Some of the Big Ideas in Quantum Mechanics

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2.1 Quantum Zeno effect

Proposition II.1. *Consider a path of states $|\psi(s)\rangle$ where $s \in [0,1]$. Assume that, for fixed d and all δ,*

$$|\langle\psi(s)|\psi(s+\delta)\rangle|^2 \geq 1 - d^2\delta^2.$$

Then

2.2 Adiabatic quantum computation

2.3 Evolution through measurement

2.3.1 Phase randomisation

$|\psi_0\rangle = |E_0(0)\rangle = \sum_i \alpha_i \, |E_i(s_1)\rangle$

$$\begin{aligned}
\rho(s_1) &= \frac{1}{T}\int_0^T e^{-iH(s_1)t}|\psi_0\rangle\langle\psi_0|e^{iH(s_1)t}\,\mathrm{d}t \\
&= \sum_{i,j} \alpha_i\alpha_j^* \frac{1}{T}\left(\int_0^T e^{-i(E_i-E_j)t}\,\mathrm{d}t\right)|E_i\rangle\langle E_j| \\
&= \sum_{i,j} \alpha_i\alpha_j^* \left(\frac{i(e^{-i(E_i-E_j)T}-1)}{T(E_i-E_j)}\right)|E_i\rangle\langle E_j|
\end{aligned}$$

Theorem II.2 (Randomised dephasing)**.** *Let $|\psi(s)\rangle$ be a nondegenerate eigenstate of $H(s)$ and $\{\omega_j\}$ the energy differences to the other eigenstates $|\psi_j(s)\rangle$. Let T be a random variable. Then, for all states ρ, we have*

$$\left\|(M_l - e^{-iH(s)T}\rho e^{-iH(s)T}\right\|_{tr} \leq \epsilon = \sup_{\omega_j}|\Phi(\omega_j)|$$

Chapter 3

Variational quantum algorithms

Chapter 4

Algorithms

4.1 Grover's search algorithm

Suppose we have a set $\mathcal{N}$ of N items, some of which are marked. WLOG we can take $\mathcal{N}$ to be the set of bit strings of length ν. The marked items form a subset $\mathcal{M}$ of size M. Suppose we have an oracle that tells us whether an object is marked or not:

$$f : \mathcal{N} \to \{0, 1\} : x \mapsto \begin{cases} 0 & (x \in \mathcal{M}) \\ 1 & (x \notin \mathcal{M}) \end{cases}$$

Now we want to find a marked item. Classically we need to check $\Theta(N/M)$ items on average.

4.1.1 Circuit model

Let $\{\, |n\rangle \mid n \in \mathcal{N} \}$ be a basis of a Hilbert space $\mathcal{H}_N$ and let $\mathcal{H}_2$ be a qubit space.

4.1.1.1 The oracle

We would like to use the operator

$$U_f : \mathcal{H}_N \to \mathcal{H}_N : |n\rangle \mapsto (-1)^{f(n)} |n\rangle \, .$$

This can be constructed from the oracle

$$O_f : \mathcal{H}_N \otimes \mathcal{H}_2 \to \mathcal{H}_N \otimes \mathcal{H}_2 : |n\rangle \otimes |q\rangle \mapsto |n\rangle \otimes |f(n) \oplus q\rangle \, ,$$

where $\oplus$ is addition modulo 2, by preparing the qubit in the superposition $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Then

$$\begin{aligned} O_f \left(|n\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right) &= |n\rangle \otimes \frac{1}{\sqrt{2}}(|f(n)\rangle - |f(n) \oplus 1\rangle) \\ &= (-1)^{f(n)} |n\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \end{aligned}$$

The operator U_f can also be written

$$U_f = \mathrm{id}_{\mathcal{H}_N} - 2 \sum_{m \in \mathcal{M}} |m\rangle\langle m|.$$

4.1.1.2 The algorithm

We construct the an initial state which is a superposition of all possible solutions

$$|\mathcal{N}\rangle = \frac{1}{\sqrt{N}} \sum_{n\in\mathcal{N}} |n\rangle\,.$$

This can be prepared by applying a Hadamard gate W to each qubit in the register.
We also define the state

$$|\mathcal{M}\rangle = \frac{1}{\sqrt{M}} \sum_{m\in\mathcal{M}} |m\rangle\,.$$

We define the operation

$$U_0 = W^{\otimes\nu}(\mathrm{id} - 2|0\rangle\langle 0|)W^{\otimes\nu} = \mathrm{id} - 2|\mathcal{N}\rangle\langle\mathcal{N}|.$$

Now both U_0 and U_f leave span$\{\}$

4.1.2 Analogue Grover

Why does analogue Grover work?

4.1.3 Adiabatic Grover

$$H_0 = \mathbf{1} - |\mathcal{N}\rangle\langle\mathcal{N}| = \mathbb{1} - \mathbb{J}/N$$
$$H_f = \mathbf{1} - \sum_{m\in\mathcal{M}} |m\rangle\langle m| = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{N-M} \end{pmatrix}.$$

Then

$$H(s) = (1-s)H_0 + sH_f = \begin{pmatrix} (1-s)\mathbb{1} + \frac{s-1}{N}\mathbb{J} & \frac{s-1}{N}\mathbb{J} \\ \frac{s-1}{N}\mathbb{J} & \mathbb{1} + \frac{s-1}{N}\mathbb{J} \end{pmatrix} = \begin{pmatrix} (1-s)\mathbb{1}_M & 0 \\ 0 & \mathbb{1}_{N-M} \end{pmatrix} + \frac{s-1}{N}\mathbb{J}.$$

Then, setting $A = \begin{pmatrix} (1-s-\lambda)\mathbb{1}_M & 0 \\ 0 & (1-\lambda)\mathbb{1}_{N-M} \end{pmatrix}$

$$\begin{aligned}
\det(H(s) - \lambda) &= \det\left(A + \frac{s-1}{N}\mathbb{J}^{n\times 1}\mathbb{J}^{1\times n}\right) \\
&= \det(A)\det\left(\mathbb{1}_N + \frac{s-1}{N}A^{-1}\mathbb{J}^{n\times 1}\mathbb{J}^{1\times n}\right) \\
&= \det(A)\det\left(1 + \frac{s-1}{N}\mathbb{J}^{1\times n}A^{-1}\mathbb{J}^{n\times 1}\right) \\
&= (1-s-\lambda)^M(1-\lambda)^{N-M}\left(1 + \frac{M(s-1)}{(1-s-\lambda)N} + \frac{(N-M)(s-1)}{(1-\lambda)N}\right) \\
&= (1-s-\lambda)^{M-1}(1-\lambda)^{N-M-1}\left(\lambda^2 - \lambda + s(1-s)\frac{N-M}{N}\right),
\end{aligned}$$

where we have used the matrix determinant lemma to simplify the calculation.

We see that there are four distinct eigenvalues:

$$\begin{aligned}
\lambda_{0,1} &= \frac{1}{2}\left(1 \pm \frac{\sqrt{N + 4(Ns^2 - Ns - Ms^2 + Ms)}}{\sqrt{N}}\right) && \text{with multiplicity } 1 \\
\lambda_2 &= 1 - s && \text{with multiplicity } M-1 \\
\lambda_3 &= 1 && \text{with multiplicity } N-M-1.
\end{aligned}$$

Next we are interested in the eigen vectors associated to $\lambda_{0,1}$. Let Q_1 be a unitary transformation that maps $\mathbb{J}^{M\times 1}$ to $\begin{pmatrix}\sqrt{M}\\0\\0\\\vdots\end{pmatrix} \in \mathbb{C}^M$. Similarly let Q_2 be a unitary transformation that maps $\mathbb{J}^{(N-M)\times 1}$ to $\begin{pmatrix}\sqrt{N-M}\\0\\0\\\vdots\end{pmatrix} \in \mathbb{C}^{(N-M)}$ and define $Q = \begin{pmatrix}Q_1 & 0\\ 0 & Q_2\end{pmatrix}$, which is also unitary. Then we have

$$\begin{aligned}
QH(s)Q^{-1} &= \begin{pmatrix}(1-s)\mathbb{1}_M & 0\\ 0 & \mathbb{1}_{N-M}\end{pmatrix} + \frac{s-1}{N} Q\mathbb{J}^{N\times 1}(\mathbb{J}^{N\times 1})^* Q^* \\
&= \begin{pmatrix}(1-s)\mathbb{1}_M & 0\\ 0 & \mathbb{1}_{N-M}\end{pmatrix} + \frac{s-1}{N}\begin{pmatrix}\sqrt{M}\\0\\\vdots\\0\\\sqrt{N-M}\\0\\\vdots\end{pmatrix}\begin{pmatrix}\sqrt{M}\\0\\\vdots\\0\\\sqrt{N-M}\\0\\\vdots\end{pmatrix}^* .
\end{aligned}$$

All but two of the eigenvectors are just elements of $\mathcal{N}$. To study the other two we can simplify by "removing the zeros". Now the eigenvalue problem becomes

$$\left(\begin{pmatrix}1-s-\lambda_{0,1} & 0\\ 0 & 1-\lambda_{0,1}\end{pmatrix} + \frac{s-1}{N}\begin{pmatrix}\sqrt{M}\\ \sqrt{N-M}\end{pmatrix}\begin{pmatrix}\sqrt{M}\\ \sqrt{N-M}\end{pmatrix}^*\right)\mathbf{v} = 0.$$

Setting $\mathbf{v} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ gives us the equations

$$0 = (1-s-\lambda_{0,1})x_1 + \frac{s-1}{N}(Mx_1 + \sqrt{M}\sqrt{N-M}x_2)0 \quad = (1-\lambda_{0,1})x_2 + \frac{s-1}{N}(\sqrt{M}\sqrt{N-M}x_1 + (N-M)x_2)$$

reshuffling and dividing these equations gives

$$\frac{(1-s-\lambda_{0,1})x_1}{(1-s-\lambda_{0,1})x_2} = \frac{-\frac{s-1}{N}(Mx_1 + \sqrt{M}\sqrt{N-M}x_2)}{-\frac{s-1}{N}(\sqrt{M}\sqrt{N-M}x_1 + (N-M)x_2)} = \frac{\sqrt{M}(\sqrt{M}x_1 + \sqrt{M}\sqrt{N-M}x_2)}{\sqrt{N-M}(\sqrt{M}x_1 + \sqrt{N-M}x_2)} = \frac{\sqrt{M}}{\sqrt{N-M}}.$$

So we have eigenvectors $\mathbf{v}_{0,1} = (\sqrt{M}(1-\lambda_{0,1}), \sqrt{N-M}(1-s-\lambda_{0,1}))^{\mathrm{T}}$ of $QH(s)Q^{-1}$. The corresponding eigenvectors of $H(s)$ are then given by

$$Q^*\mathbf{v}_{0,1} = \begin{pmatrix}(1-\lambda_{0,1})\mathbb{J}^{M\times 1}\\ (1-s-\lambda_{0,1})\mathbb{J}^{(N-M)\times 1}\end{pmatrix}.$$

For computational ease we will keep on working with the eigenvectors $\mathbf{v}_{0,1}$ of $QH(s)Q^{-1}$. We denote by $|0\rangle$ and $|1\rangle$ the normalisation of $\mathbf{v}_{0,1}$.

4.1.3.1 Poisson projective measurement

The dynamics of the system is governed by the differential equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}s} = \Lambda(|0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1| - \rho).$$

Now we can rewrite $\frac{\mathrm{d}\rho}{\mathrm{d}t}$ in the basis $|0\rangle, |1\rangle$: let $i, j, k, l = 0, 1 \mod 2$ with implicit summation

$$\begin{aligned}
\tfrac{\mathrm{d}\rho}{\mathrm{d}s} &= \tfrac{\mathrm{d}}{\mathrm{d}s}\left(|i\rangle\langle i|\rho|j\rangle\langle j|\right) \\
&= \tfrac{\mathrm{d}}{\mathrm{d}s}(\langle i|\rho|j\rangle)|i\rangle\langle j| + \langle i|\rho|j\rangle \tfrac{\mathrm{d}}{\mathrm{d}s}(|i\rangle\langle j|) \\
&= \tfrac{\mathrm{d}}{\mathrm{d}s}(\langle i|\rho|j\rangle)|i\rangle\langle j| + \langle i|\rho|j\rangle|k\rangle\langle k|\tfrac{\mathrm{d}}{\mathrm{d}s}|i\rangle\langle j| - \langle i|\rho|j\rangle|i\rangle\langle j|\tfrac{\mathrm{d}}{\mathrm{d}s}|l\rangle\langle l| \\
&= \tfrac{\mathrm{d}}{\mathrm{d}s}(\langle i|\rho|j\rangle)|i\rangle\langle j| + \langle i|\rho|j\rangle|i+1\rangle\langle i+1|\tfrac{\mathrm{d}}{\mathrm{d}s}|i\rangle\langle j| - \langle i|\rho|j\rangle|i\rangle\langle j|\tfrac{\mathrm{d}}{\mathrm{d}s}|j+1\rangle\langle j+1| \\
&= \left(\tfrac{\mathrm{d}}{\mathrm{d}s}(\langle i|\rho|j\rangle) + \langle i+1|\rho|j\rangle\langle i|\tfrac{\mathrm{d}}{\mathrm{d}s}|i+1\rangle - \langle i|\rho|j+1\rangle\langle j+1|\tfrac{\mathrm{d}}{\mathrm{d}s}|j\rangle\right)|i\rangle\langle j|.
\end{aligned}$$

For each $|i\rangle\langle j|$ we get an equation, four in total. One of these is redundant by the zero trace requirement. Writing $y_{i,j} \coloneqq \langle i|\rho|j\rangle)$ and $\omega_{ij} \coloneqq \langle i|\frac{\mathrm{d}}{\mathrm{d}s}|j\rangle$ the three remaining equations are

$$\begin{aligned}
\tfrac{\mathrm{d}\rho_{00}}{\mathrm{d}s} &= -\rho_{10}\omega_{01} + \rho_{01}\omega_{10} \\
\tfrac{\mathrm{d}\rho_{01}}{\mathrm{d}s} &= -\Lambda\rho_{01} - (1-\rho_{00})\omega_{01} + \rho_{00}\omega_{01} \\
\tfrac{\mathrm{d}\rho_{10}}{\mathrm{d}s} &= -\Lambda\rho_{10} - \rho_{00}\omega_{10} + (1-\rho_{00})\omega_{10}
\end{aligned}$$

Setting $\omega = \omega_{10} = -\omega_{01}$ and $y = \rho_{00} - 1/2$we obtain the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} = \left(\frac{\frac{\mathrm{d}\omega}{\mathrm{d}s}}{\omega} - \Lambda\right)\frac{\mathrm{d}y}{\mathrm{d}s} - 4\omega^2 y.$$

Setting $\mathbf{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$ this second order differential equation is equivalent to the first order system

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -4 & -\Lambda/\omega \end{pmatrix},$$

which is in the form $\mathbf{y}' = A\mathbf{y}$. Now A is similar to $\begin{pmatrix} \omega/\Lambda & 0 \\ -4 & -\Lambda/\omega \end{pmatrix}$, so it is bounded if both Λ and Λ^{-1} are bounded functions.

Then by the Picard-Lindelöf theorem the initial value problem has a unique solution on $[0, 1]$. In addition let y_1 be the solution obtained using Λ_1 and y_2 using Λ_2. Then

$$\Lambda_1 \leq \Lambda_2 \implies y_1 \leq y_2.$$

To see this, we may first remark that

$$\{\, t \in [0,1] \mid y_1(t) \leq y_2(t) \wedge y_1'(t) \leq y_2'(t)\} = (y_2 - y_1)^{-1}[\,[0, +\infty[\,] \cap (y_2' - y_1')^{-1}[\,[0, +\infty[\,]$$

is a closed and bounded set that contains 0. Let t_1 be its supremum. This means that $y_1(t_1) \leq y_2(t_1)$ and $y_1'(t_1) \leq y_2'(t_1)$, but $y_1(t) > y_2(t)$ or $y_1'(t) > y_2'(t)$ on some open set $]t_1, t_1 + \delta[$.