Convex Sets

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Outline of Lecture

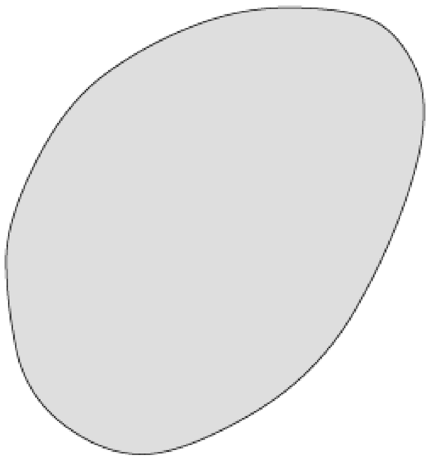
- Definition convex set
- Hyperplanes, halfspaces, polyhedra
- Balls and ellipsoids
- Convex hull
- Cones: norm cones, PSD cone
- Operations that preserve convexity
- Generalized inequalities

(Acknowledgement to Stephen Boyd for material for this lecture.)

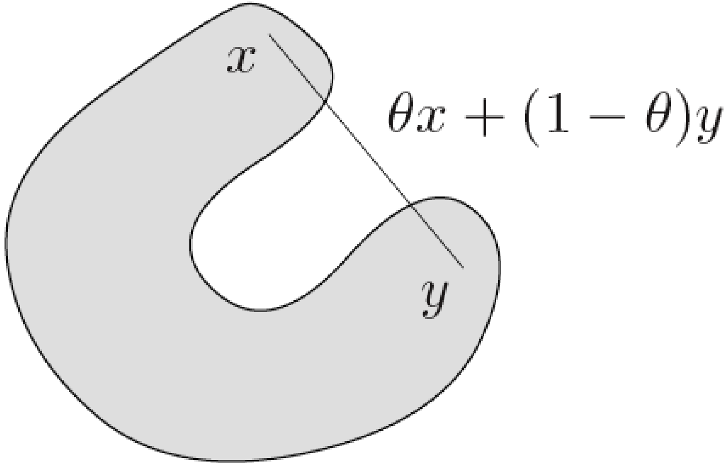
Definition of Convex Set

- A set $C \in \mathbf{R}^n$ is said to be **convex** if the line segment between any two points is in the set: for any $x, y \in C$ and $0 \leq \theta \leq 1$,

$$\theta x + (1 - \theta)\, y \in C.$$



convex



non-convex

Examples: Hyperplanes and Halfspaces

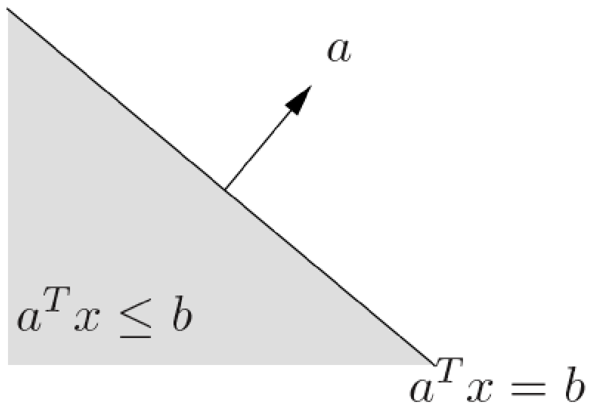
- **Hyperplane**:

$$C = \{x \mid a^T x = b\}$$

where $a \in \mathbf{R}^n$, $b \in \mathbf{R}$.

- **Halfspace**:

$$C = \{x \mid a^T x \le b\}$$

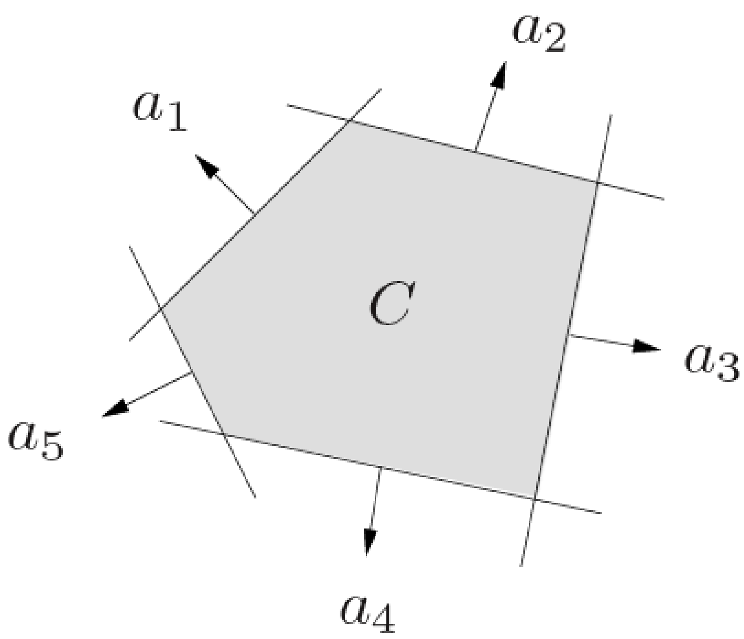


Example: Polyhedra

- **Polyhedron**:

$$C = \{x \mid Ax \leq b,\ Cx = d\}$$

where $A \in \mathbf{R}^{m \times n}$, $C \in \mathbf{R}^{p \times n}$, $b \in \mathbf{R}^m$, $d \in \mathbf{R}^p$.



Examples: Euclidean Balls and Ellipsoids

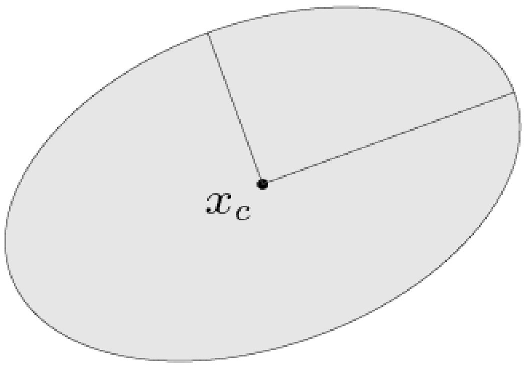
- **Euclidean ball** with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}.$$

- **Ellipsoid**:

$$E(x_c, P) = \left\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\right\} = \{x_c + Au \mid \|u\|_2 \le 1\}$$

with $P \in \mathbf{R}^{n \times n} \succ 0$ (positive definite).



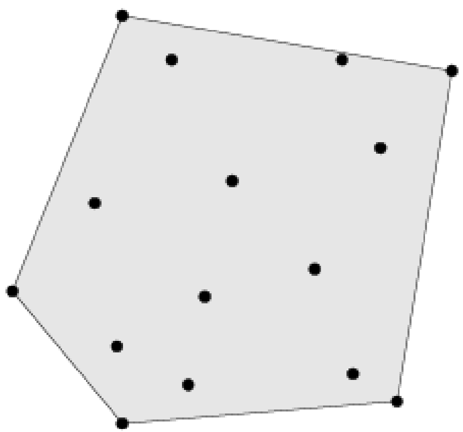
Convex Combination and Convex Hull

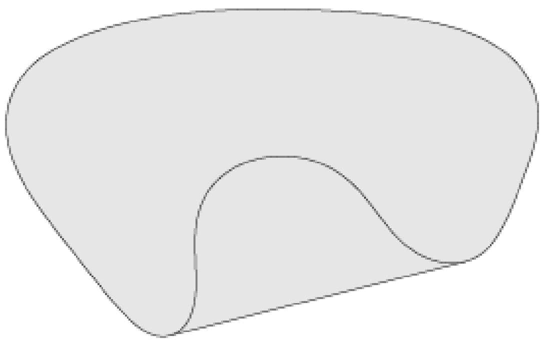
- **Convex combination** of $x_1, \ldots, x_k$: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

  with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$.

- **Convex hull** of a set: set of all convex combinations of points in the set.

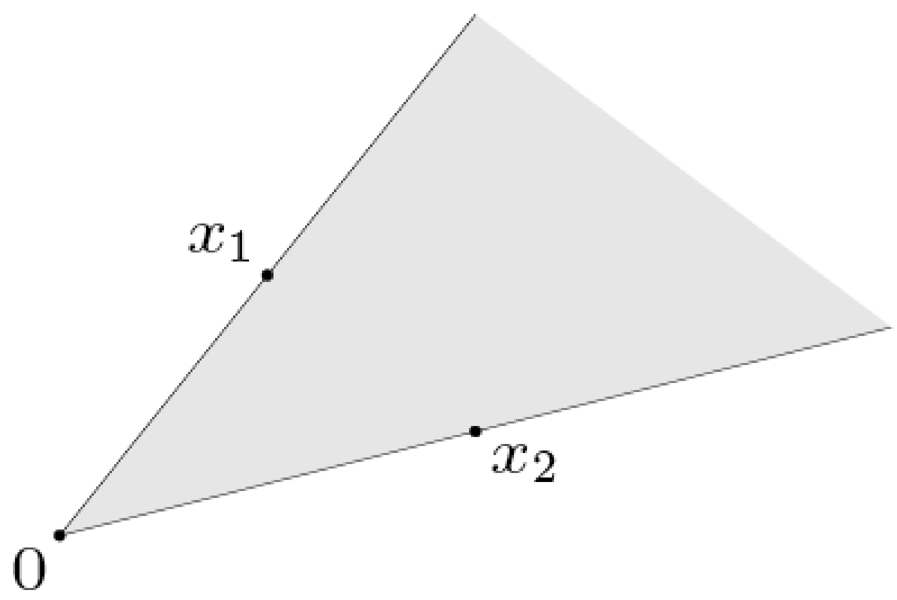




Convex Cones

- A set $C \in \mathbf{R}^n$ is said to be a **convex cone** if the ray from each point in the set is in the set: for any $x_1, x_2 \in C$ and θ_1, $\theta_2 \geq 0$,

$$\theta_1 x_1 + \theta_2 x_2 \in C.$$

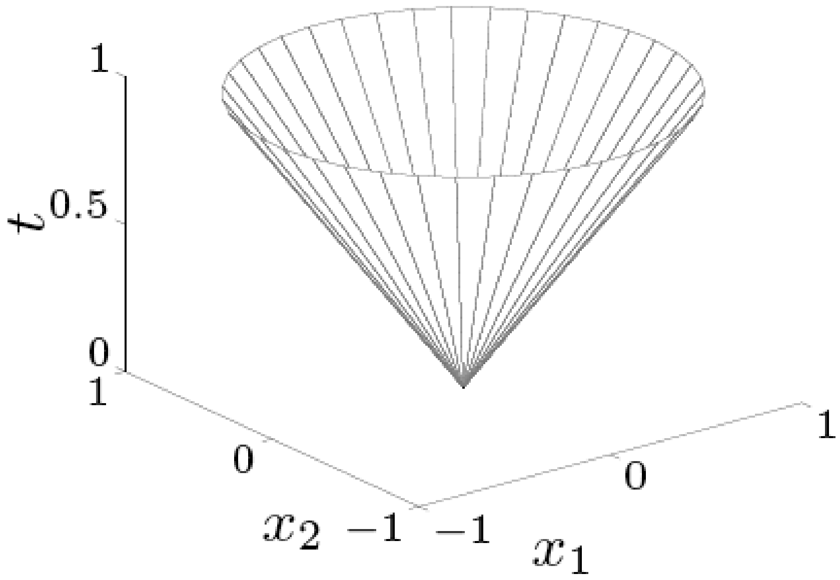


Norm Balls and Norm Cones

- **Norm ball** with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$ where $\|\cdot\|$ is a norm.

- **Norm cone**: $\left\{(x,t) \in \mathbf{R}^{n+1} \mid \|x\| \leq t\right\}$.

- Euclidean norm cone or second-order cone (aka ice-cream cone):

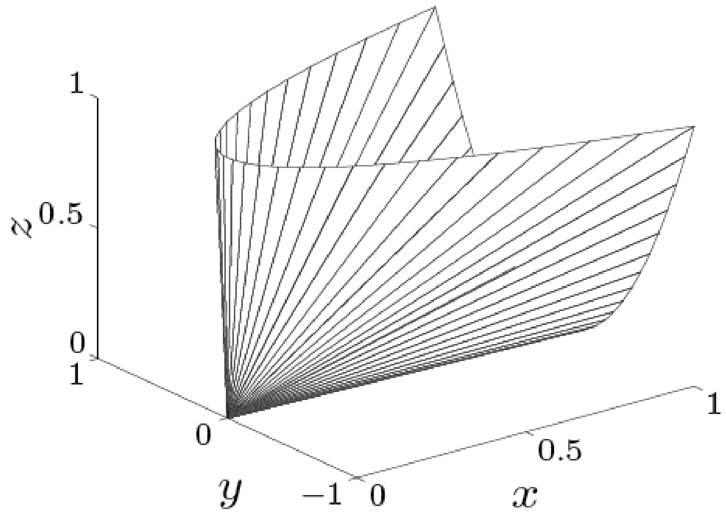


Positive Semidefinite Cone

- **Positive semidefinite (PSD) cone**:

$$\mathbf{S}_+^n = \left\{ X \in \mathbf{R}^{n \times n} \mid X = X^T \succeq 0 \right\}.$$

- Example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2$



Operations that Preserve Convexity

How do we establish the convexity of a given set?

1. Applying the definition:

$$x, y \in C,\ 0 \leq \theta \leq 1 \Longrightarrow \theta x + (1-\theta)\, y \in C$$

   which can be cumbersome.

2. Showing that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, etc.) by operations that preserve convexity:

   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

Intersection

- **Intersection**: if S_1, $S_2, \ldots, S_k$ are convex, then $S_1 \cap S_2 \cap \cdots \cap S_k$ is convex.

- Example: a polyhedron is the intersection of halfspaces and hyperplanes.

- Example:

$$S = \{x \in \mathbf{R}^n \mid |p_x(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p_x(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_n \cos nt$.

Affine Function

- **Affine composition**: the image (and inverse image) of a convex set under an affine function $f(x) = Ax + b$ is convex:

$$S \subseteq \mathbf{R}^n \text{ convex} \Longrightarrow f(S) = \{f(x) \mid x \in S\} \text{ convex.}$$

- Examples: scaling, translation, projection.

- Example: $\left\{(x,t) \in \mathbf{R}^{n+1} \mid \|x\| \le t\right\}$ is convex, so is

$$\left\{x \in \mathbf{R}^n \mid \|Ax + b\| \le c^T x + d\right\}.$$

- Example: solution set of LMI: $\{x \in \mathbf{R}^n \mid x_1 A_1 + \cdots + x_n A_n \preceq B\}$.

Perspective and Linear-Fractional Functions

Perspective function: $P : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n$:

$$P(x,t) = x/t, \qquad \mathbf{dom}\ P = \{(x,t) \mid t > 0\}.$$

- Images and inverse images of convex sets under perspective functions are convex.

Linear-fractional function: $f : \mathbf{R}^n \longrightarrow \mathbf{R}^m$:

$$f(x) = \frac{Ax+b}{c^T x + d}, \qquad \mathbf{dom}\ P = \left\{x \mid c^T x + d > 0\right\}.$$

- Images and inverse images of convex sets under linear-fractional functions are convex.

Generalized Inequalities

- A convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if it is closed, solid, and pointed.

- **Examples**:

  – nonnegative orthant:

$$K = \mathbf{R}_+^n = \{x \in \mathbf{R}^n \mid x_i \geq 0, \ i = 1, \ldots, n\}$$

  – positive semidefinite cone:

$$K = \mathbf{S}_+^n = \left\{X \in \mathbf{R}^{n \times n} \mid X = X^T \succeq 0\right\}$$

  – nonnegative polynomials on $[0, 1]$:

$$K = \left\{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\right\}.$$

- A **generalized inequality** is defined by a proper cone K:

$$y \succeq_K x \Longleftrightarrow y - x \succeq_K 0 \text{ or } y - x \in K.$$

- **Examples**:
  - componentwise inequality ($K = \mathbf{R}_+^n$):

$$y \succeq_{\mathbf{R}_+^n} x \Longleftrightarrow y_i \geq x_i, \ i = 1, \ldots, n$$

  - matrix inequality ($K = \mathbf{S}_+^n$):

$$Y \succeq_{\mathbf{S}_+^n} X \Longleftrightarrow Y - X \text{ is positive semidefinite.}$$

References

Chapter 2 of

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.

http://www.stanford.edu/~boyd/cvxbook/