Abstract

PROPERTIES AND APPLICATIONS OF GAUSSIAN QUANTUM PROCESSES

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December 2020

Gaussian states, operations, and measurements are central building blocks for continuous-variable quantum information processing which paves the way for abundant applications, especially including network-based quantum computation and communication. To make the most use of the Gaussian processes, it is required to understand and utilize suitable mathematical tools such as the symplectic space, symplectic algebra, and Wigner representation. Applying these mathematical tools to practical quantum scenarios, we developed various schemes for faithful quantum transduction, interference-based bosonic mode permutation and ultra-sensitive bosonic sensing. Quantum transduction, proposed for transferring quantum information between different bosonic platforms, will enable the construction of large-scale hybrid quantum networks. We demonstrated that generic coupler characterized by Gaussian unitary process can be transformed into a high-fidelity transducer, assuming the availability of infinite squeezing and high-precision adaptive feedforward with homodyne measurements, all of which are Gaussian operations, measurements or classical communication channels and can be easily analyzed using symplectic algebra. To address the practical limitation of finite squeezing, we also explored the potential of interference-based protocols. It turns out that these protocols can let us freely permute bosonic modes only assuming the access to single-mode Gaussian unitary operations and multiple uses of a given generic multi-mode Gaussian process. Thus, such a scheme not only enables universal decoupling for multi-mode bosonic systems, which can be useful for suppressing undesired coupling between the system and the environment, but also efficient and faithful bidirectional single-mode quantum transduction. Moreover,

noticing that the Gaussian processes are appropriate theoretical models for optical sensors, we studied the quantum noise theory for optical parameter sensing and its potential in providing great measurement precision enhancement. We also extended the Gaussian theories to discrete variable systems, with several examples such as quantum (gate) teleportation. All the analyses and conclusions originated from the fundamental quantum commutation relations, and therefore are widely applicable.

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A Dissertation

Presented to the Faculty of the Graduate School

of

Yale University

in Cnadidacy for the Degree of Doctor of Philosophy

by

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Dissertation Director: A. Douglas Stone

December 2020

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List of Abbreviations and Symbols

$\hat{q}, \hat{p}, \hat{x}$	Canonical quadrature operatos.
M^t	Transpose of a matrix M.
Id	Identity of groups and matrix algebras; the identity matrix.
$\mathrm{Tr}\left[\cdot\right]$	Trace of operators or matrices.
$\sigma\left(\cdot,\cdot\right), \Omega$	Symplectic form.
$\hat{U}_S$	Unitary operator associated with a symplectic transformation S.
$\hat{D}_u$	Displacement operation associated with a vector u in the symplectic space.
$W_{\hat{A}}$	Winger function associated with an operator $\hat{A}$.
$\hat{\Pi}_l\left(\eta\right)$	Infinitely squeezed state associated with Lagrangian plane l and position vector η.
$\hat{\rho}\left(\bar{x}, V\right)$	Gaussian state associated with first moments $\bar{x}$ and covariance matrix V.
$\mathcal{G}_{T,N}$	Gaussian channel associated with matrix data T and N.

Acknowledgement

It has been quite a pleasure and honor to work in Yale Quantum Institute (YQI). I have received a lot of help from many people from YQI, to whom I wan to express my great gratitude.

Firstly, I cannot be more grateful to Liang Jiang for being my advisor. Liang showed me the way to the fantastic field of quantum information. He can understand and accept novel theoretical tools effortlessly, and incorporate them into practical applications, which shaped my research patterns. In particular, his emphasis on experiments always keeps me from wasting my life on playing meaningless mathematical games. Liang has also created a relaxing working environment so that all the people in his can freely share their wildest ideas. My collaboration with A. Douglas Stone is absolutely one of my most valuable experiences at Yale. Doug's wonderful physical intuition and ability of clarifying complicated concepts constantly amaze me, not to mention his well-known talents for writing. Among all the courses I took at Yale, Michel Devoret's introductory lectures on quantum noise theory is the most helpful to me. His patient and detailed explanation of the elementary concepts helped me quickly master this powerful tool, which has played a central role in my doctoral research. Although I did not have many conversations with Hong Tang, his unreserved and constructive comments on my early projects and our later collaboration taught me a great deal.

Next, I must thank Changling Zou and Chao Shen for their unselfish help and fruitful discussions during their stay at YQI. Without them, the beginning stage of me being a researcher would not have been so smooth and painless. Their scientific and emotional support will always be cherished by me. I thank Chia Wei Hsu and William

Sweeney for their countless valuable scientific inputs and friendship. I am grateful to Shoumik Chowdhury, still an undergraduate student at Yale, for our pleasant collaboration due to his patience and diligence. We also owe the nice presentation of one of our most complicated ideas to him. Also, I cannot forget to mention Aashish Clerk. His comments provide me with new perspectives of my work and are influencing some of my ongoing projects.

I have enjoyed my life so much at Yale, thanks to each and every one of my friends and my fellow group members. For example, Stephan Krastanov's recognized achievement in teaching and organizing outreaching activities has not only deeply touched me and brought me a lot of fun, but also improve my communication skills.

I will forever be indebted to Yanhong Xiao, who kindly encouraged and supported me as an undergraduate to pursue the fun of physics. I was very lucky to have learnt the basics of scientific research and some still essential physical concepts from her.

Last but absolutely not the least, I would like to express my gratitude to my family. My mother and father, Yunfang Han and Lijun Zhang, have supplied me with the most needed support as always, otherwise this thesis could not have been possible.

CHAPTER 1

Introduction

In classical mechanics, states of a physical system are completely determined by the values of its position and momentum variables. Thus, each state of a classical physical system corresponds to a point in a multi-dimensional space, the phase space, which enables the geometrization of the time evolution of classical systems. The momenta and positions are conjugate pairs of variables, such that their Poisson brackets must be preserved under any physical dynamics. Therefore, phase space must be regarded as a symplectic manifold, and physical dynamics should be covariant under symplectic diffeomorphisms [**1, 2**]. Thanks to the fundamental correspondence between Poisson brackets and commutators, phase space is also compatible with quantum mechanics, along with its symplectic-geometrical structure. Furthermore, this allows us to analyze the quantum states using the Wigner function, a quasi-probabilistic function and hence the qauntum counterpart of the statistical ensemble in statistical mechanics [**3**].

These concepts–phase space, symplectic geometry and Wigner function–are extensively applied to studying problems related to Gaussian quantum processes [**4**]. In such a process, dynamics of a quantum physical system is tracked by the linear change of the expectation values of the momentum and position operators, and therefore most closely resembles its classical counterparts. As the consequence, the symplectic geometry methods greatly reduce the complexity of representation and analysis of Gaussian quantum processes, leading to many successful applications in bosonic quantum information and computation. However, the potential of these tools have not been fully exploited for solving practical bosonic quantum engineering problems, despite the ubiquitous involvement of Gaussian processes in such problems. In view of the rapid development and increasing demand of bosonic quantum engineering, I will demonstrate in this thesis how these powerful symplectic-geometry-related mathematical tools can be utilized to derive promising systematic solutions to some intriguing problems.

Firstly, applications are found in quantum transduction. Successful construction of hybrid quantum networks [**5, 6**], which often involves bosonic modes with different physical characteristics such as frequencies, polarizations or even mode carriers, relies on the performance of quantum transducers. They are devices that can efficiently and faithfully convert quantum signals between different bosonic platforms. For example, with a high-quality microwave-to-optical-frequency quantum transducer, one can transfer information between superconducting circuits and optical waveguides with 100% fidelity, which will enable the construction of large scale quantum computers with an enormous number of qubits free of size-dependent penalties.

Many theoretical and experimental efforts have been made to realize the quantum transducers, including frequency conversion between the microwave and optical modes [**7, 8, 9, 10, 11, 12, 13, 14, 15, 16**], between the microwave and mechanical/spin-wave modes [**17, 18, 19**], and between quantum processors and quantum memory [**20, 21, 22, 23, 24**]. These attempts are usually based on tuning the linear scattering process to the working point where the transmittance between two wanted modes is maximized and hence the reflectance simultaneously vanishes. However, because of the limited parameter controllability and more importantly the inevitable presence of sideband modes [**12, 25**], often ignored in theoretical proposals, such working points in the experimental environments either do not exist or cannot be achieved and thus high-fidelity quantum transducers are still far from ready.

We note that in the existing attempts using linear scattering processes, the reflected signal is usually discarded although it often carries a significant part of the information of the input signal. Then it is natural to ask: Can we collect the information from the reflected signal and restore the quantum state of the input signal? Based on this conception and by applying the symplectic geometry tools, we find that by infinitely squeezing the auxiliary modes, measuring the reflected signal homodyne measurements and displacing the output signal according to the measurement outcomes, the quantum information can be fully restored, which is universally applicable to general Gaussian processes [**26**]. In addition, all of the key components–the squeezed auxiliary modes, the homodyne measurement and the adaptive displacements–also appear in the continuous variable teleportation scheme. Actually, we will see in this thesis that our scheme for quantum transduction is a generalization of the teleportation scheme, where the special Gaussian process, the balanced beam splitters, is replaced with a general Gaussian process.

However, there is a major drawback of this scheme: large squeezing is too hard to achieve in laboratories. In the search of a solution this problem, a promising interference-based scheme came into our sight. The idea was originally proposed for transducers involving two bosonic modes [**27**]: We consider a imperfect quantum transducer corresponding to a unitary Gaussian process and assume all the single-mode Gaussian unitary processes for any mode are available. Then we let the input signal pass a sequence of copies of this unitary process with single-mode Gaussian unitary processes interspersed them. It turns out that in general, the quantum signal can be faithfully transduced if the interspersed single-mode Gaussian unitary processes are designed appropriately. However, the two bosonic modes constraint of this proposal is impractical in most practical situations as we have explained above. However, we find that this proposal enjoys a hidden underlying mathematical mechanism that can be concisely explained by some most fundamental facts in symplectic geometry and hence is extendable to the multi-mode cases [**28**]. Moreover, the flexibility of the choice of the single-mode Gaussian processes enable us to decouple arbitrary modes without the need to change the overall structure of the scheme, which may benefit noise and error reduction in quantum information [**29, 30, 31, 32**]. As the result, these observations together lead to the interference-based bosonic permutation scheme that will be presented in this thesis.

Next, symplectic geometry is also helpful for understanding exotic properties of optical systems with special spectrum structure, known as exceptional points. Exceptional points have attracted a lot of attention recently. Specifically, we consider a non-Hermitian optical Hamiltonian depending on some parameter, e.g. the coupling strength between different modes. When the parameter is tuned to a certain value which is usually called the exceptional point, the originally distinct eigenmodes will coalesce into a single eigenmode, which is mathematically equivalent to the appearance of a non-trivial Jordan normal form in the matrix representation of the Hamiltonian [**33, 34, 35, 36**]. This phenomenon inspires many novel application-oriented proposals [**37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47**], among which one of the most promising is enhancing sensing precision of optical sensors [**48, 49, 50, 51, 52**]. People found the split of eigen-frequencies near the exceptional point varies non-linearly with respect to the parameter change: The smaller the parameter change, the larger the amplification frequency-split. Therefore, considering the frequency split to be measurable signal, one may expect to see a great improvement in sensitivity when the parameter is small. However, as presented in this thesis, this optimistic outlook is challenged when the process is carefully analyzed using quantum noise theory [**53**]. The quantum noise will increase proportionately as the signal increases. Nevertheless, such a seemingly pessimistic conclusion does not lead to a complete failure of this attempt, since we find that by properly changing the measurement scheme, an even larger sensitivity enhancement can be genuinely achieved [**54**]. Moreover, with results derived from symplectic geometry, this exceptional-point sensing scheme along with other sensing schemes based on Gaussian processes can be dilated to a unitary Gaussian process and understood as a phenomenon related to squeezing, details of which will be demonstrated in this thesis.

Apart from the applications in the continuous variable systems, the concepts related to symplectic geometry can also be generalized to discrete variable systems [**55**]. In the last chapter of this thesis, I will briefly show by examples how some of the results derived for the continuous variable cases can be applied effortlessly to the discrete variable systems. Last but not the least, in the same chapter, I systematically generalized most of the symplectic geometry toolbox to arbitrary discrete variable systems, compatible with the conventional definition of generalized Pauli operators, which has not been fully revealed before.

Here is an overview of the thesis: The necessary mathematical concepts of symplectic geometry is introduced in Ch. 2, while Gaussian states and Gaussian processes are introduced in the following Ch. 3. Ch. 4 and Ch. 5 provide details of our investigation of quantum transduction using symplectic geometry. In Ch. 6, we explore the application of Gaussian processes to analyzing bosonic sensing schemes. The extension of symplectic geometry to the discrete variable system is demonstrated in the last chapter, Ch. 7.

CHAPTER 2

Symplectic geometry for bosonic systems

2.1. Introduction

In classical mechanics, a physical state of a system is completely determined by the momenta p_j and positions q_j of its components. Momenta and positions are the canonical coordinates of the phase space. Therefore, each point in the phase space represents a unique physical state and the evolution of the system geometrically corresponds to a trajectory. Poisson brackets–which plays a central role in classical mechanics–of the canonical coordinates satisfy the following conditions

$$\{q_j, q_k\} = \{p_j, p_k\} = 0,$$

and

$$\{q_j, p_k\} = \delta_{jk},$$

where δ_{jk} is the Kronecker delta. The Poisson brackets impose a special geometrical structure on the phase space: the coordinate transformations of the canonical coordinates should keep the Poisson brackets invariant. Intuitively, these coordinate transformations preserve the area enclosed by a closed trajectory in the phase space. Such Poisson-brackets-preserving coordinate transformations in the phase space are called the symplectic transformations, and the induced geometrical structure is known as the symplectic geometry.

What are the symplectic transformations? The first candidates are the length-preserving coordinate transformations, which are obviously area-preserving. However,

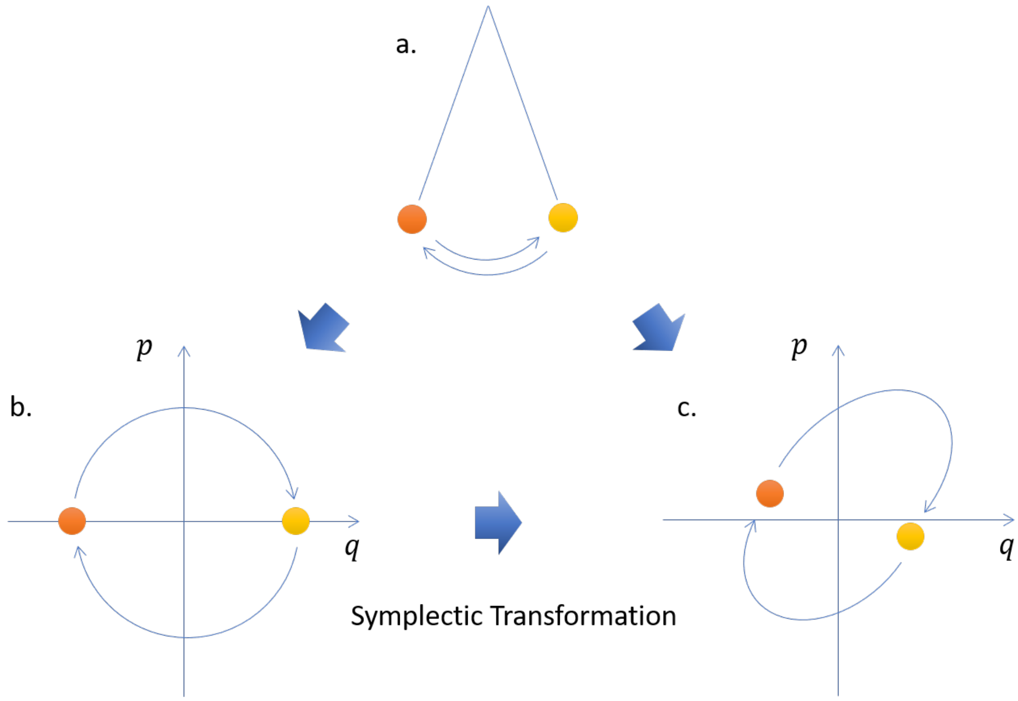

FIGURE 2.1. *Evolution of a pendulum in the phase space.*

The figure shows the phase space representation of a pendulum. As shown in **b**, the classical physical state of a pendulum (as in **a**) at each instant can be represented by a point in the phase space. The evolution of a pendulum, as a closed physical system, is thus represented by a closed trajectory, i.e. the circle in **b**. Under symplectic transformations (i.e. any physical evolution or transformation preserving the Poisson brackets), the area encircled by the closed trajectory should be preserved. Therefore, the enclosed area in **c** is equal to the enclosed area in **b**. Moreover, as for the case in **c**, the physical state of the pendulum is not directly represented by its position and momentum, but another pair of canonical variables transformed from the position and the momentum via the corresponding symplectic transformation.

not all length-preserving transformations are symplectic transformations. For example, if the canonical coordinates are transformed as

$$q_1 \to p_2,\ q_2 \to q_1,\ p_1 \to q_2,\ p_2 \to p_1,$$

then the Poisson-brackets-relations will not be broken while the area of a closed trajectory is still preserved. For the same reason, we must also exclude the mirror operations, such as

$$q_1 \to q_1,\, p_1 \to -p_1.$$

In addition to the length-preserving coordinates, the enclosed area of a closed trajectory can also be preserved by the squeezing operations, for instance

$$q_1 \to 2q_1,\, p_1 \to (1/2)p_1.$$

Later, we will see that any symplectic transformation is product of feasible rotation-like and squeezing-like coordinate transformations in the phase space.

We have seen how the Poisson brackets geometrize the classical mechanics. It turns out the quantum mechanics can also be geometrized similarly. Thanks to the canonical correspondence principles

$$\{f, g\} \to \frac{1}{i\hbar}\left[\hat{f}, \hat{g}\right]$$

with $\hat{f}, \hat{g}$ being the quantizations of the classical observables f and g, the Poisson brackets serve as a bridge connecting classical and quantum physics. Specifically, the Poisson-bracket relations of the canonical coordinates translate to canonical quantization relations (CCRs):

$$[\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0,$$

and

$$[\hat{q}_j, \hat{p}_k] = \delta_{jk}.$$

Therefore, phase space along with its symplectic geometry structure (the Poisson-bracket-preserving requirement is replaced with the commutator-preserving requirement) can be naturally established in the quantum mechanics: Each point in the phase with the canonical coordinates being u (representing the positions) and v (representing the momenta) now corresponds uniquely to a quadrature operator

$$\sum (u_j \hat{q}_j + v_j \hat{p}_j)$$

which is a linear combination of the canonical $\hat{q}$ and $\hat{p}$ quadratures.

So far, the extension from classical mechanics and quantum mechanics has been quite straightforward. However, due to the Heisenberg's uncertainty principles derived from the CCRs, states of a quantum system is always probabilistic. It turns out that we should no longer represent a physical state by its canonical coordinates, but by a quasi-probabilistic distribution, such as the Wigner function and its equivalent variants. Intuitively, this means instead of being by points, the evolution of a physical system should be described by the a trajectory of smeared blobs (the image of the Wigner function) in the phase space.

The Wigner function is covariant with the symplectic transformations insofar as these coordinate transformations in phase space will also influence the shape of the quantum states. For example, this means a rotation-like symplectic transformation can transform a coherent state to some other coherent state and a squeezing-like symplectic transformation can turn a vacuum state into a squeezed vacuum state in bosonic systems. Therefore, symplectic geometry is not only essential to capture the kinetics of a physical system but also useful for manipulating the quantum states.

To see that symplectic transformations are purely fictitious mathematical concepts, we use beamsplitter in quantum optics as an example to show they can be embodied physically. A beamsplitter can mix the two input modes through the following unitary transformation[**4**]:

$$\hat{q}_1 \to \cos\theta \hat{q}_1 + \sin\theta \hat{q}_2,$$

$$\hat{p}_1 \to \cos\theta \hat{p}_1 + \sin\theta \hat{p}_2,$$

$$\hat{q}_2 \to -\sin\theta \hat{q}_1 + \cos\theta \hat{q}_2,$$

$$\hat{p}_2 \to -\sin\theta \hat{p}_1 + \cos\theta \hat{p}_2,$$

which is apparently area-preserving in the phase space and hence symplectically transform the canonical coordinates in the phase space. Actually, all rotation-like symplectic transformations can be realized as beamsplitters or phase shifters in quantum optics. Similarly, the squeezing-like symplectic transformations can be realized by (multi-mode) squeezing operations. Later on, we will define this connection rigorously using the concept metaplectic operators, and introduce other physical ways to implement symplectic transformations.

Moreover, as coordinate transformations, the symplectic transformations can be conveniently described by matrices. For instance, the symplectic transformation corresponding to a beamsplitter as shown above can be described by the 4×4 real matrix

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}.$$

It follows that the composition of the unitary operations corresponding to the symplectic transformations can be efficiently calculated by matrix algebra. In the following sections, we introduce the notations, definitions and properties to provide a precise description of symplectic geometry in the bosonic systems, which will be necessary for proving the results in the later chapters of this thesis. We will start with the symplectic space (the mathematical term for the phase space), and then define symplectic transformations as structure-preserving linear transformations on the symplectic space. The basis properties and physical constructions of symplectic transformations will be introduced and discussed [1].

[1]Throughout this chapter, we follow the conventions in [**1**]. The detailed derivations of the statements in this chapter can also be found in the same reference.

2.2. Symplectic space

Consider an N-mode bosonic system, of which the annihilation operators are $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N$ (correspondingly the N creation operators $\hat{a}_1^\dagger, \hat{a}_2^\dagger, \ldots, \hat{a}_N^\dagger$). The annihilation and creation operators satisfy

$$[\hat{a}_j, \hat{a}_k] = [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0,$$

and

$$[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{j,k},$$

for any $j, k \in \{1, 2, \ldots, N\}$. These relations amount to the CCRs if the canonical quadrature operators are defined as follows:

$$\hat{q}_j := \frac{\hat{a}_j + \hat{a}_j^\dagger}{\sqrt{2}} \text{ and } \hat{p}_j = \frac{\hat{a}_j - \hat{a}_j^\dagger}{\sqrt{2}i}.$$

For convenience, we deonte

$$\hat{x}_{2j-1} = \hat{q}_j, \ \forall j \in \{1, 2, \ldots, N\},$$

and

$$\hat{x}_{2j} = \hat{p}_j, \ \forall j \in \{1, 2, \ldots, N\}.$$

Then the CCRs can be expressed in a more compact form:

$$[\hat{x}_j, \hat{x}_k] = i\Omega_{j,k}, \ \forall j, k \in \{1, 2, \ldots, N\}$$

where the matrix Ω on the right hand side is defined as

$$\Omega = \oplus_{j=1}^{N} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix}$$

and will be called the symplectic form.

The name, symplectic form, is obtained from the mathematical definition of symplectic space. A $2N$-dimensional real vector space E is called a symplectic space if it is equipped with a non-degenerate skew-symmetric bilinear binary map $\sigma : E \times E \to E$[2]. In such a symplectic space, we can always find $2N$ linearly independent vectors $e^{(1)}, f^{(1)}, e^{(2)}, f^{(2)}, \ldots, e^{(N)}, f^{(N)}$, such that

$$\sigma(e^{(j)}, e^{(k)}) = 0,$$

$$\sigma(f^{(j)}, f^{(k)}) = 0,$$

and

$$\sigma(e^{(j)}, f^{(k)}) = -\sigma(f^{(j)}, e^{(k)}) = \delta_{jk},$$

for all $j, k \in \{1, 2, \ldots, N\}$. This basis is called the symplectic canoncial basis, and the map σ is called the symplectic form. The canonical basis can be chosen to be the orthonormal basis of E viewed as an Euclidean space[3]. Then it is easy to check that

$$\sigma(u, v) = u^t \Omega v, \ \forall u, v \in E,$$

where Ω is the aforementioned matrix defining the CCRs.

[2]This means that given any $u, v \in E$, we have $\sigma(u, v) = -\sigma(v, u)$ and $\sigma(\lambda u, \mu v) = \lambda\mu\sigma(u, v)$for any $\lambda, \mu \in \mathbb{R}$ and $\sigma(u, v) = 0$ for every $v \in E$ if and only if $u = 0$.

[3]This perspective allows us to represent linear transformations by real matrices.

Now we relate the mathematical terms to the physical concepts. First, symplectic space is the mathematical counterpart of the phase space: Each $e^{(j)}$corresponds to a canonical quadrature operator $\hat{q}_j$ while each $f^{(j)}$ corresponds to a canonical quadrature operator $\hat{p}_j$, and each element u in E corresponds to a linear combination of the canonical operators (which is itself a genera quadrature operator)

$$\hat{u} = \sum_{j=1}^{2N} u_j \hat{x}_j.$$

Then for all $u, v \in E$, we can quickly verify that

$$[\hat{u}, \hat{v}] = i\sigma(u, v) = iu^t \Omega v,$$

which naturally leads to the CCRs. In short, now we can conveniently treat each quadrature operator as a vector in the symplectic space and the commutativity between quadrature operators can be checked easily a linear algebra operation, the symplectic form.

Before moving on to the symplectic transformations, it is helpful to understand the structure of a symplectic space in more detail. Especially, certain subspaces of a symplectic space play a crucial role in deriving the results in later chapters, and thus should be introduced here. A subspace of E is spanned by a set of linearly independent vectors. This set of vectors can be grouped as a matrix. Specifically, let $u^{(1)}, u^{(2)}, \ldots, u^{(M)}$ be M linearly independent vectors in the $2N$-dimensional symplectic space E spanning a subspace. Then these M vectors can be grouped as the following $2N \times M$ real matrix

$$F = \begin{pmatrix} u_1^{(1)} & u_1^{(2)} & \cdots & u_1^{(M)} \\ u_2^{(1)} & u_2^{(2)} & \cdots & u_2^{(M)} \\ \vdots & \vdots & \vdots & \vdots \\ u_{2N}^{(1)} & u_{2N}^{(2)} & \cdots & u_{2N}^{(M)} \end{pmatrix},$$

where the vectors $u^{(1)}, u^{(2)}, \ldots, u^{(M)}$ are the M corresponding columns. We say the subspace spanned the M vectors is determined by a such a matrix F as above. In this language, there are two kinds of subspaces of particular interest: The subspaces determined by a $2N \times M$ matrix F such that

$$F^t \Omega F = 0,$$

and the subspaces determined by a $2N \times 2M$ matrix F such that

$$F^t \Omega F = A$$

with A being a skew-symmetric matrix ($A = -A^t$) and $\mathrm{rank} A = 2M$. The former are called the isotropic subspaces and the latter are called the symplectic subspaces.

For the isotropic subspaces, specifically, the columns of F, i.e. the M linearly independent vectors spanning the this subspace, are symplectically orthogonal, which means

$$\sigma(u^{(j)}, u^{(k)}) = \left(u^{(j)}\right)^t \Omega u^{(k)} = 0, \ \forall j, k \in \{1, 2, \ldots, M\}.$$

The number of the linearly independent vectors spanning an isotropic space does not exceed the number of modes N. In particular, when $M = N$, an isotropic subspace is called a Lagrangian plane. Let l be a subspace determined by the matrix F as introduced above. (We will sometimes denote l as l_F.) The symplectic complement of l in E, denoted by $l^\perp$, will be the following subspace containing vectors that are symplectically orthogonal to every vector in l, i.e.

$$l^\perp = \{u \in E \mid \sigma(u, v) = 0, \ \forall v \in l\}.$$

Then a Lagrangian plane l can be alternatively defined as a subspace satisfying the condition $l = l^\perp$. Moreover, we call a subspace l' the symplectic conjugate of l if $l' \cap l = \{0\}$ and

$$E \cong l \oplus l'. \tag{2.2.1}$$

In other words, l' is isomorphic the quotient space E/l. It is easy to check that the symplectic conjugate of a Lagrangian plane is also a Lagrangian plane. Intuitively, we demonstrate their physical concepts in the following example:

EXAMPLE 2.1. Consider the 3-mode bosonic system with quadrature operators $\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{p}_1, \hat{p}_2, \hat{p}_3$. The subspace spanned by $\hat{q}_1, \hat{q}_2, \hat{q}_3$ with $M \leq N$ is determined by the 6×3 matrix

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

satisfying $F^t \Omega F = 0$, and is therefore an isotropic subspace. It is also a Lagrangian plane since the number of the basis of this subspace equals the number of modes. And its symplectic conjugate is spanned by all the $\hat{p}$-quadratures.

For the symplectic subspaces, the $2M$ linearly independent vectors $g^{(1)}$, $g^{(2)}$, $\ldots$, $g^{(M)}$, $h^{(1)}$, $h^{(2)}$,$\ldots$ $h^{(M)}$, which are the $2M$ columns of the determinant matrix F, satisfy

$$\sigma(g^{(j)}, g^{(k)}) = \sigma(h^{(j)}, h^{(k)}),$$

and

$$\sigma(g^{(j)}, h^{(k)}) = \delta_{jk},$$

for all $j, k \in \{1, 2, \ldots, M\}$. Therefore, such a symplectic subspace is also a $2M$-dimensional symplectic space. If $\{g^{(1)}, g^{(2)}, \ldots, g^{(M)}, h^{(1)}, h^{(2)}, \ldots, h^{(M)}\}$ happens to be a subset of the canonical basis $\{e^{(1)}, f^{(1)}, e^{(2)}, f^{(2)}, \ldots, e^{(N)}, f^{(N)}\}$ of E, they will span a canonical symplectic subspace. The symplectic conjugate of a symplectic subspace is also a symplectic subspace. For convenience, we also denote the subspace spanned by members of the canonical basis, $e^{(j)}, f^{(j)}$, for $j \in \{1, 2, \ldots, N\}$, by E_j.

Thus, we have

$$E \cong E_1 \oplus E_2 \oplus \cdots \oplus E_N. \tag{2.2.2}$$

The symplectic form of a canonical symplectic subspace E_j of E is naturally

$$\Omega_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

EXAMPLE 2.2. Consider the 3-mode bosonic system with quadrature operators $\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{p}_1, \hat{p}_2, \hat{p}_3$. The canonical quadratures $\hat{q}_2, \hat{p}_2$ span a canonical symplectic subspace which can be denoted by E_2.

Moreover, we can prove that for l being an isotropic space (but strictly not a Lagrangian plane), there is always a symplectic subspace H, such that

$$E \cong H \oplus l \oplus l^H, \tag{2.2.3}$$

where l^H is the symplectic conjugate of l in the symplectic conjugate H' of H as follows:

$$H' \cong l \oplus l^H. \tag{2.2.4}$$

The isotropic subspace l is thus a Lagrangian plane of H'.

EXAMPLE 2.3. The symplectic space of the 3-mode bosonic system can be decomposed as the direct sum of a canonical subspace spanned by $\hat{q}_1, \hat{p}_1$, an isotropic subspace spanned by $\hat{q}_2, \hat{q}_3$ and $\hat{p}_2, \hat{p}_3$. The latter two subspaces are symplectic conjugate in the symplectic conjugate, spanned by $\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2$, of the canonical symplectic subspace $\hat{q}_1, \hat{p}_1$.

Each subspace of a symplectic space is associated with a projection map. Let l be a subspace of E. We denote the corresponding projection by $\pi_l : E \to l$. Since this operation will be frequently used later, for any linear transformation $S : E \to E$and any subspaces l, l' of E, we will often denote

$$S_{l,l'} = \pi_l S \pi_{l'}.$$

EXAMPLE 2.4. Let E_j be a canonical symplectic subspace for $j \in \{1, 2, \ldots, N\}$, then

$$\Omega_j = \pi_j \Omega \pi_j,$$

just as expected.

EXAMPLE 2.5. Consider a beamsplitter, which corresponds to a linear transformation

$$S = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}$$

in the phase space. Let E_1 be the canonical symplectic subspace spanned by $\hat{q}_1, \hat{p}_1$ and l be the Lagrangian plane spanned by $\hat{q}_1, \hat{q}_2$. Then we have the submatrix

$$S_{E_1,l} = \begin{pmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{pmatrix}.$$

Last but not the least, changing the ordering of the canonical basis of a symplectic basis will change the presentation of the linear transformations, which can often improve our understanding of the derivations. Aside from the

$$\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_N, \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_N$$

ordering we have been used so far, we can also order the basis as

$$\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \ldots, \hat{q}_N, \hat{p}_N.$$

These are the two major ordering that are best suited to the later development of our theory. In consideration of their frequent use, we will refer to the former as the default ordering and the latter the alternative ordering. Without notice, the default ordering is always assumed.

EXAMPLE 2.6. Consider the linear transformation corresponding to the beam-splitter. With the default ordering, we have

$$S = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}.$$

If we rearrange the canonical basis according to the alternative ordering, the above matrix will be of the following form:

$$S = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

2.3. Symplectic transformation

As we mentioned in the first section of this chapter, certain coordinate transformations of the phase space (i.e linear transformations in the symplectic space) preserve the CCRs and therefore the symplectic geometrical structure. Such coordinate transformations are symplectic transformations, which are defined mathematically as follows:

DEFINITION 2.7. Let E be a symplectic space equipped with the symplectic form $\sigma : E \times E \to E$. A linear transformation S on E is a *symplectic transformation* if

$$\sigma\left(Su, Sv\right) = \sigma\left(u, v\right)$$

for any $u, v \in E$.

Using the matrix representation of the symplectic form, the above defining condition is equivalent to

$$S^t \Omega S = \Omega, \tag{2.3.1}$$

where Ω is the matrix representation of the symplectic form.

EXAMPLE 2.8. As we mentioned before, a beamsplitter can be described by a linear transformation

$$S = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix},$$

in the symplectic space. It is a symplectic transformation since it satisfies the condition $S^t \Omega S = \Omega$.

Just as orthogonal transformations in an Euclidean space form the orthogonal group. The set of all symplectic transformations in a symplectic space also form a group–the symplectic group: Let E be a $2N$-dimensional symplectic space; The symplectic group, denoted by $\mathrm{Sp}\,(2N, \mathbb{R})$, is the set of symplectic transformations on E equipped with the normal matrix product. The identity of the symplectic group, denoted by Id, is the identity matrix[4]. The inverse of a symplectic transformations can be directly derived from its defintion as below

$$S^{-1} = -\Omega S^t \Omega.$$

The above equation, in combination with the fact that the symplectic form Ω is itself a symplectic matrix, leads to the fact that the transpose of a symplectic matrix is

[4]As identity matrix always serves as the identity of the algebraic structures throughout this thesis, we will always denote identity matrices by Id. We will not clarify the dimensions of an identity matrix every time it appears unless necessary, since they can usually be deduced from the context.

also symplectic. Moreover, a symplectic transformation S has the following notable property:

$$\det(S) = 1.$$

The symplectic group $\mathrm{Sp}(2N, \mathbb{R})$ is a non-compact Lie group. Its Lie algebra $\mathrm{sp}(2N, \mathbb{R})$ consists of the $2N \times 2N$ matrices M satisfying the condition that

$$M\Omega + \Omega M^t = 0. \tag{2.3.2}$$

The elements in $\mathrm{sp}(2N, \mathbb{R})$ are one-to-one corresponding to the real symmetric matrices: Given any $2N \times 2N$ real symmetric matrix H, we have $\Omega H \in \mathrm{sp}(2N, \mathbb{R})$, and for any$M \in \mathrm{sp}(2N, \mathbb{R})$, ΩM is a real symmetric matrix. However, since the symplectic group is non-compact, exponential map $\mathrm{Exp} : \mathrm{sp}(2N, \mathbb{R}) \to \mathrm{Sp}(2N, \mathbb{R})$, defined by

$$\mathrm{Exp}(M) = e^M, \ \forall M \in \mathrm{sp}(2N, \mathbb{R}),$$

is not surjective[5]. Simply speaking, there exist symplectic transformations that cannot be generated by elements in $\mathrm{sp}(2N, \mathbb{R})$ through the exponential map[6].

Aside form the exponential map, we can also generate $\mathrm{Sp}(2N, \mathbb{R})$ with $\mathrm{sp}(2N, \mathbb{R})$ using symplectic Cayley transform. The symplectic Cayley transform works as follows: Let M be an element of $\mathrm{sp}(2N, \mathbb{R})$; Then

$$\begin{aligned} S &= \left(M - \frac{1}{2}\mathrm{Id}\right)\left(M + \frac{1}{2}\mathrm{Id}\right)^{-1} \\ &= \mathrm{Id} - \left(M + \frac{1}{2}\mathrm{Id}\right)^{-1} \end{aligned} \tag{2.3.3}$$

is a symplectic transformation[7]. Cayley transform is not surjective either, since it cannot generate a symplectic transformation satisfying $\det(\mathrm{Id} - S) = 0$ (e.g., $S = \mathrm{Id}$).

---

[5]A theorem in [**56**] tells us the exponential map is surjective if and only if the Lie group is compact, connected and linear.

[6]This feature marks the significant difference between symplectic groups and special orthogonal groups (so that the determinant is positive and equal to 1). Special orthogonal groups are compact Lie groups, so that they can be faithfully generated by their Lie algebras with no exception.

[7]Cayley transform is originally invented for special orthogonal groups, and later on generated to other Lie groups with proper adaptations.

However, any symplectic transformation S with $\det(\mathrm{Id} - S) = 0$ is a product of two symplectic transformations, both of which can be generated from elements in $\mathrm{sp}(2N, \mathbb{R})$ using the Cayley transform.

These properties of symplectic groups are not only mathematical concepts but also closely related to physical processes. Actually, the exponential map and symplectic Cayley transform provides for us convenient tools to systematically construct symplectic transformations using physical processes. To realize a symplectic transformation S, we must find a unitary operation (since the CCRs have to be preserved), denoted by $\hat{U}_S$, such that

$$\hat{U}_S \hat{x}_j \hat{U}_S^\dagger = \sum_{k=1}^{2N} S_{j,k} \hat{x}_k, \tag{2.3.4}$$

for any $j, k \in \{1, 2, \ldots, N\}$, so that the quadratures can be transformed as expected in the symplectic space[8]. Here, the $\hat{x}$-operators are the aforementioned universal notation for all canonical quadrature operators. Now consider the following quadratic Hamiltonian for a bosonic system

$$\hat{H} = \sum_{j,k=1}^{2N} H_{j,k} \hat{x}_j \hat{x}_k,$$

[8]In fact, any unitary operation that can transform the canonical quadrature operators linearly as above corresponds to a symplectic transformation in the symplectic space: Let $\hat{U}$ be a unitary operation with $\hat{U} \hat{x}_j \hat{U}^\dagger = \sum_{k=1}^{2N} S_{j,k} \hat{x}_k$. Then we have

$$\begin{aligned} \hat{U} \left[\hat{x}_j, \hat{x}_k\right] \hat{U}^\dagger &= \left[\hat{x}_j, \hat{x}_k\right], \\ \Leftrightarrow [\sum_{l=1}^{2N} S_{j,l} \hat{x}_l, \sum_{m=1}^{2N} S_{k,m} \hat{x}_m] &= \left[\hat{x}_j, \hat{x}_k\right], \\ \Leftrightarrow \sum_{l=1}^{2N} \sum_{m=1}^{2N} S_{j,l} S_{k,m} [\hat{x}_l, \hat{x}_m] &= \left[\hat{x}_j, \hat{x}_k\right], \\ \Leftrightarrow \left(S^T \Omega S\right)_{j,k} &= \Omega_{j,k}, \\ \Leftrightarrow S^T \Omega S &= \Omega. \end{aligned}$$

Therefore, the linear transformation S induced by the unitary operation $\hat{U}$ is a symplectic transformation. Such unitary operations form a Lie group, the metaplectic group, which is denoted by $\mathrm{Mp}(2N, \mathbb{R})$. Note that there are always more than one metaplecitc operations associated to a symplecitc transormation, since a adding a global phase to a metaplectic operation will not change the corresponding symplectic transformation. For convenience, we will refer to the metaplectic operators directly as the symplectic transformations.

where the matrix H on the right hand side is a $2N \times 2N$ real matrix[9]. Then the exponential map between $\mathrm{sp}(2N, \mathbb{R})$ and $\mathrm{Sp}(2N, \mathbb{R})$ implies that

$$\hat{U}_S = e^{-i\hat{H}t}$$

induces the symplectic transformation

$$S = e^{\Omega H t}.$$

Note that the product of $\hat{U}_S$, $\hat{U}_{S'}$ corresponding to two different symplectic transformations S and S' is a unitary operation $\hat{U}_{S'S}$ corresponds to the symplectic transformation $S'S$, i.e.

$$\hat{U}_{S'S} = \hat{U}_S \hat{U}_{S'}.$$

Example 2.9. Consider a single mode bosonic system with Hamiltonian

$$\hat{H} = \hat{q}\hat{p} + \hat{p}\hat{q} = i\left(\hat{a}^{\dagger 2} - \hat{a}^2\right),$$

which can be described by a real matrix

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the symplectic transformation corresponding to the unitary operation $e^{-i\hat{H}t}$, which is the single-mode squeezing operation, is

$$Z = e^{\Omega H t} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

We have related the exponential map to the Hamiltonain evolution of a bosonic quantum system. It turns out that the Cayley transform can also be related to a physical process–the noiseless scattering process of a bosonic system. This relation is

[9]The set of all $i\hat{H}$'s form the metaplectic Lie algebra $\mathrm{mp}(2N, \mathbb{R})$, which is the Lie algebra of the metaplectic group $\mathrm{Mp}(2N, \mathbb{R})$.

based on the observation that the symplectic Cayley transform (Eq. 2.3.3) shares a similar form with the formulas for scattering amplitudes in optics and quantum mechanics. In fact, a noiseless bosonic scattering process is governed by a dimensionless quadratic Hamiltonian, which is described by a real symmetric matrix H. Then the noiseless scattering process, yielding a unitary operation $\hat{U}_S$, induces a symplectic transformation

$$S = \mathrm{Id} - \left(\Omega H + \frac{1}{2}\mathrm{Id}\right)^{-1},$$

where we have used property that $M = \Omega H$ belongs to the symplectic algebra $\mathrm{sp}\left(2N, \mathbb{R}\right)$. More details about bosonic scattering processes can be found in Appx. A.

Sometimes the symplectic transformations obtained from a physical process can be rather complicated. The best way to intuitively understand the function of a complicated symplectic transformation is to decompose it into simple fundamental mental building blocks that we are familiar with. As we have promised, any symplectic transformation S is a composition of rotation-like symplectic transformations and squeezing-like transformaions:

$$S = RZR'$$

where R and R' are rotation-like transformations (e.g. combinations of phase-shifting and beam-splitting operations in quantum optics) and Z is a squeezing-like operation (for instance, the operation in Example. 2.9). Specifically, R and R' are special orthogonal matrices satisfying

$$RR^t = R'R'^t = \mathrm{Id},\ \det\left(R\right) = \det\left(R'\right) = 1,$$

and Z is a diagonal symplectic transformation. This decomposition is known as the polar (or Euler) decomposition of symplectic transformations.

The Euler decomposition is not the only decomposition. In general, R, R', Z may not be uniquely determined (e.g. there are infinitely many ways to decompose the identity according to the Euler decomposition). Therefore, although it will not be applied in the rest of this thesis, we would like to introduce the pre-Iwasawa decomposition, which can decompose a given symplectic transformations uniquely into three families of building blocks. The pre-Iwasawa decomposition states the following: Let us shift the ordering of the canonical basis from the default ordering to the alternative ordering $\hat{q}_1, \hat{q}_2, \ldots, \hat{p}_N, \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_N$; Then a $2N$-dimensional symplectic transformation S can be decomposed as

$$S = \begin{pmatrix} \mathrm{Id} & 0 \\ P & \mathrm{Id} \end{pmatrix} \begin{pmatrix} L^t & 0 \\ 0 & L^{-1} \end{pmatrix} \begin{pmatrix} V & W \\ -W & V \end{pmatrix}, \tag{2.3.5}$$

where P is an N-order symmetric matrix, L is a invertible N-order square matrix, and V, W are N-order square matrices satisfying that $U = V + iW$ is a unitary matrix. The three factors of this decomposition

$$\begin{pmatrix} \mathrm{Id} & 0 \\ P & \mathrm{Id} \end{pmatrix}, \begin{pmatrix} L^t & 0 \\ 0 & L^{-1} \end{pmatrix}, \begin{pmatrix} V & W \\ -W & V \end{pmatrix}$$

are all symplectic transformations. Physically, the first factor may be referred to as a quantum-non-demolition operation (e.g. a symplectic transformation generated by the Hamiltonian $\hat{H} = \hat{q}_1 \hat{q}_2$); the second factor corresponds to a multi-mode squeezing operation; the third factor is an orthogonal matrix (a combination of phase-shifting and beam-splitting). In fact, any orthogonal and symplectic transformation is of the form the third factor of the pre-Iwasawa decomposition as shown above.

CHAPTER 3

Gaussian states and Gaussian processes

3.1. Introduction

In the previous chapter, we established symplectic geometrical concepts for bosonic quantum systems. We mentioned that due to the probabilistic feature of quantum mechanics, symplectic transformations are not only tools for investigating kinetics but also important operations for preparing or manipulating quantum states. We have also seen how the linear algebraic essence of symplectic transformations allow us to calculate, operate, and understand symplectic transformations. Therefore, if quantum states can also be represented by linear algebraic objects in symplectic space such as vectors or simple real matrices, we can imagine how conveniently and efficiently the calculation of the evolution of a bosonic quantum system can be executed on a classical computer. However, as we know quantum computation is superior to classical computation and quantum circuits in general cannot be simulated efficiently on a classical computer, we must restrict our search of quantum states with such good properties to a subset of all quantum states. These states exist and known as the Gaussian states. They are the quantum states of which the Wigner functions are Gaussian functions. Since Gaussian functions are completely determined by the mean value (first moments) and the variance (the covariance matrix or the second moments), the evolution of a Gaussian state under a symplectic transformation (the state will always be Gaussian during the evolution) can be easily captured by the corresponding linear algebraic calculations. Gaussian states are thus the best quantum analogues of the classical states and will be used to simulate practical quantum noise in the following chapters.

Symplectic transformations belong to a family of quantum processes, the members of which transform Gaussian states to Gaussian states. We call such quantum processes the Gaussian processes. It turns out that if a Gaussian process is unitary, it can be either a symplectic transformation or a displacement operation in the phase space. The latter will be introduced and discussed in this chapter along with its covariance with symplectic geometry which can be used as an equivalent definition of symplectic transformations. We will also briefly discuss the noisy Gaussian processes–the Gaussian channels. They are convenient theoretical models of quantum noise, which can facilitate some analyses in the later chapters.

More importantly, Gaussian states and Gaussian processes are ubiquitous in quantum physics. For example, in quantum optics, the well-known vacuum state, coherent states, squeezed states, thermal states, etc. are all Gaussian states; common operations such as beam-splitting, squeezing, phase-shifting are unitary Gaussian processes (since they are symplectic transformations); more generally, bosonics scattering processes generated by quadratic Hamiltonians are also Gaussian processes. Gaussian process is also a widely used theoretical tool in Gaussian quantum information[**4**]. In this thesis, we hope to demonstrate that Gaussian process, especially its symplectic geometrical features, can be applied to solving more practical quantum engineering problems. This chapter will lay the ground for some of the key steps of our results.

We will start with displacement operations, and then introduce Wigner functions, Gaussian states and Gaussian channels. In the last section of this chapter, we show our systematic method of embedding a noisy Gaussian process in a unitary Gaussian process.

3.2. Displacement operation

Physical laws are independent of the choice of the coordinate systems, including the origin. We have seen the CCRs are preserved under symplectic transformations, which are coordinate transformations preserving the position of the origin. Then it is natural to wonder what are the CCR-preserving transformations that change the origin of the phase space in the meanwhile. It turns out that the simplest operations satisfying these requirements are the displacement operations transforming the canonical quadrature operators as follows

$$\hat{x}_j \to \hat{x}_j + u_j,$$

where u is a real vector. We denote this operation by a map $\mathcal{D}_u : E \to E$, where E is a $2N$-dimensional symplectic space. It follows that these maps form an Abelian group: For any pair of displacement operations $\mathcal{D}_u$ and $\mathcal{D}_{u'}$, we have

$$\mathcal{D}_u \mathcal{D}_v = \mathcal{D}_{u+v},$$

and the identity of this group is obviously the operation $\mathcal{D}_{u=0}$.

Physically, the displacement operations are realized by the well-known displacement operators. The physical embodiment of the displacement operations can be easily constructed using the quadrature operators. That is, for all $u \in \mathbb{R}^{2N}$, we have a displacement operator of the following form[1]:

$$\hat{D}_u = e^{2i \sum_{j=1}^{2N} (\Omega u)_j \hat{x}_j}.$$

It follows that any quadrature operator $\hat{v} = \sum_j v_j \hat{x}_j \in E$ is transformed under the displacement operation to

$$\hat{D}_u \hat{v} \hat{D}_u^\dagger = \hat{v} + u.$$

Also, for any $2N$-dimensional real vectors $u, v \in \mathbb{R}^{2N}$, we have

$$\begin{aligned} \hat{D}_{u+v} \hat{w} \hat{D}_{u+v}^\dagger &= \hat{D}_u \hat{D}_v \hat{w} \hat{D}_v^\dagger \hat{D}_u^\dagger \\ &= \hat{w} + u + v, \end{aligned}$$

[1]Usually in quantum optics, the displacement vector u is a complex N-dimensional vector. Here we take its real and imaginary parts separately to form a $2N$-dimensional real vector

for any quadrature operator $\hat{w} = \sum_{j=1}^{2N} w_j \hat{x}_j \in E$. In particular, we have$\hat{D}_u^\dagger = \hat{D}_{-u}$. These relations confirm that the displacement operators are the right physical realization of $\mathcal{D}_u$'s.

Moreover, the displacement operations are covariant with the symplectic transformations: Let S be a symplectic transformation in the symplectic space E; then we have

$$\begin{aligned}\hat{U}_S \hat{D}_u \hat{U}_S^\dagger &= e^{i\sum_{j,k=1}^{2N} (\Omega u)_j S_{jk} \hat{x}_k} \\ &= e^{i\sum_{j=1}^{2N} (S^t \Omega u)_j \hat{x}_k} \\ &= e^{i\sum_{j=1}^{2N} (\Omega S^{-1} u)_j \hat{x}_k} \\ &= \hat{D}_{S^{-1}u}.\end{aligned}$$

In other words, symplectic transformations preserves the groups of the displacement operations. Actually, the set of all the unitary operations that preserve the group of displacement operations also form a group, containing the unitary operator that may be best addressed as the Gaussian unitary operators. In fact, a Gaussian unitary operator $\hat{U}$ is always the product of a symplectic transformation and a displacement operation, that is

$$\hat{U} = \hat{D}_u \hat{U}_S$$

for some $u \in \mathbb{R}^{2N}$ and $S \in \mathrm{Sp}\,(2N, \mathbb{R})$. So to speak, symplectic transformations can be defined equivalently as the quotient group of the group of Gaussian unitary operators modulo the group of displacement operations.

Last but not the least, exchanging the orders of two displacement operations in their product will yield an extra phase:

$$\hat{D}_u \hat{D}_v = e^{-i\sigma(u,v)} \hat{D}_v \hat{D}_u, \tag{3.2.1}$$

where $\sigma(u,v)$ is the symplectic form. Then the covariance of the displacement operation with the symplectic transformations amounts to the preservation of this phase:

$$\hat{D}_{S^{-1}u}\hat{D}_{S^{-1}v} = e^{-i\sigma(u,v)}\hat{D}_{S^{-1}v}\hat{D}_{S^{-1}u}. \tag{3.2.2}$$

Moreover, we can view it as an equivalent definition of the symplectic transformations, which will be useful later in this thesis.

The displacement operators are crucial to define Wigner functions in the next section. Now we can proceed to discussing this important quasi-probabilistic representation of quantum states.

3.3. Wigner function

The probabilistic nature of quantum states complicates their representation in the phase space. Due to the Heisenberg's uncertainty principle, the momenta and positions of a physical system cannot be precisely measured at the same time. Therefore, a point in the phase space can no longer faithfully describe a physical state. To best reflect this probabilistic nature, a faithful representation of a quantum state in phase space should assign to each point a "probability" showing how possible the system is of the corresponding momenta and positions. It turns out these requirements give rise to the concepts of quasi-probabilistic distributions–functions mapping each point in the phase space to a real/complex value and sharing important properties with a normal probability distribution. Among possible constructions of quasi-probabilistic distributions, we are particularly interested in the Wigner function: It is real-valued distribution that is equivalent to the usual probabilistic distribution[2] for certain quantum states. In this and the succeeding sections, we discuss the general properties and useful instances of the Wigner function, which will be crucial to the rest of this thesis[3].

[2]For example, the phase representation of a statistical ensemble in classical mechanics.

[3]We will stick to the conventions in [**4**]. Rigorous proofs of the statements in this section can be found in [**1**].

Consider an N-mode bosonic system with a $2N$-dimensional symplectic space E. The set of the displacement operators is a complete basis of quantum operators, i.e., any quantum operator $\hat{A}$[4] can be represented by a linear combination of displacement operators. Specifically, given an operator $\hat{A}$, the complex coefficient of this linear combination can be obtained from the following formula:

$$\chi_{\hat{A}}(v) = \mathrm{Tr}[\hat{A}\hat{D}_v], \tag{3.3.1}$$

which is justified by the property

$$\mathrm{Tr}[\hat{D}_v] \neq 0, \text{ if and only if } v = 0, \ \forall v \in E$$

of the displacement operators [5]. Therefore, when applied to a quantum state $\hat{\rho}$, this defines a function mapping each point in the symplectic space to a complex value. We call $\chi_{\hat{A}}$ the characteristic function of $\hat{A}$. It follows that based on the characteristic function can be further converted into a function, the Wigner function $W_{\hat{A}}$, through the symplectic Fourier transform, as follows:

$$W_{\hat{A}}(u) = \int_E \frac{d^{2N}v}{(2\pi)^{2N}} e^{-iu^t\Omega v}\chi_{\hat{A}}(v), \tag{3.3.2}$$

where the integration is carried out over the whole symplectic space. In particular, when $\hat{A}$ is Hermitian (for example, $\hat{A}$ being the density operator of a quantum state), the Wigner function $W_{\hat{A}}$ is a real-valued function.

Example 3.1. The Winger functions of the canonical quadrature operators $\hat{q}_1$, $\hat{p}_1$, $\hat{q}_2$, $\hat{p}_2$, $\ldots$, $\hat{q}_N$, $\hat{p}_N$ are:

$$W_{\hat{q}_1}(u) = u_1,\, W_{\hat{p}_1}(u) = u_2,\, \ldots$$

i.e., the classical canonical coordinates in the phase space.

[4]Aside from some mathematical exceptions, we assume no constraint on the form of operator $\hat{A}$. The operator $\hat{A}$ can be Hermitian, unitary, non-Hermitian, non-unitary,etc.

[5]This property can be consider ed to be a generalization of Fourier transform, which originated from the resemblance between the displacement operators and the e^{ikx}'s in the Fourier transform.

When applied to a density operator, the Wigner function is a faithful quasi-probabilistic representation of the quantum state. This can be seen from the following properties of the Wigner function:

Let $\hat{\rho}$ be the density operator of a quantum state (no matter it is pure or mixed):

(1) The Wigner functions are normalized. For a Wigner function $W_{\hat{\rho}}$ of any density operator $\hat{\rho}$, we have

$$\int_E W_{\hat{\rho}}(v)dv = 1 \tag{3.3.3}$$

(2) Let $\hat{O}$ be an observable. Then its expectation value can be calculated by

$$\langle \hat{O} \rangle_{\hat{\rho}} = \mathrm{Tr}\left[\hat{\rho}\hat{O}\right] = \int_E W_{\hat{O}}(v) W_{\hat{\rho}}(v)dv. \tag{3.3.4}$$

Although the above properties seem to be the same as those of a usual probabilistic distribution, the Wigner function is still quasi-probabilistic insofar as the Wigner functions of certain quantum states (e.g., the cat states) can take negative values on some points in the symplectic space, in contrast to the always-positive property of a probabilistic distribution.

Moreover, Wigner functions are covariant with the displacement operations and symplectic transformations:

Let $\hat{A}$ be an operator:

(1) Action of displacement operations:

$$W_{\hat{D}_v \hat{A}) \hat{D}_v^\dagger}(u) = W_{\hat{A}}(u - v) \tag{3.3.5}$$

(2) Action of symplectic transformations:

$$W_{\hat{U}_S \hat{A} \hat{U}_S^\dagger}(u) = W_{\hat{A}}(S^{-1}u) \tag{3.3.6}$$

These properties allow us to conveniently manipulate the Wigner functions of a quantum state using symplectic transformations and displacement operations. In addition, the Wigner functions are compatible with more quantum operations that are commonly used in analyzing quantum processes:

Let $\hat{O}_1$, $\hat{O}_2$ be Hermitian operators:

(1) Trace and partial trace: Let $\mathcal{H}$ denote the N-bosonic-mode Hilbert space. Let $\mathcal{H}_\alpha$ be an M-bosonic-mode Hilbert space (thus a subspace of $\mathcal{H}$). Then

$$W_{\mathrm{tr}_{\mathcal{H}_\alpha}[\hat{O}_1\hat{O}_2]}(u) = \int_{E_\alpha} W_{\hat{A}_1}(v)W_{\hat{A}_2}(v)dv_\alpha \tag{3.3.7}$$

with the right-hand-side integrated over the phase space E_α associated to the Hilbert subspace $\mathcal{H}_\alpha$, which is a symplectic subspace of the symplectic space E.

(2) Tensor product: Let $\hat{O}_1$ and $\hat{O}_2$ be Hermitian operators on bosonic Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then

$$W_{\hat{O}_1\otimes\hat{O}_2}(u_1 \oplus u_2) = W_{\hat{O}_1}(u_1)W_{\hat{O}_2}(u_2). \tag{3.3.8}$$

The phase space associated to the tensor-product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the direct sum of the phase spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ associated with $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Intuitively, this means the tensor product of Hilbert spaces can be converted into direct sums of symplectic spaces, along with the corresponding operations.

Before proceeding to the succeeding section where the important instances, the Gaussian states, are discussed, we conclude the section here with a special family of quantum states, the infintely squeezed states (or equivalently the eigenstates of quadrature operators, for example the eigenstate of $\hat{p}$ with wave function being a propagating wave $\psi(q) \propto e^{ipq}$ and the eigenstate of $\hat{q}$ with wave function being a delta function $\psi(q) \propto \delta(q - q'))$. Physically speaking, such states do not exist in practice since their wave functions cannot be normalized. However, these states turn out to be useful theoretical tools because of the convenient mathematical properties of delta functions. Consider the infintely squeezed state which is an eigenstate of $\hat{q}$. Then the deviation of the $\hat{q}$-quadrature $\left\langle \left(\hat{q} - \langle \hat{q} \rangle^2\right) \right\rangle$ is vanishing while the deviation of the $\hat{p}$-quadrature $\left\langle \left(\hat{p} - \langle \hat{p} \rangle^2\right) \right\rangle$ blows up. This means, in the phase space, such a state should be strictly represented by a straight line (as in mathematics), parallel to the $\hat{p}$-axis. Therefore, the Wigner function of this state should be of the form a delta function. Specifically, in a N-mode bosonic system, let l_z be a Lagrangian plane (as defined in the preceding chapter) of the symplectic space, which is, simply speaking, spanned by N quadratures of a symplectic basis of the space (so the other half of the basis are conjugate with them in respect of the CCRs). Then this Lagrangian plane determines a infintely squeezed (a common eigenstate of the N quadrature operators), of which the Wigner function is given by[6]

$$W_{\hat{\Pi}_{l_z}(\eta)}\left(u\right) \propto \delta\left(u - \eta\right).$$

Here $\hat{\Pi}_{l_z}\left(\eta\right)$ denotes the "density operator" of the corresponding infinitely squeezed state, with l_z being the determinant Lagrangian plane and η an N-dimensional vector consisting of the eigenvalues of the N determinant quadrature operators.

Example 3.2. Consider a Lagrangian plane spanned by $\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_N$. Then the eigenstate $\hat{\Pi}_{l_z}\left(\eta\right)$ denotes an infinitely squeezed state satisfies that

$$\mathrm{Tr}\left[\hat{q}_j \hat{\Pi}_{l_z}\left(\eta\right)\right] = \eta_j,\ \forall j \in \left\{1, 2, \ldots, N\right\}.$$

Although the infinitely squeezed states are fictitious states introduced for theoretical convenience, they can be approximated by practical quantum states, such as the Gaussian states. In addition, Gaussian states are best analogues of classical states and are handy models of quantum noise. In the succeeding section, we introduce and discuss the definition and properties of the Gaussian states.

[6]We pretend this state can be normalized by forcing $\int_E W_{\hat{\Pi}_{l_z}(\eta)}\left(u\right) = 1$.

3.4. Gaussian states

Statistical ensemble of classical states can be described by a probabilistic distribution, assigning to each point in the phase space a (non-negative) probability. Therefore, we could expect the Winger function of a "classical" quantum state to be non-negative over the whole symplectic space. According to the findings in [**57**], a pure quantum state is "classical" in this sense if and only if its Wigner function is a Gaussian function. Therefore, we consider the quantum states, of which the Wigner functions are Gaussian functions, to be the best analogues of classical states and refer to them as the Gaussian states[7].

Specifically, the Wigner function of a Gaussian state $\hat{\rho}$ is of the following form

$$W_{\hat{\rho}}(u) = \frac{e^{-\frac{1}{2}(u-\bar{x})^t V^{-1}(u-\bar{x})}}{(2\pi)^{N/2}\sqrt{\det V}} \tag{3.4.1}$$

where the vector

$$\bar{x} = \langle \hat{x} \rangle \tag{3.4.2}$$

consists of the expectation values of the canonical quadrature operators, called the first moments, and the covariance matrix

$$V_{j,k} = \langle (\hat{x}_j - \bar{x}_j)(\hat{x}_k - \bar{x}_k) \rangle, \ \forall j,k \in \{1,2,\ldots,2N\}, \tag{3.4.3}$$

also known as the second moments, consists of the second order correlation between quadrature operators. Intuitively, we can imagine that a Gaussian state to be a smeared blob in the phase space, with the position of its center given by the vector $\bar{x}$ and its size as well as shape given by the covariance matrix[8].

---

[7]We adopt the conventions of [**4**] in this section.

[8]In classical mechanics, the inertia tensor, which is simply a real symmetric matrix, assigns to each rigid body an ellipsoid. The three principal components of the inertia tensor determines the stretches of the ellipsoid along the three principal axes. Similarly, the covariance matrix assigns to each Gaussian state a multi-dimensional ellipsoid. It can be diagonalized by rotating the the coordinate system, and the diagonal elements of the diagonalized covariance matrix thus determines the size and shape of the multi-dimensional ellipsoid.

It turns out under the displacement operations and symplectic transforamtions, a Gaussian state will still be transformed to a Gaussian state. Moreover, since the a Gaussian state is completely determined by the first moments $\bar{x}$ and the covariance matrix V, the covariance of its Wigner function with the these operations can be reflected by the change of the first moments and the covariance matrix, which can be easily calculated using simple linear algebra. Specifically, for a Gaussian state $\hat{\rho}$ with first moments being $\bar{x}$ and covariance matrix being V, according to Eqs. 3.3.5 & 3.3.6, we have

$$\begin{aligned} W_{\hat{U}_S\hat{D}_v\hat{\rho}\hat{D}_v^\dagger\hat{U}_S^\dagger}(u) &= \frac{e^{-\frac{1}{2}(S^{-1}(u-v)-\bar{x})^t V^{-1}(S^{-1}(u-v)-\bar{x})}}{(2\pi)^{N/2}\sqrt{\det V}} \\ &= \frac{e^{-\frac{1}{2}(u-v-S\bar{x})^t\left(SV\left(S^t\right)\right)^{-1}(u-v-S\bar{x})}}{(2\pi)^{N/2}\sqrt{\det\left(SVS^t\right)}} \end{aligned}$$

for any symplectic transformation S and displacement operation $\hat{D}_v$, which represents a Gaussian state $\hat{\rho}'$ with first moments

$$\bar{x}' = S\bar{x} + v \tag{3.4.4}$$

and covariance matrix

$$V' = SVS^t. \tag{3.4.5}$$

In consideration of this nice property, we denote a Gaussian state using its first moments $\bar{x}$ and covariance matrix V by $\hat{\rho}\left(\bar{x}, V\right)$.

Example 3.3. The above calculation describes the following transformation between Gaussian states:

$$\hat{\rho}\left(\bar{x}, V\right) \to \hat{\rho}\left(S\bar{x} + v, SVS^t\right).$$

We now demonstrate some examples of Gaussian states, which are important quantum states in quantum optics.

Example 3.4. (Vacuum state) The simplest example of a Gaussian state is the vacuum state $|0\rangle\langle 0|$, i.e. the pure bosonic quantum state that can be annihilated by every bosonic annihilation operator of the system. The first moments of $|0\rangle\langle 0|$ is

$$\bar{x} = 0,$$

and the covariance matrix is

$$V = \mathrm{Id}.$$

Therefore, we can also represent the vacuum state by $\hat{\rho}\,(0, \mathrm{Id})$.

EXAMPLE 3.5. (Coherent state) A coherent states $|\alpha\rangle$, which are defined as a state satisfying $\hat{a}_j|\alpha\rangle = \alpha_j|\alpha\rangle$ with α_j being complex number for all $j \in \{1, 2, \ldots, N\}$, can be created out of the vacuum state by applying the displacement operation:

$$\hat{D}_u|0\rangle = |\alpha\rangle. \tag{3.4.6}$$

Here we let $u_{2j-1} = \Re[\alpha_j]$ (the real part of α_j) and $u_{2j} = \Im[\alpha]$ (the imaginary part of α_j) for all $j = \{1, 2, \ldots, N\}$. The first moments and the covaraince matrix of such a coherent state are respectively

$$\bar{x} = u,$$

and

$$V = \mathrm{Id}.$$

We can represent such a coherent state by $\hat{\rho}\,(u, \mathrm{Id})$.

EXAMPLE 3.6. (Thermal state) Gaussian states can also be mixed states. Let $\hat{\rho}_{th}$ be a thermal state satsifying $\mathrm{Tr}[\hat{\rho}\hat{n}_j] = \bar{n}_j$ with $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$ for $j \in \{1, 2, \ldots, N\}$ (i.e. different modes are surrounded by different thermal baths). The thermal state $\hat{\rho}_{th}$ is a Gaussian state, with the first moments being

$$\bar{x} = 0,$$

and the covariance matrix being

$$V=\begin{pmatrix}2\bar{n}_1+1 & 0 & \cdots & 0 & 0\\ 0 & 2\bar{n}_1+1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & 0 & 0\\ 0 & 0 & 0 & 2\bar{n}_N+1 & 0\\ 0 & 0 & 0 & 0 & 2\bar{n}_N+1\end{pmatrix}.$$

We have seen the covaraince matrix plays a crucial role in describing Gaussian states. The following theorem allows to reveal the structure the covaraince matrix in great detail:

THEOREM 3.7. *(Williamson's theorem for strictly positve symmetric matrices[9]) Let V be a $2n$-dimensional real symmetric matrix, with strictly positive eigenvalues. There exists a symplectic matrix $S\in \mathrm{Sp}(2n,\mathbb{R})$, such that SVS^t is a diagonal matrix with positive eigenvalues $\{\bar{n}_1,\bar{n}_1,\bar{n}_2,\bar{n}_2,\ldots,\bar{n}_N,\bar{n}_N\}$.*

speaking, this theorem tells us that given a Gaussian state $\hat{\rho}\left(\bar{x},V\right)$can be transformed into a thermal state by displacement operations and symplectic transformations:

$$\hat{\rho}_{th}=\hat{U}_S\hat{D}_{-\bar{x}}\hat{\rho}_{th}\hat{D}^{\dagger}_{\bar{x}}\hat{U}^{\dagger}_S. \tag{3.4.7}$$

Here, we let S be a symplectic transformation that can diagonalize the covariance matrix V using the above theorem. And the thermal excitation numbers, i.e. $n_j=\mathrm{Tr}\left[\hat{\rho}_{th}\hat{a}^{\dagger}_j\hat{a}_j\right]$, for each mode are can be extracted from the diagonalized covariance matrix of the resulting thermal state $\hat{\rho}_{th}$.

In the end of the preceding section, we mentioned that the fictitious infinitely squeezed states can be approximated by Gaussian states. To demonstrate it, we

[9]The proof of this theorem can be found in [**1**]

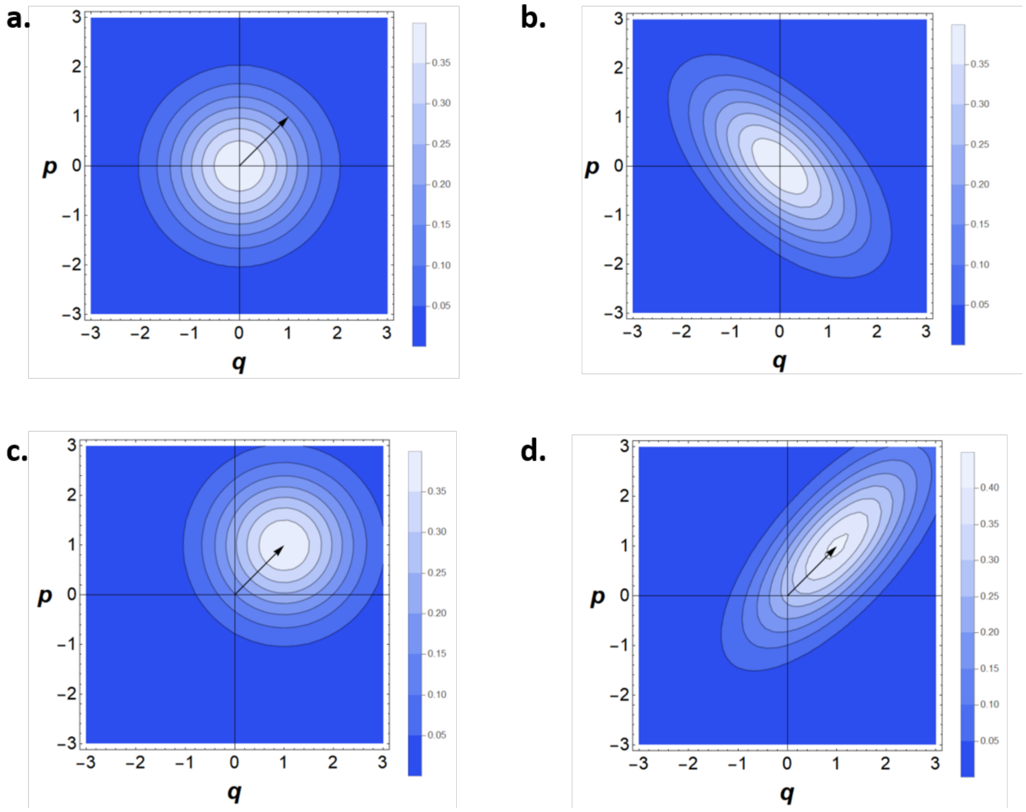

FIGURE 3.1. *Gaussian states.*

These are the illustrations of Wigner functions of some Gaussian states in the symplectic space: (a) A vacuum state. (b) A squeezed state. (c) A coherent state with first moments $(1,1)^t$. (d) A randomly generated Gaussian state.

consider the following squeezing-like symplectic transformation

$$Z(\zeta) = \begin{pmatrix} \zeta & 0 & \cdots & 0 & 0 \\ 0 & \zeta^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & \zeta^{-1} \end{pmatrix},$$

with ζ being a nonzero real number. Then given any Gaussian state $\hat{\rho}\left(\bar{x}, V\right)$, we can generate an infinitely squeezed state

$$\hat{\Pi}_{l_z}\left(\eta\right) = \lim_{\zeta \to 0} \hat{U}_{Z(\zeta)} \hat{\rho}\left(\bar{x}, V\right) \hat{U}^{\dagger}_{Z(\zeta)},$$

where l_z is a Lagrangian plane spanned by all the $\hat{q}$-quadratures (seen from the form of $Z\left(\zeta\right)$), and η is a N-dimensional vector defined by

$$\eta_j = \bar{x}_{2j-1}, \forall j \in \{1, 2, \ldots, N\}.$$

Therefore, when ζ is small enough (i.e. ζ^{-1} is large enough), the Gaussian state $\hat{U}_{Z(\zeta)} \hat{\rho}\left(\bar{x}, V\right) \hat{U}^{\dagger}_{Z(\zeta)}$ is a good approximate of the infinitely squeezed state $\hat{\Pi}_{l_z}\left(\eta\right)$. It is not difficult to realize that every infinitely squeezed state can be generated in a similar way.

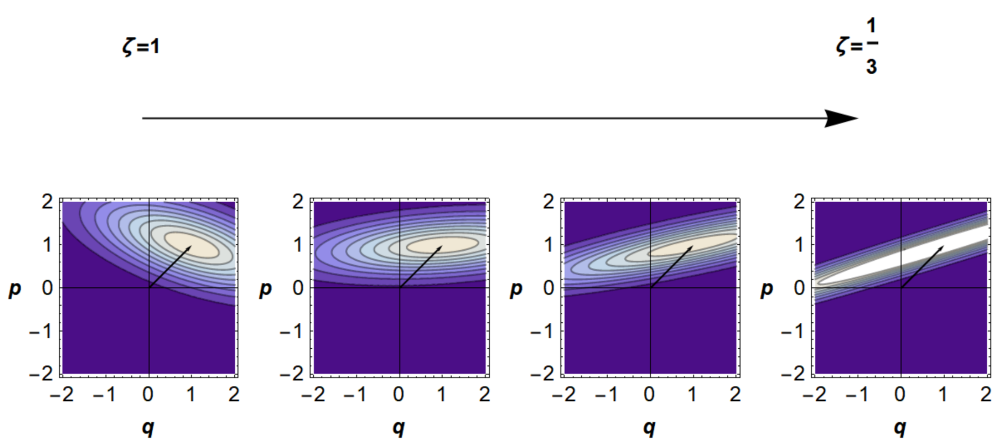

FIGURE 3.2. *Approximating an infinitely squeezed state.*

The four figures shows how a Gaussian state can be approximately converted into an infinitely squeezed state under squeezing operations with increasing amount of squeezing, quantified by the real number ζ. The initial Gaussian state and the squeezing operation are randomly chosen.

3.5. Gaussian measurement

The Wigner functions can also be applied to describing quantum measurements. In a discrete variable quantum system, a measurement scheme is described by a positive-operator-valued measure (POVM), which is a set of positive operators $\left\{\hat{P}_j\right\}$, satisfying the normalization condition $\sum_j \hat{P}_j = \mathrm{Id}$ where the subscript j of each $\hat{P}_j$ represents the corresponding measurement outcome. The probability $p\left(j\right)$ of obtaining such an outcome j by measuring a quantum state $\hat{\rho}$ is determined by

$$p\left(j\right) = \mathrm{Tr}\left[\hat{\rho}\hat{P}_j\right].$$

In continuous variable systems, the measurement outcomes can be taken from a continuum[10]. For such measurement schemes, the corresponding POVMs consist of infintely many positive operators $\hat{P}\left(\eta\right)$, with measurement outcomes η which is in general a real-valued vector, satisfying $\int \hat{P}\left(\eta\right) d\eta = \mathrm{Id}$[11]. Then the formula

$$p\left(\eta\right) = \mathrm{Tr}\left[\hat{\rho}\hat{P}\left(\eta\right)\right]$$

gives a probabilistic distribution of the measurement outcomes. According to the property of Wigner function in the preceding section, this formula is equivalent to the following

$$p\left(\eta\right) = \int_E W_{\hat{\rho}}\left(u\right) W_{\hat{P}(\eta)}\left(u\right) du,$$

where $W_{\hat{\rho}}$ and $W_{\hat{P}(\eta)}$ are the Wigner functions of the corresponding operators.

As we have seen in the preceding section, Gaussian states have easy-to-handle Wigner functions. Therefore, we are interested in measurement schemes, POVM of which consists of a family of Gaussian density operators $\hat{\rho}\left(\bar{x} - \eta, V\right)$, where η, a $2N$-dimensional real vector, denotes a measurement outcome. We call such measurement schemes Gaussian measurements. In particular, as we mentioned in the preceding

[10] For example, the outcomes of the heterodyne measurement can be considered to be real-valued vectors.

[11] The formulas for the discrete variable systems can also be expressed as integrals according to measure theory in mathematics.

section, since Gaussian states can approximate infinitely squeezed states, Gaussian measurements also include measurement schemes with POVM consisting of infinitely squeezed states $\hat{\Pi}_{l_z}(\eta)$ determined by a fixed Lagrangian plane l_z. Here the measurement outcome η is an N-dimensional real vector.

Example 3.8. (Ideal heterodyne measurement) Heterodyne measurement is an example of Gaussian measurements. The POVM of the ideal heterodyne measurement scheme consists of Gaussian states $\hat{\rho}(\eta, \mathrm{Id})$.

Example 3.9. (Ideal homodyne measurement) The ideal homodyne measurement is described by the POVM consisting of the infinitely squeezed states $\hat{\Pi}_{l_z}(\eta)$, with l_z being a Lagrangian plane.

Generally speaking, the theoretical model of a general Gaussian measurement can be decomposed into the combination of Gaussian states, symplectic transformations, and ideal homodyne measurements[12]. We start with the ideal heterodyne measurement (Fig. ??) for an instance. let $\hat{U}_{H(1/2)}$ be a symplectic transformation with

$$H(1/2) = \begin{pmatrix} \sqrt{1/2} & 0 & \sqrt{1/2} & 0 \\ 0 & \sqrt{1/2} & 0 & \sqrt{1/2} \\ -\sqrt{1/2} & 0 & \sqrt{1/2} & 0 \\ 0 & -\sqrt{1/2} & 0 & \sqrt{1/2} \end{pmatrix},$$

describing a balanced beamsplitter. Then the probability of obtaining the outcome η through the ideal heterodyne measurement is[13]

$$p(\eta) \propto \mathrm{Tr}\left[\hat{\Pi}_{l_z,0}(\eta)\, \hat{U}_{H(1/2)} \left(\hat{\rho} \otimes \hat{\rho}_{env}\right) \hat{U}^{\dagger}_{H(\tau)}\right],$$

where l_z is the Lagrangian plane spanned by the $\hat{q}$-quadratures.

[12] For a similar statement, see discussions in Sec. 4.5.2 of [**58**] about the "general dyne" measurement.
[13] For convenience, we do not track the normalization constant since the can always be obtained afterwards.

Now we look into a more general situation. Let $\hat{\rho}_{env}$ be a Gaussian state,

$$H(\tau) = \begin{pmatrix} \sqrt{1-\tau} & 0 & \sqrt{\tau} & 0 \\ 0 & \sqrt{1-\tau} & 0 & \sqrt{\tau} \\ -\sqrt{\tau} & 0 & \sqrt{1-\tau} & 0 \\ 0 & -\sqrt{\tau} & 0 & \sqrt{1-\tau} \end{pmatrix}$$

be a beamsplitter-like symplectic transformation with transmittance satisfying $0 \leq \tau \leq 1$, and l_z a Lagrangian plane. Then we have

$$p(\eta) \propto \mathrm{Tr}\left[\hat{\Pi}_{l,0}(\eta)\,\hat{U}_{H(\tau)}(\hat{\rho}\otimes\hat{\rho}_{env})\,\hat{U}^{\dagger}_{H(\tau)}\right],$$

which translate to the language of Winger functions as follows:

$$\begin{aligned}
p(\eta) &\propto \int_{E\cong E_1\oplus E_2} \delta\left(u|_l - \eta\right) W_{\mathcal{U}_H(\hat{\rho}\otimes\hat{\rho}_{env})}(u)\, du \\
&= \int_{l'} W_{\hat{\rho}}\left(\begin{pmatrix} \sqrt{1-\tau} & 0 \\ 0 & -\sqrt{\tau} \end{pmatrix} v + \begin{pmatrix} -\sqrt{\tau} & 0 \\ 0 & \sqrt{1-\tau} \end{pmatrix}\eta\right) \\
&\quad W_{\hat{\rho}_{env}}\left(\begin{pmatrix} \sqrt{\tau} & 0 \\ 0 & \sqrt{1-\tau} \end{pmatrix} v + \begin{pmatrix} \sqrt{1-\tau} & 0 \\ 0 & \sqrt{\tau} \end{pmatrix}\eta\right) dv \\
&\propto \int_{E_1} W_{\hat{\rho}}(v)\, W_{\hat{\rho}_{env}}(\begin{pmatrix} \sqrt{\tau/(1-\tau)} & 0 \\ 0 & -\sqrt{(1-\tau)/\tau} \end{pmatrix} v \\
&\quad + \begin{pmatrix} \sqrt{1/(1-\tau)} & 0 \\ 0 & \sqrt{1/\tau} \end{pmatrix}\eta)dv.
\end{aligned}$$

Back in the operator representation, it follows that the POVM of this measurement is the following set

$$\left\{\hat{\Pi}_{0,V}\left(\begin{pmatrix} \frac{1}{\sqrt{\tau}} & 0 \\ 0 & -\frac{1}{\sqrt{1-\tau}} \end{pmatrix}\eta\right) = \hat{\rho}\left(\begin{pmatrix} \frac{1}{\sqrt{\tau}} & 0 \\ 0 & -\frac{1}{\sqrt{1-\tau}} \end{pmatrix}\eta, V\right)\right\},$$

where

$$V = \begin{pmatrix} \sqrt{\frac{1-\tau}{\tau}} & 0 \\ 0 & -\sqrt{\frac{\tau}{1-\tau}} \end{pmatrix} V_{env} \begin{pmatrix} \sqrt{\frac{1-\tau}{\tau}} & 0 \\ 0 & -\sqrt{\frac{\tau}{1-\tau}} \end{pmatrix}.$$

Therefore, we see that by replacing $H(\tau)$ with other symplectic transformations and properly choosing V_{env}, we are able to make V an arbitrary covariance matrix, and therefore a arbitrary Gaussian measurement[14].

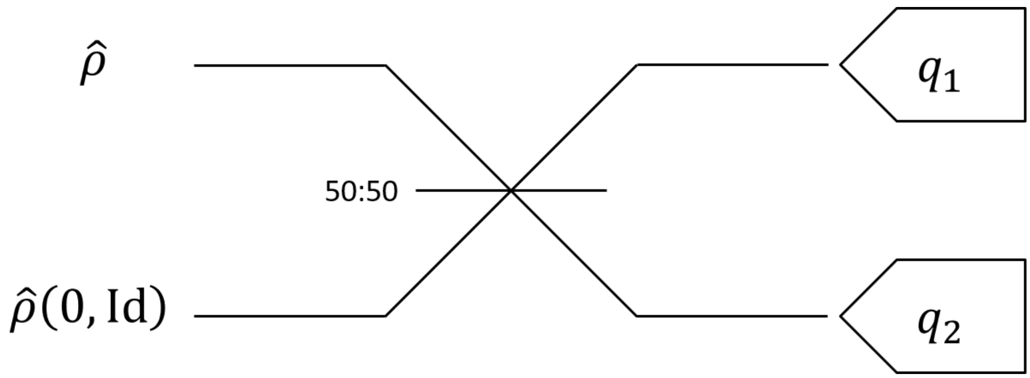

FIGURE 3.3. *The ideal heterodyne measurement.*

The density operator $\hat{\rho}$ represents the quantum state to be measured. The $\hat{\rho}\,(0, \mathrm{Id})$ is a Vacuum ancillary state. These states are mixed by a balanced beamsplitter, of which the output modes are measured by two ideal homodyne measurements measuring the $\hat{q}$-quadratures. The measurement results, grouped as a two dimensional real vector or a complex number, are then the outcome of an ideal heterodyne measurement.

3.6. Gaussian channel

We have so far focused mainly on unitary operations. Generally speaking, a quantum process is a CPTP map–quantum channel. A quantum channel $\mathcal{E}$, is a map between the spaces of bounded operators on two Hilbert spaces, transforming density

[14]This first moments can be easily engineered using displacement operations

operators to density operators. Certain noisy quantum processes can also transform Gaussian states into Gaussian states, and are therefore called the Gaussian channel[15].

We have seen that the transformation of a Gaussian state is completely determined by the change of its first moments $\bar{x}$ and covariance matrix V. It turns out that a Gaussian channel can be faithfully represented by linear transformations describing the corresponding changes. Specifically, given a Gaussian channel, a Gaussian state $\hat{\rho}\left(\bar{x}, V\right)$ is transformed to another Gaussian state $\hat{\rho}\left(\bar{x}', V'\right)$ with

$$\bar{x}' = T\bar{x} + d,$$

and

$$V' = T\bar{x}T^t + N,$$

where d is a real vector, and T, N are real matrices. All of d, T, and N are independent of the input and output states and thus completely determined by the quantum channel. For this reason, we denote such a Gaussian channel by $\mathcal{G}_{T,N,d}$. Physically speaking, the T matrix the distortion of the position, size, and shape of a Gaussian state in the phase space; and the N matrix, always a real symmetric matrix, can be considered to be the added quantum noise, which justifies the noisy nature of the process.

Example 3.10. Symplectic transformations and displacement operations are trivial examples of Gaussian channels. A symplectic transformation $\hat{U}_S$ is the Gaussian channel $\mathcal{G}_{S,0}$ while a displacement operation $\hat{D}_v$ is the Gaussian channel $\mathcal{G}_{\mathrm{Id},0,v}$.

Since the displacement operation are usually omitted in this thesis[16], we will tacitly assume $d = 0$ for the Gaussian channels to be discussed, and therefore denote them by $\mathcal{G}_{T,N}$.

[15]We follow the conventions in [**4**].

[16]The effect of the omitted displacement operation can be easily added back through a post-processing operation.

Combining two Gaussian channels is as easy as calculating simple matrix addition and multiplication. Let $\mathcal{G}_{T_1,N_1}$ and $\mathcal{G}_{T_2,N_2}$be two Gaussian channels. Then their combination $\mathcal{G}_{T_2,N_2} \circ \mathcal{G}_{T_1,N_1}$ results in a Gaussian channel $\mathcal{G}_{T_3,N_3}$ with

$$T_3 = T_2 T_1, \tag{3.6.1}$$

and

$$N_3 = T_2 N_1 N_2^t + N_2. \tag{3.6.2}$$

We can also construct a larger-size Gaussian channel by juxtaposing smaller-size Gaussian channels. Let the two Gaussian channels $\mathcal{G}_{T_1,N_1}$ and $\mathcal{G}_{T_2,N_2}$ be defined in different non-overlapping bosonic systems. Then we can juxtapose the two systems to form a larger bosonic system (through the tensor product of their Hilbert spaces), which leads to a larger symplectic space (the direct sum of the smaller symplectic spaces). Then on this resulting bosonic system, the juxtaposition of the given Gaussian channels $\mathcal{G}_{T_1,N_1} \otimes \mathcal{G}_{T_2,N_2}$ is a Gaussian channel $\mathcal{G}_{T_3,N_3}$ with

$$\begin{aligned} T_3 &= T_1 \oplus T_2 \\ &= \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \end{aligned} \tag{3.6.3}$$

and

$$\begin{aligned} N_3 &= N_1 \oplus N_2 \\ &= \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}. \end{aligned} \tag{3.6.4}$$

Gaussian channel is an efficient language noisy Gaussian quantum processes. However, it lacks the critical mathematical features of symplectic geometry, which should be deeply rooted in its construction. In the succeeding section, we discuss a systematic method of embedding non-trivial Gaussian channels in a symplectic transformation.

3.7. Symplectic dilation

I would like to conclude this chapter with my finding of the systematic way to embed a noisy Gaussian process into a symplectic transformation[17]. The idea is inspired by the fact that any quantum channel can be dilated to a unitary operation [**59**]. That is, given any quantum channel $\mathcal{E}$ defined in the Hilbert space $\mathcal{H}_a$, we can find a density operator $\hat{\rho}_b$ on another Hilbert space $\mathcal{H}_b$, and a unitary operator $\hat{U}$ defined on $\mathcal{H}_a \otimes \mathcal{H}_b$, such that for any density operator $\hat{\rho}$ on $\mathcal{H}_a$, we have

$$\mathcal{E}(\hat{\rho}) = \mathrm{Tr}_{\mathcal{H}_b}[\hat{U}(\hat{\rho} \otimes \hat{\rho}_b)\hat{U}^\dagger]. \tag{3.7.1}$$

Note that this the choices of $\mathcal{H}_b$, $\hat{U}$,and $\hat{\rho}_b$ are not unique. This method, known as the Stinespring dilation, indicates that a quantum channel can be understood as a restriction of a unitary operation to a subsystem. We will follow the convention and refer to our embedding method as the symplectic dilation. This section is written in a mathematical style in order to keep the statements short and precise. More physical discussion and applications will be demonstrated in the later chapters when we deal with practical physical problems.

Before proceeding, we first introduce how the contrary works. Let $\hat{U}_S$ be a symplectic transformation defined on the joined Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, with the joined symplectic space $E = E_a \oplus E_b$. Let $\hat{\rho}_b\,(0, V_b)$ be a density operator on $\mathcal{H}_b$. Then using Eq. 3.7.1, we obtain a Gaussian channel $\mathcal{G}_{T,N}$ with

$$T = S_{a,a}, \tag{3.7.2}$$

[17]Note that some related discussions can be found in [**4**], where it is shown that certain single-mode Gaussian channels can be embedded in symplectic transformations. In comparison, the method in section is applicable to arbitrary Gaussian channels.

and

$$N = S_{b,b} V_b S_{b,b}^t. \tag{3.7.3}$$

(Here we slightly abuse the notation by representing a symplectic space using the subscript: $S_{b,b} = S_{E_b,E_b}$.) Therefore, given a symplectic transformation, one can easily construct a Gaussian channel using submatrices of the given transformation. In fact, any Gaussian channel can be obtained in this way.

However, the symplectic dilation is not as straightforward. First, we need the following mathematical propositions: Let the symplectic space $E = E_a \oplus E_b$ be the direct sum of symplectic vector subspaces E_a and E_b. Let Ω, Ω_a, and Ω_b be the symplectic forms defined on E, E_a, and E_b. Let $\mathcal{S} : E_a \to E_a$ be a linear transformation, with $\det(\mathrm{Id} - T) \neq 0$. The we have a matrix $\mathcal{M}$, given by

$$\mathcal{M} = \Omega_a \left[\frac{1}{2} \mathrm{Id} - (\mathrm{Id} - \mathcal{S})^{-1} \right].$$

Let $\mathcal{M}_s = (\mathcal{M} + \mathcal{M}^t)/2$, $\mathcal{M}_a = (\mathcal{M} - \mathcal{M}^t)/2$ be the symmetric and skew-symmetric part of the $\mathcal{M}$.

LEMMA 3.11. *There exists a linear transformation $\mathcal{R} : E_b \to E_a$, such that*

$$\mathcal{R} \Omega_b \mathcal{R}^t = 2 \Omega_a \mathcal{M}_a \Omega_a.$$

PROOF. For any skew-symmetric matrix $\mathcal{M}_a$, there exists a matrix $\mathcal{Q}$, such that $\mathcal{M}_a = \mathcal{Q} \Omega \mathcal{Q}^t$. Then we just let $\mathcal{R} = \sqrt{2} \Omega_a \mathcal{Q} \Omega$. □

Define linear transformations $\mathcal{L} = -\Omega_b \mathcal{R}^t \Omega_a$, and $\mathcal{S}' = \mathrm{Id} - \mathcal{L} \left(\Omega_a \mathcal{M} + \frac{1}{2} \mathrm{Id} \right)^{-1} \mathcal{R}$. Then we have

LEMMA 3.12. $\mathcal{S} \Omega_a \mathcal{S}^t + (Id - \mathcal{S}) \mathcal{R} \Omega_b \mathcal{R}^t (Id - \mathcal{S})^t = \Omega_a$

PROOF. Replacing $\mathcal{S}$ with $\mathcal{M} = \Omega_a \left[\frac{1}{2}\mathrm{Id} - (\mathrm{Id} - \mathcal{S})^{-1}\right]$, we transform the equation to the equivalent form

$$\left(\Omega_a\mathcal{M} + \frac{1}{2}\mathrm{Id}\right)^{-1} \left(\mathcal{R}\Omega_b\mathcal{R}^t + \Omega_a\right) - \Omega_a = \left(\Omega_a\mathcal{M} + \frac{1}{2}\mathrm{Id}\right)^{-1} \Omega_a \left(\Omega_a\mathcal{M} + \frac{1}{2}\mathrm{Id}\right)^t.$$

Since

$$\left(\Omega_a\mathcal{M} + \frac{1}{2}\mathrm{Id}\right)^t = -\left(\mathcal{M}_s - \mathcal{M}_a + \frac{1}{2}\Omega_a\right)\Omega_a,$$

and

$$\left(\Omega_a\mathcal{M} + \frac{1}{2}\mathrm{Id}\right)^{-1} = -\left(\mathcal{M}_s + \mathcal{M}_a - \frac{1}{2}\Omega_a\right)^{-1}\Omega_a,$$

the right-hand side of the above equation is equivalent to

$$\left(\Omega_a\mathcal{M} + \frac{1}{2}\mathrm{Id}\right)^{-1} \left(2\Omega_a\mathcal{M}_a\Omega_a + \Omega_a\right) - \Omega_a,$$

which is equivalent to the left-hand side by the definition of $\mathcal{R}$. □

LEMMA 3.13. $\mathcal{S}\Omega_a \left(Id - \mathcal{S}\right)^t \mathcal{L}^t + \left(Id - \mathcal{S}\right)\mathcal{R}\Omega_b \left(\mathcal{S}'\right)^t = 0.$

PROOF. It can be easily verified by throwing in the definition of $\mathcal{L}$ and $\mathcal{S}'$, and expanding each term in the expression. □

LEMMA 3.14. $\mathcal{S}'\Omega_b \left(\mathcal{S}'\right)^t + \mathcal{L}\left(Id - \mathcal{S}\right)\Omega_a \left(Id - \mathcal{S}\right)^t \mathcal{L}^t = \Omega_b.$

PROOF. This equation can be transformed to the equivalent equation

$$\mathcal{L}\left(\Omega_a\mathcal{M} + \frac{1}{2}\mathrm{Id}\right)^{-1} \left(\mathcal{R}\Omega_b - \Omega_a\mathcal{L}^t\right) = \left(\mathcal{L}\Omega_a - \Omega_b\mathcal{R}^t\right) \left[\left(\Omega_a\mathcal{M} + \frac{1}{2}\mathrm{Id}\right)^{-1}\right]^t \mathcal{L}^t,$$

which can be verified by applying Lemma. 3.13 to it. □

THEOREM 3.15. *The linear transformation $S : E \to E$, defined as*

$$S = \begin{pmatrix} \mathcal{S} & \left(Id - \mathcal{S}\right)\mathcal{R} \\ \mathcal{L}\left(Id - \mathcal{S}\right) & \mathcal{S}' \end{pmatrix}$$

is a symplectic transformation.

Proof. The linear transformation S is a symplectic transformation if and only if $S\Omega S^t = \Omega$. This equation leads to the verification of the following set of equations

$$\mathcal{S}\Omega_a\mathcal{S}^t + (\mathrm{Id} - \mathcal{S})\,\mathcal{R}\Omega_b\mathcal{R}^t\,(\mathrm{Id} - \mathcal{S})^t = \Omega_a,$$
$$\mathcal{S}\Omega_a\,(\mathrm{Id} - \mathcal{S})^t\,\mathcal{L}^t + (\mathrm{Id} - \mathcal{S})\,\mathcal{R}\Omega_b\,(\mathcal{S}')^t = 0,$$
$$\mathcal{S}'\Omega_b\,(\mathcal{S}')^t + \mathcal{L}\,(\mathrm{Id} - \mathcal{S})\,\Omega_a\,(\mathrm{Id} - \mathcal{S})^t\,\mathcal{L}^t = \Omega_b,$$

which has been shown in Lemmas. 3.12,3.13&3.14. □

With the aforementioned construction of a Gaussian channel out of a symplectic transformation, the above theorem indicates that the Gaussian channel $\mathcal{G}_{T,N}$ with

$$T = \mathcal{S},$$

and

$$N = (\mathrm{Id} - \mathcal{S})\,\mathcal{R}V_b\mathcal{R}^t\,(\mathrm{Id} - \mathcal{S})^t$$

can be constructed using a symplectic dilation $\hat{U}_S$ and a Gaussian state $\hat{\rho}\,(0, V_b)$, with

$$S = \begin{pmatrix} \mathcal{S} & (\mathrm{Id} - \mathcal{S})\,\mathcal{R} \\ \mathcal{L}\,(\mathrm{Id} - \mathcal{S}) & \mathcal{S}' \end{pmatrix}$$

as in the above theorem, and V_b a proper covariance matrix.

The symplectic dilation is not unique, since the matrix $\mathcal{R}$ is not uniquely defined (we only showed a special construction of $\mathcal{R}$ in Lemma. 3.11). [18] Moreover, for any

[18]This can be seen by checking the determinant of each factor of the following

$$\begin{aligned} \mathrm{Id} - S &= \begin{pmatrix} \mathrm{Id} - \mathcal{S} & -(\mathrm{Id} - \mathcal{S})\,\mathcal{R} \\ -\mathcal{L}\,(\mathrm{Id} - \mathcal{S}) & \mathrm{Id} - \mathcal{S}' \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Id} \\ -\mathcal{L} \end{pmatrix} (\mathrm{Id} - \mathcal{S}) \begin{pmatrix} \mathrm{Id} & -\mathcal{R} \end{pmatrix}, \end{aligned}$$

pair of symplectic transformations $S_b, S_{b'}$ on E_b, the symplectic transformation

$$D = \begin{pmatrix} \mathrm{Id} & \\ & S_b \end{pmatrix} S \begin{pmatrix} \mathrm{Id} & \\ & S_b' \end{pmatrix} = \begin{pmatrix} \mathcal{S} & (\mathrm{Id} - \mathcal{S})\,\mathcal{R} S_{b'} \\ S_b \mathcal{L}\,(\mathrm{Id} - \mathcal{S})\, S_b' & S_b \mathcal{S}' S_b \end{pmatrix}$$

is also a symplectic dilation of the same Gaussian channel if we the Gaussian state operator $\hat{\rho}\,(0, V_b)$ to $\hat{\rho}\left(0, S_{b'}^{-1} V_b S_{b'}^{-1}\right)$. In fact this non-uniqueness of symplectic dilation implies that any Gaussian channel $\mathcal{G}_{T,N}$, with $\det\,(\mathrm{Id} - T) \neq 0$ can be dilated with a proper choice of $\mathcal{R}$ and $S_{b'}$. Moreover, for $\det\,(\mathrm{Id} - T) = 0$, we can always decompose $\mathcal{G}_{T,N}$ as the composition of $\mathcal{G}_{T_1,N_1}$ and $\mathcal{G}_{T_2,N_2}$ with non-vanishing $\det\,(\mathrm{Id} - T_1)$ and $\det\,(\mathrm{Id} - T_2)$, so that the composition of the symplectic dilations of $\mathcal{G}_{T_1,N_1}$ and $\mathcal{G}_{T_2,n_2}$ gives the symplectic dilation of $\mathcal{G}_{T,N}$.

Last, it is easy to check that the symplectic dilation obtained from the last theorem satisfies $\det\,(\mathrm{Id} - S) = 0$. Let $\det\,(\mathrm{Id} - S_b S_{b'}) \neq 0$ in the above definition of D. Then we have

THEOREM 3.16. *There exists a linear transformation $\tilde{\mathcal{M}} : E \to E$ such that $D = Id - \left(\Omega\tilde{\mathcal{M}} + \frac{1}{2} Id\right)^{-1}$ is a symplectic dilation of $\mathcal{S}$.*

PROOF. Let

$$\Omega\tilde{\mathcal{M}} + \frac{1}{2}\mathrm{Id} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$$

with

$$\begin{aligned} \mathcal{A} =& \mathcal{M} + \mathcal{R} S_{b'}\,(\mathrm{Id} - S_b S_b')^{-1}\, S_b \mathcal{L}, \\ \mathcal{B} =& \mathcal{R} S_{b'}\,(\mathrm{Id} - S_b S_{b'})^{-1}\,, \\ \mathcal{C} =& -\,(\mathrm{Id} - S_b S_{b'})^{-1}\, S_b \mathcal{L} \\ \mathcal{D} =& \,(\mathrm{Id} - S_b S_{b'})^{-1}\,, \end{aligned}$$

where $\mathrm{Id} - S_b S_{b'}$ is invertible. Then we have

$$D = \begin{pmatrix} \mathrm{Id} - \left(\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C}\right)^{-1} & \left(\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C}\right)^{-1} \mathcal{B}\mathcal{D}^{-1} \\ \mathcal{D}^{-1}\mathcal{C}\left(\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C}\right)^{-1} & \mathrm{Id} - \mathcal{D}^{-1} - \mathcal{D}^{-1}\mathcal{C}\left(\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C}\right)^{-1} \mathcal{B}\mathcal{D}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{Id} & \\ & S_b \end{pmatrix} S \begin{pmatrix} \mathrm{Id} & \\ & S_b' \end{pmatrix}$$

is a symplectic dilation of $\mathcal{S}$. □

Therefore, D is the symplectic Cayley transform of $\Omega\tilde{\mathcal{M}}$. Physically speaking, the symplectic dilation D can be viewed as a noiseless scattering process governed by a dimensionless quadratic Hamiltonian (see Appx. A) determined by the following symmetric matrix

$$\begin{pmatrix} \mathcal{A} - \frac{1}{2}\mathrm{Id} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} - \frac{1}{2}\mathrm{Id} \end{pmatrix}.$$

The symplectic dilation allows us to model and analyze the quantum noise feature of a Gaussian process with limited knowledge of the Gaussian channel. We will discuss in more detail about the physical meaning and applications in later chapters.

CHAPTER 4

Quantum teleportation and quantum transdcution

4.1. Introduction to quantum transduction

In this chapter, we demonstrate our first physical application of symplectic geometry to an intriguing field of bosonic quantum control, the quantum transduction[1]. Firstly, we brefeifly what is quantum transduction and why it is of practical importance in quantum information science.

Nowadays, quantum computation has become a high-profile physical subject due to its promising potential of drastically improving human being's ability of solving complicated computationaly problems. Realizing such prospects requires tremendous efforts to mitigate systematic computational errors caused by the probabilistic nature of quantum physics, and to suit quantum computers to running large-scale quantum programs. Considering the current limitation of controlling quantum systems in laboratories, the idea of quantum network has been come up with as a practical way to meet these requirements. A quantum network is an extendable set of processing units, i.e., small-scale quantum computers, connected by transmission channels. On a quantum network, a large-scale quantum program is decomposed into many smaller-scale sub-programs, each of which is executed on a processing unit. The outputs of these sub-programs, i.e. quantum states, are then combined with each other through the transmission channels to constitute the final output. Therefore, each processing unit or transmission only needs to be implemented on a small-scale physical platform (e.g., several entangled qubits, or several bosonic modes), so that the complexity of controlling the processing and transmission units can be reduced greatly.

[1]The discussions in this chapter are based on [**26**]

However, different physical platforms are often suitable for different tasks in quantum computation. For example, superconducting quantum circuits (working at microwave frequency) are great at processing quantum information, while optical fibers are still the best candidates of transmitting quantum information. The practical realization of quantum network may be hybrid, with different tasks executed on different platforms. Therefore, physical devices that can convert quantum states between different platforms with high fidelity and efficiency becomes a crucial addition to practical implementation of quantum networks. In particular, we are particularly interested in such devices that can faithfully transfer bosonic quantum states, since bosonic modes have widespread applications in both quantum information processing and transmission. We refer to these state-conversion devices as quantum transducers.

Many theoretical and experimental efforts have been made since the birth of quantum transduction. Many all of the promising proposals and experimental demonstrations are based on bosonic scattering processes. For example, as in [**9**], to transfer a microwave mode to an optical mode, we first couple the microwave circuit to the optical cavity directly or indirectly mediated by another mode (e.g., a mechanical mode). Then we inject the signal into the cavities, and tune the system parameters to satisfy a so-called matching condition, so that the transmission of the signal is maximized and the reflection suppressed completely. Since the physical couplings are linear, i.e. determined by a quadratic bosonic Hamiltonian, the scattering process is actually a Gaussian process and therefore can be described by a symplectic transformation.

However, the matching condition cannot always be satisfied because of the inevitable presence of side bands. Specifically, a (over-)simplified quadratic Hamiltonian for the process mentioned above is

$$\begin{aligned}\hat{H} = \omega_o \hat{a}_o^\dagger \hat{a}_o + \omega_m \hat{a}_m^\dagger \hat{a}_m + \omega_e \hat{a}_e^\dagger \hat{a}_e \\ + g_{om} \hat{a}_o^\dagger \hat{a}_m + g_{om} \hat{a}_m^\dagger \hat{a}_o + g_{em} \hat{a}_e^\dagger \hat{a}_m + g_{em} \hat{a}_m^\dagger \hat{a}_e \\ + g_{om} \hat{a}_o^\dagger \hat{a}_m^\dagger + g_{om} \hat{a}_o \hat{a}_m + g_{em} \hat{a}_e^\dagger \hat{a}_m^\dagger + g_{em} \hat{a}_e \hat{a}_m,\end{aligned}$$

with ω_o, ω_m and ω_e being the frequencies of the optical, mechanical and microwave modes, g_{om} and g_{em} being the corresponding coupling strengths. The excitation-number-non-preserving terms in the the third line of the above equation are known as the counter-rotating terms and are usually omitted in rotating wave approximation in quantum optics given that $\omega_o, \omega_m, \omega_e \gg g_{om}, g_{oe}$ (i.e. physically speaking, these terms will not make any substantial contribution to the quantum process). This approximation cannot be carried out in this scenario, since ω_m, the frequency of the mechanical mode, is comparable with the coupling strengths. Therefore, $\hat{H}$ will not preserve the total excitation number, $\hat{a}_o^\dagger \hat{a}_o + \hat{a}_m^\dagger \hat{a}_m + \hat{a}_e^\dagger \hat{a}_e$, due tot the failure of the rotating wave approximation. Since excitation-number-preserving is a prerequisite of the matching condition, we see that the previously expected perfect quantum transdcution cannot be realized using this quadratic Hamiltonian. More intuitively, the irremovability of the counter-rotating terms leads to the presence of side bands, additional bosonic modes, in the frequency spectrum. These side bands will interact with the quantum signal, causing leakage of quantum information during the scattering process.

From the perspective of Gaussian process, the above amounts to the following abstraction: Consider a bosonic system consisting of bosonic modes of different physical characteristics (frequencies, carriers, etc.). The symplectic space will be divided in the following two ways: (1) Let E be the symplectic space of this system, E_{in} be the canonical symplectic subspace associated with the modes storing the input quantum signal, and E_{anc} the canonical symplectic subspace associated with the unwanted environmental modes (e.g., the side bands); (2) Let E_{out} be the canonical symplectic subspace associated with the modes storing the output quantum signal, and E_{idl} be the canonical symplectic subspace associated with the rest of the system. Now let S be the symplectic transformation describing the Gaussian process. Then the transduction is perfect if the $S_{out,anc} = S_{idl,in} = 0$; and imperfect if the either the submatrix $S_{out,anc}$ or $S_{idl,in}$ is non-vanishing.

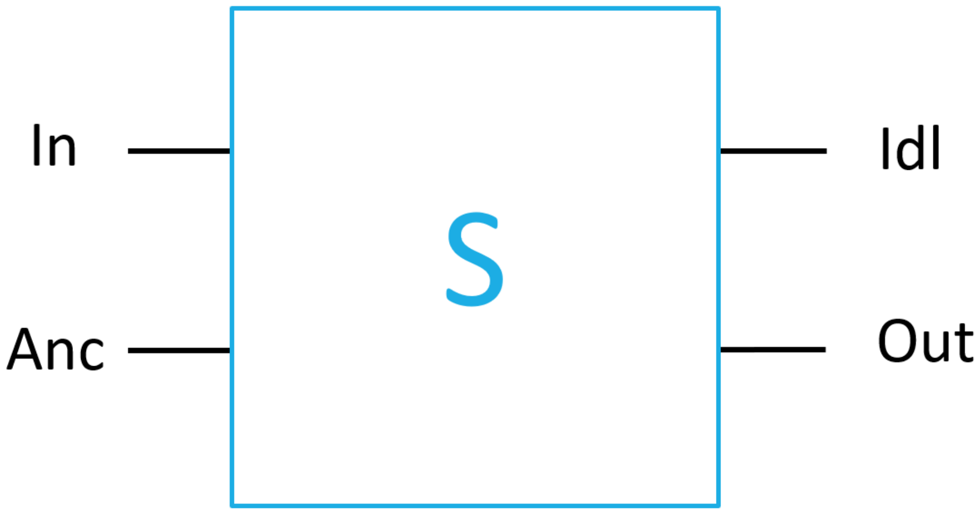

FIGURE 4.1. *Direct quantum transduction.*

The input quantum state is mixed with the with the ancillary state by a symplectic transformation S. The idler modes are dropped and the direct output modes are used as the the transduced quantum signal.

Based on this abstraction (which will be referred to as the direct quantum transduction for later convenience), in the following sections, we will demonstrate how a generalization of continuous-variable quantum teleportation protocol turns out to be a promising candidate to address the above mentioned problems concerning perfect quantum transduction and its robustness against practical system imperfections. We will also see symplectic geometry plays a crucial in inspiring such an application-oriented scheme.

4.2. Generalization of continuous-variable quantum teleportation

The task of quantum transduction, transferring quantum states between different physical platforms, reminds us of a well-known physical concept, quantum teleportation. We first briefly introduce quantum teleportation in bosonic system. Continuous-variable quantum teleportation [**60, 61**] is an analogue of the discrete-variable quantum teleportation in bosonic systems [**62**]. The system considered in quantum teleportation consists of three qubits, which can be tagged as an input qubit and two ancilla qubits or as an output qubit and two idler qubits. The input qubit, storing the quantum state to transfer, and the output qubit, storing the transferred quantum state, are different qubits, possibly located at rather distant places. To implement the quantum teleportation protocol, we first maximally entangle the two ancilla qubits, which are prepared as eigenstates of the Pauli-Z operator, using a CNOT gate; then entangle the input qubit with one of the ancillary qubits (the other one is the output qubit) using anothre CNOT gate; then measure the entangled input-ancillia pair using Pauli measurements; and at last perform adaptive Pauli operations on the output qubit, based on the syndrome measurement outcomes, to obtain a perfectly transferred output quantum state. People found the counterparts of the required components of discrete-variable quantum teleportation in continuous variable system: infinitely squeezed states for the Pauli-operator eigenstates, balanced beam-splitters for CNOT gates, homodyne measurements for Pauli measurements, and displacement operators for Pauli-operators, as shown in Fig. 4.2. By combing these counterparts in the same order as in the discrete-variable quantum teleportation, we obtain the continuous-variable quantum teleportation.]

We notice that beam-splitters are symplectic transformations. Then the composition of the two balanced beam-splitters in continuous-variable quantum teleportation can be described by a single symplectic transformation S, which can be viewed as

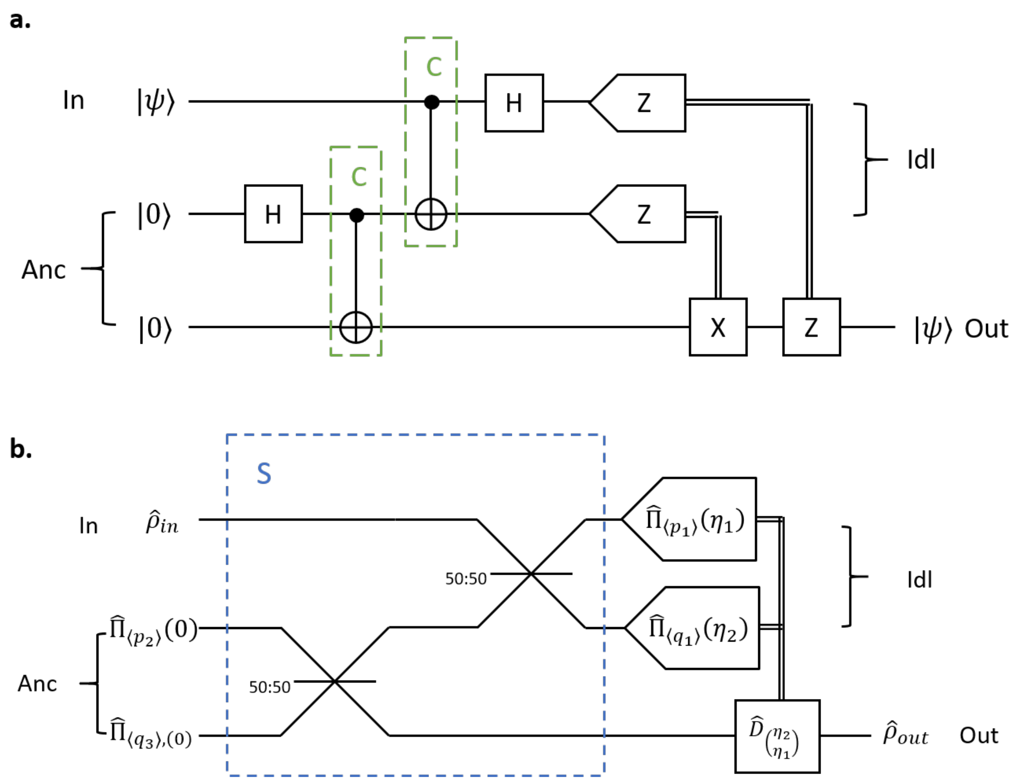

FIGURE 4.2. *Quantum teleportation circuits.*

(a) Illustration of the qubit quantum teleportation circuit. The boxes labeled by H are Hadamard gates, labled by X and Z are Pauli-X and Pauli-Z operators. The two-qubit gates, encircled by the green dashed boxes and labeled by C, are the CNOT gates. The solid double lines represent classical communication channels. The classical communication channels connects the syndrome measurement, i.e. the Pauli-Z measurements, to the corresponding Pauli operators. (b) Illustration of continuous variable quantum teleportation circuit. It shows the resemblance between the two teleportation schemes: The CNOT gates are replaced by the 50:50 (balanced) beam-splitters, the eigenstates $|0\rangle$'s by the infinitely squeezed states, the syndrome measurement by the homodyne measurements, and the Pauli operators by the displacement operator. Note that the Hadmard gates are absorbed into the basis of squeezed ancillary states and the homodyne measurements.

an imperfect quantum transducer since obviously both $S_{out,anc}$ and $S_{idl,in}$ are non-vanishing, as mentioned in the last section. In other words, we see that continuous variable quantum teleportation can be considered as a way to convert a specific imperfect quantum transducer, a combination of two beam-splitters, into a perfect transducer, using only single-mode Gaussian controls including infinite single-mode squeezing, homodyne measurements, and displacement operations. Then it is natural to ask: Can we devise a similar protocol that can be applied to general imperfect quantum transducers requiring the same single-mode Gaussian mode controls?

It turns out that such a generalization exists and can be understood intuitively. Given a symplectic transformation S defined on a multi-mode bosonic system associated with the symplectic space E. As mentioned in the previous section, we decompse E as $E_{in} \oplus E_{anc}$ and $E_{idl} \oplus E_{out}$, where E_{in} supports the input quantum signal, E_{anc} supports the ancillary modes, E_{idl} supports the modes to measure later on, and E_{out} supports the output quantum signal. Then we prepare an ancillary infinitely squeezed state $\hat{\Pi}_{l_z}(0)$ with l_z being a Lagrangian plane in E_{anc} (e.g., l_z can be spanned by all $\hat{q}$ quadratures in E_{anc}). Then, the input and the infinitely squeezed ancillary modes are mixed by the given symplectic transformation S. Then the we perform a multi-mode homodye measurement, described by the POVM $\{\hat{\Pi}_{l_h}(\eta)\}$ on the E_{idl}, with l_h being a Lagrangian plane of E_{idl} spanned by the measured quadratures (e.g., all $\hat{q}$-quadratures in E_{idl}) and η a vector of the measurement outcomes of the corresponding quadratures. At last, we transform the vector η using a real matrix F and displace the output signal using a displacement operation $\hat{D}_{F\eta}$. We found if F is chosen properly, the output state will be equivalent to the input state up to a symplectic transformation $\tilde{S}$, i.e.,

$$\hat{\rho}_{out} = \hat{U}_{\tilde{S}} \hat{\rho}_{in} \hat{U}_{\tilde{S}}^{\dagger}.$$

Then how to choose F? The idea is based on noise cancellation. By squeezing the ancillary modes, we concentrate the quantum noise to the anti-squeezed quadratures, represented by a Lagrangian plane $l_{z'}$ of E_{anc}. When mixed with the input quantum signal, the quantum noise is distributed to both the idler modes and the output modes. The proportion of the noise going into the idler modes can be quantified by the submatrix $S_{idl,z'}$ while the proportion going into the output by the submatrix $S_{out,z'}$[2]. Then the homomedyne measurement captures the proportion, quantified by $S_{h,z'}$ of noise coming into the idler modes. Therefore, the proper matrix F should be able to transform the measured noise to match the noise in the output, i.e., we require

$$FS_{h,z'} = S_{out,z'}.$$

Or equivalently, we require $F = S_{out,z'} \left(S_{h,z'}\right)^{-1}$.

This intuitive picture can be rigorously proved. Our generalization is equivalent to the following equation

$$\hat{\rho}_{out} = \hat{U}_{\tilde{S}} \hat{\rho}_{in} \hat{U}_{\tilde{S}}^{\dagger} = \int_{l_h} \hat{D}_{F\eta} \left(\mathrm{Tr}_{\mathcal{H}_{idl}} [\hat{\Pi}_{l_h}(\eta) \hat{U}_S \left(\hat{\rho}_{in} \otimes \hat{\Pi}_{l_z}(0) \right) \hat{U}_S^{\dagger}] \right) \hat{D}_{F\eta}^{\dagger} d\eta. \quad (4.2.1)$$

Note that we have abused the notation by representing symplectic subspaces using their subscripts (e.g., $S_{idl,z'} = S_{E_{idl},l_{z'}}$). For convenience, we list the subspaces that will be useful later: canonical symplectic subspaces E_{in}, E_{anc}, E_{idl} and E_{out} satisfying $E = E_{in} \oplus E_{anc}$ and $E = E_{idl} \oplus E_{out}$, Lagrangian planes l_z, $l_{z'}$, l_h, and $l_{h'}$ satisfying $E_{anc} = l_z \oplus l_{z'}$ and $E_{idl} = l_h \oplus l_{h'}$. Translating to Wigner functions by letting

$$F = \tilde{S} \left[\left(S^{-1}\right)_{in,h} - \left(S^{-1}\right)_{in,h'} \left(\left(S^{-1}\right)_{z,h'} \right)^{-1} \left(S^{-1}\right)_{z,h'} \right], \quad (4.2.2)$$

and

$$\tilde{S} = \left(S^{-1}\right)_{in,out} - \left(S^{-1}\right)_{in,h'} \left(\left(S^{-1}\right)_{z,h'} \right)^{-1} \left(S^{-1}\right)_{z,out},$$

the above equation amounts to

$$W_{\hat{\rho}_{out}}(u) = W_{\hat{U}_{\tilde{S}} \hat{\rho}_{in} \hat{U}_{\tilde{S}}^{\dagger}}(u) = W\left(\tilde{S}^{-1} u\right)$$

[2]We can imagine the original quantum noise in the ancillary modes to be a real vector, say v. Then the noise going into the the idler modes can be imagined as the vector $S_{idl,z}v$ and the noise going into the output as $S_{out,z'}v$, etc..

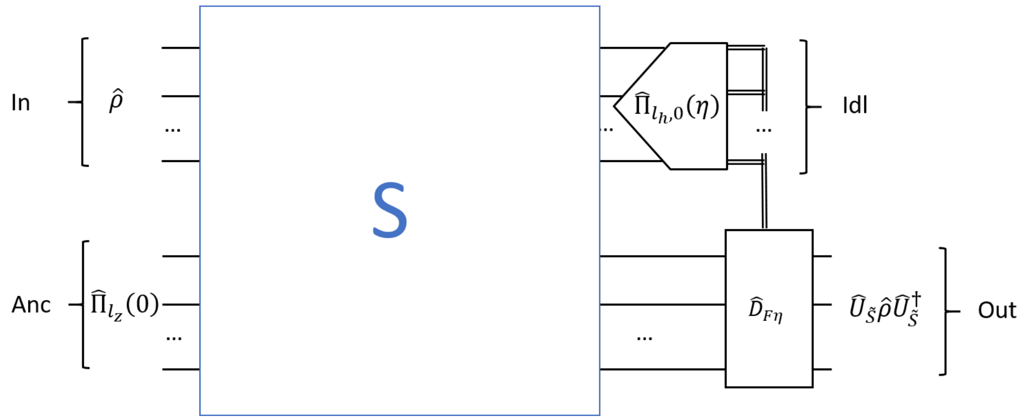

FIGURE 4.3. *Generalized teleportation protocol.*

The multi-mode input state $\hat{\rho}$ is mixed with the infinitely squeezed multi-mode ancillary state $\hat{\Pi}_{l_z}(0)$(determined by a Lagrangian plane l_z) by the multi-mode symplectic transformation S. Then the idler modes are measured by ideal homodyne measurements, described by $\hat{\Pi}_{l_h}(\eta)$. The measurement outcomes, represented by a real vector η, are used to determine the displacement operation $\hat{D}_{F\eta}$ with F the matrix given in the main text. At last, the density operator of the output state is given by $\hat{U}_{\tilde{S}}\hat{\rho}\hat{U}_{\tilde{S}}^{\dagger}$, which is unitarily equivalent to the input state $\hat{\rho}$.

if we $\tilde{S}$ is a symplectic transformation. Therefore, we conclude this section with the following theorem, where we prove F thus defined is equal to what we obtained from the noise cancellation picture, and $\tilde{S}$ is indeed a symplectic transformation.

THEOREM 4.1. *Let $S : E \to E$ be a symplectic transformation with $S_{h,z'}$ being invertible. Then the following statements hold:*

(i) $\tilde{S} := S_{out,in} - S_{out,z'}\left(S_{h,z'}\right)^{-1} S_{h,in} : E_{in} \to E_{out}$ *is a symplectic transformation;*

(ii) $\check{S} := \left(S^{-1}\right)_{in,out} - \left(S^{-1}\right)_{in,h'}\left(\left(S^{-1}\right)_{z,h'}\right)^{-1}\left(S^{-1}\right)_{z,out} : E_{out} \to E_{in}$ *is a symplectic transformation;*

(iii) $\tilde{S} = \left(\check{S}\right)^{-1}$*;*

(iv) *(forward transformation)*

$$F = -S_{out,z'}\left(S_{h,z'}\right)^{-1} = \tilde{S}\left[\left(S^{-1}\right)_{in,h} - \left(S^{-1}\right)_{in,h'}\left(\left(S^{-1}\right)_{z,h'}\right)^{-1}\left(S^{-1}\right)_{z,h'}\right]; \quad (4.2.3)$$

(v) *(backward transmission)*

$$B = -\left(S^{-1}\right)_{in,h'}\left(\left(S^{-1}\right)_{z,h'}\right)^{-1} = \tilde{S}^{-1}\left[S_{out,z} - S_{out,z'}\left(S_{h,z'}\right)^{-1}S_{h,z}\right]. \quad (4.2.4)$$

Proof. (i): For all $w, w' \in E_{in}$, let $u, u' \in E_{in} \oplus E_{anc}$ be vectors with

$$u_{in} = w,$$

$$u_z = 0,$$

$$u_{z'} = -\left(S_{h,z'}\right)^{-1} S_{h,in} w,$$

$$u'_{in} = w',$$

$$u'_z = 0,$$

$$u'_{z'} = -\left(S_{h,z'}\right)^{-1} S_{h,in} w'.$$

Then we let $v = Su$ and $v' = Su'$. It is easy to show that

$$v_{out} = \tilde{S}w,$$

$$v_h = 0,$$

$$v_{h'} = S''w,$$

$$v'_{out} = \tilde{S}w',$$

$$v'_h = 0,$$

$$v'_{h'} = S''w',$$

with $S'' = S_{h',in} - S_{h',z'}\left(S_{h,z'}\right)^{-1} S_{h,in}$. It then follows that

$$\sigma_{E_{in} \oplus E_{anc}}\left(u, u'\right) = \sigma_{E_{idl} \oplus E_{out}}\left(v, v'\right),$$

$$\begin{aligned}\sigma_{E_{idl}\oplus E_{out}}\left(v, v'\right) &= \sigma_{E_{out}}\left(\tilde{S}w, \tilde{S}w'\right) + \sigma_{E_{idl}}\left(v|_{l_{h'}}, v'|_{l_{h'}}\right)\\ &= \sigma_{E_{out}}\left(\tilde{S}w, \tilde{S}w'\right),\end{aligned}$$

and

$$\begin{aligned}\sigma_{E_{in}\oplus E_{anc}}\left(u, u'\right) &= \sigma_{E_{in}}\left(w, w'\right) + \sigma_{E_{anc}}\left(u|_{l_{z'}}, u'|_{l_{z'}}\right)\\ &= \sigma_{E_{in}}\left(w, w'\right).\end{aligned}$$

Therefore, we have

$$\sigma_{E_{out}}\left(\tilde{S}w, \tilde{S}w'\right) = \sigma_{E_{in}}\left(w, w'\right),$$

which, by definition, means that $\tilde{S}$ is a symplectic transformation.

(ii): Recalling the symplectic form Ω, the inverse of a symplectic transformation S is

$$S^{-1} = -\Omega S^t \Omega.$$

Then we have

$$\begin{aligned}\left(S^{-1}\right)_{z,h'} &= -\Omega_{z,z'}\left(S^t\right)_{z',h}\Omega_{h,h'}\\ &= -\Omega_{z,z'}\left(S_{h,z'}\right)^t\Omega_{h,h'},\end{aligned}$$

which shows the invertibility of $(S^{-1})_{z,h'}$ given the invertibility of $S_{h,z'}$. Using similar arguments as in (i), it then can be proven easily that the linear transformation $\check{S}$ is symplectic. The details of the proof are omitted here.

(iii): As what we defined in (i), $v = Su$, so we also have $u = S^{-1}v$. This leads to the following equations:

$$w = u_{in} = \left[\left(S^{-1}\right)_{in,out}\tilde{S} + \left(S^{-1}\right)_{in,h'}S''\right]w,$$

and

$$0 = u_z = \left[\left(S^{-1}\right)_{z,out}\tilde{S} + \left(S^{-1}\right)_{z,h'}S''\right]w$$

for all $w \in E_{in}$. Then, we have

$$\left(S^{-1}\right)_{in,out}\tilde{S} + \left(S^{-1}\right)_{in,h'} S'' = \mathrm{Id},$$

and

$$\left(S^{-1}\right)_{z,out}\tilde{S} + \left(S^{-1}\right)_{z,h'} S'' = 0.$$

Since $\left(S^{-1}\right)_{z,h'}$ is invertible, combining the above two equations gives the following:

$$\check{S}\tilde{S} = \mathrm{Id}.$$

Therefore, $\check{S}$ is an inverse of $\tilde{S}$.

(iv) &(v): As shown in (iii), we have

$$\left(S^{-1}\right)_{z,out}\tilde{S} + \left(S^{-1}\right)_{z,h'} S'' = 0.$$

It can be rewritten as

$$-\left(\left(S^{-1}\right)^t\right)_{out,z}\left(\left(\left(S^{-1}\right)^t\right)_{h',z}\right)^{-1} = \left(\tilde{S^t}\right)^{-1}\left(S''\right)^t.$$

We can view $\tilde{\square}$ and $\square''$ as mappings between sets of linear transformations. It is easy to check that both $\tilde{\square}$ and $\square''$ commute with the transposition operation. Thus we have

$$-\left(\left(S^t\right)^{-1}\right)_{out,z}\left(\left(\left(S^t\right)^{-1}\right)_{h',z}\right)^{-1} = \left(\tilde{S^t}\right)^{-1}\left(S^t\right)''.$$

Then (4) is proved by replacing S^t with S^{-1}, and exchanging the subscripts z with z', and h with h'; (5) is proved by replacing S^t with S, and exchanging the subscripts in with out, z with h' and z' with h. It is also easy to check these replacements will not require invertibility of the sub-linear-transformation of S other than that of $S_{h,z'}$. $\square$

4.3. Gaussian channel representation of adaptive control

In Sec. 4.2, we demonstrated the generalized teleportation protocol assuming ideal and noiseless quantum controls. In this and the following sections, we investigate the robustness of this scheme against practical experimental imperfections. First, we need to prepare necessary theoretical models of some of the key components in the language of Gaussian channel. In this section, we discuss how to represent the crucial adaptive displacement operation as a Gaussian channel.

We follow the convention in the last section, except that we assume the matrix F, governing the displacement operation. is not of the specific form as discussed previously, but rather an arbitrary linear transformation. Specifically, we will show how to represent the quantum channel defined by

$$\mathcal{A}_F(\hat{\rho}) := \int \hat{D}_{F\eta} \mathrm{Tr}_{\mathcal{H}_{idl}} \left[\hat{\Pi}_{l_h}(\eta)\, \hat{\rho} \right] \hat{D}^{\dagger}_{F\eta} \tag{4.3.1}$$

as a Gaussian channel (i.e. we will find the two matrices determining this channel). Note that here the density operator $\hat{\rho}$ is defined on the whole symplectic space.

Firstly, we represent the above equation by Winger functions, which is given by the following formula

$$W_{\mathcal{A}_F(\hat{\rho})}(u) \propto \int_{l_h} \int_{E_{idl}} W_{\hat{\rho}}\left((u - F\eta) \oplus v\right) \delta\left(v - \eta\right) dv d\eta.$$

Then we let

$$X = -\Omega_{h',h} F^t \Omega_{out}, \tag{4.3.2}$$

and

$$Y = \frac{1}{2} \Omega_{h',h} F^t \Omega_{out} F, \tag{4.3.3}$$

in the linear transformation A_F defined by

$$A_F = \begin{pmatrix} \mathrm{Id} & F & 0 \\ 0 & \mathrm{Id} & 0 \\ -X & Y & \mathrm{Id} \end{pmatrix}. \tag{4.3.4}$$

It turns out that A_F is a symplectic transformation with its inverse being

$$A_F^{-1} = \begin{pmatrix} \mathrm{Id} & -F & 0 \\ 0 & \mathrm{Id} & 0 \\ E & G & \mathrm{Id} \end{pmatrix}. \tag{4.3.5}$$

Then straightforward calculation shows that

$$\begin{aligned} W_{\mathcal{A}_F(\hat{\rho})}(u) &= \int_{E_{idl}} W_{\hat{\rho}}\left(A_F^{-1}(u \oplus v)\right) dv \\ &= \int_{E_{idl}} W_{\hat{U}_{A_F}\hat{\rho}\hat{U}_{A_F}^{\dagger}}(u \oplus v)\, dv. \end{aligned} \tag{4.3.6}$$

In other words, this shows that the quantum channel $\mathcal{A}_F$ can be equivalently defined as

$$\mathcal{A}_F(\hat{\rho}) = \mathrm{Tr}_{\mathcal{H}_{idl}}\left[\hat{U}_{A_F}\hat{\rho}\hat{U}_{A_F}^{\dagger}\right],$$

which is at the same time a Gaussian channel $\mathcal{G}_{(A_F)_{out,tot},0}$.

This will be done as follows: Let $\hat{\rho}$ be arbitrary density operator, l_h a Lagrangian plane, and $l_{h'}$ the symplectic conjugate of l_h, such that $l_h \oplus l_{h'} = E_{idl}$ which is the symplectic subspace with associated to the idle subsystem. Let $F : l_h \to E_{out}$ be an arbitrary linear transformation which will be used to construct the adaptive displacement operation. We will show how to represent the quantum channel

$$\mathcal{A}_F(\hat{\rho}) := \int \hat{D}_{-F\eta}\mathrm{Tr}_{\mathcal{H}_{idl}}\left[\hat{\Pi}_{l_h}(\eta)\,\hat{\rho}\right]\hat{D}_{-F\eta}^{\dagger} \tag{4.3.7}$$

as a Gaussian channel. First, we translate it to the Wigner function representation (we will frequently omit the constant prefactor in such computations):

$$W_{\mathcal{A}_F(\hat{\rho})}(u) \propto \int\int_{E_{idl}} W_{\hat{\rho}}\left((u - F\eta) \oplus v\right)\delta\left(v|_{l_h} - \eta\right) dv d\eta. \tag{4.3.8}$$

Then by letting

$$X = -\Omega_{h',h}F^t\Omega_{out}, \tag{4.3.9}$$

and

$$Y = \frac{1}{2}\Omega_{h',h} F^t \Omega_{out} F, \tag{4.3.10}$$

we have a linear transformation $A : E_{tot} = E_{out} \oplus E_{idl} \to E_{tot}$, with the explicit form

$$A_F = \begin{pmatrix} \mathrm{Id} & F & 0 \\ 0 & \mathrm{Id} & 0 \\ -X & Y & \mathrm{Id} \end{pmatrix}. \tag{4.3.11}$$

It is easy to check that A is a symplectic transformation, the inverse of which is the following:

$$A_F^{-1} = \begin{pmatrix} \mathrm{Id} & -F & 0 \\ 0 & \mathrm{Id} & 0 \\ E & G & \mathrm{Id} \end{pmatrix}. \tag{4.3.12}$$

After several steps of straightforward calculations, we have

$$\begin{aligned} W_{\mathcal{A}_F(\hat{\rho})}(u) &= \int_{E_{idl}} W_{\hat{\rho}}\left(A_F^{-1}(u \oplus v)\right) dv \\ &= \int_{E_{idl}} W_{\hat{U}_{A_F}\hat{\rho}\hat{U}^\dagger_{A_F}}(u \oplus v)\, dv. \end{aligned} \tag{4.3.13}$$

Therefore the map $\mathcal{A}$ is a Gaussian channel defined by

$$\mathcal{A}_F(\hat{\rho}) = \mathrm{Tr}_{\mathcal{H}_{idl}}\left[\hat{U}_{A_F}\hat{\rho}\hat{U}^\dagger_{A_F}\right], \tag{4.3.14}$$

or can equivalently be denoted as $\mathcal{G}_{(A_F)_{out,tot},0}$ with the submatrix of A_F being

$$(A_F)_{out,tot} = \begin{pmatrix} \mathrm{Id}, & -F, & 0 \end{pmatrix}. \tag{4.3.15}$$

Here *tot* represents the whole symplectic space E.

4.4. Adaptive quantum transduction

Following the discussion in the last section, we continue representing more components of the generalized teleportation by Gaussian channels and at the same time incorporate models of quantum noise into the new representation. To mark the "practicality" of this model and emphasize the final goal, we now officially rebrand the generalized teleportation as the adaptive quantum transduction.

We have seen The successful implementation of the adaptive quantum trasndcution protocol relies on the availability of infinitely-squeezed ancillary states and the availability of perfect homodyne measurements of the given transducer[3]. However, in practical experimental environments, these conditions cannot be totally satisfied: Infinitely-squeezed states are purely theoretical assumptions which can only be approximated by finitely-squeezed Gaussian states; homodyne measurements are usually inefficient; intrinsic losses are inevitable. Here, we will discuss in detail how to model these experimental imperfections and incorporate them into our adaptive quantum transduction scheme.

The conventions in Secs. 4.2&4.3 will be followed in this section, with the addition of the environmental modes, associated with the symplectic space E_{env}. These environmental modes will be used to model the inefficeicy of the homodyne measurement. Therefore, now the symplectic space E of the whole system can be decomposed as follows

$$\begin{aligned} E &= E_{in} \oplus E_{anc} \oplus E_{env} \\ &= E_{out} \oplus E_{idl} \oplus E_{env}. \end{aligned}$$

First, the imperfect ancillary state will be modeled by a possibly thermal and finitely squeezed Gaussian state, denoted by $\hat{\rho}\,(0, V_{anc})$, on E_{anc}. As we have seen in Sec. 3.5, the imperfect homodyne measurement, still a Gaussian measurement, will be modeled by the combination of a thermal state in E_{env}, a symplectic transformation on $E_{idl} \oplus E_{env}$ and perfect homodyne measurements on E_{idl}. Here we would like to explain

[3]For simplicity, we ignore other non-unitary imperfections such as the intrinsic losses or gains in the bosonic system, since they are not as an urgent issue as the these two major imperfections. However, the method introduced in this section can be easily adapted to incorporating these imperfections into our adaptive quantum transduction scheme.

in more detail how we will model the imperfect homodyne measurement. For a single-quadrature homodyne measurement, we will model it as a combination of a fictitious beamsplitter and an ideal homodyne detector, so that the signal will first be mixed with a fictitious environmental mode by the beamsplitter before being measured. Then, by changing the transmittance of the fictitious beamsplitter and the thermal excitation number of the environmental mode, this mode can neatly simulate the imperfections in a practical homodyne measurement. For the multi-quadrature situation, we will simply group many copies of this single-mode model. That is, we will use a fictitious multi-mode beamsplitter, denoted by $\tilde{H}$, and a multi-mode environmental Gaussian state $\hat{\rho}\left(0, V_{env}\right)$ to model the imperfections[4].

As explained in the last section, we model the adaptive displacement operation determined by a linear transformation F using the symplectic transformation A_F. Then the adaptive quantum transduction scheme amounts to the following Gaussian channel (as shown in Fig. 4.4):

$$\mathcal{G}_{T,N}\left(\hat{\rho}\right) = \mathrm{Tr}_{\mathcal{H}_{idl}\otimes\mathcal{H}_{ign}}\left[\mathcal{G}_{A_F\oplus\mathrm{Id},0}\circ\mathcal{G}_{\mathrm{Id}\oplus\tilde{H},0}\circ\mathcal{G}_{S\oplus\mathrm{Id},0}\left(\hat{\rho}\otimes\hat{\rho}_{anc}\otimes\hat{\rho}_{env}\right)\right]$$

with

$$T = S_{out,in} + F\tilde{H}_{h,idl}S_{idl,in}, \tag{4.4.1}$$

and

$$N = \tilde{B}V_{anv}\tilde{B}^t + \left(F\tilde{H}_{h,env}\right)V_{env}\left(F\tilde{H}_{h,env}\right)^t,$$

where

$$\tilde{B} = S_{out,anc} + F\tilde{H}_{F,idl}S_{idl,anc}.$$

[4]Specifically, we let

$$\tilde{H} = \oplus_{i=1}^{M} H(\tau)$$

with M the number of modes supported by $\mathcal{H}_{idl}$, τ the transmittance of a single two-mode beamsplitter as in Sec. 3.5.

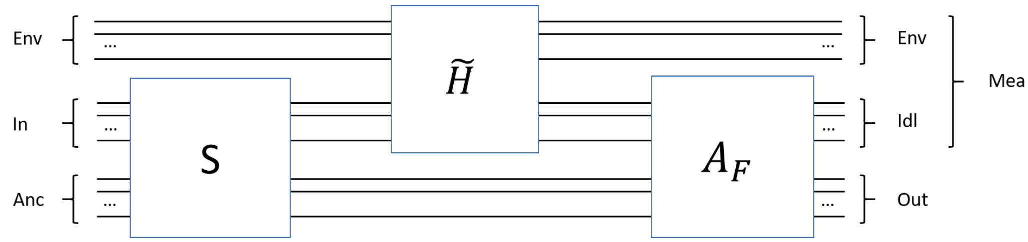

FIGURE 4.4. *Adaptive quantum transduction.*

The abstraction of adaptive quantum transduction with imperfections. The box labeled by S represents the given imperfect transducer with the corresponding symplectic transformation. The box $\tilde{H}$ represents the fictitious multi-mode beam-splitter modeling the imperfections of of the homodyne measurements. The symplectic transformation A_F represents the adaptive control as explained in the preceding section.

Furthermore, we let n_z denote the thermal excitation in the ancillary state, ξ be the degree of squeezing. Then the covariance matrix, arranged in the alternative order as mentioned in Sec. 2.3, of the ancillary state is given by

$$V_{anc} = \begin{pmatrix} e^{-2\xi}\,(2n_z+1)\,\mathrm{Id} & 0 \\ 0 & e^{2\xi}\,(2n_z+1)\,\mathrm{Id} \end{pmatrix}.$$

In addition, we assume the transmittance of the beamsplitter $\tilde{H}$ to be τ, and the thermal excitation number of the environmental state to be n_h. The the covariance matrix of the environmental state is given by

$$V_{env} = (2n_h+1)\,\mathrm{Id}.$$

Shortly after, we will see that these two imperfections can be quantified by two real numbers: (1) the ancillary imperfection coefficient ν, defined as

$$\nu = e^{-2\xi}(2n_z+1),$$

(2) and the measurement imperfection coefficient μ, defined as

$$\mu = \frac{\tau}{1-\tau}(2n_h + 1).$$

Now we put these data in the adaptive quantum transduction channel $\mathcal{G}_{T,N}$. We obtain that

$$T = S_{out,in} + \sqrt{1-\tau} F S_{h,in},$$

and

$$N = \tilde{B} V_{anc} \tilde{B}^t + \tau F F^t$$

with

$$\tilde{B} = S_{out,anc} + \sqrt{1-\tau} F S_{h,anc}.$$

According to Theorem. 4.1, if now we choose F to be $F_\star = -S_{out,z'} \left(S_{h,z'}\right)^{-1} / \sqrt{1-\tau}$, then it follows that

$$T = \tilde{S},$$

is a symplectic transformation, and

$$N = \nu \tilde{S} B_\star B_\star^t \tilde{S}^t + \mu F_\star F_\star^t,$$

with

$$B_\star = B = -\left(S^{-1}\right)_{in,h'} \left(\left(S^{-1}\right)_{z,h'}\right)^{-1}.$$

Then, composing this Gaussian channel $\mathcal{G}_{\tilde{S},N}$ with a post-processing unitary Gaussian channel $\mathcal{G}_{\tilde{S}^{-1},0}$, we obtain the Gaussian channel

$$\mathcal{G}_{\mathrm{Id},N'} = \mathcal{G}_{\tilde{S}^{-1},0} \circ \mathcal{G}_{\tilde{S},N}$$

with

$$N' = \nu B_\star B_\star^t \left(\tilde{S}\right) + \mu \left[\tilde{S}^{-1} S_{out,z'} \left(S_{h,z'}\right)^{-1}\right] \left[\tilde{S}^{-1} S_{out,z'} \left(S_{h,z'}\right)^{-1}\right]^t,$$

which will be referred to as the Gaussian channel of the adaptive quantum transduction. Now we see that this channel is completely determined by the transducer S, and the two imperfection coefficients μ and ν.

4.5. Average fidelity for single-mode Gaussian channels

In the preceding section, we defined the Gaussian channel for adaptive quantum transduction, which is the model of the noise caused by the experimental imperfections. In this section, we will use this knowledge to verify the robustness of the adaptive quantum transduction against these experimental imperfections. We will demonstrate, with respect to certain quantum benchmark, the adaptive quantum transduction is a promising quantum transduction scheme even in presence of practical experimental limitations.

The quantity we use to benchmark the adaptive quantum transduction scheme is the average fidelity. The performance of a quantum channel for quantum transduction depends largely on how close the channel is to a unitary operation. One natural way to measure this "distance" is to send an ensemble of quantum states into the given quantum channel $\mathcal{E}$, and then calculate the average overlap between the ensemble of the output quantum states and the original input ensemble. Specifically, let $\{\hat{\rho}_i\}$ be an ensemble quantum states associated with a probability distribution $\{p_i\}$, so that $\{\mathcal{E}(\hat{\rho}_i)\}$ is the output ensemble. The "overlapping" of the two ensembles, i.e. the average fidelity, is given by the following formula:

$$\bar{F}\left(\mathcal{E};p\right)=\sum_i p_i F\left(\hat{\rho}_i,\mathcal{E}\left(\hat{\rho}_i\right)\right),$$

where

$$F\left(\hat{\rho},\hat{\rho}'\right)=\mathrm{Tr}\left[\sqrt{\hat{\rho}}\hat{\rho}'\sqrt{\hat{\rho}}\right]$$

is the quantum fidelity between two arbitrary quantum states $\hat{\rho}$ and $\hat{\rho}'$. Now supposing that the input states are sampled from a uniformly-distributed ensemble of single-mode coherent states (i.e. every coherent state in the phase space has the same probability to be sampled out), and that the quantum channel $\mathcal{E}$ is a single-mode Gaussian channel $\mathcal{G}_{T,N}$, the average fidelity can be calculated by the following simple formula:

$$\bar{F}\left(\mathcal{G}_{T,N}\right) = \begin{cases} \frac{2}{\sqrt{\det(2\mathrm{Id}+N)}}, & \text{if } T = \mathrm{Id}, \\ 0, & \text{if } T \neq \mathrm{Id}, \end{cases}$$

EXAMPLE 4.2. This formula makes sense in the special situation of an identity Gaussian channel $\mathcal{G}_{\mathrm{Id},0}$, with $\bar{F}(\mathcal{G}_{\mathrm{Id},0}) = 1$ as it should be.

In addition, the average fidelity not only allows us to quantify the difference between a quantum channel and an identity channel, but also serves as a measure of the "quantumness". To see it, we consider the following process: Instead of being sent through a quantum channel $\mathcal{E}$, the ensemble of input states are measured in the beginning. Then the measurement outcomes are transferred through a classical communication channel to the output. Based only on the classical information transferred to the output, we prepare the output state to be as close to the input state as possible.We refer to such a process as classical transduction. For instance, considering the uniformly-distributed coherent-state ensemble, the average fidelity of any classical transduction scheme, regardless of what measurement scheme or state preparation method is applied, can only approach but not exceed the 1/2 threshold [**63**]. Therefore, given a quantum channel $\mathcal{E}$, the "quantumness" of $\mathcal{E}$ as a transduction protocol will only be justified if $\bar{F}(\mathcal{E}) > 1/2$. In particular, for a single-mode Gaussian channel $\mathcal{G}_{\mathrm{Id},N}$, this amounts to

$$\det\left(\mathrm{Id} + \frac{N}{2}\right) < 4.$$

Later on, we will use the average fidelity to compare our adaptive transduction protocol with the direct transduction protocol and the classical threshold, given a symplectic transformation S involving two bosonic modes. Specifically, we will need to calculate the average fidelity for the Gaussian channels associated with the adaptive quantum transduction protocol and the direct quantum transduction protocol. The former has been displayed in the previous section. For the latter, we will assume the ancillary state is the vacuum state so that the added noise to the output is minimized. This leads to the Gaussian channel $\mathcal{G}_{T,N}$ with

$$T = S_{out,in},$$

and

$$N = S_{out,anc}(S_{out,anc})^t.$$

Furthermore, when T is invertible, another Gaussian channel $\mathcal{G}_{T^{-1},N'}$ will be appended, leading to the Gaussian channel

$$\begin{aligned}\mathcal{G}_{T'',N''} =& \mathcal{G}_{T^{-1},N'} \circ \mathcal{G}_{T,N} \\ =& \mathcal{G}_{\mathrm{Id},T^{-1}N(T^{-1})^t+N'},\end{aligned}$$

which will be referred to as the Gaussian channel of the direct transduction protocol later on. Note that the matrix N' will be optimized to maximize the average fidelity $\bar{F}(\mathcal{G}_{\mathrm{Id},N''})$.

Finally, we can apply the theoretical tools established in this and the previous sections to analyze the transdcution performance of Gaussian processes. We will demonstrate the method in the following two examples:

Example 4.3. (Passive two-mode coupling.) First, we consider a two-mode bosonic system, dynamics of which are determined by the following total-excitation-number-preserving Hamiltonian

$$\hat{H} = g\left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1\right).$$

Each of the two modes is coupled to an external propagating mode. i.e. the mode $\hat{a}_{1(2)}$ is coupled to the propagating mode $\hat{A}_{1(2)}$ with coupling rate $\kappa_{1(2)}$. We identify the mode $\hat{A}_1$ as the input and $\hat{A}_2$ as the output. Then this process amounts to a Gaussian

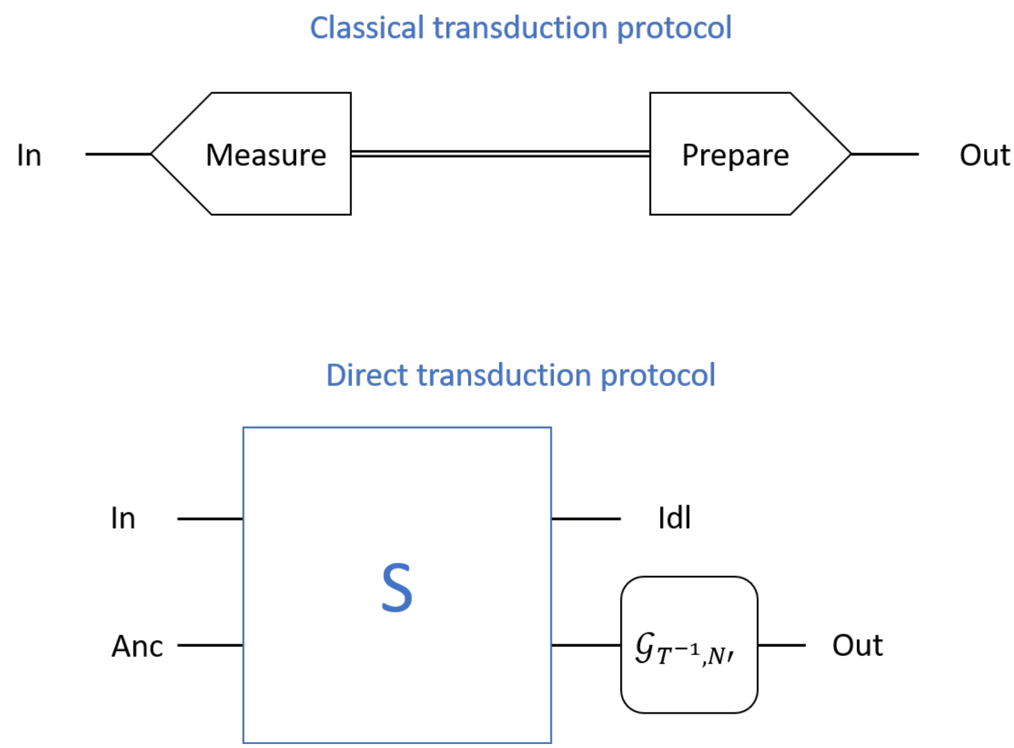

FIGURE 4.5. *Classical and direct quantum transduction protocols.* The classical transduction protocol: The input quantum state is measured by a given measurement scheme. Then the measurement outcome is transmitted to the output through a classical communication channel, and is used to prepare the output quantum state.The Direct transduction protocol: The input quantum state and the ancillary noisy environment state are mixed by the symplectic transformation S. Then the idler modes are dropped and the output modes are modified by a Gaussian channel $\mathcal{G}_{T^{-1},N'}$. This modifying Gaussian channel is applied to optimize the average fidelity at the end.

process associated with the following symplectic transformation (see Appx. A for more detail)

$$S = \begin{pmatrix} 0 & -t & r & 0 \\ t & 0 & 0 & r \\ r & 0 & 0 & -t \\ 0 & r & t & 0 \end{pmatrix}$$

with

$$r = \frac{C-1}{C+1}, \quad t = \frac{2\sqrt{C}}{C+1}$$

satisfying $r^2 + t^2 = 1$, where

$$C = \frac{g^2}{\kappa_1 \kappa_2}.$$

The details of the calculation will be elaborated in later chapters. It follows that the direct quantum transduction Gaussian channel of this process is $\mathcal{G}_{\mathrm{Id},N_D}$ with

$$N_D = 2\frac{1-t^2}{t^2}\mathrm{Id},$$

average fidelity of which is

$$\bar{F}\left(\mathcal{G}_{\mathrm{Id},N_D}\right) = t^2.$$

In the meantime, with the $\hat{q}$-quadrature of the ancillary state being squeezed and the idler mode being measured, we obtain the adaptive quantum transduction Gaussian channel (after performing the post-processing symplectic transformation $\tilde{S}$), $\mathcal{G}_{\mathrm{Id},N_A}$ with

$$N_A = \begin{pmatrix} \mu\left(1-t^2\right) & \\ & \nu\frac{1-t^2}{t^2} \end{pmatrix},$$

average fidelity of which is

$$\bar{F}\left(\mathcal{G}_{\mathrm{Id},N_D}\right) = \frac{1}{\sqrt{\left(1+\frac{1-t^2}{2}\mu\right)\left(1+\frac{1-t^2}{2t^2}\nu\right)}},$$

where μ, ν are the measurement and ancillary imperfection coefficients, as is introduced earlier in this chapter.

Example 4.4. (Active two-mode coupling.) The second system we consider is described by the following Hamiltonian:

$$\hat{H} = g\left(\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_1 \hat{a}_2\right),$$

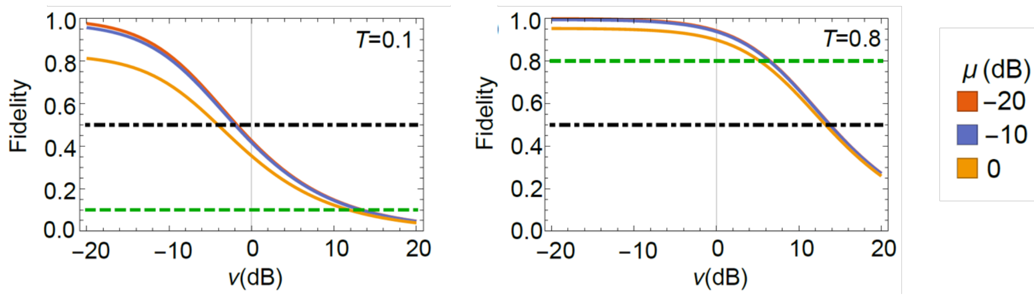

FIGURE 4.6. *Average fidelity in the passive system.*

The figures show the average fidelity of adaptive quantum transduction as a function of ancillary imperfection coefficient ν, given the measurement imperfection coefficient $\mu = -20$, -10, and 0 dB. The simulated imperfect transducers are passive two-mode scattering processes with the transmittance being $T = t^2 = 0.8$ and $T = t^2 = 0.1$ respectively, as explained in the example. The dark-dotted dashed lines correspond to the threshold fidelity of 1/2. The green-dashed lines correspond to the fidelity achieved by the optimized direct quantum transduction protocol. Therefore, we see that adaptive quantum transduction can greatly improve the average fidelity with reasonable imperfection coefficients, and can even convert a non-quantum transducer (with average fidelity lower that 1/2) into a quantum transducer.

which couples the two modes like a two-mode squeezer. With other components unchanged, the Gaussian process is associated with the symplectic matrix (see Appx. A for more detail),

$$S = \begin{pmatrix} 0 & t' & r' & 0 \\ t' & 0 & 0 & r' \\ r' & 0 & 0 & t' \\ 0 & r' & t' & 0 \end{pmatrix},$$

with

$$r' = \frac{C+1}{C-1}, \quad t' = \frac{2\sqrt{C}}{1-C},$$

satisfying $r'^2 - t'^2 = 1$, and

$$C = \frac{g^2}{\kappa_1 \kappa_2}.$$

. It follows that the direct quantum transduction Gaussian channel is $\mathcal{G}_{\mathrm{Id},N_D}$ with

$$N_D = 2\frac{1+t'^2}{t'^2}\mathrm{Id}.$$

The average fidelity of $\mathcal{G}_{\mathrm{Id},N_D}$ is

$$\bar{F}\left(\mathcal{G}_{\mathrm{Id},N_D}\right) = \frac{t'^2}{1+2t'} < \frac{1}{2}.$$

The adaptive quantum transduction Gaussian channel is $\mathcal{G}_{\mathrm{Id},N_A}$ with

$$N_A = \begin{pmatrix} \mu\left(1+t'^2\right) & \\ & \nu\frac{1+t'^2}{t^2} \end{pmatrix}.$$

Its average fidelity is

$$\bar{F}\left(\mathcal{G}_{\mathrm{Id},N_A}\right) = \frac{1}{\sqrt{\left(1+\frac{1+t'^2}{2}\mu\right)\left(1+\frac{1+t'^2}{2t^2}\nu\right)}}$$

Clearly, as shown in Figs. 4.6&4.7, when μ and ν are reasonably small, we can always obtain a “quantum” transducer from the adaptive quantum transduction protocol in both of the above examples, while the direct transdcution production protocol fails to be quantum when $t^2 \le \frac{1}{2}$ in the passive situation or for any t' in the active situation.

Furthermore, the symplectic transformation $\tilde{S}$ in each of the two examples can be viewed as a single-mode squeezing operation. Specifically, in the first example, we have

$$\tilde{S} = -Z\left(t\right)\Omega = -\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -t \\ t^{-1} & 0 \end{pmatrix}$$

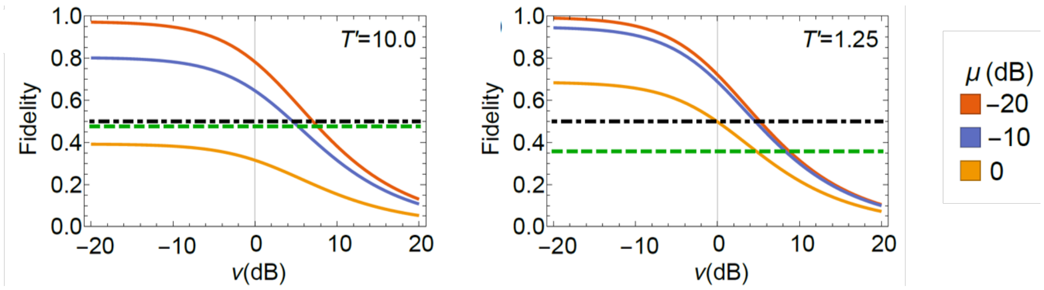

FIGURE 4.7. *Average fidelity in the active system.*

The figures show the average fidelity of adaptive quantum transduction as a function of ancillary imperfection coefficient ν, given the measurement imperfection coefficient $\mu = -20, -10$, and 0 dB. The simulated imperfect transducers are active two-mode scattering processes with the transmittance being $T = t^2 = 10.0$ and $T = t^2 = 1.25$ respectively, as explained in the example. The dark-dotted dashed lines correspond to the threshold fidelity of 1/2. The green-dashed lines correspond to the fidelity achieved by the optimized direct quantum transduction protocol. In both figures, the direct quantum transduction protocols are beaten by classical transduction protocols, since the average fidelities are below the 1/2 threshold, while the adaptive quantum transduction protocols beat both of them, given reasonable μ and ν.

while in the second example, we have

$$\tilde{S} = -Z(t')\,\Omega = \begin{pmatrix} 0 & -t' \\ t'^{-1} & 0 \end{pmatrix}.$$

Thus, the adaptive quantum transdcution protocol also provides a way to squeeze bosonic modes using squeezing resources in the other modes. In general, the generalized teleportation protocol allows us generating controllable few-body Gaussian operation out of a many-body Gaussian operation using only local Gaussian processes and classical communications.

4.6. Conclusion and outlook

In this chapter, we demonstrated a generalized interpretation of continuous variable quantum teleportation and its application in quantum transduction. This interpretation also applied to discrete variable quantum teleportation, which we will see in the last chapter of this thesis. Here we summarize the advantages of this theoretical framework: (1) It applies to, in general, arbitrary symplectic transformations; (2) It imposes no constraints on the number of bosonic modes, meaning that we can apply it to a large bosonic system comprising a huge amount of modes; (3) It requires on local Gaussian processes and classical communications. Therefore, its flexibility renders this theoretical framework a promising candidate for practical solution to quantum transduction and possibly more bosonic engineering problems.

However, as we have demonstrated, the success of this framework relies heavily on the high quality of squeezing and homodyne measurement, which are often elusive in laboratories. In the succeeding chapter, we discuss another possibility for practical quantum transduction without assuming these usually impractical resources of bosonic control.

CHAPTER 5

Interference-based Gaussian control

5.1. Introduction

Interference, an essential feature of quantum mechanics, is the fountain of countless applications. As we know, interference can be easily calculated, if we represent quantum states as vectors and quantum operations as matrices. Therefore, we are wondering if this feature can be useful in the phase space, where the quadrature operators are treated as vectors and the symplectic transformations as matrices. It turns out that interference not only exists in the phase space but is also a powerful tool for solving practical bosonic control problems, including quantum transduction[1].

Alluded to our conclusion of the preceding chapter, adaptive quantum transduction suffers the unavailability of infinite squeezing and perfect homodyne measurement. Therefore, finding a transduction scheme independent of these impractical resources is much needed. An attempt to solve this problem is demonstrated in [**27**], based on the interference of symplectic transformations. Simply speaking, suppose we have many identical copies of a given imperfect two-mode quantum transducer, which is described by a symplectic transformation, and we are able to control both of the involved modes using arbitrary single-mode unitary Gaussian operations (not including infinite squeezing); Then we can construct a sequence of symplectic transformations interspersed by carefully-chosen single-mode symplectic transformations, such that the whole sequence effectively swaps the two modes up to single-mode symplectic transformations[2], as hown in Fig. 5.1. In other words, we are able to convert

[1]This chapter is based on the results in [**28**].

[2]This means, the input quantum state $\hat{\rho}$ of one mode will be transformed to the output state $\hat{U}_S\hat{\rho}\hat{U}_S^\dagger$ of the other mode, with $\hat{U}_S$ a single-mode symplectic transformation.

an imperfect quantum transducer to a perfect quantum transducer, without requiring infinite squeezing or any measurement.

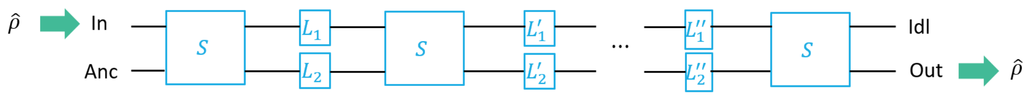

FIGURE 5.1. *Interference-based swapping of two bosonic modes.* The blue boxes labeled by S represent identical copies of a given two-mode symplectic transformation S. The other blue boxes represent the local (single-mode) symplectic transformations which are carefully designed to engineer the interference of the quadratures. Once properly constructed, this sequence will faithfully convert the input quantum state to the output port, up to a unitary transformation.

The above scheme demonstrates the power of the interference of symplectic transformations and its promising application in quantum transduction. However, we notice that scheme works only for the two-mode situation. Thus, we cannot apply this scheme to addressing the inevitable side band effects, which is a major challenge for usual transduction proposals and our motivation of adaptive quantum transduction as explained in the beginning section of the preceding chapter.

In this chapter, we will show that, by fully exploiting the mathematical properties of symplectic transformations, we can swap a pair of bosonic modes by construcing a sequence of symplectic transformations, with similar structure as of the above scheme but fundamentally different engineering strategies, for arbitrary-mode quantum transduction scenarios. Specifically, we still assume we have many copies of an imperfect quantum transducer, now described by a multi-mode symplectic transformation S, and arbitrary single-mode unitary Gaussian control of each mode involved in the process, which will be referred to as the local Gaussian operations. The whole idea will be based on a surprisingly simple but useful property of the local Gaussian operations:

We can transform any given quadrature operator to any other quadrature operator[3] using local Gaussian operations. This can be easily seen form the single-mode situation as shown in Fig. 5.2. For some technical reasons, this property provides for us great flexibility in manipulating multi-mode symplectic transformations: Now we can construct a three-component sequence $S' = SL'S$, which we call a sandwich operation, using the given symplectic transformation and a local operation L' to decouple $\hat{q}_1$ with the other canonical quadrature operators; similarly, we can construct a sandwich operation $S^* = SL^*S$ to decopule $\hat{q}_N$ (suppose there are N-modes) with the other canonical quadrature operators. Then another layer of sandwich operation, $S^{td} = S^* L^{td} S'$, will be an effective symplectic transformation perfectly transducing the first mode to the last mode. Moreover, we can obtain another sandwich operation $S^{dc} = S' L^{dc} S'$ to decouple the first mode from the rest of the system. It turns out, the sequence $\left(S^{td} L^a S^{dc}\right) L^b \left(S^{td} L^c S^{dc}\right)$ swaps the first and the last modes of system. Since which mode is the first mode and which mode is the last only depends on how we label the bosonic modes, we therefore obtain universal swap and universal decoupling (S^{dc}) for multi-mode bosonic systems.

To give a more intuitive picture of the above mechanism, we would like to demonstrate its application to the simplest two-mode situation in the following example.

Example 5.1. (Two-mode application) Consider a symplectic transformation

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}. \tag{5.1.1}$$

[3]We have to exclude some exceptional cases to make this statement rigorous: The statement may not hold if a considered quadrature does not act on every canonical quadrature operator. Specifically, the quadrature $\hat{q}_1 + \hat{p}_1$ cannot be transformed to $\hat{q}_2 + \hat{p}_2$ by local Gaussian operations, since $\hat{q}_1 + \hat{p}_1$ does not influence the second bosonic mode.

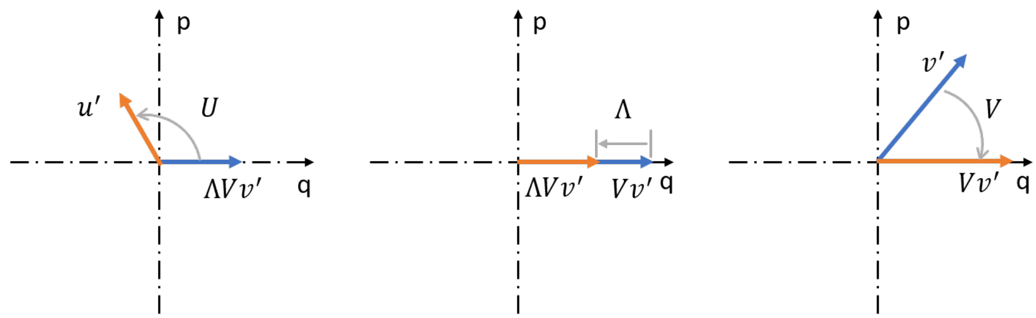

FIGURE 5.2. *Transitivity of single-mode symplectic transformations.* These figures demonstrate a random quadrature can be transformed to another random quadrature by symplectic transformations in the single-mode symplectic space. We start with a random quadrature, i.e. the blue vector in the right figure, which can be symplectically rotated to the $\hat{q}$-axis (the orange vector). Then this orange vector can be dilated to a proper length using a symplectic squeezing operation, as shown in the middle figure. At last, in the left figure, the dilated vector is transformation to the target quadrature via a simple rotation. Denoting the rotations and the squeezing operation by U, V and Λ, we conclude that the same result can be achieved by a single-mode symplectic operation $U\Lambda V$. Since a multi-mode quadrature can be decomposed as multi-mode single-mode chuncks, each of which can be freely controlled by single-mode symplectic transformations, this single-mode conclusion can be easily extended to multi-mode situations.

Now assume a local operation L' transforming the quadrature[4] $(S_{11}, S_{21}, S_{31}, S_{41})^t$ (the first column of S) to the quadrature $\Omega\,(S_{11}, S_{12}, S_{13}, S_{14})^t$ (the first row of S)[5]. Then the definition of S, i.e. the CCRs, guarantees that the sandwich operation

[4]As we know in euclidean geometry, the columns (and rows) of an orthogonal matrix form a orthonormal basis of the euclidean space. Similarly in symplectic geometry, the columns (and rows) of a symplectic transformation form a symplectic basis of the symplectic space, which is a direct result of the definition of symplectic transformation. Therefore, we can safely treat columns (and rows) of a symplectic transformation as quadratures.

[5]Here Ω is the symplectic form.

$S' = SL'S$ to be of the following form:

$$S' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & S'_{22} & S'_{23} & S'_{24} \\ 0 & S'_{32} & S'_{33} & S'_{34} \\ 0 & S'_{42} & S'_{43} & S'_{44} \end{pmatrix}.$$

Similarly, let L^* be a local operation transforming the quadrature $\Omega\left(S_{11}, S_{21}, S_{31}, S_{41}\right)^t$ (the first column of S) to $(S_{31}, S_{32}, S_{33}, S_{34})^t$ (the third row of S). The sandwich operation $S^* = SL^*S$ is of the form

$$S^* = \begin{pmatrix} 0 & S^*_{12} & S^*_{13} & S^*_{14} \\ 0 & S^*_{22} & S^*_{23} & S^*_{24} \\ 0 & 1 & 0 & 0 \\ -1 & S^*_{42} & S^*_{43} & S^*_{44} \end{pmatrix}.$$

Then using a local operation L^{td} transforming properly transforming the second column of S' to be a linear combination, multiplied by the symplectic form, of S^*, of rows of S^*, i.e.

$$\begin{pmatrix} 1 \\ S'_{22} \\ S'_{32} \\ S'_{42} \end{pmatrix} \to -2S^*_{42}\Omega \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \Omega \begin{pmatrix} -1 \\ S^*_{42} \\ S^*_{43} \\ S^*_{44} \end{pmatrix},$$

we obtain the sandwich opeartion

$$S^{td} = S^* L^{td} S' = \begin{pmatrix} 0 & 0 & S^{td}_{13} & S^{td}_{14} \\ 0 & 0 & S^{td}_{23} & S^{td}_{24} \\ 0 & 1 & 0 & 0 \\ -1 & S^{td}_{42} & 0 & 0 \end{pmatrix}.$$

Since each off-diagonal 2×2 block of S^{td} is a symplectic transformation, the symplectic transformation S^{td} swaps the two modes up to local symplectic transformations.

We will provide more technical details and rigorous proofs of this idea in the succeeding section. However, although this scheme is widely applicable to physically practical symplectic transformations, there are still some exceptional situations where more careful discussions are necessary. To stress these exceptions, we say the above scheme is applicable to generic symplectic transformations and call the non-generic symplectic transformations the edge cases. We will discuss the edge cases, which are possibly of more mathematical interest than practical importance, in detail in the third section of this chapter, to complete our discussion.

5.2. Permuting bosonic modes using generic symplectic transformations

Symplectic transformations can be viewed as coordinate changes on the phase space. Therefore, for a symplectic transformation $S : E \to E$, we have symplectic bases $\{y_j\}$ and $\{z_j\}$ of E, such that

$$\begin{aligned} S &= \sum_j y_j x_j^t, \\ &= \sum_j x_j z_j^t, \end{aligned}$$

with $\{x_j\}$ be the canonical symplectic basis of E. We allude to our previous discussions on symplectic subspaces that

$$E = \oplus_j E_j$$

with each E_j a canonical symplectic subspace, spanned by $\{x_{2j-1}, x_{2j}\}$. Then we can denote the restriction and projection of an element u in E to a symplectic subspace E_j by

$$u_{E_j} := \pi_j u.$$

We suppose the symplectic space E is $2N$-dimensional. Then we have the following:

LEMMA 5.2. *For all* $u, v \in E$ *which satisfy*

$$u_{E_j}^t u_{E_j} + v_{E_j}^t v_{E_j} = 0, \text{ if and only if } u_{E_j} = v_{E_j} = 0, \text{for all } j \in \{1, \ldots, N\},$$

there exists a symplectic transformation

$$L = \oplus_{j=1}^N L_j,$$

with each $L_j : E_j \to E_j$ *being a symplectic transformation, such that*

$$Lu = cv,$$

for any non-vanishing real number c*.*

PROOF. For each pair of non-vanishing u', v' in each symplectic subspace E_j, there are orthogonal matrices U, V and a diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

such that

$$u' = U\Lambda V v'.$$

This proves the existence of L_j for all $j \in \{1, \ldots, N\}$, and hence the existence of L. □

Symplectic transformations on E that are direct sum of symplectic transformations on the canonical symplectic subspaces, such as L in the above lemma, will be referred to as the local symplectic transformations.

Now let $\{x_j\}$ be the canonical symplectic basis, $\{y_j\}$ and $\{z_j\}$ be two other symplectic bases of E, satisfying

$$\sigma(y_j, y_k) = \Omega_{j,k}, \;\forall j, k \in \{1, 2, \ldots, 2N\},$$

and

$$\sigma(z_j, z_k) = \Omega_{j,k},\ \forall j, k \in \{1, 2, \ldots, 2N\}.$$

Then we have two symplectic transformations

$$S = \sum_{j=1}^{2N} y_j x_j^t,$$

and

$$S' = \sum_{j=1}^{2N} x_j z_j^t.$$

Furthermore, these symplectic transformations are generic if they are randomly sampled from the ensemble of all the symplectic transformations. In particular, for generic S and S', each pair of vectors y_j and z_j are expected to satisfy the condition in the lemma above, for the probability of sampling a non-generic symplectic transformation from the ensemble vanishes[6]. Then the following holds:

LEMMA 5.3. *Let S and S' be generic symplectic transformation. For any non-vanishing real number c, there is a sympletic transformation L, such that*

$$S'' = S'LS$$

is a symplectic transformation satisfying

$$(S'')_{2a-1,2b} = c^{-1},$$

$$(S'')_{2a,2b-1} = -c,$$

and

$$(S'')_{j,2b-1} = (S'')_{2a-1,k} = 0,$$

for any $j \neq 2b$ and $k \neq 2a$, for any $a, b \in \{1, 2, \ldots, N\}$.

[6]We can simply assume a generic symplectic transformation is a matrix with no vanishing element.

Proof. According to Lemma. 5.2, for any non-vanishing real number c, there always exists a local symplectic transformation L, such that

$$Ly_{2b-1} = c\Omega z_{2a-1},$$

or equivalently

$$z_{2a-1} = -c^{-1}\Omega L y_{2b-1}.$$

It follows that

$$\begin{aligned}
S'' &= S'LS \\
&= \sum_{j,k=1}^{2N} (z_j^t L y_k) x_j x_k^t \\
&= \sum_{j=1}^{2N} (z_j^t L y_{2b-1}) x_j x_{2b-1}^t + \sum_{j=2}^{2N} (z_{2a-1}^t L y_j) x_{2a-1} x_j^t + \sum_{j,k=2}^{2N} (z_j^t L y_k) x_j x_k^t \\
&= -c x_{2a} x_{2b-1}^t + c^{-1} x_{2a-1} x_{2b}^t + + \sum_{j,k=2}^{2N} (z_j^t L y_k) x_j x_k^t,
\end{aligned}$$

which leads to the following results

$$(S'')_{2a-1,2b} = x_{2a-1}^t S'' x_{2b} = c^{-1},$$

$$(S'')_{2a,2b-1} = x_{2a-1}^t S'' x_{2b} = -c,$$

$$(S'')_{2a-1,j} = x_{2a-1}^t S'' x_j = 0, \ \forall j \neq 2,$$

and

$$(S'')_{j,2b-1} = x_j^t S'' x_{2b-1} = 0, \ \forall j \neq 2.$$

This finishes the proof. $\square$

Let S, S' be symplectic transformations as introduced above, with

$$y_{2b-1} = -c x_{2b}$$

and

$$z_{2a-1} = c^{-1} x_{2a},$$

for some non-vansihing real number c, and $a, b \in \{1, 2, \ldots, N\}$. Then according to the definition of symplectic transformations, we also have

$$S_{j,2b-1} = S_{2a-1,j} = (S')_{j,2b-1} = (S')_{2a-1,j} = 0,$$

for any $j \neq 2$. Furthermore, they are generic in the sense that Lemma. 5.2 can be applied to y_{2b} and z_{2a}. We denote the conjugate symplectic subspace of E_a by $E_{a'}$ and E_b by $E_{b'}$, then we have:

LEMMA 5.4. *Consider generic symplectic transformations S and S' as introduced right above. Then there exists a symplectic transformation L such that*

$$S''_{E_a,E_b} = S_1,$$

$$S''_{E_{a'},E_{b'}} = S_r,$$

and

$$S''_{E_a,E_{b'}} = S''_{E_{a'},E_b} = 0$$

with

$$S'' = S'LS,$$

where $S_1 : E_b \to E_a$ and $S_r :\to E_{b'} \to E_{a'}$ are symplectic transformations.

PROOF. According to Lemma. 5.2, there exists a local symplectic transformation

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_N,$$

such that

$$Ly_{2b} = c^{-2}\Omega z_{2a} - (z^t_{2a-1} z_{2a} - y^t_{2b-1} y_{2b})\Omega z_{2a-1}.$$

This can be satisfied if we let $L_a = \Omega_a$ which leads to

$$Ly_{2b-1} = -c^2 \Omega z_{2a-1}.$$

The above amounts to the following

$$z_{2a-1} = c^{-2} \Omega L y_{2b-1},$$

and

$$z_{2a} = (z_{2a-1}^t z_{2a} - y_{2b-1}^t y_{2b}) \Omega L y_{2b-1} - c^2 \Omega L y_{2b}.$$

It follows that

$$\begin{aligned} x_j^t S'' x_k &= x_j^t S' L S x_k \\ &= z_j^t L y_k \\ &= 0, \end{aligned}$$

for $j \neq 2a-1, 2a$ and $k = 2b-1, 2b$ or $j = 2a-1, 2a$ $k \neq 2b-1, 2b$, which finishes the proof. Besides, we have the following explicit form:

$$\begin{aligned} S_1 &= \begin{pmatrix} S_{2a-1,2a-1} & S_{2a-1,2b} \\ S_{2a,2b-1} & S_{2a,2b} \end{pmatrix} \\ &= \begin{pmatrix} x_{2a-1}^t \\ x_{2a}^t \end{pmatrix} S \begin{pmatrix} x_{2b-1} & x_{2b} \end{pmatrix} \\ &= \begin{pmatrix} z_{2a-1}^t L y_{2b-1} & z_{2a-1}^t L y_{2b} \\ z_{2a}^t L y_{2b-1} & z_{2a}^t L y_{2b} \end{pmatrix} \\ &= \begin{pmatrix} 0 & c^{-2} \\ -c^2 & z_{2a-1}^t z_{2a} - y_{2b-1}^t y_{2b} \end{pmatrix}. \end{aligned}$$

$\square$

The lemmas we just proved directly lead to the following:

THEOREM 5.5. *Let $\{v_i\}$ be a symplectic basis on E. For generic symplectic transformations $S_a, S_b, S_c, S_d : E \to E$, there exist three local symplectic transformations $L^{(1)}$, $L^{(2)}$and $L^{(3)}$, such that*

$$S_a L^{(3)} S_b L^{(2)} S_c L^{(1)} S_d = S_1 \oplus S_r,$$

with symplectic transformations $S_1 : E_1 \to E_1$, $S_r : E_2 \oplus E_3 \oplus \cdots \oplus E_N \to E_2 \oplus E_3 \oplus \cdots \oplus E_N$.

And:

THEOREM 5.6. *Let $\{v_i\}$ be a symplectic basis on E. For generic symplectic transformations $S_a, S_b, S_c, S_d : E \to E$, there exist local symplectic trasnformations $L^{(1)}$, $L^{(2)}$, and $L^{(3)}$ such that*

$$S_a L^{(3)} S_b L^{(2)} S_c L^{(1)} S_d = \begin{pmatrix} 0 & S_r \\ S_1 & 0 \end{pmatrix},$$

where $S_1 : E_1 \to E_N$, $S_r : E_{1'} \to E_{N'}$ are symplectic transformations.

Furthermore, by combing the above two theorems we see that:

THEOREM 5.7. *Let $\{v_i\}$ be a symplectic basis on E. For generic symplectic transformations $S^{(1)}, S^{(2)}, \ldots, S^{(16)} : E \to E$, there local symplectic transformations exist $L^{(1)}, L^{(2)}, \ldots, L^{(15)}$, such that*

$$S^{(16)} L^{(15)} S^{(15)} \cdots S^{(3)} L^{(2)} S^{(2)} L^{(1)} S^{(1)} = \begin{pmatrix} 0 & 0 & S_1 \\ 0 & S_r & 0 \\ S_2 & 0 & 0 \end{pmatrix},$$

where $S_1 : E_N \to E_1$, $S_2 : E_1 \to E_N$, $S_r : E_{1'} \to E_{N'}$ are symplectic transformations.

Proof. It suffices to consider the following product of symplectic transformations

$$\begin{pmatrix} S_g & 0 \\ 0 & S_h \end{pmatrix} \begin{pmatrix} L_c & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} S_e & 0 \\ 0 & S_f \end{pmatrix} \begin{pmatrix} L_b & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} S_c & 0 \\ 0 & S_d \end{pmatrix} \begin{pmatrix} L_a & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 & S_a \\ S_b & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & S_g L_c S_e L_b S_c L_a S_a \\ S_h S_f S_d S_b & 0 \end{pmatrix}$$

with

$$S_a : E_{1'} \to E_{N'},$$

$$S_b : E_N \to E_1,$$

$$S_c, S_e, S_g : E_{N'} \to E_{N'},$$

$$S_h, S_f, S_d : E_N \to E_N,$$

being symplectic transformations and

$$L_a, L_b, L_c : E_{N'} \to E_{N'}$$

being local symplectic transformations. Therefore, $S_h S_f S_d S_b$ should be equivalent to S_2. According to the previous theorem, we then choose the proper forms of L_a, L_b, and L_c, such that the product of symplectic transformations $S_g L_c S_e L_b S_c L_a S_a$ amounts to the following transformation

$$\begin{pmatrix} 0 & S_1 \\ S_r & 0 \end{pmatrix},$$

which finishes the proof. □

This result implies that we can swap any two bosonic modes (up to local symplectic transformations) using generic symplectic transformations. Therefore, it enables us to arbitrarily permute bosonic modes using a sequence of generic symplectic transformations and local symplectic transformations, since any permutation is the product of swappings.

5.3. Edge cases: the non-generic symplectic transformations

The proofs in the last section are based on the assumption of generic symplectic transformations, which is only vaguely defined by now. Although a direct and explicit definition of generic symplectic transformations may be elusive, it is possible to specify what are non-generic symplectic transformations, which are referred to as the edge cases. A quick example of a non-generic symplectic transformations is a symplectic transformation permuting multiple bosonic modes (up to local symplectic transformations), e.g.

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Apparently, the product of such bosonic permutations cannot be of arbitrary form, which conflicts with our promise at the end of the last section.

Before proceeding, we first define a map f, mapping an arbitrary symplectic transformation S to a matrix with each of its elements equals 1. Specifically, denoting

$$\mathbb{S}_{j,k} = \begin{pmatrix} S_{2j-1,2k-1} & S_{2j-1,2k} \\ S_{2j,2k-1} & S_{2j,2k} \end{pmatrix},$$

we have

$$f(S)_{j,k} = \begin{cases} 1, & \text{if } \mathbb{S}_{j,k} \neq 0, \\ 0, & \text{if } \mathbb{S}_{j,k} = 0, \end{cases}$$

for any $j, k \in \{1, 2, \ldots, N\}$. Then we note that an arbitrary symplectic transformation S will satisfy one of the following three conditions:

(1) $f(S) = P + Q$, with P being a cyclic permutation matrix and Q a full-ranked matrix, each element of which is equal to 1;

(2) There are decomposition of the symplectic space

$$E \cong E_S^{(1)} \oplus E_S^{(2)} \oplus \cdots \oplus E_S^{(M)}$$
$$\cong E_S^{'(1)} \oplus E_S^{'(2)} \oplus \cdots \oplus E_S^{'(M)}$$

with each $E^{(j)}$ being a direct sum of canonical symplectic subspaces for all $j \in \{1, 2, \ldots, M\}$, and a permutation $\mathcal{P}$ of the set $\{1, 2, \ldots, M\}$, such that $S_{E'^{(j)}, E^{(k)}}$ is not a zero matrix if and only if $j = \mathcal{P}(k)$ for any $j, k \in \{1, 2, \ldots, M\}$. And each non-zero $S_{E'^{(j)}, E^{(k)}}$ satisfies the first condition.

This implies that we can associate any given symplectic transformation S with a pair of collectioncs of symplectic spaces decomposition $(\left\{E_S^{(j)}\right\}, \left\{E_S'^{(k)}\right\})$, and a permutation of the set $\{1, 2, \ldots, M\}$ denoted by $\mathcal{P}_S$. In particular for a symplectic transformation satisfying the first condition, we have $M = 1$ and $\mathcal{P}_S$ is the trivial identity operation.

Since we can perform local symplectic transformations before and after any symplectic transformations, from now on, we will assume S is fully randomized, i.e. each $\mathbb{S}_{j,k}$ will be considered to be a randomly-generated 2×2 matrix. Then for any such symplectic transformation S, we have

$$f(S^c) = f(f(S)^c),$$

where c is an integer.

It turns out that there will always be a large enough integer c, such that the c-th power S^c of an symplectic transformation S satisfies the following stabilizing condition: Let $(\left\{E_{S^c}^{(j)}\right\}, \left\{E_{S^c}'^{(k)}\right\})$ be the symplectic space decompositions and $\mathcal{P}_{S^c}$ the permutation associated to S^c. Then we have $E_{S^c}^{(j)} = E_{S^c}'^{(j)}$ for all $j \in \{1, 2, \ldots, M\}$ and $\mathcal{P}_{S^c}$ is the trivial identify operation. Moreover, there is an integer $d \geq c$, such that the associated permutations of S^c, S^{c+1},...,S^d forms a permutation group[7]. Also note that $S^c, S^{c+1}, \ldots, S^d$ will have the same associated symplectic space decomposition.

[7]This implies that any S^e with $e > d$ will also be associated to an element in this group.

When $N = 2$, it is easy to check the stabilizing condition can be satisfied with $c \leq 2$. When $N > 2$, a rough upper bound of such an integer c is $N(N+1)$, which is a result of the observation that the total number of non-zero $\mathbb{S}_{j,k}$ decreases as c increases.

Example 5.8. Consider the symplectic transformation

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$f(S) = Q + P = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

with

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

then we have

$$f\left(S^2\right) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

which satisfies the stabilizing condition above.

EXAMPLE 5.9. Let

$$f(S) = Q + P = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

with

$$Q = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \text{ and } P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

then we have

$$f\left(S^2\right) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

$$f\left(S^3\right) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

and

$$f\left(S^4\right) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

which satisfies the stabilizing condition above.

EXAMPLE 5.10. The last example: Let

$$f(S) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then we have

$$f(S^7) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

which satisfying the stabilizing condition.

Therefore, supposing that we only have access to a given symplectic transformation and arbitrary local symplectic transformations, the edge cases can now be easily identified, as well as the resulting bosonic mode permutation achieved via our results in the previous section. In particular, when the first stabilizing condition is satisfied, we are able to permute arbitrary pair of bosonic modes and thus construct arbitrary bosonic mode permutations. Specifcally, let S^c satisfy the stabilizing condition above, associated to a symplectic space decomposition

$$E \cong E_{S^c}^{(1)} \oplus E_{S^c}^{(2)} \oplus \cdots \oplus E_{S^c}^{(M)},$$

and let integer $d \geq c$, such that the associated permutations of S^c, S^{c+1},...,S^d form a permutation group. Then we have the following conclusions:

(1) With a proper randomized prior-processing based on the discussions in this section, any mode in the bosonic system can be decoupled using the algorithm introduced in the last section.

(2) Moreover, the algorithms in the last section can be applied to swap the jth and kth modes in a N-mode system using copies of a given symplectic transformation S, if and only if either both canonical symplectic spaces E_j and $E_k$are contained in the same direct summand $E_{S^c}^{(l)}$ for some $l \in \{1, 2, \ldots, M\}$ or there exist integers $l, m \in \{1, 2 \ldots, N\}$ and $c \leq n \leq d$, such that $E_j \subseteq E_{S^c}^{(l)}$, $E_k \subseteq E_{S^c}^{(m)}$, and $\mathcal{P}_{S^n}$ swaps the j and k.

5.4. Conclusion and outlook

In this chapter, we discussed how to construct universal swap and decoupling, and hence permutations of bosonic modes using interference of symplectic transformations. We demonstrated our scheme, derived from the fundamental CCRs of quantum mechanics, are widely applicable to bosonic systems and unitary Gaussian processes involving arbitrary many bosonic modes. When applied to quantum transduction, we obtain a scheme that convert imperfect quantum transducers to perfect quantum transducers only using reasonable single-mode Gaussian unitary operations.

This chapter concludes our attempts of applying symplectic transformation to solving bosonic quantum control, especially quantum transduction. Next, we discuss the role of symplectic transformations in another intriguing field–bosonic sensing. We will see, with the knowledge of symplectic geometry, the quantum noise model of a bosonic sensing scheme can be easily established and utilized to provide an ultimate bound of the sensing precision of the corresponding scheme.

CHAPTER 6

Applications to bosonic sensing

6.1. Introduction

We discussed the application of symplectic geometry in quantum information processing in the previous chapters. In this chapter[1], our focus is shifted to another intriguing field–exotic bosonic classical sensing[2] with drastic precision enhancement. We will utilize the theory of symplectic geometry to provide a thorough and systematic analysis for the exotic bosonic sensing schemes.

What motivated us to invest our time in this field is the recent rapid development of exceptional point sensing. Exceptional point sensing has recently attracted a lot of attention because of its potential to drastically improve signal sensing precision, which is of broad interest in physics with many important applications. Based on this, people proposed [**48, 49, 50**] and experimentally demonstrated [**51, 52**] the novel sensing scheme enhanced by exceptional points, where the whole frequency spectrum is measured and the frequency splitting is used to estimate the signal which is treated as a small perturbation to the exceptional point system. However, these theoretical proposals and experimental demonstrations lack the consideration of measurement uncertainty caused by quantum noise, which may be possibly amplified in presence of the exceptional points and hence deteriorate the sensing enhancement predicted by the classical analysis. Therefore, it is questionable that this precision enhancement is genuine in a serious quantum model of the whole process.

[1]This chapter is mainly based on our publication [**54**].
[2]For readers who are familiar with quantum sensing, here, we would like to stress that although tools of quantum metrology will be heavily used in this chapter, the schemes we will investigate remain classical insofar as the entanglement of probes are not used as a boost of the sensing precision.

To answer this question, we must first find a suitable quantum noise model for exceptional point sensing. Fortunately, as classical bosonic sensing, the physical process of exceptional point sensing can be described a Gaussian process. Specifically, the physical process of an exceptional point sensing sensing can be described as follows: People first add the quantity to estimate into a linearly-coupled bosonic system at exceptional point; and then inject classical probes into this coupled system; at last, the scattered probes will be measured and the outcomes will be used to provide an estimate of the value of the quantity. For example, the change of temperature may cause the change of the resonant frequency of optical modes. Therefore, the small temperature change can be viewed as a perturbation to the Hamiltonian of a linearly-coupled optical system. Now we put the unperturbed the system at the exceptional point. And send probing light to interact with these optical modes. Then, the amplitude and phase of the scattered probing light will depend on the small temperature change. Therefore, to estimate this temperature change, we only need to measure the scattered probing light and extract the wanted information from the outcomes. This linear scattering process picture of exceptional point sensing reveals its Gaussian process nature. Moreover, alluded to Sec. 3.7, a noisy Gaussian process can be embedded in a unitary Gaussian process, which can thus be modeled by a symplectic transformation. Physically speaking, this is equivalent to a conclusion in quantum noise theory [**53**]: bosonic losses and gains, as long as they are generated by linear coupling between the system and the reservoir, can be modeled as a linear interaction between fictitious probing modes and the system modes. This observation lays the ground of our theoretical framework in this chapter.

Now we know exceptional sensing is closely related to Gaussian processes. It remains unknown how to theoretically justify the sensing precision enhancement. Since now we model exceptional sensing as a quantum process, the justification must be done using a theory compatible with quantum mechanics. This theory is quantum metrology, originally developed for justifying sensing precision of all quantum sensing schemes. In particular, we will utilize the notion of quantum Fisher information, a quantity that is used to quantify the ultimate bound of the sensing power of a sensing scheme. The calculation of quantum Fisher information only depends on the physical process, and is by its definition optimized over all possible physical measurement schemes. Although the calculation of quantum Fisher information is complicated and non-analytical in general, for Gaussian processes and Gaussian input states, we have convenient closed-form formulas which yield analytical results. We will see in the succeeding section more details about quantum Fisher information and its calculation and demonstrate in later sections how this tool can justify the sensing enhancement induced by exceptional points.

After the discussions of exceptional point sensing, we extend our discussions to the general ideal of exotic bosonic sensing. We are interested in the following question: Is exceptional point sensing the only classical bosonic sensing scheme that can provide unexpected drastic precision enhancement, beyond any conventional classical sensing schemes? This question is partly answered in [**64**], where a non-exceptional-point classical bosonic sensing scheme is demonstrated to have a genuine precision enhancement even in presence of quantum noise. In this chapter, by using our knowledge of symplectic dilation of noisy Gaussian processes, we would like to provide a unified theoretical framework for any such exotic bosonic sensing schemes based on Gaussian process. Our investigation delivers a short message: by dilating the noisy Gaussian process to a symplectic transformation, one can see the exotic precision enhancement, if it exists, can be understood as a phenomenon of squeezing. We hope this novel perspective can help deepen our understanding of classical bosonic sensing schemes and inspire more promising applications in the future.

6.2. Gaussian Fisher information

The process of sensing a classical physical quantity θ can be described as follows: We let physical objects, the probes, interact with a system described by a Hamiltonian containing this quantity. Then we measure the scattered probes, described by the quantum state $\hat{\rho}_\theta$, to obtain a measurement outcome. Usually, we will perform multiple rounds, say N rounds, of measurement, such that there will be than one outcomes, denoted by η_j, for $j \in \{1, 2, \ldots, N\}$. Then based on our knowledge of the physical process, we will use a function $f(N, \eta_1, \eta_2, \ldots, \eta_N)$ to obtain an estimate $\hat{\theta}_N$ of the true value θ. This function will be wisely chosen, so that as N approaches infinity, the difference between the estimate and the true value tends to vanishing.

When the dependence of the measured quantum state on θ has as good properties as what we focus on in the rest of this chapter, we know how to construct the optimal estimate in general. Note that the state $\hat{\rho}_\theta$ and the measurement scheme together yields a probability distribution of the measurement outcome, $p_\theta(\eta)$. Then the optimal estimate can be given by

$$\hat{\theta}_N = \mathrm{argmax}_\phi \prod_{i=1}^{N} p_\phi\left(\eta_i\right), \tag{6.2.1}$$

which is known as the maximum likelihood estimation method [**65**].

This estimate is optimal in the following sense: With a large N, the value of $\hat{\theta}_N$ can be considered to be sampled from a normal distribution centered at the true value θ. Then letting

$$\delta\theta_N = \sqrt{\mathbb{E}(\hat{\theta}_N - \theta)} \tag{6.2.2}$$

be the deviation of the estimate from the true value, we have

$$\delta\theta_N = \frac{1}{N} I(\theta)^{-1} + o(\frac{1}{N^2}), \tag{6.2.3}$$

which is the smallest deviation one can obtain no matter how we estimate the value of θ. The function $I(\theta)$ is known as the Fisher information and is given by the following equation [**65**]

$$I(\theta) = \int \left(\frac{\partial p_\theta(\eta)}{\partial \theta} \right)^2 p_\theta(\eta)\, d\eta \tag{6.2.4}$$

which is completely determined by the quantum state $\hat{\rho}_\theta$ and the measurement scheme. Furthermore, if we are allowed tor choose the optimal measurement scheme, we can construct such a function of θ:

$$\mathcal{I}(\theta) = \sup\{I(\theta) \,|\, \text{all measurement schemes}\}. \tag{6.2.5}$$

This value is known as the quantum Fisher information [**66, 67, 68, 69**].

Therefore, the power of a physical process for sensing a quantity θ is characterized by the Cramér-Rao bound [**65**]

$$\delta\theta_N \geq \frac{1}{\sqrt{N I(\theta)}}, \tag{6.2.6}$$

if the measurement scheme is fixed, and the quantum Cramér-Rao bound

$$\delta\theta_N \geq \frac{1}{\sqrt{N \mathcal{I}(\theta)}} \tag{6.2.7}$$

if we have the freedom to perform the optimal measurement scheme.

Although the calculation of both the Fisher information and the quantum Fisher information can be difficult in general, it turns out that given a Gaussian state $\hat{\rho}_\theta$, there exist explicit formulae that allow us to compute these quantities. First of all, what can be easily proved is that the probability distribution yielded by an arbitrary Guassian state $\hat{\rho}(\bar{x}_\theta, V_\theta)$ and an arbitrary Gaussian measurement scheme is a Gaussian function with first moments μ_θ and covariance matrix Σ_θ. In particular, when the ideal heterodyne measurement is performed to measure each mode in the probes, we have

$$\mu_\theta = \bar{x}_\theta,$$

and

$$\Sigma_\theta = V_\theta + \mathrm{Id}.$$

Then the Fisher information is given by

$$I(\theta) = I_{\Sigma}(\theta) + I_{\mu}(\theta) \tag{6.2.8}$$

with

$$I_{\Sigma}(\theta) = \frac{1}{2}\mathrm{Tr}\left[\Sigma_{\theta}^{-1}\frac{d\Sigma_{\theta}}{d\theta}\Sigma_{\theta}^{-1}\frac{d\Sigma_{\theta}}{d\theta}\right] \tag{6.2.9}$$

and

$$I_{\mu}(\theta) = \left(\frac{d\mu_{\theta}}{d\theta}\right)^{t}\Sigma_{\theta}^{-1}\frac{d\mu_{\theta}}{d\theta}. \tag{6.2.10}$$

In addition, the quantum Fisher information of a Gaussian state is determined by a similar formula [**70, 71**]

$$\mathcal{I}(\theta) = \mathcal{I}_{V}(\theta) + \mathcal{I}_{\bar{x}}(\theta) \tag{6.2.11}$$

where

$$\mathcal{I}_{V}(\theta) = \frac{1}{2}\mathrm{Tr}\left[\Phi_{\theta}\frac{dV_{\theta}}{d\theta}\right] \tag{6.2.12}$$

and

$$\mathcal{I}_{\bar{x}}(\theta) = \left(\frac{d\bar{x}_{\theta}}{d\theta}\right)^{t}V_{\theta}^{-1}\left(\frac{d\bar{x}_{\theta}}{d\theta}\right). \tag{6.2.13}$$

The matrix Φ_{θ} above is implicitly determined by the following equation:

$$\frac{dV_{\theta}}{d\theta} = V_{\theta}\Phi_{\theta}V_{\theta} - \Omega\Phi_{\theta}\Omega^{t}, \tag{6.2.14}$$

with Ω the symplectic form. In general, there is no explicit analytical formula for the matrix Φ_{θ}. However, in situations where $\det(V_{\theta}) \gg 1$, Φ_{θ} can be approximated by [**71**]

$$\Phi_{\theta} \approx V_{\theta}^{-1}\left(\frac{dV_{\theta}}{d\theta}\right)V_{\theta}^{-1}. \tag{6.2.15}$$

This leads to

$$\mathcal{I}_{V}(\theta) \approx \frac{1}{2}\mathrm{Tr}\left[V_{\theta}^{-1}\left(\frac{dV_{\theta}}{d\theta}\right)V_{\theta}^{-1}\frac{dV_{\theta}}{d\theta}\right]. \tag{6.2.16}$$

As a matter of fact, under such assumptions, the value of $\mathcal{I}(\theta)$ can also be directly approximated by the value of $I(\theta)$ with all the probe modes measured by the ideal heterodyne measurement scheme.

6.3. Quantum theory of exceptional point sensing

In this section, we establish the quantum noise model for exceptional sensing and apply the results of the preceding section to bounding its ultimate sensing precision. The existence of exceptional points, as a feature of the non-Hermitian Hamiltonians, requires losses and gains from the surround environment. To establish the suitable quantum model for exceptional point sensing, we must theoretically simulate the losses and gains using quantum noise theory. The idea is that we model each loss or gain channel using a linear interaction between a fictitious probing channel and the corresponding system mode. Then we assume vacuum state inputs for these fictitious probing channel as the simulation of the quantum noise associated to the corresponding losses or gains. Then the whole process can be viewed as a symplectic transformation involving the original system modes and these fictitious modes. By doing so, the output state, which is a Gaussian state given that all the inputs are Gaussian states, can be easily calculated. The output states, which carries the information of the parameter to estimate, will be used to calculate the quantum Fisher information. Let us demonstrate this idea in the following example.

Example 6.1. Consider two bosonic modes $\hat{a}_1$ and $\hat{a}_2$ linear coupled with each other, through the Hamiltonian[3]

$$\hat{H} = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + g \hat{a}_1^\dagger \hat{a}_2 + g \hat{a}_2^\dagger \hat{a}_1.$$

Each system mode will be coupled with a probing channel–a propagating bosonic mode: $\hat{a}_1$ coupled with $\hat{A}_1$, and $\hat{a}_2$ coupled with $\hat{A}_2$. The coupling rates are given by κ_1 and κ_2, respectively. In addition, an additional loss η_1 is introduced to $\hat{a}_1$ while an additional gain η_2 is introduced to $\hat{a}_2$, as shown in Fig. 6.1. The loss is modeled by a linear coupling between the fictitious $\hat{B}_1$ channel with $\hat{a}_1$ through a

[3]The readers may remember our discussion of side bands in the previous chapters. In this example, as we may assume the bosonic modes to be optical (so the rotating wave approximation works well), the side bands can be safely omitted.

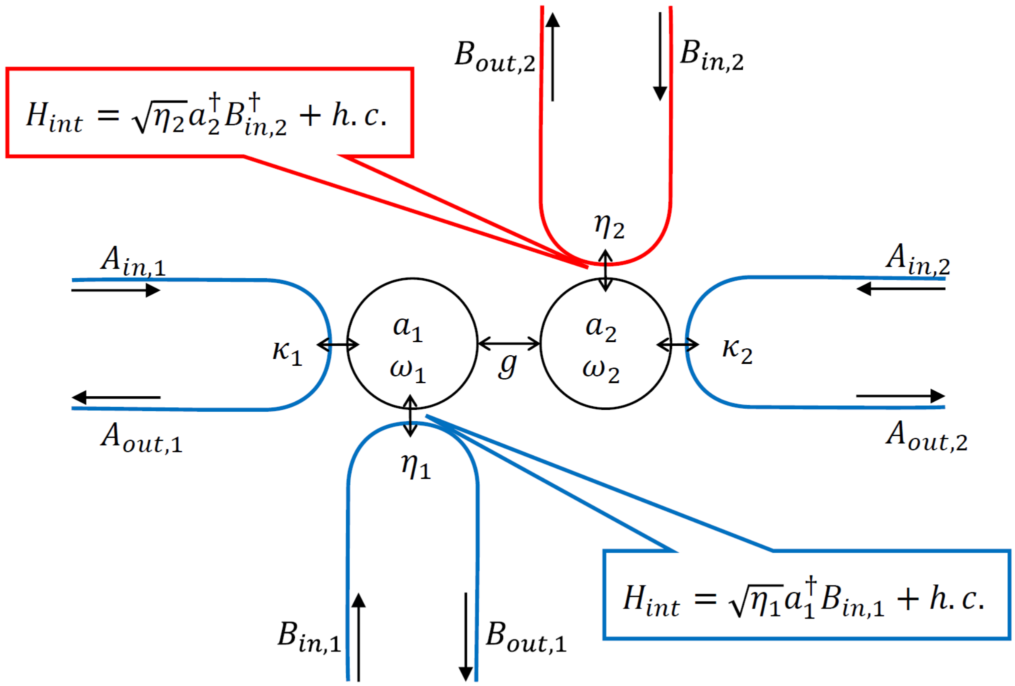

FIGURE 6.1. *Modeling a two-bosonic-mode system.*

This figure shows the schematic of a two-bosonic-mode system with a full quantum description of the losses and gains. The labeled circles represent the two system modes $\hat{a}_1$ and $\hat{a}_2$ with frequency ω_1 and ω_2. They are coupled to the two probing channels, $\hat{A}_1$ and $\hat{A}_2$ with coupling rates κ_1 and κ_2 respectively. In addition to the probing channels, the system mode $\hat{a}_1$ is intrinsically dissipated by the fictitious channel $\hat{B}_1$, and $\hat{a}_2$ intrinsically amplified by the fictitious channel $\hat{B}_2$. Here the dissipation is modeled by the fictitious interaction $\hat{H}_{int} = \sqrt{\eta_1}\hat{a}_1^\dagger \hat{B}_{in,1} + h.c.$ and the amplification is modeled by the fictitious interaction $\hat{H}_{int} = \sqrt{\eta_2}\hat{a}_2^\dagger \hat{B}_{in,2}^\dagger + h.c.$, with η_1, η_2 being the corresponding intrinsic loss and gain. The arrows in the channels which point towards the system modes represent the input modes, while those with opposite direction represent the output modes, as defined in the quantum noise theory [**53**].

passive interaction $\sqrt{\eta_1}\hat{a}_1^\dagger \hat{B}_{in,1} + h.c.$, and the gain is modeled by a linear coupling between the fictitious $\hat{B}_2$ channel through an active interaction $\sqrt{\eta_2}\hat{a}_2^\dagger \hat{B}_{in,2}^\dagger + h.c.$. The non-Hermitian Hamiltonian associated to the system is the matrix

$$\begin{pmatrix} \omega_1 - i\frac{\eta_1+\kappa_1}{2} & g \\ g & \omega_2 + i\frac{\eta_2-\kappa_2}{2} \end{pmatrix},$$

which is at an exceptional point (i.e. with coalesced eigen-vectors) when we let

$$\omega_1 = \omega_2,\ \kappa_1 = \kappa_2 = \kappa \,\text{and}\ g = \frac{\eta_1+\kappa_1}{2} = \frac{\eta_2-\kappa_2}{2}.$$

Therefore, for simplicity, from now on, we let $\omega_1 = \omega_2 = 0$[4] and $\kappa_1 = \kappa_2 = \kappa$. Now assume we want to sense a dimensionless parameter θ, which is added to the system as a perturbation of the resonance frequencies, i.e. resulting in a perturbed Hamiltonian

$$\begin{pmatrix} \theta\kappa - ig & g \\ g & \theta\kappa + ig \end{pmatrix}.$$

Then the dynamics of this system is determined by the following quantum Langevin equation[5]

$$\frac{d}{dt}\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = -i\begin{pmatrix} \theta\kappa - ig & g \\ g & \theta\kappa + ig \end{pmatrix}\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} + \sqrt{\kappa}\begin{pmatrix} \hat{A}_{1,in} \\ \hat{A}_{2,in} \end{pmatrix} + \begin{pmatrix} \sqrt{\eta_1}\hat{B}_{1,in} \\ -\sqrt{\eta_2}\hat{B}_{2,in}^\dagger \end{pmatrix}.$$

Then the symplectic transformation associated with this scattering process can be easily calculated using the formulas introduced in Appx. A. Then we denote the input probe, a Gaussian state on the joined symplectic space of $\hat{A}_1$ and $\hat{A}_2$, by $\hat{\rho}\left(\bar{x}_{in}, V_{in}\right)$, which is independent of the parameter θ; and the input noise, a Gaussian state on the joined symplectic space of $\hat{B}_1$ and $\hat{B}_2$, by $\hat{\rho}\left(0, V_{env}\right)$. Then the output of the probing channels $\hat{A}_1$ and $\hat{A}_2$, after tracing out hte noise channels $\hat{B}_1$ and $\hat{B}_2$, is a Gaussian

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[4]This is valid if we are working in the interaction picture.

[5]Note that the mode $\hat{B}_{2,in}$ appears as $\hat{B}_{2,in}^\dagger$ in the following equation, due to its active interaction with the system mode.

state $\hat{\rho}\left(\bar{x}_{out}\left(\theta\right), V_{out}\left(\theta\right)\right)$ with

$$\bar{x}_{out}\left(\theta\right) = \mathrm{Id} - G_{\theta},$$

and

$$V_{out}\left(\theta\right) = \left(\mathrm{Id} - G_{\theta}\right) V_{in} \left(\mathrm{Id} - G_{\theta}\right)^{t} + G_{\theta} R V_{in}' R^{t} G_{\theta}.$$

Here we have

$$G_{\theta} = \frac{\kappa}{2g} \begin{pmatrix} -1 & 0 & -\theta & 1 \\ 0 & 1 & 1 & -\theta \\ \theta & -1 & -1 & 0 \\ -1 & \theta & 0 & 1 \end{pmatrix}^{-1},$$

and

$$R = \begin{pmatrix} \sqrt{\frac{\eta_1}{\kappa}} & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{\eta_2}{\kappa}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\eta_1}{\kappa}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{\eta_2}{\kappa}} \end{pmatrix},$$

where the matrices are arranged according to the alternative order of the symplectic basis. Applying formulas of the quantum Fisher information to this output Gaussian state, we obtain that

$$\mathcal{I}_{\bar{x}}\left(\theta\right) = \left(\frac{d}{d\theta}\bar{x}_{out}\left(\theta\right)\right)^{t} V_{out}\left(\theta\right)^{-1} \left(\frac{d}{d\theta}\bar{x}_{out}\left(\theta\right)\right) = \mathrm{constant} \cdot \theta^{-4}$$

which implies that the smaller the parameter θ, the larger the quantum Fisher information and hence the better the sensing precision[6]. Moreover, we can check that $\bar{x}_{out}\left(\theta\right) \sim \theta^{-2}$ and $V_{out}\left(\theta\right) \sim \theta^{-4}$, which means the noise, indicated by the covariance matrix $V_{out}\left(\theta\right)$, is actually of the same magnitude as the signal, indicated by the first

[6]This is a big edge over the traditional sensing schemes, as one can check that $\mathcal{I}_{\bar{x}}\left(\theta\right)$ for a traditional sensor does not have such a singular dependence of the parameter θ. Moreover, the θ^{-4} scaling of $\mathcal{I}_{\bar{x}}\left(\theta\right)$ guarantees that the relative error $\delta\theta/\theta$ can be infinitely close to zero as the parameter θ approaching zero, which cannot happen even with the a θ^{-2} scaling.

moments $\bar{x}_{out}(\theta)$. The reason that $\mathcal{I}_{\bar{x}}(\theta)$ has such an unexpectedly singular dependence on θ is the special mathematical structure of the quantity: It has a differential-signal-to-noise ratio structure but in a inner product form. In other words, unlike the usual signal-to-noise ration which only involves scalars, both signal and noise here are represented by non-scalars, and combined by matrix multiplications. Matrix multiplication is the fountain of interference. This leads to the unexpected underlying amplification of signals and cancellation of noise, which together result in the drastic boost of sensing precision.

In the rest of this section, we further extend the protocol of the above example to the general situation, in a more rigorous and formal sense. The ideas are essentially the same. Only that in the general situation where more complicated exceptional points can be constructed in a multi-mode bosonic system, the precision boost can be further increased (i.e. $\mathcal{I}_{\bar{x}}(\theta) \sim \theta^{-2k}$), depending on the number (i.e. k) of the coalesced eigenvectors of a non-Hermitian system at an exceptional point. The readers can skip this part and move on to the next section since the following technical details will not add much to the general picture.

Now we consider non-unitary Gaussian processes associated with exceptional point systems. For any $M \in \mathbb{C}^{m\times m}$ and any integer m, there always exists a m-dimensional invertible matrix P, such that

$$M = P\Lambda P^{-1},$$

where

$$\Lambda = \oplus_{k=1}^{r} J_{n_k}(\lambda_k),$$

with $r \leq m$, $\sum_{k=1}^{r} n_k = m$, $\lambda_k \in \mathbb{C}$, and

$$J_{n_k}(\lambda_k) = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix} \in \mathbb{C}^{n_k\times n_k}.$$

This is well-known as the Jordan decomposition, and Λ is called the Jordan normal form of A and is uniquely defined insofar as any two Jordan normal forms of A have the same set of $J_{n_k}(\lambda_k)$, known as the Jordan blocks. We say a Jordan block $J_n(\lambda)$ is trivial if $n = 1$, and Λ is trivial if all its Jordan blocks are trivial.

Physical properties of the linearly-coupled bosonic oscillators are determined by matrices. The calculation of the evolution of the associated quadrature operators relies on a real dynamical matrix M as in the equation

$$\frac{d}{dt}\begin{pmatrix} \langle \hat{q}_1 \rangle \\ \langle \hat{p}_1 \rangle \\ \vdots \\ \langle \hat{q}_N \rangle \\ \langle \hat{p}_N \rangle \end{pmatrix} = M \begin{pmatrix} \langle \hat{q}_1 \rangle \\ \langle \hat{p}_1 \rangle \\ \vdots \\ \langle \hat{q}_N \rangle \\ \langle \hat{p}_N \rangle \end{pmatrix}.$$

Moreover, when probed by extrinsic bosonic modes, the scattering process is determined by a real scattering matrix $\mathcal{S}$, as in the following relation

$$\begin{pmatrix} \langle \hat{Q}_1^{(out)} \rangle \\ \langle \hat{P}_1^{(out)} \rangle \\ \vdots \\ \langle \hat{Q}_N^{(out)} \rangle \\ \langle \hat{P}_N^{(out)} \rangle \end{pmatrix} = \mathcal{S} \begin{pmatrix} \langle \hat{Q}_1^{(in)} \rangle \\ \langle \hat{P}_1^{(in)} \rangle \\ \vdots \\ \langle \hat{Q}_N^{(in)} \rangle \\ \langle \hat{P}_N^{(in)} \rangle \end{pmatrix}.$$

Therefore, there exist bosonic systems where either M or $\mathcal{S}$ have non-trivial Jordan normal forms and we say such systems are at exceptional points (EPs). Unlike the non-EP systems, where the dynamics can be well-understood by tracking the phase change of the linear combinations of the quadratures since the M or $\mathcal{S}$ matrices are diagonlizable, more complicated dynamical features may emerge in EP systems. One of them is the exotic behavior under perturbations. Let $A(s)$ be a smooth-matrix-valued function dependent on a complex parameter s. Let $A(0)$ be at the exceptional point (or equivalently, let $s = 0$ be the exceptional point). Then the eigenvalues $\epsilon(s)$

of $A(s)$ may scale as $\epsilon(s) \sim s^{\alpha}$ with $0 < \alpha < 1$ when s is near zero, which cannot be observed in any non-EP system.

Bosonic exceptional-point systems are usually constructed by adding extra losses and gains. However, these added losses or gains will inevitably introduce extra quantum noise into the physical process, which is often not taken into consideration in classical EP studies and is crucial to what we will discuss here in this chapter. These losses and gains, in most cases, can be viewed as the result of linear coupling between the system and certain bosonic modes in the environment.

Let $\mathcal{S} : E_{in} \to E_{out}$ be a scattering matrix at some exceptional point, satisfying $\det(\mathrm{Id} - \mathcal{S}) \neq 0$. In Section. 3.7, we showed for any such $\mathcal{S}$, no matter symplectic or not, there always exists a symplectic transformation $S : E_{in} \oplus E_{env} \to E_{out} \oplus E_{env}$, of which $\mathcal{S}$ is a submatrix. Denoting $\mathrm{Id} - \mathcal{S}$ by G, and let $\mathcal{R} : E_{env} \to E_{out}$ be the linear transformation as defined in Theorem. 3.15. Then for a input state Gaussian state $\hat{\rho}\left(\bar{x}_{in}, V_{in}\right) \otimes \hat{\rho}\left(0, V_{env}\right)$, we have the output state being $\hat{\rho}\left(\bar{x}_{out}, V_{out}\right)$ with

$$\bar{x}_{out} = \left(\mathrm{Id} - G\right)\bar{x}_{in}, \tag{6.3.1}$$

and

$$V_{out} = \left(\mathrm{Id} - G\right)V_{in}\left(\mathrm{Id} - G\right)^{t} + G\mathcal{R}V_{env}\mathcal{R}^{t}G^{t}. \tag{6.3.2}$$

In other words, we thus obtain a Gaussian channel $\mathcal{G}_{\mathrm{Id}-G,G\mathcal{R}V_{env}\mathcal{R}^{t}G^{t}}$.

Now we consider a multi-mode bosonic system, each mode of which is coupled with an external detectable channel. Then suppose that we send a Gaussian signal $\hat{\rho}\left(\bar{x}_{in}, V_{in}\right)$ into the system through these channels, and detect the scattered signal using the ideal heterodyne measurement. As we have mentioned, the scattering process can be described by the Gaussian channel $\mathcal{G}_{\mathrm{Id}-G,G\mathcal{R}V_{env}\mathcal{R}^{t}G^{t}}$, if we assume the source of the undetectable losses and gains are intelligible and hence can be simulated by the matrix $\mathcal{R}$ and a Gaussian state $\hat{\rho}\left(0, V_{env}\right)$. Now we further assume the system under investigation depends on a real parameter θ in the form of $G = G_{\theta} = \left(\theta\Pi - M\right)^{-1}$ with Π a full-rank matrix. Physically, this describes a system whose Hamiltonian is perturbed by a small interaction proportional to θ. Furthermore, we assume $\Pi^{-1}M$ has a non-trivial Jordan normal form $\oplus_l J_{m_l}(0)$. Particularly the unperturbed system is at an exceptional point when $\Pi = \mathrm{Id}$. Let $k = \max_l \{m_l\}$, then

$$G_\theta = \theta^{-k} C_0 + o\left(\theta^{-k}\right),$$

with C_0 constant matrix.

With these assumptions, we have the following equations

$$\begin{aligned} G_\theta^{-1} V_{out} \left(G_\theta^{-1}\right)^t =& G_\theta^{-1} \left[(\mathrm{Id} - G_\theta) V_{in} \left(\mathrm{Id} - G_\theta^t\right) + G_\theta \mathcal{R} V_{env} \mathcal{R}^t G_\theta^t \right] \left(G_\theta^{-1}\right)^t \\ =& \left(\theta\Pi - M - \mathrm{Id}\right) V_{in} \left(\theta\Pi - M - \mathrm{Id}\right)^t + \mathcal{R} V_{env} \mathcal{R}^t \\ =& C_1 + o(\theta), \end{aligned}$$

and

$$G_\theta^{-1} \left(V_{out} + \mathrm{Id}\right) \left(G_\theta^{-1}\right)^t = C_1 + M M^t + o(\theta)$$

with the positive-definite matrix

$$C_1 := (\mathrm{Id} + M) V_{in} (\mathrm{Id} + M)^t + \mathcal{R} V_{env} \mathcal{R}^t.$$

In addition, we also have

$$\begin{aligned} \frac{d\bar{x}_{out}}{d\theta} =& \frac{d\left(\mathrm{Id} - G_\theta\right)}{d\theta} \bar{x}_{in} \\ =& G_\theta \Pi G_\theta \bar{x}_{in}. \end{aligned}$$

Then it is easy to calculate

$$\mathcal{I}_{\bar{x}}(\theta) = \theta^{-2k} \bar{x}_{in}^t C_0^t C_1 C_0 \bar{x}_{in} + o\left(\theta^{-2k}\right), \tag{6.3.3}$$

$$I_\mu(\theta) = \theta^{-2k} \bar{x}_{in}^t C_0^t \left(C_1 + M M^t\right) C_0 \bar{x}_{in} + o\left(\theta^{-2k}\right). \tag{6.3.4}$$

Thus, we can already conclude that when the perturbation θ is close to zero,

$$I(\theta), \mathcal{I}(\theta) \gtrsim \theta^{-2k} \cdot \text{constant}.$$

Moreover, we can remove the "$>$" sign above since the similar calculations shows

$$I_{\Sigma}(\theta) = \theta^{-2k} \cdot \text{constant},$$

and, when the output Gaussian state is very noisy(i.e., $\det(V_{out}) \gg 1$),

$$\mathcal{I}_V(\theta) \approx I_{\Sigma}(\theta),$$

as explained in the last section. This means we may be able to infer the value of the parameter θ up to an arbitrary precision, whenever the system is still stable. The enhancement can be further boosted by sending stronger input signal (i.e., with larger $\bar{x}_{in}$), and/or by constructing higher-order exceptional points (i.e., increasing the magnitude of k). However, a strong input signal may saturate the bosonic system, causing the failure of the linearly-coupled-mode theory which serves as the bedrock of our analysis. This problem may be addressed by sending a vacuum signal with $\bar{x}_{in} = 0$ and detecting the spontaneous emission of the bosonic system afterwards, because the quantities $I_0(\theta)$ and $\mathcal{I}_0(\theta)$ does not depend on $\bar{x}_{in}$ at all.

In the scenario above, the (quantum) Fisher information is enhanced when the parameter to be inferred is approaching zero. Often times, we are allowed to tune one or more than one system parameters to improve the sensor performance. Suppose the parameter θ to be inferred has the value of zero. Let α be a controllable system parameter. We assume

$$G_{\theta=0} = (\alpha\Pi + M)^{-1} \sim \alpha^{-k}$$

with Π a full rank matrix. Then we have

$$G_0^{-1} V_{out} \left(G_0^{-1}\right)^t = C_1 + o(\alpha),$$

$$G_0^{-1}\left(V_{out}+\mathrm{Id}\right)\left(G_0^{-1}\right)^t = C_1 + MM^t + o\left(\alpha\right),$$

$$\left(\frac{d\bar{x}_{out}}{d\theta}\right)_{\theta=0} = G_0\left(\frac{dG_\theta^{-1}}{d\theta}\right)_{\theta=0} G_0\bar{x}_{in},$$

and thus

$$I_\mu\left(\theta=0\right), \mathcal{I}_{\bar{x}}\left(\theta=0\right), I_\Sigma\left(\theta=0\right) \sim \alpha^{-2k},$$

as well as,

$$\mathcal{I}_V\left(\theta=0\right) \sim \alpha^{-2k},$$

when the output state is highly thermalized.

Moreover, sometimes it is easier tune the system parameter α to an arbitrarily large magnitude. For this situation, we can design a system with

$$G_{\theta=0} = \left(\alpha M + \Pi\right)^{-1} \sim \alpha^{k-1},$$

such that

$$G_0^{-1}V_{out}\left(G_0^{-1}\right)^t = \alpha^2\cdot\mathrm{constant} + o\left(\alpha^2\right),$$

$$G_0^{-1}\left(V_{out}+\mathrm{Id}\right)\left(G_0^{-1}\right)^t = \alpha^2\cdot\mathrm{constant} + o\left(\alpha^2\right),$$

$$\left(\frac{d\bar{x}_{out}}{d\theta}\right)_{\theta=0} = G_0\left(\frac{dG_\theta^{-1}}{d\theta}\right)_{\theta=0} G_0\bar{x}_{in},$$

Thus

$$I_\mu\left(\theta=0\right), \mathcal{I}_{\bar{x}}\left(\theta=0\right), I_\Sigma\left(\theta=0\right) \sim \alpha^{2k-4}, \tag{6.3.5}$$

as well as, when the output state is very noisy,

$$\mathcal{I}_V\left(\theta=0\right) \sim \alpha^{2k-4}. \tag{6.3.6}$$

We can freely combine the techniques listed above to construct even better exceptional point sensors.

6.4. Unitary bound of non-unitary Gaussian sensing

Motivated by the surprising potential of exceptional point sensing, we are wondering whether there are other bosonic sensing schemes described by carefully engineered Gaussian processes exhibit the same ability to drastically enhance the sensing precision. We believe the answer to this question lies at its fundamental physical origin: What is the dominating physical mechanism behind this phenomenon? In the preceding section, we used the symplectic dilation to embed a specific noisy Gaussian process into a symplectic transformation. Here, we replace the specific process, exceptional point sensing, with an arbitrary Gaussian process corresponding to an arbitrary bosonic sensing scheme, and dilate it symplectically. Then we assume not only the modes involved in the orignial Gaussian process can be accessed, but also the modes added later to dilate this Gaussian process (say, the $\hat{B}$ modes in Example. 6.3). In other words, we are allowed to measure all ports of the dilated symplectic transformation. For simplicity and without loss of generality, let us assume the considered symplectic transformation S satisfies $\det(\mathrm{Id} - S) \neq 0$ and therefore can be generated by a symmetric matrix M via the symplectic Cayley transform:

$$S = \left(\Omega M - \frac{1}{2}\mathrm{Id}\right)^{-1}\left(\Omega M + \frac{1}{2}\mathrm{Id}\right).$$

That is, S is generated from a scattering process governed by a quadratic Hamiltonian described by the symmetric matrix M (see Appx. A).

Let S_θ be the symplectic dilation of a bosonic sensing scheme with θ the parameter to be sensed. Then let the θ-dependent quantum state, i.e. the output state of the process, be a Gaussian state

$$\hat{\rho}(S_\theta \bar{x}, S_\theta V S_\theta^t) = \hat{U}_{S_\theta} \hat{\rho}(\bar{x}, V) \hat{U}_{S_\theta}^\dagger.$$

It follows that

$$\mathcal{I}_{\bar{x}}(\theta) = \bar{x}^t \left[S_\theta^{-1} \left(\frac{dS_\theta}{d\theta} \right) \right]^t V \left[S_\theta^{-1} \left(\frac{dS_\theta}{d\theta} \right) \right] \bar{x}. \tag{6.4.1}$$

Therefore, the scaling of $\mathcal{I}_1(\theta)$ with respect to the parameter θ should be determined by the matrix

$$W_\theta = S_\theta^{-1} \left(\frac{dS_\theta}{d\theta} \right).$$

Supposing $S_\theta = \mathrm{Id} - \left(\Omega M_\theta + \frac{1}{2}\mathrm{Id} \right)^{-1}$ with M as symmetric matrix[7], then we have

$$W_\theta = \left(\Omega M_\theta - \frac{1}{2}\mathrm{Id} \right)^{-1} \Omega \frac{dM_\theta}{d\theta} \left(\Omega M_\theta + \frac{1}{2}\mathrm{Id} \right)^{-1}.$$

We note that for certain choices of M_θ, either of the factors

$$(\Omega M_\theta - \frac{1}{2}\mathrm{Id})\mathrm{or}\ (\Omega M_\theta + \frac{1}{2}\mathrm{Id})$$

can be singular and therefore W_θ can blow up. Specifically, W_θ blows up when

$$M_\theta = S_\theta (\oplus_j M_\theta^{(j)}) S_\theta^t$$

with S_θ being an arbitrary symplectic transformation and at least one of the direct summands, say $M_\theta^{(m)}$, corresponding to the a Hamiltonain of the following squeezing-like form [8]

$$i \sum_{kl} g_{kl}(\theta) \left(\hat{a}_k \hat{a}_l - \hat{a}_k^\dagger \hat{a}_l^\dagger \right).$$

This observation states the following: Let M_θ also depends on a bunch of other real parameters $\{\xi_j\}$with $\xi_0 := \theta$, and the limit

$$\lim_{\xi_k} M_\theta$$

be a matrix satisfying the above conditions so that either of

$$(\Omega \lim_{\xi_k} M_\theta - \frac{1}{2}\mathrm{Id})\ \mathrm{or}\ (\Omega \lim_{\xi_k} M_\theta + \frac{1}{2}\mathrm{Id})$$

[7]As explained in Sec. 3.7, such a dilation S_θ always exists

[8]We can obtain this result by writing down the matrix representation for the quadratic Hamiltonians classified in Appendix 6 of [**2**].

is singular for some parameter ξ_k. Then we have

$$\lim_{\xi_k} \mathcal{I}_{\bar{x}}(\theta) = \infty,$$

which means the estimation of θ can be infinitely precise if measured in a proper measurement scheme (e.g. the heterodyne measurement as explained in the last section). Moreover, a similar conclusion also holds for $\mathcal{I}_V(\theta)$, especially when each mode of the input probes is in a thermal state.

It is not hard to conclude that the Fisher information obtained for the symplectic dilation should be the upper bound of that for the bosonic sensing scheme, since there are less noise involved in the whole process and more modes to measure. Therefore, a bosonic sensing scheme can drastically improve the sensing precision if its symplectic dilation corresponds to a mutli-mode squeezing operation, of which the degree of squeezing can be infinitely increased. From another point of view, to devise an exotic bosonic sensing scheme, we may starting with a suitable squeezing-like symplectric transformation and then tracing out the inaccessible modes, which account for the losses and gains. As squeezing is not just a phenomenon of bosonic system–spin squeezing is also an important effect, this picture may serve as a novel and universal perspective for various exotic physical sensing schemes.

6.5. Conclusion and outlook

In this chapter, we show how a careful investigation of the quantum noise in exceptional system can shed light on developing exotic sensing schemes. The toolbox and statements can be extended to a larger family of Gaussian sensing. We see that the sensing precision enhancement in such schemes can also be studied as a phenomenon of mutli-mode squeezing. This observation may allow us to extend these ideas to other physical platforms beyond the bosonic system.

This chapter also concludes our investigation of applications symplectic geometry in solving practical physical problems in continuous variable system. Amazed by the simplicity and efficacy of symplectic geometry in continuous variable systems, in the next chapter, we try to bring this powerful tool to the seeming irrelevant physical systems–the discrete variable systems. We will see the formulas, previous derived for bosonic systems, can also be applied, with little change, to interpreting important physical phenomenon in the discrete variable systems.

CHAPTER 7

Beyond the continuous variable systems

7.1. Introduction

We have seen the applications of symplectic geometry in continuous variable systems, leading to novel and promising solutions to practical physical problems. In this chapter, we try to extend our consideration to the discrete variable systems. First, we would like to introduce how to suit the essential components of symplectic geometry to the seemingly irrelevant discrete variable systems, based on the interesting correspondence between displacement operations and Pauli operators. Then we will see extension allows effortless transplantation of the results originally derived for continuous variable systems, which will be demonstrated by examples concerning the well-know notion of quantum teleportation. After that we will show our attempts of tackling a theoretical issue about this extension, in order to overcome certain mathematical shortcomings and broaden its generality.

We start with the establishment of symplectic geometry for qubit systems in the succeeding section.

7.2. Symplectic geometry for qubit systems

Concepts in continuous variable systems such as phase space, symplectic trnansformations, and Wigner functions have their analogues in certain discrete variable systems[1], allowing extension of symplectic geometry to discrete variable systems. Here, we demonstrate the general idea for qubit systems. We will see how this extension reveals the connection between Pauli operators and displacement operations, and between symplectic transformations and Clifford operations.

[1]This section is based on the methods introduced in [**55**].

To see this, we start with the qubit systems. Consider an N-qubit system with $\hat{X}_i$ and $\hat{Z}_i$, $i \in \{1, \ldots, N\}$ being the Pauli-X and Pauli-Z operators for the i-th qubit. Now for any $u \in (\mathbb{Z}/2\mathbb{Z})^{2N} =: E$, we can define the generalized Pauli operator as

$$\hat{D}_u = \otimes_{j=1}^{N} \hat{Z}_j^{u_i^{(q)}} \hat{X}_j^{u_i^{(p)}} \tag{7.2.1}$$

where $u^{(q)}, u^{(p)} \in (\mathbb{Z}/2\mathbb{Z})^N$ are defined by

$$u_i^{(q)} := u_{2i-1},\ u_i^{(p)} := u_{2i}. \tag{7.2.2}$$

for any $a, b \in (\mathbb{Z}/2\mathbb{Z})^N$. This operator resembles the continuous variable displacement operator insofar as $\hat{D}_{-u} = \hat{D}_u^{\dagger}$ and

$$\hat{D}_u \hat{D}_v \propto \hat{D}_{u+v}. \tag{7.2.3}$$

Moreover, same as in the continuous variable systems, when an additional phase will appear when exchanging the order of two generalized Pauli operators:

$$\hat{D}_u \hat{D}_v = e^{i\pi\sigma(u,v)} \hat{D}_v \hat{D}_u.$$

Here, given the same E considered as a discrete variable symplectic space and hence a finite phase space, we define the symplectic form $\sigma(\cdot, \cdot) : E \times E \to \mathbb{Z}/2\mathbb{Z}$ by

$$\sigma(u, v) \to \sum_{j=1}^{N} u_{2j-1} v_{2j} - u_{2j} v_{2j-1}. \tag{7.2.4}$$

Therefore, the group of symplectic transformations is similarly defined as the group of linear transformations $S : E \to E$ preserving the symplectic form in the following sense:

$$\sigma(Su, Sv) = \sigma(u, v)$$

for any $u, v \in E$.

In addition, given a N-qubit quantum state $\hat{\rho}$, we can calculate the discrete characteristic function using

$$\chi_{\hat{\rho}}(u) = \frac{1}{2}\mathrm{Tr}\left[\hat{D}(-u)\hat{\rho}\right]. \tag{7.2.5}$$

Then the Wigner function can be calculated by

$$W_{\hat{\rho}}(u) = \frac{1}{2}\sum_{v\in E} e^{i\pi\sigma(u,v)}\chi_{\hat{\rho}}(v). \tag{7.2.6}$$

As expected, for each symplectic transformation $S : E \to E$, there exists a unitary operator $\hat{U}_S$ such that

$$W_{\hat{\rho}}\left(S^{-1}u\right) = W_{\hat{U}_S\hat{\rho}\hat{U}_S^\dagger}(u),$$

and for every discrete displacement operator $\hat{D}(v)$, we have

$$W_{\hat{\rho}}(u+v) = W_{\hat{D}(v)\hat{\rho}\hat{D}(v)^\dagger}(u).$$

Actually, the group generated by $\hat{U}_S$ and $\hat{D}(v)$ with every possible S and v is the group of Clifford operators. Note that $\hat{U}_S\hat{D}(v)\hat{U}_S = \hat{D}(S^t v)$.

One can easily extend these results to qu$d$its systems with d being any prime number, just by letting $E = (\mathbb{Z}/d\mathbb{Z})^N$,

$$\hat{D}(u) = \prod_{i=1}^{N} e^{-i\frac{\pi}{d}u^{(q)t}u^{(p)}}\hat{Z}_i^{u_i^{(q)}}\hat{X}_i^{u_i^{(p)}},$$

$$\chi_{\hat{\rho}}(u) = \frac{1}{d}\mathrm{Tr}\left[\hat{D}(-u)\hat{\rho}\right],$$

and

$$W_{\hat{\rho}}(u) = \frac{1}{d}\sum_{v\in E} e^{i\frac{2\pi}{d}\sigma(u,v)}\chi_{\hat{\rho}}(v).$$

Although it remains unclear how to replicate the concepts of Gaussian states in the discrete variable systems, we can still find the counterparts of infinitely squeezed states quite easily. As we know, in continuous variable systems, infinitely squeezed states are the eigenstates of the quadrature operators and hence the displacement operators. Therefore, the counterparts of the continuous variable infinitely squeezed states in the discrete variable systems are nothing but the stabilizer states, e.g, the eigenstates of the generalized Pauli operators (products of the clock and shift operators) in the discrete variable system. Moreover, the homodyne measurement can similarly be mapped to the syndrome measurement in discrete variable systems.

7.3. Understanding discrete variable quantum teleportation

Now that all the key ingredients can be extended to discrete variable systems, we show in examples how to extend our previous theorems of generalized teleportation to discrete variable systems. Note that the idea of simulating Clifford operations using linear transformations is well known [**72**]. Here we are focusing on the extending our results in the continuous variable systems to the discrete variable systems.

Example 7.1. (Quantum teleportation) We first consider the well-known quantum teleportation circuit. The circuit is composed only of Clifford operations, stabilizer states, classical transmission and syndrome measurement and thus can be understood using our theorem. To see this, we first demonstrate the symplectic transformation representation of each Clifford gate in the circuit. The Hadmard gate

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

by definition, transforms Pauli-X operator to Pauli-Z operators and vice versa. Therefore, it can be correspondingly represented by a symplectic transformation

$$S_H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The CNOT gate

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

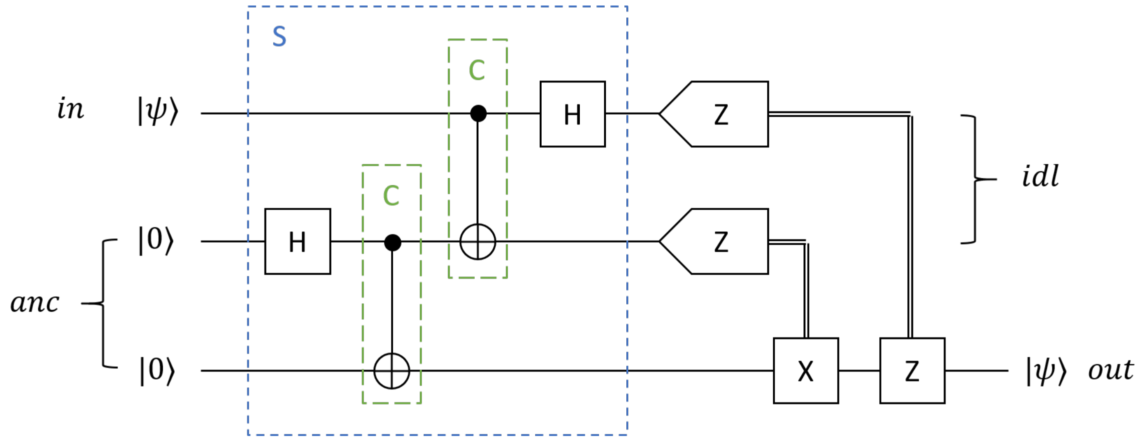

FIGURE 7.1. *Quantum teleportation circuit.*

The boxes labeled by H are Hadamard gates, labled by X and Z are Pauli-X and Pauli-Z operators. The two-qubit gates, encircled by the green dashed boxes and labeled by C, are the CNOT gates. The solid double lines represent classical communication channels. The classical communication channels connects the syndrome measurement, i.e. the Pauli-Z measurements, to the corresponding Pauli operators. As stated in the main text, the Hadmard gates and the CNOT gates, encircled by the blue dashed box, can be treated as a single symplectic transformation S. The Pauli-operators are performed to modified the output mode based on the measurement outcomes of the syndrome measurements.

transforms

$$\hat{D}(\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^t) = \hat{Z} \otimes \mathrm{Id} \text{ to } \hat{D}(\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^t) = \hat{Z} \otimes \hat{Z},$$

$$\hat{D}(\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}^t) = \hat{X} \otimes \mathrm{Id} \text{ to } \hat{D}(\begin{pmatrix}0 & 1 & 0 & 1\end{pmatrix}^t) = \hat{X} \otimes \mathrm{Id},$$

$$\hat{D}(\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}^t) = \mathrm{Id} \otimes \hat{Z} \text{ to } \hat{D}(\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}^t) = \mathrm{Id} \otimes \hat{Z},$$

and

$$\hat{D}(\begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}^t) = \mathrm{Id} \otimes \hat{X} \text{ to } \hat{D}(\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^t) = \hat{X} \otimes \hat{X}.$$

Therefore, the symplectic transformation of the CNOT gate is

$$S_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, the composition of the Clifford gates in the circuit can be represented by the symplectic transformation

$$\begin{aligned} S &= (S_H \oplus \mathrm{Id}_{4\times 4})\,(S_C \oplus \mathrm{Id}_{2\times 2})\,(\mathrm{Id}_{2\times 2} \oplus S_C)\,(\mathrm{Id}_{2\times 2} \oplus S_H \oplus \mathrm{Id}_{2\times 2}) \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Now we have

$$S_{out,in} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ S_{out,z'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ S_{h,z'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ S_{h,in} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(the syndrome measurement is determined by projections $\Pi_0 = \frac{1}{2}\left(\mathrm{Id} - Z\right), \Pi_1 = \frac{1}{2}\left(\mathrm{Id} + Z\right)$.) Then we obtain that

$$\tilde{S} = S_{out,in} - S_{out,z'}\left(S_{h,z'}\right)^{-1} S_{h,in} = \mathrm{Id}, \tag{7.3.1}$$

and

$$F_\star = S_{out,z'}\left(S_{h,z'}\right)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.3.2}$$

It is then easy to check this these results lead to the correct controlled Pauli operations in the circuit. For example, if the syndrome measurement returns 0 for the first qubit (the $|1\rangle$ state) and 1 for the second qubit (the $|0\rangle$ state), then we will apply $\hat{D}\left(F_\star\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \hat{D}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \hat{Z}$.

EXAMPLE 7.2. Then we test our theorem on a slightly more complicated circuit – a gate teleportation circuit [**73**]. The 3-qubit Clifford gate composed of the three CNOTs and the Hadmard gate in this circuit can be represented by a symplectic transformation

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

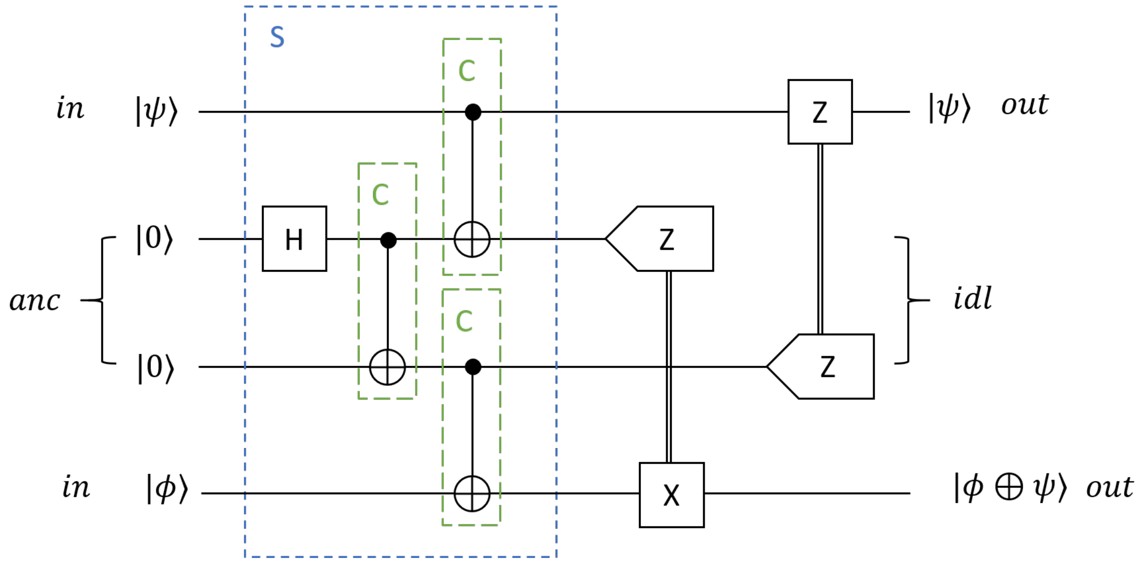

FIGURE 7.2. *Quantum gate teleportation.*

The boxes labeled by H are Hadamard gates, labled by X and Z are Pauli-X and Pauli-Z operators. The two-qubit gates, encircled by the green dashed boxes and labeled by C, are the CNOT gates. The solid double lines represent classical communication channels. The classical communication channels connects the syndrome measurement, i.e. the Pauli-Z measurements, to the corresponding Pauli operators. As stated in the main text, the Hadmard gates and the CNOT gates, encircled by the blue dashed box, can be treated as a single symplectic transformation S. The Pauli-operators are performed to modified the output mode based on the measurement outcomes of the syndrome measurements.

Then we have

$$S_{out,in} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S_{out,z'} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$S_{h,z'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$S_{h,in} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, it is easy to calculate

$$\tilde{S} = S_{out,in} - S_{out,z'} \left(S_{h,z'}\right)^{-1} S_{h,in} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = S_C \tag{7.3.3}$$

and

$$F_\star = S_{out,z'} \left(S_{h,z'}\right)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.3.4}$$

This explains why the corresponding controlled Pauli operations adapted to the syndrome measurement results can generated the CNOT gate between the first and last qubits.

7.4. Symplectic geometry for general discrete variable systems

We have so far restricted ourselves to working on qu$d$it systems with d being a prime number, or, in other words, with $\mathbb{Z}/d\mathbb{Z}$ being a field. The reason of this restriction is that a field has no zero divisors and any element of a field has an inverse, which allows us to effortlessly reproduce the constructions and proofs of symplectic space and symplectic algebra in the continuous variable systems. However, the integer d is not a prime number in general, which implies that the commutative ring $\mathbb{Z}/d\mathbb{Z}$ is not a field[2] and hence contains zero divisors for most discrete variable systems. Therefore, a straightforward extension of the symplectic algebra to general discrete variable systems seems not existing. Motivated by the successful applications of symplectic algebra in continuous variable systems, we attempt to give a solution to this problem here.

First, we briefly review the structure of generalized Pauli-operators in a N-partite d-level system. The generalized Pauli-X and Pauli-Y operators in a single d-level system are defined as the shift and clock operators as follows

$$\hat{X} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \hat{Z} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \ddots & \vdots \\ 0 & 0 & \omega^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \omega^{d-1} \end{pmatrix},$$

with

$$\omega = e^{2\pi i/d}.$$

It follows that

$$\hat{Z}\hat{X} = \omega \hat{X}\hat{Z}.$$

Now we define the generalized Pauli-operators for the N-partite d-level system with d being an arbitrary integer as

$$\hat{D}(q,p) = \otimes_{j=1}^{n} \hat{Z}_j^{q_j} \hat{X}_j^{p_j},$$

[2]Note that $\mathbb{Z}/d\mathbb{Z}$ is not a field when d is a power of a prime number.

where each q_j and p_j are elements of $\mathbb{Z}/d\mathbb{Z}$, and hence the "vectors" q, p are elements of a free of rank N module over $\mathbb{Z}/d\mathbb{Z}$[3] Then we have

$$\hat{D}(q,p)\hat{D}(q',p') = \omega^{-(q^t p' - p^t q')}\hat{D}(q',p')\hat{D}(q,p),$$

for any two generalized Pauli operators $\hat{D}(q,p)$ and $\hat{D}(q',p')$, with $\omega = e^{i2\pi/d}$. The relation amounts to the following: For any two elements

$$\begin{pmatrix} q \\ p \end{pmatrix} \text{ and } \begin{pmatrix} q' \\ p' \end{pmatrix}$$

of a free of rank $2N$ module over the ring $\mathbb{Z}/d\mathbb{Z}$, denoted by E, we have

$$\sigma\left(\begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} q' \\ p' \end{pmatrix}\right) = q^t p' - p^t q',$$

with the bilinear map $\sigma : E \times E \to \mathbb{Z}/d\mathbb{Z}$. Therefore,

$$\left[\hat{D}(q,p), \hat{D}(q',p')\right] = 0$$

if and only if

$$\sigma\left(\begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} q' \\ p' \end{pmatrix}\right) = 0.$$

It is tempting to claim that E is the symplectic space and σ is the symplectic form as we have for the continuous variable systems. But there are important issues which must be identified and addressed.

Let us now focus on p^r-level systems with an integer $r > 1$ and p a prime number. It means that we will work on the free of rank $2N$ module E over the ring $\mathbb{Z}/p^r\mathbb{Z}$. The first issue we have to solve here is how to identify or find a symplectic basis

$$\mathcal{B} = \{e_1, e_2, \ldots, e_N, f_1, f_2, \ldots, f_N\} \subset E$$

[3] A free module is the generalization of vector space when $\mathbb{Z}/d\mathbb{Z}$ is a ring. Both of a free module and a vector space are spanned by subsets of them, known as the bases [**74**].

satisfying

$$\sigma\left(e_j, e_k\right) = \sigma\left(f_j, f_k\right) = 0$$

and

$$\sigma\left(e_j, f_k\right) = -\sigma\left(f_j, e_k\right) = \delta_{j,k}$$

for all $(j,k) \in \{1, 2, \ldots, N\} \times \{1, 2, \ldots, N\}$. Let us consider the following example, and we will see this task is nontrivial:

Example 7.3. Let $N = 1$. Then

$$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

is a symplectic basis while

$$\left\{\begin{pmatrix}p\\0\end{pmatrix}, \begin{pmatrix}0\\p^{r-1}\end{pmatrix}\right\}$$

is not, since

$$\sigma(\begin{pmatrix}p\\0\end{pmatrix}, \begin{pmatrix}0\\p^{r-1}\end{pmatrix}) = \left(p \cdot p^{r-1} - 0 \cdot 0\right) \bmod p^r = 0,$$

even if the latter is obtained by multiplying each element in the former with an element in the ring $\mathbb{Z}p^r\mathbb{Z}$.

To understand this issue, here we treat the problem in an isomorphic ring of the $\mathbb{Z}/p^r\mathbb{Z}$, the quotient polynomial ring (corresponding to the coordinate ring of a singular point[**75**])

$$\mathbb{F}_p\left[x\right] / \left(x^r\right) \cong \mathbb{Z}/p^r\mathbb{Z},$$

where the isomorphism is given by

$$\sum_{j=0}^{r-1} a_j p^j \mapsto \sum_{j=0}^{r-1} a_j x^j,$$

with $a_j \in \{0, 1, \ldots, p-1\}$, for all $j \in \{0, 1, \ldots, r-1\}$.

Since the only prime ideal and hence the maximal ideal of $\mathbb{Z}/p^r\mathbb{Z}$ is $\mathfrak{p} = (p)$, $\mathbb{Z}/p^r\mathbb{Z}$ is a local ring with its localization at $\mathfrak{p}$, the residue field being [**75**]

$$k(\mathfrak{p}) = (\mathbb{F}_p[x]/(x^r))_{\mathfrak{p}} \cong \mathbb{F}_p.$$

Note that the image of any element

$$b = \sum_{j=0}^{r-1} a_j x^j \in \mathbb{F}_p[x]/(x^r)$$

is equal to a_0, denoted by $b_{\mathfrak{p}}$, which corresponds to the "function value" of b. We can also define the formal differential $d : \mathbb{F}_p[x]/(x^r) \to \mathrm{Hom}(\mathbb{F}_p[x]/(x^r), \mathbb{F}_p)$ as [**74**]

$$d\left(\sum_{j=0}^{r-1} a_j x^j\right) = \sum_{j=1}^{r-1} j a_j x^{j-1} dx.$$

It is easy to check that d satisfy the Leibniz rules.

In view of these properties of $\mathbb{F}_p[x]/(x^r)$, we can now represent each of its elements by a r-tuple. Specifically, this means the following map from $\mathbb{F}_p[x]/(x^r)$ to $k(\mathfrak{p})^{\times r} \cong \mathbb{F}_p^{\times r}$:

$$\begin{aligned} b = \sum_{j=0}^{r-1} a_j x^j &\mapsto \left(b_{\mathfrak{p}}, (d(b))_{\mathfrak{p}}, \left(d^2(b)\right)_{\mathfrak{p}}, \ldots, \left(d^{r-1}(b)\right)_{\mathfrak{p}}\right) \\ &= \left(a_0, a_1 dx, a_2 dx^2, \ldots, a_{r-1} dx^{r-1}\right). \end{aligned}$$

Now we can define the symplectic basis on the free module E over $\mathbb{F}_p[x]/(x^r)$ as a subset

$$\mathcal{B} = \{e_1, e_2, \ldots, e_N, f_1, f_2, \ldots, f_N\} \subset E$$

satisfying the following

$$\sigma(e_j, e_k)_{\mathfrak{p}} = \sigma(f_j, f_k)_{\mathfrak{p}} = 0, \tag{7.4.1}$$

$$\sigma(e_j, f_k)_{\mathfrak{p}} = -\sigma(f_j, e_k)_{\mathfrak{p}} = \delta_{j,k}, \tag{7.4.2}$$

and

$$\left(d^l\left(\sigma\left(e_j,e_k\right)\right)\right)_{\mathfrak{p}}=0 \tag{7.4.3}$$

for all $(j,k)\in\{1,2,\ldots,N\}\times\{1,2,\ldots,N\}$ and all $l\in\{1,2,\ldots,r-1\}$.[4]

EXAMPLE 7.4. Let $p=3$, $r=2$ and $N=1$. The two elements

$$u=\begin{pmatrix}x\\0\end{pmatrix},\ v=\begin{pmatrix}0\\x\end{pmatrix}\in E$$

satisfy

$$\sigma\left(u,v\right)=0$$

and therefore

$$\sigma\left(u,v\right)_{\mathfrak{p}}=0.$$

However, we have

$$\begin{aligned}\left(d\left(\sigma\left(e_j,e_k\right)\right)\right)_{\mathfrak{p}}&=\left(dx\right)x+x\left(dx\right)-0\cdot 0\\&=2x\,dx\\&\neq 0.\end{aligned}$$

Therefore, u,v cannot be contained in the same symplectic basis.

Accordingly, the symplectic homomorphism should be defined to preserve all of the following constraints for any two elements $u,v\in E$:

$$\left(d^l\left(\sigma\left(Su,Sv\right)\right)\right)_{\mathfrak{p}}=\left(d^l\left(\sigma\left(u,v\right)\right)\right), \tag{7.4.4}$$

for all $l\in\{0,1,2,\ldots,r-1\}$. An immediate corollary of this definition is the following: Given a symplectic homomorphism S, we have

$$\Omega(S^{-1}dS)^t+\left(S^{-1}dS\right)\Omega=0, \tag{7.4.5}$$

[4]Note that the differential must be computed first in $\mathbb{F}_p\left[x\right]$ before we map the result to the quotient ring $\mathbb{F}_p\left[x\right]/\left(x^r\right)$.

which resembles the definition of symplectic Lie algebra in the continuous variable system.

Now we shift our focus to the case with d being any positive integer. The whole idea is premised on Chinese remainder theorem [**74**]. Therefore, before proceeding, we first state the theorem here:

THEOREM 7.5. *Let $n_1, n_2, \ldots, n_k$ be arbitrary pairwise coprime integers, and $a_1, a_2, \ldots, a_k$ be arbitrary integers. Then there always exists an integer x such that*

$$
\begin{aligned}
x &\equiv a_1 \,(mod\; n_1) \\
x &\equiv a_2 \,(mod\; n_2) \\
&\cdots \\
x &\equiv a_k \,(mod\; n_k)\,.
\end{aligned}
$$

Moreover, the integer x is unique modulo $n_1 n_2 \cdots n_k$.

REMARK 7.6. This theorem states a fact in the scheme theory [**75**]. For example, when $n_1, n_2, \ldots, n_k$ are distinct coprime numbers, the spectrum of the ring $\mathbb{Z}/d\mathbb{Z}$ is given by

$$X := \mathrm{Spec}\mathbb{Z}/d\mathbb{Z} = \{(n_1)\,, (n_2)\,, \ldots, (n_k)\}.$$

Then we have the $\mathcal{O}_X = \mathbb{Z}/d\mathbb{Z}$ consisting of the global sections on X, and the stalks $\mathcal{O}_{X,(n_l)} \cong \mathbb{Z}/n_l\mathbb{Z}$ for all $l \in \{1, 2, \ldots, k\}$. Since all the integers n_l are coprime, the generic points in X will not overlap with each other. Therefore, we have $\mathcal{O}_X \cong \mathcal{O}_{X,(n_1)} \times \mathcal{O}_{X,(n_2)} \times \cdots \times \mathcal{O}_{X,(n_k)}$.

In other words, for any $\mathbb{Z}/d\mathbb{Z}$ with $d = n_1 n_2 \cdots n_k$ and $n_1, n_2, \ldots, n_k$ are pairwise coprime, we have the following isomorphism

$$\mathbb{Z}/d\mathbb{Z} \cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}.$$

This means every element in $\mathbb{Z}/d\mathbb{Z}$ can be represented by a k-tuple. And addition and multiplication in $\mathbb{Z}/d\mathbb{Z}$ can be carried out separately in each $\mathbb{Z}/n_l\mathbb{Z}$, for all $l \in \{1, 2, \ldots, k\}$.

EXAMPLE 7.7. Let $d = 6 = 2 \times 3$. Then each element in $\mathbb{Z}/d\mathbb{Z}$ can be represented b a 2-tuple:

$$0 \mapsto \begin{pmatrix} 0, & 0 \end{pmatrix},$$

$$1 \mapsto \begin{pmatrix} 1, & 1 \end{pmatrix},$$

$$2 \mapsto \begin{pmatrix} 0, & 2 \end{pmatrix},$$

$$3 \mapsto \begin{pmatrix} 1, & 0 \end{pmatrix},$$

$$4 \mapsto \begin{pmatrix} 0, & 1 \end{pmatrix},$$

and

$$5 \mapsto \begin{pmatrix} 1, & 2 \end{pmatrix}.$$

Then the addition of two elements, e.g. 2 and 5 can be calculated by

$$2 + 5 \equiv \begin{pmatrix} 0, & 2 \end{pmatrix} + \begin{pmatrix} 1, & 2 \end{pmatrix} \equiv \begin{pmatrix} 1, & 1 \end{pmatrix} \equiv 1 \pmod 6,$$

as well as the multiplication

$$2 \cdot 5 \equiv \begin{pmatrix} 0, & 2 \end{pmatrix} \cdot \begin{pmatrix} 1, & 2 \end{pmatrix} \equiv \begin{pmatrix} 0, & 1 \end{pmatrix} \equiv 4 \pmod 6.$$

Now it seems that we have all we need to carry out the symplectic algebraic calculation on a module E over $\mathbb{Z}/d\mathbb{Z}$ free of rank $2n$ (a generalization of the concept of vector space when $\mathbb{Z}/d\mathbb{Z}$ is not a field). But just as discussed in the previous scenario, we will encounter unsolvable problems as shown in the following example:

EXAMPLE 7.8. Consider a module E over $\mathbb{Z}/6\mathbb{Z}$ free of rank 2. Let $u, v \in E$ be

$$u = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } v = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then we have

$$\sigma\left(u,v\right)=\left(2\cdot 3 \bmod 6\right)=0,$$

while

$$\sigma\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right)=1\neq 0.$$

This problem can never be observed when E is an actual vector space and has to be fixed for otherwise we will not be able to identify the symplectic bases.

To address this problem, we utilize the Chinese remainder theorem to establish an isomoprhism

$$E\cong E_{n_1}\times E_{n_2}\times\cdots\times E_{n_k}$$

with $d=n_1n_2\cdots n_k$ and $n_1,n_2,\ldots,n_k$ pairwise coprime integers. Moreover, since each integer n_l is either a prime number or a power of a prime number, each E_{n_m} is either a $2n$-dimensional symplectic vector space over the finite field $\mathbb{Z}/n_m\mathbb{Z}$ or a module which has already been studied in the previous scenario. Naturally, any multiplication and addition in E will be conducted separately in each E_{n_m}.

Example 7.9. The same u and v in the lase examples are now represented by 2-tuples:

$$u\mapsto(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix})$$

and

$$v\mapsto(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}).$$

Then we have

$$\begin{aligned}\sigma(u,v)&\mapsto(\sigma_{E_2}(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}),\sigma_{E_3}(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}))\\&=(0,0)\\&\mapsto 0.\end{aligned}$$

At the same time, the original $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ in E are now represented by 2-tuples as the following:

$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto (\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix})$$

and

$$\begin{pmatrix}0\\1\end{pmatrix} \mapsto (\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}),$$

with

$$\begin{aligned}\sigma(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}) &\mapsto (\sigma_{E_2}(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}), \sigma_{E_3}(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}))\\ &= (1,1)\\ &\mapsto 1.\end{aligned}$$

However this time, the difference between the two skew-inner product will not cause a problem, since there is no element $(c_1, c_2) \in \mathbb{Z}/6\mathbb{Z}$ with $c_1 \neq 0$ and $c_2 \neq 0$, such that

$$(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}) = (c_1, c_2) \cdot (\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix})$$

or

$$(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}) = (c_1, c_2) \cdot (\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}).$$

Correspondingly, every homomorphism $S : E \to E$ should be represented as a k-tuple:

$$S \mapsto (S_1, S_2, \ldots, S_k)$$

with each $S_l : E_{n_l} \to E_{n_l}$ being a module homomorphism (or linear transfomration if n_l is a prime number). Now we can define the symplectic homorphism: The homomorphism S is a symplectic if each $S_l : E_{n_l} \to E_{n_l}$ is a symplectic homomorphism (with n_l not a prime number) or a symplectic transformation (with n_l a prime number).

EXAMPLE 7.10. We consider the module E over $\mathbb{Z}/6\mathbb{Z}$ free of rank 2. We see the following homomorphism is symplectic:

$$\begin{pmatrix} 5 & 4 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 2 & 5 \end{pmatrix}$$

with the 2-tuple representation

$$(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}),$$

since each component of this 2-tuple is a symplectic transformation on the corresponding symplectic space. Its symplecity can also be checked directly as follows:

$$\begin{pmatrix} 5 & 4 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 2 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 4 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 2 & 5 \end{pmatrix}^t \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{pmatrix}.$$

Note that $5 \equiv -1 \pmod 6$.

Now, using the tuple representation, the symplectic bases can be easily constructed or identified.

EXAMPLE 7.11. Let

$$\mathcal{B}_1 = \{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$$

be a symplectic basis of E_2, and

$$\mathcal{B}_2 = \{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\}$$

be a symplectic basis of E_3. Then the set

$$\{(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}), (\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix})\} \mapsto \{\begin{pmatrix}5\\3\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix}\}$$

is a symplectic basis of the module E since

$$\sigma(\begin{pmatrix}5\\3\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix}) = 1$$

and this set spanned the whole module E for obvious reason.

In summary, the symplectic basis for the most general situation may be defined as such: The subset

$$\mathcal{B} = \{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$$

of E is a symplectic basis if the set

$$\mathcal{B}_l = \{(e_1)_l, (e_2)_l, \ldots, (e_n)_l, (f_1)_l, (f_2)_l, \ldots, (f_n)_l\}$$

is a symplectic basis of $E_{n_l}$for all pair $(s,t) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ and all $l \in \{1, 2, \ldots, k\}$. Here the notation $(e_s)_k$ denotes the k-th component of e_s's tuple representation. Note that this definition simply amounts to the following familiar relations:

$$\sigma(e_s, e_t) = \sigma(f_s, f_t) = 0,$$

and

$$\sigma(e_s, f_t) = -\sigma(f_s, e_t) = \delta_{st},$$

for all pair $(s,t) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ when all of the n_l are prime numbers. Although the defintion of symplectic basis may remain the same, definitons of some important related concepts require more visible adjustments. For instance, the conventioanl defintions of the Lagrangian planes and the symplectic subspaces fail in the example below:

Example 7.12. Let

$$e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

be a symplectic basis of the free module E over $\mathbb{Z}/6\mathbb{Z}$ of rank 2. Let

$$u = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2e, \; v = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3f$$

be two elements of E. Conventionally, the subset of E spanned by e is regarded as the Lagrangian plane. However, as we know $\sigma(u, v) = 0$, according to its traditional defintion, this Lagrangian plane must also include v. This gives rise to a conflict since $\sigma(e, v) = 3 \neq 0$ which implies that v should not be included in this Lagrangian plane. This contradiction cannot be resolved unless we look into the 2-tuple representation of E. Then e and f yields the following symplectic bases

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for the vector space E_2 and

$$e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; f_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for the vector space E_3. And we know

$$u = (\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}), \; v = (\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}).$$

It now follows that $v \in l_1 \times l_2$ while $u \notin l_1 \times l_2$, where l_1 is the Lagrangian plane spanned by e_1 in E_2 and l_2 is the Lagrangian plane spanned by e_2 in E_3.

Thus, we define the module counterparts of the crucial subspaces of a symplectic space as follows: A subset $\mathcal{E} \in E$ is a Lagrangian plane (symplectic submodule, ...) if the projection of $\mathcal{E}$ to each E_{n_k} is a Lagrangian plane (symplectic submodule, ...). Naturally, the symplectic conjugate $\mathcal{E}'$ of a subset $\mathcal{E}$ in E will be the following set

$$\mathcal{E}' = \mathcal{E}_1' \times \mathcal{E}_2' \times \cdots \times \mathcal{E}_k'$$

with each $\mathcal{E}_l'$ defined as

$$\mathcal{E}_l' \cong E_{n_l}/\mathcal{E}_l$$

where $\mathcal{E}_l$ is the projection of $\mathcal{E}$ to the E_{n_l}. At last, we have all necessary ingredients to prove the symplectic Gram-Schmidt theorem for every d-level systems, which allows us to construct a symplectic basis given a subset of it. The proof of this theorem is trivial since the validity for this theorem is already known when restricted to each E_{n_l}.

Due to the relation of this construction to the scheme theory we briefly mentioned, what we obtained in this section can be further generalized. For example, the ring $\mathbb{Z}/d\mathbb{Z}$ may be generalized to a general scheme, and the module E may be generalized to coherent sheaves [**75**]. Hopefully, this picture can open the way to describing more stabilizer-based quantum processes using the powerful tools provided symplectic algebra.

7.5. Conclusion and outlook

We demonstrated a framework of defining symplectic geometry in discrete variable systems and its immediate application in understanding the quantum teleportation schemes. It is also possible to further generalize this framework to other quantum systems, in particular the systems consisting of stabilizer states. We are looking forward to seeing and understanding more interesting applications of the symplectic geometry and mathematical tools developed for classical mechanics in quantum mechanics.

APPENDIX A

Formulas for bosonic scattering process

A.1. Introduction

Linear bosonic scattering processes are Gaussian processes that are widely in theoretical models in this thesis. In this appendix, we hope to demonstrate formulas suitable for analyzing these processes in general situations. Specifically, a linear bsonic scattering process is completely determined by a quadratic Hamiltonian, describing the interactions between system modes, and the coupling between the system modes and the external propagating modes (i.e. the modes that will be scattered). By properly organizing these data as matrices, we systematically develop formulas that can be applied to general linear scattering processes (Eq. A.5.2), no matter these processes are passive or active, and which are meanwhile compatible with the symplectic geometrical nature of Gaussian process. These formulas are particularly friendly for computer programs, and have greatly simplified our analysis in this thesis. We hope they will also be useful for other studies concerning complex scattering processes.

A.2. Quadratic bosonic Hamiltonian

In Sec. 2.3, we mentioned that the elements of the metaplectic Lie algebra $\mathrm{mp}(2N, \mathbb{R})$, which generates all the metaplectic operators (unitary operators determined by symplectic transformations up to complex constants), are quadratic Hamiltonians. Required by the later derivations in the following sections, we describe the connection between the quadratic Hamiltonians and the metaplectic operators in detail.

We consider an N-mode bosonic system. A quadratic bosonic Hamiltonian for this system is a Hamiltonian of the following form

$$\hat{H} = \sum_{j,k=1}^{N} \left(\mathcal{Y}_{j,k} \hat{a}_j^\dagger \hat{a}_k + \frac{1}{2} \mathcal{W}_{j,k} \hat{a}_j^\dagger \hat{a}_k^\dagger + \frac{1}{2} \mathcal{W}_{j,k}^* \hat{a}_j \hat{a}_k \right) + \text{const.}, \tag{A.2.1}$$

with $\hat{a}_j$, $\hat{a}_j^\dagger$, $j \in \{1, 2, \ldots, N\}$ being the bosonic annihilation and creation operators for each mode, and complex matrices

$$\mathcal{Y} = \mathcal{Y}^\dagger,$$

and

$$\mathcal{W} = \mathcal{W}^t.$$

Then the dynamics of the system is given by the metaplectic unitary operator

$$\hat{U} = e^{-i\hat{H}t}$$

with the real parameter t representing time. Specifically, in the Heisenberg picture, the equation of evolution of the system is given as follows:

$$\begin{aligned}
\frac{d\hat{a}_j}{dt} &= i[\hat{H}, \hat{a}_j] \\
&= -i \sum_{k=1}^{n} \mathcal{Y}_{j,k} \hat{a}_k - \frac{i}{2} \sum_{k=1}^{n} \mathcal{W}_{j,k} \hat{a}_k^\dagger - \frac{i}{2} \sum_{k=1}^{n} \mathcal{W}_{k,j} \hat{a}_k^\dagger,
\end{aligned}$$

for all $j \in \{1, 2, \ldots, N\}$, and equivalently

$$\begin{aligned}
\frac{d\hat{a}_j^\dagger}{dt} &= i[\hat{H}, \hat{a}_j^\dagger] \\
&= i \sum_{k=1}^{n} \mathcal{Y}_{j,k}^* \hat{a}_k^\dagger + \frac{i}{2} \sum_{k=1}^{n} \mathcal{W}_{j,k}^* \hat{a}_k^\dagger + \frac{i}{2} \sum_{k=1}^{n} \mathcal{W}_{k,j}^* \hat{a}_k.
\end{aligned}$$

With the usual definition of the quadrature operators, it turns out that the

$$\begin{aligned}\hat{U} =& e^{-i\hat{H}t} \\ =& \hat{U}_S\end{aligned}$$

with the symplectic transformation

$$S = \exp\left(\begin{pmatrix} \Im[\mathcal{Y}+\mathcal{W}] & \Re[\mathcal{Y}-\mathcal{W}] \\ -\Re[\mathcal{Y}+\mathcal{W}] & \Im[\mathcal{Y}-\mathcal{W}] \end{pmatrix} t\right) \tag{A.2.2}$$

if we arrange the canonical symplectic basis as $\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_N, \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_N$. In fact, the matrix

$$\begin{pmatrix} \Im[\mathcal{Y}+\mathcal{W}] & \Re[\mathcal{Y}-\mathcal{W}] \\ -\Re[\mathcal{Y}+\mathcal{W}] & \Im[\mathcal{Y}-\mathcal{W}] \end{pmatrix}$$

is a member of the symplectic Lie algebra $\mathrm{sp}(2N, \mathbb{R})$ as it should be.

For simplicity, we denote

$$\hat{a} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_N \end{pmatrix}, \ \hat{a}^\dagger = \begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \\ \vdots \\ \hat{a}_N^\dagger \end{pmatrix}.$$

as well as

$$\hat{q} = \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \vdots \\ \hat{q}_N \end{pmatrix}, \ \hat{p} = \begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \vdots \\ \hat{p}_N \end{pmatrix}.$$

Therefore, the above can also be expressed as follows:

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}(t) = \exp\left(i\begin{pmatrix} -\mathcal{Y} & -\mathcal{W} \\ \mathcal{W}^* & \mathcal{Y}^* \end{pmatrix} t\right)\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

and

$$\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}(t) = \exp\left(\begin{pmatrix} \Im[\mathcal{Y}+\mathcal{W}] & \Re[\mathcal{Y}-\mathcal{W}] \\ -\Re[\mathcal{Y}+\mathcal{W}] & \Im[\mathcal{Y}-\mathcal{W}] \end{pmatrix} t\right) \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}.$$

A.3. Passive scattering process

Apart from the correspondence between the quadratic Hamiltonians and the symplectic transformations shown in the last section, we can also obtain symplectic transformations using scattering processes. Let $\hat{A}_j^{(in/out)}, \hat{A}_j^{\dagger(in/out)}$ be the annihilation and creation operators of each input/output propagating bosonic mode. Then the dynamics of such a scattering process is determined by the following quantum Langevin equation [**53**]:

$$\frac{d}{dt}\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}(t) = i[\hat{H}, \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}] - \frac{\mathcal{B}}{2}\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} + \mathcal{C}\begin{pmatrix} \hat{A}^{(in)} \\ \hat{A}^{(in)\dagger} \end{pmatrix},$$

and the input-output relation

$$\begin{pmatrix} \hat{A}^{(out)} \\ \hat{A}^{(out)\dagger} \end{pmatrix} = \mathcal{D}\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} + \begin{pmatrix} \hat{A}^{(in)} \\ \hat{A}^{(in)\dagger} \end{pmatrix}.$$

In the above,

$$\hat{H} = \sum_{j,k=1}^{N}\left(\mathcal{Y}_{j,k}\hat{a}_j^\dagger\hat{a}_k + \frac{1}{2}\mathcal{W}_{j,k}\hat{a}_j^\dagger\hat{a}_k^\dagger + \frac{1}{2}\mathcal{W}_{j,k}^*\hat{a}_j\hat{a}_k\right)$$

is a quadratic Hamiltonian, the complex matrices $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are introduced to describe the effects of the coupling between the system and the propagating modes.

With a fixed frequency ω, the Fourier transform of these equations leads to

$$\left(i\begin{pmatrix} \mathcal{Y}-\omega\mathrm{Id} & \mathcal{W} \\ -\mathcal{W}^* & -\mathcal{Y}^*-\omega\mathrm{Id} \end{pmatrix} + \frac{\mathcal{B}}{2}\right)\begin{pmatrix} \hat{a}[\omega] \\ \hat{a}[-\omega]^\dagger \end{pmatrix} = \mathcal{C}\begin{pmatrix} \hat{A}^{(in)}[\omega] \\ \hat{A}^{(in)}[-\omega]^\dagger \end{pmatrix}, \tag{A.3.1}$$

and

$$\begin{pmatrix} \hat{A}^{(out)}[\omega] \\ \hat{A}^{(out)}[-\omega]^\dagger \end{pmatrix} = \mathcal{D}\begin{pmatrix} \hat{a}[\omega] \\ \hat{a}[-\omega]^\dagger \end{pmatrix} + \begin{pmatrix} \hat{A}^{(in)}[\omega] \\ \hat{A}^{(in)}[-\omega]^\dagger \end{pmatrix}, \tag{A.3.2}$$

with

$$\hat{O}[\omega] := \frac{1}{\sqrt{2\pi}} \int \hat{O}(t) e^{-i\omega t} dt$$

for an operator $\hat{O}$. Then we have the following equation:

$$\begin{pmatrix} \hat{A}^{(out)}[\omega] \\ \hat{A}^{(out)}[-\omega]^\dagger \end{pmatrix} = \left[\mathrm{Id} + \mathcal{D} \left(i \begin{pmatrix} \mathcal{Y} - \omega \mathrm{Id} & \mathcal{W} \\ -\mathcal{W}^* & -\mathcal{Y}^* - \omega \mathrm{Id} \end{pmatrix} + \frac{\mathcal{B}}{2} \right)^{-1} \mathcal{C} \right] \begin{pmatrix} \hat{A}^{(in)}[\omega] \\ \hat{A}^{(in)}[-\omega]^\dagger \end{pmatrix}. \tag{A.3.3}$$

Supposing there is no intrinsic losses or gains in the system and letting $\omega = 0$

$$O = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{Id} & i\mathrm{Id} \\ \mathrm{Id} & -i\mathrm{Id} \end{pmatrix},$$

$$B = O^{-1} \mathcal{B} O, \quad C = O^{-1} \mathcal{B} O, \quad D = O^{-1} \mathcal{B} O,$$

the matrix

$$S = \mathrm{Id} + D \left(\begin{pmatrix} \Im[\mathcal{Y} + \mathcal{W}] & \Re[\mathcal{Y} - \mathcal{W}] \\ -\Re[\mathcal{Y} + \mathcal{W}] & \Im[\mathcal{Y} - \mathcal{W}] \end{pmatrix} + \frac{B}{2} \right)^{-1} C \tag{A.3.4}$$

is symplectic, since the transformation from the input propagating modes to the output porpagating modes is purely unitary and O is the matrix transforming the quadrature operators to the annihilation and creation operators.

In the above, the matrices B, C, and D must be real. Let $C \in \mathbb{R}^{2N \times 2M}$, $D \in \mathbb{R}^{2M \times 2N}$, with $N \leq M$, are full-ranked, where N is the number of the system modes, and N is the number of the propagating modes. Therefore, according to the symplectic singular-value decomposition which will be introduced later, we can conclude that there exists a symplectic transformation $\mathcal{S}$, such that

$$C = -\Omega C' \Omega \begin{pmatrix} \mathrm{Id} & 0 \end{pmatrix} \mathcal{S}, \quad D = \mathcal{S}^{-1} \begin{pmatrix} \mathrm{Id} \\ 0 \end{pmatrix} C'^t$$

with $C', D' \in \mathbb{R}^{2N \times 2M}$. and if there are no intrinsic losses or gains,

$$-\Omega \left(C' \right)^{-1} \Omega B C'^t + C' B^t \Omega \left(C'^t \right)^{-1} \Omega = 2\Omega.$$

For the situation where intrinsic losses or gains are significant, the right-hand-side of the last equation should be modified to some arbitrary skew-symmetric matrix. Because most of times the symplectic spaces where the input and output modes are defined artificially, we will assume $M = N$ from now on.

EXAMPLE A.1. Consider the quadratic Hamiltonian

$$\hat{H} = g(\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_1 \hat{a}_2),$$

and the system-environment interaction determined by

$$\mathcal{B} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},$$

and

$$\mathcal{C} = \mathcal{D} = \begin{pmatrix} \sqrt{\kappa_1} & 0 \\ 0 & \sqrt{\kappa_2} \end{pmatrix}.$$

We thus have

$$\mathcal{Y} = 0,$$

and

$$\mathcal{W} = \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix}.$$

It follows that

$$S = \begin{pmatrix} 0 & t' & r' & 0 \\ t' & 0 & 0 & r' \\ r' & 0 & 0 & t' \\ 0 & r' & t' & 0 \end{pmatrix}$$

with

$$r' = \frac{\chi + 1}{\chi - 1}, \quad t' = \frac{2\sqrt{\chi}}{1 - \chi},$$

satisfying $r'^2 - t'^2 = 1$, and

$$\chi = \frac{g^2}{\kappa_1 \kappa_2}.$$

We call a scattering process passive if there is no coupling between any two annihilation operators of the system. Then the symplectic transformations will only transform $\hat{Q}[\omega]^{(in)}$ and $\hat{P}[\omega]^{(in)}$ to $\hat{Q}[\omega]^{(out)}$ and $\hat{P}[\omega]^{(out)}$, and will be in the form of

$$S = \mathrm{Id} + D\left(\begin{pmatrix} \Im[\mathcal{Y}] & \Re[\mathcal{Y}] \\ -\Re[\mathcal{Y}] & \Im[\mathcal{Y}] \end{pmatrix} + \omega\Omega + \frac{B}{2}\right)^{-1} C.$$

EXAMPLE A.2. Consider the quadratic Hamiltonian

$$\hat{H} = g\left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1\right),$$

and the system-environment interaction given by

$$\mathcal{B} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},$$

and

$$\mathcal{C} = \mathcal{D} = \begin{pmatrix} \sqrt{\kappa_1} & 0 \\ 0 & \sqrt{\kappa_2} \end{pmatrix}.$$

Then the above equation leads to

$$S = \begin{pmatrix} 0 & -t & r & 0 \\ t & 0 & 0 & r \\ r & 0 & 0 & -t \\ 0 & r & t & 0 \end{pmatrix}$$

with

$$r = \frac{\chi - 1}{\chi + 1}, \quad t = \frac{2\sqrt{\chi}}{\chi + 1}$$

satisfying the condition that $r^2 + t^2 = 1$, and

$$\chi = \frac{g^2}{\kappa_1 \kappa_2}.$$

A.4. A matrix representation of Clifford algebra $Cl_{3,0}$

Let V be a 3-dimensional real vector space. Let $Q : V \to \mathbb{R}$ be a quadratic form (e.g. the norm $\|\cdot\|$ induced by the standard Euclidean inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$). Let A be an unital assocaitve algebra over $\mathbb{R}$. Then there exists a unital algebra over $\mathbb{R}$, denoted by B, with a linear map $i : V \to Cl_{3,0}$ satisfying $i(v)^2 = Q(v)\,\mathrm{Id}$ for all $v \in V$, such that for any linear map $j : V \to A$ satisfying $j(v)^2 = Q(v)\,\mathrm{Id}$ for all $v \in V$, there exists an unique algebra homomorphism satisfying $f \circ i = j$. Thus, the pair (B, i) is a universal object, meaning it is uniquely determined up to an algebra isomorphism, which is called a Clifford algebra [**76**].

For the case of $Q = \|\|$, we denote B as $Cl_{3,0}$. Let v_1, v_2, v_3 be the orthonormal basis of V. The $Cl_{3,0}$ is generated by $i(v_1), i(v_2), i(v_3)$. These elements satisfy

$$i(v_k)\,i(v_l) + i(v_l)\,i(v_k) = 2\langle v_k, v_l \rangle \mathrm{Id}, \text{ for all } v_k, v_l \in V.$$

As an real vector space, $Cl_{3,0}$ has the following set of elements as a basis

$$\{\mathrm{Id}, i(v_1), i(v_2), i(v_3), i(v_1)\,i(v_2), i(v_2)\,i(v_3), i(v_1)\,i(v_3), i(v_1)\,i(v_2)\,i(v_3)\}.$$

Now we introduce two matrix representations of $Cl_{3,0}$. First, we can represent any element of $Cl_{3,0}$ as a 4×4 complex matrix by letting

$$\epsilon_1 = i \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \epsilon_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.4.1}$$

be the representations of $i(v_1), i(v_2)$, and $i(v_3)$ correspondingly. We can also let the representaions of $i(v_1), i(v_2)$, and $i(v_3)$ be the 4×4 real matrices

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \text{(A.4.2)}$$

so that each element of $Cl_{3,0}$ can be represented by a 4×4 real matrix.

Let

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{Id} & i\mathrm{Id} & 0 & 0 \\ 0 & 0 & \mathrm{Id} & -i\mathrm{Id} \\ 0 & 0 & \mathrm{Id} & i\mathrm{Id} \\ \mathrm{Id} & -i\mathrm{Id} & 0 & 0 \end{pmatrix}, \quad \text{(A.4.3)}$$

it is easy to check that

$$e_k = U^{-1} \epsilon_k U, \quad \text{for } k = 1, 2, 3.$$

A.5. Active scattering process

We now resume our discussion about the scattering processes to include the situations where the couplings between creation operators are present. Such scattering processes are regarded as active scattering processes. Throughout this section, we assume $\omega > 0$ and the complex matrices $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are diagonal and full-ranked. We define

$$\Theta = \omega \mathcal{B}^{-1},$$

$$Y = \mathcal{C}^{-1} \mathcal{Y} \mathcal{D}^{-1},$$

$$W = \mathcal{C}^{-1} \mathcal{W} \mathcal{D}^{-1},$$

and

$$\Gamma = \frac{\mathcal{C}^{-1} \mathcal{B} \mathcal{D}^{-1}}{2}.$$

Then we have

$$\begin{pmatrix} \hat{A}^{(out)}[\omega] \\ \hat{A}^{(out)}[-\omega]^\dagger \\ \hat{A}^{(out)}[-\omega] \\ \hat{A}^{(out)}[\omega]^\dagger \end{pmatrix} = \mathcal{S} \begin{pmatrix} \hat{A}^{(in)}[\omega] \\ \hat{A}^{(in)}[-\omega]^\dagger \\ \hat{A}^{(in)}[-\omega] \\ \hat{A}^{(in)}[\omega]^\dagger \end{pmatrix},$$

where

$$\mathcal{S} = \mathrm{Id} - \left(\mathrm{Id} \otimes \left(\Im Y + \frac{1}{2}\mathrm{Id}\right) - \epsilon_1 \otimes \Re W + \epsilon_2 \otimes \Im W - \epsilon_1\epsilon_2 \otimes \Re Y + \epsilon_1\epsilon_2\epsilon_3 \otimes \Theta\right)^{-1}. \tag{A.5.1}$$

Since

$$\begin{pmatrix} \hat{a}[\omega] \\ \hat{a}[-\omega]^\dagger \\ \hat{a}[-\omega] \\ \hat{a}[\omega]^\dagger \end{pmatrix} = U \begin{pmatrix} \hat{q}[\omega] \\ \hat{p}[\omega] \\ \hat{q}[-\omega] \\ \hat{p}[-\omega] \end{pmatrix}$$

with

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{Id} & i\mathrm{Id} & 0 & 0 \\ 0 & 0 & \mathrm{Id} & -i\mathrm{Id} \\ 0 & 0 & \mathrm{Id} & i\mathrm{Id} \\ \mathrm{Id} & -i\mathrm{Id} & 0 & 0 \end{pmatrix},$$

we also have

$$\begin{pmatrix} \hat{Q}^{(out)}[\omega] \\ \hat{P}^{(out)}[\omega] \\ \hat{Q}^{(out)}[-\omega] \\ \hat{P}^{(out)}[-\omega] \end{pmatrix} = S \begin{pmatrix} \hat{Q}^{(in)}[\omega] \\ \hat{P}^{(in)}[\omega] \\ \hat{Q}^{(in)}[-\omega] \\ \hat{P}^{(in)}[-\omega] \end{pmatrix}$$

with

$$\begin{aligned} S =& U^{-1}\mathcal{S}U \\ =& \mathrm{Id} - (\mathrm{Id} \otimes (\Im Y + \Gamma) - e_1 \otimes \Re W + e_2 \otimes \Im W - e_1e_2 \otimes \Re Y + e_1e_2e_3 \otimes \Theta)^{-1} \end{aligned} \tag{A.5.2}$$

a symplectic transformation when there are no intrinsic losses or gains, i.e., $\Gamma = \mathrm{Id}/2$. Note that he symplectic form can be represented by

$$\Omega = e_1 e_2 \otimes \mathrm{Id}.$$

Furthermore, we also note that the symplectic transformation $\mathcal{S}$ of a scattering process can be written as

$$\begin{aligned} \mathcal{S} =& \mathrm{Id} - (\Omega\mathcal{M} + \frac{1}{2}\mathrm{Id})^{-1} \\ =& (\Omega\mathcal{M} + \frac{1}{2}\mathrm{Id})^{-1}(\Omega\mathcal{M} - \frac{1}{2}\mathrm{Id}) \end{aligned} \tag{A.5.3}$$

with $\mathcal{M}$ being a symmetric matrix, which is called the symplectic Cayley transform [**1**]. The matrix $\mathcal{M}$ can be understood as the dimensionless quadratic Hamiltonian, since it corresponds to the actual quadratic Hamiltonian normalized by the system-environment coupling rates.

A.6. Symplectic singular-value decomposition

Let E be a symplectic vector space. Let Ω be the symplectic form. Then for any H being a real symmetric positive semi-definite linear transformation on E, there exists a symplectic transformation S, such that $H = S\Lambda S^t$, where Λ is real, positive-semi-definite, and diagonal[1]. Now consider an arbitray $M \in \mathbb{R}^{2m\times n}$, with $\mathrm{rank}\,(M) = n$. Since MM^t is positve-semi-definite, there exists a symplectic matrix S, such that

$$SMM^tS^t = \begin{pmatrix} \Lambda & \\ & 0 \end{pmatrix},$$

where Λ is positive-definite and diagonal. We thus have

$$\begin{pmatrix} \Lambda^{-1/2} & \\ & \mathrm{Id} \end{pmatrix} SMM^tS^t \begin{pmatrix} \Lambda^{-1/2} & \\ & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & \\ & 0 \end{pmatrix}.$$

[1]This is a straightforward result of Appendix 6 of [**2**].

We now let

$$\begin{pmatrix} \Lambda^{-1/2} & \\ & \mathrm{Id} \end{pmatrix} SM = \begin{pmatrix} Q \\ T \end{pmatrix},$$

with $Q \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{(2m-n)\times n}$. Then we have

$$QQ^t = \mathrm{Id},$$

and

$$TT^t = 0,$$

which imply that Q is orthogonal and $T = 0$. Therefore,

PROPOSITION A.3. *For any $M \in \mathbb{R}^{2m\times n}$ with rank$(M) = n$, there exist a symplectic matrix S on $\mathbb{R}^{2m}$ and an orthognal matrix Q on $\mathbb{R}^n$, such that*

$$M = S \begin{pmatrix} \Lambda & \\ & 0 \end{pmatrix} Q, \tag{A.6.1}$$

where Λ is positive-definite and diagonal.

This result has also been proved in [**77**] using a different approach.

Index

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