Cousework1-AGAT

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1

For the given bimatrix, no pure strategy is strictly dominated by other pure strategies. However, it is easily to check that there is no a pure NE of this Game G because this two players can unilaterally switch to other pure strategies to increase their own expected payoff. By using eliminate algorithm to eliminate those pure strategies weakly dominated by other pure strategies without being afraid of eliminating a pure NE. There is a 2×2 bimatrix left:

$$\begin{bmatrix} (7,13) & (17,12) \\ (9,5) & (7,14) \end{bmatrix} \tag{1}$$

Use the following corollary for NE:In a NE x^*, if $x_i^*(j) > 0$ then $U_i(x_{-i}^*; \pi_{ij})$=$U_i(x^*)$.
Let $x^* = (x_1^*, x_2^*)$ is a mixed NE for this 2×2 bmatrix ,where $x_1^* = \begin{bmatrix} x_1^*(1) \\ x_1^*(2) \end{bmatrix}$ and $x_2^* = \begin{bmatrix} x_2^*(1) \\ x_2^*(2) \end{bmatrix}$.
if $x_1^*(1) > 0$ then $U_1(\pi_{11}, x_2^*) = U_1(x^*)$
if $x_1^*(2) > 0$ then $U_1(\pi_{12}, x_2^*) = U_1(x^*)$

$$\sum_{j=1}^{2} x_2^*(j) u_1(1,j) = \sum_{j=1}^{2} x_2^*(j) u_1(2,j) \tag{2}$$

if $x_2^*(1) > 0$ then $U_2(x_1^*, \pi_{21}) = U_2(x^*)$.
if $x_2^*(2) > 0$ then $U_2(x_1^*, \pi_{22}) = U_2(x^*)$.

$$\sum_{i=1}^{2} x_1^*(i) u_2(i,1) = \sum_{i=1}^{2} x_1^* u_2(i,2) \tag{3}$$

Equation 2 and Equation 3 give following two linear equation:

$$\begin{cases} 7x_2^*(1) + 17x_2^*(2) = 9x_2^*(1) + 7x_2^*(2) \\ x_2^*(1) + x_2^*(2) = 1 \end{cases} \tag{4}$$

$$\begin{cases} 13x_1^*(1) + 5x_1^*(2) = 12x_1^*(1) + 14x_1^*(2) \\ x_1^*(1) + x_1^*(2) = 1 \end{cases} \tag{5}$$

Therefore, the solutions for the two linear equations are $x_2^* = (x_2^*(1) = \frac{5}{6}, x_2^*(2) = \frac{1}{6})$ and $x_1^* = (x_1^*(1) = \frac{9}{10}, x_1^*(2) = \frac{1}{10})$
Therefore the NE for this game G is $x^* = (x_1^* = (\frac{9}{10}, 0, \frac{1}{10}, 0), x_2^* = (0, 0, \frac{5}{6}, 0, \frac{1}{6}))$
I can prove the profile x^* is indeed an NE of Game G by the following claim: A profile $x^* = (x_1^*, x_2^*) \in X$ is a NE if and only if, for the two player $i \in \{1, 2\}$ and every pure strategies π_{ij},$U_i(x^*) \geq U_i(x_{-i}^*; \pi_{ij})$ For player 1, $U_1(x^*) = \frac{52}{6}$, $U_1(\pi_{11}, x_2^*) = U_1(x^*)$, $U_1(\pi_{12}, x_2^*) = \frac{44}{6} < U_1(x^*)$, $U_1(\pi_{13}, x_2^*) = U_1(x^*)$, $U_1(\pi_{14}, x_2^*) = \frac{37}{6} < U_1(x^*)$. For player 2, $U_2(x^*) = \frac{122}{10}$, $U_2(x_1^*, \pi_{21}) = \frac{86}{10} < U_2(x^*)$, $U_2(x_1^*, \pi_{22}) = \frac{92}{10} < U_2(x^*)$, $U_2(x_1^*, \pi_{23}) = U_2(x^*)$, $U_2(x_1^*, \pi_{24}) = \frac{121}{10} < U_2(x^*)$, $U_2(x_1^*, \pi_{25}) = U_2(x^*)$.
As I argue before, there is no pure Nash Equilibria in the Game G. However, Every finite game has a mixed Nash Equilibria. After using iterated SDS elimination to eliminate redundant pure strategies and the useful corollary, both Equation 4 and 5 only have one unique solution which indicates that there are no other pure or mixed NEs of this Game.

2

In the 2-player zero-sum game, the minmaximizer x_1^* and maxminimizer x_2^* strategies for player 1 and player. The following payoff matrix for player 1 is:

$$A = \begin{bmatrix} 0 & 4 & 3 & 0 & 5 \\ 5 & 7 & 9 & 4 & 3 \\ 3 & 9 & 0 & 9 & 4 \\ 8 & 9 & 11 & 3 & 3 \\ 7 & 1 & 4 & 0 & 9 \end{bmatrix} \tag{6}$$

Let $x_1^* = \begin{bmatrix} x_1^*(1) \\ x_1^*(2) \\ x_1^*(3) \\ x_1^*(4) \\ x_1^*(5) \end{bmatrix}$ be the minmaximizer for player 1 and $x_2^* = \begin{bmatrix} x_2^*(1) \\ x_2^*(2) \\ x_2^*(3) \\ x_2^*(4) \\ x_2^*(5) \end{bmatrix}$ be the maxminimizer for player 2. The linear programming for solving the minmaximizer for player 1 is:
Maximize v
subject to:

$$\begin{cases} 5x_1^*(2) + 3x_1^*(3) + 8x_1^*(4) + 7x_1^*(5) & \geq v \\ 4x_1^*(1) + 7x_1^*(2) + 9x_1^*(3) + 9x_1^*(4) + x_1^*(5) & \geq v \\ 3x_1^*(1) + 9x_1^*(2) + 11x_1^*(4) + 4x_1^*(5) & \geq v \\ 4x_1^*(2) + 9x_1^*(3) + 3x_1^*(4) & \geq v \\ 5x_1^*(1) + 3x_1^*(2) + 4x_1^*(3) + 3x_1^*(4) + 9x_1^*(5) & \geq v \\ \sum_{i=1}^{5} x_1^*(i) = 1 & \\ x_1^*(i) \geq 0 \quad i = 1, 2, 3, 4, 5 & \end{cases}$$

Define A' and incorporate v variable to x_1^*

$$A' = \begin{bmatrix} 0 & -5 & -3 & -8 & -7 & 1 \\ -4 & -7 & -9 & -9 & -1 & 1 \\ -3 & -9 & 0 & -11 & -4 & 1 \\ 0 & -4 & -9 & -3 & 0 & 1 \\ -5 & -3 & -4 & -3 & -9 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The above primal form can be re-written as:

Maximize $c^T x_1^*$, where $c^T = \begin{bmatrix} 0, 0, 0, 0, 0, 1 \end{bmatrix}$ and $x_1^* = \begin{bmatrix} x_1^*(1) \\ x_1^*(2) \\ x_1^*(3) \\ x_1^*(4) \\ x_1^*(5) \\ v \end{bmatrix}$

subject to:
$(A'x_1^*)_i \leq 0, i = 1, 2, 3, 4, 5$
$(A'x_1^*)_6 = 1$
$x_1^*(1), x_1^*(2), x_1^*(3), x_1^*(4), x_1^*(5) \geq 0$
By using general recipe for LP duals, the dual of the above linear programming is:
Minimize u
subject to:

$$\begin{cases} 4x_2^*(2) + 3x_2^*(3) + 5x_2^*(5) & \leq u \\ 5x_2^*(1) + 7x_2^*(2) + 9x_2^*(3) + 4x_2^*(4) + 3x_2^*(5) & \leq u \\ 3x_2^*(1) + 9x_2^*(2) + 9x_2^*(4) + 4x_2^*(5) & \leq u \\ 8x_2^*(1) + 9x_2^*(2) + 11x_2^*(3) + 3x_2^*(4) + 3x_2^*(5) & \leq u \\ 7x_2^*(1) + x_2^*(2) + 4x_2^*(3) + 9x_2^*(5) & \leq u \\ \sum_{j=1}^{5} x_2^*(j) = 1 & \\ x_2^*(j) \geq 0 \quad j = 1, 2, 3, 4, 5 & \end{cases}$$

minimaxmizer x_1^* guarantees player 1 at last expected payoff v^* and thus player 1 want to maximize v, whereas adversary find a vector $y = \begin{bmatrix} x_2^*(1) \\ x_2^*(2) \\ x_2^*(3) \\ x_2^*(4) \\ x_2^*(5) \\ u \end{bmatrix}$ such that $c^T \leq A'y$ and in turn $c^T x_1^* = v \leq y^T b = u$. In this case, the dual of the primal is constructed. Therefore player 2 wants to make the variable u as lower as possible. $x_2^* = \begin{bmatrix} x_2^*(1) \\ x_2^*(2) \\ x_2^*(3) \\ x_2^*(4) \\ x_2^*(5) \end{bmatrix}$ is a mixed strategy for player 2 which corresponds to the minimal value of u. Use the linear programming solver package to solve the two linear programming problems. The minmaximizer for player 1 is $x_1^* = (1.848767006616221e^{-17}, 1.550026755340816e^{-17}, 0.416184971098261, 0.352601156069346, 0.231213872832393)$ and the maxminimizer for player 2 is $x_2^* = (1.058418804270293e^{-12}, 4.769086337288972e^{-13}, 0.225433525807565, 0.341040462251992, 0.433526011938904)$. When player 1 plays minmaximizer and player 2 plays maxminimizer, they both achieve $v^* = 4.803468210618121$.

3

(a) Like Q1, the following 2×2 bimatrix: $\begin{bmatrix} (1,-1) & (-1,1) \\ (-1,1) & (1,-1) \end{bmatrix}$ also does not have pure NE. Following the same procedure for computing the mixed NE of game G in Q1, I got the following two linear equations:

$$\begin{cases} x_2^*(1) - x_2^*(2) = -x_2^*(1) + x_2^*(2) \\ x_2^*(1) + x_2^*(2) = 1 \end{cases} \tag{7}$$

$$\begin{cases} -x_1^*(1) + x_1^*(2) = x_1^*(1) - x_1^*(2) \\ x_1^*(1) + x_1^*(2) = 1 \end{cases} \tag{7}$$

Therefore, the NE for the 2-player zero sum game is $x^* = (x_1^* = (\frac{1}{2}, \frac{1}{2}), x_2^* = (\frac{1}{2}, \frac{1}{2}))$

(b) The code is provided below and this code is written in Python. However, the following code is just to provide one of situation for the Matching Pennies game. More specifically, player 1 plays "Head" and player 2 plays "Tail" at the beginning. In addition, player 1 and player 2 always choose Head and Tail to break the tie. This template can be modified to verify the conclusion for any situation. **Other outcomes under different situations can be found at the end of the coursework.**

Matching Pennies

```
N1_init=1
M1_init=0
N2_init=0
M2_init=1
N1=1
M1=0
N2=0
M2=1
mixied_st_11=N1_init/(N1_init+M1_init)
mixied_st_12=M1_init/(N1_init+M1_init)
mixied_st_21=N2_init/(N2_init+M2_init)
mixied_st_22=M2_init/(N2_init+M2_init)
for Round in range(0,50):
    U2_1=mixied_st_11*(-1)+mixied_st_12
```

```
    U2_2=mixied_st_11+mixied_st_12*(-1)
    if U2_1 > U2_2:
        N2+=1
    elif U2_1<U2_2:
        M2+=1
    else:          #mxiedstradgy=(1/2,1/2)
        N2+=1           #  player 2 always choose Head
    U1_1=mixied_st_21+mixied_st_22*(-1)
    U1_2=mixied_st_21*(-1)+mixied_st_22
    if U1_1 > U1_2:
        N1+=1
    elif U1_1< U1_2:
        M1+=1
    else:
        N1+=1    #player 1 always choose Head
    mixied_st_11=N1/(N1+M1)
    mixied_st_12=M1/(N1+M1)
    mixied_st_21=N2/(N2+M2)
    mixied_st_22=M2/(N2+M2)
print('The mixed strategy  for \
player 1 is:',(mixied_st_11,mixied_st_12))
print('The mixed strategy  for \
player 2 is:',(mixied_st_21,mixied_st_22))
```

I experimentally set the round to 2,10,50,100,500,1000,10000,100000 and 1000000. The output is showed at the end of the coursework. There are 16 situations I need to consider.
From Figure 1 to Figure 16,for all possible start strategies of both players and different breaking tie rules(player 1 and player 2 randomly select Head or Tail at that moment), the "statistical mixed strategies" of the two players looks like it is converging to their NE strategies.

4

(a)
First, I prove the fact that a linear system of inequalities $Ax \leq b$ is feasible $\Longrightarrow \neg\exists y, y \geq 0$ and $y^T A = 0$ s.t. $y^T b<0$ by contradiction.
Firstly, by multiplying $y^T>0$ for both sides of inequalities, $(y^T A)x \leq y^T b$ holds. From the condition $y^T A = 0$, $0 \leq y^T b$ which contradicts with $y^T b<0$. Therefore the condition is wrong and there is no vector y satisfying y ≥ 0 and $y^T A = 0$ s.t. $y^T b<0$.

For the other direction, based on Propositional equivalence the following statement is needed to be proved: $Ax \leq b$ is infeasible $\Longrightarrow \exists y, y \geq 0$ and $y^T A = 0$ s.t. $y^T b<0$.

For the linear systems $Ax \leq b$, Fourier-Motzkin Elimination can give an equivalent system $A'x' \leq b$ where $A' = MA$ and $b' = Mb$. This is because once one variable is eliminated, it is equivalent to pre-multiplying a given system of linear inequalities by a non-negative matrix M. In order to finish the proof, the following lemma will be used: Let $Q = \{(x_1, \ldots, x_n) : Ax \leq b\}$ we can construct $Q' = \{(x_1', \ldots, x_{n-1}') : A'x' \leq b'\}$ satisfying:
(1)Q is non-empty if and only if $Q^{'}$ is non-empty
(2)Every inequality defining $Q^{'}$ is a non-negative linear combination of the inequalities defining Q.
Proof (1)$\exists x = (x_1 \cdots x_n) \in R^n s.t \text{Ax} \leq b$. $x' = (x_1' \cdots x_{n-1}')$ denotes the first $n-1$ term of x, each element in vector a_k corresponds to the element in kth row of A. Put inequalities of Q in three groups: $Z = \{i \mid a_{in} = 0\}, P = \{j \mid a_{jn} = 1\}, N = \{k \mid a_{kn} = -1\}$. $a_k x - b_k = a_k{}'x' + a_{kn}x_n - b_k = a_k{}'x' - x_n - b_k \leq 0$ for $\forall k \in N$. Therefore $a_k'x' \leq x_n + b_k = b_k'$. Similarly, $a_j x - b_j = a_j'x' + a_{jn}x_n - b_j = a_j'x' + x_n - b_j \leq 0$ for $\forall j \in P$. Therefore $a_j'x' \leq b_j - x_n = b_j'$. Another direction can be proved via reversing the proof above.Although (2) will not be used in this proof, it is easy to check that $a_i'x' \leq b_i$ for $\forall i \in Z$ and $a_j'x' + a_k'x' \leq b_j + b_k$. for $\forall j \in P$ and for $\forall k \in N$.
Therefore, if $Ax \leq b$ is infeasible, then $A'x' \leq b'$ is also infeasible. By induction, $A'x' \leq b'$ is infeasible $\Longrightarrow \exists y', y' \geq 0$ and $y'^T A' = 0$ s.t. $y'^T b' {<} 0$ where $x' = (x_1 \ldots x_{n-1})$. Let see situation: $x = (x_1, \ldots x_n)$. Define $y = M^T y'$. Then $y > 0$ because of $y' > 0$ and non-negative matrix M. $y^T A = (M^T y^{'})^T A = y^{'T} MA = y'^T (MA) = y'^T A' = 0$. Similarly, $y^T b = y'^T Mb = y'^T b^{'} \leq 0$. Finally, for $x = (x_1, \ldots x_n)$ the statement also holds.
(b)
The Primal problem is: **Maximize** $2x_1 - x_2$
subject to:

$$\begin{cases} x_1 - x_2 \leq 1 \\ -x_1 + x_2 \leq -2 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

The dual problem is:
Minimize $y_1 - 2y_2$
subject to:

$$\begin{cases} y_1 - y_2 \geq 2 \\ y_2 - y_1 \geq -1 \\ y_1 \geq 0, y_2 \geq 0 \end{cases}$$

The infeasibility can be easily checked by visualizing those constraints for both primal and dual problem. Apparently, There are no vectors $x = (x_1, x_2)$ that satisfies every constraints for primal problem and $y = (y_1, y_2)$ that satisfies every constraints for the dual problem. Therefore, for both primal and dual problem, there are not feasible solutions.

```
At the beginning of the game, player 1 has number of Head:1,Tail0
At the beginning of the game, player 2 has number of Head:1,Tail0
When there is a break tie, player 1 always choose Head and  player 2 always choose Head
*****The round is:************ 2
The mixed strategy  for player 1 is:(1.000,0.000)
The mixed strategy  for player 2 is:(0.333,0.667)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.364,0.636)
The mixed strategy  for player 2 is:(0.545,0.455)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.569,0.431)
The mixed strategy  for player 2 is:(0.529,0.471)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.554,0.446)
The mixed strategy  for player 2 is:(0.515,0.485)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.495,0.505)
The mixed strategy  for player 2 is:(0.473,0.527)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.484,0.516)
The mixed strategy  for player 2 is:(0.505,0.495)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.507,0.493)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.501,0.499)
The mixed strategy  for player 2 is:(0.498,0.502)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.501,0.499)
The mixed strategy  for player 2 is:(0.500,0.500)
```

Figure 1: At the beginning, player1 choose Head, play2 choose Head. When there is tie, player 1 choose Head and player 2 choose Head

```
At the beginning of the game, player 1 has number of Head:1,Tail0
At the beginning of the game, player 2 has number of Head:1,Tail0
When there is a tie, player 1 always choose Head and  player 2 always choose Tail
*****The round is:************ 2
The mixed strategy  for player 1 is:(1.000,0.000)
The mixed strategy  for player 2 is:(0.333,0.667)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.273,0.727)
The mixed strategy  for player 2 is:(0.455,0.545)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.490,0.510)
The mixed strategy  for player 2 is:(0.588,0.412)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.495,0.505)
The mixed strategy  for player 2 is:(0.564,0.436)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.523,0.477)
The mixed strategy  for player 2 is:(0.491,0.509)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.493,0.507)
The mixed strategy  for player 2 is:(0.485,0.515)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.493,0.507)
The mixed strategy  for player 2 is:(0.500,0.500)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.502,0.498)
The mixed strategy  for player 2 is:(0.501,0.499)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.501,0.499)
```

Figure 2: At the beginning, player1 choose Head, play2 choose Head. When there is tie, player 1 choose Head and player 2 choose Tail

```
At the beginning of the game, player 1 has number of Head:1,Tail0
At the beginning of the game, player 2 has number of Head:1,Tail0
When there is a tie, player 1 always choose Tail and  player 2 always choose Head
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.667,0.333)
The mixed strategy  for player 2 is:(0.333,0.667)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.545,0.455)
The mixed strategy  for player 2 is:(0.727,0.273)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.549,0.451)
The mixed strategy  for player 2 is:(0.451,0.549)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.525,0.475)
The mixed strategy  for player 2 is:(0.455,0.545)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.473,0.527)
The mixed strategy  for player 2 is:(0.505,0.495)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.505,0.495)
The mixed strategy  for player 2 is:(0.517,0.483)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.507,0.493)
The mixed strategy  for player 2 is:(0.501,0.499)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.498,0.502)
The mixed strategy  for player 2 is:(0.499,0.501)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.499,0.501)
```

Figure 3: At the beginning, player1 choose Head, play2 choose Head. When there is tie, player 1 choose Tail and player 2 choose Head

```
At the beginning of the game, player 1 has number of Head:1,Tail0
At the beginning of the game, player 2 has number of Head:1,Tail0
When there is a tie, player 1 always choose Tail and  player 2 always choose Tail
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.667,0.333)
The mixed strategy  for player 2 is:(0.333,0.667)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.364,0.636)
The mixed strategy  for player 2 is:(0.636,0.364)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.569,0.431)
The mixed strategy  for player 2 is:(0.529,0.471)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.554,0.446)
The mixed strategy  for player 2 is:(0.515,0.485)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.493,0.507)
The mixed strategy  for player 2 is:(0.475,0.525)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.484,0.516)
The mixed strategy  for player 2 is:(0.506,0.494)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.507,0.493)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.501,0.499)
The mixed strategy  for player 2 is:(0.498,0.502)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.501,0.499)
The mixed strategy  for player 2 is:(0.500,0.500)
```

Figure 4: At the beginning, player1 choose Head, play2 choose Head. When there is tie, player 1 choose Tail and player 2 choose Tail

```
At the beginning of the game, player 1 has number of Head:1,Tail0
At the beginning of the game, player 2 has number of Head:0,Tail1
When there is a tie, player 1 always choose Head and  player 2 always choose Head
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.333,0.667)
The mixed strategy  for player 2 is:(0.333,0.667)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.727,0.273)
The mixed strategy  for player 2 is:(0.455,0.545)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.451,0.549)
The mixed strategy  for player 2 is:(0.451,0.549)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.455,0.545)
The mixed strategy  for player 2 is:(0.475,0.525)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.505,0.495)
The mixed strategy  for player 2 is:(0.527,0.473)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.517,0.483)
The mixed strategy  for player 2 is:(0.495,0.505)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.501,0.499)
The mixed strategy  for player 2 is:(0.493,0.507)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.499,0.501)
The mixed strategy  for player 2 is:(0.502,0.498)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.499,0.501)
The mixed strategy  for player 2 is:(0.500,0.500)
```

Figure 5: At the beginning, player1 choose Head, play2 choose Tail. When there is tie, player 1 choose Head and player 2 choose Head

```
At the beginning of the game, player 1 has number of Head:1,Tail0
At the beginning of the game, player 2 has number of Head:0,Tail1
When there is a tie, player 1 always choose Head and  player 2 always choose Tail
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.333,0.667)
The mixed strategy  for player 2 is:(0.000,1.000)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.545,0.455)
The mixed strategy  for player 2 is:(0.636,0.364)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.529,0.471)
The mixed strategy  for player 2 is:(0.431,0.569)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.515,0.485)
The mixed strategy  for player 2 is:(0.446,0.554)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.473,0.527)
The mixed strategy  for player 2 is:(0.505,0.495)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.505,0.495)
The mixed strategy  for player 2 is:(0.516,0.484)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.507,0.493)
The mixed strategy  for player 2 is:(0.500,0.500)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.498,0.502)
The mixed strategy  for player 2 is:(0.499,0.501)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.499,0.501)
```

Figure 6: At the beginning, player1 choose Head, play2 choose Tail. When there is tie, player 1 choose Head and player 2 choose Tail

```
At the beginning of the game, player 1 has number of Head:1,Tail0
At the beginning of the game, player 2 has number of Head:0,Tail1
When there is a tie, player 1 always choose Tail and  player 2 always choose Head
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.333,0.667)
The mixed strategy  for player 2 is:(0.333,0.667)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.636,0.364)
The mixed strategy  for player 2 is:(0.636,0.364)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.529,0.471)
The mixed strategy  for player 2 is:(0.431,0.569)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.515,0.485)
The mixed strategy  for player 2 is:(0.446,0.554)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.475,0.525)
The mixed strategy  for player 2 is:(0.507,0.493)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.506,0.494)
The mixed strategy  for player 2 is:(0.516,0.484)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.507,0.493)
The mixed strategy  for player 2 is:(0.500,0.500)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.498,0.502)
The mixed strategy  for player 2 is:(0.499,0.501)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.499,0.501)
```

Figure 7: At the beginning, player1 choose Head, play2 choose Tail. When there is tie, player 1 choose Tail and player 2 choose Head

```
At the beginning of the game, player 1 has number of Head:1,Tail0
At the beginning of the game, player 2 has number of Head:0,Tail1
When there is a tie, player 1 always choose Tail and  player 2 always choose Tail
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.333,0.667)
The mixed strategy  for player 2 is:(0.000,1.000)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.455,0.545)
The mixed strategy  for player 2 is:(0.727,0.273)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.588,0.412)
The mixed strategy  for player 2 is:(0.510,0.490)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.564,0.436)
The mixed strategy  for player 2 is:(0.505,0.495)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.491,0.509)
The mixed strategy  for player 2 is:(0.477,0.523)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.485,0.515)
The mixed strategy  for player 2 is:(0.507,0.493)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.507,0.493)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.501,0.499)
The mixed strategy  for player 2 is:(0.498,0.502)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.501,0.499)
The mixed strategy  for player 2 is:(0.500,0.500)
```

Figure 8: At the beginning, player1 choose Head, play2 choose Tail. When there is tie, player 1 choose Tail and player 2 choose Tail

```
At the beginning of the game, player 1 has number of Head:0,Tail1
At the beginning of the game, player 2 has number of Head:1,Tail0
When there is a tie, player 1 always choose Head and  player 2 always choose Head
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.667,0.333)
The mixed strategy  for player 2 is:(1.000,0.000)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.545,0.455)
The mixed strategy  for player 2 is:(0.273,0.727)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.412,0.588)
The mixed strategy  for player 2 is:(0.490,0.510)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.436,0.564)
The mixed strategy  for player 2 is:(0.495,0.505)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.509,0.491)
The mixed strategy  for player 2 is:(0.523,0.477)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.515,0.485)
The mixed strategy  for player 2 is:(0.493,0.507)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.493,0.507)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.499,0.501)
The mixed strategy  for player 2 is:(0.502,0.498)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.499,0.501)
The mixed strategy  for player 2 is:(0.500,0.500)
```

Figure 9: At the beginning, player1 choose Tail, play2 choose Head. When there is tie, player 1 choose Head and player 2 choose Head

```
At the beginning of the game, player 1 has number of Head:0,Tail1
At the beginning of the game, player 2 has number of Head:1,Tail0
When there is a tie, player 1 always choose Head and  player 2 always choose Tail
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.667,0.333)
The mixed strategy  for player 2 is:(0.667,0.333)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.364,0.636)
The mixed strategy  for player 2 is:(0.364,0.636)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.471,0.529)
The mixed strategy  for player 2 is:(0.569,0.431)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.485,0.515)
The mixed strategy  for player 2 is:(0.554,0.446)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.525,0.475)
The mixed strategy  for player 2 is:(0.493,0.507)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.494,0.506)
The mixed strategy  for player 2 is:(0.484,0.516)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.493,0.507)
The mixed strategy  for player 2 is:(0.500,0.500)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.502,0.498)
The mixed strategy  for player 2 is:(0.501,0.499)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.501,0.499)
```

Figure 10: At the beginning, player1 choose Tail, play2 choose Head. When there is tie, player 1 choose Head and player 2 choose Tail

```
At the beginning of the game, player 1 has number of Head:0,Tail1
At the beginning of the game, player 2 has number of Head:1,Tail0
When there is a tie, player 1 always choose Tail and  player 2 always choose Head
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.667,0.333)
The mixed strategy  for player 2 is:(1.000,0.000)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.455,0.545)
The mixed strategy  for player 2 is:(0.364,0.636)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.471,0.529)
The mixed strategy  for player 2 is:(0.569,0.431)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.485,0.515)
The mixed strategy  for player 2 is:(0.554,0.446)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.527,0.473)
The mixed strategy  for player 2 is:(0.495,0.505)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.495,0.505)
The mixed strategy  for player 2 is:(0.484,0.516)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.493,0.507)
The mixed strategy  for player 2 is:(0.500,0.500)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.502,0.498)
The mixed strategy  for player 2 is:(0.501,0.499)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.501,0.499)
```

Figure 11: At the beginning, player1 choose Tail, play2 choose Head. When there is tie, player 1 choose Tail and player 2 choose Head

```
At the beginning of the game, player 1 has number of Head:0,Tail1
At the beginning of the game, player 2 has number of Head:1,Tail0
When there is a tie, player 1 always choose Tail and  player 2 always choose Tail
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.667,0.333)
The mixed strategy  for player 2 is:(0.667,0.333)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.273,0.727)
The mixed strategy  for player 2 is:(0.545,0.455)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.549,0.451)
The mixed strategy  for player 2 is:(0.549,0.451)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.545,0.455)
The mixed strategy  for player 2 is:(0.525,0.475)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.495,0.505)
The mixed strategy  for player 2 is:(0.473,0.527)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.483,0.517)
The mixed strategy  for player 2 is:(0.505,0.495)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.499,0.501)
The mixed strategy  for player 2 is:(0.507,0.493)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.501,0.499)
The mixed strategy  for player 2 is:(0.498,0.502)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.501,0.499)
The mixed strategy  for player 2 is:(0.500,0.500)
```

Figure 12: At the beginning, player1 choose Tail, play2 choose Head. When there is tie, player 1 choose Tail and player 2 choose Tail

```
At the beginning of the game, player 1 has number of Head:0,Tail1
At the beginning of the game, player 2 has number of Head:0,Tail1
When there is a tie, player 1 always choose Head and  player 2 always choose Head
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.333,0.667)
The mixed strategy  for player 2 is:(0.667,0.333)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.636,0.364)
The mixed strategy  for player 2 is:(0.364,0.636)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.431,0.569)
The mixed strategy  for player 2 is:(0.471,0.529)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.446,0.554)
The mixed strategy  for player 2 is:(0.485,0.515)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.507,0.493)
The mixed strategy  for player 2 is:(0.525,0.475)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.516,0.484)
The mixed strategy  for player 2 is:(0.494,0.506)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.493,0.507)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.499,0.501)
The mixed strategy  for player 2 is:(0.502,0.498)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.499,0.501)
The mixed strategy  for player 2 is:(0.500,0.500)
```

Figure 13: At the beginning, player1 choose Tail, play2 choose Tail. When there is tie, player 1 choose Head and player 2 choose Head

```
At the beginning of the game, player 1 has number of Head:0,Tail1
At the beginning of the game, player 2 has number of Head:0,Tail1
When there is a tie, player 1 always choose Head and  player 2 always choose Tail
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.333,0.667)
The mixed strategy  for player 2 is:(0.667,0.333)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.455,0.545)
The mixed strategy  for player 2 is:(0.273,0.727)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.451,0.549)
The mixed strategy  for player 2 is:(0.549,0.451)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.475,0.525)
The mixed strategy  for player 2 is:(0.545,0.455)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.527,0.473)
The mixed strategy  for player 2 is:(0.495,0.505)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.495,0.505)
The mixed strategy  for player 2 is:(0.483,0.517)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.493,0.507)
The mixed strategy  for player 2 is:(0.499,0.501)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.502,0.498)
The mixed strategy  for player 2 is:(0.501,0.499)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.501,0.499)
```

Figure 14: At the beginning, player1 choose Tail, play2 choose Tail. When there is tie, player 1 choose Head and player 2 choose Tail

```
At the beginning of the game, player 1 has number of Head:0,Tail1
At the beginning of the game, player 2 has number of Head:0,Tail1
When there is a tie, player 1 always choose Tail and  player 2 always choose Head
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.000,1.000)
The mixed strategy  for player 2 is:(0.667,0.333)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.727,0.273)
The mixed strategy  for player 2 is:(0.545,0.455)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.510,0.490)
The mixed strategy  for player 2 is:(0.412,0.588)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.505,0.495)
The mixed strategy  for player 2 is:(0.436,0.564)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.477,0.523)
The mixed strategy  for player 2 is:(0.509,0.491)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.507,0.493)
The mixed strategy  for player 2 is:(0.515,0.485)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.507,0.493)
The mixed strategy  for player 2 is:(0.500,0.500)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.498,0.502)
The mixed strategy  for player 2 is:(0.499,0.501)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.499,0.501)
```

Figure 15: At the beginning, player1 choose Tail, play2 choose Tail. When there is tie, player 1 choose Tail and player 2 choose Head

```
At the beginning of the game, player 1 has number of Head:0,Tail1
At the beginning of the game, player 2 has number of Head:0,Tail1
When there is a tie, player 1 always choose Tail and  player 2 always choose Tail
*****The round is:************ 2
The mixed strategy  for player 1 is:(0.000,1.000)
The mixed strategy  for player 2 is:(0.667,0.333)
*****The round is:************ 10
The mixed strategy  for player 1 is:(0.636,0.364)
The mixed strategy  for player 2 is:(0.455,0.545)
*****The round is:************ 50
The mixed strategy  for player 1 is:(0.431,0.569)
The mixed strategy  for player 2 is:(0.471,0.529)
*****The round is:************ 100
The mixed strategy  for player 1 is:(0.446,0.554)
The mixed strategy  for player 2 is:(0.485,0.515)
*****The round is:************ 500
The mixed strategy  for player 1 is:(0.505,0.495)
The mixed strategy  for player 2 is:(0.527,0.473)
*****The round is:************ 1000
The mixed strategy  for player 1 is:(0.516,0.484)
The mixed strategy  for player 2 is:(0.495,0.505)
*****The round is:************ 10000
The mixed strategy  for player 1 is:(0.500,0.500)
The mixed strategy  for player 2 is:(0.493,0.507)
*****The round is:************ 100000
The mixed strategy  for player 1 is:(0.499,0.501)
The mixed strategy  for player 2 is:(0.502,0.498)
*****The round is:************ 1000000
The mixed strategy  for player 1 is:(0.499,0.501)
The mixed strategy  for player 2 is:(0.500,0.500)
```

Figure 16: At the beginning, player1 choose Tail, play2 choose Tail. When there is tie, player 1 choose Tail and player 2 choose Tail