Multicommodity Flow on Connection Graph in Time Schedule Networks

I. All-or-Nothing Splittable Multi-Commodity Flow

In this section, we will describe in details of the proposed random rounding based approach for solving the multi-commodity flow problem.

Formulation 1: Maximize the total number of commodities that can be successfully delivered.

We use a variable f_i to represent whether or not F_i is successfully delivered. For the sake of simplicity, we rescale each commodity size and edge capacity in the network by s so that this problem thus becomes an unit flow multi-commodity problem.

$$\begin{aligned}
&\text{Input: Directed graph } G(V,E), \text{Commodities } \mathcal{F}=\{F_1,...,F_n\}\\
&\text{Source-sink pair: } (sv_i,dv_i), \qquad \forall F_i \in \mathcal{F}\\
&\text{Capacity: } c(u,v), \qquad \forall (u,v)\in E\\
&\text{Variable: Flow } f_{i,(u,v)}, \qquad \forall F_i \in \mathcal{F}, (u,v)\in E\\
&\qquad f_i, \qquad \forall F_i \in \mathcal{F}
\end{aligned}$$

$$\begin{aligned}
\text{Maximize } & \sum_{i=1}^{n} f_i\\
\text{Subject to } & \sum_{(sv_i,v)\in E} f_{i,(sv_i,v)} = f_i, \quad \forall F_i \in \mathcal{F}\\
& \sum_{(u,v)\in E} f_{i,(u,v)} = \sum_{(v,u)\in E} f^i_{v,u},\\
& \forall F_i \in \mathcal{F},\ \forall v \in V-\{sv_i,dv_i\}\\
& \sum_{i=1}^{f} f_{i,(u,v)} \le c_{(u,v)}, \quad \forall (u,v)\in E\\
& f^i_{(u,v)} \ge 0, \quad \forall F_i \in \mathcal{F}, \forall (u,v)\in E\\
& f_{i,(u,v)} \le f_i \cdot c_{(u,v)}, \quad \forall F_i \in \mathcal{F}, \forall (u,v)\in E\\
& f_i \in \{0,1\}, \quad \forall F_i \in \mathcal{F}
\end{aligned}$$

Let OPT_i be the optimal solution of the IP in Formulation 1, and let OPT_f be the total amount of commodities delivered by solving the linear relaxation LP of IP in Formulation 1 where the variables f_i are relaxed to assume any value in $[0,1]$. It is obvious to see that $OPT_f = \sum \tilde{f}_i$. Define the solution from the above algorithm as ALG, and total amount of commodities delivered by ALG as OPT_{ALG}.

We use Chernoff Bound over continuous random variables to bound the probability of achieving a fraction of the optimal solution.

Algorithm 1 Random Rounding Algorithm for Formulation 1

Input:
Directed Connection Graph $G(V,E)$;
Commodities $\mathcal{F}=\{F_1,...,F_n\}$;
Source-sink pair (sv_i,dv_i) for each $F_i \in \mathcal{F}$;
Capacity $c(u,v)$, $\forall (u,v)\in E$;

Output: OPT_{ALG}

1: Change the last constraint to be $0 \le f_i \le 1$;
2: Then it is relaxed to an LP, solve this LP and get optimal solution $\tilde{f}_i$;
3: With probability $\tilde{f}_i$, set $f_i = 1$, otherwise set it to 0;
4: Scale up the fractional flow $\tilde{f_{i,e}}$ from the LP solution on edge e for commodity i by $\frac{1}{\tilde{f}_i}$, i.e., $f_{i,e} = \tilde{f_{i,e}} \times \frac{1}{\tilde{f}_i}$, for i s.t. $f_i = 1$;
5: If the solution is within a certain fraction of the optimal solution, return this solution, otherwise, repeat step 3 and 4, at most N times.

Fact 1 (Chernoff-Bound) . Let $X = \sum_{i=1}^{n} X_i$ be a sum of n independent random variables $X_i \in [0,1], 1 \le i \le n$. Then $P(X < (1-\epsilon)\cdot E(X)) \le exp(-\epsilon^2 \cdot E(X)/2)$ holds for $0<\epsilon<1$.

Since we have already scaled down the flow by s_r, thus the variables are between 0 and 1 so that we can apply the above Chernoff Bound.

Claim 2. $Pr[OPT_{ALG} < (1-\epsilon)\cdot OPT_f] \le e^{-\epsilon^2 \cdot OPT_f/2}$

Proof: For each commodity i, the expectation of f_i is $E(f_i) = 1\cdot\tilde{f}_i + 0\cdot(1-\tilde{f}_i) = \tilde{f}_i$. Recall that $OPT_f = \sum \tilde{f}_i$, and let $OPT_{ALG} = \sum f_i$.

$$\begin{aligned}
&Pr[OPT_{ALG} < (1-\epsilon)\cdot OPT_f] \qquad (1)\\
&= Pr[\sum f_i < (1-\epsilon)\cdot OPT_f]\\
&\le e^{-\epsilon^2\cdot OPT_f/2}
\end{aligned}$$ □

Since we also assume that all commodity can be fully fractionally delivered alone, therefore, we get:

$$OPT_f \ge 1 \qquad (2)$$

Since we have proved that the optimal fractional solution is an upper bound on optimal solution for the ILP problem, by taking $\epsilon = 2/3$ in the Chernoff Bound, we get,

Theorem 3. The probability of achieving less than 1/3 of the profit of an optimal solution is upper bounded by $e^{-2/9} \approx 0.8007$.

Proof. By taking $\epsilon = 2/3$, Equation 1 becomes:

Algorithm 2 De-randomize Algorithm for Formulation 1

Input:
Directed Connection Graph $G(V, E)$;
Commodities $\mathcal{F} = \{F_1, ..., F_n\}$;
Source-sink pair (sv_i, dv_i) for each $F_i \in \mathcal{F}$;
Capacity $c(u, v),\ \forall(u, v) \in E$;

Output: OPT_{ALG}

1: Change the last constraint to be $0 \leq f_i \leq 1$;
2: Then it is relaxed to a linear programming formulation denoted by LP, solve this LP and get optimal solution $OPT_f = \sum \tilde{f}_i$;
3: Sort all f_i by descending order and remark each commodity i accordingly from 1 to n, repeat step 4 to 13 for each i:
4: Set $f_i = 1$ and get a new formulation $LP_1 = LP$.
5: In LP_1, scale up the fractional flow $\tilde{f_{i,e}}$ from the LP solution on edge e for commodity i by $\frac{1}{\tilde{f}_i}$, i.e., $f_{i,e} = \tilde{f_{i,e}} \times \frac{1}{\tilde{f}_i}$, for i s.t. $f_i = 1$;
6: In LP_1, add constraint $f_i = 1$, $f_{i,e} = \tilde{f_{i,e}} \times \frac{1}{\tilde{f}_i}$, and replace the capacity constraints $\sum_{j=i}^{n} f_{j,e} \leq c_e^*$ by $\sum_{j=i+1}^{n} f_{j,e} \leq c_e^* - f_{i,e}$ to current LP_1 formulation (note that c_e^* denote the right side of the capacity constraints and now $c_e^* = c_e^* - f_{i,e}$ is the new c_e^* of current LP_1).
7: In LP_1, check for each edge e, if $\sum_{j=1}^{i} f_{j,e} \geq \gamma \cdot c_e$ where $\gamma = (1 + \epsilon' \cdot \sqrt{2 \log |V||\Delta_{F,e}|})$ and ϵ' is as chosen in Theorem 5, then set $OPT_{f1} = 0$ and directly go to step 9.
8: Calculate the LP_1 solution $OPT_{f1} = \sum \tilde{f}_i$;
9: Set $f_i = 0$ and get a new formulation $LP_2 = LP$;
10: Add constraint $f_i = 0$, $f_{i,e} = 0$ to current LP_2 formulation.
11: Calculate the LP_2 solution $OPT_{f2} = \sum \tilde{f}_i$;
12: If $OPT_{f1} > OPT_{f2}$, set $LP = LP_1$.
13: Otherwise set $LP = LP_2$;

$$
\begin{aligned}
Pr[OPT_{ALG} < \frac{1}{3} \cdot OPT_f] &\leq e^{-(\frac{2}{3})^2 \cdot OPT_f/2} && (3)\\
&= e^{-2 \cdot OPT_f/9} && (4)
\end{aligned}
$$

By Equation 2, we know that the minimum value of OPT_f is 1, by taking $OPT_f = 1$, we get the upper bound of $e^{-\epsilon^2 \cdot OPT_f/2}$. Therefore,

$$Pr[OPT_{ALG} < \frac{1}{3} \cdot OPT_f] \leq e^{-2/9} \tag{5}$$

Since $OPT_i \leq OPT_f$, thus we get,

$$Pr[OPT_{ALG} < \frac{1}{3} \cdot OPT_i] \leq e^{-2/9} \qquad \square$$

Fact 4 (Hoeffding's Inequality)[**?**] . Let $\{X_i\}$ be independent random variables, s.t. $X_i \in [a_i, b_i]$, then $Pr(\sum_i X_i - E(\sum_i X_i) \geq t) \leq exp(-2t^2/\sum_i (b_i - a_i)^2)$ holds.

Theorem 5 Given a single edge e with capacity c_e, and let $\Delta_{F,e}$ be the commodities going through edge e. For all the commodity $i \in \Delta_{F,e}$, choose ϵ' such that $\max \frac{\tilde{f_{i,e}}}{\tilde{f}_i} \leq \epsilon' \cdot c_e$ .The probability that ALG exceeds the edge capacity constraint by a factor of $\gamma = (1 + \epsilon' \cdot \sqrt{2 \log |V||\Delta_{F,e}|})$ is bounded by $|V|^{-4}$.

We analyze the probability that our algorithm violates capacity constraints by a certain factor, by employing Hoeffding's Inequality.

Proof: Fix an edge $e \in E$, for commodity i, with probability $1 - \tilde{f}_i$, the flow on edge e for commodity i is set to 0, i.e., $f_{i,e} = 0$, with probability $\tilde{f}_i$, the flow on edge e for commodity i is set to $\tilde{f_{i,e}} \cdot \frac{1}{\tilde{f}_i}$.

Then the expectation of $f_{i,e}$ is

$$E(f_{i,e}) = \tilde{f_{i,e}} \cdot \frac{1}{\tilde{f}_i} \cdot \tilde{f}_i + 0 \cdot (1 - \tilde{f}_i) = \tilde{f_{i,e}} \tag{6}$$

Let F_e denotes the flow on edge e by ALG, then $F_e = \sum_{i, f_{i,e} \neq 0} f_{i,e}$ and the expectation of F_e is

$$E[F_e] = \sum_{i, f_{i,e} \neq 0} \tilde{f_{i,e}} \cdot \frac{1}{\tilde{f}_i} \cdot \tilde{f}_i = \sum_{i, f_{i,e} \neq 0} \tilde{f_{i,e}} \tag{7}$$

Since we relax the LP, and a feasible solution must obey edge capacity constraint, so the cumulative load on edge e is equal or less than the capacity of e:

$$\sum_{i, f_{i,e} \neq 0} \tilde{f_{i,e}} \leq c_e \tag{8}$$

Therefore, we can get the following:

$$E[F_e] \leq c_e \tag{9}$$

Let $t = \epsilon' \cdot \sqrt{2 \log |V||\Delta_{F,e}|} \cdot c_e$, by applying Hoeffding's Inequality and the assumption of $\max \frac{\tilde{f_{i,e}}}{\tilde{f}_i} \leq \epsilon' \cdot c_e$, we get:

$$
\begin{aligned}
Pr[F_e - E(F_e) \geq t] &\leq exp(\frac{-2t^2}{\sum_i (\frac{\tilde{f_{i,e}}}{\tilde{f}_i})^2}) && (10)\\
&\leq exp(\frac{-2 \cdot \epsilon'^2 \cdot \log |V| \cdot |\Delta_{F,e}| \cdot c_e^2}{\epsilon'^2 \cdot c_e^2 \cdot |\Delta_{F,e}|})\\
&= |V|^{-4}
\end{aligned}
$$

Given Equation 9, and let $\gamma = (1 + \epsilon' \cdot \sqrt{2 \log |V||\Delta_{F,e}|})$,

$$
\begin{aligned}
& Pr[F_e - c_e \geq \epsilon' \cdot \sqrt{2 \log |V||\Delta_{F,e}|}]\\
= \;& Pr[F_e \geq (1 + \epsilon' \cdot \sqrt{2 \log |V||\Delta_{F,e}|}) \cdot c_e]\\
= \;& Pr[F_e \geq \gamma \cdot c_e]\\
\leq \;& |V|^{-4}
\end{aligned}
$$

There are at most $|V|^2$ edges, by applying union bound over all edges using Theorem 5, hence we obtain the following corollary.

Corollary 6 The probability that ALG exceeds any of the edge capacity constraints by a factor of $\gamma = (1 + \epsilon' \cdot \sqrt{2 \log |V|})$ is upper bounded by $|V|^{-2}$.

Based on Theorem 3 and Corollary 6, if $|V| \geq 3$ holds, the probability of not finding a suitable solution, satisfying the objective and the capacity constraint, within a single round is therefore upper bounded by $exp(2/9) + 1/9 \leq 11/12,$. The probability to find a suitable solution within N many rounds is then lower bounded by $1 - (11/12)^N$ for $|V| \geq 3$ and hence the randomized rounding scheme yields a solution with high probability.