B|}{|A|})^{n/2} \exp(-\tfrac{n}{2} trace(B)) \\ &= \tfrac{1}{(2\pi)^{dn/2}} |A|^{-n/2} (\textstyle\prod_{i=1}^{d} \lambda_i)^{n/2} \exp(-\tfrac{n}{2} \textstyle\sum_{i=1}^{d} \lambda_i) \end{aligned} \tag{25}$$

$$\log(P(D|\Sigma)) = -\frac{dn}{2} \log(2\pi) - \frac{n}{2} \log(|A|) + \frac{n}{2} \sum_{i=1}^{d} \log(\lambda_i) - \frac{n}{2} \sum_{i=1}^{d} \lambda_i \tag{26}$$

$$\frac{\partial(\log(P(x|\Sigma)))}{\partial \lambda_i} = \frac{n}{2\lambda_i} - \frac{n}{2} = 0 \Rightarrow \lambda_i = 1 \tag{27}$$

Therefore $B \sim I$, and $\Sigma^{-1} A = I \Rightarrow \hat{\Sigma} = A = \frac{1}{n} \sum_{k=1}^{n} (x_k - \mu)(x_k - \mu)^T$.

3.2 Bayesian Estimation (BE)

Steps:

- The parameters are assumed to be random variables with some known priori distribution.
- Bayesian approach aims at estimating the posterior density $P(\theta|D)$
- The MAPE(Maximum A Posteriori Estimate) of θ is the value of θ that maximizes the posterior density $P(\theta|D)$.

Method:

- Estimate $P(x|D)$ from sample data, and assume it has a known parametric form. So $P(x|\theta)$ is completely known, but θ is random variable and has its own PDF.
- $P(x|D) = \int P(x, \theta|D) d\theta = \int P(x|\theta) P(\theta|D) d\theta$
- Posterior $P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$, where $P(D|\theta) = \prod_{k=1}^{n} P(x_k|\theta)$
- Since $P(D)$ is constant, therefore $\hat{\theta} = \arg\max_\theta \{P(D|\theta)P(\theta)\}$

3.2.1 E.g. Univariate Gaussian

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3.2.2 E.g. Multivariate Gaussian

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Back to Top

4 Principle Component Analysis (PCA)

Purpose: get projection vectors ω that maximize the variance of the projected data.

$$\max_{\omega}(Var(\omega^T x)), \text{ where } ||\omega|| = 1 \tag{28}$$

Derivation: Assume the objective function

$$\begin{array}{rcl} J(\omega) & = & Var(\omega^T x) \\ & = & E(\omega^T (x-\mu)(x-\mu)^T \omega) \\ & = & \omega^T E((x-\mu)(x-\mu)^T)\omega \\ & = & \omega^T \Sigma \omega \quad (\text{where } ||\omega|| = 1) \end{array} \tag{29}$$

Then use Lagrangian optimization method to get the ω^*

$$\begin{array}{rcl} L(\omega, \lambda) & = & \omega^T \Sigma \omega - \lambda(\omega^T \omega - 1) \\ \frac{\partial L(\omega,\lambda)}{\partial \omega} & = & 2\Sigma\omega - 2\lambda\omega = 0 \\ & \Rightarrow & \Sigma\omega = \lambda\omega \end{array} \tag{30}$$

To maximize $J(\omega)$, we need to maximize $\omega^T \Sigma \omega = \lambda \to \lambda_{max}$