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Question 1

$g(p|y) \propto p^{\Sigma y_i} (1-p)^{30-\Sigma y_i} I(0<y<p)$ $X = \Sigma y_i$

$p|x \sim Beta(X+1, 31-X)$

$x = 2$

$\therefore p|X=2 \sim Beta(3, 29)$

$\therefore Pr(p \leq 0.1) = 0.611$

Question 2

(a) $y \sim \text{Bernoulli}(p)$ $\quad \Sigma y_i = x$

$g(p|y) \propto f(y|p)g(p)$

$f(y|p) = p^y (1-p)^{1-y}$

$g(p) = \frac{\Gamma(8)}{\Gamma(3)\Gamma(5)} p^2(1-p)^4, \quad 0<p<1$

$\therefore g(p|y) \propto p^{2+\Sigma y_i} \cdot (1-p)^{24-\Sigma y_i} \, I(0<y<p)$

$\theta|y \sim \text{Beta}(\Sigma y_i + 3, 25 - \Sigma y_i)$

$\Rightarrow \theta|x \sim \text{Beta}(x+3, 25-x)$

$\Rightarrow E(\theta|X) = \frac{x+3}{28}$

When $x = 15$.

posterior Bayes' estimate of p is

$E(\theta|x=15) = \frac{18}{28} = \frac{9}{14} \approx 0.64$

(b)

$f(x) = \binom{20}{x} p^x (1-p)^{20-x}$ $\qquad x_i \sim \text{Bernoulli}(p)$

Likelihood function is:

$L(x;p) = p^{\Sigma x_i} (1-p)^{\sum_{i=1}^{20}(1-x_i)}$

$\log(L(x;p)) = \sum_{i=1}^{20} x_i \log(p) + \sum_{i=1}^{20}(1-x_i)\log(1-p)$

$\frac{\partial \log(L(x;p))}{\partial p} = \sum_{i=1}^{20}\frac{x_i}{p} - \sum_{i=1}^{20}\frac{1-x_i}{1-p} = 0$

$\Rightarrow \hat{p}_{MLE} = \frac{\Sigma x_i}{20}$, when $\Sigma x_i = 15$

$\hat{p}_{MLE} = \frac{3}{4} = 0.75$

(c) when $f(p) = 1 \quad 0<p<1$

$$g(p|y) \propto p^{\Sigma y_i} (1-p)^{20-\Sigma y_i} I(0<y<p)$$

$$\theta|y \sim Beta(\Sigma y_i + 1 \quad 21 - \Sigma y_i)$$

$$\theta|x \sim Beta(x+1, 21-x)$$

$$E(\theta|x) = \frac{x+1}{22}$$

when $x = 15$

$$E(\theta|x=15) = \frac{16}{22} = \frac{8}{11} \approx 0.73$$

(d)

maximum likelihood estimate of p is

$$a = \hat{p}_{MLE} = \frac{\Sigma x_i}{200} = \frac{150}{200} = \frac{3}{4} = 0.75$$

posterior Bayes's estimate:

① under beta (3,5)

$$\theta|y \sim Beta(\Sigma y_i + 3, 205 - \Sigma y_i)$$

$$\theta|x \sim Beta(x+3, 205-x)$$

$$E(\theta|x) = \frac{x+3}{208}$$

$$b = E(\theta|x=150) = \frac{153}{208} \approx 0.75$$

② under uniform (0,1)

$$\theta|y \sim Beta(\Sigma y_i + 1, 201 - \Sigma y_i)$$

$$\theta|x \sim Beta(x+1, 201-x)$$

$$E(\theta|x) = \frac{x+1}{202}$$

$$c = E(\theta|x=150) = \frac{151}{202} \approx 0.75$$

we got.

$a > c > b$

Question 3.

$$f(x)=\int_0^\infty f(x|\theta)\cdot g(\theta)\,d\theta$$

$$=\int_0^\infty \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x}\cdot\frac{r^\beta}{\Gamma(\beta)}\theta^{\beta-1}e^{-r\theta}\,d\theta$$

$$=\int_0^\infty \frac{x^{\alpha-1}\cdot r^\beta}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha+\beta-1}e^{-\theta(x+r)}\,d\theta$$

$$=\frac{x^{\alpha-1}\cdot r^\beta}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty \theta^{\alpha+\beta-1}e^{-\theta(x+r)}\,d\theta$$

$$\int_0^\infty \theta^{\alpha+\beta-1}e^{-\theta(x+r)}\,d\theta = \left(-\frac{1}{x+r}\right)\int_0^\infty \theta^{\alpha+\beta-1}\,de^{-\theta(x+r)}$$

$$=\left(-\frac{1}{x+r}\right)\left(\theta^{\alpha+\beta-1}\cdot e^{-\theta(x+r)}\Big|_0^\infty-\int_0^\infty e^{-\theta(x+r)}\cdot(\alpha+\beta-1)\theta^{\alpha+\beta-2}\,d\theta\right.$$

$$=\frac{\alpha+\beta-1}{x+r}\int_0^\infty e^{-\theta(x+r)}\theta^{\alpha+\beta-2}\,d\theta\,,$$

$$\vdots$$

$$=\frac{\Gamma(\alpha+\beta)}{(x+r)^{\alpha+\beta-1}}\int_0^\infty e^{-\theta(x+r)}\,d\theta$$

$$=\frac{-\Gamma(\alpha+\beta)}{(x+r)^{\alpha+\beta}}\quad e^{-\theta(x+r)}\Big|_0^\infty$$

$$=\frac{\Gamma(\alpha+\beta)}{(x+r)^{\alpha+\beta}}$$

$$\therefore f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{x^{\alpha-1}r^\beta}{(x+r)^{\alpha+\beta}}$$

Question 4.

(a) $E[\theta] = \int_\beta^\infty \gamma \beta^\gamma \theta^{-\gamma} d\theta$

$= \gamma\beta^\gamma \cdot \left(\frac{1}{-\gamma+1}\right) \theta^{-\gamma+1} \Big|_\beta^\infty$

$= \frac{\gamma\beta^\gamma}{1-\gamma} \left(0 - \beta^{-\gamma+1}\right)$

$= \frac{\gamma\beta^\gamma \cdot \beta^{-\gamma+1}}{\gamma-1}$

$= \frac{\gamma\beta}{\gamma-1}$

(b) $g(\theta|y) = \frac{f(y|\theta)\pi(\theta)}{\int f(y|\theta)\pi(\theta)d\theta}$

$\int f(y|\theta)\pi(\theta)d\theta = \int_\beta^\infty \gamma\beta^\gamma \theta^{-\gamma-1-n} d\theta$

$= \frac{\gamma\beta^\gamma}{-\gamma-n} \cdot \theta^{-\gamma-n} \Big|_\beta^\infty$

$= \frac{\gamma\beta^\gamma\beta^{-\gamma-n}}{\gamma+n} = \frac{\gamma \cdot \beta^{-n}}{\gamma+n}$

$g(\theta|y) = \frac{\gamma \cdot \beta^\gamma \theta^{-\gamma-1-n}}{\gamma \cdot \beta^{-n}} \cdot \gamma+n$

$= (\gamma+n)\beta^{\gamma+n} \theta^{-\gamma-1-n}$

$E(\theta|y) = \int_\beta^\infty \theta \cdot g(\theta|y) d\theta$

$= \int_\beta^\infty (\gamma+n)\beta^{\gamma+n} \cdot \theta^{-\gamma-n} d\theta$

$= \frac{(\gamma+n)\beta^{\gamma+n}}{-\gamma-n+1} \cdot \theta^{-\gamma-n+1} \Big|_\beta^\infty$

$= \frac{(\gamma+n)\beta}{\gamma+n-1}$

$= \frac{\gamma+n}{\gamma+n-1}\beta$

(C)

By Theorem 4.3B, the Bayes estimator is equal to the posterior Bayes estimator, a Bayes estimator, when unique, is thus always admissible.

$\therefore \frac{\gamma+n}{\gamma+n-1}\beta$ is admissible

$\because E(z)=\theta$

$\therefore \frac{r+n}{r+n-1} z$ is admissible with respect to $y_i \sim Uniform[0,\theta]$

Question 5. (Bayes' risk)

· Travel in plane:

$$R_1 = 10^6 \times 5\times10^{-6} + 5\times(1-10^{-6})$$
$$= 5 + 5\times(1-10^{-6}) \approx 10$$

Travel in car:

$$R_2 = 100\times 10^{-2} + 0\times(1-100^{-2}) = 1$$

$R_1 > R_2$

∴ Travel in car have smaller risk.

Question 6.

(a) For d_1

$$r_1(T) = 0\times\frac{1}{8} + 1\times\frac{3}{8} + 3\times\frac{1}{4} + 1\times\frac{1}{4}$$
$$= \frac{11}{8}$$

For d_2

$$r_2(T) = 2\times\frac{1}{8} + 0 + 4\times\frac{1}{4} + 2\times\frac{1}{4}$$
$$= \frac{7}{4}$$

For d_3

$$r_3(T) = 3\times\frac{1}{8} + 2\times\frac{3}{8} = \frac{9}{8}$$

(b) ∵ $r_2(T) > r_1(T) > r_3(T)$

∴ d_3 is the Bayes' decision.

(c)

	$R_T(\theta_1)$	$R_T(\theta_2)$	$R_T(\theta_3)$	$R_T(\theta_4)$	maximum	minimum.
$T=(d_1,d_1)$	0	2	6	2	6	
$T=(d_1,d_2)$	2	1	7	3	7	
$T=(d_1,d_3)$	3	3	3	1	3	✓ → Same risk for $\theta_1,\theta_2,\theta_3$. ∴ we do not select this.
$T=(d_2,d_2)$	4	0	8	4	8	
$T=(d_2,d_3)$	5	2	4	2	5	
$T=(d_3,d_3)$	6	4	0	0	6	
$T=(d_1,d_2,d_3)$	5	3	7	3	7	
$T=(d_1)$	0	1	3	1	3	✓
$T=(d_2)$	2	0	4	2	4	
$T=(d_3)$	3	2	0	0	3	✓

Therefore the minimax decisions are d_1 and d_3.

Question 7.

(a) $R_d(\theta) = E_\theta L(T;\theta)$

$= E(d-\theta)^2$

$= Ed^2 - 2\theta Ed + \theta^2$

$= d_1^2 \cdot Pr_\theta(X=1) + d_2^2 \cdot Pr(X=0) - 2\theta \cdot d_1 Pr_\theta(X=1) - 2\theta d_2 Pr_\theta(X=0) + \theta^2$

$= d_1^2 \theta + d_2^2 (1-\theta) - 2\theta^2 d_1 - 2\theta(1-\theta) d_2 + \theta^2$

$= d_1^2\theta + d_2^2 - d_2^2\theta - 2\theta^2 d_1 - 2\theta d_2 + 2\theta^2 d_2 + \theta^2$

(b) When

$$\begin{cases} d_1^2 - d_2^2 - 2d_2 = 0 \\ -2d_1 + 2d_2 + 1 = 0 \end{cases} \Rightarrow \begin{cases} d_1 = \frac{3}{4} \\ d_2 = \frac{1}{4} \end{cases},$$

the risk $R_d(\theta)$ is indepent of θ, $\therefore$ minimax decision is $d(1) = \frac{3}{4}$, $d(2) = \frac{1}{4}$.