Some of the Big Ideas in Mathematics

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TODO: Bertrand paradox (probability); Routh-Hurwitz; Marden's theorem, Sylvester-Gallai;
`https://mathoverflow.net/questions/28997/does-anyone-know-an-intuitive-proof-of-the-b`
`https://grossack.site/`

Chapter 0

Preface

The text is a collection of interesting results. If a proof is not supplied, it it probably fairly trivial, but it may also be an oversight. This text starts with set theory and attempts to develop as much mathematics as possible from there. Only a basic knowledge of logic is assumed.
In particular the following logical symbols will be used and not introduced:

Symbol	Meaning
$\wedge$	logical and
$\vee$	logical or
$\oplus$	exclusive or
$\neg$	logical not
$\forall$	universal quantifier
$\exists$	existential quantifier
$\exists!$	uniqueness quantifier
$(\,), [\,]$	parentheses
$=, \neq$	equality, non-equality

One important note for proving things: if we say $\exists x : P(x)$, we can take this x and continue the proof using it.

Part I

Material set theory

Chapter 1

Set theory

`math.colorado.edu/~monkd`
`philsci-archive.pitt.edu/1372/1/SetClassCat.PDF`
`https://mathoverflow.net/questions/22635/can-we-prove-set-theory-is-consistent`
Rethinking set theory - Leinster

1.1 Some initial ideas

1. Sets are defined by what is in them (extensionality);
2. Sets can be created by specifying a condition: for any definite condition P there is a set
$$A = \{x \mid P(x)\}$$
defined by $x \in A \iff P(x)$. This is de general comprehension principle.

1.1.1 Russell's paradox

Russell showed that the general comprehension principle cannot be valid. Consider
$$R = \{x \mid x \text{ is a set and } x \notin x\}.$$
Then $R \in R$ if and only if $R \notin R$.
There have been several proposed solutions:

1. We can restrict the general comprehension principle to only separation, which means that we can only apply comprehension to elements that are already in a set. In other words, we do not write
$$\{x \mid P(x)\} \qquad \text{but instead} \qquad \{x \in B \mid P(x)\} \qquad \text{for some set } B.$$
For this to work clearly there must not be a set containing all objects in the universe. We could say the universe is too big to fit into a set.
2. We can restrict the type of objects that are put into the same set: We can put two objects that are not sets in the same set, but not a set and an object that is not a set. Such objects have type 0. Sets of these objects have type 1. Sets of these sets have type 2 etc. We then say we can only form sets containing only objects of the same type. Such a set is of a type one higher. This is the essential idea behind type theory.

In these notes we will use the first solution.

1.1.2 The axiomatic setup

For the setup of set theory we have:

1. A domain or universe $\mathcal{W}$ of objects. Some of these objects are sets. Some of these object are not sets. These are called atoms or urelements. A universe is called pure if it contains only sets.
2. We have a (logical) language (usually first order logic) which allows us to express definite conditions using which we can define sets using comprehension. We assume this logical language has
   (a) a notion of identity, i.e. $=$;
   (b) a definite predicate giving sethood:
   $$\mathrm{Set}(x) \iff x \text{ is a set};$$
   (c) a definite binary predicate giving membership, denoted $\in$:
   $$x \in y \iff \mathrm{Set}(y) \text{ and } x \text{ is a member of } y.$$

TODO purity?

We abbreviate

- $\forall x : x \in X \implies (\ldots)$ by $\forall x \in X : (\ldots)$;
- $\exists x : x \in X \wedge (\ldots)$ by $\exists x \in X : (\ldots)$.

This can be generalised to the abbreviation, for some definite condition P, of

- $\forall x : P(x) \implies (\ldots)$ by $\forall P(x) : (\ldots)$;
- $\exists x : P(x) \wedge (\ldots)$ by $\exists P(x) : (\ldots)$.

For formal proofs it is often useful to write out the abbreviations.

1.1.2.1 Definitions of some set-theoretic operations

TODO: setbuilder (with expressions in creation part) and $\{\}$.

Let A, B be sets. We define

- the emptyset
$$\emptyset \coloneqq \{x \mid x \neq x\}\,;$$
- the set difference of A and B or relative complement of B in A
$$A \setminus B \coloneqq \{x \mid x \in A \wedge x \notin B\}\,;$$
- the powerset of A
$$\mathcal{P}(A) \coloneqq \{X \mid X \subseteq A\}\,;$$
- the union of A
$$\bigcup A \coloneqq \{x \mid \exists X \in A : x \in X\}\,;$$

- the intersection of A

$$\bigcap A \coloneqq \{x \mid \forall X \in A : x \in X\}\,;$$

We introduce some abbreviations for the union and intersection:

- $A \cup B \coloneqq \bigcup\{A, B\}$;
- $\bigcup_\Phi \sigma \coloneqq \bigcup\{\sigma(x) \mid \Phi(x)\}$.

And similarly for the intersection.

Lemma I.1. *Let A, B be collections. Then*

$$A \in B \implies A \subseteq \bigcup B$$

1.2 The Zermelo axioms

The Zermelo axioms are the following, except for the axiom of choice which will be discussed later:

(I) **Axiom of extensionality**: for any two sets A, B:

$$A = B \iff [\forall x : x \in A \iff x \in B]\,.$$

(II) **Axiom of elementary sets** or the **emptyset and pairset axioms**:

(0) There exists a set $\emptyset$ that has no members.

(1) For any object x in the domain, there exists a set $\{x\}$ containing only x.

(2) For any two objects x, y in the domain, there exists a set $A = \{x, y\}$ containing only x and y:

$$t \in A \iff [t = x \ \vee t = y].$$

A set with only one element is a singleton. A set with two elements is a doubleton. The existence of singletons follows from the existence of doubletons by setting $x = y$, so part (1) is superfluous.

(III) **Axiom of separation**: for each set A and each unitary predicate P, there exists a set B such that

$$x \in B \iff [x \in A \wedge P(x)].$$

We write

$$B = \{x \in A \mid P(x)\}.$$

When working in a first-order system we do not have variables for predicates and the axiom of separation becomes more properly an axiom schema: for every property definable by a first-order formula we add a new axiom with this formula substituted for P.

Proposition I.2. *Let A be a set. The set*

$$r(A) \coloneqq \{x \in A \mid x \notin x\}$$

is not a member of A.

Corollary I.2.1. *There is no set of sets.*

(IV) **Power set axiom**: for each set A there exists a set $\mathcal{P}(A)$ whose members are the subsets of A. This set is called the power set of A.

For this we need the definition of a subset. We define

$$X \subseteq A \quad \Leftrightarrow_{\text{def}} \quad \forall t : t \in X \implies t \in A$$

and say that X is a subset of A. Then A is a superset of X. We also write $X \subset A$. If we want to emphasise that $X \neq A$, we write $X \subsetneq A$. In this case X is a proper subset.

Lemma I.3. *Let A and B be sets. Then $A = B$* if and only if

$$(A \subseteq B) \wedge (B \subseteq A).$$

Then we can define the power set of A as

$$\mathcal{P}(A) \coloneqq \{X \mid \operatorname{Set}(X) \wedge (X \subseteq A)\}.$$

(V) **Union set axiom**: for every object $\mathcal{E}$, there exists a set $\bigcup \mathcal{E}$ whose members are the members of $\mathcal{E}$:

$$t \in \bigcup \mathcal{E} \iff \exists X \in \mathcal{E} : t \in X$$

Lemma I.4. *Let a be an atom. Then*

$$\bigcup a = \emptyset.$$

Also

$$\bigcup \emptyset = \emptyset.$$

Given two objects A, B, we define

$$A \cup B \coloneqq \bigcup \{A, B\}.$$

(VI) **Axiom of infinity**: there exists a set I which contains

- the empty set $\emptyset$ and
- the singleton of each of its members:

$$\forall x : \quad x \in I \implies \{x\} \in I.$$

A possible version of the set I is

$$I_0 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots\}$$

and indeed we need $I_0 \subset I$, but we have not excluded the possibility that I contains other elements.

1.3 Von Neumann-Bernays-Gödel and Morse-Kelley set theory

NBG / MK are axiomatic systems in first-order predicate logic with equality (TODO: remove equality, made definition (?), in which case make equivalence relation a lemma. Check definition of empty class and universe class.), whose only primitive notion is the membership relation $\in$. (TODO sometimes class is called primitive notion as well (?))
TODO: replacement, infinity and foundation.

1.3.1 Classes

We consider a universe $\mathcal{W}$. Let x be an object in $\mathcal{W}$. Then we call x

- a class if $\exists y : y \in x$;
- an element if $\exists y : x \in y$;
- a set if it is both a class and an element;
- a proper class if it is a class and not an element (or, equivalently, a class and not a set).

We define the predicates

- $\text{Class}(x) \quad \Leftrightarrow_{\text{def}} \quad \exists y : y \in x;$
- $\text{Element}(x) \quad \Leftrightarrow_{\text{def}} \quad \exists y : x \in y;$
- $\text{Set}(x) \quad \Leftrightarrow_{\text{def}} \quad \exists y, z : y \in x \wedge x \in z;$
- $\text{ProperClass}(x) \quad \Leftrightarrow_{\text{def}} \quad (\exists y : y \in x) \wedge \neg(\exists z : x \in z).$

(A) **Axiom of extensionality**: For any two classes A, B:

$$A = B \iff [\forall x : x \in A \iff x \in B] .$$

(B) **Axiom of class existence**: Let $\phi(x)$ be a proposition with free variable x. Then there exists a class $\{x \mid \phi(x)\}$ such that

$$\forall y : \ \text{Element}(y) \implies \Big(y \in \{x \mid \phi(x)\} \iff \phi(y)\Big).$$

This is for MK. In NBG the quantifiers in ϕ must be bound to elements.
Let A be a class and $\phi(x)$ a proposition with free variable x. We often abbreviate $\{x \mid x \in A \wedge \phi(x)\}$ by $\{x \in A \mid \phi(x)\}$.

Proposition I.5 (Russel's paradox)**.** *Let A be a set. The set*

$$r(A) \coloneqq \{x \in A \mid x \notin x\}$$

is not a member of A.

Corollary I.5.1. *The class of sets, $\{x \mid \mathrm{Set}(x)\}$, is not a set.*

Corollary I.5.2. *There exists a proper class.*

Proposition I.6. *There exists a proper*

1.3.1.1 Subclasses

Let A, B be classes. We say B is a subclass of A, denoted $B \subseteq A$, if

$$\forall x : \ x \in A \implies x \in B.$$

If B is a subclass of A and $A \neq B$, then we call B a strict subclass of A, denoted $B \subset A$ or $B \subsetneq A$.

Lemma I.7. *Let A, B, C be classes. Then*

1. $A = B$ if and only if $A \subseteq B$ *and* $B \subseteq A$*;*
2. *if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

(C) **Axiom of separation**: Any subclass of a set is a set.

TODO: consequence of replacement.
In particular, for any set A and proposition $\phi(x)$, the class $\{x \in A \mid \phi(x)\}$ is a set.

1.3.1.2 Class construction

We can use the axiom of class construction to build some basic classes.

Let A, B be classes. We define

- the union of A and B as $A \cup B \coloneqq \{x \mid x \in A \vee x \in B\}$;
- the intersection of A and B as $A \cap B \coloneqq \{x \mid x \in A \wedge x \in B\}$;
- the complement of A as $A^c \coloneqq \{x \mid x \notin A\}$;
- the power set of A as $\mathcal{P}(A) \coloneqq \{B \mid B \subseteq A\}$.

Let $\mathcal{E}$ be a class. We define

- the union of $\mathcal{E}$ as
$$\bigcup \mathcal{E} \coloneqq \{x \mid \exists A \in \mathcal{E} : \ x \in A\}\,;$$
- the intersection of $\mathcal{E}$ as
$$\bigcap \mathcal{E} \coloneqq \{x \mid \forall A \in \mathcal{E} : \ x \in A\}\,.$$

We also define

- the empty class $\emptyset \coloneqq \{x \mid x \neq x\}$;
- the universe class $\mathcal{U} \coloneqq \{x \mid x = x\}$.

Lemma I.8. *Let x be an element and A a class. Then*

1. *$x \notin \emptyset$ and $x \in \mathcal{U}$;*
2. *$\emptyset \subseteq A$ and $A \subseteq \mathcal{U}$.*

Lemma I.9. *The empty class is a set* if and only if *there exists a set.*

Proof. The direction $\Rightarrow$ is clear.
For the direction $\Leftarrow$, assume there exists a set A. Then $\emptyset \subseteq A$. By the axiom of separation, this means $\emptyset$ is a set. ☐

(D) **Set existence**: There exists a set.

TODO: is implied by the axiom of infinity (TODO later!)

Lemma I.10. *Let $A, B, \mathcal{E}$ be classes. Then*

1. *$\bigcap \mathcal{E}$ is either a set or $\mathcal{U}$;*
2. *if either A or B is a set, the intersection $A \cap B$ is a set.*

Proof. (1) Assume $\bigcap \mathcal{E} \neq \mathcal{U}$. Then there exists an element $x \notin \bigcap \mathcal{E}$, meaning there exists a set $A \in \mathcal{E}$ such that $x \notin A$. Now $\bigcap \mathcal{E} \subseteq A$, so it is a set by separation.
(2) This follows from the axiom of separation and the fact that $A \cap B$ is a subclass of both A and B. ☐

We also assert some other classes constructed from sets are sets.

(E) **Power set axiom**: Let A be a set. Then $\mathcal{P}(A)$ is a set.

(F) **Union set axiom**: Let $\mathcal{E}$ be a set. Then $\bigcup \mathcal{E}$ is a set.

Lemma I.11.

1. *$\emptyset \neq \mathcal{P}(\emptyset)$;*
2. *for any object a, we can find an object b such that $a \neq b$.*

Proof. (1) We have $\emptyset \in \mathcal{P}(\emptyset)$, but $\emptyset \notin \emptyset$.
(2) If $a = \emptyset$, we may take $\mathcal{P}(\emptyset)$. Otherwise take $\emptyset$. ☐

Let $a, b, c \ldots$ be objects. Then

$$\{a, b, c \ldots\} \coloneqq \{x \mid x = a \vee x = b \vee x = c \vee \ldots\}.$$

(G) **Doubleton existence axiom**: Let a, b be objects. Then $\{a, b\}$ is a set.

Lemma I.12. *Let A, B be sets. Then*

1. *$A \cup B = \bigcup\{A, B\}$;*
2. *$A \cap B = \bigcap\{A, B\}$;*

In particular $A \cup B$ is a set.

Lemma I.13. *Let $a, b, c \dots$ be an object. Then*

1. *$\{a\} = \{a, a\}$;*
2. *$\{a\}$ is a set;*
3. *$a \in \{a\}$ if and only if $\mathrm{Element}(a)$;*
4. *$\{b, c \dots\} \subseteq \{a, b, c \dots\}$;*
5. *$\{a, b, c \dots\}$ is a set.*

Proof. (1) $\{x \mid x = a \vee x = a\} = \{x \mid x = a\}$.
(2) By the doubleton existence axiom.
(3) By definition.
(4) Clear.
(5) By definition we can find sets $A, B, C \dots$ such that $a \in A, b \in B, c \in C \dots$ Then $\{a, b, c \dots\} \subseteq A \cup B \cup C \cup \dots$ and thus a set by I.12 and the axiom of separation. ☐

1.4 Potter-Scott set theory

Let $\mathcal{W}$ be a universe with urelements $\mathcal{U}$.

- The accumulation of an entity a is
$$\mathrm{acc}(a) := \{x \mid x \in \mathcal{U} \vee [\exists b \in a : x \in b \vee x \subseteq b]\}\,.$$
- A collection V is called a history if
$$\forall v \in V : v = \mathrm{acc}(V \cap v).$$
- The accumulation of a history is called a level.

We can write the accumulation as
$$\mathrm{acc}(a) = \mathcal{U} \cup \left(\bigcup a\right) \cup \left(\bigcup \{\mathcal{P}(b) \mid b \in a\}\right).$$

Lemma I.1 translates to:

Lemma I.14. *Let a, b be collections. If $b \in a$, then $b \subseteq \mathrm{acc}(a)$.*

Example

Let $\mathcal{W}$ be a universe with urelements $\mathcal{U}$.

1. $\emptyset$ is (trivially) a history (assuming it exists). Then
$$\mathrm{acc}(\emptyset) = \mathcal{U}$$

is a level.

2. $V = \{\mathcal{U}\}$ is a history:

   - $\text{acc}(V \cap \mathcal{U}) = \text{acc}(\emptyset) = \mathcal{U}$.

   Then $\text{acc}(\{\mathcal{U}\}) = \mathcal{U} \cup \mathcal{P}(\mathcal{U}) =: L_1$ is a level.

3. $V = \{\mathcal{U}, L_1\}$ is a history:

   - $\text{acc}(V \cap \mathcal{U}) = \text{acc}(\emptyset) = \mathcal{U}$;
   - $\text{acc}(V \cap L_1) = \text{acc}(\{\mathcal{U}\}) = L_1$.

   Then $\text{acc}(\{\mathcal{U}, L_1\}) = \mathcal{U} \cup \mathcal{P}(\mathcal{U}) \cup \mathcal{P}(\mathcal{P}(\mathcal{U})) =: L_2$ is a level.

4. $V = \{\mathcal{U}, L_1, L_2\}$ is a history:

   - $\text{acc}(V \cap \mathcal{U}) = \text{acc}(\emptyset) = \mathcal{U}$;
   - $\text{acc}(V \cap L_1) = \text{acc}(\{\mathcal{U}\}) = L_1$;
   - $\text{acc}(V \cap L_2) = \text{acc}(\{\mathcal{U}, L_1\}) = L_2$.

   Then $\text{acc}(\{\mathcal{U}, L_1, L_2\}) = \mathcal{U} \cup \mathcal{P}(\mathcal{U}) \cup \mathcal{P}(\mathcal{P}(\mathcal{U})) \cup \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{U}))) =: L_3$ is a level.

A level is supposed to contain all collections up to a certain "depth".
A history V is supposed to be a set of levels such that for each level L in V all previous levels contained in L are also in V.
Formally:

Proposition I.15. *1. Let V be a history, then every element l of V is a level with history $V \cap l$.*

2. Let l, L be levels and V a history of L. If $l \in L$, then $l \in V$.

3. Each level has a unique history.

4. Let L, L' be levels such that $L' \subseteq L$. If V is a history and $L \in V$, then $L' \in V$.

5. Let L, L' be levels then $L' \subseteq L$ if and only if *$L' = L$ or $L' \in L$.*

Proof. (1) Because V is a history, we have $l = \text{acc}(V \cap l)$. So we just need to show $V \cap l$ is a history. Indeed take $A \in (V \cap l)$, then $A \subseteq \text{acc}(V \cap l) = l$ (by I.14). So $A = A \cap l$ and

$$\text{acc}((V \cap l) \cap A) = \text{acc}(V \cap A) = A.$$

(2) Let L be a level. Assume there exist histories V_1, V_2 such that $\text{acc}(V_1) = L = \text{acc}(V_2)$.
(3) ☐

1.4.1 Sets

A collection is called a set or grounded if it is a subcollection of some level.

1.4.2 Axiom scheme of separation

TODO A level??

(I) **Axiom of separation**: let P be a unary predicate, then for each set A, the collection $\{x \in A \mid P(x)\}$ exists.

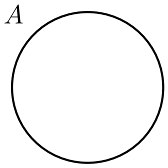
These collections are automatically sets.
Only those sets describable by predicates (a countable number). (cfr. second order).

1.4.3 Theory of levels

1.5 Working with sets

1.5.1 Venn diagrams

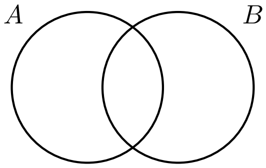
Much reasoning about sets can be simplified by drawing sets as circles. Many set-theoretic operations can be described in this way. The resulting pictures are called Venn diagrams.
For any given set A, all objects are either in A or not in A. So we divide the paper into a region inside A and a region outside A:



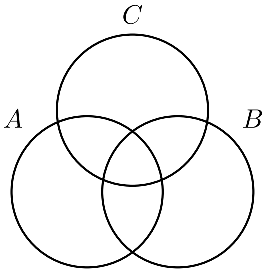
Given two sets A, B there are now four possibilities for all objects in the universe:

1. outside both A and B;
2. inside A, but not inside B;
3. inside B, but not inside A;
4. inside both A and B.

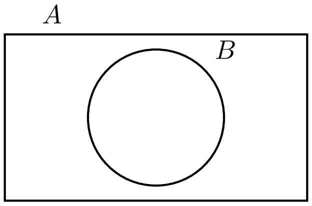
We correspondingly divide the paper into four regions:



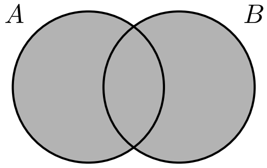
For three sets A, B, C the picture becomes:



If we want to show that one set is a subset of another set, e.g $B \subset A$, then we can represent this as follows:



Expressions talking about sets can be expressed by shading regions of Venn diagrams. For example $A \cup B$:



1.5.2 Operations on sets

1.5.2.1 Intersection

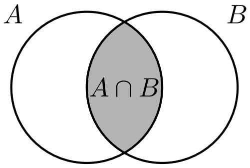
The <u>intersection</u> of an object $\mathcal{E}$ is a set $\bigcap \mathcal{E}$ defined by

$$\bigcap \mathcal{E} := \left\{ x \in \bigcup \mathcal{E} \mid \forall X \in \mathcal{E} : x \in X \right\}.$$

As before, for the union, we define

$$A \cap B := \bigcap \{A, B\}.$$

The intersection $A \cap B$ can be represented in a Venn diagram as follows:



Proposition I.16. *Let A, B be non-empty sets. Then*

1. $A \subseteq B \implies \bigcap B \subseteq \bigcap A$;
2. $\bigcap(A \cup B) = (\bigcap A) \cup (\bigcap B)$.

These statements hold for all sets A, B iff we use an unrelativised intersection (TODO!).

Let A, B be sets. We call A and B <u>disjoint</u> if $A \cap B = \emptyset$. We write $A \perp B$.
A family of sets $\mathcal{E}$ is called <u>pairwise disjoint</u> if $\forall A, B \in \mathcal{E} : A \perp B$.

The notation $A \perp B$ is not standard in general set theory, but is somewhat standard for disjoint elements in lattice theory.

Let A, B be sets. We call the union $A \cup B$ a (inner) disjoint union if A and B are disjoint.
We may write $A \uplus B$ for the union if it is disjoint.
If a family of sets $\mathcal{E}$ is pairwise disjoint, we may denote its union $\biguplus \mathcal{E}$.

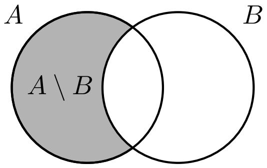
Let $\mathcal{A}, \mathcal{B}$ be families of sets. We say $\mathcal{A}$ and $\mathcal{B}$ mesh, denoted $\mathcal{A} \# \mathcal{B}$ if A and B are not disjoint, $A \cap B \neq \emptyset$, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
We write $A \# \mathcal{B}$ for $\{A\} \# \mathcal{B}$ and $A \# B$ for $\{A\} \# \{B\}$.

1.5.2.2 Difference

Given two objects A, B, we define the difference as

$$A \setminus B \coloneqq \{x \in A \mid x \notin B\}.$$

The difference $A \setminus B$ can be represented in a Venn diagram as follows:



Proposition I.17 (De Morgan's laws)**.** *Let A, B, C be sets. Then*

$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$$
$$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$$

This can be extended to arbitrary families of sets:

$$C \setminus \left(\bigcup \mathcal{E}\right) = \bigcap \{C \setminus A \mid A \in \mathcal{E}\}$$
$$C \setminus \left(\bigcap \mathcal{E}\right) = \bigcup \{C \setminus A \mid A \in \mathcal{E}\}$$

where $\mathcal{E}$ is a family of sets.

Lemma I.18. *Let $\mathcal{E}$ be a family of sets and A a set. Then*

$$\bigcup \mathcal{E} \setminus A = \bigcup \{X \setminus A \mid X \in \mathcal{E}\} \qquad \text{and} \qquad \bigcap \mathcal{E} \setminus A = \bigcap \{X \setminus A \mid X \in \mathcal{E}\}.$$

Lemma I.19. *Let A, B, C be sets. Then*

1. $(A \setminus B) \setminus C = A \setminus (B \cup C)$;
2. $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$;

and

3. $(A \setminus B) \cap C = (A \cap C) \setminus B = A \cap (C \setminus B)$;
4. $(A \setminus B) \cup C = (A \cup C) \setminus (B \setminus C)$;

and

5. $A \setminus A = \emptyset$;
6. $\emptyset \setminus A = \emptyset$;
7. $A \setminus \emptyset = A$.

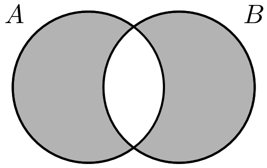
1.5.2.3 Symmetric difference

We define the symmetric difference of two sets A, B as

$$A \mathbin{\Delta} B \coloneqq (A \setminus B) \cup (B \setminus A).$$

This is equivalent to $A \mathbin{\Delta} B = \{x \in A \cup B \mid (x \in A) \oplus (x \in B)\}$.

The symmetric difference $A \mathbin{\Delta} B$ can be represented in a Venn diagram as follows:



Lemma I.20. *Let A, B, C be sets. Then*

$$\begin{aligned} A \mathbin{\Delta} \emptyset &= A \\ A \mathbin{\Delta} B = \emptyset &\iff A = B \end{aligned}$$

and

$$\begin{aligned} A \mathbin{\Delta} B &= B \mathbin{\Delta} A \\ (A \mathbin{\Delta} B) \mathbin{\Delta} C &= A \mathbin{\Delta} (B \mathbin{\Delta} C) \\ A \mathbin{\Delta} C &= (A \mathbin{\Delta} B) \mathbin{\Delta} (B \mathbin{\Delta} C). \end{aligned}$$

1.5.3 Identities and equivalences

1.5.3.1 Identities involving families of sets

Lemma I.21. *Let $\mathcal{F}, \mathcal{G}$ be families of sets such that $\mathcal{F} \subseteq \mathcal{G}$. Then*

1. $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$;
2. $\bigcap \mathcal{F} \supseteq \bigcup \mathcal{G}$.

1.5.3.2 Set operations and statements characterised

Lemma I.22. *Let A, B be sets. Then*

1. $\begin{aligned} A \cap B &= A \setminus (A \setminus B) \\ &= B \setminus (A \mathbin{\Delta} B) \\ &= A \mathbin{\Delta} (A \setminus B); \end{aligned}$

2. $A \cup B = (A \Delta B) \cup A$
   $= (A \Delta B) \Delta (A \cap B)$
   $= (A \setminus B) \cup B$;
3. $A \Delta B = (A \cup B) \setminus (A \cap B)$
   $= (A \Delta C) \Delta (C \Delta B)$
   for any set C*;*
4. $A \setminus B = A \setminus (A \cap B)$
   $= A \cap (A \Delta B)$
   $= (A \cup B) \Delta B$
   $= A \Delta (A \cap B)$;

Lemma I.23. *Let* A *be a set. Then the following are equivalent:*

1. A *is empty;*
2. $A \cup B \subseteq B$ *for every set* B*;*
3. $A \subseteq B$ *for every set* B*;*
4. $A \subseteq (B \setminus A)$ *for some set* B*;*
5. $A \subseteq (B \setminus A)$ *for every set* B*;*
6. $\emptyset \setminus A = A$.

Lemma I.24. *Let* A, B *be sets. Then following are equivalent:*

1. $A \subseteq B$*;*
2. $A \cap B = A$*;*
3. $A \cup B = B$*;*
4. $A \Delta B = B \setminus A$*;*
5. $A \Delta B \subseteq B \setminus A$
6. $A \setminus B = \emptyset$.

For any universe set Ω *such that* $A, B \subset \Omega$ *the above is also equivalent to*

5. $\Omega \setminus B \subseteq \Omega \setminus A$*;*
6. $(\Omega \cap A) \setminus B = \emptyset$*;*
7. $(\Omega \setminus A) \cup B = \Omega$.

Corollary I.24.1. *Let* A, B *be sets. Then following are equivalent:*

1. $A = B$*;*
2. $A \Delta B = \emptyset$*;*
3. $A \setminus B = B \setminus A$.

Corollary I.24.2. *Let $A, B \subseteq \Omega$ be sets. Then following are equivalent:*

1. $A \perp B$;
2. $A \subseteq \Omega \setminus B$;
3. $B \subseteq \Omega \setminus A$.

1.5.3.3 Distributivity

TODO: tables complete / correct?? TODO: only works for relativised intersection.

Lemma I.25. *We say $*$ left distributes over $\bullet$ if*

$$A * (B \bullet C) = (A * B) \bullet (A * C) \qquad \textit{for all sets } A, B, C.$$

This gives the table

$\bullet$ $*$	$\cup$	$\cap$	Δ	$\setminus$	$\times$
$\cup$	✓	✓			
$\cap$	✓	✓	✓	✓	
Δ					
$\setminus$					
$\times$	✓	✓		✓	

Lemma I.26. *We say $*$ right distributes over $\bullet$ if*

$$(A \bullet B) * C = (A * C) \bullet (B * C) \qquad \textit{for all sets } A, B, C.$$

This gives the table

$\bullet$ $*$	$\cup$	$\cap$	Δ	$\setminus$	$\times$
$\cup$	✓	✓			
$\cap$	✓	✓	✓	✓	
Δ					
$\setminus$	✓	✓	✓	✓	
$\times$	✓	✓		✓	

TODO: intersection distributes over disjoint union.

Lemma I.27. *Let $\mathcal{I}$ be a family of index sets and let A_i be a set for all $i \in \bigcup \mathcal{I}$. Then*

1. $\bigcup_{i \in \bigcup \mathcal{I}} A_i = \bigcup_{I \in \mathcal{I}} \bigcup_{i \in I} A_i$;
2. $\bigcap_{i \in \bigcap \mathcal{I}} A_i = \bigcap_{I \in \mathcal{I}} \bigcap_{i \in I} A_i$;
3. $\bigcup_{i \in \bigcap \mathcal{I}} A_i \subseteq \bigcap_{I \in \mathcal{I}} \bigcup_{i \in I} A_i$;
4. $\bigcap_{i \in \bigcup \mathcal{I}} A_i \supseteq \bigcup_{I \in \mathcal{I}} \bigcap_{i \in I} A_i$.

Proof. (1) We calculate

$$\begin{aligned} x \in \bigcup_{i \in \bigcup \mathcal{I}} A_i &\iff \exists i \in \bigcup \mathcal{I} : x \in A_i \\ &\iff \exists i : (i \in \bigcup \mathcal{I}) \wedge (x \in A_i) \\ &\iff \exists i : (\exists I \in \mathcal{I} : i \in I) \wedge (x \in A_i) \\ &\iff \exists i : \exists I : (I \in \mathcal{I}) \wedge (i \in I) \wedge (x \in A_i) \\ &\iff \exists I \in \mathcal{I} : \exists i \in I : x \in A_i \\ &\iff x \in \bigcup_{I \in \mathcal{I}} \bigcup_{i \in I} A_i \end{aligned}$$

(2) Replace $\exists$ by $\forall$ and $\wedge$ by $\Rightarrow$ in the proof of (1).
(3) We calculate

$$\begin{aligned} x \in \bigcup_{i \in \bigcap \mathcal{I}} A_i &\iff \exists i \in \bigcap \mathcal{I} : x \in A_i \\ &\iff \exists i : (i \in \bigcap \mathcal{I}) \wedge (x \in A_i) \\ &\iff \exists i : (\forall I \in \mathcal{I} : i \in I) \wedge (x \in A_i) \\ &\iff \exists i : (\forall I : (I \in \mathcal{I}) \Rightarrow (i \in I)) \wedge (x \in A_i) \\ &\implies \exists i : \forall I : (I \in \mathcal{I}) \Rightarrow ((i \in I) \wedge (x \in A_i)) \\ &\implies \forall I : \exists i : (I \in \mathcal{I}) \Rightarrow ((i \in I) \wedge (x \in A_i)) \\ &\iff \forall I : (I \in \mathcal{I}) \Rightarrow (\exists i : (i \in I) \wedge (x \in A_i)) \\ &\iff \forall I \in \mathcal{I} : \exists i \in I : x \in A_i \\ &\iff x \in \bigcap_{I \in \mathcal{I}} \bigcup_{i \in I} A_i \end{aligned}$$

(4) TODO □

For more identities, see `https://en.wikipedia.org/wiki/List_of_set_identities_and_relations`.

Chapter 2

Relations and functions

2.1 Pairs

A definition of (a,b) for all a,b is called an (ordered) pair operation if it satisfies

- $(a,b)=(x,y) \iff (a=x)\wedge(b=y)$;
- for all sets A,B, $\{(a,b) \mid a\in A\wedge b\in B\}$ is a set.

We call $\{(a,b) \mid a\in A\wedge b\in B\}$ the Cartesian product of classes A and B and denote it $A\times B$.

Note that for the second condition it is enough to check that $A\times B$ is a subset of some set S. Then, by separation, $A\times B$ is the set

$$A\times B=\{x\in S \mid \exists a\in A:\exists b\in B:x=(a,b)\}.$$

Lemma I.28. *Let A be a class. Then*

$$A\times\emptyset=\emptyset=\emptyset\times A.$$

Lemma I.29. *Let A,B,C,D be classes. Then*

1. $(A\cup B)\times C=A\times C\cup B\times C$;
2. $(A\uplus B)\times C=A\times C\uplus B\times C$;
3. $(A\cap B)\times C=A\times C\cap B\times C$;
4. $(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)$.

Let $p=(a,b)$ be a pair, we use $\pi_1(p)$ to denote the first element of p and $\pi_2(p)$ to denote the second element:

$$(a,b)=p=(\pi_1(p),\pi_2(p)).$$

We can convert a pair to a set:

$$\operatorname{set}((a,b))=\{a,b\}.$$

2.1.1 Axiomatising pairs

Consider the ternary proposition $A = (B, C)$ as primitive.

Consider a universe $\mathcal{W}$. Let p be an object in $\mathcal{W}$. We call p a pair if $\exists x, y : p = (x, y)$. We define the predicate

- $\text{Pair}(p) \quad \Leftrightarrow_{\text{def}} \quad \exists x, y : \ p = (x, y)$.

(A) **Axiom of pair identity**: Let p, x_1, x_2, y_1, y_2 be objects. Then

$$\big[p = (x_1, y_1)\big] \ \wedge \ \big[p = (x_2, y_2)\big] \qquad \Longrightarrow \qquad \big[x_1 = x_2\big] \ \wedge \ \big[y_1 = y_2\big].$$

(B) **Axiom of pair existence**: Let x, y be objects. Then $\exists p : \ p = (x, y)$ with $p \neq x$ and $p \neq y$.

Usually an axiomatic system of pairs is integrated with a set theory.

Let $\mathcal{W}$ be a set theoretic universe satisfying the pair axioms. Let A, B be classes. The Cartesian product of A and B is defined as

$$A \times B \coloneqq \{p \mid \exists a \in A : \exists b \in B : p = (a, b)\} \, .$$

In this case we impose the following additional axioms:

(C) **Axiom of element pairs**: Let A, B be classes, $a \in A, b \in B$ and $p = (a, b)$. Then $p \in A \times B$.

(D) **Cartesian product axiom**: Let A, B be sets. Then $A \times B$ is a set.

Lemma I.30. *Let x, y be elements and $p = (x, y)$. Then p is an element.*

Proof. Let A, B be classes such that $x \in A$ and $y \in B$. Then $p \in A \times B$. ☐

2.1.2 Defining pairs

TODO: pair axioms conservative extension of ZFC.

2.1.2.1 The Kuratowski pair

The Kuratowski pair is defined as

$$(a, b) \coloneqq \{\{a\}, \{a, b\}\}$$

Note that when $a = b$ we have

$$(a, a) = \{\{a\}, \{a, a\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}.$$

Lemma I.31. *Given a Kuratowski pair $p = (a,b)$, we can extract the first element $\pi_1(p)$ and the second element $\pi_2(p)$ as follows:*

$$\begin{aligned}\pi_1(p) &= \bigcup\bigcap p;\\ \pi_2(p) &= \bigcup\left\{x \in \bigcup p \mid \left[\bigcup p \neq \bigcap p\right] \implies \left[x \notin \bigcap p\right]\right\}.\end{aligned}$$

The Kuratowski pair can be converted to a set by

$$\operatorname{set}(p) = \bigcup p.$$

The construction "$[\bigcup p \neq \bigcap p] \implies$" in the formula for $\pi_2(p)$ is there so that it still works in case the first and second elements are the same.

Proposition I.32. *The Kuratowski pair is adequate, in that is satisfies*

$$(a,b) = (x,y) \iff (a = x) \wedge (b = y)$$

and the Cartesian product is a set.

Proof. If $(a = x) \wedge (b = y)$, then

$$\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$$

and thus $(a,b) = (x,y)$.
Now assume $(a,b) = (x,y)$. We consider two cases: $a = b$ and $a \neq b$.

$\boxed{a = b}$ Then $(a,b) = \{\{a\},\{a,a\}\} = \{\{a\}\} = (x,y)$ and thus $\{x\} = \{x,y\} = \{a\}$ by extensionality. This implies $a = x = y$ and thus $(a = x) \wedge (b = y)$.

$\boxed{a \neq b}$ Now $(a,b) = (x,y)$ implies

$$\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}.$$

By extensionality, either $\{x\} = \{a\}$ or $\{x\} = \{a,b\}$. In the second option $a = x = b$ and thus $a = b$ which is a contradiction. So $x = a$.

Again by extensionality, either $\{x,y\} = \{a\}$ or $\{x,y\} = \{a,b\}$. In the first case (x,y) would be a singleton and then by equality so would (a,b), yielding a contradiction. Thus $\{x,y\} = \{a,b\}$. We know $a = x$ and $b \neq a$, so by extensionality $b = y$.

To prove the Cartesian product $A \times B$ is a set, notice that

$$\begin{aligned}a \in A, b \in B &\implies \{a\},\{a,b\} \subseteq A \cup B \implies \{a\},\{a,b\} \in \mathcal{P}(A \cup B)\\ &\implies \{\{a\},\{a,b\}\} \subseteq \mathcal{P}(A \cup B) \implies \{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))\\ &\implies (a,b) \in \mathcal{P}(\mathcal{P}(A \cup B)).\end{aligned}$$

and $\mathcal{P}(\mathcal{P}(A \cup B))$ is a set. So

$$A \times B = \{x \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \exists a \in A : \exists b \in B : x = (a,b)\}$$

is a set. □

2.1.2.2 The short variant

We can also define a pair as

$$(a, b) \coloneqq \{a, \{a, b\}\}$$

The advantage of this short definition is fewer braces. Some disadvantages include:

1. If a and b have the same type, a and $\{a, b\}$ do not have the same type.
2. In order to prove adequacy, we need a new axiom, the axiom of regularity.

2.1.2.3 Using $0, 1$

Suppose we have decided on two special, distinct objects $0, 1$. Then we can define

$$(a, b) \coloneqq \{\{0, a\}, \{1, b\}\}.$$

Proposition I.33. *This definition satisfies the requirements for a pair.*

Proof. Assuming $(a = x) \land (b = y)$, it is clear that $(a, b) = (x, y)$ as sets by extensionality.
Now assume $(a, b) = (x, y)$. In this implementation a pair always has two elements as a set. The reasoning by extensionality is simple and only slightly more difficult if a, b, x, y are equal to 0 or 1.
The Cartesian product is a subset of $\mathcal{P}(\mathcal{P}(A \cup B \cup \{0, 1\}))$ and thus a set. □

2.1.2.4 Wiener pair

The Wiener definition of a pair is

$$(a, b) \coloneqq \{\{\emptyset, \{a\}\}, \{\{b\}\}\}.$$

Proposition I.34. *This definition satisfies the requirements for a pair.*

2.1.3 Structured classes and sets

A structured class is a pair $U = (A, S)$ where A is a class and S is an arbitrary object.

- A is the field or space of U, written $\text{Field}(U)$;
- S is the frame of U.

If we write $x \in U$, we mean $x \in \text{Field}(U)$.
In particular if A is a set, then we call U a structured set.

Often the frame S is a n-tuple. In this case we may write the structured class as an $n+1$-tuple by concatenating the field and the frame. e.g $(A, (S_1, S_2, S_3))$ becomes (A, S_1, S_2, S_3).

2.2 Relations

Let A, B be classes and G any subclass of the Cartesian product $A \times B$. A (binary) relation R on (A, B) is a tuple $(G, (A, B))$. The graph of the binary relation R is the class $\operatorname{graph}(R) := G$.
We write

$$xRy \quad \Leftrightarrow_{\text{def}} \quad (x, y) \in \operatorname{graph}(R)$$

and say x is left related to y or y is right related to x.
We call

- $\operatorname{dom}(R) := A$ the domain of the relation;
- $\operatorname{codom}(R) := B$ the codomain of the relation.

A relation R is homogeneous or an endorelation if $\operatorname{dom}(R) = \operatorname{codom}(R)$. If we say R is a (homogenous) relation on A, we mean $\operatorname{dom}(R) = A = \operatorname{codom}(R)$.
A relation is heterogeneous is the domain and codomain are different.

Often we will write $R \subseteq S$ as a shorthand for $\operatorname{graph}(R) \subseteq \operatorname{graph}(S)$.

Lemma I.35. *Let R, S be relations on (A, B). Then*

1. $R \cup S := (\operatorname{graph}(R) \cup \operatorname{graph}(S), (A, B))$ *is a relation on* (A, B)*;*
2. $R \cap S := (\operatorname{graph}(R) \cap \operatorname{graph}(S), (A, B))$ *is a relation on* (A, B)*.*

Let A, B be classes. We have the following relations:

- the empty relation $E_{A,B}$ on (A, B) has graph $\emptyset$;
- the universal relation $U_{A,B}$ on (A, B) has graph $A \times B$;
- the identity relation id_A on A has graph $\{(x, y) \in A \times A \mid x = y\}$.

We may also write U_A instead of $U_{A,A}$ and E_A instead of $E_{A,A}$.

The identity relation on A is also known as the diagonal relation on A.

Lemma I.36. *Let A, B, C, D be classes. Then*

1. $\operatorname{id}_{A\cup B} = \operatorname{id}_A \cup \operatorname{id}_B$ *and* $\operatorname{id}_{A\cap B} = \operatorname{id}_A \cap \operatorname{id}_B$*;*
2. $E_{A\cup C, B\cup D} = E_{A,B} \cup E_{C,D}$ *and* $E_{A\cap C, B\cap D} = E_{A,B} \cap E_{C,D}$*;*
3. $U_{A\cap C, B\cap D} = U_{A,B} \cap U_{C,D}$*.*

Note that the equalities mean the graphs are equal. The relations are not the same as they have different domains and codomains.

Proof. (1) Assume $(x, x) \in \operatorname{id}_{A\cup B}$. We have the equivalences

$$(x \in A) \vee (x \in B) \iff \Big((x, x) \in \operatorname{id}_A\Big) \vee \Big((x, x) \in \operatorname{id}_B\Big) \iff (x, x) \in \operatorname{id}_A \cup \operatorname{id}_B\,.$$

For the second part replace $\vee$ with $\wedge$.

(2) Trivial because all sets are $\emptyset$.
(3) Take $(x, y) \in U_{A\cap C, B\cap D}$. We have the equivalences

$$\begin{aligned}(x \in A\cap C) \wedge (y \in B \cap D) &\iff (x \in A) \wedge (y \in B) \wedge (x \in C) \wedge (y \in D)\\ &\iff \big((x, y) \in U_{A,B})\big) \wedge \big((x, y) \in U_{A,B}\big)\\ &\iff (x, y) \in U_{A,B} \cap U_{C,D}.\end{aligned}$$

$\square$

Let R be a binary relation. We say

- x is related to or comparable with y if xRy or yRx; we denote this $x \nparallel_R y$ or just $x \nparallel y$;
- x is unrelated to, incomparable with or parallel with y if neither xRy nor yRx; we denote this $x \parallel_R y$ or just $x \parallel y$.

2.2.1 Relations and subclasses

2.2.1.1 Images and preimages

Let R be a relation on (A, B).

- The image of a subclass $X \subset A$ under R is the class
$$X_R \coloneqq \{b \in B \mid \exists x \in X : xRb\}.$$
- The preimage of a subclass $Y \subset B$ under R is the class
$${}_RY \coloneqq \{a \in A \mid \exists y \in Y : aRy\}.$$

In particular for $X = A$ and $Y = B$:

1. The class A_R is the active codomain, codomain of definition, image or range of the relation, also denoted $\operatorname{im}(R)$.
2. The class ${}_RB$ is the active domain, domain of definition, preimage or prerange of the relation, also denoted $\operatorname{preim}(R)$.

In particular for $X = \{x\}$ and $Y = \{y\}$ we define:

- $xR \coloneqq \{x\}_R$;
- $Ry \coloneqq {}_R\{x\}$.

We call such images and preimages principle images and preimages.

A relation is completely characterised by its principal images.

Lemma I.37. *Let R be a relation on (A, B), $x \in A$ and $y \in B$. Then*

$$x \in Ry \iff xRy \iff xR \ni y.$$

Principal images atoms in lattice of images??

Lemma I.38. *Let R be a relation on (A, B), $X \subseteq A$ and $Y \subseteq B$. Then*

1. $X_R = {}_{R^{\mathrm{T}}}X$;
2. ${}_RX = X_{R^{\mathrm{T}}}$.

In particular $xR = R^{\mathrm{T}}x$ and $Ry = yR^{\mathrm{T}}$ for all $x \in A$ and $y \in B$.

TODO: atomicity / atomistic ??

Lemma I.39. *Let R be a relation on (A, B), $X \subseteq A$ and $Y \subseteq B$. Then*

1. $X_R = \bigcup_{x \in X} xR = \bigcup \{xR \mid x \in X\}$;
2. ${}_RY = \bigcup_{y \in Y} Ry = \bigcup \{Ry \mid y \in Y\}$.

Corollary I.39.1. *Let R be a relation on (A, B) and $X, Y \subset A$. Then*

1. *if $X \subseteq Y$, then $X_R \subseteq Y_R$;*
2. *if $X \subseteq Y$, then ${}_RX \subseteq {}_RY$.*

Proof. (1) We can write $Y = X \cup (Y \setminus X)$. Then

$$Y_R = \bigcup \{xR \mid x \in Y\} = \bigcup \{xR \mid x \in X\} \cup \{xR \mid x \in (Y \setminus X)\} = X_R \cup (Y \setminus X)_R \supseteq X_R.$$

(2) Similar. ☐

Corollary I.39.2. *Let R be a relation on (A, B), $X, Y \subset A$ and $Z, W \subset B$. Then*

1. $(X \cup Y)_R = X_R \cup Y_R$;
2. $(X \cap Y)_R \subseteq X_R \cap Y_R$;
3. $(X \setminus Y)_R \supseteq X_R \setminus Y_R$;
4. $(X \,\Delta\, Y)_R \supseteq X_R \,\Delta\, Y_R$.

and

1. ${}_R(Z \cup W) = {}_RZ \cup {}_RW$;
2. ${}_R(Z \cap W) \subseteq {}_RZ \cap {}_RW$;
3. ${}_R(Z \setminus W) \supseteq {}_RZ \setminus {}_RW$;
4. ${}_R(Z \,\Delta\, W) \supseteq {}_RZ \,\Delta\, {}_RW$.

Proof.

(1) $$\begin{aligned}(X \cup Y)_R &= \bigcup \{xR \mid x \in (X \cup Y)\} = \bigcup \{xR \mid x \in X\} \cup \{xR \mid x \in Y\} \\ &= \Big(\bigcup \{xR \mid x \in X\}\Big) \cup \Big(\bigcup \{xR \mid x \in Y\}\Big) = X_R \cup Y_R.\end{aligned}$$

(2) We have both $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, so $(X \cap Y)_R \subseteq X_R$ and $(X \cap Y)_R \subseteq Y_R$ by I.39.1. Thus $(X \cap Y)_R \subseteq X_R \cap Y_R$.

TODO: use lattice properties

(3), (4) TODO ☐

2.2.1.2 Left and right bounds

Let R be a relation on (A, B), $X \subseteq A$ and $Y \subseteq B$.

- A right bound (or upper bound) of X under R is an element $b \in B$ such that b is right related to all $x \in X$. We denote the class of right bounds by

$$X^R := \{b \in B \mid \forall x \in X : xRb\} .$$

- A left bound (or lower bound) of Y under R is an element $a \in A$ such that a is left related to all $y \in Y$. We denote the class of left bounds by

$$^R Y := \{a \in A \mid \forall y \in Y : aRy\} .$$

The classes X^R and $^R Y$ are also called polars.[a]
In particular for $X = A$ and $Y = B$:

1. The class A^R is referred to as the top of $(R, (A, B))$.
2. The class $^R B$ is the bottom of $(R, (A, B))$.

---

[a] According to Birkhoff (TODO ref), the term "polar" was chosen due to the link with the polars of conic sections.

Note that the definitions of the polars is similar to the definition of the image/preimage. The only difference is that "∃" is replaced by "∀".
Consequently, we can state results similar to the ones above.

Lemma I.40. *Let R be a relation on (A, B), $X \subset A$ and $Y \subset B$. Then*

1. $X^R = {}^{R^{\mathrm{T}}} X$;
2. ${}^R X = X^{R^{\mathrm{T}}}$.

Lemma I.41. *Let R be a relation on (A, B), $X \subset A$ and $Y \subset B$. Then*

1. $X^R = \bigcap_{x \in X} xR = \bigcap \{xR \mid x \in X\}$;
2. ${}^R Y = \bigcap_{y \in Y} Ry = \bigcap \{Ry \mid y \in Y\}$.

In particular for $x \in A$ and $y \in B$:

1. $\{x\}^R = xR = \{x\}_R$;
2. ${}^R\{y\} = Ry = {}_R\{y\}$.

TODO this only works for empty sets if we relativise the intersection!

Lemma I.42. *Let R be a relation on (A, B) and $X \subseteq A, Y \subseteq B$. Then*

$$Y \subseteq X^R \iff X \times Y \subseteq R.$$

Corollary I.42.1. *Let R be a relation on (A, B) and $X, Y \subset A$. Then*

1. *if $X \subseteq Y$, then $X^R \supseteq Y^R$;*

2. *if* $X \subseteq Y$*, then* ${}^{R}X \supseteq {}^{R}Y$*.*

Proof. (1) We can write $Y = X \cup (Y \setminus X)$. Then

$$Y^R = \bigcap \{xR \mid x \in Y\} = \bigcap \{xR \mid x \in X\} \cap \{xR \mid x \in (Y \setminus X)\} = X^R \cap (Y \setminus X)^R \subseteq X^R.$$

(2) Similar. ☐

Corollary I.42.2. *Let* R *be a relation on* (A, B)*,* $X, Y \subset A$ *and* $Z, W \subset B$*. Then*

1. $(X \cup Y)^R \subseteq X^R \cap Y^R$*;*
2. $(X \cap Y)^R \supseteq X^R \cup Y^R$*;*
3. $(X \setminus Y)^R ? X^R \setminus Y^R$*;*
4. $(X \,\Delta\, Y)^R ? X^R \,\Delta\, Y^R$*.*

and

1. ${}^{R}(Z \cup W) \subseteq {}^{R}Z \cup {}^{R}W$*;*
2. ${}^{R}(Z \cap W) \supseteq {}^{R}Z \cap {}^{R}W$*;*
3. ${}^{R}(Z \setminus W) ? {}^{R}Z \setminus {}^{R}W$*;*
4. ${}^{R}(Z \,\Delta\, W) ? {}^{R}Z \,\Delta\, {}^{R}W$*.*

Proof. (1) We have both $X \cup Y \supseteq X$ and $X \cup Y \supseteq Y$, so $(X \cup Y)^R \subseteq X^R$ and $(X \cup Y)^R \subseteq Y^R$ by I.39.1. Thus $(X \cup Y)^R \subseteq X^R \cap Y^R$.
(2) TODO ref order reversing function on lattice.
(3), (4) TODO ☐

2.2.1.3 Extending the relation to powersets

Let R be a relation on (A, B). Let $X \subseteq A$ and $Y \subseteq B$ be classes. Then we write XRY if

$$\forall x \in X \forall y \in Y : \; xRy.$$

Lemma I.43. *Let* R *be a relation on* (A, B)*. Let* $X \subseteq A$ *and* $Y \subseteq B$ *be classes. Then*

$$XRY \iff X \subseteq {}^{R}Y \iff Y \subseteq X^R.$$

Let R be a relation on (A, B). Let $\mathcal{E} \subseteq \mathcal{P}(A)$. Then we define

- $\mathcal{E}_R := \{X_R \mid X \in \mathcal{E}\}$;
- $\mathcal{E}^R := \{X^R \mid X \in \mathcal{E}\}$.

2.2.1.4 Greatest and least elements

Let (A, R) be a relational structure and $X \subseteq A$ a subset.

- A greatest element, largest element or maximum of X is an upper bound of X that is an element of X. We denote the class of maxima by $\max(X) \coloneqq X^R \cap X$.

- A least element, smallest element or minimum of X is a lower bound of X that is an element of X. We denote the class of minima by $\min(X) \coloneqq {}^R X \cap X$.

We also call

- an element of $\sup(S) \coloneqq \min(X^R) = X^R \cap (X^R)^{R^{\mathrm{T}}}$ a least upper bound, supremum, or join;

- an element of $\inf(S) \coloneqq \max({}^R X) = X^{R^{\mathrm{T}}} \cap (X^{R^{\mathrm{T}}})^R$ a greatest lower bound, infimum, or meet.

Lemma I.44. *If* (A, R) *is an anti-symmetric relational structure and* $X \subseteq A$*, then* $\max(X), \min(X), \sup(X)$ *and* $\inf(X)$ *are either singletons or empty.*

Proof. We prove for $\max(X)$. The other cases follow dually or a fortiori. Let $x, y \in \max(X)$. Then xRy and yRx, so $x = y$ by anti-symmetry. ☐

In this case we use max / min / sup / inf to denote the contents of the singleton rather than the singleton itself. If the set is empty, we say the max / min / sup / inf does not exist.

Lemma I.45. *Let* $(P, \precsim)$ *be a poset and* $R \subseteq S \subseteq P$.

1. *If* $\max(R)$ *and* $\max(S)$ *exists, then* $\max(R) \precsim \max(S)$.
2. *If* $\min(R)$ *and* $\min(S)$ *exists, then* $\min(R) \succsim \min(S)$.

Proof. By definition $\max(S) \succsim x$ for all $x \in S$. Now $\max(R) \in R \subseteq S$, so in particular $\max(R) \precsim \max(S)$. ☐

TODO: set to class?

Lemma I.46. *If* $(P, \prec)$ *is an ordered set and* $S \subseteq P$*, then*

1. $\max(S) \subset \sup(S)$*;*
2. $\min(S) \subset \inf(S)$.

Proof. For (1) we calculate $\max(S) = S \cap S^u \subseteq (S^u)^l \cap S^u = \sup(S)$; (2) is dual.
TODO: post Galois?? ☐

2.2.1.5 Maxima and minima

$X^{R \cup \overline{R}^{\mathrm{T}}} \cap X$ (if anti-asymmetric $= X^{\overline{R}^{\mathrm{T}}} \cap X$).
A maximal element of S is an element $x \in S$ such that no element y is strictly greater than x

2.2.2 Converse relation

Let R be a relation on (A, B). The converse R^{T} of R is the relation on $B \times A$ with graph

$$\operatorname{graph}(R^{\mathrm{T}}) = \{(y, x) \mid (x, y) \in \operatorname{graph}(R)\} \subset B \times A.$$

It is also known as the inverse, transpose, reciprocal, opposite or dual of R.

Lemma I.47. *Let R, S be relations on (A, B), $X \subseteq A$ and $Y \subseteq B$. Then*

1. $(R^{\mathrm{T}})^{\mathrm{T}} = R$;
2. $(R \cup S)^{\mathrm{T}} = R^{\mathrm{T}} \cup S^{\mathrm{T}}$;
3. $(R \cap S)^{\mathrm{T}} = R^{\mathrm{T}} \cap S^{\mathrm{T}}$;
4. $\operatorname{dom}(R^{\mathrm{T}}) = \operatorname{codom}(R)$ *and* $\operatorname{codom}(R^{\mathrm{T}}) = \operatorname{dom}(R)$;
5. ${}_{R^{\mathrm{T}}}X = X_R$ *and* $Y_{R^{\mathrm{T}}} = {}_RY$;
6. $\operatorname{im}(R^{\mathrm{T}}) = \operatorname{preim}(R)$;
7. *if* $R \subseteq S$, *then* $R^{\mathrm{T}} \subseteq S^{\mathrm{T}}$.

Lemma I.48. *Let A, B be classes. Then*

1. $U_{A,B}^{\mathrm{T}} = U_{B,A}$;
2. $E_{A,B}^{\mathrm{T}} = E_{B,A}$;
3. $\operatorname{id}_A^{\mathrm{T}} = \operatorname{id}_A$.

2.2.3 Complementary relation

Let R be a binary relation on (A, B). The complementary relation $\overline{R}$ of R is the relation on (A, B) with graph

$$\operatorname{graph}(\overline{R}) = \{(x, y) \mid \neg xRy\}.$$

Lemma I.49. *Let R, S be binary relations.*

1. $\overline{\overline{R}} = R$;
2. $\overline{R^{\mathrm{T}}} = \overline{R}^{\mathrm{T}}$;
3. $\overline{R \cup S} = \overline{R} \cap \overline{S}$;
4. $\overline{R \cap S} = \overline{R} \cup \overline{S}$;
5. $U = R \cup \overline{R}$;
6. *if* $R \subseteq S$, *then* $\overline{R} \supseteq \overline{S}$.

Lemma I.50. *Let R be a relation on (A, B), $x \in A$ and $y \in B$. Then*

1. $x\overline{R} = B \setminus (xR)$;

2. *$\overline{R}y = A \setminus (Ry)$.*

If $X \subseteq A$ and $Y \subseteq B$. Then

3. *$X_{\overline{R}} = B \setminus X^R$;*

4. *${}_{\overline{R}}Y = A \setminus {}^R Y$.*

Proof. (1) We calculate

$$y \in x\overline{R} \iff \neg xRy \iff \neg(y \in xR) \iff y \in B \setminus (xR).$$

(2) Similar.
(3) We calculate, using (1),

$$X_{\overline{R}} = \bigcup_{x \in X} x\overline{R} = \bigcup_{x \in X} B \setminus (xR) = B \setminus \left(\bigcap_{x \in X} xR \right) = B \setminus X^R.$$

(4) Similar. □

Corollary I.50.1. *Let $X \subseteq A$ be classes. Then*

$$X^c = X^{\overline{\mathrm{id}_A}},$$

where the complement is taken with respect to A.

Proof. We have $X = X_{\mathrm{id}_A} = (X^{\overline{\mathrm{id}_A}})^c$. □

2.2.4 Composition of relations

Let R be a relation on (A, B) and S a relation on (B, C). Then the composition of R and S is a new relation $R;S$ on (A, C) with graph

$$\operatorname{graph}(R;S) = \{(x, z) \in A \times C \mid \exists y \in B : xRy \wedge ySz\}.$$

If R and S are relations such that the codomain of one is the domain of the other, they are called composable.
If R and S are composable we also define the notation

$$S \circ R :\!= R;S.$$

Lemma I.51. *Let R, S, T be composable relations.*

1. *The composition is associative: $R;(S;T) = (R;S);T$.*

2. *$(R;S)^{\mathrm{T}} = S^{\mathrm{T}};R^{\mathrm{T}}$;*

3. *$(R \cup S);T = (R;T) \cup (S;T)$;*

4. *$(R \cap S);T \subseteq R;T \cap S;T$.*

Proof. TODO □

TODO: equality for $\cap$ with functions!!

Lemma I.52. *Let R, S be composable relations. Then for all x, y*

1. $x(R;S)y \iff xR \# Sy$;
2. $x(\overline{R;S})y \iff xR \perp Sy$
   $\iff xR \subseteq \overline{S}y$;
3. $x(\overline{R;\overline{S}})y \iff xR \subseteq Sy$.

Proof. TODO ☐

Corollary I.52.1. *Let R, S be composable relations. If R is right unique, then $\overline{R;\overline{S}} = R;S$.*

Lemma I.53. *Let A, B, C, D be relations such that $A \subseteq B$ and $C \subseteq D$ and both A, B and C, D are composable, then $A;C \subseteq B;D$.*

Proof. Assume the hypotheses of the lemma and let $(x, y) \in \operatorname{graph}(A;C)$. Then there exists a z such that xAz and zCy. By hypothesis this means xBz and zDy, so $x(B;D)y$. ☐

Lemma I.54. *Let R, S be composable and T a relation. Then*

$$R;S \subseteq T \qquad \iff \qquad \forall x, y, z : \; xRy \wedge ySz \implies xTz.$$

Lemma I.55. *Let A, B, X, Y be classes and R a relation on (A, B). Then*

1. $\mathrm{id}_A; R = R = R; \mathrm{id}_B$;
2. *if* $S, T \subseteq A$, *then* $\mathrm{id}_S; \mathrm{id}_T = \mathrm{id}_{S \cap T}$;
3. $E_{X,A}; R = E_{X,B}$ *and* $R; E_{B,Y} = E_{A,Y}$;
4. $U_{X,A}; R = \mathrm{U}_{X,A_R}$ *and* $R; U_{B,Y} = U_{{}_R B,Y}$;
5. $U_{X,A}; R; U_{B,Y} = \begin{cases} U_{X,Y} & \operatorname{graph}(R) \neq \emptyset \\ E_{X,Y} & \operatorname{graph}(R) = \emptyset. \end{cases}$

Lemma I.56. *Let R be a relation. Then*

1. $\mathrm{id}_{\operatorname{preim}(R)} \subseteq R; R^{\mathrm{T}} \subseteq U_{\operatorname{preim}(R)}$;
2. $\mathrm{id}_{\operatorname{im}(R)} \subseteq R^{\mathrm{T}}; R \subseteq U_{\operatorname{im}(R)}$.

Corollary I.56.1. *Let R be a relation on (A, B), then*

1. $R \subseteq R; R^{\mathrm{T}}; R$;
2. $(\mathrm{id}_A \cap R; R^{\mathrm{T}}); R = R = R; (\mathrm{id}_B \cap R^{\mathrm{T}}; R)$.

Corollary I.56.2. *Let R be a relation on (A, B). Then*

1. $\mathrm{id}_A \;\subseteq\; R; R^{\mathrm{T}} \cup \overline{R}; \overline{R}^{\mathrm{T}}$;
2. $\mathrm{id}_A \;\subseteq\; R^{\mathrm{T}}; R \cup \overline{R}^{\mathrm{T}}; \overline{R}$.

Lemma I.57. *Let R be a relation on (A, B) and S a relation on (B, C). Let $X \subset A$ and $Y \subset C$. Then*

1. $X_{R;S} = (X_R)_S = X_{S \circ R}$;
2. ${}_{R;S}Y = {}_R({}_SY) = {}_{S\circ R}Y$.

Lemma I.58. *Let R be a relation on (A, B) and S a relation on (B, C). Let $X \subseteq A$. Then $(X^R)_S \subseteq X^{R;S}$*

Proof. We have

$$z \in (X^R)_S \iff \Big[\exists y \in B : \forall x \in X : xRy \wedge ySz\Big] \implies \Big[\forall x \in X : \exists y \in B : xRy \wedge ySz\Big] \iff \Big[\forall x \in X : x(R;S)z\Big] \iff$$

☐

Proposition I.59 (Dedekind formula)**.** *Let R, S, T be compatible relations. Then*

$$(R;S) \cap T \subseteq (R \cap (T;S^{\mathrm{T}}));(S \cap (R^{\mathrm{T}};T)).$$

Proof. Take $(x, z) \in \operatorname{graph}((R;S) \cap T)$. Then xTy and $xR \,\#\, Sy$, meaning we can take a $z \in xR \cap Sy$, i.e. satisfying xRz and zSy. It is then easy to show that $x(R \cap (T;S^{\mathrm{T}}))z$ and $z(S \cap (R^{\mathrm{T}};T))y$. ☐

Lemma I.60. *Let R, S, T, Q be relations. Then $R^{\mathrm{T}};S \subseteq T$ implies $R;Q \cap S \subseteq R;(Q \cap T)$.*

Let R be a homogeneous relation on a class A. Then we can define R^n as the n-fold composition of R:

$$R^n \coloneqq \underbrace{R;R;\ldots;R}_{n \text{ times}}.$$

We call R idempotent if $R^2 = R$.

TODO: refine

Proposition I.61.

- *transitive equivalent with* $\operatorname{graph}(R^2) \subseteq \operatorname{graph}(R)$
- *reflexive implies* $\operatorname{graph}(R) \subseteq \operatorname{graph}(R^2)$

so preorder sufficient, but not necessary for idempotent.

2.2.4.1 Left and right residuals

Let R, S be relations.

- If R, S have the same codomain, we define the right residual as $R \mathbin{/} S \coloneqq \overline{\overline{R};S^{\mathrm{T}}}$.
- If R, S have the same domain, we define the left residual as $S \mathbin{\backslash} R \coloneqq \overline{S^{\mathrm{T}};\overline{R}}$.

Lemma I.62. *Let R, S, T be relations. Then*

1. $(R \mathbin{\backslash} S)^{\mathrm{T}} = R^{\mathrm{T}} \mathbin{/} S^{\mathrm{T}}$ *and* $(R \mathbin{/} S)^{\mathrm{T}} = R^{\mathrm{T}} \mathbin{\backslash} S^{\mathrm{T}}$;
2. $R^{\mathrm{T}} \mathbin{\backslash} S = \overline{R} \mathbin{/} \overline{S}^{\mathrm{T}}$ *and* $R^{\mathrm{T}} \mathbin{/} S = \overline{R} \mathbin{\backslash} \overline{S}^{\mathrm{T}}$;
3. $R \mathbin{\backslash} (T \cap S) = R \mathbin{\backslash} T \cap R \mathbin{\backslash} S$ *and* $(T \cap S) \mathbin{/} R = T \mathbin{/} R \cap S \mathbin{/} R$;

4. *$R\backslash(T\cup S) = R\backslash T \cup R\backslash S$ and $(T\cap S)/R = T/R \cup S/R$;*
5. *$(R\cap S)\backslash T = R\backslash T \cup S\backslash T$ and $T/(R\cap S) = T/R \cup T/S$;*
6. *$(R\cup S)\backslash T = R\backslash T \cap S\backslash T$ and $T/(R\cup S) = T/R \cap T/S$;*
7. *if $R\subseteq T$, then $R/S \subseteq T/S$ and $S\backslash R \subseteq S\backslash T$;*
8. *if $S\subseteq T$, then $R/S \supseteq R/T$ and $S\backslash R \supseteq T\backslash R$.*

As an aide-mémoire: the residuals are monotone in the "numerator" and antitone in the "denominator", where the numerator and denominator refer the the relation above, resp. below, the line in both residuals. The left residual has the denominator on the left; the right residual has it on the right.

Proposition I.63 (Schröder rule)**.** *Let R, S, T be relations. Then*

$$\begin{array}{ccccc} R;S\subseteq T & \iff & \overline{R}\supseteq \overline{T};S^{\mathrm{T}} & \iff & R\subseteq T/S \\ & \iff & \overline{S}\supseteq R^{\mathrm{T}};\overline{T} & \iff & S\subseteq R\backslash T \end{array}$$

Proof. The second line follows from the first by applying the first to $S^{\mathrm{T}};R^{\mathrm{T}}\subseteq T^{\mathrm{T}}$. The second equivalence is immediate by I.49 and $\overline{\overline{T};S^{\mathrm{T}}} = T/S$. For the first equivalence we only need to prove the direction ⇒: applying this implication to $\overline{T};S^{\mathrm{T}}\subseteq\overline{R}$ gives $\overline{\overline{T}}\supseteq\overline{\overline{R}};S^{\mathrm{TT}}$. i.e. $R;S\subseteq T$. So assume $R;S\subseteq T$. By I.54 this is equivalent to

$$\forall x,y,z:\ xRy\wedge ySz \implies xTz.$$

Fix arbitrary x,y,z. Assume ySz and $\neg xTz$. This means we must have $\neg xRy$, or we could also derive xTz, leading to a contradiction. Thus $ySz\wedge x\overline{T}z$ imply $x\overline{R}y$. Using I.54 again, we get $\overline{T};S^{\mathrm{T}}\subseteq\overline{R}$. □

We can also give a proof using image and preimage classes.

Proof. As before it is enough to prove the first implication. So assume $R;S\subseteq T$; we want to prove $R\subseteq T/S = \overline{\overline{T};S^{\mathrm{T}}}$.
Take x,y such that xRy. Then $yS\subseteq xT$, because

$$\forall z:\quad ySz \implies xRy\wedge ySz \implies x(R;S)z \implies xTz.$$

We have the following equivalences:

$$yS\subseteq xT \iff S^{\mathrm{T}}y\subseteq xT \iff \overline{S^{\mathrm{T}}y}\supseteq\overline{xT} \iff x(\overline{\overline{T};S^{\mathrm{T}}})y,$$

using I.52 for the last equivalence. □

Corollary I.63.1. *Let T, S be relations. Then*

1. *$(R/S);S\subseteq R$ and $S;(S\backslash R)\subseteq R$;*
2. *$(R;S)/S\supseteq R$ and $R\backslash(R;S)\supseteq S$;*
3. *$((R;S)/S);S = R;S$ and $R;(R\backslash(R;S)) = R;S$.*

Proof. (1) Setting R in the proposition to $R\diagup S$, we get that the truth $R\diagup S \subseteq R\diagup S$ implies $(R\diagup S); S \subseteq R$. The case for $S\diagdown R$ is similar.
(2) Now we set T in the proposition to $R; S$.
(3) This is a combination of (1) and (2): Set the R in (1) to $R; S$ to get $((R; S)\diagup S); S \subseteq R; S$. From (2) we see that $(R; S)\diagup S \supseteq R$, so $((R; S)\diagup S); S \supseteq R; S$. ☐

Corollary I.63.2. *Let R, S, T be relations. Then*

$$\overline{R}^{\mathrm{T}}; \overline{S}^{\mathrm{T}} \subseteq T \quad\iff\quad \overline{S}^{\mathrm{T}}; \overline{T}^{\mathrm{T}} \subseteq R \quad\iff\quad \overline{T}^{\mathrm{T}}; \overline{R}^{\mathrm{T}} \subseteq S.$$

Suppose we have relations R and T with the same domain and we are interested in finding a relation X such that

$$R; X = T.$$

It will not always be possible to find such an X. It is, however, always possible to find an X such that $R; X \subseteq T$ (for example we could take the empty relation). The Schröder rule says that $R\diagdown T$ is the largest X satisfying this inequality (i.e. for all such X we have $X \subseteq R\diagdown T$). So, if the equation $R; X = T$ has a solution, then it must be the left residual $X = R\diagdown T$. There is a similar result for the right residual.

Corollary I.63.3. *Let R, T be relations and suppose they have the same domain. Then*

$$\begin{aligned} \text{There exists an } X \text{ such that } R; X = T \quad &\iff\quad R; (R\diagdown T) = T \quad\iff\quad R; (R\diagdown T) \supseteq T \\ &\implies\quad X = R\diagdown T. \end{aligned}$$

Suppose R and T have the same codomain, then

$$\begin{aligned} \text{There exists an } X \text{ such that } X; R = T \quad &\iff\quad (T\diagup R); R = T \quad\iff\quad (T\diagup R); R \supseteq T \\ &\implies\quad X = T\diagup R. \end{aligned}$$

2.2.4.2 Symmetric quotient

Let R, S be composable relations. We define the symmetric quotient of R and S as

$$R \obar S := \overline{R; \overline{S}} \;\cap\; \overline{\overline{R}; S}.$$

Usually in the literature the first argument of the symmetric quotient transposed, i.e. it is defined as $R^{\mathrm{T}} \obar S$.

Lemma I.64. *Let R, S be composable relations. Then*

$$\begin{aligned} R \obar S &= R^{\mathrm{T}}\diagdown S \cap R\diagup S^{\mathrm{T}} \\ &= \overline{R}\diagup \overline{S}^{\mathrm{T}} \cap \overline{R}^{\mathrm{T}}\diagdown \overline{S}. \end{aligned}$$

Lemma I.65. *Let R, S be composable relations. Then*

$$x(R \obar S)y \qquad\iff\qquad xR = Sy.$$

Proof. Immediate from I.52. ☐

2.2.5 Restrictions and extensions

Let R be a relation on (A, B), $X \subseteq A$ and $Y \subseteq B$. The restriction of R to (X, Y) is the relation $R|_X^Y$ on (X, Y) with graph

$$\operatorname{graph}(R|_X^Y) = \operatorname{graph}(R) \cap (X \times Y).$$

- If $Y = B$, then the restriction is called the left-restriction of R to X and denoted $R|_X$.
- If $X = A$, then the restriction is called the right-restriction of R to Y and denoted $R|^Y$.

If S is a restriction of R, then R is called an extension of S.

Lemma I.66. *Let R be a relation on (A, B), $X \subseteq A$ and $Y \subseteq B$. Then*

$$R|_X^Y = \mathrm{id}_X; R; \mathrm{id}_Y .$$

Corollary I.66.1. *Let R be a relation on (A, B), $X_1, X_2 \subseteq A$ and $Y_1, Y_2 \subseteq B$. Then*

$$\left(R|_{X_1}^{Y_1}\right)\Big|_{X_2}^{Y_2} = R|_{X_1 \cap X_2}^{Y_1 \cap Y_2}.$$

Proof. Use I.55. □

Lemma I.67. *Let R be a relation on (A, B), $X \subseteq A$ and $Y \subseteq B$. Then*

1. $X_R = \operatorname{im}(R|_X) = \operatorname{im}(\mathrm{id}_X; R)$*;*
2. ${}_RY = \operatorname{preim}(R|^Y) = \operatorname{preim}(R; \mathrm{id}_Y)$*.*

2.2.6 Galois connections

Proposition I.68. *Consider a monoid M of relations under composition. Then for $R \in M$ the following maps form a Galois connection:*

$$\rho_R : M \to M : S \mapsto S; R \qquad \text{and} \qquad \rho_R^+ : M \to M : S \mapsto S \diagup R$$

as do the following:

$$\lambda_R : M \to M : S \mapsto R; S \qquad \text{and} \qquad \lambda_R^+ : M \to M : S \mapsto R \diagdown S.$$

Proof. The is just a restatement of I.63.1. □

Proposition I.69. *We can order relations by image, $\leq_i$ or by preimage $\leq_p$:*

$$R \leq_i S \quad \Leftrightarrow_{def} \quad \operatorname{im}(R) \subseteq \operatorname{im}(S) \qquad R \leq_p S \quad \Leftrightarrow_{def} \quad \operatorname{preim}(R) \subseteq \operatorname{preim}(S).$$

These orders are preorders, but not partial orders.

1. *If we order relations by image, then $X \mapsto \mathrm{id}_X$ and* im *form a Galois connection;*
2. *If we order relations by preimage, then $X \mapsto \mathrm{id}_X$ and* preim *form a Galois connection.*

Lemma I.70. *Let R be a relation on (A, B), $X \subseteq A$ and $Y \subseteq B$, then*

1. *$X_R = \mathrm{im}(\mathrm{id}_X; R)$ and ${}_RX = \mathrm{preim}(R; \mathrm{id}_X)$;*
2. *$X^R = B \setminus \mathrm{im}(\overline{\mathrm{id}_X}; R)$ and ${}^RX = A \setminus \mathrm{preim}(R; \overline{\mathrm{id}}_X)$.*

Proposition I.71. *Let R be a relation on (A, B). Then*

1. *$\mathcal{P}(A) \to \mathcal{P}(B) : X \to X^R$ and $\mathcal{P}(B) \to \mathcal{P}(A) : X \to {}^RX$ form a Galois connection;*
2. *$\mathcal{P}(A) \to \mathcal{P}(B) : X \to X^{\overrightarrow{R}}$ and $\mathcal{P}(B) \to \mathcal{P}(A) : X \to {}^{\overleftarrow{R}}X$ form a Galois connection.*

2.2.6.1 Closures

TODO use Galois theory

Let R be a homogeneous relation on a class A.

- The reflexive closure of R is the relation $R^=$ on A with graph

$$\begin{aligned}\mathrm{graph}(R^=) &= \bigcap \{\mathrm{graph}(R') \mid R' \text{ extends } R \text{ and is reflexive}\} \\ &= \{(x, x) \in A \times A \mid x \in A\} \cup \mathrm{graph}(R).\end{aligned}$$

- The reflexive reduction of R is the relation $R^{\neq}$ on A with graph

$$\mathrm{graph}\, R^{\neq} = \mathrm{graph}(R) \setminus \{(x, x) \in A \times A \mid x \in A\}.$$

- The transitive closure of R is the relation R^+ on A with graph

$$\mathrm{graph}(R^+) = \bigcap \{\mathrm{graph}(R') \mid R' \text{ extends } R \text{ and is transitive}\}.$$

- The symmetric closure of R is the relation $R^{\leftrightarrow}$ on A with graph

$$\begin{aligned}\mathrm{graph}(R^{\leftrightarrow}) &= \bigcap \{\mathrm{graph}(R') \mid R' \text{ extends } R \text{ and is symmetric}\} \\ &= \mathrm{graph}(R) \cup \mathrm{graph}(R^{\mathrm{T}}).\end{aligned}$$

Lemma I.72. *Let R be a homogeneous relation on a class A.*

1. *The reflexive closure $R^=$ is the smallest reflexive relation on A that extends R.*
2. *The reflexive reduction $R^{\neq}$ is the largest irreflexive relation on A that is a restriction of R.*
3. *The transitive closure R^+ is the smallest transitive relation on A that extends R.*
4. *The symmetric closure $R^{\leftrightarrow}$ is the smallest symmetric relation on A that extends R.*

Lemma I.73. *The closures (and reduction) are monotone: if R extends S, then R^a extends S^a for all $a, b \in \{=, +, \leftrightarrow, \neq\}$.*

Lemma I.74. *Let R be a homogeneous relation over a class A. Then all closures commute, i.e.*

$$R^a = R^b \qquad \forall a, b \in \{=, +, \leftrightarrow\}.$$

Proof. Only the equations involving R^+ are non-trivial. For example take $(R^{\leftrightarrow})^+ = (R^+)^{\leftrightarrow}$. Now $(R^+)^{\leftrightarrow}$ is transitive, because transitivity is preserved under taking the symmetric closure, and contains $R^{\leftrightarrow}$, so $(R^{\leftrightarrow})^+$ is extended by $(R^+)^{\leftrightarrow}$.
Conversely, take an $x \in (R^+)^{\leftrightarrow}$. Then either $x \in R^+$ or $x \in (R^{\mathrm{T}})^+$ and both $R^+ \subseteq (R^{\leftrightarrow})^+$ and $(R^{\mathrm{T}})^+ \subseteq (R^{\leftrightarrow})^+$. So $(R^+)^{\leftrightarrow}$ is extended by $(R^{\leftrightarrow})^+$ □

Lemma I.75. *Let R be a homogeneous relation over a class A.*

1. *The reflexive transitive closure R^* is the smallest preorder containing R.*
2. *The reflexive transitive symmetric closure $R^{\equiv}$ is the smallest equivalence relation containing R.*

TODO: equivalence relations are defined below.

2.2.7 Direct product

TODO: extend: heterogeneous relations + direct product of two different relations.

Let R be a homogeneous relation over a class A. We can turn R into a relation over (A, A) as follows:

$$(a, b)R(c, d) \iff aRc \wedge bRd.$$

Lemma I.76. *The following properties of binary endorelations are conserved under taking the direct product:*

1. *all forms of reflexivity;*
2. *all forms of symmetry;*
3. *all forms of transitivity;*
4. *left and right Euclideanness;*
5. *density.*

The following properties are not necessarily conserved:

1. *connexity;*
2. *semi-connexity;*
3. *trichotomy.*

2.2.8 Homogeneous relations

Lemma I.77. *Let R be a homogeneous relation on A. Then*

1. $R \cap \mathrm{id}_A = R^{\mathrm{T}} \cap \mathrm{id}_A \subseteq R^{\mathrm{T}}$*;*
2. $\mathrm{id}_A \cap R \subseteq R \cap R^{\mathrm{T}}$*;*
3. $\mathrm{id}_A \cap R \cap \overline{R}^{\mathrm{T}} \subseteq E_A$*;*
4. $R \cap \overline{R}^{\mathrm{T}} \subseteq \overline{\mathrm{id}_A}$.

Proof. (1) Assume $x(R \cap \mathrm{id}_A)y$, then $x = y$ and xRy. So xRx, meaning $xR^{\mathrm{T}}x$ and thus $xR^{\mathrm{T}}y$.
(2) Clearly $\mathrm{id} \cap R \subseteq R$. Combining this with (1) gives the result.
(3, 4) Follow from I.24.2. □

2.2.8.1 Reflexivity and fixed points

Let R be a homogeneous binary relation on a class A. The fixed points of R are the elements of the class

$$\mathrm{Fp}(R) \coloneqq \{a \in A \mid aRa\}\,.$$

Let R be a homogeneous binary relation on a class A. We say

- R is reflexive if $\mathrm{Fp}(R) = A$;
- R is irreflexive or anti-reflexive if $\mathrm{Fp}(R) = \emptyset$;
- R is quasi-reflexive if every element that is related to some element is related to itself;
- R is left quasi-reflexive if every element that is left related to some element is related to itself;
- R is right quasi-reflexive if every element that is right related to some element is related to itself;
- R is coreflexive if xRy implies $x = y$.

Lemma I.78. *Let R be a homogeneous relation on A. Then R is*

1. *reflexive* if and only if $\mathrm{id}_A \subseteq R$
   if and only if $\mathrm{id}_A \perp \overline{R}$;
2. *irreflexive* if and only if $\mathrm{id}_A \subseteq \overline{R}$
   if and only if $\mathrm{id}_A \perp R$;
3. *left quasi-reflexive* if and only if $\mathrm{id}_{\mathrm{preim}(R)} \subseteq R$*;*
4. *right quasi-reflexive* if and only if $\mathrm{id}_{\mathrm{im}(R)} \subseteq R$*;*
5. *quasi-reflexive* if and only if $\mathrm{id}_{\mathrm{preim}(R)\cup\mathrm{im}(R)} \subseteq R$*;*
6. *coreflexive* if and only if if and only if $R \subseteq \mathrm{id}_A$
   if and only if $R \perp \overline{\mathrm{id}}_A$.

Lemma I.79. *Let R be a homogeneous relation on A.*

1. *If R is left quasi-reflexive, then R^{T} is right quasi-reflexive.*
2. *If R is right quasi-reflexive, then R^{T} is left quasi-reflexive.*
3. *If R is reflexive / irreflexive / coreflexive, then R^{T} is too.*

Proof. (1) Assume $\mathrm{id}_{\mathrm{preim}(R)} \subseteq R$, then $\mathrm{id}_{\mathrm{preim}(R)} = \mathrm{id}_{\mathrm{im}(R^{\mathrm{T}})} = \mathrm{id}^{\mathrm{T}}_{\mathrm{im}(R^{\mathrm{T}})}$. So $\mathrm{id}_{\mathrm{im}(R^{\mathrm{T}})} \subseteq R^{\mathrm{T}}$.
(2) Similar.
(3) Follow simply because the converse preserves inclusions. ☐

Lemma I.80. *Let R be a homogeneous relation on A. Then R is reflexive* if and only if $\overline{R}$ *is irreflexive.*

Lemma I.81. *Let R be a relation. Then R/R and $R\backslash R$ are reflexive.*

Proof. We have $\mathrm{id}; R = R \subseteq R$, so $\mathrm{id} \subseteq R/R$ by the Schröder rule I.63. The other case is similar. □

2.2.8.2 Transitivity

Let R be a homogeneous binary relation on a class A. We say

- R is transitive if $\forall x, y, z \in A : [xRy \land yRz] \implies xRz$;
- R is intransitive if it is not transitive;
- R is anti-transitive if it is never transitive:

$$\forall x, y \in A : (xRy \land yRz) \implies \neg xRz.$$

Lemma I.82. *Let R be a homogeneous relation on A. Then R is*

1. *transitive* if and only if $R; R \subseteq R$*;*
2. *anti-transitive* if and only if $R; R \subseteq \overline{R}$*.*

Lemma I.83. *Let R be a homogeneous relation on A. If R is transitive / anti-transitive, then R^{T} is too.*

Lemma I.84. 1. *An anti-transitive relation is always irreflexive.*

2. *An irreflexive and left- (or right-) unique relation is always anti-transitive.*
3. *An anti-transitive relation on a class of more than four elements elements is never connex.*

2.2.8.3 Symmetry

Let R be a homogeneous binary relation on a class A. We say

- R is symmetric if $\forall x, y \in A : xRy \implies yRx$;
- R is asymmetric if $\forall x, y \in A : xRy \implies \neg yRx$;
- R is anti-symmetric if $\forall x, y \in A : (xRy \land yRx) \implies x = y$.

Lemma I.85. *Let R be a homogeneous relation on A. Then R is*

1. *symmetric* if and only if $R = R^{\mathrm{T}}$*;*
2. *asymmetric* if and only if $R \subseteq \overline{R}^{\mathrm{T}}$
   if and only if $R = R \cap \overline{R}^{\mathrm{T}}$
   if and only if $R \cap R^{\mathrm{T}} = E_A$;
3. *anti-symmetric* if and only if $R \cap R^{\mathrm{T}} \subseteq \mathrm{id}_A$
   if and only if $R^{\mathrm{T}} \subseteq \overline{R} \cup \mathrm{id}_A$.

Proof. (1) The inclusion $R \subseteq R^{\mathrm{T}}$ follows straight from the definition. The other inclusion is obtained by taking the converse.
(2) The third equation is a consequence of I.24.2.
(3) For the second equivalence we calculate using I.24.2:

$$R \cap R^{\mathrm{T}} \subseteq \mathrm{id}_A \iff R \cap R^{\mathrm{T}} \cap \overline{\mathrm{id}_A} = E_A \iff R^{\mathrm{T}} \subseteq \overline{R \cap \overline{\mathrm{id}_A}} = \overline{R} \cup \mathrm{id}_A\,.$$

□

Corollary I.85.1. *Asymmetry implies anti-symmetry.*

Lemma I.86. *Let R be a homogeneous relation on A.*

1. *If R is symmetric / asymmetric / anti-symmetric, then R^{T} is too.*
2. *If R is symmetric / asymmetric, then $\overline{R}$ is too.*

Lemma I.87. *Let R be a relation. Then*

1. *if R is asymmetric, then R is irreflexive;*
2. *if R is transitive, then R is asymmetric iff R is irreflexive.*

Proof. (1) We have $E_A = R \cap R^{\mathrm{T}} \supseteq \mathrm{id}_A \cap R \supseteq E_A$ from I.77. Thus $\mathrm{id}_A \cap R = E_A$, which means R is irreflexive.
(2) Assume R transitive and irreflexive. Then $R;R \subseteq R \subseteq \overline{\mathrm{id}}_A$. By Schröder's rule, I.63, we have $R;\mathrm{id}_A \subseteq \overline{R}^{\mathrm{T}}$ and thus $R \perp R^{\mathrm{T}}$ by I.24.2. □

Proposition I.88. *Let R be a homogeneous relation. Then R can be decomposed as $R = R_S \cup R_A$ where*

1. *$R_S \coloneqq R \cap R^{\mathrm{T}}$ is symmetric; and*
2. *$R_A \coloneqq R \cap \overline{R}^{\mathrm{T}}$ is asymmetric.*

Proof. We have

$$R = R \cap U = R \cap (R^{\mathrm{T}} \cup \overline{R}^{\mathrm{T}}) = (R \cap R^{\mathrm{T}}) \cup (R \cap \overline{R}^{\mathrm{T}}) = R_S \cup R_A.$$

(1) From $R_S^{\mathrm{T}} = (R \cap R^{\mathrm{T}})^{\mathrm{T}} = R^{\mathrm{T}} \cap R = R_S$, we see that R_S is symmetric.
(2) From

$$R_A \cap R_A^{\mathrm{T}} = R \cap \overline{R}^{\mathrm{T}} \cap R^{\mathrm{T}} \cap \overline{R} = (R \cap \overline{R}) \cap (R^{\mathrm{T}} \cap \overline{R}^{\mathrm{T}}) = E_A,$$

we see that R_A is asymmetric. □

Using this decomposition we can rephrase the anti-symmetry property as $R_S \subseteq \mathrm{id}_A$.

2.2.8.4 Connexity

Let R be a homogeneous binary relation on a class A. We say

- R is connex (or connected or complete) if $\forall x, y \in A : xRy \vee yRx$;
- R is semi-connex (or weakly connected or total) if $\forall x, y \in A : xRy \vee yRx \vee x = y$;

- R is trichotomous if $\forall x, y \in A$, exactly one of xRy, yRx or $x = y$ holds.

Lemma I.89. *Let R be a homogeneous relation on A. Then*

1. *R is connex* $\iff U_A = R \cup R^{\mathrm{T}} \iff E_A = \overline{R} \cap \overline{R}^{\mathrm{T}}$
   $\iff \overline{R} \subseteq R^{\mathrm{T}} \iff \overline{R}$ *is asymmetric*;
2. *R is semi-connex* $\iff U_A = R \cup R^{\mathrm{T}} \cup \mathrm{id}_A \iff \overline{\mathrm{id}}_A \subseteq R \cup R^{\mathrm{T}}$ *;*
   $\iff \overline{R} \cap \overline{R}^{\mathrm{T}} \subseteq \mathrm{id}_A \iff \overline{R}$ *is anti-symmetric*
3. *R is trichotomous* $\iff U_A = R \,\Delta\, R^{\mathrm{T}} \,\Delta\, \mathrm{id}_A$

$$\iff \begin{cases} U_A = R \cup R^{\mathrm{T}} \cup \mathrm{id}_A \\ E_A = R \cap R^{\mathrm{T}} \cap \mathrm{id}_A \end{cases} \iff R \cup R^{\mathrm{T}} = \overline{\mathrm{id}}_A$$

$$\iff \begin{cases} U_A = R \cup R^{\mathrm{T}} \cup \mathrm{id}_A \\ E_A = R \cap \mathrm{id}_A \\ E_A = R \cap R^{\mathrm{T}} \end{cases} \iff R \textit{ is } \begin{cases} \textit{semi-connex} \\ \textit{irreflexive} \\ \textit{asymmetric} \end{cases}$$

$$\iff R \textit{ is } \begin{cases} \textit{semi-connex} \\ \textit{asymmetric} \end{cases} \iff \overline{R} \textit{ is } \begin{cases} \textit{anti-symmetric} \\ \textit{connex} \end{cases} .$$

Corollary I.89.1. *Let R be a homogeneous relation. Then*

1. *if R is connex, then R is reflexive.*

Proof. (1) We have

$$R \text{ connex} \iff \overline{R} \text{ asymmetric} \implies \overline{R} \text{ irreflexive} \iff R \text{ reflexive},$$

using I.87 and I.80. ☐

Lemma I.90. *Let R be a homogeneous relation on A. If R is connex / semi-connex / trichotomous, then R^{T} is too.*

2.2.8.5 Euclideanness

Let R be a homogeneous binary relation on a class A. We say

- R is (right) Euclidean if $\forall x, y, z \in A : xRy \wedge xRz \implies yRz$;
- R is left Euclidean if $\forall x, y, z \in A : yRx \wedge zRx \implies yRz$.

Lemma I.91. *Let R be a homogeneous relation on A. Then R is*

1. *Euclidean* if and only if *$R^{\mathrm{T}}; R \subseteq R$;*
2. *left Euclidean* if and only if *$R; R^{\mathrm{T}} \subseteq R$.*

Lemma I.92. *Let R be a homogeneous relation on A. If*

1. *R is right Euclidean, then R^{T} is left Euclidean;*
2. *R is left Euclidean, then R^{T} is right Euclidean.*

2.2.8.6 Density

Let R be a homogeneous binary relation on a class A. We say

- R is dense if $\forall x, y \in A : xRy \implies [\exists z \in A : xRz \wedge zRy]$.

Lemma I.93. *A homogeneous relation R is dense* if and only if $R \subseteq R; R$.

Lemma I.94. *If R is reflexive, then R is dense.*

Proof. Assume R reflexive, i.e. $\mathrm{id}_A \subseteq R$. Then $R = \mathrm{id}_A; R \subseteq R; R$. ☐

Lemma I.95. *Let R be a homogeneous relation on A. If R is dense, then R^{T} is dense.*

2.2.9 Tolerance relations

A tolerance relation is a binary relation that is reflexive and symmetric.
If T is a tolerance relation on a class X, then (X, T) is called a tolerance space.

A tolerance relation expresses the idea of 'resembling' or 'being within tolerance'.

2.2.10 Equivalence relations

An equivalence relation is a binary relation that is reflexive, symmetric and transitive.

Lemma I.96. *Let A, B be classes. Then the empty relation on (A, B), the universal relation $U_{A,B}$ and the identity relation id_A are equivalence relations.*

2.2.10.1 Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Let $x \in A$. The equivalence class of x is the set

$$[x]_\sim \coloneqq x\sim = \sim x = \{y \in A \mid x \sim y\}.$$

Then x is called a representative of this equivalence class.
If A is a set, there is a set of all equivalence classes. This is called the quotient set of A by $\sim$

$$A/\sim \coloneqq \{[x]_\sim \in \mathcal{P}(A) \mid x \in A\}.$$

So the equivalence classes are the principle images / preimages of the equivalence relation.

Proposition I.97. *Let $\sim$ be an equivalence relation on a set A. The quotient set of A by $\sim$ defines a partition of A:*

1. *Every equivalence class is non-empty (each equivalence class has a representative);*
2. *every element of A is in an equivalence class, i.e.*
$$A = \bigcup(A/\sim);$$
3. *any two equivalence classes are either disjoint or the same:*
$$\forall x, y \in A : \qquad [x]_\sim \perp [y]_\sim \quad \vee \quad [x]_\sim = [y]_\sim.$$

Conversely, every partition defines an equivalence relation.

2.2.10.2 Egg-box diagrams and multiple equivalences

Lemma I.98. *Let A be a class and R_1, R_2 equivalence relations on A. Then $R_1 \cap R_2$ is an equivalence relation.*

Let A be a class and R_1, R_2 two equivalence relations. We can draw the elements of A in a grid such that

- two elements are in the same column if and only if they are R_1-equivalent;
- two elements are in the same row if and only if they are R_2-equivalent.

The cells in the grid are then $R_1 \cap R_2$-equivalence classes.
Such a diagram is called an egg-box diagram.

Usually the term "egg-box diagram" specifically refers to the case when R_1, R_2 are the Green's relations $\mathcal{L}$ and $\mathcal{R}$ of a semigroup.

Example

Consider the set $\mathcal{P}(\{1,2,3,4\})$ with equivalence relations

- aR_1b if subsets a and b have the number of elements;
- aR_2b if subsets a and b have the number of even elements.

Then the egg-box diagram of R_1, R_2 is

$\emptyset$	$\{1\}, \{3\}$	$\{1,3\}$		
	$\{2\}, \{4\}$	$\{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}$	$\{1,2,3\}, \{1,3,4\}$	
		$\{2,4\}$	$\{1,2,4\}, \{2,3,4\}$	$\{1,2,3,4\}$

Proposition I.99. *Let A be a class and R_1, R_2 equivalence relations on A. Then the following are equivalent:*

1. *$R_1; R_2$ is symmetric;*
2. *$R_1; R_2 = R_2; R_1$;*
3. *$R_1; R_2$ is an equivalence relation;*
4. *the egg-box diagram of R_1, R_2 decomposes into blocks, i.e.*
    - *a block consists of a rectangular region of cells;*
    - *each cell in the block is occupied;*
    - *no cell not included in the block, but in the same row / column as a cell in the block is occupied;*

    these blocks are $R_1; R_2$-equivalence classes;
5. *for any four cells that are the corners of a rectangle in the egg-diagram: if any three of these cells are occupied, all four cells are occupied.*

In this case $R_1; R_2$ *is the smallest equivalence relation containing* $R_1 \cup R_2$.

TODO $R_1; R_2$ is transitive closure of $R_1 \cup R_2$.

Proof. (1) $\Rightarrow$ (2) We have $R_1; R_2 = (R_1; R_2)^{\mathrm{T}} = R_2^{\mathrm{T}}; R_1^{\mathrm{T}} = R_2; R_1$.
(2) $\Rightarrow$ (3) We verify

- reflexivity: from $\mathrm{id}_A \subseteq R_1$ and $\mathrm{id}_A \subseteq R_2$, we get $\mathrm{id}_A = \mathrm{id}_A^2 \subseteq R_1; R_2$;
- symmetry: we have $(R_1; R_2)^{\mathrm{T}} = R_2^{\mathrm{T}}; R_1^{\mathrm{T}} = R_2; R_1 = R_1; R_2$;
- transitivity: we have

$$(R_1; R_2); (R_1; R_2) = R_1; R_1; R_2; R_1 = R_1; R_2.$$

(3) $\Rightarrow$ (1) Immediate.
(2) $\Leftrightarrow$ $(4, 5)$ On reflection, one may understand that (4) and (5) are equivalent.
First assume (2) and take any three occupied cells that lie on the corners of a rectangle. Take a, b from the diagonally opposing cells. Then $a(R_1; R_2)b$, because there exists a c in the third cell. So also $a(R_2; R_1)b$, meaning there exists an element in the fourth sell, so (5) holds.
We can run the argument in reverse for the converse.
(Final observation) We first show that $R_1 \cup R_2 \subseteq R_1; R_2$. Because $\mathrm{id}_A \subseteq R_1, R_2$, we have

$$R_1 \cup R_2 \subseteq R_1; (R_1 \cup R_2); R_2 = R_1; R_1; R_2 \cup R_1; R_2; R_2 = R_1; R_2.$$

Now assume R is an equivalence class containing $R_1 \cup R_2$, then

$$R \supseteq (R_1 \cup R_2); (R_1 \cup R_2) = R_1 \cup R_1; R_2 \cup R_2; R_1 \cup R_2 = R_1; R_2.$$

☐

2.2.11 Uniqueness and totality

Let R be a relation on (A, B). We call R

- left-total, serial or simply total if $A_R = B$;
- right-total or surjective or onto if $A = {}_R B$;
- left-unique or injective if

$$\forall x_1, x_2 \in A : \forall y \in B : \; x_1 R y \wedge x_2 R y \implies x_1 = x_2;$$

- right-unique or functional if

$$\forall x \in A : \forall y_1, y_2 \in B : \; x R y_1 \wedge x R y_2 \implies y_1 = y_2.$$

We may also call a binary relation

- **one-to-one** if it is injective and functional;
- **one-to-many** if it is injective and not functional;
- **many-to-one** if it is not injective and functional;
- **many-to-many** if it is not injective and not functional.

Lemma I.100. *Let R be a relation. Then*

1. *R is right-unique* if and only if *R^{T} is left-unique;*
2. *R is right-total* if and only if *R^{T} is left-total.*

2.2.11.1 Uniqueness

Lemma I.101. *Let R be a relation. Then*

1. *R is functional* if and only if $R^{\mathrm{T}}; R \subseteq \mathrm{id}_{\mathrm{codom}(R)}$*;*
2. *R is injective* if and only if $R; R^{\mathrm{T}} \subseteq \mathrm{id}_{\mathrm{dom}(R)}$.

We also have

3. *R is functional* if and only if $R^{\mathrm{T}}; R = \mathrm{id}_{\mathrm{im}(R)}$*;*
4. *R is injective* if and only if $R; R^{\mathrm{T}} = \mathrm{id}_{\mathrm{preim}(R)}$.

Proof. (1) Assume that there exist x, y_1, y_2 such that xRy_1 and xRy_2. This means $y_1 R^{\mathrm{T}} x$ and xRy_2, so $y_1(R^{\mathrm{T}}; R)y_2$. Then R is functional iff $y_1 = y_2$ iff $R^{\mathrm{T}}; R \subseteq \mathrm{id}_{\mathrm{codom}(R)}$.
(2) Similar.
(3, 4) Due to the inclusions in I.56. ☐

Corollary I.101.1. *Let R, S be relations. Then*

1. *if R and S are left/right-unique, then $R; S$ is left/right-unique;*
2. *R is right-unique* if and only if $R \subseteq R^{\mathrm{T}} \diagdown \mathrm{id}$*;*
3. *R is left-unique* if and only if $R \subseteq \mathrm{id} \diagup R^{\mathrm{T}}$.

Proof. (1) Assume R and S are right-unique. Then

$$(R; S)^{\mathrm{T}}; (R; S) = S^{\mathrm{T}}; R^{\mathrm{T}}; R; S \subseteq S^{\mathrm{T}}; \mathrm{id}_{\mathrm{dom}(S)}; S = S^{\mathrm{T}}; S \subseteq \mathrm{id}_{\mathrm{codom}(S)} = \mathrm{id}_{\mathrm{codom}(R;S)}\,.$$

(2, 3) Applications of Schröder's rule. ☐

Lemma I.102. *Let R, S, T be composable relations. Then*

1. *if R is right-unique, then $R; (S \cap T) = R; S \cap R; T$*
$$S \cap T; R = (S; R^{\mathrm{T}} \cap T); R$$
2. *if R is left-unique, then $(S \cap T); R = S; R \cap T; R$*
$$R; S \cap T = R; (S \cap R^{\mathrm{T}}; T).$$

Proof. (1a) The inclusion $\subseteq$ is in I.51. For the other inclusion we can use the Dedekind formula I.59:

$$\begin{aligned} R; S \cap R; T &\subseteq (R \cap (R; T; S^{\mathrm{T}})); (S \cap R^{\mathrm{T}}; R; T) \\ &\subseteq R; (S \cap R^{\mathrm{T}}; R; T) \\ &\subseteq R; (S \cap T), \end{aligned}$$

where we have used the right-uniqueness for the last inclusion: $R^{\mathrm{T}}; R; T \subseteq T$.

(1b) We use the Dedekind formula twice:

$$\begin{aligned} S \cap T;R &\subseteq (T \cap S;R^{\mathrm{T}});(R \cap T^{\mathrm{T}};S) \subseteq (T \cap S;R^{\mathrm{T}});R \\ &\subseteq (S \cap T;R);(R^{\mathrm{T}} \cap S^{\mathrm{T}};T);R \subseteq (S \cap T;R);R^{\mathrm{T}};R \\ &\subseteq S \cap T;R, \end{aligned}$$

where we have used the right-uniqueness for the last inclusion: $(S \cap T;R);R^{\mathrm{T}};R \subseteq (S \cap T;R)$.
(2a) The calculation is similar to (1a):

$$\begin{aligned} S;R \cap T;R &\subseteq (S \cap (T;R;R^{\mathrm{T}}));(R \cap S^{\mathrm{T}};T;R) \\ &\subseteq (S \cap (T;R;R^{\mathrm{T}}));R \\ &\subseteq (S \cap T);R. \end{aligned}$$

(2b) The calculation is similar to (1b):

$$\begin{aligned} R;S \cap T &\subseteq (R \cap T;S^{\mathrm{T}});(S \cap R^{\mathrm{T}};T) \subseteq R;(S \cap R^{\mathrm{T}};T) \\ &\subseteq R;(R^{\mathrm{T}} \cap S;T^{\mathrm{T}});(T \cap R;S) \subseteq R;R^{\mathrm{T}};(T \cap R;S) \\ &\subseteq T \cap R;S. \end{aligned}$$

□

Lemma I.103. *Let R, S be composable relations.*

1. *If R is right-unique, then* $\overline{R;S} = R;\overline{S} \cup \overline{R;U}$
   $R;\overline{S} = R;U \cap \overline{R;S}$
   $\overline{R;\overline{S}} = \overline{R;U} \cup R;S.$
2. *If S is left-unique, then* $\overline{R;S} = \overline{R};S \cup \overline{U;S}$
   $\overline{R};S = U;S \cap \overline{R;S}$
   $\overline{\overline{R};S} = \overline{U;S} \cup R;S.$

Lemma I.104. *Let R be a relation on (A, B), $X, Y \subset A$ and $Z, W \subset B$. If R is left-unique, then*

1. $(X \cup Y)_R = X_R \cup Y_R$*;*
2. $(X \cap Y)_R = X_R \cap Y_R$*;*
3. $(X \setminus Y)_R = X_R \setminus Y_R$*;*
4. $(X \mathbin{\Delta} Y)_R = X_R \mathbin{\Delta} Y_R$*.*

If R is right-unique, then

1. ${}_R(Z \cup W) = {}_RZ \cup {}_RW$*;*
2. ${}_R(Z \cap W) = {}_RZ \cap {}_RW$*;*
3. ${}_R(Z \setminus W) = {}_RZ \setminus {}_RW$*;*
4. ${}_R(Z \mathbin{\Delta} W) = {}_RZ \mathbin{\Delta} {}_RW$*.*

TODO: similar for totality and polars?

2.2.11.2 Totality

Lemma I.105. *Let R be a relation on (A, B). Then*

1. *the following are equivalent to R being left-total:*
   (a) $U_{A,B} = R; U_{B,B}$
   (b) $\mathrm{id}_A \subseteq R; R^{\mathrm{T}}$;
   (c) $\overline{R} \subseteq R; \overline{\mathrm{id}}_B$;
   (d) *for all relations S: $S; R = E \implies S = E$;*
2. *the following are equivalent to R being right-total:*
   (a) $U_{A,B} = U_{A,A}; R$
   (b) $\mathrm{id}_B \subseteq R^{\mathrm{T}}; R$;
   (c) $\overline{R} \subseteq \overline{\mathrm{id}}_A; R$;
   (d) *for all relations S: $R; S = E \implies S = E$.*

Proof. $(a \Rightarrow b)$ We calculate using the Dedekind rule:

$$\mathrm{id} = U \cap \mathrm{id} = R; U \cap \mathrm{id} \subseteq (R \cap \mathrm{id}; U^{\mathrm{T}}); (U \cap R^{\mathrm{T}}; \mathrm{id}) = R; R^{\mathrm{T}}.$$

□

2.2.11.3 Kernels

Let R be a relation on (A, B). The kernel of R is

$$\ker R \coloneqq \{(x, y) \in A \times A \mid xR \# yR\}.$$

Lemma I.106. *Let R be a relation on (A, B). Then*

1. $\ker R$ *is a symmetric relation on A;*
2. *if R is left total, then* ker *is a tolerance relation on A;*
3. $\ker R = R; R^{\mathrm{T}} = R^{\mathrm{T}} \circ R$.

2.2.12 Constant relations and points

Let A, B be classes and R a relation on A, B. We call R

- right-constant if $\forall x \in A, \forall y, z \in B : \; xRy \iff xRz$;
- left-constant if $\forall x, y \in A, \forall z \in B : \; xRz \iff yRz$;
- a point if $R = \{x\} \times B$ for some $x \in A$.

Let $x \in A$. We define $\vec{x} \coloneqq \{x\} \times A$.

Lemma I.107. *Let R be a relation. Then R is right-constant* if and only if *R^{T} is left-constant.*

Lemma I.108. *Let R be a relation on (A, B). Then*

1. *R is right-constant* if and only if $R = R; U_{B,B}$*;*
2. *R is left-constant* if and only if $R = U_{A,A}; R$*;*
3. *R is a point* if and only if *R is right-constant, non-zero and injective.*

Lemma I.109. *Let R be a relation on (A, B), $x \in A$ and $y \in B$. Then*

1. *$R; \vec{y}$ is a point;*
2. *$\vec{x}^{\mathrm{T}}; R$ is the converse of a point;*
3. *$Ry = \operatorname{preim}(R; \vec{y})$;*
4. *$xR = \operatorname{im}(\vec{x}^{\mathrm{T}}; R)$.*

2.3 Functions

TODO: equiv for equality!

A function is a binary relation that is functional and serial.

If a relation f on (A, B) is a function, then for every $x \in A$, there is a unique $y \in B$ such that xfy. We write $f(x)$ to denote this unique element.
Conceptually we can rephrase this as saying that for each “input” in the domain A, the function f produces a unique “output” in the codomain B.

We write

$$f : A \to B : x \mapsto f(x)$$

to show f is a function with domain A and codomain B that maps $x \in A$ to $f(x) \in B$.
We say f is a function from A to B or f maps A to B.
If A, B are sets, we can consider the set of all functions from A to B. This set is denoted $(A \to B)$.

Depending on the context, functions may also be called maps or transformations. These are synonyms with slightly different connotations.

Lemma I.110. *Let $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ be functions. Then f_1 and f_2 are the same functions* if and only if

1. *$A_1 = A_2$ and $B_1 = B_2$;*
2. *$\forall x \in A_1 : f_1(x) = f_2(x)$.*

Lemma I.111. *Let f be a relation. Then f is a function* if and only if $f; \overline{\mathrm{id}} = \overline{R}$*.*

Lemma I.112. *Let $f, g : A \to B$ be functions. Then $f = g$* if and only if $\mathrm{id}_A \subseteq f; g^{\mathrm{T}}$*.*

2.3.1 Image and preimage

2.3.1.1 Functions associated to a relation

We can associate to any relation R on (A, B)

- an image function $R^{\downarrow} : \mathcal{P}(A) \to \mathcal{P}(B) : X \mapsto X_R$;
- a preimage function $R^{\text{-}\downarrow} : \mathcal{P}(B) \to \mathcal{P}(A) : X \mapsto {}_R X$;
- a right bounds function $\mathcal{P}(A) \to \mathcal{P}(B) : X \mapsto X^R$;
- a left bounds function $\mathcal{P}(B) \to \mathcal{P}(A) : X \mapsto {}^R X$.

Clearly $R^{\text{-}\downarrow} = (R^{\mathrm{T}})^{\downarrow}$.

Lemma I.113. *Let R, S be composable relations. Then $(R;S)^{\downarrow} = R^{\downarrow}; S^{\downarrow}$.*

Proof. Take some arbitrary $A \subseteq \operatorname{dom}(R)$. Then for all $y \in \operatorname{codom}(S)$ we have

$$\begin{aligned} y \in (R;S)^{\downarrow}(A) &\iff \exists x \in A :\; x(R;S)y \\ &\iff \exists x \in A : \exists z \in \operatorname{dom}(S) = \operatorname{codom}(R) :\; xRz \;\wedge\; zSy \\ &\iff \exists z \in \operatorname{dom}(S) = \operatorname{codom}(R) : \exists x \in A :\; xRz \;\wedge\; zSy \\ &\iff \exists z \in R^{\downarrow}(A) : zSy \\ &\iff y \in S^{\downarrow}(R^{\downarrow}(A)) = (R^{\downarrow}; S^{\downarrow})(A). \end{aligned}$$

□

Corollary I.113.1. *For any relation R, $\ker(R)^{\downarrow} = R^{\downarrow}; R^{\text{-}\downarrow} = R^{\text{-}\downarrow} \circ R^{\downarrow}$.*

2.3.1.2 Applying functions inside sets

As functions are relations they have images, preimages and associated image and preimage functions.

Lemma I.114. *Let A, B be classes and $f : A \to B$ a function. Then*

1. *$f^{\downarrow} : \mathcal{P}(A) \to \mathcal{P}(B) : X \mapsto \{f(x) \in B \mid x \in X\}$;*
2. *$f^{\text{-}\downarrow} : \mathcal{P}(B) \to \mathcal{P}\,A : Y \mapsto \{x \in A \mid f(x) \in Y\}$.*

Image and preimage functions are also functions and thus also have images, preimages and associated image and preimage functions.
We have, for a function $f : A \to B$,

- $f^{\downarrow\downarrow} := (f^{\downarrow})^{\downarrow} : \mathcal{P}^2(A) \to \mathcal{P}^2\, B$;
- $f^{\text{-}\downarrow\text{-}\downarrow} := (f^{\text{-}\downarrow})^{\text{-}\downarrow} : \mathcal{P}^2(A) \to \mathcal{P}^2\, B$;
- $f^{\downarrow\text{-}\downarrow} := (f^{\downarrow})^{\text{-}\downarrow} : \mathcal{P}^2(B) \to \mathcal{P}^2\, A$;
- $f^{\text{-}\downarrow\downarrow} := (f^{\text{-}\downarrow})^{\downarrow} : \mathcal{P}^2(B) \to \mathcal{P}^2\, A$;

Note that in general $f^{\downarrow\text{-}\downarrow} \neq f^{\text{-}\downarrow\downarrow}$ and $f^{\text{-}\downarrow\text{-}\downarrow} \neq f^{\downarrow\downarrow}$.

Proposition I.115. *Let A, B be classes and $f : A \to B$ a function. Then*

1. *$(f^{\downarrow}, f^{-\downarrow})$ is a Galois connection between $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(B), \subseteq)$;*
2. *$(f^{-\downarrow}, f^{\downarrow})$ is a Galois connection between $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(B), \subseteq)$;*
3. *$(f^{\downarrow}, f^{-\downarrow})$ and $(f^{-\downarrow}, f^{\downarrow})$ are Galois connections between $\big(\mathcal{P}(A), \mathrm{id}_{\mathcal{P}(A)}\big)$ and $\big(\mathcal{P}(B), \mathrm{id}_{\mathcal{P}(B)}\big)$.*

Proof. We have

$$f^{\downarrow}(X) \subseteq Y \iff \forall x \in X : \exists y \in Y : \; f(x) = y \iff X \subseteq f^{-\downarrow}(Y)$$

and

$$f^{\downarrow}(X) \supseteq Y \iff \forall y \in Y : \exists x \in X : \; f(x) = y \iff X \supseteq f^{-\downarrow}(Y).$$

□

Lemma I.116. *Let A, B be classes, $f : A \to B$ a function, $x \in A$ and $Y \subseteq B$. Then $x \in f^{-\downarrow}(Y) \iff f(x) \in Y$.*

Lemma I.117. *Let f be a function. Then $f^{-\downarrow}\big(\biguplus \mathcal{E}\big) = \biguplus f^{-\downarrow}(\mathcal{E})$.*

Lemma I.118. *Let A, B, C be classes, $f : A \to B$ a function and R a relation on (B, C). Let $X \subseteq A$. Then $f^{\downarrow}(X)^R = X^{f;R}$.*

Proof. We calculate

$$(X_f)^R = (X_{f;\overline{R}})^c = X^{\overline{f;\overline{R}}} = X^{f;R \cup \overline{f;U_{B,C}}} = X^{f;R},$$

where we have used the uniqueness and totality of f in I.103 and I.105.
Alternatively we can give the following calculation:

$$y \in f^{\downarrow}(X)^R \iff \forall x \in X : f(x) R y \iff \forall x \in X : x(f;R)y \iff y \in X^{f;R}.$$

□

2.3.2 Injectivity, surjectivity and bijectivity

> The terms injective and surjective are commonly applied to functions. A function that is both injective and surjective is bijective.

In the context of functions the notions of injectivity and one-to-one coincide.

Lemma I.119. *Let $f : A \to B$ be a function. We say*

- *f is injective, denoted $f : A \rightarrowtail B$, if*

$$\forall x_1, x_2 \in A : f(x_1) = f(x_2) \implies x_1 = x_2;$$

- *f is surjective, denoted $f : A \twoheadrightarrow B$, if*

$$\forall y \in B : \exists x \in A : f(x) = y;$$

- *f is bijective, denoted $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$, if*

$$\forall y \in B : \exists! x \in A : f(x) = y;$$

We also say f is an injection, a surjection or a bijection if it is injective, surjective or bijective, respectively.
We will also sometimes write $A \leftrightarrow B$, instead of $A \rightarrowtail\!\!\!\!\twoheadrightarrow B$, to denote a bijection between A and B.

Lemma I.120. *If $A \subseteq B$ classes, then there exists a canonical injection $\iota : A \to B$, the inclusion map*

$$\iota : A \to B : a \mapsto a.$$

The inclusion map is often denoted $A \hookrightarrow B$.

Lemma I.121. *Let $f : A \to B$ be a function. Then $\ker f$ is an equivalence relation.*

> Let $f : A \to B$ be a function between sets. An equivalence class $[x]_{\ker f}$ is called a fibre of f.

Proposition I.122. *Let $f : A \to B$ be a function between sets. We can associate to f*

1. *a surjective function $f' : A \to f[A] : x \mapsto f(x)$;*
2. *an injective function $f'' : A/(\ker f) \to B : [x]_{\ker f} \mapsto f(x)$;*
3. *a bijective function $f''' : A/(\ker f) \to f[A] : [x]_{\ker f} \mapsto f(x)$.*

Whenever a function on a quotient set defines an image of an equivalence class using an element of said equivalence class, we need to verify this definition is well-defined, i.e. it does not depend on the chosen element in the equivalence class. (In other words it is properly a function of the equivalence class, not of the elements of the equivalence classes.)

Proof. We show f'' is well-defined. Let $[x]_{\ker f} \in A/(\ker f)$ and let $x_1, x_2 \in [x]_{\ker f}$. Then $f(x) = f(x_1) = f(x_2)$ and

$$f''([x_1]_{\ker f}) = f(x_1) = f(x_2) = f''([x_2]_{\ker f}),$$

so f'' is well-defined. □

2.3.3 Constructing new functions

Constructions defined for relations are in particular applicable to functions.

2.3.3.1 Restriction and extension

Lemma I.123. *Let $f : A \to B$ be a function and $S \subset A$. Then $f|_S$ is a function.*

When talking about the restriction of a function, this left restriction is always the one that is meant. Right restrictions $f|^K$ are sometimes called corestrictions and are in general not functions.

> Let $A \subset B$ and C be classes. Let $f : A \to C$ be a function. A function $\tilde{f} : B \to C$ such that $\tilde{f}|_A = f$ is called an extension of f.

Given a function, a different function with as codomain a superset of the original codomain, which is otherwise identical, is sometimes called a coextension of the function.
When given a function prescription, we may need to verify the function is well-defined in the sense that the prescription always gives an element in the codomain for every element in the domain.

2.3.3.2 Composition

Lemma I.124. *The functions $f : A \to B$ and $g : B \to C$ are composable as relations and the relation*

$$f; g = g \circ f$$

is a function. In particular it has functional form

$$g \circ f : A \to C : x \mapsto g(f(x)).$$

Because of the simplicity of $(g \circ f)(x) = g(f(x))$, the notation $\circ$ is almost exclusively used when dealing with functions.

Let $f, g : A \to A$ be functions. We say f and g commute if $f \circ g = g \circ f$.

Let X, Y, Z be classes. Using composition we can view any function $f : X \to Y$ also as a function

$$f : (Z \to X) \to (Z \to Y) : g \mapsto f(g) = (z \mapsto (f \circ g)(z)).$$

2.3.3.3 Pre- and post-composition

Let A, B, C be sets and $f \in (A \to B)$. Then we define

- the post-composition function of f as $f_* : (C \to A) \to (C \to B) : g \mapsto f \circ g$.
- the pre-composition function of f as $f^* : (B \to C) \to (A \to C) : g \mapsto g \circ f$.

The post-composition function is also known as the pointwise application or extension of f.

Lemma I.125. *Let A, B, C, D be sets, $f \in (A \to B)$ and $g \in (B \to C)$. Then*

1. $(g \circ f)_* = g_* \circ f_*$;
2. $(g \circ f)^* = f^* \circ g^*$.

Proof. (1) Let $h \in (D \to A)$. Then

$$(g \circ f)_*(h) = (g \circ f) \circ h = g \circ (f \circ h) = g_*(f_*(h)) = (g_* \circ f_*)(h).$$

(2) Let $h \in (C \to D)$. Then, similarly,

$$(g \circ f)^*(h) = h \circ (g \circ f) = (h \circ g) \circ f = (g^*(h)) \circ f = f^*(g^*(h)) = (f^* \circ g^*)(h).$$

□

2.3.3.4 Inverses of functions

All identity relations are functions.

Let $f : X \to Y$ be a function.

- A left inverse (or retraction) of f is a function $g : Y \to X$ such that
$$g \circ f = \mathrm{id}_X .$$
- A right inverse (or section) of f is a function $h : Y \to X$ such that
$$f \circ h = \mathrm{id}_Y .$$
- A (two-sided) inverse of f is a function that is both a left and a right inverse.

Lemma I.126. *Let $f : X \to Y$ be a function. The following are equivalent:*

1. *f has a left inverse g;*
2. *f^{T} is a partial function from X to Y;*
3. *$\mathrm{graph}(f^{\mathrm{T}}) \subseteq \mathrm{graph}(g)$;*
4. *f is injective.*

There is also a result linking the existence of a right inverse and surjectivity, but this in general only holds assuming the axiom of choice. (TODO ref)

Lemma I.127. *If f has a left inverse g and a right inverse h, then $g = h$.*

Proof. By the simple calculation
$$g = g \circ (f \circ h) = (g \circ f) \circ h = h.$$
□

Thus the two-sided inverse of f is unique and exists if and only if f has a left and a right inverse. The unique two-sided inverse is denoted f^{-1}.

Lemma I.128. *Let $f : X \to Y$ be a function. The following are equivalent:*

1. *the inverse exists;*
2. *the inverse exists and $f^{-1} = f^{\mathrm{T}}$;*
3. *f^{T} is a function from X to Y;*
4. *f is a bijection;*
5. *f^{T} is a bijective function from X to Y.*

Lemma I.129. *Let $f, g : A \to A$ be commuting and bijective functions. Then f^{-1} and g^{-1} commute.*

Proof. We calculate
$$f^{-1} \circ g^{-1} = f^{-1} \circ g^{-1} \circ (f \circ g \circ g^{-1} \circ f^{-1}) = f^{-1} \circ g^{-1} \circ (g \circ f) \circ g^{-1} \circ f^{-1} = g^{-1} \circ f^{-1}$$
□

2.3.3.5 Constant functions

Let X, Y be classes. A function $f : X \to Y$ is called constant if there exists a y_0 such that $\forall x \in X : f(x) = y_0$.
We denote this function $\underline{y_0}$.

A function is constant if and only if its range is a singleton.

Lemma I.130. *Let X, Y be classes and $y \in Y$. Then*

$$\operatorname{graph}(\underline{y}) = X \times \{y\}.$$

2.3.3.6 Tuples of functions

Let $f : A \to B$ and $g : X \to Y$ be functions. Then the tuple function of f and g is the function

$$(f, g) : A \times X \to G \times Y : (a, x) \mapsto (f(a), g(x)).$$

In particular if $f = g$, we say the tuple function is the pointwise application of f. We write f instead of (f, f).

2.3.4 Partial functions

A partial function is a relation that is functional (but not necessarily serial). If we wish to emphasise a function is both functional and serial, we may call it a total function.

If $f \subset A \times B$ is a partial function, we write $A \not\to B$. The set of all partial functions from A to B is

$$(A \not\to B) = \bigcup_{S \subseteq A} (S \to B).$$

For all $a \in A$ we write

$$f(a)\downarrow \quad \Leftrightarrow_{\text{def}} \quad a \in \operatorname{dom}(f), \qquad f(a)\uparrow \quad \Leftrightarrow_{\text{def}} \quad a \notin \operatorname{dom}(f)$$

We can read $f(a)\downarrow$ as "f converges at a" and $f(a)\uparrow$ as "f diverges at a".

Lemma I.131. *Let $f : A \not\to B$ be a partial function. Then we can construct a total function*

$$\widehat{f} : A \to B^+$$

where $B^+ = B \cup \{e\}$ for some $e \notin B$, such that $\widehat{f}|^B_{\operatorname{preim}(f)} = f|_{\operatorname{preim}(f)}$.

Proof. We can always find an element e by I.5. Then we let $\widehat{f}$ map all elements in $A \setminus \operatorname{preim}(f)$ to e. ☐

Proposition I.132. *Let A, B be sets and $f, g \in (A \not\to B)$. Then the following are equivalent:*

1. $f \subseteq g$;
2. $\ker(f) \subseteq \ker(g)$ *and* $\operatorname{im}(f) \subseteq \operatorname{im}(f \cap g)$;
3. $\ker(f \cup g) \subseteq \ker(g)$ *and* $\operatorname{im}(f) \subseteq \operatorname{im}(g)$.

TODO: give proper meaning to $\ker(f \cup g)$

Proposition I.133. *(1) clearly implies (2) and (3).*
(2) $\Rightarrow$ (1). *Take $(x, f(x)) \in f$. Now $f(x) \in \mathrm{im}(f)$, so $f(x) \in \mathrm{im}(f \cap g)$, meaning there exists a $u \in A$ such that $(u, f(x)) \in f \cap g$. Because $(u, x) \in \ker(f) \subseteq \ker(g)$, we have $(x, f(x)) \in g$.*
(3) $\Rightarrow$ (1). *Take $(x, f(x)) \in f$. Now $f(x) \in \mathrm{im}(f)$, so $f(x) \in \mathrm{im}(g)$ and we can find a $u \in A$ such that $(u, f(x)) \in g$. Then $(x, f(x)) \in f \cup g$ and $(u, f(x)) \in f \cup g$, so $(x, u) \in \ker(f \cup g)$ and thus $(x, u) \in \ker(g)$. This means $(x, f(x)) \in g$.*

2.4 Binary functions

We can easily give a function multiple inputs from multiple domains by first joining them into an n-tuple, i.e. considering the Cartesian product of the domains as the domain of the function.

> Example
>
> A function f that takes an input in A and one in B to generate an output in C can be written as
> $$f : A \times B \to C : (a, b) \mapsto f(a, b).$$

We can consider the class of binary functions between sets.

Lemma I.134. *There does not exist a class of binary functions between classes.*

Proof. There exists a proper class $\mathcal{U}$ by I.5.2. Then

$$\mathcal{U} \times \{\emptyset\} \to \mathcal{U} : (x, \emptyset) \mapsto x$$

is a proper class by replacement (TODO ref! + necesary?) and thus an element. However, it would have to be an element of a class of binary functions between classes. This is a contradiction. ☐

2.4.1 Input permutation

Let A, B be classes. Then we can define a swap function as follows:

$$\mathrm{swap}_{A,B} : A \times B \to B \times A : (a, b) \mapsto (b, a).$$

We may write swap instead of $\mathrm{swap}_{A,B}$ is the classes are clear from the context.

Lemma I.135. *Let A, B be classes. Then $\mathrm{swap}_{B,A} \circ \mathrm{swap}_{A,B} = \mathrm{id}_{A \times B}$.*

> Let A, B, C be classes and $f : A \times B \to C$ a binary function. We define the dual function $f^d : B \times A \to C$ of f as $f^d := f \circ \mathrm{swap}$.

If we restrict ourselves to the class of binary functions between sets, then swap can be considered a function.

Lemma I.136. *Let A, B, C be classes and $f : A \times B \to C$ a binary function. Then $(f^d)^d = f$.*

2.4.2 Currying

TODO cfr residuation

Let A, B, C be sets.
Given a function $f : A \times B \to C$, we can curry it in the first argument to obtain a new function

$$\text{curry}(f)_1 : A \to (B \to C) : a \mapsto f(a, -) \qquad \text{where } f(a, -) : B \to C : b \mapsto f(a, b)$$

or in the second argument to obtain

$$\text{curry}(f)_2 : B \to (A \to C) : b \mapsto f(-, b) \qquad \text{where } f(-, b) : A \to C : a \mapsto f(a, b).$$

Lemma I.137. *For given sets A, B, C the act of currying defines two bijective functions*

$$\begin{aligned} \text{curry}_1 &: (A \times B \to C) \rightarrowtail\!\!\!\!\twoheadrightarrow (A \to (B \to C)); \\ \text{curry}_2 &: (A \times B \to C) \rightarrowtail\!\!\!\!\twoheadrightarrow (B \to (A \to C)) \end{aligned}$$

2.4.2.1 Partial application

Let $f : A \times B \to C$ be a function, $a \in A$ and $b \in B$. We write

- $f(a, -)$ to mean $\text{curry}_1(f)(a)$; and
- $f(-, b)$ to mean $\text{curry}_2(f)(b)$.

Any function of the form $f(a, -)$ or $f(-, b)$ is called a partial application of f.

2.4.3 Homogenous binary operators

Let A be a class and $f : A \times A \to A$ a binary function. We call f

- associative if $\forall x, y, z \in A : \; f(f(x, y), z) = f(x, f(y, z))$;
- commutative if $\forall x, y \in A : \; f(x, y) = f(y, x)$;
- idempotent if $\forall x \in A : \; f(x, x) = x$.

If something is equal to an undefined quantity, we require it to be undefined.

2.4.3.1 Duality

Let A be a class and $f : A \times A \to A$ a homogeneous binary function. Then $f^d : A \times A \to A$ is also a homogeneous binary function.
Often a property of f can be translated to a different property of f^d. In this case we say the properties are dual to each other.

Proposition I.138. *If a certain logical dependence holds between certain properties of binary homogenous functions, for all binary homogenous functions, then the same logical dependence holds between the duals of these properties.*

Proof. If the logical dependence holds for all functions, it in particular holds for all functions of the form f^d. □

2.4.3.2 Notation for binary operators

Prefix, infix, postfix, Polish, necessity of brackets.

2.4.3.3 Identity and absorbing elements

Let A be a class and $f : A \times A \to A$ a binary function. We say

- f has a left-identity e_L if $\forall x \in A : \ f(e_L, x) = x$;
- f has a right-identity e_R if $\forall x \in A : \ f(x, e_R) = x$;
- f has an identity e if e is both a left- and a right-identity of f.

We say

- f has a left-absorbing element u_L if $\forall x \in A : \ f(u_L, x) = u_L$;
- f has a right-absorbing element u_R if $\forall x \in A : \ f(x, u_R) = u_R$;
- f has an absorbing element u if u is both a left- and a right-absorbing element of f.

Left and right identity are dual. Left and right absorbing are also dual.
TODO require that absorbing element is no identity?

Lemma I.139. *Let A be a class and $f : A \times A \to A$ a binary operator.*

1. *If f has both a left-identity e_L and a right-identity e_R, then f has an identity e and*
$$e = e_L = e_R.$$
2. *If f has both a left-absorbing element u_L and a right-absorbing element u_R, then f has an absorbing element u and*
$$u = u_L = u_R.$$

Proof. (1) Assume f has a left- and a right-identity. Then $e_L = f(e_L, e_R) = e_R$.
(2) Assume f has a left- and a right-absorbing element. Then $u_L = f(u_L, u_R) = u_R$. □

Corollary I.139.1. *A binary operator may have multiple left-identities or multiple right-identities, but if it has both, then the identity is unique.*
An absorbing element is similarly unique.

Let A be a class and $f : A \times A \to A$ a binary operator. We define

$$\widetilde{A} := \begin{cases} A & \text{if } f \text{ has an identity} \\ A \uplus \{e\} & \text{if } f \text{ has no identity.} \end{cases}$$

and also

$$\widetilde{f} \quad := \quad \widetilde{A} \times \widetilde{A} \to \widetilde{A} : (a, b) \mapsto \begin{cases} f(a, b) & a, b \in A \\ b & a = e \\ a & b = e. \end{cases}$$

Let A be a class and $f : A \times A \to A$ a binary operator. We define

$$\widehat{A} \coloneqq \begin{cases} A & \text{if } A \text{ has an absorbing element} \\ A \uplus \{u\} & \text{if } A \text{ has no absorbing element.} \end{cases}$$

and also

$$\widehat{f} \quad \coloneqq \quad \widehat{A} \times \widehat{A} \to \widehat{A} : (a, b) \mapsto \begin{cases} f(a, b) & a, b \in A \\ u & (a = u \vee b = u. \end{cases}$$

Lemma I.140. *Let A be a class and $f : A \times A \to A$ an associative partial binary function with absorbing element u. Then we can extend f to an associative total function $\widehat{f} : A \times A \to A$ by setting*

$$(x, y) \in A \times A \setminus \operatorname{preim}(f) \qquad \Longrightarrow \qquad \widehat{f}(x, y) = u.$$

Proof. We just need to verify associativity.
We have $\widehat{f}(\widehat{f}(x, y), z) = \widehat{f}(x, \widehat{f}(y, z))$ if $f(f(x, y), z)$ and $f(x, f(y, z))$ are defined.
If $f(f(x, y), z)$ is not defined, we claim $\widehat{f}(\widehat{f}(x, y), z) = u$. Indeed,

- if $f(x, y)$ is defined, then $f(\widehat{f}(x, y), z) = f(f(x, y), z)$ is undefined and thus equal to u;
- if $f(x, y)$ is undefined, then $\widehat{f}(\widehat{f}(x, y), z) = \widehat{f}(u, z) = f(u, z) = u$.

Similarly, if $f(x, f(y, z))$ is not defined, we have $\widetilde{f}(x, \widetilde{f}(y, z)) = u$. □

2.4.3.4 Closure

TODO Galois connection.

Let $f : A \times A \to A$ be a binary function and $B \subseteq A$. Then we call B closed under f if $f(B, B) \subseteq B$.

2.4.3.5 Distributivity

Let A be a class and $f, g : A \times A \to A$ two binary functions. We say

- f is left-distributive over g if

  $$\forall x, y, z \in A : \; f(x, g(y, z)) = g(f(x, y), f(x, z));$$

- f is right-distributive over g if

  $$\forall x, y, z \in A : \; f(f(x, y), z) = g(f(x, z), f(y, z));$$

- f is distributive over g if it is left- and right-distributive;
- f is self-distributive if it is distributive over itself.

2.4.3.6 The absorption law

Let A be a class and $f, g : A \times A \to A$ two binary functions. We say f, g are linked by the absorption law if

$$\forall x, y \in A : \; f(x, g(x, y)) = x = g(x, f(x, y)).$$

Lemma I.141. *Let A be a class and $f, g : A \times A \to A$ binary functions. If f and g are linked by the absorption law, then they are both idempotent.*

Proof. For all $x \in A$ we have $f(x, x) = f(x, g(x, f(x, x))) = x$, where the last equality follows from the absorption law with $y = f(x, x)$. ☐

2.4.4 The evaluation map

Let A, B be sets. We define the evaluation map

$$\mathrm{ev} : (A \to B) \times A \to B : (f, x) \mapsto f(x).$$

Often we will consider partial applications of the evaluation map in the second argument, i.e. for $x \in A$

$$\mathrm{ev}_x : (A \to B) \to B : f \mapsto f(x),$$

which is also called an evaluation map.

We have $\mathrm{ev}_x = \mathrm{curry}_2(\mathrm{ev})(x)$.

2.5 Associative classes

Let A be a class and $f : A \times A \to A$ an associative binary function. We call (A, f) an associative class.
We will often abbreviate $f(x, y)$ by xy.

TODO all functions are $\lambda - \rho$ in larger space.

2.5.1 Inverses and cancellation

Let A be a class, $f : A \times A \to A$ a binary function and $x \in A$. We call x

- left-cancellative if $f(x, \cdot)$ is injective;
- f has a right-cancellative if $f(\cdot, x)$ is injective.

Let f have an identity e, then we say an element $x \in A$

- has a left-inverse y if $f(y, x) = e$;
- has a right-inverse y if $f(x, y) = e$;
- has an (two-sided) inverse y if $f(x, y) = e = f(y, x)$.

We call x invertible if it has an inverse.

Left and right cancellative are dual. Left and right inverse are also dual.

Proposition I.142. *Let (A, f) be an associative class with identity e and x in A. If x has both a left-inverse l and a right-inverse r, then $l = r$.*

Proof. Using associativity, we have

$$l = f(l, e) = f(l, f(x, r)) = f(f(l, x), r) = f(e, r) = r.$$

□

Corollary I.142.1. *If an element has a two-sided inverse, it is unique.*

If x is invertible, we denote the unique inverse by x^{-1}.

Proposition I.143. *Let (A, f) be an associative class with identity e and x, y in A. If x and y have inverses, then xy is invertible with $(xy)^{-1} = y^{-1}x^{-1}$.*

Proof. We calculate

$$(xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = xx^{-1} = e \quad \text{and} \quad (y^{-1}x^{-1})(xy) = y^{-1}(x^{-1}x)y = y^{-1}y = e.$$

□

Lemma I.144. *Let (A, f) be an associative class and x, y in A.*

1. *If x and y are left-, resp. right-,cancellative, then $f(x, y)$ is left-, resp. right-,cancellative.*
2. *If $f(x, y)$ is left-cancellative, then y is left-cancellative.*
3. *If $f(x, y)$ is right-cancellative, then x is left-cancellative.*

Proof. Let $z_1, z_2 \in A$.
(1) Assume that $f(f(x, y), z_1) = f(f(x, y), z_2)$. By associativity, we have $f(x, f(y, z_1)) = f(x, f(y, z_2))$. Thus by injectivity we get $z_1 = z_2$.
Right-cancellation is similar.
(2) Assume $f(y, z_1) = f(y, z_2)$. Then $f(x, f(y, z_1)) = f(x, f(y, z_2))$. By associativity, we get $f(f(x, y), z_1) = f(f(x, y), z_2)$ and thus $z_1 = z_2$ because $f(x, y)$ is left-cancellative.
(3) Assume $f(z_1, x) = f(z_2, x)$. Then $f(f(z_1, x), y) = f(f(z_2, x), y)$. By associativity, we get $f(z_1, f(x, y)) = f(z_2, f(x, y))$ thus $z_1 = z_2$ because $f(x, y)$ is right-cancellative. □

Lemma I.145. *Let (A, f) be an associative class and x, y in A. Then*

1. *if x has a left inverse, it is left-cancellative;*
2. *if x has a right inverse, it is right-cancellative;*
3. *if x has a left inverse and is right-cancellative, it is invertible;*
4. *if x has a right inverse and is left-cancellative, it is invertible.*

Proof. (1) Let l be a left inverse of x and assume $f(x, z_1) = f(x, z_2)$, then $f(l, f(x, z_1)) = f(l, f(x, z_2))$ and thus

$$z_1 = f(e, z_1) = f(f(l, x), z_1) = f(l, f(x, z_1)) = f(l, f(x, z_2)) = f(f(l, x), z_2) = f(e, z_2) = z_2.$$

(2) Similar.
(3) Let l be a left inverse of x. It is enough to show that l is also a right inverse of x. We calculate

$$f(f(x, l), x) = f(x, f(l, x)) = f(x, e) = x = f(e, x).$$

Beacuse x is right-cancellative, this means $f(x, l) = e$ and thus that l is a right inverse.
(4) Similar. □

2.5.2 Left and right relation

Let (A, f) be an associative class and $x, y \in A$. Then

- let L be the relation defined by $xLy \quad \Leftrightarrow_{\text{def}} \quad \exists a : f(a, x) = y$;
- let R be the relation defined by $xRy \quad \Leftrightarrow_{\text{def}} \quad \exists a : f(x, a) = y$.

The relations L and R are dual.

Lemma I.146. *The relations L and R are transitive.*

Proof. Let (A, f) be an associative class and $x, y, z \in A$ such that xLy and yLz. Then there exist $a, b \in A$ such that $f(a, x) = y$ and $f(b, y) = z$. Then

$$z = f(b, y) = f(b, f(a, x)) = f(f(b, a), x),$$

so xLz. The statement for R is dual. □

Lemma I.147. *Let (A, f) be an associative class. Then*

1. $L = \bigcup_{a \in A} f(a, -)$*;*
2. $R = \bigcup_{a \in A} f(-, a)$*.*

For given and fixed f, we will often write

- $\lambda_a : A \to A$ for $f(a, -)$;
- $\rho_a : A \to A$ for $f(-, a)$.

Lemma I.148. *For all $a, b \in A$, we have $\lambda_a \circ \rho_b = \rho_b \circ \lambda_a$.*

Proof. We calculate, for arbitrary $x \in A$,

$$\lambda_a(\rho_b(x)) = f(a, f(x, b)) = f(f(a, x), b) = \rho_b(\lambda_a(x)).$$

$\square$

Corollary I.148.1. *Let A be a class and $f : A \times A \to A$ an associative binary function. Then $L; R = R; L$.*

Proof. Let $x, y \in A$. Then $x(L; R)y$ iff there exist $a, b \in A$ such that the top path in

$$\begin{tikzcd}
x \arrow[r, mapsto, "\lambda_a"] \arrow[d, mapsto, "\rho_b"'] & f(a, x) \arrow[d, mapsto, "\rho_b"] \\
f(x, b) \arrow[r, mapsto, "\lambda_a"'] & y
\end{tikzcd}$$

holds. By I.148 this is equivalent to the bottom path holding. The bottom path implies $x(R; L)y$. $\square$

Proposition I.149. *Let g, h be functions in the assocative class of functions with composition $\circ$. Then*

1. gLh if and only if $\ker g \subseteq \ker h$;
2. gRh if and only if $\operatorname{im} g \supseteq \operatorname{im} h$.

2.5.3 Principal ideals

Let (A, f) be an associative class and $x, y \in A$. Then

- the left principal ideal generated by x is $f(\widetilde{A}, x) = f(A, x) \cup \{x\}$;
- the right principal ideal generated by x is $f(x, \widetilde{A}) = f(x, A) \cup \{x\}$.

Lemma I.150. *Let (A, f) be an associative class and $x, y \in A$. Then*

1. $f(A, f(x, y)) \subseteq f(A, y)$;
2. $f(f(x, y), A) \subseteq f(x, A)$.

Proof. (1) Take $z \in f(A, f(x, y))$. Then there exists $z' \in A$ such that

$$z = f(z', f(x, y)) = f(f(z', x), y) \in f(A, y).$$

(2) By duality. $\square$

Lemma I.151. *LLet (A, f) be an associative class and $x, y \in A$. Then*

1. $f(A, x) \subseteq f(A, y)$ if and only if $\forall a \in A : \exists b \in A : \ f(a, x) = f(b, y)$;
2. $f(x, A) \subseteq f(y, A)$ if and only if $\forall a \in A : \exists b \in A : \ f(x, a) = f(y, b)$.

If A contains an identity e, then

3. $f(A, x) \subseteq f(A, y)$ if and only if $\exists b \in A : \ x = f(b, y)$;

4. $f(x, A) \subseteq f(y, A)$ if and only if $\exists b \in A : \; x = f(y, b)$.

Proposition I.152. *Let (A, f) be an associative class. Then f has an identity and every $x \in A$ is invertible* if and only if

$$\forall x \in A : \quad f(x, A) = A = f(A, x).$$

Proof. $\Rightarrow$ We clearly have $f(x, A) \subseteq A$. The other inclusion follows from I.150: $A = f(A, e) = f(A, f(x^{-1}, x)) \subseteq f(A, x)$.
$\Leftarrow$ Pick some $x \in A$, so $x \in f(x, a)$, meaning there exists an $a \in A$ such that $x = xa$. We claim a is a right-identity for f. Indeed, take arbitrary $y \in A$. Then $y = f(b, x)$ for some $b \in A$ and so

$$f(y, a) = f(f(b, x), a) = f(b, f(x, a)) = f(b, x) = y.$$

In the same way we can also find a left-identity. So A contains an identity $e := a$ by I.139. Now for all $x \in A$ we have $e \in A = f(x, A)$, so we can find a right-inverse of x. Similarly, we can find a left-inverse of x. This means x is invertible by I.142. $\square$

2.5.3.1 Green's relations

Let (A, f) be an associative class. The Green's relations of f are relations on A defined as

- $\mathcal{L}$ is the reflexive closure of the symmetric part of L;
- $\mathcal{R}$ is the reflexive closure of the symmetric part of R;
- $\mathcal{H} \;:=\; \mathcal{L} \cap \mathcal{R}$;
- $\mathcal{D} \;:=\; \mathcal{L}; \mathcal{R}$.

The relations $\mathcal{L}$ and $\mathcal{R}$ are dual.

Lemma I.153. *Let (A, f) be an associative class. Then the relations $\mathcal{L}$ and $\mathcal{R}$ are equivalence relations.*

Lemma I.154. *Let (A, f) be an associative class and $x, y \in A$. Then the following are equivalent:*

1. $x \mathcal{L} y$;
2. $x\big((L \cap L^{\mathrm{T}}) \cup \mathrm{id}_A\big)y$
3. $\exists a, b \in \widetilde{A} : (f(a, x) = y) \wedge (f(b, y) = x)$;
4. $$\begin{tikzcd} x \arrow[r, mapsto, shift left, "\exists a:\lambda_a"] & y \arrow[l, mapsto, shift left, "\exists b:\lambda_b"] \end{tikzcd}$$
5. $f(\widetilde{A}, x) = f(\widetilde{A}, y)$;
6. $f(A, x) = f(A, y)$;

as are

1. $x \mathcal{R} y$;

2. *$x\big((R \cap R^{\mathrm{T}}) \cup \mathrm{id}_A\big)y$*
3. *$\exists a, b \in \widetilde{A} : (f(x,a) = y) \wedge (f(y,b) = x)$;*
4. $$\begin{tikzcd} x \arrow[r, mapsto, shift left, "\exists a:\rho_a"] & y \arrow[l, mapsto, shift left, "\exists b:\rho_b"] \end{tikzcd}$$
5. *$f(x, \widetilde{A}) = f(y, \widetilde{A})$;*
6. *$f(x, A) = f(y, A)$.*

Lemma I.155. *Let (A, f) be an associative class and $x, y, z \in A$.*

1. *If $x\mathcal{L}y$, then $\rho_z(x)\mathcal{L}\rho_z(y)$.*
2. *If $x\mathcal{R}y$, then $\lambda_z(x)\mathcal{R}\lambda_z(y)$.*

2.5.3.2 Egg-box diagrams

Proposition I.156. *The relation $\mathcal{D}$ is an equivalence relation.*

This is equivalent to $\mathcal{L};\mathcal{R} = \mathcal{R};\mathcal{L}$ by I.99.

Proof. Assume $x(\mathcal{R};\mathcal{L})z$, meaning $\exists y : x\mathcal{R}y$ and $y\mathcal{L}z$. Then there exist $a, b, c, d \in \widetilde{A}$ such that

$$\begin{tikzcd} x \arrow[r, mapsto, shift left, "\rho_a"] & y \arrow[l, mapsto, shift left, "\rho_b"] \arrow[r, mapsto, shift left, "\lambda_c"] & z \arrow[l, mapsto, shift left, "\lambda_d"] \end{tikzcd} .$$

Using I.148, we can rearrange such that we also get the mappings along the left and bottom sides of

$$\begin{tikzcd} x \arrow[r, mapsto, shift left, "\rho_a"] \arrow[d, mapsto, shift left, "\lambda_c"] & y \arrow[l, mapsto, shift left, "\rho_b"] \arrow[d, mapsto, shift left, "\lambda_c"] \\ y' \arrow[u, mapsto, shift left, "\lambda_d"] \arrow[r, mapsto, shift left, "\rho_a"] & z \arrow[l, mapsto, shift left, "\rho_b"] \arrow[u, mapsto, shift left, "\lambda_d"] \end{tikzcd}$$

for some $y' \in A$. Thus $x(\mathcal{L};\mathcal{R})y$. The other inclusion is similar. □

From I.99 we also know that the $\mathcal{L}, \mathcal{R}$-egg-box diagram decomposes into blocks, which are $\mathcal{D}$-equivalence classes. The columns are $\mathcal{L}$-equivalence classes, the rows are $\mathcal{R}$-equivalence classes, and the cells are $\mathcal{H}$-equivalence classes.

Example

Let $A = (\{1,2,3\} \to \{1,2,3\})$ with the binary function $A \times A \to A : (f, g) \mapsto f;g$. We can represent an element f of A as $(f(1)f(2)f(3))$. We have

- $f\mathcal{L}g$ if f and g have the same image;
- $f\mathcal{R}g$ if f and g have the same kernel.

An egg-box diagram can be drawn as follows:

(**111**)	(**222**)	(**333**)				
			(**122**), (211)	(**133**), (311)	(233), (322)	
			(212), (**121**)	(313), (131)	(**323**), (232)	
			(221), (112)	(331), (**113**)	(332), (**223**)	
						(**123**), (231), (312) (132), (213), (321)

The bold elements are idempotents.

Proposition I.157 (Green's lemma)**.** *Let (A, f) be an associative class and $x, y \in A$.*

1. *If $x\mathcal{L}y$ with $x \underset{\lambda_b}{\overset{\lambda_a}{\rightleftarrows}} y$, then*
   - *(a)* $\lambda_a|_{[x]_\mathcal{R}} : [x]_\mathcal{R} \to [y]_\mathcal{R}$ *is a bijection with inverse* $\lambda_b|_{[y]_\mathcal{R}} : [y]_\mathcal{R} \to [x]_\mathcal{R}$*;*
   - *(b)* $\lambda_a|_{[x]_\mathcal{H}} : [x]_\mathcal{H} \to [y]_\mathcal{H}$ *is a bijection with inverse* $\lambda_b|_{[y]_\mathcal{H}} : [y]_\mathcal{H} \to [x]_\mathcal{H}$*.*
2. *If $x\mathcal{R}y$ with $x \underset{\rho_b}{\overset{\rho_a}{\rightleftarrows}} y$, then*
   - *(a)* $\rho_a|_{[x]_\mathcal{L}} : [x]_\mathcal{L} \to [y]_\mathcal{L}$ *is a bijection with inverse* $\rho_b|_{[y]_\mathcal{L}} : [y]_\mathcal{L} \to [x]_\mathcal{L}$*;*
   - *(b)* $\rho_a|_{[x]_\mathcal{H}} : [x]_\mathcal{H} \to [y]_\mathcal{H}$ *is a bijection with inverse* $\rho_b|_{[y]_\mathcal{H}} : [y]_\mathcal{H} \to [x]_\mathcal{H}$*.*

Proof. Take some arbitrary $x' \in [x]_\mathcal{R}$. Then there exist $c, d \in \widetilde{A}$ such that $x \underset{\rho_d}{\overset{\rho_c}{\rightleftarrows}} x'$. Then, by I.148,

$$x' \underset{\rho_c}{\overset{\rho_d}{\rightleftarrows}} x \underset{\lambda_b}{\overset{\lambda_a}{\rightleftarrows}} y \qquad \text{implies that} \qquad x' \underset{\lambda_b}{\overset{\lambda_a}{\rightleftarrows}} y' \underset{\rho_c}{\overset{\rho_d}{\rightleftarrows}} y$$

Thus, for all $x' \in [x]_\mathcal{R}$, we have

- $\lambda_a(x') \in [y]_\mathcal{R}$, meaning that $\lambda_a|_{[x]_\mathcal{R}} : [x]_\mathcal{R} \to [y]_\mathcal{R}$ is well-defined;
- by similar reasoning, we can see that $\lambda_b|_{[y]_\mathcal{R}} : [y]_\mathcal{R} \to [x]_\mathcal{R}$ is also well-defined;
- $\lambda_b(\lambda_a(x')) = x'$, so the functions are inverse of each other.

If $x' \in [x]_\mathcal{H}$, then $\lambda_a(x')\mathcal{L}\lambda_a(x) = y$ by I.155, so $\lambda_a(x') \in [y]_\mathcal{H}$. This means that $\lambda_a|_{[x]_\mathcal{H}} : [x]_\mathcal{H} \to [y]_\mathcal{H}$ is well-defined.
Point (2) is dual. □

Corollary I.157.1. *Let (A, f) be an associative class and $x, y \in A$ such that $x\mathcal{D}y$. Then there exist $a, b, c, d \in \widetilde{A}$ such that*

$$\lambda_a \circ \rho_b|_{[x]_\mathcal{H}} : [x]_\mathcal{H} \to [y]_\mathcal{H} \qquad \textit{is a bijection with inverse} \qquad \lambda_c \circ \rho_d|_{[y]_\mathcal{H}} : [y]_\mathcal{H} \to [x]_\mathcal{H}.$$

Proof. There exists a $z \in A$ such that $x\mathcal{L}z$ and $z\mathcal{R}y$. We then just compose the bijections in Green's lemma, keeping in mind that λ and ρ commute. $\square$

Corollary I.157.2. *Let (A, f) be an associative class and $x \in A$. If x is an idempotent, then*

$$\lambda_x|_{[x]_\mathcal{R}} = \mathrm{id}_{[x]_\mathcal{R}} \qquad \text{and} \qquad \rho_x|_{[x]_\mathcal{L}} = \mathrm{id}_{[x]_\mathcal{L}} .$$

Thus x is a left identity for $[x]_\mathcal{R}$ and a right identity for $[x]_\mathcal{L}$.

Proof. Take $y \in [x]_\mathcal{R}$. Then there exist $a, b \in \widetilde{A}$ such that $x \underset{\rho_b}{\overset{\rho_a}{\rightleftarrows}} y$. Then we have

$$xy = xxb = xb = y.$$

The other claim is dual. $\square$

Theorem I.158 (Green's theorem)**.** *Let (A, f) be an associative class and H an $\mathcal{H}$-class in A. Then either*

1. *$H^2 \perp H$; or*
2. *$H^2 = H$, $f|_{H\times H}$ has an identity and each $x \in H$ is invertible.*

Proof. Suppose $H^2 \cap H \neq \emptyset$, then there exist $a, b \in H$ such that $ab = c \in H$. By the Green's lemma II.27 we have that $\rho_b : H \to H$ and $\lambda_a : H \to H$ are bijections.
Then for all $h \in H$, $\rho_b(h) = hb \in H$. Again by the Green's lemma, this means that $\lambda_h : H \to H$ is a bijection. Similarly $\rho_h : H \to H$ is a bijection for all h. So for all $h \in H$ we have $hH = H = Hh$. The result follows by I.152. $\square$

Corollary I.158.1. *Let (A, f) be an associative class and H an $\mathcal{H}$-class in A.Then*

1. *if x is an idempotent in H, then we have the second case;*
2. *no $\mathcal{H}$-class can contain more than one idempotent.*

2.5.4 Regular elements and generalised inverses

Let (A, f) be an associative class and $x, y \in A$. We call

- x regular if $\exists a \in A :\ x = f(f(x, a), x)$
- x and y generalised inverses if $x = f(f(x, y), x)$ and $y = f(f(y, x), y)$.

Proposition I.159. *Let (A, f) be an associative class. If $x \in A$ is regular, then every element in $[x]_\mathcal{D}$ is regular.*

So it makes sense to call a $\mathcal{D}$-class regular if it consists of regular elements and irregular otherwise.

Proof. Let x be regular with $x = xx'x$ and $x\mathcal{D}y$. Then we have $a, b, c, d \in \widetilde{A}$ such that $\lambda_a \circ \rho_b|_{[x]_\mathcal{H}} : [x]_\mathcal{H} \to [y]_\mathcal{H}$ is a bijection with inverse $\lambda_c \circ \rho_d|_{[y]_\mathcal{H}} : [y]_\mathcal{H} \to [x]_\mathcal{H}$, as in I.157.1. Then we have

$$y = (\lambda_a \circ \rho_b)(x) = (\lambda_a \circ \rho_b)(xx'x) = axx'xb = a(\lambda_c \circ \rho_d)(y)x'(\lambda_c \circ \rho_d)(y)b = ydx'cy.$$

So y is regular. $\square$

Corollary I.159.1. *If there is an idempotent $x \in [a]_{\mathcal{D}}$, then $[a]_{\mathcal{D}}$ is regular.*

Proof. An idempotent is regular: $x = f(x,x) = f(f(x,x),x)$. ☐

Proposition I.160. *Let (A,f) be an associative class. Then x is regular* if and only if *it has a generalised inverse.*

Proof. Clearly every element with a generalised inverse is regular. Conversely, assume x regular with $x = xax$. Then $y = axa$ is a generalised inverse of x: $x(axa)x = xax = x$ and $(axa)x(axa) = a(xax)axa = axaxa = axa$. ☐

Note that we do not have that $x = xyx$ implies $y = yxy$.

Proposition I.161. *Let (A,f) be an associative class and $x \in A$ a regular element with $x = f(f(x,y),x)$. Then*

1. *$f(x,y)$ and $f(y,x)$ are idempotent;*
2. *$f(y,x)\mathcal{L}x$ and $x\mathcal{R}f(x,y)$.*

Proof. (1) We calculate

$$f(f(x,y),f(x,y)) = f(f(f(x,y),x),y) = f(x,y)$$

and

$$f(f(y,x),f(y,x)) = f(y,f(x,f(y,x))) = f(y,x).$$

(2) Using I.150, we have

$$f(A,x) = f(A,f(x,f(y,x))) \subseteq f(A,f(y,x)) \subseteq f(A,x).$$

Thus $f(A,x) = f(A,f(y,x))$. The second part is dual. ☐

Corollary I.161.1. *In a regular $\mathcal{D}$-class each $\mathcal{L}$-class and each $\mathcal{R}$-class contains at least one idempotent.*

Proof. Let $[x]_{\mathcal{L}}$ be an $\mathcal{L}$-class in a regular $\mathcal{D}$-class. By regularity there exists a $y \in A$ such that $xyx = x$. From the proposition, we have that $[x]_{\mathcal{L}}$ contains the idempotent yx and $[x]_{\mathcal{R}}$ the idempotent xy. ☐

Corollary I.161.2. *If $x, x' \in A$ are generalised inverses, then $[xx']_{\mathcal{H}} = [x]_{\mathcal{R}} \cap [x']_{\mathcal{L}}$ and $[x'x]_{\mathcal{H}} = [x']_{\mathcal{R}} \cap [x]_{\mathcal{L}}$.*

Proof. From $x\mathcal{R}(xx')$ and $(xx')\mathcal{L}x'$, we get the first equality. The second is dual. ☐

We can depict the situation in the corollary as follows:

$$\exists a,b,c,d \in \widetilde{A}: \qquad \begin{tikzcd}
x \arrow[r, shift left, mapsto, "\rho_a"] \arrow[d, shift left, mapsto, "\lambda_c"] & xx' \arrow[l, shift left, mapsto, "\rho_b"] \arrow[d, shift left, mapsto, "\lambda_c"] \\
x'x \arrow[r, shift left, mapsto, "\rho_a"] \arrow[u, shift left, mapsto, "\lambda_d"] & x' \arrow[l, shift left, mapsto, "\rho_b"] \arrow[u, shift left, mapsto, "\lambda_d"]
\end{tikzcd}$$

So generalised inverses along one diagonal imply idempotents along the other. In fact, the other direction also holds:

Proposition I.162. *Let (A, f) be an associative class and e, f idempotents in A such that $e\mathcal{D}f$. Then there exist $x \in [e]_{\mathcal{R}} \cap [f]_{\mathcal{L}}$ and $x' \in [e]_{\mathcal{L}} \cap [f]_{\mathcal{R}}$ such that*

- *x, x' are generalised inverses;*
- *$e = xx'$ and $f = x'x$.*

Proof. Because $e\mathcal{D}f$, we can find $x, x' \in A$ such that

$$\exists a, b, c, d \in \widetilde{A} : \qquad \begin{tikzcd}
e \arrow[r, mapsto, shift left, "\rho_a"] \arrow[d, mapsto, shift left, "\lambda_c"] & x \arrow[l, mapsto, shift left, "\rho_b"] \arrow[d, mapsto, shift left, "\lambda_c"] \\
x' \arrow[u, mapsto, shift left, "\lambda_d"] \arrow[r, mapsto, shift left, "\rho_a"] & f \arrow[l, mapsto, shift left, "\rho_b"] \arrow[u, mapsto, shift left, "\lambda_d"]
\end{tikzcd}$$

Then

$$\begin{aligned} xx' &= (df)(fb) = dfb = e \\ x'x &= (ce)(ea) = cea = f \\ xx'x &= ex = eea = ea = x \\ x'xx' &= fx' = ffb = fb = x', \end{aligned}$$

which completes the proof. □

Corollary I.162.1. *Let (A, f) be an associative class, $y \in A$ and e, f idempotents in A. Then $e\mathcal{D}f$* if and only if *there exist generalised inverses x, x' such that $e = xx'$ and $f = x'x$.*

Proof. The direction ⇒ follows from the proposition. The converse from I.161. □

Lemma I.163. *Let (A, f) be an associative class, $x \in A$. Then no $\mathcal{H}$-class contains more than one generalised inverse of x.*

Proof. Assume x has two generalised inverses, x'_1 and x'_2. From I.161.2 and II.28.1 we get that $xx'_1 = xx'_2$ and $x'_1x = x'_2x$. Thus

$$x'_1 = x'_1(xx'_1) = x'_1xx'_2 = (x'_1x)x'_2 = x'_2xx'_2 = x'_2.$$

□

Proposition I.164. *Let (A, f) be an associative class and $x, y \in A$. Then $xy \in [x]_{\mathcal{R}} \cap [y]_{\mathcal{L}}$* if and only if *$[x]_{\mathcal{L}} \cap [y]_{\mathcal{R}}$ contains an idempotent.*

Proof. First assume $[x]_{\mathcal{L}} \cap [y]_{\mathcal{R}}$ contains an idempotent e. We can depict the situation as

$$\exists a, b, c, d \in \widetilde{A} : \qquad \begin{tikzcd}
x \arrow[r, mapsto, shift left, "\rho_a"] \arrow[d, mapsto, shift left, "\lambda_c"] & {} \arrow[l, mapsto, shift left, "\rho_b"] \arrow[d, mapsto, shift left, "\lambda_c"] \\
e \arrow[u, mapsto, shift left, "\lambda_d"] \arrow[r, mapsto, shift left, "\rho_a"] & y \arrow[l, mapsto, shift left, "\rho_b"] \arrow[u, mapsto, shift left, "\lambda_d"]
\end{tikzcd}$$

Then we can calculate

$$xy = deea = dea = xa \in [x]_{\mathcal{R}} \cap [y]_{\mathcal{L}}.$$

Now assume $xy \in [x]_{\mathcal{R}} \cap [y]_{\mathcal{L}}$. We can depict the situation as

$$\exists a, b, c, d \in \widetilde{A}: \qquad \begin{tikzcd} x \arrow[r, mapsto, shift left, "\rho_a"] \arrow[d, mapsto, shift left, "\lambda_c"] & xy \arrow[l, mapsto, shift left, "\rho_b"] \arrow[d, mapsto, shift left, "\lambda_c"] \\ e \arrow[r, mapsto, shift left, "\rho_a"] \arrow[u, mapsto, shift left, "\lambda_d"] & y \arrow[l, mapsto, shift left, "\rho_b"] \arrow[u, mapsto, shift left, "\lambda_d"] \end{tikzcd}$$

Now we need to show that e is idempotent. Indeed, starting from the three other corners, we see that $e = cx$ and $e = yb$ and $e = c(xy)b = (cx)(yb) = ee$. □

2.5.5 Commutation

Let (A, f) be an associative class and $x, y \in A$. We say x and y commute if $f(x, y) = f(y, x)$. We write $x \leftrightarrow y$.

2.5.5.1 Centraliser or commutant

Let (A, f) be an associative class and $B \subseteq A$ a subclass. The centraliser or commutant of B is defined as $Z_A(B) \coloneqq B^{\leftrightarrow}$.
In particular we define the centre of A as the centraliser of all of A: $Z_A \coloneqq Z_A(A)$.

Thus

$$Z_A(B) = \{x \in A \mid \forall b \in B : \; f(x, b) = f(b, x)\}.$$

.
Note that taking the commuting forms a Galois connection. In particular $B \subseteq B^{\leftrightarrow\leftrightarrow}$.

Lemma I.165. *Let (A, f) be an associative class and $B \subseteq A$. If f is commutative, then $Z_A(B) = A$.*

Proposition I.166. *Let (A, f) be an associative class and $B \subseteq A$. Then $Z_A(B)$ is closed under f.*

Proof. Take arbitrary $x, y \in Z_A(B)$. Take arbitrary $b \in B$. Then

$$f(f(x, y), b) = f(x, f(y, b)) = f(x, f(b, y)) = f(f(x, b), y) = f(f(b, x), y) = f(b, f(x, y)),$$

which means that $f(x, y) \in Z_A(B)$. □

2.5.6 Normaliser

Let (A, f) be an associative class and $B \subseteq A$ a subclass. An element $x \in A$ is said to normalise B if $f(x, B) = f(B, x)$.
The normaliser of B in A is the set of all elements in A that normalise B:

$$N_A(B) \coloneqq \{x \in A \mid f(x, B) = f(B, x)\}.$$

Proposition I.167. *Let (A, f) be an associative class and $B \subseteq A$. Then*

1. *$N_A(B)$ is closed under f;*

2. *$Z_A(B) \subseteq N_A(B)$;*

3. *$Z_A(\{a\}) = N_A(\{a\})$ for all $a \in A$.*

Proof. (1) Take arbitrary $x, y \in N_A(B)$. Then

$$f(f(x,y),B) = f(x,f(y,B)) = f(x,f(B,y)) = f(f(x,B),y) = f(f(B,x),y) = f(B,f(x,y)),$$

which means that $f(x,y) \in N_A(B)$.
(2) If $f(x,b) = f(b,x)$ for all $b \in B$, then $f(x,B) = f(B,x)$.
(3) $f(x,\{a\}) = f(x,a)$ and $f(\{a\},x) = f(a,x)$. □

Chapter 3

The natural numbers

TODO = 1:n!

A system of natural numbers or Peano system is a structured set $(N, (0, S)) = (N, 0, S)$ which satisfies

1. N is a set containing 0;
2. S is an injective function on the set N;
3. for all $n \in N : S(n) \neq 0$;
4. the induction principle: $\forall X \subset N$

$$[(0 \in X) \wedge (\forall n \in N : n \in X \implies Sn \in X)] \implies X = N.$$

We call 0 the zero and S the successor function. These axioms are the Peano axioms. An object $n \in N$ is called a successor if $\exists m \in N : n = S(m)$.

The most involved axiom is the induction principle. Note that it is not formulated in first order logic. Thus the Peano axioms are not subject to the Löwenheim-Skolem theorem and we can obtain a uniqueness result (despite the fact that, as we will see, N is infinite).

Lemma I.168. *Let $(N, 0, S)$ be a Peano system. Then every element $n \neq 0$ is a successor and for each $n \in N : S(n) \neq n$.*

Proof. We wish to prove that the set

$$X = \{n \in N \mid (n = 0) \vee (\exists m \in N : n = S(m))\}$$

equals N. It is obvious that both $0 \in X$ and $\forall n \in N : n \in X \implies Sn \in X$, so we can apply the induction principle to obtain $X = N$. The second claim then follows by injectivity. $\square$

3.1 Recursion and induction

3.1.1 Recursion

Theorem I.169 (Recursion theorem)**.** *Assume $(N, 0, S)$ is a Peano system, E is some set, $a \in E$, and $h : E \to E$ is some function.*

There is exactly one function $f : N \to E$ *which satisfies*

$$\begin{cases} f(0) = a, \\ f(Sn) = h(f(n)) \quad (n \in N). \end{cases}$$

Functions defined using this theorem are said to be defined recursively.[1]

Proof. We consider the set $\mathcal{A}$ of "approximations" of the function f:

$$\begin{aligned} \mathcal{A} = \{p : X \to E \mid\ & (X \subset N) \wedge \\ & (0 \in X) \wedge \\ & [\forall n \in N : Sn \in X \implies (n \in X \wedge p(Sn) = h(p(n)))]\}. \end{aligned}$$

In words we may say that X is a downwards closed subset of N and p satisfies the recursion conditions. Some examples of elements of $\mathcal{A}$ include

$$\begin{aligned} & \{(0, a)\} \\ & \{(0, a), (S0, h(a))\} \\ & \{(0, a), (S0, h(a)), (SS0, h(h(a)))\} \\ & \ \ \dots \end{aligned}$$

Any function f with domain N satisfying the recursion conditions, as in the theorem must be an element of $\mathcal{A}$. Thus to prove the theorem we need to prove that $\mathcal{A}$ contains exactly one function with domain N.
We prove this using a lemma.

Lemma. *For all* $p, q \in \mathcal{A}$ *and* $n \in N$*,*

$$n \in \operatorname{dom}(p) \cap \operatorname{dom}(q) \implies p(n) = q(n).$$

Proof of lemma. We need to prove that the set of $n \in N$ for which this is true,

$$Y = \{n \in N \mid \forall p, q \in \mathcal{A} : [n \in \operatorname{dom}(p) \cap \operatorname{dom}(q) \implies p(n) = q(n)]\}$$

is exactly N. We prove this with the principle of induction.
Clearly $0 \in Y$, because every $p \in \mathcal{A}$ satisfies $p(0) = a$. Now let $n \in Y$ and $p, q \in \mathcal{A}$ such that $Sn \in \operatorname{dom}(p) \cap \operatorname{dom}(q)$. Then

$$p(Sn) = h(p(n)) = h(q(n)) = q(Sn)$$

so $Sn \in Y$. By the induction principle $Y = N$. $\dashv$ (Lemma)

The lemma immediately implies there is at most one suitable function f with domain N. We show such a function exists. Indeed it is given by $f = \bigcup \mathcal{A}$. Using the lemma we see this must be a function. It is not difficult to see that $f \in \mathcal{A}$. We then just need to verify that $\operatorname{dom}(f) = N$. This is again an application of the induction principle: $0 \in \operatorname{dom}(f)$ and if $n \in \operatorname{dom}(f)$, then there exists some $p \in \mathcal{A}$ with $n \in \operatorname{dom}(p)$ and we have

$$q = p \cup \{(Sn, h(p(n)))\} \in \mathcal{A},$$

so that $Sn \in \operatorname{dom}(q) \subseteq \operatorname{dom}(f)$. $\square$

---

[1] The terms "recursion" and "induction" are often used synonymously. We will usually distinguish recursive definitions from inductive proofs (using the induction principle).

Corollary I.169.1 (Recursion with parameters)**.** *Let* $(N, 0, S)$ *be a Peano system,* Y, E *sets and functions*

$$g : Y \to E, \qquad h : E \times Y \to E.$$

There is exactly one function $f : N \times Y \to E$ *which satisfies*

$$\begin{cases} f(0, y) = g(y), & (y \in Y) \\ f(Sn, y) = h(f(n, y), y) & (y \in Y, n \in N). \end{cases}$$

Proof. For any $y \in Y$ we can apply the normal recursion theorem the obtain a function $f_y : N \to E$. Then set $f(n, y) = f_y(n)$. ☐

Corollary I.169.2 (Recursion with the argument as parameter)**.** *Let* $(N, 0, S)$ *be a Peano system,* E *a set,* $a \in E$ *and* $h : E \times N \to E$ *a function.*
There is exactly one function $f : N \to E$ *which satisfies*

$$\begin{cases} f(0) = a, \\ f(Sn) = h(f(n), n) & (n \in N). \end{cases}$$

Proof. Consider the function $\phi : N \to N \times E$ which returns both the required result and the successor of its argument. As the argument is returned by the function, it can be used in the normal recursion theorem if we replace E by $N \times E$. Then just define $f(n)$ to be the second component of $\phi(n)$. ☐

More version of recursion can be cooked up, such as

- complete recursion where at each step of the recursion. h is passed not just the latest function value, but the whole partial function created so far:
$$h : (\mathbb{N} \not\rightarrow E) \to E.$$
- recursion with both the argument as parameter and other parameters;
- simultaneous recursion that defines two functions f_1, f_2 and the next step depends on the current step of both functions:
$$f_1(Sn) = h_1(f_1(n), f_2(n)) \qquad \text{and} \qquad f_2(Sn) = h_2(f_1(n), f_2(n)).$$

3.1.2 Induction

Lemma I.170 (Mathematical induction)**.** *We can prove a definite statement* $P(n)$ *holds for all* $n \in \mathbb{N}$ *by proving*

1. *the base case* $P(0)$*; and*
2. *the induction step* $\forall k \in \mathbb{N} : P(k) \implies P(Sk)$.

We call "$\forall k \in \mathbb{N} : P(k)$*" the induction hypothesis.*

Proof. Consider the set

$$X = \{n \in \mathbb{N} \mid P(n)\}.$$

The statement $P(n)$ holds for all $n \in \mathbb{N}$ if $X = \mathbb{N}$. Using the base case we have $0 \in X$ and using the induction step we have

$$\forall k \in \mathbb{N} : k \in X \implies Sn \in X.$$

By induction we conclude $X = \mathbb{N}$ and thus $P(n)$ for all $n \in \mathbb{N}$. ☐

Corollary I.170.1. *We can prove a definite statement $P(n)$ holds for all $n \geq n_0$ by proving*

1. *the base case $P(n_0)$; and*
2. *the induction step $\forall k \geq n_0 : P(k) \implies P(Sk)$.*

Lemma I.171 (Complete (strong) induction)**.** *We can prove a definite statement $P(n)$ holds for all $n \in \mathbb{N}$ by proving*

1. *the base case $P(0)$; and*
2. *the strong induction step $\forall k \in \mathbb{N} : [\forall j \leq k : P(j)] \implies P(Sk)$.*

The definition of $\leq$ follows later.

Proof. A strong inductive proof (i.e. proving these two steps) is a normal inductive proof of

$$Q(n) \quad \Leftrightarrow_{\text{def}} \quad \forall m \leq n : P(m)$$

for all n. Clearly $Q(n)$ implies $P(n)$. □

Conversely we can prove nothing new using strong induction: every strong inductive proof is in particular also a (weak) inductive one.

3.2 Existence and uniqueness of the natural numbers

Theorem I.172. *There exists a Peano system $(N, 0, S)$. For any two Peano systems $(N_1, 0_1, S_1)$ and $(N_2, 0_2, S_2)$ there exists a unique bijection $\pi : N_1 \to N_2$ such that*

$$\begin{cases} \pi(0_1) = 0_2, \\ \pi(S_1 n) = S_2 \pi(n) \quad (n \in N_1). \end{cases}$$

Such a bijection is called an isomorphism of Peano systems. The theorem says any two Peano systems are (uniquely) isomorphic. So all Peano systems are essentially the same. We denote the all in the same way: $(N, 0, S)$.

Proof. We first prove existence and then uniqueness.

Existence By the axiom of infinity we have the set I with properties

$$\emptyset \in I \qquad \text{and} \qquad \forall n \in I : \{n\} \in I.$$

We then define

$$\mathcal{J} = \{X \subseteq I \mid [\emptyset \in X] \wedge [\forall n \in X : \{n\} \in X]\}$$

Then we set

$$\mathbb{N} = \bigcap \mathcal{J}, \qquad 0 = \emptyset, \qquad S : \mathbb{N} \to \mathbb{N} : n \mapsto \{n\}.$$

Note that by constructing $\mathcal{J}$ and then taking the intersection, we have removed possible other elements of I and are left with

$$\mathbb{N} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots\}.$$

With these definitions verifying the Peano axioms is not hard. The induction principle follows from the intersection.

Uniqueness By the recursion theorem we can a unique function π satisfying the identities. What remains the be shown is that π is bijective.

Surjectivity We need to show that $\pi[N_1] = N_2$. To that end we use the induction principle on $(N_2, 0_2, S_2)$.

Obviously $0_2 \in \pi[N_1]$, since $0_2 = \pi(0_1)$.

Let $m \in \pi[N_1]$. Then $\exists n \in N_1 : m = \pi(n)$. Applying S_2 gives

$$S_2 m = S_2\pi(n) = \pi(S_1 n).$$

This implies $S_2 m \in \pi[N_1]$.

The induction principle then gives $\pi[N_1] = N_2$.

Injectivity We verify that the set

$$X = \{n \in N_1 \mid \forall m \in N_1 : \pi(m) = \pi(n) \implies m = n\}$$

equals the set N_1. This is again done using the induction principle.

Firstly, if $m \neq 0_1$, then $m = S_1 m'$ for some $m' \in N_1$, by lemma I.168. This implies

$$\pi(m) = \pi(S_1 m') = S_2\pi(m') \neq 0_2$$

and so $0_1 \in X$.

We need to show that, if $n \in X$,

$$\pi(m) = \pi(S_1 n) \implies m = S_1 n.$$

Assume the antecendent. By hypothesis $\pi(m) = \pi(S_1 n) = S_2\pi(n) \neq 0_2$ and thus $m \neq 0_1$. Again by lemma I.168 $\exists m' \in N_1 : m = S_1 m'$. So

$$S_2\pi(n) = \pi(m) = \pi(S_1 m') = S_2\pi(m')$$

implying $n = m'$ and thus $m = S_1 m' = S_1 n$.

The induction principle then gives $X = N_1$ and thus the surjectivity of π.

□

3.2.1 Zermelo ordinals

The natural numbers constructed in the existence proof are called Zermelo ordinals. They are defined by

$$0 = \emptyset \qquad \text{and} \qquad S(a) = \{a\}.$$

Then

$$\begin{aligned}
&0 = \emptyset \\
&1 = \{0\} = \{\emptyset\} \\
&2 = \{1\} = \{\{\emptyset\}\} \\
&3 = \{2\} = \{\{\{\emptyset\}\}\} \\
&\quad \ldots
\end{aligned}$$

3.2.2 (Finite) Von Neumann ordinals

The Von Neumann ordinals are an alternate construction. They are defined by

$$0 = \emptyset \qquad \text{and} \qquad S(a) = a \cup \{a\}.$$

Then

$$\begin{aligned}
&0 = \emptyset \\
&1 = 0 \cup \{0\} = \{\emptyset\} \\
&2 = 1 \cup \{1\} = \{0, 1\} = \{\emptyset, \{\emptyset\}\} \\
&3 = 2 \cup \{2\} = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
&\ \ \ldots
\end{aligned}$$

Unlike the Zermelo ordinals, the Von Neumann ordinals can be readily generalised to infinite ordinals (see later).

3.3 Operations and relations on natural numbers

We can use the recursion theorem to define addition and multiplication.

The addition function $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ on the natural numbers is defined by the recursion (with parameter)

$$\begin{cases} +(0, n) = n \\ +(Sm, n) = S(m + n) \end{cases}.$$

The multiplication function $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ on the natural numbers is defined by the recursion (with parameter)

$$\begin{cases} \cdot(0, n) = 0 \\ \cdot(Sm, n) = (m \cdot n) + n \end{cases}.$$

We will usually write $m + n$ and $m \cdot n$ or mn instead of $+(m, n)$ and $\cdot(m, n)$.

Proposition I.173. *Addition has the following properties:*

1. *It is associative:* $(k + n) + m = k + (n + m)$*;*
2. 0 *is a neutral element:* $0 + n = n$ *and* $n + 0 = n$*;*
3. *for all* $m, n \in \mathbb{N} : Sm + n = m + Sn$*;*
4. *it is commutative:* $n + m = m + n$*.*

Proof. All are proven by induction on m. The proofs of later claims make use of earlier ones. ☐

Lemma I.174. *For all* $n \in \mathbb{N}$*, the function* $\mathbb{N} \to \mathbb{N} : s \mapsto n + s$ *is 1-1, so*

$$n + s = n + t \implies s = t.$$

The binary relation $\leq$ defined by

$$n \leq m \quad \Leftrightarrow_{\text{def}} \quad \exists s \in \mathbb{N} : n + s = m$$

is called the ordering of the natural numbers.
We abbreviate $\neg(m \leq n)$ by $n < m$.

Lemma I.175. *Let $\mathbb{N}$ be ordered by $\leq$. Then $\forall n \in \mathbb{N}$*

1. $0 \leq n$;
2. *there is no m such that $n < m < n + 1$.*

TODO proof

Proposition I.176. *The endorelation $\leq$ on the natural numbers has the following properties:*

1. *it is transitive;*
2. *it is reflexive;*
3. *it is anti-symmetric;*
4. *it is connex (i.e. any two numbers are comparable);*
5. *every non-empty subset S of $\mathbb{N}$ contains an element $x \in S$ that is left related to all elements of S: $\forall y \in S : x \leq y$.*

These are exactly the properties of a well-ordering.

Proof. Take a non-empty set $S \subset \mathbb{N}$ and define the set

$$L = \{n \in \mathbb{N} \mid \forall m \in S : n \leq m\}.$$

We need to show that $L \cap S$ is non-empty. Assume, towards a contradiction, that $L \cap S = \emptyset$. Now

- $0 \in L$.
- If $n \in L$, then $n + 1 \in L$. Indeed if $n + 1 \notin L$, then there exists a $z \in S$ such that $n \leq z < n + 1$. But by I.175 this would mean that $z = n$ and $n \in L \cap S = \emptyset$.

By induction $L = \mathbb{N}$, so $S = L \cap S = \emptyset$. This is a contradiction. □

The element x in the last property is called the least element or minimum of S. It is unique by anti-symmetry and denoted $\min(S)$.
Some subsets S of $\mathbb{N}$ contain an element $x \in S$ that is right related to all elements of S: $\forall y \in S : y \leq x$. If it exists, it is called the greatest element or maximum of S. It is again unique by anti-symmetry and denoted $\max(S)$.

3.3.1 Enumerating pairs

It will turn out to be useful to have a bijection

$$\rho : \mathbb{N} \times \mathbb{N} \to \mathbb{N}.$$

We give two examples, the first due to Gödel, the second due to Zermelo.

Lemma I.177. *The functions*

$$\rho_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} : (m, n) \mapsto \frac{(m+n)(m+n+1)}{2} + m$$

and

$$\rho_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} : (m, n) \mapsto \begin{cases} (m+1)^2 - 1 & (m = n) \\ n^2 + m & (m < n) \\ m^2 + m + n & (m > n) \end{cases}$$

are bijections.

TODO pictures!

3.4 Sequences and strings

TODO: lower!

A sequence in a set A is a function

$$x : I \subseteq \mathbb{N} \to A.$$

The domain I is called the index set.
We usually write x_i instead of $x(i)$ and we denote the function x as $\langle x_i \rangle_{i \in I}$ (or $\langle x_i \rangle$ if the index is clear).

The choice of the index set is irrelevant up to a bijection. Thus which theorems hold depends on the cardinality of the index set (see later).
TODO: allow arbitrary countable index sets?

Let $\langle x_i \rangle_{i \in I}$ be a sequence. Let $f : I \to J \subset \mathbb{N}$ be an order-preserving bijection. Then we call $\left\langle x_{f(i)} \right\rangle_{i \in I}$ a subsequence of the sequence $\langle x_i \rangle_{i \in I}$.
A tail of the sequence $\langle x_i \rangle_{i \in I}$ is a subsequence where f is the identity restricted to a set of the form

$$\{j \in I \mid j \geq n\}$$

for some $n \in I$.

3.4.1 Finite sequences

A finite sequence or word or string in a set A is a sequence whose domain is a finite set.

This n is the length of the sequence, denoted $\operatorname{len}(x)$. We write

$$A^{(n)} \coloneqq ([0,n[\to A)$$
$$A^* \coloneqq \bigcup \left\{ A^{(n)} \in \mathcal{P}(\mathbb{N} \times A) \mid n \in \mathbb{N} \right\}.$$

We call A^* the Kleene closure of A.
Let $a_0, \dots, a_{n-1} \in A$. Then we have the string

$$\langle a_0, \dots, a_{n-1} \rangle \coloneqq \{(0, a_0), \dots, (n-1, a_{n-1})\} \in A^{(n)}.$$

- If $u, v \in A^*$ and $u \subseteq v$, then u is an initial segment of v, denoted $u \sqsubseteq v$.
- Let $u \in A^{(n)}, v \in A^{(m)}$ be strings. The concatenation of u and v is the string

$$u \star v \coloneqq \langle u_0, \dots, u_{n-1}, v_0, \dots, v_{m-1} \rangle \in A^{(n+m)}.$$

Notice that we can view tuples as strings of length two (i.e. there is a bijection $A \times A \leftrightarrow A^{(2)}$ for all sets A). Similarly an n-tuple can be seen as a string of length n. So sometimes we write

$$A^* = \bigcup_{n=0}^{\infty} A^n = A^{<\omega}$$

where $A^{<\omega}$ is just a notational equivalent.
Notice that if n is viewed as a Von Neumann ordinal, $A^{(n)} = A^n$.
We can think of A^* as a generalisation of $\mathbb{N}$, with $\langle\rangle$ instead of 0 and appending operators

$$S_a(u) = u \star \langle a \rangle$$

for all $a \in A$.

Theorem I.178 (String recursion theorem)**.** *Let A, E be sets, $a \in E$, and $h : E \times A \to E$ some function.*
There is exactly one function $f : A^* \to E$ which satisfies

$$\begin{cases} f(\langle\rangle) = a, \\ f(u \star \langle x \rangle) = h(f(u), x) \quad (u \in A^*, x \in A). \end{cases}$$

Proof. Define a function $\phi : \mathbb{N} \times A^* \to E$ recursively such that it satisfies

$$\phi(0, u) = a$$
$$\phi(n+1, u) = h(\phi(n, u), u(n))$$

and set $f(u) = \phi(\operatorname{len}(u), u)$. Proving that f satisfies the second equality and the uniqueness of f goes by induction on $\operatorname{len}(u)$. □

Like before, a version of the theorem can also be stated for recursion with parameters.

3.4.1.1 Functions on finite sequences

Let $f : A^{(m)} \to A^{(n)}$. We write

$$f_k(a) = f(a)(k)$$

to denote the k^{th} component of $f(a)$.

3.4.2 Inverse and complementary sequences

TODO: Hofstadter Figure-Figure sequence

3.4.2.1 Inverse sequences

`https://www.math.hkust.edu.hk/excalibur/v4_n1.pdf` `https://www.jstor.org/stable/2308078`

Let $s : \mathbb{N} \to \mathbb{N}$ be a non-decreasing sequence of natural numbers.

Chapter 4

Comparing sets

4.1 Equinumerosity: comparing sets in size

Two sets A, B are equinumerous or equal in cardinality if there exists an bijection between them:

$$A =_c B \quad \Leftrightarrow_{\text{def}} \quad \exists f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B.$$

The set A is less than or equal to B in size if it is equinumerous with some subset of B:

$$A \leq_c B \quad \Leftrightarrow_{\text{def}} \quad \exists C : C \subseteq B \wedge A =_c C.$$

We might think $=_c$ and $\leq_c$ are relations, but there is no set of all sets, so they are not defined on any set. If we restrict A, B to be subsets of a set U, they become relations on $\mathcal{P}(U)$.

Proposition I.179. *For all sets A, B, C:*

1. $A =_c A$*;*
2. *if* $A =_c B$*, then* $B =_c A$*;*
3. *if* $A =_c B$ *and* $B =_c C$*, then* $A =_c C$*.*

Restricted to U, equinumerosity is an equivalence relation on $\mathcal{P}(U)$.

Note that $\emptyset =_c \emptyset$, because $\emptyset$ is a function $\emptyset \to \emptyset$ which is bijective.

Lemma I.180. *Let A_1, A_2, B_1, B_2 be sets such that $A_1 =_c A_2$ and $B_1 =_c B_2$. Then*

1. $A_1 \sqcup B_1 =_c A_2 \sqcup B_2$*;*
2. $A_1 \times B_1 =_c A_2 \times B_2$*;*
3. $(A_1 \to B_1) =_c (A_2 \to B_2)$*.*

Proposition I.181. *Let A, B be sets. Then*

$$A \leq_c B \iff \exists f : f : A \rightarrowtail B.$$

TODO rewrite + add surjectivity

Corollary I.181.1. *If $A \subseteq B$, then $A \leq_c B$.*

This follows from the existence of the inclusion map $A \hookrightarrow B$.

Proposition I.182. *For all sets A, B, C*

1. *$A \leq_c A$;*
2. *if $A \leq_c B$ and $B \leq_c C$, then $A \leq_c C$.*

Restricted to U, $\leq_c$ is a preorder on $\mathcal{P}(U)$.

Even restricting to U, $\leq_c$ is not anti-symmetric and so not generally a partial order, but the Schröder-Bernstein theorem (theorem I.183) does give a result in this vein (with $=_c$ instead of $=$!).

Theorem I.183 (Schröder-Bernstein)**.** *Let A, B be sets. If there exist injective functions $f : A \rightarrowtail B$ and $g : B \rightarrowtail A$, then there exists a bijective function $h : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$.*

Proof. In general $f[A] \subset B$ and $g[B] \subset A$. First we identify subsets $A^* \subset A$ and $B^* \subset B$ such that $f[A^*] = A^*$ and $g[B^*] = B$. To that end we define A_n, B_n recursively

$$\begin{cases} A_0 = A \\ A_{n+1} = (g \circ f)[A_n], \end{cases} \qquad \begin{cases} B_0 = B \\ B_{n+1} = (f \circ g)[B_n]. \end{cases}$$

Then we can take

$$A^* = \bigcap_{n=0}^{\infty} A_n \qquad B^* = \bigcap_{n=0}^{\infty} B_n.$$

By induction we can see that we have chains of inclusions:

$$A_n \subseteq g[B_n] \subseteq A_{n+1}, \qquad B_n \subseteq g[A_n] \subseteq B_{n+1}.$$

Then $B^* = \bigcup_{n=0}^{\infty} f[A_n]$ by

$$B^* = \bigcup_{n=0}^{\infty} B_n \subseteq \bigcup_{n=0}^{\infty} f[A_n] \subseteq \bigcup_{n=0}^{\infty} B_{n+1} = B^*.$$

So, because f is injective, we have

$$f[A^*] = f[\bigcup_{n=0}^{\infty} A_n] = \bigcup_{n=0}^{\infty} f[A_n] = B^*$$

as required. Similarly $g[B^*] = A^*$.
Then notice that

$$A = A^* \cup \left[\bigcup_{n=0}^{\infty} (A_n \setminus g[B_n]) \cup (g[B_n] \setminus A_{n+1})\right]$$
$$B = B^* \cup \left[\bigcup_{n=0}^{\infty} (B_n \setminus f[A_n]) \cup (f[A_n] \setminus B_{n+1})\right]$$

and these are partitions of A and B.
Finally we can construct the bijection $\pi : A \rightarrowtail\!\!\!\!\rightarrow B$

$$\pi(x) = \begin{cases} f(x) & x \in A^* \vee \exists n \in \mathbb{N} : x \in (A_n \setminus g[B_n]) \\ g^{-1}(x) & x \notin A^* \wedge \exists n \in \mathbb{N} : x \in (g[B_n] \setminus A_{n+1}). \end{cases}$$

□

We can also give a more condensed proof TODO write proof (this is not yet a proof!) and express this as fixed point?:

Proof. Define $Q = f[A] \setminus (g \circ f)[A]$,

$$\mathcal{J} = \{X \in \mathcal{P}(A) \mid Q \cup (g \circ f)[X] \subseteq X\}$$

and $T = \bigcap \mathcal{J}$. We can show that $T = Q \cup (g \circ f)[T]$. Then

$$f[A] = Q \cup (g \circ f)[A] = Q \cup (g \circ f)[T] \cup [(g \circ f)[A] \setminus (g \circ f)[T]] = T \cup [(g \circ f)[A] \setminus (g \circ f)[T]]$$

□

Corollary I.183.1. *If $A \leq_c B$ and $B \leq_c A$, then $A =_c B$.*

Theorem I.184 (Cantor's theorem)**.** *For every set A,*

$$A <_c \mathcal{P}(A)$$

i.e. $A \leq_c \mathcal{P}(A)$ but $A \neq_c \mathcal{P}(A)$.

Proof. That $A \leq_c \mathcal{P}(A)$ follows from the existence of the injection $A \to \mathcal{P}(A) : x \mapsto \{x\}$.
Assume, towards a contradiction, that there exists a surjection

$$\pi : A \twoheadrightarrow \mathcal{P}(A),$$

and define the set $B = \{x \in A \mid x \notin \pi(x)\}$. Now B is a subset of A and π is a surjection, so there must exist some $b \in A$ such that $B = \pi(b)$. We get

$$b \in B \iff b \notin B.$$

A contradiction. □

4.1.1 Cantor's paradox

We can use Cantor's theorem to disprove the existence of a set of all sets. It is similar to Russell's paradox, but predates it.
Assume there was a set V of all sets. Consider the power set $\mathcal{P}(V)$. Since every element of $\mathcal{P}(V)$ is a set, $\mathcal{P}(V) \subseteq V$ and thus $\mathcal{P}(V) \leq_c V$. This contradicts Cantor's theorem, yielding a paradox.
Looking at the proof of Cantor's theorem, we see a strong analogy with Russell's paradox.

4.1.2 Countability

TODO: this definition excludes $\emptyset$ (i.e 0) from being countable. Is this ok?

Let A be a set. We say

- A is finite if there exists some $n \in \mathbb{N}$ such that
$$A =_c [0, n[= \{i \in \mathbb{N} \mid i < n\},$$
otherwise A is infinite.
- A is countable if it is equinumerous to a subset of $\mathbb{N}$, otherwise it is uncountable.

Notice the link between $[0, n[$ and the Von Neumann ordinals.
If we have a hundred pigeons, a hundred pigeonholes and are not allowed to put two pigeons in the same hole, all pigeonholes must be filled. This principle is formalised in the pigeonhole principle.

Theorem I.185 (Pigeonhole principle)**.** *Every injection $f : A \rightarrowtail A$ of a finite set into itself is also a surjection, i.e. $f[A] = A$.*

Proof. It is enough to prove that every injection $g : [0, m[\rightarrowtail [0, m[$ is surjective. (By finiteness A is bijectively related to some $[0, m[$.) The proof is by induction on m to prove the assertion:

$$\forall g : (g : [0, m[\rightarrowtail [0, m[) \implies g[\,[0, m[\,] = [0, m[.$$

Basis step If $m = 1$, there is only one such function, namely $0 \mapsto 0$, which is bijective.

Induction step It is easy to see that $[0, Sm[= [0, m[\cup\{m\}$. Taking a $g : [0, m[\rightarrowtail [0, m[$ and letting $h = g \setminus \{(m, g(m))\}$ (which is still injective), we have three cases:

$\boxed{m \notin \operatorname{im}(g)}$ By the induction hypothesis, $h[\,[0, m[\,] = [0, m[$. Because also $g(m) \in [0, m[$, g would not be injective, making this case impossible.

$\boxed{g(m) = m}$ Now $h[\,[0, m[\,] = [0, m[$ and $g(m) = m$, so $g[\,[0, Sm[\,] = [0, m[\cup\{m\} = [0, Sm[$. Thus g is a surjection.

$\boxed{\exists u < m : g(u) = m}$ In this case there must be a $v < m$ such that $g(m) = v$. Now we apply the induction hypothesis to

$$h' : [0, m[\to [0, m[: i \mapsto \begin{cases} g(i) & i \neq u \\ v & i = u. \end{cases}$$

So h' is surjective and $g[\,[0, m[\,] = [0, m[\cup\{m\} = [0, Sm[$, so g is surjective.

□

Corollary I.185.1. *The set $\mathbb{N}$ of natural numbers is infinite.*

Proof. The function $\mathbb{N} \to \mathbb{N} \setminus \{0\} : n \mapsto Sn$ is injective. □

Corollary I.185.2. *For each finite set A, there exists exactly one $n \in \mathbb{N}$ such that $A =_c [0, n[$. We call this n the number of elements of A, $\#(A)$.*

Corollary I.185.3. *Every surjection $f : A \twoheadrightarrow A$ of a finite set into itself is also an injection.*

Proof. Let $\#(A) = n$ and f be surjective. Fix a bijection $\pi : [0, n[\rightarrowtail\!\!\!\!\twoheadrightarrow A$. Construct the sets

$$X = \{j \in \mathbb{N} \mid \exists i < j : f(\pi(i)) = f(\pi(j))\} \qquad \text{and} \qquad Y = \{j \in \mathbb{N} \mid \forall i < j : f(\pi(i)) \neq f(\pi(j))\}.$$

Then $X \cup Y = [0, n[$. Now $f' : A/\sim \to A$ is a bijection and we have a bijection $A/\sim \leftrightarrow Y$ given by

$$[a] \quad \leftrightarrow \quad \min\{i \in \mathbb{N} \mid \pi(i) \in [a]\},$$

so we have a bijection $A \leftrightarrow Y$ and thus $\#(Y) = n$. Assume f not injective, in which case X is not empty and let $\#(X) = m \neq 0$.
Now because X, Y disjunct, $\#(X \cup Y) = \#(X) + \#(Y) = m + n > n$, by the uniqueness of $\#$. This is a contradiction. $\square$

Proposition I.186. *The following are equivalent for every set A:*

1. *A is countable;*
2. *there is a surjection $\pi : \mathbb{N} \twoheadrightarrow A$;*
3. *A is finite or equinumerous with $\mathbb{N}$.*

We call such a π an enumeration. Then

$$A = \pi[\mathbb{N}] = \{\pi(0), \pi(1), \pi(2), \ldots\}.$$

Proof. The proof is cyclic:

$\boxed{(1) \to (2)}$ If $A = \emptyset$, any $\pi : \mathbb{N} \to A$ is surjective. Assume $A \neq \emptyset$ and choose an $a_0 \in A$. By lemma I.181, there is an $f : A \rightarrowtail$. Define

$$\pi : \mathbb{N} \to A : i \mapsto \begin{cases} a_0 & (i \notin f[A]) \\ f^{-1}(i) & (i \in f[A]). \end{cases}$$

$\boxed{(2) \to (3)}$ Assume 2, so we have a surjective $\pi : \mathbb{N} \twoheadrightarrow A$. We need to prove that if A is not finite, it is equinumerous with $\mathbb{N}$. Define the function $f : \mathbb{N} \to A$ recursively by

$$\begin{cases} f(0) = \pi(0) \\ f(n+1) = \pi\left(\text{ the least } m \text{ such that } \pi(m) \neq \{f(0), \ldots, f(n)\}\right). \end{cases}$$

Note that because A is infinite, the set $\{m \in \mathbb{N} \mid \pi(m) \notin \{f(0), \ldots, f(n)\}$ is not empty. Due to the well-ordering on $\mathbb{N}$, every such set has a least element.

Then f is a bijection. Injectivity is obvious. Surjectivity follows from the fact that $\forall n \in \mathbb{N} : \pi(n) \in f[\mathbb{N}]$.

$\boxed{(3) \to (1)}$ All intervals $[0, n[$ are subsets of $\mathbb{N}$ and $\mathbb{N}$ is a subset of $\mathbb{N}$.

$\square$

By Cantor's theorem:

Corollary I.186.1. *There exists no set A such that $\mathcal{P}(A) =_c \mathbb{N}$.*

Lemma I.187. *We have the following characterisation of countable infinity:*

$$\begin{aligned} A =_c \mathbb{N} \quad \Leftrightarrow \quad & (\exists \mathcal{E})[A = \bigcup \mathcal{E} \\ & \&\emptyset \in \mathcal{E} \\ & \&(\forall u \in \mathcal{E})(\exists! y \notin u)[u \cup \{y\} \in \mathcal{E}] \\ & \&(\forall Z)[[\emptyset \in Z \& (\forall u \in Z)(\exists! y \notin u) u \cup \{y\} \in Z \cap \mathcal{E} \Rightarrow \mathcal{E} \subseteq Z]]. \end{aligned}$$

This is directly in terms of the membership relation with no appeal to the defined notions of $\mathbb{N}$ *and function.*

Proposition I.188 (Cantor)**.** *A countable union of countable sets is countable:*
For each sequence $A_0, A_1, \ldots$ *of countable sets, the union*

$$A = \bigcup_{n=0}^{\infty} A_n$$

is countable.

Proof. We may assume none of the A_n are empty (otherwise we can just take a superset and if this is countable, the subset will be as well). We can find an enumeration $\pi^n : \mathbb{N} \twoheadrightarrow A_n$ for each A_n. Let ρ^{-1} be an enumeration of $\mathbb{N} \times \mathbb{N}$, as in lemma I.177. Then

$$\mathbb{N} \twoheadrightarrow \bigcup_{n=0}^{\infty} A_n : m \mapsto \pi^x(y)$$

where $x(m) = \rho_0^{-1}(m)$ and $y(m) = \rho_1^{-1}(m)$, is an enumeration. ☐

Proposition I.189. *The set of infinite, binary sequences*

$$\Delta = \{(a_i)_{i \in \mathbb{N}} \mid \forall i \in \mathbb{N} : a_i = 0 \vee a_i = 1\}$$

is uncountable.

Proof. Assume, towards a contradiction, that there exists an enumeration $(\alpha_n)_{n \in \mathbb{N}}$ of Δ. The diagonal argument is then to construct the binary sequence

$$\alpha_0(0), \alpha_1(1), \alpha_2(2), \alpha_3(3), \alpha_4(4), \ldots$$

and make every 0 a 1 and vice versa. This new is not an element of the enumeration by construction (it is different from every element of the enumeration in at least one digit). ☐

4.2 Comparing well-ordered sets in length

Let U, V be well-ordered sets. We say U is less than or equal to V in length, $U \leq_o V$ if U is order isomorphic to an initial segment of V.

$$U \leq_o V \quad \Leftrightarrow_{\text{def}} \quad \exists I \sqsubseteq V : U =_o I.$$

We also write

$$U <_o V \quad \Leftrightarrow_{\text{def}} \quad U \leq_o V \wedge U \neq_o V.$$

Every proper initial segment is of the form seg(x), so using corollary III.175.1 gives

Lemma I.190. *Let U, V be well-ordered sets.*

$$U <_o V \quad \iff \quad \exists x \in V : U =_o \operatorname{seg}_V(x).$$

Clearly $=_o$ and $\leq_o$ imply $=_c$ and $\leq_c$.

Lemma I.191. *For all well-ordered sets U, V, W:*

1. $U \leq_o U$;
2. $[U \leq_o V \wedge V \leq_o W] \implies U \leq_o W$;
3. $[U \leq_o V \wedge V \leq_o U] \implies U =_o V$.

Proof. For point 3., if $U \neq_o V$, then composing the order isomorphisms would yield an isomorphism between U and a proper initial segment of U. Because such an isomorphism must be expansive we have a contradiction. ☐

Theorem I.192 (Comparability of well-ordered sets)**.** *For any two well-ordered sets U, V: either $U \leq_o V$ or $V \leq_o U$.*

Proof. The result is trivial if $V = \emptyset$, so we may assume the minimum 0_V exists.
Define, by transfinite recursion the function $f : U \to V$ such that $f(x) = h(f|_{\operatorname{seg}(x)})$ where h sends each partial function to the least element of V not in the image:

$$h : (U \not\rightarrow V) \to V : \sigma \mapsto \begin{cases} \min_V\{v \in V \mid v \notin \sigma[U]\} & (\{v \in V \mid v \notin \sigma[U]\} \neq \emptyset) \\ 0_V & (\{v \in V \mid v \notin \sigma[U]\} = \emptyset). \end{cases}$$

We immediately note two properties of f:

1. It is order-preserving. Indeed, assume $x \leq_U y$, which implies

$$\begin{aligned} \operatorname{seg}(x) \sqsubseteq \operatorname{seg}(y) &\implies f[\operatorname{seg}(x)] \subseteq f[\operatorname{seg}(y)] \implies f|_{\operatorname{seg}(x)}[U] \subseteq f|_{\operatorname{seg}(y)}[U] \\ &\implies \{v \in V \mid v \notin f|_{\operatorname{seg}(x)}[U]\} \supset \{v \in V \mid v \notin f|_{\operatorname{seg}(y)}[U]\} \\ &\implies \min\{v \in V \mid v \notin f|_{\operatorname{seg}(x)}[U]\} \leq \min\{v \in V \mid v \notin f|_{\operatorname{seg}(y)}[U]\} \\ &\implies f(x) \leq f(y). \end{aligned}$$

2. Every point in V, other than 0_V, can only be the image of at most one point in U.

We distinguish three cases for 0_V: it may be the image of zero, one or multiple points in U. If 0_V is the image of no points in U, there must be no points in U and the result is trivial.

- If 0_V is the image of one point in U, then f is injective. Then the function $f : U \to f[U]$ is bijective and order-preserving, so an order isomorphism by lemma III.10. We just need to show $f[U]$ is an initial segment of V. Take an arbitrary $x \in V$ and $y \in f[U]$ and assume $x \leq y$. There is a u such that $f(u) = y$. If x were not in $f[U]$, then $x \in \{v \in V \mid v \notin f|_{\operatorname{seg}(u)}[U]\}$, but x is smaller than the minimum yielding a contradiction. In this case $U \leq_o V$.
- If 0_V is the image of multiple points in U, there is an $x \in U$ such that

$$\{v \in V \mid v \notin f|_{\operatorname{seg}(x)}[U]\} = \emptyset.$$

We take the least such x and then $f|_{\operatorname{seg}(x)}$ is bijective. In this case $V \leq_o U$.

□

Corollary I.192.1. *For all well-ordered set U, V,*

$$U \leq_o V \quad \iff \quad \exists : \textit{order-preserving injection}\, U \rightarrowtail V.$$

Proof. If $U \leq_o V$, then U is order isomorphic to an initial segment of V; this order isomorphism is an order-preserving injection into V.
Conversely, assume there is an order-preserving injection $f : U \rightarrowtail V$. Assume, towards a contradiction that $U \nleq_o V$; by the theorem this means $V <_o U$. Then $V =_o \operatorname{seg}_U(x)$ for some $x \in U$ and composing f with this isomorphism gives an order-preserving injection $U \rightarrowtail \operatorname{seg}_U(x)$. This is obviously not expansive, so by proposition III.175 we have a contradiction. □

Corollary I.192.2 (Wellfoundedness of $\leq_o$)**.** *Every non-empty class $\mathcal{E}$ of well-ordered sets has a $\leq_o$-least member.*

i.e. for some $U_0 \in \mathcal{E}$ and all $U \in \mathcal{E}$: $U_0 \leq_o U$.

Proof. Because $\mathcal{E}$ is non-empty, we have a $W \in \mathcal{E}$. If W is $\leq_o$-least in $\mathcal{E}$, we are finished. If W is not $\leq_o$-least, there are sets U in $\mathcal{E}$ such that $U \leq_o W$ and the set

$$J \coloneqq \{x \in W \mid \exists U \in \mathcal{E} : U =_o \operatorname{seg}_W(x)\}$$

is not empty. Take the least element of J; it is easy to prove that the corresponding set U is the $\leq_o$-least member of $\mathcal{E}$. □

Corollary I.192.3. *Let U, V be well-ordered sets. Then $U \nleq_o V \iff V <_o U$.*

4.2.1 Hartogs number

The Hartogs number of any set X is a bigger well-ordered set.
A set X may not be well-orderable itself, but it definitely has well-orderable subsets. Some subsets may even have inequivalent well-orderings.

Let X be a set. Let $\mathrm{WO}(X)$ be the set

$$\mathrm{WO}(X) = \{(U, \leq_U) \in \mathcal{P}(X) \times \mathcal{P}(X \times X) \mid \leq_U \text{ is a well-ordering of } U\}.$$

Then Hartogs number of X is the set

$$\aleph(X) \coloneqq \mathrm{WO}(X)/ =_o .$$

Here $=_o$ is restricted to $\mathcal{P}(X)$ and thus is an equivalence relation (see lemma II.44).

If the natural numbers are viewed as Von Neumann ordinals, we have the following:

$$\aleph(0) = 1, \quad \aleph(1) = 2, \quad \aleph(2) = 3, \quad \ldots$$

This follows because each finite set can be well-ordered exactly one way, up to order isomorphism and well-ordered finite sets of the same size are order isomorphic.

Lemma I.193. *Let X be a set and $\aleph(X)$ its Hartogs number. Then $\leq$ defined by*

$$\forall \alpha, \beta \in \aleph(X) : \quad \alpha \leq \beta \quad \Leftrightarrow_{def} \quad \alpha = [U] \wedge \beta = [V] \wedge U \leq_o V$$

makes $(\aleph(X), \leq)$ a well-ordered set.

Proof. First we show $\leq$ is well-defined: let $[U] = [U']$ and $[V] = [V']$. Then $U =_o U'$ and thus $U' \leq_o U$; also $U \leq_V$ and $V \leq_o V'$. By point 2 of lemma I.191 we have $U' \leq_o V'$, showing the definition is well-defined.
A simple application of lemma I.191 shows us $(\aleph(X), \leq)$ is a poset.
The order is total by the comparability of well-ordered sets, theorem I.192, and well-founded by its corollary, corollary I.192.2. ☐

Lemma I.194. *Let X be a set and $\alpha = [U] \in \aleph(X)$. Then*

$$\operatorname{seg}_{\aleph(X)}(\alpha) = \{[\operatorname{seg}_U(x)] \mid x \in U\} =_o U.$$

Proof. The identity is clear by the previous lemma I.193. The isomorphism follows from proposition III.172 and the fact that each $[\operatorname{seg}_U(x)]$ contains exactly one initial segment of U, by lemma III.171. ☐

Theorem I.195 (Hartogs' lemma)**.** *Let X be a set. There is no injection $\aleph(X) \rightarrowtail X$, i.e. $\aleph(X) \not\leq_c X$.*

Proof. Suppose, towards a contradiction, that there exists an injection

$$f : \aleph(X) \rightarrowtail X$$

and let $Y = f[\aleph(X)] \subseteq X$ be its image. Then $f : \aleph(X) \to Y$ is a bijection, meaning Y is well-ordered by lemma III.169. So $Y =_o \aleph(X)$. But also $[Y] \in \aleph(X)$, because $Y \subseteq X$, and by lemma I.194 Y is similar to a proper initial segment of $\aleph(X)$. So $\aleph(X)$ would seem to be similar to a proper initial segment, but this contradicts the expansiveness of order embeddings on well-ordered sets, see corollary III.175.1. ☐

Proposition I.196. *Let X be a set. The Hartogs number $\aleph(X)$ is the $\leq_o$-least well-ordered set not smaller than or equal to X in size. i.e.*

$$\forall \text{ well-ordered sets } U : \quad U \not\leq_c X \implies \aleph(X) \leq_o U.$$

Proof. We prove the contrapositive:

$$W <_o \aleph(X) \implies W \leq_c X.$$

Assume $W <_o \aleph(X)$. Then $W =_o \operatorname{seg}_{\aleph(X)}(\alpha)$ for some $\alpha = [U] \in \aleph(X)$, so $W =_o U$ and thus $W =_c U \subseteq X$. So $W \leq_c X$. ☐

4.2.1.1 Burali-Forti's paradox

Burali-Forti's paradox is like Cantor's paradox for well-ordered sets. It shows we cannot have a set of well-ordered sets.
One way to put it is as follows: assume we have a set WO of all well ordered sets. Then $\aleph(WO)$, which is a set of well-ordered sets, must be a subset of WO, so $\aleph(WO) \leq_c WO$. This contradicts Hartogs' lemma.
A (slightly) more historical[1] approach: let $\Omega \coloneqq WO/=_o$. The elements of Ω are well-ordered by $\leq_o$ (like in lemma I.193). Consider $\Omega + 1 \coloneqq \operatorname{Succ}(\Omega)$. Because $\Omega = \operatorname{seg}(t_\Omega)$, we have $[\Omega] < [\Omega + 1]$. On the other hand, by proposition III.172 we see that $\Omega =_o \operatorname{seg}[\Omega]$, so that each element of Ω is comparable to Ω. Thus all well-ordered sets are $\leq_o \Omega$ and in particular $[\Omega + 1] \leq [\Omega]$. By $[\Omega] < [\Omega + 1]$ and $[\Omega + 1] \leq [\Omega]$ we see that the elements of Ω are not well-ordered, which is a contradiction.
The paradox is nowadays more commonly stated for (von Neumann) ordinals (see later).

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[1] See `https://zenodo.org/record/2362091/files/article.pdf`

Chapter 5

Choice

TODO: Ultrafilter lemma, Szpilrajn extension theorem

5.1 The axiom and equivalent formulations

The axiom of choice deals with the following situation: given a family of (non-empty) sets, we want to be able to pick one element from each set. The axiom of choice posits that this is possible.
The function that takes a set in the family and returns our pick is called a choice function:

Let $\mathcal{E}$ be a collection of non-empty sets. A choice function is any function

$$f : \mathcal{E} \to \bigcup \mathcal{E}$$

such that $\forall X \in \mathcal{E} : f(X) \in X$.

If there is a criterion by which to choose the object, then clearly we can find a choice function, without needing to appeal to the axiom of choice.
For example, if all sets in $\mathcal{E}$ are well-ordered, we can choose the least element from each set. The choice function is then

$$f = \left\{ (X, x) \in \mathcal{E} \times \bigcup \mathcal{E} \;\middle|\; x \in X \land \forall y \in X : x \leq X \right\}.$$

This is a well defined choice function and we did not need the axiom of choice.
For some collections of sets $\mathcal{E}$ we do need the axiom of choice to find a choice function.
There is a well-known analogy due to Russell: if we have a collection $\mathcal{E}$ of pairs of shoes, it is easy to give a choice function, just always take the left shoe for example. If we have a collection $\mathcal{E}'$ of pairs of socks this choice function does not work. We cannot construct a choice function because the socks are indistinguishable. In order to pick one sock from each pair we need to axiom of choice.
We now properly state the axiom:

(VII) **Axiom of choice (AC)**: for any non-empty set of sets $\mathcal{E}$ there is a choice function:

$$\emptyset \notin \mathcal{E} \implies \exists f \in \left(\mathcal{E} \to \bigcup \mathcal{E} \right) : \forall X \in \mathcal{E} : f(X) \in X.$$

A set S is a choice set for a family of sets $\mathcal{E}$ if

- $S \subseteq \bigcup \mathcal{E}$;
- $\forall X \in \mathcal{E} : S \cap X$ is a singleton.

Proposition I.197. *The following statements are equivalent to the axiom of choice:*

1. *The Cartesian product of any family of non-empty sets is non-empty.*
2. Zermelo Postulate *Every family $\mathcal{E}$ of non-empty and pairwise disjoint sets admits a choice set.*
3. *For every set A the family $\mathcal{P}(A) \setminus \emptyset$ admits a choice function.*
4. *For any sets A, B and binary relation $P \subseteq A \times B$,*
$$[\forall x \in A : \exists y \in B : P(x,y)] \implies [\exists f \in (A \to B) : \forall x \in A : P(x, f(x))] .$$

5.2 Some equivalent theorems

Theorem I.198. *The following results are equivalent to the axiom of choice:*

1. *Zorn's lemma: If every chain in a poset P has an upper bound, then P has a maximal element.*
2. *Zorn's lemma (dual): If every chain in a poset P has an lower bound, then P has a minimal element.*
3. *Hypothesis of cardinal comparability: for all sets A, B: $A \leq_c B$ or $B \leq_c A$.*
4. *Well-ordering theorem: every set is well-orderable.*

Proof. We proceed round-robin-style:

$\boxed{\text{(AC)} \Rightarrow (1)}$ Assume every chain in a poset P has an upper bound. Let f be a choice function on $\mathcal{P}(P) \setminus \emptyset$. We define a function $g : \aleph(P) \to P$ by transfinite recursion as follows:

$$g(x) = \begin{cases} f\left(\{\text{upper bounds of } g[\text{seg}(x)]\} \setminus g[\text{seg}(x)]\right) & \text{(if defined, i.e. we are not choosing from } \emptyset) \\ f\left(\{\text{upper bounds of } g[\text{seg}(x)]\}\right) & \text{(else).} \end{cases}$$

This is well defined because $g[\text{seg}(x)]$ is a chain in P and thus $\{\text{upper bounds of } g[\text{seg}(x)]\}$ cannot be empty by assumption, so the second case always works.

By Hartogs' lemma, theorem I.195, the function g cannot be injective, so there exists an $m \in P$ that is the image of multiple elements $x_1, x_2 \in \aleph(P)$. Assume $x_1 < x_2$, then $g(x_2)$ must have been defined by the second case in the recursion, meaning m is a maximal element of P.

$\boxed{(1) \Leftrightarrow (2)}$ By duality.

$\boxed{(1) \Rightarrow (3)}$ Every chain in the set $(A \not\rightarrowtail B)$ of injective partial functions from A to B, ordered by inclusion, is inductive by proposition III.32, and thus has a maximal element. If this maximal element is a total function, then $A \leq_c B$. If it is not a total function, it is surjective and we have a bijection from a subset of A to be, i.e. $B \leq_c A$.

$\boxed{(3) \Rightarrow (4)}$ Let A be a set. By Hartogs' lemma, theorem I.195, $\aleph(A) \not\leq_c A$. By cardinal comparability this implies $A \leq_c \aleph(A)$. The bijection between A and a subset of $\aleph(A)$ determines a well-ordering on A.

$\boxed{(4) \Rightarrow (\text{AC})}$ Let A be a set. Then we can define a well-ordering $\leq$ on A. We can then define a choice function on $\mathcal{P}(A) \setminus \emptyset$ by returning the $\leq$-least member of each subset.

□

Lemma I.199. *Every surjective function has a right inverse* if and only if *the axiom of choice holds.*

A family of sets $\mathcal{E}$ is of finite character if

$$X \in \mathcal{E} \qquad \iff \qquad \text{every finite subset of } X \text{ belongs to } \mathcal{E}.$$

An immediate property of families of finite character: for each $X \in \mathcal{E}$, every (finite or infinite) subset of X belongs to $\mathcal{E}$.

Lemma I.200. *Any family of sets $\mathcal{E}$ of finite character ordered by inclusion is inductive.*

Proof. Let S be a chain in $\mathcal{E}$, then we claim $\bigcup S \in \mathcal{E}$. Indeed every (finite) subset of $\bigcup S$ is a subset of an element of S and so $\bigcup S \in \mathcal{E}$. □

Some more principles reminiscent of (and equivalent to) Zorn's lemma:

Theorem I.201. *The following results are equivalent to the axiom of choice:*

1. *Teichmüller-Tukey lemma: Every non-empty collection of finite character has a maximal element with respect to inclusion.*
2. *Hausdorff maximal principle: Let P be a poset. Every chain in P is contained in a maximal chain in P.*
3. *Maximal chain principle: Every non-empty poset has a maximal chain.*

The Hausdorff maximal principle is also known as the "Kuratowski lemma". The name "Hausdorff maximal principle" may be reserved for the specific case where P is a family of sets ordered by inclusion. These formulations are equivalent, by III.128.

Proof. We prove equivalence with Zorn's lemma round-robin-style:

$\boxed{(\text{Zorn}) \Rightarrow (1)}$ Assume Zorn's lemma. Let $\mathcal{E}$ be a non-empty collection of finite character, which is partially ordered by inclusion. Because $\mathcal{E}$ is inductive, by lemma I.200, each chain has an upper bound and thus $\mathcal{E}$ has a maximal element.

$\boxed{(1) \Rightarrow (2)}$ Being a chain is a property of finite character. Let C be a chain in P. Then the family of all chains in P containing C is a family of finite character and P is contained in the maximal element.

$\boxed{(2) \Rightarrow (3)}$ Take one element in the poset. The set of this element is a chain.

$\boxed{(3) \Rightarrow (\text{Zorn})}$ Assume there is a maximal chain and every chain has an upper bound. Any upper bound of a maximal chain is a maximal element.

□

5.3 Weaker axioms

5.3.1 Countable choice

It is in general clear we can make a finite number of choices, by the definition of the existence quantifier (TODO). The axiom of choice says we can make an arbitrary number of choices. The axiom of countable choice is weaker, it says we can make a countable number of choices. Compare also to point 4. of I.197.

(VII') **Axiom of countable choice ($\mathbf{AC}_{\mathbb{N}}$)**: for any set B and binary relation $P \subseteq \mathbb{N} \times B$,

$$[\forall n \in \mathbb{N} : \exists y \in B : P(n, y)] \implies [\exists f \in (\mathbb{N} \to B) : \forall n \in \mathbb{N} : P(n, f(n))].$$

5.3.2 Dependent choice

The axioms

Proposition I.202. *Assume the axiom of dependent choice and let P be a poset. Then*

1. *P satisfies the descending chain condition* if and only if *P is well-founded;*
2. *P satisfies the ascending chain condition* if and only if *P is converse well-founded.*

Proof. TODO ☐

Corollary I.202.1. *A poset P has no infinite chains* if and only if *it satisfies both the ascending and the descending chain condition.*

Proof. Clearly if P has no infinite chains, then it satisfies both the ascending and the descending chain condition.
Suppose, towards a contradiction, that P satisfies both the ascending and the descending chain condition and contains an infinite chain C. Then C has both a maximal en and a minimal element by the proposition. Because C is totally ordered, these maximal and minimal elements are greatest and least elements. Then C is finite by the ascending and the descending chain conditions. ☐

Chapter 6

Replacement

Maximal antichain principle: Every non-empty poset has a maximal antichain.

Chapter 7

Cardinals and ordinals

`http://euclid.colorado.edu/~monkd/monk11.pdf`

7.1 Cardinals

The idea behind the cardinals is to have a set of objects that witnesses the size, or potency, of sets. These two conditions define the cardinal assignment.

Define an operation $A \mapsto |A|$ on the class of sets such that

- $A =_c |A|$;
- for each set of sets $\mathcal{E}$, $\{|X| \mid X \in \mathcal{E}\}$ is a set.

Such an operation is called a (weak) cardinal assignment. The class of cardinal numbers(relative to a given cardinal assignment), denoted Card, is the image of the cardinal assigment:

$$\kappa \in \mathrm{Card} \quad \Leftrightarrow_{\mathrm{def}} \quad \exists A : \kappa = |A|.$$

If the cardinal assignment also satisfies

- if $A =_c B$, then $|A| = |B|$,

then it is called a strong cardinal assignment.

In particular, notice that cardinals are sets. Indeed the assignment $A \mapsto |A|$ is a weak cardinal assignment, so practically all results in this section hold in particular for sets.

Lemma I.203. *If cardinal numbers are defined using a strong cardinal assignment, then for all cardinals κ, λ:*

$$|\kappa| = \kappa \quad \textit{and} \quad \kappa =_c \lambda \iff \kappa = \lambda$$

Lemma I.204. *For any cardinal assignment and any two sets A, B,*

1. $|A| = |B| \implies A =_c B$;
2. $|\emptyset| = \emptyset$.

7.1.1 Cardinal arithmetic without choice

Fix a specific, possibly weak, cardinal assignment. We define the arithmetic operations on the cardinal numbers κ, λ as

$$\begin{aligned} \kappa + \lambda &:= |\kappa \sqcup \lambda| && =_c \kappa \sqcup \lambda, \\ \kappa \cdot \lambda &:= |\kappa \times \lambda| && =_c \kappa \times \lambda, \\ \kappa^\lambda &:= |(\lambda \to \kappa)| && =_c (\lambda \to \kappa). \end{aligned}$$

Also

$$\begin{aligned} \sum_{i \in I} \kappa_i &:= \left| \bigsqcup_{i \in I} \kappa_i \right|, \\ \prod_{i \in I} \kappa_i &:= \left| \prod_{i \in I} \kappa_i \right|. \end{aligned}$$

We fix the following symbols for some empty set, singleton and doubleton cardinals:

$$0 := |\emptyset| = \emptyset \qquad 1 := |\{0\}| \qquad 2 := |\{0, 1\}|.$$

The definition of $0, 1, 2$ does not conflict with the natural numbers.

Lemma I.205. *Let $\kappa_1, \kappa_2, \lambda_1, \lambda_2$ be cardinal numbers such that $\kappa_1 =_c \kappa_2$ and $\lambda_1 =_c \lambda_2$. Then*

1. $\kappa_1 + \lambda_1 =_c \kappa_2 + \lambda_2$;
2. $\kappa_1 \cdot \lambda_1 =_c \kappa_2 \cdot \lambda_2$;
3. $\kappa_1^{\lambda_1} =_c \kappa_2^{\lambda_2}$.

The proof is the same as for lemma I.180.

Lemma I.206. *For all sets A, B and thus for all cardinals κ, λ:*

$$\prod_{i \in A} B = (A \to B), \qquad \prod_{i \in \lambda} \kappa = \kappa^\lambda.$$

Notice that there is equality, $=$, not just equinumerosity $=_c$.
Cardinal arithmetic has properties reminiscent of a commutative semiring (i.e. ring where additive inverses are not guaranteed, see later). Of course cardinal arithmatic is not defined on a set, but on a class, so this is not completely true.

Lemma I.207. *Let κ, λ, μ be cardinal numbers. Then $+$ is like a commutative monoid:*

1. $\kappa + (\lambda + \mu) =_c (\kappa + \lambda) + \mu$;
2. $0 + \kappa =_c \kappa + 0 =_c \kappa$;
3. $\kappa + \lambda =_c \lambda + \kappa$;

and $\cdot$ is also like a commutative monoid:

4. $\kappa \cdot (\lambda \cdot \mu) =_c (\kappa \cdot \lambda) \cdot \mu$;

5. $1 \cdot \kappa =_c \kappa \cdot 1 =_c \kappa$;

6. $\kappa \cdot \lambda =_c \lambda \cdot \kappa$.

Multiplication distributes over addition:

7. $\kappa \cdot (\lambda + \mu) =_c \kappa \cdot \lambda + \kappa \cdot \mu$.

Multiplication by 0 *annihilates:*

8. $0 \cdot \kappa =_c 0$.

There are no zero divisors:

9. $\kappa \neq 0 \neq \lambda \implies \kappa \cdot \lambda \neq 0$.

Lemma I.208. *Let κ_i be cardinals for all $i \in I$. Then*

$$\exists i \in I : \kappa_i = 0 \implies \prod_{i \in I} \kappa_i.$$

The other implication depends on the axiom of choice!

Lemma I.209. *Let κ be a cardinal number. Then*

$$\kappa^0 =_c 1, \qquad \kappa^1 =_c \kappa, \qquad \kappa^2 =_c \kappa \cdot \kappa.$$

Lemma I.210. *Let κ, λ, μ be cardinal numbers. Then*

1. $(\kappa \cdot \lambda)^\mu =_c \kappa^\mu \cdot \lambda^\mu$;

2. $\kappa^{(\lambda+\mu)} =_c \kappa^\lambda \cdot \kappa^\mu$;

3. $\left(\kappa^\lambda\right)^\mu =_c \kappa^{\lambda \cdot \mu}$.

By Cantor's theorem I.184, this also gives $\kappa \leq_c 2^\kappa$.

Lemma I.211. *Let κ, λ_i be cardinals for all $i \in I$. Then*

$$\kappa \cdot \sum_{i \in I} \lambda_i =_c \sum_{i \in I} \kappa \cdot \lambda_i.$$

Proof. We compute

$$\begin{aligned} \kappa \cdot \sum_{i \in I} \lambda_i &=_c \kappa \times \left(\bigsqcup_{i \in I} \lambda_i \right) = \kappa \times \left(\bigcup_{i \in I} \{i\} \times \lambda_i \right) = \bigcup_{i \in I} \kappa \times (\{i\} \times \lambda_i) \\ &=_c \bigcup_{i \in I} \{i\} \times (\kappa \times \lambda_i) = \bigsqcup_{i \in I} \kappa \times \lambda_i =_c \sum_{i \in I} \kappa \cdot \lambda_i. \end{aligned}$$

□

TODO: expand on such computations?

Lemma I.212. *Let κ, λ, μ be cardinal numbers. Then*

1. $\kappa \leq_c \mu \implies \kappa + \lambda \leq_c \mu + \lambda$;

2. $\kappa \leq_c \mu \implies \kappa \cdot \lambda \leq_c \mu \cdot \lambda$;

3. $\kappa \leq_c \mu \implies \kappa^\lambda \leq_c \mu^\lambda$;

4. $\kappa \leq_c \mu \implies \lambda^\kappa \leq_c \lambda^\mu$ *if* $\lambda \neq 0$.

These implications do not necessarily hold for strict inequalities.
For the last implication: if $\lambda = 0$, then

$$\forall \kappa \in \text{Card} : \lambda^\kappa = (\kappa \to \emptyset) = \begin{cases} \emptyset & \kappa \neq 0 \\ \{\emptyset\} & \kappa = 0. \end{cases}$$

So if $\kappa = 0$ and $\mu \neq 0$, there exists an injection $\kappa \to \mu$, namely $\emptyset$ and so $\kappa \leq_c \mu$, but there is no injection (in fact no function) $\{\emptyset\} \to \emptyset$, so $\lambda^\kappa \not\leq_c \lambda^\mu$.

7.1.2 Cardinal arithmetic with choice

$$|\mathbb{F}|^{|\beta|} > |\beta|$$

In this section we assume the axiom of choice.

Given any cardinal κ, we can define the successor cardinal as

$$\kappa^+ \coloneqq |\aleph(\kappa)|\,.$$

Lemma I.213. *For any cardinal κ, the cardinal κ^+ is $\leq_c$-least among the cardinals bigger than κ.*

Proof. By Hartogs' lemma and cardinal comparability, we know $\kappa <_c \kappa^+$. By proposition I.196 κ^+ is $\leq_o$-least (and thus $\leq_c$-least) with this property. ☐

Tarski's theorem about choice: For every infinite set A, there is a bijective map between the sets A and $A \times A$.

7.1.3 The cardinality of natural numbers

We define

$$\aleph_0 \coloneqq |\mathbb{N}|.$$

This is the cardinality of countably infinite sets. If we have a strong cardinal assignment is is uniquely so.
We also define for each $n \in \mathbb{N}$:

$$\kappa^n \coloneqq |\kappa^{(n)}|.$$

We also define $\aleph_1 \coloneqq \aleph_0^+, \aleph_2 \coloneqq \aleph_1^+, \ldots$
If we take the natural numbers to be Von Neumann ordinals, then for all finite sets A, $\#(A) =_c A$. Then $\#$ is a strong cardinal assignment on the finite sets and all the previous results apply.

Proposition I.214. *For each countably infinite set A and each $n > 0$,*

$$A =_c A \times A =_c A^{(n)} =_c A^*.$$

The equivalent expression in cardinal arithmetic is

$$\aleph_0 =_c \aleph_0 \cdot \aleph_0 =_c \aleph_0^n =_c |\aleph_0^*|.$$

Proof. If all $=_c$ are replaced by $\leq_c$, the claim is trivial, thus, by the Schröder-Bernstein theorem I.183, it is enough to show $\mathbb{N}^* \leq_c \mathbb{N}$.
Choose a bijection $\rho : \mathbb{N} \times \mathbb{N} \rightarrowtail\!\!\!\!\!\twoheadrightarrow \mathbb{N}$ as in lemma I.177.
Define by recursion a function $f : \mathbb{N} \to (\mathbb{N}^{(n+1)} \rightarrowtail \mathbb{N}) : n \mapsto \pi_n$, such that

$$\begin{aligned} \pi_0(u) &= u(0) \\ \pi_{n+1}(u) &= \rho(\pi_n(u|_{[0,n+1[}), u(n+1)). \end{aligned}$$

Then the function

$$\pi(u) = (\operatorname{len}(u) - 1, \pi_{\operatorname{len}(u)-1}(u))$$

is injective, proving

$$\bigcup_{n=0}^{\infty} \mathbb{N}^{(n+1)} \leq_c \mathbb{N} \times \mathbb{N}.$$

Using ρ we see that $\bigcup_{n=0}^{\infty} \mathbb{N}^{(n+1)} \leq_c \mathbb{N}$. □

Lemma I.215. *For all cardinals κ, $2^\kappa \neq_c \aleph_0$.*

Proof. We split into two cases κ countable and uncountable:

1. If κ is countable, then either $\kappa =_c \aleph_0$ and $2^\kappa \neq_c \aleph_0$ by Cantor's theorem, I.184, or κ is finite in which case $2^\kappa =_c \#(2^\kappa) = 2^{\#(\kappa)}$ which is finite.

2. If κ is uncountable and $2^\kappa =_c \aleph_0$, then $\kappa \leq_c 2^\kappa =_c \aleph_0$ and κ would be countable. A contradiction.

□

This is an expression of the fact that we can compare cardinals to countable cardinals, even without choice.

7.1.4 The continuum

We define the continuum

$$\mathfrak{c} := |\mathcal{P}(\mathbb{N})| =_c 2^{\aleph}_0.$$

The continuum hypothesis is that there are no cardinals between $\aleph_0$ and $\mathfrak{c}$. This is independent of ZFC.
From cardinal arithmetic we can immediately obtain some results, like

$$\mathfrak{c} \cdot \mathfrak{c} =_c 2^{\aleph_0} \cdot 2^{\aleph_0} =_c 2^{\aleph_0 + \aleph_0} =_c 2^{\aleph_0} =_c \mathfrak{c}$$

and

$$\mathfrak{c} =_c 2^{\aleph_0} \leq_c \aleph_0^{\aleph_0} \leq_c \mathfrak{c}^{\aleph_0} =_c \left(2^{\aleph_0}\right)^{\aleph_0} =_c 2^{\aleph_0 \cdot \aleph_0} =_c \mathfrak{c}.$$

7.2 Ordinals

A set S is a (von Neumann) ordinal if it is transitive and all its members are transitive.
The class of all ordinals is denoted Ord.

The class Ord is a transitive class.

Chapter 8

Going from two to many

8.1 Functions on ordinals

Let α be an ordinal and A a set. We introduce special notation in this case:

- $A^\alpha := (\alpha \to A)$;
- if $a \in A^\alpha$, then we write $a_0, a_1, a_2 \ldots$ instead of $a(0), a(1), a(2) \ldots$

8.1.1 Pointwise extensions

Let $\alpha \in \mathrm{Ord}$, $A, B \in \mathrm{Set}$ and $f \in (A \to B)$. Then

$$f^\alpha : A^\alpha \to B^\alpha : a \mapsto f \circ a$$

is the pointwise extension of f to A^α.

This is a particular instance of post-composition, f_*.

8.2 Finite Cartesian proucts: Tuples

Let $a_1, \ldots, a_n$ be objects. A definition of $(a_1, \ldots, a_n)$ is called an n-tuple operation (or just tuple operation) if it satisfies

- $(a_1, \ldots, a_n) = (b_1, \ldots, b_n) \iff \forall i \in (1:n) : a_i = b_i$;
- for all sets $A, \ldots, A_n$, $\{(a_1, \ldots, a_n) \mid \forall i \in (1:n) : a_i \in A_i\}$ is a set.

We call

$$\bigtimes_{i=1}^{n} A_i = \{(a_1, \ldots, a_n) \mid \forall i \in (1:n) : a_i \in A_i\}$$

the Cartesian product of $A_1, \ldots, A_n$.

Proposition I.216. *Let $a_1, \ldots, a_n$ be objects. Defining $(a_1, \ldots, a_n)$ as the string $\langle a_1, \ldots, a_n \rangle$ of length n is a valid n-tuple operation.*

We can also directly use a pair operation to define a n-tuple operation.

Proposition I.217. *Let $a_1, \ldots, a_n$ be objects. Define the function f_a recursively by*

$$\begin{cases} f_a(0) = \emptyset \\ f_a(i+1) = \begin{cases} (f_a(i), a_{i+1}) & i < n \\ (f_a(i), \emptyset) & i \geq n \end{cases} \end{cases}$$

Defining $(a_1, \ldots, a_n)$ as $f_a(n)$ is a valid n-tuple operation.

Informally, this construction can be described as follows:

- The 0-tuple is defined as the empty set $\emptyset$;
- A 1-tuple containing a is defined as $(\emptyset, a)$;
- An n-tuple, with $n > 1$, is defined as an ordered pair of its last entry and an $(n-1)$-tuple which contains the preceding entries:

$$(a_1, \ldots, a_n) = ((a_1, \ldots, a_{n-1}), a_n) = ((\ldots((\emptyset, a_1), a_2), \ldots), a_n).$$

For example $(1, 2, 3, 4) = ((((\emptyset, 1), 2), 3), 4)$.

8.2.1 Association relations

TODO: $((a, b), c) \approx (a, (b, c))$.

8.2.2 n-ary relations

Let $A_1, \ldots, A_n$ be sets and $G \subseteq \bigtimes_{i=1}^{n} A_i$. We call $R = (G, (A_1, \ldots, A_n))$ an n-ary relation on $(A_1, \ldots, A_n)$ and $\mathrm{graph}(R) \coloneqq G$ the graph of R.

In particular a ternary relation on (A, B, C) is a structured sets $(G, (A, B, C))$ where $G \subset A \times B \times C$.

Proposition I.218. *Let $A_1, \ldots, A_n$ be sets and $G \subseteq \bigtimes_{i=1}^{n} A_i$. Using the definition of n-tuple in I.216, $R = (G, (A_1 \times \ldots \times A_{n-1}), A_n))$ is a binary relation.*

8.3 Operations on sequences of sets

Let I be an arbitrary index set, A some set and let there be a surjective function $a : I \twoheadrightarrow A : i \mapsto a_i$.
Then we say A is indexed by I and we write $A = \{a_i\}_{i \in I}$. In this case we call A an (indexed) family.
In particular if $A \subseteq \mathcal{P}(X)$ for some set X, A is an indexed family of sets.

Notice we do not require the function a to be injective and thus multiple indexed elements may be the same.
Sometimes the notation $(a_i)_{i \in I}$ is used, if I is ordered, to emphasise the ordering of $\{a_i\}_{i \in I}$ by I.

8.3.1 Union and intersection of indexed families of sets

Let $\{A_i\}_{i\in I}$ be an indexed family of sets. Then we write

$$\bigcup_{i\in I} A_i \coloneqq \bigcup A[I]$$
$$\bigcap_{i\in I} A_i \coloneqq \bigcap A[I]$$

If $I = [0, n[$, we write $\{A_i\}_{i=1}^{n} \coloneqq \{A_i\}_{i\in[0,n[}$,

$$\bigcup_{i=1}^{n-1} A_i \coloneqq \bigcup_{i\in[0,n[} A_i \qquad \text{and} \qquad \bigcap_{i=1}^{n-1} A_i \coloneqq \bigcap_{i\in[0,n[} A_i$$

and if $I = \mathbb{N}$, we write $\{A_i\}_{i=1}^{n} \coloneqq \{A_i\}_{i\in\mathbb{N}}$,

$$\bigcup_{i=1}^{\infty} A_i \coloneqq \bigcup_{i\in\mathbb{N}} A_i \qquad \text{and} \qquad \bigcap_{i=1}^{\infty} A_i \coloneqq \bigcap_{i\in\mathbb{N}} A_i.$$

Lemma I.219. *Let A be a set and $\{B_i\}_{i\in I}$ an indexed family of sets. Then*

$$A \times \bigcup_{i\in I} B_i = \bigcup_{i\in I} A \times B_i;$$
$$A \times \bigcap_{i\in I} B_i = \bigcap_{i\in I} A \times B_i.$$

8.3.1.1 Multiple indices

If the index set I is a Cartesian product $I = J \times K$, then we also write

$$\bigcup_{(j,k)\in I} A_{j,k} = \bigcup_{\substack{j\in J\\ k\in K}} A_{j,k} \qquad \text{and} \qquad \bigcap_{(j,k)\in I} A_{j,k} = \bigcap_{\substack{j\in J\\ k\in K}} A_{j,k}.$$

If $\{A_{j,k}\}_{(j,k)\in J\times K}$ is such an indexed family of sets, then $\{A_{j,k'}\}_{j\in J}$ is an indexed family of sets for each $k \in K$by partial application of k' to the second argument. This allows us to apply union and intersection pointwise: we define

$$\bigcup_{j\in J} A_{j,k} \coloneqq \left(k' \mapsto \bigcup_{j\in J} A_{j,k'}\right) \qquad \text{and} \qquad \bigcap_{j\in J} A_{j,k} \coloneqq \left(k' \mapsto \bigcap_{j\in J} A_{j,k'}\right)$$

as well as something similar for the first argument.

8.3.1.2 Associativity and commutativity

Lemma I.220. *Let $\{A_{j,k}\}_{(j,k)\in J\times K}$ be an indexed family of sets. Then*

$$\bigcup_{j\in J}\left(\bigcup_{k\in K} A_{j,k}\right) = \bigcup_{\substack{j\in J\\ k\in K}} A_{j,k} = \bigcup_{k\in K}\left(\bigcup_{j\in J} A_{j,k}\right) \qquad \textit{and}$$

$$\bigcap_{j\in J}\left(\bigcap_{k\in K} A_{j,k}\right) = \bigcap_{\substack{j\in J\\ k\in K}} A_{j,k} = \bigcap_{k\in K}\left(\bigcap_{j\in J} A_{j,k}\right).$$

Corollary I.220.1. *Let $\{A_i\}_{i\in I}$ be an indexed family of sets and B a set. Then*

$$\left(\bigcup_{i\in I} A_i\right) \cup B = \bigcup_{i\in I}(A_i \cup B) \qquad \textit{and} \qquad \left(\bigcap_{i\in I} A_i\right) \cap B = \bigcap_{i\in I}(A_i \cap B).$$

Proof. Set $J \times K = I \times \{0,1\}$ and $A_{j,k} = \begin{cases} A_j & (k=0) \\ B & (\text{else}) \end{cases}$. ☐

Corollary I.220.2. *Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be indexed families of sets. Then*

$$\left(\bigcup_{i\in I} A_i\right) \cup \left(\bigcup_{j\in J} B_j\right) = \bigcup_{\substack{i\in I\\ j\in J}}(A_i \cup B_j) \qquad \textit{and} \qquad \left(\bigcap_{i\in I} A_i\right) \cap \left(\bigcap_{j\in J} B_j\right) = \bigcap_{\substack{i\in I\\ j\in J}}(A_i \cap B_j).$$

If both families are indexed by the same index set I, we may take the union/intersection over just I, not $I \times I$.

8.3.1.3 Distributivity

Lemma I.221. *Let $\{A_i\}_{i\in I}, \{B_j\}_{j\in J}$ be indexed families of sets and C a set. Then*

$$\left(\bigcup_{i\in I} A_i\right) \cap B = \bigcup_{i\in I}(A_i \cap B) \qquad \textit{and} \qquad \left(\bigcap_{i\in I} A_i\right) \cup B = \bigcap_{i\in I}(A_i \cup B).$$

Also

$$\left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{j\in J} B_j\right) = \bigcup_{\substack{i\in I\\ j\in J}}(A_i \cap B_j) \qquad \textit{and} \qquad \left(\bigcap_{i\in I} A_i\right) \cup \left(\bigcap_{j\in J} B_j\right) = \bigcap_{\substack{i\in I\\ j\in J}}(A_i \cup B_j).$$

Proposition I.222. *Let $\{A_{j,k}\}_{(j,k)\in J\times K}$ be an indexed family of sets. Then*

$$\bigcap_{j\in J}\bigcup_{k\in K} A_{j,k} = \bigcup_{f\in K^J}\bigcap_{j\in J} A_{j,f(j)}$$
$$\bigcup_{j\in J}\bigcap_{k\in K} A_{j,k} = \bigcap_{f\in K^J}\bigcup_{j\in J} A_{j,f(j)}$$

Proof. We have

$$\begin{aligned} x \in \bigcap_{j\in J}\bigcup_{k\in K} A_{j,k} &\iff \forall j \in J : \exists k \in K : x \in A_{j,k} \\ &\iff \exists f \in J^K : \forall j \in J : x \in A_{j,f(j)} \\ &\iff \bigcup_{f\in K^J}\bigcap_{j\in J} A_{j,f(j)}. \end{aligned}$$

☐

The $f \in J^K$ encodes which k is the "good one" for each j.

Corollary I.222.1. *Let* $\{A_{j,k}\}_{(j,k)\in J\times K}$ *be an indexed family of sets. Then*

$$\bigcup_{j\in J}\left(\bigcap_{k\in K} A_{j,k}\right) \subseteq \bigcap_{k\in K}\left(\bigcup_{j\in J} A_{j,k}\right).$$

Proof. We restrict the f in the proposition to the set of constant functions. □

In general these two sets are not equal!

8.3.1.4 Union and intersection of index sets

Lemma I.223. *Let* $\mathcal{I}$ *be a family of index sets and let* A_i *be a set for all* $i \in \bigcup\mathcal{I}$*. Then*

1. $\bigcup_{i\in\bigcup\mathcal{I}} A_i = \bigcup_{I\in\mathcal{I}}\bigcup_{i\in I} A_i$;
2. $\bigcap_{i\in\bigcap\mathcal{I}} A_i = \bigcap_{I\in\mathcal{I}}\bigcap_{i\in I} A_i$;
3. $\bigcup_{i\in\bigcap\mathcal{I}} A_i \subseteq \bigcap_{I\in\mathcal{I}}\bigcup_{i\in I} A_i$;
4. $\bigcap_{i\in\bigcup\mathcal{I}} A_i \supseteq \bigcup_{I\in\mathcal{I}}\bigcap_{i\in I} A_i$.

Proof. (1) We calculate

$$\begin{aligned}
x \in \bigcup_{i\in\bigcup\mathcal{I}} A_i &\iff \exists i \in \bigcup\mathcal{I} : x \in A_i \\
&\iff \exists i : (i \in \bigcup\mathcal{I}) \wedge (x \in A_i) \\
&\iff \exists i : (\exists I \in \mathcal{I} : i \in I) \wedge (x \in A_i) \\
&\iff \exists i : \exists I : (I \in \mathcal{I}) \wedge (i \in I) \wedge (x \in A_i) \\
&\iff \exists I \in \mathcal{I} : \exists i \in I : x \in A_i \\
&\iff x \in \bigcup_{I\in\mathcal{I}}\bigcup_{i\in I} A_i
\end{aligned}$$

(2) Replace $\exists$ by $\forall$ and $\wedge$ by $\Rightarrow$ in the proof of (1).
(3) We calculate

$$\begin{aligned}
x \in \bigcup_{i\in\bigcap\mathcal{I}} A_i &\iff \exists i \in \bigcap\mathcal{I} : x \in A_i \\
&\iff \exists i : (i \in \bigcap\mathcal{I}) \wedge (x \in A_i) \\
&\iff \exists i : (\forall I \in \mathcal{I} : i \in I) \wedge (x \in A_i) \\
&\iff \exists i : (\forall I : (I \in \mathcal{I}) \Rightarrow (i \in I)) \wedge (x \in A_i) \\
&\implies \exists i : \forall I : (I \in \mathcal{I}) \Rightarrow ((i \in I) \wedge (x \in A_i)) \\
&\implies \forall I : \exists i : (I \in \mathcal{I}) \Rightarrow ((i \in I) \wedge (x \in A_i)) \\
&\iff \forall I : (I \in \mathcal{I}) \Rightarrow (\exists i : (i \in I) \wedge (x \in A_i)) \\
&\iff \forall I \in \mathcal{I} : \exists i \in I : x \in A_i \\
&\iff x \in \bigcap_{I\in\mathcal{I}}\bigcup_{i\in I} A_i
\end{aligned}$$

(4) TODO □

8.3.2 Arbitrary Cartesian products

Let $\{A_i\}_{i\in I}$ be an arbitrary indexed family of sets, then we define the Cartesian product of $\{A_i\}_{i\in I}$ to be

$$\prod_{i\in I} A_i \coloneqq \left\{ f \in \left(I \to \bigcup_{i\in I} A_i \right) \,\middle|\, \forall i \in I : f(i) \in A_i \right\}.$$

For each $j \in I$, the function

$$\pi_j : \prod_{i\in I} A_i \to A_j : f \mapsto f(j)$$

is called the j^{th} projection map.

A Cartesian product of an indexed family of sets $\{A_i\}_{i\in I}$ is called a Cartesian power of A if for all $i \in I$, A_i is the same set A. This is denoted A^I.

If $I = [0, n[$ for some $n \in \mathbb{N}$, we write $A^n = A^I$.
Note that

$$A^I = \prod_{i\in I} A = (I \to A).$$

TODO IMPORTANT ↑!

Lemma I.224. *There exists a bijection* $A_0 \times A_1 \leftrightarrow \prod_{i\in\{0,1\}} A_i$.

Proof. The bijection is given by

$$(a, b) \quad \leftrightarrow \quad \{(0, a), (1, b)\} \qquad \forall a \in A_0, b \in A_1.$$

□

8.3.2.1 Distributing over unions and intersections

Lemma I.225. *Let* $\{A_i\}_{i\in I}$ *and* $\{B_j\}_{j\in J}$ *be indexed families of sets, indexed over the same index family* I. *Then*

$$\left(\prod_{i\in I} A_i\right) \cap \left(\prod_{i\in I} B_i\right) = \prod_{i\in I} (A_i \cap B_i) \qquad \textit{but} \qquad \left(\prod_{i\in I} A_i\right) \cup \left(\prod_{i\in I} B_i\right) \subset \prod_{i\in I} (A_i \cup B_i).$$

Lemma I.226. *Let* $\{A_{i,j}\}_{(i,j)\in I\times J}$ *be an indexed family of sets. Then*

$$\bigcap_{i\in I} \left(\prod_{j\in J} A_{i,j}\right) = \prod_{j\in J} \left(\bigcap_{i\in I} A_{i,j}\right) \qquad \textit{but} \qquad \bigcup_{i\in I} \left(\prod_{j\in J} A_{i,j}\right) \subset \prod_{j\in J} \left(\bigcup_{i\in I} A_{i,j}\right).$$

8.3.3 Disjoint union

The (outer) disjoint union of a family of sets $\{A_i\}_{i\in I}$ is defined as

$$\bigsqcup_{i\in I} A_i \coloneqq \bigcup_{i\in I} \{i\} \times A_i.$$

If A, B are sets, then we define

$$A \sqcup B \coloneqq (\{0\} \times A) \cup (\{1\} \times B).$$

Notice the difference between the inner and outer disjoint union: one is a normal union that happens to be disjoint, while the other is a separate operation that may be applied to any indexed family of sets.

Lemma I.227. *Let $\{A_i\}_{i\in I}$ be an indexed family of sets. If $\{A_i\}_{i\in I}$ is pairwise disjoint, then*

$$\bigsqcup_{i\in I} A_i \rightarrowtail\!\!\!\twoheadrightarrow \biguplus_{i\in I} A_i : (i,a) \mapsto a.$$

is a bijection.

Proof. The function is clearly surjective. Now assume, towards a contradiction, that it is not injective. Then there exist distinct (i,a) and (j,a) in $\bigsqcup_{i\in I} A_i$, which means $a \in A_i$ and $a \in A_j$. So $a \in A_i \cap A_j$ and $\{A_i\}_{i\in I}$ is not pairwise disjoint. □

Lemma I.228. *Let $\{A_i\}_{i\in I}$ be a family of sets.*

$$\bigsqcup_{i\in I} A_i = \left\{ (i,a) \in I \times \bigcup_{i\in I} A_i \mid a \in A_i \right\}$$

8.3.4 Images and preimages

Lemma I.229. *Let $R \subseteq A \times B$ be a relation, $\{X_i\}_{i\in I}$ a family of subsets of A and $\{Y_j\}_{j\in J}$ a familiy of subsets of B. Then*

1. $R\left(\bigcup_{j\in J} Y_j\right) = \bigcup_{j\in J} RY_j$;
2. $R\left(\bigcap_{j\in J} Y_j\right) \subseteq \bigcap_{j\in J} RY_j$;
3. $\left(\bigcup_{i\in I} X_i\right) R = \bigcup_{i\in I} X_i R$;
4. $\left(\bigcap_{i\in I} X_i\right) R \subseteq \bigcap_{i\in I} X_i R$.

also

5. *if R is functional, then* $R\left(\bigcap_{j\in J} Y_j\right) = \bigcap_{j\in J} RY_j$;
6. *if R is injective, then* $\left(\bigcap_{i\in I} X_i\right) R = \bigcap_{i\in I} X_i R$.

In particular these result hold for functions.

Part II

Algebra

Chapter 1

Universal algebra

TODO: forgetful functors.

1.1 Algebras and terms

A signature or operational type or operator domain is a pair (Ω, α) where Ω is a set whose elements are called operator symbols or just operators and $\alpha : \Omega \to \mathrm{Ord}$ is a function.

- We call $\alpha(\omega)$ the arity of the operator $\omega \in \Omega$.
- If the arity of $\omega \in \Omega$ is n, then we say ω is an n-ary operator.
- If the arity of $\omega \in \Omega$ is the first limit ordinal (i.e. ω, confusingly), we say ω has countable arity.

We also say unary instead of 1-ary, binary instead of 2-ary and ternary instead of 3-ary.

A structure of type (Ω, α), an Ω-structure or an Ω-algebra, is a set A, called the carrier, equipped with a function

$$\omega_A : A^{\alpha(\omega)} \to A$$

for each $\omega \in \Omega$. We call ω_A the interpretation of ω in A.
If we allow the interpretations ω_A to be partial functions, we call A a partial Ω-structure.
If $\alpha(\omega) = 0$ we take ω_A to be a constant.

Let A be an Ω-algebra. An Ω-subalgebra of A is a subset that is closed under the operations of Ω.

Lemma II.1. *Let A be an Ω-algebra. Let $\mathcal{E}$ be a family of subalgebras. Then $\bigcap \mathcal{E}$ is also a subalgebra.*

Let A be an Ω-algebra and X a subset of A. The subalgebra of A generated by X is the intersection of all subalgebras containing X. We call X the generating set of this subalgebra.

Every algebra has a (non-unique) generating set. For example the algebra itself.

The trivial Ω-algebra is the algebra generated by $\emptyset$.

1.1.1 Homomorphisms

Let A, B be Ω-algebras. A homomorphism of Ω-algebras is a function $f : A \to B$ such that

$$\forall \omega \in \Omega : \qquad f \circ \omega_A = \omega_B \circ f^{\alpha(\omega)}.$$

Proposition II.2. *Let $f, g : A \to B$ be two homomorphisms between Ω-algebras A, B. If f, g agree on a generating set of A, then they are equal.*

Proof. The set $\{x \in A \mid f(x) = g(x)\}$ is a subalgebra of A. By hypothesis it contains a generating set of A and thus it is all of A. □

Proposition II.3. *Let $f : A \to B$ be a homomorphism of Ω-algebras. Then $\operatorname{im} f$ is a subalgebra of B.*

Proof. Take some arbitrary $\omega \in \Omega$. Let $b \in (\operatorname{im} f)^{\alpha(\omega)}$. Then there exists an $a \in A^{\alpha(\omega)}$ such that $b = f^{\alpha(\omega)}$. So we have $\omega_B(b) = \omega_B(f^{\alpha(\omega)}(a)) = f(\omega_A(a)) \in \operatorname{im} f$. □

Proposition II.4. *Let (Ω, α) be a signature. Then the Ω-algebras form a category with homomorphisms as arrows.*

Consequently we have the concepts of isomorphism, endomorphism and automorphism.

Proposition II.5. *Let $f : A \to B$ be a bijective homomorphism. Then f^{-1} is also a homomorphism and thus f is an isomorphism.*

Proof. Take arbitrary $\omega \in \Omega$. Then we compute, using I.125,

$$\omega_A(f^{-1})^{\alpha(\omega)} = f^{-1}f\omega_A(f^{-1})^{\alpha(\omega)} = f^{-1}\omega_B f^{\alpha(\omega)}(f^{-1})^{\alpha(\omega)} = f^{-1}\omega_B(ff^{-1})^{\alpha(\omega)} = f^{-1}\omega_B.$$

□

1.2 Relations on algebras

1.2.1 Direct product

Let $\{A_i\}_{i\in I}$ be a family of Ω-algebras. The direct product $\prod_{i\in I} A_i$ is the Ω-algebra whose carrier is the Cartesian product of $\{A_i\}_{i\in I}$ and where operations are carried out componentwise and relations are verified pointwise.
Similarly a direct power of A is a Cartesian power of A with operations and relations defined componentwise.

The direct product of $\{A, B\}$ is simply written $A \times B$.

1.2.2 Relations as algebras

Let A, B be Ω-algebras and R a relation on (A, B) is called Ω-compatible (or just compatible) if R is a subalgebra of $A \times B$.

Lemma II.6. *Let A, B, C be Ω-algebras and Γ, Δ Ω-compatible relations on (A, B) and (B, C), respectively. Then*

1. *$\Gamma^{\mathrm{T}} \subset B \times A$ is an Ω-algebra;*
2. *$\Gamma ; \Delta \subset A \times C$ is an Ω-algebra;*
3. *for any subalgebra A' of A, $A'\Gamma \subset B$ is an Ω-algebra;*
4. *for any subalgebra B' of B, $\Gamma B' \subset A$ is an Ω-algebra.*

1.2.3 Congruences

Let A be an Ω-algebra. A congruence $\mathfrak{q}$ on A is an Ω-compatible equivalence relation.

Example

Any algebra has the trivial congruences I_A and A^2.

An algebra is simple if there are no congruences on it other than the trivial ones. We assume a simple algebra is non-trivial.

Lemma II.7. *Let $\mathfrak{q}$ be a congruence on an Ω-algebra A and B an Ω-subalgebra of A. Then*

1. *$\mathfrak{q}^n$ is a congruence on A^n for all $n \in \mathbb{N}$,*
2. *$\mathfrak{q}|_B^B$ is a congruence on B.*

Proof. (1) The extension of $\mathfrak{q}$ is an equivalence relation by I.76.
TODO subalgebra of $(A^n)^2$ with canonical isomorphism.
(2) The restriction is clearly still reflexive, symmetric and transitive. It is an Ω-algebra by II.1. ☐

For simplicity we may write $B/\mathfrak{q}$ instead of $B/(\mathfrak{q}|_B^B)$.

Proposition II.8. *Let $f : A \to B$ be a homomorphism. Then $\ker f$ is a congruence on A.*

1.2.3.1 Lattice of congruences

Congruences closed under intersection, so complete sublattice.
TODO universal property.

1.2.3.2 Quotient algebras

TODO: index free notation

Proposition II.9. *Let A be an Ω-algebra and $\mathfrak{q}$ an equivalence relation. Then there exists an interpretation of $A/\mathfrak{q}$ such that the function*

$$A \to A/\mathfrak{q} : a \mapsto [a]_{\mathfrak{q}}$$

is a homomorphism if and only if $\mathfrak{q}$ is a congruence.
Explicitly, this interpretation is unique and given by

$$\omega_{A/\mathfrak{q}}([a_1]_{\mathfrak{q}}, \ldots, [a_{\alpha(\omega)}]_{\mathfrak{q}}) = [\omega_A(a_1, \ldots, a_{\alpha(\omega)})]_{\mathfrak{q}} \qquad \forall \omega \in \Omega.$$

Proof. The requirement that $[\cdot]_{\mathfrak{q}}$ be a homomorphism forces the interpretation $\omega_{A/\mathfrak{q}}$ of ω to be the one given.
We just need to show that $\omega_{A/\mathfrak{q}}$ is well-defined if and only if $\mathfrak{q}$ is a congruence. To that end, choose arbitrary $a_1, \ldots, a_{\alpha(\omega)}$ and $a'_1, \ldots, a'_{\alpha(\omega)}$ such that $a'_1 \in [a_1]_{\mathfrak{q}}, \ldots, a'_{\alpha(\omega)} \in [a_{\alpha(\omega)}]_{\mathfrak{q}}$. This is equivalent to choosing $(a_1, a'_1), \ldots, (a_{\alpha(\omega)}, a'_{\alpha(\omega)}) \in \mathfrak{q}$. Then

$$\begin{aligned} [\omega_A(a_1, \ldots, a_{\alpha(\omega)})]_{\mathfrak{q}} = [\omega_A(a'_1, \ldots, a'_{\alpha(\omega)})]_{\mathfrak{q}} &\iff (\omega_A(a_1, \ldots, a_{\alpha(\omega)}), \omega_A(a'_1, \ldots, a'_{\alpha(\omega)})) \in \mathfrak{q} \\ &\iff \omega_{A^2}((a_1, a'_1), \ldots, (a_{\alpha(\omega)}, a'_{\alpha(\omega)})) \in \mathfrak{q} \end{aligned}$$

where the first statement is the requirement of being well-defined and the last is the requirement for being a subalgebra of A^2. $\square$

The Ω-algebra $A/\mathfrak{q}$ is called the quotient algebra of A by $\mathfrak{q}$. The function $A \to A/\mathfrak{q} : a \mapsto [a]_{\mathfrak{q}}$ is known as the quotient map.

Proposition II.10 (Factor theorem)**.** *Let $f : A \to B$ be a homomorphism of Ω-algebras and $\mathfrak{q}$ a congruence on A such that $\mathfrak{q} \subseteq \ker f$. Then*

$$f' : A/\mathfrak{q} \to B : [a]_{\mathfrak{q}} \mapsto f'([a]_{\mathfrak{q}}) = f(a)$$

is a well-defined homomorphism with $\operatorname{im} f' = \operatorname{im} f$. Further, f' is injective if and only if *$\mathfrak{q} = \ker f$.*

Note that $\ker f$ is a congruence by II.8. TODO: universal property.

Proof. To show the function is well defined, take $a, a' \in A$ such that $[a]_{\mathfrak{q}} = [a']_{\mathfrak{q}}$, i.e. $(a, a') \in \mathfrak{q}$. This implies $(a, a') \in \ker f$, so $f(a) = f(a')$ and f' is well-defined.
We see that f' is a homomorphism by the calculation

$$\begin{aligned} f'(\omega_{A/\mathfrak{q}}([a_1], \ldots, [a_{\alpha(\omega)}])) = f'([\omega_A(a_1, \ldots, a_{\alpha(\omega)})]) &= f(\omega_A(a_1, \ldots, a_{\alpha(\omega)})) \\ &= \omega_B(f(a_1), \ldots f(a_{\alpha(\omega)})) = \omega_B(f'([a_1]), \ldots, f'([a_{\alpha(\omega)}])). \end{aligned}$$

Finally f' is injective iff no two distinct $\mathfrak{q}$-classes are identified by f', which is exactly the condition $\mathfrak{q} = \ker f$. $\square$

Lemma II.11. *Let A be an Ω-algebra and $\mathfrak{q}$ a congruence on A. Then*

$$[(x, y)]_{\mathfrak{q}^2} = [x]_{\mathfrak{q}} \times [y]_{\mathfrak{q}}.$$

In particular $A^2/\mathfrak{q}^2 = \{[x]_{\mathfrak{q}} \times [y]_{\mathfrak{q}} \mid x, y \in A\}$.

Proof. We calculate

$$(a, b) \in [(x, y)]_{\mathfrak{q}^2} \iff a\mathfrak{q}x \wedge b\mathfrak{q}y \iff a \in [x]_{\mathfrak{q}} \wedge b \in [y]_{\mathfrak{q}} \iff (a, b) \in [x]_{\mathfrak{q}} \times [y]_{\mathfrak{q}}.$$

$\square$

1.2.3.3 Isomorphism theorems

Theorem II.12 (First isomorphism theorem)**.** *Let $f : A \to B$ be a homomorphism of Ω-algebras. Then we have the isomorphism*

$$A/\ker f \cong \operatorname{im} f.$$

Proof. From the factor theorem II.10 we get an injective homomorphism $f' : A/\ker f \to B$ which is made surjective by restricting the codomain to $\operatorname{im} f$. By II.5 this is an isomorphism. □

Theorem II.13 (Second isomorphism theorem)**.** *Let A be an Ω-algebra, B an Ω-subalgebra of A and $\mathfrak{q}$ a congruence on A. Then we have the isomorphism*

$$(\mathfrak{q}B)/\mathfrak{q} \cong B/(\mathfrak{q} \cap B^2).$$

Note that $\mathfrak{q}B = B\mathfrak{q}$ because $\mathfrak{q}$ is a congruence. Also $\mathfrak{q} \cap B^2 = \mathfrak{q}|_B^B$, so the quotient is well-defined by II.7. Further, $\mathfrak{q}$ should really be restricted in $(\mathfrak{q}B)/\mathfrak{q}$, as in II.7.

Proof. Take the homomorphism $[\cdot]_\mathfrak{q} : A \to A/\mathfrak{q}$ as defined in II.9 and restrict it to B. Applying the first isomorphism theorem II.12 yields the required result. □

Theorem II.14 (Third isomorphism theorem)**.** *Let A be an Ω-algebra and $\mathfrak{q}, \mathfrak{r}$ congruences on A such that $\mathfrak{q} \subseteq \mathfrak{r}$. Then $\mathfrak{r}/\mathfrak{q}$ is a congruence on $A/\mathfrak{q}$ and we have the isomorphism*

$$(A/\mathfrak{q})/(\mathfrak{r}/\mathfrak{q}) \cong A/\mathfrak{r}.$$

This is a slight abuse of notation: clearly $\mathfrak{q}$ is a congruence on A, but $\mathfrak{r} \subseteq A^2$. What we mean is that we take the quotient "pointwise":

$$\mathfrak{r}/\mathfrak{q} = \{([x]_\mathfrak{q}, [y]_\mathfrak{q}) \mid (x, y) \in \mathfrak{r}\}.$$

Proof. Applying the factor theorem II.10 to the homomorphism $A \to A/\mathfrak{r}$ from II.9. We get a surjective homomorphism

$$f : A/\mathfrak{q} \to A/\mathfrak{r} : [a]_\mathfrak{q} \mapsto [a]_\mathfrak{r}.$$

We apply the first isomorphism theorem II.12 to this homomorphism to get $(A/\mathfrak{q})/\ker f \cong A/\mathfrak{r}$. We just need to show that $\ker f = \mathfrak{r}/\mathfrak{q}$.
Indeed

$$([x]_\mathfrak{q}, [y]_\mathfrak{q}) \in \ker f \iff [x]_\mathfrak{r} = [y]_\mathfrak{r} \iff (x, y) \in \mathfrak{r} \iff ([x]_\mathfrak{q}, [y]_\mathfrak{q}) \in \mathfrak{r}/\mathfrak{q}.$$

□

In particular, we see that $A/\mathfrak{q}$ is simple if and only if $\mathfrak{q}$ is a maximal proper congruence on A.

1.3 Free algebras and varieties

Chapter 2

Magmas

A magma or groupoid is an algebra whose signature contains a single ginary operator. Let A be the carrier of the algebra and $\cdot$ the interpretation of the operator. We denote the magma as $(A, \cdot)$. We call $(A, \cdot)$

- a semigroup if $\cdot$ is associative;
- a monoid if $\cdot$ is associative and has an identity;
- a group if $\cdot$ is associative, has an identity and every $a \in A$ has an inverse.

We call a magma commutative if its operation is commutative.
If $(A, \cdot)$ is a monoid or group with identity e, we denote the monoid/group as $(A, \cdot, e)$.

TODO: change of signature for monoid!
Let $x, y \in A$. We usually write $x \cdot y$ instead of $\cdot(x, y)$. (TODO ref infix notation)

We write x^n to abbreviate $\underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ times}}$.
If $x^2 = x$, we say x is an idempotent.

Lemma II.15. *Let $(M, \cdot)$ be a magma and $a \in M$. Then $a \cdot M \subseteq M$.*

Let $(A, \cdot)$ be a magma of finite cardinality. A table containing all possible outputs of the function $\cdot$, i.e. of the form

$\cdot$	a_1	a_2	$\ldots$	a_n
a_1	$a_1 \cdot a_1$	$a_1 \cdot a_2$	$\ldots$	$a_1 \cdot a_n$
a_2	$a_2 \cdot a_1$	$a_2 \cdot a_2$	$\ldots$	$a_2 \cdot a_n$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
a_n	$a_n \cdot a_1$	$a_n \cdot a_2$	$\ldots$	$a_n \cdot a_n$

is called a Cayley table.

A magma is completely determined by its Cayley table.

Proposition II.16. *Let $(M, \cdot, e)$ be a monoid and $a \in M$. If a has both a left inverse l and a right inverse r, then $l = r$.*

Proof. We calculate

$$l = l \cdot e = l \cdot (a \cdot r) = (l \cdot a) \cdot r = e \cdot r = r.$$

□

2.1 Semigroups

Lemma II.17. *Let S be a semigroup. If $a \in S$ is idempotent, then $\{a\}$ is a subsemigroup.*

Corollary II.17.1. *Let S be a semigroup.*

1. *If S contains an identity e, then $\{e\}$ is a subsemigroup.*
2. *If S contains an absorbing element u, then $\{u\}$ is a subsemigroup.*

Proposition II.18. *Let S be a semigroup. Then S is a group* if and only if *for all $a \in S$*

$$aS = S = Sa.$$

Proof. If S is a group, then we have for all $a \in S$

$$S \supseteq aS \supseteq aa^{-1}S = S.$$

Similarly we also have $S = Sa$.
For the converse: from $aS = S$, we het that there exists an $x \in S$ such that $ax = a$. We claim x is a right-identity for S. Indeed, take arbitrary $b \in S$. Then $b = ya$ for some y and so $bx = yax = yab$. In the same way we can also find a left-identity. So S contains an identity e by I.139.
Then for all a we can find $u, v \in S$ such that $au = e = va$. This means a has an inverse by II.16. □

2.1.1 Adjoining identities and absorbing elements

2.1.1.1 Adjoining identity

Let $(S, \boldsymbol{\cdot})$ be a semigroup. We define

$$\widetilde{S} := \begin{cases} S & \text{if } S \text{ has an identity} \\ S \uplus \{e\} & \text{if } S \text{ has no identity.} \end{cases}$$

and also

$$\widetilde{\boldsymbol{\cdot}} \quad := \quad \widetilde{S} \times \widetilde{S} \to \widetilde{S} : (a, b) \mapsto \begin{cases} a \boldsymbol{\cdot} b & a, b \in S \\ b & a = e \\ a & b = e. \end{cases}$$

2.1.1.2 Adjoining an absorbing element

Let $(S, \cdot)$ be a semigroup. We define

$$\widehat{S} := \begin{cases} S & \text{if } S \text{ has an absorbing element} \\ S \uplus \{u\} & \text{if } S \text{ has no absorbing element.} \end{cases}$$

and also

$$\widehat{\cdot} \quad := \quad \widehat{S} \times \widehat{S} \to \widehat{S} : (a, b) \mapsto \begin{cases} a \cdot b & a, b \in S \\ u & (a = u \vee b = u. \end{cases}$$

TODO: link with partial semigroup.

2.1.2 Subsets and subsemigroups

2.1.2.1 Ideals

Let S be a semigroup and $A \subseteq S$ a non-empty subset. Then A is called

- a left ideal if $SA \subseteq A$;
- a right ideal if $AS \subseteq A$;
- a (two-sided) ideal if A is both a left and a right ideal.

Clearly S is an ideal. If $u \in S$ is an absorbing element, then $\{u\}$ is an ideal. If an ideal is neither of these two, it is called proper.

Lemma II.19. *Let $(S, \cdot)$ be a semigroup and $a \in S$. Then*

1. $\widetilde{S}a = Sa \cup \{a\}$ *is a left ideal;*
2. $a\widetilde{S} = aS \cup \{a\}$ *is a right ideal;*
3. $\widetilde{S}a\widetilde{S} = SaS \cup Sa \cup aS \cup \{a\}$ *is an ideal.*

In particular $\widetilde{S}a \subseteq S$, $a\widetilde{S} \subseteq S$ *and* $\widetilde{S}a\widetilde{S} \subseteq S$.

We call

- $\widetilde{S}a$ the principal left ideal generated by a;
- $a\widetilde{S}$ the principal right ideal generated by a;
- $\widetilde{S}a\widetilde{S}$ the principal ideal generated by a.

2.1.2.2 Generated semigroups

TODO!

2.1.2.3 Periodic semigroups

Period and index.

2.1.3 Homomorphism

Proposition II.20. *Let S be a semigroup. Then the functions*

$$\lambda : S \to (\widetilde{S} \to \widetilde{S}) : a \mapsto (\lambda_a : x \mapsto ax)$$
$$\rho : S \to (\widetilde{S} \to \widetilde{S}) : a \mapsto (\rho_a : x \mapsto xa)$$

are injective homomorphisms.

Note that for all $s \in S$ we view λ_a as a function $\widetilde{S} \to \widetilde{S}$. This is necessary for injectivity.

Proof. The functions λ, ρ are homomorphisms by associativity.
For injectivity, let $\lambda_a = \lambda_b$. Then $a = a \cdot e = \lambda_a(e) = \lambda_b(e)b \cdot e = b$. ☐

We call λ (ρ) the extended left (right) regular representation of S.

2.1.3.1 Congruences

Let $(S, \boldsymbol{\cdot})$ be a semigroup and R a relation on S. Then R is called

- left compatible if it is Ω-compatible with $\Omega = \{\lambda_a \mid a \in S\}$;
- right compatible if it is Ω-compatible with $\Omega = \{\rho_a \mid a \in S\}$;
- compatible if it is Ω-compatible with $\Omega = \{\boldsymbol{\cdot}\}$.

If the relation R is additionally an equivalence relation, then R is called a (left / right) congruence.

In other words:

- R is left compatible if $\forall x, y, a \in S : \ xRy \implies (ax)R(ay)$;
- R is right compatible if $\forall x, y, a \in S : \ xRy \implies (xa)R(ya)$;
- R is compatible if $\forall x, y, v, w \in S : \ xRy \land vRw \implies (xv)R(yw)$.

Proposition II.21. *A relation R on a semigroup S is a congruence* if and only if *it is a left and a right congruence.*

Proof. ($\Rightarrow$) If R is a congruence, then aRa by reflexivity. So xRy implies $(ax)R(ay)$ and $(xa)R(ya)$, meaning that R is a left and a right congruence.
($\Leftarrow$) Assume R a left and a right congruence. Take x, y, v, w such that xRy and vRw. Then $(xv)R(yv)$ and $(yv)R(yw)$ by left and right compatibility. We conclude $(xv)R(yw)$ by transitivity. ☐

2.1.4 Green's relations

Let $(S, \cdot)$ be a semigroup. Let

- $\mathcal{L}$ be a relation on S defined by $a\mathcal{L}b \quad \Leftrightarrow_{\text{def}} \quad \widetilde{S}a = \widetilde{S}b$;
- $\mathcal{R}$ be a relation on S defined by $a\mathcal{R}b \quad \Leftrightarrow_{\text{def}} \quad a\widetilde{S} = b\widetilde{S}$;

- $\mathcal{H} \coloneqq \mathcal{L} \cap \mathcal{R}$;
- $\mathcal{D} \coloneqq \mathcal{L} \vee \mathcal{R}$; TODO: which lattice?
- $\mathcal{J}$ be a relation on S defined by $a\mathcal{J}b \quad \Leftrightarrow_{\text{def}} \quad \widetilde{S}a\widetilde{S} = \widetilde{S}b\widetilde{S}$.

These five relations are known as Green's relations.

Clearly we have $\mathcal{D} \subseteq \mathcal{J}$.

Lemma II.22. *Let $(S, \cdot)$ be a semigroup and $a, b \in S$. Then*

1. $a\mathcal{L}b$ if and only if $\exists x, y \in \widetilde{S} : (xa = b) \wedge (yb = a)$;
2. $a\mathcal{R}b$ if and only if $\exists x, y \in \widetilde{S} : (ax = b) \wedge (by = a)$;
3. $a\mathcal{J}b$ if and only if $\exists x, y, u, v \in \widetilde{S} : (xay = b) \wedge (ubv = a)$.

Proof. We prove (1), (2) is analogous.
($\Rightarrow$) $a \in \widetilde{S}a = \widetilde{S}b$, so there exists $y \in \widetilde{S}$ such that $a = yb$. The other equation is similar.
($\Leftarrow$) Because $\widetilde{S}x \subseteq \widetilde{S}$, we have $\widetilde{S}a \subseteq \widetilde{S}xa = \widetilde{S}b$. Similarly $\widetilde{S}b \subseteq \widetilde{S}a$. □

Corollary II.22.1. *$\mathcal{L}$ is a right congruence and $\mathcal{R}$ a left congruence.*

Proposition II.23. *The relations $\mathcal{L}$ and $\mathcal{R}$ commute.*

Proof. Assume $a(\mathcal{L};\mathcal{R})b$, meaning $\exists c : a\mathcal{L}c$ and $c\mathcal{R}b$. Then there exist $x, y, u, v \in \widetilde{S}$ such that

$$(xa = c) \wedge (yc = a) \;\wedge\; (cu = b) \wedge (bv = c).$$

Now define $d = ycu$. Then $a\mathcal{R}d$ because

$$d = (yc)u = au \text{ and } a = yc = ybv = ycuv = dv$$

and $d\mathcal{L}b$ because

$$d = ycu = yb \text{ and } b = cu = xau = xycu = xd.$$

So $a(\mathcal{R};\mathcal{L})b$. The other inclusion is similar. □

Corollary II.23.1. $\mathcal{D} = \mathcal{L};\mathcal{R} = \mathcal{R};\mathcal{L}$.

In other words, $a\mathcal{D}b \iff a\mathcal{L} \# \mathcal{R}b \iff a\mathcal{R} \# \mathcal{L}b$.

Proposition II.24. *Let S be a periodic semigroup. Then $\mathcal{D} = \mathcal{J}$.*

Proof. The inclusion $\mathcal{D} \subseteq \mathcal{J}$ is generally true. We want to prove the other inclusion.
Suppose $a\mathcal{J}b$. Then we can find $x, y, u, v \in \widetilde{S}$ such that

$$xay = b, \quad ubv = a.$$

We see that $a = (ux)a(yu) = (ux)^2a(yu)^2 = \ldots$. By periodicity (TODO ref) we can find an $m \in \mathbb{N}$ such that $(ux)^m$ is idempotent. Then set $c = xa$, so that

$$a = (ux)^m a(yu)^m = (ux)^m(ux)^m a(yu)^m = (ux)^m a = (ux)^{m-1}uc,$$

which means that $a\mathcal{L}c$.

Similarly we can choose $n \in \mathbb{N}$ such that $(vy)^n$ is idempotent, so from $b = (xu)b(vy) = (xu)^2 b(vy)^2 = \dots$ we get

$$\begin{aligned} c = xa &= x(ux)^{n+1}a(yu)^{n+1} = (xu)^{n+1}xay(vy)^n v \\ &= (xu)^{n+1}b(vy)^{2n}v = (xu)^{n+1}b(vy)^{n+1}(vy)^{n-1}v \\ &= b(vy)^{n-1}v. \end{aligned}$$

This along with $cy = xay = b$, gives $c\mathcal{R}b$. Thus $a\mathcal{L};\mathcal{R}b$, i.e. $a\mathcal{D}b$. ☐

Corollary II.24.1. *If the semigroup is finite, then* $\mathcal{D} = \mathcal{J}$*.*

Proposition II.25. *Let* S *be a semigroup. If* $(S/\mathcal{L}, \subseteq)$ *and* $(S/\mathcal{R}, \subseteq)$ *are well-founded, then* $\mathcal{D} = \mathcal{J}$*.*

Proof. TODO ☐

2.1.4.1 Egg-box diagrams

In an egg-box diagram a semigroup S is depicted as a grid. Each element is put in this grid such that

- the rows are $\mathcal{R}$-classes; and
- the columns are $\mathcal{L}$-classes.

Thus for $a, b \in S$, $[a]_{\mathcal{L}} = [b]_{\mathcal{L}}$ if and only if a and b are in the same column. Similarly, $[a]_{\mathcal{R}} = [b]_{\mathcal{R}}$ if and only if a and b are in the same row.

Lemma II.26. *Let* S *be a semigroup and* $a, b \in S$*. Then*

1. *the cells in the egg-box diagram are* $\mathcal{H}$*-equivalence classes;*
2. $a\mathcal{D}b$ if and only if *there is an element in the intersection of the row of* a *and the column of* b *or vice versa.*

Example

Consider the semigroup $(\{1,2,3\} \to \{1,2,3\})$ with composition ;. We can represent an element f of this semigroup as $(f(1)f(2)f(3))$. We have

- $f\mathcal{L}g$ if f and g have the same image;
- $f\mathcal{R}g$ if f and g have the same kernel.

An egg-box diagram can be drawn as follows:

(**111**)	(**222**)	(**333**)				
			(**122**), (211)	(**133**), (311)	(233), (322)	
			(212), (**121**)	(313), (131)	(**323**), (232)	
			(221), (112)	(331), (**113**)	(332), (**223**)	
						(**123**), (231)(312) (132), (213)(321)

The bold elements are idempotents.

Proposition II.27 (Green's lemma)**.** *Let S be a semigroup and $a, b, x \in S$.*

1. *If $ax = b$ and $a\mathcal{R}b$, then*
   - *(a) $\rho_x|_{[a]_\mathcal{L}} : [a]_\mathcal{L} \to [b]_\mathcal{L}$ is a bijection;*
   - *(b) if $by = a$ for some $y \in \widetilde{S}$, then $\rho_y|_{[b]_\mathcal{L}}$ is the inverse;*
   - *(c) $\rho_x|_{[a]_\mathcal{L}}$ preserves $\mathcal{R}$-classes.*
2. *If $xa = b$ and $a\mathcal{L}b$, then*
   - *(a) $\lambda_x|_{[a]_\mathcal{R}} : [a]_\mathcal{R} \to [b]_\mathcal{R}$ is a bijection with inverse $\lambda_y|_{[b]_\mathcal{R}}$;*
   - *(b) if $yb = a$ for some $y \in \widetilde{S}$, then $\lambda_y|_{[b]_\mathcal{R}}$ is the inverse;*
   - *(c) $\lambda_x|_{[a]_\mathcal{R}}$ preserves $\mathcal{L}$-classes.*

In particular

1. *if $a\mathcal{R}b$, then there exists $x \in \widetilde{S}$ such that $ax = b$ and thus $|[a]_\mathcal{L}| = |[b]_\mathcal{L}|$;*
2. *if $a\mathcal{L}b$, then there exists $x \in \widetilde{S}$ such that $xa = b$ and thus $|[a]_\mathcal{R}| = |[b]_\mathcal{R}|$.*

Proof. The function ρ_x is well-defined: let $u \in [a]_\mathcal{L}$ so that $u = va$ for some $v \in \widetilde{S}$. Then $\rho_x(u) = vax = vb \in [b]_\mathcal{L}$.
Because $a\mathcal{R}b$, we can always find a $y \in \widetilde{S}$ such that $by = a$. The function $\rho_y|_{[b]_\mathcal{L}}$ is clearly the inverse: $\rho_y(\rho_x(u)) = \rho_y(vb) = vby = va = u$. This shows that ρ_x is bijective.
It is clear that ρ_x preserves $\mathcal{R}$-classes, because it operates on the right.
The arguments for λ_x and λ_y are completely analogous. ☐

Corollary II.27.1. *Let S be a semigroup and $a, b \in S$ such that $a\mathcal{D}b$, then $|[a]_\mathcal{H}| = |[b]_\mathcal{H}|$.*

Proof. If $a(\mathcal{L};\mathcal{R})b$, then there exists a $c \in S$ such that $c = sa$ and $b = ct$. Then $\lambda_s \circ \rho_t$ is the required bijection. ☐

Theorem II.28 (Green's theorem)**.** *Let S be a semigroup and H an $\mathcal{H}$-class in S. Then either*

1. *$H^2 \perp H$; or*

2. *$H^2 = H$ and H is a subgroup of S.*

Proof. Suppose $H^2 \cap H \neq \emptyset$, then there exist $a, b \in H$ such that $ab = c \in H$. By the Green's lemma II.27 we have that $\rho_b : H \to H$ and $\lambda_a : H \to H$ are bijections.
Then for all $h \in H$, $\rho_b(h) = hb \in H$. Again by the Green's lemma, this means that $\lambda_h : H \to H$ is a bijection. Similarly $\rho_h : H \to H$ is a bijection for all h. So for all $h \in H$ we have $hH = H = Hh$. This means $H^2 = H$ and H is a group by II.18. ☐

Corollary II.28.1. *Let S be a semigroup and H an $\mathcal{H}$-class in S. Then*

1. *if x is an idempotent in H, then H is a subgroup of S;*

2. *no $\mathcal{H}$-class can contain more than one idempotent.*

2.1.4.2 Regular elements $\mathcal{D}$-classes

Let S be a semigroup. An element $a \in S$ is called regular if there exists $x \in S$ such that $axa = a$.

Proposition II.29. *Let S be a semigroup. If $a \in S$ is regular, then every element in $[a]_{\mathcal{D}}$ is regular.*

So it makes sense to call a $\mathcal{D}$-class regular if it consists of regular elements and irregular otherwise.

Corollary II.29.1. *If there is an idempotent $x \in [a]_{\mathcal{D}}$, then $[a]_{\mathcal{D}}$ is regular.*

Proof. An idempotent is regular: $x = xx = x(xx) = xxx$. ☐

Proposition II.30. *Let S be a semigroup.*

1. *If $x \in S$ is an idempotent, then x is a left identity for $[x]_{\mathcal{R}}$ and x is a right identity for $[x]_{\mathcal{L}}$.*

2. *If a is regular with $axa = a$, then xa and ax are idempotents.*

3. *If a is regular with $axa = a$, then $xa\mathcal{L}a$ and $ax\mathcal{R}a$.*

4. *In a regular $\mathcal{D}$-class each $\mathcal{L}$-class and each $\mathcal{R}$-class contains at least one idempotent.*

Proof. (1) Let $a \in [x]_{\mathcal{R}}$. Then there exists a $y \in \widetilde{S}$ such that $a = xy$ and thus $xa = xxy = xy = a$. The claim of right-identity is similar.
(2) We have $(xa)(xa) = x(axa) = xa$ and $(ax)(ax) = (axa)x = ax$.
(3) If a is regular with $axa = a$, then xa is idempotent: $(xa)(xa) = x(axa) = xa$ and $xa \in [a]_{\mathcal{L}}$:

$$xa = (xa) \qquad a(xa) = a$$

Similarly $ax \in [a]_{\mathcal{R}}$ and is idempotent.
(4) Let $[a]_{\mathcal{L}}$ be an $\mathcal{L}$-class in a regular $\mathcal{D}$-class. By regularity there exists an $x \in S$ such that $axa = a$. From (2) and (3), we have that $[a]_{\mathcal{L}}$ contains the idempotent xa and $[a]_{\mathcal{R}}$ the idempotent ax. ☐

Proposition II.31. *Let $a, b \in S$ such that $a\mathcal{D}b$. Then $ab \in [a]_{\mathcal{R}} \cap [b]_{\mathcal{L}}$* if and only if $[b]_{\mathcal{R}} \cap [a]_{\mathcal{L}}$ *contains an idempotent.*

Proof. Suppose $[b]_{\mathcal{R}} \cap [a]_{\mathcal{L}}$ contains an idempotent x. Then $xb = b$ by II.30. Then ρ_b maps $[x]_{\mathcal{L}} = [a]_{\mathcal{L}}$ to $[b]_{\mathcal{L}}$ by II.27. It preserves $\mathcal{R}$-classes, so in particular it maps $[a]_{\mathcal{H}}$ to $[a]_{\mathcal{R}} \cap [b]_{\mathcal{L}}$. Thus $ab = \rho_b(a) \in [a]_{\mathcal{R}} \cap [b]_{\mathcal{L}}$.
Now assume $ab \in [a]_{\mathcal{R}} \cap [b]_{\mathcal{L}}$. Then there exists a $c \in \widetilde{S}$ such that $abc = a$. By II.27 this means $\rho_c \in ([ab]_{\mathcal{L}} \to [a]_{\mathcal{L}}) = ([b]_{\mathcal{L}} \to [a]_{\mathcal{L}})$. In particular this restricts to $([b]_{\mathcal{H}} \to [b]_{\mathcal{R}} \cap [a]_{\mathcal{L}})$. Because $ab = (ab)$, ρ_b is the inverse of ρ_c by the same lemma.
So $\rho_c(b) = bc \in [b]_{\mathcal{R}} \cap [a]_{\mathcal{L}}$ and bc is idempotent because

$$bcbc = (\rho_c \circ \rho_b \circ \rho_c)(b) = \rho_c(b) = bc.$$

□

2.1.4.3 Generalised inverses

Let S be a semigroup and $a \in S$. We call a' a (generalised) inverse of a if

$$aa'a = a, \qquad a'aa' = a'.$$

Lemma II.32. *Let S be a semigroup and $a \in S$. Then a has a generalised inverse* if and only if *it is regular.*

Proof. Clearly every element with a generalised inverse is regular. Conversely, assume a regular with $axa = a$. Then $a' = xax$ is a generalised inverse of a. □

Proposition II.33. *Let a be an element of a regular $\mathcal{D}$-class D in a semigroup S.*

1. *If a' is a generalised inverse of a, then*
   - *(a) $a' \in D$;*
   - *(b) $[aa']_{\mathcal{H}} = [a]_{\mathcal{R}} \cap [a']_{\mathcal{L}}$;*
   - *(c) $[a'a]_{\mathcal{H}} = [a']_{\mathcal{R}} \cap [a]_{\mathcal{L}}$.*
2. *If $b \in S$ is such that $[a]_{\mathcal{R}} \cap [b]_{\mathcal{L}}$ contains an idempotent x and $[b]_{\mathcal{R}} \cap [a]_{\mathcal{L}}$ contains an idempotent y, then $[b]_{\mathcal{H}}$ contains a generalised inverse a' of a such that $aa' = x$ and $a'a = y$.*
3. *No $\mathcal{H}$-class contains more than one generalised inverse of a.*

Proof. (1) Firstly notice that $[a]_{\mathcal{R}} \cap [a']_{\mathcal{L}}$ is an $\mathcal{H}$-class. Also $a\mathcal{R}aa'$ and $a'\mathcal{L}aa'$ by II.30.
(2) From $a\mathcal{R}x$, we deduce that there exists a $u \in \widetilde{S}$ such that $au = x$.
Set $a' = yux$. Then, using that x is a left-identity on $[a]_{\mathcal{R}}$ and y a right-identity on $[a]_{\mathcal{L}}$ by II.30,

$$\begin{aligned} aa'a &= a(yux)a = (ay)u(xa) = aua = (au)a = xa = a \\ a'aa' &= (yux)a(yux) = (yu)(xay)(ux) = (yu)a(ux) = (yu)(au)x = yux^2 = yux = a' \\ aa' &= a(yux) = ((ay)u)x = (au)x = x^2 = x. \end{aligned}$$

For the last equality, we need that $a\mathcal{L}y$ implies the existence of $v \in \widetilde{S}$ such that $va = y$. Then

$$a'a = (yux)a = (va)u(xa) = (va)(ua) = v(au)a = v(xa) = va = y.$$

We now just need to show that $a' \in [b]_{\mathcal{H}}$. By II.30 we have $a' \in [aa']_{\mathcal{L}}$ and $a' \in [a'a]_{\mathcal{R}}$, so

$$a' \in [aa']_{\mathcal{L}} \cap [a'a]_{\mathcal{R}} = [x]_{\mathcal{L}} \cap [y]_{\mathcal{R}} = [b]_{\mathcal{L}} \cap [b]_{\mathcal{R}} = [b]_{\mathcal{H}}.$$

(3) Suppose $[b]_{\mathcal{H}}$ is an $\mathcal{H}$-class containing two generalised inverses a'_1, a'_2 of a. Then $aa'_1 = aa'_2$ and $a'_1 a = a'_2 a$ by II.28.1 and thus

$$a'_1 = a'_1(aa'_1) = a'_1(aa'_2) = (a'_2 a)a'_2 = a'_2.$$

□

Corollary II.33.1. *Let S be a semigroup and $x, y \in S$ idempotents. Then $x\mathcal{D}y$* if and only if *we can write $x = aa'$ and $y = a'a$ for some $a \in [x]_{\mathcal{R}} \cap [y]_{\mathcal{L}}$ and $a' \in [y]_{\mathcal{R}} \cap [x]_{\mathcal{L}}$ that are generalised inverses of each other.*

Proof. We have $x\mathcal{D}y$ iff there exist $a, b \in S$ such that $x\mathcal{R}a$, $a\mathcal{L}y$ and $x\mathcal{L}b$, $b\mathcal{R}y$. This means $x \in [a]_{\mathcal{R}} \cap [b]_{\mathcal{L}}$ and $y \in [b]_{\mathcal{R}} \cap [a]_{\mathcal{L}}$. So by the proposition we can find an $a' \in [b]_{\mathcal{H}}$ that satisfies the requirements. □

Proposition II.34. *Let S be a semigroup and let H, K be $\mathcal{H}$-classes that are subgroups and members of the same $\mathcal{D}$-class. Then H and K are isomorphic groups.*

Proof. Let e be the idempotent (and thus identity) in H and f the identity in K. By II.33.1 we have mutually inverse $a \in [x]_{\mathcal{R}} \cap [y]_{\mathcal{L}}$ and $a' \in [y]_{\mathcal{R}} \cap [x]_{\mathcal{L}}$ such that $x = aa'$ and $y = a'a$. We claim $\lambda_{a'} \circ \rho_a|_H$ is an isomorphism $H \to K$. Indeed it is bijective by II.27. We need to show that it is a homomorphism: take $u, v \in H$

$$(\lambda_{a'} \circ \rho_a)(u)(\lambda_{a'} \circ \rho_a)(v) = a'uaa'va = a'uxva = a'uva = (\lambda_{a'} \circ \rho_a)(uv),$$

where we have used that x is the identity for H. □

2.1.4.4 Regular semigroups

A regular semigroup is a semigroup where every element is regular.

Lemma II.35. *Let S be a regular semigroup. Then for all $a \in S$*

$$\widetilde{S}a = Sa, \qquad a\widetilde{S} = aS \qquad \text{and} \qquad \widetilde{S}a\widetilde{S} = SaS.$$

So we can drop all mention of $\widetilde{S}$ when working with ideal. In particular

- $a\mathcal{L}b$ if and only if $Sa = Sb$;
- $a\mathcal{R}b$ if and only if $aS = bS$;
- $a\mathcal{J}b$ if and only if $SaS = SbS$.

2.1.5 Inverse semigroups

2.2 Monoids

Lemma II.36. *A locally small category with a single object is a monoid.*

TODO: delooping $\mathsf{B}M$.

2.2.1 Ordered monoids

An ordered monoid is a monoid $(M, \cdot, 0)$ on which a partial order $\preceq$ is defined that is compatible, i.e. $\forall x, y, z \in M$

$$x \preceq y \implies x \cdot z \preceq y \cdot z \wedge z \cdot x \preceq z \cdot y.$$

Positive: $x > 0$.

2.2.1.1 The Archimedean property

Let $(M, +, \leq)$ be a totally ordered monoid and $x, y \in M$ positive. Then

- x is infinitesimal w.r.t. y or y is infinite w.r.t. x if $nx < y$ for all $n \in \mathbb{N}$;
- M is Archimedean if there is no pair (x, y) such that x is infinitesimal w.r.t. y.

every submonoid is Archimedean.
abelian??

2.2.1.2 Regular ordering for commutative monoids

Let $(M, +, 0)$ be a commutative monoid. Consider the function $m : \mathcal{P}(M \to M) \to \mathcal{P}(X^2)$ from II.67. (TODO define higher).
The regular order on M is $m\{\lambda_a \mid a \in M\}$.

Lemma II.37. *The regular order is a preorder on M that is compatible with M. It is a partial order* if and only if *M is cancellative.*

Example

The regular order on $\mathbb{N}$ is the standard order on $\mathbb{N}$.

2.3 Divisibility

$m|n$ order relation.
$\sup\{n, m\} = kgv(n, m)$ and $\inf\{n, m\} = ggd(n, m)$

Chapter 3

Relational structures

A relational structure is a structured set (A, R) where R is a homogeneous relation on the set A.

3.1 Duality

Let (A, R) be a relational structure.
The relational structure dual to (A, R) is $(A, R)^o \coloneqq \left(A^o, R^{\mathrm{T}}\right) \coloneqq \left(A, R^{\mathrm{T}}\right)$.

3.2 Functions on relational structures

Lemma II.38. *Let $f : A \to A$ be a function on a relational structure (A, R). Then*

$$f \subseteq R \iff \mathrm{id}_A \subseteq R; f^{\mathrm{T}}.$$

Proof. We have

$$f \subseteq R \implies \mathrm{id}_A \subseteq f; f^{\mathrm{T}} \subseteq R; f^{\mathrm{T}} \implies f \subseteq R; f^{\mathrm{T}}; f \subseteq R.$$

□

Let (A, R) and (B, S) be relational structures and $f : A \to B$ a function. We say

- f is relation-preserving if

$$\forall x, y \in A : xRy \implies f(x) S f(y);$$

- f is relation-reflecting if

$$\forall x, y \in A : f(x) S f(y) \implies xRy;$$

- f is a relation embedding if it is relation-preserving and -reflecting:

$$\forall x, y \in A : xRy \iff f(x) S f(y).$$

Proposition II.39. *Let (X, R) and (Y, S) be relational structures and $f : X \to Y$ a function. Then the following are equivalent:*

1. *f is relation-preserving;*
2. *$R \subseteq f; S; f^{\mathrm{T}}$;*
3. *$R; f \subseteq f; S$.*

Proof. (1) $\Leftrightarrow$ (2) Is clear from the definition.
(2) $\Rightarrow$ (3) We calculate

$$R; f \subseteq (f; S; f^{\mathrm{T}}); f = f; S; (f^{\mathrm{T}}; f) \subseteq f; S; \mathrm{id} = f; S.$$

(3) $\Rightarrow$ (2) We calculate

$$R \subseteq R; (f; f^{\mathrm{T}}) = (R; f); f^{\mathrm{T}} \subseteq f; S; f^{\mathrm{T}}.$$

□

Corollary II.39.1. *Let (X, R) and (Y, S) be relational structures and $f : X \to Y$ a function. Then the following are also equivalent to f being relation-preserving:*

1. *$f : X^o \to Y^o$ is relation preserving;*
2. *$R^{\mathrm{T}} \subseteq f; S^{\mathrm{T}}; f^{\mathrm{T}}$;*
3. *$R^{\mathrm{T}}; f \subseteq f; S^{\mathrm{T}}$;*
4. *$f^{\mathrm{T}}; R \subseteq S; f^{\mathrm{T}}$.*

Proof. (2) Is equivalent to $R \subseteq f; S; f^{\mathrm{T}}$ by taking the transpose. The equivalence of (1), (2) and (3) is then given by the proposition.
(4) Is equivalent to (3) by taking the transpose. □

Corollary II.39.2. *Let (A, R) and (B, S) be relational structures and $f : A \to B$ a function. Then the following are equivalent:*

1. *f is order-preserving;*
2. *for all $X \subseteq A$: $f[X_R] \subseteq f[X]_S$;*
3. *for all $x \in A$: $f[xR] \subseteq f(x)S$.*

They are also equivalent to:

4. *for all $Y \subseteq B$: $f^{-1}[Y]_R \subseteq f^{-1}[Y_S]$*
5. *for all $y \in B$: $f^{-1}[\{y\}]_R \subseteq f^{-1}[yS]$.*

TODO: this implies R-closure for f^{-1}: by preservation of union we have $f^{-1}[yS]_R \subseteq f^{-1}[yS^2] = f^{-1}[yS]$.

Proof. ($1 \Leftrightarrow 2$) Follows by taking images of X under $R; f \subseteq f; S$ and the fact that relations are completely determined by their images.
($2 \Leftrightarrow 3$) This is the particular case of principal images. A relation is completely characterised by its principal images. See I.37.
($4, 5$) Here we are taking images of $f^{\mathrm{T}}; R \subseteq S; f^{\mathrm{T}}$. □

Useful exercise: give elementary proof of II.39.2. This can be based on the calculations

$$y \in xR \iff xRy \implies f(x)Rf(y) \iff f(y) \in f(x)R$$

and

$$\begin{aligned} x \in f^{-1}[\{y\}]_R &\iff \exists z : \ f(z) = y \ \wedge \ zRx \\ &\implies \exists z : \ f(z) = y \ \wedge \ f(z)Rf(x) \\ &\implies ySf(x) \iff f(x) \in yS \iff x \in f^{-1}[yS]. \end{aligned}$$

Lemma II.40. *Let (A,R) and (B,S) be relational structures and $f : A \to B$ a function. Then*

1. *f is relation-reflecting* if and only if *$R \supseteq f; S; f^{\mathrm{T}}$;*
2. *f is a relation embedding* if and only if *$R = f; S; f^{\mathrm{T}}$.*

A function $f : A \to A$ on a relational structure (A,R) is

- left-restrictive if $f; R \subseteq R$;
- right-restrictive if $R; f^{\mathrm{T}} \subseteq R$;
- left-expansive if $R \subseteq f; R$;
- right-expansive if $R \subseteq R; f^{\mathrm{T}}$.

TODO: expansive, contractive, extensive.

Lemma II.41. *Let $f : A \to B$ be a function and R a transitive relation on B. Then $f \subseteq R$ implies $f; R \subseteq R$ and $R; f \subseteq R$.*

Proof. We have $f; R \subseteq R^2 \subseteq R$ and $R; f \subseteq R^2 \subseteq R$. □

Lemma II.42. *Let $f : (A,R) \to (A,R)$ be a relation-preserving function. Then*

$$f; R \subseteq R \implies R \subseteq R; f^{\mathrm{T}}$$

Proof. We calculate

$$R \subseteq f; R; f^{\mathrm{T}} \subseteq R; f^{\mathrm{T}}.$$

□

3.2.1 Relation isomorphisms

Let (A_1, R_1) and (A_2, R_2) be relational structures. A bijection $\phi : A_1 \rightarrowtail\!\!\!\!\twoheadrightarrow A_2$ such that

$$\forall (s_1, t_1) \in R_1 : (\phi(s_1), \phi(t_1)) \in R_2 \qquad \text{and} \qquad \forall (s_2, t_2) \in R_2 : (\phi^{-1}(s_2), \phi^{-1}(t_2)) \in R_1,$$

is called a (relation) isomorphism. If there exists a relation isomorphism $A_1 \rightarrowtail\!\!\!\!\twoheadrightarrow A_2$, then (A_1, R_1) and (A_2, R_2) are isomorphic, denoted $A_1 \cong A_2$.

Lemma II.43. *A map is a relation isomorphism* if and only if *it is a bijective relation embedding.*

Proof. Let $f : (A, R) \to (B,)$ be a bijective map. Then f is relation-reflecting iff f^{-1} is relation-preserving. Indeed

$$f; R; f^{-1} \subseteq R \iff R = f^{-1}; f; R; f^{-1}; f \subseteq f^{-1}; R; f.$$

$\square$

Lemma II.44. *Let* (A_1, R_1), (A_2, R_2) *and* (A_3, R_3) *be relational structures and* $f : A_1 \rightarrowtail\!\!\!\!\twoheadrightarrow A_2$, $g : A_2 \rightarrowtail\!\!\!\!\twoheadrightarrow A_3$ *isomorphisms. Then*

1. $I_{A_1} : A_1 \to A_1 : a \mapsto a$ *is an isomorphism;*
2. $f^{-1} : A_2 \to A_1$ *is an isomorphism;*
3. $(g \circ f) : A_1 \to A_3$ *is an isomorphism.*

Consequently, relation isomorphism would be an equivalence relation on all structured sets, except there is no set of all structured sets, by Russell's paradox. Relation isomorphism can be an equivalence relation on a (restricted) set of structured sets.

Lemma II.45. *Let* (A_1, R_1) *and* (A_2, R_2) *be isomorphic relational structures. Then* R_1 *has the same properties as* R_2*.*

e.g reflexivity, symmetry, transitivity, being an equivalence relation etc.

3.3 The semigroup of relation-preserving functions

Lemma II.46. *Let* $\{(A_i, R_i)\}_{i \in I}$ *be a set of relational structures. Then the set of relation-preserving functions together with* $\emptyset$ *forms a semigroup under composition.*

TODO: Full semigroup, i.e. if relational structure is involved, all morphisms must be present.

Proposition II.47. *Let* M *be a semigroup of relation-preserving functions and* $f : (A, R) \to (B, S)$, $g : (A, R) \to (C, T)$ *relation-preserving functions. Then*

$$f \mathcal{R} g \iff (f; S; f^{\mathrm{T}} = g; T; g^{\mathrm{T}}) \wedge (\ker f = \ker g).$$

Proof. First assume $f\mathcal{R}g$, then there exist $x, y \in \tilde{M}$ such that $f = g; x$ and $g = f; y$. Now x and y are relation preserving: $T \subseteq x; S; x^{\mathrm{T}}$ and $S \subseteq y; T; y^{\mathrm{T}}$. So

$$f; S; f^{\mathrm{T}} = g; x; S; x^{\mathrm{T}}; g^{\mathrm{T}} \supseteq g; T; g^{\mathrm{T}} \quad \text{and} \quad g; T; g^{\mathrm{T}} = f; y; T; y^{\mathrm{T}}; f^{\mathrm{T}} \supseteq f; S; f^{\mathrm{T}}.$$

For the converse, we can find functions x, y such that $f = g; x$ and $g = f; y$ because of the equality of the kernels. We just need to show that x and y are relation-preserving. Because $g^{\mathrm{T}}; f = g^{\mathrm{T}}; g; x \subseteq x$, we have

$$T \subseteq g^{\mathrm{T}}; g; T; g^{\mathrm{T}}; g = g^{\mathrm{T}}; f; S; f^{\mathrm{T}}; g \subseteq x; S; x^{\mathrm{T}}.$$

The argument for y is similar. $\square$

Corollary II.47.1. *Let* a, a' *be generalised inverses in the semigroup of relation-preserving functions. Then*

1. $aa'Ra'^{\mathrm{T}} = aSa^{\mathrm{T}}a'^{\mathrm{T}}$;

2. $a'Ra'^{\mathrm{T}}a^{\mathrm{T}} = a'aSa^{\mathrm{T}}$;

3. $a|_{\mathrm{im}\,a'}$ *and* $a'|_{\mathrm{im}\,a}$ *are relation embeddings.*

Proof. Let $a : (A,R) \to (B,S)$ and $a' : (B,S) \to (A,R)$ be generalised inverses. From II.30 we have $a\mathcal{R}aa'$ and $a'\mathcal{R}a'a$. The proposition then gives us $aSa^{\mathrm{T}} = aa'Ra'^{\mathrm{T}}a^{\mathrm{T}}$.
(1) Multiplying this equality on the right by a'^{T} and using $a'^{\mathrm{T}} = (a'aa')^{\mathrm{T}} = a'^{\mathrm{T}}a^{\mathrm{T}}a'^{\mathrm{T}}$ gives us

$$aSa^{\mathrm{T}}a'^{\mathrm{T}} = aa'R(a'^{\mathrm{T}}a^{\mathrm{T}}a'^{\mathrm{T}}) = aa'Ra'^{\mathrm{T}}.$$

(2) Similar to (1), except multiplying on the left by a'.
(3) By assumption a and a' are relation-preserving. Also, using $aSa^{\mathrm{T}} = aa'Ra'^{\mathrm{T}}a^{\mathrm{T}}$, we have

$$S \supseteq S|^{\mathrm{im}(a)}_{\mathrm{im}(a)} = a^{\mathrm{T}}aSa^{\mathrm{T}}a = a^{\mathrm{T}}(aa'Ra'^{\mathrm{T}}a^{\mathrm{T}})a = (a^{\mathrm{T}}a)a'Ra'^{\mathrm{T}}(a^{\mathrm{T}}a) = a'|_{\mathrm{im}(a)}R(a'|_{\mathrm{im}(a)})^{\mathrm{T}}.$$

This means that $a'|_{\mathrm{im}(a)}$ is also relation-reflecting. The argument for $a|_{\mathrm{im}(a')}$ is similar. ☐

3.3.1 Galois connections

Let $a : (A,R) \to (B,S)$ and $a' : (B,S) \to (A,R)$ be relation-preserving generalised inverses. We say (a,a') is a Galois connection between (A,R) and (B,S) if

- aa' is left-restrictive for R, i.e. $aa'R \subseteq R$; and
- $a'a$ is left-restrictive for S^{T}, i.e. $a'aS^{\mathrm{T}} \subseteq S^{\mathrm{T}}$.

We call a the lower adjoint or residuated map and a' the upper adjoint or residual. Sometimes a Galois connection between (A,R) and $(B,S)^o$ is referred to as an antitone Galois connection between (A,R) and (B,S). In this situation we may refer to the ordinary Galois connection as a monotone Galois connection to highlight the difference.

Some authors exclusively use the term "Galois connection" to refer to antitone Galois connections. They my call monotone Galois connections residuated mappings.

Lemma II.48. *Let* (A,R) *and* (B,S) *be relational structures, and* $a : (A,R) \to (B,S)$ *and* $a' : (B,S) \to (A,R)$ *functions. Then* (a,a') *is an antitone Galois connection* if and only if

1. a, a' *are relation-reversing;*
2. a, a' *are generalised inverses;*
3. $aa'R \subseteq R$ *and* $a'aS \subseteq S$.

For any relation-preserving involution $f : (A,R) \to (A,R)$, (f,f) is a Galois connection. For any relation-reversing involution $f : A \to A$, (f, f^o) is a Galois connection between (A,R) and $(A,R)^o$, where we consider f as a function $f : (A,R) \to (A,R)^o$ and f^o as the function $f^o : (A,R)^o \to (A,R)$ with the same graph.

Example

- Let (A, id_A) and (B, id_B) be discrete relational structures. Then $f : A \to B$ and $g : B \to A$ form a Galois connection if and only if they are invertible and $f = g^{-1}$.

- Let A be a set. Then complementation $c : \mathcal{P}(A) \to \mathcal{P}(A)$ is a relation-reversing involution, so it gives a Galois connection between $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(A), \subseteq)^o$. In particular we have the identity

$$\forall X, Y \in \mathcal{P}(A) : \qquad X^c \subseteq Y \iff X \supseteq Y^c,$$

which is the Galois identity II.50.

Lemma II.49. *If (a, a') is a Galois connection between (A, R) and (B, S), then*

1. $R^{\mathrm{T}} \subseteq aa'R^{\mathrm{T}}$ *and* $S \subseteq a'aS$;
2. $Raa' \subseteq R$ *and* $S^{\mathrm{T}}a'a \subseteq S^{\mathrm{T}}$.

Proof. (1) This is a direct application of II.42. I can also be obtained as a corollary to II.50 (later).
(2) Follows from the Galois condition and the fact $a'a, aa'$ are relation-preserving (II.39):

$$Raa' \subseteq aa'R \subseteq R \qquad \text{and} \qquad S^{\mathrm{T}}a'a \subseteq a'aS^{\mathrm{T}} \subseteq S^{\mathrm{T}}.$$

□

In particular, if the relations are reflexive, we have

$$\mathrm{id}_A \subseteq R^{\mathrm{T}} \subseteq aa'R^{\mathrm{T}} \qquad \text{and} \qquad \mathrm{id}_B \subseteq S \subseteq a'aS.$$

This means that for all $x \in A$ we have $xRa'(a(x))$ and for all $y \in B$ we have $a(a'(y))Sy$.

3.3.1.1 The Galois identity

Proposition II.50. *If (a, a') is a Galois connection between (A, R) and (B, S), then*

$$aS = Ra'^{\mathrm{T}}.$$

This identity is particularly important because it gives also gives a sufficient condition for a pair of maps between preorders to be a Galois connection, see II.53.

Proof. By II.47.1 we have $aa'Ra'^{\mathrm{T}} = aSa^{\mathrm{T}}a'^{\mathrm{T}}$. Because a, a' are relation-preserving, we have

$$aS \subseteq aa'Ra'^{\mathrm{T}} = aSa^{\mathrm{T}}a'^{\mathrm{T}} = aS(a'a)^{\mathrm{T}} \subseteq aS \quad \text{and} \quad Ra'^{\mathrm{T}} \subseteq aSa^{\mathrm{T}}a'^{\mathrm{T}} = aa'Ra'^{\mathrm{T}} \subseteq Ra'^{\mathrm{T}},$$

which implies $aS = aSa^{\mathrm{T}}a'^{\mathrm{T}} = aa'Ra'^{\mathrm{T}} = Ra'^{\mathrm{T}}$. □

Corollary II.50.1. *Let (a, a') be a Galois connection between (A, R) and (B, S).*

1. *If S is reflexive, then $aa' \subseteq R$.*
2. *If R is reflexive, then $a'a \subseteq S^{\mathrm{T}}$.*

Proof. (1) We calculate $aa' \subseteq aSa' = Ra'^{\mathrm{T}}a' \subseteq R$.
(2) Similarly, $a'a \subseteq a'R^{\mathrm{T}}a = S^{\mathrm{T}}a^{\mathrm{T}}a \subseteq S^{\mathrm{T}}$. □

Lemma II.51. *Let (A, R) and (B, S) be relationals structures and $f : A \to B$, $g : B \to A$ functions. The following are equivalent:*

1. *$f; S = R; g^{\mathrm{T}}$;*

2. *for all x: $f(x)S = g^{-1}[xR]$;*

3. *for all y: $f^{-1}[Sy] = Rg(y)$.*

Proof. Relations are completely characterised by their principal images / preimages. See I.37. □

Proposition II.52. *Let $a : (A, R) \to (B, S)$ and $a' : (B, S) \to (A, R)$ be order-preserving generalised inverses on preorders. Then the following are equivalent:*

1. *(a, a') is a Galois connection;*

2. *$aa' \subseteq R$ and $a'a \subseteq S^{\mathrm{T}}$;*

3. *$\mathrm{id}_A \subseteq aa'R^{\mathrm{T}}$ and $\mathrm{id}_B \subseteq a'aS$.*

If the orders are partial orders, then we do not need the additional assumption that a, a' are generalised inverses.

Proof. (1) ⇒ (2) is given by II.50.1.
(2) ⇒ (1) follows by transitivity, see II.41.
(2) ⇔ (3) is given by II.38.
Now assume the orders are anti-symmetric. We need to show that $(2, 3)$ imply that a, a' are generalised inverses. First, by transitivity, we have

$$R^{\mathrm{T}} \subseteq aa'(R^2)^{\mathrm{T}} \subseteq aa'R^{\mathrm{T}} \qquad \text{and} \qquad S \subseteq a'aS^2 \subseteq a'aS.$$

Then, using the fact that a and a' are relation preserving,

$$\begin{aligned}
&\mathrm{id} \subseteq R \subseteq aSa^{\mathrm{T}} \subseteq aa'aSa^{\mathrm{T}} \\
&\mathrm{id} \subseteq R^{\mathrm{T}} \subseteq aa'R^{\mathrm{T}} \subseteq aa'aS^{\mathrm{T}}a^{\mathrm{T}} \\
&\mathrm{id} \subseteq S \subseteq a'aS \subseteq a'aa'Ra'^{\mathrm{T}} \\
&\mathrm{id} \subseteq S^{\mathrm{T}} \subseteq a'R^{\mathrm{T}}a'^{\mathrm{T}} \subseteq a'aa'R^{\mathrm{T}}a'^{\mathrm{T}}.
\end{aligned}$$

By anti-symmetry, we have

$$\mathrm{id} \subseteq a'aa'(R \cap R^{\mathrm{T}})a'^{\mathrm{T}} \subseteq a'aa'a'^{\mathrm{T}} \qquad \text{and} \qquad \mathrm{id} \subseteq aa'a(S \cap S^{\mathrm{T}})a^{\mathrm{T}} \subseteq aa'aa^{\mathrm{T}}.$$

Finally I.112 gives $a = aa'a$ and $a' = a'aa'$. □

Proposition II.53. *Let (A, R) and (B, S) be preorders. Let $a : A \to B$ and $a' : B \to A$ be functions. If a, a' are generalised inverses and*

$$aS = Ra'^{\mathrm{T}},$$

then (a, a') is a Galois connection.
If (A, R) and (B, S) are partial orders, then the assumption of generalised inverses is superfluous.

So the identity $aS = Ra'^{\mathrm{T}}$ is sufficient for two functions a, a' between posets to form a Galois connection. Indeed this is often taken as the definition of a Galois connection for posets.

Proof. We need to prove that a, a' are order-preserving, generalised inverses and that $aa', a'a$ are properly restrictive.
We first verify restrictivity. By reflexivity we have

$$aa' \subseteq aSa' = Ra'^{\mathrm{T}}a' \subseteq R \qquad \text{and} \qquad a'a \subseteq a'R^{\mathrm{T}}a = S^{\mathrm{T}}a^{\mathrm{T}}a \subseteq S^{\mathrm{T}}.$$

By II.41 aa' and $a'a$ are properly restrictive.
Next we prove order-preservation $aa' \subseteq R$. We first prove the identities of II.42:

$$R^{\mathrm{T}} \subseteq aa'(aa')^{\mathrm{T}}R^{\mathrm{T}} \subseteq aa'R^{\mathrm{T}}(aa')^{\mathrm{T}}R^{\mathrm{T}} = aa'(aa'R)^{\mathrm{T}}R^{\mathrm{T}} \subseteq aa'R^{\mathrm{T}}R^{\mathrm{T}} = aa'R^{\mathrm{T}}$$

$$S \subseteq a'a(a'a)^{\mathrm{T}}S \subseteq a'aS(a'a)^{\mathrm{T}}S = a'a(a'aS^{\mathrm{T}})^{\mathrm{T}}S \subseteq a'a(S^{\mathrm{T}})^{\mathrm{T}}S = a'aS.$$

From this relation-preservation follows easily:

$$R \subseteq R(aa')^{\mathrm{T}} = (Ra'^{\mathrm{T}})a^{\mathrm{T}} = aSa^{\mathrm{T}} \qquad \text{and} \qquad S \subseteq a'aS = a'(aS) = a'Ra'^{\mathrm{T}}.$$

Finally, assuming R, S are anti-symmetric, then a, a' are generalised inverses by II.52. □

Proposition II.54. *Let (A, R) and (B, S) be relational structures.*

1. *If A is a poset, then every residuated map $a : A \to B$ has a unique residual.*
2. *If B is a poset, then every residual map $a' : B \to A$ is the residual of a unique residuated map.*

Proof. (1) Let a be a residuated map with residuals a_1' and a_2'. Then

$$\mathrm{id}_B \subseteq S \subseteq a_1'Ra_1'^{\mathrm{T}} = a_1'aS = a_1'Ra_2'^{\mathrm{T}}.$$

Similarly $\mathrm{id}_B \subseteq a_2'Ra_1'^{\mathrm{T}}$, which we can transpose to get $\mathrm{id}_B \subseteq a_1'R^{\mathrm{T}}a_2'^{\mathrm{T}}$. Together this gives

$$\mathrm{id}_B \subseteq a_1'(R \cap R^{\mathrm{T}})a_2'^{\mathrm{T}} = a_1'a_2'^{\mathrm{T}}.$$

We conclude using I.112.
(2) Similar. □

Proposition II.55. *Let (A, R) and (B, S) be partial orders. Then every residuated map $a : A \to B$ has a unique residual and every residual map $a' : B \to A$ is the residual of a unique residuated map.*

Proof. Let a be a residuated map with residuals a_1' and a_2'. It is enough to show that $aa_1' = aa_2'$ and $a_1'a = a_2'a$. Indeed II.30 then shows that $a_1'\mathcal{H}a_2'$ and by II.33 this means that $a_1' = a_2'$.
To this end we combine $\mathrm{id}_A \subseteq aa_1'R^{\mathrm{T}}$, $\mathrm{id}_A \subseteq$ □

3.3.1.2 Derived Galois connections

Lemma II.56. *Let (a, a') be a Galois connection between (A, R) and (B, S). Then (a', a) is a Galois connection between the dual structures $\big(B^o, S^{\mathrm{T}}\big)$ and $\big(A^o, R^{\mathrm{T}}\big)$.*

Lemma II.57. *Let (a, a') be a Galois connection between posets (A, R) and (B, S) and (b, b') a Galois connection between posets (B, S) and (A, R). Then*

1. *$(ab, b'a')$ is a Galois connection between (A, R) and (A, R);*
2. *$(ba, a'b')$ is a Galois connection between (B, S) and (B, S).*

Proof. We prove (1). The proof of (2) is identical. By II.53 we just need to verify the Galois identity:

$$abR = aRb'^{\mathrm{T}} = Ra'^{\mathrm{T}}b'^{\mathrm{T}} = R(b'a')^{\mathrm{T}}.$$

□

3.3.2 Monads and comonads

Let (A, R) be a relational structure and $f : A \to A$ a function. We call f a

- monad if there exists a relational structure (B, S) and functions $a : A \to B$ and $a' : B \to A$ such that (a, a') is a Galois connection and $f = a'a$;
- comonad if $f = aa'$.

Lemma II.58. *Let (A, R) be a relational structure and $f : A \to A$ a function. Then*

1. *f is a monad* if and only if
   - *(a) f is relation-preserving;*
   - *(b) f is idempotent: $f^2 = f$;*
   - *(c) f is right-restrictive: $R; f^{\mathrm{T}} \subseteq R$.*
2. *f is a comonad* if and only if
   - *(a) f is relation-preserving;*
   - *(b) f is idempotent: $f^2 = f$;*
   - *(c) f is left-restrictive: $f; R \subseteq R$;*

Proof. (1) The direction $\Rightarrow$ is clear. Conversely, $\Leftarrow$, we claim the inclusion $\iota : \mathrm{im}(f) \hookrightarrow A$ is residuated with residual $f : A \to \mathrm{im}(f)$.
Indeed, ι is clearly relation-preserving. Take arbitrary $f(x) \in \mathrm{im}(f)$ and arbitrary $y \in A$. Then

$$\begin{gathered}(\iota f \iota)(f(x)) = \iota(f^2(x)) = \iota(f(x)) \\ (f \iota f)(y) = f^2(y) = f(y),\end{gathered}$$

so $\iota f \iota = \iota$ and $f \iota f = f$, meaning f and ι are generalised inverses.
Finally we check

$$\begin{aligned}\iota f(R|^{\mathrm{im}\, f}_{\mathrm{im}(f)}) \subseteq \iota f \iota(R|^{\mathrm{im}\, f}_{\mathrm{im}(f)}) = \iota(R|^{\mathrm{im}\, f}_{\mathrm{im}(f)}) \subseteq R|^{\mathrm{im}\, f}_{\mathrm{im}(f)} \\ f \iota R^{\mathrm{T}} = f R^{\mathrm{T}} = (R; f^{\mathrm{T}})^{\mathrm{T}} \subseteq R^{\mathrm{T}}.\end{aligned}$$

(2) Similar. □

Idempotent =¿ image set is set of fixed points.

Proposition II.59. *When is a subset $X \subseteq A$ the image of a (co)monad? For posets: enough that $\uparrow x \cap X$ has bottom.*

Proposition II.60. *Bijection monads - images of monads.*

Completeness

3.3.3 Inclusion-preserving functions on powersets

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20565)%20Marcel%20Ern%C3%A9%20(auth.),%20K.%20Denecke,%20M.%20Ern%C3%
A9,%20S.%20L.%20Wismath%20(eds.)%20-%20Galois%20Connections%20and%20Applications-Spri
20Netherlands%20(2004).pdf
https://sciendo.com/pdf/10.2478/ausm-2014-0019

Proposition II.61. *Let R be a relation on (A, B). Then the polar functions*

$$\mathcal{P}(A)^o \to \mathcal{P}(B) : X \mapsto X^R \qquad \textit{and} \qquad \mathcal{P}(A)^o \to \mathcal{P}(B) : X \mapsto {}^R X$$

form a Galois connection. Every antitone Galois connection between powersets is of this form, for some relation.

Proof. As $(\mathcal{P}(A), \subseteq)^o$ and $(\mathcal{P}(B), \subseteq)$ are posets, by II.53 we just need to verify the Galois identity: $\forall X \in \mathcal{P}(A), Y \in \mathcal{P}(B)$: we have

$$X^R \subseteq^{\mathrm{T}} Y \iff X^R \supseteq Y \iff \Big[\forall x \in X : \forall y \in Y : xRy\Big] \iff X \subseteq {}^R Y.$$

We can give the same calculation using the language of I.42:

$$X^R \subseteq^{\mathrm{T}} Y \iff X \times Y \subseteq R \iff Y \times X \subseteq R^{\mathrm{T}} \iff X \subseteq Y^R.$$

TODO semilattice morphism ☐

Corollary II.61.1. *Let R be a transitive homogeneous relation on A. Then the principal image and preimage functions*

$$A \to \mathcal{P}(B) : X \mapsto X^R \qquad \textit{and} \qquad \mathcal{P}(A)^o \to \mathcal{P}(B) : X \mapsto {}^R X$$

Proposition II.62. *Let R be a homogeneous relation on A. Then*

1. *the image function is part of a Galois connection*

$$\mathcal{P}(A) \to \mathcal{P}(A) : X \mapsto X_R \qquad \textit{and} \qquad \mathcal{P}(A) \to \mathcal{P}(A) : X \mapsto {}^{\overline{R}}(X^c);$$

2. *the preimage function is part of a Galois connection*

$$\mathcal{P}(A) \to \mathcal{P}(A) : X \mapsto {}_R X \qquad \textit{and} \qquad \mathcal{P}(A) \to \mathcal{P}(A) : X \mapsto (X^c)^{\overline{R}}.$$

These Galois connections are equivalent by replacing $R \leftrightarrow R^{\mathrm{T}}$. Every isotone Galois connection between powersets is of this form, for some relation.

Proof. We compose the polar Galois connection of $\overline{R}$ with the Galois connection of complementation to get

$$X_R = (X^{\overline{R}})^c \subseteq Y \iff X^{\overline{R}} \supseteq Y^c \iff X \subseteq {}^{\overline{R}}(Y^c).$$

Similarly we have

$${}_R X = ({}^{\overline{R}} X)^c \subseteq Y \iff {}^{\overline{R}} X \supseteq Y^c \iff X \subseteq (Y^c)^{\overline{R}}.$$

TODO semilattice morphism ☐

Proposition II.63. *Let $f : A \to B$ be a function. Then the preimage and image functions*

$$\mathcal{P}(B) \to \mathcal{P}(A) : X \mapsto f^{-1}[X] \qquad \textit{and} \qquad \mathcal{P}(A) \to \mathcal{P}(B) : X \mapsto f[X]$$

form a Galois connection.

Proof. We verify the Galois identity for II.53 by writing $f^{-1}[X] = {}_fX$ and $f[X] = X_f$:

$${}_fX \subseteq Y \implies X \subseteq {}_{f^\mathrm{T};f}X \subseteq {}_{f^\mathrm{T}}Y \implies {}_fX \subseteq {}_{f;f^\mathrm{T}}Y \subseteq Y.$$

□

3.3.3.1 Closure and dual closure

Corollary II.63.1. *Let $(P, \prec)$ be an ordered set and $\mathcal{A} \subseteq \mathcal{P}(P)$. Then*

1. $\uparrow\bigcup\mathcal{A} = \bigcup_{A\in\mathcal{A}}\uparrow A$ *and* $\downarrow\bigcup\mathcal{A} = \bigcup_{A\in\mathcal{A}}\downarrow A$;
2. $\uparrow\bigcap\mathcal{A} \subseteq \bigcap_{A\in\mathcal{A}}\uparrow A$ *and* $\downarrow\bigcap\mathcal{A} \subseteq \bigcap_{A\in\mathcal{A}}\downarrow A$;
3. $\uparrow\bigcap_{A\in\mathcal{A}}\uparrow A = \bigcap_{A\in\mathcal{A}}\uparrow A$ *and* $\downarrow\bigcap_{A\in\mathcal{A}}\downarrow A = \bigcap_{A\in\mathcal{A}}\downarrow A$.

Proof. (1) We calculate using I.223:

$$\uparrow\bigcup\mathcal{A} = \bigcup_{x\in\bigcup\mathcal{A}}\uparrow x = \bigcup_{A\in\mathcal{A}}\bigcup_{x\in A}\uparrow x = \bigcup_{A\in\mathcal{A}}\uparrow A.$$

(2) Again we calculate using I.223:

$$\uparrow\bigcap\mathcal{A} = \bigcup_{x\in\bigcap\mathcal{A}}\uparrow x \subseteq \bigcap_{A\in\mathcal{A}}\bigcup_{x\in A}\uparrow x = \bigcap_{A\in\mathcal{A}}\uparrow A.$$

(3) We calculate using (2) and the fact that closures are idempotent:

$$\bigcap_{A\in\mathcal{A}}\uparrow A = \bigcap_{A\in\mathcal{A}}\uparrow\uparrow A \supseteq \uparrow\bigcap_{A\in\mathcal{A}}\uparrow A \supseteq \bigcap_{A\in\mathcal{A}}\uparrow A.$$

□

Also t

Corollary II.63.2. *Let P be an ordered set and $\{Q_i\}_{i\in I}$ be a set of down sets in P. Then*

1. $\bigcup Q_i$ *is a down set;*
2. $\bigcap Q_i$ *is a down set.*

The same is true for up sets.

Proof. This follows from $\downarrow\bigcup Q_i = \bigcup\downarrow Q_i = \bigcup Q_i$ and $\downarrow\bigcap Q_i \subseteq \bigcap\downarrow Q_i = \bigcap Q_i$ by **??**. □

3.3.3.2 Maps and polars

Proposition II.64. *Let (A, R) and (B, S) be relational structures, $f : A \to B$ a relation-preserving function and $X \subseteq A$ a subset. Then*

1. $f[X^R] \subseteq f[X]^S \cap \mathrm{im}(f)$;
2. $f[{}^R X] \subseteq {}^S f[X] \cap \mathrm{im}(f)$;
3. $f[\max(X)] \subseteq \max(f[X])$;
4. $f[\min(X)] \subseteq \min(f[X])$.

For relation-reflecting functions we have reversed inclusions. For relation embeddings the inclusions become equalities.

Proof. (1) We calculate, using I.41 and II.39.2,

$$f[X^R] = f\left[\bigcap_{x\in X} xR\right] \subseteq \bigcap_{x\in X} f[xR] \subseteq \bigcap_{x\in X} Sf(x) = \bigcap_{y\in f[X]} Sy = f[S]^S.$$

(b) Dual to (a).
(c) We calculate

$$f[\max(X)] = f[X\cap X^R] \subseteq f[X]\cap f[X^R] \subseteq f[X]\cap f[X]^S \cap \mathrm{im}(f) = f[X]\cap f[X]^S = \max(f[X]).$$

(d) Dual to (c). ☐

We cannot say anything about the supremum or infimum for general relation-preserving functions, because $f[X^R] \subseteq f[X]^S$ implies $f[X^R]^{S^{\mathrm{T}}} \supseteq (f[X]^S)^{S^{\mathrm{T}}}$, so the calculation would be

$$f[\sup(X)] = f[X^R \cap (X^R)^{R^{\mathrm{T}}}] \subseteq f[X]^S \cap f[(X^R)]^{S^{\mathrm{T}}} \supseteq f[X]^S \cap (f[X]^S)^{S^{\mathrm{T}}} = \sup(f[X]),$$

from which we cannot conclude anything.

Proposition II.65. *Let (A, R) and (B, S) be relational structures, $X \subseteq A$ a subset and $f, g : A \to B$ functions.*

1. *If f is a residuated map, then $f[\sup(X)] \subseteq \sup(f[X])$.*
2. *If g is a residual map, then $g[\inf(X)] \subseteq \inf(g[X])$.*

Proof. (1) Let $f' : B \to A$ be a residual of f. From above we see that it is enough to have $f[(X^R)^{R^{\mathrm{T}}}] \subseteq (f[X]^S)^{S^{\mathrm{T}}}$.
We use $ff'R \subseteq R$ and $f'fS^{\mathrm{T}} \subseteq S^{\mathrm{T}}$, to get $X^{ff'R} \subseteq X^R$ and $X^{f'fS^{\mathrm{T}}} \subseteq X^{S^{\mathrm{T}}}$. This allows us to make the following calculation:

$$\begin{aligned} f[(X^R)^{R^{\mathrm{T}}}] &\subseteq f[X^R]^{S^{\mathrm{T}}} \subseteq f[X^{ff'R}]^{S^{\mathrm{T}}} \\ &\subseteq f[f'[f[X]^S]]^{S^{\mathrm{T}}} = (f[X]^S)^{f'fS^{\mathrm{T}}} \subseteq (f[X]^S)^{S^{\mathrm{T}}}. \end{aligned}$$

(2) The proof in this case is similar. Let $g^- : B \to A$ be a residuated map of which g is a residual. Now it is enough to have $g[(X^{R^{\mathrm{T}}})^R] \subseteq (g[X]^{S^{\mathrm{T}}})^S$.
Using $X^{g^- gS} \subseteq X^S$ and $X^{gg^- R^{\mathrm{T}}} \subseteq X^{R^{\mathrm{T}}}$, we calculate

$$\begin{aligned} g[(X^{R^{\mathrm{T}}})^R] &\subseteq g[X^{R^{\mathrm{T}}}]^S \subseteq f[X^{gg^- R^{\mathrm{T}}}]^S \\ &\subseteq g[g^-[g[X]^{S^{\mathrm{T}}}]]^S = (g[X]^{S^{\mathrm{T}}})^{g^- gS} \subseteq (g[X]^{S^{\mathrm{T}}})^S. \end{aligned}$$

☐

Proposition II.66. *Let P, Q be partial orders.*

1. *If P is a complete $\vee$-semilattice, then an order-preserving function $f : P \to Q$ is residuated* if and only if *f preserves joins. The residual is given by*

$$f^+ : Q \to P : y \mapsto \bigvee f^{-1}[\downarrow y].$$

2. *If Q is a complete $\wedge$-semilattice, then an order-preserving function $g : Q \to P$ is a residual map* if and only if *g preserves meets. It is the residual of the residuated map*

$$g^- : P \to Q : x \mapsto \bigwedge g^{-1}[\uparrow x].$$

Proof. Both direction $\Rightarrow$ are given by II.65.
By II.52 it is enough to verify that f^+ and g^- are order-preserving and for all $x \in P, y \in Q$:

$$f^+(f(x)) \leq x, \quad f(f^+(y)) \geq y, \quad g(g^-(x)) \leq x, \quad \text{and} \quad g^-(g(y)) \geq y.$$

To prove f^+ is order-preserving, take arbitrary $y_1, y_2 \in Q$. Then

$$y_1 \leq y_2 \implies \downarrow y_1 \subseteq \downarrow y_2 \implies f^{-1}[\downarrow y_1] \subseteq f^{-1}[\downarrow y_2] \implies \bigvee f^{-1}[\downarrow y_1] \leq \bigvee f^{-1}[\downarrow y_2].$$

The proof for g^- is similar. Now we prove $f^+(f(x)) \leq x$, for any arbitrary $x \in P$:

todo

$$f(f^+(y)) = f\left(\bigvee f^{-1}[\downarrow y]\right) = \bigvee f\left[^{-1}[\downarrow y]\right] \geq y,$$

where the last inequality follows from $\{y\} \subseteq f\left[^{-1}[\downarrow y]\right]$. TODO □

TODO anti-symmetry, means preservation of inf/sup means equality.

Corollary II.66.1. *Let P be a partial order and $D \subseteq P$ a subset.*

1. *If P is a complete lattice and $\forall S \subseteq D : \bigvee_P S \in D$, then D is a complete sublattice and there exists a monad $f : P \to D$ such that for all $S \subseteq D$*

$$\bigvee_D S = \bigvee_P S \qquad \text{and} \qquad \bigwedge_D S = f\left[\bigwedge_P S\right].$$

2. *If P is a complete lattice and $\forall S \subseteq D : \bigwedge_P S \in D$, then D is a complete sublattice and there exists a comonad $f : P \to D$ such that for all $S \subseteq D$*

$$\bigvee_D S = f\left[\bigvee_P S\right] \qquad \text{and} \qquad \bigwedge_D S = \bigwedge_P S.$$

Proof. (1) Saying D is closed under arbitrary joins is equivalent to saying the inclusion $D \hookrightarrow P$ preservis joins. By the proposition this means the inclusion is residuated. The rest follows from TODO □

TODO: swap monad - comonad??

3.3.3.3 Sets of functions and relations

Proposition II.67. *Let X be a set. The function*

$$m : (\mathcal{P}(X \to X), \subseteq) \to \big(\mathcal{P}(X^2), \subseteq\big) : F \mapsto \big\{(x, y) \in X^2 \;\big|\; \exists f \in F : \; f(x) = y\big\}$$

is residuated.

Proof. It preserves joins ($\cup$). $\square$

Proposition II.68. *Consider the function*

$$m : \mathcal{P}(X \to X) \to \mathcal{P}(X^2) : F \mapsto \big\{(x, y) \in X^2 \;\big|\; \exists f \in F : \; f(x) = y\big\}.$$

1. *$m(F)$ is transitive* if and only if *F is closed under composition;*
2. *$m(F)$ is reflexive* if and only if *F contains* id.

Chapter 4

Graphs

Let $G = (V, E)$ be a relational structure. We call G

- a directed graph or digraph if it is irreflexive;
- an undirected graph or just graph if it is irreflexive and asymmetric.

In either of these cases the elements of V are called vertices, nodes or points. We identify E with its graph and elements of this are called edges. If G is a digraph, we also call elements of E arrows.
Take $x, y \in V$.

- If $\langle x, y \rangle$ is an edge in E, then we write $xy \coloneqq \langle x, y \rangle$.
- Take $e \in E$. We say x is
  - the initial vertex of e if there exists some $a \in V$ such that $e = xa$;
  - the terminal vertex of e if there exists some $a \in V$ such that $e = ax$;
  - an end or endvertex of e if it is either an initial of terminal vertex.
- If $x \nparallel y$, we say x and y are adjacent or neighbours. This is equivalent to saying xy is an edge.
- Take $e_1, e_2 \in E$. We say e_1 and e_2 are adjacent if they have a common endvertex.
- We call G a complete graph if $G = U_V$ for some set V.

We call $|G| \coloneqq |V|$ the order of the graph. We also define $\|G\| \coloneqq |E|$.

Lemma II.69. *A graph (V, R) can equivalently be encoded as (V, E) where each element of E is a doubleton subset of V. TODO functor*

$$\langle x, y \rangle \mapsto \{x, y\}$$

TODO graphs are the same if we can identify them by flipping edges.
TODO subgraph

4.1 Properties of graphs and elements

4.1.1 Edge sets and paths

Let (V, E) be a digraph and $X, Y \subseteq V$. We define the edge set

$$E(X, Y) \coloneqq \{e \in E \mid \exists x \in X, \exists y \in Y : \ e = xy\}.$$

In particular, for $x \in V$, we define $E(x, Y) \coloneqq E(\{x\}, Y)$ and $E(Y, x) \coloneqq E(Y, \{x\})$.

4.1.1.1 Degrees of vertices

Let (V, E) be a digraph and $x \in V$. We define

- the indegree of x as $\deg^-(x) \coloneqq |E(V, x)|$;
- the outdegree of x as $\deg^+(x) \coloneqq |E(x, V)|$;
- the degree or valency of x as $\deg(x) \coloneqq \deg^-(x) + \deg^+(x)$.

If $\deg^-(x) = \deg^+(x)$ for all $x \in V$, then G is called a balanced digraph.
If $\deg(x) = 0$, then x is called isolated.

Lemma II.70 (Degree sum formula)**.** *Let $G = (V, E)$ be a digraph. Then*

$$\|G\| = \sum_{x \in V} \deg^-(x) = \sum_{x \in V} \deg^+(x)$$

and thus

$$\|G\| = \frac{1}{2} \sum_{x \in V} \deg(x).$$

Corollary II.70.1. *Let (V, E) be a finite digraph. Then the number of vertices of odd degree is even.*

Proof. As $\sum_{x \in V}$ is an integer, $\sum_{x \in V} \deg(x)$ must be even. $\square$

Let $G = (V, E)$ be a finite digraph. We define

- the average degree of G *deg*$(G) \coloneqq |G|^{-1} \sum_{x \in V} \deg(x)$;
- the minimum degree of G $\delta(G) \coloneqq \min\{\deg(x) \mid x \in V\}$;
  item the maximum degree of G $\Delta(G) \coloneqq \max\{\deg(x) \mid x \in V\}$.

We also define $\deg^\pm(G), \delta^\pm(G)$ and $\Delta^\pm(G)$ for in-/outdegree.

Lemma II.71. *Let $G = (V, E)$ be a finite digraph. Then*

$$\deg(G) = 2\frac{\|G\|}{|G|}$$

Proof. By the degree sum formula II.70. $\square$

Proposition II.72. *Let $G = (V, E)$ be a finite digraph with at least one edge. Then there exists a subdigraph $H \subseteq G$ such that*

$$\delta(H) > \frac{\deg(H)}{2} \geq \frac{\deg(G)}{2}.$$

Proof. We constuct a sequence of subgaphs

$$G = G_0 \supseteq G_1 \subseteq \ldots$$

by deleting a vertex at each step. At each step we choose a vertex $x \in G_i$ such that $\deg(x) \leq \deg(G_i)/2$ and delete it. We stop when no such vertices remain.
Notice that $\deg(G_{i+1}) \geq \deg(G_i)$. Let v be the deleted vertex. Then deleting v removes $2\deg(v)$ from the total degree $\sum_{x\in V_i} \deg(x)$ (because we remove $\deg(v)$ and remove $\deg(v)$ edges that connect v). Then

$$|G_i| \deg(G_i) = |G_{i+1}| \deg(G_{i+1}) + 2\deg(v) = (|G_i| - 1) \deg(G_{i+1}) + 2\deg(v) \leq |G_{i+1}| \deg(G_{i+1}) + \deg(G_i).$$

Rearranging gives $\deg(G_i) \leq \deg(G_{i+1})$.
Now this algorithm is well-defined, no G_i is empty, because $\deg((V, \emptyset)) = 0$ (and we can always rerun the algorithm with an extra unconnected vertex).
As the algorithms must terminate, we must have that $\deg(x) > \deg(H)/2$ for all $x \in H$ at the end. In particular $\delta(H) > \frac{\deg(H)}{2}$. ☐

4.1.1.2 Paths and cycles

Let $G = (V, E)$ be a digraph. A path is a non-empty sequence $\langle x_i \rangle_{i=1}^k$ of distinct vertices such that $x_i x_{i+1} \in E$ for all $i \in 1..(k-1)$.
If G is a graph, we allow for either $x_i x_{i+1} \in E$ or $x_{i+1} x_i \in E$.
We call

- x_1 the initial vertex of the path;
- x_k the terminal vertex of the path;
- x_1 and x_k the ends or end vertices of the path;
- $k - 1$ the length of the path, which is the number of edges in the path;
- the path an $x_1 - x_k$ path.

We say $\langle x_i \rangle_{i=0}^k$ is a path from x_1 to x_k.
A set of paths is independent if it is disjoint.

Let $G = (V, E)$ be a digraph. A cycle is a path $\langle x_i \rangle_{i=1}^k$ such that $x_k x_1 \in E$ (or, if G is a graph, $x_1 x_k \in E$).
Let $P = \langle x_i \rangle_{i=1}^k$ be a cycle. We call

- k the length of the cycle;
- $\min \{\text{len}(P) \mid P \text{ is a cycle}\}$ the girth of G;
- $\max \{\text{len}(P) \mid P \text{ is a cycle}\}$ the circumference of G;

- any edge xy such that $x, y \in P$, but x and y are not adjacent in P a chord.

Proposition II.73. *Let $G = (V, E)$ be a finite digraph. Then*

1. *G contains a path of at least length $\delta^+(G)$;*
2. *G contains a cycle of at least length $\delta^+(G) + 1$, if $\delta^+(G) \geq 2$;*
3. *if G is a graph, we may replace $\delta^+(G)$ by $\delta(G)$.*

Proof. Let $\langle x_i \rangle_{i=1}^k$ be a path of maximal length. Then all v such that $x_k x \in E$ must lie on the path, otherwise we could extend the path. Thus $k - 1 \geq \deg^+(v) \geq \delta^+(G)$. Connecting to any of these v yields a cycle. ☐

4.1.1.3 Distance

Let $G = (V, E)$ be a digraph and $x, y \in V$. The distance between x and y is

$$d(x, y) \coloneqq \min \{\operatorname{len}(P) \mid P \text{ is an } x - y \text{ path}\}.$$

If there do not exist any $x - y$ paths, then we set $d(x, y) \coloneqq \infty$.
We also define

- the diameter $\operatorname{diam}(G) \coloneqq \max_{x,y \in V} d(x, y)$;
- the radius $\operatorname{rad}(G) \coloneqq \min_{x \in V} \max y \in V d(x, y)$.

Lemma II.74. *Let $G = (V, E)$ be a digraph. Then G is symmetric* if and only if *$d(x, y) = d(y, x)$ for all $x, y \in V$.*

Proof. We have $xy \in E$ iff $d(x, y) = 1$. ☐

Lemma II.75. *Let $G = (V, E)$ be a digraph. Then*

$$\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2 \operatorname{rad}(G).$$

Proof. The first inequality is clear.
For the second inequality: for all $x \in V$ we have ☐

Lemma II.76. *Let $G = (V, E)$ be a digraph that contains a cycle. Then*

$$\textit{girth of } G \leq 2 \operatorname{diam}(G) + 1.$$

4.1.2 Trees and forests

4.2 r-partite graphs

4.3 Matching, covering and packing

4.4 Connectivity

4.5 Planar graphs

4.6 Colourings

4.7 Flows

Chapter 5

Groups

`https://www.maths.ed.ac.uk/~tl/gt/gt.pdf`

5.1 Basic definitions

A group is a structured set $(G, \cdot)$ where $\cdot$ is a binary operation on G

$$\cdot : G \times G \to G : (g, h) \mapsto g \cdot h$$

such that

1. $\cdot$ is associative:

$$\forall g_1, g_2, g_3 \in G : \quad g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$$

2. there exists an identity e:

$$\forall g \in G : \quad g \cdot e = e \cdot g = g$$

3. every element has an inverse:

$$\forall g \in G : \exists h \in G : \quad gh = hg = e$$

   We write the inverse h as g^{-1}.

If $\cdot$ is satisfies

$$\forall g_1, g_2 \in G : \quad g_1 \cdot g_2 = g_2 \cdot g_1$$

then the group is called commutative or abelian.
The cardinality of G is the order of the group, denoted $|G|$.

Example

The Klein 4-group has carrier $A = \{e, a, b, c\}$ and is defined by the Cayley table

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

It is commutative (which can be seen by noting that the Cayley table equals its transpose). It is also

- the unique commutative 4-element group with $a^2 = b^2 = c^2 = e$;
- the unique commutative 4-element semigroup with identity $e = a^2 = b^2 = c^2$, and $ab = c$.

Lemma II.77. *Let S be a semigroup. Then S is a group* if and only if

$$\forall x \in S: \; xS = S = Sx.$$

Proposition II.78. *A group is a structure of type $(e, (\cdot)^{-1}, \cdot)$ with arity defined by*

$$\alpha(e) = 0, \qquad \alpha((\cdot)^{-1}) = 1, \qquad \alpha(\cdot) = 2.$$

Conversely, a $(e, (\cdot)^{-1}, \cdot)$-algebra is a group if

- *$\cdot$ is associative;*
- *$\forall g \in G : g \cdot e = e \cdot g = g$;*
- *$\forall g \in G : g \cdot g^{-1} = g^{-1} \cdot g = e$.*

In particular the concepts of homomorphism and isomorphism apply.

5.1.1 Notations

We can use whatever symbols we want to denote the group operation, but there are two main conventions:

1. In multiplicative notation the group operation is denoted by $\cdot$, $*$ or just by concatenation (i.e. we write gh instead of $g \cdot h$). In this case the inverse of g is written g^{-1}, the neutral element e is denoted 1 and we can define

$$g^n \coloneqq \underbrace{gg\ldots g}_{n \text{ factors}}$$

which is unambiguous due to associativity. Also

$$g^{-n} \coloneqq (g^{-1})^n = (g^n)^{-1}.$$

2. Additive notation is mainly used for abelian groups. Conversion between multiplicative and additive notation is as follows:

$$g \cdot h \longleftrightarrow g + h$$

$$1 \longleftrightarrow 0$$

$$g^{-1} \longleftrightarrow -g$$

$$g^n \longleftrightarrow ng.$$

Lemma II.79. *Let G be a group, $g \in G$ and $m, n \in \mathbb{Z}$. Then*

1. *in multiplicative notation we have*
$$g^m g^n = g^{m+n} \qquad (g^m)^n = g^{mn};$$
2. *in additive notation we have*
$$mg + ng = (m+n)g \qquad n(mg) = (mn)g.$$

These statements are equivalent.

5.1.2 Translation invariance

Let G be a group, X a set and $f : G \times G \to X$ a binary function. Then f is called

- left translation invariant if $\forall x, y, z \in G : \; f(x,y) = f(zx, zy)$;
- right translation invariant if $\forall x, y, z \in G : \; f(x,y) = f(xz, yz)$;
- translation invariant if f is left and right translation invariant.

TODO: just for relations??

Proposition II.80 (Universal property translation invariance)**.** *Let G be a group. Define*

$$\Delta_r : G \times G \to G : (x,y) \to xy^{-1} \qquad \Delta_l : G \times G \to G : (x,y) \to x^{-1}y.$$

1. *For any set X and right translation invariant function $f : G \times G \to X$, there exists a unique $\widetilde{f} : G \to X$, such that*

$$\begin{tikzcd} G \times G \arrow[r, "\Delta_r"] \arrow[dr, "f"'] & G \arrow[d, dashed, "\widetilde{f}"] \\ & X \end{tikzcd} \qquad \textit{commutes.}$$

2. *For any set X and left translation invariant function $f : G \times G \to X$, there exists a unique $\widetilde{f} : G \to X$, such that*

$$\begin{tikzcd} G \times G \arrow[r, "\Delta_l"] \arrow[dr, "f"'] & G \arrow[d, dashed, "\widetilde{f}"] \\ & X \end{tikzcd} \qquad \textit{commutes.}$$

Proof. TODO
(1) $\widetilde{f} = f(-, e)$;
(2) $\widetilde{f} = f(e, -)$; $\square$

If a function $G \times G \to X$ is both left and right invariant, then we can choose either factorisation. We typically choose the right invariant factorisation $f = \widetilde{f} \circ \Delta_r$. We often write just Δ to denote Δ_r.

Lemma II.81. *A binary relation is*

1. *left translation invariant* if and only if *it is left compatible;*
2. *right translation invariant* if and only if *it is right compatible.*

Proof. TODO $\square$

For a right compatible binary relation R, we have

$$xRy \qquad \iff \qquad xy^{-1} \in \widetilde{R}.$$

Proposition II.82. *Let G be a group.*

1. *Let $f : G \to H$ be a group homomorphism. The kernel* $\ker f$ *is translation invariant as a Boolean-valued function. We have*

$$\widetilde{\ker} f = \{ g \in G \mid f(g) = e_H \}.$$

2. *A $\mathfrak{q}$ on G is translation invariant, when viewed as a Boolean-valued function.*

Proof. (1) We have

$$(x, y) \in \ker f \iff f(x) = f(y) \iff f(x)f(z) = f(y)f(z) \iff f(xz) = f(yz) \iff (xz, yz) \in \ker.$$

Left translation invariance is dual.
(2) TODO $\square$

When dealing with groups, we will redefine ker to mean $\widetilde{\ker}$.

5.1.3 Subgroups

Let $(G, \cdot)$ be a group. We call $(H, *)$ a subgroup if it is a group and $H \subseteq G$ and $* = \cdot|_H$.

Lemma II.83 (Subgroup criterion)**.** *Let $(G, \cdot)$ be a group and H a non-empty subset of G. The following are equivalent:*

1. *$(H, \cdot|_H)$ is a subgroup;*
2. *for all $a, b \in H$:*
    - *$a \cdot b \in H$,*
    - *$a^{-1} \in H$;*
3. *for all $a, b \in H : a \cdot b^{-1} \in H$.*

Lemma II.84. *Let G be a group and H_1, H_2 be subgroups. Then $H_1 \cap H_2$ is again a subgroup of G.*

Lemma II.85. *Let $f : G \to H$ be a group homomorphism. Then* $\ker(f)$ *is a subgroup.*

5.1.3.1 Cosets

Let G be a group and $H \subseteq G$ a subgroup. We call a subset of the form

- $g \cdot H$ for some $g \in G$ a left coset;
- $H \cdot g$ for some $g \in G$ a right coset.

A coset is a subset that is either a left coset or a right coset.

Lemma II.86. *Let G be a group and $H \subseteq G$ a subgroup. Any two left (resp. right) cosets are either identical or disjoint.*

Proof. Take $g, h \in G$. Assume $x \in gH \cap hH$. Then there exist $x_1, x_2 \in H$ such that $gx_1 = x = hx_2$. Thus $g = hx_2x_1^{-1}$ and $h = gx_1x_2^{-1}$, meaning $gH = hx_2x_1^{-1}H = hH$ by II.18. $\square$

5.1.3.2 Lagrange's theorem

Theorem II.87. *Let G be a group and H a subgroup of G. Then*

$$|G| = [G : H] \cdot |H|.$$

If G is finite, $|G|$ and $|H|$ are natural numbers. If G is infinite, the theorem still holds, but the orders and index are cardinals.

5.1.3.3 Normal subgroups

Let G be a group. A subgroup $N \subseteq G$ is called normal or self-conjugate if $gNg^{-1} \subseteq N$ for all $g \in G$.
We write $N \lhd G$.

Proposition II.88. *Let G be a group. A translation invariant binary relation $\mathfrak{q}$ on G is a congruence* if and only if *$\widetilde{\mathfrak{q}}$ is a normal subgroup.*

Proof. First assume $\mathfrak{q}$ is a congruence and take $z \in \widetilde{\mathfrak{q}}$. We can find $(x, y) \in \mathfrak{q}$ such that $z = xy^{-1}$. Take arbitrary $g \in G$. We need to show that $g(xy^{-1})g^{-1} \in \widetilde{\mathfrak{q}}$. Because $\mathfrak{q}$ is reflexive, we have $(g, g) \in \mathfrak{q}$. Because is it is a subalgebra of G^2, we have $(gx, gy) = (g, g) \cdot (x, y) \in \mathfrak{q}$. So $g(xy^{-1})g^{-1} = gx(gy)^{-1} \in \widetilde{\mathfrak{q}}$.
Now assume $\widetilde{\mathfrak{q}}$ is a normal subgroup. We first check that $\mathfrak{q}$ is an equivalence relation:

- reflexivity: $e = gg^{-1} \in \widetilde{\mathfrak{q}}$, so $(g, g) \in \mathfrak{q}$ for all $g \in G$;
- symmetry: if $xy^{-1} \in \widetilde{\mathfrak{q}}$, then $yx^{-1} = (xy^{-1})^{-1} \in \widetilde{\mathfrak{q}}$;
- transitivity: if $xy^{-1}, yz^{-1} \in \widetilde{\mathfrak{q}}$, then $xy^{-1}yz^{-1} = xz^{-1} \in \widetilde{\mathfrak{q}}$.

Now we need to show that $\mathfrak{q}$ is a subalgebra. Take $(x, y) \in \mathfrak{q}$. Then $x^{-1}(y^{-1})^{-1} = x^{-1}y = y^{-1}(xy^{-1})^{-1}y$, so $\widetilde{\mathfrak{q}}$ is closed inder taking the inverse.
Take $(x, y), (u, v) \in \mathfrak{q}$. Then uv^{-1}, xy^{-1} and $y^{-1}x = y^{-1}(xy^{-1})y$ are elements of $\widetilde{\mathfrak{q}}$. Then

$$xu(yv)^{-1} = xuv^{-1}y^{-1} = x(uv^{-1})(y^{-1}x)x^{-1} \in \widetilde{\mathfrak{q}},$$

so $\mathfrak{q}$ is closed under the group operation. $\square$

Corollary II.88.1. *Let $f : G \to H$ be a group homomorphism. Then $\ker f$ is a normal subgroup.*

If $N \subseteq G$ is a normal subgroup, we define the quotient algebra

$$G/N \coloneqq G/(\{(x,y) \in G^2 \mid xy^{-1} \in N\}).$$

This is a group because homomorphisms preserve associativity, inverses and identity (TODO ref). We call such a group a quotient group.
We denote the equivalence classes by $[x]_N$.

5.1.4 Conjugation

Let G be a group and $g \in G$ an element. Then the mapping

$$\mathrm{Ad}_g : G \to G : x \mapsto g^{-1}xg$$

is called conjugation by g. We also write $h^g \coloneqq \mathrm{Ad}_g(h) = g^{-1}hg$.

Thus a subgroup is normal if and only if $\mathrm{Ad}_g[N] \subseteq N$ for all $g \in G$.

5.1.4.1 Conjugacy

Let G be a group. Elements $g, h \in G$ are called conjugate if $\exists x : \ \mathrm{Ad}_x(g) = h$.

Proposition II.89. *Conjugacy is an equivalence relation.*

The equivalence classes under conjugation are called conjugacy classes.

5.1.4.2 Centraliser and normaliser

Lemma II.90. *Let G be a group and $A \subseteq G$ a subset. Then*

1. $Z_G(A) = \{g \in G \mid \forall a \in A : \ \mathrm{Ad}_g(a) = a\}$*;*
2. $N_G(A) = \{g \in G \mid \mathrm{Ad}_g[A] = A\}$*.*

Proposition II.91. *Let G be a group and $A \subseteq G$ a subset. Then*

1. $Z_G(A) \lhd N_G(A) \lhd G$*;*
2. $N_G(A)$ *is the largest subgroup of G in which A is normal.*

Corollary II.91.1. $Z_G \lhd G$*.*

5.1.4.3 Inner and outer automorphisms

Lemma II.92. *Let G be a group and $g \in G$ an element. Then Ad_g is an automorphism.*

Let G be a group. Automorphisms of the form Ad_g for some $g \in G$ are called inner automorphisms. Automorphisms that are not of this form are called outer automorphisms.

The set of inner automorphisms forms a group, denoted $\mathrm{Inn}(G)$.

Theorem II.93 (N/C theorem)**.** *Let G be a group and $H \subseteq G$ a subgroup. Then*

$$N_H(H)/Z_G(H) \cong \mathrm{Inn}(H).$$

Proof. TODO `https://proofwiki.org/wiki/Centralizer_is_Normal_Subgroup_of_Normalizer` ☐

Corollary II.93.1. *Let G be a group. Then $G/Z_G \cong \mathrm{Inn}(G)$.*

5.1.5 Direct product

The direct product $G \equiv H \otimes F$ of two groups H and F is defined with the following operation:

$$(H \otimes F) \times (H \otimes F) \to (H \otimes F) : ((h_1, f_1), (h_2, f_2)) \mapsto (h_1 \cdot h_2, f_1 \cdot f_2)$$

The direct product is a group with

$$\begin{cases} e_G = (e_H, e_F) \\ g^{-1} = (h^{-1}, f^{-1}) \qquad \forall g = (h, f) \in G. \end{cases}$$

The groups F and H are subgroups of G and can be recovered by considering, respectively the elements of G of the form (e_H, g) and (g, e_F).

5.1.6 Semidirect product

TODO

5.2 Types of groups

Example

Examples of Groups:

1. The trivial group $\{e\}$.
2. $\mathbb{Z}_n = 0 : (n-1)$ with addition modulo n, is a group of order n.
3. $\mathbb{Z}_n$, the group of all n^{th} roots of 1 with the ordinary product, is of order n.
    - $Z_2 = \{1, -1\}$
    - $Z_3 = \{1, e^{i2/3\pi}, e^{i1/3\pi}\}$
4. S_n, the group of all permutations of n elements, is of order $n!$.
5. Integers with addition.
6. $\mathbb{R} \setminus \{0\}$ with multiplication.
7. The square (i.e. $n \times n$) invertible matrices with matrix multiplication form a group.

Proposition II.94. *The groups Z_n and $\mathbb{Z}_n$ are isomorphic.*

We use Z_n to denote the group if we are using multiplicative notation and $\mathbb{Z}_n$ if we are using additive notation.
In particular we have $\mathbb{Z}_2 = (\{0, 1\}, +)$ and $Z_2 = (\{1, -1\}, \cdot)$.

Lemma II.95. *All groups of order 2 are isomorphic to $\mathbb{Z}_2$.*

Proof. Let $G = (\{e, g\}, \cdot)$ be a groups of order 2, with e the identity. We must have $g \cdot g = e$. Indeed, from $g \neq e$, we get $g \cdot g \neq g \cdot e = g$ and $g \cdot g = e$ is the only other option.
Consider the function

$$f : G \to \mathbb{Z}_2 : \begin{cases} e \mapsto 0 \\ g \mapsto 1 \end{cases} .$$

This functions is clearly bijective. We just need to see that it is a homomorphism. Indeed we have

$$\begin{cases} f(e \cdot e) = f(e) = 0 = 0 + 0 = f(e) + f(e) & f(e \cdot g) = f(g) = 1 = 0 + 1 = f(e) + f(g) \\ f(g \cdot e) = f(g) = 1 = 1 + 0 = f(g) + f(e) & f(g \cdot g) = f(e) = 0 = 1 + 1 = f(g) + f(g). \end{cases}$$

☐

5.2.1 Words, relations and presentations

Example

Quaternion group

$$\mathbb{H} := \mathrm{gp}\left\{a, b \mid a^4 = e, a^2 = b^2, b^{-1}ab = a^{-1}\right\}$$

5.2.2 Cyclic groups

A group is called cyclic if it is generated by a single element.

Lemma II.96.

1. *The group $(\mathbb{Z}, +)$ is cyclic.*
2. *Every cyclic group is a an image of $\mathbb{Z}$ by a homomorphism.*
3. *Every cyclic group is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$.*

We write $\mathbb{Z}_m$ or C_m for $\mathbb{Z}/m\mathbb{Z}$.

5.2.3 Torsion groups and orders of elements

Let G be a group. An element a satisfying $a^n = 1$ for some n is said to be of finite order.
In this case the order of the element a is n.
A group in which every element is of finite order is called a torsion group or a periodic group.

Lemma II.97. *Every finite group is a torsion group. The converse is not true.*

Proof. Let G be a finite group. Assume G is not a torsion group. Then we can find an element $g \in G$ that is not of finite order. Consider the mapping $\mathbb{N} \to G : n \mapsto g^n$. The image of this mapping is a subset of G and thus finite, so the mapping is not injective, so we can find $n < m \in \mathbb{N}$ such that $g^n = g^m$. Then $g^{m-n} = 1$ with $m - n \in \mathbb{N}$, so g is of finite order, which is a contradiction. This falsity of the converse is shown by the following examples. ☐

Example

The set

$$\{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}\}$$

together with complex multiplication forms an infinite torsion group.

Lemma II.98. *Let G be an abelian group. Then the set of all elements of finite order forms a subgroup, called the torsion subgroup.*

5.2.4 Permutation groups

Proposition II.99.

1. *Let X be a set. The set of bijections $X \to X$ forms a group;*
2. *Let X, Y be sets. The groups of bijections on X and Y are isomorphic* if and only if *X and Y are equinumerous.*

We call the group of bijections on a set X the symmetric group of X, denoted $S(X)$.

- The degree of $S(X)$ is the cardinality of X.
- For any cardinal n, we denote the unique permutation group of degree n by S_n.
- Elements of S_n are called permutations and subgroups of S_n are called permutation groups.

TODO: cfr. Clifford algebra with V containing the transpositions.

Proposition II.100. *For all sets X, we have $S(X) = S_{|X|}$.*

In general we state and prove results for S_n, without loss of generality.

Lemma II.101. *For all cardinals $\kappa >_c 2$, the permutation group S_κ is non-abelian.*

5.2.4.1 Cycles

5.2.4.2 Transpositions, parity and the alternating group

Let S_n be a permutation group and $x \in S_n$. Consider the set $\operatorname{Fp}(x)$. If $|\operatorname{Fp}(x)| = 2$, then x is called a transposition.

Let S_n be a finite permutation group. We call the function

$$p_n : S_n \to Z_2 : x \mapsto \begin{cases} 1 & |\operatorname{Fp}(x)| \text{ is even} \\ -1 & |\operatorname{Fp}(x)| \text{ is odd} \end{cases}$$

the parity homomorphism.
The group $A_n \coloneqq \ker p_n$ is called the alternating group of degree n.

Proposition II.102. *The parity homomorphism is a homomorphism.*

Lemma II.103. *The alternating group A_n has order $n!/2$.*

5.2.5 Dihedral groups

Dihedral group of order $2n$.

$$D_n \coloneqq \operatorname{gp}\left\{a, b \;\middle|\; a^n = b^2 = e, b^{-1}ab = a^{-1}\right\}.$$

Proposition II.104. *Let $n \in \mathbb{N}$. Then*

$$Z_G = \begin{cases} \{e, a^{n/2}\} & n \textit{ is even} \\ \{e\} & n \textit{ is odd.} \end{cases}$$

Corollary II.104.1. *For all $n \in \mathbb{N}$ there is a short exact sequence*

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}_2 \arrow[r] & D_{2n} \arrow[r] & D_n \arrow[r] & 1. \end{tikzcd}$$

5.2.5.1 Full dihedral group

TODO: full dihedral group D of isometries of $\mathbb{C}$ that fix the origin.

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{T} \arrow[r] & D \arrow[r] & Z_2 \arrow[r] & 1 \end{tikzcd} \qquad \text{is short exact.}$$

5.3 Short exact sequences

5.3.1 Quotient sequences

Proposition II.105. *Let G be a group and $N \lhd G$ a normal subgroup. Then*

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, hook, "\subseteq"] & G \arrow[r, "{[\cdot]_N}"] & G/N \arrow[r] & 1 \end{tikzcd}$$

is a short exact sequence.

Proof. Clearly the inclusion $N \hookrightarrow G$ is injective and $[\cdot]_N$ is surjective. Finally note that $x \in \ker[\cdot]_N \iff [x]_N = [e]_N \iff xe^{-1} = x \in N$. □

Proposition II.106. *For any short exact sequence of groups*

$$\begin{tikzcd} 1 \arrow[r] & H_1 \arrow[r, "\alpha"] & G \arrow[r, "\beta"] & H_2 \arrow[r] & 1 \end{tikzcd},$$

there exist isomorphisms f, g and subgroup $N \lhd G$ such that

$$\begin{tikzcd}
1 \arrow[r] & H_1 \arrow[r, "\alpha"] \arrow[d, "f"] & G \arrow[r, "\beta"] \arrow[d, "\mathrm{id}_G"] & H_2 \arrow[r] \arrow[d, "g"] & 1 \\
1 \arrow[r] & N \arrow[r, hook] & G \arrow[r, "{[\cdot]_N}"] & G/N \arrow[r] & 1
\end{tikzcd}$$

commutes.

Proof. We can take $N = \mathrm{im}(\alpha) = \ker(\beta)$, which is a normal subgroup by II.88.1. Because α is injective, $\alpha|^{\mathrm{im}(\alpha)} : H_1 \to N$ is bijective. We take f to be this.
The function $\beta' : G/N \to H_2$ defined in the factor theorem II.10 is bijective because β is surjective.
Both constituent squares clearly commute. So the rectangle commutes by IV.6. $\square$

So in some sense all short exact sequences of groups are of the form

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, hook] & G \arrow[r] & G/N \arrow[r] & 1 .
\end{tikzcd}$$

Given G and either N or G/N, we can easily find the third group in the sequence. Given N and G/N, there may be several inequivalent ways to complete the short exact sequence. Groups that fit in the middle of a short exact sequence are called group extensions.

5.3.2 Group extensions

Let N, Q be groups. An extension of Q by N is a group G such that

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "\iota"] & G \arrow[r, "\pi"] & Q \arrow[r] & 1 .
\end{tikzcd}$$

is a short exact sequence.

Lemma II.107. *If G is an extension of Q by N, then G is a group (TODO: closure), $\iota(N)$ is a normal subgroup of G and Q is isomorphic to Q.*

Example

The real numbers $\mathbb{R}$ are an extension of the unit complex numbers by the integers $\mathbb{Z}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\subseteq"] & \mathbb{R} \arrow[r, "\theta \mapsto e^{2\pi i \theta}"] & \mathbb{T} \arrow[r] & 1
\end{tikzcd}$$

5.3.2.1 Equivalent group extensions

Two extensions G, G' of Q by N are equivalent if there is a homomorphism $T : G \to G'$ making the following diagram commutative:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "\iota"] \arrow[d, equal] & G \arrow[r, "\pi"] \arrow[d, "T"] & Q \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N \arrow[r, "\iota"] & G' \arrow[r, "\pi"] & Q \arrow[r] & 1.
\end{tikzcd}$$

Lemma II.108. *If G, G' are equivalent extensions, then they are isomorphic. So equivalence of extension is an equivalence relation.*

Proof. The short five lemma (TODO). ☐

The converse is not true! TODO: For instance, there are 8 inequivalent extensions of the Klein four-group by $\mathbb{Z}/2\mathbb{Z}$, but there are, up to group isomorphism, only four groups of order 8 containing a normal subgroup of order 2 with quotient group isomorphic to the Klein four-group.

5.3.2.2 Split exact sequences

`https://kconrad.math.uconn.edu/blurbs/grouptheory/splittinggp.pdf`

5.3.2.3 Double covers

Let G_1, G_2 be groups. If G_1 is an extension of G_2 by $\mathbb{Z}_2$, i.e.

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow 1$$

is a short exact sequence, then we call G_1 a double cover of G_2.

Example

For all $n \in \mathbb{N}$, the dihedral group D_{2n} is a double cover of D_n:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow D_{2n} \longrightarrow D_n \longrightarrow 1$$

See II.104.1.

Quaternionic group gives inequivalent extension.

5.4 Group action

An action of a group G on a set X is a mapping

$$\cdot : G \times X \to X : (g, x) \mapsto g \cdot x$$

satisfying

1. $e \cdot x = x$ for all $x \in X$ where e is the neutral element of G;
2. $g(h \cdot x) = (gh) \cdot x$ for all $x \in X$ and $g, h \in G$.

We will often just right gx instead of $g \cdot x$.
A set X with a given action of G on it is called a G-set.

This definition can be reformulated using the curried form of π, namely

$$\rho \coloneqq \mathrm{curry}(\pi) : G \to (X \to X).$$

Then the rest of the definition of group action amounts to the statement that

$$\rho : G \to S(X) \quad \text{is a group homomorphism.}$$

We may then specify G-sets by the data (X, ρ), where X is a set and $\rho : G \to S(X)$ a group homomorphism.
TODO: opposite action.

Given two G-sets X, Y, a G-equivariant mapping or intertwiner is a map $f : X \to Y$ such that

$$f(gx) = gf(x)$$

for all $g \in G$ and $x \in X$.

We can express this by saying $f : (X, \rho_1) \to (Y, \rho_2)$ is a map between G-sets such that

$$f \circ \rho_1(g) = \rho_2(g) \circ f$$

for all $g \in G$.

Lemma II.109. *The G-sets form a locally small category with G-equivariant maps as morphisms.*

5.4.1 Orbits and stabilisers

Let G be a group acting on a set X. We define the orbit of $x \in X$ as the set

$$Gx = G \cdot x \coloneqq \{g \cdot x \in X \mid g \in G\}$$

and the stabiliser of $x \in X$ as the set

$$G_x \coloneqq \{g \in G \mid g \cdot x = x\}.$$

Proposition II.110 (Orbit-stabiliser theorem)**.** *Let X be a G-set and $x \in X$. Then*

1. $|Gx| = [G : G_x]$;
2. $|G| = |Gx| \cdot |G_x|$.

5.4.2 Actions of groups on themselves

5.4.2.1 Regular actions

A group G has a natural left action on the set G:

$$G \times \text{Field}(G) \to \text{Field}(G) : (g, h) \mapsto gh.$$

This action of G is called the left-regular group action.
Similarly, the natural right action on the set G is called the right-regular group action:

$$\text{Field}(G) \times G \to \text{Field}(G) : (h, g) \mapsto hg.$$

5.4.2.2 Conjugation

Let G be a group. The conjugation mapping

$$\text{Ad} : G \times \text{Field}(G) \to \text{Field}(G) : (g, h) \mapsto \text{Ad}_g(h) = g^{-1}hg$$

is a group action by G on itself.

The orbits under this action are the conjugacy classes.

The stabiliser of $a \in G$ when G is acting on itself by conjugation, is the centraliser of $\{a\}$.

5.5 Group action

We have seen that symmetry transformations naturally form a group. Based on the concrete set of transformations that are symmetries we saw they form this abstract structure which we called a group. The advantage of working with this abstract entity is that it contains exactly the relevant details about the symmetry. We need not worry ourselves about the peculiarities of the particular system and we can easily make use of results others have obtained solving other problems.

Once we have thoroughly studied the symmetries of our system, we will want a way to move back from studying abstract groups to studying transformations of the system we are actually interested in.

Sometimes there is a natural correspondence between the set of group elements and the set of transformations. If this is the case the group can be interpreted as acting on the system in a canonical (or natural) way.

Example

- Dihedral group D_4 acts quite naturally on a blanc, square piece of paper.
- The symmetric group $\mathcal{S}_n$ of all permutations of a set of n elements acts naturally on a set of n elements.
- The group of $n \times n$ matrices acts naturally on n-dimensional vectors through matrix multiplication.

In general the transition back may not be so clear, simple or natural. For instance there may be a subset of the n-dimensional vectors with a symmetry group isomorphic to D_4. To what transformations do these group elements correspond? We cannot just rotate and flip these vectors. It is for understanding these cases that the concept of a group action is useful.

5.5.1 Definition

We start with a group G and a set X. The set X is frequently the set of configurations of the system and thus transformations of the system are functions of the type $f : X \to X$; to keep things general, we only assume we have set and we are agnostic as to its origins. A group action quite simply associates a transformation of the set to every element of the group.

We do however require that this association has some fairly natural features, so that the nature and essence of the group is not lost in transition: the group action must respect the identity element and and group operation. This leads us to the following definition:

Let G be a group and X a set, then a (left) group action φ of G on X is a function

$$\varphi : G \times X \to X : (g, x) \mapsto \varphi(g, x) = g \cdot x$$

with the properties:

1. For the identity element e and all $x \in X$:

$$e \cdot x = x$$

2. For all $g, h \in G$ and $x \in X$:

$$(gh) \cdot x = g \cdot (h \cdot x)$$

Notice that we have introduced the notation $g \cdot x$ meaning apply the transformation attributed to g through the group action to the element x.

The above definition is for a *left* group action. We can analogously define a right group action. The only difference between the two is that in the right group action in the transformation

$$x \cdot (gh) = (x \cdot g) \cdot h$$

the transformation associated with g gets applied first. Using the formula $(gh)^{-1} = h^{-1}g^{-1}$ we can always construct a left group action from a right one and vice versa, so typically we only consider left group actions.
An important property is immediately apparent from the definition:

The transformation associated with g (i.e. $x \mapsto g \cdot x$) is always a bijection because the inverse is given by $x \mapsto g^{-1} \cdot x$.

5.5.2 Types of action

What follows is simply an enumeration of some properties group actions may have. The action of G on X is called

1. transitive if X is non-empty and for each x, y in X there exists a $g \in G$ such that $g \cdot x = y$.
2. faithful if for every distinct g, h in G there exists an $x \in X$ such that $g \cdot x \neq h \cdot x$. In other words the mapping of elements of G to transformations of X is 1-to-1 or injective.

5.5.3 Orbits and stabilizers

Consider a group G acting on a set X. The orbit of an element x of X is denoted $G \cdot x$.

$$G \cdot x = \{g \cdot x | g \in G\}.$$

The stabilizer subgroup of G with respect to an element x of X is the set of all elements in G that fix x and is denoted G_x.

$$G_x = \{g \in G | g \cdot x = x\}$$

5.5.4 Continuous group action

A continuous group action on a topological space X is a group action of a topological group G that is continuous: i.e.

$$G \times X \to X : \ (g, x) \mapsto g \cdot x$$

is a continuous map.
This is the proper type of group action to use with topological groups, if their topologicalness is relevant and to be preserved.

5.5.5 Representations

If the group action is the action of a group on a vector space such that the transformations the group elements are mapped to are linear transformations, we call this group action a representation.

A representation of a group G on an n-dim vector space V is a mapping of the elements of G to the set of invertible linear operations acting on V:

$$D : G \to GL(V) : g \mapsto D(g)$$

Such that

- $D(e) = \mathbb{1}_V$
- $D(g_1 \cdot g_2) = D(g_1)D(g_2) = D(g_3)$

Example

- Representations of $Z_3 = \{e, \omega, \omega^2\}$ $\qquad (\omega = e^{i2/3\pi})$
  - Trivial representations

    $$D(e) = D(\omega) = D(\omega^2) = \mathbb{1}_V$$

  - Representation $\mathrm{GL}(1, \mathbb{C})$

    $$D(e) = 1, \quad D(\omega) = e^{i\frac{2}{3}\pi}, \qquad D(\omega^2) = e^{i\frac{1}{3}\pi}$$

  - Regular representation:

    $$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D(\omega) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D(\omega^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

    In general a we can define a regular representation for any finite group G as follows: Let V be a vector space with basis e_t indexed by the elements of G, $t \in G$. The mapping $D : e_t \mapsto e_{ts}$ defines the (left) regular representation of G. This notion can be extended to groups of infinite order.

- The standard representation of a subgroups H of $\mathrm{GL}(n, \mathbb{C})$ on the vector space $\mathbb{C}^n$ is given by the inclusion:

  $$D : H \to \mathrm{GL}(\mathbb{C}^n) = \mathrm{GL}(n, \mathbb{C}) : h \mapsto h$$

Two representations are equivalent if there exists a linear operator S such that

$$D(g) \mapsto D'(g) = S^{-1}D(g)S$$

In other words there exists a similarity transformation S

A representation is unitary if $\forall g \in G$

$$D(g)D^{\dagger}(g) = D^{\dagger}(g)D(g) = \mathbb{1}_V$$

Consider a representation D of a group G on a vector space V

1. A subspace W of V is called invariant if $D(g)w$ is in W for all $w \in W$ and all $g \in G$. An invariant subspace W is called nontrivial if $W \neq \{0\}$ and $W \neq V$.
2. We call D reducible if there exists a nontrivial subspace U of V that is invariant under D.
3. D is irreducible if the only subspaces invariant under all elements of the image of D are $\emptyset$ and V
4. D is completely reducible if we can decompose V into invariant subspaces:

   $$V = U_1 \oplus U_2 \oplus \ldots \oplus U_n$$

   There then exists a similarity transformation such that

   $$\forall g : D(g) = \begin{pmatrix} D_1(g) & 0 & \ldots & 0 \\ 0 & D_2(g) & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & D_n(g) \end{pmatrix} \qquad \text{with} \quad D \equiv D_1 \oplus D_2 \oplus \ldots \oplus D_n$$

Example

The regular representation of Z_3 is completely reducible. The linear operators $D(e), D(\omega)$ and $D(\omega^2)$ have eigenvalues $1, \omega, \omega^2$ with eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}.$$

Each eigenvector generates an invariant subspace. We can then apply the following coordinate transformation

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$$

in order to get the following matrices

$$D'(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D'(\omega) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad D'(\omega^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

$$D' = D_1 \oplus D_2 \oplus D_3 = \mathrm{diag}\{1, 1, 2\} \oplus \mathrm{diag}\{1, \omega, \omega^2\} \oplus \mathrm{diag}\{1, \omega^2, \omega\}$$

5.5.5.1 Projective representations

Bargmann theorem

5.6 Topological groups

A group is a set with an extra structure layered on top: the group operation that satisfies the group axioms. A topological space is also a set with an extra structure layered on top: the topology, as discussed in a previous part. Now here's a novel idea: let's layer both of these structures on a set at once. This gives no new mathematics because the two structures do not interact in any way; in order for interesting things to occur, we must pose some additional requirements.

A topological group G is a topological space that is also a a group such that the group operations of

1. product
$$G \times G \to G : \ (x, y) \mapsto xy$$
2. and taking inverses
$$G \to G : \ x \mapsto x^{-1}$$

are **continuous**.

TODO also need that points are closed?

Lemma II.111. *The continuity of the product and inverse is equivalent to the continuity of* $G \times G \to G : (s, r) \mapsto sr^{-1}$.

TODO; reframe as criterion?

Lemma II.112. *Let G be a topological group. The following are homeomorphisms:*

1. $G \to G : s \mapsto s^{-1}$;
2. $G \to G : s \mapsto rs$ *for any* $r \in G$.

An important consequence of this is that the topology of G is determined by the topology near the identity e.
Topological groups are also sometimes called continuous groups.

5.7 Grothendieck group

Given a commutative monoid M, the Grothendieck group $G(M)$ is the “most general” Abelian group that arises from M. Intuitively it is formed by adding additive inverses for all elements of M.
TODO Grothendieck construction for Abelian monoids: $G(M)$. Universality, functoriality
Cancellation property: simplified construction.
Grothendieck map $M \to G(M)$ is injective if and only if M has cancellation.

5.7.1 The integers

$\mathbb{Z}$

5.8 Ordered groups

Lemma II.113. *Let* $(G, +, \leq)$ *be an ordered group and* $x, y \in G$. *Then*

$$(\forall \varepsilon > 0 : x < y + \varepsilon) \implies x \leq y.$$

Proof. The proof is by contraposition. Assume $x > y$, then we can take $\varepsilon = x - y > 0$. This implies $x = y + \varepsilon$ and so $x \geq y + \varepsilon$. $\square$

Chapter 6

Rings and fields

TODO: signature $(R, +, \cdot, 0, 1)$. TODO: addition, multiplication and scalar multiplication of functions: pointwise.
TODO: unital homomorphisms; unital subalgebra.

Lemma II.114. *Let R be a ring and $a \in R$.*

1. *If a has a left and a right inverse, they are equal. Thus a has an inverse.*
2. *The inverse of a is unique, if it exists.*

Proof. Let l be a left inverse of a and r a right inverse. Then

$$l = l(ar) = (la)r = r.$$

The unicity of the inverse is an easy consequence. □

Lemma II.115. *Let R be a ring and $a, b \in R$. Then a and b are invertible* if and only if *ab and ba are invertible.*

Proof. Assume a, b invertible. Then $b^{-1}a^{-1}$ is an inverse for ab and $a^{-1}b^{-1}$ is an inverse for ba. Assume both ab and ba have inverses. Then from

$$\begin{aligned} a[b(ab)^{-1}] &= \mathbf{1} \qquad & [(ba)^{-1}b]a &= \mathbf{1} \\ [(ab)^{-1}a]b &= \mathbf{1} \qquad & b[a(ba)^{-1}] &= \mathbf{1} \end{aligned}$$

we see that both a and b have left and right inverses. □

Proposition II.116. *Every proper ideal is contained in a maximal ideal.*

Lemma II.117. *Let R be a unital ring. If $a \in R$ is non-invertible, then the generated ideal (a) is not the whole ring.*

Proof. If $(a) = R$, then $1 \in (a)$, implying $ab = 1$ for some $b \in R$. A contradiction. □

Proposition II.118. *Let f be a ring homomorphism. If f is invertible as a function (i.e. bijective), its inverse f^{-1} is also a ring homomorphism.*

Proposition II.119. *Kernel of Ring Homomorphism is Ideal*

A $*$-rng is a structured set $(R,+,\cdot,*)$, where R is a rng and $*:R\to R$ is an involutive anti-automorphism. That is, $\forall x,y\in R$:

- $(xy)^* = y^*x^*$;
- $(x+y)^* = x^*+y^*$;
- $(x^*)^* = x$.

This is also known as an involutive rng or rng with involution.

Lemma II.120. *If R is a unital ring with involution, then $1^*=1$.*

Proof. From $1^*x = (x^*1)^* = (x^*)^* = x$, we see that 1^* is a multiplicative identity, which is unique. ☐

An element of a $*$-rng is self-adjoint if $x^*=x$.

6.1 Ideals

TODO $((X))$ is ideal generated by X.
ALSO $((x\otimes y - b(x,y) | x\in X, y\in Y))$

6.2 Types of elements

6.2.1 Zero divisors

Let R be a rng. An element $x\in R$ is called

- a left zero divisor if $\exists y\in R\setminus\{0\} : xy=0$;
- a right zero divisor if $\exists y\in R\setminus\{0\} : yx=0$;
- a zero divisor if it is either a left or a right zero divisor.

Lemma II.121. *Let R be a ring and $x\in R$.*

1. *If x is a left zero divisor, it has no left inverse.*
2. *If x is a right zero divisor, it has no right inverse.*
3. *If x is a zero divisor, it has no inverse.*

Proof. Assume x is a left zero divisor, with $y\neq 0$ such that $xy=0$. Assume, towards a contradiction, that z is a left inverse of x. Then

$$y = 1y = (zx)y = z(xy) = 0,$$

which is disallowed by definition.
The rest follows by duality. ☐

6.2.2 Nilpotents

Let R be a rng. An element $x \in R$ is called nilpotent if there exists $k \in \mathbb{N}$ such that $x^k = 0$. The least such k is called the index of the nilpotent.

6.3 Group rings

Let G be a finite group and R a r(i)ng. The group ring RG is the set of functions $(G \to R)$ with pointwise addition and the convolution product

$$(x \star y)(g) = \sum_h x(h)y(h^{-1}g) = \sum_{g=hk} x(h)y(k)$$

for all $x, y \in RG$ and $g \in G$.

The a group ring can be seen as a free module generated by G. (TODO: this as definition?)

6.4 Integral domains

6.4.1 Bézout domains

Theorem II.122 (Bézout-Bachet)**.** *TODO*

Also known as Bézout's identity.

6.5 Fields

6.5.1 Totally ordered fields

Let K be a set with binary operations $+, \cdot$ such that $(K, +, \cdot)$ is a field and a binary relation $\leq$ such that $(F, \leq)$ is a total order. Then the structured set $(K, +, \cdot, \leq)$ is a totally ordered field if $\forall a, x, y \in K$:

1. $x \leq y \implies x + a \leq y + a$;
2. $x \leq y \land a \geq 0 \implies ax \leq ay$;
3. $x \leq y \land a \leq 0 \implies ax \geq ay$.

Lemma II.123. *Let $(K, +, \cdot, \leq)$ be a totally ordered field and $a, b, c, d \in K$. Then*

1. $a \leq b \land c \leq d \implies (a + c) \leq (b + d)$;
2. $a \leq b \implies -b \leq -a$;
3. $a \geq 0 \iff -a \leq 0$;
4. $a \geq 0 \land b \geq 0 \implies ab \geq 0$;
5. $a \geq 0 \implies a^n \geq 0$ *for all* $n \in \mathbb{N}$;

6. *$a \leq 0 \implies (a^{2n} \geq 0 \wedge a^{2n+1} \leq 0)$ for all $n \in \mathbb{N}$;*

7. *$a > 0 \iff a^{-1} > 0$ and $a < 0 \iff a^{-1} < 0$;*

8. *if $b \geq a > 0$, then $b^{-1} \leq a^{-1}$;*

9. *$0 < 1$.*

Proof. (1) By applying point 1. of the definition twice, we get $(a+c) \leq (b+c) \leq (b+d)$.
(2) This is the result of multiplying by -1.
(3) Idem, using $-0 = 0$.
(4) Special case of point 2. of the definition.
(5) By induction on n and point 2. of the definition.
(6) By induction on n and point 3. of the definition.
(7) By multiplying $a \geq 0$ with $(a^{-1})^2$, which is positive, we get $a^{-1} \geq 0$. Also $a \neq 0 \iff a^{-1} \neq 0$.
(8) By point 7. and point 4. of the lemma $a^{-1}b^{-1}$ is positive, so multiplying $b \geq a$ by $a^{-1}b^{-1}$ yields $b^{-1} \leq a^{-1}$.
(9) Assume, towards a contradiction, that this is false, so $1 \leq 0$. Then 1 is negative and multiplying the inequality with 1 yields $1 \cdot 1 \geq 1 \cdot 0$, or $1 \geq 0$. Taking both inequalities gives $1 = 0$ by anti-symmetry, which is prohibited for fields. ☐

Let $F \subset K$ be totally ordered fields. Then we call F dense in K if

$$\forall a, b \in K : \exists x \in F : a < x < b.$$

TODO: topology definition?

6.5.2 Field extensions

6.5.2.1 Complex numbers

We write

- $\mathbb{C}^{\uparrow} := \{z \in \mathbb{C} \mid \mathfrak{Im}(z) \geq 0\}$ for the (open) upper half plane;
- $\mathbb{C}^{\downarrow} := \{z \in \mathbb{C} \mid \mathfrak{Im}(z) \leq 0\}$ for the (open) lower half plane;
- $\mathbb{C}^{\leftarrow} := \{z \in \mathbb{C} \mid \mathfrak{Re}(z) \leq 0\}$ for the (open) left half plane;
- $\mathbb{C}^{\rightarrow} := \{z \in \mathbb{C} \mid \mathfrak{Re}(z) \geq 0\}$ for the (open) right half plane.

6.6 Polynomials over rings and fields

`https://www.math.uci.edu/~ndonalds/math120b/2poly.pdf`
TODO notation $p[x]$.

Proposition II.124. *Let R be a ring. Then $R[x]$ is a PID* if and only if *R is a field.*

Corollary II.124.1. *Bézout's identity.*
...

Proposition II.125. *Polynomials over fields have unique prime decompositions.*

6.6.1 Rings and fields of functions

TODO notation p[X](z).

6.6.1.1 Algebroid functions

Chapter 7

Valuation theory

Absolute values on integral domains.

Part III

Order theory

Chapter 1

Ordered sets

1.1 Order relations

All types of orders are transitive.

- A preorder or quasiorder $\precsim$ is also reflexive.
- A total preorder is also connex.
- A partial order $\preceq$ is also reflexive and anti-symmetric.
- A strict partial order $\prec$ is also irreflexive (or, equivalently, asymmetric).
- A total order $\leq$ is also reflexive, anti-symmetric and connex. Connex means all elements are comparable.
- A strict total order $<$ is also trichotomous.

An ordered set is a pair $(P, \prec)$ such that P is a set and $\prec$ is an order on P. We call this ordered set

- a proset if $\prec$ is a preorder;
- a poset if $\prec$ is a partial order;

A total preorder is necessarily also reflexive by I.89.1.
A strict partial order is necessarily also anti-symmetric by I.87 and I.85.1 (which is why there is no separate notion of "strict preorder").
A strict total order is like a total order that is strict (i.e. irreflexive instead of reflexive). However by I.89.1, no irreflexive relation can be connex, so we relax the requirement to semi-connexity:

$$\begin{cases} \text{transitive} \\ \text{trichotomous} \end{cases} \iff \begin{cases} \text{transitive} \\ \text{irreflexive} \\ \textit{semi}\text{-connex} \end{cases}.$$

If we say $(P, \prec)$ is an ordered set without any other qualifiers, we only assume $\prec$ is transitive.

Example

- Every equivalence relation is a preorder. Conversely, symmetric preorders are equivalence relations.
- Let X be a set. Then $(\mathcal{P}(X), \subseteq)$ is a poset.
- Let A be a set and id_A the identity relation on A. Then (A, id_A) is a poset. Such posets are called discrete posets. Two elements $x, y \in A$ are comparable if and only if $x = y$.
- Let B be a set and $\bot \in B$. Then the order $\preceq$ defined by

  $$x \preceq y \quad \Leftrightarrow_{\mathrm{def}} \quad (x = \bot) \vee (x = y)$$

  is a partial order. Such posets are called flat posets.

Lemma III.1. *For any binary homogeneous relation R,*

1. *the reflexive transitive closure, $R^{+=}$, is a preorder;*
2. *the left residual, $R \backslash R = \overline{R^{\mathrm{T}}; \overline{R}}$ is a preorder.*

Lemma III.2. *Let $(P, \precsim)$ be a proset. Then $\precsim \cap \precsim^{\mathrm{T}}$ is an equivalence relation.*

Let $(P, \prec)$ be an ordered set. We can decompose $\prec = \prec_S \cup \prec_A$, as in I.88, into a symmetric and asymmetric part.

- By I.87 an order is strict (i.e. irreflexive) if and only if it is asymmetric: $\prec_S = E_P$ and $\prec_A = \prec$.
- For non-strict partial and total orders we have $\prec_S = \mathrm{id}_P$.

So we can turn a partial / total order into a strict partial / total order by removing id_P. Conversely, we can turn a strict partial / total order into a partial / total order by adding id_P.

Proposition III.3.

1. *Let $(P, \precsim)$ be a proset. The relation $\precsim \cap \overline{\precsim}^{\mathrm{T}}$ is a strict partial order.*
2. *Let $(P, \preceq)$ be a poset. The relation $\preceq \cap \overline{\preceq}^{\mathrm{T}} = \preceq \cap \overline{\mathrm{id}}_P$ is a strict partial order.*
3. *Let $(P, \prec)$ be a strict partial order. Then $\prec \cup \mathrm{id}_P$ is a partial order.*
4. *Let $(P, \leq)$ be a totally ordered set. The relation $\leq \cap \overline{\leq}^{\mathrm{T}} = \leq \cap \overline{\mathrm{id}}_P = \overline{\leq}^{\mathrm{T}}$ is a strict total order.*
5. *Let $(P, <)$ be a strict total order. Then $< \cup \mathrm{id}_P$ is a total order.*

Multiple different preorders are associated to the same strict partial orders. For partial and total orders the association is one-to-one.

1.2 Covering relations

Let $(P, \prec)$ be a an order relation and $x, y \in P$. We say y covers x if

- $x \prec y$;
- $x \neq y$
- $\nexists z \in P \setminus \{x, y\} : x \prec z \prec y)$.

We write $x \lessdot y$.

Lemma III.4. *Let* $(P, \prec)$ *be an ordered set. Let* $\prec_A$ *be the asymmetric part of* $\prec$. *Then* $\lessdot = \prec_A \cap \overline{\prec_A^2}$.

Lemma III.5. *Let* P *be a preordered set. Then* y *covers* x if and only if $x \neq y$ *and* $[x, y] = \{x, y\}$.

1.2.1 Hasse diagrams

A Hasse diagram is a graphical depiction of an order relation. Each element of the ordered set is a point and points are connected such that:

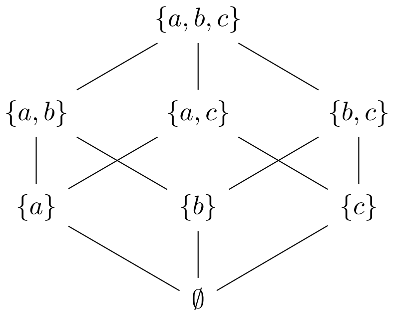
1. if $x \prec y$, then the point for x is drawn lower than the point for y;
2. two elements x, y are connected if y covers x or x covers y.

TODO is transitive (reflective) reduction! (+ link Galois)

Lemma III.6. *The Hasse diagrams of partial orders are acyclic due to anti-symmetry.*

Example

The power set $\mathcal{P}\{a, b, c\}$ can be ordered by the inclusion relation $\subseteq$. The following is a Hasse diagram for this ordered set:



1.3 The dual of an ordered set

For any ordered set $(P, \prec)$ the dual of P is the ordered set $(P^o, \prec^o)$, where $P^o := P$ and $\prec^o := \prec^{\mathrm{T}} = \succ$.

This means $\forall a, b \in P : a \prec b \iff b \prec^o a$.

Lemma III.7. *The dual of an ordered set of a particular type is an ordered set of the same type.*

Corollary III.7.1. *All statements about all ordered sets also hold for all duals of ordered sets.*

1.4 Functions on ordered sets

TODO define for all relations

Let $(P, \prec_P)$ and $(Q, \prec_Q)$ be ordered sets and $f : P \to Q$ a function.

- f is order-preserving, isotone, monotonically increasing or just increasing if f is relation-preserving;
- f is order-reversing, antitone, monotonically decreasing or just decreasing if $f : P \to Q^o$ is relation-preserving;
- f is monotone or monotonic if f is relation-preserving as either a function $P \to Q$ or $P \to Q^o$.

Similarly we say f is (reverse) order-reflecting or a (reverse) order embedding if f is relation-reflecting or a relation embedding as a function $P \to Q$ (resp. $P \to Q^o$).
The adjectives "strict" and "weak" can be added to any of these cases to identify whether the (weak, i.e. reflexive) order or the associated strict order is meant.

The concept of order isomorphism (or similarity) is similarly inherited. Let U, V be ordered sets. We write $U =_o V$ if U and V are order isomorphic and $U \neq_o V$ if not.

Lemma III.8. *Let P, Q be posets and $f : P \to Q$. The following are sufficient conditions for f to be injective:*

1. *f is order-reflecting;*
2. *f is strictly order-preserving and P is totally ordered.*

Proof. (1) If $f(x) = f(y)$, then $x \preceq y$ and $y \preceq x$, so $x = y$ by anti-symmetry.
(2) Let f be the strict order-preserving function and $x, y \in P$. Assume $f(x) = f(y)$. Assume, towards a contradiction, that $x \neq y$, then either (by totality) $x < y$ and $f(x) < f(y)$ or $y < x$ and $f(y) < f(x)$. Both cases contradict $f(x) = f(y)$. ☐

Lemma III.9. *Let $f : P \to Q$ be a function between posets. Then*

1. *If f preserves strict order, then it is order-preserving.*
2. *If f is order-reflecting, then it reflects strict order.*

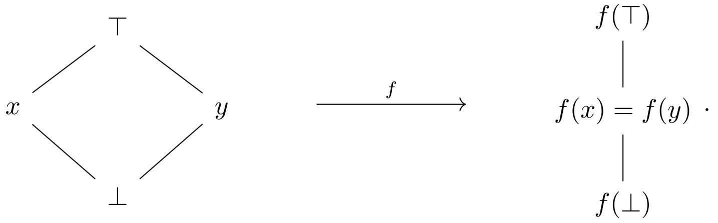
If f is injective, the opposite implications also hold.

Corollary III.9.1. *Let $f : P \to Q$ be a function between posets.*

1. *If f is an order embedding, then it is a strict order embedding.*
2. *If P is totally ordered, the converse implication holds as well.*

Proof. Use III.8. ☐

The converse does not hold in general if P is a poset because a strict order embedding between posets is not necessarily injective. A counterexample is



Lemma III.10. *Let $f : P \to Q$ be a function between totally ordered sets. Then*

1. *If f preserves strict order, then it is order-preserving.*
2. *If f is order-reflecting, then it reflects strict order.*

If f is injective, the opposite implications also hold.

Proof. (1) Assume f order-reflecting. Assume $x \leq y$. Either $f(x) \leq f(y)$ or $f(y) \leq f(x)$. If $f(y) \leq f(x)$, then

$$y \leq x \implies x = y \implies f(x) = f(y) \implies f(x) \leq f(y).$$

So in both cases $f(x) \leq f(y)$.
(2) Assume f preserves strict order. Assume $f(x) < f(y)$. Either $x < y, y < x$ or $x = y$. If either $y < x$ or $x = y$ hold, preservation of strict order yields a contradiction. So $x < y$.
(1') Assume f order-preserving and injective. Assume $f(x) \leq f(y)$. Either $x \leq y$ or $y \leq x$. If $y \leq x$, then

$$f(y) \leq f(x) \implies f(x) = f(y) \implies x = y \implies x \leq y.$$

So in both cases $x \leq y$.
(2') Assume f is injective reflects strict order. Assume $x < y$. Either $f(x) < f(y), f(y) < f(x)$ or $f(x) = f(y)$. If $f(y) < f(x)$, preservation of strict order yields a contradiction. If $f(x) = f(y)$, injectivity yields a contradiction. So $f(x) < f(y)$. ☐

Proposition III.11. *Let (P, R) and $(Q, \prec)$ be relational structures and $f : P \to Q$ a function. Assume $\prec$ is a preorder. Then the following are equivalent:*

1. *f is relation-preserving;*
2. *$R; f; \prec \subseteq f; \prec$;*
3. *the inverse image of every principal down-set is closed under R^{T};*
4. *$\prec; f^{\mathrm{T}}; R \subseteq \prec; f^{\mathrm{T}}$*
5. *the inverse image of every principal up-set is closed under R.*

Proof. ($1 \Rightarrow 2$) By II.39 we have $R; f \subseteq f; \prec$. Then $R; f; \prec \subseteq f; \prec^2 \subseteq f; \prec$ by transitivity.
($2 \Rightarrow 1$) By reflexivity we have $\mathrm{id}_Q \subseteq \prec$ and so $R; f \subseteq R; f; \prec \subseteq f; \prec$. Thus f is relation-preserving by II.39.
(3) Is simply a translation of $\forall x : R; f; Sx \subseteq f; Sx$.
(4) Equivalent to (2) by taking transposed and using that f is $\prec$-preserving iff f is $\prec^{\mathrm{T}}$-preserving.
(3) Is simply a translation of $\forall x : xS; f^{\mathrm{T}}; R \subseteq xS; f^{\mathrm{T}}; f^{\mathrm{T}}$. ☐

Proposition III.12. *Let* $(P, \leq_P)$ *and* $(Q, \leq_Q)$ *be posets and* $f : P \to Q$ *a function. Then* f *is residuated* if and only if *the inverse image of every principal down-set is a principal down set.*

Clearly we can replace "down-set" with "up-set" by considering the dual order.

Proof. This follows directly from II.51 and II.53. ☐

A function $f : P \to P$ on an ordered set $(P, \prec)$ into itself is

- expansive if $\forall x \in P : x \prec f(x)$;
- contractive if $\forall x \in P : f(x) \prec x$.

Lemma III.13. *Let* $f : P \to P$ *be a function on an ordered set* $(P, \prec)$*. Consider* $(P \to P)$ *to be ordered by pointwise order. Then the following are equivalent:*

1. f *is expansive;*
2. $\mathrm{id}_P \prec f$*;*
3. $\mathrm{graph}(f) \subseteq \mathrm{graph}(\prec)$.

The following are also equivalent:

1. f *is contractive;*
2. $f \prec \mathrm{id}_P$*;*
3. $\mathrm{graph}(f) \subseteq \mathrm{graph}(\prec^{\mathrm{T}})$.

1.4.1 Closure

1.4.1.1 Moore closure

Let A be a set.

- A (Moore) closure is a monad on the poset $(\mathcal{P}(A), \subseteq)$.
- A (Moore) dual closure is a comonad on the poset $(\mathcal{P}(A), \subseteq)$.

Example

Let (A, R) be a relational structure. Then

$$\mathrm{Cl}_R : \mathcal{P}(A) \to \mathcal{P}(A) : X \mapsto (X^R)^{R^{\mathrm{T}}}$$

is a closure due to II.61.

Proposition III.14. *Let* A *be a set and* $f : \mathcal{P}(A) \to \mathcal{P}(A)$ *a function. Then* f *is a closure* if and only if f *is*

- *extensive:* $\forall X \subseteq A : X \subseteq f(X)$*;*
- *isotone:* $\forall X, Y \subseteq A :$ *if* $X \subseteq Y$*, then* $f(X) \subseteq f(Y)$*;*

- *idempotent:* $\forall X \subseteq A : f(f(X)) = f(X)$.

And f *is a dual closure* if and only if f *is*

- *contractive:* $\forall X \subseteq A : X \supseteq f(X)$*;*
- *isotone:* $\forall X, Y \subseteq A :$ *if* $X \subseteq Y$*, then* $f(X) \subseteq f(Y)$*;*
- *idempotent:* $\forall X \subseteq A : f(f(X)) = f(X)$.

The contents of this proposition is often given as the definition of a (Moore) closure.

Proof. Sufficiency $\Rightarrow$ is clear. In particular extensivity/contractivity follows from II.52. Upper adjoint can be taken to be $\mathrm{id}_{\mathrm{im}(f)}$. □

The family of saturated sets is a Moore family. The For any $x \in P$, the downset $\downarrow x$ is saturated.

Let $(P, \prec)$ be an ordered set and let $f : P \to P$ be a function. We say f is a (Moore) closure on P if it is

- extensive: $x \prec f(x)$;
- monotone: if $A \prec B$, then $\mathrm{Cl}(A) \prec \mathrm{Cl}(B)$;
- idempotent: $\mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A)$.

We say f is a dual closure if it is

- contractive: $f \leq \mathrm{id}$
- monotone
- idempotent

Let A be a subset of an ordered set P. If $A = (A^u)^l$, the A is called saturated.

Let (A, R) be a relational structure. We say a set $X \subseteq A$ is R-closed if $X_R \subseteq X$.

Lemma III.15. *Let* (A, R) *be a relational structure. A set* $X \subseteq A$ *is* R*-closed* if and only if $X^c \subseteq X^{\overline{R}}$.

Proof. We have $X_R \subseteq X \iff X^c \subseteq (X_R)^c = X^{\overline{R}}$. □

Proposition III.16. *Let* (A, R) *be a relational structure.*

1. *For all* $X \subseteq A$*, the* R*-closure* $\mathrm{Cl}_R(X)$ *is* R*-closed.*
2. *The closure function* Cl_R *is identical to the function*

$$\mathcal{P}(A) \to \mathcal{P}(A) : X \mapsto \bigcap \{Y \mid X \subseteq Y \subseteq A \text{ and } Y \text{ is } R\text{-closed}\}.$$

Proof. (1) We have $\mathrm{Cl}_R(X)_R = X_R$ due to the generalised inverses of II.62.
(2) □

Lemma III.17. *Every Moore closure is an R-closure for some homogeneous relation R.*

Proposition III.18. *Let L be a lattice and f a closure. Then $f[L]$ is a lattice with lattice operations given by*

$$a \wedge b = a \wedge_L b, \qquad a \vee b = f(a \vee_L b).$$

Proof. Take $a, b \in f[L]$. Because f is extensive, we have $a \wedge_L b \leq f(a \wedge_L b)$ and because f is monotone, we have $f(a \wedge_L b) \leq f(a) \wedge_L f(b)$ by III.54, so

$$f(a) \wedge_L f(b) = a \wedge_L b \leq f(a \wedge_L b) \leq f(a) \wedge_L f(b),$$

meaning that $a \wedge_L b \in f[L]$ and thus $f[L]$ is a $\wedge$-subsemilattice.
From $a \vee_L b \leq f(a \vee_L b)$, it is clear that $f(a \vee_L b)$ is an upper bound of $\{a, b\}$. Let c be any other upper bound in $f[L]$. Clearly $a \vee_L b \leq c$, so $f(a \vee_L b) \leq f(c) = c$, meaning $f(a \vee_L b)$ is the least upper bound in $f[L]$. ☐

Proposition III.19. *Let L be a complete lattice and f a closure. Then $f[L]$ is a complete lattice with lattice operations given by*

$$\bigwedge A = \bigwedge\nolimits_L A, \qquad \bigvee A = f(\bigvee\nolimits_L A).$$

Proof. TODO ☐

Theorem III.20 (Dedekind-MacNeille)**.** *Every ordered set E can be embedded in a Dedekind complete lattice L such that meets and joins that exist in E are preserved in L.*

Proof. We use the closure $f : \mathcal{P}(E) \to \mathcal{P}(E) : A \mapsto A^{ul}$. ☐

1.4.1.2 Closure under a relation

Proposition III.21. *Let $(P, \leq)$ be a partially ordered set and $(Q, \leq)$ a complete meet semi-sublattice. Then $\uparrow : P \to \mathcal{P}(Q)$ and $\bigwedge : \mathcal{P}(Q) \to P$ are antitone generalised inverses.*

Let R be a homogeneous binary relation on a set X. Let $A \subseteq X$ be a subset.

- We call A R-closed if $AR \subseteq A$.
- We define the R-closure of A in X as

$$\mathrm{Cl}_R(A) \coloneqq \bigcap \{B \mid A \subseteq B \subseteq X \wedge\ B \text{ is } R\text{-closed}\}.$$

Proposition III.22. *Let R be a homogeneous binary relation on a set X and $A \subseteq X$ a subset. Then Cl_R is a proper closure operator:*

1. $A \subseteq \mathrm{Cl}_R(A)$;
2. *if* $A \subseteq B$, *then* $\mathrm{Cl}_R(A) \subseteq \mathrm{Cl}_R(B)$;
3. $\mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A)$;

and

4. $\mathrm{Cl}_R(A)$ *is* R*-closed;*

5. $\mathrm{Cl}_R(A)$ *is the smallest* R*-closed superset of* A *in the poset* $(\mathcal{P}(X), \subseteq)$*;*

6. A *is* R*-closed* if and only if $A = \mathrm{Cl}_R(A)$.

Proof. (1) This is clear.
(2) This follows because $\{C \mid A \subseteq C \subseteq X \wedge\ C \text{ is } R\text{-closed}\} \supseteq \{C \mid B \subseteq C \subseteq X \wedge\ C \text{ is } R\text{-closed}\}$.
(3) This follows because $\mathrm{Cl}_R(A) \in \{C \mid \mathrm{Cl}_R(A) \subseteq C \subseteq X \wedge\ C \text{ is } R\text{-closed}\}$.
(4) We calculate

$$\begin{aligned}\mathrm{Cl}_R(A)R &= \Big(\bigcap\{B \mid A \subseteq B \subseteq X \wedge\ B \text{ is } R\text{-closed}\}\Big)\, R \\ &\subseteq \bigcap\{BR \mid A \subseteq B \subseteq X \wedge\ B \text{ is } R\text{-closed}\} \\ &\subseteq \bigcap\{B \mid A \subseteq B \subseteq X \wedge\ B \text{ is } R\text{-closed}\} = \mathrm{Cl}_R(A).\end{aligned}$$

(5) Intersection is infimum in $(\mathcal{P}(X), \subseteq)$. (TODO terminology higher??)
(6) The direction $\Leftarrow$ is clear because $\mathrm{Cl}_R(A)$ is R-closed. The converse follows from (5). ☐

Lemma III.23. *Let* R *be a homogeneous binary relation on a set* X *and* $A \subseteq X$ *a subset. Then*

1. $\mathrm{Cl}_R(AR) \subseteq \mathrm{Cl}_R(A)$*;*

2. $\mathrm{Cl}_R(A) = A \cup \mathrm{Cl}_R(AR)$*;*

3. $\mathrm{Cl}_R(AR) = \mathrm{Cl}_R(A)R$.

Proof. (1) We calculate $AR \subset \mathrm{Cl}_R(A)R \subseteq \mathrm{Cl}_R(A)$, using I.39.1 and the fact that $\mathrm{Cl}_R(A)$ is R-closed.
(2) The inclusion $\mathrm{Cl}_R(A) \supseteq A \cup \mathrm{Cl}_R(AR)$ is given by III.22 and point (1).
For the converse it is enough to see that $A \cup \mathrm{Cl}_R(AR)$ is R-closed:

$$\big(A \cup \mathrm{Cl}_R(AR)\big)R = AR \cup \mathrm{Cl}_R(AR)R \subseteq \mathrm{Cl}_R(AR) \subseteq A \cup \mathrm{Cl}_R(AR),$$

where we have used that $\mathrm{Cl}_R(AR)$ is R-closed.
(3) First we calculate

$$\mathrm{Cl}_R(A)R = \big(A \cup \mathrm{Cl}_R(AR)\big)R = AR \cup \mathrm{Cl}_R(AR)R \subseteq \mathrm{Cl}_R(AR)R \subseteq \mathrm{Cl}_R(AR)$$

where we have used point (2) and the fact that $\mathrm{Cl}_R(AR)$ is closed.
For the converse it is enough to prove that $AR \subseteq \mathrm{Cl}_R(A)R$ and $\mathrm{Cl}_R(A)R$ is R-closed. The first follows from I.39.1 as does the second, with

$$\mathrm{Cl}_R(A)R \subseteq \mathrm{Cl}_R(A) \implies \big(\mathrm{Cl}_R(A)R\big)R \subseteq \mathrm{Cl}_R(A)R.$$

☐

Lemma III.24. *Let* R *be a homogeneous binary relation on a set* X*. Let* im_R *denote the function* $\mathcal{P}(X) \to \mathcal{P}(X) : A \mapsto AR$*. Then for all* $A \subseteq X$*:*

$$\mathrm{Cl}_R(A) = \bigcup \mathrm{Cl}_{\mathrm{im}_R}(\mathcal{P}(A)).$$

IS THIS TRUE?

1.4.1.3 Closure under a function

Lemma III.25. $\mathrm{Cl}_f(A) = \{f(a) \mid a \in A\}$.
$\mathrm{Cl}_g(A) = \{g(a, b) \mid a, b \in A\}$.

TODO im and Cl.

1.5 Subsets of ordered sets

1.5.1 Up and down sets

Lemma III.26. *Let $(P, \prec)$ be an ordered set. Then the closure function $\mathrm{Cl}_\prec$ equals the image function $\mathrm{im}_{\prec^=}$ of the reflexive closure $\prec^=$.*

Let $(P, \prec)$ be an ordered set and $Q \subseteq P$. We denote

- the upward closure of Q as $\uparrow Q \coloneqq \mathrm{Cl}_\prec(Q) = Q_\prec \cup Q$;
- the downward closure of Q as $\downarrow Q \coloneqq \mathrm{Cl}_{\prec^\mathrm{T}}(Q) = Q_{\prec^\mathrm{T}} \cup Q$.

We call Q

- upwards closed or an up set if $\uparrow Q = Q$;
- downwards closed or a down set if $\downarrow Q = Q$.

For $x \in P$,

- $\uparrow x \coloneqq \uparrow\{x\}$ is the principal up set generated by x;
- $\downarrow x \coloneqq \downarrow\{x\}$ is the principal down set generated by x.

Lemma III.27. *Let $(P, \prec)$ be an ordered set and $Q \subseteq P$. Then*

1. *Q is upwards closed* if and only if *$Q_\prec \subseteq Q$;*
2. *Q is downwards closed* if and only if *$Q_\succ \subseteq Q$;*

also

3. *if $\prec$ is a preorder, then $Q_\prec = \uparrow Q$ and $Q_\succ \subseteq \downarrow Q$.*

Proof. This follows directly from III.26. ☐

1.5.2 Upper and lower bounds

Lemma III.28. *Let $(P, \prec)$ be an ordered set and S a non-empty subset of P, then*

1. *$\uparrow S^u \subseteq S^u$ and $(\downarrow S)^u \subseteq S^u$;*
2. *$\uparrow S^u \subseteq S^u$ and $(\uparrow S)^l \subseteq S^l$.*

In particular S^u is an up set and S^l a down set.

Corollary III.28.1. *Let $(P, \prec)$ be a preordered set, $x \in P$ and S a non-empty subset of P, then*

1. *$x \in \max(\downarrow x)$;*
2. *$x \in \min(\uparrow x)$;*
3. *if $\max(S) \neq \emptyset$, then $\downarrow S = \downarrow \max(S)$;*
4. *if $\min(S) \neq \emptyset$, then $\uparrow S = \uparrow \min(S)$.*

1.5.3 Chains

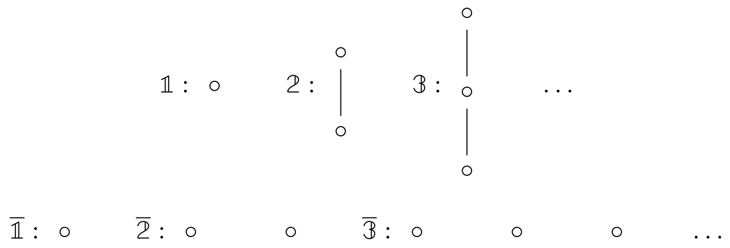
Let $(P, \prec)$ be an ordered set. A chain in P is a linearly ordered subset S of P, i.e.

$$\forall x, y \in S : x \prec y \vee y \prec x.$$

An antichain is a subset A such that no two elements of A are comparable.

Example

For every $n \in \mathbb{N}$ there exists a chain $\mathbb{n}$ of n elements and an antichain $\overline{\mathbb{n}}$ of n elements:



Lemma III.29. *Let P be an ordered set. If C is a non-empty, finite chain in P, then*

1. *$\sup(C) \supseteq \max(C) \neq \emptyset$;*
2. *$\inf(C) \supseteq \min(C) \neq \emptyset$.*

Proof. The $\supseteq$-relation is due to I.46.
The proof that $\max(C)$ is non-empty is by induction on the cardinality of C. For the base case, assume C has one element x. Then $x \leq x$, so $x \in \max(C)$.
Now assume all chains with one fewer element than C have a maximal element. Pick some $x_0 \in C$, then $y = \max(C \setminus \{x_0\})$ exists. We can compare x_0 and y because C is a chain. If $x_0 \leq y$, then $\max(C) = y$. If $y \leq x_0$, then $\max(C) = x_0$. This exhausts the possibilities. ☐

An ordered set P

- is chain-complete or inductive if every chain in P has a least upper bound;
- satisfies the ascending chain condition (ACC) if each chain in P that has a least element is finite;
- satisfies the descending chain condition (DCC) if each chain in P that has a greatest element is finite.

Lemma III.30. *Every inductive poset has a least element.*

Proof. The empty set $\emptyset$ is a chain in any poset P. Then $\sup(\emptyset) = \min(\emptyset^u) = \min(P)$. ☐

Lemma III.31. *Let P be an ordered set.*

1. *If P satisfies to ascending chain condition, then for each non-empty chain C in P we have $\max(C) \neq \emptyset$;*
2. *If P satisfies to descending chain condition, then for each non-empty chain C in P we have $\min(C) \neq \emptyset$,*

Proof. Assume P satisfies the ascending chain condition and take a non-empty chain $C \subset P$. Take some $x_0 \in C$ and define $C' = \{c \in C \mid x_0 \prec c\}$. Clearly x_0 is a leat element of C', so C' is finite by ACC and $\max(C) = \max(C') \neq \emptyset$ by III.29. ☐

Corollary III.31.1. *Let P be an ordered set. If P satisfies the ascending chain condition and has a least element, then P is inductive.*

Proof. We have $\sup(C) \supseteq \max(C)$ by I.46. ☐

Example

The set $\mathbb{N}$ is not inductive, because $\mathbb{N}$ itself does not have a least upper bound. It therefore does also not satisfy the ascending chain condition.
It does satisfy the descending chain condition.

Proposition III.32. *Let A, B be sets and P an ordered set. Then the following posets are inductive:*

1. *$\mathcal{P}(A)$, ordered by inclusion;*
2. *$(A \not\rightarrow B)$, ordered by inclusion;*
3. *the set of chains in P, ordered by inclusion.*

Proof. In all cases the least upper bound of a chain S is given by $\bigcup S$. ☐

1.5.4 Intervals

Let $(P, \prec)$ be an ordered set and $m, n \in P$. We define

- the closed interval $[m, n] \coloneqq \{k \in P \mid m \prec k \wedge k \prec n\}$;
- the open interval $]m, n[\coloneqq [m, n] \setminus \{m, n\}$;
- the half-open intervals

$$\begin{aligned} [m, n[&\coloneqq [m, n] \setminus \{n\}; \\ ]m, n] &\coloneqq [m, n] \setminus \{m\}. \end{aligned}$$

Lemma III.33. *Let $(P, \prec)$ be an ordered set and $m, n \in P$.*

1. *If $m \not\prec n$, then $[m, n] = \emptyset$.*
2. *If $\prec$ is a preorder, then*

(a) $m \not\prec n$ *if and only if* $[m, n] = \emptyset$*;*

(b) either $[m, n] = \emptyset$ *or* $\{m, n\} \subseteq [m, n]$.

Proof. (1) We prove by contraposition. Assume $k \in [m, n]$, then $m \prec k$ and $k \prec n$, so $m \prec n$ by transitivity.
(2) (a) The direction $\Rightarrow$ is given by III.33. Now assume $m \prec n$. Also $m \prec m$ and $n \prec n$ by reflexivity. So $m, n \in [m, n]$ by definition. This also gives point (b). ☐

1.6 Completeness

An ordered set P is

- order complete (or simply complete) if each subset has a supremum and an infimum;
- order σ-complete if each countable subset has a supremum and an infimum;
- finitely order complete if each finite subset has a supremum and an infimum;
- Dedekind complete if
  - each non-empty subset that is bounded above has a supremum; and
  - each non-empty subset that is bounded below has an infimum;
- Dedekind σ-complete if
  - each non-empty countable subset that is bounded above has a supremum; and
  - each non-empty countable subset that is bounded below has an infimum.
- finitely Dedekind complete if
  - each non-empty finite subset that is bounded above has a supremum; and
  - each non-empty finite subset that is bounded below has an infimum.

Some authors use "order completeness" to mean what we have called Dedekind completeness.

Example

The closed unit disk in $\mathbb{R}^2$ with coordinatewise ordering is Dedekind complete, but not order complete and is not a lattice.

Lemma III.34. *Let P be an ordered set that contains a least and a greatest element. Then P is order complete* if and only if *P is Dedekind complete.*

Proof. Every set is bounded (by the least and greatest elements). ☐

Proposition III.35. *Let $(P, \precsim)$ be an ordered set. Then the following are equivalent:*

1. *P is order complete;*
2. *each subset of P has a supremum;*
3. *each subset of P has an infimum;*

4. *each non-empty subset of P has a supremum and P has a least element;*
5. *each non-empty subset of P has an infimum and P has a greatest element.*

The following are also equivalent:

1. *P is Dedekind complete*
2. *each non-empty set that is bounded above has a supremum;*
3. *each non-empty set that is bounded below has an infimum.*

Proof. We show that (2) and (3) are equivalent. The other assertions then follow simply. Suppose each subset has a supremum and let S be a subset. Then in particular S^l has a supremum. This supremum of S^l is an infimum of S:

$$\sup(S^l) = (S^l)^u \cap ((S^l)^u)^l = (S^l)^u \cap S^l = \inf(S).$$

The converse is similar.
For equivalence with (3) and (4), we just need to remark that $\sup(\emptyset) = \min(P)$ and $\inf(\emptyset) = \max(P)$.
For Dedekind completeness the argument is similar, we just need to remark that if S is non-empty and bounded below, then S^l is non-empty and bounded above. ☐

Lemma III.36. *The ordered natural numbers $(\mathbb{N}, \leq)$ are complete.*

Proof. Any set of natural numbers has a least element by I.176. Any non-empty set S of natural numbers that is bounded above, we can take $\max(S) = \min(\mathbb{N} \setminus S) - 1$. ☐

1.6.1 Directed sets

An ordered set $(D, \prec)$ is called

- (upward) directed if every finite subset has an upper bound;
- downward directed if every finite subset has a lower bound.

If $(D, \prec)$ is a directed set, we call the relation $\prec$ a direction.
Let $(P, \prec)$ be an ordered set. The set of directed subsets of P is denoted $\mathcal{D}(P)$.

TODO: use $\text{Cl}_\vee$ and $\text{Cl}_\wedge$.
TODO: important note: $\text{Cl}_\vee$ only implies closure under finite $\vee$

Example

- Any non-empty chain is directed.
- An antichain is directed iff it is a singleton.

TODO: move:

Proposition III.37. *Let $\{(D_i, \prec_i)\}_{i\in I}$ be a family of directed sets. Then $D = \prod_{i\in I} D_i$ is a directed set with direction defined by*

$$(a_i)_{i\in I} \prec (b_i)_{i\in I} \iff \forall i \in I : a_i \prec_i b_i.$$

The directed set $(D, \prec)$ is called a product direction.

Proposition III.38. *Let $(D, \prec)$ a directed set. Every maximal element in D is a greatest element.*

Proof. Let $a \in D$ be a maximal element. Take an arbitrary $x \in D$. By directedness, $\{a, x\}$ has an upper bound, say c. Because a is maximal, $a \leq c$ implies $c \leq a$. So $x \leq c \leq a$, which means a is a greatest element. ☐

1.7 Ordered sets of subsets

1.7.1 Refinement

Let $(P, \prec)$ be an ordered set and $A, B \subseteq P$. We say A refines B (or A is a refinement of B) if $A \subseteq \uparrow B$. We write $A \preceq B$.
If A refines B and B refines A, we write $A \approx B$ and say A and B are equally fine.

Alternatively, we say A refines B iff

$$\forall a \in A : \exists b \in B : \ b \prec a.$$

Lemma III.39. *Let $(P, \prec)$ be an ordered set and $A, B, C \subseteq P$. Then*

1. *if A refines B and B refines C, then A refines C;*
2. *if $\prec$ is a preorder, then A refines A.*

Proof. (1) Assume A refines B and B refines C. Then $A \subseteq \uparrow B$ and $B \subseteq \uparrow C$, so $\uparrow B \subseteq \uparrow\uparrow C$ and thus $A \subseteq \uparrow\uparrow C \subseteq \uparrow C$.
(2) from ref TODO. ☐

Corollary III.39.1. *Let $(P, \prec)$ be a preordered set.*

1. *The relation $\preceq$ is a preorder on $\mathcal{P}(P)$.*
2. *The relation $\approx$ is an equivalence relation.*

The equivalence relation is the equivalence from III.2.

Lemma III.40. *Let $(P, \prec)$ be an ordered set and $A \subseteq P$. Then*

1. *A is a refinement of $\uparrow A$;*
2. *if $\prec$ is a preorder, then $A \approx \uparrow A$;*
3. *$A \approx B$ implies $\uparrow A = \uparrow B$;*
4. *if $\prec$ is a preorder, then $A \approx B$* if and only if *$\uparrow A = \uparrow B$.*

Proof. (1) is equivalent to $\uparrow A \subseteq \uparrow A$.
(2) We need to show that $\uparrow A$ is refines A, i.e. $A \subseteq \uparrow\uparrow A$. This follows from ref TODO.
(3) Assume $A \approx B$. Then $A \subseteq \uparrow B$ and $B \subseteq \uparrow A$. This implies $\uparrow A \subseteq \uparrow\uparrow B \subseteq \uparrow B$ and $\uparrow B \subseteq \uparrow\uparrow A \subseteq \uparrow A$.
(4) Assume $\uparrow A = \uparrow B$. Then A refines $\uparrow B$ and B is refines $\uparrow A$. The result then follows using (2). ☐

1.7.2 The ordered set of downsets

Let P be an ordered set. The set of down sets in P is denoted $\mathcal{O}(P)$.

Lemma III.41. *Let P be an ordered set. If P is a discrete poset, then $\mathcal{O}(P) = \mathcal{P}(P)$.*

Proposition III.42. *Let P be a preordered ordered set. Then $x \mapsto \downarrow x$ is an order embedding between P and the principal down sets in P.*
If P is a partial order, then the map is bijective and thus an order isomorphism.

Proof. The first part is just **??**.
For the second part, $x \mapsto \downarrow x$ is surjective by definition. For injectivity: the inverse map is given by $Q \mapsto \max(Q)$. This is a function by I.44 and an inverse by III.28.1. □

Lemma III.43. *Let P_1, P_2 be prosets and $\mathbb{1}$ a singleton. Then*

1. $\mathcal{O}(P)^o \cong \mathcal{O}(P^o)$;
2. $\mathcal{O}(P \oplus \mathbb{1}) \cong \mathcal{O}(P) \oplus \mathbb{1}$;
3. $\mathcal{O}(\mathbb{1} \oplus \mathbb{1}) \cong \mathbb{1} \oplus \mathcal{O}(P)$;
4. $\mathcal{O}(P_1 \sqcup P_2) \cong \mathcal{O}(P_1) \times \mathcal{O}(P_2)$.

1.8 Join- and meet-density

Let P be a poset and let $Q \subset P$ be a subset. Then Q is called join-dense in P if for every $x \in P$, there exists a subset $S \subset Q$ such that $x = \bigvee_P S$.
The dual of join-dense is meet-dense.

1.9 Atoms

1.9.1 Atomic elements

Let $(P, \leq)$ be a poset.

- An element $a \in P$ is called an atom if a is minimal in $P \setminus \min(P)$.
- An element $a \in P$ is called a coatom if a is maximal in $P \setminus \max(P)$.

We denote the set of atoms of P as $\mathcal{A}(P)$. The set of coatoms is denoted $\mathcal{CA}(P)$.

If P has a least element $\bot$, then a is an atom if and only if it covers $\bot$.

1.9.2 Atomic posets

Let $(P, \leq)$ be a poset. Then

- P is atomic if $\forall x \in L \setminus \min(L) : \exists a \in \mathcal{A}(L) : a \leq x$;
- P is coatomic if $\forall x \in L \setminus \min(L) : \exists a \in \mathcal{CA}(L) : a \geq x$;

also

- P is strongly atomic if $\forall x, y \in L$ the poset $[x, y]$ is atomic;
- P is strongly coatomic if $\forall x, y \in L$ the poset $[x, y]$ is coatomic;

and

- P is atomistic if $\mathcal{A}(L)$ is join-dense in L;
- P is coatomistic if $\mathcal{CA}(L)$ is meet-dense in L.

`https://www.emis.de/journals/PM/51f4/pm51f409.pdf`

1.10 Combining ordered sets

Let P, Q be ordered sets. Then

- the disjoint union $P \sqcup Q$ is ordered by

$$x \precsim y \iff \begin{cases} x, y \in P \text{ and } x \precsim y \\ x, y \in Q \text{ and } x \precsim y \end{cases}$$

- the linear sum $P \oplus Q$ is the disjoint union ordered by

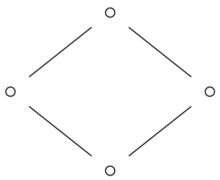
$$x \precsim y \iff \begin{cases} x, y \in P \text{ and } x \precsim y \\ x, y \in Q \text{ and } x \precsim y \\ x \in P \text{ and } y \in Q \end{cases}$$

Example

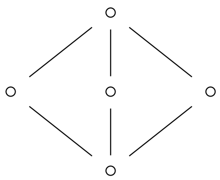
We define $\mathbf{M}_n \coloneqq \mathbb{1} \oplus \overline{\mathbb{n}} \oplus \mathbb{1}$:

$\mathbf{M}_1$:

$\mathbf{M}_2$:



$\mathbf{M}_3$:



. . .

Lemma III.44. *For ordered sets taking the disjoint union or linear sum is associative.*

Lemma III.45. *Let $m, n \in \mathbb{N}$. If $p = m + n$, then $\mathbb{m} \oplus \mathbb{n} = \mathbb{p}$.*

Let P, Q be ordered sets. Then

- the standard order on $P \times Q$ is defined by

$$(x_1, x_2) \precsim (y_1, y_2) \iff x_1 \precsim y_1 \text{ and } x_2 \precsim y_2$$

- the lexicographic order on $P \times Q$ is defined by

$$(x_1, x_2) \precsim (y_1, y_2) \iff (x_1 \precsim y_1) \text{ or } (x_1 = y_1 \text{ and } x_2 \precsim y_2).$$

Chapter 2

Lattices

2.1 Semilattice

A semilattice is an algebraic structure $(S, \star)$ where $\star$ is a binary operation on the set S satisfying

Associativity $x \star (y \star z) = (x \star y) \star z$;

Commutativity $x \star y = y \star x$;

Idempotency $x \star x = x$.

We call a semilattice bounded if it contains an identity e for $\star$. We write $(S, \star, e)$.

In other words a semilattice is a commutative idempotent semigroup or commutative band.

Lemma III.46. *Let $(P, \leq)$ be a poset and let $\{x, y, z\} \subseteq P$ be such that each subset has a supremum. Then*

$$\sup\{\sup\{x, y\}, z\} = \sup\{x, y, z\} = \sup\{x, \sup\{y, z\}\}$$

and, dually,

$$\inf\{\inf\{x, y\}, z\} = \inf\{x, y, z\} = \inf\{x, \inf\{y, z\}\}.$$

Corollary III.46.1. *Let $(P, \leq)$ be a poset.*

1. *If $\sup\{x, y\}$ exists for all $x, y \in P$, then $(P, \sup)$ is a semilattice.*
2. *If $\inf\{x, y\}$ exists for all $x, y \in P$, then $(P, \inf)$ is a semilattice.*

Proposition III.47. *Let $(S, \star)$ be a semilattice. Define the relations $\leq_1, \leq_2$ on S by*

$$\forall x, y \in S : \qquad x \leq_1 y \iff x = x \star y \iff y \leq_2 x.$$

Then

1. *$(S, \leq_1)$ is a poset such that $\forall x, y \in S : \ x \star y = \inf_{\leq_1}\{x, y\}$;*
2. *$(S, \leq_2)$ is a poset such that $\forall x, y \in S : \ x \star y = \sup_{\leq_2}\{x, y\}$;*

3. *$(S, \leq_1)$ and $(S, \leq_2)$ are dual order structures, i.e.* $\leq_1 = \leq_2^{\mathrm{T}}$.

Proof. (1) First we prove $(S, \leq_1)$ is a poset:

- Reflexivity follows from idempotency.
- Antisymmetry follows from commutativity:

$$x \leq_1 y, y \leq_1 x \quad \Longrightarrow \quad x = x \star y = y \star x = y.$$

- For transitivity: assume $x \leq_1 y$ and $y \leq_1 z$. This implies $x = x \star y$ and $y = y \star z$, and so $x = x \star (y \star z) = (x \star y) \star z = x \star z$. This implies $x \leq_1 z$.

Next we prove that $\forall x, y \in S : \ x \star y = \inf_{\leq_1}\{x, y\}$.
First we note that $x \star y$ is a lower bound of $\{x, y\}$: we have $(x \star y) \leq_1 x$ because $x \star (x \star y) = (x \star x) \star y = x \star y$. Let u be any other lower bound of $\{x, y\}$. Then $u = x \star u = x \star (y \star u) = (x \star y) \star u$. So $u \leq_1 x \star y$, meaning $x \star y$ is the infimum.
(2) Similar to (1).
(3) Immediate. □

Often we will want to interpret a semilattice as a partially ordered set. This means we need to choose $\leq_1$ or $\leq_2$.

Let $(S, \star)$ be a semilattice. We call S

- a meet-semilattice or $\wedge$-semilattice if we consider it ordered by $\leq_1$, i.e. $x \star y = \inf\{x, y\}$;
- a join-semilattice or $\vee$-semilattice if we consider it ordered by $\leq_2$, i.e. $x \star y = \sup\{x, y\}$.

By convention, we represent the semilattice operation $\star$ of a $\wedge$-semilattice by $\wedge$ and of a $\vee$-semilattice by $\vee$.
We usually denote the identity of a $\wedge$-semilattice by $\bot$ and the identity of a $\vee$-semilattice by $\top$.

We call a poset $(P, \leq)$

- a $\wedge$-semilattice if $\inf\{x, y\}$ exists for all $x, y \in P$;
- a $\vee$-semilattice if $\sup\{x, y\}$ exists for all $x, y \in P$.

2.1.1 Subsets of semilattices

2.1.1.1 Disjoint elements and meshing elements

TODO compare with set-theoretical definitions.

Let S be a $\wedge$-semilattice with identity $\bot$.
We say $x, y \in S$

- are disjoint, denoted $x \perp y$, if $x \wedge y = \bot$;
- mesh, denoted $x \# y$, if $x \wedge y \neq \bot$.

Let $D \subset S$. Then

- the polar $D^\perp$ is called the disjoint complement of D;
- the polar $D^\#$ is called the grill of D.

2.1.1.2 Containment

Let S be a bounded $\wedge$-semilattice and let $D, E \subseteq S$ be subsets. We say D is contained in E if $D \subseteq E \subseteq D^\#$. We write $D \lhd E$.

Lemma III.48. *Let S be a bounded $\wedge$-semilattice. Then containment $\lhd$ is transitive.*

Proof. Assume $D \lhd E \lhd F$. Then $D \subseteq E \subseteq F$, so $D \subseteq F$. Also $D \subseteq E \subseteq F^\#$, so $D \subseteq F^\#$. ☐

Proposition III.49. *Let S be a bounded $\wedge$-semilattice, let $D \subseteq S$ be a subset and $E \subseteq S$ a proper filter. Then $D \subseteq E$ if and only if $D \lhd E$.*

Proof. By definition $D \lhd E$ implies $D \subseteq E$. For the opposite inclusion we need to show that $E \subseteq D^\#$. Assume, towards a contradiction, that this is not the case. Then there exists $a \in E$ and $b \in D$ such that $a \wedge b = \bot$. Now because $D \subseteq E$ and E is $\wedge$-closed, this means that $\bot \in E$. Because E is upward directed, this means $E = S$, which contradicts the assumption of proper filter. ☐

Lemma III.50. *Let S be a bounded $\wedge$-semilattice. Let $F, G \subseteq S$ be subsets.*

1. *If G is a filter, then $F \lhd G$ if and only if $\uparrow F \lhd G$.*
2. *If G is a proper filter, then $F \lhd G$ if and only if $\mathfrak{F}\{F\} \lhd G$.*
3. *If F is an ultafilter, then $F \lhd G$ if and only if $F = G$.*
4. *If G is an ultrafilter, then $F \lhd G$ if and only if $F \subseteq G$.*

Proof. TODO ☐

TODO: move to filters??

2.1.2 Complete semilattices

Lemma III.51. *Let $(S, \wedge)$ be a semilattice. For every finite set $D \subset S$, $\inf(D)$ exists.*

Let $(S, \wedge)$ be a semilattice. We call S a complete semilattice if each subset $D \subseteq S$ has a supremum. We write

$$\sup(D) = \bigvee D.$$

If we want to emphasise that the supremum of D is taken as a subset of S, we write $\bigvee_S D$.

We of course have dual definitions for $\vee$-semilattices. In this case we consider suprema and write $\sup(D) = \bigvee D$.

Lemma III.52. *Let $(S, \wedge)$ be a semilattice and $F \subseteq S$ a finite subset. Then $\bigvee S$ exists.*

In particular every finite semilattice is complete.

Theorem III.53 (Knaster-Tarski fixed-point theorem for semilattices)**.** *Let* $(S, \wedge)$ *be a complete semilattice and* $f : S \to S$ *an order-preserving map. Then*

$$\bigwedge \{x \in S \mid f(x) \leq x\}$$

is the least fixed point of f*.*

Proof. Let P be the set of fixed points of f. Set $H = \{x \in L \mid f(x) \leq x\}$ and $\alpha = \bigwedge H$. It is clear that α is a lower bound of P because $P \subset H$. So we just need to show that α is a fixed point.
Now $f(\alpha)$ is a lower bound of H due to f being order preserving: for all $x \in H$ we have $\alpha \leq x$, so $f(\alpha) \leq f(x)$ and also $f(\alpha) \leq f(x) \leq x$. So $f(\alpha) \leq \alpha$ because α is the greatest lower bound. Conversely, $f(f(\alpha)) \leq f(\alpha)$ because f is order preserving. This means $f(\alpha) \in H$, so $f(\alpha) \leq \alpha$. We have thus shown that $\alpha = f(\alpha)$. ☐

2.1.3 Semilattice homomorphisms

Proposition III.54. *Let* $(P, \leq)$, $(Q, \leq)$ *be posets and* $f : P \to Q$ *an order-preserving function.*

1. *If* P, Q *are* $\wedge$*-semilattices, then*

$$\forall x, y \in P : f(x \wedge y) \leq f(x) \wedge f(y) \quad \iff \quad f \text{ is order-preserving.}$$

2. *If* P, Q *are complete* $\wedge$*-semilattices, then*

$$\forall D \subseteq P : f(\bigwedge D) \leq \bigwedge f[D] \quad \iff \quad f \text{ is order-preserving.}$$

3. *If* P, Q *are* $\vee$*-semilattices, then*

$$\forall x, y \in P : f(x \vee y) \geq f(x) \vee f(y) \quad \iff \quad f \text{ is order-preserving.}$$

4. *If* P, Q *are complete* $\vee$*-semilattices, then*

$$\forall D \subseteq P : f(\bigvee D) \geq \bigvee f[D] \quad \iff \quad f \text{ is order-preserving.}$$

Proof. (1) Assume $f(x \wedge y) \leq f(x) \wedge f(y)$. Then $x \leq y$ implies $x = x \wedge y$, which implies $f(x) = f(x \wedge y) \leq f(x) \wedge f(y) \leq f(y)$.
For the converse: From $x \wedge y \leq x$ we get $f(x \wedge y) \leq f(x)$. Similarly $f(x \wedge y) \leq f(y)$. Together this gives $f(x \wedge y) \leq f(x) \wedge f(y)$.
(2) The direction $\Rightarrow$ follows from (1). Assume f order-preserving. Take arbitrary $D \subseteq P$. For all $x \in D$ we have $\bigwedge D \leq x$, so $f(\bigwedge D) \leq f(x)$. This means $f(\bigwedge D)$ is a lower bound of $f[D]$, so $f(\bigwedge D) \leq \bigwedge f[D]$.
(3,4) Similar. ☐

Corollary III.54.1. *Let* $(S, \wedge), (T, \wedge)$ *be a semilattices and* $f : S \to T$ *a function. If* f *is a semilattice homomorphism, then it is order-preserving. The converse does not hold in general.*

Proof. We just need to show the converse does not hold. TODO ☐

Lemma III.55. *Let $(P, \leq)$, $(Q, \leq)$ be posets and $f : P \to Q$ a bijective order-embedding. Then f preserves all meets and joins, i.e. $\forall D \subseteq P$:*

$$f(\bigwedge D) = \bigwedge f[D] \qquad \text{and} \qquad f(\bigvee D) = \bigvee f[D]$$

if the relevant meets and joins exist.

TODO does existence of one side of an equals mean existence of the other???

Proof. We have already proved an inequality in both cases. By bijectivity we can find a $c \in P$ such that $f(c) = \bigwedge f[D]$. By order-reflection, we have that c is a lower bound of D, so $c \leq \bigwedge D$. By order-preservation we have $\bigwedge f[D] = f(c) \leq f(\bigwedge D)$.
The other equality is dual. □

Corollary III.55.1. *Let $(P, \leq)$, $(Q, \leq)$ be posets and $f : P \to Q$ a reverse bijective order-embedding. Then $\forall D \subseteq P$*

$$f(\bigwedge D) = \bigvee f[D] \qquad \text{and} \qquad f(\bigvee D) = \bigwedge f[D]$$

if the relevant meets and joins exist.

Proposition III.56. *Let $(S, \wedge)$ be a semilattice and X a set. Then the pointwise ordering of $(X \to S)$ makes it a semilattice. For all $f, g \in (X \to S)$, the function $f \wedge g$ is defined by*

$$f \wedge g : X \to S : x \mapsto (f \wedge g)(x) = f(x) \wedge g(x).$$

If S is a complete semilattice, then so is $(X \to S)$.

Proposition III.57. *Let $(S, \wedge), (T, \wedge)$ be (complete) semilattices. The set of (complete) semilattice homomorphisms is a (complete) subsemilattice of $(S \to T)$.*

Proof. We calculate for all $x, y \in S$:

$$\begin{aligned}(f \wedge g)(x \wedge y) &= f(x \wedge y) \wedge g(x \wedge y) = f(x) \wedge f(y) \wedge g(x) \wedge g(y) \\ &= (f(x) \wedge g(x)) \wedge (f(y) \wedge g(y)) = (f \wedge g)(x) \wedge (f \wedge g)(y).\end{aligned}$$

TODO complete notation. □

TODO link universal algebra.

2.1.4 Join- and meet-irreducible elements

Let L be a $\vee$-semilattice. We call $x \in L$ join-irreducible if

- x is not a least element of L,
- for all $a, b \in L$: $x = a \vee b$ implies $x = a$ or $x = b$.

The definition of meet-irreducible is dual on a $\wedge$-semilattice.
We denote the set of join-irreducible elements in L as $\mathcal{J}(L)$ and the set of meet-irreducible elements in L as $\mathcal{M}(L)$.

Lemma III.58. *Let S be a $\vee$-semilattice and $x \in S$ not a least element. Then the following are equivalent:*

1. *x is join-irreducible;*
2. *for all $a, b \in S$: $x = a \vee b$ implies $x \leq a$ or $x \leq b$;*
3. *for all $a, b \in S$: $x > a$ and $x > b$ implies $x > a \vee b$;*
4. *for all finite $F \subseteq S$: $x = \bigvee F$ implies $x \in F$.*

Lemma III.59. *In a finite lattice L, an element is join-irreducible if and only if it has exactly one lower cover.*

> Example
>
> Consider the lattice $\langle \mathbb{N}, \mathrm{lcm}, \mathrm{gcd} \rangle$. A non-zero element of $\mathbb{N}$ is join irreducible if and only if it is of the form p^r for some prime p and $r \in \mathbb{N}$.

Proposition III.60. *x is join-irreducible iff $\downarrow x$ is linearly ordered.*

TODO: isolated points link with boundary
`http://www.m-hikari.com/ijcms-password2008/9-12-2008/xuluoshanIJCMS9-12-2008.pdf`
`http://imar.ro/journals/Revue_Mathematique/pdfs/2015/2/5.pdf`
`https://link.springer.com/content/pdf/10.1007/BF01349957.pdf`

Lemma III.61. *Let $(P, \leq)$ be a poset. Then*

1. *$\mathcal{A}(P) \subseteq \mathcal{J}(P)$;*
2. *$\mathcal{CA}(P) \subseteq \mathcal{M}(P)$;*

Proof. (1) Assume $a \in \mathcal{A}(P)$. Then a is not a least element of P. Assume there exist $x, y \in P$ such that $a = x \vee y$. Then $x, y \in \downarrow a \subseteq \{\bot, a\}$. If $x = \bot = y$, then $a = \bot$, which is not an atom. So either $x = a$ or $y = a$.
(2) Dual. □

2.1.5 Generalised inverse of inf

Proposition III.62. *Let $(P, \leq)$ be a poset and $S \subseteq P$ a subset such that $(S, \leq)$ is a complete $\wedge$-semilattice. Then*

$$f : P \to \mathcal{P}(S)^o : x \mapsto \uparrow x \cap S \qquad \text{and} \qquad g : \mathcal{P}(S)^o \to P : D \mapsto \bigwedge_S D$$

are order-preserving generalised inverses. Additionally, they form a Galois connection if and only if *for all $D \subseteq S$: $\bigwedge_P D$ exists and is equal to $\bigwedge_S D$.*

Proof. TODO first part.
Assume for all $D \subseteq S$: $\bigwedge_P D$ exists and $\bigwedge_S D = \bigwedge_P D$. It is enough to verify, for all $x \in P$ and $B \subseteq S$

$$(\uparrow x) \cap S \supseteq B \iff x \leq \bigwedge_S B.$$

Assume $(\uparrow x) \cap S \supseteq B$. Then $\uparrow x \supseteq B$, so $x \bigwedge_P \leq \bigwedge_P B = \bigwedge_S B$. Conversely, assume $x \leq \bigwedge_S B$. Then $B \subseteq \uparrow_S \bigwedge_S B \subseteq \uparrow_S x = \uparrow x \cap S$.

Assume these functions form a Galois connection. In order to obtain a contradiction, assume there exists a $D \subseteq S$ such that $\bigwedge_P D$ does not exist or is larger than $\bigwedge_S D$ (it can never be smaller). In both cases this means there exists a lower bound y of D in P that is larger than $\bigwedge_S D$.
Then $(\uparrow y) \cap S \supseteq (\uparrow \bigwedge_S D) \cap S \supseteq D$, but $y \geq \bigwedge_S D$. □

2.1.5.1 Projection onto a complete subsemilattice

Let $(P, \leq)$ be a poset and $S \subseteq P$ a subset such that $(S, \leq)$ is a complete $\wedge$-semilattice. Then the (upper) order projection on S is the function

$$P^u_S : P \to S : x \mapsto \bigwedge(\uparrow x \cap S).$$

Lemma III.63. *Let $(P, \leq)$ be a poset and $S \subseteq P$ a subset that is a complete $\wedge$-semilattice. The upper order projection P^u_S is*

1. *idempotent;*
2. *order-preserving.*

Also P^u_S is expansive if and only if *for all $D \subseteq S$: $\bigwedge_S D = \bigwedge_P D$.*

Proof. (1) Assume $x \in S$. Then $x \in (\uparrow x \cap S)$ and x is a lower bound of $(\uparrow x \cap S)$, so $P^u_S(x) = x$.
(2) Assume $x \leq y$. Then $\uparrow x \supseteq \uparrow y$, so $\uparrow x \cap S \supseteq \uparrow y \cap S$ and $\bigwedge(\uparrow x \cap S) \subseteq \bigwedge(\uparrow y \cap S)$. □

2.2 Lattices

`file:///C:/Users/user/Downloads/Gr%C3%A4tzer,%20George%20-%20General%20lattice%20theory-Birkh%C3%A4user%20(2007).pdf` `file:///C:/Users/user/Downloads/R.%20Padmanabhan,%20S.%20Rudeanu%20-%20Axioms%20for%20lattices%20and%20Boolean%20algebras-World%20Scientific%20(2008).pdf`

A lattice is an algebraic structure $\langle L, \vee, \wedge \rangle$, where $\vee, \wedge$ are binary operations on the set L such that $\langle L, \vee \rangle$ and $\langle L, \wedge \rangle$ are semilattices and $\vee, \wedge$ are linked by the absorption law:

$$\forall a, b \in L: \; a \vee (a \wedge b) = a = a \wedge (a \vee b).$$

We call

- $(L, \vee)$ the join-semilattice and $a \vee b$ the join of a and b;
- $(L, \wedge)$ the meet-semilattice and $a \wedge b$ the meet of a and b.

We call a lattice bounded if both the join- and the meet-semilattice are bounded. We denote

- the identity of the join-semilattice by $\perp$;
- the identity of the meet-semilattice by $\top$.

We denote the bounded lattice $(L, \vee, \wedge, \perp, \top)$.

By I.141 the absortion law renders the axiom of idempotency of the semilattices redundant. So we just need that $\langle L, \vee\rangle$ and $\langle L, \wedge\rangle$ are commutative semigroups that are linked by the absorption law.

Proposition III.64. *Observations from universal algebra:*

1. *Subset closed under* $\vee, \wedge$ *is sublattice.*
2. *Product lattices*
3. *Inverse of homomorphism is homomorphism*

Example

- Let X be a set. Then $\mathcal{P}(X)$ ordered by inclusion is a lattice and $\cup, \cap$ are the corresponding join and meet operations.

  In particular, for all $A, B \in \mathcal{P}(X)$:

  1. $\inf\{A, B\} = A \cap B$;
  2. $\sup\{A, B\} = A \cup B$.
- The natural numbers forms a lattice if ordered by division. The meet and join are given by
  $$m \vee n = \operatorname{lcm}\{m, n\} \qquad \text{and} \qquad m \wedge n = \gcd\{m, n\}.$$

2.2.1 Lattices and order

As for semilattices, we can equivalently characterise lattices as posets with certain conditions.

Proposition III.65. *Let L be a set.*

1. *If $\langle L, \vee, \wedge\rangle$ is a lattice, then $\langle L, \leq\rangle$ is a poset such that every two element set has a supremum and an infimum, where*
   $$\forall x, y \in L:\ x \leq y \qquad \iff \qquad x \vee y = y$$
   or, equivalently,
   $$\forall x, y \in L:\ x \leq y \qquad \iff \qquad x \wedge y = x.$$
2. *If $\langle L, \leq\rangle$ is a poset such that every two element set has a supremum and an infimum, then $\langle L, \vee, \wedge\rangle$ is a lattice, where*
   $$\forall x, y \in L:\ x \vee y = \sup\{x, y\} \quad \textit{and} \quad x \wedge y = \inf\{x, y\}.$$

The order can also be defined by

$$\forall x, y \in L:\ x \leq y \qquad \iff \qquad x \wedge y = x.$$

Proof. Mostly this follows from III.47. We just need to show the two definitions of order are equivalent. This follows from the absorption law:

$$\begin{aligned} x \vee y = y &\implies x \wedge y = x \wedge (x \vee y) = x \\ x \wedge y = x &\implies x \vee y = (x \wedge y) \vee y = y. \end{aligned}$$

□

Lemma III.66. *Let L be a lattice and $a, b, c, d \in L$. If $a \leq b$ and $c \leq d$, then*

$$a \vee c \leq b \vee d \qquad \text{and} \qquad a \wedge c \leq b \wedge d.$$

Proof. If $a \leq b$ and $c \leq d$, then we have

$$\begin{cases} a = a \wedge b \\ c = c \wedge d \end{cases} \qquad \text{and} \qquad \begin{cases} b = a \vee b \\ d = c \vee d. \end{cases}$$

We calculate

$$(a \vee c) \wedge (b \vee d) = (a \vee c) \wedge (a \vee b \vee c \vee d) = (a \vee c) \wedge ((a \vee c) \vee (b \vee d)) = a \vee c,$$

so $a \vee c \leq b \vee d$. Similarly

$$(a \wedge c) \vee (b \wedge d) = (a \wedge b \wedge c \wedge d) \vee (b \wedge d) = ((a \wedge c) \wedge (b \wedge d)) \vee (b \wedge d) = b \wedge d,$$

so $a \wedge c \leq b \wedge d$. ☐

Corollary III.66.1. *Let L be a lattice and $a, b, c, d \in L$. Then*

1. *if $a \leq b$ then $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$;*
2. *if $a \leq c$ and $b \leq c$, then $a \vee b \leq c$;*
3. *if $a \leq b$ and $a \leq c$, then $a \leq b \wedge c$.*

Lemma III.67. *Let L be a lattice and $a, b \in L$. If for all $x \in L$: $a \vee x = b \vee x$ and $a \wedge x = b \wedge x$, then $a = b$.*

Proof. Setting $x = a$ gives $a = a \vee b$ and $a = a \wedge b$. Thus $a \leq b$ and $b \leq a$, meaning $a = b$. ☐

Proposition III.68 (Mini-max theorem)**.** *Let L be a lattice and let $\langle a_{i,j} \rangle \subset L$ be indexed by $i, j \in \mathbb{N}$. Then*

$$\bigvee_{j=1}^{n} \left(\bigwedge_{i=1}^{m} a_{i,j} \right) \leq \bigwedge_{i=1}^{m} \left(\bigvee_{j=1}^{n} a_{i,j} \right).$$

Proof. For all k, l we have $a_{k,l} \leq \bigvee_{j=1}^{n} a_{k,j}$. This implies $\bigwedge_{i=1}^{m} a_{i,l} \leq \bigwedge_{i=1}^{m} \left(\bigvee_{j=1}^{n} a_{i,j} \right)$ for all l. Taking the supremum over l gives the result. ☐

Corollary III.68.1 (Median inequality)**.** *Let L be a lattice and $a, b, c \in L$, then*

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

Proof. Use $a_{i,j} = \begin{pmatrix} a & b & a \\ b & b & c \\ a & c & c \end{pmatrix}$. ☐

Corollary III.68.2 (Distributive inequalities)**.** *Let L be a lattice and $a, b, c \in L$, then*

$$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c);$$
$$(a \vee b) \wedge (a \vee c) \geq a \vee (b \wedge c).$$

In particular this also means

$$c \leq a \implies (a \wedge b) \vee c \leq a \wedge (b \vee c).$$

Proof. Use $a_{i,j} = \begin{pmatrix} a & a \\ b & c \end{pmatrix}$ and $a_{i,j} = \begin{pmatrix} a & b \\ a & c \end{pmatrix}$. The particular cases follow because in this case $a \wedge c = c$. This statement is self-dual. ☐

The distributive inequalities are fairly elementary and do not need to be derived from minimax theorem. In fact they are corollary to the following lemma which can be obtained by more elementary means:

Lemma III.69. *Let L be a lattice $x \in L$ and $S \subseteq L$ a subset. Then*

1. $S^u \vee x \subseteq (S \vee x)^u$ *and* $S^l \vee x \subseteq (S \vee x)^l$;
2. $S^u \wedge x \subseteq (S \wedge x)^u$ *and* $S^l \wedge x \subseteq (S \wedge x)^l$.

Proof. The maps $y \mapsto y \vee x$ and $y \mapsto y \wedge x$ are order-preserving, so we can use II.64. ☐

Corollary III.69.1 (Infinite distributive inequalities)**.** *Let L be a lattice $x \in L$ and $S \subseteq L$ a subset. Assume all relevant suprema exist, then*

1. $\sup(S) \wedge x \geq \sup(S \wedge x)$;
2. $\inf(S) \vee x \leq \inf(S \vee x)$.

Proof. Because $\sup(S) \in S^u$, we have $\sup(S) \wedge x \in S^u \wedge x$, so $\sup(S) \wedge x \in (S \wedge x)^u$ and $\sup(S) \wedge x \geq \min((S \wedge x)^u) = \sup(S \wedge x)$. The second part is dual. ☐

In particular if S has two elements, we recover the distributive inequalities.

2.2.2 Complete lattices

Lemma III.70. *Let L be a lattice. For every finite set $S \subset L$, $\sup(S)$ and $\inf(S)$ exist.*

Let L be a lattice. We call L a complete lattice if each subset $S \subseteq L$ has both a supremum and an infimum. We write

$$\sup(S) = \bigvee S \qquad \inf(S) = \bigwedge S.$$

If we want to emphasise that the supremum/infimum of S is taken as a subset of L, we write $\bigvee_L S$ and $\bigwedge_L S$.

Clearly every finite lattice is complete.

Lemma III.71. *Let L be a lattice and $F \subseteq L$ a finite subset. Then $\bigvee F$ and $\bigwedge F$ exist.*

Example

For any set X, $\mathcal{P}(X)$ is a complete lattice.

Lemma III.72. *Let P be an ordered set such that all relevant suprema and infima exist and $S, T \subseteq P$. Then*

1. $\bigvee S \leq \bigwedge T$ *if and only if* $s \leq t$ *for all* $s \in S, t \in T$;
2. *if* $S \subseteq T$, *then* $\bigvee S \leq \bigvee T$ *and* $\bigwedge S \geq \bigwedge T$;

3. $\bigvee(S \cup T) = (\bigvee S) \vee (\bigvee T)$ *and* $\bigwedge(S \cup T) = (\bigwedge S) \wedge (\bigwedge T)$.

Proposition III.73 (Mini-max theorem for complete lattices)**.** *Let L be a complete lattice, I, J index sets and $\{a_{i,j} \in L \mid i \in I, j \in J\}$. Then*

$$\bigvee_{j\in J}\bigwedge_{i\in I} a_{i,j} \leq \bigwedge_{i\in I}\bigvee_{j\in J} a_{i,j}.$$

Proof. For all k, l we have $a_{k,l} \leq \bigvee_{j\in J} a_{k,j}$. This implies $\bigwedge_{i\in I} a_{i,l} \leq \bigwedge_{i\in I}\bigvee_{j\in J} a_{i,j}$ for all l. Taking the supremum over l gives the result. □

Proposition III.74. *Let P be a non-empty ordered set. Then the following are equivalent:*

1. *P is a complete lattice;*
2. *$\bigvee S$ exists for all subsets $S \subseteq P$;*
3. *$\bigwedge S$ exists for all subsets $S \subseteq P$;*
4. *P has a bottom element $\bot$ and $\bigvee S$ exists for all non-empty $S \subseteq P$;*
5. *P has a top element $\top$ and $\bigwedge S$ exists for all non-empty $S \subseteq P$;*
6. *for all $x \in P$ both $\uparrow x$ and $\downarrow x$ are complete lattices.*

Proof. The only difficult implication is (5) $\Rightarrow$ (1). All infima exist because $\inf(\emptyset) = \top \in P$. New each non-empty set S in P has an upper bound, $\top$, so $\bigvee S$ exists in P by III.35. Finally $\bigvee \emptyset = \bot = \bigwedge P \in P$. □

Theorem III.75 (Knaster-Tarski fixed-point theorem)**.** *Let L be a complete lattice and $f : L \to L$ an order-preserving map. Then*

$$\bigvee \{x \in L \mid x \leq f(x)\} \qquad \textit{and} \qquad \bigwedge \{x \in L \mid x \geq f(x)\}$$

are, resp., the greatest and the least fixed point of f.

Proof. This is just two applications of the semilattice formulation of the Knaster-Tarski theorem, III.53, once for the join-semilattice and once for the meet-semilattice. □

Corollary III.75.1. *The set of fixed points of f forms a complete lattice.*

Proof. Let P be the set of fixed points. To show P is a complete lattice, take any subset $S \subset P$. Set $w = \bigvee S$, where S is considered as a subset of L. For all $x \in W$: $x \leq w$, which implies $x = f(x) \leq f(w)$. As w is the least upper bound, we have $w \leq f(w)$. This implies $f[\uparrow w] \subseteq \uparrow w$, meaning we can view f as a function on the complete lattice $\uparrow w$. In particular $f|_{\uparrow w}$ has a least fixed point by the theorem, so S has a supremum in P. The existence of the infimum is dual. □

Corollary III.75.2 (Banach decomposition theorem)**.** *Let X, Y be sets and $f : X \to Y$ and $g : Y \to X$ functions. There exist partitions $X_1, X2$ and Y_1, Y_2 of X and Y such that*

$$f[X_1] = Y_1 \qquad \textit{and} \qquad g[Y_2] = X_2.$$

Proof. Consider the map $F : \mathcal{P}(X) \to \mathcal{P}(X) : S \mapsto X \setminus g[Y \setminus f[S]]$. By the theorem this map has a fixed point, which we call X_1. We then need to set $Y_1 = f[X_1], X_2 = X \setminus X_1$ and $Y_2 = Y \setminus Y_1$. The fact X_1 is a fixed point means that $X_1 = X \setminus g[Y \setminus f[X_1]] = g[Y \setminus Y_1] = g[Y_2]$. □

Corollary III.75.3 (Schröder-Bernstein)**.** *Let X, Y be sets and $f : X \rightarrowtail Y$ and $g : Y \rightarrowtail X$ injective functions. Then there exists a bijective function $h : X \mathrel{\rightarrowtail\kern-1.9ex\twoheadrightarrow} Y$.*

Proof. Use the Banach decomposition theorem to obtain partitions X_1, X_2 and Y_1, Y_2. Then $f|_{X_1} : X_1 \mathrel{\rightarrowtail\kern-1.9ex\twoheadrightarrow} Y_1$ and $g|_{Y_2} : Y_2 \mathrel{\rightarrowtail\kern-1.9ex\twoheadrightarrow} X_2$ are bijective, so we can construct

$$h : X \mathrel{\rightarrowtail\kern-1.9ex\twoheadrightarrow} Y : x \mapsto \begin{cases} f(x) & x \in X_1 \\ (g|_{Y_2})^{-1}(x) & x \in X_2 \end{cases}.$$

□

The Schröder-Bernstein theorem was already proven in I.183.

2.2.2.1 Upper and lower order projection

Let L be a complete lattice and $S \subseteq L$ a subset. We define

- the upper projection on S as the map $P^U_S : L \to S : x \mapsto \bigwedge(\uparrow x \cap S)$;
- the lower projection on S as the map $P^L_S : L \to S : x \mapsto \bigvee(\downarrow x \cap S)$.

Lemma III.76. *Let L be a complete lattice and $S \subseteq L$ a subset. Then the projections P^U_S, P^L_S are*

1. *idempotent;*
2. *order-preserving.*

Proof. □

2.2.2.2 Chain conditions

The following requires dependent choice:

Proposition III.77. *Let L be a lattice. Then*

1. *if L satisfies the ascending chain condition, then for all non-empty subsets $S \subset L$ there exists a finite set $F \subset L$ such that $\bigvee S = \bigvee F$;*
2. *if L satisfies the descending chain condition, then for all non-empty subsets $S \subset L$ there exists a finite set $F \subset L$ such that $\bigwedge S = \bigwedge F$;*
3. *if L has a bottom element and satisfies the ascending chain condition, then L is complete;*
4. *if L has a top element and satisfies the descending chain condition, then L is complete;*
5. *if L has no infinite chains, then L is complete.*

Proof. (1) Assume L satisfies the ascending chain condition and let $S \subset L$ be non-empty. Define

$$B = \left\{ \bigvee G \;\middle|\; G \text{ is a finite, non-empty subset of } S \right\}.$$

This is well-defined by III.71. Then B has a maximal element $m = \bigvee F$ for some finite F by I.202.

Now m is an upper bound of S. Indeed, let $x \in S$. Then $m = \bigvee F \leq \bigvee(F \cup \{x\})$ because $F \subseteq (F \cup \{x\})$. Since m is maximal in B, we have $m = \bigvee(F \cup \{x\}) \geq x$. It is clearly also the least upper bound, otherwise it was not the least upper bound of F.
(2) Dual of 1.
(3) This follows from 1. and III.74.
(4) Dual of 3.
(5) A lattice with no infinite chains satisfies the ascending chain condition. Also a lattice with no infinite chains has a bottom element. (TODO: need dependent/countable choice?) ☐

Proposition III.78. *Let L be a lattice satisfing the descending chain condition. Then*

1. $\forall a, b \in L : \; a \nleq b \implies \exists x \in \mathcal{J}(L) : \; x \leq a$ *and* $x \nleq b$;
2. $\forall a \in L : \; a = \bigvee \{x \in \mathcal{J}(L) \mid x \leq a\}$.

Proof. (1) Set $S = \{x \in L \mid x \leq a \text{ and } x \nleq b\}$, which is non-empty and thus contains a minimal element m by I.202. We claim m is join-irreducible. Assume, towards a contradiction, $x = c \vee d$ and $c < x > d$. By minimality of x, $c, d \notin S$. As $c, d < x \leq a$, we must have $c, d \leq b$. But this means $x \leq b$, so $x \notin S$ which is a contradiction.
(2) Set $T = \{x \in \mathcal{J}(L) \mid x \leq a\}$. Clearly a is an upper bound of T. To see that it is the least upper bound, take a different upper bound c. Assume, towards a contradiction, that $a \nleq c$. Then $a \nleq a \wedge c$. By point 1. there exists an $x \in \mathcal{J}(L)$ such that $x \leq a$ (meaning $x \in T$) and $x \nleq a \wedge c$. But if c were an upper bound of T, then $x \leq a \wedge c$, which is a contradiction. ☐

Proposition III.79. *Let L be a lattice.*

1. *If L satisfies the descending chain condition, then any subset $Q \supseteq \mathcal{J}(L)$ is join-dense in L.*
2. *If L satisfies the ascending chain condition and Q is join-dense in L, then for all $a \in L$ there exists a finite subset F of Q such that $a = \bigvee F$.*

Proof. (1) is a corollary of III.78. (2) is a corollary of III.77. ☐

Corollary III.79.1. *Let L be a lattice with no infinite chains. Then*

1. *for each $a \in L$, there exists a finite subset F of $\mathcal{J}(L)$ such that $a = F$.*
2. *$Q \subseteq L$ is join-dense in L if and only if $Q \supseteq \mathcal{J}(L)$.*

Proof. If L has no finite chains, then it satisfies the ascending and descending chain conditions. Only the $\Rightarrow$ direction of (2) is not immediately obvious. Assume Q is join-dense and let $x \in \mathcal{J}(L)$. By the proposition there exists a finite $F \subseteq Q$ such that $x = \bigvee F$. Since x is join-irreducible, we have $x \in F$ and hence $x \in Q$. Thus, $\mathcal{J}(L) \subseteq Q$. ☐

2.2.3 Distributive and modular lattices

2.2.3.1 Distributive lattices

For all lattices L the distributive inequalities, III.68.2, hold: $\forall a, b, c \in L$:

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c);$$
$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c).$$

The two corresponding equalities are equivalent:

Proposition III.80. *Let L be a lattice. Then the following are equivalent:*

1. $\forall a, b, c \in L: \; a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$*;*
2. $\forall a, b, c \in L: \; a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$*.*

These equivalent equalities are known as the distributive laws.

Proof. We show (1) ⇒ (2). Then other implication follows by duality.
Assume (1). Then, for all $a, b, c \in L$:

$$\begin{aligned} (a \wedge b) \vee (a \wedge c) &= ((a \wedge b) \vee a) \wedge ((a \wedge b) \vee c) && \text{by (1)} \\ &= (a \wedge (c \vee (a \wedge b)) && \text{by the absorption law} \\ &= (a \wedge ((c \vee b) \wedge (c \vee a)) && \text{by (1)} \\ &= a \wedge (b \vee c) && \text{by the absorption law.} \end{aligned}$$

□

Note that it is not true that

$$\forall a, b, c \in L: \; a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \iff a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

A lattice L is called distributive if it satisfies the distributive laws.

Lemma III.81. *A lattice is distributive* if and only if *its dual is distributive.*

Example

- Any totally ordered set is a distributive lattice.
- For any set X, the poset $(\mathcal{P}(X), \subseteq)$ is a distributive lattice.

Lemma III.82. *Let L be a lattice. The following are equivalent:*

1. *L is distributive;*
2. *for all* $x, y, z, w \in L: \quad x \wedge y \leq w$ *and* $x \wedge z \leq w \implies x \wedge (y \vee z) \leq w$*;*
3. *for all* $x, y, z, w \in L: \quad x \vee y \geq w$ *and* $x \vee z \geq w \implies x \vee (y \wedge z) \geq w$*.*

Proof. Point (1) is self-dual and points (2) and (3) are dual, so it is enough to show that (1) and (2) are equivalent.
Assume L distributive. By III.66.1, $x \vee y \geq w$ and $x \vee z \geq w$ imply $(x \wedge y) \vee (x \wedge z) \leq w$. By distributivity, we get (2).
Conversely, we can take $w = (x \wedge y) \vee (x \wedge z)$. Then (2) gives $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ and the distributive inequality III.68.2 gives the other inequality. □

Proposition III.83 (Cancellation law for distributive lattices)**.** *Let L be a distributive lattice and $a, b, c \in L$. Then*

$$a \vee c = b \vee c \text{ and } a \wedge c = b \wedge c \text{ implies } a = b.$$

Note that we only assume the equalities hold for some c, not all c. If these equalities hold for all c, then we have cancellation in all lattices, not just distributive ones. See III.67.

Proof. We calculate, using the absorption law and distributivity,

$$a = a \vee (a \wedge c) = a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge (b \vee c) = b \vee (a \wedge c) = b \vee (b \wedge c) = b.$$

□

Lemma III.84. *Let L be a distributive lattice with a bottom $\bot$ and $S \subset L$. Then $S^\perp$ is an ideal.*

Proof. Let $a, b \in S^\perp$. Then for all $x \in S$, we have

$$(a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) = \bot \vee \bot = \bot,$$

so $a \vee b \in S^\perp$.
It is also clear $S^\perp$ must be a down set by III.66.1. □

2.2.3.2 Modular lattices

For all lattices L the modular inequality holds: $\forall a, b, c \in L$:

$$a \leq c \implies a \vee (b \wedge c) \leq (a \vee b) \wedge c.$$

See III.68.2.

Proposition III.85. *Let L be a lattice. Then the following are equivalent:*

1. *$\forall a, b, c \in L$: $a \vee (b \wedge c) = (a \vee b) \wedge c$ if $a \leq c$;*
2. *$\forall a, b, c \in L$: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ if $a \leq b$ or $a \leq c$; the dual of (2);*
3. *$\forall a, b, c \in L$: $a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$; the dual of (3);*
4. Shearing identity*: $\forall a, b, c \in L$: $(a \vee b) \wedge c = (a \vee (b \wedge (a \vee c))) \wedge c$; the dual of the shearing identity.*

The first of these is referred to as the modular law. Notice that it is self-dual.

Proof. (1) is equivalent to its dual by replacing $a \leftrightarrow c$. This will imply all statements are equivalent to their duals once the equivalence with (1) has been established.
$\boxed{(1) \Rightarrow (2)}$ Assume (1). Assume $a \leq b$ or $a \leq c$. By relabelling we can assume $a \leq c$. Then $a \vee c = c$ and (2) clearly follows from (1).
$\boxed{(2) \Rightarrow (3)}$ Apply (2) to $a \leq a \vee c$.
$\boxed{(3) \Rightarrow (1)}$ Assume $a \leq c$. Then $a \vee c = c$ and $a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$ reduces to $a \vee (b \wedge c) = (a \vee b) \wedge c$.
$\boxed{(3) \Rightarrow (4)}$ We calculate, using (3),

$$(a \vee (b \wedge (a \vee c))) \wedge c = ((a \vee b) \wedge (a \vee c)) \wedge c = (a \vee b) \wedge (a \vee c) \wedge c = (a \vee b) \wedge c.$$

$\boxed{(4) \Rightarrow (3)}$ TODO???? □

A lattice L is called modular if it satisfies the modular law.

Lemma III.86. *If a lattice is distributive, it is also modular.*

Proof. If the lattice is distributive, point (2) of III.85 holds unconditionally, and so in particular also conditionally. □

2.2.3.3 Derived lattices

Proposition III.87. *(i) If L is a modular (distributive) lattice, then every sublattice of L is modular (distributive).*
(ii) If L and K are modular (distributive) lattices, then $L \times K$ is modular (distributive).
(iii) If L is modular (distributive) and K is the image of L under a homomorphism, then K is modular (distributive).

Corollary III.87.1. *If a lattice is isomorphic to a sublattice of a product of distributive (modular) lattices,then it is distributive (modular).*

Theorem III.88 ($\mathbf{M}_3 - \mathbf{N}_5$ theorem)**.** *Let L be a lattice.*

1. *L is non-modular* if and only if $\mathbf{N}_5 \rightarrowtail L$;
2. *L is non-distributive* if and only if $\mathbf{N}_5 \rightarrowtail L$ *or* $\mathbf{M}_3 \rightarrowtail L$.

Proof. TODO! □

2.2.3.4 Infinitely distributive lattices

Let L be a lattice. We call L

- join-infinitely distributive if L is complete as a $\vee$-semilattice and

$$\forall x \in L, \forall D \subseteq L : \qquad x \wedge \bigvee D = \bigvee \{x \wedge y \mid y \in D\};$$

- meet-infinitely distributive if L is complete as a $\wedge$-semilattice and

$$\forall x \in L, \forall D \subseteq L : \qquad x \vee \bigwedge D = \bigwedge \{x \vee y \mid y \in D\};$$

- infinitely distributive if L is both join-infinitely distributive and meet-infinitely distributive.

TODO cfr Heyting algebras
The inequalities $x \wedge \bigvee D \geq \bigvee x \wedge D$ and $x \vee \bigwedge D \leq \bigwedge x \vee D$ hold in any lattice, see III.69.1.

2.2.3.5 Completely distributive lattices

2.3 Complementation

2.3.1 Complementation

Let L be a bounded lattice and $x \in L$. We call y a complement of x if

$$x \vee y = \top \qquad \text{and} \qquad x \wedge y = \bot.$$

Proposition III.89. *Let L be a bounded lattice. If L is distributive, then any $x \in L$ has at most one complement.*

Proof. This follows from the cancellation law III.83. Let y, y' be complements of x. Then $y \vee x = \top = y' \vee x$ and $y \wedge x = \bot = y' \wedge x$, so $y = y'$. □

A lattice element may also have no complements.

Example

The only elements with a complement in a bounded chain are $\top$ and $\bot$.

2.3.1.1 Complemented lattices

A complemented lattice is a bounded lattice with a function $c : L \to L$, called the complementation, such that $c(x)$ is a complement of x for all $x \in L$.
A lattice in which every element has exactly one complement is called a uniquely complemented lattice.
A lattice with the property that every interval (viewed as a sublattice) is complemented is called a relatively complemented lattice.

A distributive complemented lattice is uniquely complemented.

Lemma III.90. *Let L be a complemented lattice with complementation $c : L \to L$. Then $c(\top) = \bot$ and $c(\bot) = \top$.*

Proof. For $c(\top)$ to be a complement of $\top$, we need $\top \wedge c(\top) = \bot$. But for all $x \in L$ we have $x \leq \top$, so $\top \wedge x = x$. This means we have $c(\top) = \bot$. ☐

Lemma III.91. *Let L be a uniquely complemented lattice. Then the complementation $c : L \to L$ is an involution.*

Proof. Clearly if $c(x)$ is a complement of x, then x is a complement of $c(x)$. By uniqueness $c(c(x)) = x$. ☐

2.3.1.2 Orthocomplemented lattices

Let L be a bounded lattice. An orthocomplementation is a function $L \to L : x \to \overline{x}$ that maps each element $x \in L$ to an orthocomplement $\overline{x}$ such that

- x and $\overline{x}$ are complements;
- $x \mapsto \overline{x}$ is an involution: $\overline{\overline{x}} = x$;
- $x \mapsto \overline{x}$ is order-reversing: $x \leq y \implies \overline{y} \leq \overline{x}$.

An orthocomplemented lattice or ortholattice is a bounded lattice equipped with an orthocomplementation.

An orthocomplemented lattice is not necessarily uniquely complemented.

Lemma III.92. *Let L be an ortholattice with completement c. Then $c : L \to L$ is a reverse order-isomorphism.*

Corollary III.92.1. *The dual lattice of an ortholattice is an ortholattice. We can view an orthocomplementation as an isomorphism between an ortholattice and its dual.*

Corollary III.92.2. *Let $A \subseteq L$. Then*

1. $c\left[\bigvee A\right] = \bigwedge c[A]$;
2. $c\left[\bigwedge A\right] = \bigvee c[A]$.

Proof. See III.55.1. ☐

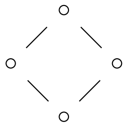
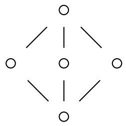
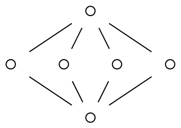
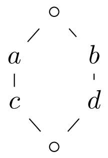
Proposition III.93. *The cardinality of any finite ortholattice is either even or* 1.

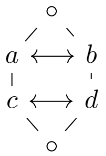
Proof. Let L be a finite ortholattice. The orthocomplementation pairs elements x, y such that $\overline{x} = y$ and $\overline{y} = x$. If for all such pairs we have $x \neq y$, then the cardinality of L is even. Now assume there exists an $x \in L$ such that $\overline{x} = x$. Then

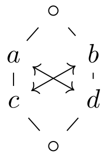
$$\bot = x \wedge \overline{x} = x \wedge x = x = x \vee x = x \vee \overline{x} = \top.$$

So $\bot = \top$, which is only possible if $L = \{\bot\}$. ☐

Example

- The lattice $\mathbf{M}_2$ =  admits a unique orthocomplementation.
- The lattice $\mathbf{M}_3$ =  admits no orthocomplementations.
- The lattice $\mathbf{M}_4$ =  admits three orthocomplementations.
- The hexagon lattice 

admits a unique orthocomplementation, but it is not uniquely complemented. Indeed both of the functions f : 

and g : 

are complementations. Only g is an orthocomplementation, because f does not reverse order: $a \geq c$ and $f(a) = b \geq d = f(c)$.

Theorem III.94 (De Morgan's laws)**.** *Let L be an ortholattice, then for all $x, y \in L$*

1. *$\overline{(x \vee y)} = \overline{x} \wedge \overline{y}$;*
2. *$\overline{(x \wedge y)} = \overline{x} \vee \overline{y}$.*

Proof. From $x \leq x \vee y$ and $y \leq x \vee b$, we have $\overline{(x \vee b)} \leq \overline{x}$ and $\overline{(x \vee b)} \leq \overline{y}$. By III.66.1 we have $\overline{(x \vee y)} \leq \overline{x} \wedge \overline{y}$. For the other inequality we start with $\overline{x} \geq \overline{x} \wedge \overline{y}$ and $\overline{y} \geq \overline{x} \wedge \overline{y}$ to obtain $x \vee y \leq \overline{(\overline{x} \wedge \overline{y})}$, which implies $\overline{x} \wedge \overline{y} \leq \overline{(x \vee y)}$. ☐

Proposition III.95. *Let L be a complemented lattice.*
If the complement is an involution and satisfies either of the de Morgan laws, then L is an ortholattice.

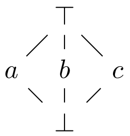
Proof. Assume the de Morgan law $\overline{(x \vee y)} = \overline{x} \wedge \overline{y}$ holds. Assume $x \leq y$. Then $x \vee y = y$, so

$$\overline{y} = \overline{(x \vee y)} = \overline{x} \wedge \overline{y}$$

meaning $\overline{y} \leq \overline{x}$. ☐

The requirement that the complement be an involution is important. There are lattices in which the de Morgan laws hold that are not ortholattices.

Example

The lattice $\mathbf{M}_3 =$  admits a complementation $'$ such that $a' = b$, $b' = c$ and $c' = a$ that is clearly not an orthocomplementation, but does satisfy the de Morgan laws.

TODO Ockham algebras, De Morgan algebras, Kleene algebras, Stone algebras.

2.3.2 Heyting algebras

2.3.2.1 Pseudocomplementation

Let S be a bounded $\wedge$-semilattice and $x \in S$. The pseudocomplement of x in S is defined as

$$x^* \coloneqq \max \{y \in S \mid x \wedge y = \bot\},$$

if the maximum exists.
If for all $x \in S$ the pseudocomplement exists, we say S is pseudocomplemented.

Lemma III.96.

2.3.3 Boolean lattices

A distributive complemented lattice is called a Boolean lattice or Boolean algebra.

We will use $\overline{x}$ to denote the (necessarily unique, III.89) completement of x.

Lemma III.97. *Let L be a Boolean lattice and $x \in L$. Then $\{x\}^{\perp} = {\downarrow}\overline{x}$.*

Proof. From the requirement $x \wedge \overline{x} = \bot$, we see that $\overline{x} \in \{x\}^{\perp}$. Now $\{\overline{x}\} \subseteq \{x\}^{\perp}$ implies $\downarrow\overline{x} \subseteq \downarrow\{x\}^{\perp} = \{x\}^{\perp}$. For the last equality we have used the fact that $\{x\}^{\perp}$ is downwards closed (in fact even an ideal), see III.84.
Now we prove the other inclusion: $\{x\}^{\perp} \subseteq \downarrow\overline{x}$. Take $y \in \{x\}^{\perp}$. We just need to show that $y \leq \overline{x}$, or, equivalently, $y = y \wedge \overline{x}$. Indeed

$$y = y \wedge \top = y \wedge (x \vee \overline{x}) = (y \wedge x) \vee (y \wedge \overline{x}) = \bot \vee (y \wedge \overline{x}) = y \wedge \overline{x}.$$

□

Corollary III.97.1. *Let L be a Boolean lattice and $x, y \in L$. Then the following are equivalent:*

1. $x \leq y$;
2. $x \wedge \overline{y} = \bot$;
3. $\overline{x} \vee y = \top$.

Corollary III.97.2. *Every Boolean lattice is an ortholattice.*

Proof. By III.89 we know that a Boolean lattice is uniquely complemented, so its complement is an involution by III.91. We just need to check the complementation reverses order.
Let $x \leq y$. Then $\overline{y} \wedge x \leq \overline{y} \wedge y = \bot$, so $\overline{y} \wedge x = \bot$ and thus $\overline{y}$ is disjoint from x. Then $\overline{y} \leq \overline{x}$ follows from III.97. □

Corollary III.97.3. *The laws of de Morgan hold in Boolean lattices.*

Lemma III.98. *Let L be a Boolean lattice and $x, y \in L$. Then*

1. $y = (x \vee y) \wedge (\overline{x} \vee y)$;
2. $y = (x \wedge y) \vee (\overline{x} \wedge y)$;
3. $x \wedge y = x \wedge (\overline{x} \vee y)$;
4. $x \vee y = x \vee (\overline{x} \wedge y)$.

Proof. (1, 2) We calculate

$$y = y \wedge \top = y \wedge (x \vee \overline{x}) = (x \vee y) \wedge (\overline{x} \vee y).$$

The second statement is dual.
(3, 4) We calculate $x \wedge (\overline{x} \vee y) = (x \wedge \overline{x}) \vee (x \wedge y) = \bot \vee (x \wedge y) = x \wedge y$. The fourth statement is dual. □

Lemma III.99. *Let L be a Boolean lattice. Then*

1. $\mathcal{A}(L) = \mathcal{J}(L)$;
2. $\mathcal{CA}(L) = \mathcal{M}(L)$;

Proof. We have the inclusion $\subseteq$ from III.61. Assume x is an atom. Take $y \in \downarrow x \setminus \{x\}$, which is non-empty due to the existence of $\bot \neq x$. In fact we need to prove $y = \bot$. By III.98 we have $x = x \vee y = (x \wedge \overline{y}) \vee y$. By join-irreducibility this means that $x = x \wedge \overline{y}$ (because $x \neq y$ by assumption). So $x \leq \overline{y}$, meaning $y = x \wedge y \leq \overline{y} \wedge y = \bot$. □

Lemma III.100. *Let L be a lattice, $x \in L$ and $A \subseteq L$. Then*

1. $\uparrow x \wedge A^u = (x \wedge A)^u$*;*

2. $\downarrow x \vee A^l = (x \vee A)^l$*.*

In any lattice we have the inclusions $x \wedge A^u \subseteq (x \wedge A)^u$, $x \wedge A^l \subseteq (x \wedge A)^l$, $x \vee A^u \subseteq (x \vee A)^u$ and $x \vee A^l \subseteq (x \vee A)^l$. See III.69.

Proof. Because of III.69, we have $x \wedge A^u \subseteq (x \wedge A)^u$, so $\uparrow x \wedge A^u \subseteq \uparrow(x \wedge A)^u = (x \wedge A)^u$.
We just need to prove $\uparrow x \wedge A^u \supseteq (x \wedge A)^u$. Take some $z \in (x \wedge A)^u$, we need to find a $y \in x \wedge A^u$ such that $y \leq z$. We claim that $y = x \wedge (\overline{x} \vee z)$ is such a y. We just need to verify that $y \leq z$ and $\overline{x} \vee z \in A^u$.
The first claim is immediate from $y = x \wedge (\overline{x} \vee z) = x \wedge z$, using III.98.
For the second claim, take an arbitrary $a \in A$. Then $x \wedge a \leq z$ by definition and so $\overline{x} \vee z \geq \overline{x} \vee (x \wedge a) = \overline{x} \vee a \geq a$. $\square$

Corollary III.100.1 (Infinite distributive laws)**.** *Let L be a Boolean lattice. Then for all $x \in L$ and all $A \subseteq L$:*

1. if $\bigvee A$ exists, then $x \wedge \bigvee A = \bigvee x \wedge A$;

2. if $\bigwedge A$ exists, then $x \vee \bigwedge A = \bigwedge x \vee A$.

Thus any complete Boolean lattice is infinitely distributive.

The inequalities $x \wedge \bigvee A \geq \bigvee x \wedge A$ and $x \vee \bigwedge A \leq \bigwedge x \vee A$ hold in any lattice, see III.69.1.

Proof. We calculate

$$\begin{aligned} x \wedge \sup(A) &= x \wedge (A^u \cap A^{ul}) \\ &\subseteq (x \wedge A^u) \cap (x \wedge A^{ul}) \\ &\subseteq (x \wedge A)^u \cap (x \wedge A^u)^l = (x \wedge A)^u \cap (\uparrow(x \wedge A^u))^l = (x \wedge A)^u \cap (x \wedge A))^{ul} = \sup(x \wedge A). \end{aligned}$$

We have used that $x \wedge (A^{\cap} A^{ul})$ is an image of the function $y \mapsto x \wedge y$ for the first inclusion and III.69 for the second. We have also used III.28 for $(x \wedge A^u)^l = (\uparrow(x \wedge A^u))^l$.
The left-hand set is a singleton. The right is either a singleton or empty. Thus both sets are singletons with the same content.
Point (2) is dual. $\square$

Proposition III.101. *Let L be a Boolean lattice with complement and $a, b \in L$. Then $[a, b]$ is a Boolean lattice with complementation $x \mapsto \widetilde{x} = (\overline{x} \wedge b) \vee a$.*

Proof. Clearly $[a, b]$ inherits distributivity from L. All we need to show is that for all $x \in [a, b]$ the complement of x in $[a, b]$ is $\widetilde{x}$. We calculate

$$x \wedge \widetilde{x} = x \wedge ((\overline{x} \wedge b) \vee a) = (x \wedge (\overline{x} \wedge b)) \vee (x \wedge a) = ((x \wedge \overline{x}) \wedge b) \vee (x \wedge a) = (\bot \wedge b) \vee a = \bot \vee a = a$$
$$x \vee \widetilde{x} = x \vee ((\overline{x} \wedge b) \vee a) = ((x \vee \overline{x}) \wedge (x \vee b)) \vee a = (\top \wedge b) \vee a = b \vee a = b.$$

$\square$

Proposition III.102. *An algebra of sets is a Boolean algebra with as top the unit Ω, as bottom the empty set $\emptyset$ and as complement $A \mapsto A^c = \Omega \setminus A$.*

2.3.3.1 Duality and complementation

TODO: dual expression can be obtained by taking the complement? Dual statement of equality is equality of complements?
TODO: general for ortholattices?

2.3.3.2 Boolean rings

TODO: define ring above!

Let $(R, +, \cdot, 0, 1)$ be a ring. We call R a Boolean ring if each $x \in R$ is idempotent: $x^2 = x$.

Lemma III.103. *Let* $(R, +, \cdot, 0, 1)$ *be a Boolean ring. Then*

1. $x = -x$ *for all* $x \in R$*;*
2. R *is commutative.*

Proof. (1) We calculate

$$x + x = (x + x)^2 = x^2 + x^2 + x^2 + x^2 = x + x + x + x.$$

Subtracting $x + x$ from both sides gives $x + x = 0$.
(2) Let $x, y \in R$. We then have

$$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y.$$

So $xy = -yx = yx$, using point (1). □

Lemma III.104. *Every subring of a Boolean ring is a Boolean ring.*

Proposition III.105. *Let* R *be a set and* $0, 1 \in R$.

1. *If* $(R, \vee, \wedge, 0, 1, c)$ *is a Boolean algebra, then* $(R, +, \cdot, 0, 1)$ *is a Boolean ring with operations defined by*

$$\begin{aligned} + &: (x, y) \mapsto (x \wedge y^c) \vee (x^c \wedge y) \\ \cdot &: (x, y) \mapsto x \wedge y. \end{aligned}$$

2. *If* $(R, +, \cdot, 0, 1)$ *is a Boolean ring, then* $(R, \vee, \wedge, 0, 1, c)$ *is a Boolean algebra with operations defined by*

$$\begin{aligned} \vee &: (x, y) \mapsto x + y - x \cdot y \\ \wedge &: (x, y) \mapsto x \cdot y \\ c &: x \mapsto x^c = 1 + x \end{aligned}$$

Note we need the existence of 1 to define the complement.

Proof. TODO □

TODO: `https://en.wikipedia.org/wiki/Boolean_ring` (Maybe later in ring section?)

2.3.3.3 Identities in Boolean algebras

TODO rewrite!!!!
Given any set U we can form the family $\mathcal{P}(U)$ for which U is a universe set.
The set theoretic operations of union, intersection, difference, symmetric difference and complementation can be restricted to $\mathcal{P}(U)$. In other words $\mathcal{P}(U)$ is closed w.r.t. these operations.

Proposition III.106. *Given a set U, the operations $\cap, \cup, ^c$ form a Boolean algebra with bottom $\emptyset$ and top U: $\forall A, B, C \subset U$: $\mathcal{P}(U)$ is closed under $\cap, \cup, ^c$ and*

Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Identity	$A \cup \emptyset = A$	$A \cap U = A$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Complements	$A \cup A^c = U$	$A \cap A^c = \emptyset$

The two columns are duals of each other.
TODO: distributivity for arbitrary union and intersection.

Corollary III.106.1. *Let $A, B \subseteq U$ be sets. Then*

Idempotency	$A \cup A = A$	$A \cap A = A$
Domination	$A \cup U = U$	$A \cap \emptyset = \emptyset$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$

Proof. Using set theory the proof of these statements is simple. It is also possible to prove the equalities using only the properties of Boolean algebras listed in lemma III.106 and the properties derived here. We only prove the first column. The proof of the second column can be obtained easily by duality.
(1) $A = A \cup (A \cap A^c) = (A \cup A) \cap (A \cup A^c) = (A \cup A) \cap U = A \cup A$.
(2) $U = U \cup (U \cap A) = (U \cup U) \cap (U \cup A) = U \cap (U \cup A) = (A \cup U) \cap U = (A \cup U)$.
(3) $A \cup (A \cap B) = (A \cap U) \cup (A \cap B) = A \cap (U \cup B) = A \cap U = A$.
(4) TODO `https://proofwiki.org/wiki/Operations_of_Boolean_Algebra_are_Associative` □

2.4 Residuated lattices

2.5 Completions

TODO: Dedekind-MacNeille completion.

2.6 Formal concept analysis

`file:///C:/Users/user/Downloads/978-3-540-31881-1.pdf` `file:///C:/Users/user/Downloads/978-3-662-49291-8.pdf`

A context is a triple $\langle G, M, I\rangle$ where G is a set of objects, M is a set of attributes and $I \subseteq G \times M$ is a binary relation.
For $g \in G, m \in M$ we interpret gIm as "the object g has the attribute m".

A concept is a pair $\langle A, B\rangle$ where $A \subset G$ is a set of objects and $B \subset M$ is a set of attributes such that

- $A = \{g \in G \mid \forall m \in B : gIm\}$;
- $B = \{m \in M \mid \forall g \in A : gIm\}$

The letters G and M come from the German: Gegenstände and Merkmale. The I is for "incidence relation" (I think).

2.7 Filters and ideals

Let $(P, \prec)$ be an ordered set. A subset $F \subseteq P$ is called

- an order filter if
  - F is downward directed;
  - F is an up set in P;
- an order ideal if
  - F is upward directed;
  - F is a down set in P.

In addition F is called proper if $F \neq P$.
The set of ideals on P is denoted $\mathcal{I}(P)$ and the set of filters on P is denoted $\mathcal{F}(P)$.

Many authors also require filters and ideals to be non-empty.

Lemma III.107. *Let P be an ordered set and $x \in P$.*

1. *The principal up set $\uparrow x$ is an order filter.*
2. *The principal down set $\downarrow x$ is an order ideal.*

If the order is not a preorder, then a principal filter/ideal may be empty.

Example

Let X be a set.

- The set of finite subsets of X is an ideal.
- The set of countable subsets X is an ideal.
- A subset A of X is called cofinite if $X \setminus A$ is finite. The set of cofinite subsets of X is a filter.
- A subset A of X is called cocountable if $X \setminus A$ is countable. The set of cocountable subsets of X is a filter.

Lemma III.108. *Let P be an ordered set.*

1. *If P has a least element l, then a filter F is proper* if and only if *it does not contain l.*
2. *If P has a greatest element g, then an ideal I is proper* if and only if *it does not contain g.*

Lemma III.109. *Let $(S, \leq)$ be a poset $J \subseteq S$ a subset.*

1. *If S is a $\wedge$-semilattice, then J is a filter* if and only if
   - (a) *J is an up set;*
   - (b) *J is a $\wedge$-subsemilattice.*
2. *If S is a $\vee$-semilattice, then J is an ideal* if and only if
   - (a) *J is a down set;*
   - (b) *J is a $\vee$-subsemilattice.*

Proof. TODO □

Lemma III.110. *Let L be a lattice and $J \subseteq L$ a subset. Then*

1. *J is a filter* if and only if
   - (a) $\forall a, b \in J : \qquad a \wedge b \in J;$
   - (b) $\forall x \in L, \forall b \in J : \quad x \vee b \in J;$
2. *J is an ideal* if and only if
   - (a) $\forall a, b \in J : \qquad a \vee b \in J;$
   - (b) $\forall x \in L, \forall b \in J : \quad x \wedge b \in J.$

2.7.1 Filter bases and subbases

Lemma III.111. *Let $(P, \prec)$ be an ordered set and $\{F_i\}_{i \in I}$ a family of subsets.*

1. *If F_i is a filter for all $i \in I$, then $\bigcap_{i \in I} F_i$ is a filter;*
2. *If F_i is an ideal for all $i \in I$, then $\bigcap_{i \in I} F_i$ is an ideal.*

Let $(P, \prec)$ be an ordered set and $B \subseteq P$ a subset. Then

- the filter generated by B is

$$\mathfrak{F}\{B\} \coloneqq \bigcap \{B \subseteq S \subseteq P \mid S \text{ is a filter}\}$$

  we call B a filter subbasis of F. If B is downward directed, it is called a filter base of F;

- the filter base generated by B as

$$\mathfrak{F}\mathfrak{B}\{B\} \coloneqq \bigcap \{B \subseteq S \subseteq P \mid S \text{ is downward directed}\}.$$

If P is not downward directed, there may not exists any $S \supseteq B$ that is downward directed, in which case $\mathfrak{F}\mathfrak{B}\{B\} = \text{TODO}$.

Proposition III.112. *Let $(P, \prec)$ be an ordered set and $B \subseteq P$ a subset. Then*

1. *if P is downward directed, then $\mathfrak{F}\mathfrak{B}\{B\}$ is a filter base;*
2. *$\mathfrak{F}\{B\} = \uparrow \mathfrak{F}\mathfrak{B}\{B\}$.*

In particular, if S is a filter base, then $\uparrow S$ is a filter.

Lemma III.113. *Let $(P, \prec)$ be an ordered set.*
If P is downward directed and $\exists \bot \in \min(P)$ and $B \subseteq P$ a filter base, then $\mathfrak{F}\{B\}$ is a proper filter if and only if *$\bot \notin B$;*

Proposition III.114. *Let $(P, \prec)$ be an ordered set. Then for all filter bases B, C in P: B is finer than C* if and only if *$\mathfrak{F}\{B\} \supseteq \mathfrak{F}\{C\}$.*

Proposition III.115. *Let S be a $\wedge$-semilattice and $B \subseteq S$ a subset. Then*

1. *$\mathfrak{F}\mathfrak{B}\{B\} = \mathrm{Cl}_\wedge(B)$;*
2. *$\mathfrak{F}\{B\} = \uparrow \mathrm{Cl}_\wedge(B)$.*

2.7.2 Meshing and trace

Lemma III.116. *Let F be a filter on a bounded $\wedge$-semilattice S and $a \subseteq S$ an element. If $\{a\} \# F$, then*

$$F|_a \coloneqq \{x \wedge a \mid x \in F\}$$

is a filter. It is in particular a filter on $\downarrow a$.

The filter $F|_a$ of III.116 is called the trace filter of F on a.

2.7.3 Ordering filters and ideals

Proposition III.117 (The lattices of filters and ideals)**.** *Let P be an ordered set and I an index set.*

1. *If P is a $\wedge$-semilattice, the set $\mathcal{F}(P)$ of filters on P is a complete bounded sublattice of $(\mathcal{P}(P), \subseteq)$, with top P and bottom $\{\top\}$. Let $\{F_i\}_{i \in I} \in \mathcal{F}(L)^I$ be a set of filters. Then*

$$\bigwedge \{F_i\}_{i\in I} = \bigcap_{i \in I} F_i \qquad \textit{and} \qquad \bigvee \{F_i\}_{i \in I} = \mathrm{Cl}_\wedge \left(\bigcup_{i \in I} F_i \right).$$

2. *If P is a $\vee$-semilattice, the set $\mathcal{I}(P)$ of ideals on P is a complete bounded sublattice of $(\mathcal{P}(P), \subseteq)$, with top P and bottom $\{\bot\}$. Let $\{J_i\}_{i \in I} \in \mathcal{F}(L)^I$ be a set of ideals. Then*

$$\bigwedge \{J_i\}_{i\in I} = \bigcap_{i \in I} J_i \qquad \textit{and} \qquad \bigvee \{J_i\}_{i \in I} = \mathrm{Cl}_\vee \left(\bigcup_{i \in I} J_i \right).$$

Proof. This is an application of II.66.1 (TODO show this is the correct residual). We just need to prove that $\uparrow \mathrm{Cl}_\wedge \left(\bigcup_{i \in I} F_i \right) = \mathrm{Cl}_\wedge \left(\bigcup_{i \in I} F_i \right)$ TODO is distributive □

Proposition III.118. *Let L be a $\wedge$-semilattice and $\{F_i\}_{i \in I} \in \mathcal{F}(L)^I$ be set of filters. Then $\bigvee \{F_i\}_{i \in I}$ is a proper filter* if and only if *$\{F_i\}_{i \in I} \# \{F_i\}_{i \in I}$.*

In particular, for $F, G \in \mathcal{F}(L)$, we have that $F \vee G$ is a proper filter if and only if $F \# G$.

2.7.3.1 Maximal filters and ideals

Let $(P, \prec)$ be an ordered set.

- A maximal filter or ultrafilter is a coatom in $\mathcal{F}(P)$.
- A maximal ideal is a coatom in $\mathcal{I}(P)$.

We denote the set of ultrafilters on P as $\mathbb{U}(P)$.

Proposition III.119. *Let S be a bounded $\wedge$-semilattice. A filter $F \in \mathcal{F}(S)$ is an ultrafilter* if and only if *$\forall x \in S$:*

$$x \notin F \iff \exists y \in F : x \wedge y = \bot.$$

Proof. First assume F is an ultrafilter. Then for all $x \in F$, the filter $\mathfrak{F}\big\{F \cup \{x\}\big\}$ is either F or S. In other words, $x \notin F$ is equivalent to $\mathfrak{F}\big\{F \cup \{x\}\big\} = S$. Now a filter is equal to S iff $\bot$ is an element and, by III.115 we have that $\bot \in \mathfrak{F}\big\{F \cup \{x\}\big\}$ iff it is the intersection of x and some $y \in F$. This concludes the forward direction.
Now for the other direction, take some filter F satisfying the condition. Then F is automatically proper ($\bot$ is not in F). Assume $G \supseteq F$ is another proper filter. Assume, towards a contradiction, that $G \setminus F \neq \emptyset$. Then take $x \in G \setminus F$. By hypothesis there exists a $y \in F$ (and thus also in G) such that $x \wedge y = \bot$, meaning $\bot \in G$ by $\wedge$-closure. Thus G is not proper. ☐

Proposition III.120 (Ultrafilter lemma)**.** *Every proper filter is contained in an ultrafilter.*

Corollary III.120.1. *Let S be a bounded $\wedge$-semilattice. Then $\mathcal{F}(S)$ is a coatomistic lattice.*

In other words, for every $F \in \mathcal{F}(S)$: we have

$$F = \bigcap_{\substack{U \in \mathbb{U}(S) \\ U \geq F}} U.$$

Proof. TODO! Need Boolean?????? ☐

2.7.3.2 Prime filters and ideals

Let $(P, \prec)$ be an ordered set.

- A prime filter is a meet-irreducible element in $\mathcal{F}(L)$.
- A prime ideal is a meet-irreducible element in $\mathcal{I}(L)$.

Proposition III.121. *Let $(P, \prec)$ be an ordered set.*

1. *A filter F is prime* if and only if *F^c is an ideal.*
2. *An ideal I is prime* if and only if *I^c is a filter.*

Proof. ☐

Lemma III.122. *Every maximal is prime. If Boolean, then every prime is maximal.*

Proof. TODO III.61 ☐

Proposition III.123. *Let L be a lattice.*

1. *A filter $F \in \mathcal{F}(L)$ is prime* if and only if
   - (a) *F is proper;*
   - (b) *$\forall x, y \in L$: $x \vee y \in F \implies x \in F$ or $y \in F$.*
2. *An ideal $I \in \mathcal{I}(L)$ is prime* if and only if
   - (a) *I is proper;*
   - (b) *$\forall x, y \in L$: $x \wedge y \in I \implies x \in I$ or $y \in I$.*

If L is only a $\wedge$-semilattice (resp. $\vee$-semilattice), then only the direction $\Rightarrow$ holds in general.

Proof. (1) Assume F is a prime filter. Then $F \neq \top = \mathcal{P}(L)$, so it is proper. Now assume $x \vee y \in F$, meaning $\uparrow x \vee y \subseteq F$. Thus

$$F = \mathrm{Cl}_\wedge(F \cup \uparrow(x \vee y)) = (F \vee \uparrow x) \cap (F \vee \uparrow y).$$

Because F is prime, we have $F = F \vee \uparrow x$ or $F \vee \uparrow y$.
Now assume TODO `http://www.ascent-journals.com/IJPEM/Vol3No3/1-sagi.pdf` using III.58. ☐

2.7.4 Free and principal filters

Let L be a complete lattice and $F \subseteq P$ a filter.

- The kernel of the filter F is $\ker F := \bigwedge F$.
- A filter is called principal if $\ker F \in F$;
- A filter is called free if $\ker F = \bot$.

The set of principal filters in P is denoted $\mathcal{F}_*(P)$ and the set of free filters in P is denoted $\mathcal{F}_0(P)$.

The only filter that is both principal and free is L. A proper filter cannot be both principal and free.

Lemma III.124. *Let L be a lattice and F a principal filter in L. Then $F = \uparrow \ker F$.*

Lemma III.125. *Let L be a finite lattice. Then every filter in L is principal.*

Proof. Every filter F in a finite lattice is finite, so $\bigwedge F$ is a finite meet, and thus must be in F. ☐

Proposition III.126. *Let X be a set and F a free filter in $\mathcal{F}_0(\mathcal{P}(X))$. If $A \in F$, then F is finer than the cofinite filter of A.*

Proof. As filters are upwards closed, "F finer than the cofinite filter of A" means the cofinite filter of A is a subset of F. So take some cofinite subset C of A, such that $A \setminus C = \{x_1, \ldots, x_n\}$ is finite. Now for each x_i we can find a set $f_i \in F$ such that $x_i \notin f_i$, because if this was not possible, then $x_i \in \bigcap F$, but $\bigcap F = \emptyset$ due to F being free. Also $\bigcap_{i=1}^n f_i \in F$, because F is closed under finite intersections.
So $(A \setminus C) \cap (\bigcap_{i=1}^n f_i) = \emptyset$, or equivalently $\bigcap_{i=1}^n f_i \subseteq X \setminus (A \setminus C)$, meaning $X \setminus (A \setminus C) \in F$ and thus $C = A \cap (X \setminus (A \setminus C)) \in F$. ☐

Corollary III.126.1. *A filter of subsets of X is free* if and only if *it is finer than the cofinite filter of X.*

Proposition III.127. *Let $(P, \prec)$ be a complete ordered set (TODO: do we need Boolean lattice??) and $F \subseteq P$ a filter. Then there exists a unique pair of filters F_* and F_0 such that F_* is principal, F_0 is free and*

$$F = F_* \wedge F_0 \qquad P = F_* \vee F_0.$$

We call F^* the principal part and F^0 the free part of the filter.

Proof. TODO $\square$

Chapter 3

The poset of subsets

Let U be a set. Then $(\mathcal{P}(U), \subseteq)$ is a partially ordered set. We call U the universe of this poset.

Any family of sets $\mathcal{F}$ may be seen as a subset of the poset with universe $\bigcup \mathcal{F}$.

Proposition III.128. *Let $\mathcal{F}$ be a family of sets. Then $(\mathcal{F}, \subseteq)$ is a poset.*
Conversely, every poset $(P, \preceq)$ is isomorphic to $(\mathcal{F}, \subseteq)$ for some family of sets $\mathcal{F}$ with universe P.

Proof. This is just a reformulation of III.42. ☐

Corollary III.128.1. *Let $\mathcal{F}$ be a family of subsets of U. If*

- *$\mathcal{F}$ is closed under arbitrary intersections; and*
- *$U \in \mathcal{F}$;*

then $(\mathcal{F}, \subseteq)$ is a complete lattice.
Conversely, every complete lattice is isomorphic to such a lattice.

Note that in general the join in such a lattice is not given by the union, $\bigvee \neq \bigcup$!

Proof. Such a family of sets is a complete lattice by III.74.
For the converse, the isomorphism is given by $x \mapsto \downarrow x$, as in III.42. We just need to show that the meet translates to the intersection, i.e. $\downarrow \bigwedge Q = \bigcap_{x \in Q} \downarrow x$. We calculate, using III.28.1 and I.41:

$$\downarrow \bigwedge Q = \downarrow \max(Q^l) = \downarrow Q^l = \downarrow \bigcap_{x \in Q} \downarrow x \subseteq \bigcap_{x \in Q} \downarrow x.$$

For the other inclusion we just note that a lattice order is in particular a preorder and so we can use **??**. ☐

A family of sets $\mathcal{F}$ satisfying the hypothesis of III.128.1, i.e.

- $\mathcal{F}$ is closed under arbitrary intersections; and
- $U \in \mathcal{F}$;

is called a Moore family or topped intersection structure.

TODO: join is given by union if directed?? (+CPO)??

3.1 The complete Boolean lattice of subsets

Lemma III.129. *Let U be a universe set. Consider the poset $(\mathcal{P}(U), \subseteq)$ and let $\mathcal{F} \subseteq \mathcal{P}(U)$. Then*

1. $\sup(\mathcal{F}) = \bigcup \mathcal{F}$;
2. $\inf(\mathcal{F}) = \bigcap \mathcal{F}$.

(Assuming relativised intersection TODO!)

Corollary III.129.1. *Let U be a universe set. Then $(\mathcal{P}(U), \subseteq)$ is a bounded, complete, distributive lattice with top U and bottom $\emptyset$.*

Proof. TODO ref distributivity. □

3.1.1 Complementation

Let U be a universe and $A \subseteq U$. The complement of A w.r.t. U is

$$A^c := U \setminus A.$$

Lemma III.130. *The complement ^{c} is a lattice-theoretical complement.*

Proof. For all $A \subseteq U$ we have $A \cup A^c = U$ and $A \cap A^c = \emptyset$ from I.19. □

Corollary III.130.1. *Let U be a universe set. Then $(\mathcal{P}(U), \subseteq)$ is a Boolean lattice.*

In particular de Morgan's laws can be formulated in this context as:

Proposition III.131. *Let U, A, B be sets, then*

$$(A \cup B)^c = (A^c) \cap (B^c);$$
$$(A \cap B)^c = (A^c) \cup (B^c).$$

Where complementation is with respect to U.
This can be extended to arbitrary families of sets:

$$\left(\bigcup \mathcal{E}\right)^c = \bigcap \{A^c \mid A \in \mathcal{E}\}$$
$$\left(\bigcap \mathcal{E}\right)^c = \bigcup \{A^c \mid A \in \mathcal{E}\}$$

where $\mathcal{E}$ is a family of sets.

3.1.2 Expressing set theoretic operations with $\cup, \cap, ^c$

Proposition III.132. *Let $A, B \subseteq U$ be sets. Then*

1. $A \setminus B = A \cap B^c$;
2. $A \,\Delta\, B = (A \cup B) \cap (A^c \cup B^c)$.

Corollary III.132.1. *Let $A, B \subseteq U$ be sets. Then*

1. $A \setminus B = B^c \setminus A^c$;
2. $A \,\Delta\, B = A^c \,\Delta\, B^c$;
3. $A \,\Delta\, A^c = U$.

3.1.3 Filters and ideals

Proposition III.133. *Let U be a set. If U is infinite, then there exists a non-principal ultrafilter in $\mathcal{FP}(U)$.*

This proposition assumes the ultrafilter lemma III.120.

Proof. Consider the cofinite filter $F = \{A \subseteq U \mid U \setminus A \text{ is finite}\}$. By the ultrafilter lemma, III.120, we can extend F to an ultrafilter $\mathcal{U}$. Now $\mathcal{U}$ does not contain any sets of the form $\{x\}$ with $x \in U$. If it did, then it would contain both $\{x\}$ and $U \setminus \{x\}$, meaning it would also contain $\emptyset$ and thus would not be proper and in particular not an ultrafilter. $\square$

Proposition III.134. *Let $f : X \to Y$ be a function between sets and $F \in \mathcal{FP}(X)$ a filter. Then $\uparrow f[F]$ is a filter on $\mathcal{P}(Y)$. If F is an ultrafilter on $\mathcal{P}(X)$, then $\uparrow f[F]$ is an ultrafilter on $\mathcal{P}(Y)$.*

We call $\uparrow f[F]$ the image filter of F under f.

TODO principal filter $\dot{x}$.

3.2 Grills

TODO for general filters.

Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}(X)$. The grill of $\mathcal{A}$ is defined as the polar $\mathcal{A}^{\#}$.

Lemma III.135. *Let X be a set, $\mathcal{P}(X)$ be ordered by inclusion, $A, B \in \mathcal{P}(X)$ and $\mathcal{A} \subseteq \mathcal{P}(X)$. Then*

1. $\uparrow A \,\# = A \,\#$;
2. $\uparrow \mathcal{A}^{\#} = \mathcal{A}^{\#} = (\uparrow \mathcal{A})^{\#}$;
3. $((\mathcal{A}^{\#})^{\#})^{\#} = \mathcal{A}^{\#}$;
4. $(\mathcal{A}^{\#})^{\#} = \uparrow \mathcal{A}$.

Proof. (1) If B meshes with A, then any superset of B also meshes with A. The opposite inclusion is given by **??**.
(2) We calculate using II.63.1

$$\uparrow \mathcal{A}^{\overrightarrow{\#}} = \uparrow \bigcap_{A \in \mathcal{A}} A \,\# = \uparrow \bigcap_{A \in \mathcal{A}} \uparrow A \,\# = \bigcap_{A \in \mathcal{A}} \uparrow A \,\# = \bigcap_{A \in \mathcal{A}} A \,\# = \mathcal{A}^{\overrightarrow{\#}}$$

TODO
(3) TODO (general for polars using $\#^{\mathrm{T}} = \#$ (?)).
(4) By II.28.1 this is a consequence of (2) and (3). $\square$

3.3 Indicator functions

The family $\mathcal{P}(U)$ can be bijectively mapped to the family of functions $(U \to \{0, 1\})$, by mapping each set A to its indicator function χ_A.

Let U be a set and $A \subseteq U$. The indicator function or characteristic function of A as an element of $\mathcal{P}(U)$ is defined as

$$\chi_A : U \to \{0,1\} : x \mapsto \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Lemma III.136. *Let A, B be elements of $\mathcal{P}(U)$. Then*

1. $\chi_{A\cap B} = \min\{\chi_A, \chi_B\} = \chi_A \cdot \chi_B$;
2. $\chi_{A\cup B} = \max\{\chi_A, \chi_B\} = \chi_A + \chi_B - \chi_A \cdot \chi_B$;
3. $\chi_{A^c} = \underline{1} - \chi_A$; *and thus* $\chi_A + \chi_{A^c} = \underline{1}$;
4. $$\begin{aligned} \chi_{A\Delta B} &= \chi_A + \chi_B - \underline{2} \cdot \chi_A \cdot \chi_B \qquad\qquad . \\ &= |\chi_A - \chi_B| \\ &= \chi_A + \chi_B \mod 2 \coloneqq \begin{cases} 1 & (\chi_A + \chi_B = 1) \\ 0 & \textit{(else)} \end{cases} \end{aligned}$$

Where all operations are defined point-wise.

Proposition III.137. *The indicator functions define a bijection between $\mathcal{P}(U)$ and $(U \to \{0,1\})$.*

3.4 Families of sets

3.4.1 Algebra and lattice structures

Let U be a set. A family of subsets of U is an element $\mathcal{F} \subseteq \mathcal{P}(U)$, i.e. $\mathcal{F} \in \mathcal{P}^2(U)$.
A type of (partial) subsetalgebra $\boldsymbol{T}$ is a set of set theoretical operations.

Most commonly we have

$$\boldsymbol{T} \subseteq \{\cup, \cap, \bigcup_\omega, \bigcap_\omega, \bigcup, \bigcap, \biguplus, \bigcup_{\text{monotone}}, \bigcap_{\text{monotone}}, \setminus, {}^c, \Delta\}$$

Proposition III.138. *Let U be a set and $\boldsymbol{T}$ a type of (partial) subsetalgebra. Then the $\boldsymbol{T}$-subalgebras of $\mathcal{P}(U)$ form a $\wedge$-subsemilattice of $\mathcal{P}^2(U)$.*

3.4.2 Closure under set operations

We say a family of sets $\mathcal{F}$ is closed under an operation if the result of this operation acting on sets in $\mathcal{F}$ is again in $\mathcal{F}$.

A family of sets $\mathcal{F} \subseteq \mathcal{P}(U)$ is called

- closed under complementation if $A^c \in \mathcal{F}$ for all $A \in \mathcal{F}$;
- closed under relative complements if $A \setminus B \in \mathcal{F}$ for all $A \supset B \in \mathcal{F}$;

- closed under set difference if $A \setminus B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$;

and

- closed under finite unions if $A \cup B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$;
- closed under finite intersections if $A \cap B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$;
- closed under disjoint unions if $\biguplus_{i\in I} A_i \in \mathcal{F}$ for any indexed family of disjoint sets $\{A_i\}_{i\in I}$;
- closed under countable monotone unions if $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ for any indexed family of sets $\{A_i\}_{i\in\mathbb{N}}$ such that $i \leq j \implies A_i \subseteq A_j$;
- closed under countable monotone intersections if $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$ for any indexed family of sets $\{A_i\}_{i\in\mathbb{N}}$ such that $i \leq j \implies A_i \supseteq A_j$.

Our definition of closure under relative complements is not the same as closure under set difference! Our definition is non-standard as they are usually taken to be the same thing.

For all these set-theoretical operations, "closed under X" is the same as "being X-subalgebra".

3.4.2.1 Complementation, relative complementation and set difference

In general the notions of closure under complementation, relative complementation and set difference are distinct, the only implication being from set difference to relative complementation.

Lemma III.139. *Let $\mathcal{F}$ be a family of sets that is closed under finite (disjoint) unions. Then*

$$\begin{tikzcd} \mathcal{F} \text{ is closed under set differences} \arrow[dr, Rightarrow] & \\ \mathcal{F} \text{ is closed under complementation} \arrow[r, Rightarrow] & \mathcal{F} \text{ is closed under relative complements.} \end{tikzcd}$$

Proof. Assume $A \supset B$, then $A \setminus B = A^c \uplus B$. This is a disjoint union. ☐

Lemma III.140. *Let $\mathcal{F}$ be a family of sets that is closed under finite intersections. Then*

$$\text{closure under complements} \quad\implies\quad \text{closure under set difference} \quad\iff\quad \text{closure under relative complements.}$$

All three are equivalent if $\mathcal{F}$ contains the universe set.

Proof. $A \setminus B = A \cap B^c$ and $A \setminus B = A \setminus (B \cap A)$. ☐

Lemma III.141. *Let $\mathcal{F}$ be a family of sets. Then*

$$\text{closure under set differences} \quad\implies\quad \text{closure under finite intersections.}$$

Proof. $A \cap B = A \setminus (A \setminus B)$. ☐

Any non-empty family of sets that is closed under set differences is also an order-theoretic ring.

3.4.2.2 Types of closure for unions and intersections

Lemma III.142. *Let $\mathcal{F}$ be a family of sets. Then we have the following implications for closure under unions:*

$$\begin{tikzcd} & & \textit{countable disjoint } \uplus \\ \textit{arbitrary } \cup \arrow[r, Rightarrow] & \textit{countable } \cup \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \textit{countable monotone } \cup \\ & & \textit{finite } \cup \end{tikzcd}$$

and for closure under intersections:

$$\begin{tikzcd} & & \textit{countable monotone } \cap \\ \textit{arbitrary } \cap \arrow[r, Rightarrow] & \textit{countable } \cap \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\ & & \textit{finite } \cap \end{tikzcd}$$

The implications in III.142 for unions can be simplified, if $\mathcal{F}$ is closed under relative complementation:

Lemma III.143. *Let $\mathcal{F}$ be a family of sets that is closed under relative complementation. Then we have the following implications for closure under unions:*

$$\begin{tikzcd} \textit{arbitrary } \cup \arrow[r, Rightarrow] & \textit{countable } \cup \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \textit{countable disjoint } \uplus \arrow[r, Rightarrow] & \textit{countable monotone } \cup \\ & & \textit{finite } \cup & \end{tikzcd}$$

Proof. We need to prove that closure under countable disjoint unions implies closure under countable monotone unions.
Assume $\mathcal{F}$ closed under countable disjoint unions. Let $\{A_i\}_{i\in\mathbb{N}}$ be a monotonically increasing family of sets. Then we can recursively define a family $\{D_i\}_{i\in\mathbb{N}}$ by $D_0 = \emptyset$ and

$$D_{i+1} = A_{i+1} \setminus D_i.$$

This is allowed because $A_{i+1} \supset A_i \supset D_i$. By induction we see that $\{D_i\}_{i\in\mathbb{N}}$ is a disjoint family and has the same union as $\{A_i\}_{i\in\mathbb{N}}$. ☐

These implications can be further simplified, if $\mathcal{F}$ is closed under set differences:

Lemma III.144. *Let $\mathcal{F}$ be a family of sets that is closed under set differences. Then we have the following implications for closure under unions:*

$$\begin{tikzcd} \textit{arbitrary } \cup \arrow[r, Rightarrow] & \textit{countable } \cup \arrow[r, Leftrightarrow] & \textit{countable disjoint } \uplus \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \textit{countable monotone } \cup \\ & & & \textit{finite } \cup \end{tikzcd}$$

Proof. We just need to prove that closure under countable disjoint unions implies closure under countable unions.
This can be done with the same construction of $\{D_i\}_{i\in\mathbb{N}}$ as before because now the assignment $D_{i+1} = A_{i+1} \setminus D_i$ works for arbitrary families $\{A_i\}_{i\in\mathbb{N}}$, not just monotone ones. ☐

3.4.3 Filters

Proposition III.145. *Let X be a set and $F \in \mathcal{FP}(X)$ a proper filter. Then $F \subseteq F^{\overrightarrow{\#}}$.*

Proof. Every set in F meshes with every other set in F. If not, i.e. $f, g \in F$ such that $f \cap g = \emptyset$, then $\emptyset \in F$ and F would not be proper. ☐

3.4.3.1 Ultrafilters

Let X be a set and $F \in \mathcal{FP}(X)$ a filter. We call F an ultrafilter if $F = F^{\overrightarrow{\#}}$.

TODO characterisations. Ultrafilters are maximally fine (-¿ Zorns lemma).

Proposition III.146. *An ultrafilter is either principal or free.*

Proposition III.147. *An ultrafilter is sequential* if and only if *it is principal.*

3.4.4 Monotone classes

A family of sets $\mathcal{F}$ is called a monotone class if it is closed under both countable monotone unions and countable monotone intersections.

3.4.4.1 Dynkin systems

A Dynkin system of sets (also known as a λ-system or d-system) is a pair of a set Ω and a collection of sets $D \subset \mathcal{P}(\Omega)$ such that

- $\Omega \in D$;
- if $A \in D$, then $A^c \in D$;
- if $(A_i)_{i\in\mathbb{N}}$ is a countable sequence of pairwise disjoint sets in D, then $\biguplus_{i=1}^{\infty} A_i \in D$.

Lemma III.148. *A pair of a set Ω and a family of subsets D is a Dynkin system* if and only if

- *$\Omega \in D$;*
- *D is closed under relative complements: if $A, B \in D$ and $A \supset B$, then $A \setminus B \in D$;*
- *D is closed under countable monotone unions.*

Proof. Call the original set of axioms Ax1 and this set of axioms Ax2.
$\boxed{\text{Ax1} \implies \text{Ax2}}$ Point (2) follows from III.139 and point (3) follows from III.143.
$\boxed{\text{Ax2} \implies \text{Ax1}}$ Point (2) follows immediately. For point (3): let A, B be disjoint sets. Then $A^c \supset B$ and so $A \cup B = (A^c \setminus B)^c \in D$, meaning D is closed under finite unions of disjoint sets. Now let $(A_i)_{i\in\mathbb{N}}$ be a countable sequence of pairwise disjoint sets. Then

$$\biguplus_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty}\left(\biguplus_{j=1}^{i} A_j\right)$$

which is a countable monotone union. □

Lemma III.149. *A Dynkin system is a monotone class.*

Proof. If $\bigcap_{i=1}^{\infty} A_i$ is a countable monotone intersection, then $\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c$ is a countable monotone union. □

3.4.5 π-systems

TODO: directed set!

A π-system is a collection of sets P such that

- P is not empty;
- if $A, B \in P$, then $A \cap B \in P$.

TODO $\{\cap\}$-subalgebra of $\mathcal{P}(U)$.

Lemma III.150. *Let $\mathcal{F}$ be a collection of sets. If $\mathcal{F}$ is closed under set differences, then it is a π-system.*

Proof. By III.141. ☐

For π-systems the intersection implications in III.142 reduce to:

Lemma III.151. *Let $\mathcal{F}$ be a π-system. Then we have the following implications for closure under intersections:*

$$\textit{arbitrary} \cap \quad \implies \quad \textit{countable} \cap \quad \iff \quad \textit{countable monotone} \cap$$

Proof. We need to prove that closure under countable monotone intersections implies closure under countable intersections.
Assume $\mathcal{F}$ is a π-system closed under countable monotone intersections. Let $\{A_i\}_{i\in\mathbb{N}}$ be an indexed family of sets. Then we can define the family $\{B_i\}_{i\in\mathbb{N}}$ recursively by $B_1 = A_1$ and

$$B_{i+1} = B_i \cap A_{i+1}.$$

By induction we see that $\{B_i\}_{i\in\mathbb{N}}$ is monotone and has the same intersection as $\{A_i\}_{i\in\mathbb{N}}$. ☐

3.4.5.1 Intersections structures

Let $\mathcal{F} \subseteq \mathcal{P}(U)$ be a family of sets. We call $\mathcal{F}$ an intersection structure if it is closed under arbitrary intersections.
If an intersection structure contains the universe set, it is called a topped intersection structure or closure system.

TODO $\{\cap_\omega\}$-subalgebra of $\mathcal{P}(U)$.

3.4.5.2 Rings

Let U be a set. Then

- an order-theoretic ring on U is a $\{\cup, \cap\}$-subalgebra of $\mathcal{P}(U)$;
- a measure-theoretic ring on U is a $\{\cap, \setminus\}$-subalgebra of $\mathcal{P}(U)$;
- a σ-ring in U is a $\{\cap, \cup_\omega, \setminus\}$-subalgebra of $\mathcal{P}(U)$;
- a δ-ring in U is a $\{\cap_\omega, \setminus\}$-subalgebra of $\mathcal{P}(U)$.

TODO semi-ring!

Order-theoretic ring TODO: sublattice!

An order-theoretic ring of sets is a non-empty collection of sets $\mathcal{R}$ such that

- if $A, B \in \mathcal{R}$, then $A \cup B \in \mathcal{R}$;
- if $A, B \in \mathcal{R}$, then $A \cap B \in \mathcal{R}$.

Semi-rings

A semi-ring is a non-empty collection of sets $\mathcal{S}$ such that

- if $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$;
- if $A, B \in \mathcal{S}$, then $A \setminus B$ is a finite disjoint union of sets in $\mathcal{S}$.

Lemma III.152. *Let $\mathcal{S}$ be a semi-ring. Then $\emptyset \in \mathcal{S}$.*

Proof. Because $\mathcal{S}$ is non-empty, we can take $A \in \mathcal{S}$. Then $A \setminus A = \emptyset$ is a finite disjoint union of sets in $\mathcal{S}$, so $\emptyset \in \mathcal{S}$. ☐

Measure-theoretic rings

A (measure-theoretic) ring of sets is a non-empty collection of sets $\mathcal{R}$ such that

- if $A, B \in \mathcal{R}$, then $A \cup B \in \mathcal{R}$;
- if $A, B \in \mathcal{R}$, then $A \setminus B \in \mathcal{R}$.

By III.141 a measure-theoretic ring is in particular a π-system and an order-theoretic ring.

Lemma III.153. *A non-empty collection of sets $\mathcal{R}$ is a measure-theoretic ring* if and only if

- *it is closed under finite intersections;*
- *it is closed under symmetric differences.*

Proof. By the identities $A \cup B = (A \,\Delta\, B) \,\Delta\, (A \cap B)$ and $A \setminus B = A \,\Delta\, (A \cap B)$. ☐

Lemma III.154. *Let $\mathcal{S}$ be a semi-ring. Then the smallest ring $\mathcal{R}$ containing $\mathcal{S}$ is*

$$\mathfrak{R}\{\mathcal{S}\} = \{E_1 \uplus \ldots \uplus E_n \mid E_i \in \mathcal{S} \text{ are pairwise disjoint}\}.$$

We call this the ring generated by $\mathcal{S}$.

Proof. Every ring containing $\mathcal{S}$ must contain $\mathfrak{R}\{\mathcal{S}\}$, so if it is a ring it is automatically the smallest. We just need to show it is a ring. Let E, F be arbitrary elements of $\mathfrak{R}\{\mathcal{S}\}$. We need to show that both $E \setminus F$ and $E \cup F$ are in $\mathfrak{R}\{\mathcal{S}\}$.
Now $E \cup F = (E \setminus F) \uplus F$ can be written as a disjoint union, so we just need to write $E \setminus F$ as a pairwise disjoint union of elements of $\mathcal{S}$. To that end write $E = \biguplus_{i=0}^n E_i$ and $\biguplus_{j=0}^m F_j$. Then

$$E \setminus F = \biguplus_{i=0}^{n} \Big[\Big(\big((E_j \setminus F_1) \setminus F_2 \big) \setminus \ldots \Big) \setminus F_m \Big].$$

By industion and using the semi-ring property we can see that this is expressible as a finite pairwise disjoint union of elements of $\mathcal{S}$, and thus is an element of $\mathfrak{R}\{\mathcal{S}\}$. ☐

σ- and δ-rings

A σ-ring of sets is a collection of sets $\mathcal{R}$ such that

- $\mathcal{R}$ is a measure-theoretic ring;
- $\mathcal{R}$ is closed under countable unions.

A δ-ring of sets is a collection of sets $\mathcal{R}$ such that

- $\mathcal{R}$ is a measure-theoretic ring;
- $\mathcal{R}$ is closed under countable intersections.

3.4.5.3 Algebras of sets

An algebra of sets on a set Ω (also known as a field of sets) is a ring that contains Ω.

In principle we can thus define

- order-theoretic algebra;
- semi-algebra;
- measure-theoretic algebra;
- σ-algebra;
- δ-algebra.

Some of these notions coincide.

Lemma III.155. *Let $\mathcal{A}$ be a family of sets. Then*

1. *$\mathcal{A}$ is an order-theoretic algebra* if and only if *it is a measure-theoretic algebra;*
2. *$\mathcal{A}$ is a σ-algebra* if and only if *it is a δ-algebra.*

So we define define semi-algebra, algebra and σ-algebra.

Lemma III.156. *A family of sets $\mathcal{A}$ is a semi-algebra on Ω* if and only if

- *$\Omega \in \mathcal{A}$;*
- *if $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$;*
- *if $A \in \mathcal{A}$, then A^c is a finite disjoint union of sets in $\mathcal{A}$.*

A family of sets $\mathcal{A}$ is an algebra on Ω if and only if

- *$\Omega \in \mathcal{A}$;*
- *if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;*
- *if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.*

A family of sets $\mathcal{A}$ is a σ-algebra on Ω if and only if

- *$\Omega \in \mathcal{A}$;*
- *if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;*
- *if $(A_i)_{i\in\mathbb{N}}$ is a countable sequence of sets in $\mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.*

Example

For any set Ω, the power set $\mathcal{P}(\Omega)$ is a σ-algebra on Ω.

Lemma III.157. *An algebra $\mathcal{A}$ is a σ-algebra* if and only if *$\mathcal{A}$ is a monotone class.*

Lemma III.158. *A Dynkin system is a σ-algebra* if and only if *it is a π-system.*

Lemma III.159.

1. *The countable monotone union of a sequence of σ-algebras is an algebra, but not necessarily a σ-algebra.*
2. *Any arbitrary intersection of σ-algebras is a σ-algebra.*

Lemma III.160. *Let $\mathcal{A}$ be a σ-algebra on Ω and $B \subset \Omega$. Then $B \cap \mathcal{A} = \{B \cap A \mid A \in \mathcal{A}\}$ is a σ-algebra on B.*

3.4.6 Generators

TODO: $\boldsymbol{T}$-generators, denoted $\boldsymbol{T}\{\mathcal{F}\}$.

Let $\mathcal{F}$ be a family of subsets of Ω. Then we define

- the σ-algebra generated by $\mathcal{F}$, $\sigma\{\mathcal{F}\}$;
- the monotone class generated by $\mathcal{F}$, $\mathfrak{M}\{\mathcal{F}\}$; and
- the Dynkin system generated by $\mathcal{F}$, $\mathfrak{D}\{\mathcal{F}\}$;

the intersection of all such families $\subseteq \mathcal{P}(\Omega)$ that contain $\mathcal{F}$.
In each case we call $\mathcal{F}$ the generator of the system.

These intersections are again σ-algebras, monotone classes and Dynkin systems, respectively. So these generated families are the smallest such families containing $\mathcal{F}$.

Example

- If $\mathcal{A}$ is a σ-algebra, then $\sigma\{\mathcal{A}\} = \mathcal{A}$.
- If $\mathcal{A} = \{A\}$, a single set, then $\sigma\{\mathcal{A}\} = \{\emptyset, A, A^c, \Omega\}$.

Lemma III.161. *A universe set for $\mathcal{F}$ is also a universe set for $\sigma\{\mathcal{F}\}$, $\mathfrak{M}\{\mathcal{F}\}$ and $\mathfrak{D}\{\mathcal{F}\}$.*

Proposition III.162 (Monotone class theorem)**.** *Let $\mathcal{A}$ be an algebra. Then*

$$\mathfrak{M}\{\mathcal{A}\} = \sigma\{\mathcal{A}\}.$$

Proof. Every σ-algebra is a monotone class, so $\mathfrak{M}\{\mathcal{A}\} \subset \sigma\{\mathcal{A}\}$.
For the other inclusion it is enough to show that $\mathfrak{M}\{\mathcal{A}\}$ is an algebra: using III.157 we have

$$\mathfrak{M}\{\mathcal{A}\} \text{ is an algebra} \implies \mathfrak{M}\{\mathcal{A}\} \text{ is a } \sigma\text{-algebra} \implies \sigma\{\mathcal{A}\} \subset \mathfrak{M}\{\mathcal{A}\}.$$

In particular, due to III.156, we verify $\Omega \in \mathfrak{M}\{\mathcal{A}\}$, $B^c \in \mathfrak{M}\{\mathcal{A}\}$ and $B \cup C \in \mathfrak{M}\{\mathcal{A}\}$.
$\boxed{\Omega \in \mathfrak{M}\{\mathcal{A}\}}$ By III.161.

$\boxed{B^c \in \mathfrak{M}\{\mathcal{A}\}}$ Define

$$\mathcal{E}_1 = \{B \in \mathfrak{M}\{\mathcal{A}\} \mid B^c \in \mathfrak{M}\{\mathcal{A}\}\}$$

which is a monotone class by De Morgan's laws:

$$(B_i)_{i=1}^{\infty} \subset \mathcal{E}_1 \implies (B_i^c)_{i=1}^{\infty} \subset \mathfrak{M}\{\mathcal{A}\} \implies \bigcup_{i=1}^{\infty} B_i^c = \left(\bigcap_{i=1}^{\infty} B_i\right)^c \in \mathfrak{M}\{\mathcal{A}\} \implies \bigcap_{i=1}^{\infty} B_i \in \mathcal{E}_1.$$

Also $\mathcal{E}_1 \subset \mathfrak{M}\{\mathcal{A}\}$, so $\mathcal{E}_1 = \mathfrak{M}\{\mathcal{A}\}$ by minimality, so

$$B \in \mathfrak{M}\{\mathcal{A}\} \iff B \in \mathcal{E}_1 \implies B^c \in \mathfrak{M}\{\mathcal{A}\}.$$

$\boxed{B \cup C \in \mathfrak{M}\{\mathcal{A}\}}$ Define

$$\begin{aligned} \mathcal{E}_2 &= \{B \in \mathfrak{M}\{\mathcal{A}\} \mid \forall C \in \mathcal{A} : B \cup C \in \mathfrak{M}\{\mathcal{A}\}\} \\ \mathcal{E}_3 &= \{B \in \mathfrak{M}\{\mathcal{A}\} \mid \forall C \in \mathfrak{M}\{\mathcal{A}\} : B \cup C \in \mathfrak{M}\{\mathcal{A}\}\}. \end{aligned}$$

Now $\mathcal{E}_2$ and $\mathcal{E}_3$ are monotone classes: by I.220 and I.221, for $k = 1, 2$

$$(B_i)_{i=1}^{\infty} \subset \mathcal{E}_k \implies \forall C : (B_i \cup C)_{i=1}^{\infty} \subset \mathfrak{M}\{\mathcal{A}\} \implies \forall C : \bigcup_{i=1}^{\infty} B_i \cup C = \left(\bigcup_{i=1}^{\infty} B_i\right) \cup C \in \mathfrak{M}\{\mathcal{A}\} \implies \bigcup_{i=1}^{\infty} B_i \in \mathcal{E}_k.$$

Now clearly $\mathcal{A} \subseteq \mathcal{E}_2$, so by minimality $\mathcal{E}_2 = \mathfrak{M}\{\mathcal{A}\}$. Moreover,

$$D \in \mathcal{A} \implies \forall C \in \mathcal{E}_2 : C \cup D \in \mathfrak{M}\{\mathcal{A}\} \implies \forall C \in \mathfrak{M}\{\mathcal{A}\} : D \cup C \in \mathfrak{M}\{\mathcal{A}\} \implies D \in \mathcal{E}_3.$$

So $\mathcal{A} \subseteq \mathcal{E}_3$ and by minimality $\mathcal{E}_3 = \mathfrak{M}\{\mathcal{A}\}$, which means that

$$B \in \mathfrak{M}\{\mathcal{A}\} \iff B \in \mathcal{E}_3 \implies \forall C \in \mathfrak{M}\{\mathcal{A}\} : B \cup C \in \mathfrak{M}\{\mathcal{A}\}.$$

□

Corollary III.162.1. *Let $\mathcal{A}$ be an algebra and M a monotone class with $\mathcal{A} \subseteq M$, then $\sigma\{\mathcal{A}\} \subseteq M$.*

Proposition III.163. *If $\mathcal{F}$ is a π-system on Ω, then*

$$\mathfrak{D}\{\mathcal{F}\} = \sigma\{\mathcal{F}\}.$$

Proof. Every σ-algebra is a Dynkin system, so $\mathfrak{D}\{\mathcal{F}\} \subset \sigma\{\mathcal{F}\}$.
For the other inclusion it is enough to show that $\mathfrak{D}\{\mathcal{F}\}$ is a π-system: using III.158, we have

$$\mathfrak{D}\{\mathcal{F}\} \text{ is a } \pi\text{-system} \implies \mathfrak{D}\{\mathcal{F}\} \text{ is a } \sigma\text{-algebra} \implies \sigma\{\mathcal{F}\} \subset \mathfrak{D}\{\mathcal{F}\}.$$

To this end we define

$$\mathcal{D}_B = \{A \subset \Omega \mid A \cap B \in \mathfrak{D}\{\mathcal{F}\}\} \qquad \text{for some } B \in \mathfrak{D}\{\mathcal{F}\},$$

which we claim is a Dynkin system.

$\boxed{\Omega \in \mathcal{D}_B}$ Because $\Omega \cap B = B$.

$\boxed{A^c \in \mathcal{D}_B}$ Let $A \in \mathcal{D}_B$. Then

$$A^c \cap B = (\Omega \setminus A) \cap B = (\Omega \cap B) \setminus (A \cap B) \in \mathfrak{D}\{\mathcal{F}\},$$

so $A^c \in \mathcal{D}_B$.

$\boxed{\biguplus_{i\in\mathbb{N}} A_i \in \mathcal{D}_B}$ Let $(A_i)_{i=1}^{\infty}$ be a disjoint family of sets in $\mathcal{D}_B$. Then, using I.221,

$$\left(\bigcup_{i=1}^{\infty} A_i\right) \cap B = \bigcup_{i=1}^{\infty}(A_i \cap B) \in \mathfrak{D}\{\mathcal{F}\},$$

so $\bigcup_{i=1}^{\infty} A_i \in \mathcal{D}_B$.

Now because $\mathcal{F}$ is a π-system, we have $\mathcal{F} \subset \mathcal{D}_B$ and thus $\mathcal{D}_B \subset \mathfrak{D}\{\mathcal{F}\}$.
Now for all $B \in \mathcal{F}$, we have

$$A \in \mathcal{F} \implies A \cap B \in \mathcal{F} \implies A \cap B \in \mathfrak{D}\{\mathcal{F}\} \implies A \in \mathcal{D}_B.$$

So $\mathcal{F} \subset \mathcal{D}_B$ if $B \in \mathcal{F}$. In this case we then also have $\mathfrak{D}\{\mathcal{F}\} \subset \mathcal{D}_B$.
In fact this holds for all $B \in \mathfrak{D}\{\mathcal{F}\}$:

$$B \in \mathfrak{D}\{\mathcal{F}\} \implies \forall A \in \mathcal{F} : B \in \mathcal{D}_A \implies \forall A \in \mathcal{F} : B \cap A \in \mathfrak{D}\{\mathcal{F}\} \implies \mathcal{F} \subset \mathcal{D}_B \implies \mathfrak{D}\{\mathcal{F}\} \subset \mathcal{D}_B.$$

Consequently,

$$B, C \in \mathfrak{D}\{\mathcal{F}\} \implies C \in \mathcal{D}_B \implies C \cap B \in \mathfrak{D}\{\mathcal{F}\},$$

meaning $\mathfrak{D}\{\mathcal{F}\}$ is a π-system. □

Corollary III.163.1 (π-λ theorem)**.** *Let P be a π-system and D a Dynkin system with $P \subseteq D$, then $\sigma\{P\} \subseteq D$.*

Corollary III.163.2. *If $\mathcal{A}$ is an algebra, then $\mathfrak{M}\{\mathcal{A}\} = \mathfrak{D}\{\mathcal{A}\} = \sigma\{\mathcal{A}\}$.*

3.4.6.1 Product structures

3.4.6.2 Finite products

Let X, Y be sets, $\mathcal{A} \subseteq \mathcal{P}(X), \mathcal{B} \subseteq \mathcal{P}(Y)$ and $\boldsymbol{T}$ the signature of a set-theoretical algebra. Then the product $\boldsymbol{T}$-algebra of $\mathcal{A}$ and $\mathcal{B}$ is

$$\mathcal{A} \otimes \mathcal{B} \coloneqq \boldsymbol{T}\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Notice that in general for $\boldsymbol{T}$-algebras $\mathcal{A}, \mathcal{B}$, the set $\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ is not a $\boldsymbol{T}$-algebra. It is necessary to take the closure.

Example

Take $\Omega = \{a, b, c\}$. Define the σ-algebras $\mathcal{A}, \mathcal{B}$ on Ω by

$$\mathcal{A} = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$$
$$\mathcal{B} = \{\emptyset, \{a, b\}, \{c\}, \Omega\}.$$

Then $\{a\} \times \{a,b\} \in \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$, but

$$(\{a\} \times \{a,b\})^c = \Big(\{b,c\} \times \{a,b,c\}\Big) \cup \Big(\{a,b,c\} \times \{c\}\Big)$$

is not in $\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$, so it is not a σ-algebra.

Lemma III.164. *Let X, Y be sets and $A \subseteq X, B \subseteq Y$ subsets. Then*

$$(A \times B)^c = A^c \times B \uplus A \times B^c \uplus A^c \times B^c = A^c \times Y \cup X \times B^c.$$

Proof. We have $X \times Y = (A \uplus A^c) \times (B \uplus B^c)$ and the lemma follows by I.29. $\square$

3.4.6.3 Infinite products

Let I be an index set, $\{X_i\}_{i\in I}$ a set of sets and $\{\mathcal{A}_i\}_{i\in I}$ a set of $\boldsymbol{T}$-algebras such that for all $i \in I$, $\mathcal{A}_i$ is a $\boldsymbol{T}$-algebra on X_i. We define the product $\boldsymbol{T}$-algebra $\bigotimes_{i\in I}\mathcal{A}_i$ on $\prod_{i\in I} X_i$ as

$$\bigotimes_{i\in I}\mathcal{A}_i \coloneqq \boldsymbol{T}\left\{\prod_{i\in I} A_i \;\middle|\; \forall i \in I : A_i \in \mathcal{A}_i \text{ and } \{i \in I \mid A_i \neq X_i\} \text{ is finite}\right\}.$$

Proposition III.165. *We may replace "is finite" with "is countable" in the definition.*

Proof. TODO (can we??) $\square$

3.5 Filter-valued functions

Let X be a set and $f : X \to \mathcal{FP}(X) \subseteq \mathcal{P}^2(X)$ a filter-valued function. Then the contour of f along $A \in \mathcal{P}(X)$ is defined as

$$f(A) \coloneqq \bigcap_{a\in A} f(a)$$

and the contour of f along $\mathcal{A} \in \mathcal{P}^2(X)$ is defined as

$$f(\mathcal{A}) \coloneqq \bigcup_{A\in\mathcal{A}} \bigcap_{a\in A} f(a).$$

Lemma III.166. *Let X be a set, $f : X \to \mathcal{FP}(X) \subseteq \mathcal{P}^2(X)$ a filter-valued function, $A \subseteq X$ and $\mathcal{A} \in \mathcal{P}^2(X)$. Then*

1. $f(\mathcal{A}) = f(\uparrow\mathcal{A})$;
2. $f(A) = f(\{A\}) = f(\dot{A})$.

Proof. (1) We note that if $A \subseteq B$, then $\bigcap_{a\in A} f(a) \supseteq \bigcap_{b\in B} f(b)$. Thus $f(\{A,B\}) = f(\{A\})$.
(2) $f(A) = \bigcap_{a\in A} f(a) = \bigcup_{A\in\{A\}} \bigcap_{a\in A} f(a)$. $\square$

Proposition III.167. *Let* X *be a set and* $f : X \to \mathcal{FP}(X) \subseteq \mathcal{P}^2(X)$ *a filter-valued function. If* $\mathcal{A}$ *is a filter, then* $f(\mathcal{A})$ *is a filter.*

Proof. TODO ☐

Chapter 4

Well-founded ordered sets

A poset is called

- well-founded if every non-empty subset has a minimal element;
- converse well-founded if every non-empty subset has a maximal element.

A well-ordering on a set U is a total order $\leq$ on U such that $(U, \leq)$ is well-founded.
A set A is well-orderable if it admits a well-ordering.

It turns out a well-order is what is needed to do recursion and induction.

Lemma III.168. *Every well-ordering has a least element.*

Proof. A minimal element for a total order is always a least element. $\square$

Lemma III.169. *Let $(U, \leq_U)$ be a well-ordered set and $f : W \rightarrowtail U$ an injection. Then W is well-ordered by*

$$\forall x, y \in W : x \leq_W y \quad \Leftrightarrow_{def} \quad f(x) \leq_U f(y).$$

In particular, if $W \subseteq U$ is a subset, then $\leq_W$ is the left- and right-restriction of $\leq$ to W, $\leq|_W^W$.

4.1 Succession

Every well-ordered set U must have a least element and at its low end it looks like $\mathbb{N}$:

- let 0_U denote the least element of U;
- we can define $S_U(x) \coloneqq \min\{y \in P \mid x < y\}$.

This successor function is defined for all $x \in U$, except the maximum (if it exists).

Let $(U, \leq)$ be a well-ordered set.

- The values of the partial function $S : U \not\rightarrow U$ are the successor points of U.
- A limit point is an element $x \in U$ that is neither 0_U nor a successor. The first limit point (i.e. the least point in the set of limit points) is denoted ω or ω_U.
- The points below ω are called finite points and the points above, and including, ω

are the infinite points of U.

If P is a poset, we can always add a point on top of all the rest: We can take, e.g the set

$$t_P = \{x \in P \mid x \notin x\}.$$

This is guaranteed, by proposition I.5, not to be in P. The poset $P \cup t_P$ is called the successor $\operatorname{Succ}(P)$ of P.

4.2 Initial segments

TODO: streamline

Let $(U, \leq)$ be a well-ordered set. An initial segment I of U is a downward closed subset:

$$\forall y \in I : \forall x \in U : x \leq y \implies x \in I.$$

We write $I \sqsubseteq U$.

Each element y of U determines a proper initial segment of points strictly below y:

$$\operatorname{seg}(y) \coloneqq \{x \in U \mid x < y\} \sqsubsetneq U.$$

We have $\operatorname{seg}(S_U(y)) = \operatorname{seg}(y) \cup \{y\}$.
Conversely, each proper initial segment is of the form $\operatorname{seg}(x)$:

Proposition III.170. *Let $(U, \leq)$ be a well-ordered set and W a subset of U. Then W is an initial segment* if and only if *either $W = U$ or $\exists! x \in U : W = \operatorname{seg}(x)$.*

Proof. The direction $\Rightarrow$ is trivial. For the other direction, assume $W \sqsubsetneq U$ and let $x = \min(U \setminus W)$. Showing that $W = \operatorname{seg}(x)$ is not difficult. ☐

We may then, in some sense, view x as the length of $\operatorname{seg}(x)$. We identify t_U as the length of U.
Let U be a well-ordered set. We define len_U which maps initial segments of U to $\operatorname{Succ}(U)$ by

$$\operatorname{len}_U(V) = \begin{cases} x & \exists x \in U : V = \operatorname{seg}(x) \\ t_U & V = U. \end{cases}$$

Each well-ordered set U can be viewed as a proper initial segment of another:

$$U = \operatorname{seg}_{\operatorname{Succ}(U)}(t_U) \sqsubsetneq \operatorname{Succ}(U).$$

Lemma III.171. *Let $(U, \leq)$ be a well-ordered set and $x, y \in U$, then*

$$\begin{aligned} \operatorname{seg}(x) = \operatorname{seg}(y) &\iff x = y; \\ \operatorname{seg}(x) \sqsubseteq \operatorname{seg}(y) &\iff x \leq y; \\ \operatorname{seg}(x) \sqsubsetneq \operatorname{seg}(y) &\iff x < y. \end{aligned}$$

Proposition III.172. *Any well-ordered set $(U, \leq)$ is order isomorphic to the set of its proper initial segments ordered by inclusion, $(\operatorname{seg}_U[U], \sqsubseteq)$.*

Proof. The function seg_U is an order embedding by lemma III.171. By lemma III.8 it must be injective and thus $U =_o \mathrm{seg}_U[U]$. ☐

Lemma III.173. *The family of initial segments of a well-ordered set U is*

1. *well-ordered by $\sqsubseteq$; and*
2. *closed under arbitrary unions.*

Lemma III.174. *Let $(U, \leq)$ be a well-ordered set and $t \in U$. Then $\bigcup\{\mathrm{seg}(u) \mid u < t\}$ is an initial segment and thus equal to $\mathrm{seg}(v)$ for some v. Also*

$$\mathrm{seg}(v) \leq \mathrm{seg}(t) \leq \mathrm{seg}(S_U(v)).$$

Proof. For the first inequality: let $x \in \mathrm{seg}(v)$, so $\exists u < t : x \in \mathrm{seg}(u)$ so $x < t$ and $x \in \mathrm{seg}(t)$. For the second inequality: assume, towards a contradiction, that $\mathrm{seg}(t) > \mathrm{seg}(S_U(v))$. Then $S_U(v) < t$ and so $\mathrm{seg}(S_U(v)) \subseteq \mathrm{seg}(v)$ and thus $v \in \mathrm{seg}(v)$, a contradiction. ☐

Proposition III.175. *Every order-preserving injection $f : U \rightarrowtail U$ of a well-ordered set into itself is expansive.*

Proof. Assume $f : U \rightarrowtail U$ injective but not expansive, i.e. $\exists x \in U : f(x) < x$ then let

$$x^* = \min\{x \in U \mid f(x) < x\}.$$

Then $f(x^*) < x^*$ and $f(f(x^*)) < f(x^*)$ by order preservation. Then $f(x^*)$ is a smaller element in the set, yielding a contradiction. ☐

Corollary III.175.1. *No well-ordered set is isomorphic with one of its proper initial segments, and hence no two distinct initial segments of a well-ordered set are isomorphic.*

4.3 Transfinite induction and recursion

The principles of induction and recursion can be generalised to well-ordered sets.
In general induction and recursion use the predecessor to define / prove a property of the successor. In general well-ordered sets, there are limit points that have no predecessor. For this reason it is easiest to generalize the principles of proof by *complete induction* and definition by *complete recursion.* Then we take the set of all predecessors, not the one predecessor that may or may not exist.

Theorem III.176 (Transfinite induction)**.** *Let U be a well-ordered set and P a unary definite predicate. We can prove $P(x)$ holds for all $x \in U$ by proving the strong induction step $\forall x \in U : \forall y < x : P(y) \implies P(x)$.*
Or, in other symbols,

$$\textit{if} \quad \forall x \in U : [\forall y < x : P(y) \implies P(x)] \quad \textit{then} \quad \forall x \in U : P(x)$$

Proof. Assume, towards a contradiction, the induction step and that $\exists x \in P : \neg P(x)$. Then the set of all such x has a least element (due to U being well-ordered). Let

$$x^* = \min\{x \in U \mid \neg P(x)\},$$

then all elements smaller than x^* must not be in this set: $\forall y < x^* : P(y)$, so that by the induction step $P(x^*)$. ☐

Notice that the "base step" ($\nexists y \in U : y < 0$, so $\forall y < 0 : P(y) \implies P(x)$) is vacuously true. It is often as easy to repeat this argument as appeal to the theorem.

Theorem III.177 (Transfinite recursion)**.** *Let U be a well-ordered set, E some non-empty set and $h : (U \nrightarrow E) \to E$ some function.*
There is exactly one function $f : U \to E$ which satisfies

$$f(x) = h(f|_{\operatorname{seg}(x)}) \qquad \forall x \in U.$$

Proof. Like when proving recursion on $\mathbb{N}$, we will consider "approximations" of the function f, i.e. functions $\operatorname{seg}(t) \to E$ which satisfy the requirement for all $x < t$.
This is the subject of the following lemma:

Lemma. *Let U be a well-ordered set, E some non-empty set and $h : (U \nrightarrow E) \to E$ some function.*
For all $t \in U$, there is exactly one function $\sigma_t : \operatorname{seg}(t) \to E$ which satisfies

$$\sigma_t(x) = h(\sigma_t|_{\operatorname{seg}(x)}) \qquad \forall x < t.$$

Proof of lemma. The proof goes by transfinite induction. Fix an arbitrary $t \in U$. Assume the induction hypothesis:

$$\forall u < t : \exists! \sigma_u \in (\operatorname{seg}(u) \to E) : \forall x < u : \sigma_u(x) = h(\sigma_u|_{\operatorname{seg}(x)}).$$

We need to prove that this implies there exists exactly one σ_t satisfying the condition. We consider three cases: t is the least point 0_U, a successor point or a limit point.

$\boxed{t = 0_U}$ Then $\operatorname{seg}(t) = \emptyset$ and we must have $\sigma_t = \emptyset$.

$\boxed{t = S_U(v)}$ If t is the successor of v, we can set

$$\sigma_t = \sigma_v \cup \{(v, h(\sigma_v))\}.$$

$\boxed{t \in \operatorname{Limit}(U)}$ The set of functions $\{\sigma_u \mid u < t\}$ is a chain in the poset $((U \nrightarrow E), \subseteq)$ which is inductive, see III.32. Let σ_t be the least upper bound.

To prove $\{\sigma_u \mid u < t\}$ is a chain, assume not i.e.

$$x < u < v < t \implies \sigma_u(x) = \sigma_v(x)$$

fails for some $x < u < v$. Take the least such x (we are effectively doing transfinite induction) so then

$$\sigma_u|_{\operatorname{seg}(x)} = \sigma_v|_{\operatorname{seg}(x)},$$

and by the induction hypothesis

$$\sigma_u(x) = h(\sigma_u|_{\operatorname{seg}(x)}) = h(\sigma_v|_{\operatorname{seg}(x)}) = \sigma_v(x).$$

This is a contradiction, proving we do indeed have a chain.

Finally we verify

- the domain of σ_t is $\mathrm{seg}(t)$; indeed

  $$\mathrm{dom}(\sigma_t) = \bigcup\{\mathrm{dom}(\sigma_u) \mid u < t\} = \bigcup\{\mathrm{seg}(u) \mid u < t\}$$

  which is an initial segment and thus equal to $\mathrm{seg}(v)$ for some v. By lemma III.174

  $$v \leq t \leq S_U(v).$$

  Then either $t = v$ or $t = S_U(v)$. The latter is excluded because t was a limit point.
- σ_t satisfies the condition (easily by transfinite induction);
- σ_t is unique (also easily by transfinite induction).

⊣ (Lemma)

Now consider the well-ordered set $\mathrm{Succ}(U)$ and extend h to $h' : (\mathrm{Succ}(U) \not\twoheadrightarrow E) \to E$ by

$$h'(\sigma) = \begin{cases} h(\sigma) & \sigma \in (U \not\twoheadrightarrow E) \\ \text{an arbitrary element of } E & \sigma \notin (U \not\twoheadrightarrow E). \end{cases}$$

We can then apply the lemma to $\mathrm{Succ}(U)$ and h'. Because $\mathrm{seg}(t_U) = U$, this gives a unique function $\sigma_{t_U} : U \to E$. We take this as our f. □

4.3.1 Recursion invariants

TODO recursively defining some function with certain property (+usng this property in recursive definition)

Chapter 5

Fixed points

A monotone mapping $\pi : P \to Q$ on a inductive posets is countably continuous if for every non-empty, countable chain $S \subseteq P$:

$$\pi(\sup S) = \sup \pi[S].$$

Let $(P, \leq)$ be a poset and $f : P \to P$ a function from P to P.

- A fixed point is an element $x^* \in P$ such that

  $$f(x^*) = x^*.$$

- A strongly least fixed point is a fixed point such that

  $$\forall y \in P : f(y) \leq y \implies x^* \leq y.$$

- The orbit of an element p of P is a sequence $\mathbb{N} \to P$ defined recursively:

  $$\begin{aligned} p_0 &= p \\ p_{n+1} &= f(p). \end{aligned}$$

  Sometimes we use orbit to mean the set $\{p_n \in P \mid n \in \mathbb{N}\}$.

Theorem III.178 (Continuous least fixed point theorem)**.** *Let $\pi : P \to P$ be a countably continuous, monotone mapping on an inductive poset $(P, \leq)$. Then π has a unique strongly least fixed point $x^* \in P$.*

Proof. As P is inductive, it has a least element $\bot$. The orbit $\{x_n \in P \mid n \in \mathbb{N}\}$ of $\bot$ is a chain: $\bot \leq \pi(\bot)$ and the rest follows by induction on n, using the monotonicity of π. Thus the orbit has a supremum. Let x^* be this supremum.
Then, by countable continuity,

$$\pi(x^*) = \pi(\sup\{x_n \in P \mid n \in \mathbb{N}\}) = \sup \pi[\{x_n \in P \mid n \in \mathbb{N}\}] = \sup\{x_{n+1} \in P \mid n \in \mathbb{N}\} = x^*.$$

To prove x^* is a strongly least fixed point, let $y \in P$ assume $\pi(y) \leq y$. Then we apply induction on n:

Basis step $x_0 = \bot \leq y$.

Induction step $x_n \leq y \implies x_{n+1} = \pi(x_n) \leq \pi(y) \leq y$.

□

Iteration lemma.
Fixed point theorem.
Least fixed point theorem.
Hitchhiker's guide: Knaster-Tarski fixed point; Tarksi fixed point

Chapter 6

Graphs

Chapter 7

Trees

Part IV

Category theory

`https://arxiv.org/pdf/1912.10642.pdf` `https://arxiv.org/pdf/0810.1279.pdf` `http://katmat.math.uni-bremen.de/acc/acc.pdf`

Chapter 1

Basic concepts

1.1 Categories

1.1.1 Definitions and examples

A category C consists of

1. a class, denoted $\mathrm{ob}(\mathsf{C})$, of objects $X, Y, Z, \dots$
2. a class, denoted $\mathrm{mor}(\mathsf{C})$, of morphisms or arrows $f, g, h, \dots$
3. a partial binary function $\circ : \mathrm{mor}(\mathsf{C}) \times \mathrm{mor}(\mathsf{C}) \nrightarrow \mathrm{mor}(\mathsf{C})$ of composition
4. functions $\mathrm{dom} : \mathrm{mor}(\mathsf{C}) \to \mathrm{ob}(\mathsf{C})$ and $\mathrm{codom} : \mathrm{mor}(\mathsf{C}) \to \mathrm{ob}(\mathsf{C})$; we call $\mathrm{dom}(f)$ the domain and $\mathrm{codom}(f)$ the codomain of the morphism f; we write

   $$f : X \to Y \qquad \text{or} \qquad X \xrightarrow{f} Y$$

   to mean f is a morphism with $\mathrm{dom}(f) = X$ and $\mathrm{codom}(f) = Y$;
5. a function $\mathrm{id} : \mathrm{ob}(\mathsf{C}) \to \mathrm{mor}(\mathsf{C}) : X \mapsto (\mathrm{id}_X : X \to X)$, whose images are called identity morphisms;

such that

- $g \circ f$ is defined if and only if $\mathrm{codom}(f) = \mathrm{dom}(g)$; in this case

  $$g \circ f : \mathrm{dom}(f) \to \mathrm{codom}(g);$$

- for any $f : X \to Y$, we have $\mathrm{id}_Y \circ f = f \circ \mathrm{id}_X = f$;
- $\circ$ is associative.

We also write gf or $g \cdot f$ instead of gf.

Two arrows are called parallel if they have the same domain and codomain.
TODO: we may characterise categories as typed associative classes.

Example

The terminology suggests that we can think of the objects as sets and the morphisms as functions. We will indeed often consider categories where the objects are sets (potentially with extra structure) and the morphisms are (a certain class of) functions (e.g. structure preserving functions).
Some examples of such categories are

- All sets are objects in the category Set and all functions are morphisms.
- The category Poset has partially ordered sets as objects and order-preserving functions as morphisms.
- A set may be identified with the set of its singleton subsets. One way to make this a category is by considering only the identity functions of the form

  $$\mathrm{id}_{\{a\}} : \{a\} \to \{a\} : a \mapsto a.$$

  A category is discrete if every morphism is an identity.

There are also categories that are not of this type:

- A preorder $(P, \precsim)$ can be regarded as a category. The elements of P are the objects of the category and for $x, y \in P$ there exists a unique morphism $x \to y$ if and only if $x \precsim y$. Transitivity implies the existence of composites and reflexivity the existence of identity morphisms.
- In particular (TODO: Von Neumann??) ordinals are preorders and thus define categories. We use the notation $\mathbb{0}, \mathbb{1}, \mathbb{2}, \dots$ for the categories defined by $0, 1, 2, \dots$. Also $\mathbb{\omega}$ is the category defined by ω.

If the category is small enough, we can depict it using a diagram. Points are objects and arrows are morphisms.

Example

$$\begin{tikzcd} \mathbb{1} : & 0 \arrow[loop left] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{2} : & 0 \arrow[loop left] \arrow[r] & 1 \arrow[loop right] \end{tikzcd}$$

walking arrow or free arrow

$$\begin{tikzcd} \mathbb{I} : & 0 \arrow[loop left] \arrow[r, shift left] & 1 \arrow[loop right] \arrow[l, shift left] \end{tikzcd}$$

walking isomorphism or free isomorphism

$$\begin{tikzcd} \mathbb{3} : & 0 \arrow[loop left] \arrow[r] \arrow[rr, bend left] & 1 \arrow[loop below] \arrow[r] & 2 \arrow[loop right] \end{tikzcd}$$

1.1.2 Hom sets and sizes of categories

A category C is small if its collection of morphisms is a set.
A category C is locally small for any objects X, Y if the collection of morphisms from X to Y is a set. This set is called a hom-set and is denoted $\mathrm{Hom}(X, Y)$ or $\mathsf{C}(X, Y)$.

Lemma IV.1. *If* C *is a small category, then* $\operatorname{ob}(\mathsf{C})$ *is a set.*

Proof. TODO: replacement with $X \mapsto \operatorname{id}_X$. ☐

1.1.2.1 Thin categories

A category with at most one morphism in each category is called thin.

Lemma IV.2. *A small category*

- *with at most one element in every hom-set defines a preorder;*
- *such that for all objects* X, Y*, the set* $\operatorname{Hom}(X,Y) \cup \operatorname{Hom}(Y,X)$ *is at most a singleton defines a partial order.*

We call such categories preorder categories and poset categories.

1.1.2.2 Skeletal categories

The isomorphism class of an object in a category is the collection of objects isomorphic to the object.
A category C is skeletal if it contains one object in each isomorphism class.

Lemma IV.3. *Let* C *be a category. There is a unique - up to isomorphism - skeletal category equivalent to* C*.*

This category is called the skeleton of C, denoted skC.

Proof. TODO ☐

1.1.2.3 Initial, terminal and zero objects; zero morphisms

Let C be a category.

- An object $X \in \operatorname{ob}(\mathsf{C})$ is called initial if $\mathsf{C}(X,Y)$ contains one element for each $Y \in \operatorname{ob}(\mathsf{C})$.
- An object $Y \in \operatorname{ob}(\mathsf{C})$ is called terminal if $\mathsf{C}(X,Y)$ contains one element for each $X \in \operatorname{ob}(\mathsf{C})$.
- An object $Z \in \operatorname{ob}(\mathsf{C})$ is called a zero object if Z is both initial and terminal.

Initial and terminal objects are dual notions.

Proposition IV.4. *Let* C *be a category. Any two initial objects are isomorphic. Any two terminal objects are isomorphic.*

Proof. Let X_1, X_2 be initial objects let $f : X_1 \to X_2$ and $g : X_2 \to X_1$ be the unique morphisms with these domains / codomains. Then $g \circ f : X_1 \to X_1 \in \mathsf{C}(X_1, X_1) = \{\operatorname{id}_{X_1}\}$. Similarly f is also a left inverse of g. Thus X_1 and X_2 are isomorphic.
The case for terminal objects is dual. ☐

Let C be a category with zero object 0. A zero morphism $f : X \to Y$ is a morphism that can be factored as $f = h \circ g$, where $g : X \to 0$ and $h : 0 \to Y$.

Proposition IV.5. *Let C be a category with zero object 0.*

1. *Every hom-set contains exactly one zero morphism.*
2. *The composition of a zero morphism and any other morphism is a zero morphism.*

1.1.3 (Commuting) diagrams

Informally a diagram in a category is a drawing with points and arrows.
We always assume that all identity arrows and composite arrows are present in the diagram, even if they are rarely drawn.
We say that such a diagram commutes if, for any two points x, y in the diagram, every path from x to y defines the same morphism.

Example

Let $(N_1, 0_1, S_1)$ and $(N_2, 0_2, S_2)$ be Peano systems and π the unique isomorphism defined in I.172. The condition

$$\forall n \in N_1 : \pi(S_1 n) = S_2 \pi(n)$$

is equivalent to saying

$$\begin{tikzcd} N_1 \arrow[r, "\pi"] \arrow[d, "S_1"] & N_2 \arrow[d, "S_2"] \\ N_1 \arrow[r, "\pi"] & N_2 \end{tikzcd}$$

commutes.

The two paths in the diagram correspond to the two sides of the equation.

More formally we have the following definition:

Let J be a category. A diagram of type (or shape) J in a category C is a (covariant) functor

$$D : \mathsf{J} \to \mathsf{C}.$$

The category J is called the index category or scheme of the diagram D.
Usually we are interested in diagrams where J is small or even finite. In these cases we call the diagram small or finite.
If the index category J is a preordered set, then we call the diagram a commutative diagram.
Let I be a subcategory of J. Then the functor D restricted to I is a subdiagram of D.

To actually draw such a diagram, you draw each object and morphism in the index category and name it using its image under D.
There is no requirement of injectivity, so the same object or morphism in C may appear several times in the diagram.
Every path in the index category with the same start X and end Y defines a morphism in the hom-set $\operatorname{Hom}(X, Y)$. In a preorder category there is at most one morphism in each hom-set, so each such path must correspond to the same morphism. Thus the informal and formal definitions of commutative diagram agree.

Example

The commutative square shown in the last example has the shape

$$\mathbb{S}: \begin{tikzcd} \mathbf{1} \arrow[loop left] \arrow[r] \arrow[d] \arrow[dr] & \mathbf{2} \arrow[loop right] \arrow[d] \\ \mathbf{3} \arrow[loop left] \arrow[r] & \mathbf{4} \arrow[loop right] \end{tikzcd} .$$

All morphisms in the category have been drawn. We also call this preorder category the commutative square.

Any two paths in a commutative diagram with the same start and end yield an equation of morphism. Constructing proofs with these equations is known as diagram chasing or abstract nonsense.

Example

Warning! When drawing the diagram you draw the index category and label with the object category. You do not draw the image in the object category! This is not necessarily a preorder category when the index category is a preorder category. In fact it is not necessarily even a category. So, for example, the functor F that maps

$$\begin{tikzcd} X_1 \arrow[d, "f"] & X_2 \arrow[d, "g"] & \text{to} & X = F(X_1) = F(X_2) \arrow[d, shift right, "f'=F(f)"'] \arrow[d, shift left, "g'=F(g)"] \\ Y_1 & Y_2 & & Y = F(Y_1) = F(Y_2) \end{tikzcd}$$

should be drawn as

$$\begin{tikzcd} X \arrow[d, "f'"] & X \arrow[d, "g'"] & \text{not} & X \arrow[d, shift right, "f'"'] \arrow[d, shift left, "g'"] \\ Y & Y , & & Y \end{tikzcd}$$

1.1.3.1 Diagram chasing results

Lemma IV.6. *Consider the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "k"] & Y \arrow[r, "g"] \arrow[d, "h"] & Z \arrow[d, "l"] \\ A \arrow[r, "p"] & B \arrow[r, "q"] & C \end{tikzcd} .$$

Suppose the left and right squares commute (as subdiagrams), then the whole diagram commutes.

Proof. There are two pairs of nodes that may have more than one morphism between them: (X, C) and (A, Z). We show that there is only one morphism in $\mathrm{Hom}(X, C)$. The other case is similar.
The question boils down to whether $qhf = qpk$. Now we know that $hf = pk$ because the left square commutes, so the identity is automatic. ☐

1.1.4 Categories and associative classes

Lemma IV.7. *Let* C *be a category. We may view* C *as an associative class* $(\mathrm{mor}(\mathsf{C}) \uplus \{u\}, \widehat{\circ})$*, where*

$$\widehat{\circ} : (\mathrm{mor}(\mathsf{C}) \uplus \{u\}) \times (\mathrm{mor}(\mathsf{C}) \uplus \{u\}) \to (\mathrm{mor}(\mathsf{C}) \uplus \{u\}) : (f, g) \mapsto \begin{cases} u & (f = u \textit{ or } g = u \textit{ or } fg \textit{ is undefined}) \\ fg & (\textit{otherwise}). \end{cases}$$

This means we can use the results about associative classes in this context.

1.1.4.1 Duality

Intuitively category-theoretic duality is obtained by "flipping all the arrows".

Let C be a category. The opposite category C^{op} is obtained by

- replacing the composition $\circ$ with its dual $\circ^d$;
- swapping the domain and codomain functions

$$\mathrm{dom}_{\mathsf{C}^{\mathsf{op}}} \coloneqq \mathrm{codom}_{\mathsf{C}} \qquad \text{and} \qquad \mathrm{codom}_{\mathsf{C}^{\mathsf{op}}} = \mathrm{dom}_{\mathsf{C}} .$$

In particular $\mathrm{ob}(\mathsf{C}) = \mathrm{ob}(\mathsf{C}^{\mathsf{op}})$ and $\mathrm{mor}(\mathsf{C}) = \mathrm{mor}(\mathsf{C}^{\mathsf{op}})$.

As we often suppress the composition $\circ$, and both the original and dual category have the same objects, it may sometimes be unclear in which category we are working. We then may write f^o the emphasis that we are working in the dual category.

Lemma IV.8. *The category* $(\mathsf{C}^{op})^{op}$ *is again the category* C.

A statement about the category C^{op} can often be interpreted as a statement about C. This is referred to as the "dual statement". In other words: a statement in C is true if and only if the dual statement is true in C^{op}.
If we prove that something is true in a collection of categories including C and C^{op}, we have in effect proven both the statement and the dual statement. We then say the second follows from the first by duality.

1.1.4.2 Mono- and epimorphisms

Let C be a category. Then a morphism $f \in \mathrm{mor}(\mathsf{C})$ is called

- a monomorphism or monic if it is left-cancellative.
- an epimorphism or epic if it is right-cancellative.

1.1.5 Left and right inverses

Let C be a category and $f : X \to Y$ a morphism.

- A left inverse (or retraction) of f is a morphism $g : Y \to X$ such that

$$g \circ f = \mathrm{id}_X .$$

In this case we call f a split monomorphism or split monic.

- A right inverse (or section) of f is a morphism $h : Y \to X$ such that

$$f \circ h = \mathrm{id}_Y .$$

In this case we call f a split epimorphism or split epic.

- A (two-sided) inverse of f is a morphism that is both a left and a right inverse. In this case we call f an isomorphism.

If an isomorphism exists from X to Y (or equivalently from Y to X), we say X and Y are isomorphic, and write $X \cong Y$.

An endomorphism is a morphism f such that $\mathrm{dom}(f) = \mathrm{codom}(f)$. An automorphism is an endomorphism that is an isomorphism.
Note that a left or right inverse in the sense of category theory is not the same as in the sense of an associative class.

Lemma IV.9. *Let $f : X \to Y$ be a morphism. If $g : Y \to X$ is a left inverse and $h : Y \to X$ a right inverse of f, then $g = h$ and f is an isomorphism. In particular an isomorphism can have at most one inverse.*

Proof. The proof is identical to I.142:

$$g = g\,\mathrm{id}_Y = g(fh) = (gf)h = \mathrm{id}_X\, h = h.$$

□

Lemma IV.10.

1. *Every split monomorphism is a monomorphism.*
2. *Every split epimorphism is a epimorphism.*

The converses are not true.

Lemma IV.11. *In the category* Set

1. *every monomorphism is split;*
2. *the assertion that every epimorphism is split is equivalent to the axiom of choice.*

Proof. This is a restatement of I.126 and I.199. □

Lemma IV.12. *If a monomorphism is split epic, it is also split monic, and thus an isomorphism.*
If an epimorphism is split monic, it is also split epic, and thus an isomorphism.

Proof. Assume a morphism $f : X \to Y$ is monic and has a right inverse r. We claim r is also a left inverse. Indeed

$$fr = \mathrm{id}_Y \implies frf = \mathrm{id}_Y f = f\,\mathrm{id}_X \implies rf = \mathrm{id}_x$$

where the second implication is because f is monic. The second claim of the lemma follows by duality. □

Lemma IV.13. *In a preorder category, every morphism that is monic or epic, is an isomorphism.*
In a poset category, every morphism that is monic or epic, is an identity.

Lemma IV.14. *Let $f : X \to Y$ and $g : Y \to X$ be morphisms in a category C. Then*

1. *g is a left inverse of f if and only if the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}_X"'] & Y \arrow[r, "g"] & X \end{tikzcd} \qquad \textit{commutes;}$$

2. *g is a right inverse of f if and only if the diagram*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & X \arrow[r, "f"] & Y \end{tikzcd} \qquad \textit{commutes.}$$

1.1.5.1 Groups and groupoids

A groupoid is a category in which every morphism is an isomorphism.
A group is a groupoid with one object.

Lemma IV.15. *For any category C there exists a subcategory containing all the objects of C and only the morphisms of C that are isomorphisms. This subcategory is called the maximal groupoid inside C.*

1.1.5.2 The core of a category

Let C be a category.

- The core of C is the groupoid whose objects are the objects of C and whose morphisms are the isomorphisms of C.

Thus a category is a groupoid if and only if it equals its core.

1.1.6 Subcategories

Let C be a category. We call D a subcategory of C if

- $\mathrm{ob}(\mathsf{D}) \subseteq \mathrm{ob}(\mathsf{C})$;
- $\mathrm{mor}(\mathsf{D}) \subseteq \mathrm{mor}(\mathsf{C})$;
- $\forall f \in \mathrm{mor}(\mathsf{D}) : \ \mathrm{dom}(f) \in \mathrm{ob}(\mathsf{D}) \wedge \mathrm{codom}(f) \in \mathrm{ob}(\mathsf{D})$;
- $\forall X \in \mathrm{ob}(\mathsf{D}) : \ \mathrm{id}_X \in \mathrm{mor}(\mathsf{D})$;
- $\forall f, g \in \mathrm{mor}(\mathsf{D}) : \ fg \in \mathrm{mor}(\mathsf{D})$.

We call the subcategory full if for all $X, Y \in \mathrm{ob}(\mathsf{D}) : \ \mathsf{C}(X, Y) = \mathsf{D}(X, Y)$.

1.2 Functors

A (covariant) functor $F : \mathsf{C} \to \mathsf{D}$ consists of

1. a function $F : \mathrm{ob}(\mathsf{C}) \to \mathrm{ob}(\mathsf{D})$;
2. a function $F : \mathrm{mor}(\mathsf{C}) \to \mathrm{mor}(\mathsf{D})$;

satisfying the functorial properties:

- $F(a \xrightarrow{f} b) = F(a) \xrightarrow{F(f)} F(b)$;
- for any composable pair g, f in C, $F(g \circ f) = F(g) \circ F(f)$;
- for each object c in C, $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.

A contravariant functor $F : \mathsf{C} \to \mathsf{D}$ is similar, except it satisfies

- $F(a \xrightarrow{f} b) = F(b) \xrightarrow{F(f)} F(a)$
- for any composable pair g, f in C, $F(g \circ f) = F(f) \circ F(g)$;
- for each object c in C, $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.

An endofunctor is a functor between a category and itself.

Lemma IV.16. *The composition of two contravariant functors is covariant.*

Lemma IV.17. *Let* C *be a category. The functions*

$$\begin{cases} \mathrm{id}_{\mathrm{ob}(\mathsf{C})} = \mathrm{id}_{\mathrm{ob}(\mathsf{C}^{\mathsf{op}})} \\ \mathrm{id}_{\mathrm{mor}(\mathsf{C})} = \mathrm{id}_{\mathrm{mor}(\mathsf{C}^{\mathsf{op}})} \end{cases} \qquad \textit{form a contravariant functor } \mathrm{OP} : \mathsf{C} \to \mathsf{C}^{\mathsf{op}}.$$

We can compose any contravariant functor $F : \mathsf{C} \to \mathsf{D}$ *with the functor* OP *to obtain a covariant functor* $F : \mathsf{C} \to \mathsf{D}^{\mathsf{op}}$ *or* $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{D}$.

Example

Examples of covariant functors:

- For any category C, there is an identity functor id_{C} that maps objects and morphisms to themselves.
- There is an endofunctor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ that maps sets to their powerset and functions $f : A \to B$ to their image function $f[\cdot] : \mathcal{P}(A) \to \mathcal{P}(B)$.
- Forgetful functors are functors from the category of some type of structured set to a category of structured set with less (or) no structure. For example, we have a forgetful functor $U : \mathsf{Poset} \to \mathsf{Set}$ that maps the cateory Poset into the category Set by forgetting the order.

Examples of contravariant functors:

- The contravariant endofunctor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ that maps sets to their powerset and functions $f : A \to B$ to their inverse image function $f^{-1}[\cdot] : \mathcal{P}(B) \to \mathcal{P}(A)$.

Lemma IV.18. *Functors preserve split monomorphisms, split epimorphisms and isomorphisms.*

Functors do not necessarily preserve monomorphisms and epimorphisms.

Corollary IV.18.1. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. Let $f : X \to Y$ be monic (resp. epic). If $F(f)$ is not monic (resp. epic), then f is not split monic (resp. split epic).*

Lemma IV.19. *Functors preserve commutative diagrams: Let $D : \mathsf{J} \to \mathsf{C}$ be a commutative diagram and $F : \mathsf{C} \to \mathsf{B}$ a functor. Then $F \circ D$ is a commutative diagram.*

1.2.1 Functors as morphisms

We define Cat as the category whose objects are small categories and whose morphisms are functors between them.
We define CAT as the category whose objects are locally small categories and whose morphisms are functors between them.

HOW DOES CAT MAKE SENSE?

Lemma IV.20. Cat *is not small, but it is locally small.*
CAT *is not locally small.*

There is an inclusion functor Cat $\hookrightarrow$ CAT.
We can obviously consider isomorphisms in the category Cat, but this is not a very natural notion to work with. For example, a category is not typically isomorphic to its opposite category.
The more natural concept is equivalence of categories.

Lemma IV.21. *Let F, G be composable morphisms in* CAT*, i.e. functors. Then FG is contravariant if exactly one of F, G is contravariant and covariant otherwise.*

1.2.2 Properties of functors

A functor $F : \mathsf{C} \to \mathsf{D}$ is called

- full if for all $x, y \in \mathsf{C}$, the map $\mathsf{C}(x, y) \to \mathsf{D}(Fx, Fy)$ is surjective;
- faithful if for all $x, y \in \mathsf{C}$, the map $\mathsf{C}(x, y) \to \mathsf{D}(Fx, Fy)$ is injective;
- and essentially surjective on objects if for every object $d \in \mathsf{D}$ there is some $c \in \mathsf{C}$ such that d is isomorphic to Fc.

These are local definitions. We also call F

- an embedding if it is injective on objects;
- a full embedding if it is full, faithful and an embedding.

Lemma IV.22. *If $F : \mathsf{C} \to \mathsf{D}$ is a full and faithful functor, the F reflects isomorphisms: If f is a morphism in C so that Ff is an isomorphism in D, then f is an isomorphism.*

Thus, if x and y are objects in C such that Fx and Fy are isomorphic in D, then x and y are isomorphic in C.
By IV.18 any functor preserves isomorphisms.

Proposition IV.23. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor.*

1. *If F is an embedding, then the image of F is a subcategory of D.*
2. *If F is a full embedding, then the image of F is a full subcategory of D.*

1.2.2.1 Abstract and concrete categories

A category C is called concrete if there exists a faithful functor $F : \mathsf{C} \to \mathsf{Set}$. If a category is not concrete, it is called abstract.

Proposition IV.24. *In any concrete category*

1. *f is an injective function implies f is monomorphic;*
2. *f is a surjective function implies f is epimorphic.*

In the category Set, *the opposite implications also hold.*

Proof. Set $f : X \to Y$.

1. Assume f injective. Let $g, h : Z \to X$ such that $g \neq h$. Because we are in a concrete category, this means there exists a $z \in Z$ such that $g(z) \neq h(z)$. By injectivity of f this means $f(g(z)) \neq f(h(z))$ and so $fg \neq fh$. We conclude f is monic by contraposition.
2. Assume f surjective. Let $g, h : Y \to Z$ such that $g \neq h$. Because we are in a concrete category, this means there exists a $y \in Y$ such that $g(y) \neq h(y)$. By surjectivity of f we can find an $x \in X$ such that $f(x) = y$. So $g(f(x)) \neq h(f(x))$ and we conclude $gf \neq hf$, meaning f is epic by contraposition.

Suppose we are now in the category Set.

1. Assume f monic. To prove injectivity, take arbitrary $x_1, x_2 \in X$ and assume $f(x_1) = f(x_2)$. Now define the constant functions $x_1 : Z \to X : z \mapsto x_1$ and $x_2 : Z \to X : z \mapsto x_2$. Then
$$fx_1 = f(x_1) = f(x_2) = f(x_2) \qquad \Longrightarrow \qquad x_1 = x_2$$
because f monic.
2. Assume f epic. Left compose f with both the characteristic function $\chi_{f[X]}$ and the constant function $1 : Y \to \{1\} : y \mapsto 1$. It is clear $\chi_{f[X]} f = 1f$, so $\chi_{f[X]} = 1$ meaning f is surjective.

□

1.2.3 Arrow and comma categories

1.2.3.1 Arrow category

Let C be a category the arrow category of C is the category with

- as objects tuples of the form $((A, B), h)$, such that $A \xrightarrow{h} B$ in C;
- as morphisms pairs $(f, g) : (A \xrightarrow{h} B) \to (A' \xrightarrow{h'} B')$ where $A \xrightarrow{f} A'$, $B \xrightarrow{g} B'$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"] & A' \arrow[d, "h'"] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$ commutes.

  In other words, $hg = fh'$.

The arrow category of C is denoted $\mathsf{C}^{\to}$.

Lemma IV.25. *A functor* $\mathsf{C} \to \mathsf{D}$ *is a function* $\mathrm{ob}(\mathsf{C}^{\to}) \to \mathrm{ob}(\mathsf{D}^{\to})$ *that preserves composition.*

1.2.3.2 Comma category

Let A, B and C be categories and S, T functors of type $\mathsf{A} \xrightarrow{S} \mathsf{C} \xleftarrow{T} \mathsf{B}$. The comma category $S \downarrow T$ is the category with

- as objects tuples of the form $((A, B), h)$, where $A \in \mathrm{ob}(\mathsf{A}), B \in \mathrm{ob}(\mathsf{B})$ and $h : S(A) \to T(B)$ a morphism in C;
- as morphisms pairs $(f, g) : ((A, B), h) \to ((A', B'), h')$ where $f : A \to A'$, $g : B \to B'$ such that

$$\begin{tikzcd} S(A) \arrow[r, "S(f)"] \arrow[d, "h"] & S(A') \arrow[d, "h'"] \\ T(B) \arrow[r, "T(g)"] & T(B') \end{tikzcd}$$ commutes.

Composition is defined by $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$ and the identity on $((A, B), h)$ is $(\mathrm{id}_A, \mathrm{id}_B)$.

The composition $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$ is a morphism in the comma category by IV.6. The comma category is the arrow category containing arrows f such that $\mathrm{dom}(f) \in \mathrm{im}(S)$ and $\mathrm{codom}(f) \in \mathrm{im}(T)$.

Lemma IV.26. *An arrow category is a comma category of type* $\mathsf{C} \xrightarrow{\mathrm{id}_{\mathsf{C}}} \mathsf{C} \xleftarrow{\mathrm{id}_{\mathsf{C}}} \mathsf{C}$.

1.2.3.3 Slice and coslice categories

Let A be a category and a an object in A.

- The slice category over a, denoted A/a, is the comma category of $\mathsf{A} \xrightarrow{\mathrm{id}_{\mathsf{A}}} \mathsf{A} \xleftarrow{(0 \mapsto a)} \mathbb{1}$.
- The coslice category under a, denoted a/A, is the comma category of

$$\mathbb{1} \xrightarrow{(0\mapsto a)} \mathsf{A} \xleftarrow{\mathrm{id}_{\mathsf{A}}} \mathsf{A} .$$

Objects in a slice category A/a are always of the form $((X,a), h : X \to a)$ and morphisms are always of the form (f, id_a). So we may drop the dependence on a.

Lemma IV.27. *Let A be a category and a an object in A.*

1. *The objects of the slice category A/a may be written as (X,h). A morphism $f : X \to Y$ in A may be seen as a morphism $f : (X,h) \to (Y,h')$ in A/a if*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "h'"] \\ & a & \end{tikzcd} \qquad \textit{commutes.}$$

2. *The objects of the coslice category a/A may be written as (X,h). A morphism $f : X \to Y$ in A may be seen as a morphism $f : (X,h) \to (Y,h')$ in A/a if*

$$\begin{tikzcd} & a & \\ X \arrow[ur, "h"] \arrow[rr, "f"'] & & Y \arrow[ul, "h'"'] \end{tikzcd} \qquad \textit{commutes.}$$

1.2.4 Constructions of categories

1.2.4.1 Subcategories

Let C be a category. A category D is a subcategory of C if all its objects are objects of C and all its morphisms are morphisms of C.

1.2.4.2 Product categories

The product $\mathsf{C} \times \mathsf{D}$ of categories C and D is a category consisting of

1. ordered pairs (c, d) where $c \in \mathsf{C}$ and $d \in \mathsf{D}$ as objects;
2. ordered pairs $(f, g) : (c, d) \to (c', d')$ where $f : c \to c'$ and $g : d \to d'$ as morphisms.

Composition and identities are defined component-wise.

1.3 Naturality

1.3.1 Natural transformations

Let C, D be categories and $F, G : \mathsf{C} \to \mathsf{D}$ functors. A natural transformation $\alpha : F \Rightarrow G$ consists of

1. an arrow $\alpha_c : Fc \to Gc$ in D, called a component of the natural transformation, for each object $c \in \mathsf{C}$

such that, for any morphism $f : c \to c'$ in C,

$$\begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\alpha_{c'}"] & Gc' \end{tikzcd}$$

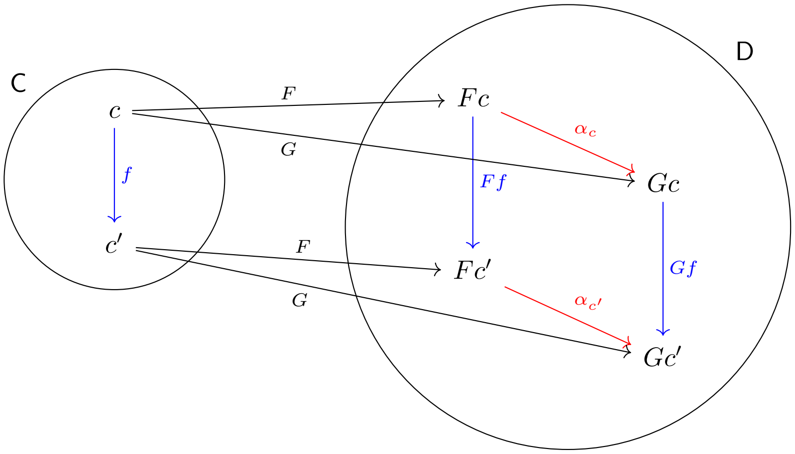
commutes.

A natural isomorphism is a natural transformation in which every component is an isomorphism. We write $F \cong G$ in this case.

We represent the natural transformation $\alpha : F \Rightarrow G$ between $F, G : \mathsf{C} \to \mathsf{D}$ diagrammatically as

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=L}] & \mathsf{D} \arrow[Rightarrow, from=U, to=L, "\alpha"] \end{tikzcd}$$

The origin of this commutative square can be found in the following picture:



Thus a natural transformation is one that shifts the image of a functor along the morphisms in the target category.
This leads us to the following proposition, which is sometimes used to define natural transformations:

Proposition IV.28. *Let $F, G : \mathsf{C} \to \mathsf{D}$ be functors. The natural transformations $\alpha : F \Rightarrow G$ correspond bijectively to functors $H : \mathsf{C} \times \mathbb{2} \to \mathsf{D}$ such that*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "i_0"] \arrow[dr, "F"'] & \mathsf{C} \times \mathbb{2} \arrow[d, "H"] & \mathsf{C} \arrow[l, "i_1"'] \arrow[dl, "G"] \\ & \mathsf{D} & \end{tikzcd}$$

commutes.

Here i_0, i_1 are the functors that map objects c to $(c, 0)$ and $(c, 1)$ respectively.
The same holds for natural isomorphisms, if $\mathbb{2}$ is replaced by $\mathbb{I}$.

Example

Let M be a monoid. A dynamical system is a functor $F : \mathsf{B}M \to \mathsf{C}$ from the delooping space to some category. Often this category is Top or Vect. Then the single object of $\mathsf{B}M$ is mapped to an object c in C and the morphisms of $\mathsf{B}M$ become actions on the object: each element $m \in M$ specifies a morphism in $\mathsf{B}M$, which is mapped to a morphism in C

$$m : c \to c : x \mapsto m \cdot x.$$

A morphism of dynamical systems, or intertwiner, is a map between natural transformation $\alpha : F \to G$. This can be seen as a map between objects in C that is covariant, in the sense that

$$\alpha(m \cdot x) = m \cdot \alpha(x).$$

1.3.1.1 Natural transformations as canonical maps

A canonical map α is a map between objects that arises naturally from the definition or the construction of the objects.
TODO: make rigourous! explicate for universal algebra.
In some sense the canonical map works at the level of structure and thus must commute with morphisms.
A canonical map is a functor such that there exists a natural transformation from the identit functor to it.
TODO: what about between different categories?

1.3.2 Equivalence of categories

An equivalence of categories consists of functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that we have natural isomorphisms $\mathrm{id}_{\mathsf{C}} \cong GF$ and $\mathrm{id}_{\mathsf{D}} \cong FG$.
The categories C, D are equivalent if there exists an equivalence between them. We write $\mathsf{C} \simeq \mathsf{D}$.

Except for considerations of size, this equivalence is an equivalence relation.

Lemma IV.29. *The notion of equivalence is reflexive, symmetric and transitive.*

Proposition IV.30. *A functor defining an equivalence of categories is full, faithful, and essentially surjective on objects.*
Assuming the axiom of choice, any functor with these properties defines an equivalence of categories.

Lemma IV.31. *An equivalence between skeletal categories is an isomorphism of categories*

Corollary IV.31.1. *Two categories are equivalent* if and only if *their skeletons are isomorphic.*

A category C is called

- essentially small if it is equivalent to a small category (i.e. its skeleton is small);
- essentially discrete if it is equivalent to a discrete category (i.e. its skeleton is discrete).

Lemma IV.32. *Every preorder category is equivalent to a poset category.*

1.4 Monoidal categories

A monoidal category is a category C equipped with

1. a functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ called the tensor product;
2. an object $1 \in \mathsf{C}$ called the unit object or the tensor unit

We write $(\mathsf{C}, \otimes, 1, \alpha, \lambda, \rho)$.

1.4.1 Enrichment

Let $(\mathsf{V}, \otimes, 1, \alpha, \lambda, \rho)$ be a monoidal category. A V-category C (or V-enriched category or category enriched over V) contains

1.

such that

Chapter 2

Higher category theory

2.1 2-categories

2.1.1 Functor categories

Proposition IV.33. *Let* C, D *be categories. There is a category of functors* $\mathsf{C} \to \mathsf{D}$ *with as morphisms natural transformations called the functor category* $[\mathsf{C}, \mathsf{D}]$.

Proof. There are clearly identity natural transformations $I : F \Rightarrow F$ for each functor $F : \mathsf{C} \to \mathsf{D}$ with components given by I_{Fc} for all $c \in \mathsf{C}$.
To verify composition, let $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be natural transformations between parallel functors $F, G, H : \mathsf{C} \to \mathsf{D}$. Then the composition $\beta\alpha : F \Rightarrow H$ has components

$$(\beta\alpha)_c = \beta_c \alpha_c.$$

We just need to show $\beta\alpha$ is a natural transformation. Consider the diagram

$$\begin{tikzcd}
Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"] & Gc \arrow[r, "\beta_c"] \arrow[d, "Gf"] & Hc \arrow[d, "Hf"] \\
Fc' \arrow[r, "\alpha_{c'}"] & Gc' \arrow[r, "\beta_{c'}"] & Hc'
\end{tikzcd}$$

Both squares commute by naturality of α, β. The rectangle then commutes: the only non-trivial part is the equality of paths $Fc \to Hc'$, but any such path can be deformed into another by flipping over squares (TODO better reference). ☐

2.1.2 The 2-category of categories

Chapter 3

Representability and universal properties

3.1 The Yoneda lemma

`https://math.stackexchange.com/questions/149376/excessive-use-of-the-yoneda-lemma`
`https://qchu.wordpress.com/2012/04/02/the-yoneda-lemma-i/`

Lemma IV.34. *Let* C, D *be categories. For every* $c \in \mathsf{C}$*, there is a functor defined by*

$$\mathrm{ev}_c : [\mathsf{C}, \mathsf{D}] \to \mathsf{D} : F \mapsto Fc.$$

Proof. A natural transformation $\alpha : F \Rightarrow G$ is mapped to its component $\alpha_c : Fc \to Gc$. By IV.33, for all F, $I_F \mapsto I_{Fc}$ and the composition of composable natural transformations β, α has the component $\beta_c \alpha_c$ at c. $\square$

So this gives us a functor

$$\mathrm{ev} : (\mathsf{Set}^{\mathsf{C}}, \mathsf{C}) \to \mathsf{Set} : \quad (F, c) \mapsto Fc.$$

Let C be a small category. Then for a given $c \in \mathsf{C}$, $\mathsf{C}(c, -)$ is the covariant functor represented by c and thus an object in $[\mathsf{C}, \mathrm{Set}]$, which is locally small. Then we can consider the covariant functor represented by $\mathsf{C}(c, -)$,

$$[\mathsf{C}, \mathrm{Set}] \to \mathsf{Set} : \quad F \mapsto \mathrm{Hom}(\mathsf{C}(c, -), F).$$

By currying, we can consider the two-sided represented functor $\mathsf{C}(-, -) : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Set}$ as a contravariant functor $\mathsf{C} \to \mathsf{Set}^{\mathsf{C}} : c \mapsto \mathsf{C}(c, -)$. Composing this with the contravariant functor represented by F gives the covariant functor

$$\mathsf{C} \to \mathsf{Set} : \quad c \mapsto \mathrm{Hom}(\mathsf{C}(c, -), F).$$

Combining these two functors we have a bifunctor

$$(\mathsf{Set}^{\mathsf{C}}, \mathsf{C}) \to \mathsf{Set} : \quad (F, c) \mapsto \mathrm{Hom}(\mathsf{C}(c, -), F).$$

If we now let C be only locally small, not necessarily small, $\mathsf{Set}^{\mathsf{C}}$ is no longer necessarily locally small. It turns out however that $\mathrm{Hom}(\mathsf{C}(c, -), F)$ still always yields a set. This assertion is part of the Yoneda lemma. The other part is the assertion that there is a natural isomorphism Φ:

$$\begin{tikzcd}
\mathsf{C} \times \mathsf{Set}^{\mathsf{C}} \arrow[rr, bend left, "\mathrm{Hom}(\mathsf{C}(c,-),F)"] \arrow[rr, bend right, "\mathrm{ev}_c(F)"'] \arrow[rr, phantom, "\Phi \Downarrow\cong"] & & \mathsf{Set}
\end{tikzcd} .$$

Theorem IV.35 (Yoneda lemma)**.** *Let* C *be a locally small category.*
Given a functor $F : \mathsf{C} \to \mathsf{Set}$ *and an object* c *in* C*, the function*

$$\Phi_{c,F} : \mathrm{Hom}(\mathsf{C}(c,-),F) \to Fc : \alpha \mapsto \alpha_c(I_c)$$

is a bijection.
Also Φ *is a natural transformation with components* $\Phi_{c,F}$*.*

Corollary IV.35.1 (Yoneda embedding)**.** *Let* C *be a locally small category. The functors*

$$\begin{tikzcd}
\mathsf{C} \arrow[r, hook] & \mathsf{Set}^{\mathsf{C}^{\mathsf{op}}} & & \mathsf{C}^{\mathsf{op}} \arrow[r, hook] & \mathsf{Set}^{\mathsf{C}} \\
c \arrow[r, mapsto] \arrow[d, "f"'] & \mathsf{C}(o(-),c) \arrow[d, "f_*"] & \textit{and} & c \arrow[r, mapsto] & \mathsf{C}(c,-) \\
{} \arrow[r, mapsto] & {} & & {} \arrow[r, mapsto] & {} \\
d \arrow[r, mapsto] & \mathsf{C}(o(-),d) & & d \arrow[r, mapsto] \arrow[uu, "f^o"] & \mathsf{C}(d,-) \arrow[uu, "f^*"']
\end{tikzcd}$$

define full and faithful embeddings.

Proof. The functors are clearly injective on objects as morphisms with different domain or codomain are different.
Consider first the left embedding. We want the transformation f_* to be natural (this is the transformation with all components equal to f_*). Indeed the functors $\mathsf{C}(o(-),c)$ and $\mathsf{C}(o(-),c)$ map morphisms g^o in C^{op} to g^*. Now f_* and g^* clearly commute, proving the naturality of f_*.
Then the function $\mathsf{C}(c,d) \to \mathrm{Hom}(\mathsf{C}(-,c),\mathsf{C}(-,d)) : f \mapsto f_*$ is the inverse of the Yoneda map $\Phi_{c,\mathsf{C}(-,d)}$:

$$\Phi_{c,\mathsf{C}(-,d)}(f_*) = f_*(I_c) = f \circ I_c = f.$$

Thus by the Yoneda lemma it is bijection, meaning

$$\mathsf{C}(c,d) \cong \mathrm{Hom}(\mathsf{C}(-,c),\mathsf{C}(-,d))$$

and the left functor is full and faithful by definition.
The statement for the right functor can be proven analogously. Alternatively, we can see that it is a dual statement in the following way: taking the category to be C^{op}, we get

$$\mathsf{C}^{\mathsf{op}}(c,d) \cong \mathrm{Hom}(\mathsf{C}^{\mathsf{op}}(-,c),\mathsf{C}^{\mathsf{op}}(-,d))$$

or

$$o[\mathsf{C}(d,c)] \cong \mathrm{Hom}(o \cdot \mathsf{C}(c,-) \cdot o, o \cdot \mathsf{C}(d,-) \cdot o)$$

now for every $\alpha \in \mathrm{Hom}(\mathsf{C}(c,-),\mathsf{C}(d,-))$, there is an α^o in $\mathrm{Hom}(o \cdot \mathsf{C}(c,-) \cdot o, o \cdot \mathsf{C}(d,-) \cdot o)$, so we have an isomorphism

$$\mathsf{C}(d,c) \cong \mathrm{Hom}(\mathsf{C}(c,-),\mathsf{C}(d,-)).$$

$\square$

It is common to refer both to the Yoneda lemma IV.35 and its corollary on the Yoneda embedding, IV.35.1, as the Yoneda lemma.

Proposition IV.36. *Let C be a locally small category. The following are equivalent:*

1. $f : x \to y$ *is an isomorphism in* C*;*
2. $f_* : \mathsf{C}(-, x) \Rightarrow \mathsf{C}(-, y)$ *is a natural isomorphism;*
3. $f^* : \mathsf{C}(y, -) \Rightarrow \mathsf{C}(x, -)$ *is a natural isomorphism.*

Proof. We have already shown f_* and f^* are natural transformations in IV.35.1. The proof follows from the Yoneda embedding IV.35.1 and IV.22.
Alternatively, a partial proof is as follows: If f has an inverse f^{-1}, then $(f^{-1})^*$ is an inverse of f^* and $(f^{-1})_*$ an inverse of f_*. ☐

3.2 Representable functors

3.2.1 Functors represented by objects

Let C be a locally small category and c an object in C.
Given a morphism $f : X \to Y$ in C, we can define the functions

1. $f_* : \mathsf{C}(c, X) \to \mathsf{C}(c, Y) : g \mapsto fg$ defined by post-composition;
2. $f^* : \mathsf{C}(Y, c) \to \mathsf{C}(X, c) : g \mapsto gf$ defined by pre-composition.

These are morphisms in the category Set.
Based on this we can define the covariant functor represented by c $\mathsf{C}(c, -)$ and the contravariant functor represented by c $\mathsf{C}(-, c)$:

$$\mathsf{C} \xrightarrow{\mathsf{C}(c,-)} \mathsf{Set} \qquad\qquad \mathsf{C} \xrightarrow{\mathsf{C}(-,c)} \mathsf{Set}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[dd, "f"'] & \mathsf{C}(c, x) \arrow[dd, "f_*"] \\
{} \arrow[r, mapsto] & {} \\
y \arrow[r, mapsto] & \mathsf{C}(c, y)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[r, mapsto] & \mathsf{C}(x, c) \arrow[dd, "f^*"] \\
{} \arrow[r, mapsto] & {} \\
y \arrow[r, mapsto] \arrow[uu, "f"] & \mathsf{C}(y, c)
\end{tikzcd}$$

We also have the two-sided represented functor $\mathsf{C}(-, -) : \mathsf{C} \times \mathsf{C} \to \mathsf{Set}$, which is contravariant in the first argument and covariant in the second:

$$\mathsf{C} \times \mathsf{C} \xrightarrow{\mathsf{C}(-,-)} \mathsf{Set}$$

$$\begin{tikzcd}
(x, y) \arrow[r, mapsto] \arrow[dd, shift left] & \mathsf{C}(x, y) \arrow[dd, "f^* h_* : g \mapsto hgf"] \\
{} \arrow[r, mapsto] & {} \\
(w, z) \arrow[r, mapsto] \arrow[uu, shift left, "{(f,h)}"] & \mathsf{C}(w, z)
\end{tikzcd}$$

The placement of the asterisk indicates the placement of the function being composed with f: below means to the right and above to the left. This is consistent with the notation for indicating bases in the part on matrix representations of linear functions (TODO ref).

Proposition IV.37. *Let* C *be a locally small category and* $f : X \to Y$ *a morphism. Then* f_* *and* f^* *are dual as follows:*

$$\begin{aligned}(f^o)^* &= of_*o = (f^*)^o;\\ (f^o)_* &= of^*o = (f^*)^o.\end{aligned}$$

Proof. We start with the locally small category C^{op}, the morphism $f^{\mathrm{o}} : Y \to X$ and an object c. Then

$$(f^{\mathrm{o}})^* : \mathsf{C}^{\mathsf{op}}(X,c) \to \mathsf{C}^{\mathsf{op}}(Y,c) : g^{\mathrm{o}} \mapsto g^{\mathrm{o}} f^{\mathrm{o}}.$$

We can rewrite this as

$$(f^{\mathrm{o}})^* : o[\mathsf{C}(c,X)] \to o[\mathsf{C}(c,Y)] : g^o \mapsto (fg)^o$$

or

$$o|_{o[\mathsf{C}(Y,c)]}(f^{\mathrm{o}})^*o|_{\mathsf{C}(c,X)} : \mathsf{C}(c,X) \to \mathsf{C}(c,Y) : g \mapsto fg.$$

This last function is exactly f_*. The other equality follows by duality: replacing C with C^{op} and $f : X \to Y$ with $f^o : Y \to X$ gives

$$(f^o)_* = o|_{o[\mathsf{C}^{\mathsf{op}}(X,c)]}(f^{\mathrm{oo}})^*o|_{\mathsf{C}^{\mathsf{op}}(c,Y)} = o|_{\mathsf{C}(c,X)}f^*o|_{o[\mathsf{C}(Y,c)]}$$

so

$$f^* = o|_{o[\mathsf{C}(c,X)]}(f^o)_*o|_{\mathsf{C}(Y,c)}.$$

□

Corollary IV.37.1. *The co- and contravariant represented functors are dual in the following sense:*

$$\begin{aligned}o \cdot \mathsf{C}^{\mathsf{op}}(c,-) \cdot o &= \mathsf{C}(-,c)\\ o \cdot \mathsf{C}^{\mathsf{op}}(-,c) \cdot o &= \mathsf{C}(c,-)\end{aligned}$$

Note that o still works as a function on sets of morphisms in C, even though they are now said to be in the category Set. We also use that o acts on transformations as $o(\alpha) = o\alpha o$ (TODO!).

Proof. We verify the equality for all objects and morphisms:

$$(o \cdot \mathsf{C}^{\mathsf{op}}(c,-) \cdot o)(x) = (o \cdot \mathsf{C}^{\mathsf{op}}(c,-))(x) = o(\mathsf{C}^{\mathsf{op}}(c,x)) = o(o(\mathsf{C}(x,c))) = \mathsf{C}(x,c)$$

and

$$(o \cdot \mathsf{C}^{\mathsf{op}}(c,-) \cdot o)(f) = (o \cdot \mathsf{C}^{\mathsf{op}}(c,-))(f^o) = o((f^o)_*) = o(of^*o) = oof^*oo = f^*.$$

□

Proposition IV.38. *Let* C *be a locally small category and* $f : X \to Y$ *a morphism. Then*

1. $f : X \to Y$ *is monic* if and only if *for all* c *in* C, $f_* : \mathsf{C}(c,X) \to \mathsf{C}(c,Y)$ *is injective;*

and, dually,

2. $f : X \to Y$ *is epic* if and only if *for all c in* C, $f^* : \mathsf{C}(Y, c) \to \mathsf{C}(X, c)$ *is injective.*

Proof. Assume f monic. To show injectivity, assume $f_*(g) = f_*(h)$, which means $fg = fh$. By monicity $g = h$, showing f^* is injective.
The converse is equally direct, as is the second statement.
The second statement also follows by duality, recognising that because o is bijective when restricted to the relevant sets, of^*o is injective if and only if f^* is, by IV.24 and **??**. ☐

3.2.2 Representable functors

Let C be a locally small category.
A covariant / contravariant functor $F : \mathsf{C} \to \mathsf{Set}$ is representable if there is an object $c \in \mathsf{C}$ such that F is naturally isomorphic to the covariant / contravariant functor represented by c, i.e.

$$F \cong \begin{cases} \mathsf{C}(c, -) & F \text{ covariant} \\ \mathsf{C}(-, c) & F \text{ contravariant.} \end{cases}$$

We say F is represented by the object $c \in \mathsf{C}$.
A representation of a functor $F : \mathsf{C} \to \mathsf{Set}$ is given by an object $c \in \mathsf{C}$ and a natural isomorphism α from F to the functor represented by c.

A functor is represented by at most one object, up to isomorphism, by IV.36.
By the Yoneda lemma, every natural isomorphism α corresponds to en element $\alpha_c(I_c) \in Fc$.

A univeral property of an object $c \in \mathsf{C}$ is expressed by a representable functor F together with a universal element $x \in Fc$ that corresponds to a natural isomorphism between F and the functor represented by c.

3.3 The category of elements

Chapter 4

Limits and colimits

Chapter 5

Adjunctions

Chapter 6

Monads

Chapter 7

Abelian categories

Five lemma. Short exact sequences. Split exact. Exact functors.
`https://mathoverflow.net/questions/363720/short-exact-sequences-every-mathematician-s`

Part V

Model theory

Part VI

Discrete mathematics

Chapter 1

Summation

$$a \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a x_i$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} = \sum_{j=1}^{m} \sum_{i=1}^{n} x_{ij}$$

δ_{ij}
Introduce $a : b$ for sequence. (bounds inclusive)
$-$ means everything.

Chapter 2

Combinatorics

`https://www.cut-the-knot.org/` `https://math.stackexchange.com/questions/785624/show-that-sum-limits-sigma-in-s-n-mboxnumber-of-fixed-points-of-si/785652#785652`
todo: inclusion-exclusion principle + move after analysis

2.1 Permutations

Permutation group S_n.
Factorials

2.2 Combinations

$$\binom{n}{k} = \frac{n!}{k!(n-k!)}$$

(k, m) shuffle
Binomial theorem. Fill in $x = 1, y = -1$.

2.3 Finite sets

We call a set X an n-set if its cardinality is n.

2.3.1 Covering finite sets

Let X be a finite set.

- A cover is a collection $\mathcal{F}$ of sets such that $\bigcup \mathcal{F} = X$;
- A k-cover is a cover containing k sets.

`https://core.ac.uk/download/pdf/82680601.pdf`

Part VII

Elements of mathematics

Chapter 1

Elements of Euclidean geometry

TODO: Tusi couple
In this section we will give a practical rundown of some of the classic results of Euclidean geometry, especially those results that have elementary proofs not needing more involved machinery. We will focus on practical things like calculating angles, surface areas and volumes. A more general discussion will follow in the section about spaces.

1.1 Flat shapes

Pick theorem

1.1.1 Circles

are shapes consisting of all the points at a certain fixed distance from a central point. We call this distance the radius, denoted r. We can then calculate the circumference

$$C_\circ = 2\pi r$$

and the surface area

$$A_\circ = \pi r^2.$$

Chord. Diameter.

1.1.2 Rectangles

Rectangles are made up of four lines which intersect at right angles. Rectangles have a length l and a width w. The surface area is

$$A_\square = lw.$$

Squares are rectangles with the same length and width.

1.1.3 Triangles

Triangles are shapes with three sides. To calculate the surface area we choose one side to be our base b. The shortest possible distance from the line that extends this side to the point not on this line is called the height h. The surface area is then

$$A_\triangle = \frac{1}{2} bh.$$

TODO picture of proof: box around triangle. Draw line from point to base that splits box in two. In each half the triangle takes up half the area.

Proposition VII.1. *A triangle, inscribed in a circle, consisting of two chords and a diameter always has a right angle.*

Proof. Rotate the triangle 180°. □

1.2 Solids

1.2.1 Spheres

are shapes consisting of all the points at a certain fixed distance from a central point. We call this distance the radius, denoted r. The surface area is

$$S_{\text{sphere}} = 4\pi r^2$$

and the volume is

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3.$$

1.2.2 Prisms

are extrusions of a polygonal base, not necessarily in a direction orthogonal to the plane of the base. If A is the surface area of the base, the volume of the prism is

$$V_{\text{prism}} = Ah.$$

1.2.3 Cylinders

are like prisms, but with plane curves instead of polygons as their base. Again, if A is the surface area of the base, the volume is

$$V_{\text{cylinder}} = Ah.$$

1.2.4 Triangular pyramids

have a volume

$$V_{\text{pyramid}} = \frac{1}{3} Ah$$

1.3 Angles

We often use greek letters like α, β or γ to denote angles. For quantifying angles we can use degrees. Or we can identify each direction with a point on a circle of radius 1 and use the distance along the edge of the edge of the circle to represent the angle. We call that distance the angle in radians. In that case 360 degrees (or 360°) is the full circumference, or 2π radians (also written as 2π rad or just 2π); 180° is half that, or π.

So we can convert any angle in degrees to radians by dividing by 360° and multiplying by 2π. To convert the other way we divide by 2π and multiply by °.
Radians are often useful to work with because the arc length L of a circular arc with radius r and subtending an angle θ (measured in radians), is

$$L = \theta r.$$

We will usually use radians, and it should be assumed that all angles are expressed in radians, unless degrees or other units are explicitly stated.

1.4 Some classic results and theorems

TODO similar triangles, inner angles
Viviani's theorem

1.5 Trigonometry

TODO

1.5.1 Defining sine and cosine

With triangle, but not all numbers, so circle.
Domain and image
The squaring notation $\sin^2\theta$
Table of angles

1.5.1.1 Some useful identities

- **Pythagorean identity**

  $$\cos^2\theta + \sin^2\theta = 1$$

  Corollary: $\sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha$.

- **Periodicity**

  $$\begin{cases}\cos(\theta + 2\pi) = \cos\theta \\ \sin(\theta + 2\pi) = \sin\theta\end{cases}$$

- Cosine is **even**, sine is **odd**

  $$\begin{cases}\cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta\end{cases}$$

- **Complementary angles**. Two angles are complementary if their sum is $\pi/2$.

  $$\begin{cases}\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \\ \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\end{cases}$$

- **Supplementary angles**. Two angles are supplementary if their sum is π.

  $$\begin{cases}\cos\left(\pi - \theta\right) = -\cos\theta \\ \sin\left(\pi - \theta\right) = \sin\theta\end{cases}$$

TODO fig.

1.5.1.2 Addition formulae

$$\begin{aligned}\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi \\ \cos(\theta-\phi) &= \cos\theta\cos\phi + \sin\theta\sin\phi \\ \sin(\theta-\phi) &= \sin\theta\cos\phi - \cos\theta\sin\phi\end{aligned}$$

1.5.1.3 Double- and half-angle formulae

Double-angle

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

Half-angle

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \text{and} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}.$$

1.5.2 Other trigonometric functions

tangent, cotangent, secant, cosecant (primary / secondary)

1.5.3 Angles and sides in triangles.

TODO figure vertices A, B, C and sides a, b, c opposite.

1.5.3.1 Law of sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

1.5.3.2 Law of cosines

$$\begin{aligned}a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C\end{aligned}$$

1.5.4 Waves

1.5.5 Plane waves

frequency , wavelength, angular frequency, period

1.5.6 The wave equation

1.5.7 Group and phase velocity

1.6 Cyclometric functions

Name places me geographically.
Restrict domain to make bijective. Both notations $\sin^{-1}$ and arcsin
TODO + continuity of $\cos^{-1}$

1.7 Hyperbolic functions

Definition using e.

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

Origin of name:

$$\cosh^2 t - \sinh^2 t = 1$$

(Proof:)

$$\begin{aligned}\cosh^2 t - \sinh^2 t &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{1}{4}\left(e^{2t} + 2 + e^{-2t} - (e^{2t} - 2 + e^{-2t})\right) \\ &= \frac{2+2}{4} = 1\end{aligned}$$

Cosh is catenary curve.
Many properties similar to regular sine and cosine:

- $\cosh 0 = 1$ and $\sinh 0 = 0$;
- The hyperbolic cosine is even ($\cosh(-x) = \cosh x$) and the hyperbolic sine is odd ($\sinh(-x) = \sinh x$).
- Addition formulae (notice sign difference with cosh):

$$\begin{aligned}\cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \\ \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y\end{aligned}$$

- Double angle formulae (again sign difference for cosh):

$$\begin{aligned}\sinh 2x &= 2 \sinh x \cosh x \\ \cosh 2x &= \cosh^2 x + \sinh^2 x \\ &= 2\cosh^2 x - 1 \\ &= 1 + 2\sinh^2 x\end{aligned}$$

Other hyperbolic functions:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Inverse hyperbolic functions:

$$\sinh^{-1} x = \log_e\left(x + \sqrt{x^2+1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \qquad (-1 < x < 1)$$

$$\cosh^{-1} x = \log_e\left(x + \sqrt{x^2-1}\right) \qquad (x \geq 1)$$

with restriction because cosh is not automatically bijective. (tanh?)

Part VIII

Convergence and topology

https://en.wikipedia.org/wiki/Cauchy_space
https://en.wikipedia.org/wiki/Cauchy-continuous_function
https://en.wikipedia.org/wiki/Proximity_space
https://www.bioinf.uni-leipzig.de/~studla/Publications/PREPRINTS/01-pfs-007-subl1.pdf

Chapter 1

Convergence

1.1 Convergence spaces

Intuition: directed sets and refinement.

Let X be a set and let ξ be a relation between downward directed sets in $\mathcal{P}(X)$ and elements of X. Then

- we denote the image function of ξ as $\lim_\xi : \mathcal{D}(\mathcal{P}(X)) \to \mathcal{P}(X) : F \mapsto F\xi$; we call $\lim_\xi F$ the ξ-limit of F;
- ξ is called a preconvergence on X if $\lim_\xi$ is order-preserving when $\mathcal{D}(\mathcal{P}(X))$ is ordered by refinement:
$$F \preceq G \implies \lim_\xi F \subseteq \lim_\xi G;$$
- ξ is called a convergence if it is a preconvergence and it is centered:
$$\forall x \in X : \quad x \in \lim_\xi \big\{\{x\}\big\}.$$

If ξ is a convergence, then we call (X, ξ) a convergence space.
If $\lim_\xi F \neq \emptyset$, we say the directed set F converges.

We write ${\lim_\xi}^{-1}$ for the preimage function of ξ restricted to directed sets $\nsim \mathcal{P}(X)$. Thus ${\lim_\xi}^{-1}(x)$ is the set of all directed sets D such that

- $D \xrightarrow{\xi} x$;
- $D \nsim \mathcal{P}(X)$.

Lemma VIII.1. *Let X be a set and $x \in X$. A preconvergence ξ on X is a convergence* if and only if $\forall x \in X : \{x\} \in {\lim_\xi}^{-1}(x)$.

Lemma VIII.2. *Let X be a set, ξ a convergence on X and $x \in X$. Then $\lim_\xi^{-1}(x)$ is upwards closed.*

1.1.1 Filters and convergence

Lemma VIII.3. *Let X be a set and ξ a convergence on X. Let $A, B \in \mathcal{D}(X)$ be downward directed sets. If $A \approx B$, then $\lim_\xi A = \lim_\xi B$.*

Proof. We have $A \preceq B$ and $B \preceq A$, so $\lim_\xi A \subseteq \lim_\xi B$ and $\lim_\xi B \subseteq \lim_\xi A$. ☐

This lemma means we can view $\lim_\xi$ as a function on the quotient $\mathcal{D}(X)/\approx$, which is order isomorphic to $\mathcal{F}(X)$ (indeed there is one filter in each equivalence class in $\mathcal{D}(X)/\approx$ and for filters the refinement relation simplifies to inclusion).
Consequently we will usually think of a convergence on a set X as a relation between filters on $\mathcal{P}(X)$ and elements of X. The axioms then become

- ξ is a preconvergence on X if $\lim_\xi$ is order-preserving when $\mathcal{FP}(X)$ is ordered by inclusion:

$$F \subseteq G \implies \lim_\xi F \subseteq \lim_\xi G;$$

- ξ is called a convergence if it is a preconvergence and it is centered:

$$\forall x \in X : \quad x \in \lim_\xi \dot{x}.$$

Any convergence defined for filters uniquely extends to a convergence defined for all downward directed sets.

Lemma VIII.4. *Let X be a set and ξ a preconvergence on X. Then*

1. $\forall F, G \in \mathcal{FP}(X) : \quad \lim_\xi(F \cap G) \subseteq \lim_\xi F \cap \lim_\xi G$;
2. $\forall F, G \in \mathcal{FP}(X) : \quad \lim_\xi(F \cup G) \supseteq \lim_\xi F \cup \lim_\xi G$.

Proof. Reformulation of III.54, because a preconvergence is order-preserving. ☐

Corollary VIII.4.1. *For any convergence ξ on X we have $\lim_\xi \mathcal{P}(X) = X$.*

Proof. Clearly $\dot{x} \subseteq \mathcal{P}(X)$ for all $x \in X$, so

$$\lim_\xi(\mathcal{P}(X)) \supseteq \lim_\xi \left(\bigcup_{x \in X} \dot{x} \right) \supseteq X.$$

☐

In the sequel we will usually consider convergence as a property of filters, however sometimes it will be easier to consider downward directed sets (e.g. for continuity).

1.1.1.1 Approaches

Let (X, ξ) be a convergence space. An approach on X is a function $F : X \to \mathcal{FP}(X)$ such that $F(x) \xrightarrow{\xi} x$ for all $x \in X$.

1.1.2 Finite depth and other requirements

TODO Kent convergence.

Let (X, ξ) be a convergence space. Take arbitrary $F, G \in \mathcal{FP}(X)$ and $x \in X$. The convergence space is called

- finitely deep or a limit space if $\lim_\xi (F \cap G) = \lim_\xi F \cap \lim_\xi G$;
- a Kent space if $F \to x \implies F \cap \dot{x} \to x$.

TODO replace finite depth with limit space.

Lemma VIII.5. *Let X be a set and ξ a preconvergence on X. Then ξ is finitely deep* if and only if

$$\forall F, G \in \mathcal{FP}(X) : \quad \lim_\xi (F \cap G) \supseteq \lim_\xi F \cap \lim_\xi G.$$

In other words, a preconvergence is finitely deep if and only if $\lim_\xi^{-1}(x)$ is directed (and thus a filter) for all $x \in X$.
TODO finitely deep = quotient of (pre)topologies; inf of topologies; prime topologies dense in it (?)

1.1.3 The lattices of preconvergences and convergences

Let X be a set and ξ, ζ preconvergences on X. We say ξ is finer (or stronger) than ζ, denoted $\xi \leq \zeta$, if $\lim_\xi F \subseteq \lim_\zeta F$ for all $F \in \mathcal{FP}(X)$. We also say ζ is coarser (or weaker) than ξ.

We can think of strength as the ability to stop a filter converging. So ξ is strictly stronger that ζ if there are filters that converge in ζ, but not in ξ.

Lemma VIII.6. *Let X be a set and ξ a preconvergence on X. Then*

1. $E_{\mathcal{FP}(X),X} \leq \xi \leq U_{\mathcal{FP}(X),X}$;
2. *ξ is a convergence* if and only if $\iota_X \leq \xi$.

Let X be a set.

- The empty preconvergence on X is $E_{\mathcal{FP}(X),X}$, i.e. the limit of all filters is the empty set.
- The discrete convergence ι_X on X is defined by $x \in \lim_\iota F \iff F = \dot{x}$ if F is a proper filter. If $F = \mathcal{P}(X)$, then $\lim_\iota F = X$.
- The chaotic convergence on X is $U_{\mathcal{FP}(X),X}$, i.e. the limit of all filters is X.

Proposition VIII.7. *Let X be a set and and let Ξ be a set of (pre)convergences on X. The sets of preconvergences and of convergences are complete bounded lattices and for all $F \in \mathcal{FP}(X)$:*

$$\lim_{\bigwedge \Xi} F = \bigcap_{\xi \in \Xi} \lim_\xi F \quad \text{and} \quad \lim_{\bigvee \Xi} F = \bigcup_{\xi \in \Xi} \lim_\xi F.$$

- *The top of both is the chaotic convergence.*
- *The bottom of the lattice of preconvergences is the empty preconvergence.*

- *The bottom of the lattice of convergences is the discrete convergence.*

Lemma VIII.8. *A preconvergence is a convergence* if and only if *it is coarser than the discrete convergence.*

1.1.4 Directional convergence

Let (X, ξ) be a convergence space, $D \subseteq X$ and $x \in X$. Then a filter $F \in \mathcal{FP}(X)$ is said to converge to x in the direction D if $F \xrightarrow{\xi} x$ and $D \in F$.
We write $F \xrightarrow{\xi, D} x$.

Lemma VIII.9. *Let (X, ξ) be a convergence space, $D \subseteq X$ and $x \in X$. Then F converges to x in the direction D* if and only if *there exists a net $\langle x_i \rangle_{i \in I} \subseteq D$ that converges to x such that $F = \mathrm{TailsFilter}(\langle x_i \rangle_{i \in I})$.*

Proof. If there exists such a net, then F converges to x by definition of net convergence. Every tail of $\langle x_i \rangle$ is a subset of D. So $D \in F$ by upward closure.
Now assume F converges to x in the direction D. Consider the index set $I_F = \{(A, x) \in F \times X \mid x \in A\}$ as in VIII.201. Now define $I_F^D = I_F \setminus \{(A, x) \in F \times X \mid x \notin D\}$. We claim this set if directed and the associated filter of $I_F^D \to X : (A, x) \mapsto x$ is F.
Directedness follows because for all $(A, x), (B, y) \in I_F^D$, the set $A \cap B \cap D$ is not empty as all three sets are elements of the filter F and the filter is not trivial.
Finally we show that F is generated by the tails of $I_F^D \to X : (A, x) \mapsto x$. Take $A \in F$. Then $A \cap D \in F$ and by VIII.202

$$A \cap D = \{y \in X \mid (B, y) \geq (A \cap D, x)\}.$$

Then $y \in B \subseteq A \cap D \subseteq D$, so this is also a tail of $I_F^D \to X : (A, x) \mapsto x$. □

1.2 Pointwise properties

A property $\mathbf{P}$ of preconvergences is called pointwise if there exists a property $\mathbf{Q}$ of classes of filters such that for all preconvergences ξ on X,

$$\xi \in \mathbf{P} \iff \forall x \in X : \lim_{\xi}{}^{-1}(x) \in \mathbf{Q}.$$

1.2.1 Isolation and primeness

Let (X, ξ) be a convergence space.

- A point $x \in X$ is called isolated if $\lim_{\xi}{}^{-1}(x) = \left\{\dot{x}\right\}$.
- The convergence space (X, ξ) is called prime if it contains at most one non-isolated point. Such a point is called distinguished.

A convergence space is discete if and only if each point is isolated. A discrete convergence space is prime.

1.2.2 Pavement

Let (X, ξ) be a convergence space.

- A pavement of ξ at a point $x \in X$ is a family of filters $\mathcal{H}$ such that

$$\lim_{\xi}{}^{-1}(x) = \uparrow \mathcal{H}.$$

- The paving number of ξ at x is the least cardinality κ such that there is a pavement of ξ of cardinality κ at x.
- The paving number λ of ξ is

$$\lambda = \sup \{ \kappa \mid \kappa \text{ is the paving number of } \xi \text{ at } x \text{ for some } x \in X \}.$$

  We say ξ is κ-paved if $\lambda \leq \kappa$.
- We call the convergence ξ pretopological if it is 1-paved.

Being κ-paved is a pointwise property.

Proposition VIII.10. *A finitely deep convergence that is finitely paved is pretopological.*

1.2.3 A base of a convergence

Let (X, ξ) be a convergence space. A set $\mathcal{Z} \subseteq \mathcal{P}(X)$ is called a base of the converence ξ if for each point $x \in X$ and each convergent filter $F \in \lim_{\xi}^{-1}(x)$ there exists a filter base $G \subseteq \mathcal{Z}$ such that $x \in \lim_{\xi} G$.
We also say ξ is based in $\mathcal{Z}$.
The weight of ξ is the least cardinal of a base of ξ.

TODO filter based in set.

1.3 The vicinity filter

Let X be a set, ξ a preconvergence on X and $x \in X$. The vicinity filter of ξ at x is

$$\mathcal{V}_{\xi}(x) := \bigcap \overset{-1}{\lim_{\xi}}(x) = \bigcap \left\{ F \in \mathcal{FP}(X) \;\middle|\; x \in \lim_{\xi} F \right\}.$$

A subset $V \subseteq X$ is called a vicinity of x for ξ if $V \in \mathcal{V}_{\xi}(z)$.
We extend the vicinity filters to be defined for elements of $\mathcal{P}(X)$ and $\mathcal{P}^2(X)$ by contours.

Lemma VIII.11. *Let X be a set, ξ a preconvergence on X and $A \subseteq X$. Then*

$$\begin{aligned}\mathcal{V}_{\xi}(A) &= \bigcap \left\{ F \in \mathcal{FP}(X) \;\middle|\; \exists x \in A : \; x \in \lim_{\xi} F \right\} \\ &= \bigcap \left\{ F \in \mathcal{FP}(X) \;\middle|\; A \mathrel{\#} \lim_{\xi} F \right\}.\end{aligned}$$

Lemma VIII.12. *Let (X,ξ) be a convergence space. Then $\mathcal{V}_\xi$ is an approach* if and only if *ξ is pretopological.*

Lemma VIII.13. *Let (X,ξ) be a convergence space and $A \subseteq X$. Then $\mathcal{V}_\xi(A) \subseteq {\uparrow}\{A\}$. In particular for $x \in X$, $\mathcal{V}_\xi(x) \subseteq \dot{x}$.*

This also means $\{x\} \in \mathcal{V}_\xi(x)^{\#}$.

Proof. We have $\mathcal{V}_\xi(A) \subseteq \bigcap_{x\in A} \dot{x} = {\uparrow}\{A\}$. ☐

Lemma VIII.14. *Let X be a set, ξ a preconvergence on X and $x \in X$. Then*

$$\mathcal{V}_\xi(x) = \bigcap_{F\in \lim_\xi{}^{-1}(x)\,\cap\, \mathbb{U}(\mathcal{P}(X))} F.$$

Proof. TODO ☐

Lemma VIII.15. *Let (X,ξ) be a convergence space and $x \in X$. Then*

1. *if $F \to x$, then $\mathcal{V}_\xi(x) \lhd F$*

Proof. (1) Simple application of III.49. ☐

Lemma VIII.16. *Let X be a set and ζ, ξ convergences on X. If $\zeta \leq \xi$, then $\mathcal{V}_\zeta(x) \supseteq \mathcal{V}_\xi(x)$ for all $x \in X$.*

Proof. Assume $\zeta \leq \xi$. For all $F \in \mathcal{FP}(X)$ we have $\lim_\zeta^{-1}(x) \subseteq \lim_\xi^{-1}(x)$, so

$$\mathcal{V}_\zeta(x) = \bigcap \overset{-1}{\lim_\zeta}(x) \supseteq \bigcap \overset{-1}{\lim_\xi}(x) = \mathcal{V}_\xi(x).$$

☐

1.3.1 Pretopological convergence

1.4 Adherence and inherence

Let (X,ξ) be a convergence space and $\mathcal{A} \subseteq \mathcal{P}(X)$ a family of subsets. We define

- the adherence of $\mathcal{A}$ as
$$\operatorname{adh}_\xi(\mathcal{A}) = \bigcup_{\mathcal{A}\lhd F} \lim_\xi F;$$
- the inherence of $\mathcal{A}$ as
$$\operatorname{inh}_\xi(\mathcal{A}) = \left\{ x \in X \;\middle|\; x \in \lim_\xi F \implies F \,\#\, \mathcal{A} \right\}.$$

Lemma VIII.17. 1. *If $F \to x$, then $x \in \operatorname{adh}_\xi(F)$;*

2. *If $F = \mathfrak{F}(\mathcal{A})$, then $\operatorname{adh}_\xi(F) = \operatorname{adh}_x i(\mathcal{A})$.*

Lemma VIII.18. *Let (X,ξ) be a convergence space and $x \in X$. Then $\operatorname{adh}_\xi(\{x\}) = \lim_\xi \dot{x}$.*

Proof. If F is a filter such that $\{x\} \lhd F$, then $F = \dot{x}$ by III.50. So $\mathrm{adh}_\xi\{x\} = \lim_\xi \dot{x}$. □

Proposition VIII.19. *Let* (X,ξ) *be a convergence space and* $F \subseteq \mathcal{FP}(X)$ *a filter. Then* $\ker(F) \subseteq \mathrm{adh}_\xi(F)$

Proof. Take $x \in \ker(F)$. Then $\dot{x} \mathrel{\#} F$, so $F \vee \dot{x}$ is a proper filter that converges to x. Now $F \lhd F \vee \dot{x}$ by III.49, so $x \in \lim_\xi F \vee \dot{x} \subseteq \mathrm{adh}_\xi(F)$. □

Let X be a set, ξ a convergence on X and $A \subseteq X$ a subset. We define

- the adherence $\mathrm{adh}_\xi(A)$ of A by
$$x \in \mathrm{adh}_\xi(A) \quad \Leftrightarrow_{\mathrm{def}} \quad A \in \mathcal{V}_\xi(x)^\#;$$
- the inherence $\mathrm{inh}_\xi(A)$ of A by
$$x \in \mathrm{inh}_\xi(A) \quad \Leftrightarrow_{\mathrm{def}} \quad A \in \mathcal{V}_\xi(x).$$

Proposition VIII.20. *Let* (X,ξ) *be a convergence space and* $A \subseteq X$ *a subset. Then*

1. $x \in \mathrm{adh}_\xi(A)$ *if and only if* $A \in \mathcal{V}_\xi(x)^\#$*;*
2. $x \in \mathrm{inh}_\xi(A)$ *if and only if* $A \in \mathcal{V}_\xi(x)$*.*

Proof. TODO □

Corollary VIII.20.1. *Let* (X,ξ) *be a convergence space and* $A, B \subseteq X$ *subsets. Then*
$$\mathrm{adh}_\xi(A) \mathrel{\#} B \qquad \iff \qquad \{A\} \mathrel{\#} \mathcal{V}_\xi(B).$$

Proof. We have
$$\begin{aligned}
\mathrm{adh}_\xi(A) \mathrel{\#} B &\iff \exists b \in B : b \in \mathrm{adh}_\xi(A) \\
&\iff \exists b \in B : A \in \mathcal{V}_\xi(b)^\# \\
&\iff A \in \bigcup_{b \in B} \mathcal{V}_\xi(b)^\# = \left(\bigcap_{b\in B} \mathcal{V}_\xi(b)\right)^\# = \mathcal{V}_\xi(B)^\#.
\end{aligned}$$
□

Proposition VIII.21. *Let* X *be a set,* ξ *a convergence on* X *and* $A \subseteq X$ *a set. Then*

1. $x \in \mathrm{adh}_\xi(A) \iff A \in \bigcup_{F \in \lim_\xi{}^{-1}(x)} F^\# \iff A \in \bigcup_{F \in \lim_\xi{}^{-1}(x)} F$*;*
2. $\mathrm{adh}_\xi(A) = \bigcup_{A \in F^\#} \lim_\xi F = \bigcup_{\substack{G \text{ proper filter} \\ A \in G}} \lim_\xi G$*;*
3. $x \in \mathrm{inh}_\xi(A) \iff A \in \bigcap_{F \in \lim_\xi{}^{-1}(x)} F$*;*
4. $\mathrm{inh}_\xi(A) = \bigcap_{A \in F} \lim_\xi F$*.*

Proof. TODO □

Corollary VIII.21.1. *Let X be a set, ξ a convergence on X and $A \subseteq X$. Then*

$$(\operatorname{inh}_\xi A)^c = (\operatorname{adh}_\xi A^c)^c.$$

Corollary VIII.21.2. *Let X be a set, ξ a convergence on X and $A, B \subseteq X$. Then*

1. $\operatorname{adh}_\xi \emptyset = \emptyset$;
2. *if* $A \subseteq B$, *then* $\operatorname{adh}_\xi A \subseteq \operatorname{adh}_\xi B$;
3. $A \subseteq \operatorname{adh}_\xi A$;
4. $\operatorname{adh}_\xi(A \cup B) = \operatorname{adh}_\xi A \cup \operatorname{adh}_\xi B$.

and

5. $\operatorname{inh}_\xi X = X$;
6. *if* $A \subseteq B$, *then* $\operatorname{inh}_\xi A \subseteq \operatorname{inh}_\xi B$;
7. $\operatorname{inh}_\xi A \subseteq A$;
8. $\operatorname{inh}_\xi(A \cap B) = \operatorname{inh}_\xi A \cap \operatorname{inh}_\xi B$.

Lemma VIII.22. *Let X be a set, ξ a convergence on X and $A \subseteq X$ a set. Then*

$$\operatorname{adh}_\xi A = \bigcup_{A \in F^\#} \lim_\xi F = \bigcup_{\substack{G \text{ proper filter} \\ A \in G}} \lim_\xi G.$$

Proof. The first equality is obvious. For the second equality: Let F be a filter such that $A \in F^\#$ then $F \vee \uparrow A$ contains A and is a proper filter by III.118, so $\lim_\xi F \subseteq \lim_x F \vee \uparrow A \subseteq \bigcup_{A \in G} \lim_\xi G$. For the other inclusion: $A \in G$ iff $\uparrow\{A\} \subseteq G$, so $G = G \vee \uparrow A$. But G is a proper filter, so it must mesh with $\uparrow A$. In particular it must mesh with A. □

Proposition VIII.23. *Let X be a set, ξ a convergence on X and $A \subseteq X$. Then the following are equivalent:*

1. $x \in \operatorname{adh}_\xi A$;
2. $A \in \bigcup_{F \in \lim_\xi{}^{-1}(x)} F^\#$;
3. $A \# \mathcal{V}_\xi(x)$.

Proof. (1) ⇔ (2) We have

$$x \in \bigcup_{A \in F^\#} \lim_\xi F \iff \exists F : A \in F^\# \wedge x \in \lim_\xi F \iff A \in \bigcup_{F \in \lim_\xi{}^{-1}(x)} F^\#.$$

(2) ⇔ (3) We have, using III.55.1,

$$A \# \mathcal{V}_\xi(x) \iff A \in (\mathcal{V}_\xi(x))^\# = \left(\bigcap_{F \in \lim_\xi{}^{-1}(x)} F \right)^\# = \bigcup_{F \in \lim_\xi{}^{-1}(x)} F^\#.$$

□

Lemma VIII.24. *Let X be a set, ζ, ξ convergences on X such that $\zeta \leq \xi$ and $A \subseteq X$ a subset. Then*

$$\operatorname{inh}_\xi(A) \subseteq \operatorname{inh}_\zeta(A) \subseteq A \subseteq \operatorname{adh}_\zeta(A) \subseteq \operatorname{adh}_\xi(A).$$

Proof. We only need to show the first and last inclusion.
From VIII.16 we have $\mathcal{V}_\xi(x) \subseteq \mathcal{V}_\zeta(x)$ and $\mathcal{V}_\zeta(x)^\# \subseteq \mathcal{V}_\xi(x)^\#$ for all $x \in X$.
So we have the implications

$$x \in \operatorname{inh}_\xi(A) \implies A \in \mathcal{V}_\xi(x) \implies A \in \mathcal{V}_\zeta(x) \implies x \in \operatorname{inh}_\zeta(A)$$

and

$$x \in \operatorname{adh}_\zeta(A) \implies A \in \mathcal{V}_\zeta(x)^\# \implies A \in \mathcal{V}_\xi(x)^\# \implies x \in \operatorname{adh}_\xi(A).$$

□

1.4.1 General adherence (TODO)

Let X be a set, ξ a convergence on X and $\mathcal{A} \subseteq \mathcal{P}(X)$. The adherence of $\mathcal{A}$ is defined as

$$\operatorname{adh}_\xi \mathcal{A} := \bigcup_{\substack{F \in \mathcal{FP}(X) \\ F \# \mathcal{A}}} \lim_\xi F.$$

The (principal) adherence of a set $A \subseteq X$ is the adherence of $\{A\}$.

Clearly if ξ, ζ are convergences on X, then $\xi \leq \zeta$ implies $\operatorname{adh}_\xi \mathcal{A} \subseteq \operatorname{adh}_\zeta \mathcal{A}$.

Lemma VIII.25. *Let X be a set, ξ a convergence on X and $\{A_i\}_{i \in I} \subseteq \mathcal{P}(X)$. Then*

$$\operatorname{adh}_\xi \{A_i\}_{i \in I} \subseteq \bigcap_{i \in I} \operatorname{adh}_\xi A_i.$$

Proof. We have $x \in \operatorname{adh}_\xi \{A_i\}_{i \in I}$ iff $\exists F : x \in \lim F \land F \# \{A_i\}_{i \in I}$ iff $\exists F : \forall i \in I : x \in \lim F \land F \# A_i$. This implies $\forall i \in I : \exists F : x \in \lim F \land F \# A_i$ iff $\forall i \in I : x \in \operatorname{adh}_\xi A_i$. □

Lemma VIII.26. *Let X be a set and F be a filter in $\mathcal{FP}(X)$. Then $\ker F = \operatorname{adh}_{\iota_X} F$.*

Proof. Let $x \in X$. Then $\dot{x} \# F$ iff $\Big\{\{x\}\Big\} \# F$ iff $x \in \ker F$. □

Corollary VIII.26.1. *Let ξ be a convergence on a set X and F be a filter in $\mathcal{FP}(X)$. Then $\ker F \subseteq \operatorname{adh}_\xi F$.*

Proof. This follows from $\iota_X \leq \xi$. □

1.4.2 Dense sets

Let (X, ξ) be a convergence space and A a subset. The subset A is called dense in X if $\operatorname{adh}_\xi(A) = X$.

1.4.2.1 Strict density

TODO strict density (Colebunders)

1.4.3 Inherence

Let X be a set, ξ a convergence on X and $A \subseteq X$. The (principal) inherence of A, denoted $\operatorname{inh}_\xi A$, is defined by

$$x \in \operatorname{inh}_\xi A \quad \Leftrightarrow_{\text{def}} \quad A \in \mathcal{V}_\xi(x).$$

Lemma VIII.27. *Let X be a set, ξ a convergence on X and $A \subseteq X$. Then*

1. $\operatorname{inh}_\xi A = (\operatorname{adh}_\xi A^c)^c$;
2. $x \in \operatorname{inh}_\xi A \iff (x \in \lim_\xi F \implies A \in F)$.

Proposition VIII.28. *Let X be a set, ξ a convergence on X and $A, B \subseteq X$. Then*

1. $\operatorname{inh}_\xi X = X$;
2. *if* $A \subseteq B$*, then* $\operatorname{inh}_\xi A \subseteq \operatorname{inh}_\xi B$;
3. $\operatorname{inh}_\xi A \subseteq A$;
4. $\operatorname{inh}_\xi(A \cap B) = \operatorname{inh}_\xi A \cap \operatorname{inh}_\xi B$.

Lemma VIII.29. *Let (X, ξ) be a convergence space and $A, B \subseteq X$ subsets. If for every $a \in A$ there exists a vicinity of a that is a subset of B, then $A \subseteq \operatorname{inh}_\xi(B)$.*

Proof. Assume that for every $a \in A$ there exists a $U_a \in \mathcal{V}_\xi(a)$ such that $U_a \subseteq B$. Then $B \in \mathcal{V}_\xi(a)$ for all a in A and thus $a \in \operatorname{inh}_\xi(B)$ for all $a \in A$. □

1.4.4 Topology

1.4.4.1 Open and closed sets

Let X be a set and ξ a convergence on X.

- A subset $O \subseteq X$ is called open if $\operatorname{inh}_\xi(O) = O$.
- A subset $C \subseteq X$ is called closed if $\operatorname{adh}_\xi(C) = C$.

The set of all open sets in (X, ξ) is called the topology of (X, ξ) and is denoted $\mathcal{T}_\xi$.

Lemma VIII.30. *Let X be a set, ξ a convergence on X and $A \subseteq X$ a subset. Then A is open* if and only if *A^c is closed.*

Proof. Assume $\operatorname{inh}_\xi(A) = A$. Then $A^c = \operatorname{inh}_\xi(A)^c = \operatorname{adh}_\xi(A^c)$. □

Lemma VIII.31. *Let X be a set and ξ a convergence on X. Then*

1. *the topology $\mathcal{T}_\xi$ is closed under arbitrary unions;*
2. *the set of closed sets in (X, ξ) is closed under arbitrary intersections.*

Proof. TODO □

Corollary VIII.31.1. *The topology $\mathcal{T}_\xi$ is a complete sublattice of $\mathcal{P}(X)$.*

Lemma VIII.32. *Let (X, ξ) be a convergence space and $O, C \subseteq X$ subsets. The following are equivalent:*

1. *O is open* if and only if *or all $x \in O$ there exists $U_x \in \mathcal{V}(x)$ such that $U_x \subseteq O$;*

Proof. TODO VIII.29 □

1.4.4.2 Interior, closure and boundary

Let (X, ξ) be a convergence space.

- The closure mapping of $A \in \mathcal{P}(X)$ into $\mathcal{T}_\xi$ is called the interior of A, denoted $\operatorname{int}(A)$ or A°.
- The closure mapping of $A \in \mathcal{P}(X)$ into the set of closed sets in X is called the closure of A, denoted $\operatorname{cl}(A)$ or $\overline{A}$.

The boundary of $A \in \mathcal{P}(X)$ is $\partial A \coloneqq \overline{A} \setminus A^\circ$

Proposition VIII.33. *???*
$\operatorname{int}(\operatorname{cl}(A))$ *and* $\operatorname{cl}(\operatorname{int}(A))$*?*

1.4.4.3 Neighbourhoods

Let (X, ξ) be a convergence space and $x \in X$. We call a subset $A \subseteq X$ a neighbourhood of x is there exists an open set O such that $x \in O \subseteq A$.
The set of all neighbourhoods of x is denoted $\mathcal{N}_\xi(x)$.

Proposition VIII.34. *Let (X, ξ) be a convergence space, $A \subseteq X$ and $x \in X$. Then*

1. $x \in \operatorname{int}_\xi(A) \iff A \in \mathcal{N}_\xi(x)$;
2. $x \in \operatorname{cl}_\xi(A) \iff A \in \mathcal{N}_\xi(x)^{\#}$.

Lemma VIII.35. *Let (X, ξ) be a convergence space, $A \subseteq X$ and $x \in X$. Then*

1. $A \in \mathcal{N}(x)$ if and only if $\operatorname{int}(A) \in \mathcal{N}(x)$;
2. *if $\mathcal{A} \in \mathcal{P}^2(X)$ is a base for $\mathcal{N}(x)$, then $\operatorname{int}^\downarrow(\mathcal{A})$ is also a base for $\mathcal{N}(x)$.*

Proof. (1) We have $A \in \mathcal{N}(x) \iff x \in \operatorname{int}(A) = \operatorname{int}^2(A) \iff \operatorname{int}(A) \in \mathcal{N}(x)$.
(2) First $\operatorname{int}^\downarrow(\mathcal{A})$ is a filter base, because it is closed under finite intersections: for all $\operatorname{int}(A), \operatorname{int}(B) \in \operatorname{int}^\downarrow(\mathcal{A})$ we have $\operatorname{int}(A) \cap \operatorname{int}(B) = \operatorname{int}(A \cap B) \in \operatorname{int}^\downarrow(\mathcal{A})$ because $A \cap B \in \mathcal{A}$.
We clearly have $\mathcal{A} \preceq \operatorname{int}^\downarrow(\mathcal{A})$ because $\operatorname{int}(A) \subseteq A$. For the opposite inequality, we need to show that for all $A \in \mathcal{A}$, $\operatorname{int}(A)$ is a neighbourhood of x. This is point (1). ☐

1.4.5 Accumulation points

Let (X, ξ) be a convergence space and $\mathcal{A} \subseteq \mathcal{P}(X)$ a family of subsets. A point $x \in X$ is called an accumulation point of $\mathcal{A}$ if $\mathcal{A} \# \mathcal{V}_\xi(x)$.

TODO $x \in \operatorname{adh}_\xi(A \setminus \{x\})$.

Proposition VIII.36. *Let (X, ξ) be a pretopological convergence space, $F \in \mathcal{FP}(X)$ and $x \in X$. If x is an accumulation point of F, then there exists a proper filter $G \geq F$ such that $G \xrightarrow{\xi} x$.*

Proof. By III.118, we have that $F \cup \mathcal{V}_\xi(x)$ is a proper filter. We can take this filter to be G. ☐

1.4.6 Cover

Let X be a set, ξ a convergence on X, $A \subseteq X$ and $\mathcal{A} \subseteq \mathcal{P}(X)$. We say $\mathcal{A}$ is an ξ-cover (or simply cover) of A if every filter converging to a point in A contains an element of $\mathcal{A}$. We write $\mathcal{A} \succ_\xi A$.

So we have

$$\mathcal{A} \succ_\xi A \iff \forall F \in \mathcal{FP}(X) : \left(\lim_\xi F \# A \implies F \# \mathcal{A} \right).$$

Proposition VIII.37. *Let X be a set, ξ a convergence on X, $A \subseteq X$ and $\mathcal{A} \subseteq \mathcal{P}(X)$. Then*

$$\mathcal{A} \succ_\xi A \qquad \iff \qquad \mathrm{adh}_\xi[\mathcal{A}]^c \perp A$$

1.5 Examples of (pre)convergences

1.5.1 Preconvergences on two-point sets

1.5.2 Preconvergences on thee-point sets

1.5.3 Convergences on ordered sets

1.5.3.1 Order convergence

`https://core.ac.uk/download/pdf/82382859.pdf` TODO move down!

Let $(P, \leq)$ be a poset and $F \in \mathcal{FP}(P)$. Then F converges to x in the order convergence on P if there exist nets $\langle l_i \rangle_{i \in I}$ and $\langle u_i \rangle_{i \in I}$ in P such that

- $\langle l_i \rangle_{i \in I}$ is increasing and $\sup_{i \in I} l_i = x$;
- $\langle u_i \rangle_{i \in I}$ is decreasing and $\inf_{i \in I} u_i = x$;
- $\{[l_i, u_i] \mid i \in I\} \subseteq F$.

We say F is bounded by $\langle l_i \rangle_{i \in I}$ and $\langle u_i \rangle_{i \in I}$.

TODO: initial or final??

Example

Order convergence not necessarily topological.

Let $(P, \leq)$ be a poset equipped with order convergence, (X, ξ) a convergence space, $x_0 \in P$ and $f : P \to X$ a function. Then

- f is called left continuous at x_0 if $f|_{\downarrow x_0}$ is continous at x_0;
- f is called right continuous at x_0 if $f|_{\uparrow x_0}$ is continous at x_0.

Proposition VIII.38. *Let $(P, \leq)$ be a poset equipped with order convergence, (X, ξ) a convergence space of finite depth and $f : P \to X$ a function. Then f is continuous at $x_0 \in P$* if and only if *it is left and right continuous at x_0.*

Proof. If f is continuous at $x_0 \in P$, then it is left and right continuous at x_0 by VIII.65. Conversely, assume f is left and right continuous and take F bounded by $\langle l_i \rangle_{i\in I}$ and $\langle u_i \rangle_{i\in I}$ such that $F \to x_0$. Define $F_0 = \mathfrak{F}\{[l_i, u_i] \mid i \in I\} = \uparrow\{[l_i, u_i] \mid i \in I\}$ and note that for any $A \in F_0$, there exists an $i \in I$ such that $[l_i, u_i] \subseteq A$.
Then $F_0|_{\downarrow x_0}$ is bounded by $\langle l_i \rangle_{i\in I}$ and $\langle x_0 \rangle$ and $F_0|_{\uparrow x_0}$ is bounded by $\langle x_0 \rangle$ and $\langle u_i \rangle_{i\in I}$. So $f[F_0|_{\downarrow x_0}] \to f(x_0)$ and $f[F_0|_{\uparrow x_0}] \to f(x_0)$, meaning $f[F_0|_{\downarrow x_0}] \cap f[F_0|_{\uparrow x_0}] \to f(x_0)$ by finite depth.
We conclude by showing that $f[F_0|_{\downarrow x_0}] \cap f[F_0|_{\uparrow x_0}] \preceq f[F]$. To that end, take $A \in f[F_0|_{\downarrow x_0}] \cap f[F_0|_{\uparrow x_0}]$. We need to show that there is a subset of A in $f[F]$.
Indeed, let $A_1 \in F_0|_{\downarrow x_0}$ be such that $f[A_1] = A$ and $A_2 \in F_0|_{\uparrow x_0}$ be such that $f[A_2] = A$. Then there exist $i, j \in I$ such that $[l_i, x_0] \subseteq A_1$ and $[x_0, u_j] \subseteq A_2$. Set $k = \max\{i, j\}$.
Now $f\Big[[l_k, u_k]\Big] = f\Big[[l_k, x_0]\Big] \cup f\Big[[x_0, u_k]\Big] \subseteq f[A_1] \cup f[A_2] = A$ and also $f\Big[[l_k, u_k]\Big] \in F$ by order convergence. □

1.5.3.2 Scott convergence

Let L be a complete lattice.

- The Scott convergence, or lower convergence, S_* on L is defined by
$$x \in \lim_{S_*} F \iff x \leq \bigvee_{A\in F} \bigwedge_{a\in A} a = \liminf F.$$
- The upper convergence, S^* on L is the Scott convergence on the dual L^o. This means it is defined by
$$x \in \lim_{S^*} F \iff x \geq \bigwedge_{A\in F} \bigvee_{a\in A} a = \limsup F.$$
- The convergence $S_* \wedge S^*$ is called convergence in order.

Note that F is a filter in $\mathcal{FP}(L)$, not a filter in $\mathcal{F}(L)$.

Proposition VIII.39. *Convergence in order is Hausdorff.*

Proof. This is due to the distributive inequality TODO ref. □

1.6 Compactness

A convergence space (X, ξ) is called compact if every ultrafilter converges.

Proposition VIII.40. *Let (X, ξ) be a convergence space and $A \subseteq X$ a subspace.*

1. *If X is compact and A is closed, then A is compact.*
2. *If X id a Hausdorff space and A is compact, then A is closed.*

Theorem VIII.41 (Tychonoff). *Let $\{(X_i, \xi_i)\}_{i\in I}$ be a family of compact convergence spaces. Then $\prod_{i\in I} X_i$ is compact.*

1.6.1 Relative compactness

Let (X, ξ) be a convergence space and $A \subseteq X$ a subset. We call A relatively compact if adh_ξ is compact.

1.6.2 Local compactness

A Hausdorff convergence space (X, ξ) is called locally compact if each filter which converges in X contains a compact set.

Chapter 2

Continuity

2.1 Continuous functions

Let (X, ξ) and (Y, ζ) be (pre)convergence spaces. A function $f : X \to Y$ is called continuous if it preserves limits: for all $D \in \mathcal{DP}(X)$ and $x \in X$:

$$D \xrightarrow{\xi} x \implies f^{\downarrow\downarrow}(D) \xrightarrow{\zeta} f(x).$$

The set of all continuous functions $(X, \xi) \to (Y, \zeta)$ is denoted $\mathcal{C}(\xi, \zeta)$ or $\mathcal{C}(X, Y)$, if the convergence is clear. If $X = Y$, we also write $\mathcal{C}(X)$.
If ξ, ζ are preconvergences we write $\mathcal{C}_{\text{pre}}(\xi, \zeta)$ for the set of continuous functions.

In other words, a function is continuous if it is relation-preserving as a function

$$\left(X \cup \mathcal{P}^2(X), \xrightarrow{\xi}\right) \to \left(Y \cup \mathcal{P}^2(Y), \xrightarrow{\zeta}\right).$$

Note that for filters $F \in \mathcal{FP}(X)$, $f[F]$ is in general a directed set, but not necessarily a filter.

Lemma VIII.42. *Let (X, ξ) and (Y, ζ) be (pre)convergence spaces. A function $f : X \to Y$ is continuous* if and only if *for all $D \in \mathcal{DP}(X)$*

$$f^{\downarrow}\left[\lim_{\xi} D\right] \subseteq \lim_{\zeta} f^{\downarrow\downarrow}[D].$$

Proof. Immediate from II.39. ☐

Lemma VIII.43. *Let (X, ξ), (Y, σ) and (Z, ζ) be (pre)convergence spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f$ is continuous.*

Proof. Let $F \to x \in X$. Then $f[F] \to f(x)$ by continuity of f and $g[f[F]] \to g(f(x))$ by continuity of g. So $g \circ f$ is continuous. ☐

Lemma VIII.44. *Let $(X, \xi), (Y, \zeta)$ be (pre)convergence spaces and $f \in \mathcal{C}_{(pre)}(\xi, \zeta)$.*

1. *Let σ be a (pre)convergence on X such that $\sigma \leq \xi$, then $f \in \mathcal{C}_{(pre)}(\sigma, \zeta)$.*
2. *Let τ be a (pre)convergence on Y such that $\tau \geq \zeta$, then $f \in \mathcal{C}_{(pre)}(\xi, \tau)$.*

Proof. (1) Let $F \xrightarrow{\sigma} x \in X$, then $F \xrightarrow{\xi} x$ (because $\sigma \leq \xi$), so $f[F] \xrightarrow{\zeta} f(x)$, meaning $f \in \mathcal{C}_{(\text{pre})}(\sigma, \zeta)$.
(2) Let $F \xrightarrow{\xi} x \in X$, then $f[F] \xrightarrow{\zeta} f(x)$, so $f[F] \xrightarrow{\tau} f(x)$ (because $\zeta \leq \tau$), meaning $f \in \mathcal{C}_{(\text{pre})}(\xi, \tau)$. □

Proposition VIII.45. *Let (X, ξ) and (Y, ζ) be convergence spaces, $f : X \to Y$ a continuous function and $A \subseteq X$ a subset. Then*

$$f[\operatorname{adh}_\xi(A)] \subseteq \operatorname{adh}_\zeta(f[A]).$$

Proof. We calculate

$$\begin{aligned} f[\operatorname{adh}_\xi(A)] &= f\left[\bigcup\left\{\lim_\xi F \;\middle|\; A \lhd F\right\}\right] = \bigcup f\left[\left\{\lim_\xi F \;\middle|\; A \lhd F\right\}\right] = \bigcup\left\{f\left[\lim_\xi F\right] \;\middle|\; A \lhd F\right\} \\ &\subseteq \bigcup\left\{\lim_\zeta f[F] \;\middle|\; A \lhd F\right\} \subseteq \bigcup\left\{\lim_\zeta f[F] \;\middle|\; f[A] \lhd f[F]\right\} \subseteq \bigcup\left\{\lim_\zeta G \;\middle|\; f[A] \lhd G\right\} \\ &= \operatorname{adh}_\zeta(f[A]), \end{aligned}$$

where we have used that $A \lhd B$ implies $f[A] \lhd f[B]$ (TODO ref). □

Proposition VIII.46. *Let (X, ξ) and (Y, ζ) be convergence spaces, $f : X \to Y$ a function and $x \in X$. If ξ and ζ are pretopological, then the following are equivalent:*

1. *f is continuous at x;*
2. *$\mathcal{V}_\zeta(f(x)) \supseteq \uparrow f[\mathcal{V}_\xi(x)]$;*
3. *for all $U \in \mathcal{V}_\zeta(f(x))$, there exists $V \in \mathcal{V}_\xi(x)$ such that $f[V] \subseteq U$;*
4. *for all $U \in \mathcal{V}_\zeta(f(x))$, $f^{-1}[U] \in \mathcal{V}_\xi(x)$.*

Proof. (1) ⇔ (2) We have that $\mathcal{V}_\xi(x)$ converges to x, so by continuity $f[\mathcal{V}_\xi(x)]$ converges to $f(x)$ and thus $\uparrow f[\mathcal{V}_\xi(x)] \supseteq \mathcal{V}_\zeta(f(x))$.
Conversely, take $F \to x$. Then $F \supseteq \mathcal{V}_\xi(x)$ and thus $f[F] \supseteq f[\mathcal{V}_\xi(x)]$. By assumption this means $\uparrow f[F] \supseteq \uparrow f[\mathcal{V}_\xi(x)] \supseteq \mathcal{V}_\zeta(f(x))$. So $f[F] \to f(x)$.
(2) ⇔ (3) Immediate.
(3) ⇔ (4) Because $V \subseteq f^{-1}[U]$ and $\mathcal{V}_\xi(x)$ is a filter, we have $f^{-1}[U] \in \mathcal{V}_\xi(x)$.
Conversely, we may take $V = f^{-1}[U]$. □

Note that the downward implications rely on the pretopologicity of ξ and the upward implication on the pretopologicity of ζ.

Lemma VIII.47. *Let ξ and ζ be two convergences on the same set X. Then $\operatorname{id}_X : (X, \xi) \to (X, \zeta)$ is continuous* if and only if *$\xi \leq \zeta$. I.e. ξ is finer than ζ.*

Proof. This is essentially a restatement of definitions:

$$\begin{aligned} \operatorname{id}_X : (X, \xi) \to (X, \zeta) \text{ is continuous } &\iff \forall F \in \mathcal{FP}(X) : \operatorname{id}_X\left[\lim_\xi F\right] \subseteq \lim_\zeta \operatorname{id}_X[F] \\ &\iff \forall F \in \mathcal{FP}(X) : \lim_\xi F \subseteq \lim_\zeta F \\ &\iff \xi \leq \zeta. \end{aligned}$$

□

Lemma VIII.48. *Let (X,ξ) and (Y,ζ) be a convergence space.*

1. (Identity function) *The identity function $\mathrm{id}_X : X \to X$ is continuous.*
2. (Constant function) *For all y in Y, the constant function $\underline{y} : X \to Y$ is continuous.*

Proof. (1) Let $F \to x \in X$. Then $x \in \lim_\xi(\mathrm{id}_X[F]) = \lim_\xi(F)$.
(2) Let $F \to x \in X$. Then $\underline{y}[F] = \dot{y} \to y = \underline{y}(x)$. ☐

Lemma VIII.49. *Let (X,ξ) and (Y,ζ) be preconvergence spaces and $f : X \to Y$ a function. Then f is continuous* if and only if *$\forall F \in \mathcal{FP}(Y)$ and $y \in Y$*

$$\exists x \in f^{-1}[y] :\ f^{-1}[F]\overset{\xi}{x} \quad\implies\quad F \xrightarrow{\zeta} y.$$

Proof. For $\Rightarrow$, assume f continuous and that there exists $x \in f^{-1}[y]$ such that $f^{-1}[F] \to x$. By continuity we have $f[f^{-1}[F]] \to y$. Now $f[f^{-1}[F]] \leq F$ by I.101, so $F \xrightarrow{\zeta} y$.
For $\Leftarrow$, take arbitrary $G \xrightarrow{\xi} x \in X$. Then $f^{-1}[f[G]] \geq G$ by I.105, so $f^{-1}[f[G]] \xrightarrow{\xi} x$ and $x \in f^{-1}[f(x)]$. This means $f[G] \xrightarrow{\zeta} f(x)$ and thus f is continuous. ☐

Proposition VIII.50. *Let (X,ξ) and (Y,ζ) be convergence spaces. Let Ξ be a set of convergences on X and Z a set of convergences on Y. Then*

1. $\mathcal{C}\left(\xi, \bigwedge Z\right) = \bigcap_{\sigma\in Z} \mathcal{C}(\xi,\sigma)$;
2. $\mathcal{C}\left(\bigvee \Xi, \zeta\right) = \bigcap_{\sigma\in \Xi} \mathcal{C}(\sigma,\zeta)$;
3. $\mathcal{C}\left(\bigwedge \Xi, \zeta\right) \supseteq \bigcup_{\sigma\in Z} \mathcal{C}(\xi,\sigma)$;
4. $\mathcal{C}\left(\xi, \bigvee Z\right) \supseteq \bigcup_{\sigma\in \Xi} \mathcal{C}(\sigma,\zeta)$;

Proof. (1) We calculate, using VIII.7:

$$\begin{aligned} f \in \mathcal{C}\left(\xi, \bigwedge Z\right) &\iff f\left[\lim_{\xi} F\right] \subseteq \lim_{\bigwedge Z}\!\uparrow f[F] = \bigcap_{\sigma\in Z} \lim_{\sigma}\!\uparrow f[F] \\ &\iff \forall \sigma \in Z :\ f\left[\lim_{\xi} F\right] \subseteq \lim_{\sigma}\!\uparrow f[F] \\ &\iff \forall \sigma \in Z :\ f \in \mathcal{C}(\xi,\sigma) \\ &\iff f \in \bigcap_{\sigma\in Z} \mathcal{C}(\xi,\sigma). \end{aligned}$$

(2) Similarly, we have

$$\begin{aligned} f \in \mathcal{C}\left(\bigvee \Xi, \zeta\right) &\iff f\left[\lim_{\bigvee\Xi} F\right] = \bigcup_{\sigma\in\Xi} f\left[\lim_{\sigma} F\right] \subseteq \lim_{\zeta}\!\uparrow f[F] \\ &\iff \forall \sigma \in \Xi :\ f\left[\lim_{\sigma} F\right] \subseteq \lim_{\zeta}\!\uparrow f[F] \\ &\iff \forall \sigma \in \Xi :\ f \in \mathcal{C}(\sigma,\zeta) \\ &\iff f \in \bigcap_{\sigma\in\Xi} \mathcal{C}(\sigma,\zeta). \end{aligned}$$

(3) Now we have

$$\begin{aligned} f \in \mathcal{C}\left(\bigwedge \Xi, \zeta\right) &\iff f\left[\lim_{\bigwedge \Xi} F\right] = \bigcap_{\sigma \in \Xi} f\left[\lim_{\sigma} F\right] \subseteq \lim_{\zeta}\uparrow f[F] \\ &\Longleftarrow \exists \sigma \in \Xi : \ f\left[\lim_{\sigma} F\right] \subseteq \lim_{\zeta}\uparrow f[F] \\ &\iff \exists \sigma \in \Xi : \ f \in \mathcal{C}(\sigma, \zeta) \\ &\iff f \in \bigcup_{\sigma \in \Xi} \mathcal{C}(\sigma, \zeta). \end{aligned}$$

(4) Finally we have

$$\begin{aligned} f \in \mathcal{C}\left(\xi, \bigvee Z\right) &\iff f\left[\lim_{\xi} F\right] \subseteq \lim_{\bigvee Z}\uparrow f[F] = \bigcup_{\sigma \in Z} \lim_{\sigma}\uparrow f[F] \\ &\Longleftarrow \exists \sigma \in Z : \ f\left[\lim_{\xi} F\right] \subseteq \lim_{\sigma}\uparrow f[F] \\ &\iff \exists \sigma \in Z : \ f \in \mathcal{C}(\xi, \sigma) \\ &\iff f \in \bigcup_{\sigma \in Z} \mathcal{C}(\xi, \sigma). \end{aligned}$$

□

2.1.0.1 Homeomorphisms

Let (X, ξ) and (Y, ζ) be convergence spaces. A function $f : X \to Y$ is called a homeomorphism if

- f is bijective;
- both f and f^{-1} are continuous.

Proposition VIII.51. *If f homeomorphism, then*

1. $f[\lim F] = \lim f[F]$;
2. $f[\mathcal{V}(x)] = \mathcal{V}(f[x])$;
3. $f[\operatorname{adh}(A)] = \operatorname{adh}(f[A])$.

2.1.1 Continuous convergence structure

Let (X, ξ) and (Y, ζ) be convergence spaces. The continuous convergence on $(X \to Y)$ is the coarses convergences such that the evaluation map $\text{ev} : (X \to Y) \times X \to Y$ is continuous.

We denote the congergence space $(X \to Y)$ equipped with the continuous convergence by $(X \to Y)_c$.

Proposition VIII.52. *Concrete description of continuous convergence.*

Proposition VIII.53 (Universal property of the continuous convergence structure)**.** *Let* $(X, \xi), (Y, \sigma)$ *and* (Z, ζ) *be convergences spaces. A function* $h : X \to (Y \to Z)_c$ *if continuous* if and only if $\mathrm{curry}_1^{-1}(h) : (X \times Y) \to Z$ *is continuous.*

Corollary VIII.53.1. *The category of convergence spaces is cartesian closed.*

Proposition VIII.54. $\mathcal{C}_c(X, Y)$ *is closed subset of* $(X \to Y)_c$*??*

Proposition VIII.55. *Let* $(X, \xi), (Y, \sigma)$ *and* (Z, ζ) *be convergences spaces. The composition operation*

$$\circ : (Y \to Z)_c \times (X \to Y)_c \to (X \to Z)_c$$

is continuous.

Proof. By VIII.53, this is equivalent to the continuity of $\mathrm{curry}_1^{-1}(\circ) : (Y \to Z)_c \times (X \to Y)_c \times X \to Z$, which follows from the commutativity of the following diagram:

$$\begin{tikzcd}
(Y \to Z)_c \times (X \to Y)_c \times X \arrow[d, "\mathrm{id}_{(Y \to Z)} \times \mathrm{ev}"'] \arrow[dr, "\mathrm{curry}_1^{-1}(\circ)"] & \\
(Y \to Z)_c \times Y \arrow[r, "\mathrm{ev}"] & Z.
\end{tikzcd}$$

□

2.1.2 Directional continuity

Let $f : (X, \xi) \to (Z, \zeta)$ be a function between convergence spaces, $x \in X$ and $D \subseteq X$. We called f directionally continuous at x in the direction D if for all $F \in \mathcal{FP}(X)$

$$F \xrightarrow{\xi, D} x \qquad \Longrightarrow \qquad f[F] \xrightarrow{\zeta} f(x).$$

Lemma VIII.56. *Let* $f : (X, \xi) \to (Z, \zeta)$ *be a function between convergence spaces,* $x_0 \in X$ *and* D *a vicinity of* x_0*. Then* f *is directionally continuous at* x_0 *in* D if and only if f *is continuous at* x_0*.*

Proof. TODO inherence □

2.2 Initial and final convergences

Let Y be a set.

- Given a set of (pre)convergence spaces $\{(Z_i, \zeta_i)\}_{i \in I}$ and a set of functions $\{f_i : Y \to Z_i\}_{i \in I}$, we define the initial (pre)convergence μ on Y w.r.t. $\{f_i : Y \to Z_i\}$ as the coarsest (pre)convergence on Y that makes all functions in $\{f_i : Y \to Z_i\}$ continuous:
$$\mu = \bigvee \left\{ \sigma \;\middle|\; \forall i \in I : f_i \in \mathcal{C}_{(\mathrm{pre})}(\sigma, \zeta_i) \right\}.$$
- Given a set of convergence spaces $\{(X_i, \xi_i)\}_{i \in I}$ and a set of functions $\{g_i : X_i \to Y\}_{i \in I}$, we define the final convergence ν on Y w.r.t. $\{g_i : X_i \to Y\}$ as the finest

convergence on Y that makes all functions in $\{g_i : X_i \to Y\}$ continuous:

$$\nu = \bigwedge \left\{ \sigma \;\middle|\; \forall i \in I : g_i \in \mathcal{C}_{(\mathrm{pre})}(\xi_i, \sigma) \right\}.$$

Proposition VIII.57. *Let Y be a set.*

1. *Let $\{f_i : Y \to (Z_i, \zeta_i)\}_{i\in I}$ be set of functions to convergence spaces and μ the initial $\underline{\text{pre}}$convergence on Y w.r.t. this set. Then μ is also the initial convergence w.r.t. $\{f_i : Y \to (Z_i, \zeta_i)\}$.*
2. *Let $\{g_i : (X_i, \xi_i) \to Y\}_{i\in I}$ be set of functions from convergence spaces and ν the final $\underline{\text{pre}}$convergence on Y w.r.t. this set. Then $\nu \vee \iota_Y$ is the final convergence w.r.t. $\{g_i : (X_i, \xi_i) \to Y\}$.*

Proof. TODO convergence modification □

Proposition VIII.58. *Let Y be a set, $F \in \mathcal{FP}(Y)$ and $y \in Y$.*

1. *Let $\{f_i : Y \to (Z_i, \zeta_i)\}_{i\in I}$ be set of functions to preconvergence spaces and μ the initial preconvergence on Y w.r.t. this set. Then*

$$F \xrightarrow{\mu} y \qquad \iff \qquad \forall i \in I :\; f_i[F] \xrightarrow{\zeta_i} f_i(y).$$

2. *Let $\{g_i : (X_i, \xi_i) \to Y\}_{i\in I}$ be set of functions from convergence spaces and ν the final preconvergence on Y w.r.t. this set. Then*

$$F \xrightarrow{\nu} y \qquad \iff \qquad \exists i \in I : \exists x \in g_i^{-1}[y] :\; g_i^{-1}[F] \xrightarrow{\xi_i} x.$$

Note that point (1) still holds true for convergences, by VIII.57. In the convergence case, point (2) needs to be modified to

$$F \xrightarrow{\nu} y \qquad \iff \qquad \exists i \in I : \exists x \in g_i^{-1}[y] :\; g_i^{-1}[F] \xrightarrow{\xi_i} x \quad \vee \quad F = \dot{y}.$$

Proof. (1) The direction $\Rightarrow$ is clear: μ makes all f_i continuous by VIII.44.
For $\Leftarrow$, assume F such that $\forall i \in I :\; f_i[F] \xrightarrow{\zeta_i} f_i(y)$, but $F \overset{\mu}{\not\to} y$. Then define the preconvergence μ' with the same limits as μ, but with the addition of $G \xrightarrow{\mu'} y$ for all $G \geq F$. Now μ' makes all f_i continuous (because $G \geq F$ implies $f_i[G] \geq f_i[F]$), so $\mu' \leq \mu$. Thus $F \xrightarrow{\mu} y$, which is a contradiction.
(2) By VIII.44, g_i is continuous for all $i \in I$. Then the direction $\Leftarrow$ follows from VIII.49.
For $\Rightarrow$, assume $F \xrightarrow{\nu} y$ and $\forall i \in I : \forall x \in g_i^{-1}[y] :\;\; g_i^{-1}[F] \not\to x$. Define the ν' from ν by removing all limits of the form $G \to y$ for $G \leq F$. Now we have $g_i^{-1}[G] \leq g_i^{-1}[F]$, so $\forall x \in g_i^{-1}[y] :\; g_i^{-1}[G] \not\to x$. By VIII.49, ν' still makes all g_i continuous, meaning $\nu' \geq \nu$. Thus $F \xrightarrow{\nu'} y$, which is a contradiction. □

Corollary VIII.58.1 (Characteristic property of initial and final convergence)**.** *Let Y be a set, (X, ξ) and (Z, ζ) (pre)convergence spaces.*

1. *Let* $\{f_i : Y \to (Z_i, \zeta_i)\}_{i\in I}$ *be set of functions to (pre)convergence spaces and* μ *the initial (pre)convergence on* Y *w.r.t. this set. A function* $g : (X, \xi) \to Y$ *is continuous* if and only if $f_i \circ g$ *is continuous for all* $i \in I$*.*

$$\begin{tikzcd}
Y \arrow[r, "f_i"] & Z_i \\
X \arrow[u, "g"] \arrow[ur, "f_i\circ g"'] &
\end{tikzcd}$$

2. *Let* $\{g_i : (X_i, \xi_i) \to Y\}_{i\in I}$ *be set of functions from (pre)convergence spaces and* ν *the final (pre)convergence on* Y *w.r.t. this set. A function* $f : Y \to (Z, \zeta)$ *is continuous* if and only if $f \circ g_i$ *is continuous for all* $i \in I$*.*

$$\begin{tikzcd}
X_i \arrow[r, "g_i"] \arrow[dr, "f\circ g_i"'] & Y \arrow[d, "f"] \\
& Z
\end{tikzcd}$$

Proof. (1) Take arbitrary $F \xrightarrow{\xi} x \in X$. Then the continuity of g is equivalent to the convergence $g[F] \xrightarrow{\mu} g(x)$. By the proposition this is equivalent to

$$\forall i \in I : \; f_i[g[F]] \xrightarrow{\zeta_i} f_i(g(x)),$$

which is equivalent to the continuity of $f_i \circ g$ for all $i \in I$.

(2) Take arbitrary $F \in \mathcal{FP}(Z)$ and $z \in Z$ such that $F \overset{\zeta}{\not\to} z$. Then, by VIII.49, the continuity of f is equivalent to

$$\forall y \in f^{-1}[z] : \; f^{-1}[F] \overset{\nu}{\not\to} y$$

By the proposition this is equivalent to

$$\forall y \in f^{-1}[z] : \forall i \in I : \forall x \in g_i^{-1}[y] : \; g_i^{-1}[f^{-1}[F]] \overset{\xi_i}{\not\to} x.$$

Using the equality $g_i^{-1}[f^{-1}[F]] = (f \circ g_i)^{-1}[F]$, we can rewrite this as

$$\forall i \in I : \forall x \in (f \circ g_i)^{-1}[z] : \; (f \circ g_i)^{-1}[F] \overset{\xi_i}{\not\to} x.$$

By VIII.49, this is equivalent to the continuity of all $f \circ g_i$. □

TODO: final convergence does not preserve finite depth! (what about Kent space??)
TODO: initial/final not universal property, but product is.
TODO $\prod, \coprod$

2.2.1 Initial pretopological convergence

TODO

Proposition VIII.59. *Let* X *be a set and* ξ *the initial convergence on* X *w.r.t. a set of functions* $\{f_i : X \to (Y_i, \zeta_i)\}_{i\in I}$ *from* X *to topological spaces* Y_i*. Then* ξ *is topological and the topology is given by*

$$\mathcal{T}_X = \bigcup_{i\in I} f_i^{-\downarrow\downarrow}(\mathcal{T}_{Y_i}) = \bigcup_{i\in I} \left\{ f_i^{-\downarrow}(U) \;\middle|\; U \in \mathcal{T}_{Y_i} \right\}.$$

Proof. See Dolecki-Mynard p.143 □

2.2.2 Constructions

2.2.2.1 Product convergence

Let (X_i, ξ_i) be a convergence space for all $i \in I$. The product convergence space $\prod_{i \in I} X_i$ is the initial convergence on $\bigtimes_{i \in I} X_i$ w.r.t. the set of projections $p_i : \bigtimes_{i \in I} X_i \to X_i$.

Proposition VIII.60. *Let (X_i, ξ_i) be a (pre)convergence space for all $i \in I$, $F \in \mathcal{FP}(\prod_{i \in I} X_i)$ and $x \in \prod_{i \in I} X_i$. Then $F \to x$ if and only if $\forall i \in I : \exists F_i \in \mathcal{FP}(X_i) : F_i \xrightarrow{\xi_i} p_i(x)$ and*

$$F \geq \bigotimes_{i \in I} F_i \coloneqq \left\{ \bigtimes_{i \in I} A_i \;\middle|\; \forall i \in I : A_i \in F_i \;\wedge\; A_i = X_i, \textit{ except for finitely many } A_i \right\}.$$

Proof. The direction $\Leftarrow$ follows straight form VIII.58 because $p_i(F) = F_i \to p_i(x)$.
TODO adjoint of $\coprod_{i \in I} p_i$. □

Corollary VIII.60.1. *Let (X_i, ξ_i) be a (pre)convergence space for all $i \in I$ and $x \in \prod_{i \in I} X_i$. Then*

$$\mathcal{V}_{\prod \xi_i}(x) = \uparrow \left\{ \bigtimes_{i \in I} A_i \;\middle|\; \forall i \in I : A_i \in \mathcal{V}_{\xi_i}(p_i(x)) \right\}.$$

In particular $\mathcal{V}_{\xi \otimes \zeta}((x, y)) = \uparrow \mathcal{V}_\xi(x) \otimes \mathcal{V}_\zeta(y)$.

Corollary VIII.60.2. *Let (X_i, ξ_i) be a (pre)convergence space and $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\operatorname{adh}_{\prod_{i \in I} X_i} \left(\bigtimes_{i \in I} A_i \right) = \bigtimes_{i \in I} \operatorname{adh}_{\xi_i}(A_i).$$

In particular $\operatorname{adh}_{\xi \otimes \zeta}(A \times B) = \operatorname{adh}_\xi(A) \times \operatorname{adh}_\zeta(B)$.

Proof. We have

$$\begin{aligned}
\langle x_i \rangle_{i \in I} \in \operatorname{adh}_{\prod_{i \in I} X_i} \left(\bigtimes_{i \in I} A_i \right) &\iff \bigtimes_{i \in I} A_i \in \mathcal{V}_{\prod \xi_i}(\langle x_i \rangle_{i \in I}) = \uparrow \left\{ \bigtimes_{i \in I} B_i \;\middle|\; \forall i \in I : B_i \in \mathcal{V}_{\xi_i}(x_i) \right\} \\
&\iff \forall i \in I : \exists B_i \in \mathcal{V}_{\xi_i}(x_i) : B_i \subseteq A_i \\
&\iff \forall i \in I : A_i \in \mathcal{V}_{\xi_i}(x_i) \\
&\iff \forall i \in I : x_i \in \operatorname{adh}_{\xi_i}(A_i) \\
&\iff \langle x_i \rangle_{i \in I} \in \bigtimes_{i \in I} \operatorname{adh}_{\xi_i}(A_i).
\end{aligned}$$

□

Let $\{X_i\}_{i \in I}$ be a set of sets and $\{F_i \in \mathcal{FP}(X_i) \mid i \in I\}$ a set of filters. Then the product filter is defined by

$$\bigotimes_{i \in I} F_i \coloneqq \left\{ \bigtimes_{i \in I} A_i \;\middle|\; \forall i \in I : A_i \in F_i \;\wedge\; A_i = X_i, \text{ except for finitely many } A_i \right\}.$$

Lemma VIII.61. *Let X, Y be sets and $F \in \mathcal{FP}(X \times Y)$. Then $p_1^{\downarrow\downarrow}(F) \otimes p_2^{\downarrow\downarrow}(F) \subseteq F$.*

Proof. Take some $A \in p_1^{\downarrow\downarrow}(F) \otimes p_2^{\downarrow\downarrow}(F)$. Then there exist $B, C \in F$ such that $A = p_1^{\downarrow}(B) \times p_2^{\downarrow}(C)$, which means that $B \cap C \subseteq A$. Now $B \cap C \in F$, so $A \in F$. ☐

Lemma VIII.62. *Let (X, ξ) and (Y, ζ) be convergence spaces, $F \in \mathcal{FP}(X \times Y)$, $G \in \mathcal{FP}(X)$ and $H \in \mathcal{FP}(Y)$. Then*

1. *$F \xrightarrow{\xi \otimes \zeta} (x, y)$* if and only if *$p_1(F) \xrightarrow{\xi} x$ and $p_2(F) \xrightarrow{\zeta} y$;*
2. *$G \otimes H \xrightarrow{\xi \otimes \zeta} (x, y)$* if and only if *$G \xrightarrow{\xi} x$ and $H \xrightarrow{\zeta} y$.*

Proof. Point (2) follows from point (1) because $p_1(F \otimes G) = F$ and $p_2(F \otimes G) = G$. TODO ref. ☐

Lemma VIII.63. *Let $(X, \xi), (Y, \sigma)$ and (Z, ζ) be convergence spaces and $f : X \to Y \times Z$ be a function of the form*

$$f : X \to Y \times Z : x \mapsto f(x) = (f_Y(x), f_Z(x)).$$

The f is continuous if and only if *f_Y and f_Z are continuous.*

Proof. Follows immediately from VIII.58.1, because $f_Y = p_1 \circ f$ and $f_Z = p_2 \circ f$. ☐

Corollary VIII.63.1. *Let (X, ξ) and (Y, σ) be convergence spaces. Then for all $y \in Y$, the function $X \to X \times Y : x \mapsto (x, y)$ is continuous.*

Proof. The functions id_X and $\underline{y}$ are continuous. ☐

Corollary VIII.63.2. *Let (X, ξ) and (Y, σ) be convergence spaces and $f_X : X \to X, f_Y : Y \to Y$ be continuous functions. Then*

$$f_X \times f_Y : X \times Y \to X \times Y : (x, y) \mapsto (f_X(x), f_Y(y))$$

is continuous.

Proof. The functions $p_1 \circ f_X : (x, y) \mapsto f_X(x)$ and $p_2 \circ f_Y : (x, y) \mapsto f_Y(y)$ are continuous. ☐

2.2.2.2 Subspace convergence

> Let (X, ξ) be a convergence space and $A \subseteq X$ a subset. The subspace convergence on A is the initial convergence w.r.t. $\{\iota : A \hookrightarrow X : a \mapsto a\}$.

Lemma VIII.64. *Restrictions on continuous functions are continuous (domain + codomain).*

Proposition VIII.65. *Let (X, ξ), (Y, σ) and (Z, ζ) be convergence spaces.*

1. (Restricting the domain) *If $f : X \to Y$ is continuous and A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.*
2. (Restricting the range) *Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f[X]$, then $f : X \to Z$ is continuous.*
3. (Expanding the range) *Let $f : X \to Y$ be continuous. If Y is a subspace of Z, then $f : X \to Z$ is continuous.*

Proof. (1) Composition of continuous maps: $f|_A = f \circ \iota$.
(2) Composition of continuous maps: $f : X \to Z = \iota \circ (f : X \to Y)$.
(3) Characteristic property VIII.58.1: if $f : X \to Y = \iota \circ (f : X \to Z)$ is continuous, then $f : X \to Z$ is too. ☐

2.2.2.3 Quotient convergence

Proposition VIII.66. *Each convergence space is the quotient convergence space of a topological space.*

2.2.3 Projective and injective limits

Let $(I, \prec)$ be an upwards directed set.

- Let $\{sSetX_i, \xi_i\}_{i \in I}$ be a set of convergence spaces and for each $j \succ i$, let $p_{j,i} : X_j \to X_i$ be a continuous mapping. Then the structure $(I, \{(X_i, \xi_i)\}_{i \in I}, \{p_{j,i}\}_{j \succ i})$ is called a projective system (or inverse system) if for all $k \succ j \succ i \in I$ the diagram

$$\begin{tikzcd} X_k \arrow[rr, "p_{k,i}"] \arrow[dr, "p_{k,j}"] & & X_i \\ & X_j \arrow[ur, "p_{j,i}"] & \end{tikzcd}$$

commutes.

  Let (X, ξ) be a convergence space and $p_i : X \to X_i$ a continuous mapping for all $i \in I$.

Chapter 3

Separation axioms and other properties of convergences spaces

3.1 Distinguishability, separation and regularity

3.1.1 Distinguishable points

Let (X, ξ) be a convergence space and $x, y \in X$. We call x and y distinguishable if $\lim_\xi{}^{-1}(x) \neq \lim_\xi{}^{-1}(y)$.

In orther words, $x, y \in X$ are distinguishable if there exists a filter $F \in \mathcal{FP}(X)$ such that

$$\Big(x \in \lim_\xi F \wedge y \notin \lim_\xi F\Big) \ \vee \ \Big(x \notin \lim_\xi F \wedge y \in \lim_\xi F\Big).$$

We say F distinguishes x and y.

Proposition VIII.67. *Let (X, ξ) be a Kent convergence space and $x, y \in X$. Then x and y are indistinguishable* if and only if

$$\dot{x} \to y \qquad \textit{and} \qquad \dot{y} \to x.$$

Proof. The direction $\Rightarrow$ is clear: from $\dot{x} \to x$, we get $\dot{x} \to y$ by indistinguishability.
The direction $\Leftarrow$ is proved by contradiction. Assume x and y are distinguishable, so there exists a filter F such that $F \to x$ but $F \not\to y$. Then $F \cap \dot{x} \to y$ by the definining property of Kent spaces. Now $F \cap \dot{y} \subseteq F$, so $F \to y$. This is a contradiction. □

3.1.2 Separation

Let (X, ξ) be a convergence space and $A, B \subseteq X$. We say A and B are separated if $\mathrm{adh}_\xi(A) \perp B$ and $A \perp \mathrm{adh}_\xi(B)$.
We call two points $x, y \in X$ separated if $\{x\}$ and $\{y\}$ are separated.

Proposition VIII.68. *Let X be a set, ξ a convergence on X and $x, y \in X$. Then x and y are separated* if and only if

$$\dot{x} \not\to y \qquad \textit{and} \qquad \dot{y} \not\to x.$$

Proof. By VIII.18 we have $\mathrm{adh}_\xi(\{x\}) = \lim_\xi \dot{x}$ and $\mathrm{adh}_\xi(\{y\}) = \lim_\xi \dot{y}$. ☐

In other words, x and y are not separated iff $\dot{x} \to y$ or $\dot{y} \to x$.

Lemma VIII.69. *Let (X, ξ) be a convergence space and $x, y \in X$. If x, y are separated, then they are distinguishable.*

Proof. Assume x, y separated. WLOG we can assume $\dot{x} \not\to y$. So $\dot{x}$ distinguishes x and y. ☐

3.1.2.1 Separation by vicinities

Let (X, ξ) be a convergence space and $A, B \subseteq X$. We say A and B are separated by vicinities if there exist $U \in \mathcal{V}_\xi(A)$ and $V \in \mathcal{V}_\xi(B)$ such that $U \perp V$.
We call two points $x, y \in X$ separated by vicinities if $\{x\}$ and $\{y\}$ are separated by vicinities, i.e. there exist vicinities U, V of x, y, resp., such that $U \perp V$.

Proposition VIII.70. *Let (X, ξ) be a convergence space.*

1. *If $A, B \subseteq X$ are separated by vicinities, then .*
2. *If ξ is pretopological and $\lim^{-1}(x) \perp \lim^{-1}(y)$ for some $x, y \in X$, then x, y are separated by vicinities.*

Proof. (1) Assume A, B are separated by vicinities U and V. Assume, towards a contradiction, that $\lim^{-1}(A) \# \lim^{-1}(B)$, i.e. there exists a filter F that converges to $x \in A$ and $y \in B$. Then $U, V \in F$ and thus $U \cap V = \emptyset \in F$, meaning F is not a proper filter.
(2) Assume x, y are not separated by vicinities. Then $\mathcal{V}_\xi(x) \# \mathcal{V}_\xi(y)$, meaning x is an accumulation point of $\mathcal{V}_\xi(y)$. By VIII.36, there exists a filter $G \geq \mathcal{V}_\xi(y)$ such that $G \to x$. Then $G \in \lim^{-1}(x) \cap \lim^{-1}(y)$. ☐

3.1.2.2 Separation by convergent filters

Let (X, ξ) be a convergence space and $A, B \subseteq X$. We say A and B are separated by convergent filters if for all approaches $F : X \to \mathcal{FP}(X)$, the contours satisfy $\neg(F(A) \# F(B))$.
We call two points $x, y \in X$ separated by convergent filters if $\{x\}$ and $\{y\}$ are separated by convergent filters.

Lemma VIII.71. *Let (X, ξ) be a convergence space. For $x, y \in X$ the following are equivalent*

1. *x, y are separated by convergent filters;*
2. *$\lim^{-1}(x) \perp \lim^{-1}(y)$.*

Proof. (1) $\Rightarrow$ (2) Assume $\lim^{-1}(x) \# \lim^{-1}(y)$, i.e. there exists some filter $F \in \lim^{-1}(x) \cap \lim^{-1}(y)$. Then the constant approach $x, y \mapsto F$ has $F(x) = F(y)$, meaning $F(x) \# F(y)$ and thus x, y are not separated by convergent filters.
(2) $\Rightarrow$ (1) Assume $\lim^{-1}(x) \perp \lim^{-1}(y)$. Pick an arbitrary approach F. If $F(x) \# F(y)$, then $F(x) \vee F(y)$ is proper by III.118 and in $\lim^{-1}(x) \cap \lim^{-1}(y)$ by monotonicity, which is a contradiction. ☐

Proposition VIII.72. *Let (X, ξ) be a convergence space and $A, B \subseteq X$.*

1. *If there exist disjoint vicinities of A and B, then A and B are separated by convergent filters.*
2. *If ξ is pretopological, the converse also holds.*

Proof. (1) Assume A, B are separated by vicinities U and V. Assume, towards a contradiction, that $F(A) \# F(B)$ for some approach F. Then for all $a \in A$, $U \in F(a)$, so $U \in F(A)$. Similarly $V \in F(B)$. This means $U \# V$, which is a contradiction.
(2) If ξ is pretopological, then $\mathcal{V}_\xi$ is an approach. Thus $\neg(\mathcal{V}_\xi(A) \# \mathcal{V}_\xi(B))$, meaning there exist disjoint vicinities of A, B. □

Thus in the pretopological case all approaches F satisfy $\neg(F(A) \# F(B))$ iff the vicinity filter satisfies $\neg(\mathcal{V}(A) \# \mathcal{V}(B))$.

3.1.2.3 Separation by neighbourhoods

Let (X, ξ) be a convergence space and $A, B \subseteq X$. We say A and B are separated by neighbourhoods if there exist disjoint neighbourhoods of A and B.
We call two points $x, y \in X$ separated by neighbourhoods if $\{x\}$ and $\{y\}$ are separated by neighbourhoods.

Lemma VIII.73. *Let (X, ξ) be a convergence space and $A, B \subseteq X$. Then A, B are separated by neighbourhoods* if and only if *there exists an open set U such that $A \subseteq U \subseteq \overline{U} \subseteq B^c$.*

Proof. WLOG we may take $\overline{U}^c$ to be the neighbourhood of B. □

3.1.2.4 Separation by closed vicinities

3.1.2.5 Separation by functions

`https://en.wikipedia.org/wiki/Separated_sets`

3.1.3 Regularity

Let (X, ξ) be a convergence space and $\mathcal{Z} \subseteq \mathcal{P}(X)$. The convergence ξ is called $\mathcal{Z}$-regular if for all $x \in X, F \in \lim_\xi^{-1}(x)$ there exists a filter base $G \subseteq \mathcal{Z}$ such that

- $G \preceq F$;
- $G \xrightarrow{\xi} x$.

3.2 Separation properties

3.2.1 T_0 or Kolmogorov

We call a convergence space (X, ξ) Kolmogorov or T_0 if every pair of distinct points in X is distinguishable.

3.2.2 R_0 or symmetric

Let X be a set and ξ a convergence on X. Then ξ is called symmetric or R_0 if all distinguishable pairs of points are separated.

Proposition VIII.74. *Let X be a set and ξ a convergence on X. Then the following are equivalent:*

1. *ξ is an R_0 convergence;*
2. *x and y are indistinguishable* if and only if *they are separated;*
3. *for all $x, y \in X$, x and y are indistinguishable* if and only if *$\dot{x} \to y$;*
4. *$\operatorname{adh}_\xi(\{x\})$ is the set of points that are indistinguishable from x for all $x \in X$;*

The following are consequences of the above. If ξ is a Kent convergence, then they are also equivalent:

5. *the set $\{\operatorname{adh}_\xi(\{x\}) \mid x \in X\}$ is a partition of X;*
6. *for all $x, y \in X$: $\dot{x} \to y$* if and only if *$\dot{y} \to x$;*

Proof. (1) ⇒ (2) The converse to the R_0 condition, that separated points are distinguishable, is automatic (see VIII.69).
(2) ⇒ (3) All points that are indistinguishable from x are in $\operatorname{adh}_\xi(\{x\}) = \lim_\xi\{x\}$ by construction. Assume y is distinguishable from x. Then y is separated from x, so $\dot{x} \not\to x$ by VIII.68.
(3) ⇒ (4) $\operatorname{adh}_\xi(\{x\}) = \lim_\xi \dot{x}$ by VIII.18.
(4) ⇒ (1) Assume $x, y \in X$ are distinguishable. Then $y \notin \operatorname{adh}_\xi(\{x\})$ and $x \notin \operatorname{adh}_\xi(\{y\})$. This means $\dot{x} \not\to y$ and $\dot{y} \not\to x$ and we conclude by VIII.68.
(4) ⇒ (5) Indistinguishability is an equivalence relation.
(5) ⇒ (6) Equivalence relations are symmetric.
(6) ⇒ (1) Assume ξ of a Kent convergence and x, y distinguishable. Then $\dot{x} \not\to y$ or $\dot{y} \not\to x$ by VIII.67. Because of (6), the "or" becomes an "and" and we conclude with VIII.68. ☐

TODO: characterisation "every open set is a union of closed sets" or "every closed set is an intersection of open sets".
?? $\operatorname{adh}_\xi(\{x\})$ is closed for all $x \in X$??

3.2.3 T_1 or Fréchet

Let X be a set and ξ a preconvergence on X. Then ξ is called Fréchet or T_1 if all distinct points in X are separated.

`https://en.wikipedia.org/wiki/T1_space`

Proposition VIII.75. *Let X be a set and ξ a convergence on X. Then the following are equivalent:*

1. *ξ is a T_1 convergence;*
2. *ξ is both T_0 and R_0;*

3. *all singletons are closed, i.e.* $\forall x \in X : \operatorname{adh}_\xi(\{x\}) = \{x\}$*;*

4. $\forall x \in X : \lim_\xi \dot{x} = \{x\}$*;*

5. $\forall x \in X : \lim_\xi \dot{x} \subseteq \{x\}$*;*

Proof. (1) $\Leftrightarrow$ (2) Definitions together with point (2) of VIII.74.
(1) $\Leftrightarrow$ (3) From point (4) of VIII.74.
(3) $\Leftrightarrow$ (4) From VIII.18.
(4) $\Leftrightarrow$ (5) Convergences are centered. ☐

TODO: every singleton in X is closed; every finite subset of X is closed.

Corollary VIII.75.1. *Let X be a finite set, then the only T_1 convergence on X is the discrete convergence ι_X.*

Proof. Let ξ be a T_1 convergence on X and F a proper filter in $\mathcal{P}(X)$. Now F is principal (TODO ref), so $F = \uparrow \ker F$. If $\ker F$ is a singleton, then $\lim F = \ker F$ by point (2). Otherwise $\lim F = \emptyset$ by point (4). This is discrete convergence. ☐

Lemma VIII.76. *If (X, ξ) is a T_1 convergence on X, then ξ is also T_0.*

Proof. The filters $\dot{x}$ and $\dot{y}$ distinguish $x, y \in X$. ☐

Proposition VIII.77. *Let ξ be a T_1 preconvergence on a set X and F a filter in $\mathcal{FP}(X)$. If $x \in \lim F$, then $\ker F \subseteq \{x\}$.*

Proof. Suppose $\ker F$ in non-empty and take $y \in \ker F$. Then $F \subseteq \dot{y}$ and so

$$x \in \lim F \subseteq \lim \dot{y} \subseteq \{y\}.$$

This means that $y = x$. ☐

Corollary VIII.77.1. *The kernel of a convergent proper filter in a T_1 space is empty or a singleton.*

In T_2 spaces the <u>limit</u> of a convergent proper filter is empty or a singleton.
TODO: big question marks:

Proposition VIII.78. *Let X be a set and ξ a convergence on X. If ξ is T_1, then $\forall x \neq y \in X : \exists F, G \in \mathcal{FP}(X)$ such that*

$$x \in \lim_\xi F \;\wedge\; y \in \lim_\xi G \;\wedge\; x \notin \lim_\xi G \;\wedge\; y \notin \lim_\xi F.$$

If ξ is topological, the converse also holds.

Proof. $\Rightarrow$ We may take $F = \dot{x}$ and $G = \dot{y}$.
$\Leftarrow$ If ξ is topological, we may take $F = \mathcal{V}_\xi(x)$ and $G = \mathcal{V}_\xi(y)$. It is enough to show that $y \notin \operatorname{adh}_\xi(\{x\})$.
Assume, towards a contradiction, that $y \in \operatorname{adh}_\xi(\{x\})$. Then $\{x\} \in \mathcal{V}_\xi(y)^\#$ ☐

3.2.4 R_1 or reciprocal

Let (X, ξ) be a convergence space. Then ξ is called R_1, reciprocal or preregular if all distinguishable points are separated by convergent filters.

TODO review definition.
`https://gdz.sub.uni-goettingen.de/id/PPN235181684_0187?tify={%22pages%22:[191],%22panX%22:0.893,%22panY%22:0.579,%22view%22:%22info%22,%22zoom%22:0.894}`

Proposition VIII.79. *Let X be a set and ξ a convergence on X. Then the following are equivalent:*

1. *ξ is an R_1 convergence;*
2. *if x and y are distinguishable, then $\lim_\xi^{-1}(x) \perp \lim_\xi^{-1}(y)$;*
3. *x and y are distinguishable* if and only if *$\lim_\xi^{-1}(x) \perp \lim_\xi^{-1}(y)$;*
4. *the set $\left\{\lim_\xi^{-1}(x) \;\middle|\; x \in X\right\}$ is a partition of the set on convergent filters on X;*
5. *if there exists a filter F such that $F \to x$ and $F \to y$, then x and y are indistinguishable.*

Proof. (1) $\Leftrightarrow$ (2) By VIII.71.
(2) $\Rightarrow$ (3) If $\lim_\xi^{-1}(x) \perp \lim_\xi^{-1}(y)$, then x and y are definitely distinguishable, e.g. by $\dot{x}$.
(3) $\Rightarrow$ (4) All pairs of points are either distinguishable, or indistinguishable. In the first case we have $\lim_\xi^{-1}(x) \perp \lim_\xi^{-1}(y)$, in the second $\lim_\xi^{-1}(x) = \lim_\xi^{-1}(y)$.
(4) $\Rightarrow$ (5) From $F \to x$ and $F \to y$, we get $F \in \lim_\xi^{-1}(x)$ and $F \in \lim_\xi^{-1}(y)$, so $\lim_\xi^{-1}(x) \# \lim_\xi^{-1}(y)$. Using (3) this implies $\lim_\xi^{-1}(x) = \lim_\xi^{-1}(y)$.
(5) $\Rightarrow$ (2) Clear. ☐

Corollary VIII.79.1. *Any R_1 convergence space is also R_0.*

Proof. Compare point (2) with point (3) of VIII.74 and note that $\dot{x} \to y$ implies $\lim_\xi^{-1}(x) \# \lim_\xi^{-1}(y)$. ☐

3.2.5 T_2 or Hausdorff

Let X be a set and ξ a preconvergence on X. Then ξ is called Hausdorff or T_2 if every proper ξ-limit contains at most one point.

By "proper ξ-limit" we mean we exclude from this condition the degenerate filter $\mathcal{P}(X)$. Otherwise there would be no T_2 convergences by VIII.4.1.

Proposition VIII.80. *Let X be a set and ξ a convergence on X. Then the following are equivalent:*

1. *ξ is a T_2 convergence;*
2. *if $x \neq y$, then $\lim_\xi{}^{-1}(x) \perp \lim_\xi{}^{-1}(y)$;*
3. *ξ is T_0 and R_1;*
4. *ξ is T_1 and R_1;*

Proof. (1) ⇔ (2) If $F \in \lim_{\xi}^{-1}(x) \cap \lim_{\xi}^{-1}(y)$, then $F \to x$ and $F \to y$, so ξ would not be T_2.
(2) ⇔ (3) Clear.
(3) ⇔ (4) R_1 implies R_0 and $R_0 + T_0$ is equivalent with T_1. ☐

Proposition VIII.81. *Every T_2 convergence is also T_1. If the space is finite, the converse also holds.*

Proof. Let (X, ξ) be a T_2 convergence space and $x \in X$. By T_2, $\lim_{\xi} \dot{x}$ is a singleton. Definition of convergence this singleton is $\{x\}$.
Now let X be a finite set and let F be a proper filter in $\mathcal{FP}(X)$. Then F is principal by III.125 and not free because it is proper. So we can take $x \in \ker F$ and $F \subseteq \dot{x}$. Thus $\lim F \subseteq \lim \dot{x} \subseteq \{x\}$, meaning the convergence is T_2. ☐

Corollary VIII.81.1. *Let X be a finite set, then the only T_2 convergence on X is the discrete convergence ι_X.*

Proof. By VIII.75.1. ☐

Proposition VIII.82. *TODO: move: topological Hausdorff implies regular. This does not hold for non-topological convergence in general.*

3.2.6 R_2 or regular

Let (X, ξ) be a convergence space. Then ξ is called regular or R_2 if it is based in $\mathrm{adh}_{\xi}^{\downarrow}(\mathcal{P}^2(X))$.

`https://en.wikipedia.org/wiki/Regular_space`

Proposition VIII.83. *Let (X, ξ) be a convergence space. Then the following are equivalent:*

1. *ξ is an R_2 convergence;*
2. *for all $F \in \mathcal{FP}(X)$, $F \xrightarrow{\xi} x$ implies $\mathrm{adh}_{\xi}[F] = \{\mathrm{adh}_{\xi}(A) \mid A \in F\} \xrightarrow{\xi} x$.*
3. *for all $F \in \mathcal{FP}(X)$: $F \xrightarrow{\xi} x$* if and only if *$\mathrm{adh}_{\xi}[F] = \{\mathrm{adh}_{\xi}(A) \mid A \in F\} \xrightarrow{\xi} x$.*

Proof. (1) ⇔ (2) Assume (1), then there exists a filter G based in $\{\mathrm{adh}_{\xi}(A) \mid A \in \mathcal{P}(X)\}$ such that $G \to x$ and $G \subseteq F$. We just need to show that $G \subseteq \mathrm{adh}[F]$. Indeed take $A \in G$, then $A = \mathrm{adh}(B)$ for some $B \subseteq X$. Now $B \in F$ TODO: is this wrong?
Because $\mathrm{adh}_{\xi}[F] \preceq F$ Then $\mathrm{adh}_{\xi}(A) \supseteq A$, so adh
(2) ⇔ (3) One direction is given by definition. The other follows by monotonicity because $\mathrm{adh}_{\xi}[F] \preceq F$. ☐

Lemma VIII.84. *Any R_2 convergence is also R_1.*

Proposition VIII.85. *A pretopology is regular* if and only if *it separates points from the complements of their vicinities by convergent filters.*

Proof. A pretopology is regular iff $\mathcal{V}_\xi(x) \subseteq {\uparrow}\mathrm{adh}_\xi[\mathcal{V}_\xi(x)]$ for all $x \in X$. This means

$$\begin{aligned}
&\forall x \in X : \forall V \in \mathcal{V}_\xi(x) : \exists U \in \mathcal{V}_\xi(x) : \quad \mathrm{adh}_\xi(U) \subseteq V \\
\iff &\forall x \in X : \forall V \in \mathcal{V}_\xi(x) : \exists U \in \mathcal{V}_\xi(x) : \quad (\mathrm{adh}_\xi(U) \perp V^c) \\
\iff &\forall x \in X : \forall V \in \mathcal{V}_\xi(x) : \exists U \in \mathcal{V}_\xi(x) : \quad \neg(\mathrm{adh}_\xi(U) \# V^c) \\
\iff &\forall x \in X : \forall V \in \mathcal{V}_\xi(x) : \exists U \in \mathcal{V}_\xi(x) : \quad \neg(\{U\} \# \mathcal{V}_\xi(V^c)) \\
\iff &\forall x \in X : \forall V \in \mathcal{V}_\xi(x) : \quad \neg(\forall U \in \mathcal{V}_\xi(x) : \{U\} \# \mathcal{V}_\xi(V^c)) \\
\iff &\forall x \in X : \forall V \in \mathcal{V}_\xi(x) : \quad \neg(\mathcal{V}_\xi(x) \# \mathcal{V}_\xi(V^c)).
\end{aligned}$$

We have used VIII.20.1. The last line is equivalent to the separation of points from the complements of their vicinities by VIII.72. □

Proposition VIII.86. *Let (X, ξ) be a topological space. Then the following are equivalent:*

1. *ξ is regular;*
2. *for any $x \in X$ and any base $\mathcal{B}$ of $\mathcal{N}_\xi(x)$, $\mathrm{cl}^{\downarrow}(\mathcal{B})$ is also a base of $\mathcal{N}_\xi(x)$;*
3. *for any closed set C there exist disjoint open sets U, V such that $x \in U$ and $C \subseteq V$;*
4. *for any open set $O \subseteq X$ and $x \in O$ there exists an open set U such that $x \subseteq U \subseteq \overline{U} \subseteq O$.*

Proof. (1) ⇔ (2) TODO ref.
(1) ⇔ (3) By VIII.85.
(2) ⇔ (4) By VIII.73. □

3.2.7 T_3 or regular Hausdorff

Let (X, ξ) be a convergence space. Then ξ is called T_3 if it is regular and Hausdorff.

Proposition VIII.87. *Let X be a set and ξ a convergence on X. Then the following are equivalent:*

1. *ξ is a T_3 convergence, i.e. R_2 and T_2;*
2. *ξ is R_2 and T_0.*

3.2.8 R_3 or normal

Let (X, ξ) be a convergence space. Then ξ is called normal or R_3 if all disjoint closed sets are separated by convergent filters.

Proposition VIII.88. *Let (X, ξ) be a pretopological convergence space and $A, B \subseteq X$. Then* $\lim^{-1}(\mathrm{adh}_\xi(A)) \perp \lim^{-1}(\mathrm{adh}_\xi(B))$ if and only if *there exist disjoint vicinities of* $\mathrm{adh}_\xi(A)$ *and* $\mathrm{adh}_\xi(B)$.

Proposition VIII.89. *Any R_3 convergence is also R_2.*

3.2.9 T_4 or normal Hausdorff

Let (X, ξ) be a convergence space. Then ξ is called T_4 if it is normal and Hausdorff.

Proposition VIII.90. *Let X be a set and ξ a convergence on X. Then the following are equivalent:*

1. *ξ is a T_4 convergence, i.e. R_3 and T_2;*
2. *ξ is R_3 and T_4;*
3. *ξ is R_3 and T_0.*

3.3 Countability properties

3.3.1 $C1$ or first countable

A convergence space (X, ξ) is called first countable or C_1 if for all $x \in X$ the vicinity filter $\mathcal{V}_\xi(x)$ has a countable base.

3.3.1.1 Strongly first countable

3.3.2 $C2$ or second countable

A convergence space (X, ξ) is called second countable or C_2 if ξ has a countable base.

Lemma VIII.91. *Second countable implies first countable.*

Proposition VIII.92. *Let (X, ξ) be a topological space. If ξ is regular and second countable, then ξ is normal.*

Proof. TOOD eg `https://www.math.auckland.ac.nz/~gauld/750-05/section3.pdf` ☐

3.4 Comparison with reals and metrisability

3.4.1 Functional convergence properties

3.4.1.1 Functional closure

Let (X, ξ) be a convergence space and $A \subseteq X$. We call A functionally closed if there exists a continuous function $f : X \to \mathbb{R}$ and a closed set $C \subseteq \mathbb{R}$ such that $A = f^{-1}[C]$.

Proposition VIII.93. *Every functionally closed set if closed. In a metric space the converse holds.*

Proof. TODO + see VIII.210 ☐

Lemma VIII.94. *Let (X, ξ) be a convergence space and $A \subseteq X$. Then A is functionally closed* if and only if *there exists a continuous function $g : X \to \mathbb{R}$ such that $A = g^{-1}[\{0\}]$.*

This means we may take $C = \{0\}$ in the definition of functionally closed.

Proof. If there exists such a function g, then A is clearly functionally closed. For the converse, fix a continuous function $f : X \to \mathbb{R}$ and a closed set $C \subseteq \mathbb{R}$ such that $A = f^{-1}[C]$. Then we use the continuity of $d_C : \mathbb{R} \to \mathbb{R} : x \mapsto \inf_{c\in C} d(x, c)$ (see VIII.210) and set $g = d_C \circ f$. ☐

3.4.1.2 Functional separation

Let (X, ξ) be a convergence space and $A, B \subseteq X$. Then A and B are functionally separated if there exists a continuous function $f : X \to [0, 1] \subseteq \mathbb{R}$ with $f[A] = \{0\}$ and $f[B] = \{1\}$.

Proposition VIII.95. *Two sets are functionally separated* if and only if *they are included in disjoint functionally closed sets.*

Proof. First assume A, B are functionally separated by f. Then $f^{-1}[\ \{0\}\ ]$ and $f^{-1}[\ \{1\}\ ]$ are disjoint functionally closed sets containing A, resp. B.
Conversely, assume $A \subseteq f^{-1}[\{0\}]$ and $B \subseteq g^{-1}[\{0\}]$ (we can take $C = \{0\}$ by VIII.94) with $f^{-1}[\{0\}] \perp g^{-1}[\{0\}]$. Then

$$h : X \to \mathbb{R} : x \mapsto \frac{|f(x)|}{|f(x)| + |g(x)|}$$

is well-defined everywhere because $f^{-1}[\{0\}] \perp g^{-1}[\{0\}]$ and functionally separates A and B. ☐

Lemma VIII.96. *Let (X, ξ) be a topological space and $D \subseteq \mathbb{R}$ a dense subset of the reals. Suppose $(U_d)\, d \in D$ is a set of open subsets of X satisfying*

1. *if $d < t$, then $\overline{U_d} \subseteq U_t$;*
2. *$\bigcup_{d\in D} U_d = X$;*
3. *$\bigcap_{d\in D} U_d = \emptyset$.*

Then $f : X \to \mathbb{R} : x \mapsto \inf \{d \in D \mid x \in U_d\}$ is continuous.

Proof. The function f is well-defined, because $\{d \in D \mid x \in U_d\}$ is never empty.
We will prove continuity at $x \in X$ using VIII.46. To that end, take some $U \in \mathcal{V}_{\mathbb{R}}(f(x))$. By density, we can find some $d, t \in D$ such that $[d, t] \subseteq U$ and $d < f(x) < t$.
Now set $V = U_t \setminus \overline{U_d}$. It is then enough to prove that $V \in \mathcal{V}_\xi(x)$ and $f[V] \subseteq [d, t]$. For the first point, note that $V = U_t \cap \overline{U_d}^c$, so V is open.
Next we show $x \in V$. Indeed, from $f(x) < t$, we get that there exists $f(x) \leq s < t$ such that $x \in U_s$. Thus $x \in U_t$. From $d < f(x)$, we get that there exists $s > d$ such that $x \notin U_s$. By assumption ($d < t \implies \overline{U_d} \subseteq U_t$) this means that $x \notin \overline{U_d}$.
Finally note that if $y \in V \subseteq U_t$, then $f(y) \leq t$ and if $y \notin \overline{U_d}$, then $f(y) > s$ for all $s < d$. Thus $d \leq f(y)$ and $f[V] \subseteq [d, t]$. ☐

Theorem VIII.97 (Urysohn's lemma)**.** *Let (X, ξ) be a normal topological space. If C_1, C_2 are disjoint closed sets, then they are functionally separated.*

Proof. We will construct a family $\{U_r \mid r \in Q\}$ of open sets satisfying the conditions of VIII.96.
First set $U_r = \emptyset$ for all $r < 0$ and $U_r = X$ for all $r > 1$.
Now set $U_1 = C_2^c$. By normality (TODO ref) there exists an open set U_0 such that $C_1 \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$.

We can enumerate the rationals in a sequence $\langle r_n \rangle_{n\in\mathbb{N}}$ with $r_0 = 0$ and $r_1 = 1$. We now recursively define U_{r_n} with recursion invariant $d < t \implies \overline{U_d} \subseteq U_t$ as follows: Set

$$V = U_i \qquad \text{where } i = \max_{\substack{k<n \\ r_k<r_n}} k \qquad \text{and} \qquad W = U_j \qquad \text{where } j = \min_{\substack{k<n \\ r_k>r_n}} k.$$

By the recursion invariant, we have $\overline{U_i} \subseteq U_j$, so by normality we can define a U_{r_k} such that $\overline{U_i} \subseteq U_{r_k} \subseteq \overline{U_{r_k}} \subseteq U_j$. This assignment satisfies the recursion invariant.
Then by VIII.96 the function $f : X \to \mathbb{R} : x \mapsto \inf\{r \in \mathbb{Q} \mid x \in U_r\}$ is continuous. It functionally separates C_1 and C_2. □

Corollary VIII.97.1 (Urysohn's metrisation theorem)**.** *Every regular and second countable topological space is pseudometrisable.*

Proof. Let (X, ξ) be such a topological space. By VIII.92, ξ is automatically normal. TODO embedding into $[0, 1]^F$. □

3.4.1.3 Functional regularity

Let (X, ξ) and (Y, ζ) be convergence spaces. We call ξ ζ-functionally regular if ξ is the initial convergence on X w.r.t. $\mathcal{C}(\xi, \zeta)$.
In particular we call ξ completely regular if it is $\mathbb{R}$-functionally regular.

Chapter 4

Pretopological and Choquet convergence

4.1 Pretopological convergence

A convergence space (X,ξ) is pretopological if it 1-paved, i.e. at each point x there is a filter G that converges to x such that

$$F \in \lim_{\xi}^{-1}(x) \implies G \leq F.$$

Lemma VIII.98. *Let ξ be a convergence on a set X. Then the following are equivalent:*

1. *ξ is a pretopology;*
2. *$\lim_{\xi}^{-1}(x)$ has a least element for all $x \in X$;*
3. *$\mathcal{V}_{\xi}(x) \in \lim_{\xi}^{-1}(x)$ for all $x \in X$;*
4. *$\mathcal{V}_{\xi}(x) \leq F \implies F \in \lim_{\xi}^{-1} x$;*
5. *$x \in \lim_{\xi} F \iff \mathcal{V}_{\xi}(x) \leq F$.*

Proof. (1) ⇔ (2). Because G converging to x is equivalent to $G \in \lim_{\xi}^{-1}(x)$, we see that G is a least element of $\lim_{\xi}^{-1}(x)$.
(2) ⇔ (3). The vicinity filter $\mathcal{V}_{\xi}(x)$ is the infimum of $\lim_{\xi}^{-1}(x)$, and a set contains a least element iff it contains an infimum.
(3) ⇔ (4). The set $\lim_{\xi}^{-1} x$ is upwards closed.
(4) ⇔ (5). The opposite implication to (4) is immediate because $\mathcal{V}_{\xi}(x)$ is the infimum of $\lim_{\xi}^{-1}$. □

Thus in a pretopological convergence space we can determine whether x is in the limit of a filter F by simply comparing F to the vicinity filter of x.

Proposition VIII.99. *Let ξ be a convergence on a set X. Then ξ is a pretopological convergence* if and only if *$\lim_{\xi} : \mathcal{FP}(X) \to \mathcal{P}(X)$ is a complete meet-semilattice homomorphism, i.e. for all families of filters $\mathcal{F} \subseteq \mathcal{P}(\mathcal{FP}(X))$,*

$$\lim_{\xi} \bigwedge_{F\in\mathcal{F}} F = \bigcap_{F\in\mathcal{F}} \lim_{\xi} F.$$

Proof. We have

$$x \in \lim_{\xi} \bigwedge_{F\in\mathcal{F}} F \iff \bigwedge_{F\in\mathcal{F}} F \geq \mathcal{V}_{\xi}(x) \iff \forall F \in \mathcal{F} : F \geq \mathcal{V}_{\xi}(x)$$
$$\iff \forall F \in \mathcal{F} : x \in \lim_{\xi} F \iff x \in \bigcap_{F\in\mathcal{F}} \lim_{\xi} F.$$

□

Corollary VIII.99.1. *The set of pretopological convergences on a set X forms a complete meet-subsemilattice of the lattice of convergences.*

Proof. Complete meet-semilattice homomorphism form a complete meet-subsemilattice, III.57. □

Corollary VIII.99.2. *There exists a closure operator S_0 that maps a convergence ξ to the finest pretopological convergence coarser than ξ.*

4.1.1 Pretopological modification

Let X be a set. The closure operator S_0 on the lattice of convergences on X is called the pretopologiser. For any convergence ξ on X, the pretopological convergence $S_0\xi$ is called the pretopological modification of ξ.

Proposition VIII.100. *Let X be a set and ξ a convergence on X. Then $S_0\xi$ is defined by*

$$x \in \lim_{S_0\xi} F \iff F \geq \mathcal{V}_{\xi}(x).$$

Proposition VIII.101. *Let (X, ξ) and (Y, ζ) be two pretopological convergence spaces. Then the following are equivalent for a function $f : X \to Y$:*

1. $f \in \mathcal{C}(\xi, \zeta)$;
2. *for all* $x \in X$: $f[\mathcal{V}_{\xi}(x)] \geq \mathcal{V}_{\zeta}(f(x))$;
3. *for all* $A \subseteq X$: $f(\mathrm{adh}_{\xi}) \subseteq \mathrm{adh}_{\zeta} f[A]$;
4. *for all* $B \subseteq Y$: $f^{-1}[\mathrm{inh}_{\zeta} B] \subseteq \mathrm{inh}_{\xi}(f^{-1}[B])$.

4.2 Choquet convergence spaces

A convergence space (X, ξ) is called a Choquet space if

$$x \in \lim_{\xi} F \quad\iff\quad \forall U \in \mathbb{U}(\mathcal{P}(X)) : U \geq F \implies x \in \lim_{\xi} U.$$

$$\lim_{\xi} F = \bigcap_{U\in\mathbb{U}(\mathcal{P}(X))\wedge F\subseteq U} \lim_{\xi} U.$$

A pretopological convergence is completely determined by the vicinity filters. A Choquet convergence is completely determined by the convergence of ultrafilters.

Lemma VIII.102. *Every pretopological convergence space is a Choquet space.*

4.2.1 Compact Choquet spaces

Chapter 5

Related types of spaces

5.1 Cauchy spaces

A Cauchy space is a set X together with a set of directed sets that could be the set of convergent directed sets in some convergence space.

Let X be a set and $\mathcal{F}$ a family of filters in $\mathcal{D}(\mathcal{P}(X))$ such that

- $\dot{x} \in \mathcal{F}$ for all $x \in X$;
- $\mathcal{F}$ is upwards closed.

We call $(X, \mathcal{F})$ a Cauchy space.

As with convergence spaces we may impose additional axioms.

5.1.1 Equivalence and convergence

TODO finite depth?

Let $(X, \mathcal{F})$ be a Cauchy space. Define the relation $\sim$ on $(\mathcal{F}, \mathcal{F})$ by

$$F \sim G \qquad \Leftrightarrow_{\text{def}} \qquad F \cap G \in \mathcal{F}.$$

We call the filters F, G equivalent.
We define the Cauchy convergence on X by

$$F \to x \qquad \Leftrightarrow_{\text{def}} \qquad F \sim \dot{x}.$$

Lemma VIII.103. *The Cauchy convergence is a convergence. In particular it is a Kent space.*

Proof. Clearly $\dot{x} \cap \dot{x} = \dot{x} \in \mathcal{F}$, so $\dot{x} \to x$.
Let $F \to x$ and $F \subseteq G$. Then $G \in \mathcal{F}$ and $G \cap \dot{x} \supseteq F \cap \dot{x} \in \mathcal{F}$, so $G \to x$.
The Kent property is immediate. □

Proposition VIII.104. *Let $(X, \mathcal{F})$ be a Cauchy space of finite depth. Then*

1. *the relation $\sim$ is an equivalence relation;*

2. *the Cauchy convergence is of finite depth;*

3. *the Cauchy convergence is* R_1.

5.1.1.1 Cauchy continuity

TODO: equivalent to continuity of Cauchy convergence?

5.1.2 Completeness

Let $(X, \mathcal{F})$ be a Cauchy space. We call X Cauchy complete (or just complete) if every $F \in \mathcal{F}$ converges in the Cauchy convergence.

Proposition VIII.105. 1. *Each closed subspace of a complete Cauchy space is complete.*

2. *A subspace of a complete Hausdorff Cauchy space is complete* if and only if *it is closed.*

3. *The product of complete Cauchy spaces is complete (TODO products!).*

4. *Each compact uniform convergence is complete.*

Proof. TODO ☐

5.1.2.1 Completion

Let $(X, \mathcal{F})$ be a Cauchy space. A completion of $(X, \mathcal{F})$ is another Cauchy space $(Y, \mathcal{G})$ and a function $k : X \to Y$ such that

- $k : X \to Y$ is a Cauchy embedding;
- $k[X]$ is dense in Y.

5.2 Closure spaces

`https://www.researchgate.net/profile/Peter-Stadler-2/publication/239066337_Higher_Separation_Axioms_in_Generalized_Closure_Spaces/links/53d1cf440cf2a7fbb2e95303/Higher-Separation-Axioms-in-Generalized-Closure-Spaces.pdf?origin=publication_detail`

5.3 Merotopic and nearness spaces

`https://en.wikipedia.org/wiki/Proximity_space`

Chapter 6

Uniform spaces

`file:///C:/Users/user/Downloads/(12)%20(Mathematical%20Surveys%20and%20Monographs)%20J.%20R.%20Isbell%20-%20Uniform%20Spaces-American%20Mathematical%20Society%20(1964).pdf` `file:///C:/Users/user/Downloads/(London%20Mathematical%20Society%20Lecture%20Note%20Series)%20I.%20M.%20James%20-%20Introduction%20to%20Uniform%20Spaces-Cambridge%20University%20Press%20(1990).pdf`

6.1 Uniform convergence

Let X be a set. A uniform convergence structure (or just uniform structure) is a family $\mathcal{J}$ of filters on $X \times X$ such that

- $\dot{x} \otimes \dot{x} \in \mathcal{J}$ for all $x \in X$;
- $F \cap G \in \mathcal{J}$ for all $F, G \in \mathcal{J}$;
- if $F \in \mathcal{J}$ and $F \subseteq G$, then $G \in \mathcal{J}$;
- if $F \in \mathcal{J}$, then $F^{\mathrm{T}} \in \mathcal{J}$;
- $F;G \in \mathcal{J}$ if $F, G \in \mathcal{J}$ and $F;G$ exists.

TODO product filter notation $\otimes$
$(F \otimes G)^{\mathrm{T}} = G \otimes F$.

6.1.1 Induced Cauchy structure

Let $(X, \mathcal{J})$ be a uniform structure. Then

$$\mathcal{F} := \{F \in \mathcal{FP}(X) \mid F \otimes F \in \mathcal{J}\}$$

makes $(X, \mathcal{F})$ a Cauchy space, called the induced Cauchy space. We call the elements of $\mathcal{F}$ Cauchy filters.

6.1.1.1 Induced convergence

Let $(X, \mathcal{J})$ be a uniform structure. We say a filter $F \in \mathcal{FP}(X)$ converges uniformly to $x \in X$ if it is Cauchy and converges to x in the induced Cauchy structure. We write $F \overset{u}{\to} x$.
The induced convergence on X is defined by

$$F \to x \qquad \Leftrightarrow_{\text{def}} \qquad .$$

We say F converges uniformly to x and write $F \overset{u}{\to} x$.

Proposition VIII.106. *Let $(X, \mathcal{J})$ be a (finite depth TODO!) uniform structure, $F \in \mathcal{FP}(X)$ and $x \in X$. Then $F \overset{u}{\to} x$* if and only if *$F \otimes \dot{x} \in \mathcal{J}$.*

Proof. TODO

$$(F \cap \dot{x}) \otimes (F \cap \dot{x}) = F \otimes F \cap \dot{x} \otimes \dot{x} \cap F \otimes \dot{x} \cap \dot{x} \otimes F.$$

□

6.1.1.2 Complete uniform spaces

A uniform space is called complete if the associated Cauchy space is complete.

6.1.2 The Arzelà-Ascoli theorem

6.1.2.1 Equicontinuity

6.2 Uniform structure

Let X be a set. A uniform structure or uniformity on X is a family of sets $\mathcal{U} \subseteq \mathcal{P}(X \times X)$ such that

- for all $E \in \mathcal{U}$: $\mathrm{id}_X \subseteq E$;
- $\mathcal{U}$ is a filter in $\mathcal{P}(X \times X)$;
- for all $E \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V ; V \subseteq E$;
- if $E \in \mathcal{U}$, then $E^{\mathrm{T}} \in \mathcal{U}$.

The elements $E \in \mathcal{U}$ are called entourages. We call $(X, \mathcal{U})$ a uniform space.

6.2.1 Topology induced by uniform structure

Let $(X, \mathcal{U})$ be a uniform space. The pretopological convergence ξ induced by $\mathcal{U}$ is defined by

$$x \in \lim_{\xi} F \iff \{xE \mid E \in \mathcal{U}\} \leq F.$$

In other words, the vicinity filter at x is given by

$$\mathcal{V}_\xi(x) = \{xE \mid E \in \mathcal{U}\}\,.$$

Proposition VIII.107. *The pretopological convergence induced by a uniform structure is topological.*

TODO diagonality. Also generalise uniform structure to pretopologies.
??? diagonal iff $\{x\} \# \mathcal{V}_\xi(x)$??? `https://open.library.ubc.ca/media/download/pdf/831/1.0080496/1`

6.2.2 Uniform continuity

6.3 Function spaces

Let $(Y,\mathcal{U})$ be a uniform space and X a set. We define the following uniformities on $X \to Y$:

- the uniformity $\mathcal{U}_p$ of pointwise convergence is the product uniformity:

$$\mathcal{U}_p = \uparrow\Big\{\big\{(f,g) \in (X \to Y)^2 \;\big|\; \forall x \in S : f(x)Eg(x)\big\} \;\Big|\; E \in \mathcal{U}, \emptyset \neq S \subseteq X \text{ finite}\Big\}\,;$$

- the uniformity $\mathcal{U}_u$ of uniform convergence:

$$\mathcal{U}_u \coloneqq \uparrow\Big\{\big\{(f,g) \in (X \to Y)^2 \;\big|\; \forall x \in X : f(x)Eg(x)\big\} \;\Big|\; E \in \mathcal{U}\Big\}\,.$$

We say a filter, net or sequence of functions in $(X \to Y)$ converges pointwise if it converges w.r.t. the uniformity of pointwise convergence. We say it converges uniformly if it converges w.r.t. the uniformity of uniform convergence.

TODO: compare box topology!

Lemma VIII.108. *Let $(Y,\mathcal{U})$ be a uniform space and X a set. The uniformities of pointwise and uniform convergence are uniformities.*

Proof. TODO □

`https://arxiv.org/pdf/1209.1619.pdf`

Proposition VIII.109. *Let $(Y,\mathcal{U})$ be a uniform space and X a set. Let I be a directed set and $\{f_n\}_{n\in I}$ a net of functions in $(X \to Y)$. Then*

1. *f_n converges uniformly to f* if and only if

$$\forall E \in \mathcal{U} : \exists n_0 \in I : \forall n \geq n_0 : \forall x \in X : \; (f(x), f_n(x)) \in E;$$

2. *f_n converges pointwise to f* if and only if

$$\forall x \in X : \forall E \in \mathcal{U} : \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \; (f(x), f_n(x)) \in E.$$

Chapter 7

Topological convergence and topological spaces

A topology specifies which points are close to each other and which are not. This is useful for determining continuity and the existence of holes for example.
Obviously one way to get an idea of which points are close to other points is by explicitly supplying a notion of distance. In fact this a particular type of topological space called a metric space. This way of describing the topology will turn out to be too restrictive, however.
Another way of describing topology is saying that it is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing. Those continuous deformations do not change *which* points are close to each other, just exactly how close they are. Tearing separates points that were close and gluing makes points that were not in each others neighbourhoods suddenly neigbours.
A famous example of such a continuous deformation is the deformation between a doughnut and a coffee cup. Because a topologist is only interested in properties that are preserved under such a transformation, the joke goes that for him as doughnut and a mug is the same thing.
All this can be achieved by defining which subsets of the space are *neighbourhoods* of each point. A neighbourhood of a point is an open set around that set and can be thought of as a sort of generalisation of the open intervals on the real line. TODO motivate definition.
`https://en.wikipedia.org/wiki/List_of_topologies`
`http://www.dynamics-approx.jku.at/lena/Cooper/riesz.pdf` For order convergence!!!

7.1 Axiomatisations and basic concepts

TODO Motivation.

7.1.1 Building blocks

The basic building blocks of topology are neighbourhoods, open sets, closed sets, interior, closure, boundaries, limit points, convergence and nearness. Each of these concepts can be axiomatised and given any one, the others are uniquely fixed.

We first describe how these concepts are related, and then give each axiomatisation and show they are equivalent.
`https://en.wikipedia.org/wiki/Characterizations_of_the_category_of_topological_spaces` `https://mathoverflow.net/questions/19152/why-is-a-topology-made-up-of-open-sets`
`19173#19173`

7.1.2 Neighbourhoods, open sets, closed sets

TODO: function spaces: closed sets point-wise topology ¿¡ uniform topology (bounded by functions v bounded by horizontals)
Let X be a set.
Neighbourhoods: sets that "completely surround" x.
For every $x \in X$ we specify which subsets of X are neighbourhoods of x. This family of sets is denoted $\mathcal{N}(x)$. We would like the following to hold:

1. Every neighbourhood of x must contain x.
2. Every set that contains a neighbourhood of x is a neighbourhood of x.
3. The intersection of two neighbourhoods is again a neighbourhood.
4. The point x must in some sense be in the interior of each of its neighbourhoods. TODO: `https://math.stackexchange.com/questions/2692678/why-does-the-definition-of-a-top`

Proposition VIII.110. *Let $(X, \mathcal{T})$ be a topological space and $x \in X$. Then*

1. *if $N \in \mathcal{N}(x)$, then $x \in N$;*
2. *if $M \subset X$ and there exists $N \in \mathcal{N}(x)$ such that $N \subset M$, then $M \in \mathcal{N}(x)$;*
3. *if $M, N \in \mathcal{N}(x)$, then $M \cap N \in \mathcal{N}(x)$;*
4. *$\forall N \in \mathcal{N}(x) : \exists M \in \mathcal{N}(x) : \forall y \in M : N \in \mathcal{N}(y)$.*

Conversely, every function $X \to \mathcal{P}(X)$ that satisfies these properties is the neighbourhood topology for some topology on X.

The fourth point is the least obvious. It essentially says that if y is sufficiently close to x (i.e. $y \in M(x)$), then x is also close to y.

Proof. TODO ☐

> A topology on a set X is a collection $\mathcal{T} \in \mathcal{P}(X)$ of subsets of X having the following properties:
>
> 1. Both $\emptyset$ and X are in $\mathcal{T}$.
> 2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
> 3. The intersection of the elements on any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
>
> A set X for which a topology $\mathcal{T}$ has been specified is called a topological space.

These axioms formalise an idea of open subset.

We call a subset U of T an open set if U is in $\mathcal{T}$. A closed set is any set that can be constructed as $X \setminus U$ for some open set U. In a given topology a set may be open, closed, both or neither. If a set is both open and closed, it is called clopen.
We say U is an open neighbourhood of x if U is an open set containing x. This is sometimes denoted $U(x)$.
A neighbourhood of x is a set containing an open neighbourhood of x. We denote by $\mathcal{N}(x)$ the set of neighbourhoods of x. The function $\mathcal{N}$ is called the neighbourhood topology.

Lemma VIII.111. *Let X be a topological space. Every open set O can be written as $X \setminus K$ for some closed K.*

Proof. Lemma III.106.1. $\square$

Example

- Let X be a three-element set, $X = \{a, b, c\}$. There are many possible topologies on X, to name a few:
  - $\{\emptyset, X\}$
  - $\{\emptyset, \{a\}, \{a, b\}, X\}$
  - $\{\emptyset, \{a\}, X\}$
  - $\{\emptyset, \{a, b\}, X\}$
  - $\{\emptyset, \{a, b\}, \{a, c\}, \{b\}, X\}$
  - $\{\emptyset, \{a, b\}, \{c\}, X\}$
  - $\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$
  - ...
- For any set X, the collection of all subsets of X is a topology, called the discrete topology.
- For any set X, the topology $\mathcal{T} = \{\emptyset, X\}$ is called the trivial topology.
- In any topology, both X and $\emptyset$ are both open and closed.
- Let X be a set. Let $\mathcal{T}_f$ be a collection of all subsets U of X such that $X \setminus U$ is finite or $U = \emptyset$. Then $\mathcal{T}_f$ is a topology on X called the finite complement topology.

We can also characterise the topology with closed sets or with neighbourhoods:

Proposition VIII.112. *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{T}_c$ be the family of closed subsets of X. Then*

1. *Both $\emptyset$ and X are in $\mathcal{T}_c$.*
2. *Let $\mathcal{E}$ be a subset of $\mathcal{T}_c$. Then $\bigcap \mathcal{E} \in \mathcal{T}_c$.*
3. *If $A, B \in \mathcal{T}_c$, then $A \cup B \in \mathcal{T}_c$.*

Conversely, given any set X and any family $\mathcal{T}_c \subset \mathcal{P}(X)$ that satisfies these propserties, the family

$$\mathcal{T} = \{O \subset X \mid X \setminus O \in \mathcal{T}_c\}$$

is a topology on X.

Obviously a set can have different topologies.

Sometimes we can compare them. Two topologies $\mathcal{T}$ and $\mathcal{T}'$ are comparable if either $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T} \supseteq \mathcal{T}'$.

$\boxed{\mathcal{T} \subseteq \mathcal{T}'}$ In topology $\mathcal{T}'$ there are more open sets. This allows more granular specification of neighbourhoods. We say $\mathcal{T}'$ is finer than $\mathcal{T}$. If $\mathcal{T}'$ is a proper superset, we say it is strictly finer than $\mathcal{T}$.

$\boxed{\mathcal{T} \supseteq \mathcal{T}'}$ In this case $\mathcal{T}'$ is (strictly) coarser than $\mathcal{T}$.

7.1.3 Closure and interior of a set

`https://en.wikipedia.org/wiki/Kuratowski_closure_axioms`

Given any subset A of a topological space X,

- The interior of A, denoted A°, is the union of all open sets contained in A;
- The closure of A, denoted $\bar{A}$, is the intersection of all closed sets containing A.

The boundary of A is $\partial A := \bar{A} \setminus A^\circ$.

We immediately have the inclusions

$$A^\circ \subset A \subset \bar{A}$$

Lemma VIII.113. *The interior and closure are dual in the sense that*

$$A^\circ = X \setminus \overline{(X \setminus A)} = \overline{(A^c)}^c \qquad \bar{A} = X \setminus (X \setminus A)^\circ = ((A^c)^\circ)^c$$

where X is a topological space and A is a subset.

Proposition VIII.114. *Let A be a subset of the topological space X, then*

$$x \in \bar{A} \qquad \text{if and only if} \qquad \textit{every open set } U \textit{ containing } x \textit{ intersects } A.$$

Proof. We prove the contrapositive.

$$x \notin \bar{A} \iff \text{there exists an open set } U \text{ containing } x \text{ that does not intersect } A.$$

$\boxed{\Rightarrow}$ The set $U = X \setminus \bar{A}$ is an open set containing x that does not intersect A.

$\boxed{\Leftarrow}$ If there exists such a U, then $X \setminus U$ is a closed set containing A, so $X \setminus U \supset \bar{A}$. Therefore x cannot be in $\bar{A}$.

□

Proposition VIII.115. *Let A be a subset of the topological space X, then*

$$x \in A^\circ \qquad \text{if and only if} \qquad \textit{there exists an open set } U \textit{ such that } x \in U \subset A.$$

Proof. The interior is the union of all open sets $U \subset A$. Thus if $x \in A^\circ$, then x is in such a U. □

Lemma VIII.116. *Given any subset A of X,*

- *$\bar{A}$ is the smallest closed set containing A;*
- *A° is the largest open set contained in A.*

Consequently the closure and interior are idempotent:

$$\overline{\overline{A}} = \bar{A} \qquad \text{and} \qquad (A^\circ)^\circ = A^\circ.$$

Lemma VIII.117. *Let A, B be subsets of a topological space X. Then*

1. *$\overline{A \cup B} = \overline{A} \cup \overline{B}$;*
2. *$(A \cap B)^\circ = A^\circ \cap B^\circ$.*

These properties do not hold for arbitrary unions and intersections.

Proof. TODO ☐

Lemma VIII.118. *TODO: Intersection of Interiors contains Interior of Intersection and Closure of Union contains Union of Closure and Closure of Intersection is Subset of Intersection of Closures and Union of Interiors is Subset of Interior of Union*

Lemma VIII.119. *Let $A \subseteq B$ be sets in a topological space X. Then*

1. *$\overline{A} \subseteq \overline{B}$;*
2. *$A^\circ \subseteq B^\circ$.*

7.1.4 Boundaries

`https://math.stackexchange.com/questions/2254363/definitions-of-a-topological-space-r`
`https://math.stackexchange.com/questions/4398247/axiomatizations-of-the-boundary-oper`
`https://mathoverflow.net/questions/175800/which-sets-occur-as-boundaries-of-other-set`

7.1.5 Metrics

Quantales and continuity spaces: `https://link.springer.com/content/pdf/10.1007/s000120050018.pdf` `https://arxiv.org/abs/1311.4940` All Topologies Come From Generalized Metrics - Kopperman

7.1.6 Limit points

> If A is a subset of the topological space X and if x is a point of X (not necessarily of A), we say x is a limit point (also sometimes called cluster point or point of accumulation) of A if every (open) neighbourhood of x intersects A in some point other than x itself.
> The set A' of all limit points of A is called the derived set of A.
> An isolated point of A is a point $x \in A$ that is not an accumulation point for A.

So x is a limit point of A if it belongs to the closure of $A \setminus \{x\}$.

Example

Consider $\mathbb{R}$. If $A =]0,1]$, then the point 0 is a limit point of A. In fact every point in $[0,1]$ is a limit point and no other points of $\mathbb{R}$ are limit points.

This motivates the following assertion:

Proposition VIII.120. *Let A be a subset of a topological space X. Then*

$$\bar{A} = A \cup A' = A^\circ \cup A'$$

where A' is the derived set of A.

Corollary VIII.120.1. *A topological space is closed if and only if it contains all its limit points.*

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subset X$ is perfect in X if it is closed and every point of A is an accumulation point of A.

Lemma VIII.121. *If A has no isolated points, then $\overline{A}$ is perfect in X.*

7.1.7 Special subsets

Let $(X, \mathcal{T})$ be a topological space. A set $A \subset X$ is called

- a $\mathcal{G}_\delta$-set if it is a countable intersection of open sets;
- an $\mathcal{F}_\sigma$-set if it is a countable union of closed sets.

7.2 Topologies

7.2.1 The basis of a topology

For many topologies specifying *all* the open sets can be challenging. In this section we give a way to specify a smaller collection of subsets of X, called a basis, and generate the topology in terms of that.

If X is a set, a basis is a subset $\mathcal{B}$ of the powerset of X such that

1. For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$ containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element $B_3 \in \mathcal{B}$ containing x such that $B_3 \subset B_1 \cap B_2$.

The topology $\mathcal{T}$ generated by $\mathcal{B}$ is defined as follows: A subset U of X is said to be open in X if, for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Each basis element is itself an open set. It is not too difficult to check that $\mathcal{T}$ is indeed a topology.

Example

- For any set X, the collection of all one-point subsets of X is a basis for the discrete topology.
- The collection of all open intervals on the real line is a basis for the standard topology on the real line $\mathbb{R}$.

There is an easier way to obtain the topology $\mathcal{T}$ from a basis $\mathcal{B}$:

Lemma VIII.122. *The topology $\mathcal{T}$ equals the collection of all unions of elements in $\mathcal{B}$.*

Here is a way to obtain a basis from a topology on X.

Lemma VIII.123. *Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

We can link the basis to the coarseness of the topology.

Lemma VIII.124. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. The following are equivalent:*

1. *$\mathcal{T}'$ is finer than $\mathcal{T}$.*
2. *For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Closures of sets can also be described using a basis.

Lemma VIII.125. *Let A be a subset of X which has a topology generated by a basis $\mathcal{B}$, then $x \in \bar{A}$ if and only if every basis element $B \in \mathcal{B}$ containing x intersects A.*

7.2.1.1 Subbasis

If X is a set, a subbasis is a subset $\mathcal{S}$ of the powerset of X such that $X = \bigcup \mathcal{S}$.
The topology $\mathcal{T}$ generated by $\mathcal{S}$ is the collection of all unions of finite intersections of elements of $\mathcal{S}$.

The topology $\mathcal{T}$ is exactly the coarsest topology that makes all sets in the subbasis open.

7.2.2 The subspace topology

Let X be a topological space with topology $\mathcal{T}$. Let Y be a subspace of X. The collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y called the subspace topology. With this topology, Y is called a subspace of X.

Lemma VIII.126. *Let Y be a subspace of X. A set A is closed in Y* if and only if *it equals the intersection of a closed set of X with Y.*

Lemma VIII.127. *If $\mathcal{B}$ is a basis for the topology of X, then*

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Lemma VIII.128. *Let Y be a subspace of X.*

1. *If A is open in Y and Y is open in X, then A is open in X.*
2. *If A is closed in Y and Y is closed in X, then A is closed in X.*

We reserve the notation $\overline{A}$ to stand for the closure of A in X, not Y.

Lemma VIII.129. *Let X be a topological space and $Y \subset X$ a subspace. Let A be a subset of Y, then the closure of A in Y is*

$$\mathrm{Cl}_Y(A) = \overline{A} \cap Y.$$

Lemma VIII.130. *Let $A \subseteq X$ be a subspace of X. Then $a \in A \setminus A'$, then $\{a\}$ is open in A.*

Proof. Assume such an a. Then there exists an open neighbourhood U of a in X that does not intersect A in any other point. By definition of the subspace topology $\{a\}$ is open. ☐

7.2.3 Topology and order

`https://planetmath.org/orderedspace` `https://www.jstor.org/stable/2032122?seq=2#metadata_info_tab_contents` `http://www.math.wm.edu/~lutzer/drafts/PragueSurveyFinal.pdf` `https://ncatlab.org/nlab/show/pospace` `file:///C:/Users/user/Downloads/order-topological-lattices.pdf`

7.2.3.1 Specialisation preorder

Let $(X, \mathcal{T})$ be a topological space and $x, y \in X$. We say x

Alexandrov topology `https://planetmath.org/inducedalexandrofftopologyonaposet` `https://arxiv.org/pdf/0708.2136.pdf` `https://ncatlab.org/nlab/show/specialization+topology` `http://math.uchicago.edu/~may/REU2018/REUPapers/Asness.pdf`

7.2.3.2 Order topology on totally ordered sets

Let $(X, \leq)$ be a linearly ordered set. Let $\mathcal{B}$ be the collection of all sets of the following type:

1. All open intervals $]a, b[$ in X;
2. All intervals of the form $[a_0, b[$, where a_0 is the smallest element (if any) of X;
3. All intervals of the form $]a, b_0]$, where b_0 is the largest element (if any) of X;

The collection $\mathcal{B}$ is a basis for a topology, called the order topology.

Product of linearly ordered topology

7.3 Separation axioms

7.3.1 T_0

7.3.1.1 Kolmogorov quotient

7.3.2 T_1

Proposition VIII.131. *Let X be a topological space satisfying T_1; let A be a subset of X. Then the point x is a limit point of A* if and only if *every neighbourhood of x contains infinitely many points of A.*

7.3.3 Hausdorff spaces

> A topological space X is called a Hausdorff space if for each pair x_1, x_2 of distinct points in X, their exist neighbourhoods $U(x_1)$, $U(x_2)$ that are disjoint.

In Hausdorff spaces distinct points can be told apart topologically, hence Hausdorff spaces are also called separated spaces. In particular the Hausdorff condition implies the uniqueness of limits, which is not otherwise guaranteed.

Proposition VIII.132. *Every finite point set in a Hausdorff space is closed. TODO: T1*

Proof. It suffices to show that every one-point set $\{x_0\}$ is closed. Indeed if $\{x_0\}$ was not closed, the closure of $\{x_0\}$ would contain another point. This other point has a disjoint neighbourhood by Hausdorff, so this fails by proposition VIII.114. ☐

Proposition VIII.133. *Limits are unique (sequences, filters, nets)*

Lemma VIII.134. *1. A subspace of a Hausdorff space is Hausdorff.*

2. Every totally ordered set is Hausdorff in the order topology.

3. Every metric topology is Hausdorff.

7.4 Functions on topological spaces

7.4.1 Open and closed maps

TODO

7.4.2 Continuity and continuous functions

Intuitively, a continuous map is a map between topological spaces that does not make jumps. In particular let $f : X \to Y$ be a potentially continuous function. Say we want to stay in a neighbourhood $V(f(x_0))$, then we want there to be a neighbourhood $U(x_0)$ such that points inside $U(x_0)$ map to points in V, i.e.

$$x \in U(x_0) \implies f(x) \in V(f(x_0)).$$

That is $f(U(x_0)) \subset V$, or $U(x_0) \subset f^{-1}(V)$. So we conclude that for any point x_0 and neighbourhood $V(f(x_0))$ in Y, $f^{-1}(V)$ must contain a neighbourhood of x_0. Thus $f^{-1}(V)$ can be written as a union of open sets, $\bigcup_{x \in f^{-1}(V)} U(x)$, and therefore must be open. This motivates the definition:

Let X, Y be topological spaces.

- A function $f : X \to Y$ is continuous if for each open set V of Y, $f^{-1}[V]$ is an open subset of X.
- The function f is continuous at x_0 if for each open neighbourhood V of $f(x_0)$, their is an open neighbourhood U of x_0 such that $f[U] \subset V$.

If a function is not continuous, it is discontinuous.
The set of all continuous functions $X \to Y$ is denoted $\mathcal{C}(X, Y)$. If $X = Y$, we also write $\mathcal{C}(X)$.

Lemma VIII.135. *A function $f : X \to Y$ is continuous* if and only if *it is continuous at every point.*

Lemma VIII.136. *Let $f : X \to Y$ be a function between topological spaces. If $\{x_0\} \subset X$ is open, then f is continuous at x_0.*

Lemma VIII.137.

1. *If the topology of Y is given by a basis $\mathcal{B}$, then to prove continuity of f it suffices to show that the inverse image of every basis element is open.*
2. *If the topology of Y is given by a subbasis $\mathcal{S}$, then to prove continuity of f it suffices to show that the inverse image of every subbasis element is open.*

Proposition VIII.138. *Let X, Y be topological spaces; $f : X \to Y$. The following are equivalent:*

1. *f is continuous;*
2. *$f[\bar{A}] \subset \overline{f[A]}$;*
3. *for every closed set B of Y, the set $f^{-1}[B]$ is closed in X. TODO f closed.*

Proof. We proceed round-robin-style.

$\boxed{(1) \Rightarrow (2)}$ Let $x \in \bar{A}$ and V a neighbourhood of $f(x)$. Then $f^{-1}[V]$ is an open set containing x, so it must intersect A in some point y by proposition VIII.114. Then V intersects $f[A]$ in $f(y)$, so $f(x) \in \overline{f[A]}$ as desired.

$\boxed{(2) \Rightarrow (3)}$ Let B be closed in Y. We observe that $f[f^{-1}[B]] \subset B$. Choose some $x \in \overline{f^{-1}[B]}$, then

$$f(x) \in f\left[\overline{f^{-1}[B]}\right] \subset \overline{f[f^{-1}[B]]} \subset \bar{B} = B,$$

so that $x \in f^{-1}[B]$. Thus $\overline{f^{-1}[B]} \subset f^{-1}[B]$, meaning $f^{-1}[B]$ is closed.

$\boxed{(3) \Rightarrow (1)}$ Let V be an open set in Y. Set $B = Y \setminus V$. Then $V = Y \setminus B$ and

$$f^{-1}[V] = f^{-1}[Y \setminus B] = f^{-1}[Y] \setminus f^{-1}[B] = X \setminus f^{-1}[B]$$

using lemma I.104. Thus $f^{-1}[V]$ is open.

□

7.4.2.1 Homeomorphisms *or* topological isomorphisms

Let X, Y be topological spaces and $f : X \to Y$ a bijection. Then f is a homeomorphism if both f and f^{-1} are continuous.

Lemma VIII.139. *A homeomorphism is a bijection f such that $f(U)$ is open* if and only if *U is open.*

7.4.2.2 Constructing continuous functions

Proposition VIII.140. *Let X, Y and Z be topological spaces.*

1. (Identity function) *The identity function $I : X \to X$ is continuous.*
2. (Constant function) *If $f : X \to Y$ maps all of X into a single y_0 of Y, then f is continuous.*
3. (Inclusion) *Let A be a subspace of X, then the inclusion $A \hookrightarrow X$ is continuous.*
4. (Composites) *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*
5. (Restricting the domain) *If $f : X \to Y$ is continuous and A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.*
6. (Restricting the range) *Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f[X]$, then $f : X \to Z$ is continuous.*
7. (Expanding the range) *Let $f : X \to Y$ be continuous. If Y is a subspace of Z, then $f : X \to Z$ is continuous.*
8. (Local formulation of continuity) *The map $f : X \to Y$ is continuous is X can be written as the union of open sets U_α such that $f|_{U_\alpha}$ is continuous for each α.*

Proposition VIII.141 (The pasting lemma)**.** *Let $X = A \cup B$ where A, B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous such that $f(x) = g(x)$ for all $x \in A \cap B$. Then the function defined by*

$$h : X \to Y : x \mapsto h(x) = \begin{cases} f(x) & (x \in A) \\ g(x) & (x \in B) \end{cases}$$

is continuous.

Proof. Let C be a closed subset of Y, then $f^{-1}[C]$ and $g^{-1}[C]$ are both closed. So

$$h^{-1}[C] = f^{-1}[C] \cup g^{-1}[C]$$

is closed, meaning h is continuous, all by proposition VIII.138. □

TODO: complex conjugation continuous.

7.4.3 Limits of functions

Let X, Y be topological spaces. Let p be a limit point of $A \subseteq X$ and $f : A \to Y$. We say

$L \in Y$ is a limit of $f(x)$ as x approaches p if

$$\forall \text{ open neighbourhood } V(L) : \exists \text{ open neighbourhood } U(p) : \; f[(U \cap A) \setminus \{p\}] \subseteq V.$$

We write $f(x) \to L$ as $x \to p$ or

$$\lim_{x \to p} f(x) = L.$$

Note that the value of f at p is irrelevant to the definition of the limit. The domain of f does not even need to contain p.

Proposition VIII.142. *Let $f : X \to Y$ be a functions between topological spaces. Then f is continuous at $p \in X$ if and only if $\lim_{x \to p} f(x) = f(p)$.*

Limits may or may not exist and may or may not be unique, but uniqueness is guaranteed if Y is Hausdorff.

Proposition VIII.143. *Let $f : A \subseteq X \to Y$ be a function and X, Y be topological spaces. If Y is Hausdorff, then there is a most one limit of f at any point $p \in X$.*

Proof. Assume L_1 and L_2 are two distinct limits of $f(x)$ as $x \to p$. Because Y is Hausdorff there are two disjoint open neighbourhoods V_1, V_2 of L_1, L_2. Let U_1, U_2 be the corresponding open neighbourhoods of p. Then $U_1 \cap U_2$ must be an open neighbourhood of p, so that $U_1 \cap U_2 \cap A$ contains a point other than p, by virtue of p being a limit point. This however means that $f[(U_1 \cap A) \setminus \{p\}]$ and $f[(U_2 \cap A) \setminus \{p\}]$ are not disjoint, so neither are V_1, V_2: a contradiction. ☐

7.4.4 Initial and final topologies

7.4.5 Sets of functions

Let X, Y be topological spaces.

- The set of continuous functions in $(X \to Y)$ is denoted $\mathcal{C}(X, Y)$.
- The set of continuous functions in $(X \to Y)$ which vanish at infinity is denoted $\mathcal{C}_0(X, Y)$.
- The set of continuous functions in $(X \to Y)$ with compact support is denoted $\mathcal{C}_c(X, Y)$.
- The set of bounded continuous functions in $(X \to Y)$ is denoted $\mathcal{C}_b(X, Y)$.

If we omit Y, we generally mean $Y = \mathbb{R}$.

TODO: define these notions in general! TODO: ideals and multiplier algebras.

7.5 The product topology

7.5.1 Finite Cartesian products

The product topology on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Lemma VIII.144. *If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ a basis for the topology of Y, then*

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Proposition VIII.145. *Let A be a subspace of X and B a subspace of Y. The product topology on $A \times B$ is the same as the subspace topology on $A \times B$, when viewed as a subset of $X \times Y$.*

Let X, Y be topological spaces. The maps

$$\pi_1 : X \times Y \to X : (x, y) \mapsto x \qquad\qquad \pi_2 : X \times Y \to Y : (x, y) \mapsto y$$

are called the projections of $X \times Y$ onto its first and second factors, respectively.

Proposition VIII.146. *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$; let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$.

$\boxed{\mathcal{T}' \subset \mathcal{T}}$ We need to prove all elements of $\mathcal{S}$ are open. Indeed $\pi_1^{-1}(U) = U \times Y$ is open and $\pi_2^{-1}(V) = X \times V$ is also open.

$\boxed{\mathcal{T} \subset \mathcal{T}'}$ Let $B \times C$ be an element of the basis, in other words $B \subset X$ and $C \subset Y$ are open. Then $B \times C = \pi_1^{-1}(B) \cap \pi_2^{-1}(C)$.

□

In particular π_1 and π_2 are continuous.

Proposition VIII.147. *Let A, X, Y be topological spaces and let*

$$f : A \to X \times Y : a \mapsto f(a) = (f_1(a), f_2(a)).$$

Then f is continuous if and only if *the functions f_1 and f_2 are continuous.*

There is no useful criterion for the continuity of a map $f : A \times B \to X$.

Proposition VIII.148. *Let X, Y be metrisable topological spaces with metrics d_X and d_Y. Then $X \times Y$ is metrisable. Possible, equivalent, metrics include*

$$d_{max} = \max \circ \{d_X, d_Y\}$$

and

$$d_{graph} = d_X \circ \pi_1 + d_Y \circ \pi_2.$$

Proof. We first prove that the product topology and the metric topology generated by $d_{\max}$ are the same using VIII.124.
First take an element of a basis for the product topology, which by VIII.144 can be taken of the form

$$B = B_{d_X}(x, \epsilon_1) \times B_{d_Y}(y, \epsilon_2) \qquad \text{for some } x \in X, y \in Y, \epsilon_1, \epsilon_2 > 0.$$

Then we can find a basiselement $B_{d_{\max}}((x, y), \min\{\epsilon_1, \epsilon_2\})$ of the metric topology generated by $d_{\max}$ that is a subset.
Conversely, take $B_{d_{\max}}((x, y), \epsilon)$. Then $B_{d_X}(x, \epsilon) \times B_{d_Y}(y, \epsilon)$ is a subset.
The equivalence of the two metrics can then be seen by applying VIII.207 twice:

$$B_{d_{\max}}((x, y), \epsilon) \subset B_{d_{\text{graph}}}((x, y), \epsilon) \qquad B_{d_{\text{graph}}}((x, y), \epsilon/2) \subset B_{d_{\max}}((x, y), \epsilon).$$

□

Corollary VIII.148.1. *A sequence $(x_n, y_n)_n$ converges to (x, y) in the product topology* if and only if *$(x_n)_n$ converges to x and $(y_n)_n$ converges to y.*

TODO:also nets?

7.5.2 Arbitrary Cartesian products

Let $X = \prod_{\alpha \in J} X_\alpha$ and define

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\} \qquad \text{and} \qquad \mathcal{S} = \bigcup_{\beta \in J}.$$

Then the topology on X generated by the subbasis $\mathcal{S}$ is the product topology and then X is called a product space.

Lemma VIII.149.

- *The product topology on $\prod X_\alpha$ has as a basis all sets of the form $\prod_\alpha U_\alpha$, where U_α is open in X_α for all α and $U_\alpha = X_\alpha$ except for finitely many values of α.*
- *If each X_α has a basis $\mathcal{B}_\alpha$, a basis for the product topology is given by all the sets of the form $\prod_{\alpha \in J} B_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many values of α and $B_\alpha = X_\alpha$ for the rest.*

If we remove the condition that $U_\alpha = X_\alpha$ except for finitely many values of α, we get the box topology.
Some results that held for finite Cartesian product also hold for arbitrary products:

Lemma VIII.150. *Let the topology on $\prod X_\alpha$ be the product topology.*

- *If each space X_α is Hausdorff, then $\prod X_\alpha$ is Hausdorff.*
- *Let A_α be subsets of X_α, then*
$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}.$$
- *Let A_α be subspaces of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the product topology.*

Proposition VIII.151. *Let $\prod X_\alpha$ have the product topology. Let $f : A \to \prod X_\alpha$ be given by*

$$f(a) = (f_\alpha(a))_{\alpha \in J} \qquad \text{where } f_\alpha = \pi_\alpha \circ f : A \to X_\alpha \text{ for each } \alpha \in J.$$

Then f is continuous if and only if *each function f_α is continuous.*

This does not hold for the box topology. TODO: Universal mapping property; generalise to initial topologies.

Corollary VIII.151.1. *Assume we have some points $c_i \in X_i$ for all $i \in J$. Then the functions*

$$i_\alpha : X_\alpha \to X : p \mapsto \left(\begin{cases} c_i & (i \neq \alpha) \\ p & (i = \alpha) \end{cases} \right)_{i \in J}$$

are continuous.

Proof. Consider i_α. Then for $i = \alpha$, the function $\pi_i \circ i_\alpha : X_\alpha \to X_i$ is the identity in X_α and thus continuous, VIII.140. For $i \neq \alpha$, the function $\pi_i \circ i_\alpha : X_\alpha \to X_i$ is a constant function $p \mapsto c_i$ and thus continuous, VIII.140. □

Lemma VIII.152. *Given points $\mathbf{x} = (x_i)_{i \in \mathbb{N}}$ and $\mathbf{y} = (y_i)_{i \in \mathbb{N}}$ of $\mathbb{R}^{\mathbb{N}}$, define the metric*

$$D(\mathbf{x}, \mathbf{y}) = \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \mid i \in \mathbb{N} \right\},$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. Then D induces the product topology on $\mathbb{R}^{\mathbb{N}}$.

Proof. Let $\mathcal{T}$ denote the product topology on $\mathbb{R}^{\mathbb{N}}$ and $\mathcal{T}_D$ the topology induced by D. We prove two inclusions using lemma VIII.124.

$\boxed{\mathcal{T}_D \subset \mathcal{T}}$ Choose arbitrary basis element $B_D(\mathbf{x}, \epsilon)$. Then choose an $N \in \mathbb{N}$ such that $1/N < \epsilon$. Take the basis element

$$V =]x_1 - \epsilon, x_1 + \epsilon[\times]x_1 - \epsilon, x_1 + \epsilon[\times \ldots \times]x_N - \epsilon, x_N + \epsilon[\times \mathbb{R} \times \mathbb{R} \times \ldots$$

for the product topology. We assert that $V \subset B_D(\mathbf{x}, \epsilon)$. Indeed, for all $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{i} \leq \frac{1}{N} \qquad \text{if } i \geq N.$$

Therefore,

$$D(\mathbf{x}, \mathbf{y}) \leq \max \left\{ \frac{\bar{d}(x_1, y_1)}{1}, \frac{\bar{d}(x_2, y_2)}{2}, \ldots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N} \right\}.$$

So if $\mathbf{y} \in V$, then $D(\mathbf{x}, \mathbf{y}) < \epsilon$ and $V \subset B_D(\mathbf{x}, \epsilon)$.

$\boxed{\mathcal{T} \subset \mathcal{T}_D}$ Choose an arbitrary basis element $U = \prod U_i$. Let $U_i = \mathbb{R}$ if $i \notin \{\alpha_1, \ldots, \alpha_n\}$. For each $i \in \{\alpha_1, \ldots, \alpha_n\}$ choose an interval $]x_i - \epsilon_i, x_i + \epsilon_i[\subset U_i$ and define

$$\epsilon = \min\{\epsilon_i / i \mid i = \alpha_1, \ldots, \alpha_n\}.$$

We can easily see that $B_D(\mathbf{x}, \epsilon) \subset U$.

Corollary VIII.152.1. *Countable products of metrisable spaces are metrisable.*

Corollary VIII.152.2. *Countable products of Hausdorff spaces are Hausdorff.*

□

Lemma VIII.153. *The product $\mathbb{R}^J$, with J an uncountable index set, is not metrisable.*

Proof. In a metrisable space, by TODO ref, we have that if $x \in \bar{A}$, then there exists a sequence of points in A converging to x. We construct a counterexample. Let A be the subset of $\mathbb{R}^J$ containing all points $(x_i)_{i \in J}$ such that $x_i = 1$ for all but finitely many i. Now the point $(0)_{i \in J}$ is in the closure of A, but has no sequence in A converging to it. To see that it is in the closure, let $\prod U_\alpha$ be a basis element containing $(0)_{i \in J}$. The intersection $A \cap \prod U_\alpha$ is never empty. Indeed for only finitely many α, $U_\alpha \neq \mathbb{R}$. Set $x_\alpha = 0$ for these α and $x_i = 1$ for the rest. □

7.5.3 Box topology

Let $X = \prod_{\alpha \in J} X_\alpha$ and take as a basis for a topology the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha \qquad (U_\alpha \text{ open in } X_\alpha).$$

The topology generated by this basis is the box topology.

The following properties hold, like in the product topology:

Lemma VIII.154. *Let the topology on $\prod X_\alpha$ be the box topology.*

- *If each space X_α is Hausdorff, then $\prod X_\alpha$ is Hausdorff.*
- *Let A_α be subsets of X_α, then*
$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}.$$
- *Let each X_α have a basis $\mathcal{B}_\alpha$. The collection of all the sets of the form*
$$\prod_{\alpha \in J} B_\alpha \qquad B_\alpha \in \mathcal{B}_\alpha$$
serves as a basis for the box topology.
- *Let A_α be subspaces of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology.*

7.5.3.1 Failure of metrisability

Lemma VIII.155. *$\mathbb{R}^\omega$ is not metrisable in the box topology.*

7.5.3.2 Failure of continuity

7.5.3.3 Failure of compactness

7.6 The quotient topology

Let X be a topological space. A quotient set can always be defined by a surjective function $f : X \to A$ to a set A. Then A can be identified with a partition X^* of X. Now we would like to define a topology on the partition. We can think of the quotient as shrinking each partition to a single point. Thus it is natural to call a subset of A open if the union of the corresponding partitions is open:

$$V \text{ is open in } A \quad \Leftrightarrow_{\text{def}} \quad p^{-1}(V) \text{ is open in } X.$$

This gives us the following definition:

Let X, Y be topological spaces and $p : X \to Y$ a surjective map. The map p is a quotient map if

$$V \text{ is open in } Y \iff p^{-1}(V) \text{ is open in } X$$

This condition is stronger than continuity.

Let X be a topological space.

- Let a be A a subset, and $p : X \to A$ a surjective map. There exists exactly one topology on A relative to which p is a quotient map; it is called the quotient topology induced by p.
- Let X^* be a partition of X and $p : X \to X^*$ the surjective map that carries each point of X to its partition. In the quotient topology induced by p, the space X^* is called a quotient space of X.

We can also characterise the notion of quotient map in another way, starting from the following definition:

A subset C of a topological space X is saturated with respect to a surjective map $p : X \to Y$ if C is the complete inverse image of a subset of Y, i.e. it contains every set $p^{-1}(\{y\})$ that it intersects.

Lemma VIII.156. *A surjective map p is a quotient map* if and only if *p is continuous and maps saturated open sets of X to open sets of Y.*

Corollary VIII.156.1.

- *Surjective continuous open maps are quotient maps.*
- *Surjective continuous closed maps are quotient maps.*

There are quotient maps that are neither open or closed.

Proposition VIII.157. *Let $p : X \to Y$ be a quotient map; let A be a subset of X that is saturated w.r.t. p; let $q : A \to p(A) = p|_A$.*

1. *If A is either open or closed in X, then q is a quotient map.*
2. *If p is either an open map or a closed map, then q is a quotient map.*

Lemma VIII.158. *Let p, q be quotient maps.*

1. *A composite $q \circ p$ of quotient maps is a quotient map.*
2. *The product $p \times q$ is not necessarily a quotient map.*
3. *A quotient space of a Hausdorff space is not necessarily Hausdorff.*

Proof. Point (1) follows from

$$p^{-1}(q^{-1}(U)) = (q \circ p)^{-1}(U).$$

□

TODO: theorem 22.2 + corollary

7.7 Density

Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X. Then A is called

1. dense in X if the closure of A is the whole of X: $\overline{A} = X$;
2. rare or nowhere dense if its closure has empty interior: $(\overline{A})^\circ = \emptyset$;
3. meagre (or a set of first category) if it is a countable union of rare subsets of X;
4. nonmeagre (or a set of second category) if it is not meagre;
5. comeagre if its complement $X \setminus A$ is meagre in X.

Lemma VIII.159. *Let A be a subset of a topological space X. A being is dense in X is equivalent to any of the following:*

1. *every element of X either lies in A or is a limit point of A;*
2. *A^c has empty interior.*

Proof. For the first point: $\overline{A} = A \cup A' = X$.
For the second point:

$$X = \overline{A} \iff X = ((A^c)^\circ)^c \iff (A^c)^\circ = X^c = \emptyset.$$

□

Lemma VIII.160. *Let X be a topological space and $A \subset X$ a subset. Then A is nowhere dense* if and only if $\overline{A}^c$ *is dense.*

Proof. We calculate:

$$(\overline{A})^\circ = \emptyset \iff \overline{(\overline{A}^c)}^c = \emptyset \iff \overline{(\overline{A}^c)} = X.$$

□

Lemma VIII.161. *Any subset of a meagre set is meagre.*

Proof. Let $A = \bigcup_k R_k$ be meagre and $B \subseteq A$. Then

$$B = B \cap A = B \cap \left(\bigcup_k R_k\right) = \bigcup_k B \cap R_k.$$

Now for each k, $B \cap R_k \subset R_k$. So $\overline{B \cap R_k}^\circ \subseteq \overline{R_k}^\circ = \emptyset$, using lemma VIII.119, and thus $B \cap R_k$ is nowhere dense. □

Lemma VIII.162. *Let Y be a dense subspace of a topological space X. Let S be a dense subset of Y. Then S is dense in X.*

Proof. Let $\overline{S}$ be the closure of S in X. Then by VIII.129 we have $Y = Y \cap \overline{S}$, so $\overline{S} \supseteq Y \supseteq S$, which, taking the closure, implies $\overline{S} \supseteq \overline{Y} \supseteq \overline{S}$. Thus $\overline{S} = \overline{Y} = X$. ☐

A topological space Y has the unique extension property if for any topological space X, any continuous functions $f, g : X \to Y$ and any dense subset $E \subset X$ we have

$$\forall x \in E : f(x) = g(x) \qquad \Longrightarrow \qquad f = g.$$

TODO:

Proposition VIII.163. 1. *Assume X Hausdorff and quotient map open, then $X/\sim$ is Hausdorff iff $\sim$ is closed in $X \times X$.*

2. *X Hausdorff iff diagonal is closed*

3. *Let $f, g : A \to B$ be continuous functions. If B is Hausdorff, then $\{x \in A \mid f(x) = g(x)\}$ is closed. (Pre-image of diagonal set)*

Proposition VIII.164. *A topological space Y has the unique extension property* if and only if *Y is Hausdorff.*

Proof. First assume Y Hausdorff. Take functions $f, g : E \subset X \to Y$ that agree on E. They must agree on a closed set (TODO ref), thus at least on $\overline{E} = X$.
Now suppose Y is not Hausdorff. TODO `https://www.jstor.org/stable/2315068?seq=1#metadata_info_tab_contents` ☐

Lemma VIII.165. *Let X be a topological space and $E \subset X$ a .*

1. *If for any dense subspace $E \subset X$ the only continuous extension of id_E to X is id_X, then X is T_0.*

2. *If X is T_2, then for any dense subspace $E \subset X$ the only continuous extension of id_E to X is id_X.*

T_1 is neither necessary nor sufficient.

Proof. TODO `https://math.stackexchange.com/questions/1592144/does-the-identity-map-on-a` `1592169` ☐

7.7.1 The Baire property

A topological space X has the Baire property if it satisfies either of the following equivalent conditions:

1. every countable union of closed nowhere dense sets has empty interior;

2. every countable intersection of open dense sets is dense.

These properties are equivalent because a subset has empty interior if and only if its complement is dense, see lemma VIII.159.

Lemma VIII.166. *A topological space X is Baire* if and only if *either of the following equivalent conditions:*

1. *every meagre subset of X is either empty or not open;*
2. *every non-empty open subset of X is a nonmeagre subset of X;*
3. *every comeagre subset of X is dense in X.*

Proof. We prove the characterisation of spaces with the Baire property using countable unions implies the first point, the last point implies the countable intersection Baire condition.

$\boxed{\text{Baire} \Rightarrow (1)}$ Every meagre set $A = \bigcup_k R_k$ (where all R_k are nowhere dense) is a subset of $\bigcup_k \overline{R_k}$ where $\overline{R_k}$ are closed nowhere dense sets. Thus if the Baire property holds, $\bigcup_k \overline{R_k}$ has empty interior, meaning A has empty interior. So either A is empty or not open.

$\boxed{(1) \Leftrightarrow (2)}$ By contraposition.

$\boxed{(1) \Rightarrow (3)}$ Suppose A is a meagre set. Then A° must also be meagre, by VIII.161. Now A° is certainly open, so by (1) it must be empty. Thus A^c is dense, by lemma VIII.159.

$\boxed{(3) \Rightarrow \text{Baire}}$ Let $A = \bigcap_k O_k$ where all O_k are open dense sets. Then $A^c = \bigcup_k O_k^c$. Now for each k, O_k^c is nowhere dense by lemma VIII.160, because $\overline{O_k^c}^c = O_k^\circ$ is still dense. Thus A^c is meagre and A is comeagre, so A is dense in X.

□

A topological space has the Baire property if and only if it has the property locally, in the following sense:

Lemma VIII.167. *A topological space X has the Baire property* if and only if *every point in X has a neighbourhood with the Baire property.*

Proof. If X is Baire, the neighbourhood can simply be taken to be X.
Assume every point in X has a neighbourhood with the Baire property. We will prove point (2) in lemma VIII.166 holds. Take a non-empty open subset A of X. As A is non-empty, we can take a point $x \in A$ and find a neighbourhood U of x with the Baire property. Then $A \cap U$ is a non-empty open subset of U and thus must not be meagre in U. By contraposition of lemma VIII.161, we see that A must be non-meagre in X, proving the Baireness of X. □

Theorem VIII.168 (Baire category theorem)**.**

1. *Every complete pseudometric space has the Baire property.*
2. *Every locally compact Hausdorff space has the Baire property.*

Proof. TODO + relocate □

7.8 Connectedness

Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X.
The space X is said to be connected if there does not exist a separation of X.

Lemma VIII.169. *A space X is connected* if and only if *the only subsets of X that are both open and closed in X are $\emptyset$ and X.*

Proof. We prove the contrapositive of both implications.

$\boxed{\Rightarrow}$ Let A be a nonempty proper subset of X that is both open and closed in X. The sets A and $X \setminus A$ form a separation.

$\boxed{\Leftarrow}$ Let U, V be a separation. Then U is open. It is also closed, because its complement in X is V, which is open.

☐

The following lemma characterises separations in the subspace topology.

Lemma VIII.170. *Let Y be a subspace of a topological space X. A pair of disjoint nonempty sets A, B constitute a separation of the subspace Y* if and only if *neither set contains a limit point of the other in X.*

Proof. ☐

7.8.1 Path connectedness

7.9 Compactness

TODO relatively compact.
TODO every compact set in (pseudo??)metric space is closed and bounded. Converse is not automatic (Heine-Borel property).

Proposition VIII.171. *The continuous image of a compact set is compact.*

Corollary VIII.171.1. *If $f : X \to Y$ is a continuous function from a topological space to a metric space and $K \subset X$ a compact set, then $f[K]$ is closed and bounded.*

Corollary VIII.171.2. *Let $f : X \to V$ be a function from a compact topological space to a TVS with Heine Borel property. Then $\operatorname{im}(f)$ has a maximum and minimum.*

Proposition VIII.172. *A compact set in a Hausdorff space is closed.*

Proposition VIII.173. *Let $f : X \to Y$ be a continuous bijection between topological spaces. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. We just need to prove f^{-1} is continuous. This is equivalent to f being closed by VIII.138.
TODO ☐

Proposition VIII.174. *Let X be a topological space. Then X is compact* if and only if *every ultrafilter converges.*

Proof. We prove the contrapositive. Assume U is an ultrafilter that does not converge. Then for all $x \in X$ we have $\mathcal{N}(x) \nsubseteq U$, so we can find some $V_x \in \mathcal{N}(x) \setminus U$ and thus also an open set $U_x \setminus V_x \in \mathcal{N}(x) \setminus U$. (If U_x were in U, then also $V_x \in U$, which is not possible.) The set $\{U_x\}_{x \in X}$ is an open cover. By compactness, we can find a finite subcover $\{U_{x_i}\}_{i=0}^n$. Then $U_{x_0} \cup \ldots \cup U_{x_n} = X \in U$. Because every ultrafilter on $\mathcal{P}(X)$ is a prime filter **??** we can find a U_{x_i} such that $U_{x_i} \in U$, which is a contradiction.
Now assume X is not compact. Then there exists an open cover $\{U_i\}_{i \in I}$ that does not have a finite subcover. Consider the family $\{U_i^c\}_{i \in I}$. Every finite intersection of sets in $\{U_i^c\}_{i \in I}$ is nonempty. (If there was such an empty intersection, then taking the complement would give a finite union equaling X.) This means the $\cap$-closure of $\{U_i^c\}_{i \in I}$ does not contain $\emptyset$, meaning $F = \mathfrak{F}\{U_i^c\}_{i \in I}$ is a proper filter. Assume, towards a contradiction, that F converges to $a \in X$. We can find a U_a in the cover $\{U_i\}_{i \in I}$ such that $a \in U_a$. Because $F \geq \mathcal{N}(x)$, we have $U_a \in F$. But U_a^c is an element of the subbasis of F, so $U_a^c \in F$. Thus $U_a \cap U_a^c = \emptyset \in F$, meaning F is not proper. □

`https://fa.ewi.tudelft.nl/~hart/37/publications/the_papers/betaR.pdf`

7.9.1 Limit point compactness

7.9.2 Local compactness

Chapter 8

Sequences, nets and filters

Sequences, nets and filters can be seen as probes of the topology.

8.1 Sequences

8.1.1 Sequential filters

Lemma VIII.175. *Let X be a set and $\langle x_n\rangle$ a sequence in X. Then*

$$\mathrm{Tails}(\langle x_n\rangle) \coloneqq \{\{x_n \mid n \geq k\} \mid k \in \mathbb{N}\} = \{\langle x_n\rangle\,[k:\infty] \mid k \in \mathbb{N}\}$$

is a downward directed subset of $\mathcal{P}(X)$.

Proof. Take $A, B \in \mathrm{Tails}(\langle x_n\rangle)$. If $A = \{x_n \mid n \geq a\}$ and $B = \{x_n \mid n \geq b\}$, then A is a lower bound of $\{A, B\}$ if $a \geq b$. Otherwise B is a lower bound. ☐

Let X be a set and $\langle x_n\rangle$ a sequence in X. We call

- $\mathrm{Tails}(\langle x_n\rangle)$ the sequential filter base of $\langle x_n\rangle$;
- the filter generated by $\mathrm{Tails}(\langle x_n\rangle)$ in $\mathcal{P}(X)$ the (sequential) filter of $\langle x_n\rangle$, which we denote $\mathrm{TailsFilter}(\langle x_n\rangle)$;
- any filter that is generated by a sequence a sequential filter;
- the kernel of the sequence the kernel of the associated sequential filter.

We have

$$\mathrm{TailsFilter}(\langle x_n\rangle) = {\uparrow}\,\mathrm{Tails}(\langle x_n\rangle).$$

Proposition VIII.176. *Let X be a set. A filter $F \in \mathcal{FP}(X)$ is sequential* if and only if *F contains a countable set and admits a countable base consisting of almost equal sets.*

Proof. $\Rightarrow$ Assume $F = \mathrm{TailsFilter}(\langle x_n\rangle)$ for some sequence $\langle x_n\rangle$. Then every set in $\mathrm{Tails}(\langle x_n\rangle)$ is countable and an element of F.
$\Leftarrow$ Assume F contains a countable set. Consider the intersection of all countable sets in F. ☐

8.1.2 Limits and convergence

Let $(X, \mathcal{T})$ be a topological space and $(a_n)_{n\in\mathbb{N}}$ a sequence in X. We can view this sequence as a function $\mathbb{N} \subset (\mathbb{N} \cup \{\infty\}) \to X$.
We define limit of the sequence as a limit of this function at ∞ if $\mathbb{N} \cup \{\infty\}$ is equipped with the order topology.

By VIII.143 a sequence has at most one limit $L \in X$ if X is Hausdorff. In this case we call it the limit of the sequence and write

$$\lim_{n\to\infty} a_n = L.$$

We call a sequence divergent if it does not have a limit and convergent if it does have a limit L. In this last case we say the sequence converges to L.

Proposition VIII.177. *Let $(X, \mathcal{T})$ be a topological space and $(a_n)_{n\in\mathbb{N}}$ a sequence in X. Then the sequence converges to $L \in X$* if and only if

$$\forall \textit{ open neighbourhood } V(L) : \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : a_n \in V(L).$$

Proof. Assume the sequence converges to L and take an arbitrary open neighbourhood $V(L)$. Then there exists an open neighbourhood $U(\infty)$ such that $a[U \setminus \{\infty\}] \subseteq V$. Now by definition of the order topology, there exists an interval $]m, \infty] \subseteq U(\infty)$. Then $a[\,]m, \infty[\,] \subseteq V$ and by setting $n_0 = m + 1$ we get the criterion of the proposition.
Conversely assume the criterion and fix $V(L)$. Then we can take $U(\infty) =]n_0, \infty]$. □

Lemma VIII.178. *Let $(X, \mathcal{T})$ be a topological space and $(a_n)_{n\in\mathbb{N}}$ a sequence in X that converges to L. Then all subsequences converge to L.*

TODO: Bolzano-Weierstrass (sequence version + accumulation point version)

8.1.3 Sequential spaces

8.1.3.1 The sequential topology

Let $(X, \mathcal{T})$ be a topological space and $S \subseteq X$ a subset.

- The sequential closure of S in X is the set
$$\mathrm{SeqCl}(S) \coloneqq \{x \in X \mid \exists \text{ a sequence in } S \text{ that converges to } x \text{ in } X\}.$$
- The sequential interior of S in X is the set
$$\mathrm{SeqInt}(S) \coloneqq \{s \in S \mid \text{every sequence in } X \text{ that converges to } s \text{ has a tail in } S\}.$$

We call S

- sequentially open if $S = \mathrm{SeqInt}(S)$;
- sequentially closed if $S = \mathrm{SeqCl}(S)$;
- a sequential neighbourhood of a point $x \in X$ if $x \in \mathrm{SeqInt}(S)$.

A sequentially closed set is a set S such that all limits of sequences in S are also in S.

Lemma VIII.179. *Let $(X, \mathcal{T})$ be a topological space and $R, S \subseteq X$ subsets. Then*

1. $\operatorname{SeqInt}(S) = (\operatorname{SeqCl}(S^c))^c$;
2. $\operatorname{SeqCl}(\emptyset) = \emptyset$ *and* $\operatorname{SeqCl}(X) = X$;
3. $S \subseteq \operatorname{SeqCl}(S)$;
4. $\operatorname{SeqCl}(R \cup S) = \operatorname{SeqCl}(R) \cup \operatorname{SeqCl}(S)$;
5. $\operatorname{SeqCl}(S) \subseteq \bar{S}$ *and* $\operatorname{SeqInt}(S) \supseteq S^\circ$.

Proof. (1) Both sides of the equation are equivalent to

$$\{s \in S \mid \nexists \text{ a sequence in } X \setminus S \text{ that converges to } s \text{ in } X\}.$$

(2) There are no sequences that converge to a point in $\emptyset$ and all points $x \in X$ are the limit of a constant sequence $n \mapsto x$.
(3) The constant sequence $n \mapsto x$ converges to x.
(4) We can find a subsequence in R or in S. All subsequences converge by VIII.178.
(5) Let $x \in \operatorname{SeqCl}(S)$, so there is a sequence (a_n) in S that converges to x. Take an arbitrary open neighbourhood V of x. Then by convergence there is a subsequence of (a_n) that is a sequence in V. In particular V intersects S. So $x \in \bar{S}$ by VIII.114. □

Proposition VIII.180. *Let $(X, \mathcal{T})$ be a topological space. The set of all sequentially open sets forms a topology $\mathcal{T}_{seq}$ on X. This topology is finer than the original topology.*

Proof. First note that sequentially closed sets are complements of sequentially open sets by point 1. of VIII.179.
By point 2. of VIII.179, $\emptyset$ and X are both clopen.
We will prove the rest using closed sets. By point 4. of VIII.179 finite unions of sequentially closed sets are sequentially closed.
Let $\bigcap_{i \in I} K_i$ be an arbitrary intersection of sequentially closed sets K_i. We only need to prove

$$\operatorname{SeqCl}\left(\bigcap_{i \in I} K_i\right) \subseteq \bigcap_{i \in I} K_i$$

because the other inclusion is immediate. Take an $x \in \operatorname{SeqCl}\left(\bigcap_{i \in I} K_i\right)$. Then there is a sequence in $\bigcap_{i \in I} K_i$ that converges to x. Because of the intersection this sequence is in each K_i and thus so is x.
The fineness of the topology follows from point 5. of VIII.179. □

Corollary VIII.180.1. *Let $(X, \mathcal{T})$ be a topological space and $S \subseteq X$ a subset. Then*

open/closed $\implies$ sequentially open/closed.

Proposition VIII.181. *Let $(X, \mathcal{T})$ be a topological space and (x_n) a sequence in X. Then $x_n \to x$ in $(X, \mathcal{T})$* if and only if *$x_n \to x$ in $(X, \mathcal{T}_{seq})$.*

Proof. Now $\mathcal{T}_{\text{seq}}$ is finer than $\mathcal{T}$, so the $\Leftarrow$ direction is evident. For the $\Rightarrow$ direction, assume $x_n \to x$ in the original topology. Let $V(x)$ be an open neighbourhood in the sequential topology. By definition of the sequential topology (x_n) has a tail in V. This means $x_n \to x$ in the sequential topology by VIII.177. □

8.1.3.2 Transfinite sequential closure

It is possible that the sequential closure is not idempotent (unlike the normal topological closure), i.e.

$$\operatorname{SeqCl}(\operatorname{SeqCl}(S)) \neq \operatorname{SeqCl}(S).$$

8.1.3.3 Sequential continuity

A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is called sequentially continuous if

$$f : (X, \mathcal{T}_{\text{seq}}) \to (Y, \mathcal{T}'_{\text{seq}})$$

is continuous. i.e. f is continuous when X, Y are equipped with their sequential topologies.

Proposition VIII.182. *A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is sequentially continuous* if and only if *for every sequence $(x_n)_{n\in\mathbb{N}}$ in X and $x \in X$*

$$x_n \to x \ \textit{ in } (X, \mathcal{T}) \qquad \implies \qquad f(x_n) \to f(x) \ \textit{ in } (Y, \mathcal{T}').$$

Proof. First assume this property holds and we want to prove sequential continuity. Let $S \subset Y$ be sequentially closed. Then we need to prove $f^{-1}[S]$ is also sequentially closed. Indeed take a converging sequence (x_n) in $f^{-1}[S]$ with limit x. Then $(f(x_n))$ converges to $f(x)$ and $f(x) \in S$. This implies $x \in f^{-1}[S]$, meaning it is sequentially closed.
Conversely, assume f is sequentially continuous. Let (x_n) be a sequence in X that converges to x. Let $V(f(x)) \in \mathcal{T}'$ be an open neighbourhood of $f(x)$; V is also sequentially open. Then by continuity we have a $U(x) \in \mathcal{T}_{\text{seq}}$ such that $f[U] \subseteq V$. Because U is sequentially open, there is an $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 : x_n \in U$. This implies $\forall n \geq n_0 : f(x_n) \in f[U] \subseteq V$ and so $(f(x_n))$ converges to $f(x)$. ☐

Proposition VIII.183. *Every continuous function is sequentially continuous.*

Proof. We use the characterisation of sequential continuity in VIII.182. Let $x_n \to x$. Let V be an open neighbourhood of $f(x)$. Then there exists an open neighbourhood $U(x)$ such that $f[U] \subset V$. By VIII.177 U contains all but finitely many elements of the sequence (x_n). Thus V contains all but finitely many of the elements of the sequence $(f(x_n))$. Take n_0 larger than the indices of all elements of $(f(x_n))$ omitted from V. By VIII.177 $f(x_n) \to f(x)$. ☐

8.1.3.4 Sequential spaces

A topological space $(X, \mathcal{T})$ is called a sequential space if $\mathcal{T} = \mathcal{T}_{\text{seq}}$.

Lemma VIII.184. *A topological space $(X, \mathcal{T})$ is a sequential space if every sequentially open set is open.*

Proof. We already know $\mathcal{T} \subseteq \mathcal{T}_{\text{seq}}$ from VIII.180. The hypothesis of the lemma is that $\mathcal{T} \supseteq \mathcal{T}_{\text{seq}}$. Together this gives $\mathcal{T} = \mathcal{T}_{\text{seq}}$. ☐

Lemma VIII.185. *Let $(X, \mathcal{T})$ be a topological space. Then X equipped with its sequential topology is a sequential space.*

Proof. It is enough to show that a sequentially closed set in $(X, \mathcal{T}_{\text{seq}})$ is also sequentially closed in $(X, \mathcal{T})$. (i.e. that passing to the finer topology does not introduce even more sequentially open sets). By VIII.181 the definition of SeqCl is the same in both topologies, yielding the proof. ☐

Proposition VIII.186. *Let $(X, \mathcal{T})$ be a topological space. Then the following are equivalent:*

1. *$(X, \mathcal{T})$ is a sequential space;*
2. *for every subset $S \subset X$ that is not closed in X, there exists some $x \in \bar{S} \setminus S$ for which there exists a sequence in S that converges to x;*
3. *$(X, \mathcal{T})$ is the quotient of a first countable space;*
4. *$(X, \mathcal{T})$ is the quotient of a metric space.*

TODO: relocate observation about metric spaces.

Proposition VIII.187 (Universal property of sequential spaces)**.** *Let $(X, \mathcal{T})$ be a topological space. Then X is sequential* if and only if *for every topological space Y, a function $f : X \to Y$ is continuous $\Leftrightarrow$ f is sequentially continuous.*

8.1.3.5 T-sequential and N-sequential spaces

8.1.3.6 Fréchet–Urysohn spaces

A topological space $(X, \mathcal{T})$ is called a Fréchet–Urysohn space if for every subset $S \subseteq X$

$$\text{SeqCl}(S) = \bar{S}.$$

Clearly every Fréchet–Urysohn space is a sequential space.

Proposition VIII.188. *Let $(X, \mathcal{T})$ be a topological space. Then the following are equivalent:*

1. *$(X, \mathcal{T})$ is a Fréchet–Urysohn space;*
2. *every subspace of X is a sequential space;*
3. *for every subset $S \subset X$ that is not closed in X and for all $x \in \bar{S} \setminus S$ there exists a sequence in S that converges to x.*

8.1.4 Sequences in ordered space

In this section we will be considering sequences in a totally ordered set $(X, \leq)$ equipped with the order topology.

Lemma VIII.189. *A convergent sequence in a totally ordered space has an upper and a lower bound.*

Proof. Let $x_n \to x$. Choose a basis element containing x. If it is of the form $]a, b_0]$ for some greates element b_0, then b_0 is the upper bound. If not, it is of the form $[a_0, b[$ or $]a, b[$. Find an n_0 corresponding to this basis element. Then an upper bound is given by

$$\max(x[\,[0, n_0]\,] \cup \{b\}).$$

The lower bound is analogous. ☐

Proposition VIII.190. *Let (a_n) and (b_n) be convergent sequences in a totally ordered space such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

Proof. Let $a_n \to a$ and $b_n \to b$. If $a = b$ then the proposition is valid. Now assume $a \neq b$. If a or b are either the greatest or the least element, the proposition is valid. Now assume this is not the case.
Assume towards a contradiction that $a > b$. Then we can find open neighbourhoods of a and b of the form $]b,d[$ and $]c,a[$, respectively. Now find n_0, n_1 such that $\forall n \geq n_0 : a_n \in]b,d[$ and $\forall n \geq n_1 : b_n \in]c,a[$. Then for all $n \geq \max\{n_0, n_1\}$ we have $a_n \in]b,d[$ and $b_n \in]c,a[$, implying $a_n > b_n$ which is a contradiction. ☐

It is easy to show that this does not in general hold for the strict inequality $<$.

Proposition VIII.191 (Squeeze theorem for sequences)**.** *Let (a_n), (b_n) and (c_n) be sequences in a totally ordered space such that*

$$\forall n \in \mathbb{N} : a_n \leq b_n \leq c_n.$$

If (a_n) and (c_n) are convergent with the same limit L, then

$$\lim_{n\to\infty} b_n = L.$$

Proof. Let $V(L)$ be an open neighbourhood of L. By definition of the order topology there is an interval $I =]x,y[\subset V$ such that $L \in I$. Then find n_0 and n_1 such that $\forall n \geq n_0 : a_n \in I$ and $\forall n \geq n_1 : c_n \in I$. Then set $n_2 = \max\{n_0, n_2\}$ and we have $\forall n \geq n_2$:

$$x \leq a_n \leq b_n \quad \text{and} \quad b_n \leq c_n \leq y.$$

By transitivity we have $b_n \in I \subset V$. ☐

Proposition VIII.192. *Let X be an ordered space and A a subspace. Assume the axiom of dependent choice.*

1. *If A has a supremum a, then there exists a sequence in A that converges to a in X.*
2. *If A has an infimum b, then there exists a sequence in A that converges to b in X.*

Proof. Assume the supremum a of A exists. If $a \in A$ we can take the constant sequence $(a)_{n\in\mathbb{N}}$. If $a \notin A$, we can find for each $x_i \in A$ an x_{i+1} satisfying $x_i < x_{i+1} < a$. The sequence thus defined converges by monotone convergence. ☐

In many cases the axiom of dependent choice is superfluous, if the details of the spaces X, A allow for the construction of x_{i+1} from x_i.

8.1.4.1 Divergence to ±∞

Let (x_n) be a sequence in a totally ordered space X. Then

- (x_n) diverges to $+\infty$ if $\forall M \in X : \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : x_0 > M$; and
- (x_n) diverges to $-\infty$ if $\forall M \in X : \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : x_0 < M$.

We write $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} x_n = -\infty$, respectively.

Lemma VIII.193. *Let (x_n) be a sequence in a totally ordered space X. Then*

1. *if (x_n) is increasing, but not bounded above, it diverges to $+\infty$;*
2. *if (x_n) is decreasing, but not bounded below, it diverges to $-\infty$.*

8.1.5 Sequences in complete ordered space

8.1.5.1 Monotone convergence

Proposition VIII.194 (Monotone convergence)**.** *Let $(X, \leq)$ be a complete totally ordered space and let (x_n) be a sequence in X.*

1. *If (x_n) is increasing and bounded above, then it is convergent with limit $\sup_n x_n$.*
2. *If (x_n) is decreasing and bounded below, then it is convergent with limit $\inf_n x_n$.*

Proof. We prove the first point. The second is analogous.
Let $V(\sup_n x_n)$ be an open and $]x, y[\subset V$ such that $\sup_n x_n \in]x, y[$. Now because $x < \sup_n x_n$ it is not an upper bound of the sequence and there exists an $x_{n_0} > x$. Because the sequence is increasing (and y is a strict upper bound), all x_n where $n \geq n_0$ are in $]x, y[\subset V$. □

8.1.5.2 Limes superior and inferior

Let $(X, \leq)$ be a complete totally ordered space and let (x_n) be a sequence in X. We define

- the limes superior or limit superior or limsup of (x_n) as
$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup \{x_m \mid m \geq n\};$$
- the limes inferior or limit inferior or liminf of (x_n) as
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf \{x_m \mid m \geq n\}.$$

The liminf and limsup may not exist.

Lemma VIII.195. *Let $(X, \leq)$ be a complete totally ordered space and let (x_n) be a sequence in X. The limsup and liminf exist* if and only if *(x_n) is bounded above and below.*

Proof. The sequences $\sup\{x_m \mid m \geq n\}$ and $\inf\{x_m \mid m \geq n\}$ are bounded if the limsup and liminf exist and bound (x_n).
The converse follows because the sequences $\sup\{x_m \mid m \geq n\}$ and $\inf\{x_m \mid m \geq n\}$ are monotone. □

Proposition VIII.196. *Let $(X, \leq)$ be a complete totally ordered space and let (x_n) be a bounded sequence in X. Then*

1. $L_s = \limsup_{n\to\infty} x_n$ if and only if

$$\forall b > L_s : \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : x_n < b \qquad \textit{and}$$
$$\forall a < L_s : \forall n_0 \in \mathbb{N} : \exists n \geq n_0 : a < x_n$$

2. $L_i = \liminf_{n\to\infty} x_n$ if and only if

$$\forall a < L_i : \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : a < x_n \qquad \textit{and}$$
$$\forall b > L_i : \forall n_0 \in \mathbb{N} : \exists n \geq n_0 : x_n < b.$$

Proposition VIII.197. *Let $(X, \leq)$ be a complete totally ordered space and let (x_n) be a sequence in X. Then (x_n) is convergent* if and only if

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

In this case

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Proof. Assume (x_n) is a sequence with identical liminf and limsup. Now

$$\inf \{x_m \mid m \geq n\} \leq x_n \leq \sup \{x_m \mid m \geq n\}$$

so we can apply the squeeze theorem for sequences.
For the converse we use VIII.196. ☐

Lemma VIII.198. *Let (a_n) and (b_n) be bounded sequences in a totally ordered space such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then*

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n \quad \textit{and} \quad \liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$

Lemma VIII.199. *There exist sequences converging to supremum and infimum.*

8.1.6 Completeness

8.2 Nets

Let $(I, \leq)$ be a directed set and X a set. Then a net in X is a function $I \to X$. The directed set I is called the index set.

In particular sequences are nets because $(\mathbb{N}, \leq)$ is a directed set.
Note we do not require I to be a partial order.

8.2.1 Relating filters and nets

8.2.1.1 From nets to filters

Lemma VIII.200. *Let X be a set, I a directed index set and $\langle x_i \rangle_{i\in I}$ a net in X. Then*

$$\mathrm{Tails}(\langle x_i \rangle_{i\in I}) \coloneqq \{\{x_i \mid i \geq j\} \mid j \in I\}$$

is a downward directed subset of $\mathcal{P}(X)$.

Proof. Take $A, B \in \mathrm{Tails}(\langle x_i \rangle_{i\in I})$. If $A = \{x_i \mid i \geq a\}$ and $B = \{x_i \mid i \geq b\}$, then we can find some $k \in I$ such that $k \geq a$ and $k \geq b$. Then $\{x_i \mid i \geq k\}$ is a subset of both A and B. ☐

Let X be a set, I a directed index set and $\langle x_i \rangle_{i \in I}$ a net in X. We call

- $\text{Tails}(\langle x_i \rangle_{i \in I})$ the filter base of $\langle x_i \rangle_{i \in I}$;
- the filter generated by $\text{Tails}(\langle x_i \rangle_{i \in I})$ in $\mathcal{P}(X)$ the associated filter of $\langle x_i \rangle_{i \in I}$, which we denote $\text{TailsFilter}(\langle x_i \rangle_{i \in I})$;
- the kernel of the net the kernel of the associated filter.

We have

$$\text{TailsFilter}(\langle x_i \rangle_{i \in I}) = \uparrow \text{Tails}(\langle x_i \rangle_{i \in I}).$$

8.2.1.2 From filters to nets

Lemma VIII.201. *Let X be a set and $F \in \mathcal{FP}(X)$ a proper filter. Then*

$$I_F \coloneqq \{(A, x) \in F \times X \mid x \in A\}$$

is a directed proset when ordered by $(A, x) \leq (B, y) \iff B \subseteq A$.

Proof. Transitivity and reflexivity follow from the properties of $\subseteq$.
Take (A, x) and (B, y) in I_F. Then $A \cap B \in F$ and $A \cap B \neq \emptyset$ because F is proper. So we can find $z \in A \cap B$. Then $(A, x) \geq (A \cap B, z)$ and $(B, y) \geq (A \cap B, z)$. ☐

Let X be a set, $F \in \mathcal{FP}(X) \setminus \mathcal{P}(X)$ and I_F the directed set of F as in VIII.201. Then

$$I_F \to X : (A, x) \mapsto x$$

is the associated net of F.

Lemma VIII.202. *Let X be a set, $F \in \mathcal{FP}(X) \setminus \mathcal{P}(X)$ and I_F the directed set of F as in VIII.201. Then for any $(A, x) \in I_F$,*

$$A = \{y \mid (B, y) \geq (A, x)\}.$$

Proof. $\boxed{\subseteq}$ For all $a \in A$ we have $(A, a) \geq (A, x)$.
$\boxed{\supseteq}$ For any $(B, y) \geq (A, x)$, we have $y \in B \subseteq A$, so $y \in A$. ☐

Corollary VIII.202.1. *Let X be a set and $F \in \mathcal{FP}(X)$. Then F equals the filter associated to its associated net.*

Proof. By the lemma we have that $A \subseteq X$ is a tail of the net iff it is an element of the filter. ☐

8.2.2 Convergence

Let (X, ξ) be a convergence space $\langle x_i \rangle_{i \in I}$ a net on X and $x \in X$. The net $\langle x_i \rangle_{i \in I}$ converges to x in ξ, denoted $\langle x_i \rangle_{i \in I} \xrightarrow{\xi} x$, if the associated filter converges to x.

Proposition VIII.203. *A topological space is Hausdorff* if and only if *every net converges to at most one point.*

8.2.3 Subnets

TODO: generalise:

Proposition VIII.204. *Let X be a topological space and $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$. The converse holds if X is metrisable.*

Chapter 9

Some topologies

These are the topologies that these object usually posses. If nothing else is said, these topologies will be assumed.

9.1 The metric topology

A metric on a set X is a function

$$d : X \times X \to \mathbb{R}$$

with the properties:

1. $\forall x, y \in X : d(x, y) \geq 0$ and equality holds if and only if $x = y$;
2. $\forall x, y \in X : d(x, y) = d(y, x)$;
3. $\forall x, y, z \in X : d(x, y) + d(y, z) \geq d(x, z)$.

The number $d(x, y)$ is often called the distance between x and y in the metric d.

- The ϵ-ball centered at x is the set

  $$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

  of all points y whose distance to x is less than ϵ.
- The closed ϵ-ball centered at x is the set

  $$\overline{B}_d(x, \epsilon) = \{y \mid d(x, y) \leq \epsilon\}$$

  of all points y whose distance to x is less than or equal to ϵ.
- The ϵ-sphere centered at x is the set

  $$S_d(x, \epsilon) = \{y \mid d(x, y) = \epsilon\}.$$

TODO: tolerance space for give ϵ !!

Lemma VIII.205. *Let X be a set and d a metric on X. Then the collection of all ϵ-balls forms a basis for a topology on X.*

- A metric space (X, d) is a set X together with a metric d.
- The topology generated by the ϵ-balls in X is called the metric topology generated by d.
- If $(X, \mathcal{T})$ is a topological space such that there exists a metric d on X such that $\mathcal{T}$ is the metric topology, then X is called metrisable.

TODO: metric is continuous + use this to define topology????
Only the local behaviour of the metric is important:

Proposition VIII.206. *Let (X, d) be a metric space. Define*

$$\bar{d} : X \times X \to \mathbb{R} : (x, y) \mapsto \min\{d(x, y), 1\}.$$

Then $\bar{d}$ is a metric that induces the same topology as d.

The metric $\bar{d}$ is called the standard bounded metric corresponding to d.

Proposition VIII.207. *Let d, d' be metrics on the set X, inducing $\mathcal{T}$ and $\mathcal{T}'$, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$* if and only if

$$\forall x \in X : \forall \epsilon > 0 : \exists \delta > 0 : \quad B_{d'}(x, \delta) \subset B_d(x, \epsilon).$$

Proof. Application of lemma VIII.124. □

Let (X, d) be a metric space and $S \subset X$ a subset. We call S bounded if there exists a ball $B_d(x, \epsilon)$ such that $S \subseteq B_d$.

9.1.1 Continuous functions in metric spaces

For maps between metric spaces, the continuity requirement is equivalent to the $\epsilon - \delta$ formulation:

Proposition VIII.208. *Let $(X, d_X), (Y, d_Y)$ be metric spaces. The continuity of $f : X \to Y$ is equivalent to the condition that, for all $x \in X$:*

$$\forall \epsilon > 0 : \exists \delta > 0 : \ d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space (Y, d). The sequence converges uniformly to the function $f : X \to Y$ if

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n > N : \forall x \in X : \quad d(f_n(x), f(x)) < \epsilon.$$

Theorem VIII.209 (Uniform limit theorem)**.** *Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.*

Proof. We show f is continuous at every point, so choose a point x_0 and a neighbourhood V of $f(x_0)$. We then need to show that we can find a neighbourhood U of x_0 such that $f(U) \subset V$ Now choose an $\epsilon > 0$ such that $B(f(x_0), \epsilon) \subset V$. By uniform convergence, we can find an $N \in \mathbb{N}$ such that

$$\forall n > N : \forall x \in X : \quad d(f_n(x), f(x)) < \epsilon/3.$$

By continuity of f_N, choose a neighbourhood U of x_0 such that $f(U) \subset B(f_N(x_0), \epsilon/3)$. We claim this is the U we need: take an arbitrary $x \in U$, then

$$\begin{aligned} d(f(x), f(x_0)) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

So $f(U) \subset B(f(x_0), \epsilon) \subset V$. □

TODO: convex sets; distance point to set (as special case of distance set to set).
TODO: A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

Proposition VIII.210. *Let (X, d) be a metric space and $A \subseteq X$. Then the function*

$$d_A : X \to \mathbb{R} : x \mapsto d(x, A) = \inf \{d(x, a) \mid a \in A\}$$

is continuous.

Proof. TODO □

TODO: We have that $x \in \overline{A}$ iff $d_A(x) = 0$.

9.1.2 Maps between metric spaces

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is an isometry or distance preserving if

$$\forall a, b \in X : \quad d_Y(f(a), f(b)) = d_X(a, b).$$

Lemma VIII.211. *An isometry is automatically injective.*

Proof. Let $f : X \to Y$. Assume $f(a) = f(b) = y$, then

$$0 = d_Y(y, y) = d_Y(f(a), f(b)) = d_X(a, b).$$

By non-degeneracy of the metric we have $a = b$, meaning f is injective. □

Lemma VIII.212. *An isometry is automatically continuous.*

Proof. For an isometry $f : X \to Y$, we have

$$f[B(x, \epsilon)] = B(f(x), \epsilon),$$

so f is continuous at each point x. It is then globally continuous by VIII.135. □

TODO: merge lemmas?

Lemma VIII.213. *Let $f : X \to Y$ be an isometry and X a complete metric space. Then f is a closed map.*

Proof. Take $K \subset X$ closed and $y \in \overline{f[K]}$. Then there exists a sequence $(f(x_n))$ in $f[K]$ converging to y. The sequence $(f(x_n))$ is convergent and thus Cauchy. Because $d(f(x_n), f(x_m)) = d(x_n, x_m)$, the sequence (x_n) must also be Cauchy. It is convergent because X is complete and the limit x lies in K because it is complete (TODO ref). By continuity of f, VIII.212, we have

$$y = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x) \in f[K].$$

So $\overline{f[K]} = f[K]$ and f is closed. ☐

9.1.3 Cauchy sequences and completeness

Let (X, d) be a metric space and (x_n) a sequence in X. Then (x_n) is called a Cauchy sequence if

$$\forall 0 < \epsilon \in \mathbb{R} : \exists n_0 \in \mathbb{N} : \forall m, n \geq n_0 : d(x_m, x_n) < \epsilon.$$

Lemma VIII.214. *Let (X, d) be a metric space. Cauchy sequences in X are bounded.*

Proposition VIII.215. *Let (X, d) be a metric space. Convergent sequences in X are Cauchy.*

Spaces in which the converse holds are special.

Let (X, d) be a metric space, then X is called (Cauchy) complete if all Cauchy sequences converge.

Proposition VIII.216. *Let (X, d) be a metric space and $(a_n), (b_n)$ sequences in X. If (b_n) is Cauchy and there exists some $A \in \mathbb{R}$ such that*

$$\forall m, n \in \mathbb{N} : d(a_n, a_m) \leq A d(b_n, b_m),$$

then (a_n) is also Cauchy.

Proposition VIII.217 (Completeness criterion)**.** *Let (X, d) be a metric space and $S \subset X$ a dense subset. If every Cauchy sequence in S converges in X, then X is complete.*

This proposition depends on the axiom of countable choice.

Proof. TODO ☐

Lemma VIII.218. *The real numbers with the standard topology are complete.*

9.1.3.1 Completion

Proposition VIII.219. *Let (X, d_X) be a metric space. There exists a complete metric space (Y, d_Y) and an isometry $\pi : X \hookrightarrow Y$ such that $\pi[X]$ is a dense subspace of Y.*

We view X as a subspace of Y through π and call Y the completion of X.

Proof. Let Y' be the space of Cauchy sequences in X. Introduce the equivalence relation on $\langle x_i \rangle, \langle y_j \rangle \in Y'$:

$$\langle x_i \rangle \sim \langle y_i \rangle \qquad \iff \qquad \lim_{i\to\infty} d_X(x_i, y_i) = 0.$$

Let Y be the set of equivalence classes in Y' under this equivalence relation. Define

$$d_Y : Y \times Y \to \mathbb{R} : ([\langle x_i \rangle], [\langle y_i \rangle]) \mapsto \lim_{i\to\infty} d_X(x_i, y_i) \qquad \text{and} \qquad \pi : X \to Y : x \mapsto \langle x \rangle_i .$$

We need to show that d_Y is well-defined, that it is a metric on Y, that $\pi[X]$ is dense in Y and that (Y, d_Y) is complete:

- Let $[\langle x_i' \rangle] = [\langle x_i \rangle]$. Then

  $$\begin{aligned} d_Y([\langle x_i' \rangle], [\langle y_i \rangle]) = \lim_{i\to\infty} d_X(x_i', y_i) &= \lim_{i\to\infty} d_X(x_i', y_i) + \lim_{i\to\infty} d_X(x_i, x_i') \\ &= \lim_{i\to\infty} d_X(x_i, x_i') + d_X(x_i', y_i) \\ &\geq \lim_{i\to\infty} d_X(x_i, y_i) = d_Y([\langle x_i \rangle], [\langle y_i \rangle]). \end{aligned}$$

  Similarly we can show $d_Y([\langle x_i' \rangle], [\langle y_i \rangle]) \leq d_Y([\langle x_i \rangle], [\langle y_i \rangle])$, so $d_Y([\langle x_i' \rangle], [\langle y_i \rangle]) = d_Y([\langle x_i \rangle], [\langle y_i \rangle])$.

  We must also show that the domain and codomain of d_Y make sense, i.e. the limit exists and does not diverge. It is enough to show that $(d_X(x_i, y_i))$ is a Cauchy sequence, due to the completeness of $\mathbb{R}$. To this end, let $\epsilon > 0$. As $\langle x_i \rangle$ and $\langle y_i \rangle$ are Cauchy, we can find $N_x, N_y \in \mathbb{N}$ such that $d_X(x_m, x_n) < \epsilon/2$ and $d_X(y_m, y_n) < \epsilon/2$ for all $m, n \geq N_x, N_y$. Then $\forall m, n \geq \max\{N_x, N_y\}$:

  $$\begin{aligned} |d_X(x_m, y_m) - d_X(x_n, y_n)| &\leq |d_X(x_m, x_n) + d_X(x_n, y_m) - d_X(x_n, y_n)| \\ &\leq |d_X(x_m, x_n) + d_X(y_m, y_n) + d_X(y_n, x_n) - d_X(x_n, y_n)| \\ &= |d_X(x_m, x_n) + d_X(y_m, y_n)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

  So $(d_X(x_i, y_i))$ is Cauchy and thus converges in $\mathbb{R}$.

- That d_Y is a metric is easy to check.

- To prove $\pi[X]$ is dense in Y, we just need to show that every element $y = [\langle x_i \rangle] \in Y$ is the limit of a sequence in $\pi[X]$, because all metric spaces are sequential. We claim $\langle \pi(x_j) \rangle_j$ converges to y.

  Let $\epsilon > 0$. Because $\langle x_i \rangle$ is Cauchy, we can find an $N \in \mathbb{N}$ such that $\forall m, n > N : d_X(x_m, x_n) < \epsilon/2$. Take $j \geq N$ arbitrary. Then

  $$\forall i \geq N : d_X(x_i, x_j) < \epsilon/2 \qquad \implies \qquad \lim_{i\to\infty} d_X(x_i, x_j) = d_Y([\langle x_i \rangle], \langle \pi(x_j) \rangle_j) \leq \epsilon/2 < \epsilon.$$

- For completeness, it is enough, by VIII.217, to show that Cauchy sequences in $\pi[X]$ converge in Y.

  Let $\langle \pi(x_j) \rangle_j$ be a Cauchy sequence in $\pi[X]$, then $\langle x_i \rangle$ is Cauchy in X because π is isometric. So $\langle \pi(x_j) \rangle_j$ converges to $\langle x_i \rangle$ by the previous point. Let $\langle \pi(x_j) \rangle_j$ be a Cauchy sequence in $\pi[X]$, then $\langle x_i \rangle$ is Cauchy in X because π is isometric. So $\langle \pi(x_j) \rangle_j$ converges to $\langle x_i \rangle$ by the previous point.

□

Proposition VIII.220. *Let (X, d) be a metric space. The completion (Y, π) of X is unique in the following sense: for any other such completion (Y', π'), there exists a unique isometric isomorphism $\theta : Y \to Y'$ satisfying $\theta \circ \pi = \pi'$.*

Proof. Since π is an isometry, it is injective, so $\pi^{-1} : \pi[X] \to X$ is a surjective isometry and so $\pi' \circ \pi^{-1} : \pi[X] \to \pi'[X]$ is too. Now we must have $\theta|_{\pi[X]} = \pi' \circ \pi^{-1} : \pi[X] \to \pi'[X]$
TODO universal property!! □

9.1.4 Equicontinuity

Cfr uniform limit theorem

Let (X, d_X) and (Y, d_Y) be metric spaces and let $\mathcal{F}$ be a family of functions in $(X \to Y)$. We say

- $\mathcal{F}$ is an equicontinuous family at $x' \in X$ if
$$\forall \epsilon > 0 : \forall x \in X : \exists \delta > 0 : \forall f \in \mathcal{F} : \; d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon;$$
- $\mathcal{F}$ is a uniform equicontinuous family at $x' \in X$ if
$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in X : \forall f \in \mathcal{F} : \; d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon.$$

We say $\mathcal{F}$ is (uniformly) equicontinuous if it is (uniformly) equicontinuous at all $x' \in X$.

- For continuity, δ may depend on ϵ, f, x_0;
- For uniform continuity, δ may depend on ϵ and f;
- For pointwise equicontinuity, δ may depend on ϵ and x_0;
- For uniform equicontinuity, δ may depend only on ϵ.

TODO: generalise to X general topological space (esp for TVSs).

Proposition VIII.221. *Let $(f_n)_{n\in\mathbb{N}}$ be an equicontinuous family between metric spaces. If $f_n \to f$ pointwise, then f is continuous.*

Proof. TODO ☐

TODO Reed / Simon

9.1.5 Hölder and Lipschitz continuity

Let $f : X \to Y$ be a map between metric spaces, then f is called α-Hölder continuous, where $0 < \alpha \leq 1$, if there exists a constant M such that
$$d_Y(f(x), f(y)) \leq M d_X(x, y)^\alpha \qquad \forall x, y \in X.$$

If $\alpha = 1$, then f is called Lipschitz continuous.
The set of α-Hölder continuous functions $X \to Y$ is denoted $\mathcal{C}^{0,\alpha}(X, Y)$.

The 0 in $\mathcal{C}^{0,\alpha}(X, Y)$ appears because we are not considering any derivatives.

Lemma VIII.222. *Let $f : X \to Y$ be a map between metric spaces and $0 < \alpha \leq \beta \leq 1$. If f is β-Hölder continuous, then it is α-Hölder continuous.*

TODO consolidate lemmas + uniform continuity.

Lemma VIII.223. *A Lipschitz continuous function between metric spaces is continuous.*

Proof. Let f be a Lipschitz continuous function and $\langle x_n \rangle$ a sequence such that $x_n \to x$. Then

$$\begin{aligned} d\left(\lim_{n\to\infty} f(x_n), f(x)\right) &= \lim_{n\to\infty} d(f(x_n), f(x)) \\ &\leq \lim_{n\to\infty} Md(x_n, x) = Md\left(\lim_{n\to\infty} x_n, x\right) = Md(x,x) = 0. \end{aligned}$$

□

The converse in not necessarily true. TODO example.

9.1.5.1 Contractions

Let $f : X \to Y$ be a map between metric spaces, then f is called a contraction if it is Lipschitz continuous with Lipschitz constant $M < 1$.

Proposition VIII.224. *Let $f : X \to X$ be a contraction. If X is a complete metric space, then f has a unique fixed point.*

Proof. Uniqueness is easy: assume that f has two fixed points x_1, x_2. Then $d(x_1, x_2) = d(fx_1, fx_2) \leq Md(x_1, x_2)$. Since $M < 1$ this is only possible if $d(x_1, x_2) = 0$, meaning $x_1 x_2$. For existence: take some $x_0 \in X$. Then define the sequence $\langle T^n(x_0)\rangle_n$. This is a Cauchy sequence because, for $m > n > 1$

$$d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq \sum_{i=n}^{m-1} M^{i-1} d(x_2, x_1) = \frac{M^{n-1}(1 - M^{m-n+1})}{1-M} d(x_2, x_1) \leq \frac{M^{n-1}}{1-M}.$$

By completeness it has a limit. Now T is continuous by VIII.223. Then

$$T\left(\lim_{n\to\infty} T^n(x_0)\right) = \lim_{n\to\infty} T(T^n(x_0)) = \lim_{n\to\infty} T^{n+1}(x_0) = \lim_{n\to\infty} T^n(x_0),$$

so the limit is a fixed point. □

The construction of the sequence $\langle T^n(x_0)\rangle_n$ is called fixed point iteration. Following the proof of the proposition it is clear we can obtain the fixed point starting the iteration from any point in X.

Corollary VIII.224.1. *Let $f : X \to X$ be a function on a complete metric space such that f^n is a contration for some $n \in \mathbb{N}$. Then f has a unique fixed point.*

Proof. By the proposition we know that f^n has a unique fixed point: $f^n(x) = x$. Applying f to both sides gives

$$f(f^n(x)) = f^n(f(x)) = f(x),$$

so $f(x)$ is also a fixed point of f^n. By uniqueness $f(x) = x$. This shows f has a fixed point. For uniqueness it is enough to note that any fixed point of f is a fixed point of f^n, **??**. □

9.1.6 Pseudometric spaces

Let X be a set.

- A map $p : X \times X \to \mathbb{R}_{\geq 0}$ is called a pseudometric or semimetric if

  - $\forall x \in X : p(x, x) = 0$;
  - $\forall x, y \in X : p(x, y) = p(y, x)$;
  - $\forall x, y, z \in X : p(x, z) \leq p(x, y) + p(y, z)$.

  Unlike a metric space, points in a pseudometric space need not be distinguishable; that is, one may have $d(x, y) = 0$ for distinct values $x \neq y$.

- The pair (X, p) is called a pseudometric space.
- The pseudometric topology is the topology generated by the basis of open balls

$$B_p(x_0, \epsilon) = \{x \in X \mid p(x_0, x) < \epsilon\} .$$

  A topological space is said to be a pseudometrizable space if it can be given a pseudometric such that the pseudometric topology coincides with the given topology on the space.

Proposition VIII.225. *The notions of compactness, limit point compactness and sequential compactness are equivalent in a pseudometric space.*

`https://link.springer.com/article/10.1007/BF01351999`

9.2 Topologies of $\mathbb{R}$

The lower limit topology on $\mathbb{R}$ is the topology generated by the basis

$$\{[a, b[\mid a < b\}.$$

- Let K denote the set
$$K = \{1/n \mid n \in \mathbb{N}_0\}.$$
- The K-topology on $\mathbb{R}$ is the topology generated by the basis
$$\{]a, b[\mid a < b\} \,\cup\, \{]a, b[\setminus K \mid a < b\}.$$

Lemma VIII.226. *The lower limit and K-topologies are strictly finer than then standard topology on $\mathbb{R}$, but are not comparable with one another.*

Proposition VIII.227. *Let X be a topological space and $f, g : X \to \mathbb{R}$ continuous functions, then*

1. *$f + g, f - g$ and $f \cdot g$ are continuous.*
2. *If $g(x) \neq 0$ for all $x \in X$, then f/g is continuous.*

Proof. First define the map

$$h : X \to \mathbb{R} \times \mathbb{R} : x \mapsto (f(x), g(x))$$

which is continuous by proposition VIII.147. The functions $f + g, f - g, f \cdot g, f/g$ are the composition of h and the continuous functions $+, -, \cdot, /$. ☐

9.3 Uniform topology

Given an index set J and points $\mathbf{x} = (x_i)_{i \in J}$ and $\mathbf{y} = (y_i)_{i \in J}$ of $\mathbb{R}^J$. Define the metric $\bar{\rho}$ on $\mathbb{R}^J$ by

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_i, y_i) \mid i \in J\},$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. The metric space $(\mathbb{R}^J, \bar{\rho})$ is the uniform topology on $\mathbb{R}^J$ and $\bar{\rho}$ is the uniform metric.

Proposition VIII.228. *On the set $\mathbb{R}^J$, the box topology is finer than the uniform topology is finer than the product topology. These topologies are all different* if and only if *J is infinite.*

9.4 Set-theoretic topology

`https://en.wikipedia.org/wiki/Set-theoretic_limit` `https://math.stackexchange.`
`com/questions/3384916/topology-of-set-theoretic-limits` `https://math.stackexchange.`
`com/questions/2799181/is-there-a-way-to-express-the-set-theoretic-limit-in-terms-of-t`
`https://math.stackexchange.com/questions/97440/do-limits-of-sequences-of-sets-come-fr`
`https://math.stackexchange.com/questions/1947171/the-topology-of-sets`

9.5 Initial and final topologies

Chapter 10

More topological constructions

10.1 Fibre bundles

10.2 Cones and suspensions

10.3 Wedge sum and smash product

Chapter 11

Convergence and topology on algebraic structures

11.1 Convergence relational structures

11.2 Convergence order structures

11.2.1 Difference operators

TODO

11.3 Convergence groups

11.3.1 Convergence

Let G be a set and

- $\cdot : G \times G \to G$ a binary operation such that $(G, \cdot)$ is a group;
- ξ a relation on $(\mathcal{P}(\mathcal{P}(G)), G)$ such that (G, ξ) is a convergence space;

such that

- $\cdot : G \times G \to G$ is continuous; and
- ${}^{-1} : G \to G : x \mapsto x^{-1}$ is continuous.

Then we call $(G, \cdot, \xi)$ a convergence group.

Lemma VIII.229. *A group G with a convergence structure is a convergence group* if and only if

$$G \times G \to G : (x, y) \mapsto xy^{-1}$$

is continuous.

Proof. If G is a convergence group, the function $(x, y) \mapsto xy^{-1}$ is the composition of two continuous functions and thus continuous.

Conversely, assume $(x, y) \mapsto xy^{-1}$ continuous. Then $y \mapsto 1y^{-1} = y^{-1}$ is continuous by VIII.63.1. Then $\cdot : (x, y) \mapsto xy = x(y^{-1})^{-1}$ is a composition of continuous functions. ☐

Lemma VIII.230. *Let $(G, \cdot, \xi)$ be a convergence group and A, B subsets of G. Then*

$$\operatorname{adh}_\xi(A) \cdot \operatorname{adh}_\xi(B) \subseteq \operatorname{adh}_\xi A \cdot B.$$

Proof. From the continuity of $\cdot$ and VIII.45 together with $\operatorname{adh}_{\xi\otimes\xi}(A \times B) = \operatorname{adh}_\xi(A) \times \operatorname{adh}_\xi(B)$ (TODO ref). ☐

Lemma VIII.231. *Let $(G, \cdot, 1, \xi)$ be a convergence group.*

1. *For all $a \in G$, both*

$$\lambda_a : G \to G : x \mapsto ax \qquad \text{and} \qquad \rho_a : G \to G : x \mapsto xa$$

   are homeomorphisms.

2. $F \to x$ if and only if $F \cdot x^{-1} \to 1$ if and only if $x^{-1} \cdot F \to 1$.
3. $\mathcal{V}_\xi(x) = \mathcal{V}_\xi(1) \cdot x$.
4. *Let $f : (G, \cdot, 1, \xi) \to (H, \cdot, 1, \zeta)$ be a group homomorphism. Then f is continuous* if and only if *it is continuous at* 1.

Proof. (1) The functions are clearly bijective. They are also continuous because the constant function $\underline{a}$ is continuous as well as the multiplication $(x, y) \mapsto xy$. Thus the composition is continuous.
(2) Assume $F \to x$, then by continuity of $\rho_{x^{-1}}$, we get $F \cdot x^{-1} \to 1$. The converse follows from continuity of ρ_x. The second equivalence follows from the continuity of λ^x and $\lambda_{x^{-1}}$.
(3) We calculate

$$\begin{aligned}
\mathcal{V}_\xi(x) &= \bigcap \left\{ F \in \mathcal{FP}(X) \;\middle|\; x \in \lim_\xi F \right\} \\
&= \bigcap \left\{ F \in \mathcal{FP}(X) \;\middle|\; 1 \in \lim_\xi F \cdot x^{-1} \right\} \\
&= \bigcap \left\{ G \cdot x \in \mathcal{FP}(X) \;\middle|\; 1 \in \lim_\xi G \right\} \\
&= \bigcap \left\{ G \in \mathcal{FP}(X) \;\middle|\; 1 \in \lim_\xi G \right\} \cdot x = \mathcal{V}_\xi(1) \cdot x.
\end{aligned}$$

TODO ρ_x homeomorphism.
(4) If f is continuous, it is automatically continuous at 1. Now assume f continuous at 1 and let $x \in G$. Then

$$F \to x \iff F \cdot x^{-1} \to 1 \implies f[F \cdot x^{-1}] = f[F] \cdot f(x)^{-1} \to f(1) = 1 \iff f[F] \to f(x).$$

☐

The convergence structure of a convergence group is completely determined by $\lim^{-1}(1)$. Thus the following lemma gives a way to generate convergence groups.

Proposition VIII.232. *Let $(G, +, 0)$ be a commutative group. And $\mathcal{G} \subseteq \mathcal{FP}(G)$ a family of filters. There exists a convergence ξ on G such that $\mathcal{G} = \lim_\xi^{-1}(0)$* if and only if

1. $\dot{0} \in \mathcal{G}$*;*

2. if $F \in \mathcal{G}$ *and* $G \supseteq F$*, then* $G \in \mathcal{G}$*;*

3. if $F, G \in \mathcal{G}$*, then* $F - G \in \mathcal{G}$*.*

The group convergence is completely determined by $\lim_{\xi}^{-1}(0)$ due to the translation homeomorphism VIII.231.

Proof. It is clear that $F \to x$ iff $F - x \in \mathcal{G}$ determines a convergence. We just need to show that $u : (x, y) \mapsto x - y$ is continuous.
Let $F \to (x, y) \in G \times G$, so by VIII.60 there exist $F_1 \to x$ and $F_2 \to y$ such that $F_1 \otimes F_2 \leq F$. Then $F_1 - x \in \mathcal{G}$ and $F_2 - y \in \mathcal{G}$. From point (3) (and commutativity) we get $F_1 - F_2 - (x - y) \in \mathcal{G}$, so $F_1 - F_2 \to x - y$. Now $F_1 - F_2 = u[F_1 \otimes F_2] \leq u[F]$ by VIII.61, so $u[F] \to x - y$ and thus u is continuous. □

Proposition VIII.233. *Let* $(G, \cdot, 1, \xi)$ *be a convergence group. Then* G *is Hausdorff* if and only if $\{1\}$ *is closed.*

Proof. The direction ⇒ is clear, since every Hausdorff convergence is T_1 and in a T_1 convergence all singletons are closed.
Conversely, assume $F \to x, y$ in G. Then $FF^{-1} \to xy^{-1}$. Now $FF^{-1} \subseteq \dot{1}$, so $\dot{1} \to xy^{-1}$ and thus $xy^{-1} \in \operatorname{adh}_\xi(\dot{1}) = \operatorname{adh}_\xi(\{1\}) = \{1\}$, meaning $x = y$. □

Lemma VIII.234. *Let* $(G, \cdot, 1, \xi)$ *be a pretopological convergence group and* $x, y \in G$*. If* $U \in \mathcal{V}_\xi(xy)$*, then there exist* $V \in \mathcal{V}_\xi(x)$ *and* $W \in \mathcal{V}_\xi(y)$ *such that* $V \cdot W \subseteq U$*.*
If $x = y$*, then we can take* $V = W$*.*

Proof. Consider the function $f : G \times G \to G : (x, y) \mapsto xy$. Then by VIII.46 and VIII.60.1 we have

$$\mathcal{V}_\xi(xy) \subseteq \uparrow f[\mathcal{V}_{\xi\otimes\xi}((x, y))] = \uparrow f[\uparrow \mathcal{V}_\xi(x) \otimes \mathcal{V}_\xi(y)] = \uparrow f[\mathcal{V}_\xi(x) \otimes \mathcal{V}_\xi(y)] = \uparrow(\mathcal{V}_\xi(x) \cdot \mathcal{V}_\xi(y))$$

This implies the first result.
If $x = 1 = y$, then consider $V \cap W$. This is still a neighbourhood of $x = y$ and $(V \cap W) \cdot (V \cap W) \subseteq V \cdot W \subseteq U$. □

Lemma VIII.235. *Let* $(G, \cdot, 1, \xi)$ *be a pretopological convergence group, then* $\mathcal{V}_\xi(1)$ *is based in the symmetric subsets.*

Symmetric subsets are subsets V such that $V^{-1} = V$.

Proof. Because $^{-1}$ is a group homeomorphism, we have $\mathcal{V}_\xi(1) = \mathcal{V}_\xi(1^{-1}) = (\mathcal{V}_\xi(1))^{-1}$. So V is a vicinity of 1 iff V^{-1} is a vicinity of 1. Thus $V \cap V^{-1} \subseteq V$ is a vicinity of 1 and $V \cap V^{-1}$ is also a symmetric set. □

Proposition VIII.236. *Each pretopological convergence group is topological.*

Proof. Let $(G, \cdot, 1, \xi)$ be a pretopological convergence group. To prove the convergence it topological, it is enough to prove that $\operatorname{inh}_\xi(A) \subseteq \operatorname{inh}_\xi^2(A)$ for all $A \subseteq X$. Fix such an A and take an arbitrary $x \in \operatorname{inh}_\xi(A)$. Then $A \in \mathcal{V}_\xi(x)$ by VIII.20.
By VIII.234 there exist $V \in \mathcal{V}_\xi(x)$ and $W \in \mathcal{V}_\xi(1)$ such that $V \cdot W \subseteq A$.
Now for all $y \in V$, $y \cdot W$ is a vicinity of y by VIII.231, so $V \subseteq \operatorname{inh}_\xi(A)$ by VIII.29. By the upward closure of the vicinity filter, $V \in \mathcal{V}_\xi(x)$ implies $\operatorname{inh}_\xi(A) \in \mathcal{V}_\xi(x)$. Thus $x \in \operatorname{inh}_\xi^2(A)$ by VIII.20. □

Proposition VIII.237. *Every topological group is regular.*

Proof. By VIII.86 we check that for all open U and $x \in U$ there exists an open set V such that $x \in V \subseteq \overline{V} \subseteq U$. In fact it is enough to check this for $e = 1$.
Because $1 \cdot 1 = 1$, we can find $W \in \mathcal{N}(1)$ such that $W \cdot W \subseteq U$ by VIII.234. We claim $V = W \cap W^{-1}$ works. Indeed it is an open neighbourhood of 1 and clearly $V \subseteq U$. We just need to show that $\overline{V} \subseteq U$. Take $y \in \overline{V}$. Then $yV \in \mathcal{V}_\xi(y)$ by VIII.231 and $V \in \mathcal{V}_\xi(y)^\#$ by VIII.20. So $yV \# V$ and we can find $v_1, v_2 \in V$ such that $v_1 = yv_2$. Thus

$$y = v_1 v_2^{-1} \in V \cdot V^{-1} = V \cdot V \subseteq U.$$

□

TODO: in fact completely regular.

Proposition VIII.238. *Let G be a convergence group. Then the pseudotopological modification $\chi(G)$ is a convergence group.*

(This is in general not true for the pretopological modification).

Proposition VIII.239. *Let G be a group, $\{G_i\}_{i\in I}$ a set of convergence groups and $\{f_i : G \to G_i\}_{i\in I}$ a set of group homomorphisms. Then the initial convergence on G w.r.t. $\{f_i : G \to G_i\}_{i\in I}$ makes G a convergence group.*

Proof. By VIII.229 we just need to verify that $u : G \times G \to G : (x, y) \mapsto x \cdot y^{-1}$ is continuous. Using VIII.58.1, we need to verify that $f_i \circ u$ is continuous for all $i \in I$. Because the f_i are group homomorphisms, we have

$$f_i(x \cdot y^{-1}) = f_i(x) \cdot f_i(y)^{-1}$$

for all $x, y \in G$. This means that $f_i \circ u = u_i \circ (f_i \times f_i)$, where $u_i : G_i \times G_i \to G_i : (x, y) \mapsto x \cdot y^{-1}$. By VIII.63.2 this is a composition of continuous functions and thus continuous. □

Proposition VIII.240. *Let G be a convergence group. Any subgroup $H \in \mathcal{V}(1)$ is closed.*

Proof. Take $g \in \operatorname{adh}(H)$, so $H \in \mathcal{V}(g)^\#$ by VIII.20. Now, by VIII.51, $gH \in \mathcal{V}(g)$ because λ_g is a homeomorphism by VIII.231. Thus $gH \# H = 1 \cdot H$. As cosets are either the same or disjoint (by II.86), we have $gH = H$ and in particular $g = g \cdot 1 \in gH = H$. So $\operatorname{adh}(H) = H$. □

11.3.2 Cauchy structure

Proposition VIII.241. *Let $(G, \cdot, 1, \xi)$ be a convergence group. Consider the family*

$$\mathcal{F} := \left\{ F \in \mathcal{FP}(G) \;\middle|\; F \cdot F^{-1} \xrightarrow{\xi} 1 \right\}.$$

Then

1. *$(G, \mathcal{F})$ is a Cauchy space;*
2. *if $F \in \mathcal{FP}(G)$ Cauchy converges to x, then it converges to x;*
3. *if $F \in \mathcal{FP}(G)$ converges to x, then $F \in \mathcal{F}$; if G is a Kent space, then F Cauchy converges to x.*

Proof. (1) We have $\dot{x}\cdot\dot{x}^{-1} \supseteq \dot{1}$, so $\dot{x}\cdot\dot{x}^{-1} \to 1$ and thus $\dot{x} \in \mathcal{F}$.
Take $F \in \mathcal{F}$ and consider $H \supseteq F$. Clearly $F\cdot F^{-1} \subseteq H\cdot H^{-1}$, so $H\cdot H^{-1} \to 1$ and thus $H \in \mathcal{F}$.
(2) Notice that

$$(F\cap\dot{x})\cdot(F\cap\dot{x})^{-1} = F\cdot F^{-1}\cap F\cdot\dot{x}^{-1}\cap\dot{x}\cdot F^{-1}\cap\dot{x}\cdot\dot{x}^{-1}.$$

So if F Cauchy converges to x, then $F\cap\dot{x}\in\mathcal{F}$ and this filter converges to 1. In particular this means that $F\cdot\dot{x}^{-1}\to 1$. Then we use $F\cdot\dot{x}^{-1} = F\cdot x^{-1}$ (TODO ref?) to conclude that $F\to x$.
(3) Assume $F\to x$. Then by VIII.62, $F\otimes F\to(x,x)$. By continuity of $(x,y)\mapsto x\cdot y^{-1}$, we have $F\cdot F^{-1}\to x\cdot x^{-1} = 1$, so F is a Cauchy filter.
Now assume G is a Kent space. Then $F\cap\dot{x}\to x$, meaning $F\cap\dot{x}\in\mathcal{F}$ by the previous argument and so F Cauchy converges to x. □

Let $(G,\cdot,1,\xi)$ be a convergence group. The family $\mathcal{F}$ of VIII.241 is the associated Cauchy structure of the convergence group.

Proposition VIII.242. *Every locally compact convergence group of finite depth is complete.*

TODO: finite depth necessary?

Proof. Let $(G,\cdot,1,\xi)$ be a locally compact convergence group and let F be a proper Cauchy filter. Then $F\cdot F^{-1}\to 1$, so there exists a compact set $K\in F\cdot F^{-1}$. This means there exist $A,B\in F$ such that $A\cdot B^{-1}\subseteq K$. Take $x_0\in B$, then $F_0\subseteq K\cdot x_0$ and $K\cdot x_0$ is compact by TODO ref. Now by the ultrafilter lemma III.120 we can take an ultrafilter $G\supseteq F$. Then $K\cdot x_0\in G$, so G converges to some y.
Then $G\cap\dot{y}$ is a Cauchy filter. By finite depth, $F\cap G\cap\dot{y} = F\cap\dot{y}$ is also a Cauchy filter, so F Cauchy converges to y. □

11.3.3 Metrisability and norm

Theorem VIII.243 (Birkhoff-Kakutani)**.** *Let $(G,\cdot,1,\xi)$ be a convergence group. Then the following are equivalent:*

1. *G is pseudometrisable;*
2. *the topology on G is induced by a left translation invariant pseudometric;*
3. *the topology on G is induced by a right translation invariant pseudometric;*
4. *G is first countable and topological.*

Let $(G,\cdot,1)$ be a group. We call a function $\|\cdot\| : G\to\mathbb{R}^+$ a group norm on G if $\forall x,y\in G$ the following hold

1. triangle inequality / subadditivity: $\|xy\|\leq\|x\|+\|y\|$;
2. positivity $\|x\|>0$ if $x\neq 1$;
3. inversion $\|x^{-1}\| = \|x\|$.

We call the norm cyclically permutable if $\forall x,y\in G$: $\|xy\| = \|yx\|$.
If $\|\cdot\|$ is a group norm for G, then we call $(G,\cdot,1,\|\cdot\|)$ a normed group.

Part IX

Number systems

Chapter 1

The integers $\mathbb{Z}$

Chapter 2

The rational numbers $\mathbb{Q}$

Proposition IX.1. *The ordered set* $(\mathbb{Q}, \leq)$ *is not complete.*

Proof. Consider the set

$$A = \left\{ x \in \mathbb{Q} \;\middle|\; x < 0 \wedge x^2 < 2 \right\}.$$

□

Proposition IX.2. *The rational numbers are embedded in any ordered field* K *by*

$$\mathbb{Q} \cong \mathbb{Q} \cdot 1 \subset K.$$

Any totally ordered field has characteristic zero.

2.1 The Archimedean property

A totally ordered field $(K, +, \cdot, \leq)$ is said to have the Archimedean property if $(K, +, \leq)$ is an Archimedean monoid. We say

- x is infinitesimal if x is infinitesimal w.r.t. 1;
- x is infinite if x is infinite w.r.t. 1;

Lemma IX.3. *Let* $(K, +, \cdot, \leq)$ *be a totally ordered field. Then*

1. *if* $x \in K$ *is infinitesimal, then* $1/x$ *is infinite, and vice versa;*
2. *if* $x \in K$ *is infinitesimal and* $r \in \mathbb{Q} \cdot 1$*, then* rx *is also infinitesimal.*

Proposition IX.4 (Axiom of Archimedes)**.** *Let* $(K, +, \cdot, \leq)$ *be a totally ordered field. Then the following are equivalent:*

1. K *is Archimedean;*
2. *for all* $x \in K$ *there exists* $n \in \mathbb{N}$ *such that* $x < n \cdot 1$*;*
3. *for all positive* $\varepsilon \in K$ *there exists* $n \in \mathbb{N}$ *such that* $(n \cdot 1)^{-1} < \varepsilon$*.*

Lemma IX.5. *Let* $(K, +, \cdot, \leq)$ *be an Archimedean totally ordered field. Then for all* $x \in K$ *there exists an* $n \in \mathbb{N} \cdot 1$ *such that*

$$n \cdot 1 \leq x < n \cdot 1 + 1.$$

Chapter 3

The real numbers $\mathbb{R}$

Proposition IX.6. *The field of real numbers is Archimedean.*

Proof. Assume, towards a contradiction, that the field of real numbers is Archimedean. Then $\mathbb{N}$ has an upper bound, and by completeness a least upper bound $s = \sup \mathbb{N}$. Then $s - 1$ is not an upper bound and thus there exists and $n \in \mathbb{N}$ such that $s - 1 < n$. But then $s < n + 1 \in \mathbb{N}$ so that s is not an upper bound of $\mathbb{N}$, yielding a contradiction. ☐

Proposition IX.7. *There exists a totally ordered field and it is unique up to isomorphism.*

Proposition IX.8. *The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$.*

Proof. Let $x < y$ be real numbers. By the Archimedean property there exists a natural number $n > (y - x)^{-1}$. TODO: complete. ☐

3.1 Affinely extended real number system

Dedekind–MacNeille completion of the real numbers

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

3.2 Functions on the real numbers

3.2.1 Functions from reals to integers

3.2.1.1 Rounding

3.2.1.2 Floor and ceiling

$\lfloor x \rfloor$ and $\lceil x \rceil$

3.2.1.3 Fractional and integer part

3.3 Irrational numbers

3.3.1 Beatty sequences

TODO `https://en.wikipedia.org/wiki/Lambek%E2%80%93Moser_theorem`

Let r be a positive irrational number. The Beatty sequence generated by r is the sequence

$$\mathcal{B}_r : \mathbb{N} \to \mathbb{N} : n \mapsto \lfloor nr \rfloor$$

Lemma IX.9. *Let r be a positive irrational number. For a positive integer m the following are equivalent:*

1. *m is a term in the Beatty sequence $\mathcal{B}_r$;*
2. *$1 - \frac{1}{r} < \operatorname{frac}\left(\frac{m}{r}\right)$;*
3. *$m = \lfloor\left(\lfloor\frac{m}{r}\rfloor + 1\right)\rfloor$.*

Theorem IX.10 (Rayleigh)**.** *TODO*

Theorem IX.11 (Uspensky)**.** *If $r_1, \ldots, r_n$ are positive numbers such that the Beatty sequences $\mathcal{B}_{r_1}, \ldots, \mathcal{B}_{r_n}$ partition the positive integers, then $n \leq 2$.*

This means there is no equivalent to Rayleigh's theorem for more than two sequences.

Proof. TODO `https://mathweb.ucsd.edu/~fan/ron/papers/63_01_uspensky.pdf` □

3.3.1.1 Beatty series

`https://math.stackexchange.com/questions/2052179/how-to-find-sum-i-1n-left-lfloor-i-s`

3.3.2 Games

Chapter 4

Complex numbers

TODO: prove $\mathbb{C}$ cannot be a totally ordered field. TODO

$$-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1$$

The set of the complex numbers is denoted $\mathbb{C}$.

Lemma IX.12. *Let $z \in \mathbb{C}$. Suppose there is a $C \geq 0$ such that*

$$\forall t \in \mathbb{R} : \quad |z + it|^2 \leq C + t^2,$$

then $z \in \mathbb{R}$.

Proof. Write $z = a + bi$ for some $a, b \in \mathbb{R}$. Then

$$|z + it|^2 - t^2 = a^2 + (b + t)^2 - t^2 = a^2 + b^2 + 2bt.$$

The left side is bounded by C for all $t \in \mathbb{R}$. If $b > 0$, the right side is unbounded for $t \to +\infty$. If $b < 0$, the right side is unbounded for $t \to -\infty$. So we need $b = 0$ and thus $z = a \in \mathbb{R}$. ☐

4.1 Solutions to quadratic equations

We define[1]

$$i \equiv \sqrt{-1}$$

We call i the imaginary unit. The choice of terminology is not great here; it creates an artificial divide between the "real" numbers and this new "imaginary" thing. Many a maths teacher has taken great pains to explain that imaginary and complex numbers are no more imaginary than real numbers. I would tend to make the opposite case: all numbers are imaginary mathematical constructs that happen to be quite useful. We are just more familiar with real numbers.

So let us, arguendo, accept this new mathematical object. We are now faced with two questions: is it useful and what are the consequences? We will start by exploring the consequences.

We have already remarked upon the fact that the real numbers form a field. It makes some sense to try to find a field that contains both the real numbers and this new imaginary unit. There may be many such fields. We will try to find the smallest, which we will call the complex numbers, notated $\mathbb{C}$. Obviously this new field contains $\mathbb{R} \cup \{i\}$ as a subset, but this

---

[1] In engineering i is often called j, because i is used to denote electric current

in itself is not a field. As stated above, in a field both multiplication and addition work well and in particular give results that are still part of the field. For instance, imagine multiplying i with a real number, like 2 (something that we can do in a field by definition). We can show that $2i$ is not an element of $\mathbb{R} \cup \{i\}$. Now $2i$ cannot be a real number, because

$$\begin{aligned}(2i)^2 &= 2i2i \\ &= 4i^2 \qquad \text{by commutativity} \\ &= -4\end{aligned}$$

and we know that the square of a real number cannot be negative. It might be the case that $2i = i$, but subtracting i from both sides of the equation would give us $i = 0$, which can quite easily be seen to be contradictory. This means $2i$ is an entirely new number not yet considered. We could give it a new name (like we did for $\sqrt{-1}$) if we wanted to, but given we can multiply i by any real number and get a new object, we will not start naming them just yet. So we can sum up our current knowledge as follows

$$\mathbb{C} \supset \mathbb{R} \cup \{a \cdot i \mid a \in \mathbb{R}_0\}$$

We call the set $\{a \cdot i \mid a \in \mathbb{R}_0\}$ the imaginary numbers. We are not finished yet, because there are still more numbers that must be included in the set of complex numbers. Take for instance $i + 1$. Again this must be a complex number if we wish the complex numbers to be a field. Performing simple calculations as before we can see that $i + 1$ cannot be either a real number (as that would mean that i was a real number minus 1) or a purely imaginary one (assuming $i + 1 = a \cdot i$ imaginary, we would see that $1 = (a - 1)i$ which can easily be seen to be absurd). Consequently we can see that if we add a real number to an imaginary number, the result is a new complex number that we have not considered yet. The complex numbers must therefore include all numbers of the form $a + bi$ and for each $a, b \in \mathbb{R}$ we have a unique complex number. Finally we can remark that the set of complex numbers that we have created so far is in fact a field. We can verify for example that the result of addition or multiplication of two numbers of the form $a + bi$ can also be written in that form (making use of the fact that the addition and multiplication operations fulfill the requirements to be part of a field):

$$\begin{aligned}(a_1 + b_1 i) + (a_2 + b_2 i) &= (a_1 + a_2) + (b_1 + b_2)i \\ (a_1 + b_1 i)(a_2 + b_2 i) &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i\end{aligned}$$

Because we were looking for the smallest field that contains both the real numbers and the imaginary unit, we can stop here.

The treatment so far has included the essential arguments necessary to prove that all complex numbers can be written as

$$a + bi \qquad \text{with} \qquad a, b \in \mathbb{R}.$$

In other words we can say

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}.$$

4.2 How to represent complex numbers

In the previous section we saw that every complex number can be written as $a + bi(a, b \in \mathbb{R})$. Conversely for every a, b in $\mathbb{R}$ there is a unique complex number $a + bi$. Thus we can see that every complex number can be constructed using two real numbers. We give those real numbers

special names: For a complex number $z = a + bi$, we call a the real part (denoted $\Re\mathrm{e}(z)$) and b the complex part (denoted $\Im\mathrm{m}(z)$).
TODO complex plane
modulus argument Euler formula?? Conversions

4.3 Practical calculations

The following methods give a practical way to perform calculations with complex numbers. Assume we have two complex numbers z_1 and z_2.

4.3.1 Addition

is usually easiest if the complex numbers are in the form $a + bi$. Then we have

$$\begin{aligned} z_1 + z_2 &= (a_1 + b_1 i) + (a_2 + b_2 i) \\ &= (a_1 + a_2) + (b_1 + b_2)i \end{aligned}$$

4.3.2 Multiplication

is usually easiest if the complex numbers are in the form $re^{i\phi}$. Then we have

$$\begin{aligned} z_1 \cdot z_2 &= r_1 e^{i\phi_1} \cdot r_2 e^{i\phi_2} \\ &= (r_1 \cdot r_2) e^{i(\phi_1 + \phi_2)} \end{aligned}$$

So we multiply the moduli and add the arguments.

4.3.3 Exponentiation

with an integer (or real) exponent is again usually easiest if the complex number is in the form $re^{i\phi}$.

$$\begin{aligned} z^n &= (re^{i\phi})^n \\ &= r^n e^{in\phi} \end{aligned}$$

4.4 Trigonometry revisited

4.4.1 Waves and complex numbers

Cayley-Dickinson

Part X

Linear Algebra

Chapter 1

Vector spaces

Gauss-Jordan reduction
TODO projective transformations
orientation `https://en.wikipedia.org/wiki/Orientation_(vector_space)` also for fixed set of n vectors
`http://www.physics.rutgers.edu/~gmoore/618Spring2018/GTLect2-LinearAlgebra-2018.pdf`

1.1 Formal definition

A vector space is a collection of vectors, which are objects that have a natural addition and scalar multiplication.

A vector space over a field $\mathbb{F}$ is a set V together with an addition

$$+ : V \times V \to V$$

and a scalar multiplication

$$\cdot : \mathbb{F} \times V \to V$$

such that $(V, +)$ is a commutative group and the following properties hold:

Distributivity 1 $\lambda \cdot (v + w) = \lambda v + \lambda w$ for all $\lambda \in \mathbb{F}$ and all $v, w \in V$.

Distributivity 2 $(\lambda_1 + \lambda_2) \cdot v = \lambda_1 v + \lambda_2 v$ for all $\lambda_1, \lambda_2 \in \mathbb{F}$ and all $v \in V$.

Mixed associativity $\lambda_1 \cdot (\lambda_2 \cdot v) = (\lambda_1 \lambda_2) \cdot v$ for all $\lambda_1, \lambda_2 \in \mathbb{F}$ and all $v \in V$.

Multiplicative identity $1 \cdot v = v$ for all $v \in V$.

This vector space can be denoted $(\mathbb{F}, V, +)$.

In the definition we have used the following convention: for all $v, w \in V$ and $\lambda \in \mathbb{F}$, we denote $+(v, w)$ as $v + w$ and $\cdot(\lambda, v)$ as $\lambda \cdot v$ or λv.
We call the elements of the field scalars and the elements of the set V vectors. The zero of the group is known as the zero vector.
Almost always we will actually be interested in $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

1.1.1 Examples

1. The n-tuples in $\mathbb{F}^n$ with pointwise addition and multiplication. If the entries of the n-tuples are written one above the other in a column, it is called a column vector.
2. The polynomials in $\mathbb{F}[X]$.
3. The polynomials in $\mathbb{F}[X]_{\leq n}$ of maximally degree n.
4. For any set S, the functions $(S \to \mathbb{F})$, denoted $\mathbb{F}^S$, with pointwise addition and multiplication.
5. For any topological space X, the continuous functions in $(X \to \mathbb{C})$, denoted $\mathcal{C}(X)$.
6. The trivial vector space $\{0\}$. A vector space can never be empty, because a commutative group always has a neutral element.
7. The set of all possible *displacements* in (Euclidean) space forms a vector space. Once we have chosen an origin, we can view space as a vector space.

1.1.2 Some elementary lemmas

Lemma X.1. *Given the vector space* $(\mathbb{F}, V, +)$ *and arbitrary* $u, v, w \in V$ *and* $\lambda \in \mathbb{F}$*, we have*

1. $0v = 0 = \lambda \cdot 0$;
2. $(-1)v = -v = 1(-v)$;
3. $(-\lambda)v = -(\lambda v) = \lambda(-v)$;
4. $u + v = w + v \implies u = w$.

By $-v$ *we mean the additive inverse of* v.

Proof.

1. First, use distributivity to get

$$0v = (0+0)v = 0v + 0v.$$

The apply the previous lemma to $0 + 0v = 0v = 0v + 0v$ to get $0 = 0v$. The equality $\lambda \cdot 0 = 0$ is proved analogously.

2. To show that $(-1) \cdot v$ is the additive inverse of v, i.e. $-v$, we simply add $(-1) \cdot v + v$ and observe the result is 0.

$$(-1) \cdot v + v = (-1) \cdot v + 1 \cdot v = (1 + (-1)) \cdot v = 0 \cdot v = 0.$$

3. Similar to the previous point.
4. The additive inverse $-v$ exists, so we can just add it left and right.

□

1.1.3 Subspaces

A *subset* U of a vector space V is called a subspace of V if U is also a vector space.

The subset U automatically inherits a lot of the structure of V. We only need to verify a couple of conditions.

Proposition X.2 (Subspace criterion)**.** *A subset U of a vector space V is a subspace of V* if and only if *U satisfies the following conditions:*

1. ***Additive identity:*** $0 \in U$. *Alternatively it is enough to show that U is not empty.*
2. ***Closed under addition:*** $v, w \in U$ *implies* $v + w \in U$*;*
3. ***Closed under scalar multiplication:*** $\lambda \in \mathbb{F}$ *and* $u \in U$ *implies* $\lambda u \in U$.

Alternatively the last two criteria are equivalent to:

$$v, w \in U; \lambda \in \mathbb{F} \qquad \text{implies} \qquad v + \lambda w \in U.$$

If the question is whether a set is a subspace, this criterion is almost always the answer. An elementary application:

Proposition X.3. *Any arbitrary intersection of subspaces is a subspace.*

Corollary X.3.1. *Let V be a vector space. Then the subspaces of V form a complete sublattice of $(\mathcal{P}(V), \subseteq)$.*

The closure operator into the complete lattice of subspaces of V is called the span.
If D is a subset of V such that $V = \operatorname{span}(D)$, then D spans V.

- A vector space is called finite-dimensional if it is spanned by a finite set of vectors.
- A vector space is infinite-dimensional if it is not finite-dimensional.

Let V be a vector space. A hyperplane in V is a coatom in the lattice of subspaces of V.

1.2 Basis and dimension

1.2.1 Linear combinations and span

A (finite) linear combination of vectors $v_1, \ldots, v_n$ is a vector of the form

$$a_1 v_1 + \ldots + a_n v_n$$

where $a_1, \ldots, a_n \in \mathbb{F}$.

Proposition X.4. *Let V be a vector space over a field $\mathbb{F}$ and $D \subseteq V$ a subset. Then* $\operatorname{span}(D)$ *is the set of all finite linear combinations of vectors in D if $D \neq \emptyset$. If $D = \emptyset$, then* $\operatorname{span}(D) = \{0\}$.

1.2.2 Linear independence

A set of vectors D is linearly independent if the only linear combinations in D that equal 0 are the trivial ones with all scalars zero. i.e.

$$\sum_{i=1}^{n} a_i v_i = 0 \qquad \Longrightarrow \qquad a_1 = \ldots = a_n = 0,$$

assuming the v_i are vectors in D and the a_i are scalars.
Linear dependence is the opposite of linear independence.

The empty set $D = \emptyset$ is taken as linearly independent. No non-trivial combinations of vectors in $\emptyset$ are equal to zero, because there are no non-trivial combinations of vectors in $\emptyset$.

Lemma X.5. *Let D be a linearly dependent set of vectors. Then there exists a vector $v \in D$ such that*

1. *v is a linear combination of other vectors in D;*
2. *$v \in \operatorname{span}(D \setminus \{v\})$;*
3. *$\operatorname{span}(D) = \operatorname{span}(D \setminus \{v\})$.*

Proof. Take a linear combination of vectors in D equalling zero,

$$\sum_i a_i v_i = 0.$$

By linear dependence such a combination can be found such that not all a_i are zero. In particular at least two must be non-zero. Take $a_j \neq 0$. Then

$$v_j = \sum_{i \neq j} \frac{a_i v_i}{a_j}.$$

To prove the last point, take a $u \in \operatorname{span}(D)$. Then

$$u = \sum_i b_i v_i = b_j v_j + \sum_{i \neq j} b_i v_i = b_j \sum_{i \neq j} \frac{a_i v_i}{a_j} + \sum_{i \neq j} b_i v_i = \sum_{i \neq j} \left(\frac{b_j a_i}{a_j} + b_i \right) v_i.$$

So $u \in \operatorname{span}(D \setminus \{v\})$. The opposite inclusion is obvious. □

1.2.3 Bases

A basis of a vector space V is a set of vectors in V that spans V and is linearly independent.

Example

The standard basis or natural basis of $\mathbb{F}^n$ is given by

$$\begin{gathered}(1,0,0,\dots,0),\\(0,1,0,\dots,0),\\(0,0,1,\dots,0),\\\dots\\(0,0,0,\dots,1).\end{gathered}$$

We will denote it $\mathcal{E}$ or $\mathcal{E}_n$.

1.2.3.1 In finite-dimensional spaces

Proposition X.6. *A finite set* $\{v_1,\dots,v_n\}$ *of vectors in* V *is a basis of* V if and only if *every* $v \in V$ *can be written uniquely in the form*

$$v = a_1v_1 + \dots + a_nv_n,$$

where $a_1,\dots,a_n \in \mathbb{F}$.

Proof. We prove both directions.

$\boxed{\Rightarrow}$ Suppose $\{v_1,\dots,v_n\}$ is a basis of V. Then any vector v can be written as $a_1v_1+\dots+a_nv_n$, because the basis spans the space. We just need to show the decomposition is unique. To that end, assume there was another decomposition $v = b_1v_1 + \dots + b_nv_n$. Subtracting both decompositions gives

$$0 = (a_1 - b_1)v_1 + \dots + (a_n - b_n)v_n.$$

Because $\{v_1,\dots,v_n\}$ is linearly independent, $a_i = b_i$ for all i.

$\boxed{\Leftarrow}$ Now suppose every vector has such a decomposition. Clearly $\{v_1,\dots,v_n\}$ spans V. The unique decomposition of 0 gives linear independence.

□

Theorem X.7 (Steinitz exchange lemma)**.** *Let* V *be a vector space. If* $U = \{u_1,\dots,u_m\}$ *is a linearly independent set of* m *vectors in* V*, and* $W = \{w_1,\dots,w_n\}$ *spans* V*, then:*

1. $m \leq n$;
2. *There is a set* $\{u_1,\dots,u_m,w'_{m+1},\dots,w'_n\} \supset U$ *that spans* V *where* $w'_{m+1},\dots,w'_n \in W$.

Proof. We obtain the set $\{u_1,\dots,u_m,w'_{m+1},\dots,w'_n\}$ by starting with the list $B_0 = (w_1,\dots,w_n)$ and applying the following steps for each element $u_i \in U$, in the process defining sets $B_1,\dots,B_m$. Each of these sets spans V.

1. Add u_i to B_{i-1}. The set is now linearly dependent, because B_{i-1} spans V.
2. By lemma X.5, we can find a vector v that is a linear combination of $B_{i-1}\setminus\{v\}$. Because $u_1,\dots,u_i$ are linearly independent, we can choose this vector to be an element of W. Define $B_i = B_{i-1}\setminus\{v\}$. By lemma X.5, B_i still spans V, as required.

This process only stops when we have had all elements of U. □

Corollary X.7.1. *If a vector space V has a basis with n vectors, then any basis of V has n vectors.*

Theorem X.8. *Suppose V is a finite-dimensional vector space spanned by $D = \{v_1, \ldots, v_n\}$.*

1. *We can find a subset of D that is a basis of V, i.e. D can be reduced to a basis;*
2. *Each linearly independent set of vectors can be expanded to a basis.*

Proof.

1. Remove 0 from D, if it is an element. If D is not linearly independent, find a vector in D that is a linear combination of other vectors in D. Repeat until the set is linearly independent. This process stops due to the finite number of vectors. The set spans V at every step.
2. Follows easily from the Steinitz exchange lemma, taking W to be a basis.

□

Corollary X.8.1. *Every finite-dimensional vector space has a basis.*

Thanks to corollaries X.7.1 and X.8.1, the following definition makes sense:

The dimension of a finite-dimensional vector space is the length of any basis of the vector space. The dimension of V (if V is finite-dimensional) is denoted by $\dim V$ or $\dim_{\mathbb{F}} V$.[a] If $V = \{0\}$, we take $\dim V = 0$.

---

[a] The latter notation is particularly useful if when distinguishing between real and complex vector spaces, because every complex vector space can be seen as a real vector space. In this case $\dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} V$, because v and iv are linearly independent over $\mathbb{R}$.

Corollary X.8.2. *Every linearly independent set of vectors in V with length $\dim V$ is a basis of V.*

Corollary X.8.3. *Every spanning set of vectors in V with length $\dim V$ is a basis of V.*

Proposition X.9. *Let V be a finite-dimensional vector space and U a subspace of V. Then*

1. *U is finite-dimensional and $\dim U \leq \dim V$;*
2. *$\dim U = \dim V \iff U = V$.*

Proof. We construct a basis for U using the following process:

1. If U={0}, then we can take the basis $\emptyset$ and we are done. If $U \neq \{0\}$, we choose a nonzero vector $v_1 \in U$.
2. If U is the span of all the vectors we have chosen, we are done. If not choose a vector in U, not in the span of the other vectors.
3. Repeat step (2).

By construction, the chosen set of vectors is linearly independent. By the Steinitz exchange lemma this process must stop. In particular it must stop before reaching $\dim V$ vectors.
If the process reaches this upper bound, then by corollary X.8.2, the set of vectors in U is also a basis for V. □

We now have two tools for proving equalities of finite-dimensional vector spaces: either by proving both inclusions, or by leveraging point (2) of the previous proposition.

1.2.3.2 In infinite-dimensional spaces

Our definition of a basis of a vector spaces still makes sense for infinite-dimensional vector spaces, and many results of the previous section still make sense for infinite-dimensional vector spaces.

For infinite-dimensional vector spaces, there are, however, other notions of basis we might be interested in. In particular, our definition of basis requires all vectors to be constructible as finite linear combinations of basis elements. In some contexts we might want to relax this to allow infinite combinations as well. For that, of course, we need some notion of infinite sum. Often we construct infinite sums as the limit of a sequence of finite sums, in which case we need a topology on our vector space that allows us to take limits.[1]

In order to distinguish our purely algebraic definition of basis from these other notions of basis, a basis in the sense defined above is sometimes known as an algebraic basis of Hamel basis.

We will be discussing Hamel bases in this section.

Theorem X.10. *Let V be a vector space.*

1. *Any spanning set contains a basis.*
2. *Any linearly independent subset can be expanded to a basis.*

Proof. Requires the axiom of choice. We will use Zorn's lemma twice.

1. Let S be a spanning subset of V. Define

$$\mathcal{A} = \{D \subset S \mid D \text{ is linearly independent}\}$$

ordered by inclusion. It is easy to see that any chain on $\mathcal{A}$ has an upper bound on $\mathcal{A}$, by just taking the union which is still linearly independent. It follows from Zorn's lemma that $\mathcal{A}$ has a maximal element R. We show that $\text{span}(R) \supset S$ by contradiction. If $\text{span}(R) \not\supset S$, we can consider $R \cup \{v\}$ for some $v \in S$ that is not in $\text{span}(R)$ and we obtain an element of $\mathcal{A}$ which is greater than a maximal element. This is a contradiction. Then from $\text{span}(R) \supset S$ we conclude

$$\text{span}(R) = \text{span}(\text{span}(R)) \supset \text{span}(S) = V$$

from which it follows that $\text{span}(R) = V$.

2. Let S be a linearly independent subset of V. Define

$$\mathcal{A} = \{D \subset V \mid S \subset D \text{ and } D \text{ is linearly independent}\}$$

ordered by inclusion. It is easy to see that any chain on $\mathcal{A}$ has an upper bound on $\mathcal{A}$, by just taking the union. It follows from Zorn's lemma that $\mathcal{A}$ has a maximal element R. We show that $\text{span}(R) = V$ by contradiction. If $\text{span}(R) \neq V$, we can consider $R \cup \{v\}$ for some $v \notin \text{span}(R)$ and we obtain an element of $\mathcal{A}$ which is greater than a maximal element. This is a contradiction.

□

Corollary X.10.1. *Every vector space has a Hamel basis*

---

[1] Although other options exist, such as taking sums over hyperintegers.

Theorem X.11 (Dimension theorem for vector spaces)**.** *Given a vector space V, any two bases have the same cardinality.*

Proof. The finite-dimensional case has already been proved. Suppose A is a basis of V with $|A| \geq \aleph_0$. Let B be another basis of V. Each element $a \in A$ can be written as a finite combination of elements in B. Collect all the elements that go into the finite linear combination in a finite set $B_a \subset B$. We claim

$$B = \bigcup_{a \in A} B_a.$$

Indeed, assume $b \in B \setminus (\cup_{a\in A} B_a)$. Since A spans V, so does $\cup_{a\in A} B_a$. Thus b can be written as a non-trivial combination of vectors in $\cup_{a\in A} B_a \subset B$, contradicting the linear independence of B. Then we have

$$|B| = \left| \bigcup_{a\in A} B_a \right| \leq \aleph_0 \cdot |A| = |A|$$

A similar argument gives

$$|A| \leq \aleph_0 \cdot |B| = |B|.$$

By the Schröder–Bernstein theorem I.183, we conclude $|A| = |B|$. □

TODO: does this proof work with only the ultrafilter lemma?
Thus the notion of dimension (also known as Hamel-dimension) also makes sense for infinite-dimensional vector spaces, except it is a cardinality, not a number.
TODO: do we need a strong cardinality assignment? (Assumed for now)
Many textbooks state results using dimensions only for the finite-dimensional case. As we will see, these results almost always generalise directly to the infinite-dimensional case as well, if we assume the axiom of choice.

> The inverse of this theorem (i.e. the infinite-dimensional analogue of proposition X.9) does not hold: infinite-dimensional vector spaces always have proper subspaces with a basis of the same cardinality. This is obvious because dropping one vector in the Hamel basis of an infinite-dimensional vector space will not change the cardinality, but will make it a proper subspace.

Corollary X.11.1. *Let V and W be vector spaces.*

1. *If $\dim V > \dim W$, then no linear map from V to W is injective.*
2. *If $\dim V < \dim W$, then no linear map from V to W is surjective.*

Lemma X.12. *Let V be an infinite-dimensional vector space over a field $\mathbb{F}$. Assume $|\mathbb{F}| \leq \dim_{\mathbb{F}} V$, then $\dim_{\mathbb{F}} V = |V|$.*

Proof. Let B be a basis of V. It is supposed infinite. There is a surjection

$$\bigcup_{n\in\mathbb{N}} (\mathbb{F} \times B)^n \to V : (a_i, v_i)^{i<n} \mapsto \sum_{i<n} a_i v_i.$$

So we have

$$|V| \leq \left| \bigcup_{n\in\mathbb{N}} (F \times B)^n \right| = \sum_{n\in\mathbb{N}} |F \times B|^n \leq \aleph_0 \cdot |\mathbb{F}| \cdot |B| = \max\{\aleph_0, |\mathbb{F}|, |B|\} = |B|.$$

Thus $|V| \leq \dim_{\mathbb{F}} V$. The other inequality is obvious. By the Schröder–Bernstein theorem I.183, we conclude $\dim_{\mathbb{F}} V = |V|$. □

1.3 Constructing vector spaces

1.3.1 Sums of subspaces

Suppose $\{U_i\}_{i\in I}$ a set of subspaces of a vector space V. The sum of these subspaces, denoted $\sum_{i\in I} U_i$, is the set of all finite linear combinations of elements in $\bigcup_{i\in I} U_i$:

$$\sum_{i\in I} U_i = \operatorname{span}\left(\bigcup_{i\in I} U_i\right) = \left\{ \sum_{i\in J} u_i \;\middle|\; J \subset I \text{ finite}, u_i \in \bigcup_{i\in I} U_i \right\}.$$

For finite sums this reduces to

$$U_1 + \ldots + U_m = \left\{ \sum_{i=1}^{m} u_i \;\middle|\; u_1 \in U_1, \ldots, u_m \in U_m \right\}.$$

Proposition X.13. *Let $\{U_i\}_{i\in I}$ be a set of subspaces of a vector space V and β_i a basis of U_i for all $i \in I$. Then*

$$\sum_{i\in I} U_i = \operatorname{span}\left(\bigcup_{i\in I} \beta_i\right).$$

Proof. From $\bigcup_{i\in I} \beta_i \subseteq \bigcup_{i\in I} U_i$, we get $\operatorname{span}\left(\bigcup_{i\in I} \beta_i\right) \subseteq \operatorname{span}\left(\bigcup_{i\in I} U_i\right) = \sum_{i\in I} U_i$. Conversely, take $u \in \sum_{i\in I} U_i$. Then $u = \sum_{j\in J} u_j$ where J is finite subset of I and $u_i \in U_i$. Now each u_j can be written as $\sum_k a_{j,k}v_{j,k}$, where $a_{j,k}$ are scalars and $v_{j,k}$ are vectors in β_j. So

$$u = \sum_{j,k} a_{j,k}v_{j,k},$$

which is a finite linear combination of vectors in $\bigcup_{i\in I} \beta_i$. So $u \in \operatorname{span}\left(\bigcup_{i\in I} \beta_i\right)$. □

Proposition X.14. *Let V be a vector space and A, B, C subspaces. Then*

1. $A + (B \cap C) \subseteq (A + B) \cap (A + C)$;
2. $(A + B) \cap C \supseteq (A \cap C) + (B \cap C)$.

Proof. (1) Take $v = v_1 + v_2 \in A + (B \cap C)$ where $v_2 \in B$ and $v_2 \in C$, so $v_1 + v_2 \in A + B$ and $v_1 + v_2 \in A + C$.
(2) Take $v = v_1 + v_2 \in (A \cap C) + (B \cap C)$. Then $v_1, v_2 \in C$ and thus $v \in (A + B) \cap C$. □

Theorem X.15 (Dimension of a sum)**.** *Let U_1 and U_2 be subspaces of a finite-dimensional vector space, then*

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

Proof. Let $\dim U_1 = r, \dim U_2 = s$ and $\dim(U_1 \cap U_2) = t$. Then $t \leq r$ and $t \leq s$. Take a basis $\{v_1, \ldots, v_t\}$ of $U_1 \cap U_2$. This can be expanded to a basis $\beta_{U_1} = \{v_1, \ldots, v_t, u_{t+1}, \ldots u_r\}$ of U_1 and also to a basis $\beta_{U_2} = \{v_1, \ldots, v_t, u'_{t+1}, \ldots u'_s\}$ of U_2. We will show that $\{v_1, \ldots, v_t, u_{t+1}, \ldots u_r, u'_{t+1}, \ldots, u'_s\}$ is a basis of $U_1 \cap U_2$. This completes the proof because

$$\begin{aligned}\dim(U_1 + U_2) &= t + (s - t) + (r - t) = s + r - t \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).\end{aligned}$$

The spanning property is easy. Linear independence is slightly more difficult: Take a linear combination

$$\sum_{i=1}^{t} \alpha_i v_i + \sum_{j=t+1}^{r} \beta_j u_j + \sum_{k=t+1}^{s} \beta'_k u'_k = 0.$$

We must show this combination is trivial. Indeed observe that

$$\sum_{i=1}^{t} \alpha_i v_i + \sum_{j=t+1}^{r} \beta_j u_j = - \sum_{k=t+1}^{s} \beta'_k u'_k.$$

The left-hand side is a vector in U_1, the right-hand side is a vector in U_2, so it must lie in $U_1 \cap U_2$, so we rewrite the left-hand side as

$$\sum_{i=1}^{t} \lambda_i v_i = - \sum_{k=t+1}^{s} \beta'_k u'_k.$$

Due to β_{U_2} being a basis, this linear combination must be trivial and all β'_k are zero. This leaves us

$$\sum_{i=1}^{t} \alpha_i v_i + \sum_{j=t+1}^{r} \beta_j u_j = 0$$

from our original linear combination. Due to β_{U_2} being a basis this combination must also be trivial. ☐

If $\dim(U_1 \cap U_2) < \dim U_1$ and $\dim(U_1 \cap U_2) < \dim U_2$, this proof generalises to infinite-dimensional vector spaces.

1.3.2 (Internal) direct sum

Suppose $\{U_i\}_{i \in I}$ is a set of subspaces of V. The sum $\sum_{i \in I} U_i$ is called a direct sum if each element u of the sum can be uniquely written as

$$u = \sum_{i \in I} u_i \qquad (u_i \in U_i)$$

where only finitely many of the u_i are nonzero.
In this case we write $\bigoplus_{i \in I} U_i$, or $U_1 \oplus \ldots \oplus U_m$ if $I = \{1, \ldots, m\}$.

Proposition X.16 (Conditions for a direct sum)**.** *Let $\{U_i\}_{i \in I}$ be a set of subspaces of a vector space V and β_i a basis of U_i for all $i \in I$. Let $U, W \subseteq V$ also be subspaces of V.*

1. *The sum $\sum_{i \in I} U_i$ is direct* if and only if *0 has the unique decomposition as in the definition.*
2. *The sum $\sum_{i \in I} U_i$ is direct* if and only if *the union $\bigcup_{i \in I} \beta_i$ is disjoint and linearly independent.*
3. *The sum $U + W$ is direct* if and only if *$U \cap W = \{0\}$.*

Proof. TODO ☐

Corollary X.16.1. *Let $\{U_i\}_{i\in I}$ be a set of subspaces of a vector space V and β_i a basis of U_i for all $i \in I$. Then*

$$\dim\left(\bigoplus_{i\in I} U_i\right) = \sum_{i\in I} \dim U_i$$

In a vector space V, a subspace W is a complementary subspace (or a complement) of the subspace U if $V = U \oplus W$.

Proposition X.17. *Let V be a vector space, then each subspace of V has a complement.*

Proof. Let U be a subspace of V. Then, by X.10.1, we can find a basis B of U and, by X.10, we can extend it to a basis D of V. Now $V = U \oplus \operatorname{span}(D \setminus B)$ by X.13 and X.16. ☐

Note this requires the axiom of choice, and is in fact equivalent with it.

Corollary X.17.1. *Suppose V is finite-dimensional and $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \ldots + U_m$ is a direct sum* if and only if

$$\dim(U_1 + \ldots + U_m) = \dim U_1 + \ldots \dim U_m.$$

1.3.3 External direct sum

Let U, W be vector spaces over the same field $\mathbb{F}$. We define the vector space $U \oplus W$, called the (external) direct sum, as the set $U \times W$ with the operations

$$\begin{cases} (u_1, w_1) + (u_2, w_2) = (u_1 +_U u_2, w_1 +_W w_2) & (u_1, u_2 \in U; w_1, w_2 \in W) \\ r \cdot (u, w) = (ru, rw) & (r \in \mathbb{F}; u \in U; w \in W) \end{cases}$$

In general we can define a direct sum of an arbitrary collection of vector spaces $\{U_i\}_{i\in I}$, denoted

$$\bigoplus_{i\in I} U_i$$

as the vector space with as field the subset of the Cartesian product $\prod_{i\in I} U_i$ where all but finitely many of the terms are zero. The operations are defined point-wise.

Proposition X.18. *Suppose $V_1, \ldots V_m$ are vector spaces over $\mathbb{F}$. Then*

$$\dim(V_1 \oplus \ldots \oplus V_m) = \dim V_1 + \ldots + \dim V_m$$

Proof. We construct a basis β of $V_1 \oplus \ldots \oplus V_m$ from bases β_{V_i} of V_i:

$$\beta = (\beta_{V_1} \times \{0\} \times \ldots \times \{0\}) \cup (\{0\} \times \beta_{V_2} \times \{0\} \times \ldots \times \{0\}) \cup \ldots \cup (\{0\} \times \ldots \times \{0\} \times \beta_{V_m}).$$

All these unions are disjunct, so

$$\begin{aligned} |\beta| &= |(\beta_{V_1} \times \{0\} \times \ldots \times \{0\}) \cup \ldots \cup (\{0\} \times \ldots \times \{0\} \times \beta_{V_m})| \\ &= |(\beta_{V_1} \times \{0\} \times \ldots \times \{0\})| + \ldots + |(\{0\} \times \ldots \times \{0\} \times \beta_{V_m})| \\ &= |\beta_{V_1}| + \ldots + |\beta_{V_m}| \\ &= \dim V_1 + \ldots + \dim V_m. \end{aligned}$$

☐

Proposition X.19. *Let U, W be subspaces of V. Then the external direct sum of U and W is isomorphic to the internal direct sum of U and W.*

Proof. The map $f : U \times W \to V : (u, w) \mapsto u + w$ is an isomorphism. □

For this reason we use the same symbol for both.

Let V, W, X, Y be vector spaces over $\mathbb{F}$. Let $S : V \to X$ and $T : W \to Y$ be linear maps. Then the direct sum of S and T is a linear map

$$S \oplus T : V \oplus W \to X \oplus Y : (v, w) \mapsto (S(v), T(v)).$$

Lemma X.20. *Let V, W be vector spaces over a field $\mathbb{F}$ and $A, C \in \mathcal{L}(V)$ and $B, D \in \mathcal{L}(W)$. Then*

1. $a(A \oplus B) + b(C \oplus D) = (aA + bC) \oplus (aB + bD)$;
2. $(A \oplus B)(C \oplus D) = AC \oplus BD$;
3. $(A \oplus B)^k = A^k \oplus B^k$.

1.3.3.1 Matrix representation

TODO: move Assume V and W are finite-dimensional vector spaces with resp. bases $\{\mathbf{e}_i\}_{i=1}^m$ and $\{\mathbf{f}_j\}_{j=1}^n$. As in the proof of proposition X.18, we can take the basis $\{\mathbf{e}_i\}_i \times \{0\} \cup \{0\} \times \{\mathbf{f}_j\}_j$ of $V \oplus W$.
We can naturally fit the basis into a list of $m + n$ elements:

$$(\mathbf{e}_1, 0), \ldots (\mathbf{e}_m, 0), (0, \mathbf{f}_1), \ldots, (0, \mathbf{f}_n)$$

1.3.3.2 Linear maps

TODO: also move Let $S : V \to X$ and $T : W \to Y$ be linear maps, with matrix representations A and B, respectively. The matrix representation of $S \oplus T$ is given by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

with respect to the basis $\{\mathbf{e}_i\}_i \times \{0\} \cup \{0\} \times \{\mathbf{f}_j\}_j$.

1.4 Linear maps

Let $(\mathbb{R}, V, +)$ and $(\mathbb{R}, W, +)$ be vector spaces over the same field. A linear map or linear transformation is a function $L : V \to W$ with the following properties:

Additivity $L(u + v) = L(u) + L(v)$ for all $u, v \in V$;

Homogeneity $L(\lambda v) = \lambda L(v)$ for all $\lambda \in \mathbb{R}$ and all $v \in V$.

These conditions are equivalent to the condition that

$$L(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 L(v_1) + \lambda_2 L(v_2) \qquad \text{for all } \lambda_1, \lambda_2 \in \mathbb{F} \text{ and all } v_1, v_2 \in V.$$

We denote the set of all linear maps from V to W as $\mathcal{L}_{\mathbb{F}}(V, W)$, or $\mathcal{L}(V, W)$. The set of endomorphisms on V is denoted $\mathcal{L}(V) \coloneqq \mathrm{End}(V) = \mathcal{L}(V, V)$.

Lemma X.21. *Let $L \in \mathcal{L}(V, W)$.*

1. *$L(0) = 0$ and $L(-v) = -L(v)$*
2. *$L\left(\sum_{i=1}^{n} \lambda_i v_i\right) = \sum_{i=1}^{n} \lambda_i L(v_i)$.*
3. *A linear map is completely determined by the images of a basis of V.*
4. *Let D be a set of vectors. Then $L[D]$ is linearly independent* if and only if *D is linearly independent.*

1.4.1 Examples

1. The zero map that maps everything to zero.
2. Identity maps.
3. Differentiation of polynomials.
4. Integration of polynomials.
5. Shifting elements in a list.
6. Projections.

A (linear) operator between two vector spaces V and W is a linear partial function $T : V \nrightarrow W$ such that the domain $\mathrm{dom}(T)$ is a vector space.
We also say an operator is a function $T : \mathrm{dom}(T) \subseteq V \to W$.

The requirement that $\mathrm{dom}(T)$ be a subspace of V is necessary for linearity to make sense! Some authors (e.g Axler) use the word "operator" to mean a linear endomorphism.

1.4.2 Image and kernel

Let $L \in \mathcal{L}(V, W)$. The kernel or null space of L is the set of vectors that L maps to zero:

$$\ker(L) = \{v \in V \mid L(v) = 0\}.$$

Proposition X.22. *The kernel of $L \in \mathcal{L}(V, W)$ is a subspace of V.*

The dimension of the kernel of a linear map is its nullity.

Proposition X.23. *Let $L \in \mathcal{L}(V, W)$. Then L is injective if and only if $\ker(L) = 0$.*

TODO: generalise to groups

Proof. We show both implications.

$\boxed{\Rightarrow}$ We know $\{0\} \subset \ker(L)$ by lemma X.21. Suppose $v \in \ker(L)$, then $L(v) = 0 = L(0)$. So $v = 0$ by injectivity and $\{0\} \supset \ker(L)$.

$\boxed{\Leftarrow}$ Suppose $u, v \in V$ such that $L(u) = L(v)$. Then

$$0 = L(u) - L(v) = L(u - v).$$

Thus $u - v \in \ker(L)$, meaning $u - v = 0$ and $u = v$.

$\square$

Let $L \in \mathcal{L}(V, W)$. The image or range of L is the set of vectors that are of the form $L(v)$ for some $v \in V$:

$$\operatorname{im}(L) = \{L(v) \mid v \in V\}.$$

Proposition X.24. *The range of $L \in \mathcal{L}(V, W)$ is a subspace of W.*

The dimension of the image of a linear map is its rank.

Theorem X.25. *Every short exact sequence of vector spaces splits.*

Proof. Let

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "S"] & V \arrow[r, "T"] & W \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence of vector spaces. By the splitting lemma TODO ref, it is enough to find a left inverse of S. Pick a basis β of U. Because S is injective, $S[\beta]$ is linearly independent and we can extend it to a basis β'. We can now define the left inverse by specifying how the basis elements are mapped, by X.21. To wit: $\beta' \setminus S[\beta]$ is mapped to 0 and each element $S[\beta]$ has exactly one origin be injectivity and it is to this origin that it is now mapped. $\square$

Corollary X.25.1. *Let $L \in \mathcal{L}(V, W)$. Then*

$$V \cong \ker L \oplus \operatorname{im} L.$$

Proof. Given L we have the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker L \arrow[r, hook] & V \arrow[r, "L"] & \operatorname{im} L \arrow[r] & 0. \end{tikzcd}$$

The isomorphism then follows from the splitting lemma TODO ref. $\square$

Corollary X.25.2 (Dimension theorem for linear maps)**.** *Let $L \in \mathcal{L}(V, W)$. Then*

$$\dim(V) = \dim(\ker L) + \dim(\operatorname{im} L).$$

This corollary is also known as the rank-nullity theorem or the fundamental theorem of linear maps.

Proof. By $\dim(V) = \dim(\ker L \oplus \operatorname{im} L) = \dim(\ker L) + \dim(\operatorname{im} L)$.
Alternatively this can be proven directly as follows:
Take a basis β_0 of $\ker(L)$. We can expand this to a basis β of V, by theorem X.10. It is easy to show that $L[\beta \setminus \beta_0]$ is a basis of $\operatorname{im}(L)$. Now $L[\beta \setminus \beta_0] =_c \beta \setminus \beta_0$ and $(\beta \setminus \beta_0) \cap \beta_0 = \emptyset$. Thus $|\beta| = |(\beta \setminus \beta_0) \cup \beta_0| = |\beta \setminus \beta_0| + |\beta_0|$. This proves the assertion. $\square$

Corollary X.25.3. *Let*

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \ldots \longrightarrow V_n \longrightarrow 0$$

be an exact sequence of vector spaces, then

$$\sum_{i=1}^{n} (-1)^i \dim(V_i) = 0.$$

Proof. Let f_i be the map $V_i \to V_{i+1}$. By exactness $\operatorname{im} f_i = \ker f_{i+1}$ and $\dim(\operatorname{im} f_i) = \dim(\ker f_{i+1})$. By the previous corollary $\dim(V_i) = \dim(\ker f_i) + \dim(\operatorname{im} f_i)$. Then

$$\sum_{i=1}^{n}(-1)^i \dim(V_i) = \sum_{i=1}^{n}(-1)^i \dim(\ker f_i) + \sum_{i=1}^{n}(-1)^i \dim(\ker f_{i+1}) = \sum_{i=2}^{n}(-1)^i \dim(\ker f_i) - \sum_{i=2}^{n}(-1)^i \dim(\ker f_i) = 0.$$

☐

Corollary X.25.4. *Let $L \in \mathcal{L}(V, W)$. Then*

$$\dim(\operatorname{im} L) \leq \dim(V).$$

Proof. TODO ref cardinal arithmetic. ☐

Lemma X.26. *Let S, T be compatible linear maps. Then*

$$\text{rank of } ST \ \leq\ \min\{\text{rank of } S,\ \text{rank of } T\}.$$

If T is invertible, then the rank of ST equals the rank of S. Similarly if S is invertible, then the rank of ST equals the rank of T.

Proof. Clearly $\operatorname{im}(ST) \subset \operatorname{im}(S)$, so $\dim \operatorname{im}(ST) \leq \dim \operatorname{im}(S)$. We also have $ST = S|_{\operatorname{im} T} T$, where $S|_{\operatorname{im} T}$ is S restricted to $\operatorname{im} T$. Then corollary X.25.4 applied to $S|_{\operatorname{im} T}$ gives $\dim \operatorname{im}(ST) \leq \dim \operatorname{im} T$. Together these inequalities give the result.
To show equality in the invertible case, first assume T invertible:

$$\dim \operatorname{im} ST \leq \dim \operatorname{im} STT^{-1} = \dim \operatorname{im} S.$$

Together with the first inequality this gives an equality. The case for S invertible is similar. ☐

Proposition X.27. *Let S, T be compatible linear maps. Then*

1. $\ker(ST) \supseteq \ker(T)$*;*
2. $\dim \ker(ST) = \dim \ker(T) + \dim(\operatorname{im}(T) \cap \ker(S))$*.*

Proof. (1) $x \in \ker(T) \implies (ST)x = S(Tx) = S(0) = 0 \implies x \in \ker(ST)$. (2) Consider the restriction $T|_{\ker(ST)}$. Applying the dimension theorem gives

$$\dim \ker(ST) = \dim \ker(T|_{\ker(ST)}) + \dim \operatorname{im}(T|_{\ker(ST)}) = \dim \ker(ST) = \dim \ker(T) + \dim \operatorname{im}(T|_{\ker(ST)}),$$

so it is enough to show $\operatorname{im}(T|_{\ker(ST)}) = \operatorname{im}(T) \cap \ker(S)$. First take $v \in \operatorname{im}(T|_{\ker(ST)}$, then there exists some $w \in \ker(ST)$ such that $v = Tw$, meaning $v \in \operatorname{im}(T)$. Also $Sv = STw = 0$, meaning $v \in \ker(S)$.
Then take $v \in \operatorname{im}(T) \cap \ker(S)$, so we can find a w such that $v = Tw$. Also $Sv = STw = 0$, so $w \in \ker(ST)$ and $v \in \operatorname{im}(T|_{\ker ST})$. ☐

1.4.3 Algebraic operations on linear maps

Suppose $K, L \in \mathcal{L}_{\mathbb{F}}(V, W)$ and $\lambda \in \mathbb{F}$.

- The sum $K + L$ is defined by $(K + L)(v) = Kv + Lv$ for all $v \in V$;
- The scalar product is defined by $(\lambda K)(v) = \lambda K(v)$ for all $v \in V$.

Proposition X.28. • *The sum of linear maps is again a linear maps. Scalar multiples of linear maps are linear maps.*

- *With addition and scalar multiplication defined as above, $\mathcal{L}_{\mathbb{F}}(V, W)$ is a vector space.*

Let $K \in \mathcal{L}_{\mathbb{F}}(U, V)$ and $L \in \mathcal{L}_{\mathbb{F}}(V, W)$. The product LK is defined as the composition

$$(LK)(u) = L(K(u)) \qquad \text{for all } u \in U.$$

If the product of two linear maps K, L makes sense, we call the linear maps compatible.

Proposition X.29. *The product of two (compatible) linear maps is a linear map.*

Proposition X.30 (Algebraic properties of linear maps)**.** *The product of linear maps has the following properties.*

Associativity *Let L_1, L_2, L_3 be compatible linear maps, then*

$$(L_1L_2)L_3 = L_1(L_2L_3)$$

Identity *Let $L \in \mathcal{L}(V, W)$. The identity maps $I_V : V \to V$ and $I_W : W \to W$ are linear and have the property that*

$$LI_V = I_WL = L.$$

Distributive properties $(S_1 + S_2)T = S_1T + S_2T$ *and* $S(T_1 + T_2) = ST_1 + ST_2$ *whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.*

These properties mean that for any vector space V, $\mathcal{L}(V)$ forms a unital algebra.

Note that multiplication of linear maps is not commutative, not even for maps that are compatible both ways.

1.4.4 Invertibility and isomorphisms

Proposition X.31. *Let L be a linear map. If L is invertible as a function (i.e. bijective), its inverse L^{-1} is linear.*

Proof. We calculate for x, y vectors and $a \in \mathbb{F}$

$$L^{-1}(ax + y) = L^{-1}(aLL^{-1}x + LL^{-1}y) = L^{-1}L(aL^{-1}x + L^{-1}y) = aL^{-1}x + L^{-1}y.$$

□

- An invertible linear map is called an isomorphism.
- Two vector spaces V, W are isomorphic if there is an isomorphism between them. This is denoted $V \cong W$.

Proposition X.32. *Let V, W be vector spaces over the same field $\mathbb{F}$ and $n \in \mathbb{N}$. Then*

1. $V \cong W \iff \dim V = \dim W$;
2. $V \cong \mathbb{F}^n \iff \dim V = n$;
3. $\mathbb{F}^n \cong \mathbb{F}^m \iff n = m$.

Proof. We prove the first statement. The second and third follow easily, using $\dim_{\mathbb{F}} \mathbb{F}^n = n$.

$\boxed{\Rightarrow}$ Let $T : V \to W$ be an isomorphism. Then $\ker T = \{0\}$ and $\operatorname{im} T = W$. Thus

$$\dim V = \dim \ker T + \dim \operatorname{im} T = 0 + \dim W = \dim W.$$

$\boxed{\Leftarrow}$ Assume $\dim V = \dim W$. Thus there exists an invertible function from a basis of V to a basis of W. This can be extended by linearity to a function on V, because it is defined on a Hamel basis. It is easy to see this function is linear and bijective.

□

Proposition X.33. *Let $L \in \mathcal{L}(V, W)$ be an isomorphism. Let β be a basis of V, then $L[\beta]$ is a basis of W.*

Proposition X.34. *Suppose V is a finite-dimensional vector space and $L \in \mathcal{L}(V)$ is a linear map on V, then*

$$L \text{ is invertible} \iff L \text{ is injective} \iff L \text{ is surjective}$$

Proof. All we need to prove is

$$L \text{ is injective} \iff L \text{ is surjective}$$

$\boxed{\Rightarrow}$ Assume L injective. Then $\ker L = \{0\}$. By the dimension theorem for linear maps, theorem X.25.2

$$\dim \operatorname{im} L = \dim V - \dim \ker L = \dim V.$$

Because $\operatorname{im} L \subset V$ and using proposition X.9, we conclude that $\operatorname{im} L = V$ and thus L is surjective.

$\boxed{\Leftarrow}$ Assume L surjective. Then, by the dimension theorem for linear maps,

$$\dim \ker L = \dim V - \dim \operatorname{im} L = 0,$$

which means L is injective.

□

Remark that the proof of the first implication uses proposition X.9, and thus cannot be generalised to infinite-dimensional vector spaces. In the proof of the second implication the subtraction of infinite cardinals is only uniquely defined if $\dim V > \dim \operatorname{im} L$, which is clearly not the case.

Example

Counterexamples to the previous theorem in the infinite-dimensional case are given by the left shift map on $\mathbb{F}^{\mathbb{N}}$ (which is injective, but not surjective) and the right shift map on $\mathbb{F}^{\mathbb{N}}$ (which is surjective, but not injective).

1.4.5 Types of linear maps

1.4.5.1 Finite-rank operators

A linear map $T : V \to V$ is said to be a finite-rank operator if it has finite rank.

1.4.5.2 Idempotents

Lemma X.35. *Let V be a vector space and $U \subseteq V$ a subspace. Then for any complement W of U in V,*

$$P : U \oplus W = V \to V : u + w \mapsto u$$

is an idempotent such that $\operatorname{im} P = U$*.*

Proposition X.36. *Let V be a vector space and P an idempotent linear map. Then*

$$V = \operatorname{im} P \oplus \ker P.$$

Proof. For any $v \in V$, we can write $v = (v - Pv) + Pv$ where $Pv \in \operatorname{im} P$ and $(v - Pv) \in \ker P$ because

$$P(v - Pv) = Pv - P^2 v = Pv - Pv = 0.$$

So we have $V = \operatorname{im} P + \ker P$. To show that the sum is direct, we take $u \in \operatorname{im} P \cap \ker P$. Then $u = Pw$ for some $w \in V$ and applying P gives $0 = Pu = P^2 w = Pw = 0$. So the sum is direct by X.16. □

Lemma X.37. *Let V be a vector space, P an idempotent linear map and $v \in V$. Then $v \in \operatorname{im} P$* if and only if *$v = Pv$.*

Proof. We have that $v \in \operatorname{im} P$ iff $\exists u \in V : Pu = v$. Then we have $v = Pu = P^2 u = Pv$. □

Lemma X.38. *Let V be a vector space and P an idempotent linear map. Then $P' \coloneqq \mathrm{id}_V - P$ is an idempotent linear map such that*

$$\operatorname{im} P' = \ker P \qquad \textit{and} \qquad \ker P' = \operatorname{im} P.$$

Consequently, $V = \operatorname{im} P \oplus \operatorname{im} P'$.

Proof. Clearly $\mathrm{id}_V - P$ is idempotent: $(\mathrm{id}_V - P)^2 = \mathrm{id}_V - P - P + P^2 = \mathrm{id}_V - 2P + P = \mathrm{id}_V - P$.
It is enough to show that $\operatorname{im} P' = \ker P$, because $P = \mathrm{id}_V - P'$.
Assume $v \in \ker P$. Then $P'v = v - Pv = v - 0 = v$, so $v \in \operatorname{im} P'$.
Assume $v \in \operatorname{im} P'$. Then, by X.37, $v = P'v = v - Pv$, so $Pv = 0$.
The last remark follows from X.36 □

TODO trace:

Lemma X.39. *Let V be a vector space and P an idempotent linear map. Then*

$$\operatorname{Tr}(P) = \dim(\operatorname{im}(P)).$$

1.4.5.3 Invariant, reducing and irreducible subspaces

Let V be a vector space, T a linear operator on V and $U \subseteq V$ a subspace. Then U is called

- invariant under T is $T[U] \subseteq U$;
- reducing for T if $V = U \oplus W$ and both U and W are invariant under T;
- irreducible w.r.t. T if for all $W \subseteq U$ such that W is reducing for T, we have $W = U$ or $W = \emptyset$.

Lemma X.40. *Let V be a vector space, T a linear operator on V and P an idempotent operator on V with image $U = P[V]$. Then*

1. *U is invariant under T* if and only if *$PTP = TP$;*
2. *the following are equivalent:*
    (a) *U is reducing for T;*
    (b) *$P[V]$ and $(\mathrm{id}_V - P)[V]$ are invariant under T;*
    (c) *$PT = PTP = TP$;*
    (d) *$PT = TP$.*

   if and only if .

Proof. (1) The invariance of U under T can be stated as $TP[V] \subseteq \operatorname{im} P$. By X.37 this can be restated as $TPv = PTPv$ for all $v \in V$.
(2) Points (a) and (b) are equivalent by X.37.
Points (b) and (c) are equivalent by point (1) and $(\mathrm{id}_V - P)T(\mathrm{id}_V - P) = T - PT - TP + PTP = T(\mathrm{id}_V - P) + PTP - PT$.
Point (d) follows immediately from (c). The converse follows from $P(TP) = P(PT) = PT$ and $(PT)P = (TP)P = TP$. □

1.4.5.4 Irreducible operators

Let V be a vector space and T an operator on V. Then T is called irreducible if V is irreducible w.r.t. T.

1.5 Sets of vectors

Proposition X.41. *Let V be vector space and consider a function $f : \mathcal{P}V \to \mathcal{P}V$ and define*

$$\mathcal{X} = \{X \subseteq V \mid A \subseteq X \implies f(A) \subseteq X\}.$$

Then $\mathcal{X}$ is closed under arbitrary intersections and thus a complete sublattice of $\mathcal{P}(V)$. The closure operator into $\mathcal{X}$ is given by the intersection of all supersets in $\mathcal{X}$.

Most of the types of sets of vectors in this section are of this form.

1.5.1 Star-shaped sets

A subset S of a real or complex vector space V is called

- star-shaped at $a \in V$ if for all $x \in S$ and $0 \leq r \leq 1$, $rx + (1-r)a \in C$;
- absolutely star-shaped at $a \in V$ if for all $x \in S$ and $|r| \leq 1$, $rx + (1-r)a \in C$.

1.5.2 Affine sets

A subset A of a real or complex vector space V is called affine if for all $x, y \in A$ and $\lambda \in \mathbb{F}$, $\lambda x + (1-\lambda)y \in A$.
The closure of a set X into the lattice of affine sets is called the affine hull of X and is denoted $\operatorname{aff}(X)$.

Proposition X.42. *Let V be a vector space and $A \subseteq V$ a subset. Then following are equivalent:*

1. *A is affine;*
2. *for all $x \in A$: the set $A - x$ is a vector subspace;*
3. *for some $x \in A$: the set $A - x$ is a vector subspace.*

Proof. (1) ⇒ (2) Assume A affine and take arbitrary $x \in A$. We verify the subspace criterion X.2: clearly $0 \in A - x$ because $x \in A$.
Take $y - x, z - x \in A - x$. Then $y - x + z - x = 2\left(\frac{1}{2}y + \frac{1}{2}z\right) - x - x \in A - x$.
Take $y - x \in A - x$ and $\lambda \in \mathbb{R}$. Then $\lambda(y-x) = \lambda y + (1-\lambda)x - x \in A - x$.
(2) ⇒ (3) Immediate.
(3) ⇒ (1) Assume $A - x$ is a vector subspace for some $x \in A$. Take $y, z \in A$ and $\lambda \in \mathbb{F}$. Then $A - x \ni \lambda(y-x) + (1-\lambda)(z-x) = \lambda y + (1-\lambda)z - x$, so $\lambda y + (1-\lambda)z \in A$. $\square$

1.5.3 Balanced set

A subset B of a vector space V over a field $\mathbb{F}$ with valuation $|\cdot|$ is called balanced if for all $|r| \leq 1$, $rC \subseteq C$.

- The closure of a set $X \subseteq V$ into the lattice of balanced sets is called the balanced hull of X and is denoted $\operatorname{bal}(X)$.
- The dual closure of a set $X \subseteq V$ into the lattice of balanced sets is called the balanced core of X and is denoted $\operatorname{balcore}(X)$.

Note that 0 is an element of any balanced set. The lattice of balanced sets is closed under unions and thus a complete sublattice of $\mathcal{P}(X)$.

Lemma X.43. *Let V be a vector space and $B \subseteq V$ a subset. Then*

1. $\operatorname{bal}(B) = \bigcup_{|r|\leq 1} rB = \overline{\mathrm{B}}(0,1) \cdot B$;
2. $\operatorname{balcore}(B) = \begin{cases} \bigcap_{|r|\geq 1} rB & 0 \in B \\ \emptyset & 0 \notin B \end{cases}$.

Lemma X.44. *Let V be a vector space and $B \subseteq V$ a subset. Then the following are equivalent:*

1. *B is balanced;*
2. *$\mathrm{bal}(B) = \overline{\mathrm{B}}(0,1) \cdot B \subseteq B$;*
3. *$\mathrm{bal}(B) = \overline{\mathrm{B}}(0,1) \cdot B = B$;*
4. *$B \subseteq \mathrm{balcore}(B)$;*
5. *for all $|r| \geq 1$, $C \subseteq rC$;*
6. *B is symmetric and star-shaped at 0.*

Lemma X.45. *Let V be a vector space and $B \subseteq V$ a balanced subset. Then*

1. *for all $\lambda \in \mathbb{F}$: $\lambda B = |\lambda| B$.*

Lemma X.46. *The balanced core of a convex set is convex.*

Proof. Let $B \subseteq V$ be a convex subset of a vector space V. Then $\mathrm{balcore}(B) = \begin{cases} \bigcap_{|r|\geq 1} rB & 0 \in B \\ \emptyset & 0 \notin B \end{cases}$. The empty set is convex. For all $r \in \mathbb{F}$, rB is convex by X.49 and arbitrary intersections of convex sets are convex. □

1.5.4 Convex sets

A subset C of a real or complex vector space V is called convex if for all $x, y \in C$ and $0 \leq r \leq 1$, $rx + (1-r)y \in C$.
The closure of a set $X \subseteq V$ into the lattice of convex sets is called the convex hull of X and is denoted $\mathrm{conv}(X)$.

Note that this is a stronger property than metric convexity!

Example

Let C be the set of all vectors with norm in $\mathbb{Q} \cap [0,1]$. The is metrically convex, but not a convex set of vectors.

Lemma X.47. *Let V be a vector space and $X \subseteq V$ a subset. Then* $\mathrm{conv}(X) = \{rx + (1-r)y \mid 0 \leq r \leq 1, x, y \in B\}$.

Lemma X.48. *Let V be a vector space an $X \subseteq V$ a subset. Then the following are equivalent:*

1. *X is convex;*
2. *for all $0 \leq r \leq 1$, $rX + (1-r)X \subseteq X$.*

Lemma X.49. *Let V be a vector space over $\mathbb{F}$, $v \in V$, $\lambda \in \mathbb{F}$ and $X \subseteq V$ a convex subset subset. Then $v + \lambda X$ is convex.*

Proof. Take $v + \lambda x_1, v + \lambda x_2 \in v + \lambda X$ and $r \in [0,1]$. Then

$$r(v + \lambda x_1) + (1-r)(v + \lambda x_2) = v + \lambda(rx_1 + (1-r)x_2) \in v + \lambda X.$$

□

1.5.4.1 Absolutely convex sets

A subset B of a vector space V over a field $\mathbb{F}$ with valuation $|\cdot|$ is called absolutely convex or disked if for all $x, y \in C$ and $|r| \leq 1$, $rx + (1-r)y \in C$.
The closure of a set $X \subseteq V$ into the lattice of absolutely convex sets is called the absolute convex hull or disked hull of X and is denoted $\operatorname{cobal}(X)$.

Lemma X.50. *Let V be a vector space and $X \subseteq V$ a subset. Then the following are equivalent:*

1. *X is absolutely convex;*
2. *X is convex and balanced;*
3. *for all $x, y \in X$ and $|r| \leq 1$: $rx + (1-r)y \in X$;*
4. *for all $|a| + |b| \leq 1$, $aX + bX \subseteq X$;*
5. *for all $|a| + |b| \leq |c|$, $aX + bX \subseteq cX$.*

Lemma X.51. *Let V be a vector space and $X \subseteq V$ a subset. Then* $\operatorname{cobal}(X) = \operatorname{conv}(\operatorname{bal}(X))$.

In general $\operatorname{cobal}(X) \neq \operatorname{bal}(\operatorname{conv}(X))$.

1.5.5 Cones

A subset C of a real or complex vector space V is called a cone if for all real $r > 0$, $rC \subseteq C$. A cone is called

- pointed if it contains the origin and blunt if not;
- flat if $\exists x \neq 0 : x \in C \wedge -x \in C$, and salient if not.

The closure of a set X into the lattice of cones is called the conic hull of X and is denoted $\operatorname{cone}(X)$.

Lemma X.52. *Let V be a vector space and $X \subseteq V$ a subset. Then* $\operatorname{cone}(X) = \mathbb{R}^{>0} \cdot X$.

The closure of a set $X \subseteq V$ into the lattice of cones is given by $\mathbb{R} \cdot X$.

Lemma X.53. *A subset C of a vector space V is a cone* if and only if *$rC = C$ for all $r > 0$.*

Lemma X.54. *A cone C is convex if and only if $C + C \subseteq C$.*

Proof. Assume C convex. Take $v, w \in C$, then $v/2 + w/2 \in C$ by convexity and so $v + w = 2(v/2 + w/2) \in C$.
Assume C closed under addition. Take $v, w \in C$ and $\lambda \in [0, 1]$. Then $(1-\lambda)v$ and λw are elements of C and so the convex combination $(1-\lambda)v + \lambda w$ is too. ☐

1.5.6 Absorbing sets

Let V be a vector space and $A, B \subseteq V$. The A absorbs B if there exists a real $r > 0$ such that for all $|c| \geq r$: $B \subseteq cA$.
The set A is called absorbing if it absorbs $\{v\}$ for all $v \in V$.

Lemma X.55. *Let V be a vector space and $A \subseteq V$ a subset. Then the following are equivalent:*

1. *A is absorbing;*
2. *for all $v \in V$ there exists an $\epsilon > 0$ such that $\overline{\mathrm{B}}(0, \epsilon) \cdot v \in A$;*
3. *for all $v \in V$ there exists an $\epsilon > 0$ such that $\mathrm{B}(0, \epsilon) \cdot v \in A$.*

1.5.7 Translation invariance

TODO Unique factorisation through $(x, y) \mapsto y - x$. (Universal property)
eg kernel, commutator, metric

1.5.7.1 Quotient spaces

TODO: need closed U? For quotient map to be continuous? TODO show quotient topology.

Proposition X.56. *Let V be a vector space. Then $\mathfrak{q} \subset V \times V$ is a congruence* if and only if *the set*

$$U_{\mathfrak{q}} = \{w - v \mid (v, w) \in \mathfrak{q}\}$$

is a vector space.

Proof. Then
- $\mathfrak{q}$ is reflexive iff $0 \in U_{\mathfrak{q}}$;
- $\mathfrak{q}$ is symmetric iff $U_{\mathfrak{q}}$ is closed under multiplication with -1;
- $\mathfrak{q}$ is transitive iff $U_{\mathfrak{q}}$ is closed under addition;
- $\mathfrak{q}$ is a subalgebra of $V \oplus V$ iff $U_{\mathfrak{q}}$ is closed under addition and scalar multiplication.

As $U_{\mathfrak{q}}$ is a subset of V, we use the subspace criterion. □

Then the equivalences

$$[v]_{\mathfrak{q}} = [w]_{\mathfrak{q}} \iff (v, w) \in \mathfrak{q} \iff w - v \in U_{\mathfrak{q}} \iff w + U_{\mathfrak{q}} = v + U_{\mathfrak{q}}$$

motivate the following definition:

Let V be a vector space.

- An affine subset of V is a subset of V of the form $v + U$ for some $v \in V$ and some subspace U of V.
- An affine subset $v + U$ is parallel to U.

Suppose U subspace of V. The quotient vector space V/U is the vector space of all affine subsets of V parallel to U:

$$V/U = \{v + U \mid v \in V\},$$

which is a vector space by virtue of being a quotient algebra.
We call the dimension of V/U the codimension of U in V:

$$\operatorname{codim}(U) = \dim(V/U).$$

Proposition X.57. *Let U be a subspace of a vector space V. Then*

$$\dim V = \dim U + \dim V/U = \dim U + \operatorname{codim} U.$$

Proof. Apply the dimension theorem for linear maps to the quotient map. □

Let $f : V \to W$ be a linear map of vector spaces. The cokernel of f is the quotient space

$$\operatorname{coker}(f) = W/\operatorname{im}(f).$$

The dimension of the cokernel is called the corank.

Lemma X.58. *Let U be a subspace of a vector space V. The codimension of U is the corank of the inclusion $U \hookrightarrow V$:*

$$\operatorname{codim}(U) = \dim \operatorname{coker}(U \hookrightarrow V).$$

Proposition X.59. *Let $T \in \mathcal{L}(V, W)$. Then T induces a linear map*

$$\tilde{T} : V/\ker(T) \to W : v + \ker(T) \mapsto Tv$$

with the following properties:

1. *$\tilde{T}$ is injective;*
2. *$\operatorname{im} \tilde{T} = \operatorname{im} T$;*
3. *$\tilde{T}$ is an isomorphism from $V/\ker(T)$ to $\operatorname{im} T$.*

TODO each short exact sequence of vector spaces splits `https://en.wikipedia.org/wiki/Rank%E2%80%93nullity_theorem`

Chapter 2

Modules

TODO: $*$-modules

2.1 Representation theory

A representation $G \to \mathrm{GL}(V)$ gives V the structure of a G-module.
A map φ between two representations V, W of G is a vector space map $\varphi : V \to W$ such that

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd} \text{ commutes for every } g \in G.$$

In other symbols $\forall g \in G : g\varphi = \varphi g$. We call such a map G-linear.
Define isomorphism and isomorphic.
Eg trivial representation: $gv = v$.
A subrepresentation of a representation V is a vector subspace W of V that is invariant under G: $\forall g \in G : g[W] \subseteq W$.
$\ker\varphi$, $\operatorname{coker}\varphi$, $\mathfrak{Im}\,\varphi$ subrepresentations.
Irrep has no proper, non-trivial subrepresentation.

Lemma X.60. *Every representation has an irreduciple subrepresentation.*

Proof. If V has no non-trivial subrepresentations, then V is simple and we are done. Otherwise take the set of non-trivial subrepresentations. This forms a poset ordered by inclusion and by the maximal chain principle I.201 this poset has a maximal chain C and it is clear that $\bigcap C$ is a simple subrepresentation. In particular $\bigcap C$ is closed because it is an intersection of closed subspaces. $\square$

Representation gives representation on dual.
Direct sums and tensor products of representations. Also symmetric and exterior powers.
$\mathrm{Hom}(V, W)$ has representation via $V^* \otimes W$.
A space is irreducible if and only if it is completely reducible and indecomposable.

Proposition X.61. *Let φ be a G-linear map. If φ is invertible as a function, its inverse is also G-linear.*

Proof. Assume φ invertible and take an arbitrary $g \in G$. Then

$$g\varphi = \varphi g \implies \varphi^{-1} g = g\varphi^{-1}$$

by multiplying left and right by φ^{-1}. So

$$\forall g \in G : g\varphi^{-1} = \varphi^{-1} g$$

meaning φ^{-1} is G-linear. □

Proposition X.62. *Let (V_1, ρ_1) and (V_2, ρ_2) be isomorphic, then V_1 is irreducible* if and only if *V_2 is irreducible.*

Proposition X.63. *Let (V, ρ) be a representation of G. Every element of the centre $Z(G)$ of G defines an isomorphism $V \to V$.*

Proof. Every $g_0 \in Z(G)$ defines a G-linear map $\rho(g_0)$:

$$\forall g \in G : \rho(g_0)\rho(g) = \rho(g_0 g) = \rho(g g_0) = \rho(g)\rho(g_0).$$

The map $\rho(g_0)$ has an inverse $\rho(g_0^{-1})$. □

Proposition X.64. *Let G be an Abelian group. The irreps of G are 1-dimensional and thus homomorphisms*

$$\rho : G \to \mathrm{GL}(\mathbb{C}).$$

Proof. Let V be an irrep. The action of each $g \in G$ is an isomorphism and thus a scalar multiple by Schur's lemma. Thus every subspace of V must be invariant, so also a subrepresentation. This means V may not have any proper, non-trivial subspaces, meaning it is 1-dimensional. □

2.2 Hilbert modules

Let B be a C^*-algebra. A Hilbert B-module E is essentially a B-module with a B-valued inner product $\langle \cdot, \cdot \rangle_B : E \times E \to B$.

To be more precise: a (right) Hilbert B-module is a complex Banach space E equipped with a right B-module structure and a positive[1] definite B-valued inner product which is linear in the second and anti-linear in the first and satisfies, for all $\xi, \eta \in E$ and $b \in B$

$$(\langle \xi, \eta \rangle_B)^* = \langle \eta, \xi \rangle_B, \qquad \langle \xi, \eta \rangle_B b = \langle \xi, \eta \cdot b \rangle_B, \qquad \text{and} \qquad \|\xi\|^2 = \|\langle \xi, \xi \rangle_B\|.$$

We can also define left Hilbert B-modules analogously. The Hilbert $\mathbb{C}$-modules are precisely the complex Hilbert spaces.

Any C^*-algebra B can be seen as a Hilbert B-module by equipping it with the following inner product:

$$\langle \cdot, \cdot \rangle_B : B \times B \to B : (a, b) \mapsto a^* b.$$

If E and F are Hilbert B-modules, then a map $T : E \to F$ is called adjointable if there exists a map $T^* : F \to E$ such that for all $\xi \in E, \eta \in F : \langle T\xi, \eta \rangle_B = \langle \xi, T^*\eta \rangle_B$. Adjointable operators are bounded and B-linear.

---

[1] i.e. $\langle \xi, \xi \rangle_B$ is an element of the positive cone B^+.

Chapter 3

Algebras

TODO GL(A) which forms group under multiplication.
Multiplicative map: preserves multiplication.
Anti-commute
representation = algebra homomorphism with linear operators on a vector space.
Envelope of a representation: module to algebra.

3.1 Definition

Vector space with associative bilinear function.

3.2 Semisimple algebra

Proposition X.65. *Let V be a Hilbert space with a subspace U. Let A be a bounded linear operator on V. If U is stable under A, then $U^\perp$ is stable under A^*.*

Proof. Let $u \in U$ and $v \in U^\perp$. Then from

$$\langle u, A^* v\rangle = \langle Au, v\rangle = 0$$

we see that $A^* v \in U^\perp$. Thus $U^\perp$ is stable under A^*. □

Corollary X.65.1. *Let D be a ring of bounded linear operators on the Hilbert space V. If $D = D^*$, then V is a semisimple D-module.*

Proof. Let S be the set of direct sums of simple subrepresentations of V:

$$S = \left\{ \bigoplus_{i\in I} V_i \;\middle|\; V_i \text{ simple subrepresentations of } V \text{ and all } V_i \text{ are orthogonal} \right\}.$$

Then S is a poset ordered by inclusion. Now any chain C in S has an upper bound $\bigcup C$ and $\bigcup C$ is in S because every $v \in \bigcup C$ can be written uniquely as a finite linear sum

$$v = \sum_{\substack{i\in J\\ J \text{ finite}}} v_i \qquad v_i \in V_i.$$

By Zorn's lemma S has a maximal element U, which is closed by XIII.142. We now claim that $U = V$. Assume, towards a contradiction, that $U \neq V$. Then $U^\perp$ is stable under D by the proposition, closed by X.118 and thus contains a simple subrepresentation W by X.60. Then $U \subset U \oplus W \in S$, meaning U is not a maximal element. This is a contradiction. $\square$

Note that for the corollary it is important that V be a Hilbert space, not only for the condition $D = D^*$ which could also be fulfilled by a set of symmetric operators or a group of unitary operators.

3.3 Graded and filtered algebras

TODO; move to rings. TODO move filtration:

Let X be a set. A filtration on X is a family of subsets $\langle X_i\rangle_{i=0}^\infty$ such that $X_i \subseteq X_{i+1}$ and $X = \bigcup_{i=0}^\infty X_i$.

3.3.1 Graded algebras

Let A be an algebra and let S be a semigroup. An S-grading on A is a set $\{A_s\}_{s\in S}$ of vector subspaces of A indexed by S such that $A_sA_t \subseteq A_{st}$ and $A = \bigoplus_{s\in S} A_s$.

Proposition X.66. *Let A be an algebra over a field $\mathbb{F}$, $f : A \to A$ a diagonalisable algebra homomorphism. Then*

1. *$\sigma(f)$ is a multiplicative subsemigroup S of $\mathbb{F}$;*
2. *$A_s \coloneqq \{a \in A \mid f(a) = sa\}$ defines an S-grading on $\mathbb{F}$.*

TODO!!!

Corollary X.66.1. *Let A be an algebra and f and involutive algebra homomorphism. Then this involution defines a Z_2-grading.*

3.3.1.1 Grade operator

Let $A = \bigoplus_{i=0}^\infty A_k$ be a graded algebra. Then, for $r \in \mathbb{N}$, we call the projection $A \to A_r : a \mapsto \langle a\rangle_r$ respecting this decomposition the grade operator.

3.3.2 $\mathbb{Z}_2$-graded or superalgebras

3.3.3 Filtered algebra

A filtered algebra is an algebra F together with a filteration $\langle F_i\rangle_{i=0}^\infty$ of subspaces such that $F_i \cdot F_j \subseteq F_{i+j}$.

3.3.3.1 Associated graded algebra

Let $(F, \langle F_i \rangle_{i=0}^{\infty})$ be a filtered algebra. The associated graded algebra is defined as

$$G = \bigoplus_{i=0}^{\infty} G_i \qquad \text{where} \qquad G_i = \begin{cases} F_0 & (i = 0) \\ F_i / F_{i-1} & (i \geq 1). \end{cases}$$

TODO define the multiplication!
TODO: G is isomorphic to F as vector space, but not as an algebra!!

3.4 Tensor algebra

$$\mathcal{T}(V) := \mathbb{R} \bigoplus_{n=1}^{\infty} V^n = \mathbb{R} \bigoplus_{n=1}^{\infty} \underbrace{V \otimes \ldots \otimes V}_{n \text{ times}}.$$

Transpose: $v \otimes w \to w \otimes v$.

3.4.1 Tensor product

+ Graded tensor product

3.5 Matrix algebras

TODO

3.5.1 Natural isomorphism

Remove parentheses block matrix.
Also $A^{n \times n} \cong \mathbb{C}^{n \times} \otimes A$.

Chapter 4

Lie groups and Lie algebras

4.1 Definitions

$$g(x) = \exp(ix^a X_a)$$

Lie algebra has the operation $[X_a, X_b] = X_a X_b - X_b X_a$.

4.2 Matrix groups

We now consider some extremely important examples of topological groups: the matrix groups. If we take the set of real, $N \times N$ matrices with a non-zero determinant, it turns out that they form a group with the matrix multiplication:

1. The matrix multiplication is associative;
2. The identity is the identity matrix $\mathbb{1}$;
3. Because their determinant is not zero, every matrix in this set has an inverse.
4. Because the matrices are square, the multiplication of two matrices gives a matrix of the same dimensions. In other words the matrix multiplication is a closed operation.

We call this group the real general linear group $\mathrm{GL}(N, \mathbb{R})$. It also has a complex counterpart, the complex general linear group $\mathrm{GL}(N, \mathbb{C})$.
TODO: topological We can also immediately see that the operations of matrix multiplication and inversion are smooth. (For inversion this is obviously only true after restriction to the open subset of invertible matrices, which luckily all matrix Lie groups are in turn a subset of). This follows quite readily because both operations are in effect comprised of addition and multiplication operations, which are infinitely differentiable. (e.g $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$)

> Example
>
> TODO: $A^2 = \mathbb{1}$

These groups, along with all their subgroups, are known as the matrix groups and are very important in physics.

4.2.0.1 Continuous parameters

It is sometimes interesting to know how many degrees of freedom a particular set of transformations has. For example, rotations in the 2D plane are characterized with one parameter: the angle of rotation. In 3D we need three parameters. This notion of continuous parameter is formalised below.

A function $A : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ is called a one-parameter subgroup of $\mathrm{GL}(n, \mathbb{C})$ if

1. A is continuous,
2. $A(0) = \mathbb{1}_n$,
3. $A(t + s) = A(t)A(s)$ for all $t, s \in \mathbb{R}$.

We also call the image of A a one-parameter subgroup.

A one-parameter subgroup has one continuous parameter. A subgroup of $\mathrm{GL}(n, \mathbb{C})$ with m continuous parameters, is a function $A : \mathbb{R}^m \to \mathrm{GL}(n, \mathbb{C})$ such that each function of the form

$$x \mapsto A(a_1, a_2, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m)$$

gives a one-parameter subgroup for fixed $a_1, \ldots, a_m$.
We can speak of an m-parameter subgroup because, while different parametrisations may be found, any subgroup of $\mathrm{GL}(n, \mathbb{C})$ constructed in this way must always be constructed with the same number of parameters. To see that this must be the case, consider two parametrised subgroups $A : \mathbb{R}^m \to \mathrm{GL}(n, \mathbb{C})$ and $B : \mathbb{R}^{m'} \to \mathrm{GL}(n, \mathbb{C})$ with the same image.
TODO !! + dimension of manifold

4.2.0.2 Examples

We now give names to the most important matrix groups, and list the number of continuous parameters.

1. General linear group

$$\mathrm{GL}(N, \mathbb{R}) = \{N \times N \text{ real matrices}, \ \det M \neq 0\}$$

   - We have N^2 independent parameters (= the entries of the matrix), so $\dim \mathrm{GL}(N, \mathbb{R}) = N^2$
   - Each complex number can be described with two real ones, so $\dim \mathrm{GL}(N, \mathbb{C}) = 2N^2$

2. Special linear group

$$\mathrm{SL}(N, \mathbb{R}) = \{M \in \mathrm{GL}(N, \mathbb{R}), \ \det M = 1\}$$

   - $\dim \mathrm{SL}(N, \mathbb{R}) = N^2 - 1$: 1 dimension is used to fix determinant.
   - $\dim \mathrm{SL}(N, \mathbb{C}) = 2(N^2-1)$: 1 dimension is used to fix the real part of the determinant, and 1 to fix the imaginary part.

3. Unitary matrices

$$\mathrm{U}(N) = \{U \in \mathrm{GL}(N, \mathbb{C}), U^\dagger \mathbb{1}_N U = \mathbb{1}_N\}$$

   - $U^\dagger U$ is Hermitian, meaning that the complex transpose of U is U.

- $U^\dagger U = \mathbb{1}_N$ yields only N^2 independent equations, not $2N^2$ because of the Hermiticity of the equation.
- $\dim \mathrm{U}(N) = 2N^2 - N^2 = N^2$

4. Special unitary groups
$$\mathrm{SU}(N) = \{U \in \mathrm{U}(N),\ \det U = 1\}$$
    - For unitary matrices we have that $|\det U| = 1$. This fixes one continuous parameter and thus one dimension.
    - $\dim \mathrm{SU}(N) = N^2 - 1$
5. Orthogonal groups
    - $\mathrm{O}(N) = \{O \in \mathrm{GL}(N, \mathbb{R}),\ O^\intercal \mathbb{1}_N O = \mathbb{1}_N\}$
        - $O^\intercal O$ is symmetric, so $\frac{N(N+1)}{2}$ independent equations (half the matrix already fixed by the other half)
        - $\dim \mathrm{O} = N^2 - \frac{N(N+1)}{2} = \frac{N(N-1)}{2}$
    - $\mathrm{SO}(N) = \{O \in \mathrm{O}(N),\ \det O = 1\}$
        - For orthogonal matrices $\det O = \pm 1$. This does not fix any continuous parameters.
        - $\dim \mathrm{SO} = \dim \mathrm{O} = \frac{N(N-1)}{2}$
6. Using a non definite metric $\eta = \mathrm{diag}(\mathbb{1}_p, -\mathbb{1}_q)$
    - $\mathrm{U}(p, q) = \{U \in \mathrm{GL}(N, \mathbb{C}), U^\dagger \eta U = \eta\}$
    - $\mathrm{O}(p, q) = \{O \in \mathrm{GL}(N, \mathbb{R}), O^\intercal \eta O = \eta\}$
    In particular $\mathrm{SO}(1, 3)$ is the Lorentz group (with mostly minus convention).

Here are some of the most important examples written more explicitly in terms of their continuous parameters:

- $\mathrm{U}(1) \equiv \{z \in \mathbb{C}|\ |z| = 1\}, \cdot$ has one real parameter. Every element z of this group can be written $z = e^{i\alpha}$ for a real α.
- $\mathrm{SO}(2)$ has one real parameter.
$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$
- $\mathrm{SO}(3)$ has three real parameters.
$$R(\theta_{12}, \theta_{13}, \theta_{23}) = R_1(\theta_{12}) R_2(\theta_{13}) R_3(\theta_{23})$$
where
$$R_1(\theta_{12}) = \begin{pmatrix} \cos(\theta_{12}) & -\sin(\theta_{12}) & 0 \\ \sin(\theta_{12}) & \cos(\theta_{12}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_2(\theta_{13}) = \begin{pmatrix} \cos(\theta_{13}) & 0 & -\sin(\theta_{13}) \\ 0 & 1 & 0 \\ \sin(\theta_{13}) & 0 & \cos(\theta_{13}) \end{pmatrix}$$
$$R_3(\theta_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{23}) & -\sin(\theta_{23}) \\ 0 & \sin(\theta_{23}) & \cos(\theta_{23}) \end{pmatrix}$$

- SU(2) has three real parameters and its elements can be seen as complex 2 × 2 rotations.

$$\mathrm{U}(\alpha, \beta, \gamma) = \begin{pmatrix} \cos\theta e^{i\alpha} & -\sin\theta e^{i\beta} \\ \sin\theta e^{-i\beta} & \cos\theta e^{-i\alpha} \end{pmatrix}$$

Chapter 5

Representation theory

5.1 Finite groups

5.1.1 Character tables

5.1.1.1 For $\mathbb{Z}_n$

Denoting $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$

g_i	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\ldots$	$\overline{n-1}$
$\lvert\mathrm{Cl}\rvert$	1	1	1	$\ldots$	1
χ_0	1	1	1	$\ldots$	1
χ_1	1	ω_n	ω_n^2	$\ldots$	ω_n^{n-1}
χ_2	1	ω_n^2	ω_n^4	$\ldots$	$\omega_n^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
χ_{n-1}	1	ω_n^{n-1}	$\omega_n^{2(n-1)}$	$\ldots$	$\omega_n^{(n-1)(n-1)}$

5.1.2 Complete reducibility of complex representations

5.1.3 Schur's lemma, isotypic decomposition and duals

5.1.4 Orthogonality in the character tables

5.1.5 The sum of squares formula

5.1.6 The number of irreps is the number of conjugacy classes

5.1.7 Dimensions of irreps divide the order of the group

Chapter 6

Normed spaces and inner product spaces

In this chapter we will always use either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.
TODO: `https://math.stackexchange.com/questions/2151779/normed-vector-spaces-over-finit`
`2568231`

6.1 Normed spaces

A normed space is a vector space equipped with a norm.

Lemma X.67. *A subspace of a normed vector space is a normed space, with the norm given by the restriction of the norm in the larger space.*

A vector with norm 1 is called a unit vector. Unit vectors are often written with a hat:

$$\|\hat{\mathbf{v}}\| = 1.$$

6.1.1 The topology of a normed space

Let $(V, \|\cdot\|)$ be a normed space. The initial vector space convergence w.r.t. the norm is called the norm convergence.

The norm convergence is topological TODO ref(!). Its topology is called the norm topology.

Lemma X.68. *The norm convergence is not the initial convergence w.r.t. to the norm.*

Proof. In the initial convergence w.r.t. to the norm, all vectors of the same norm are indistinguishable, so this convergence space is not T_0.
On the other hand, $\{0\}$ is closed in $\mathbb{R}$ and thus its preimage $\{0\} \subset V$ is closed in the norm topology (TODO ref preimage closed is closed). By VIII.233, we have that the norm convergence must be Hausdorff, or T_2. $\square$

Proposition X.69. *The norm convergence is topological and metric.*
Every normed space can be viewed as a metric space with the metric $d : V \times V \to [0, \infty[$ given by

$$d(x, y) = \|x - y\|.$$

This metric has the properties of

Translation invariance $d(x + a, y + a) = d(x, y)$;

Scaling $d(\lambda x, \lambda y) = |\lambda| d(x, y)$.

Conversely, any metric with translation invariance and scaling determines a norm:

$$\|x\| = d(x, \mathbf{0}).$$

Passing from norm to metric back to norm, we recover the original norm.

Lemma X.70. *A linear map $L : V \to W$ between normed spaces is an isometry for the metric* if and only if *it preserves the norm, i.e.*

$$\forall v \in V : \quad \|v\|_V = \|L(v)\|_W .$$

Proof. Assume L is an isometry, then

$$\|v\| = d(v, \mathbf{0}) = d(L(v), L(\mathbf{0})) = \|L(v) - L(\mathbf{0})\| = \|L(v) - \mathbf{0}\| = \|L(v)\|.$$

Assume L preserves the norm, then

$$d(L(v_1), d(v_2)) = \|L(v_1) - L(v_2)\| = \|L(v_1 - v_2)\| = \|v_1 - v_2\| = d(v_1, v_2).$$

□

Proposition X.71. *Let V be a normed vector space, then the norm $\|\cdot\| : V \to \mathbb{R}$ is a continuous map.*

Proof. The reverse triangle inequality, $|\|v\| - \|w\|| \leq \|v - w\|$, implies that the norm is Lipschitz continuous with Lipschitz constant 1, so we can use VIII.223. □

6.1.1.1 Continuous operators

Theorem X.72. *Let L be a linear operator between normed spaces V, W. The following are equivalent:*

1. *L is continuous everywhere in* $\operatorname{dom}(L)$*;*
2. *L is continuous at $x_0 \in \operatorname{dom}(L)$;*
3. *L is continuous at 0;*
4. *L is Lipschitz continuous.*

Proof. We proceed round-robin-style:

$\boxed{(1) \Rightarrow (2)}$ Trivial.

$\boxed{(2) \Rightarrow (3)}$ Let $\langle x_n \rangle$ converge to 0, then

$$\lim_{n\to\infty} L(x_n) = \lim_{n\to\infty} L(x_n+x_0)-L(x_0) = L(\lim_{n\to\infty} x_n+x_0)-L(x_0) = L(x_0)-L(x_0) = 0.$$

Continuity follows because normed vector spaces are sequential spaces.

$\boxed{(3) \Rightarrow (4)}$ From continuity at zero, there exists a $\delta > 0$ such that $\|L(h)\| = \|L(h) - L(0)\| \leq 1$ for all $h \in \operatorname{dom}(L)$ with $\|h\| \leq \delta$. Thus for all nonzero $v \in \operatorname{dom}(L)$

$$\|L(v)\| = \left\| \frac{\|v\|}{\delta} L(\delta \frac{v}{\|v\|}) \right\| = \frac{\|v\|}{\delta} \left\| L(\delta \frac{v}{\|v\|}) \right\| \leq \frac{\|v\|}{\delta}.$$

$\boxed{(4) \Rightarrow (1)}$ Lipschitz continuity implies continuity VIII.223.

□

Corollary X.72.1. *A linear operator $L : V \to W$ between normed spaces is a homeomorphism* if and only if *there exists $C_1, C_2 > 0$ such that*

$$\forall x \in V : \qquad C_1\|x\| \leq \|L(x)\| \leq C_2\|x\|.$$

An operator L between normed vector spaces is called bounded if it is (Lipschitz) continuous.
The set of bounded operators from V to W is denoted $\mathcal{B}(V, W)$. If $V = W$, we write $\mathcal{B}(V)$.

In other words, bounded means there exists an $M > 0$ such that $\forall v \in \operatorname{dom}(L)$

$$\|L(v)\| \leq M\|v\|.$$

Proposition X.73. *An anti-linear map between complex vector spaces is continuous* if and only if *it is bounded.*

Proof. An anti-linear map $A : V \to W$ is an $\mathbb{R}$-linear map $A : V_\mathbb{R} \to W_\mathbb{R}$. Now $V_\mathbb{R}, W_\mathbb{R}$ have the same norms as V, W and thus the same topology. So $A : V \to W$ is continuous if and only if $A : V_\mathbb{R} \to W_\mathbb{R}$ is continuous. □

6.1.1.2 Equicontinuity

Proposition X.74. *Let $\mathcal{F}$ be a set of linear operators in $(V \to W)$. Then $\mathcal{F}$ is equicontinuous at $x_0 \in V$* if and only if *there exists $C > 0$ such that*

$$\forall f \in \mathcal{F} : \qquad \|f(x_0)\| \leq C\|x_0\|.$$

6.1.1.3 Comparison of norm topologies

Let V be a vector space and $\|\cdot\|_1$, $\|\cdot\|_2$ two norms on V. We say

- $\|\cdot\|_1$ is bounded by $\|\cdot\|_2$ if there exists $C \in \mathbb{R}$ such that $\forall v \in V : \|v\|_1 \leq C\|v\|_2$;
- $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if each is bounded by the other.

Lemma X.75. *Let V be a vector space and $\|\cdot\|_1$, $\|\cdot\|_2$ two norms on V. These norms are equivalent* if and only if *there exists a $C > 0$ such that*

$$\frac{1}{C}\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1.$$

Proposition X.76. *Let V be a vector space and $\|\cdot\|_1$, $\|\cdot\|_2$ two norms on V. Then the following are equivalent:*

1. *$\|\cdot\|_2$ is bounded by $\|\cdot\|_1$;*
2. *$\mathrm{id}_V : (V, \|\cdot\|_1) \to (V, \|\cdot\|_2)$ is continuous;*
3. *the norm topology of $\|\cdot\|_1$ is finer than the norm topology of $\|\cdot\|_2$.*

Proof. (1) $\Leftrightarrow$ (2) By X.72 both are equivalent to

$$\exists C > 0 : \forall v \in V : \qquad \|v\|_2 = \|\mathrm{id}_V((v))\|_2 \leq C\|v\|_1.$$

(2) $\Leftrightarrow$ (3) Follows straight from VIII.47. ☐

Corollary X.76.1. *Equivalent norms induce the same topology.*

6.1.1.4 Infinite linear combinations

Because normed product spaces are metric spaces, we have a notion of convergence and can define infinite sums:

In a normed space V, we can define a infinite linear combination as an infinite sum

$$\sum_{i \in I} c_i v_i$$

where $\{v_i\}_{i \in I}$ is a set of vectors and $\{c_i\}_{iI}$ a set of scalars, if that sum converges in the norm topology.

This finite sum is defined using nets: Ordered by inclusion, the set $J = \{I' \subset I \mid I' \text{ is finite}\}$ is a directed set. This means

$$\left(\sum_{i \in A} c_i v_i\right)_{A \in J}$$

is a net. The infinite sum is defined if this net converges.

Lemma X.77. *Every proper subspace U of a normed vector space V has empty interior.*

Proof. Suppose U has a non-empty interior. Then it contains some ball $B(u, \epsilon)$. Now every vector in V can be translated and rescaled to fit inside the ball $B(u, \epsilon)$. Indeed let $v \in V$ and set $u' = u + \frac{\epsilon}{2\|v\|}v \in B(u, \epsilon)$. Then, since U is a subspace $v = \frac{2\|v\|}{\epsilon}(u' - u) \in U$. So $U = V$. ☐

Lemma X.78 (Riesz's lemma)**.** *Let V be a normed vector space. Given a non-dense subspace X and a number $\theta < 1$, there exists a unit vector $v \in V$ such that*

$$\theta \leq d(X, v) = \inf_{x \in X} \|x - v\|.$$

Proof. Take a vector v_1 not in the closure of X and put $a = \inf_{x\in X}\|x - v_1\|$. Then $a > 0$ by lemma VIII.199. For $\epsilon > 0$, let $x_1 \in X$ be such that $\|x_1 + v_1\| < a + \epsilon$. Then take

$$v = \frac{v_1 - x_1}{\|v_1 - x_1\|} \qquad \text{so} \qquad \|v\| = 1.$$

And

$$\inf_{x\in X}\|x - v\| = \inf_{x\in X}\left\|x - \frac{v_1 - x_1}{\|v_1 - x_1\|}\right\| = \inf_{x\in X}\left\|\frac{x - v_1 + x_1}{\|v_1 - x_1\|}\right\| = \frac{\inf_{x\in X}\|x - v_1\|}{\|v_1 - x_1\|} \geq \frac{a}{a+\epsilon}.$$

By choosing $\epsilon > 0$ small, $a/(a+\epsilon)$ can be made arbitrarily close to 1. □

For finite-dimensional spaces we can even take $\theta = 1$.

6.1.2 Linear independence and bases in normed spaces

`https://math.stackexchange.com/questions/1518029/are-uncountable-schauder-like-bases-`

6.1.3 Finite-dimensional normed (sub)spaces

Lemma X.79. *Let V be a normed vector space and $\{x_1, \ldots, x_n\}$ a linearly independent set of vectors. There exists a $c > 0$ such that $\forall \alpha_1, \ldots, \alpha_n \in \mathbb{F}$:*

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \geq c(|\alpha_1| + \ldots + |\alpha_n|).$$

Proof. TODO ref locally convex spaces? Local compactness? □

TODO This is equivalent with continuity of coordinate functions.

Proposition X.80. *Every finite-dimensional subspace of a normed vector space is complete.*

Proof. Take a basis $\{e_i\}_{i=1}^n$ and let c be as in lemma X.79. Consider an arbitrary Cauchy sequence $(v_k)_{k\in\mathbb{N}}$. We can write

$$v_k = \alpha_{k,1}e_1 + \ldots + \alpha_{k,n}e_n.$$

We claim that $(\alpha_{k,i})_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$ for all $1 \leq i \leq n$. Indeed, take an $\epsilon > 0$. By the Cauchy nature of $(v_k)_{k\in\mathbb{N}}$ we can find a k_0 such that $\forall k', k'' > k_0$:

$$c\epsilon > \|v_{k'} - v_{k''}\| \geq \left\|\sum_{i=1}^n (\alpha_{k',i} - \alpha_{k'',i})e_i\right\| \geq c\sum_{i=1}^n |\alpha_{k',i} - \alpha_{k'',i}| \geq c|\alpha_{k',i} - \alpha_{k'',i}|.$$

Dividing left and right by c gives exactly the Cauchy condition for each $1 \leq i \leq n$. By the completeness of $\mathbb{R}$ or $\mathbb{C}$, each of these sequences has a limit α_i. Then $v = \sum_{i=1}^n \alpha_i e_i$ is an element of the subspace. The sequence (v_k) converges to v because

$$\|v_k - v\| = \left\|\sum_{i=1}^n (\alpha_{k,i} - \alpha_i)e_i\right\| \leq \sum_{i=1}^n |\alpha_{k,i} - \alpha_i|\|e_i\|$$

and the right-hand side goes to zero as $k \to \infty$. □

Corollary X.80.1. *Every finite-dimensional subspace of a normed vector space is closed.*

TODO ref for proof.

Proposition X.81. *On a finite-dimensional vector space all norms are equivalent.*

Proof. Let $\{e_i\}_{i=1}^n$ be a basis and take an arbitrary vector $v = \sum_{i=1}^n v_i e_i$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms. We calculate

$$\|v\|_1 \leq \sum_{i=1}^n |v_i| \|e_i\|_1 \leq k \sum_{i=1}^n |v_i| \leq \frac{k}{c_2} \|v\|_2$$

where the first inequality is the triangle inequality, the second comes from $k = \max \|e_i\|_1$ and the third is lemma X.79. A similar calculation gives the other necessary inequality. ☐

Proposition X.82. *In a finite-dimensional normed space V, any subset $M \subseteq V$ is compact if and only if M is closed and bounded.*

Proof. TODO + ref Heine Borel property ☐

TODO: move up?

Proposition X.83. *The closed unit ball of a vector space is compact* if and only if *the vector space is finite-dimensional.*

Proof. One direction is given by the previous proposition. For the other direction, we show the contrapositive: let the vector space be infinite-dimensional. We define a sequence of unit vectors $(e_i)_{i \in \mathbb{N}}$ recursively as follows:

- e_1 is just a unit vector;
- for e_{n+1} apply Riesz's lemma X.78 to the subspace $\operatorname{span}\{e_i\}_{i=1}^n$ and $\theta = 1/2$. This subspace cannot be dense, because it is a closed (by corollary X.80.1) finite-dimensional subspace of an infinite-dimensional vector space.

This yields a sequence such that for all m, n

$$\|e_m - e_n\| \geq \frac{1}{2}.$$

This sequence is not Cauchy and thus not convergent. ☐

6.1.4 Norms on constructed vector spaces

6.1.4.1 Direct sum

$$\|x \oplus y\|_{X \oplus Y} = \|x\|_X + \|y\|_Y$$

TODO + arbitrary direct sums.

6.1.4.2 The graph norm

Let $L : V \to W$ be a linear map between normed spaces. The graph of L

$$\{(v, w) \in V \oplus W \mid w = Lv\}$$

has a natural norm inherited from the direct sum:

$$\|(v, Lv)\| = \|v\|_V + \|Lv\|_W.$$

This norm can also be seen as a norm on V: the graph norm induced by L is defined as

$$\|v\|_L := \|v\|_V + \|Lv\|_W.$$

6.2 Operators on normed spaces

6.2.1 Bounded operators

Proposition X.84. *Let V, W be normed spaces. Then $T : V \to W$ is bounded* if and only if *$T^{-1}[B(\mathbf{0}, 1)]$ has nonempty interior.*

Proof. TODO! ☐

Lemma X.85. *Let T be a bounded linear operator. Then $\ker(T)$ is closed.*

Proof. Suppose T bounded and thus continuous. Then $\ker L = L^{-1}[\{0\}]$ and thus closed, by proposition VIII.138. ☐

Proof. Let $v \in \overline{\ker(T)}$. Then find a sequence (v_n) in $\ker(T)$ that converges to v. Then by continuity (Tv_n) converges to Tv, but for all $n \in \mathbb{N} : Tx_n = 0$, so the limit is $Tv = 0$. Thus $v \in \ker(T)$, making it closed. ☐

Proposition X.86. *Let $L : V \to W$ be a linear map between normed spaces.*

1. *If V is finite-dimensional, then L is continuous.*
2. *If W is finite-dimensional, then L is continuous* if and only if *$\ker L$ is closed.*

TODO: true for general TVS

Proof.

1. This follows from a consideration of the graph norm $\|v\|_L = \|v\| + \|Lv\|$ and the fact that on a finite-dimensional space any two norms are equivalent: for all v we can choose an M such that
$$\|Lv\| \leq \|v\|_L \leq M\|v\|.$$
2. Assume W finite-dimensional. Consider the map $\bar{L} : V/\ker L \to W : v + \ker L \mapsto L(v)$, defined in proposition X.59. Then $V/\ker L$ is isomorphic to a subspace of W and thus is finite-dimensional. By the first point, $\bar{L}$ must be continuous. Let $\pi : V \to V/\ker L$ denote the quotient map, which is continuous (TODO is this where closure of $\ker L$ is used?). Then $L = \bar{L} \circ \pi$ is a composition of continuous maps and thus continuous.

   Conversely, we have the lemma X.85.

☐

> Example
>
> Let $\langle e_n \rangle$ be a basis of unit vectors of an infinite dimensional real vector space. Then consider the map $e_n \mapsto n$ and extend by linearity. This is an unbounded linear operator with finite dimensional codomain.

6.2.1.1 The normed algebra of bounded operators

Lemma X.87. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces and $L \in \mathcal{L}(V, W)$. Then L is bounded* if and only if
$$\sup\left\{ \frac{\|Lx\|_W}{\|x\|_V} \;\middle|\; x \in V \setminus \{0\} \right\}$$
exists.

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces and $L \in \mathcal{L}(V, W)$ bounded. Then

$$\|L\| \coloneqq \sup\left\{ \frac{\|Lx\|_W}{\|x\|_V} \;\middle|\; x \in V \setminus \{0\} \right\}$$

is called the operator norm of L.

Proposition X.88. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces. Then the set $\mathcal{B}(V, W)$ of bounded linear maps is a normed subspace of $\mathcal{L}(V, W)$ equipped with the operator norm.*

Lemma X.89. *Let S, T be compatible bounded operators. Then*

$$\|ST\| \leq \|S\|\|T\|.$$

Proof. $\|ST\| = \sup\left\{ \frac{\|STx\|}{\|x\|} \;\middle|\; \|x\| = 1 \right\} \leq \sup\left\{ \frac{\|S\|\|Tx\|}{\|x\|} \;\middle|\; \|x\| = 1 \right\} \leq \|S\| \, \|T\|$. ☐

Proposition X.90. *Let $L \in \mathcal{B}(V, W)$ be a bounded operator and let $B(\mathbf{0}, \epsilon)$ be an open ball centered at $\mathbf{0}$. Then*

$$\begin{aligned}\|L\| &= \frac{\sup L[B(\mathbf{0}, \epsilon)]}{\epsilon} \\ &= \frac{\sup L[\overline{B}(\mathbf{0}, \epsilon)]}{\epsilon} \\ &= \sup\left\{ \|Lx\| \mid \|x\| = 1 \right\}.\end{aligned}$$

Proof. TODO ☐

6.2.1.2 Operators bounded below

TODO: also unbounded operators!

Let T be a bounded linear operator. We say T is bounded below if

$$\exists b > 0 : \forall v \in \mathrm{dom}(T) : \quad \|Tv\| \geq b\|v\|$$

Proposition X.91. *Let $T \in \mathcal{L}(V, W)$ be an operator. Then T has a bounded inverse T^{-1} :* $\mathrm{im}(T) \to V$ if and only if T *is bounded below by some constant b.*
In this case

$$\left\|T^{-1}\right\| = \left(\inf_{x \neq 0} \frac{\|Tx\|}{\|x\|} \right)^{-1} \leq \frac{1}{b}.$$

Proof. First assume T bounded below. To show T is injective, take $x_1, x_2 \in \mathrm{dom}\, T$ such that $Tx_1 = Tx_2$. Then

$$0 = \|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \geq b\|x_1 - x_2\| \geq 0.$$

So $\|x_1 - x_2\| = 0$ and thus $x_1 = x_2$. The existence of T^{-1} is then clear. For boundedness notice that $T^{-1}y \in \mathrm{dom}(T)$, so because T is bounded below,

$$b\left\|T^{-1}y\right\| \leq \left\|TT^{-1}y\right\| = \|y\| \quad \Longrightarrow \quad \left\|T^{-1}y\right\| \leq \frac{1}{b}\|y\|.$$

This also shows that $\|T^{-1}\| \leq 1/b$ for all lower bounds b. In other words $1/\|T^{-1}\| \geq \inf_{x\neq 0}\|Tx\|/\|x\|$.
Now assume T^{-1} bounded. Then for all $x \in \operatorname{dom}(T)$: $\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\|\|Tx\|$, so T is bounded below by $1/\|T^{-1}\|$.
This also shows that $1/\|T^{-1}\|$ is a lower bound, so $1/\|T^{-1}\| \leq \inf_{x\neq 0}\|Tx\|/\|x\|$. □

6.2.2 Closed operators

> Let $T : \operatorname{dom}(T) \subseteq X \to Y$ be an operator. Then T is a closed operator if $\operatorname{graph}(T)$ is closed in $X \oplus Y$.

This is not the same as a closed map in the topological sense!
The most important property of closed operators is given by the following proposition. It is sometimes taken as the definition.

Proposition X.92. *Let X, Y be normed spaces and $T : \operatorname{dom}(T) \subset X \to Y$ be a linear operator. Then the following are equivalent:*

1. *T is a closed operator;*
2. *if $(x_n)_{n\in\mathbb{N}} \subset \operatorname{dom}(T)$ converges to $x \in X$ and $(Tx_n)_{n\in\mathbb{N}}$ converges to y, then $x \in \operatorname{dom}(T)$ and $Tx = y$;*
3. *$\operatorname{dom}(T)$ is complete w.r.t. the graph norm.*

TODO: (? If domain is closed?) I.e. does this work outside the realm of Banach operators??

Corollary X.92.1. *All bounded operators have closed graph. (? If domain is closed?)*

The converse is not true in general.
`https://en.wikipedia.org/wiki/Unbounded_operator#Closed_linear_operators`
`https://en.wikipedia.org/wiki/Closed_graph_theorem_(functional_analysis)`

Proposition X.93. *Let T be a closed and S a bounded operator, then*

1. *$S + T$ is closed;*
2. *TS is closed;*
3. *if T is injective, then $T^{-1} : \operatorname{im}(T) \to \operatorname{dom}(T)$ is closed.*

Proof. (1) TODO
(2) TODO
(3) We use X.92. Take $\langle y_n\rangle \subset \operatorname{dom}(T^{-1})$ such that $y_n \to y$ and $T^{-1}y_n \to x$. Set $x_n = T^{-1}y_n$, so then $Tx_n = TT^{-1}y_n = y_n \to y$. Because T is closed it follows that $Tx = y$, so $T^{-1}y = x$, meaning T^{-1} is closed. □

TODO example ST need not be closed.

Lemma X.94. *Let T be a closed operator, then $\ker(T)$ is closed.*

Proof. Let $\langle x_n\rangle \subset \ker(T)$ be a convergent sequence. Then $\langle Tx_n\rangle$ is identically zero and thus converges to 0. By closedness of T, $Tx = 0$ and thus $x \in \ker(T)$. □

We have already proven this for bounded operators, see X.85.

Proposition X.95. *Let $T \in \mathcal{L}(X,Y)$ be a closed operator between Banach spaces that is bounded below. Then* $\operatorname{im}(T)$ *is closed.*

Proof. Let T be bounded below by b and let $\langle Tx_n \rangle$ be a Cauchy sequence in $\operatorname{im}(T)$. Then $\|x_m - x_n\| \leq \frac{1}{b}\|T(x_m - x_n)\|$, so $\langle x_n \rangle$ is also Cauchy by VIII.216.
So we can find $x \in X, y \in Y$ such that $x_n \to x$ and $Tx_n \to y$. By closedness of T, we have $Tx = y$ and thus $y \in \operatorname{im}(T)$. □

6.2.2.1 Closable operators

A linear operator is called closable if it has closed extension.

Proposition X.96. *A linear operator T is closable* if and only if *for all sequences* $\langle x_n \rangle \subset \operatorname{dom}(T)$

$$(x_n \to 0 \wedge T(x_n) \to v) \qquad \Longrightarrow \qquad v = 0.$$

Proof. TODO □

Lemma X.97. *A closable operator T has a minimal closed extension $\overline{T}$, which is given by the closure of the graph of T.*

Proof. TODO □

6.2.3 Compact operators

A linear map $L : V \to W$ between normed spaces is called compact if $L[\overline{B}(\mathbf{0},1)]$ is relatively compact.
i.e. the image of the closed unit ball has compact closure.
The space of compact maps from V to W is denoted $\mathcal{K}(V,W)$.

These operators were introduced to study equations of the form

$$(T - \lambda I)x(t) = p(t).$$

Proposition X.98. *Let $L \in \mathcal{L}(V,W)$. The following are equivalent:*

1. *L is compact;*
2. *the image of any bounded subset of V is relatively compact in W;*
3. *there exists a neighbourhood U of 0 in V such that the image of U is a subset of a compact set in W;*
4. *for any bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq V$, then sequence $(Lx_n)_{n\in\mathbb{N}}$ contains a converging subsequence.*

Proof. TODO □

Corollary X.98.1. *All maps of finite rank are compact.*

Proof. Closed balls in $\mathbb{C}^n$ are compact. □

Proposition X.99. *Let V be a normed space. Then $\mathcal{K}(V)$ is a closed two-sided ideal in $\mathcal{B}(V)$.*

Lemma X.100. *The identity map on X is compact* if and only if *X is finite-dimensional.*

Proof. The unit ball is compact iff X is finite-dimensional, by X.83. ☐

Corollary X.100.1. *Let $T \in \mathcal{K}(X, Y)$. If T is injective and T^{-1} bounded, then X is finite-dimensional.*

Proof. In this case $\mathrm{id}_X = T^{-1}T$ is compact by TODO ref. ☐

6.3 Inner product spaces

An inner product on a vector space V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

that has the following properties:

Linearity in the second[a] component

$$\langle v, \lambda_1 w_1 + \lambda_2 w_2 \rangle = \lambda_1 \langle v, w_1 \rangle + \lambda_2 \langle v, w_2 \rangle ,$$

where $\lambda_1, \lambda_2 \in \mathbb{F}$ and $v, w_1, w_2 \in V$.

Conjugate symmetry[b] $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$.

Positivity[c] $\langle v, v \rangle \geq 0$ for all $v \in V$.

Definiteness $\langle v, v \rangle = 0$ if and only if $v = 0$.

An inner product space or pre-Hilbert space $(\mathbb{F}, V, +, \langle \cdot, \cdot \rangle)$ is a vector space $(\mathbb{F}, V, +)$ together with an inner product $\langle \cdot, \cdot \rangle$ on V.
A real finite-dimensional inner product space is called a Euclidean space.

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[a] Some authors take linearity in the first component.
[b] This is for $\mathbb{F} = \mathbb{C}$. For $\mathbb{F} = \mathbb{R}$ this reduces to normal symmetry $\langle v, w \rangle = \langle w, v \rangle$.
[c] By conjugate symmetry we know that $\langle v, v \rangle$ is a real number, so this condition makes sense.

Lemma X.101. *An inner product over a complex vector space V is anti-linear in the first component.*

Lemma X.102. *Definiteness implies the inner product on V is non-degenerate:*

$$[\forall u \in V : \langle u, v \rangle = 0] \implies v = 0.$$

The converse is not true.
There are some generalised notions of inner product:

Let V be a complex vector space.

1. A sesquilinear form is a function $V \times V \to \mathbb{C}$ that is linear in the second component and anti-linear in the first.
2. A Hermitian form is a conjugate symmetric sesquilinear form.

3. A pre-inner product is a positive Hermitian form, i.e. an inner product without the requirement of definiteness.

Example

1. The standard inner product on $\mathbb{R}^n$ is given by

$$\langle a, b\rangle = \left\langle \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\rangle = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a^{\mathrm{T}} b$$

   This is also known as the dot product $a \cdot b$.

2. The standard inner product on $\mathbb{C}^n$ is given by

$$\langle a, b\rangle = \left\langle \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\rangle = \begin{bmatrix} \bar{a}_1 & \dots & \bar{a}_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \bar{a}^{\mathrm{T}} b$$

3. The Frobenius inner product on $\mathbb{C}^{m\times n}$ is given by

$$\langle A, B\rangle_F = \operatorname{Tr}(\overline{A}^{\mathrm{T}} B) = \overline{\operatorname{vec}_C(A)}^{\mathrm{T}} \operatorname{vec}_C(B)$$

4. On the vector space $\mathcal{C}[a, b]$ of continuous real functions on $[a, b]$, we can take the inner product

$$\langle f, g\rangle = \int_a^b f(x) \cdot g(x) \, \mathrm{d}x.$$

Two vectors $u, v \in V$ are orthogonal if $\langle u, v\rangle = 0$. This is denoted $u \perp v$.

Lemma X.103. *Let V be an inner product space.*

1. *0 is the only vector orthogonal to itself.*
2. *0 is orthogonal to all $v \in V$;*
3. *Let $x, y \in V$. If, for all $v \in V$, $\langle v, x\rangle = \langle v, y\rangle$, then $x = y$.*

Proof. The first is a consequence of definiteness, the second a consequence of linearity: $\langle v, 0\rangle = \langle v, 0 \cdot 0\rangle = 0 \langle v, 0\rangle = 0$.
The third is also a consequence of linearity: assume $\forall v \in V : \langle v, x\rangle = \langle v, y\rangle$, then $\langle v, x - y\rangle = 0$ and $x - y$ is orthogonal to all $v \in V$ and in particular to 0. Thus $x - y$ must be zero. ☐

Proposition X.104. *Every inner product gives rise to a norm, defined by*

$$\|\cdot\| = \sqrt{\langle \cdot, \cdot\rangle}.$$

Proof. The only non-trivial part is the triangle inequality. This will be proved later using the Cauchy-Schwarz inequality. ☐

Lemma X.105. *Let V be an inner product space. Then*

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2 + 2\,\mathfrak{Re}\,\langle v, w\rangle$$

Lemma X.106. *Let $v, w \in V$, with $w \neq 0$. We can decompose v as a multiple of w and a vector u orthogonal to w:*

$$v = cw + u = \left(\frac{\langle v, w\rangle}{\|w\|^2}\right) w + \left(v - \frac{\langle v, w\rangle\, w}{\|w\|^2}\right).$$

Proof. The only thing to check is $\left\langle w, v - \frac{\langle v,w\rangle w}{\|w\|^2}\right\rangle = 0$, which is a simple calculation. ☐

6.3.1 Pythagoras and Cauchy-Schwarz

Theorem X.107 (Pythagorean theorem)**.** *Suppose $u \perp v$. Then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.*

Proof.

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle = \|u\|^2 + \|v\|^2.$$

☐

Theorem X.108 (Cauchy-Schwarz-Bunyakovsky inequality.)**.** *Let V be a vector space with a pre-inner product $\langle\cdot,\cdot\rangle$. Let $v, w \in V$. Then*

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle \cdot \langle w, w\rangle .$$

Suppose $\langle\cdot,\cdot\rangle$ is definite (i.e. an inner product), then this is an equality if and only if *v and w are scalar multiples.*

This result is also known as the Cauchy-Schwarz inequality, or the CSB inequality.

Proof. Consider

$$\langle v - \lambda w, v - \lambda w\rangle = \langle v, v\rangle - \lambda\langle v, w\rangle - \overline{\lambda}\langle w, v\rangle + |\lambda|^2\langle w, w\rangle \geq 0.$$

Suppose $\langle v, w\rangle = re^{i\theta}$ (if $\mathbb{F} = \mathbb{R}$, then $\theta = 0$ or $\theta = \pi$). The inequality must still hold for all λ of the form $te^{-i\theta}$ for some $t \in \mathbb{R}$. The inequality thus becomes

$$0 \leq \langle v, v\rangle - te^{-i\theta}re^{i\theta} - te^{i\theta}re^{-i\theta} + t^2\langle w, w\rangle = \langle v, v\rangle - 2rt + t^2\langle w, w\rangle .$$

On the right we have a quadratic formula in t. This may never be negative and the discriminant may therefore not be positive. Calculating the discriminant gives $(2r)^2 - 4\langle v, v\rangle\langle w, w\rangle$. Thus

$$0 \geq r^2 - \langle v, v\rangle\langle w, w\rangle = |\langle v, w\rangle|^2 - \langle v, v\rangle\langle w, w\rangle .$$

☐

In the case of an inner product, there is a simpler proof:

Proof. Take the decomposition from lemma X.106 and apply the Pythagorean theorem to obtain

$$\|v\|^2 = \frac{|\langle v, w\rangle|^2}{\|w\|^2} + \|u\|^2 \geq \frac{|\langle v, w\rangle|^2}{\|w\|^2}.$$

This also shows the claim about scalar multiples. ☐

Corollary X.108.1. *Let V be an inner product space. The functions*

$$\langle v, \cdot \rangle : V \to \mathbb{F} : x \mapsto \langle v, x \rangle$$

are bounded linear functionals for all $v \in V$.

Corollary X.108.2. *Let V be a vector space with a pre-inner product $\langle \cdot, \cdot \rangle$. Then*

$$\langle x, x \rangle = 0 \vee \langle y, y \rangle = 0 \qquad \implies \qquad \langle x, y \rangle = 0.$$

The Cauchy-Schwarz inequality allows us to define the angle θ between two vectors v, w by

$$\cos\theta = \frac{\langle v, w \rangle}{\|v\|\|w\|}.$$

Lemma X.109. *If $v \perp w$, then the angle between them is $\pi/2 + k\pi$.*

TODO CS special case of Hölder inequality.

Theorem X.110 (Triangle inequality)**.** *Let $v, w \in V$. Then*

$$\|v + w\| \leq \|v\| + \|w\|$$

This inequality is an equality if and only if one of u, v is a nonnegative multiple of the other. Also

1. $\big|\|v\| - \|w\|\big| \leq \|v - w\|$;
2. $\big|\|v\| - \|w\|\big| \leq \|v + w\| \leq \|v\| + \|w\|$.

Proof. We calculate

$$\begin{aligned} \|v + w\|^2 &= \|v\|^2 + \|w\|^2 + 2\,\Re\mathrm{e}\,\langle v, w \rangle \\ &\leq \|v\|^2 + \|w\|^2 + 2|\langle v, w \rangle| \\ &\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| \\ &= (\|v\| + \|w\|)^2. \end{aligned}$$

The other inequalities are the reverse triangle inequalities XIII.15. □

6.3.2 Parallelogram law and polarisation

Theorem X.111 (Parallelogram law)**.** *Let V be an inner product space and $v, w \in V$. Then*

$$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2).$$

Proof. We calculate

$$\|v + w\|^2 + \|v - w\|^2 = \langle v + w, v + w \rangle + \langle v - w, v - w \rangle = 2(\|v\|^2 + \|w\|^2).$$

□

Corollary X.111.1 (Appolonius' identity)**.** *Let V be an inner product space and $x, y, z \in V$. Then*

$$\|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\|z - \frac{1}{2}(x + y)\|^2.$$

Proof. Apply the parallelogram law to $u = \frac{1}{2}(z - x)$ and $v = \frac{1}{2}(z - y)$. ☐

Proposition X.112 (Ptolemy's inequality)**.** *Let* V *be an inner product space. Then the norm satisfies* $\forall u, v, w \in V$

$$\|u - v\| \, \|w\| + \|v - w\| \, \|u\| \geq \|u - w\| \, \|v\|.$$

Polarisation identities allow us to recover the inner product from the norm.

Theorem X.113 (Polarisation identities)**.**

1. *For real inner product spaces,* $\mathbb{F} = \mathbb{R}$*:*

$$\begin{aligned}\langle v, w\rangle &= \frac{1}{2}(\|v + w\|^2 - \|v\|^2 - \|w\|^2) \\ &= \frac{1}{2}(\|v\|^2 + \|w\|^2 - \|v - w\|^2) \\ &= \frac{1}{4}(\|v + w\|^2 - \|v - w\|^2) = \frac{1}{4}\sum_{k=0}^{1}(-1)^k \left\|v + (-1)^k w\right\|^2.\end{aligned}$$

2. *For complex inner product spaces,* $\mathbb{F} = \mathbb{C}$*:*

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \left\|i^k x + y\right\|^2.$$

3. *For general sesquilinear forms:*

$$S(x, y) = \frac{1}{4}\sum_{k=0}^{3} i^k S(i^k x + y, i^k x + y).$$

Corollary X.113.1. *A sesquilinear form* S *is Hermitian* if and only if $S(v, v)$ *is real for all* $v \in V$*.*

Proof. The direction $\Rightarrow$ is obvious (conjugate symmetry gives $S(v, v) = \overline{S(v, v)}$). For the other direction, consider

$$\begin{cases} S(u + iv, u + iv) = S(u, u) + S(v, v) + i\Big(S(u, v) - S(v, u)\Big) \\ S(u + v, u + v) = S(u, u) + S(v, v) + \Big(S(u, v) + S(v, u)\Big). \end{cases}$$

Taking the imaginary part gives

$$\begin{cases} \mathfrak{Im}\, S(u + iv, u + iv) = 0 = \mathfrak{Re}\left(S(u, v) - S(v, u)\right) \\ \mathfrak{Im}\, S(u + v, u + v) = 0 = \mathfrak{Im}\left(S(u, v) + S(v, u)\right). \end{cases}$$

☐

Alternative proof. We can also prove the direction $\Leftarrow$ by direct calculation:

$$\begin{aligned}
\overline{S(u,v)} &= \frac{1}{4}\sum_{k=0}^{3}(-i)^k S(u+i^k v, u+i^k v) && \text{Using the fact that } S(u+i^k v, u+i^k v) \text{ is real} \\
&= \frac{1}{4}\sum_{k=0}^{3}(-i)^k S\Big((i^k)(v+(-i)^k u), (i^k)(v+(-i)^k u)\Big) && \text{Using } i^k(-i)^k = 1 \\
&= \frac{1}{4}\sum_{k=0}^{3}(-i)^k \cancel{(i^k)}\cancel{(-i^k)} S(v+(-i)^k u, v+(-i)^k u) && \text{Using (conjugate) linearity} \\
&= \frac{1}{4}\sum_{k=0}^{3} i^k S(v+i^k u, v+i^k u) && \text{Substituting } k \to k+2 \\
&= S(v,u).
\end{aligned}$$

□

Not all norms on vector spaces can be obtained from an inner product. If a norm can be obtained from an inner product, we can use polarisation to recover the inner product. If a norm cannot be obtained from an inner product, the putative inner product suggested by polarisation will turn out not to be an inner product.

Theorem X.114 (Jordan-von Neumann)**.** *Let* $(V, \|\cdot\|)$ *be a real or complex normed space. The following are equivalent:*

1. *the polarisation yields an inner product;*
2. *the parallelogram law holds;*
3. *Appolonius' identity holds;*
4. *Ptolemy's inequality holds.*

TODO! (For other fields??)

Proof. If polarisation yields an inner product, then we have an inner product space and thus the parallelogram law and Ptolemy's inequality hold by X.111 and X.112.
The polarisation identities immediately imply

- Conjugate symmetry:

$$\langle x,y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \big\|i^k x + y\big\|^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \big\|x + i^{-k} y\big\|^2 = \frac{1}{4}\sum_{k'=0}^{3} \overline{i^{k'}}\Big\|i^{k'} y + x\Big\|^2 = \overline{\langle y,x\rangle}.$$

- Positivity and definiteness:

$$\langle x,x\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \big\|i^k x + x\big\|^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \big\|(i^k+1)x\big\|^2 = \frac{\|x\|^2}{4}\sum_{k=0}^{3} i^k \cdot |1+i^k|^2 = \|x\|^2$$

Now assume Appolonius' identity, X.111.1, holds. We need to show linearity in second component. We can calculate

$$\begin{aligned}
\langle x, y_1\rangle + \langle x, y_2\rangle &= \frac{1}{4}\sum_{k=0}^{3} i^k\left\|i^k x + y_1\right\|^2 + \frac{1}{4}\sum_{k=0}^{3} i^k\left\|i^k x + y_2\right\|^2 = \frac{1}{4}\sum_{k=0}^{3} i^k\Big(\left\|i^k x + y_1\right\|^2\frac{1}{4} + \left\|i^k x + y_2\right\|^2\Big) \\
&= \frac{1}{4}\sum_{k=0}^{3} i^k\left(\frac{1}{2}\|y_1 - y_2\|^2 + 2\left\|i^k + \frac{y_1 + y_2}{2}\right\|\right) \\
&= 2\frac{1}{4}\sum_{k=0}^{3} i^k\left(\left\|i^k + \frac{y_1 + y_2}{2}\right\|\right) \\
&= 2\left\langle x, \frac{y_1 + y_2}{2}\right\rangle.
\end{aligned}$$

Setting $y_2 = 0$ and $y_1 = 2y$, this yields $\langle x, 2y\rangle = 2\langle x, y\rangle$, which also means that

$$\langle x, y_1 + y_2\rangle = 2\left\langle x, \frac{y_1 + y_2}{2}\right\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle.$$

By induction, we can prove that this putative inner product is linear for all positive rational scalars. By continuity this result extends to all positive scalars.
Finally we check

$$\begin{aligned}
\langle x, -y\rangle &= \frac{1}{4}\sum_{k=0}^{3} i^k\left\|i^k x - y\right\|^2 = \frac{1}{4}\sum_{k=0}^{3} i^k\left\|i^{k-2} x + y\right\|^2 = -\frac{1}{4}\sum_{k'=0}^{3} i^{k'}\left\|i^{k'} x + y\right\|^2 = -\langle x, y\rangle \\
\langle x, iy\rangle &= \frac{1}{4}\sum_{k=0}^{3} i^k\left\|i^k x + iy\right\|^2 = \frac{1}{4}\sum_{k=0}^{3} i^k\left\|i^{k-1} x + y\right\|^2 = i\frac{1}{4}\sum_{k'=0}^{3} i^{k'}\left\|i^{k'} x + y\right\|^2 = i\langle x, y\rangle.
\end{aligned}$$

TODO Ptolemy inequality. □

Corollary X.114.1. *The space l^p is an inner product space* if and only if $p = 2$.

Proof. The inner product on l^2 is defined by $\langle x_n, y_n\rangle = \sum_{n=1}^{\infty}\overline{x_n}y_n$.
If $p \neq 2$ we can find a counterexample to the parallelogram law: let $x = (1, 1, 0, 0, \ldots) \in l^p$ and $y = (1, -1, 0, 0, \ldots) \in l^p$. Then

$$\|x\|_p = \|y\|_p = 2^{1/p} \qquad \text{and} \qquad \|x + y\| = \|x - y\| = 2$$

and the parallelogram law is then not valid if $p \neq 2$. □

6.4 Orthogonal and orthonormal sets of vectors

6.4.1 Orthogonal complements

Let A be a subset of an inner product space V. The orthogonal complement $A^\perp$ of A is the set of vectors in V that are orthogonal to every vector in A:

$$A^\perp = \{v \in V \mid \langle v, a\rangle = 0\ \forall a \in A\}.$$

Proposition X.115. *Let A, B be subsets of an inner product space V.*

1. *$A^\perp$ is a subspace of V;*
2. *$A^\perp = \operatorname{span}(A)^\perp$;*
3. *$\{0\}^\perp = V$;*
4. *$V^\perp = \{0\}$;*
5. *$A \cap A^\perp \subset \{0\}$;*
6. *If $A \subset B$, then $B^\perp \subset A^\perp$.*

We can also consider the orthogonal complement of a subspace with respect to another subspace, not the full space.

> Let $A \subseteq B$ be subsets of an inner product space V. The orthogonal complement of A with respect to B is the set of vectors in B that are orthogonal to every vector in A:
>
> $$B \ominus A = \{b \in B \mid \langle b, a\rangle = 0 \; \forall a \in A\}.$$

Lemma X.116. *Let V be an inner product space and $A \subseteq B \subseteq V$ subsets. Then $B \ominus A = B \cap A^\perp$.*

Proof. Take $v \in B \ominus A$. This is equivalent to $v \in B \wedge \forall u \in A : \langle v, u\rangle = 0$ and thus equivalent to $v \in B \wedge v \in A^\perp$. $\square$

Proposition X.117. *Let V be an inner product space, let $A \subseteq B \subseteq V$ be subsets and $T : V \to V$ an isometry. Then*

1. *if $A \perp B$, then $T[A] \perp T[B]$;*
2. *$T[B \ominus A] = T[B] \ominus T[A]$.*

Proof. (1) If $\langle a, b\rangle = 0$ for all $a \in A, b \in B$, then $\langle T(a), T(b)\rangle = 0$, meaning $T[A] \perp T[B]$.
(2) Take $v \in T[B \ominus A]$. This is equivalent to the existence of $x \in B$ such that $T(x) = v$ and $\langle x, y\rangle = 0$ for all $y \in A$. By isometry $\langle x, y\rangle = 0 \iff \langle T(x), T(y)\rangle = 0$ for all $y \in A$. So, equivalently, $v \in T[B] \ominus T[A]$. $\square$

Proposition X.118. *Let A be a subset of an inner product space V. Then $A^\perp$ is closed and $\overline{A}^\perp = A^\perp$. This can be rephrased as*

$$\overline{A}^\perp = \overline{A^\perp} = A^\perp.$$

Also

$$\overline{A} \subset (A^\perp)^\perp.$$

Proof. Let $x \in \overline{A^\perp}$. Then there exists a sequence (x_i) in $A^\perp$ that converges to x. For all $a \in A$, the functional $\langle a, \cdot\rangle : y \mapsto \langle a, y\rangle$ is bounded (by Cauchy-Schwarz). Thus all these functionals are continuous. Applying any one to the sequence x_i gives a sequence of zeros. Thus $\langle a, x\rangle = 0$ for all $a \in A$. Thus $x \in A^\perp$ and hence $A^\perp \supset \overline{A^\perp}$ meaning $A^\perp$ is closed.
Now $\overline{A} \supset A$, so $\overline{A}^\perp \subset A^\perp$. For the other inclusion, take an $x \in A^\perp$. Take an arbitrary $y \in \overline{A}$. Then there exists a sequence (y_i) in A that converges to y. Apply the bounded functional $\langle x, \cdot\rangle$ to the sequence (y_i), yielding a sequence of zeros. Thus $\langle x, y\rangle = 0$. Thus $x \in \overline{U}^\perp$.
Finally let $a \in \overline{A}$. Take a sequence $a_i \to a$. Take an arbitrary element $x \in A^\perp$. As before $\langle x, a\rangle = \lim_i \langle x, a_i\rangle = 0$. So $a \in (A^\perp)^\perp$. $\square$

Corollary X.118.1. *Let A be a subset of V. If* $\operatorname{span}(A)$ *is dense in V, then $A^{\perp} = \{0\}$.*

Proof.

$$A^{\perp} = \operatorname{span}(A)^{\perp} = \overline{\operatorname{span}(A)}^{\perp} = V^{\perp} = \{0\}.$$

□

Corollary X.118.2. *Let $x \in V$. If there exists a dense set S such that $x \perp y$ for all $y \in S$, then $x = 0$.*

Proof. If such an S exists, then $x \in S^{\perp} = \overline{S}^{\perp} = V^{\perp} = \{0\}$. □

Proposition X.119. *Let U be a finite-dimensional subspace of an inner product space V.*

1. *$V = U \oplus U^{\perp}$;*
2. *$U = (U^{\perp})^{\perp}$.*

Notice that V may be infinite dimensional!

Proof. We start with the first point. The sum $U+U^{\perp}$ is definitely direct, $U \oplus U^{\perp}$, by proposition X.115 and the criterion for a direct sum, proposition X.16. Clearly $U \oplus U^{\perp} \subseteq V$, so we just need to show that $V \subseteq U \oplus U^{\perp}$.
To that end, take a vector $v \in V$. Let $\{e_i\}_{i=1}^{n}$ be an orthonormal basis of U. We can write

$$v = \left(v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i \right) + \left(\sum_{i=1}^{n} \langle v, e_i \rangle e_i \right).$$

The first part is an element of $U^{\perp}$, the second of U, so $v \in U \oplus U^{\perp}$.
For the second point: any finite-dimensional subspace U is automatically closed, so $U = \overline{U} \subset (U^{\perp})^{\perp}$, by proposition X.118. For the other inclusion, take $v \in (U^{\perp})^{\perp}$. By the first point, we can write $v = v_1 + v_2$ where $v_1 \in U$ and $v_2 \in U^{\perp}$. Because $v \in (U^{\perp})^{\perp}$ and $v_2 \in U^{\perp}$, we must have

$$0 = \langle v_2, v \rangle = \langle v_2, v_1 + v_2 \rangle = \langle v_2, v_1 \rangle + \langle v_2, v_2 \rangle = \|v_2\|.$$

So $v = v_1 \in U$. □

TODO all projection results for projection onto finite dim? See proposition before Bessel inequality. In fact better: projection onto summand of direct sum! Put under decompositions.

Proposition X.120. *Let W_1, W_2 be subspaces of an inner product space V. Then*

$$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}.$$

Proof. For a vector $v \in V$,

$$v \in (W_1 + W_2)^{\perp} \implies \forall x \in W_1 \cup W_2 : \langle v, x \rangle = 0 \implies v \in W_1^{\perp} \cap W_2^{\perp}$$

and

$$\begin{aligned} v \in W_1^{\perp} \cap W_2^{\perp} &\implies \forall x \in W_1, y \in W_2 : \langle v, x \rangle = 0 = \langle v, y \rangle \\ &\implies \forall x \in W_1, y \in W_2 : \langle v, x + y \rangle = 0 \implies v \in (W_1 + W_2)^{\perp}. \end{aligned}$$

□

A result dual to proposition X.120 holds for finite-dimensional spaces:

Proposition X.121. *Let W_1, W_2 be subspaces a finite-dimensional space V. Then*

$$(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp.$$

Proof. We start by applying proposition X.120 to $W_1^\perp$ and $W_2^\perp$:

$$(W_1^\perp + W_2^\perp)^\perp = (W_1^\perp)^\perp \cap (W_2^\perp)^\perp = W_1 \cap W_2.$$

Taking the orthogonal complement of both sides gives the result. In infinite dimensions $(W_1^\perp + W_2^\perp)$ is not necessarily closed. □

6.4.2 Orthogonal sets and sequences

- A set of vectors D is called orthogonal if for any two vectors $v, w \in D$, $v \perp w$ if and only if $v \neq w$.
- A set of vectors D is called orthonormal if for any two vectors $v, w \in D$,

$$\langle v, w \rangle = \begin{cases} 0 & (v \neq w) \\ 1 & (v = w) \end{cases}.$$

In particular an orthonormal set is an orthogonal set of unit vectors.

Lemma X.122. *Every orthogonal set of vectors is linearly independent.*

Lemma X.123. *Every subset of an orthogonal (resp. orthonormal) set is orthogonal (resp. orthonormal).*

Theorem X.124 (Gram-Schmidt procedure)**.** *Every finite set of linearly independent vectors $D = \{v_1, \ldots, v_n\}$ can be transformed into an orthonormal set $D' = \{e_1, \ldots, e_n\}$ with the same number of vectors such that the spans are the same:* $\operatorname{span}(D') = \operatorname{span}(D)$.

Proof. The procedure goes as follows:

$$\begin{aligned} e_1 &= \frac{v_1}{\|v_1\|} \\ e_2 &= \frac{v_2 - \langle e_1, v_2 \rangle e_1}{\|v_2 - \langle e_1, v_2 \rangle e_1\|} \\ &\ldots \\ e_j &= \frac{v_j - \langle e_1, v_j \rangle e_1 - \ldots - \langle e_{j-1}, v_j \rangle e_{j-1}}{\|v_2 - \langle e_1, v_2 \rangle e_1 - \ldots - \langle e_{j-1}, v_j \rangle e_{j-1}\|} \\ &\ldots \end{aligned}$$

□

If we only need an orthogonal set $\{y_1, \ldots, y_n\}$, not an orthonormal one, we can use the procedure

$$y_{k+1} = v_{k+1} - \sum_{i=1}^{k} \frac{\langle v_{k+1}, y_i \rangle}{\langle y_i, y_i \rangle} y_i.$$

Lemma X.125. *Let* $(\mathbb{F}, V, +, \langle\cdot,\cdot\rangle)$ *be an inner product space. Then*

$$\langle v, w\rangle = 0 \qquad \iff \qquad \forall a \in \mathbb{F} : \ \|v\| \leq \|v + aw\|.$$

Proof. The implication $\Rightarrow$ is a consequence of the Pythagorean theorem. For the other implication, assume $\forall a \in \mathbb{F} : \ \|v\| \leq \|v + aw\|$. Then

$$\|v\|^2 \leq \|v - aw\|^2 = \|v\|^2 - 2\,\Re\mathrm{e}\,\langle v, aw\rangle + \|aw\|^2$$

which implies $2\,\Re\mathrm{e}\,\langle v, aw\rangle \leq a^2\|w\|^2$. Let $\langle v, w\rangle = re^{i\theta}$. (If $\mathbb{F} = \mathbb{R}$, then $\theta = 0$.) Then in particular the inequality holds for all $a = te^{i\theta}$ with $t \in \mathbb{R}$. This yields

$$2\,\Re\mathrm{e}(te^{-i\theta}re^{i\theta}) \leq t^2\|w\|^2 \qquad \text{or} \qquad 2rt \leq t^2\|w\|^2.$$

Letting $t \geq 0$, we can divide out a t: $2r \leq t\|w\|^2$. Then letting $t \to 0$ gives $r = 0$ and thus $\langle v, w\rangle = 0$. ☐

Proposition X.126. *Let V be an inner product space and $D = \{e_1, \ldots, e_n\}$ a finite orthonormal set of vectors. Then $\forall v \in V$*

$$\inf_{c_i \in \mathbb{F}} \left\| v - \sum_{i=1}^n c_i e_i \right\| = \left\| v - \sum_{i=1}^n \langle e_i, v\rangle\, e_i \right\|$$

Proof. We calculate

$$\begin{aligned}
\left\| v - \sum_{i=1}^n c_i e_i \right\|^2 &= \left\langle v - \sum_{i=1}^n c_i e_i, v - \sum_{j=1}^n c_j e_j \right\rangle \\
&= \|v\| - \sum_{j=1}^n c_j \langle v, e_j\rangle - \sum_{i=1}^n \bar{c}_i \langle e_i, v\rangle + \sum_{i,j=1}^n \bar{c}_i c_j \langle e_i, e_j\rangle \\
&= \|v\| - 2\,\Re\mathrm{e}\left(\sum_{i=1}^n c_i \overline{\langle e_i, v\rangle}\right) + \sum_{i=1}^n |c_i|^2 \\
&= \sum_{i=1}^n \left(|c_i|^2 - 2\,\Re\mathrm{e}\left(\sum_{i=1}^n c_i \overline{\langle e_i, v\rangle}\right) + |\langle e_i, v\rangle|^2\right) + \|v\| - \sum_{i=1}^n |\langle e_i; v\rangle|^2 \\
&= \sum_{i=1}^n |c_i - \langle e_i, v\rangle|^2 + \|v\| - \sum_{i=1}^n |\langle e_i, v\rangle|^2.
\end{aligned}$$

This is clearly minimised when $c_i = \langle e_i, v\rangle$. ☐

Corollary X.126.1. *Let $v \in \mathrm{span}(D)$, then $v = \sum_{i=1}^n \langle e_i, v\rangle\, e_i$.*

We call the numbers $\langle e_i, v\rangle$ the Fourier coefficients of v w.r.t. D.

Proof. In this case $\inf_{c_i \in \mathbb{F}} \|v - \sum_{i=1}^n c_i e_i\| = 0$. ☐

Corollary X.126.2 (Bessel inequality)**.** *Let $\{e_i\}_{i\in I}$ be an orthonormal family and $v \in V$, then*

$$\sum_{i\in I} |\langle e_i, v\rangle|^2 = \sup\left\{ \sum_{\substack{i \in I' \subset I \\ I' \text{ finite}}} |\langle e_i, v\rangle|^2 \right\} \leq \|v\|^2.$$

Proof. In the previous proof,

$$0 \leq \left\| v - \sum_{i=1}^{n} c_i e_i \right\|^2 = \sum_{i=1}^{n} |c_i - \langle e_i, v \rangle|^2 + \|v\| - \sum_{i=1}^{n} |\langle e_i, v \rangle|^2 = \|v\| - \sum_{i=1}^{n} |\langle e_i, v \rangle|^2.$$

Where we have set $c_i = \langle e_i, v \rangle$. Thus the supremum must also be $\leq \|v\|$. ☐

Corollary X.126.3. *For any $v \in V$, $\langle e_i, v \rangle = 0$ except for countably many $i \in I$.*

Proof. Ref TODO. `https://proofwiki.org/wiki/Uncountable_Sum_as_Series`. ☐

TODO: link with metric topology being sequential?

Corollary X.126.4 (Riemann-Lebesgue lemma)**.** *For any sequence $\langle e_i \rangle_{i \in J \subset I}$, we have*

$$\lim_{i \in J} \langle e_i, v \rangle = 0.$$

Corollary X.126.5. *We can also obtain the Cauchy-Schwarz inequality from the Bessel inequality.*

Proof. Let $x, y \in V$. Then $\{x/\|x\|\}$ is an orthonormal set. Applying the Bessel inequality for y gives $\|y\|^2 \geq |\langle x/\|x\|, y \rangle|^2 \implies |\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2 \implies |\langle x, y \rangle| \leq \|x\| \, \|y\|$. ☐

6.4.3 Orthonormal bases

Let D be an orthonormal set of vectors in an inner product space V, then D is said to be

1. maximal, if it is a maximal element in the set of orthonormal sets ordered by inclusion;
2. total, if the smallest closed subspace that includes D is V (i.e. span(D) is dense in V);
3. an orthonormal basis (o.n. basis) or a Hilbert basis if any vector in V can be written as a (possibly infinite) linear combination of elements of D.

Hilbert bases are in general not Hamel bases. e.g take $\mathbb{R}^{\mathbb{N}}$. Then

$$\begin{gathered}(1,0,0,\ldots),\\ (0,1,0,\ldots),\\ (0,0,1,\ldots),\\ \cdots\end{gathered}$$

is an orthonormal basis, but not a Hamel basis (consider $(1,1,1,\ldots)$).

Proposition X.127.
- *Every vector space has a maximal orthonormal set.*
- *Every orthonormal set can be extended to a maximal orthonormal set.*

Proof. The first statement follows easily from the second. The second statement is proved using Zorn's lemma. Let S be an orthonormal set. Define

$$\mathcal{A} = \{D \subset V \mid S \subset D \text{ and } D \text{ is orthonormal}\}$$

ordered by inclusion. It is easy to see that any chain on $\mathcal{A}$ has an upper bound on $\mathcal{A}$, by just taking the union which is still orthonormal. It follows from Zorn's lemma that $\mathcal{A}$ has a maximal element R. This is by definition an orthonormal basis.
In the finite-dimensional case this can also be proved using the Gram-Schmidt procedure. □

Proposition X.128. *If V is finite-dimensional, then the notions of maximal orthonormal set, total orthonormal set and orthonormal set coincide. Such an orthonormal set is also a (Hamel) basis of V.*

Proof. Corollaries of Gram-Schmidt. □

Lemma X.129. *Let V be an inner product space and D an orthonormal set. Then*

1. *D is maximal* if and only if *$D^\perp = \{0\}$;*
2. *if D is an orthonormal basis, then D is maximal.*

Proof. (1) All possible vectors with which to extend D are elements of $D^\perp$.
(2) Assume D an o.n. basis. Then $D^\perp = (\operatorname{span}(D))^\perp = V^\perp = \{0\}$, using X.115 and X.118.1. □

There are maximal orthonormal families that are not bases.

Example

Consider the space $l^2(\mathbb{N})$ and take the subspace X generated by the family of elements

$$\left(\sum_{n=1}^{\infty} n^{-1}e_n, e_2, e_3, e_4, \ldots\right)$$

with the inner product induced by the inner product of l^2. In this space $F = \{e_2, e_3, \ldots\}$ is orthonormal and maximal, but not an orthonormal basis.

Maximal orthonormal families are easy to construct, but do not have the nice properties of orthonormal bases (see below). We would really like the concepts of orthonormal basis and maximal orthonormal family to coincide. We will see they coincide exactly in Hilbert spaces (see XIII.140).

Proposition X.130. *Let V be an inner product space and $D = \{e_i\}_{i\in I}$ an orthonormal set. The following are equivalent:*

1. *D is an orthonormal basis of V;*
2. *D is total in V;*
3. *for all $v, w \in V$,*
$$\langle v, w\rangle = \sum_{i\in I} \langle v, e_i\rangle \langle e_i, w\rangle\,;$$

4. (Parseval's identity) *for all* $v \in V$*,*

$$\|v\|^2 = \sum_{i \in I} |\langle e_i, v\rangle|^2;$$

5. for all $v \in V$*: if* $v \perp D$*, then* $v = 0$*;*

6. (Plancherel formula) *for all* $v \in V$*,*

$$v = \sum_{i \in I} \langle e_i, v\rangle\, e_i.$$

Proof. We proceed round-robin-style.

$\boxed{(1) \Rightarrow (2)}$ Assume D an o.n. basis. Then there exists a net of partial sums converging to any element $v \in V$. Each of these partial sums is a finite linear combination of elements in D and thus this net is a net in $\operatorname{span}(D)$. This means $v \in \overline{\operatorname{span}(D)}$.

$\boxed{(2) \Rightarrow (3)}$ Fix $v, w \in V$. Because V is a metric spaces and thus sequential, we can find sequences $(v_j)_{j\in J}$ and $(w_k)_{k \in K}$ in $\operatorname{span}(D)$ converging to v and w. Now the linear maps $u \mapsto \overline{\langle u, e_i\rangle}$ and $u \mapsto \langle e_i, u\rangle$ are bounded by Cauchy-Schwarz and thus continuous by theorem X.72 (TODO corollary CSB). Then we can calculate, using the fact that each v_j and w_k is a finite linear combination of e_i,

$$\begin{aligned}
\langle v, w\rangle &= \left\langle \lim_j v_j, \lim_k w_k \right\rangle = \lim_j \lim_k \langle v_j, w_k\rangle \\
&= \lim_j \lim_k \left\langle \sum_{i=1}^{N_j} \langle e_i, v_j\rangle\, e_i, \sum_{i'=1}^{N_k} \langle e_{i'}, w_k\rangle\, e_{i'} \right\rangle \\
&= \lim_j \lim_k \sum_{i=1}^{N_j}\sum_{i'=1}^{N_k} \langle v_j, e_i\rangle \langle e_{i'}, w_k\rangle \langle e_i, e_{i'}\rangle = \lim_j \lim_k \sum_{i=1}^{N_j}\sum_{i'=1}^{N_k} \langle v_j, e_i\rangle \langle e_{i'}, w_k\rangle\, \delta_{i,i'} \\
&= \lim_j \lim_k \sum_{i=1}^{\min\{N_j, N_k\}} \langle v_j, e_i\rangle \langle e_i, w_k\rangle \\
&= \lim_j \lim_k \sum_{i\in I} \langle v_j, e_i\rangle \langle e_i, w_k\rangle \\
&= \sum_{i \in I} \lim_j \lim_k \langle v_j, e_i\rangle \langle e_i, w_k\rangle \\
&= \sum_{i \in I} \langle v, e_i\rangle \langle e_i, w\rangle\,.
\end{aligned}$$

For the interchange of the limits and the summation in the penultimate equality we can use Tannery's theorem, XI.30. Indeed $|\langle e_i, w_k\rangle|$ is bounded by $\|w_k\|$ by the Bessel inequality. By the continuity of the norm we have $\lim_k \|w_k\| = \|w\|$, so the sequence $\|w_k\|$ is bounded.

$\boxed{(3) \Rightarrow (4)}$ Set $v = w$.

$\boxed{(4) \Rightarrow (5)}$ If $v \perp D$, then

$$\|v\|^2 = \sum_{i \in I} |\langle e_i, v\rangle|^2 = 0 \qquad \text{which implies } v = 0.$$

$\boxed{(5) \Rightarrow (6)}$ The vector $v - \sum_{i \in I} \langle e_i, v\rangle e_i$ is perpendicular to D:

$$\forall e_j \in D : \quad \left\langle e_j, v - \sum_{i \in I} \langle e_i, v\rangle e_i \right\rangle = \langle e_j, v\rangle - \sum_{i \in I} \langle e_i, v\rangle \langle e_j, e_i\rangle = \langle e_j, v\rangle - \langle e_j, v\rangle = 0.$$

So $v - \sum_{i \in I} \langle e_i, v\rangle e_i = 0$ and the Plancherel formula holds.

$\boxed{(6) \Rightarrow (1)}$ By definition of o.n. basis.

□

Lemma X.131. *Let V be an inner product space. If D is an orthonormal basis of V, then it is also an orthonormal basis of $\overline{V}$, the completion of V.*

Proof. Let D be an o.n. basis. By X.130 span(D) is dense in V, meaning it is also dense in $\overline{V}$, by VIII.162. Thus D is total in $\overline{V}$ and an o.n. basis by X.130. □

6.4.3.1 Cardinality and separable inner product spaces

`https://arxiv.org/pdf/1606.03869.pdf`

> An inner product space is separable if it is separable as a metric space, i.e. it admits a countable dense subset.

Proposition X.132. *Given a vector space V, any two maximal orthonormal sets have the same cardinality.*

Proof. Take $D = \{e_i\}_{i \in I}$ and $D' = \{f_j\}_{j \in J}$ maximal orthonormal sets. □

Proposition X.133. *An inner product space is separable* if and only if *it admits an orthonormal basis with at most countably many vectors.*

Proof. TODO infinite-dimensional analog of the Gram-Schmidt process □

Corollary X.133.1. *Any separable inner product space has an orthonormal basis.*

Proposition X.134. *Not every inner product space has an orthonormal basis.*

Proof. `https://en.wikipedia.org/wiki/Inner_product_space#Orthonormal_sequences` `https://groups.google.com/g/sci.math.research/c/1SA_3h1whQo?pli=1` `https://www.angelfire.com/journal/mathematics/innerproduct.pdf` `https://arxiv.org/pdf/1009.1441.pdf` □

6.5 Maps on inner product spaces

Lemma X.135 (Continuity of inner product)**.** *Let V be an inner product space. Then the inner product is a continuous function $V \times V \to \mathbb{F}$.*

Proof. We show that if $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$. By the triangle and Cauchy-Schwarz inequalities

$$\begin{aligned} |\,\langle x_n, y_n\rangle - \langle x, y\rangle\,| &= |\,\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle\,| \\ &\leq |\,\langle x_n, y_n - y\rangle\,| + |\,\langle x_n - x, y\rangle\,| \\ &\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\|. \end{aligned}$$

Because the right-hand side converges to 0, the left-hand side must too. ☐

Lemma X.136. *Let V be an inner product space and $S, T \in$* Hom(V)*. Then $S = T$* if and only if

$$\forall v, w \in V : \langle Tv, w\rangle = \langle Sv, w\rangle\,.$$

Proof. The direction $\boxed{\Rightarrow}$ is obvious. For the other direction, use

$$0 = \langle Tv, w\rangle - \langle Sv, w\rangle = \langle (T - S)v, w\rangle$$

for all v, w. In particular set w equal to $(T - S)v$. Then $\|(T - S)v\| = 0$ for all $v \in V$. By the definiteness of the norm we have $(T - S)v = 0$, meaning $Tv = Sv$. ☐

6.5.1 Bounded operators

Lemma X.137. *Let $T \in \mathcal{B}(V, W)$, then*

$$\begin{aligned} \|T\| &= \sup_{w\in\operatorname{im}(T), v\in\operatorname{dom}(T)} \frac{|\,\langle w, Tv\rangle\,|}{\|w\|\,\|v\|} \\ &= \sup\{|\,\langle w, Tv\rangle\,| \mid w \in \operatorname{im}(T) \;\wedge\; v \in \operatorname{dom} T \;\wedge\; \|w\| = 1 = \|v\|\} \\ &= \sup_{w\in W, v\in\operatorname{dom}(T)} \frac{|\,\langle w, Tv\rangle\,|}{\|w\|\,\|v\|} \\ &= \sup\{|\,\langle w, Tv\rangle\,| \mid w \in W \;\wedge\; v \in \operatorname{dom} T \;\wedge\; \|w\| = 1 = \|v\|\}\,. \end{aligned}$$

Proof. We prove

$$\|T\| \leq \sup_{w\in\operatorname{im}(T), v\in\operatorname{dom}(T)} \frac{|\,\langle w, Tv\rangle\,|}{\|w\|\,\|v\|} \leq \sup_{w\in W, v\in\operatorname{dom}(T)} \frac{|\,\langle w, Tv\rangle\,|}{\|w\|\,\|v\|} \leq \|T\|.$$

The first two inequalities follow from the characterisation X.90

$$\|T\| = \sup_{v\in\operatorname{dom}(T)} \frac{\|Tv\|}{\|v\|} = \sup_{v\in\operatorname{dom}(T)} \frac{\langle Tv, Tv\rangle}{\|Tv\|\,\|v\|}$$

and the inclusions

$$\begin{aligned} \left\{ \frac{|\,\langle w, Tv\rangle\,|}{\|w\|\,\|v\|} \;\middle|\; v \in \operatorname{dom}(T), w = Tv \right\} &\subseteq \left\{ \frac{|\,\langle w, Tv\rangle\,|}{\|w\|\,\|v\|} \;\middle|\; v \in \operatorname{dom}(T), w \in \operatorname{im}(T) \right\} \\ &\subseteq \left\{ \frac{|\,\langle w, Tv\rangle\,|}{\|w\|\,\|v\|} \;\middle|\; v \in \operatorname{dom}(T), w \in V \right\}. \end{aligned}$$

The last equality follows from the Cauchy-Schwarz inequality X.108:

$$\frac{|\left\langle w, Tv\right\rangle|}{\|w\|\,\|v\|} \leq \frac{\|w\|\,\|Tv\|}{\|w\|\,\|v\|} = \frac{\|Tv\|}{\|v\|} \leq \frac{\|T\|\,\|v\|}{\|v\|} = \|T\|$$

for all $v \in \mathrm{dom}(T), w \in V$. □

6.5.2 Isometries

Lemma X.138. *Let V, W be inner product spaces. Let $f : V \to W$ be a function. Then f preserves the metric (i.e. is an isometry)* if and only if *f also preserves the inner product:*

$$\forall x, y \in V : \quad \left\langle f(x), f(y)\right\rangle_W = \left\langle x, y\right\rangle_V .$$

The proof is a simple application of the polarisation identities.

Let V, W be an inner product spaces. A linear map $U \in \mathrm{Hom}(V, W)$ is called unitary if it is an isometry and invertible.
Unitary operators on real vector spaces are also called orthogonal operators.

Because every isometry is injective (see lemma VIII.211), it is enough for a linear map to be isometric and surjective to be unitary.

Lemma X.139. *Every unitary map is bounded and has norm* 1.

Proof. Let $U : V \to W$ be a unitary map between inner product spaces. Then $\forall v \in V : \|U(v)\| = \|v\|$. □

Unitary operators transform orthonormal bases to orthonormal bases:

Proposition X.140. *Let $T \in \mathrm{Hom}(V, W)$ with V, W inner product spaces and let V have an orthonormal basis $\{e_i\}_{i\in I}$. Then T is unitary* if and only if *$\{Te_i\}_{i\in I}$ is an orthonormal basis of W.*

Proof. Assume T unitary. The family $\{Te_i\}_{i\in I}$ is certainly orthonormal, by preservation of the inner product. Now let $w \in W$ and so $T^{-1}w \in V$. By the Plancherel formula, proposition X.130, we can write

$$T^{-1}w = \sum_{n=1}^{\infty} \left\langle e_{i_n}, T^{-1}w\right\rangle e_{i_n} = \lim_{N\to\infty} \sum_{n=1}^{N} \left\langle e_{i_n}, T^{-1}w\right\rangle e_{i_n}$$

and so

$$w = TT^{-1}w = T \lim_{N\to\infty} \sum_{n=1}^{N} \left\langle e_{i_n}, T^{-1}w\right\rangle e_{i_n} = \lim_{N\to\infty} \sum_{n=1}^{N} \left\langle e_{i_n} T^{-1}w\right\rangle Te_{i_n}$$

because T is bounded and thus continuous, by theorem X.72. Thus $\{Te_i\}_{i\in I}$ is an orthonormal basis of W.
Conversely, assume $\{Te_i\}_{i\in I}$ is an orthonormal basis of W. We first prove T is bounded, which is a simple application of Parseval's identity, proposition X.130:

$$\|Tv\|^2 = \sum_{i\in I} |\left\langle Te_i, Tv\right\rangle|^2 = \sum_{i\in I} |\left\langle e_i, v\right\rangle|^2 = \|v\|^2.$$

The rest of the proof is again an application of the Plancherel formula. □

Lemma X.141. *Let U be a unitary map. If λ is an eigenvalue of U, then $|\lambda| = 1$.*

Proof. Let v be an eigenvector associated to the eigenvalue λ. Then

$$\langle v, v\rangle = \langle L(v), L(v)\rangle = \langle \lambda v, \lambda v\rangle = \lambda^2 \langle v, v\rangle ,$$

so $\lambda^2 = 1$. □

6.5.3 Symmetric operators

Let $(\mathbb{F}, V, +, \langle\cdot,\cdot\rangle)$ be an inner product space. A linear operator L is called symmetric if, $\forall v, w \in \text{dom}(L)$

$$\langle L(v), w\rangle = \langle v, L(w)\rangle .$$

Proposition X.142. *Let V be an inner product space and L a symmetric operator on V. Then eigenvectors of L associated to different eigenvalues are orthogonal.*

Proof. Let v, w be eigenvectors of L with eigenvalues λ, μ such that $\lambda \neq \mu$. Then

$$\lambda \langle v, w\rangle = \langle \lambda v, w\rangle = \langle L(v), w\rangle = \langle v, L(w)\rangle = \langle v, \mu w\rangle = \mu \langle v, w\rangle$$

and consequently $\langle v, w\rangle = 0$. □

6.5.4 Impact on subspaces

6.5.4.1 Invariant and reducing subspaces

Let V be an inner product space and T a linear operator on V.

- A subspace $U \subseteq V$ is said to be invariant under T if $T[U] \subset U$.
- A subspace $U \subseteq V$ is said to be reducing for T if both U and $U^\perp$ are invariant under T.

6.6 Energy forms

Let T be an operator on an inner product space V. The energy form of T is the map

$$\langle\cdot,\cdot\rangle_T : \text{dom}(T) \times \text{dom}(T) \to \mathbb{F} : (x, y) \mapsto \langle x, Ty\rangle .$$

We also define the associated quadratic form

$$Q_T : \text{dom}(T) \to \mathbb{F} : x \mapsto \langle x, x\rangle_T = \langle x, Tx\rangle .$$

Energy forms are clearly sesquilinear.

Lemma X.143. *Let T be an invertible operator. Then*

$$Q_{T^{-1}}(x) = \overline{Q_T(T^{-1}(x))}.$$

Proof. For all $x \in V$

$$Q_{T^{-1}}(x) = \langle x, T^{-1}x \rangle = \langle TT^{-1}x, T^{-1}x \rangle = \overline{\langle T^{-1}x, T(T^{-1}x) \rangle} = \overline{Q_T(T^{-1}(x))}.$$

☐

Lemma X.144. *Two operators $T_1, T_2 \in \mathcal{L}(V)$ have the same energy form* if and only if $T_1 = T_2$.

Proof. This is a consequence of X.103. ☐

Lemma X.145. *The energy form of an operator T is Hermitian* if and only if *T is symmetric.*

Proof. For all x, y: $\langle x, Ty \rangle = \overline{\langle y, Tx \rangle}$ iff $\langle x, Ty \rangle = \langle Tx, y \rangle$. ☐

Corollary X.145.1. *If T is symmetric, then Q_T is real-valued.*

Proof. Assume T symmetric, then for all $u \in \operatorname{dom}(T)$

$$Q_T(u) = \langle u, Tu \rangle = \langle Tu, u \rangle = \overline{\langle u, Tu \rangle} = \overline{Q(u)}.$$

☐

So the energy form associated to a symmetric operator is Hermitian. We typically would like our energy forms to be pre-inner products. This is exactly the case for positive operators.

6.6.1 Positive operators

Let T be an operator on an inner product space V. Then T is called positive if the associated energy form is positive: for all $x \in V$

$$Q_T(x) = \langle x, x \rangle_T = \langle x, Tx \rangle \geq 0.$$

We write $A \geq 0$. We also say

- A is strictly positive, denoted $A > 0$, if $Q_T(u) > 0$;
- A is negative if $-A$ is positive;
- A is positive definite, strongly positive or coercive if there exists a constant $k > 0$ such that
$$Q_T(x) \geq k\|x\|^2 > 0.$$

Lemma X.146. *If T is a positive operator on a complex inner product space, then T is symmetric.*

Proof. If T is a positive operator on a complex inner product space V, then $Q_T(x)$ is in particular real for all $x \in V$. This is equivalent to $\langle \cdot, \cdot \rangle_T$ being Hermitian by X.113.1, which is in turn equivalent to the symmetry of T by X.145. ☐

On a real inner product space there may exist positive operators that are not symmetric.

Example

Let $V = \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (y, -x)$. Then

$$\forall (x,y) \in V : \quad Q_T\big((x,y)\big) = \langle (x,y), (y,-x)\rangle = xy - xy = 0 \geq 0,$$

so T is positive. But T is not symmetric. Indeed $\langle (0,y), T(x,0)\rangle = -xy$ and $\langle T(0,y), (x,0)\rangle = xy$.

Lemma X.147. *Let A be an bounded operator on a Hilbert space H. Then A^*A and AA^* are positive. Also A^*A is strictly positive* if and only if *A is injective.*

Proof. For all $x \in H$:

$$\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0 \qquad \langle AA^*x, x\rangle = \langle A^*x, A^*x\rangle = \|A^*x\|^2 \geq 0.$$

If A is injective, then its kernel is $\{0\}$ and thus $\|Ax\|^2 > 0$ for all $x \in H \setminus \{0\}$. ☐

Lemma X.148. *Let T be an invertible operator on an inner product space V. Then $Q_T[V] = Q_{T^{-1}}[V]$.*

Proof. Immediate from X.143. ☐

Corollary X.148.1. *Let T be an invertible operator. Then T is positive (definite)* if and only if *T^{-1} is positive (definite).*

Lemma X.149. *Let T be an operator. The energy form $\langle \cdot, \cdot \rangle_T$ is positive (and thus a pre-inner product)* if and only if *T is a positive operator.*

6.6.1.1 Energy norm

Let T be a positive operator. Then

$$\|\cdot\|_T : \mathrm{dom}(T) \to [0, +\infty[: x \mapsto \|x\|_T = \sqrt{Q_T(x)}$$

is the energy norm associated to T.

Lemma X.150. *The energy norm of a positive operator determines a pseudometric topology.*

The topology generated by the energy norm is called the energy topology and convergence in the energy topology is called convergence in energy.

Proposition X.151. *Let T be a positive operator. Then*

1. *the energy topology is coarser than the norm topology;*
2. *the topologies are the same on* dom(T) *if T is positive definite.*

6.6.1.2 The partial order on operators

We define an operator partial order by

$$A \leq B \qquad \iff \qquad B - A \geq 0.$$

TODO: restrict to bounded operators??

Lemma X.152. *The operator partial order is a partial order on the set of operators on an inner product space.*

6.6.1.3 Induced topology

We consider the topology induced by an energy norm $\|\cdot\|_T$.

Proposition X.153. *Let V be an inner product space and T a positive operator on V. Then*

1. *if T is bounded, then $\|\cdot\|_T$ is bounded by $\|\cdot\|$;*
2. *$\|\cdot\|$ is bounded by $\|\cdot\|_{T+\mathrm{id}}$.*

Proof. (1) If T is bounded, then $\forall v \in V$

$$\|v\|_T = \sqrt{\langle v, Tv\rangle} = \sqrt{|\langle v, Tv\rangle|} \leq \sqrt{\|v\|^2\|T\|} = \sqrt{\|T\|}\|v\|,$$

where we have used the Cauchy-Schwarz inequality X.108.
(2) For all $v \in V$ we have

$$\|v\| \leq \|v\| + \|v\|_T = \langle v, v\rangle + \langle v, Tv\rangle = \langle v, (T + \mathrm{id})v\rangle = \|v\|_{T+\mathrm{id}}.$$

□

Corollary X.153.1. *Let V be an inner product space and T a positive operator on V. Then*

1. *if T is bounded, then the topology induced by $\|\cdot\|$ is finer than the topology induced by $\|\cdot\|_T$;*
2. *the topology induced by $\|\cdot\|_{T+\mathrm{id}}$ is finer than the topology induced by $\|\cdot\|$.*

Proof. This follows straight from X.76. □

6.6.2 Dissipative operators

Let T be an operator on H. Then T is dissipative if, for all $x \in \mathrm{dom}(T)$

$$\mathfrak{Im}\,\langle x, Tx\rangle \geq 0.$$

6.6.3 Rayleigh quotient

Let T be a linear operator on an inner product space V. The Rayleigh quotient for T is

$$J_T : \mathrm{dom}(T) \setminus \{0\} \to \mathbb{F} : u \mapsto \frac{Q(u)}{\|u\|^2} = \frac{\langle u, Tu\rangle}{\|u\|^2}.$$

We may also write just J if the intended operator T is clear.

Lemma X.154. *Let $T \in \mathcal{L}(V)$ be a linear operator and J_T the associated Rayleigh quotient. Then for all $u \in V$:*

$$J_T(u) = J_T\left(\frac{u}{\|u\|}\right).$$

6.6.3.1 Numerical range

`https://users.math.msu.edu/users/shapiro/pubvit/downloads/numrangenotes/numrange_notes.pdf`
`https://pskoufra.info.yorku.ca/files/2016/07/Numerical-Range.pdf`
`http://www.math.wm.edu/~ckli/nrnote`
`https://link-springer-com.ezproxy.ulb.ac.be/content/pdf/10.1007%2F978-3-319-01448-7.pdf`
`https://projecteuclid.org/journalArticle/Download?urlId=10.1307%2Fmmj%2F1028997958`

Let T be a linear operator on an inner product space V and J_T the Rayleigh quotient of T. The range $\mathrm{W}(T) \coloneqq \mathrm{im}(J_T)$ is known as the numerical range.

The numerical range of T can equivalently be defined as the image of the unit sphere under the quadratic form associated to T.

Lemma X.155. *Let T be a linear operator on an inner product space V and J_T the Rayleigh quotient of T. Then*

$$\begin{aligned}\mathrm{W}(T) &= J_T[\{u \in V \mid \|u\| = 1\} \cap \mathrm{dom}(T)] \\ &= Q_T[\{u \in V \mid \|u\| = 1\} \cap \mathrm{dom}(T)].\end{aligned}$$

Lemma X.156. *Let V be an inner product space over a field $\mathbb{F}$, $\lambda, \mu \in \mathbb{F}$ and T an operator on V. Then*

$$W(\lambda T + \mu) = \lambda W(T) + \mu.$$

Theorem X.157 (Toeplitz-Hausdorff theorem)**.** *Let V be an inner product space and T an operator on V. Then $W(T)$ is convex.*

Proof. TODO `https://www.ams.org/journals/proc/1970-025-01/S0002-9939-1970-0262849-9/S0002-9939-1970-0262849-9.pdf`
`https://www.cambridge.org/core/services/aop-cambridge-core/content/view/BA251EBB1E1DE08DBD3D84964F65938B/S0008439500058197a.pdf/the-toeplitz-hausdorff-theore`
`pdf` ☐

Proposition X.158. *Let V be an inner product space and T an operator on V. If V is finite dimensional, then $W(T)$ is compact.*

Proof. Heine-Borel. TODO. ☐

Lemma X.159. *Let V be an inner product space and T a bounded symmetric operator on V. Then*

1. *the directional derivative $\partial_v(J_T(u))$ exists if $u \neq 0$ and is equal to (TODO remove and place in proof?)*

$$\partial_v(J_T)|_u = \frac{\langle u,u\rangle\Big(\langle v,Tu\rangle + \langle u,Tv\rangle\Big) - \langle u,Tu\rangle\Big(\langle u,v\rangle + \langle v,u\rangle\Big)}{\langle u,u\rangle^2};$$

2. *$u \in V\backslash\{0\}$ is a critical point of J_T* if and only if *u is an eigenvector of T with corresponding eigenvalue $\lambda = J_T(u)$.*

Proof. TODO: critical point in $\mathbb{C}$ v $\mathbb{R}$?? (For symmetric operators J is real valued) XI.9 □

6.6.3.2 Numerical radius

Let T be a linear operator on an inner product space V. Then

$$\operatorname{nr}(T) \coloneqq \sup_{u\in\operatorname{dom}(T)\backslash\{0\}} |J_T(u)|$$

is the numerical radius.

If Q_T is the quadratic form associated to an operator T, we have

$$|Q_T(u)| \leq \|u\|^2 \operatorname{nr}(T).$$

Lemma X.160. *Let T be a linear operator on an inner product space V and J_T the Rayleigh quotient of T. Then*

$$\begin{aligned}\operatorname{nr}(T) &= \sup_{\substack{u\in\operatorname{dom}(T)\backslash\{0\}\\ \|u\|=1}} |J_T(u)| \\ &= \sup_{\substack{u\in\operatorname{dom}(T)\backslash\{0\}\\ \|u\|=1}} |Q_T(u)|.\end{aligned}$$

Proposition X.161. *Let T be an operator on an inner product space V.*

1. *If T is bounded, then $\forall u \in \operatorname{dom}(T) \setminus \{0\}$*

$$|J_T(u)| \leq \operatorname{nr}(T) \leq \|T\|.$$

2. *If T is symmetric, then T is bounded with $\|T\| = \operatorname{nr}(T)$.*

Proof. (1) The first claim follows simply from the Cauchy-Schwarz inequality X.108

$$|J(u)| \leq \frac{\|u\|\,\|Tu\|}{\|u\|^2} = \frac{\|Tu\|}{\|u\|} \leq \frac{\|T\|\|u\|}{\|u\|} = \|T\|.$$

(2) For the second claim we need to also show the inverse inequality. By X.137 it is enough to show that $|\langle w, Tv\rangle| \leq \operatorname{nr}(T)$ for all $v \in \operatorname{dom}(T)$ and $w \in \operatorname{im}(T)$ with $\|v\| = 1 = \|w\|$.
Take arbitrary unit vectors $v, w \in V$ and let θ be such that $|\langle w, Tv\rangle| = e^{i\theta}\langle w, Tv\rangle$. Then $\left\langle e^{-i\theta}w, Tv\right\rangle$ is real, so, viewing it as a sesquilinear form, the imaginary parts of the polarisation identity X.113 cancel:

$$\begin{aligned}\left\langle e^{-i\theta}w, Tv\right\rangle &= \frac{1}{4}\sum_{k=0}^{3} i^k \left\langle (i^k e^{-i\theta}w + v), T((i^k e^{-i\theta}w + Tv))\right\rangle \\ &= \frac{1}{4}\Big(\left\langle v + e^{-i\theta}w, T(v + e^{-i\theta}w)\right\rangle - \left\langle v - e^{-i\theta}w, T(v - e^{-i\theta}w)\right\rangle \Big),\end{aligned}$$

where we have used that the quadratic form is real by X.145.1.

Thus

$$\begin{aligned}
|\langle w, Tv\rangle| &= |\left\langle e^{-i\theta}w, Tv\right\rangle| \\
&= \frac{1}{4}\Big(\left\langle v+e^{-i\theta}w, T(v+e^{-i\theta}w)\right\rangle - \left\langle v-e^{-i\theta}w, T(v-e^{-i\theta}w)\right\rangle\Big) \\
&\leq \frac{1}{4}\Big(|\left\langle v+e^{-i\theta}w, T(v+e^{-i\theta}w)\right\rangle| + |\left\langle v-e^{-i\theta}w, T(v-e^{-i\theta}w)\right\rangle|\Big) \\
&\leq \frac{1}{4}\operatorname{nr}(T)\Big(\left\|v+e^{-i\theta}w\right\|^2 + \left\|v-e^{-i\theta}w\right\|^2\Big) \\
&= \frac{1}{4}\operatorname{nr}(T)\Big(2\|v\|^2 + 2\|w\|^2\Big) = \operatorname{nr}(T),
\end{aligned}$$

where we have used the fact that v, w are unit vectors and the parallelogram law X.111. ☐

Corollary X.161.1. *If T is a symmetric operator; it is bounded iff J_T is bounded above and below:*

$$\forall u \in \operatorname{dom}(T) : \; k \leq J_T(u) \leq K$$

for some $k, K \in \mathbb{R}$.

Corollary X.161.2. *If T is symmetric and bounded, then*

$$\|T\| = \sup_{\|u\|\leq 1} |\langle u, Tu\rangle|.$$

Chapter 7

Bilinear and multilinear maps

TODO: bilinear maps, bilinear forms = bilinear functionals orthogonality

7.1 Bilinear form

Let V be a vector space over a field $\mathbb{F}$. A function $B : V \times V \to \mathbb{F}$ is called a bilinear form on V if for all $v \in V$, both $B(v, -)$ and $B(-, v)$ are linear.

7.1.1 Quadratic forms

Let V be a vector space over a field $\mathbb{F}$ and B a bilinear form on V. The function

$$q_B : V \to \mathbb{F} : v \mapsto B(v, v)$$

is called the associated quadratic form.

Proposition X.162. *Let V be a vector space over a field $\mathbb{F}$ and B a bilinear form on V. Then*

$$B(v, w) + B(w, v) = q_B(v + w) - q_B(v) - q_B(w).$$

Corollary X.162.1. *If B is symmetric and $\mathbb{F}$ is not of characteristic 2, then we can recover the bilinear form from the associated quadratic form:*

$$B(v, w) = \frac{1}{2}\Big(q_B(v + w) - q_B(v) - q_B(w)\Big).$$

So there is a bijection between symmetric and bilinear forms over fields not of characteristic 2.

Proposition X.163. *Let V be a vector space over a field $\mathbb{F}$ and $q : V \to \mathbb{F}$ a function. Then q is the quadratic form associated to some bilinear form* if and only if

- $\forall \lambda \in \mathbb{F} \forall v \in V : \; q(\lambda v) = \lambda^2 q(v)$*;*
- *the parallelogram law holds:* $\forall v, w \in V$

$$q(v + w) + q(v - w) = 2(q(v) + q(w)).$$

Proof. First assume q is the quadratic form associated with the bilinear form B. Then $q(\lambda v) = B(\lambda v, \lambda v) = \lambda^2 B(v,v) = \lambda^2 q(v)$. The proof of the parallelogram law is the same as in an inner product space.
For the converse, we need to show that $q(v+w) - q(v) - q(w)$ is bilinear. TODO! (only real / complex??) ☐

7.1.1.1 Finite dimensional quadratic forms

TODO matrix representation of q.

Let V be a finite dimensional vector space and q a quadratic form on V. A basis $\{e_i\}_{i\in I}$ is called

- q-orthogonal if $q(e_i + e_j) = q(e_i) + q(e_j)$ for all $i \neq j \in I$;
- q-orthonormal if it is orthogonal and $q(e_i) \in \{-1, 0, 1\}$ for all $i \in I$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis and

- $p = |\,\{e_i \mid q(e_i) = 1\}\,|$;
- $n = |\,\{e_i \mid q(e_i) = -1\}\,|$;
- $z = |\,\{e_i \mid q(e_i) = 0\}\,|$.

Then we call the triple (p, n, z) the signature of the basis $\{e_i\}_{i\in I}$.

Theorem X.164 (Sylvester's law of inertia)**.** *Let V be a finite dimensional vector space and q a quadratic form on V.*

1. *If V is a real vector space, then any orthonormal basis has the same signature.*
2. *If V is a complex vector space, then for any orthonormal basis, the pair $(p+n, z)$ is the same.*

Proof. TODO ☐

Proposition X.165. *Let V be a finite dimensional vector space and q a quadratic form on V.*

1. *If $\mathbb{F}$ is not of characteristic 2, then V has an orthogonal basis.*
2. *If $\mathbb{F}$ is a spin field, then V has an orthonormal basis.*

TODO spin field.

Proof. TODO ☐

7.2 Tensor product

`https://kconrad.math.uconn.edu/blurbs/linmultialg/tensorprod.pdf`

7.2.1 Free vector space

Given any set, we can construct a vector space by viewing each element in the set as a (linearly) independent (basis) vector. The vector space then consists of formal linear combinations of these vectors.
To be more precise:

Let S be a set and K a field. Then define

$$F_K(S) \coloneqq \left\{ f \in (S \to K) \;\middle|\; f^{-1}[K \setminus \{0\}] \text{ is finite} \right\}.$$

Define the following operations on $F_K(S)$:

$$\begin{aligned} + &: F(S) \times F(S) \to F(S) : (f, g) \mapsto (f + g : x \mapsto f(x) + g(x)) \\ \cdot &: K \times F(S) \to F(S) : (\lambda, f) \mapsto (\lambda f : x \mapsto \lambda f(x)) \end{aligned}$$

The operations $+, \cdot$ are well-defined and make $F(S)$ into a vector space, called the free vector space over S.

Proposition X.166. *Let S be a set. Then we can identify S with a subset of $F(S)$ by*

$$\iota : S \hookrightarrow F(S) : x \mapsto \chi_{\{x\}}.$$

With this identification S forms a basis for $F(S)$.

Lemma X.167. *Let V be a vector space over K and β a basis for V. Then $V \cong F_K(V)$.*

7.2.1.1 The free functor

Proposition X.168. *The operation F of finding the free vector space over a set can be extended to an contravariant functor*

$$F : \mathsf{Set} \to \mathsf{Vect}.$$

Proof. Let $f : X \to Y$ be a function between sets. □

TODO: just specific instance up to isomorphism?? covariant?? isomorphism class of all vector spaces with basis S?? Is this why tensor product only up to isomorphism??

7.2.1.2 Universal property

Proposition X.169. *Let ϕ be an arbitrary function from S to a vector space W over a field K, then there exists a unique linear map $\overline{\phi} : F(S) \to W$ such that the diagram*

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "\phi"'] & F(S) \arrow[d, dashed, "\overline{\phi}"] \\ & W \end{tikzcd}$$

commutes.

Furthermore, $F(S)$ is the unique K-vector space with this property.

7.2.2 Abstract definition

The idea behind the tensor product of two vector spaces V, W over a field K is to create the most general set of pairings that is a vector space and such that the pairings are bilinear: $\forall \lambda \in K : \forall v_1, v_2, v \in V : \forall w_1, w_2, w \in W$:

$$(\lambda v_1 + v_2, w) = \lambda(v_1, w) + (v_2, w) \qquad \text{and} \qquad (v, \lambda w_1 + w_2) = \lambda(v, w_1) + (v, w_2).$$

This will be realised as a quotient of a free vector space.
To be more precise:

Let V, W be vector spaces over a field K. Consider the set $\text{Field}(V) \times \text{Field}(W)$, which we will refer to as $V \times W$. Construct the sets

$$\begin{aligned} R_1 &= \{(\lambda(v_1, w) + (v_2, w), (\lambda v_1 + v_2, w)) \in F_K(V \times W) \mid \lambda \in K; v_1, v_2 \in V; w \in W\} \\ R_2 &= \{(\lambda(v, w_1) + (v, w_2), (v, \lambda w_1 + w_2)) \in F_K(V \times W) \mid \lambda \in K; v_1, v_2 \in V; w \in W\} \\ R &= R_1 \cup R_2 \end{aligned}$$

The tensor product $V \otimes W$ of the vector spaces V and W is the quotient vector space

$$V \otimes W := F(V \times W)/R^{\equiv}$$

where $R^{\equiv}$ is the reflexive symmetric transitive closure of R.
The equivalence class $[(v, w)]$ is denoted $v \otimes w$. An element of $V \otimes W$ that can be written as $v \otimes w$ is called a pure tensor or simple tensor.

In order for the definition to be well-defined, we need for $R^{\equiv}$ to be a congruence on F.
The tensor product $V \otimes W$ of two vector spaces V and W over a common field K is the quotient vector space

$$V \otimes W := F(V \times W)/\sim$$

where $\sim$ is the equivalence relation over the the free vector space $F(V \times W)$ with the properties of

- *Distributivity*: $(v + v', w) \sim (v, w) + (v', w)$ and $(v, w + w') \sim (v, w) + (v, w')$.
- *Scalar multiples*: $c(v, w) \sim (cv, w) \sim (v, cw)$.

TODO definition via bases: $V \otimes W = F(\beta_V \times \beta_W)$

7.2.3 Universal property

See also proposition X.249.1.

7.2.4 Tensor product of linear maps

The tensor product also operates on linear maps between vector spaces.

Given two linear maps $S : V \to X$ and $T : W \to Y$, then the tensor product of the linear maps S and T is the linear map

$$S \otimes T : V \otimes W \to X \otimes Y$$

defined by

$$(S \otimes T)(v \otimes w) = S(v) \otimes T(w).$$

For vectors that are not pure tensors, this definition is extended by linearity.

Lemma X.170. *The tensor product of linear maps is well-defined.*

With this definition the tensor product becomes a bifunctor from the category of vector spaces to itself, covariant in both arguments.
TODO: functional calculus on tensor product. TODO: tensor product of operator algebras

7.2.5 Operator-valued matrices

Proposition X.171. *Let A be an algebra over a field $\mathbb{F}$ and β any set. Consider the direct sum $A^\beta = \bigoplus_{i\in\beta} A$. Then $A^\beta \cong F_\mathbb{F}(\beta) \otimes A$.*

Proof. TODO $\square$

7.2.6 Matrix representation

7.2.6.1 Finding a basis

Assume V and W are finite-dimensional vector spaces with resp. bases $\{\mathbf{e}_i\}_i$ and $\{\mathbf{f}_j\}_j$. Then the set $\{\mathbf{e}_i \otimes \mathbf{f}_j\}_{i,j}$ forms a basis for $V \otimes W$. Indeed,

- Take two arbitrary vectors $\mathbf{v} = \sum_i a_i\mathbf{e}_i \in V$ and $\mathbf{w} = \sum_j b_j\mathbf{f}_j \in W$. Using the distributivity and scalar multiples properties of $\sim$, we can write the tensor product $\mathbf{v} \otimes \mathbf{w}$ as
$$\mathbf{v} \otimes \mathbf{w} = (\sum_i a_i\mathbf{e}_i) \otimes (\sum_j b_j\mathbf{f}_j) = \sum_{i,j} a_ib_j(\mathbf{e}_\mathbf{i} \otimes \mathbf{f}_j). \tag{7.1}$$
So any pure tensor can be written as the sum of vectors of the form $\mathbf{e}_i \otimes \mathbf{f}_j$. In general a vector in $V \otimes W$ can be written as a finite sum of pure tensors, meaning the set of vectors $\{\mathbf{e}_i \otimes \mathbf{f}_j\}_{i,j}$ spans $V \otimes W$.
- For linear independence we, observe that for any linearly independent v_1, v_2, w_1, w_2, the vector $v_1 \otimes w_1 + v_2 \otimes w_2$ cannot be written as a pure tensor.

Clearly it follows that

$$\dim(V \otimes W) = \dim(V) \cdot \dim(W)$$

7.2.6.2 Coordinates and the outer product

The coordinates of a vector with respect to the basis $\{\mathbf{e}_i \otimes \mathbf{f}_j\}_{i,j}$ can naturally be put into a matrix. Taking the tensor product of two vectors corresponds to taking the outer product of their coordinate vectors. That is, setting $\operatorname{co}(v) = \mathbf{v}$ and $\operatorname{co}(w) = \mathbf{w}$, we get

$$\operatorname{co}(v \otimes w)_{i,j} = a_ib_j = (\mathbf{v}\mathbf{w}^\mathrm{T})_{i,j}$$

which follows from (7.1) above.
For this reason $\otimes$ is also used to denote the outer product.
If we want a proper column vector as our coordinate vector, we can apply row-by-row vectorisation to this matrix.

$$\operatorname{co}(v \otimes w) = \operatorname{vec}_R(\mathbf{v}\mathbf{w}^\mathrm{T}) = \mathbf{v} \otimes \mathbf{w} = \operatorname{co}(v) \otimes \operatorname{co}(w).$$

where $\otimes$ is also used to denote the Kronecker product.
Coordinates for vectors that are not pure tensors can easily be found by the linearity of the coordinate map.

7.2.6.3 Linear maps and the Kronecker product

Letting the coordinates be columns, we can hope to find a matrix for the linear map $S \otimes T$. Fix bases for the spaces V, W, X, Y. Let A and B be the matrices of S and T with respect to these bases. Use these bases to fix the bases for $V \otimes W$ and $X \otimes Y$.

> The matrix of the map $S \otimes T$ with respect to these bases is the matrix $A \otimes B$, where $\otimes$ is the Kronecker product.

This follows from a simple calculation:

$$\begin{aligned}
\operatorname{co}(S \otimes T(v \otimes w)) &= \operatorname{co}(S(v) \otimes T(w)) \\
&= \operatorname{co}(S(v)) \otimes \operatorname{co}(T(w)) \\
&= A \operatorname{co}(v) \otimes B \operatorname{co}(v) \\
&= (A \otimes B) \operatorname{co}(v) \otimes \operatorname{co}(w) \qquad \text{(using the mixed product)} \\
&= (A \otimes B) \operatorname{co}(v \otimes w).
\end{aligned}$$

Again this calculation can be extended to non-pure tensors by linearity.

7.2.7 Properties

TODO currying. `https://math.stackexchange.com/questions/679584/why-is-texthomv-w-the-sar`
And reference later!

7.2.8 Multilinear maps

> Let $V^k = V \times \ldots \times V$. A function $f : V^k \to \mathbb{R}$ is k-linear if it is linear in each of its arguments.
>
> - A k-linear function $f : V^k \to \mathbb{R}$ is symmetric if for all permutations $\sigma \in S_k$
>
> $$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = f(v_1, \ldots, v_k).$$
>
> - A k-linear function $f : V^k \to \mathbb{R}$ is alternating if for all permutations $\sigma \in S_k$
>
> $$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\operatorname{sgn}\sigma) f(v_1, \ldots, v_k).$$
>
> We call the space of all alternating k-linear maps $A_k(V)$.

In particular $A_1(V) = V^*$.

> Given a k-linear function f and a permutation $\sigma \in S_k$, we define the k-linear function σf by
>
> $$(\sigma f)(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Then a symmetric map is one such that $\sigma f = f$ for all $\sigma \in S_k$ and an alternating map is one such that $\sigma f = (\operatorname{sgn}\sigma)f$ for all $\sigma \in S_k$.

Lemma X.172. *Let $\sigma, \tau \in S_k$ and f a k-linear map on V. Then $\tau(\sigma f) = (\tau\sigma)f$.*

7.2.8.1 The symmetrising and alternating maps

Let f be a k-linear map on a vector space V.

- The symmetrisation of f, Sf, is the map
$$Sf = \frac{1}{k!}\sum_{\sigma\in S_k}\sigma f.$$
- The anti-symmetrisation or skew-symmetrisation of f, Af, is the map
$$Af = \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\sigma f.$$

Lemma X.173. *1. The k-linear map Sf is symmetric. If f is symmetric, then $Sf = f$.*

2. The k-linear map Af is alternating. If f is alternating, then $Af = f$.

Lemma X.174. *Let f be a k-linear functional and g an l-linear functional on V. Then*
$$A(A(f)\otimes g) = A(f\otimes g) = A(f\otimes A(g)).$$

7.2.8.2 The wedge product

Let $f \in A_k(V)$ and $g \in A_l(V)$. The wedge product of f and g is given by
$$f\wedge g = \frac{(k+l)!}{k!l!}A(f\otimes g).$$

We can also write
$$(f\wedge g)(v_1,\ldots,v_{k+l}) = \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)f(v_{\sigma(1)},\ldots,v_{\sigma(k)})g(v_{\sigma(k+1)},\ldots,v_{\sigma(k+l)}).$$

We can reduce redundancies in this definition in the following way: We call $\sigma \in S_{k+l}$ a (k,l)-shuffle if
$$\sigma(1) < \ldots < \sigma(k) \qquad \text{and} \qquad \sigma(k+1) < \ldots < \sigma(k+l).$$

The we write
$$(f\wedge g)(v_1,\ldots,v_{k+l}) = \sum_{(k,l)\text{-shuffles } \sigma}(\operatorname{sgn}\sigma)f(v_{\sigma(1)},\ldots,v_{\sigma(k)})g(v_{\sigma(k+1)},\ldots,v_{\sigma(k+l)}).$$

Proposition X.175. *Let $f \in A_k(V)$ and $g \in A_l(V)$. Then*
$$f\wedge g = (-1)^{kl}g\wedge f.$$

Lemma X.176. *The wedge product is associative:*

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Proof using X.174.

Lemma X.177. *Let* $\alpha^1, \ldots, \alpha^k$ *be linear functionals on* V *and* $v_1, \ldots, v_k \in V$*, then*

$$(\alpha^1 \wedge \alpha^k)(v_1, \ldots, v_k) = \det[\alpha^i(v_j)].$$

7.2.9 Tensors

A (p,k)-tensor is a multilinear function $V^k \to V^p$.

7.3 Real, complex and quaternionic vector spaces

A function f between complex vector spaces is anti-linear (or conjugate-linear) in the first component:

$$f(\lambda_1 v_1 + \lambda_2 v_2) = \overline{\lambda_1} f(v_1) + \overline{\lambda_2} f(v_2),$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ and $v_1, v_2 \in \operatorname{dom}(f)$.

7.3.1 Complex structure on a real vector space

Let V be a real vector space. A complex structure on V is a linear map $J : V \to V$ such that $J^2 = -I_V$.

7.3.2 The real vector spaces associated to a complex vector space

Let $V = (\mathbb{C}, V, +)$. Then define $V_\mathbb{R} \coloneqq (\mathbb{R}, V, +)$.
every anti-linear map $A : V \to W$ is an $\mathbb{R}$-linear map $A : V_\mathbb{R} \to W_\mathbb{R}$. (They are equal as sets).

7.4 Quotient algebras of dual systems

Let (X, Y, b) be a dual system over a field $\mathbb{F}$. The algebra generated by this dual system is

$$A(X,Y,b) \coloneqq \bigoplus_{n\in\mathbb{N}} (X \oplus Y)^{\otimes n} / \left((x \otimes y - b(x,y)\mathbf{1} | x \in X, y \in Y) \right).$$

TODO universal algebra presented by generators and relation.

7.4.1 The $\mathbb{Z}_2$-grading

TODO parity grading grading respects ideal!

Proposition X.178. *Let (X, Y, b) be a dual system. There is a faithfull superalgebra representation*

$$A(X,Y,b) \hookrightarrow \operatorname{End}\left(\bigoplus_{n\in\mathbb{N}} Y^{\otimes n}\right)$$

given by the extension (TODO universal property x2) of

$$X \oplus Y \to \operatorname{End}(Y^{\otimes n}) : x + y \mapsto \Big(y_1 \otimes \ldots y_n \mapsto b(x, y_1)y \otimes y_2 \otimes \ldots \otimes y_n\Big).$$

7.5 Clifford algebras

Let V be a vector space over a field $\mathbb{F}$ and q a quadratic form defined on V. Let $\mathcal{T}(V)$ be the tensor algebra

$$\mathcal{T}(V) \coloneqq \mathbb{F} \oplus \bigoplus_{n=1}^{\infty} V^n = \mathbb{F} \oplus \bigoplus_{n=1}^{\infty} \underbrace{V \otimes \ldots \otimes V}_{n \text{ times}}.$$

Let $\mathcal{I}(V, q)$ be the (two-sided) ideal in $\mathcal{T}(V)$ generated by

$$\{\mathbf{v} \otimes \mathbf{v} - q(v)\mathbf{1} \mid \mathbf{v} \in V\}.$$

Then the Clifford algebra $\operatorname{Cl}(V, q)$ associated with V and q is the quotient

$$\operatorname{Cl}(V, q) \coloneqq \mathcal{T}(V)/\mathcal{I}(V, q).$$

We call

- elements of the Clifford algebra multivectors;
- elements of $\operatorname{span}(V^k)$ k-vectors;
- elements of V vectors; we use bold face to denote these elements (e.g. $\mathbf{v}$);
- elements of $\mathbb{F}\mathbf{1}$ scalars.

Elements of the Clifford algebra are called multivectors.

Let π_q be the canonical projection

$$\pi_q : \mathcal{T}(V) \to \operatorname{Cl}(V, q).$$

TODO: make all vectors bold!

Lemma X.179. *The embedding $V \hookrightarrow \operatorname{Cl}(V, q)$ is faithful, i.e. $\pi_q|_V$ is injective.*

Proof. TODO □

Clearly $\pi_q|_V(\mathbf{v})^2 = q(\mathbf{v})\mathbf{1}$ for all $\mathbf{v} \in V$.

Lemma X.180. *Let $\operatorname{Cl}(V, q)$ be a Clifford algebra. Then*

$$\operatorname{Cl}^\times(V, q) \cap V = \{\mathbf{v} \in V \mid q(v) \neq 0\}.$$

The inverse of $\mathbf{v} \in \operatorname{Cl}^\times(V, q) \cap V$ is $\mathbf{v}^{-1} = \mathbf{v}/q(\mathbf{v})$.

Proof. First take $\mathbf{v} \in \{\mathbf{v} \in V \mid q(\mathbf{v}) \neq 0\}$. This definition of $\mathbf{v}^{-1}$ is indeed a multiplicative inverse: $\mathbf{v}\mathbf{v}^{-1} = \mathbf{v}^2/q(\mathbf{v}) = q(\mathbf{v})/q(\mathbf{v})\mathbf{1} = \mathbf{1}$.
Conversely, take $\mathbf{v} \in \mathrm{Cl}^{\times}(V,q) \cap V$. If $q(\mathbf{v}) = 0$, then $\mathbf{v}$ would be a zero divisor (as $\mathbf{v}\mathbf{v} = q(\mathbf{v})\mathbf{1} = 0$). No zero divisor can have an inverse (II.121), ☐

Lemma X.181. *The algebra $Cl(V,q)$ is generated by the vector space V and $\mathbf{1}$, subject to the relations*

$$\mathbf{v}\mathbf{v} = q(\mathbf{v})\mathbf{1} \qquad \forall \mathbf{v} \in V.$$

TODO generators of an algebra!
Clifford algebras can also be defined by their universal property:

Proposition X.182 (Universal property of Clifford algebras)**.** *Let V be a vector space over a field $\mathbb{F}$ and q a quadratic form on V.*
Then for any unital associative algebra A over $\mathbb{F}$ and linear map $j : V \to A$ such that

$$j(\mathbf{v})^2 = q(\mathbf{v})\mathbf{1} \qquad \forall \mathbf{v} \in V$$

there exists a unique algebra homomorphism $\widetilde{j} : \mathrm{Cl}(V,q) \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "\pi_q|_V"] \arrow[dr, "j"'] & \mathrm{Cl}(V,q) \arrow[d, dashed, "\widetilde{j}"] \\
 & A
\end{tikzcd}$$

Furthermore, $\mathrm{Cl}(V,q)$ is the unique associative $\mathbb{F}$-algebra with this property.

Corollary X.182.1. *Let (V,q) and (V',q') be vector spaces with quadratic forms. If a linear map $f : V \to V'$ preserves to quadratic form, $q' \circ f = q$, then f extends to a unique algebra homomorphism*

$$\widetilde{f} : \mathrm{Cl}(V,q) \to \mathrm{Cl}(V',q').$$

Now let (V'',q'') be another vector space equipped with a quadratic form and let $g : V' \to V''$ be a linear map preserving the quadratic form. Then

$$\widetilde{g \circ f} = \widetilde{g} \circ \widetilde{f}.$$

Also isomorphisms of vector spaces extend to isomorphisms of Clifford algebras.

Proof. Let the algebra A of the proposition be $\mathrm{Cl}(V',q')$. Then $\pi_q|_V \circ f$ satisfies the requirement for j:

$$[(\pi_{q'}|_V \circ f)(\mathbf{v})]^2 = q'(f(\mathbf{v}))^2\mathbf{1} = q(\mathbf{v})^2\mathbf{1}.$$

Thus by the proposition, there is a unique extension of $f : V \to V'$ to a map $\mathrm{Cl}(V,q) \to \mathrm{Cl}(V',q')$.
The composition relation follows from uniqueness. ☐

Corollary X.182.2. *The orthogonal group*

$$\mathrm{O}(V,q) = \{g \in \mathrm{GL}(V) \mid q \circ g = q\}$$

extends canonically to a group of automorphisms of $\mathrm{Cl}(V,q)$:

$$\mathrm{O}(V,q) \subset \mathrm{Aut}(\mathrm{Cl}(V,q)).$$

7.5.1 Scalar and outer products

Lemma X.183. *Let* $\mathrm{Cl}(V,q)$ *be a Clifford algebra and* $\mathbf{v},\mathbf{w}\in V$*. Then* $\mathbf{vw}+\mathbf{wv}$ *is a scalar multiple of the identity.*

Proof. We can calculate

$$q(\mathbf{v}+\mathbf{w})\mathbf{1} = (\mathbf{v}+\mathbf{w})^2 = \mathbf{v}^2+\mathbf{vw}+\mathbf{wv}+\mathbf{w}^2 = \mathbf{vw}+\mathbf{wv}+q(\mathbf{v})\mathbf{1}+q(\mathbf{w})\mathbf{1},$$

so $vw+wv = [q(\mathbf{v}+\mathbf{w})-q(\mathbf{v})-q(\mathbf{w})]\mathbf{1}$. ☐

Let $\mathbb{F}$ be a field whose characteristic is not 2 and $\mathrm{Cl}(V,q)$ a Clifford algebra over $\mathbb{F}$. We can then write, for $\mathbf{v},\mathbf{w}\in V$

$$\begin{aligned}\mathbf{vw} &= \frac{\mathbf{vw}+\mathbf{wv}}{2}+\frac{\mathbf{vw}-\mathbf{wv}}{2}\\ &\coloneqq \mathbf{v}\cdot\mathbf{w}+\mathbf{v}\wedge\mathbf{w}.\end{aligned}$$

We call the symmetric part $\mathbf{v}\cdot\mathbf{w}$ the scalar product (sometimes also called the inner product), and the antisymmetric part $\mathbf{v}\wedge\mathbf{w}$ the outer product.

In the sequel, whenever we talk about the scalar and outer product, we will always assume the characteristic of the field is not 2.

Lemma X.184. *Let* $\mathrm{Cl}(V,q)$ *be a Clifford algebra and* $\mathbf{v}\in V$*. We have* $\mathbf{v}\cdot\mathbf{v}=\mathbf{vv}=\mathbf{v}^2=q(\mathbf{v})\mathbf{1}$*.*

So $\mathbf{v}^2$ can cause no confusion.

Lemma X.185. *Let* $\mathrm{Cl}(V,q)$ *be a Clifford algebra and* $\mathbf{v},\mathbf{w}\in V$*. Then*

1. $\mathbf{wv}=2(\mathbf{w}\cdot\mathbf{v})-\mathbf{vw}$;
2. $\mathbf{wv}=2(\mathbf{w}\wedge\mathbf{v})+\mathbf{vw}=-2(\mathbf{v}\wedge\mathbf{w})+\mathbf{vw}$;
3. $\mathbf{vw}^2=2(\mathbf{v}\cdot\mathbf{w})\mathbf{vw}-\mathbf{v}^2\mathbf{w}^2$;
4. $\mathbf{vwv}=2(\mathbf{v}\cdot\mathbf{w})\mathbf{v}-q(\mathbf{v})\mathbf{w}$.

Lemma X.186. *Let* $\mathrm{Cl}(V,q)$ *be a Clifford algebra and* $\mathbf{v},\mathbf{w}\in V$*. Then*

1. $\mathbf{v}(\mathbf{v}\wedge\mathbf{w})=(\mathbf{w}\wedge\mathbf{v})\mathbf{v}$;

Proof. (1) We calculate

$$2\mathbf{v}(\mathbf{v}\wedge\mathbf{w}) = q(\mathbf{v})\mathbf{w}-\mathbf{vwv} = \mathbf{w}q(\mathbf{v})-\mathbf{vwv} = (\mathbf{wv}-\mathbf{vw})\mathbf{v} = 2(\mathbf{w}\wedge\mathbf{v})\mathbf{v}.$$

☐

Let $\mathrm{Cl}(V,q)$ be a Clifford algebra. We call $\mathbf{v},\mathbf{w}\in V$

- orthogonal or perpendicular if $\mathbf{v}\cdot\mathbf{w}=0$;
- parallel if $\mathbf{v}\wedge\mathbf{w}=0$.

Lemma X.187. *Let* $\mathrm{Cl}(V,q)$ *be a Clifford algebra and* $\mathbf{v},\mathbf{w}\in V$*. The following are equivalent:*

1. $\mathbf{v}$ *and* $\mathbf{w}$ *are orthogonal;*
2. $\mathbf{vw}=\mathbf{v}\wedge\mathbf{w}$*;*
3. $\mathbf{vw}=-\mathbf{wv}$*;*
4. $q(\mathbf{v}+\mathbf{w})=q(\mathbf{v})+q(\mathbf{w})$*;*

as are

5. $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel;*
6. $\mathbf{vw}=\mathbf{v}\cdot\mathbf{w}$*.*

7.5.2 Involutions

7.5.2.1 Grade involution

Given a Clifford algebra $\mathrm{Cl}(V,q)$, consider the map $\alpha : V\to V : \mathbf{v}\mapsto -\mathbf{v}$ on the vector space V. Now α is always an element of $\mathrm{O}(V,q)$, so by X.182.2 it extends to a map on the Clifford algebra.

$$\widetilde{\alpha} : \mathrm{Cl}(V,q)\to\mathrm{Cl}(V,q).$$

Since $\alpha^2=I_V$, we have that

$$\widetilde{\alpha}^2=\widetilde{\alpha^2}=\widetilde{I_V}=I_{\mathrm{Cl}(V,q)}$$

meaning $\widetilde{\alpha}$ is an involution on the Clifford algebra. From now on we drop the tilde and just write $\alpha : \mathrm{Cl}(V,q)\to\mathrm{Cl}(V,q)$ for the grade involution.

Lemma X.188. *The grade involution* $\alpha : \mathrm{Cl}(V,q)\to\mathrm{Cl}(V,q)$

1. *is an algebra homomorphism and thus multiplicative:*
$$\alpha(xy)=\alpha(x)\alpha(y)\qquad\forall x,y\in\mathrm{Cl}(V,q);$$
2. *is unital,* $\alpha(\mathbf{1})=\mathbf{1}$*, and thus preserves inverses:*
$$\alpha(x^{-1})=\alpha(x)^{-1}\qquad\forall x\in\mathrm{Cl}(V,q);$$
3. *generates a* $\mathbb{Z}_2$*-grading*
$$\mathrm{Cl}(V,q)=\mathrm{Cl}^0(V,q)\oplus\mathrm{Cl}^1(V,q).$$

7.5.2.2 Transpose

The transpose map defined on the tensor algebra $\mathcal{T}(V)$, i.e. the linear map that reverses to order of homogeneous elements:

$$v_1\otimes\ldots\otimes v_r\mapsto v_r\otimes\ldots\otimes v_1,$$

preserves the ideal $\mathcal{I}(V,q)$, and so determines a well-defined map on the Clifford algebra $\mathrm{Cl}(V,q)$:

$$(-)^t : \mathrm{Cl}(V,q)\to\mathrm{Cl}(V,q).$$

Lemma X.189. *The transpose* $(-)^t : \mathrm{Cl}(V, q) \to \mathrm{Cl}(V, q)$ *is*

1. *an involution;*
2. *an anti-automorphism:*
$$\forall x, y \in \mathrm{Cl}(V, q) : \quad (xy)^t = y^t x^t;$$
3. *unital.*

7.5.2.3 Clifford conjugation

The composition of the grade involution and the transpose is called Clifford conjugation:

$$x \mapsto \overline{x} := \alpha(x^t).$$

Lemma X.190. *The grade involution and transpose commute*

$$\alpha \circ (-)^t = (-)^t \circ \alpha$$

and Clifford conjugation is thus equal to both.

Lemma X.191. *Clifford conjugation is*

1. *an involution;*
2. *an anti-automorphism:*
$$\forall x, y \in \mathrm{Cl}(V, q) : \quad (xy)^t = y^t x^t;$$
3. *unital.*

7.5.2.4 Quaternion types of Clifford algebra types

7.5.3 The norm mapping

We define the norm mapping N by

$$N : \mathrm{Cl}(V, q) \to \mathrm{Cl}(V, q) : x \mapsto x\overline{x}.$$

Lemma X.192. 1. *If* $v \in V$*, then* $N(v) = q(v)$*.*

2. $\alpha \circ N = N \circ \alpha$*.*

Proposition X.193. *Let* $\mathrm{Cl}(V, q)$ *be a Clifford algebra and* $\mathbf{v}, \mathbf{w} \in V$*. Then*

1. $N(\mathbf{vw}) = \mathbf{v}^2\mathbf{w}^2$*;*
2. $N(\mathbf{v} \wedge \mathbf{w}) = \mathbf{v}^2\mathbf{w}^2 - (\mathbf{v} \cdot \mathbf{w})^2$*;*

Proof. (1) Because $\mathbf{vw} \in \mathrm{Cl}(V, q)^0$, we have $N(\mathbf{vw}) = (\mathbf{vw})(\mathbf{wv}) = \mathbf{v}^2\mathbf{w}^2$.
(2) Because $\mathbf{v} \wedge \mathbf{w} \in \mathrm{Cl}(V, q)^0$, we have $N(\mathbf{v} \wedge \mathbf{w}) = (\mathbf{v} \wedge \mathbf{w})(\mathbf{w} \wedge \mathbf{v})$. We use $\mathbf{v} \wedge \mathbf{w} = -\mathbf{w} \wedge \mathbf{v} = \mathbf{vw} - \mathbf{v} \cdot \mathbf{w}$ to calculate

$$N(\mathbf{v} \wedge \mathbf{w}) = -(\mathbf{vw} - \mathbf{v} \cdot \mathbf{w})(\mathbf{vw} - \mathbf{v} \cdot \mathbf{w}) = -\mathbf{vwvw} + 2(\mathbf{v} \cdot \mathbf{w})\mathbf{vw} - (\mathbf{v} \cdot \mathbf{w})^2$$

Then X.185 gives $\mathbf{vwvw} = 2(\mathbf{v} \cdot \mathbf{w})\mathbf{vw} - \mathbf{v}^2\mathbf{w}^2$. Plugging this back in gives

$$N(\mathbf{v} \wedge \mathbf{w}) = -2(\mathbf{v} \cdot \mathbf{w})\mathbf{vw} + \mathbf{v}^2\mathbf{w}^2 + 2(\mathbf{v} \cdot \mathbf{w})\mathbf{vw} - (\mathbf{v} \cdot \mathbf{w})^2 = \mathbf{v}^2\mathbf{w}^2 - (\mathbf{v} \cdot \mathbf{w})^2.$$

□

7.5.4 Clifford algebras as filtered algebras

Proposition X.194. *A Clifford algebra* $\mathrm{Cl}(V,q)$ *has a filtration* $F_k \coloneqq \operatorname{span}(V^k)$.

We then have an associated graded algebra and a grade operator.

Proposition X.195. *Let* V *be a vector space and* q *a quadratic form on* V.

1. *As graded algebras,* $\mathrm{Cl}(V,q)$ *is naturally isomorphic to the exterior algebra* $\bigwedge^* V$.
2. *As algebras* $\bigwedge^* V \cong \mathrm{Cl}(V,0)$.
3. *As vector spaces, there is an isomorphism*

$$\bigwedge\nolimits^* V \to \mathrm{Cl}(V,q) : v_1 \wedge \ldots \wedge v_n \mapsto \frac{1}{r!}\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) v_{\sigma(1)} \ldots v_{\sigma(r)}$$

compatible with the fibrations.

Lemma X.196. *Let* $\mathrm{Cl}(V,q)$ *be a Clifford algebra and* $\mathbf{v},\mathbf{w} \in V$. *Then* $\langle \mathbf{vw} \rangle_1 = 0$.

Proof. We have $\alpha(\mathbf{vw}) = (-\mathbf{v})(-\mathbf{w}) = \mathbf{vw}$, so $\mathbf{vw} \in \mathrm{Cl}^0(V,q)$, while $V \subseteq \mathrm{Cl}^1(V,q)$. ☐

Proposition X.197. *Let* $\mathrm{Cl}(V,q)$ *be a Clifford algebra and* $\mathbf{v}_1,\ldots,\mathbf{v}_k \in V$. *Then* $\mathbf{v}_1 \ldots \mathbf{v}_k \in V^{k-1}$ if and only if $\mathbf{v}_1,\ldots,\mathbf{v}_k$ *are linearly dependent.*

Proof. If $\mathbf{v}_1,\ldots,\mathbf{v}_k \in V$ are linearly dependent, then we can eliminate one of the $\mathbf{v}_j$ and then $\mathbf{v}_1 \ldots \mathbf{v}_k \in V^{k-1}$.
Now assume $\mathbf{v}_1,\ldots,\mathbf{v}_k \in V$ are linearly independent. ☐

7.5.5 Orthogonal decomposition

As $q(u,v)$ is a bilinear form, we can consider orthogonal subspaces with respect to it. Then $V = V_1 \oplus V_2$ is a q-orthogonal decomposition if and only if $\forall v_1 \in V_1, v_2 \in V_2$:

$$q(v_1,v_2) = 0 \qquad \iff \qquad q(v_1+v_2) = q(v_1) + q(v_2).$$

Proposition X.198. *Let* $V = V_1 \oplus V_2$ *be a* q*-orthogonal decomposition. Then there is a natural isomorphism of Clifford algebras*

$$\mathrm{Cl}(V,q) \;\to\; \mathrm{Cl}(V_1, q|_{V_1}) \hat{\otimes} \mathrm{Cl}(V_2, q|_{V_2}) : \quad v_1 + v_2 \mapsto v_1 \otimes \mathbf{1} + \mathbf{1} \otimes v_2.$$

7.6 Subgroups of a Clifford algebra

Proposition X.199. *Let* V *be a finite-dimensional real or complex vector space of dimension* $\dim V = n$. *Then the group* $\mathrm{Cl}^\times(V,q)$ *of multiplicative units in the Clifford algebra is a Lie group of dimension* 2^n *and the corresponding Lie algebra* $\mathfrak{cl}^\times(V,q)$ *is the full Clifford algebra* $\mathrm{Cl}(V,q)$ *with the Lie bracket*

$$[x,y] = xy - yx.$$

7.6.1 Inner automorphisms of $\mathrm{Cl}(V, q)$

Characteristic for field not 2!!

Proposition X.200. *Let* $\mathbf{v} \in V \cap \mathrm{Cl}^\times(V, q)$*. Then for all* $\mathbf{w} \in V$*:*

$$\mathrm{Ad}_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{vwv}}{q(\mathbf{v})} = \frac{2\mathbf{v} \cdot \mathbf{w}}{q(\mathbf{v})}\mathbf{v} - \mathbf{w}.$$

In particular $\mathrm{Ad}_{\mathbf{v}}[V] = V$*.*

Proof. From X.180 we have $q(\mathbf{v}) \neq 0$ and $\mathbf{v}^{-1} = \mathbf{v}/q(\mathbf{v})$. We then calculate

$$q(\mathbf{v})\,\mathrm{Ad}_{\mathbf{v}}(\mathbf{w}) = q(\mathbf{v})\mathbf{vwv}^{-1} = \mathbf{vwv} = (\mathbf{vw} + \mathbf{wv} - \mathbf{wv})\mathbf{v} = (\mathbf{vw} + \mathbf{wv})\mathbf{v} - q(\mathbf{v})\mathbf{w}.$$

□

Lemma X.201. *Let* $\mathbf{v} \in V$ *such that* $q(\mathbf{v}) \neq 0$*. Then* $\mathrm{Ad}_{\mathbf{v}} \in \mathrm{O}(V, q)$*.*

TODO renew notation

Proof. Clearly $\mathrm{Ad}_{\mathbf{v}}$ is invertible. We then calculate using X.200

$$\begin{aligned} q(\mathrm{Ad}_v(w)) &= q\left(\frac{q(v,w)}{q(v)}v - w\right) = q\left(\frac{q(v,w)}{q(v)}v, -w\right) + q\left(\frac{q(v,w)}{q(v)}v\right) + q(w) \\ &= -\frac{q(v,w)}{q(v)}q(v,w) + \left(\frac{q(v,w)}{q(v)}\right)^2 q(v) + q(w) = q(w). \end{aligned}$$

□

7.6.2 Pin and Spin groups

Let $P(V, q)$ be the subgroup of $\mathrm{Cl}^\times(V, q)$ generated by elements $v \in V$ with $q(v) \neq 0$. We also define the group

$$\Gamma(V, q) \coloneqq \left\{x \in \mathrm{Cl}^\times(V, q) \;\middle|\; \mathrm{Ad}_x[V] = V\right\}.$$

which is called the Clifford group, Lipschitz group or Clifford-Lipschitz group.

That $\Gamma(V, q)$ is a group follows from the following observation: If $\mathrm{Ad}_x[V] = V$ and $\mathrm{Ad}_y[V] = V$, then

$$\mathrm{Ad}_{xy}[V] = \mathrm{Ad}_x[\mathrm{Ad}_y[V]] = \mathrm{Ad}_x[V] = V.$$

Lemma X.202. *There is an inclusion*

$$P(V, q) \subset \Gamma(V, q).$$

Proof. The generators of $P(V, q)$ are in $\Gamma(V, q)$ by X.200 and $\Gamma(V, q)$ is a group. □

Lemma X.203. *The* Ad *function defines a representation*

$$\mathrm{Ad} : P(V, q) \to \mathrm{O}(V, q).$$

Proof. This mapping is well-defined by X.201 and the identity

$$\mathrm{Ad}_{xy} = \mathrm{Ad}_x \circ \mathrm{Ad}_y\,.$$

This identity also shows that the mapping is a group homomorphism, and thus that it is a representation. ☐

Lemma X.204. *Let $x \in \Gamma(V,q)$. Then*

1. $\alpha(x) \in \Gamma(V,q)$;
2. $x^t \in \Gamma(V,q)$.

Proof. We calculate

$$V = \alpha[V] = \alpha[\Gamma(V,q)] = \alpha(\alpha(x))V\alpha(x)^{-1} = \mathrm{Ad}_{\alpha(x)}[V]$$

and

$$V = (\alpha[V])^t = \alpha(x^t)V(x^t)^{-1} = \mathrm{Ad}_{x^t}[V]$$

and the third follows by multiplicative closure. ☐

A consequence of this lemma is that $N(x) \in \Gamma(V,q)$ for all $x \in \Gamma(V,q)$. But we will show that something stronger holds, namely $N(x) \in \mathbb{F}^\times$.

The Pin group of (V,q) is the subgroup $\mathrm{Pin}(V,q)$ of $P(V,q)$ generated by the elements $v \in V$ with $q(v) = \pm 1$.
The Spin group of (V,q) is defined by

$$\mathrm{Spin}(V,q) = \mathrm{Pin}(V,q) \cap \mathrm{Cl}^0(V,q)$$

where $\mathrm{Cl}^0(V,q)$ is the even subalgebra of $\mathrm{Cl}(V,q)$.

7.6.3 The twisted adjoint representation

Consider the map Ad_v acting on the q-orthogonal decomposition (TODO ref)

$$V = \mathrm{span}\{v\} \oplus \mathrm{span}\{v\}^\perp \qquad \text{for some } v \in P(V,q),$$

where $\mathrm{span}\{v\}^\perp = \{w \in V \mid q(v,w) = 0\}$. Then, by the formula

$$\mathrm{Ad}_v(w) = \frac{q(v,w)}{q(v)}v - w,$$

we see that elements of $\mathrm{span}\{v\}$ are mapped to themselves:

$$\begin{aligned}\mathrm{Ad}_v(\lambda v) &= \frac{q(v,\lambda v)}{q(v)}v - \lambda v = \lambda\frac{q(2v) - 2q(v)}{q(v)}v - \lambda v\\ &= 2\lambda v - \lambda v = \lambda v.\end{aligned}$$

and that elements $w \in \mathrm{span}\{v\}^\perp$ are mapped to $-w$.
This means that Ad_v is orientation-preserving if $\dim(V)$ is odd and orientation-reversing otherwise.
We would prefer the action of Ad_v to do the opposite: fix the hyperplane $\mathrm{span}\{v\}^\perp$ and invert $\mathrm{span}\{v\}$. To that end we introduce the twisted adjoint representation.

The twisted adjoint representation $\widetilde{\mathrm{Ad}} : \mathrm{Cl}^\times(V,q) \to \mathrm{GL}(\mathrm{Cl}(V,q))$ is defined by

$$\widetilde{\mathrm{Ad}}_x(y) = \alpha(x)yx^{-1} \qquad \forall x \in \mathrm{Cl}^\times(V,q), \forall y \in \mathrm{Cl}(V,q)$$

where α is the grade involution.

Lemma X.205. *Let $x, y \in \mathrm{Cl}^\times(V,q)$ and $v, w \in V$. Then*

1. $\widetilde{\mathrm{Ad}}_{xy} = \widetilde{\mathrm{Ad}}_x \circ \widetilde{\mathrm{Ad}}_y$;
2. $\widetilde{\mathrm{Ad}}_x = \mathrm{Ad}_x$ *if* $x \in \mathrm{Cl}^0(V,q)$;
3. $\widetilde{\mathrm{Ad}}_v(w) = w - \frac{q(v,w)}{q(v)}v$.

We have

$$\Gamma(V,q) = \left\{ x \in \mathrm{Cl}^\times(V,q) \;\middle|\; \widetilde{\mathrm{Ad}}_x[V] = V \right\}.$$

Proposition X.206. *Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and q non-degenerate. Then the kernel of the homomorphism*

$$\widetilde{\mathrm{Ad}} : \Gamma(V,q) \to \mathrm{GL}(V)$$

is exactly the group $\mathbb{F}^\times$.

Proof. Choose an orthogonal basis $v_1, \ldots, v_n$ for V w.r.t. the bilinear form $q(-,-)$ (TODO ref; also proof here only finite-dim: can it generalise?). Suppose $x \in \mathrm{Cl}^\times(V,q)$ is in the kernel of $\widetilde{\mathrm{Ad}}$, then

$$\alpha(x)v = vx \qquad \text{for all } v \in V.$$

Now we can write $x = x_0 + x_1$ where x_0 is even and x_1 is odd and both are polynomial expressions in $v_1, \ldots, v_n$. Making use of $v_iv_j = \pm v_jv_i$, we can write $x_0 = a_0 + v_1a_1$ where a_0, a_1 are polynomial expressions in $v_2, \ldots, v_n$. Then a_0 is even and a_1 is odd, so

$$v_1a_0 + v_1^2a_1 = v_1(a_0 + v_1a_1) = (a_0 + v_1a_1)v_1 = a_0v_1 + v_1a_1v_1 = v_1a_0 - v_1^2a_1.$$

Thus $v_1^2a_1 = -q(v_1)a_1 = 0$, so $a_1 = 0$ and x_0 does not involve v_1. By induction x_0 does not involve any of $v_1, \ldots, v_n$ and thus $x_0 = \lambda\mathbf{1}$ for $\lambda \in \mathbb{F}$.
A similar argument shows that x_1 is independent of $v_1, \ldots, v_n$ and thus $x_1 = 0$. Here it is important that $v_1x_1 = -x_1v_1$, because in $x_1 = a_0 + v_1a_1$, a_0 is now odd and a_1 even.
Thus $x = x_0 + x_1 = \lambda\mathbf{1}$ and $x \neq 0$, so $x \in \mathbb{F}^\times$. ☐

This proof only works for the *twisted* adjoint representation, not the adjoint representation. It is also clearly important that q be non-degenerate.

Corollary X.206.1. *The restriction of the norm N to $\Gamma(V,q)$ gives a homomorphism*

$$N : \Gamma(V,q) \to \mathbb{F}^\times.$$

Proof. If we can show that $N[\Gamma(V,q)] \subset \mathbb{F}^\times$, then the multiplicativity of N follows from

$$N(xy) = xy\alpha((xy)^t) = xy\alpha(y^t)\alpha(x^t) = xN(y)\alpha(x^t) = x\alpha(x^t)N(y) = N(x)N(y).$$

Thus by the proposition it is enough to show that for all $x \in \Gamma(V,q)$, $N(x) \in \ker(\widetilde{\mathrm{Ad}})$. Because $\alpha(x)vx^{-1} \in V$, the transpose leaves it unchanged:

$$\alpha(x)vx^{-1} = (\alpha(x)vx^{-1})^t = (x^t)^{-1}v\alpha(x^t).$$

This can be rewritten as

$$v = x^t\alpha(x)vx^{-1}(\alpha(x^t))^{-1} = \alpha(\alpha(x^t)x)v(\alpha(x^t)x)^{-1} = \widetilde{\mathrm{Ad}}_{N(x)}(v).$$

Thus $N(x) \in \ker(\widetilde{\mathrm{Ad}})$. □

Corollary X.206.2. *Let $x \in \Gamma(V,q)$, then $\widetilde{\mathrm{Ad}}_x \in \mathrm{O}(V,q)$. Thus there is a group homomorphism*

$$\widetilde{\mathrm{Ad}} : \Gamma(V,q) \to \mathrm{O}(V,q).$$

Proof. First assume $v \in V^\times = \{v \in V \mid q(v) \neq 0\} \subset \Gamma(V,q)$. Then by X.192 and the previous corollary

$$q(\widetilde{\mathrm{Ad}}_x(v)) = N(\widetilde{\mathrm{Ad}}_x(v)) = N(\alpha(x)vx^{-1}) = N(\alpha(x))N(v)N(x^{-1}) = N(v)N(x)N(x^{-1}) = N(v) = q(v).$$

Now assume $q(v) = 0$. If $q(\widetilde{\mathrm{Ad}}_x(v))$ were not zero, then $\widetilde{\mathrm{Ad}}_x(v) \in V^\times$ and thus

$$q(v) = q(\widetilde{\mathrm{Ad}}_{x^{-1}} \circ \widetilde{\mathrm{Ad}}_x(v)) = q(\widetilde{\mathrm{Ad}}_x(v)) \neq 0$$

which is a contradiction. □

7.6.4 Double coverings

We define $SP(V,q) \coloneqq P(V,q) \cap \mathrm{Cl}^0(V,q)$.

Theorem X.207. *The homomorphisms*

$$\widetilde{\mathrm{Ad}} : P(V,q) \to \mathrm{O}(V,q) \qquad \text{and} \qquad \widetilde{\mathrm{Ad}} : SP(V,q) \to \mathrm{SO}(V,q)$$

are surjective. So we have the short exact sequence

$$1 \longrightarrow \mathbb{F}^\times \longrightarrow \Gamma(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V,q) \longrightarrow 1.$$

Proposition X.208. *The images $\widetilde{\mathrm{Ad}}(\mathrm{Pin}(V,q))$ and $\widetilde{\mathrm{Ad}}(\mathrm{Spin}(V,q))$ are both normal subgroups of $\mathrm{O}(V,q)$.*

Proof. By a simple calculation we have $\widetilde{\mathrm{Ad}}_{f(v)} = f \circ \widetilde{\mathrm{Ad}}_v \circ f^{-1}$ for all $v \in V$ and $f \in \mathrm{O}(V,q)$. □

> A field $\mathbb{F}$ of characteristic $\neq 2$ is called spin if for all $a \in \mathbb{F}^\times$ at least one of the equations $t^2 = a$ and $t^2 = -a$ has a solution t in $\mathbb{F}$.

Thus $\mathbb{F}$ is spin if

$$\mathbb{F}^\times = (\mathbb{F}^\times)^2 \cup (-(\mathbb{F}^\times)^2).$$

Lemma X.209. *The following fields are spin:*

1. *$\mathbb{R}$;*

2. $\mathbb{C}$*;*

3. $\mathbb{F}_p$ *with* p *prime and* $p \equiv 3 \mod 4$*.*

Theorem X.210. *Let* V *be a finite-dimensional vector space over a spin field* $\mathbb{F}$*, and suppose* q *is a non-degenerate quadratic form on* V*. Then there are short exact sequences*

$$\begin{tikzcd} 1 \arrow[r] & F \arrow[r] & \mathrm{Spin}(V,q) \arrow[r, "\widetilde{\mathrm{Ad}}"] & \mathrm{SO}(V,q) \arrow[r] & 1 \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & F \arrow[r] & \mathrm{Pin}(V,q) \arrow[r, "\widetilde{\mathrm{Ad}}"] & \mathrm{O}(V,q) \arrow[r] & 1 \end{tikzcd}$$

where

$$F = \begin{cases} \mathbb{Z}_2 = \{1, -1\} & \sqrt{-1} \notin \mathbb{F} \\ \mathbb{Z}_4 = \{\pm 1, \pm\sqrt{-1}\} & \textit{otherwise.} \end{cases}$$

This result holds for general fields if $\mathrm{SO}(V,q)$ *and* $\mathrm{O}(V,q)$ *are replaced by appropriate normal subgroups of* $\mathrm{O}(V,q)$*.*

Proof. Suppose $x = v_1 \dots v_r \in \mathrm{Pin}(V,q)$ is in the kernel of $\widetilde{\mathrm{Ad}}$. Then $x \in \mathbb{F}^\times$ and so

$$x^2 = N(x) = N(v_1)\dots N(v_r) = \pm 1$$

by X.206.1. ☐

Proposition X.211. *Let* $\mathbb{F}$ *be a spin field. Then either*

$$\Gamma(V,q) = P(v,q) \qquad \textit{or} \qquad \Gamma(V,q)/P(V,q) \cong \mathbb{Z}_2.$$

Proof. TODO

We have a group homomorphism $\widetilde{\mathrm{Ad}} : \Gamma(V,q) \to \mathrm{O}(V,q)$ with $\ker(\widetilde{\mathrm{Ad}}) = \mathbb{F}^\times$. ☐

7.7 Real and complex Clifford algebras

q-orthonormal basis.

Proposition X.212. *There is an algebra isomorphism*

$$\mathrm{Cl}_{r,s} \cong \mathrm{Cl}^0_{r,s+1} \qquad \forall r,s \in \mathbb{N}.$$

In particular $\mathrm{Cl}_n \cong \mathrm{Cl}^0_{n+1}$*.*

Proof. Take a q-orthonormal basis $\{e_i\}_{i=1}^{r+s+1}$ of $\mathbb{R}^{r+s+1}$ and let $\mathbb{R}^{r+s}$ be spanned by the basis $\{e_i\}_{i=1}^{r+s}$. Then define a linear map $f : \mathbb{R}^{r+s} \to \mathrm{Cl}^0_{r,s+1}$ by

$$f(e_i) = e_{r+s+1}e_i \qquad (i = 1, \dots, r+s).$$

We hope to apply the universal property X.182 to extend it to a map $\widetilde{f} : \mathrm{Cl}_{r,s} \to \mathrm{Cl}^0_{r,s+1}$. So we check $f(v)^2 = q(v)\mathbf{1}$. Indeed, let $v = \sum_{i=1}^{r+s} v_i e_i$, then

$$f(v)^2 = \sum_{i,j=1}^{r+s} v_i v_j e_{r+1} e_i e_{r+1} e_j = -\sum_{i,j=1}^{r+s} v_i v_j e_{r+1} e_{r+1} e_i e_j = \sum_{i,j=1}^{r+s} v_i v_j e_i e_j = q(v)\mathbf{1}.$$

It is easy to see $\widetilde{f}$ is bijective. ☐

7.7.1 Geometric algebra

A geometric algebra is a Clifford algebra $\mathrm{Cl}(\mathbb{R}^n, \|\cdot\|^2)$, where $\|\cdot\|$ is the standard norm. We denote the n-dimensional geometric algebra by $\mathbb{G}^n$.

In particular we have $u \cdot v = \langle u, v \rangle$.

7.7.1.1 Calculations

Proposition X.213. *Let* $\mathbf{v}$ *be a vector in* $\mathbb{G}^n$ *and* $\hat{\mathbf{e}}$ *a unit vector. Then*

$$\mathbf{v} = \langle \mathbf{v}, \hat{\mathbf{e}} \rangle \hat{\mathbf{e}} + (\mathbf{v} \wedge \hat{\mathbf{e}})\, \hat{\mathbf{e}}.$$

The first term commutes with $\hat{\mathbf{e}}$ *and the second term anti-commutes with* $\hat{\mathbf{e}}$*.*

Proof. We calculate

$$\mathbf{v} = \mathbf{v}\hat{\mathbf{e}}^2 = (\mathbf{v}\hat{\mathbf{e}})\hat{\mathbf{e}} = \Big(\langle \mathbf{v}, \hat{\mathbf{e}} \rangle + \mathbf{v} \wedge \hat{\mathbf{e}} \Big)\hat{\mathbf{e}} = \langle \mathbf{v}, \hat{\mathbf{e}} \rangle \hat{\mathbf{e}} + (\mathbf{v} \wedge \hat{\mathbf{e}})\hat{\mathbf{e}}.$$

The first term obviously commutes with $\hat{\mathbf{e}}$. The anti-commutivity of the second term follows from X.186. ☐

7.7.1.2 The pseudoscalar

TODO orientations and stuff

7.7.1.3 Rotors and rotations

`https://marctenbosch.com/quaternions/#h_16` -¿ why composition of rotors works.

Proposition X.214. $e^\theta \leftrightarrow \hat{\mathbf{a}}\hat{\mathbf{b}}$

Lemma X.215. *Let* $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ *be unit vectors in a geometric algebra* $\mathbb{G}^n$ *and let* $\{\mathbf{e}_1, \mathbf{e}_2\}$ *be an orthonormal basis of* $\operatorname{span}\{\hat{\mathbf{u}}, \hat{\mathbf{v}}\}$*. Then*

$$\hat{\mathbf{v}}\hat{\mathbf{u}} = \cos\theta + \sin\theta \hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 = e^{\theta\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2},$$

for some $\theta \in [0, 2\pi]$*.*

Proof. TODO ☐

In the plane $\operatorname{span}\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$, a rotation that maps $\hat{\mathbf{u}}$ to $\hat{\mathbf{v}}$ should act as $\lambda_{\hat{\mathbf{v}}\hat{\mathbf{u}}}$. On vectors perpendicular to the plane it should act as the identity.

Proposition X.216. *Let* $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ *be unit vectors in a geometric algebra* $\mathbb{G}^n$ *and let* $\{\mathbf{e}_1, \mathbf{e}_2\}$ *be an orthonormal basis of* $\operatorname{span}\{\hat{\mathbf{u}}, \hat{\mathbf{v}}\}$*. Set* $\mathbf{i} \coloneqq \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2$*. Then the rotation* $R_{\hat{\mathbf{v}}\leftarrow\hat{\mathbf{u}}}$ *that maps* $\hat{\mathbf{u}}$ *to* $\hat{\mathbf{v}}$ *can be written as*

$$R_{\hat{\mathbf{v}}\leftarrow\hat{\mathbf{u}}} : V \to V : \mathbf{w} \mapsto e^{\mathbf{i}\theta/2}\mathbf{w}e^{-\mathbf{i}\theta/2}.$$

Thus $R_{\hat{\mathbf{v}}\leftarrow\hat{\mathbf{u}}} = \mathrm{Ad}_{e^{\mathbf{i}\theta/2}} = \widetilde{\mathrm{Ad}}_{e^{\mathbf{i}\theta/2}}$*.*

Proof. We write $\mathbf{w} = \mathbf{w}_\| + \mathbf{w}_\perp \in \operatorname{span}\{\hat{\mathbf{u}}, \hat{\mathbf{v}}\} \oplus \operatorname{span}\{\hat{\mathbf{u}}, \hat{\mathbf{v}}\}^\perp$. Then

$$\begin{aligned} R_{\hat{\mathbf{v}}\leftarrow\hat{\mathbf{u}}}(\mathbf{w}) &= e^{\mathbf{i}\theta}\mathbf{w}_\| + \mathbf{w}_\perp \\ &= e^{\mathbf{i}\theta/2}e^{\mathbf{i}\theta/2}\mathbf{w}_\| + e^{\mathbf{i}\theta/2}e^{-\mathbf{i}\theta/2}\mathbf{w}_\perp \\ &= e^{\mathbf{i}\theta/2}\mathbf{w}_\| e^{-\mathbf{i}\theta/2} + e^{\mathbf{i}\theta/2}\mathbf{w}_\perp e^{-\mathbf{i}\theta/2} \\ &= e^{\mathbf{i}\theta/2}(\mathbf{w}_\| + \mathbf{w}_\perp)e^{-\mathbf{i}\theta/2} = e^{\mathbf{i}\theta/2}\mathbf{w}e^{-\mathbf{i}\theta/2}. \end{aligned}$$

We have used that elements of $\mathrm{span}\{\hat{\mathbf{u}}, \hat{\mathbf{v}}\}$ commute with $\mathbf{i}$ and elements of $\mathrm{span}\{\hat{\mathbf{u}}, \hat{\mathbf{v}}\}^\perp$ anticommute with $\mathbf{i}$. ☐

Corollary X.216.1. *Let $\hat{\mathbf{n}}$ be a unit vector in $\mathbb{G}^3$ and $\mathbf{I}$ a unit pseudoscalar. Then a rotation of θ radians around $\hat{\mathbf{n}}$ is given by*

$$R_{\hat{\mathbf{n}},\theta} : V \to V : \mathbf{w} \mapsto e^{\mathbf{nI}\theta/2}\mathbf{w}e^{-\mathbf{nI}\theta/2}.$$

Thus $R_{\hat{\mathbf{n}},\theta} = \mathrm{Ad}_{e^{\mathbf{nI}\theta/2}} = \widetilde{\mathrm{Ad}}_{e^{\mathbf{nI}\theta/2}}$. The direction is determined by the orientation of $\mathbf{I}$.

Proof. TODO ☐

We call an element of $\mathbb{G}^n$ of the form $e^{\mathbf{i}\theta/2}$ a rotor.

TODO Spin^+!

7.8 Representations

Let $K \subseteq k$ be fields, V a vector space over k and q a quadratic form on V. Then a K-representation of the Clifford algebra $\mathrm{Cl}(V, q)$ is a k-algebra homomorphism

$$\rho : \mathrm{Cl}(V, q) \to \mathrm{Hom}_K(W, W)$$

where W is a finite dimensional vector space over K. The space W is then a $\mathrm{Cl}(V, q)$-module over K.

Usually we will take the field K to be $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

Lemma X.217. 1. *A complex representation of $\mathrm{Cl}_{r,s}$ automatically extends to a representation of*

$$\mathrm{Cl}_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\mathrm{l}_{r+s}\,.$$

2. *A quaternionic representation of $\mathrm{Cl}_{r,s}$ is automatically complex.*

7.9 Lie algebra structures

Proposition X.218. *The Lie subalgebra of $(\mathrm{Cl}_n, [\cdot,\cdot])$ corresponding to the subgroup $\mathrm{Spin}_n \subset \mathrm{Cl}_n^\times$ is*

$$\mathfrak{spin}_n = \textstyle\bigwedge^2 \mathbb{R}^n.$$

In particular, $\bigwedge^2 \mathbb{R}^n$ is closed under the bracket operation.

Proof. We are looking for tangent vectors to the submanifold Spin_n at $\mathbf{1}$. Fix an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ and consider the curve

$$\gamma(t) = (e_i \cos t + e_j \sin t)(-e_i \cos t + e_j \sin t) = (\cos^2 t - \sin^2 t) + 2e_ie_j \sin t \cos t = \cos(2t)\sin(2t)e_ie_j.$$

This curve lies in Spin_n, satisfies $\gamma(0) = \mathbf{1}$ and its tangent vector at $\gamma(0)$ is $2e_ie_j$. Hence $\mathfrak{spin}_n$ contains $\mathrm{span}_{\mathbb{R}}\{e_ie_j\} = \bigwedge^2 \mathbb{R}^n$. Since $\dim_{\mathbb{R}}(\mathfrak{spin}_n)$ ☐

7.10 Geometry and geometric algebra

`http://www.faculty.luther.edu/~macdonal/GAConstruct.pdf`

7.10.1 Definitions

A geometric algebra $\mathfrak{G}$ is a real unital associative algebra of the form

$$\mathfrak{G} = \bigoplus_{r \in \mathbb{N}} \mathfrak{G}_r$$

such that

$$\begin{aligned} &\mathfrak{G}_0 = \operatorname{span}\{\mathbf{1}\} \\ &\mathfrak{G}_r = \operatorname{span}\{\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_r \mid \mathbf{a}_1, \ldots, \mathbf{a}_r \in \mathfrak{G}_1,\ \forall i, j \leq r : \mathbf{a}_i\mathbf{a}_j = -\mathbf{a}_j\mathbf{a}_i\} \quad \text{for all } r > 1. \end{aligned}$$

We also assume that the multiplication satisfies

$$\forall \mathbf{a} \in \mathfrak{G}_1 : \quad \mathbf{a}^2 = \mathbf{aa} = \lambda\mathbf{1} \in \mathfrak{G}_0 \qquad \text{for some } \lambda \in \mathbb{R}^{>0}$$

and that for each element a of $\mathfrak{G}_r \setminus \{0\}$ there exists a vector $\mathbf{a} \in \mathfrak{G}_1$ such that $\mathbf{a}a \in \mathfrak{G}_{r+1} \setminus \{0\}$.
We then call

- $\sqrt{\lambda}$ the magnitude of $\mathbf{a}$, denoted $|\mathbf{a}|$;
- the projection $\mathfrak{G} \to \mathfrak{G}_r$ the grade operator, denoted $\langle\cdot\rangle_r$;
- the multiplication of $\mathfrak{G}$ the geometric product on $\mathfrak{G}$;
- elements of $\mathfrak{G}$ multivectors;
- elements of $\mathfrak{G}_r$ r-vectors or homogenous multivectors; in particular 0-vectors are called scalars, 1-vectors vectors, 2-vectors bivectors ...
- r-vectors of the form $\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_r$ where $\mathbf{a}_1, \ldots, \mathbf{a}_r \in \mathfrak{G}_1$ anti-commute are called simple r-vectors or r-blades.

We use lowercase letters $a, b, c \ldots$ to denote multivectors, Greek letters $\mu, \nu, \lambda \ldots$ for scalars and bold letters $\mathbf{u}, \mathbf{v}, \mathbf{w} \ldots$ for vectors. Often we will use subscripts to denote the grade of a multivector, e.g a_r is an r-vector. Capital letters with subscript, e.g A_r, will be used to denote r-blades.

So for any $a \in \mathfrak{G}$, we can write

$$a = \langle a\rangle_0 + \langle a\rangle_1 + \ldots + \langle a\rangle_n = \sum_{r=0}^{n} \langle a\rangle_i$$

for some $n \in \mathbb{N}$.
We make the convention that negative grades are always zero.
Because of the assumption that no $\mathfrak{G}_r$ is trivial, we need $\mathfrak{G}_1$ to be infinite-dimensional.

Let $\mathfrak{G}$ be a geometric algebra. We define the reverse operation $\dagger$ on $\mathfrak{G}$ as the unique linear operation such that

$$\begin{aligned} \mathbf{1}^\dagger &= \mathbf{1} \\ \mathbf{u}^\dagger &= \mathbf{u} \qquad \mathbf{u} \in \mathfrak{G}_1 \\ (\mathbf{v}_1\mathbf{v}_2\ldots\mathbf{v}_r)^\dagger &= \mathbf{v}_r\ldots\mathbf{v}_2\mathbf{v}_1 \qquad \mathbf{v}_1,\ldots,\mathbf{v}_r \in \mathfrak{G}_1. \end{aligned}$$

Note that we have specified $\dagger$ on all basis elements of $\mathfrak{G}$, so it is well-defined and uniquely determined, cfr. X.21.

Lemma X.219. *Let $a, b \in \mathfrak{G}$. Then*

1. $(a^\dagger)^\dagger = a$;
2. $(ab)^\dagger = b^\dagger a^\dagger$;
3. $\langle a^\dagger \rangle_r = \langle a \rangle_r^\dagger = (-1)^{r(r-1)/2} \langle a \rangle_r$;
4. $\langle a \rangle_r = (-1)^{r(r-1)/2} \langle a \rangle_r^\dagger$;
5. $\langle a_r b_s \rangle_t = (-1)^{\frac{1}{2}(r(r-1)+s(s-1)+t(t-1))} \langle b_s a_r \rangle_t$.

Notice we are only interested in the exponent of (-1) modulo 2.

Let $\mathfrak{G}$ be a geometric algebra. We define the inner product $\cdot$ on homogeneous multivectors by

$$a_r \cdot b_s = \begin{cases} 0 & r = 0 \text{ or } s = 0 \\ \langle a_r b_s \rangle_{|r-s|} & \text{else.} \end{cases}$$

The inner product is bilinear on homogeneous multivectors and can thus be extended linearly to arbitrary multivectors.

So for arbitrary multivectors we have

$$a \cdot b = \sum_r \sum_s \langle a \rangle_r \cdot \langle b \rangle_s .$$

Let $\mathfrak{G}$ be a geometric algebra. We define the outer product $\wedge$ on homogeneous multivectors by

$$a_r \wedge b_s = \langle a_r b_s \rangle_{r+s}$$

The outer product is bilinear on homogeneous multivectors and can thus be extended linearly to arbitrary multivectors.

So for arbitrary multivectors we have

$$a \wedge b = \sum_r \sum_s \langle a \rangle_r \wedge \langle b \rangle_s .$$

For scalars $\lambda \in \mathfrak{G}^0$, we have

$$a \wedge \lambda = \lambda \wedge a = \lambda a \qquad a \in \mathfrak{G}.$$

We have explicitly excluded this in the inner product.

Lemma X.220. *If $a = \mathbf{v}_1\mathbf{v}_2\dots\mathbf{v}_r$, then $a \in \bigoplus_{i\leq r}\mathfrak{G}_i$.*
If a is an r-blade and β an orthogonal basis for $\mathfrak{G}_1$, then there exist $\mathbf{e}_1, \dots, \mathbf{e}_r \in \beta$ such that

$$a = \lambda\mathbf{e}_1\dots\mathbf{e}_r$$

be an r-blade, then

$$\mathbf{v}_1\mathbf{v}_2\dots\mathbf{v}_r = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \dots \wedge \mathbf{v}_r.$$

We introduce an order of operations (from highest priority to lowest):

1. outer product;
2. inner product;
3. geometric product.

Lemma X.221. *Let a_r, b_s be homogeneous vectors in $\mathfrak{G}$. Then*

1. $a_r \cdot b_s = (-1)^{s(r-1)}b_s \cdot a_r$ *for* $r \geq s$;
2. $a_r \wedge b_s = (-1)^{rs}b_s \wedge a_r$.

Proof. (1) We calculate

$$a_r \cdot b_s = \langle a_r b_s\rangle_{|r-s|} = (-1)^{\frac{1}{2}(r(r-1)+s(s-1)+|r-s|(|r-s|-1))}\langle a_r b_s\rangle_{|r-s|}.$$

We can simplify the exponent, assuming $r \geq s$, to

$$r^2 + s^2 - r - sr \equiv r + s + r + sr \equiv s + sr \equiv sr - s \mod 2.$$

(2) We calculate

$$a_r \wedge b_s = \langle a_r b_s\rangle_{r+s} = (-1)^{\frac{1}{2}(r(r-1)+s(s-1)+(r+s)((r+s)-1))}\langle a_r b_s\rangle_{r+s}.$$

We can simplify the exponent to

$$r^2 - r + s^2 - s + rs \equiv r - r + s - s + rs \equiv rs \mod 2.$$

□

By the fact that the definitions of inner and outer product make sense it is obvious that the geometric product does not preserve grade, or even homogeneity. It does, however, preserve a $\mathbb{Z}_2$-grading: we can split

$$\mathfrak{G} = \mathfrak{G}_{\text{even}} \oplus \mathfrak{G}_{\text{odd}} \qquad \text{where} \quad \begin{cases} \mathfrak{G}_{\text{even}} := \bigoplus_{r\in\mathbb{N}}\mathfrak{G}_{2r} \\ \mathfrak{G}_{\text{odd}} := \bigoplus_{r\in\mathbb{N}}\mathfrak{G}_{2r+1}. \end{cases}$$

We have the linear projection operators $\langle\cdot\rangle_+ : \mathfrak{G} \to \mathfrak{G}_{\text{even}}$ and $\langle\cdot\rangle_- : \mathfrak{G} \to \mathfrak{G}_{\text{odd}}$. We also write $\mathfrak{G}_+$ instead of $\mathfrak{G}_{\text{even}}$ and $\mathfrak{G}_-$ instead of $\mathfrak{G}_{\text{odd}}$.

Proposition X.222. *Let $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$ be a geometric algebra and let $p, q \in \{+, -\} \cong \mathbb{Z}_2$. Then*

$$\mathfrak{G}_p\mathfrak{G}_q \subset \mathfrak{G}_{pq}.$$

Proof. The grading operators $\langle\cdot\rangle_\pm$ are linear maps and thus determined by their action on basis elements. Thus it is enough to show that

$$\langle A_r B_s\rangle_+ = \begin{cases} A_r B_s & (r+s \equiv 0 \mod 2) \\ 0 & (r+s \equiv 1 \mod 2) \end{cases} \qquad \langle A_r B_s\rangle_- = \begin{cases} 0 & (r+s \equiv 0 \mod 2) \\ A_r B_s & (r+s \equiv 1 \mod 2) \end{cases}$$

for any r-blade A_r and s-blade B_s. In fact by associativity of the geometric product, it is enough to show

$$\mathbf{v}A_r$$

□

Lemma X.223. *Let $\mathbf{u}, \mathbf{v} \in \mathfrak{G}_1$. Then*

$$\mathbf{u}\cdot\mathbf{v} = \langle\mathbf{uv}\rangle_0 = \frac{1}{2}(\mathbf{uv}+\mathbf{vu}).$$

Proof. We start from $(\mathbf{u}+\mathbf{v})^2 = \mathbf{u}^2 + \mathbf{uv} + \mathbf{vu} + \mathbf{v}^2$ and rearrange to get

$$\mathbf{uv}+\mathbf{vu} = (\mathbf{u}+\mathbf{v})^2 - \mathbf{u}^2 - \mathbf{v}^2 = |\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2$$

which is scalar. Since $\langle\mathbf{uv}\rangle_0 = \langle\mathbf{vu}\rangle_0$, we have

$$\frac{1}{2}(\mathbf{uv}+\mathbf{vu}) = \frac{1}{2}\langle\mathbf{uv}+\mathbf{vu}\rangle_0 = \langle\mathbf{uv}\rangle_0 = \mathbf{u}\cdot\mathbf{v}.$$

□

Corollary X.223.1. *The geometric inner product restricted to $\mathfrak{G}_1$ is bilinear, symmetric and positive definite. It is thus an inner product as previously defined.*

The associated definitions are thus also applicable here. In particular two vectors $\mathbf{u}, \mathbf{v}$ are called orthogonal if $\mathbf{u}\cdot\mathbf{v} = 0$. By the lemma this is the case when $\mathbf{uv} = -\mathbf{vu}$. This means that the r vectors making up r-blades are linearly independent, X.122.

Lemma X.224. *For any algebra satisfying the other axioms, the direct sum $\mathfrak{G} = \bigoplus_{r\in\mathbb{N}}\mathfrak{G}_r$ is well-defined.*

Proof. We need to show that for all $r > s \in \mathbb{N}$, we have $\mathfrak{G}_r \cap \mathfrak{G}_s = \{0\}$. Assume, towards a contradiction, that here exist a_r, b_s such that $a_r = b_s$. Then both a_r and b_s can be written as sums of blades. Now let D be the set of all vectors featured in a blade in this sum. By Gram-Schmidt, we can find an orthogonal basis for D and rewrite a_r and b_s in this basis. As $r > s$, we can find elements of this orthogonal basis to multiply a_r with such that it becomes zero, but b_s remains non-zero (unless it already was zero). TODO: improve proof. □

Lemma X.225. *Let $\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in $\mathfrak{G}_1$. Then*

$$\mathbf{u}\cdot(\mathbf{v}_1\mathbf{v}_2\ldots\mathbf{v}_n) = \sum_{i=1}^{n}(-1)^{k+1}(\mathbf{u}\cdot\mathbf{v}_i)\mathbf{v}_1\ldots\breve{\mathbf{v}}_i\ldots\mathbf{v}_n,$$

where the breve indicates the vector under it is omitted from the product.

Proof. □

Lemma X.226.

$$\mathbf{u}a_r = \mathbf{u} \cdot a_r + \mathbf{u} \wedge a_r = \langle \mathbf{u}a_r \rangle_{r-1} + \langle \mathbf{u}a_r \rangle_{r+1}.$$

Proposition X.227. *Let* $\mathbf{v} \in \mathfrak{G}_1$ *and* $a_r \in \mathfrak{G}_r$*. Then*

Lemma X.228. *The outer product is associative:*

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

The inner product is not associative, but homogeneous multivectors obey

$$\begin{aligned} a_r \cdot (b_s \cdot c_t) &= (a_r \wedge b_s) \cdot c_t && \text{for } r + s \leq t \text{ and } r, s > 0 \\ a_r \cdot (b_s \cdot c_t) &= (a_r \cdot b_s) \cdot c_t && \text{for } r + t \leq s \end{aligned}$$

Lemma X.229.

$$\mathbf{u} \wedge a \wedge \mathbf{v} \wedge b = -\mathbf{v} \wedge a \wedge \mathbf{u} \wedge b$$

$\mathbf{v} \wedge a \wedge \mathbf{v} \wedge b = 0$

7.10.2 Affine spaces

7.10.3 Projections on 1D spaces

$$\sin(\theta) = \|a_\perp\| / \|a\| \qquad \cos(\theta) = \left\|a_\parallel\right\| / \|a\|.$$

7.10.4 The geometric product

7.10.5 Hodge duality

7.10.6 Cross product and triple product

Cross product not associative
Triple product nice way to find normal vectors with specific orientation.
`file:///C:/Users/user/Downloads/2013%20Matrix%20Computations%204th(1).pdf` `file:///C:/Users/user/Downloads/(Cambridge%20mathematical%20textbooks)%20Garcia,%20Stephan%20Ramon_%20Horn,%20Roger%20A.%20-%20A%20Second%20Course%20in%20Linear%20Algebra-Cambridge%20University%20Press%20(2017)(1).pdf`

Chapter 8

Coordinates and matrices

TODO Haynsworth inertia additivity formula

8.1 Coordinates

In this chapter we will purely be interested in finite-dimensional spaces. From proposition X.32 we know that for any n-dimensional vector space V over a field $\mathbb{F}$, $V \cong \mathbb{F}^n$. If we choose a basis β, we can explicitly give an isomorphism.

Let V be an n-dimensional vector space with basis $\beta = \{e_1, \ldots, e_n\}$. Because β is a basis, we can uniquely write every $v \in V$ as $a_1e_1 + \ldots + a_ne_n$. Then we define the coordinate map w.r.t. β as

$$\mathrm{co}_\beta : V \to \mathbb{F}^n : v \mapsto \mathrm{co}_\beta(v) = \mathrm{co}_\beta(a_1e_1 + \ldots + a_ne_n) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

The vector $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is called a coordinate vector.

The vector $\mathrm{co}_\beta(v)$ is also denoted $[v]_\beta$.

This coordinate map is indeed an isomorphism.

We conventionally write coordinate vectors as column vectors in $\mathbb{F}^n$. We will represent such vectors in bold type: $\mathbf{v} \in \mathbb{F}^n$.

Lemma X.230. *Let $\mathcal{E}$ be the standard basis of $\mathbb{F}^n$. Then*

$$\mathrm{co}_\mathcal{E} = \mathrm{id} = \mathrm{co}_\mathcal{E}^{-1}.$$

8.2 Matrices

A matrix is a rectangular grid of numbers. If it has m rows and n columns, we call it a $(m \times n)$-matrix. If the numbers are elements of the field $\mathbb{F}$, we denote the set of $(m \times n)$-matrices as $\mathbb{F}^{m\times n}$.
If $m = n$,we call the matrix a square matrix.

Example

An example of a (2×4)-matrix:

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$$

Sometimes square brackets are used, sometimes parentheses. It's just a matter of style.

Matrices are usually denoted using capital letters.

Let $n, m \in \mathbb{N}_0$.

- The zero matrix of dimension $n \times m$ is the $(n \times m)-$matrix

$$\mathbb{0}^{n\times m} = \begin{pmatrix}0 & \dots & 0\\ \vdots & \ddots & \\ 0 & \dots & 0\end{pmatrix}.$$

- The matrix of ones or all-ones matrix of dimension $n \times m$ is the $(n \times m)-$matrix

$$\mathbb{J}^{n\times m} = \begin{pmatrix}1 & \dots & 1\\ \vdots & \ddots & \\ 1 & \dots & 1\end{pmatrix}.$$

If $m = n$, we abbreviate these matrices as $\mathbb{0}_n$ and $\mathbb{J}_n$.

Lemma X.231. *The $(m \times n)$-matrices in $\mathbb{F}^{m\times n}$ naturally form a vector space with point-wise addition and scalar multiplication.*

We call matrices A, B conformal for a certain operation if the operation is defined on these matrices. So A, B are conformal for addition if they have the same dimensions.

8.2.1 Components of matrices

The element on the i^{th} row and j^{th} column of a matrix A is denoted $[A]_{i,j}$ or $a_{i,j}$. These numbers are known as the components of the matrix.
Vectors in $\mathbb{F}^n$ can be seen as matrices by writing them as column vectors. In this way we identify $\mathbb{F}^n$ with $\mathbb{F}^{n\times 1}$.
Let $\mathbf{v} \in \mathbb{F}^n$. The components of $\mathbf{v}$ are of the form $[\mathbf{v}]_{i,1}$. We abbreviate this to $[\mathbf{v}]_i$.

Lemma X.232. *The functions*

$$[-]_{ij} : A \mapsto [A]_{ij} \qquad \text{and} \qquad [-]_i : \mathbf{v} \mapsto [\mathbf{v}]_i$$

are linear.

Conversely, consider a set of numbers $a_{i,j}$ where $i \in (1 : m), j \in (1 : n)$. Then by $[a_{i,j}]$ we mean the matrix consisting of those numbers.
We can also consider components after applying a function. For some linear map f, we often write

$$f_i \coloneqq [-]_i \circ f.$$

Let A be an $(m \times n)$-matrix.

- If $1 \leq j \leq m$, then $[A]_{j,-}$ denotes the $(1 \times n)$-matrix consisting of row j of A.
- If $1 \leq k \leq n$, then $[A]_{-,k}$ denotes the $(m \times 1)$-matrix consisting of column k of A.

Lemma X.233. *Let $A, B \in \mathbb{F}^{m\times n}$ and $\lambda \in \mathbb{F}$, then*

- $[A + B]_{ij} = [A]_{ij} + [B]_{ij}$;
- $[\lambda A]_{ij} = \lambda [A]_{ij}$.

Let $A \in \mathbb{F}^{m\times n}$ be a matrix. A component $[A]_{i,j}$ is

- on the diagonal if $i = j$;
- off-diagonal $i \neq j$;
- on the k^{th} superdiagonal if $j = i + k$;
- on the k^{th} subdiagonal if $j = i - k$.

8.2.1.1 Submatrices

Let $A \in \mathbb{F}^{m\times n}$ be a matrix and $I \subseteq 1 : m$ and $J \subseteq 1 : n$ sets, then $[A]_{I,J}$ is the matrix consisting only of those entries whose row number is in I and whose column number is in J. A matrix of this form is called a submatrix.

Example

Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix},$$

then

$$[A]_{1:2,1:3} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix}.$$

8.2.1.2 Types of matrices

Let $A \in \mathbb{F}^{n \times n}$. We say

- A is upper triangular if $i > j \implies [A]_{i,j} = 0$;
- A is strictly upper triangular if $i \geq j \implies [A]_{i,j} = 0$;
- A is lower triangular if $i < j \implies [A]_{i,j} = 0$;
- A is strictly lower triangular if $i \leq j \implies [A]_{i,j} = 0$;
- A is triangular if it is upper or lower triangular.

We say

- A is diagonal if $i \neq j \implies [A]_{i,j} = 0$; in this case we write $A = \operatorname{diag}([A]_{11}, \ldots, [A]_{nn})$;
- A is tridiagonal if $|i \neq j| \geq 2 \implies [A]_{i,j} = 0$;
- A is bidiagonal if it is tridiagonal and triangular.

We say

- A is a permutation matrix if exactly one entry in each row and in each column is 1; all other entries are 0.

8.2.2 Matrix multiplication

Assume we have n vectors $v_1, \ldots, v_n$ in $\mathbb{F}^m$. We may be interested in linear combinations of these vectors, say $a_1 v_1 + \ldots + a_n v_n$. We can collect the coefficients a_i in a column vector in $\mathbb{F}^n$. The vectors v_i can be written as columns and placed in a matrix.

Consider the action that pairs such a matrix of column vectors with the element in its column space determined by a column matrix. This action is called matrix multiplication and is denoted by juxtaposing the matrix and the vector (sometimes separated by a dot).

Example

Let $v_1 = (1, 3, 4)$ and $v_2 = (2, 5, 6)$ be vectors in $\mathbb{R}^3$. These can be placed as columns in a matrix:

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 6 \end{pmatrix}$$

Consider the linear combination $2v_1 + v_2$, we can write this as the matrix multiplication

$$2v_1 + v_2 = \begin{pmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 1 \cdot 2 \\ 2 \cdot 3 + 1 \cdot 5 \\ 2 \cdot 4 + 1 \cdot 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \\ 14 \end{pmatrix}$$

Example

A very important case (and one we will explore in more detail later) is given by systems of linear equations. We might have the following equations:

$$\begin{cases} 2x + y - z = 3 \\ -x + y + 3z = 2 \\ x + y = -2 \end{cases}$$

This can be rewritten as

$$x \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + z \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

where each row is an equation. In this case the coefficients are the unknowns x, y, z. So using the notation of matrix multiplication, the equations become

$$\begin{pmatrix} 2 & 1 & -1 \\ -1 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}.$$

We can view matrix multiplication as a function

$$\mathbb{F}^{m\times n} \times \mathbb{F}^n \to \mathbb{F}^m : (A, \mathbf{v}) \mapsto \begin{pmatrix} \sum_{i=1}^n [A]_{1,i}[\mathbf{v}]_i \\ \vdots \\ \sum_{i=1}^n [A]_{m,i}[\mathbf{v}]_i \end{pmatrix}.$$

Let B be an $(m \times n)$-matrix and $\mathbf{v}$ a vector in $\mathbb{F}^n$. Then the matrix multiplication $B \cdot \mathbf{v}$ gives a vector in $\mathbb{F}^m$. This can be used as the input for another matrix multiplication, if multiplied by a $(k \times m)$ matrix A. So the expression $A(B\mathbf{v})$makes sense.
Now we would like to define matrix multiplication between the matrices B, A by the condition that

$$(A \cdot B)\mathbf{v} = A(B\mathbf{v}).$$

In other words we are asserting the associativity of the matrix multiplication.
Consider the component equations: $[A(B\mathbf{v})]_i = [(B \cdot A)\mathbf{v}]_i$. We can then calculate:

$$\begin{aligned} [A(B\mathbf{v})]_i &= \sum_{j=1}^m [A]_{i,j}[B\mathbf{v}]_j = \sum_{j=1}^m [A]_{i,j}(\sum_{k=1}^n [B]_{j,k}[\mathbf{v}]_k) = \sum_{j=1}^m \sum_{k=1}^n [A]_{i,j}[B]_{j,k}[\mathbf{v}]_k \\ &= \sum_{k=1}^n \left(\sum_{j=1}^m [A]_{i,j}[B]_{j,k} \right) [\mathbf{v}]_k =: [(A \cdot B)\mathbf{v}]_i \end{aligned}$$

The last equation can only be satisfied for all $[v]_i$ if the matrix multiplication is defined such that

$$[A \cdot B]_{i,k} := \sum_{j=1}^m [A]_{i,j}[B]_{j,k}.$$

Of course our construction only works if the dimensions of A, B are such that $A(B\mathbf{v})$ is well-defined.

Let A, B be matrices.

- The matrices A, B are conformal for multiplication if $A \in \mathbb{F}^{k\times m}$ and $B \in \mathbb{F}^{m\times n}$ for some $k, m, n \in \mathbb{N}_0$.
- If A and B are conformal, we define the product AB by

$$[AB]_{i,k} = \sum_{j=1}^{m} [A]_{i,j}[B]_{j,k}.$$

Thus matrix multiplication can be thought of as a map $\mathbb{F}^{k\times m} \times \mathbb{F}^{m\times n} \to \mathbb{F}^{k\times n}$

The compatibility requirement can be abbreviated by

$$[k \times m] \cdot [m \times n] = [k \times n].$$

Notice that the matrix multiplication $\mathbb{F}^{m\times n} \times \mathbb{F}^{n} \to \mathbb{F}^{m}$ we originally defined is a special case of this more general matrix multiplication if we identify $\mathbb{F}^{n}$ with $\mathbb{F}^{n\times 1}$ and $\mathbb{F}^{m}$ with $\mathbb{F}^{m\times 1}$ (that is, we view $\mathbb{F}^{n}, \mathbb{F}^{m}$ as column vectors). In this case we have the multiplication

$$[m \times n] \cdot [n \times 1] = [m \times 1].$$

Lemma X.234. *The matrix multiplication map $\mathbb{F}^{k\times m} \times \mathbb{F}^{m\times n} \to \mathbb{F}^{k\times n}$ is linear in both arguments.*

Proof. For linearity in the first argument, assume $A_1, A_2 \in \mathbb{F}^{k\times m}$ and $\lambda \in \mathbb{F}$. Then

$$[(\lambda A_1 + A_2)B]_{ij} = \sum_{j=1}^{m} [\lambda A_1 + A_2]_{i,j}[B]_{j,k} = \lambda \left(\sum_{j=1}^{m} [A_1]_{i,j}[B]_{j,k} \right) + \left(\sum_{j=1}^{m} [A_2]_{i,j}[B]_{j,k} \right).$$

The proof of linearity in second argument is similar. ☐

The identity matrix of dimension n is the $(n \times n)$-matrix

$$\mathbb{1}_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

The components of $\mathbb{1}_n$ are given by

$$[\mathbb{1}_n]_{ij} = \delta_{ij}.$$

Lemma X.235. *Let $A \in \mathbb{F}^{m\times n}$. Then*

$$A = \mathbb{1}_m \cdot A = A \cdot \mathbb{1}_n.$$

Proof. By a simple calculation in components:

$$[\mathbb{1}_m \cdot A]_{ij} = \sum_{k=1}^{m} \delta_{ik}[A]_{kj} = [A]_{ij}.$$

The other equation is similar. □

Lemma X.236. *Let $l, m, n \in \mathbb{N}_0$, then*

$$\mathbb{J}^{l\times m}\mathbb{J}^{m\times n} = m\mathbb{J}^{l\times n}.$$

Proof.

$$[\mathbb{J}^{l\times m}\mathbb{J}^{m\times n}]_{i,j} = \sum_{k=1^m}[\mathbb{J}^{l\times m}]_{i,k}[\mathbb{J}^{m\times n}]_{k,j} = \sum_{k=1^m} 1 = m.$$

□

Corollary X.236.1. *Let $a, b, c, d \in \mathbb{R}$, then*

$$(a\mathbb{1}_n + b\mathbb{J}_n)(c\mathbb{1}_n + d\mathbb{J}_n) = ac\mathbb{1}_n + (ad + bc + bdn)\mathbb{J}_n$$

8.2.2.1 Left and right inverses

Let $A \in \mathbb{F}^{m\times n}$. A matrix B is a left inverse of A if $BA = \mathbb{1}_n$. A matrix B is a right inverse of A if $AB = \mathbb{1}_m$.

Not all matrices have a left and/or right inverses.

8.2.2.2 Square matrices

Lemma X.237. *For any $n \in \mathbb{N}_0$, the vector space $\mathbb{F}^{n\times n}$ of square matrices is a monoid with as operation matrix multiplication and as neutral element the identity matrix $\mathbb{1}_n$.*

In particular we can define integer powers of matrices. We set $A^0 = \mathbb{1}_n$ by convention.
We also have that if square matrices have both a left inverse and a right inverse, they are the same. See II.16.
In fact, we have something stronger:

Lemma X.238. *Let $A \in \mathbb{F}^{n\times n}$ be a square matrix. Then A has a left inverse* if and only if *A has a right inverse. Both inverses are the same.*

Proof. Consider the map $f : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n} : B \mapsto AB$. (This will later be called the left regular representation of A).
We follow a chain of implications.

- *A has left inverse $\Rightarrow$ f is injective.* Assume there exist matrices B_1, B_2 such that $AB_1 = AB_2$, then

$$0 = A^{-1}0 = A^{-1}(AB_1 - AB_2) = A^{-1}A(B_1 - B_2) = B_1 - B_2.$$

- *f is injective $\Rightarrow$ f is surjective.* By X.34.
- *f is surjective $\Rightarrow$ A has right inverse.* By surjectivity there exists a matrix B such that $f(B) = AB = \mathbb{1}_n$. Then B is a right inverse by definition.

We have proven that the existence of a left inverse implies the existence of a right inverse. The opposite implication is obtained by considering the right regular representation $f : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n} : B \mapsto BA$. ☐

A square matrix is called invertible or nonsingular or nondegenerate if it has a left and right inverse.
If it is not invertible, it is called singular or degenerate.
We denote the inverse of A as A^{-1}.

Lemma X.239. *Let $A, B \in \mathbb{F}^{n\times n}$ be invertible matrices. Then AB is invertible with inverse*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Lemma X.240. *Let $a, b \in \mathbb{R}$. Then*

$$(a\mathbb{1}_n + b\mathbb{J}_n)^{-1} = \frac{1}{a(a+nb)}[(a+nb)\mathbb{1}_n - b\mathbb{J}_n] = \frac{1}{a}\mathbb{1}_n - \frac{b}{a(a+nb)}\mathbb{J}_n.$$

Proof. By multiplication, using X.236. ☐

Let $A \in \mathbb{F}^{n\times n}$. We say A is strictly diagonally dominant if

$$\sum_{\substack{j\in 1:n \\ j\neq i}} |[A]_{ij}| < |[A]_{ii}|.$$

Proposition X.241. *If $A \in \mathbb{F}^{n\times n}$ is strictly diagonally dominant, then it is invertible.*

Proof. We prove the contraposition. Assume, then, that A is not invertible, so there exists an $x \neq 0$ such that $Ax = 0$ (TODO ref). So then for all $i \in 1 : n$ we have

$$\sum_{j=1}^{n}[A]_{ij}x_j.$$

Setting $|x_m| = \max_{i\in 1:n} |x_i|$, we have

$$[A]_{mm}x_m = -\sum_{\substack{j\in 1:n \\ j\neq m}} [A]_{mj}x_j$$

and so

$$|[A]_{mm}| \cdot |x_m| = |\sum_{\substack{j\in 1:n \\ j\neq m}} [A]_{mj}x_j| \leq \sum_{\substack{j\in 1:n \\ j\neq m}} |[A]_{mj}| \cdot |x_j| \leq \sum_{\substack{j\in 1:n \\ j\neq m}} |[A]_{mj}| \cdot |x_m|.$$

In particular this means A cannot be strictly diagonally dominant. ☐

8.2.3 Matrices and linear maps

8.2.3.1 Matrices as maps $\mathbb{F}^n \to \mathbb{F}^m$

The matrix multiplication $\mathbb{F}^{m\times n} \times \mathbb{F}^n \to \mathbb{F}^m$ can be curried to produce a map ℓ. The input of ℓ, i.e. a matrix in $\mathbb{F}^{m\times n}$ is typically written as a subscript. So, for a matrix $A \in \mathbb{F}^{m\times n}$ we have

$$\ell_A : \mathbb{F}^n \to \mathbb{F}^m : \mathbf{v} \mapsto \ell_A(\mathbf{v}) = A\mathbf{v}.$$

Because the matrix multiplication is linear in the second argument (see X.234), the map $\ell_A : \mathbb{F}^n \to \mathbb{F}^m$ is linear for all matrices $A \in \mathbb{F}^{m\times n}$. So we have

$$\ell : \mathbb{F}^{m\times n} \to \operatorname{Hom}(\mathbb{F}^n, \mathbb{F}^m).$$

Additionally this map ℓ is linear due to the matrix multiplication being linear in the first argument (see again X.234).

Proposition X.242. *For all $n, m \in \mathbb{N}_0$, the map*

$$\ell : \mathbb{F}^{m\times n} \to \operatorname{Hom}(\mathbb{F}^n, \mathbb{F}^m)$$

is an isomorphism.

Proof. We will explicitly construct an inverse. Take some $L \in \operatorname{Hom}(\mathbb{F}^n, \mathbb{F}^m)$. Now L is completely determined by the images of the elements in the standard basis $\mathcal{E} = \{\mathbf{e}_i\}_{i=1}^n$, so we just need to find a matrix A such that ℓ_A maps the basis elements to the same elements as L.
Now $A\mathbf{e}_1$ is just the first column of A. So we set the first column of A to be $L(\mathbf{e}_1)$. Similarly $A\mathbf{e}_i$ is the i^{th} column of A and we set it equal to $L(\mathbf{e}_i)$. This gives the required matrix. $\square$

The matrix A that satisfies $\ell_A = L$ is often denoted A_L.
The construction in the previous proof is important for practically finding matrices associated with linear maps. The construction can be recapped as follows:

$$A_L = \begin{pmatrix} L(\mathbf{e}_1) & L(\mathbf{e}_2) & \dots & L(\mathbf{e}_n) \end{pmatrix} = \begin{pmatrix} L_1(\mathbf{e}_1) & L_1(\mathbf{e}_2) & \dots & L_1(\mathbf{e}_n) \\ L_2(\mathbf{e}_1) & L_2(\mathbf{e}_2) & & \\ \vdots & & \ddots & \\ L_m(\mathbf{e}_1) & & & L_m(\mathbf{e}_n) \end{pmatrix} \qquad [A_L]_{ij} = L_i(\mathbf{e}_j)$$

where $\mathcal{E} = \{\mathbf{e}_i\}_{i=1}^n$ is the standard basis of $\mathbb{F}^n$ and $L_i(\mathbf{e}_j) = [L(\mathbf{e}_j)]_i$ is the i^{th} component of $L(\mathbf{e}_j)$.
If $\mathbb{F}^n$ is equipped with the standard inner product, then this can also be written as $L_i(\mathbf{e}_j) = \langle \mathbf{e}_i, L(\mathbf{e}_j)\rangle$, so

$$A_L = \begin{pmatrix} \langle \mathbf{e}_1, L(\mathbf{e}_1)\rangle & \langle \mathbf{e}_1, L(\mathbf{e}_2)\rangle & \dots & \langle \mathbf{e}_1, L(\mathbf{e}_n)\rangle \\ \langle \mathbf{e}_2, L(\mathbf{e}_1)\rangle & \langle \mathbf{e}_2, L(\mathbf{e}_2)\rangle & & \\ \vdots & & \ddots & \\ \langle \mathbf{e}_m, L(\mathbf{e}_1)\rangle & & & \langle \mathbf{e}_m, L(\mathbf{e}_n)\rangle \end{pmatrix}.$$

Proposition X.243. *The map ℓ translates matrix multiplication into function composition:*

$$\ell_{AB} = \ell_A \circ \ell_B \qquad \text{and} \qquad \ell^{-1}(f \circ g) = \ell^{-1}(f)\ell^{-1}(g).$$

Proof. Let $A \in \mathbb{F}^{k\times m}$ and $B \in \mathbb{F}^{m\times n}$. Let $\mathbf{v} \in \mathbb{F}^n$, then

$$\ell_{AB}(\mathbf{v}) = (AB)\mathbf{v} = A(B\mathbf{v}) = A(\ell_B(\mathbf{v})) = \ell_A(\ell_B(\mathbf{v})) = (\ell_A \circ \ell_B)(\mathbf{v}).$$

$\square$

Lemma X.244. *For all $n \in \mathbb{N}$ we have $\operatorname{id}_{\mathbb{F}^{n\times n}} = \ell_{\mathbb{1}_n}$.*

Lemma X.245. *Let A be a matrix over $\mathbb{F}$. Then ℓ_A is invertible* if and only if *A is square and invertible.*

Proof. Assume $\ell_A : \mathbb{F}^n \to \mathbb{F}^m$ invertible. Then $\mathbb{F}^n \cong \mathbb{F}^m$, so $m = n$ by X.32. Also $\ell^{-1}((\ell_A)^{-1})$ is an inverse of A because

$$\ell^{-1}((\ell_A)^{-1}) \cdot A = \ell^{-1}((\ell_A)^{-1}) \cdot \ell^{-1}(\ell_A) = \ell^{-1}[(\ell_A)^{-1}\ell_A] = \ell^{-1}(\mathrm{id}_{\mathbb{F}^n}) = \mathbb{1}_n.$$

Conversely, assume A invertible with inverse A^{-1}. Then $\ell_{A^{-1}}$ is the inverse of ℓ_A:

$$\ell_{A^{-1}}\ell_A = \ell_{\mathbb{1}_n} = \mathrm{id}_{\mathbb{F}^n} \qquad \text{and} \qquad \ell_A\ell_{A^{-1}} = \ell_{\mathbb{1}_n} = \mathrm{id}_{\mathbb{F}^n}\,.$$

□

8.2.3.2 Linear maps as matrices

We can associate a matrix to any linear map by passing to coordinates. Let $L : V \to W$ be a linear map from an n-dimensional vector space V to an m-dimensional vector space W. If we fix bases $\mathcal{V}$ of V and $\mathcal{W}$ of W, then ℓ^{-1} associates a unique matrix with the linear map

$$\mathrm{co}_{\mathcal{W}} \circ L \circ \mathrm{co}_{\mathcal{V}}^{-1} : \mathbb{F}^n \to \mathbb{F}^m.$$

In other words, A is the unique matrix such that

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\mathrm{co}_{\mathcal{V}}"'] & W \arrow[d, "\mathrm{co}_{\mathcal{W}}"] \\ \mathbb{F}^n \arrow[r, "\ell_A"] & \mathbb{F}^m \end{tikzcd} \qquad \text{commutes.}$$

We call this matrix $A := \ell^{-1}(\mathrm{co}_{\mathcal{W}} \circ L \circ \mathrm{co}_{\mathcal{V}}^{-1})$ the matrix of the linear map L w.r.t. the bases $\mathcal{V}$ and $\mathcal{W}$. This matrix is denoted

$$(L)_{\mathcal{V}}^{\mathcal{W}} \qquad \text{or} \qquad {}^{\mathcal{W}}(L)^{\mathcal{V}} \qquad \text{or} \qquad (L)_{\mathcal{W}\leftarrow\mathcal{V}}.$$

The commutativity of the diagram translates to the following lemma:

Lemma X.246. *Let $L : V \to W$ be a linear map and $\mathcal{V}, \mathcal{W}$ bases of V, W respectively. Then*

$$(L)_{\mathcal{V}}^{\mathcal{W}} \circ \mathrm{co}_{\mathcal{V}} = \mathrm{co}_{\mathcal{W}} \circ L.$$

A practical way to calculate matrices associated with linear maps is given by the following lemma.

Lemma X.247. *Let $L : V \to W$ be a linear map and $\mathcal{V} = \{\mathbf{v}_i\}_{i=1}^n, \mathcal{W} = \{\mathbf{w}_i\}_{i=1}^m$ bases of V, W respectively. Then*

$$(L)_{\mathcal{V}}^{\mathcal{W}} = \begin{pmatrix}\mathrm{co}_{\mathcal{W}}(L(\mathbf{v}_1)) & \mathrm{co}_{\mathcal{W}}(L(\mathbf{v}_2)) & \ldots & \mathrm{co}_{\mathcal{W}}(L(\mathbf{v}_n)).\end{pmatrix}$$

Proof. The i^{th} column of $(L)_{\mathcal{V}}^{\mathcal{W}}$ is equal to $(L)_{\mathcal{V}}^{\mathcal{W}}\mathbf{e}_i = (L)_{\mathcal{V}}^{\mathcal{W}}(\mathrm{co}_{\mathcal{V}}(\mathbf{v}_i))$, where $\mathbf{e}_i$ is the i^{th} element of the standard basis $\mathcal{E}$. This is equal to $\mathrm{co}_{\mathcal{W}}(L(\mathbf{v}))$ by X.246. □

Proposition X.248. *Let U, V, W be vector spaces with bases $\mathcal{U}, \mathcal{V}, \mathcal{W}$, resp., and $S : V \to W, T : U \to V$ linear maps. Then*

$$(S)_{\mathcal{V}}^{\mathcal{W}}(T)_{\mathcal{U}}^{\mathcal{V}} = (S \circ T)_{\mathcal{U}}^{\mathcal{W}}.$$

Proof. We calculate

$$\begin{aligned}\ell^{-1}(\mathrm{co}_{\mathcal{W}} \circ S \circ \mathrm{co}_{\mathcal{V}}^{-1})\ell^{-1}(\mathrm{co}_{\mathcal{V}} \circ T \circ \mathrm{co}_{\mathcal{U}}^{-1}) &= \ell^{-1}(\mathrm{co}_{\mathcal{W}} \circ S \circ \mathrm{co}_{\mathcal{V}}^{-1} \circ \mathrm{co}_{\mathcal{V}} \circ T \circ \mathrm{co}_{\mathcal{U}}^{-1}) \\ &= \ell^{-1}(\mathrm{co}_{\mathcal{W}} \circ (S \circ T) \circ \mathrm{co}_{\mathcal{U}}^{-1}) = (S \circ T)_{\mathcal{U}}^{\mathcal{W}}.\end{aligned}$$

□

Proposition X.249. *Let V, W be finite-dimensional vector spaces over a field $\mathbb{F}$ with bases $\mathcal{V}$ and $\mathcal{W}$, respectively. The mapping*

$$(-)_{\mathcal{V}}^{\mathcal{W}} : \mathrm{Hom}_{\mathbb{F}}(V, W) \to \mathbb{F}^{m \times n} : L \mapsto (L)_{\mathcal{V}}^{\mathcal{W}}$$

is an isomorphism.

Proof. It is equal to

$$\ell^{-1} \circ (\mathrm{co}_{\mathcal{W}})_* \circ (\mathrm{co}_{\mathcal{V}}^{-1})^*.$$

Now ℓ is an isomorphism by X.242 and thus ℓ^{-1} is one by X.31. The maps $(\mathrm{co}_{\mathcal{W}})_*$ and $(\mathrm{co}_{\mathcal{V}}^{-1})^*$ are injective maps between finite-dimensional spaces by IV.38 and thus isomorphisms by X.34. So we have a composition of isomorphisms, which is an isomorphism. □

Corollary X.249.1. *Let V, W be finite-dimensional vector spaces over a field $\mathbb{F}$, then*

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(V, W) = (\dim_{\mathbb{F}} V) \cdot (\dim_{\mathbb{F}} W).$$

Lemma X.250. *Let $A \in \mathbb{F}^{m \times n}$ be a matrix and let $\mathcal{E}_m$ and $\mathcal{E}_n$ be the standard bases of $\mathbb{F}^m$ and $\mathbb{F}^n$. Then*

$$(\ell_A)_{\mathcal{E}_n}^{\mathcal{E}_m} = A.$$

8.2.3.3 Changing basis with matrices

In particular we can apply all the theory of the previous section to the identity map $\mathrm{id} : V \to V$. Let β, β' be two bases of V. Then X.246 gives

$$\mathrm{co}_{\beta'}(v) = (\mathrm{id})_{\beta}^{\beta'} \mathrm{co}_{\beta}(v)$$

which captures the effect of transforming from one basis to another.

Matrices of the form $(\mathrm{id})_{\beta}^{\beta'}$ are called transition matrices or change-of-basis matrices.

Lemma X.251. *Let β, β' be two bases of a vector space V. Then*

$$\left((\mathrm{id})_{\beta}^{\beta'}\right)^{-1} = (\mathrm{id})_{\beta'}^{\beta}.$$

Proof. This follows from X.248 which gives

$$(\mathrm{id})_{\beta'}^{\beta}(\mathrm{id})_{\beta}^{\beta'} = (\mathrm{id})_{\beta}^{\beta} = \mathbb{1}_n.$$

□

Let $L \in \mathrm{Hom}(V)$. If we know $(L)_{\beta}^{\beta}$, we can calculate $(L)_{\beta'}^{\beta'}$ using

$$\begin{aligned}(L)_{\beta'}^{\beta'} &= (\mathrm{id})_{\beta}^{\beta'}(L)_{\beta}^{\beta}(\mathrm{id})_{\beta'}^{\beta} \\ &= \left((\mathrm{id})_{\beta'}^{\beta}\right)^{-1}(L)_{\beta}^{\beta}(\mathrm{id})_{\beta'}^{\beta}\end{aligned}$$

Let $A, B \in \mathbb{F}^{n\times n}$, then A and B are called similar if there exists an invertible matrix $P \in \mathbb{F}^{n\times n}$ such that

$$B = P^{-1}AP$$

Any similar matrices may be seen as matrices of the same linear transformation w.r.t. different bases.

8.2.4 The transpose

Let $A \in \mathbb{F}^{m\times n}$. The transpose of A, denoted A^{T}, is defined by

$$[A^{\mathrm{T}}]_{ij} = [A]_{ji}.$$

Lemma X.252. *The transpose is a linear operation.*

Lemma X.253. *Let A, B be matrices such that AB is defined, then*

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}.$$

Proof. We simply calculate

$$[(AB)^{\mathrm{T}}]_{ij} = [AB]_{ji} = \sum_k [A]_{jk}[B]_{ki} = \sum_k [B]_{ki}[A]_{jk} = \sum_k [B^{\mathrm{T}}]_{ik}[A^{\mathrm{T}}]_{kj} = [B^{\mathrm{T}}A^{\mathrm{T}}]_{ij}.$$

□

- A square matrix A such that $A = A^{\mathrm{T}}$ is called symmetric.
- A square matrix A such that $A = -A^{\mathrm{T}}$ is called skew symmetric.

8.2.4.1 The standard inner product

Let $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$. Then the standard inner product is given by

$$\langle \mathbf{v}, \mathbf{w}\rangle := \overline{\mathbf{v}}^{\mathrm{T}}\mathbf{w}.$$

This reduces to $\mathbf{v}^{\mathrm{T}}\mathbf{w}$ if $\mathbb{F} = \mathbb{R}$.
In the sequel we will always assume $\mathbb{F}^n$ is equipped with the standard inner product, unless otherwise specified.
The inner product also induces a norm. There are multiple norms that may be of interest. To avoid confusion we may denote the norm that arises from the inner product with a subscript 2:

$$\|\mathbf{v}\| = \|\mathbf{v}\|_2 = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}.$$

8.2.4.2 Adjoint

From the definition of the standard inner product, it is clear that $\overline{\mathbf{v}}^{\mathrm{T}}$ is an important operation. We give this operation its own symbol:

$$\mathbf{v}^* := \overline{\mathbf{v}}^{\mathrm{T}} \qquad \text{so} \qquad \langle \mathbf{v}, \mathbf{w}\rangle = \mathbf{v}^*\mathbf{w}.$$

In fact this definition makes sense for all matrices:

Let $A \in \mathbb{F}^{m \times n}$. The conjugate transpose A^* of A is the matrix

$$A^* := \overline{A}^{\mathrm{T}} = \overline{(A^{\mathrm{T}})}.$$

It is also called the adjoint of A.

Lemma X.254. *Let A, B be conformal matrices. Then*

1. $(A^*)^*$;
2. *the conjugate transpose is antilinear:*
$$(cA + B)^* = \overline{c}A^* + B^* \qquad \forall c \in \mathbb{F};$$
3. *if A is invertible, then $(A^*)^{-1} = (A^{-1})^*$.*
4. *if $\mathbb{F} = \mathbb{R}$, then $A^* = A^{\mathrm{T}}$.*

Proposition X.255. *Let $A \in \mathbb{F}^{n \times n}$. Then for all $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$,*

$$\langle A\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, A^*\mathbf{w} \rangle .$$

Proof. $\langle A\mathbf{v}, \mathbf{w} \rangle = (A\mathbf{v})^*\mathbf{w} = \mathbf{v}^*A^*\mathbf{w} = \langle \mathbf{v}, A^*\mathbf{w} \rangle$. □

Let $A \in \mathbb{F}^{n \times n}$ be a square matrix

- if $A = A^*$, then A is called Hermitian;
- if $A = -A^*$, then A is called skew Hermitian;
- if $A^*A = \mathbb{1}_n$, then A is called unitary; a real unitary matrix is called orthogonal;
- if $A^*A = AA^*$, then A is called normal;
- if $A^2 = A = A^*$, then A is called an orthogonal projection.

The name orthogonal projection is due to the following: it is a projection because applying it multiple times is the same as applying it once; it is orthogonal because for all $\mathbf{v} \in \mathbb{F}^n$, $P\mathbf{v}$ and $\mathbf{v} - P\mathbf{v} = (\mathbb{1} - P)\mathbf{v}$ are orthogonal:

$$\langle P\mathbf{v}, \mathbf{v} - P\mathbf{v} \rangle = \langle P\mathbf{v}, \mathbf{v} \rangle - \langle P\mathbf{v}, P\mathbf{v} \rangle = \langle P\mathbf{v}, \mathbf{v} \rangle - \langle P^*P\mathbf{v}, \mathbf{v} \rangle = \langle P\mathbf{v}, \mathbf{v} \rangle - \left\langle P^2\mathbf{v}, \mathbf{v} \right\rangle = \langle P\mathbf{v}, \mathbf{v} \rangle - \langle P\mathbf{v}, \mathbf{v} \rangle = 0.$$

Lemma X.256. *Let $U \in \mathbb{F}^{n \times n}$ be a matrix. Then the following are equivalent*

1. *U is unitary, i.e. $U^*U = \mathbb{1}_n$;*
2. *the columns of U are orthonormal;*
3. *$UU^* = \mathbb{1}_n$;*
4. *U^* is unitary;*
5. *the row of U are orthonormal;*
6. *U is invertible and $U^{-1} = U^*$.*

Notice that $U^*U = \mathbb{1} \iff UU^* = \mathbb{1}$ only holds for square matrices.

Corollary X.256.1. *If α, β are orthonormal bases of $\mathbb{F}^n$, then the transition matrix $(\mathrm{id})^{\beta}_{\alpha}$ is unitary.*

8.2.5 Block matrices

Any given matrix can be *interpreted* as consisting of submatrices. Eg,

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} \quad \text{can be viewed as} \quad \begin{bmatrix}\begin{pmatrix}1 & 2\end{pmatrix} & 3\\ \begin{pmatrix}4 & 5\\ 7 & 8\end{pmatrix} & \begin{pmatrix}6\\ 9\end{pmatrix}\end{bmatrix} \quad \text{with partitioning} \quad \left[\begin{array}{cc|c}1 & 2 & 3\\ \hline 4 & 5 & 6\\ 7 & 8 & 9\end{array}\right]$$

A matrix consisting of submatrices (also known as blocks) is called a block matrix or partitioned matrix.
If $A \in \mathbb{F}^{n\times n}, \mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ and $c \in \mathbb{F}$, we call the block matrices

$$\begin{bmatrix}c & \mathbf{x}^{\mathrm{T}}\\ \mathbf{y} & A\end{bmatrix}, \begin{bmatrix}\mathbf{x}^{\mathrm{T}} & c\\ A & \mathbf{y}\end{bmatrix}, \begin{bmatrix}A & \mathbf{x}\\ \mathbf{y}^{\mathrm{T}} & c\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}\mathbf{x} & A\\ c & \mathbf{y}^{\mathrm{T}}\end{bmatrix}$$

bordered matrices. They have been obtained from A by bordering.

The partition can be specified at the level of the dimensions: let A be an $(m \times n)$-matrix, let $m_1, \ldots, m_k \leq m$ be the number of rows in each horizontal partition and let $n_1, \ldots, n_l \leq n$ be the number of columns in each vertical partition. Clearly

$$m = \sum_{i=1}^{k} m_i \qquad \text{and} \qquad n = \sum_{i=1}^{l} n_i.$$

We then say the block matrix has dimensions $(m_1|\ldots|m_k)\times(n_1|\ldots|n_l)$, or A is a $(m_1|\ldots|m_k)\times(n_1|\ldots|n_l)$-matrix.

Example

The block matrix considered above is a $(1|2) \times (2|1)$-matrix.

We write $A_{i,j}$ or $(A)_{i,j}$ for the block in the i^{th} horizontal partition and the j^{th} vertical partition. So if A is a $(m_1|\ldots|m_k) \times (n_1|\ldots|n_l)$-matrix, we can write

$$A = \begin{bmatrix}A_{1,1} & \ldots & A_{1,l}\\ \vdots & \ddots & \vdots\\ A_{k,1} & \ldots & A_{k,l}\end{bmatrix} \qquad \text{where} \qquad A_{i,j} \in \mathbb{F}^{m_i \times n_j}.$$

Any $m \times n$-matrix can be partitioned into columns, which means identifying it with an $m \times (1|1|\ldots|1)$-matrix.
Similarly, any $m \times n$-matrix can be partitioned into rows, which means identifying it with a $(1|1|\ldots|1) \times n$-matrix.

Lemma X.257. *Let A be a partitioned matrix of dimensions $(m_1|\ldots|m_k) \times (n_1|\ldots|n_l)$, then A^{T} is a partitioned matrix of dimensions $(n_1|\ldots|n_l) \times (m_1|\ldots|m_k)$ and*

$$(A)_{i,j} = (A^{\mathrm{T}})_{j,i}.$$

Let A, B be matrices. We can then form the direct sum matrix $A \oplus B$ as follows:

$$A \oplus B = \begin{pmatrix} A & \mathbb{0} \\ \mathbb{0} & B \end{pmatrix}.$$

Proposition X.258. *Let A be a partitioned matrix of dimensions $(m_1|\ldots|m_k) \times (n_1|\ldots|n_l)$ and B a partitioned matrix of dimensions $(n_1|\ldots|n_l) \times (p_1|\ldots|p_q)$.*
The matrix product of A and B is an $(m_1|\ldots|m_k) \times (p_1|\ldots|p_q)$-matrix with blocks

$$(AB)_{i,j} = \sum_t A_{i,t}B_{t,j}.$$

That is, multiplication of two block matrices can be carried out as if their blocks were scalars.

We can abbreviate the dimension requirements as

$$[(m_1|\ldots|m_k) \times (n_1|\ldots|n_l)] \cdot [(n_1|\ldots|n_l) \times (p_1|\ldots|p_q)] = [(m_1|\ldots|m_k) \times (p_1|\ldots|p_q)].$$

Corollary X.258.1. *Let A, B be matrices of dimensions $k \times l$ and $l \times m$. Then*

$$AB = [AB]_{-,-} = \sum_k [A]_{-,k}[B]_{k,-};$$

and

- *we can partition A into rows and B into columns to get*

$$[AB]_{i,j} = [A]_{i,-}[B]_{-,j};$$

- *we can partition B into columns to get This can also be written as*

$$[AB]_{-,j} = [A]_{-,-}[B]_{-,j} = A[B]_{-,j};$$

- *we can partition A into rows to get This can also be written as*

$$[AB]_{i,-} = [A]_{i,-}[B]_{-,-} = [A]_{i,-}B.$$

In particular this means we can write

$$AB = \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} [B]_{-,1} & \ldots & [B]_{-,m} \end{pmatrix} = \begin{pmatrix} A[B]_{-,1} & \ldots & A[B]_{-,m} \end{pmatrix}.$$

8.2.5.1 Identities and inverses

Lemma X.259. *Let X, Y, Z be conformal matrices and Y, Z invertible, then*

$$\begin{bmatrix} Y & X \\ \mathbb{0} & Z \end{bmatrix}^{-1} = \begin{bmatrix} Y^{-1} & -Y^{-1}XZ^{-1} \\ \mathbb{0} & Z^{-1}. \end{bmatrix}.$$

In particular

$$\begin{bmatrix} \mathbb{1} & X \\ \mathbb{0} & \mathbb{1} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbb{1} & -X \\ \mathbb{0} & \mathbb{1} \end{bmatrix}.$$

Corollary X.259.1. *If A is an upper triangular matrix with nonzero diagonal entries, the A is invertible and* $[A^{-1}]_{i,i} = [A]_{i,i}^{-1}$

Proof. Induction on dimension. □

Lemma X.260. *Let X, Y, Z be conformal matrices. Then*

$$\begin{bmatrix} Y & X \\ \mathbb{0} & Z \end{bmatrix} = \begin{bmatrix} \mathbb{1} & A \\ \mathbb{0} & \mathbb{1} \end{bmatrix} \begin{bmatrix} Y & X - AZ + YA \\ \mathbb{0} & Z \end{bmatrix} \begin{bmatrix} \mathbb{1} & -A \\ \mathbb{0} & \mathbb{1} \end{bmatrix}$$

for all conformal A.

Lemma X.261. *Let A, B, C, D be conformal matrices. Then, if D is invertible*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathbb{1} & BD^{-1} \\ \mathbb{0} & \mathbb{1} \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & \mathbb{0} \\ \mathbb{0} & D \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ D^{-1}C & \mathbb{1} \end{bmatrix}$$

and if A is invertible,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ CA^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} A & \mathbb{0} \\ \mathbb{0} & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} \mathbb{1} & A^{-1}B \\ \mathbb{0} & \mathbb{1} \end{bmatrix}.$$

The matrix $M/D \coloneqq A - BD^{-1}C$ is the Schur complement of D in M.
Similarly $M/A \coloneqq D - CA^{-1}B$ is the Schur complement of A in M.

Lemma X.262. *Let A, B, C, D be conformal matrices such that all requisite matrices are invertible. Then*

$$\begin{aligned} M^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} &= \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ -D^{-1}C & \mathbb{1} \end{bmatrix} \begin{bmatrix} (M/D)^{-1} & \mathbb{0} \\ \mathbb{0} & D^{-1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & -BD^{-1} \\ \mathbb{0} & \mathbb{1} \end{bmatrix} \\ &= \begin{bmatrix} (M/D)^{-1} & -(M/D)^{-1}BD^{-1} \\ -D^{-1}C(M/D)^{-1} & D^{-1} + D^{-1}C(M/D)^{-1}BD^{-1} \end{bmatrix} \\ M^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} &= \begin{bmatrix} \mathbb{1} & -A^{-1}B \\ \mathbb{0} & \mathbb{1} \end{bmatrix} \begin{bmatrix} A^{-1} & \mathbb{0} \\ \mathbb{0} & (M/A)^{-1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ -CA^{-1} & \mathbb{1} \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} + A^{-1}B(M/A)^{-1}CA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{bmatrix}. \end{aligned}$$

Lemma X.263. *Let A, B be conformal matrices. Then*

$$\begin{bmatrix} AB & A \\ \mathbb{0} & \mathbb{0} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \mathbb{0} & A \\ \mathbb{0} & BA \end{bmatrix}$$

are similar

Proof. $\begin{bmatrix} \mathbb{1} & \mathbb{0} \\ B & \mathbb{1} \end{bmatrix} \begin{bmatrix} AB & A \\ \mathbb{0} & \mathbb{0} \end{bmatrix} = \begin{bmatrix} \mathbb{0} & A \\ \mathbb{0} & BA \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ B & \mathbb{1} \end{bmatrix}.$ □

Proposition X.264. *Let A, B, C, U, V be conformal matrices. Then*

1. (Push-through identity) $(\mathbb{1} + UV)^{-1}U = U(\mathbb{1} + VU)^{-1}$;
2. $$\begin{aligned} (\mathbb{1} + A)^{-1} &= \mathbb{1} - (\mathbb{1} + A)^{-1}A \\ &= \mathbb{1} - A(\mathbb{1} + A)^{-1} \end{aligned}$$

3. $(\mathbb{1}+UV)^{-1} = \mathbb{1} - U(\mathbb{1}+VU)^{-1}V$*;*

4. (Woodbury identity) $(B+UCV)^{-1} = B^{-1} - B^{-1}U(C^{-1}+VB^{-1}U)VB^{-1}$.

Proof. (1) From $U(\mathbb{1}+VU) = (\mathbb{1}+UV)U$.
(2) From $\mathbb{1} = (\mathbb{1}+A)(\mathbb{1}+A)^{-1} = (\mathbb{1}+A)^{-1} + A(\mathbb{1}+A)^{-1}$.
(3)
$$\begin{aligned}(\mathbb{1}+UV)^{-1} &= \mathbb{1} - (\mathbb{1}+UV)^{-1}UV \quad \text{using (2)}\\ &= \mathbb{1} - U(\mathbb{1}+VU)^{-1}V \quad \text{using (1).}\end{aligned}$$
(4) □
$$\begin{aligned}(B+UCV)^{-1} &= (B(\mathbb{1}+B^{-1}UCV))^{-1}\\ &= (\mathbb{1}+(B^{-1}U)(CV))^{-1}B^{-1}\\ &= (\mathbb{1}-(B^{-1}U)(\mathbb{1}+(CV)(B^{-1}U))^{-1}(CV))B^{-1} \quad \text{using (3)}\\ &= B^{-1} - B^{-1}U(\mathbb{1}+CVB^{-1}U)^{-1}CVB^{-1}\\ &= B^{-1} - B^{-1}U(C^{-1}(\mathbb{1}+CVB^{-1}U))^{-1}VB^{-1}\\ &= B^{-1} - B^{-1}U(C^{-1}+VB^{-1}U)^{-1}VB^{-1}.\end{aligned}$$

Corollary X.264.1 (Sherman–Morrison formula)**.** *Let* $A \in \mathbb{F}^{n\times n}$ *and* $\mathbf{u},\mathbf{v} \in \mathbb{F}^n$*. Then* $A+\mathbf{u}\mathbf{v}^{\mathrm{T}}$ *is invertible iff* $1+\mathbf{v}^{\mathrm{T}}A^{-1}\mathbf{u} \neq 0$*. In this case*

$$(A+\mathbf{u}\mathbf{v}^{\mathrm{T}})^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}A^{-1}}{1+\mathbf{v}^{\mathrm{T}}A^{-1}\mathbf{u}}.$$

Corollary X.264.2 (Hua's identity)**.** *Let* A, B *be conformal matrices. Then*

$$\begin{aligned}(A+B)^{-1} &= A^{-1} - A^{-1}(B^{-1}+A^{-1})^{-1}A^{-1}\\ &= A^{-1} - A^{-1}(AB^{-1}+\mathbb{1})^{-1}\\ (A-B)^{-1} &= A^{-1} + A^{-1}B(A-B)^{-1}\\ &= \sum_{k=0}^{\infty}(A^{-1}B)^kA^{-1}.\end{aligned}$$

8.2.6 Vector spaces associated with a matrix

8.2.6.1 Row and column space

Let A be an $(n\times m)$-matrix. It can be partitioned into both rows and columns. Let $R_1,\ldots,R_n$ be the rows of A and $C_1,\ldots,C_m$ the columns of A:

$$A = \begin{pmatrix}R_1\\ \vdots \\ R_n\end{pmatrix} = \begin{pmatrix}C_1 & \ldots & C_m\end{pmatrix}.$$

Then

- $\operatorname{span}\{R_1,\ldots,R_n\}$ is the row space $\operatorname{row}(A)$ of A; and
- $\operatorname{span}\{C_1,\ldots,C_m\}$ is the column space $\operatorname{col}(A)$ of A.

We call $\dim\operatorname{row}(A)$ the row rank and $\dim\operatorname{col}(A)$ the column rank.

Clearly $\operatorname{col}(A) = \operatorname{row}(A^{\mathrm{T}})$.

Lemma X.265. *Let $A \in \mathbb{F}^{m\times n}$ and $\mathbf{b} \in \mathbb{F}^m$. Then*

$$\exists \mathbf{x} \in \mathbb{F}^n : A\mathbf{x} = \mathbf{b} \qquad \iff \qquad \mathbf{b} \in \operatorname{col}(A).$$

Moreover, if the columns of A are linearly independent, then the $\mathbf{x} \in \mathbb{F}^n$ is unique.

Proof. By X.258.1

$$\begin{aligned} A\mathbf{x} = [A\mathbf{x}]_{-,-} &= \sum_{j=1}^{n}[A]_{-,j}[\mathbf{x}]_{j,-} = \sum_{j=1}^{n}[A]_{-,j}[\mathbf{x}]_j \\ &= [A]_{-,1}[\mathbf{x}]_1 + \dots [A]_{-,n}[\mathbf{x}]_n \\ &= C_1[\mathbf{x}]_1 + \dots C_n[\mathbf{x}]_n. \end{aligned}$$

□

Proposition X.266. *Let A and B be matrices. Then*

1. $\operatorname{col}(B) \subseteq \operatorname{col}(A) \iff B = AX$ *for some matrix X;*
2. $\operatorname{row}(B) \subseteq \operatorname{row}(A) \iff B = YA$ *for some matrix Y.*

Note that for point (1) to hold, A and B must have the same number of rows. For point (2) they must have the same number of columns.

Proof. (1) $\boxed{\Rightarrow}$ Assume $\operatorname{col}(B) \subseteq \operatorname{col}(A)$. Then by X.265, for each column $\mathbf{b}_j = [B]_{-,j}$ of B we can find an $\mathbf{x}_j \in \mathbb{F}^n$ such that $A\mathbf{x}_j = \mathbf{b}_j$. Then

$$B = \begin{pmatrix}\mathbf{b}_1 & \dots & \mathbf{b}_k\end{pmatrix} = \begin{pmatrix}A\mathbf{x}_1 & \dots & A\mathbf{x}_k\end{pmatrix} = A\begin{pmatrix}\mathbf{x}_1 & \dots & \mathbf{x}_k\end{pmatrix} = AX.$$

$\boxed{\Leftarrow}$ By X.258.1 we can write

$$AB = \begin{pmatrix}A[B]_{-,1} & \dots & A[B]_{-,m}\end{pmatrix},$$

so every column in AB is of the form $A[B]_{-,i}$, which is a linear combination of the columns in A.
(2) We simply calculate using point (1):

$$\operatorname{row}(B) \subseteq \operatorname{row}(A) \iff \operatorname{col}(B^{\mathrm{T}}) \subseteq \operatorname{col}(A^{\mathrm{T}}) \iff B^{\mathrm{T}} = A^{\mathrm{T}}X \iff B = X^{\mathrm{T}}A.$$

□

Corollary X.266.1. *Let A, B be conformal matrices. Then*

1. $\operatorname{col}(AB) \subseteq \operatorname{col}(A)$*;*
2. $\operatorname{row}(AB) \subseteq \operatorname{row}(B)$*.*

Corollary X.266.2. *A matrix A is invertible* if and only if $\operatorname{col}(A) = \operatorname{row}(A) = \mathbb{F}^n$.

Proof. $\boxed{\Rightarrow}$ From $A = \mathbb{1}_n A$ we see that $\operatorname{col}(A) \subseteq \operatorname{col}(\mathbb{1}_n) = \mathbb{F}^n$.
From $\mathbb{1}_n = AA^{-1}$ we see that $\operatorname{col}(\mathbb{1}_n) \subseteq \operatorname{col}(A)$, so $\operatorname{col}(\mathbb{1}_n) = \operatorname{col}(A)$. The calculation of the row space is similar.
$\boxed{\Leftarrow}$ From $\operatorname{col}(A) = \operatorname{col}(\mathbb{1}_n)$, there exists an X such that $\mathbb{1}_n = AX$, so X is the inverse of A. □

Corollary X.266.3. *Let $A \in \mathbb{F}^{m\times n}$. Then*

$$\dim \operatorname{row}(A) = \dim \operatorname{col}(A)$$

i.e. the row rank equals the column rank.

Proof. Take a basis for $\operatorname{col}(A)$ and let X have these vectors as columns. Then $\operatorname{col}(X) = \operatorname{col}(A)$, so $A = XY$ for some $Y \in \mathbb{F}^{k\times n}$.
By point 2. of the proposition, we have $\operatorname{row}(A) \subseteq \operatorname{row}(Y)$ and due to the dimensions, we have $\dim \operatorname{row}(Y) \leq k$. So

$$\dim \operatorname{row}(A) \leq \dim \operatorname{row}(Y) \leq k = \dim \operatorname{col}(A).$$

Consider A^{T}, which can be factorised as before by taking a basis of its column space and putting the vectors in the columns of X'. Then $A^{\mathrm{T}} = X'Y'$. As before, we have

$$\dim \operatorname{col}(A) = \dim \operatorname{row}(A^{\mathrm{T}}) \leq \dim \operatorname{col}(A^{\mathrm{T}}) = \dim \operatorname{row}(A).$$

Combining the inequalities gives $\dim \operatorname{col}(A) = \dim \operatorname{row}(A)$. □

> We can unambiguously call $\dim \operatorname{col}(A) = \dim \operatorname{row}(A)$ the rank of the matrix A. We write $\operatorname{rank}(A)$.

Lemma X.267. *Let $A \in \mathbb{F}^{m\times n}$.*

1. *If $\operatorname{rank}(A) = m$, then $n \geq m$ and there exists $X \in \mathbb{F}^{(n-m)\times n}$ such that*

$$\begin{bmatrix} A \\ X \end{bmatrix} \in \mathbb{F}^{n\times n} \quad \text{is invertible.}$$

2. *If $\operatorname{rank}(A) = n$, then $m \geq n$ and there exists $Y \in \mathbb{F}^{m\times(m-n)}$ such that*

$$\begin{bmatrix} A & Y \end{bmatrix} \in \mathbb{F}^{m\times m} \quad \text{is invertible.}$$

Proof. (1) In this case the rows are linearly independent and elements of $\mathbb{F}^n$, so they can be extended to a basis of $\mathbb{F}^n$. This extension is the matrix X.
(2) Transpose and apply (1). □

Lemma X.268 (Full-rank factorisation)**.** *Let $A \in \mathbb{F}^{m\times n}$ and $k = \dim \operatorname{col}(A)$. Then A can be factorised as $A = XY$ where $X \in \mathbb{F}^{m\times k}$, $Y \in \mathbb{F}^{k\times n}$ and*

$$k = \operatorname{rank}(A) = \operatorname{rank}(X) = \operatorname{rank}(Y).$$

Moreover, for any matrix X, the following are equivalent:

1. *the columns of X form a basis of $\operatorname{col}(A)$;*
2. *there is a unique $Y \in \mathbb{F}^{k\times n}$ such that $A = XY$.*

Clearly considering A^{T} yields dual equivalences.

Proof. By X.265 $Xv = [A]_{-,j}$ has a unique solution $v = y_j$ for all j if and only if the columns of X are linearly independent and $[A]_{-,j}$ is in their span, i.e. they from a basis for $\operatorname{col}(A)$. □

Proposition X.269. *Let L be a linear map. Then*

$$\mathrm{im}(L) = \mathrm{col}(A_L).$$

This implies that the rank of L is the rank of A.

Proposition X.270. *Let $A, B \in \mathbb{F}^{m\times n}$. Then*

$$\mathrm{col}(A) = \mathrm{col}(B) \iff \exists \text{ invertible } X \text{ such that } A = BX.$$

Proof. $\boxed{\Leftarrow}$ follows from X.266.
$\boxed{\Rightarrow}$ Let C be a matrix whose columns form a basis of $\mathrm{col}(A) = \mathrm{col}(B)$. Then we can find matrices S, T such that $CS = A$ and $CT = B$ are full-rank factorisations and these can be extended to invertible matrices by X.267

$$X_1 = \begin{bmatrix} S \\ U \end{bmatrix} \in \mathbb{F}^{n\times n} \qquad X_2 = \begin{bmatrix} T \\ V \end{bmatrix} \in \mathbb{F}^{n\times n}.$$

Now we claim $X = X_2^{-1}X_1$ fulfils the requirements:

$$BX = (CT + 0V)X_2^{-1}X_1 = \begin{bmatrix} C & \mathbb{0} \end{bmatrix} X_2(X_2^{-1}X_1) = \begin{bmatrix} C & \mathbb{0} \end{bmatrix} X_1 = CS = A.$$

□

8.2.6.2 Null space

Let $A \in \mathbb{F}^{m\times n}$ be a matrix. The null space $\mathrm{null}(A)$ of A is the kernel of ℓ_A. The dimension of $\mathrm{null}(A)$ is called the nullity of A.

In other words:

$$\mathrm{null}(A) = \{\mathbf{v} \in \mathbb{F}^n \mid A\mathbf{v} = 0\}.$$

Proposition X.271. *Let $A \in \mathbb{F}^{m\times n}$ be a matrix, then*

$$\mathrm{null}(A) = \mathrm{col}(A^*)^\perp.$$

Proof. $\mathbf{v} \in \mathrm{null}(A) \iff A\mathbf{v} = 0 \iff \forall \mathbf{w} \in \mathbb{F}^n : \langle A\mathbf{v}, \mathbf{w}\rangle = 0 \iff \forall \mathbf{w} \in \mathbb{F}^n : \langle \mathbf{v}, A^*\mathbf{w}\rangle = 0 \iff \mathbf{v} \in \mathrm{col}(A^*)^\perp$. □

Lemma X.272. *Let $A \in \mathbb{F}^{m\times n}$ be a matrix. Then*

$$\mathrm{rank}(A) + \dim \mathrm{null}(A) = n.$$

Proof. This is the dimension theorem applied to ℓ_A, using $\mathrm{im}(\ell_A) = \mathrm{col}(A)$. □

We can also formulate X.27 for matrices:

Proposition X.273. *Let A, B be conformal matrices. Then*

1. $\mathrm{null}(AB) \supseteq \mathrm{null}(B)$;
2. $\dim \mathrm{null}(AB) = \dim \mathrm{null}(B) + \dim(\mathrm{col}(B) \cap \mathrm{null}(A))$.

Note that (1) is the opposite inclusion to $\mathrm{col}(AB) \subseteq \mathrm{col}(A)$ and $\mathrm{row}(AB) \subseteq \mathrm{row}(B)$.

Corollary X.273.1 (Sylvester's law of nullity)**.** *Let A, B be square matrices. Then*

$$\max\{\dim \mathrm{null}(A), \dim \mathrm{null}(B)\} \leq \dim \mathrm{null}(AB) \leq \dim \mathrm{null}(A) + \dim \mathrm{null}(B).$$

8.2.6.3 Rank equalities and inequalities

Proposition X.274. *Let A, B, C, D be conformal matrices, then*

1. $\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$;
2. $\max\{\operatorname{rank}(A), \operatorname{rank}(C)\} \leq \operatorname{rank}\begin{bmatrix} A & C \end{bmatrix}$;
3. $\max\{\operatorname{rank}(A), \operatorname{rank}(D)\} \leq \operatorname{rank}\begin{bmatrix} A \\ D \end{bmatrix}$.

Proof. (1) This is the matrix form of X.26. It is also an immediate consequence of X.266.1. □

Lemma X.275. *Let $A \in \mathbb{F}^{m\times n}$, then*

$$\operatorname{rank}(A) \leq \min\{m, n\}.$$

Let $A \in \mathbb{F}^{m\times n}$.

- If $\operatorname{rank}(A) = \min\{m, n\}$, we say A has full rank.
- If $\operatorname{rank}(A) = m$, we say A has full row rank.
- If $\operatorname{rank}(A) = n$, we say A has full column rank.

Lemma X.276. *Let X, A, Y be conformal matrices. Then*

1. *if X has full column rank, then* $\operatorname{rank}(A) = \operatorname{rank}(XA)$;
2. *if Y has full row rank, then* $\operatorname{rank}(A) = \operatorname{rank}(AY)$;
3. *A is invertible* if and only if *it has full row rank and full column rank.*

Proof. (1) By X.272, $\operatorname{null}(X) = \{0\}$. So $\mathbf{v} \in \operatorname{null}(XA) \iff \mathbf{v} \in \operatorname{null}(A)$, so $\operatorname{null}(XA) = \operatorname{null}(A)$. Then X.272 implies $\operatorname{rank}(XA) = \operatorname{rank}(A)$.
(2) $\operatorname{rank}(AY) = \operatorname{rank}(Y^{\mathrm{T}}A^{\mathrm{T}}) = \operatorname{rank}(A^{\mathrm{T}}) = \operatorname{rank}(A)$.
(3) Consequence of X.266.2. □

Proposition X.277 (Sylvester's rank inequality)**.** *Let $A \in \mathbb{F}^{m\times k}$ and $B \in \mathbb{F}^{k\times n}$, then*

$$\operatorname{rank}(AB) \geq \operatorname{rank}(A) + \operatorname{rank}(B) - k.$$

In addition, the following are equivalent:

1. $\operatorname{null}(A) \subseteq \operatorname{col}(B)$;
2. $\operatorname{rank}(AB) = \operatorname{rank}(A) + \operatorname{rank}(B) - k$.

In particular if AB is a full-rank factorisation, i.e. $\operatorname{rank}(AB) = k$, *then (1) and (2) hold.*

Proof. Let $AB = XY$ be a full-rank factorisation of AB and set $r = \operatorname{rank}(AB)$. Then define

$$C = \begin{bmatrix} A & X \end{bmatrix} \in \mathbb{F}^{m\times(k+r)} \quad \text{and} \quad D = \begin{bmatrix} B \\ -Y \end{bmatrix} \in \mathbb{F}^{(k+r)\times n}$$

so $CD = \begin{bmatrix} A & X \end{bmatrix} \begin{bmatrix} B \\ -Y \end{bmatrix} = AB - XY = 0$. This means that $\operatorname{col}(D) \subseteq \operatorname{null}(C)$, so $\operatorname{rank}(D) \leq \dim\operatorname{null}(C)$. Then

$$\operatorname{rank}(A) + \operatorname{rank}(B) \leq \operatorname{rank}(C) + \operatorname{rank}(D) \leq \operatorname{rank}(C) + \dim\operatorname{null}(C) = k + \operatorname{rank}(AB).$$

using X.272 for $\operatorname{rank}(C) + \dim\operatorname{null}(C) = k + r$.
Now for the equivalent statements:
$\boxed{(1) \Rightarrow (2)}$ In this case X.273 becomes

$$\dim\operatorname{null}(AB) = \dim\operatorname{null}(A) + \dim\operatorname{null}(B).$$

Using X.272, this becomes

$$n - \operatorname{rank}(AB) = k - \operatorname{rank}(A) + n - \operatorname{rank}(B),$$

which can be arranged to give (2).
$\boxed{(2) \Rightarrow (1)}$ Assume, towards contraposition, $\operatorname{null}(A) \nsubseteq \operatorname{col}(B)$. Then we can find $\mathbf{v} \in \operatorname{null}(A)$ such that $\mathbf{v} \notin \operatorname{col}(B)$. Then

$$\operatorname{rank}\begin{bmatrix} B & \mathbf{v} \end{bmatrix} = \operatorname{rank}(B) + 1 \qquad \text{and} \qquad \operatorname{rank}(A\begin{bmatrix} B & \mathbf{v} \end{bmatrix}) = \operatorname{rank}\begin{bmatrix} AB & A\mathbf{v} \end{bmatrix} = \operatorname{rank}(AB).$$

And by the inequality

$$\operatorname{rank}(AB) \geq \operatorname{rank}(A) + \operatorname{rank}(B) + 1 - k,$$

so in particular $\operatorname{rank}(AB) > \operatorname{rank}(A) + \operatorname{rank}(B) - k$ and thus $\operatorname{rank}(AB) \neq \operatorname{rank}(A) + \operatorname{rank}(B) - k$.
Finally:
$\boxed{(\operatorname{rank}(AB) = k) \Rightarrow (1)}$ From $\operatorname{rank}(AB) \leq \operatorname{rank}(A) \leq k$ and $\operatorname{rank}(AB) \leq \operatorname{rank}(B) \leq k$ we get

$$\operatorname{rank}(A) = \operatorname{rank}(B) = \operatorname{rank}(AB) = k$$

and so

$$\operatorname{rank}(A) + \operatorname{rank}(B) - k = k = \operatorname{rank}(AB).$$

□

Corollary X.277.1. *Let $A \in \mathbb{F}^{m\times k}$ and $B \in \mathbb{F}^{k\times n}$ such that* $\operatorname{null}(A) \subseteq \operatorname{col}(B)$, *then*

$$AB = \mathbb{0}_{m\times n} \iff \operatorname{col}(B) \subseteq \operatorname{null}(A) \iff \operatorname{rank}(A) + \operatorname{rank}(B) = k$$

Proposition X.278 (Frobenius's rank inequality)**.** *Let A, B, C be conformal matrices, then*

$$\operatorname{rank}(ABC) \geq \operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank}(B).$$

Proof. Let $B = XY$ be a full-rank factorisation. Then

$$\begin{aligned} \operatorname{rank}(ABC) &= \operatorname{rank}(AXYC) \geq \operatorname{rank}(AX) + \operatorname{rank}(YC) - \operatorname{rank}(B) \\ &\geq \operatorname{rank}(AXY) + \operatorname{rank}(XYC) - \operatorname{rank}(B) = \operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank}(B) \end{aligned}$$

using Sylvester's rank inequality X.277. □

Proposition X.279. *Let A, B be conformal matrices. Then*

$$|\operatorname{rank}(A) - \operatorname{rank}(B)| \leq \operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

Proof. Let $A = X_1Y_1$ and $B = X_2Y_2$ be full-rank factorisations with $r = \operatorname{rank}(A)$ and $s = \operatorname{rank}(B)s$. Define

$$C = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}.$$

Then $CD = X_1Y_1 + X_2Y_2 = A + B$.
The second inequality follows from X.274:

$$\operatorname{rank}(A+B) = \operatorname{rank}(CD) \leq \min\{\operatorname{rank}(C), \operatorname{rank}(D)\} \leq r + s.$$

The first inequality follows from X.274, which gives $\operatorname{rank}(C) \geq \max\{r, s\}$ and $\operatorname{rank}(D) \geq \max\{r, s\}$, Sylvester's rank inequality X.277, which gives

$$\operatorname{rank}(A+B) \geq \operatorname{rank}(C) + \operatorname{rank}(D) - (r+s) \geq r + r - (r+s) = r - s$$

and

$$\operatorname{rank}(A+B) \geq \operatorname{rank}(C) + \operatorname{rank}(D) - (r+s) \geq s + s - (r+s) = s - r.$$

These combine to give the first inequality. □

Proposition X.280 (Guttman rank additivity formula)**.** *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *and* A *or* D *invertible, then*

$$\begin{aligned} \operatorname{rank}(M) &= \operatorname{rank}(D) + \operatorname{rank}(A - BD^{-1}C) \\ &= \operatorname{rank}(A) + \operatorname{rank}(D - CA^{-1}B). \end{aligned}$$

Proof. This uses the Schur complement and the fact that the transformation matrices are invertible. □

8.2.6.4 The index of matrices

Proposition X.281. *Let* $A \in \mathbb{F}^{n \times n}$ *be a square matrix. Then*

1. $\operatorname{rank}(A^k) \geq \operatorname{rank}(A^{k+1})$ *for all* $k \in \mathbb{N}$*;*
2. *if* $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$*, then for all* $p \in \mathbb{N}$

$$\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+p}) \qquad \text{and} \qquad \operatorname{col}(A^k) = \operatorname{col}(A^{k+p});$$

3. *there is a least integer* $q \in [0, n]$ *such that* $\operatorname{rank}(A^q) = \operatorname{rank}(A^{q+1})$.

These assertions remain true for exponent zero so long as we define $A^0 = \mathbb{1}_n$.

> The index of a square matrix is the least positive integer q such that $\operatorname{rank}(A^q) = \operatorname{rank}(A^{q+1})$.

In particular, invertible matrices have index zero.

8.3 Inner products and matrices

TODO: adjoint A^*!

8.3.1 Orthogonal matrices

Proposition X.282. *Let $A \in \mathbb{R}^{n\times n}$ and $\mathbb{R}^n$ have the standard inner product. The following are equivalent:*

1. *The columns of A form an orthonormal basis of $\mathbb{R}^n$;*
2. *The rows of A form an orthonormal basis of $\mathbb{R}^n$;*
3. $A^{\mathrm{T}}A = \mathbb{1}_n$*;*
4. $A^{-1} = A^{\mathrm{T}}$*;*

A matrix $A \in \mathbb{R}^{n\times n}$ satisfying any of the above is called an orthogonal matrix.

We can formulate the spectral theorem as follows:

Proposition X.283. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. The there exists an orthogonal matrix $P \in \mathbb{R}^{n\times n}$ such that $P^{-1}AP = P^{\mathrm{T}}AP$ is a diagonal matrix.*

$\det(A) = \pm 1$. Orthogonal matrix means orthogonal transformation. For orthogonal matrix P: $\|A\|_F = \|PA\|_F$ s

8.4 Matrix operations

All functions on $\mathbb{F}$ can of course be extended component-wise to functions on $\mathbb{F}^{m\times n}$. In particular, let $\overline{A}$ be the component-wise complex conjugate of $A \in \mathbb{F}^{m\times n}$.

8.4.1 Elementary row and column operations

Let A be an $(m \times n)$-matrix. The following are elementary row operations (EROs):

$\boxed{R_i \to \lambda R_i}$ Replacing a row by a non-zero multiple (i.e. $\lambda \neq 0$).

$\boxed{R_i \leftrightarrow R_j}$ Swapping two rows.

$\boxed{R_i \to R_i + \lambda R_j}$ Adding a multiple of a row to a different row.

The elementary column operations (ECO) are these operations applied to the columns. The are the operations given by transposing the matrix, applying an ERO and transposing again.
Two matrices A, B are called row equivalent if it is possible to obtain B from A by applying EROs. We write $A \sim_R B$ or just $A \sim B$.
Two matrices A, B are called column equivalent if it is possible to obtain B from A by applying ECOs. We write $A \sim_C B$.
Two matrices A, B are called row+column equivalent if it is possible to obtain B from A by applying EROs and ECOs. We write $A \sim_{R+C} B$.

In general the theory will be developed for EROs. The relevant results for ECOs are obtained by transposition.

Lemma X.284. *The relations $\sim_R, \sim_C$ and $\sim_{R+C}$ are equivalence relations.*

Lemma X.285. *Applying the elementary row operations to an $(n \times m)$-matrix is the same as multiplying from the left by the matrices obtained by applying the ERO to the $(n \times n)$-unit matrix:*

$\boxed{R_i \to \lambda R_i}$

$$\begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& \lambda &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix}$$

$\boxed{R_i \leftrightarrow R_j}$

$$\begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 0 && 1 && \\ &&& \ddots &&& \\ && 1 && 0 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix}$$

$\boxed{R_i \to R_i + \lambda R_j}$

$$\begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& \ddots &&& \\ && m && 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix}$$

Corollary X.285.1. *ECOs can be performed by multiplying by an invertible matrix from the right.*

Corollary X.285.2. *Let $A, B \in \mathbb{F}^{m \times n}$ be matrices.*

1. *If $A \sim_R B$, then* $\operatorname{row}(A) = \operatorname{row}(B)$ *and* $\operatorname{null}(A) = \operatorname{null}(B)$.
2. *If $A \sim_C B$, then* $\operatorname{col}(A) = \operatorname{col}(B)$.

Corollary X.285.3. *Let $A, B \in \mathbb{F}^{m \times n}$ be matrices such that $A \sim_{R+C} B$. Then* $\operatorname{rank}(A) = \operatorname{rank}(B)$ *and* $\dim \operatorname{null}(A) = \dim \operatorname{null}(B)$.

Proof. The dimension of the null space is preserved because $\dim \operatorname{null}(A) = n - \operatorname{rank}(A)$, by X.272. □

8.4.1.1 Gauss-Jordan elimination

Let A be an $(n \times m)$-matrix.

- The first non-zero element from the left on each row is called the leading coefficient of pivot of that row.
- The matrix A is in row echelon form (ref) if
  - rows of all zeros are at the bottom;
  - the leading coefficients of all other rows are one;
  - the leading coefficients of each non-zero row is strictly to the right of the leading coefficient of the row above it.
- The matrix A is in reduced row echelon form (rref) if
  - it is in echelon form;
  - all the coefficients in the column above a leading coefficient are zero.

Proposition X.286 (Gauss-Jordan elimination)**.** *Any matrix is row equivalent to a matrix in reduced echelon form.*

The proof gives an algorithm for computing the reduced echelon form. Sometimes the part of the algorithm that brings the matrix to an (unreduced) echelon form is called Gaussian elimination. We first give an algorithm for Gaussian elimination and then use this for full Gauss-Jordan elimination.

Proof (Gaussian elimination). Let $A \in \mathbb{F}^{m\times n}$ be the input-matrix. The algorithm is recursive, we call it ref.

1. If A is empty (i.e. $m = 0$ or $n = 0$), then return A.
2. Find the entry in the first column of largest non-zero absolute value. Swap the corresponding row with the first row.
   - If all the entries in the first column are zero, return
   $$\begin{pmatrix} 0 & \operatorname{ref}\left([A]_{1:m,2:n}\right) \end{pmatrix}.$$
   - The algorithm also works if we choose any non-zero entry, not necessarily the largest.
3. Divide the first row by $[A]_{1,1}$.
4. Perform the ERO $R_i \to R_i - [A]_{i,1}R_1$ for each row $i \in 2 : m$.
5. Return
$$\begin{pmatrix} 1 & [A]_{1,2:n} \\ 0 & \operatorname{ref}\left([A]_{2:m,2:n}\right) \end{pmatrix}.$$

The algorithm terminates because each subsequent call involves a matrix of strictly smaller dimensions. □

Proof (Obtaining the reduced echelon form). For each leading coefficient $[A]_{ij}$, perform the EROs $R_k \to R_k - [A]_{k,j}R_i$ for all rows $k \in 1 : i - 1$. □

Corollary X.286.1. *Let A be any matrix. Then*

$$A \sim_{R+C} \begin{pmatrix} \mathbb{1}_k & 0^{k\times p} \\ 0^{q\times k} & 0^{q\times p} \end{pmatrix}.$$

The row and column ranks are both equal to k and the nullity is equal to p.
This also means we can write

$$A = P\begin{pmatrix} \mathbb{1}_k & 0^{k\times p} \\ 0^{q\times k} & 0^{q\times p} \end{pmatrix} Q$$

for some invertible matrices P, Q.

This factorisation of A is called the rank normal form.
We can use Gauss-Jordan elimination to find bases for $\operatorname{row}(A), \operatorname{col}(A)$ and $\operatorname{null}(A)$:

$\boxed{\operatorname{row}(A)}$ The row space is preserved by row equivalence. So we can perform Gauss-Jordan elimination, discard the null rows and the remaining rows will form a basis of $\operatorname{row}(A)$.

$\boxed{\operatorname{col}(A)}$ Applying an ERO to a matrix A is the same as multiplying from the left by some invertible matrix E. Due to X.258.1,

$$EA = \begin{pmatrix} E[A]_{-,1} & \dots & E[A]_{-,m} \end{pmatrix},$$

so $\operatorname{col}(EA) = \ell_E[\operatorname{col}(A)]$ and because ℓ_E is an isomorphism, we have $\operatorname{col}(A) \cong \operatorname{col}(EA)$.

In the echelon form it is easy to see that the columns containing leading elements form a basis. By applying the inverse map we see that the corresponding columns in the original matrix form a basis of the original column space $\operatorname{col}(A)$, see X.33.

$\boxed{\operatorname{null}(A)}$ The row space is preserved by row equivalence, so we first perform Gauss-Jordan elimination $A \sim_R U$. Then solve $Ux = 0$ for x (see later).

Proposition X.287. *Let $L : \mathbb{F}^n \to \mathbb{F}^m$ be a linear map. Then for any bases β_n, β_m of $\mathbb{F}^n$ and $\mathbb{F}^m$, we have*

$$(L)_{\beta_n}^{\beta_m} \sim_{R+C} \begin{pmatrix} \mathbb{1}_k & 0^{k\times q} \\ 0^{p\times k} & 0^{p\times q} \end{pmatrix}$$

and

1. *L is injective* if and only if $p = 0$*;*
2. *L is surjective* if and only if $q = 0$.

8.4.1.2 Calculating inverse matrices

Any invertible square matrix A is row equivalent to $\mathbb{1}_n$. We want to find the matrix associated to this row reduction. One way to keep track of this matrix is to simultaneously apply the EROs to A and $\mathbb{1}_n$:

Lemma X.288. *Let $A \in \mathbb{F}^{n\times n}$ be a square matrix. Then*

$$\left(\begin{array}{c|c} A & \mathbb{1}_n \end{array}\right) \sim_R \left(\begin{array}{c|c} \mathbb{1}_n & A^{-1} \end{array}\right).$$

Proof. $A^{-1}\begin{pmatrix} A & \mathbb{1}_\mathbb{n} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_n & A^{-1} \end{pmatrix}$. □

8.4.2 The trace

Let $A \in \mathbb{F}^{n\times n}$ be a square matrix. The trace of A, denoted $\operatorname{Tr}(A)$, is the sum of the diagonal entries of A:

$$\operatorname{Tr}(A) = \sum_{i=1}^{n} (A)_{i,i}.$$

Proposition X.289. *Let $A \in \mathbb{F}^{n\times n}$ be a square matrix, then*

1. $\operatorname{Tr}(cA + B) = c\operatorname{Tr}(A) + \operatorname{Tr}(B)$ *for all* $c \in \mathbb{F}$;
2. $\operatorname{Tr}(\overline{A}) = \overline{\operatorname{Tr}(A)}$;
3. $\operatorname{Tr}(A^{\mathrm{T}}) = \operatorname{Tr}(A)$;
4. $\operatorname{Tr}(A^{*}) = \overline{\operatorname{Tr}(A)}$.

Proposition X.290. *Let $A, B, C \in \mathbb{F}^{n\times n}$ be square matrices. Then*

1. $\operatorname{Tr}(AB) = \sum_{i,j=1}^{n} [A]_{i,j}[B]_{j,i}$;
2. $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$;
3. $\operatorname{Tr}(ABC) = \operatorname{Tr}(CAB) = \operatorname{Tr}(BCA)$.

Proof. Point (1) follows by direct computation

$$\operatorname{Tr}(AB) = \sum_{i=1}^{n} (AB)_{i,i} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}B_{j,i}.$$

(2) We use (1) and the fact that $[A]_{i,j}[B]_{j,i} = [B]_{i,j}[A]_{j,i}$.
(3) Follows straight from (2): $\operatorname{Tr}\big((AB)C\big) = \operatorname{Tr}\big(C(AB)\big)$ etc. □

The trace of a product of matrices is invariant under any cyclic permutation of the matrices. The trace of a product of matrices is not invariant under noncyclic permutation of the matrices:

$$\operatorname{Tr}(ABC) \neq \operatorname{Tr}(BAC).$$

Lemma X.291. *Let $A \in \mathbb{F}^{n\times n}$. Then A is similar to a matrix B with*

$$[B]_{i,i} = \frac{1}{n}\operatorname{Tr}(A) \qquad \forall i \leq n.$$

Proof. It is enough to show that this holds for matrices with zero trace: if A is any matrix, then $A - \frac{\operatorname{Tr}(A)}{n}\mathbb{1}$ is a matrix with zero trace. If this is similar to a matrix $S(A - \frac{\operatorname{Tr}(A)}{n}\mathbb{1})S^{-1}$ with zeros on the diagonal, then SAS^{-1} has $\frac{\operatorname{Tr}(A)}{n}$ on the diagonal.
So assume WLOG that $\operatorname{Tr}(A) = 0$ and $A \neq 0$. We can find $\mathbf{v} \in \mathbb{F}^n$ such that $\{\mathbf{v}, A\mathbf{v}\}$ is linearly independent: assume, towards contraposition, that $A\mathbf{v}$ is a scalar multiple of $\mathbf{v}$, for all $\mathbf{v}$. This must be the same multiple λ for all $\mathbf{v}$. If not, i.e. there exist $\mathbf{v}, \mathbf{w}$ such that $A\mathbf{v} = \lambda_1\mathbf{v}$ and $A\mathbf{w} = \lambda_2\mathbf{w}$ with $\lambda_1 \neq \lambda_2$, then $\mathbf{v} + \mathbf{w}$ is not mapped to a multiple. In this case A is $\lambda\,\mathrm{id}$, but $\operatorname{Tr}(A) = 0$ fixes $\lambda = 0$, which we excluded.
Now we can extend $\{\mathbf{v}, A\mathbf{v}\}$ to a basis of $\mathbb{F}^n$ and put these as columns in a matrix $S = \begin{bmatrix}\mathbf{v} & A\mathbf{v} & S_1\end{bmatrix}$. We can partition $S^{-1} = \begin{bmatrix}\mathbf{x}^{\mathrm{T}} \\ S_2\end{bmatrix}$. Then we have

$$S^{-1}S = \begin{bmatrix}\mathbf{x}^{\mathrm{T}}\mathbf{v} & \mathbf{x}^{\mathrm{T}}A\mathbf{v} & \mathbf{x}^{\mathrm{T}}S_1 \\ S_2\mathbf{v} & S_2A\mathbf{v} & S_2S_1\end{bmatrix} = \mathbb{1}.$$

In particular $\mathbf{x}^{\mathrm{T}}A\mathbf{v} = 0$. Then

$$S^{-1}AS = \begin{bmatrix} \mathbf{x}^{\mathrm{T}}A\mathbf{v} & \mathbf{x}^{\mathrm{T}}A^2\mathbf{v} & \mathbf{x}^{\mathrm{T}}AS_1 \\ S_2A\mathbf{v} & S_2A^2\mathbf{v} & S_2AS_1 \end{bmatrix} = \begin{bmatrix} 0 & \star \\ \star & S_2AS_1 \end{bmatrix}.$$

Now S_2AS_1 has trace zero and we can repeat the argument. By induction we can make all elements on the diagonal zero. □

Proposition X.292. *The trace of a matrix of a linear map is independent of the choice of basis:*

$$\mathrm{Tr}(L)^{\beta}_{\beta} = \mathrm{Tr}(L)^{\beta'}_{\beta'}.$$

Proof.

$$\mathrm{Tr}(L)^{\beta'}_{\beta'} = \mathrm{Tr}\left[((I)^{\beta}_{\beta'})^{-1}(L)^{\beta}_{\beta}(I)^{\beta}_{\beta'}\right] = \mathrm{Tr}\left[(L)^{\beta}_{\beta}(I)^{\beta}_{\beta'}((I)^{\beta}_{\beta'})^{-1}\right] = \mathrm{Tr}(L)^{\beta}_{\beta}$$

□

This allows us to make the following definition:

> The trace of a linear map on a finite-dimensional vector space is the trace of any matrix representation of that map.

8.4.3 The determinant

> A map
>
> $$f : \mathbb{F}^{n\times n} \to \mathbb{F}$$
>
> with the following properties
>
> **D-1** $f(\mathbb{1}_n) = 1$;
>
> **D-2** $f(A)$ changes sign if two rows in A are swapped;
>
> **D-3** f is linear in the first row:
>
> $$f(\begin{pmatrix} \lambda A_{1,-} + \mu A'_{1,-} \\ A_{2,-} \\ \vdots \\ A_{n,-} \end{pmatrix}) = \lambda f(\begin{pmatrix} A_{1,-} \\ A_{2,-} \\ \vdots \\ A_{n,-} \end{pmatrix}) + \mu f(\begin{pmatrix} A'_{1,-} \\ A_{2,-} \\ \vdots \\ A_{n,-} \end{pmatrix});$$
>
> is called a determinant map.

We will show that there exists one and only one determinant map. We are therefore justified in calling it the determinant.
The determinant of A is often denoted $\det(A)$ or $|A|$.

Lemma X.293. *Let $f : \mathbb{F}^{n\times n} \to \mathbb{F}$ be a determinant map.*

1. *f is linear in each row;*
2. *if a matrix A has a row of zeros, or two identical rows, then $f(A) = 0$;*
3. *the ERO $R_i \to R_i + \lambda R_j$ does not change the determinant;*

4. *the ERO $R_i \leftrightarrow R_j$ changes the sign of the determinant;*
5. *the ERO $R_i \to \lambda R_i$ multiplies the determinant by λ;*
6. *$f(A)$ is non-zero* if and only if *$A \sim_R \mathbb{1}_n$;*
7. *$f(A)$ is non-zero* if and only if *A is invertible.*

8.4.3.1 Leibniz formula

Proposition X.294 (Liebniz formula)**.** *There is one and only one determinant map. It is given by*

$$\det : \mathbb{F}^{n\times n} \to \mathbb{F} : A \mapsto \sum_{\sigma\in S^n} \operatorname{sgn}(\sigma)\prod_{i=1}^{n}[A]_{i,\sigma(i)}.$$

Proof. Let f be a determinant map and $\mathcal{E} = \{\mathbf{e}_i\}_{i=1}^n$ the standard basis of $\mathbb{F}^n$, considered as rows and $A \in \mathbb{F}^{n\times n}$. Then

$$f(A) = f(\begin{pmatrix} \sum_{j_1=1}^n [A]_{1j_1}\mathbf{e}_{j_1} \\ \vdots \\ \sum_{j_n=1}^n [A]_{nj_n}\mathbf{e}_{j_n} \end{pmatrix}) = \sum_{j_1,\ldots,j_n=1} [A]_{1j_1}\ldots[A]_{nj_n} f(\begin{pmatrix} \mathbf{e}_{j_1} \\ \vdots \\ \mathbf{e}_{j_n}\end{pmatrix}).$$

Now $f(\begin{pmatrix} \mathbf{e}_{j_1} \\ \vdots \\ \mathbf{e}_{j_n}\end{pmatrix})$ is only non-zero if all j_i are different, i.e. $j_i = \sigma(i)$ for some permutation $\sigma \in S^n$. Also

$$f(\begin{pmatrix} \mathbf{e}_{\sigma(1)} \\ \vdots \\ \mathbf{e}_{\sigma(n)}\end{pmatrix}) = \operatorname{sgn}(\sigma) f(\mathbb{1}_n) = \operatorname{sgn}(\sigma).$$

So the only possible candidate for a determinant map is

$$f(A) = \sum_{\sigma\in S^n} \operatorname{sgn}(\sigma)\prod_{i=1}^{n}[A]_{i,\sigma(i)}.$$

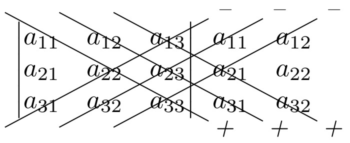
This satisfies **D-1**, **D-2** and **D-3**. ☐

In the case of 3×3 the Leibniz formula reduces to

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{23}a_{12}.$$

The rule of Sarrus is the following mnemonic:

Corollary X.294.1 (Rule of Sarrus)**.** *The determinant of a 3×3 matrix can be seen as the sum of the following diagonals (with the correct sign):*



8.4.3.2 Laplace expansion

Let $A \in \mathbb{F}^{n\times n}$ be a square matrix. A minor determinant or minor is the determinant of a submatrix. In particular the (i,j)-minor $M_{i,j}$ is the minor

$$M_{i,j} \coloneqq \det([A]_{(1:n)\setminus\{i\},(1:n)\setminus\{j\}}).$$

The (i,j)-cofactor $C_{i,j}$ is defined as

$$C_{i,j} \coloneqq (-1)^{i+j} M_{i,j}.$$

Proposition X.295 (Laplace expansion)**.** *Let $A \in \mathbb{F}^{n\times n}$. Then*

$$\det(A) = \sum_{i=1}^{n} [A]_{i,j} C_{i,j} = \sum_{i=1}^{n} [A]_{j,i} C_{j,i}$$

for all $j \in 1:n$.

This is also known as "expansion along the j^{th} column", resp., row.

Proof. Fix some $j \in 1:n$. Then we can calculate

$$\begin{aligned}\det(A) &= \sum_{\sigma\in S^n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} [A]_{k,\sigma(k)} = \sum_{i=1}^{n} \sum_{\substack{\sigma\in S^n \\ \sigma(i)=j}} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} [A]_{k,\sigma(k)} \\ &= \sum_{i=1}^{n} [A]_{i,j} \sum_{\substack{\sigma\in S^n \\ \sigma(i)=j}} \operatorname{sgn}(\sigma) \prod_{k\in(1:n)\setminus\{j\}} [A]_{k,\sigma(k)} = \sum_{i=1}^{n} [A]_{i,j} C_{i,j}.\end{aligned}$$

The expression for column expansion can be obtained by transposition. □

8.4.3.3 Volume

8.4.3.4 Properties

Lemma X.296. *Let $A \in \mathbb{F}^{n\times n}$. Then*

$$\det(A^{\mathrm{T}}) = \det(A).$$

Proof. We calculate using the Leibniz formula:

$$\begin{aligned}\det(A^{\mathrm{T}}) &= \sum_{\sigma\in S^n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} [A^{\mathrm{T}}]_{i,\sigma(i)} = \sum_{\sigma\in S^n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} [A]_{\sigma(i),i} \\ &= \sum_{\sigma\in S^n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} [A]_{i,\sigma^{-1}(i)} = \sum_{\sigma\in S^n} \operatorname{sgn}(\sigma^{-1}) \prod_{i=1}^{n} [A]_{i,\sigma^{-1}(i)} = \det(A).\end{aligned}$$

□

Lemma X.297. *Let $A \in \mathbb{F}^{n\times n}$. Then*

1. $\det(\lambda A) = \lambda^n \det(A)$;
2. $\det(\overline{A}) = \overline{\det(A)}$;

3. $\det(A^*) = \overline{\det(A)}$*;*

4. if A *is triangular, then*
$$\det(A) = \prod_{i=1}^{n} [A]_{i,i}.$$

Proposition X.298. *Let* $A, B \in \mathbb{F}^{n\times n}$*. Then*
$$\det(AB) = \det(A)\det(B).$$

Proof. First assume $\det(B) = 0$. Then AB is not invertible, for if AB were invertible, B would have inverse $(AB)^{-1}A$. So $\det(AB) = 0$ and the formula holds.
Now assume $\det(B) \neq 0$, then $A \mapsto \det(AB)/\det(B)$ satisfies the definition of a determinant map and thus is equal to det. ☐

Corollary X.298.1. *If* U *is unitary, then* $|\det(U)| = 1$*.*

Proof. $1 = \det \mathbb{1} = \det(U^*U) = \det(U^*)\det(U) = \overline{\det(U)}\det(U) = |\det(U)|$. ☐

Corollary X.298.2. *Let* $A, B \in \mathbb{F}^{n\times n}$*. Then*
$$\det(AB) = \det(BA).$$

In fact we can arbitrarily commute any matrix product inside the determinant.

Corollary X.298.3. *The determinant of a matrix of a linear map is independent of the choice of basis:*
$$\det(L)^{\beta}_{\beta} = \det(L)^{\beta'}_{\beta'}.$$

This allows us to make the following definition:

> The determinant of a linear map on a finite-dimensional vector space is the determinant of any matrix representation of that map.

Lemma X.299. *Let* $A \in \mathbb{F}^{m\times m}$ *and* $n \in \mathbb{N}$*. Then*
$$\det\begin{bmatrix} \mathbb{1}_n & \mathbb{0} \\ \mathbb{0} & A \end{bmatrix} = \det(A) = \det\begin{bmatrix} A & \mathbb{0} \\ \mathbb{0} & \mathbb{1}_n \end{bmatrix}.$$

Proof. By induction on n. ☐

Lemma X.300. *Let* A, B, C *be conformal matrices and* A, D *square. Then*
$$\det\begin{bmatrix} A & B \\ \mathbb{0} & D \end{bmatrix} = \det(A)\det(D).$$

Proof. This follows from
$$\begin{bmatrix} A & B \\ \mathbb{0} & D \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & D \end{bmatrix}\begin{bmatrix} \mathbb{1} & B \\ \mathbb{0} & \mathbb{1} \end{bmatrix}\begin{bmatrix} A & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \end{bmatrix},$$

the product rule, the previous lemma and the fact that the central matrix is triangular with only ones on the diagonal. ☐

Lemma X.301. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a partitioned matrix. Then*

$$\begin{aligned} \det(M) &= \det(A)\det(D - CA^{-1}B) \\ &= \det(D)\det(A - BD^{-1}C) \end{aligned}$$

if A, *resp.* D, *is invertible.*

Proof. This follows straight from the Schur complement. □

Corollary X.301.1 (Cauchy expansion of the determinant)**.** *Let* $A \in \mathbb{F}^{n\times n}, \mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ *and* $c \in \mathbb{F}$. *Then*

$$\det \begin{bmatrix} c & \mathbf{x}^{\mathrm{T}} \\ \mathbf{y} & A \end{bmatrix} = (c - \mathbf{x}^{\mathrm{T}} A^{-1}\mathbf{y})\det(A) = c\det(A) - \mathbf{x}^{\mathrm{T}}\operatorname{adj}(A)\mathbf{y}.$$

Corollary X.301.2. *Let* $A, B, C, D \in \mathbb{F}^{n\times n}$.

1. *If* A *or* D *is invertible and commutes with* B, *then*

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(DA - CB).$$

2. *If* A *or* D *is invertible and commutes with* C, *then*

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC).$$

Lemma X.302. *Let* $a, b \in \mathbb{R}$. *Then*

$$\det(a\mathbb{1}_n + b\mathbb{J}_n) = a^{n-1}(a + nb).$$

Proof. We can partition $a\mathbb{1}_n + b\mathbb{J}_n$ and use the ERO $R_i \to R_i - R_1$ for $1 < i \leq n$ to obtain:

$$a\mathbb{1}_n + b\mathbb{J}_n = \begin{pmatrix} a+b & b\mathbb{J}^{1\times(n-1)} \\ b\mathbb{J}^{(n-1)\times 1} & a\mathbb{1}_{n-1} + b\mathbb{J}_{n-1} \end{pmatrix} = \begin{pmatrix} a+b & b\mathbb{J}^{1\times(n-1)} \\ -a\mathbb{J}^{(n-1)\times 1} & a\mathbb{1}_{n-1} \end{pmatrix}.$$

Then by X.301, we have

$$\det(a\mathbb{1}_n + b\mathbb{J}_n) = a^{n-1}(a + b + b\mathbb{J}^{1\times(n-1)}\mathbb{J}^{(n-1)\times 1}) = a^{n-1}(a + nb).$$

□

Lemma X.303 (Weinstein-Aronszajn identity)**.** *Let* $A \in \mathbb{F}^{m\times n}$ *and* $B \in \mathbb{F}^{n\times m}$. *Then*

$$\det(\mathbb{1}_m + AB) = \det(\mathbb{1}_n + BA).$$

Also, for any $\lambda \in \mathbb{R}_0$,

$$\det(AB - \lambda\mathbb{1}_m) = (-\lambda)^{m-n}\det(BA - \lambda\mathbb{1}_n).$$

This is also sometimes referred to as the Sylvester determinant identity.

Proof. Applying the two equalities in X.301 to the matrix

$$M = \begin{bmatrix} \mathbb{1}_m & -A \\ B & \mathbb{1}_n \end{bmatrix}$$

give

$$\begin{aligned} M &= \det(\mathbb{1}_m)\det(\mathbb{1}_n - B\mathbb{1}_m^{-1}(-A)) = \det(\mathbb{1}_n + BA) \\ &= \det(\mathbb{1}_n)\det(\mathbb{1}_m - (-A)\mathbb{1}_n^{-1}B) = \det(\mathbb{1}_m + AB). \end{aligned}$$

□

Corollary X.303.1 (Matrix determinant lemma). *Let $A \in \mathbb{F}^{n\times n}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{F}^n$. Then*

$$\begin{aligned} \det(A + \mathbf{u}\mathbf{v}^{\mathrm{T}}) &= \det(A)\det(\mathbb{1}_n + A^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}) \\ &= \det(A)(\mathbb{1}_n + \mathbf{v}^{\mathrm{T}}A^{-1}\mathbf{u}) \\ &= \det(A) + \mathbf{v}^{\mathrm{T}}\operatorname{adj}(A)\mathbf{u}, \end{aligned}$$

which is interesting because $\mathbf{v}^{\mathrm{T}}A^{-1}\mathbf{u} \in \mathbb{F}$ and $\mathbf{v}^{\mathrm{T}}\operatorname{adj}(A)\mathbf{u} \in \mathbb{F}$ are scalars.

TODO

$$\log\det M = \operatorname{Tr}\log M$$

8.4.4 Adjugate

Let $A \in \mathbb{F}^{n\times n}$ be a square matrix. The adjugate matrix or classical adjoint $\operatorname{adj}(A)$ is the transposed cofactor matrix:

$$[\operatorname{adj}(A)]_{ij} = C_{ji}$$

where C_{ij} is the $(i,j)-$cofactor.

Lemma X.304. *Let $A \in \mathbb{F}^{n\times n}$, $\mathbf{b} \in \mathbb{F}^n$ and $k \in (1:n)$. Then*

$$\det(\begin{bmatrix} \begin{cases} [A]_{ij} & (j \neq k) \\ [\mathbf{b}]_i & (j = k) \end{cases} \end{bmatrix}) = [\operatorname{adj}(A)b]_k$$

Proof. We calculate

$$[\operatorname{adj}(A)\mathbf{b}]_k = \sum_l [\operatorname{adj}(A)]_{kl}[\mathbf{b}]_l = \sum_l C_{lk}[\mathbf{b}]_l.$$

Now in the definition of C_{lk}, the k^{th} column is excluded. So the $(l,k)-$cofactor of A is the same as the $(l,k)-$cofactor of

$$\begin{bmatrix} \begin{cases} [A]_{ij} & (j \neq k) \\ [\mathbf{b}]_i & (j = k) \end{cases} \end{bmatrix}$$

which is the matrix where the k^{th} column of A is replaced by $\mathbf{b}$. Then $\sum_l C_{lk}[\mathbf{b}]_l$ is the determinant of this matrix by X.295. □

Proposition X.305. *Let $A \in \mathbb{F}^{n\times n}$. Then*

$$A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det(A)\mathbb{1}_n.$$

Proof.

$$[\operatorname{adj}(A)\cdot A]_{ij} = \sum_k [A]_{ik} C_{jk}$$

Clearly if $i = j$, we have the Laplace expansion X.295 and the expression equals $\det(A)$. If $i \neq j$, then the j^{th} row does not enter into the expression (it is left out of the $(i, j)-$minor) and thus may just as well be replaced by a copy of the i^{th} row. In this case we get the expression for the determinant of a matrix with two identical rows. This must be 0. ☐

Corollary X.305.1. *Let $A \in \mathbb{F}^{n\times n}$ be invertible. Then*

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

Proposition X.306. *Let $A, B \in \mathbb{F}^{n\times n}$ be square matrices. Then*

1. $\operatorname{adj}(\mathbb{0}_n) = \mathbb{0}_n$;
2. $\operatorname{adj}(\mathbb{1}_n) = \mathbb{1}_n$;
3. $\operatorname{adj}(\lambda A) = \lambda^{n-1}\operatorname{adj}(A)$ *for any* $\lambda \in \mathbb{F}$;
4. $\operatorname{adj}(A^{\mathrm{T}}) = \operatorname{adj}(A)^{\mathrm{T}}$;
5. $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$;
6. $\operatorname{adj}(AB) = \operatorname{adj}(A)\operatorname{adj}(B)$.

Proof. Points 2. and 5. are direct consequences of X.305. TODO rest. ☐

Cauchy-Binet

8.4.5 Generalised inverses or pseudoinverses

8.4.5.1 Moore-Penrose pseudoinverse

8.4.6 Pfaffian

8.4.7 Vectorisation

For calculations it is often useful to put the matrix of coordinates into a column vector. The process of fitting a matrix into a column vector is known as the vectorisation of a matrix. Two obvious ways to do this are by going row-by-row or column by column.

- Column-by-column we get

  $$\operatorname{vec}_C : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn\times 1} : A \mapsto \operatorname{vec}_C(A) = [a_{1,1}, \ldots, a_{m,1}, a_{1,2}, \ldots, a_{m,2}, \quad \ldots \quad, a_{1,n}, \ldots, a_{m,n}]^{\mathrm{T}}$$

  Example

  $$\text{If} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{then} \qquad \operatorname{vec}_C(A) = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}.$$

- Row-by-row we get

  $$\operatorname{vec}_R : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn\times 1} : A \mapsto \operatorname{vec}_R(A) = [a_{1,1}, \ldots, a_{1,n}, a_{2,1}, \ldots, a_{2,n}, \quad \ldots \quad, a_{m,1}, \ldots, a_{m,n}]^{\mathrm{T}}$$

Example

$$\text{If} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{then} \qquad \mathrm{vec}_R(A) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Obviously these are related by

$$\mathrm{vec}_C(A) = \mathrm{vec}_R(A^{\mathrm{T}})$$

Vectorisation is a self-adjunction in the monoidal closed structure of any category of matrices. (TODO)

8.4.8 The Hadamard product

$$\mathrm{vec}(A \circ B) = \mathrm{vec}(A) \circ \mathrm{vec}(B)$$

This works for both vec_C and vec_R.

8.4.9 The outer product

Given two column vectors $\mathbf{u} = [u_1 \dots u_m]^{\mathrm{T}}$ and $\mathbf{v} = [v_1 \dots v_n]^{\mathrm{T}}$, the outer product is defined by

$$\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^{\mathrm{T}} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} = \begin{bmatrix} u_1v_1 & \dots & u_1v_n \\ \vdots & \ddots & \vdots \\ u_mv_1 & \dots & u_mv_n \end{bmatrix}$$

8.4.10 The Kronecker product

Given two matrices A, B of dimensions $m \times n$ and $p \times q$, the Kronecker product yields a $(pm \times qn)$-matrix:

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \dots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \dots & a_{m,n}B \end{bmatrix}$$

The same symbol is used as for the outer product, because the Kronecker product can be seen as a generalisation of the outer product. Indeed for column vectors $\mathbf{u}, \mathbf{v}$ we have the identities

$$\begin{aligned} \mathbf{u} \otimes_{\mathrm{Kron}} \mathbf{v} &= \mathrm{vec}_R(\mathbf{u} \otimes_{\mathrm{outer}} \mathbf{v}) && (8.1) \\ &= \mathrm{vec}_C(\mathbf{v} \otimes_{\mathrm{outer}} \mathbf{u}) && (8.2) \end{aligned}$$

and

$$\mathbf{u} \otimes_{\mathrm{outer}} \mathbf{v} = \mathbf{u} \otimes_{\mathrm{Kron}} \mathbf{v}^{\mathrm{T}}$$

In terms of matrix elements we can write, assuming the indices start at zero

$$(A \otimes B)_{i,j} = a_{\lfloor i/p \rfloor, \lfloor j/q \rfloor} b_{i\%p, j\%q}.$$

where % denotes the remainder. For indices starting from 1, we have

$$(A \otimes B)_{i,j} = a_{\lceil i/p \rceil, \lceil j/q \rceil} b_{(i-1)\%p+1, (j-1)\%q+1}.$$

8.4.10.1 Properties

The Kronecker product is bilinear and associative.

Transpose

$$(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$$

Determinant

$$\det(A \otimes B) = \det(A)^m \det(B)^n$$

if A is an $n \times n$ matrix and B an $m \times m$ matrix.

Trace

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$$

Mixed product Let A, B, C, D be conformal matrices, then

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

The proof is as follows:

$$(A \otimes B)(C \otimes D) = \begin{bmatrix} a_{1,1}B & \dots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \dots & a_{m,n}B \end{bmatrix} \begin{bmatrix} c_{1,1}D & \dots & c_{1,n}D \\ \vdots & \ddots & \vdots \\ c_{m,1}D & \dots & c_{m,n}D \end{bmatrix} \tag{8.3}$$

$$= \begin{bmatrix} \sum_k a_{1,k}c_{k,1}BD & \dots & \sum_k a_{1,k}c_{k,p}BD \\ \vdots & \ddots & \vdots \\ \sum_k a_{m,k}c_{k,1}BD & \dots & \sum_k a_{m,k}c_{k,p}BD \end{bmatrix} = \begin{bmatrix} (AC)_{1,1}BD & \dots & (AC)_{1,p}BD \\ \vdots & \ddots & \vdots \\ (AC)_{m,1}BD & \dots & (AC)_{m,p}BD \end{bmatrix} \tag{8.4}$$

$$= (AC) \otimes (BD) \tag{8.5}$$

Using the fact that multiplication of two block matrices can be carried out as if their blocks were scalars.

As an immediate consequence:

$$A \otimes B = (I_n \otimes B)(A \otimes I_k) = (A \otimes I_k)(I_n \otimes B).$$

Inverse The product $A \otimes B$ is invertible iff A and B are invertible. In that case the inverse is given by

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

This follows easily from the mixed product.

e-Penrose pseudoinverse

$$(A \otimes B)^{+} = A^{+} \otimes B^{+}.$$

A vectorisation trick Let A, B, C be matrices of dimensions $k \times l, l \times m$ and $m \times n$. Then

$$\mathrm{vec}(ABC) = (C^{\mathrm{T}} \otimes A)\,\mathrm{vec}(B).$$

From this we obtain some other formulations:

$$\mathrm{vec}(ABC) = (I_n \otimes AB)\,\mathrm{vec}(C) \tag{8.6}$$

$$= (C^{\mathrm{T}}B^{\mathrm{T}} \otimes I_k)\,\mathrm{vec}(A) \tag{8.7}$$

$$\mathrm{vec}(AB) = (I_m \otimes A)\,\mathrm{vec}(B) = (B^{\mathrm{T}} \otimes I_k)\,\mathrm{vec}(A) \tag{8.8}$$

8.4.11 The commutator

Theorem X.307 (Shoda's theorem)**.** *Let* $A \in \mathbb{F}^{n\times n}$*. Then there exists* X, Y *such that* $A = [X, Y] = XY - YX$ if and only if $\operatorname{Tr}(A) = 0$.

Proof. Assume $A = [X, Y]$. Then $\operatorname{Tr}(A) = \operatorname{Tr}(XY) - \operatorname{Tr}(YX) = \operatorname{Tr}(XY) - \operatorname{Tr}(XY) = 0$. Conversely, assume $\operatorname{Tr}(A) = 0$. By X.291 A is similar to a matrix B that has zeros on the diagonal. Let $X = \operatorname{diag}(1, 2, \ldots, n)$ and

$$[Y]_{i,j} = \begin{cases} (i-j)^{-1}[B]_{i,j} & (i \neq j) \\ 1 & (i = j). \end{cases}$$

Then

$$[X, Y]_{i,j} = [XY - YX]_{i,j} = i[Y]_{i,j} - j[Y]_{i,j} = (i-j)[Y]_{i,j} = [B]_{i,j}.$$

So B is the commutator $[X, Y]$. Then A is the commutator $[SXS^{-1}, SYS^{-1}]$. ☐

8.4.12 Kruskal rank and spark

> Let $A \in \mathbb{F}^{m\times n}$. The Kruskal rank is the largest number $\operatorname{K}(A)$ such that all $\operatorname{K}(A)$-sets of columns are linearly independent.

Lemma X.308. *Let* $A \in \mathbb{F}^{m\times n}$*. Then* $\operatorname{K}(A) \leq \operatorname{rank}(A)$*.*

Proof. If $\operatorname{rank}(A) = k$, then there exists a k-set of columns that spans $\operatorname{col}(A)$. Then every linearly independent set is smaller than k by the Steinitz exchange lemma X.7 and $\operatorname{K}(A) \leq k$. ☐

> Let $A \in \mathbb{F}^{m\times n}$. Then the spark of A is defined as
>
> $$\operatorname{spark}(A) \coloneqq \min\left\{\|x\|_0 \mid x \in \operatorname{null}(A)\right\}$$
>
> where $\|x\|_0$ is the number of non-zero elements of x.

TODO $\|\cdot\|_0$ norm for finite fields.

Lemma X.309. *Let* $A \in \mathbb{F}^{m\times n}$*. Then*

$$\operatorname{spark}(A) = \operatorname{K}(A) + 1.$$

Proof. TODO ☐

Lemma X.310. *Let* $A \in \mathbb{F}^{m\times n}$*. If* $\operatorname{rank}(A) = n$*, then* $\operatorname{K}(A) = n$*.*

Proposition X.311. *Let* $A \in \mathbb{F}^{m\times n}$*. Then*

$$\operatorname{K}(A) \geq \frac{1}{\mu(A)}$$

where $\mu(A) = \max_{i\neq j} \frac{|\langle [A]_{,i}, [A]_{-,j}\rangle|}{\left\|[A]_{,i}\right\| \left\|[A]_{,j}\right\|}$.

Proof. TODO ☐

8.5 Eigenvalues and eigenvectors

8.5.1 The spectrum

In this section we study vectors that are mapped to multiples of themselves by a given matrix A, i.e. vectors $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v} \qquad \text{for some } \lambda \in \mathbb{F}.$$

Clearly for this to be possible, A needs to be square.

Suppose $A \in \mathbb{F}^{n\times n}$.

- A scalar $\lambda \in \mathbb{F}$ is called an eigenvalue of A if there exists a $\mathbf{v} \in \mathbb{F}^n$ such that $\mathbf{v} \neq 0$ and $A\mathbf{v} = \lambda v$.
- Such a vector $\mathbf{v}$ is called an eigenvector.
- The set of all eigenvectors associated with an eigenvalue λ is called the eigenspace $E_\lambda(A)$. Because

  $$E_\lambda(A) = \ker(\ell_{\lambda\mathbb{1}_n - A}),$$

  it is indeed a vector space.

  The dimension of $E_\lambda(A)$ is the geometric multiplicity of λ.

The set of all eigenvalues is called the spectrum of A.

Proposition X.312. *Let $A \in \mathbb{F}^{n\times n}$ and $\lambda \in \mathbb{F}$, then*

$$\lambda \text{ is an eigenvalue of } A \qquad \iff \qquad \lambda\mathbb{1}_n - A \text{ is invertible.}$$

Proof. The equation $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $(A - \lambda\mathbb{1}_n)\mathbf{v} = 0$. So there exist eigenvectors associated to λ iff the kernel of $\ell_{A-\lambda\mathbb{1}_n}$ is not trivial iff $\ell_{A-\lambda\mathbb{1}_n}$ is injective (X.23) iff $\ell_{A-\lambda\mathbb{1}_n}$ is invertible (X.34) iff $A - \lambda\mathbb{1}_n$ is invertible (X.245). ☐

Proposition X.313 (Gerschgorin circle theorem)**.** *Let $A \in \mathbb{F}^{n\times n}$. If λ is an eigenvalue of A, then there is an $i \in 1 : n$ such that*

$$|\lambda - [A]_{ii}| \leq \sum_{\substack{j\in 1:n\\ j\neq i}} |[A]_{ij}|.$$

Proof. If $|\lambda - [A]_{ii}| > \sum_{\substack{j\in 1:n\\ j\neq i}} |[A]_{ij}|$ for all i, then $(A - \lambda\mathbb{1})$ is strictly diagonally dominant and thus invertible by X.241. ☐

Proposition X.314. *Let $A \in \mathbb{F}^{n\times n}$ be a matrix. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of A and $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are corresponding eigenvectors. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent.*

Proof. The proof goes by contradiction. Assume $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly dependent. Let k be the smallest positive integer such that

$$\mathbf{v}_k \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}.$$

So there exists a nontrivial linear combination

$$\mathbf{v}_k = a_1\mathbf{v}_1 + \ldots + a_{k-1}\mathbf{v}_{k-1}.$$

Multiplying by A gives

$$\lambda_k \mathbf{v}_k = a_1 \lambda_k \mathbf{v}_1 + \ldots + a_{k-1}\lambda_k \mathbf{v}_{k-1}.$$

Multiplying the previous combination by λ_k and subtracting both equations gives

$$0 = a_1(\lambda_k - \lambda_1)\mathbf{v}_1 + \ldots + a_{k-1}(\lambda_k - \lambda_{k-1})\mathbf{v}_{k-1}.$$

By assumption of linear independence of $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$ this combination must be trivial, however none of the $(\lambda_k - \lambda_i)$ can be zero, so all the a_i must be zero. This is a contradiction with the assumption of linear dependence. ☐

Corollary X.314.1. *The matrix $A \in \mathbb{F}^{n\times n}$ has at most n linearly independent eigenvalues.*

Corollary X.314.2. *Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of A. Then*

$$E_{\lambda_1}(A) \oplus \ldots \oplus E_{\lambda_m}(A)$$

is a direct sum. Furthermore, the sum of geometric multiplicities is less than or equal to the dimension of V:

$$\dim E_{\lambda_1}(A) + \ldots + \dim E_{\lambda_m}(A) \leq \dim V.$$

8.5.1.1 The characteristic equation

Let $A \in \mathbb{F}^{n\times n}$. The characteristic polynomial $p_A(x)$ of A. Is the polynomial

$$p_A(x) \coloneqq \det(x\mathbb{1}_n - A).$$

The characteristic polynomial is also sometimes defined as $\det(A - x\mathbb{1}_n)$. This differs by a sign $(-1)^n$.

Lemma X.315. *The characteristic polynomial of any square matrix is a monic polynomial.*

Lemma X.316. *The characteristic polynomials of similar matrices are identical.*

Proposition X.317. *Let $A \in \mathbb{F}^{n\times n}$. Then $p_A(x)$ can be factorised as*

$$p_A(x) = \prod_{i=1}^{m}(x - \lambda_i)^{m_i}$$

where λ_i are the eigenvalues of A and the multiplicities m_i are positive integers such that $\sum_{i=1}^{m} m_i = n$.

Corollary X.317.1. *The eigenvalues of A are the solutions of the equation*

$$p_A(x) = 0.$$

Corollary X.317.2. *The determinant of a matrix is the product of its eigenvalues, counting algebraic multiplicity: for $A \in \mathbb{F}^{n\times n}$*

$$\det(A) = \prod_{i=1}^{m} \lambda_i^{m_i}.$$

Proof. We have

$$\det(A) = (-1)^n \det(0 - A) = (-1)^n p_A(0) = (-1)^n \prod_{i=1}^{m}(0 - \lambda_i)^{m_i} = \prod_{i=1}^{m} \lambda_i^{m_i}.$$

☐

Let $A \in \mathbb{F}^{n\times n}$. The equation

$$p_A(x) = \det(x\mathbb{1}_n - A) = 0$$

is called the characteristic equation of A.
Let λ be a solution of the characteristic equation. The multiplicity of λ as a root of $p_A(x)$ is the algebraic multiplicity of λ.

Lemma X.318. *Let $A \in \mathbb{F}^{n\times n}$ and λ be an eigenvalue of A.
The geometric multiplicity of λ is less than or equal to the algebraic multiplicity of λ.*

Proof. Set $k = \dim E_\lambda$. Take a basis of $E_\lambda(A)$ and extend it to a basis β of $\mathbb{F}^n$. With respect to this basis the matrix of ℓ_A is of the form

$$(\ell_A)^\beta_\beta = \begin{pmatrix} \lambda\mathbb{1}_k & B \\ 0 & C \end{pmatrix} = P^{-1}(\ell_A)^{\mathcal{E}}_{\mathcal{E}}P = P^{-1}AP$$

for some matrices B, C and some invertible matrix P where $\mathcal{E}$ is the standard basis of $\mathbb{F}^n$. Then

$$p_A(x) = p_{P^{-1}AP} = p_{\lambda\mathbb{1}_k}(x)p_C(x) = (\lambda - x)^k p_C(x),$$

so the algebraic multiplicity of λ is at least the geometric multiplicity k. It may be greater if λ is also an eigenvector of C, but in this case the eigenvector is a linear combination of the eigenvectors already chosen for β. ☐

8.5.1.2 Diagonalisable matrices

A matrix $A \in \mathbb{F}^{n\times n}$ is called diagonalisable if $\mathbb{F}^n$ has a basis of eigenvectors of A.

Proposition X.319. *Let $A \in \mathbb{F}^{n\times n}$ and let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of A. The following are equivalent:*

1. *A is diagonalisable;*
2. *there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of A, each invariant under ℓ_A, such that*

$$\mathbb{F}^n = U_1 \oplus \ldots \oplus U_n;$$

3. *$\mathbb{F}^n = E_{\lambda_1}(A) \oplus \ldots \oplus E_{\lambda_m}(A)$;*
4. *$n = \dim E_{\lambda_1}(A) + \ldots + \dim E_{\lambda_m}(A)$;*
5. *for each λ_i the geometric multiplicity is equal to the algebraic multiplicity and the sum of algebraic multiplicities is n.*

In the case of complex vector spaces, the sum of algebraic multiplicities is always n by the fundamental theorem of algebra.

Corollary X.319.1. *If $A \in \mathbb{F}^{n\times n}$ has n distinct eigenvalues, then A is diagonalisable.*

So a matrix may fail to be diagonalisible for two reasons: not enough geometric multiplicity or not enough geometric and algebraic multiplicity

8.5.2 Spectral theorem

Theorem X.320. *Let V be a complex finite-dimensional inner product space. Let L be an operator on V. Then*

1. *there exists an orthonormal basis of V consisting of eigenvectors of L* if and only if *L is normal;*
2. *if L is self-adjoint, then the eigenvalues of L are real.*

TODO: replace following: + in real case we need self-adjoint!

Theorem X.321 (Spectral theorem for matrices)**.** *Let V be a finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$. Let $L = L^*$ be self-adjoint. Then*

1. *there exists an orthonormal basis of V consisting of eigenvectors of L;*
2. *the eigenvalues of L are real.*

Proof. We first prove the theorem for complex vector spaces. The proof is by finite induction: By the fundamental theorem of algebra the characteristic polynomial has at least 1 root λ_1. Choose a corresponding eigenvector $\mathbf{v}_1$. Then by

$$\lambda_1 \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \langle \mathbf{v}_1, L\mathbf{v}_1 \rangle = \langle L\mathbf{v}_1, \mathbf{v}_1 \rangle = \overline{\lambda_1} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle ,$$

λ_1 is real.
Now $\operatorname{span}\{\mathbf{v}_1\}^\perp$ is invariant under L:

$$\mathbf{x} \in \operatorname{span}\{\mathbf{v}_1\}^\perp \quad \iff \quad \langle \mathbf{x}, \mathbf{v}_1 \rangle = 0 \quad \implies \quad \langle L\mathbf{x}, \mathbf{v}_1 \rangle = \langle \mathbf{x}, L\mathbf{v}_1 \rangle = \lambda_1 \langle \mathbf{x}, \mathbf{v}_1 \rangle = 0.$$

We can now apply the same argument to $L|_{\operatorname{span}\{\mathbf{v}_1\}^\perp} : \operatorname{span}\{\mathbf{v}_1\}^\perp \to \operatorname{span}\{\mathbf{v}_1\}^\perp$, whose eigenvector are orthogonal to v_1. Finite induction then finishes the proof in the complex case.
In the real case: we can linearly extend L to be an operator $L_\mathbb{C}$ on the complexification $V_\mathbb{C}$. Then $L_\mathbb{C}$ has formally the same characteristic polynomial as L, except now interpreted as a function $\mathbb{C} \to \mathbb{C}$, not $\mathbb{R} \to \mathbb{R}$. Now we know the roots of $p_{L_\mathbb{C}}(x)$ are real, so they are also roots of $p_L(x)$. The rest of the proof can be completed in the same way. $\square$

The spectral decomposition is a special case of both the Schur decomposition and the singular value decomposition.
TODO: Kronecker product: multiply eigenvalues: all there (by multiplicity)

8.5.3 Computing eigenvalues and vectors

8.5.3.1 Power method

+ inverse

8.5.3.2 Deflation

`https://quickfem.com/wp-content/uploads/IFEM.AppE_.pdf`

8.5.3.3 QR

8.6 Matrix classes and decompositions

8.6.1 Matrix classes

8.6.1.1 Rank-1 projections

Proposition X.322. *Let $P \in \mathbb{F}^{n\times n}$. Then P is a rank-1 (orthogonal) projection* if and only if *there is a unit vector $\mathbf{u} \in \mathbb{F}^n$ such that $P = \mathbf{u}\mathbf{u}^*$.*

Proof. Due to P being rank-1, we can find a unit vector $\mathbf{u}$ such that $\operatorname{col}(P) = \operatorname{span}\{\mathbf{u}\}$. So the columns of P are all multiples of $\mathbf{u}$, meaning we can write P as $\mathbf{u}\mathbf{v}^*$ for some $\mathbf{v} \in \mathbb{F}^n$.
Now $P = P^* = (\mathbf{u}\mathbf{v}^*)^* = \mathbf{v}\mathbf{u}^*$, so $\mathbf{u}\mathbf{v}^* = \mathbf{v}\mathbf{u}^*$ and thus $\operatorname{col}(P) = \operatorname{span}\{\mathbf{v}\}$, meaning $\mathbf{v} = \lambda\mathbf{u}$.
Also $P^2 = \mathbf{u}\mathbf{v}^*\mathbf{u}\mathbf{v}^* = \mathbf{u}\langle v, u\rangle \mathbf{v}^* = \langle v, u\rangle \mathbf{u}\mathbf{v}^*$, so $1 = \langle \mathbf{v}, \mathbf{u}\rangle = \overline{\lambda}\langle \mathbf{u}, \mathbf{u}\rangle = \overline{\lambda}$.
So $\mathbf{v} = \mathbf{u}$ and $P = \mathbf{u}\mathbf{u}^*$. ☐

We write $P_{\mathbf{u}}$ to denote $\mathbf{u}\mathbf{u}^*$. Then in particular $P_{\mathbf{u}}\mathbf{v} = \langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u}$.

8.6.1.2 Householder matrices

Let $\mathbf{w} \in \mathbb{F}^n$ be a non-zero vector. Set $\mathbf{u} = \mathbf{w}/\|\mathbf{w}\|$. Then the corresponding Householder matrix is

$$U_{\mathbf{w}} \coloneqq \mathbb{1}_n - 2P_{\mathbf{u}} = \mathbb{1}_n - 2\frac{\mathbf{w}\mathbf{w}^*}{\langle \mathbf{w}, \mathbf{w}\rangle} = \mathbb{1}_n - 2\frac{\mathbf{w}\mathbf{w}^*}{\mathbf{w}^*\mathbf{w}}.$$

The corresponding transformation $\ell_{U_{\mathbf{w}}}$ is called a Householder transformation.

The Householder transformation reflects vectors across a hyperplane orthogonal to $\mathbf{w}$.

Lemma X.323. *Householder matrices are unitary, Hermitian and involutive.*

For any two vectors of the same length, we can construct a unitary matrix that maps one to the other, using Householder matrices.

Proposition X.324. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ such that $\|\mathbf{x}\| = \|\mathbf{y}\| \neq 0$. Let*

$$\sigma = \begin{cases} 1 & (\langle \mathbf{x}\mathbf{y}\rangle = 0) \\ -\overline{\langle \mathbf{x}, \mathbf{y}\rangle}/|\langle \mathbf{x}, \mathbf{y}\rangle| & (\langle \mathbf{x}\mathbf{y}\rangle \neq 0), \end{cases}$$

and let $\mathbf{w} = \mathbf{y} - \sigma\mathbf{x}$. Then $\sigma U_{\mathbf{w}}$ is unitary and $\sigma U_{\mathbf{w}}\mathbf{x} = \mathbf{y}$.

The use of σ is purely to improve numerical stability.

8.6.1.3 Upper Hessenberg matrices

A matrix A is called an upper Hessenberg matrix if $i > j + 1 \implies [A]_{i,j} = 0$.

This is a matrix of the form

$$\begin{bmatrix} \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \end{bmatrix}$$

Every square matrix is unitarily similar to an upper Hessenberg matrix, and the unitary similarity can be constructed from a sequence of Householder matrices and complex rotations.

8.6.2 Matrix decompositions

8.6.2.1 LU and LDU factorisation

8.6.2.2 QR factorisation

The QR factorization of an $m \times n$ matrix A is a factorisation $A = QR$ where $Q \in \mathbb{F}^{m\times n}$ has orthonormal columns and $R \in \mathbb{F}^{n\times n}$ is square upper triangular. This requires that $m \geq n$.

Proposition X.325 (QR factorisation)**.** *Let $A \in \mathbb{F}^{m\times n}$ and $m \geq n$. Then*

1. *there exists a unitary $U \in \mathbb{F}^{m\times m}$ and an upper triangular matrix $R \in \mathbb{F}^{n\times n}$ such that*

$$A = U \begin{bmatrix} R \\ \mathbb{0} \end{bmatrix}$$

   we can take R to have real, non-negative values on the diagonal;

2. *writing $U = \begin{bmatrix} Q & Q' \end{bmatrix}$, we get the decomposition*

$$A = QR$$

   where Q has orthonormal columns;

3. *if* $\operatorname{rank}(A) = n$*, then fixing the values on the diagonal of R to be positive makes the factorisation unique; all values on the diagonal are non-zero.*

Proof. We can find a (unitary) Householder transformation U_1 that maps the first columns $[A]_{-,1}$ to $\|[A]_{-,1}\|\mathbf{e}_1$. So

$$U_1 A = \begin{bmatrix} \|[A]_{-,1}\| & \star \\ \mathbb{0} & A_1 \end{bmatrix}$$

Then we can do the same for A_1, meaning $U_2 = \mathbb{1} \oplus U'$ transforms A as

$$U_2 U_1 A = \begin{bmatrix} \|[A]_{-,1}\| & \star & \star \\ 0 & \|[A_1]_{-,1}\| & \star \\ \mathbb{0} & \mathbb{0} & A_2 \end{bmatrix}.$$

Repreating this gives the required factorisation. $\square$

The factorisation $A = U \begin{bmatrix} R \\ \mathbb{0} \end{bmatrix}$ is called the wide QR factorisation and $A = QR$ the (narrow) QR factorisation.

8.6.3 Polar decomposition

8.6.3.1 Singular value decomposition

spectral decomposition is a special case

8.6.3.2 Schur decomposition

spectral decomposition is a special case

8.7 Systems of linear equations

`https://encyclopediaofmath.org/wiki/Motzkin_transposition_theorem` TODO
A homogeneous system of linear equations with more variables than equations has non-zero solutions.
An inhomogeneous system of linear equations with more equations thanvariables has no solution for some choice of the constant terms.
Calculation of inverse via row reduction.
free and bounded variables.

Lemma X.326. *If $Ax = b$ is consistent for all $b \in \mathbb{F}^n$, then A has a right inverse B, i.e. $AB = \mathbb{1}$.*

Proof. For each e_i in the standard basis we can find a c_i such that $Ac_i = e_i$. Then

$$A\begin{pmatrix} c_1 & c_2 & \dots & c_n \end{pmatrix} = \begin{pmatrix} Ac_1 & Ac_2 & \dots & Ac_n \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} = \mathbb{1}_n.$$

□

8.7.1 Cramer's rule

Proposition X.327.

$$x_i = \frac{\det(A_i)}{\det(A)}$$

Proof. $x = \begin{pmatrix} x_1 \dots x_n \end{pmatrix}^{\mathrm{T}}$

$$\begin{aligned} x_i &= \det\begin{pmatrix} e_1 & \dots & e_{i-1} & x & e_{i+1} & \dots & e_n \end{pmatrix} \\ &= \det\begin{pmatrix} A^{-1}a_1 & \dots & A^{-1}a_{i-1} & A^{-1}b & A^{-1}a_{i+1} & \dots & A^{-1}a_n \end{pmatrix} \\ &= \det(A^{-1}\begin{pmatrix} a_1 & \dots & a_{i-1} & b & a_{i+1} & \dots & a_n \end{pmatrix}) = \det(A^{-1}A_i) \\ &= \frac{\det(A_i)}{\det(A)}. \end{aligned}$$

□

8.8 Polynomials applied to endomorphisms

8.9 The spectra of matrices

What are eigenvectors of rotation? -¿ complex eigenvalues.

Real matrix: complex conjugate eigenvalues have complex conjugate eigenvectors
Finite order endomorphisms are diagonalisable over $\mathbb{C}$ (or any algebraically closed field where the characteristic of the field does not divide the order of the endomorphism) with roots of unity on the diagonal. This follows since the minimal polynomial is separable, because the roots of unity are distinct.
See `https://en.wikipedia.org/wiki/Minimal_polynomial_(linear_algebra)`

Chapter 9

Indices and symbols

9.1 Contravariant and covariant vectors and tensors

When working in finite-dimensional spaces with specified bases, we often get expressions of the form

$$v = \sum_{i=1}^{n} a_i \mathbf{e}_i.$$

We may replace this expression with

$$v = a^i \mathbf{e}_i$$

if we take the convention that if an index is repeated once up and once down, then there is a sum over all values of that index. This is the Einstein summation convention.

Note that coordinates have their indices up, and basis vectors have their indices down.

Now in the dual space, we have the dual basis $\{\varphi^j\}_j$. In the dual space we take the opposite convention: coordinates have their indices down, and dual basis vectors have their indices up, so

$$\varphi = b_j \varphi^j.$$

This allows us to write

$$\varphi(v) = \varphi(a_i \mathbf{e}_i) = a_i b^j \varphi^j(\mathbf{e}_i) = a_i b^i$$

where for the last equality we have used that $\varphi^j(\mathbf{e}_i)$ only does not vanish if $i = j$ and is 1 in this case.

We call vectors in V contravariant vectors and vectors in V^* covariant vectors, or covectors.

Per convention we put the coordinates of contravariant vectors in column vectors. This means, by proposition XIII.53, we must put the coordinates of covariant vectors in row vectors. Indeed, let $v = a^i \mathbf{e}_i \in V$ and $\varphi = b_j \varphi^j \in V^*$, then

$$\varphi(v) = a^i b_i = \begin{bmatrix} a^1 & \ldots & a^n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

We can view covariant vectors as functions that take a contravariant vector and produce a number, and we can view contravariant vectors as functions that take a covariant vector and produce a number. In general we may have linear functions that accept several co- and contravariant vectors and produce a number. By (TODO), such functions are tensor products of various co- and contravariant vectors. They would have multiple up- and down-indices. e.g

$$\mathbf{T} = T^{i}_{\;jk}{}^{lm}(\mathbf{e}_i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}_l \otimes \mathbf{e}_m)$$

Where $T^{i}_{\;jk}{}^{lm}$ are the coordinates w.r.t. the basis vectors $\mathbf{e}_i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}_l \otimes \mathbf{e}_m$.
For example, once the basis has been chosen, matrices map contravariant vectors to contravariant vectors. And contravariant vectors map covariant vectors to numbers, so by reverse currying a matrix is maps a contravariant and a covariant vector to a number.
For the indices of matrices we have taken the convention that the first index is for rows and the second for the columns. For a constant row index, the column index spells out a covariant vector, so the column index is down. Conversely, the row index is up. A matrix A with components $(A)_{i,j}$ becomes

$$A^{i}_{\;j}(\mathbf{e}_i \otimes \mathbf{e}^j).$$

This is consistent with the observation that the matrix sends a vector to a function on covectors, in other words is a function which accepts vectors in first place and covectors in second place.
In the expressions so far only repeated indices were present. Such repeated indices are called bound indices or dummy indices. They may be replaced in the expression by other letters, so long as there is no clash. If an index is not repeated, it is a free index and may not just be changed.

9.1.1 "Tensors are objects that transform as tensors"

Variants:

- "N arbitrary numbers are not the components of a vector" (Peres p.65)

9.2 Covectors

9.2.1 Multi-index notation

Let $e_1, \ldots, e_n$ be a basis for a real vector space V. Let $\alpha^1, \ldots, \alpha^n$ be the dual basis for V^*. A multi-index

$$I = (i_1, \ldots, i_k)$$

is a k-tuple of numbers $\in (1, \ldots, n)$. We write

$$\begin{cases} e_I \coloneqq e_{i_1} \otimes \ldots \otimes e_{i_k} \\ \alpha^I \coloneqq \alpha^{i_1} \wedge \ldots \wedge \alpha^{i_k} \end{cases}.$$

The covector α^I is completely determined by the values in I, the order only changes the sign.
A multi-index $I = (i_1, \ldots, i_k)$ is ascending if

$$1 \leq i_1 < \ldots < i_k \leq n.$$

Proposition X.328. *Let I, J be ascending multi-indices of length k, then*

$$\alpha^I(e_J) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}.$$

Proposition X.329. *The covectors α^I, with I an ascending multi-index of length k, form a basis of $A_k(V)$.*

Corollary X.329.1. *If $\dim V = n$, then*

$$\dim A_k(V) = \binom{n}{k}.$$

9.3 Symmetrisation and anti-symmetrisation of indices

$$T_{\{a_1 \dots a_n\}} = \frac{1}{n!}\sum_{\sigma \in S_n} T_{a_{\sigma(1)}\dots a_{\sigma(n)}}$$

$$T_{[a_1 \dots a_n]} = \frac{1}{n!}\sum_{\sigma \in S_n} (\operatorname{sgn}\sigma) T_{a_{\sigma(1)}\dots a_{\sigma(n)}}$$

9.4 Symbols

9.4.1 Kronecker delta

The Kronecker delta is defined by

$$\delta_{ij} = \delta^i_j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

9.4.2 Levi-Civita symbol

The Levi-Civita symbol is defined by

$$\varepsilon_{a_1 \dots a_n} = \begin{cases} +1 & (a_1, \dots, a_n) \text{ is an even permutation of } (1, \dots, n) \\ -1 & (a_1, \dots, a_n) \text{ is an odd permutation of } (1, \dots, n) \\ 0 & \text{otherwise} \end{cases}.$$

The indices may be placed up or down.

Lemma X.330. *The Levi-Civita symbol is given by the explicit expression*

$$\varepsilon_{a_1 \dots a_n} = \prod_{1 \le i < j \le n} \operatorname{sgn}(a_j - a_i).$$

Proposition X.331. *Working in n dimensions, when all $i_1, \dots i_n; j_1, \dots, j_n$ take values in $\{1, \dots, n\}$:*

1. $\varepsilon_{i_1 \dots i_n}\varepsilon^{j_1 \dots j_n} = n!\delta^{j_1}_{[i_1} \dots \delta^{j_n}_{i_n]} = \sum_{\sigma \in S_n} (-1)^{\operatorname{sgn}(\sigma)} \delta^{j_1}_{i_{\sigma(1)}} \dots \delta^{j_n}_{i_{\sigma(n)}}$
2. $\varepsilon_{i_1 \dots i_n}\varepsilon^{i_1 \dots i_n} = n!$
3. $\varepsilon_{i_1 \dots i_k\ i_{k+1} \dots i_n}\varepsilon^{i_1 \dots i_k\ j_{k+1} \dots j_n} = k!(n-k)!\ \delta^{j_{k+1}}_{[i_{k+1}} \dots \delta^{j_n}_{i_n]}.$

Proof. 1. Both sides of the equation are a sum over the same indices. We consider each term in the sum separately and show that the sums are equal term-by-term. We split the terms into two categories.

(a) First consider the case that $j_1 \dots j_n$ is not a permutation of $(1, \dots, n)$, i.e. a number is repeated. Then $\varepsilon_{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n}$ is automatically zero. The right-hand side is definitely zero if the is do not take the same values as the js. If they do take the same values, there is a number that is repeated at least twice. For every term in the sum over permutations, there is another term with the repeated is swapped, which also adds a minus due to the change of sign of the permutation. Hence the sum over permutations is zero.

(b) Now assume that $j_1 \dots j_n$ is a permutation of $(1, \dots, n)$. Then either the is are also a permutation, or $\delta^{j_1}_{i_{\sigma(1)}} \dots \delta^{j_n}_{i_{\sigma(n)}}$ is always zero. The only possible non-zero term is with a $\sigma \in S_n$ such that $j_k = i_{\sigma(k)}$ for all k. If $\operatorname{sgn}(i_1, \dots, i_n) = \operatorname{sgn}(j_1, \dots, j_n)$, then $\operatorname{sgn}(\sigma) = 1$ and both sides match. If $\operatorname{sgn}(i_1, \dots, i_n) = -\operatorname{sgn}(j_1, \dots, j_n)$, then $\operatorname{sgn}(\sigma) = -1$ and both sides again match.

So, in fact, we have shown something slightly stronger, namely

$$\varepsilon_{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n} = n! \delta^{j_1}_{[i_1} \dots \delta^{j_n}_{i_n]}$$

where there is no sum over indices.

2. The number of permutations of any n-element set number is exactly $n!$. Every permutation is either even or odd and $(+1)^2 = (-1)^2 = 1$. Non-permutations do not contribute to the sum.

3. The sum on the left only has terms where the is and js are permutations of $(1, \dots, n)$. In each such term we can bring the indices with values $1 - k$ to the first k spots, each by a transposition. Because both Levi-Civita symbols have the same first k indices, each will need the same number of transpositions and thus the sign does not change. Then by considering lemma X.330 we see that we have obtained a product of cases 1. and 2. This yields the answer.

□

Corollary X.331.1. *In two dimensions, where all i, j, m, n each take values in $\{1, 2\}$,*

1. $\varepsilon_{ij} \varepsilon^{mn} = {\delta_i}^m {\delta_j}^n - {\delta_i}^n {\delta_j}^m$
2. $\varepsilon_{ij} \varepsilon^{in} = {\delta_j}^n$
3. $\varepsilon_{ij} \varepsilon^{ij} = 2.$

Corollary X.331.2. *In three dimensions, where all i, j, k, m, n each take values in $\{1, 2, 3\}$,*

1. $\varepsilon_{ijk} \varepsilon^{imn} = {\delta_j}^m {\delta_k}^n - {\delta_j}^n {\delta_k}^m$
2. $\varepsilon_{jmn} \varepsilon^{imn} = {\delta_j}^i$
3. $\varepsilon_{ijk} \varepsilon^{ijk} = 6.$

Proposition X.332. *Working in 3 dimensions,*

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix}\delta_{il} & \delta_{im} & \delta_{in}\\ \delta_{jl} & \delta_{jm} & \delta_{jn}\\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{vmatrix}\\ &= \delta_{il}\left(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km}\right)-\delta_{im}\left(\delta_{jl}\delta_{kn}-\delta_{jn}\delta_{kl}\right)+\delta_{in}\left(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}\right).\end{aligned}$$

This can directly be generalised to n dimensions.

9.5 Writing matrix operations using using tensor notation

A matrix A with components $(A)_{i,j}$ becomes

$$A^i{}_j(\mathbf{e}_i \otimes \mathbf{e}^j).$$

9.5.1 Trace

The trace of $A^i{}_j$ is $A^i{}_i$.

9.5.2 Matrix multiplication

$$(AB)^i{}_k = A^i{}_j B^j{}_k$$

which in particular for matrix-vector multiplication becomes

$$(Av)^i = A^i{}_j v^j.$$

9.5.3 Transpose

The transpose of $A^i{}_j$ is $(A^{\mathrm{T}})^j{}_i$.
Or: $(A^{\mathrm{T}})_{ab} = A_{ba}$ and $(A^{\mathrm{T}})_i^{\ j} = A^j{}_i$?

9.5.4 Determinant

$$\begin{aligned}\det(A) &= \varepsilon^{j_1\dots j_n} A^1{}_{j_1}\dots A^n{}_{j_n}\\ &= \frac{1}{n!}\varepsilon_{i_1\dots i_n}\varepsilon^{j_1\dots j_n} A^{i_1}{}_{j_1}\dots A^{i_n}{}_{j_n}\end{aligned}$$

Chapter 10

Ordered vector spaces

TODO link ordered groups.

Let $(\mathbb{R}, V, +)$ be a real vector space and $\precsim$ a preorder on the set V. Then $\precsim$ is a vector preorder if it is compatible with the vector space structure as follows: $\forall x, y, z \in V, \lambda \in \mathbb{R}$

1. $x \precsim y$ implies $x + z \precsim y + z$;
2. if $\lambda \geq 0$, then $x \precsim y$ implies $\lambda x \precsim \lambda y$.

We call $(\mathbb{R}, V, +, \precsim)$ a preordered vector space.

- If $\precsim$ is a partial order, we call $(\mathbb{R}, V, +, \precsim)$ a partially ordered vector space or simply a ordered vector space.
- If $(V, \precsim)$ is a lattice, we call $(\mathbb{R}, V, +, \precsim)$ a vector lattice or a Riesz space.

Lemma X.333. *Let $(\mathbb{R}, V, +)$ be a real vector space and $\precsim$ a preorder on the set V. The compatibility of the order can equivalently be expressed by: $\forall x, y, z \in V, \lambda \in \mathbb{R}$*

$$\begin{cases} x \precsim y \text{ implies } x + z \precsim y + z; \\ \text{if } \lambda \geq 0 \text{ and } 0 \precsim x, \text{ then } 0 \precsim \lambda x. \end{cases}$$

Lemma X.334. *Let V be a preordered vector space. For all $v, w \in V$ we have*

$$v \precsim w \iff 0 \precsim w - v \iff -w \precsim -v.$$

Proof. We get the implications

$$v \precsim w \implies 0 \precsim w - v \implies -w \precsim -v \implies v - w \precsim 0 \implies v \precsim w$$

by subsequently adding $-v, -w, v, w$ to both sides by compatibility of the order. ☐

Corollary X.334.1. *Let V be a preordered vector space and $\alpha \in \mathbb{R} \setminus \{0\}$. Then for all $v, w \in V$*

$$v \precsim w \qquad \iff \qquad \begin{cases} \alpha v \precsim \alpha w & (0 < \alpha) \\ \alpha v \succsim \alpha w & (\alpha > 0) \end{cases}.$$

Lemma X.335. *Let V be a preordered vector space. For all $v, w, x, y \in V$ we have*

$$\begin{cases} v \precsim w \\ x \precsim y \end{cases} \implies v + x \precsim w + y.$$

Proof. We calculate $v + x \leq w + x \leq w + y$. $\square$

Example

- The finite-dimensional vector spaces $\mathbb{R}^n$ with coordinate-wise addition, scalar multiplication and order are Riesz spaces.
- The finite-dimensional vector spaces $\mathbb{R}^n$ with coordinate-wise addition, scalar multiplication and lexicographical order are Riesz spaces.
- Let X be a set. The set $(X \to \mathbb{R})$ is a real vector space with point-wise addition and scalar multiplication. If the order is also defined point-wise, i.e. $f \leq g$ iff $\forall x \in X : f(x) \leq g(x)$, then $(X \to \mathbb{R})$ is a Riesz space.

Lemma X.336. *Let X be a topological space. The spaces*

1. $\mathcal{C}(X, \mathbb{R})$*;*
2. $\mathcal{C}_0(X, \mathbb{R})$*;*
3. $\mathcal{C}_c(X, \mathbb{R})$*; and*
4. $\mathcal{C}_b(X, \mathbb{R})$

with point-wise operations are Riesz spaces.

Proof. In all these cases the join and meet of f, g are given by

$$\begin{aligned} f \vee g &= \frac{1}{2}(f+g) + \frac{1}{2}|f-g| \\ f \wedge g &= \frac{1}{2}(f+g) - \frac{1}{2}|f-g|. \end{aligned}$$

So the join and meet are still continuous and have the same properties as f, g. $\square$

10.1 Upsets and downsets

Lemma X.337. *Let V be a preordered vector space, $S \subseteq V$ a subset, $v \in V$ and $\alpha \in \mathbb{R}$. Then*

1. *if $\alpha > 0$, then $(\alpha S)^u = \alpha S^u$ and $(\alpha S)^l = \alpha S^l$;*
2. *if $\alpha < 0$, then $(\alpha S)^u = \alpha S^l$ and $(\alpha S)^l = \alpha S^u$;*
3. $(S + v)^u = S^u + v$ *and* $(S + v)^l = S^l + v$.

Corollary X.337.1. *Let V be a preordered vector space, $S \subseteq V$ a subset, $v \in V$ and $\alpha \in \mathbb{R}$. Then*

1. $\sup(S + v) = \sup(S) + v$ *and* $\inf(S + v) = \inf(S) + v$*;*
2. *if $\alpha > 0$, then* $\sup(\alpha S) = \alpha \sup(S)$ *and* $\inf(\alpha S) = \alpha \inf(S)$*;*
3. *if $\alpha < 0$, then* $\sup(\alpha S) = \alpha \inf(S)$ *and* $\inf(\alpha S) = \alpha \sup(S)$.

10.2 The positive cone

Let V be a preordered vector space. The subset

$$V^+ := \{v \in V \mid 0 \precsim v\}$$

is called the positive cone of V. The elements of the positive cone V^+ are called the positive elements of V.

That the positive cone is in fact a cone follows from X.333

Proposition X.338. *Let V be a vector space.*

1. *A vector preorder on V is uniquely determined by its positive cone:*
$$x \precsim y \qquad \iff \qquad y - x \in V^+.$$
2. *The positive cone of a vector preorder is pointed and convex.*
3. *Any pointed convex cone in V determines a (unique) vector preorder.*
4. *A vector preorder is a partial order* if and only if *the positive cone is salient.*

Convexity is equivalent to closure under addition (see X.54)

Proof. (1) This is just X.334.
(2) V^+ is pointed by reflexivity: $0 \precsim 0$. It is closed under addition by X.335.
(3) Compatibility with addition is immediate from the definition of the order. Compatibility with scalar multiplication is due to it being a cone (see X.333). Reflexivity is equivalent with pointedness. Finally transitivity follows from closure under addition:

$$\begin{cases} x \precsim y \\ y \precsim x \end{cases} \iff \begin{cases} 0 \precsim y - x \\ 0 \precsim z - y \end{cases} \iff \begin{cases} y - x \in V^+ \\ z - y \in V^+ \end{cases}$$
$$\implies (z - y) + (y - x) = z - x \in V^+ \iff x \precsim z.$$

(4) We have $x \precsim y$ and $y \precsim x$ iff $(y - x) \in V^+$ and $-(y - x) \in V^+$. Thus both salience and anti-symmetry are equivalent to this situation implying $x - y = 0$. □

Lemma X.339. *Let V be a preordered vector space, $v \in V^+$ and $\alpha \in \mathbb{R}$.*

1. *If $\alpha \geq 1$, then $\alpha v \succsim v$.*
2. *If $\alpha \leq 1$, then $\alpha v \precsim v$.*

Proof. (1) We have $v \succsim 0$ and $(\alpha - 1) \geq 0$, so $(\alpha - 1)v \succsim 0$ and $\alpha v \succsim v$.
(2) We have $v \succsim 0$ and $(\alpha - 1) \leq 0$, so $(\alpha - 1)v \precsim 0$ and $\alpha v \precsim v$. □

10.3 Riesz spaces

Lemma X.340. *Let V be a Riesz space, $u, v, w \in V$ and $\alpha \in \mathbb{R}$, then*

1. $-(v \wedge w) = (-v) \vee (-w)$ *and* $-(v \vee w) = (-v) \wedge (-w)$*;*
2. *if* $\alpha \geq 0$*, then* $\alpha(v \wedge w) = (\alpha v) \wedge (\alpha w)$ *and* $\alpha(v \vee w) = (\alpha v) \vee (\alpha w)$*;*
3. *if* $\alpha \leq 0$*, then* $\alpha(v \wedge w) = (\alpha v) \vee (\alpha w)$ *and* $\alpha(v \vee w) = (\alpha v) \wedge (\alpha w)$*;*
4. $u + (v \wedge w) = (u + v) \wedge (u + w)$ *and* $u + (v \vee w) = (u + v) \vee (u + w)$*.*

Proof. We apply II.64 to
(1) the reverse order-embedding $v \mapsto -v$;
(2) the order-embedding $v \mapsto \alpha v$ for $\alpha > 0$; (if $\alpha = 0$ the result is trivial);
(3) the reverse order-embedding $v \mapsto \alpha v$ for $\alpha > 0$; (if $\alpha = 0$ the result is trivial);
(4) the order-embedding $v \mapsto u + v$. ☐

Proposition X.341 (Riesz decomposition)**.** *Let V be a Riesz space and $v, w_1, w_2 \in V^+$ such that $v \leq w_1 + w_2$. Then $\exists v_1, v_2 \in V^+$ such that $v = v_1 + v_2$ and $v_1 \leq w_2, v_2 \leq w_2$.*

Proof. Set $v_1 = v \wedge w_1$ and $v_2 = v - v_1$. These satisfy all the properties. We verify the inequality $v_2 \leq w_2$: from $w_2 \geq v - w_1$ we get

$$w_2 = 0 \vee w_2 \geq 0 \vee (v - w_1) = v + (-v) \vee (-w_1) = v - v \wedge w_1 = v - v_1 = v_2.$$

☐

Proposition X.342. *Let V be a Riesz space and $v, w \in V$, then*

$$(v \vee w) + (v \wedge w) = v + w.$$

Proof. We calculate

$$(v \vee w) + (v \wedge w) = v + 0 \vee (w - v) + w + (v - w) \wedge 0 = (v + w) + 0 \vee (w - v) - 0 \vee (w - v) = v + w.$$

☐

Proposition X.343. *Let V be a Riesz space, $u, v, w \in V$ and $x, y, z \in V^+$. Then*

1. $(u + v) \vee (2w) \leq u \vee w + v \vee w$*;*
2. $(u + v) \vee z \leq u \vee z + v \vee z$*;*
3. $(x + y) \wedge z \leq x \wedge z + y \wedge z$*.*

Proof. (1) From $u \leq u \vee w$ and $v \leq v \vee w$, we get $u + v \leq u \vee w + v \vee w$. Similarly from $w \leq u \vee w$ and $w \leq v \vee w$, we get $2w \leq u \vee w + v \vee w$. Together this gives (1).
(2) We have $2z \geq z$ by X.339.
(3) TODO (use Birkhoff inequality??) ☐

Proposition X.344 (Infinite distributivity in Riesz spaces)**.** *Let V be a Riesz space, $v \in V$ and $S \subseteq V$ a subset. Then*

1. *if* $\bigvee S$ *exists, then* $(\bigvee S) \wedge v = \bigvee(S \wedge v)$*;*
2. *if* $\bigwedge S$ *exists, then* $(\bigwedge S) \vee v = \bigwedge(S \vee v)$*;*

Proof. We already have the inequality $(\bigvee S) \wedge v \geq \bigvee(S \wedge v)$ from III.69.1. To show the other inequality, it is enough to show that for ☐

Corollary X.344.1. *Riesz spaces are distributive lattices.*

10.3.1 Positive elements

10.3.1.1 Positive and negative parts

Let V be a Riesz space and $v \in V$. Then we define

$$\begin{aligned} v^+ &:= v \vee 0 \\ v^- &:= (-v) \vee 0 = -(v \wedge 0). \end{aligned}$$

We call v^+ the positive part of v and v^- the negative part of v.

Lemma X.345. *Let V be a Riesz space and $v, w \in V$. Then*

1. $v \vee w = (v-w)^+ + w = (v-w)^- + v$;
2. $v \wedge w = v - (v-w)^+ = w - (v-w)^-$.

Proof. We calculate $v \vee w = (v-w) \vee 0 + w = (v-w)^+ + w$. The other equalities are similar. □

Proposition X.346. *Let V be a Riesz space and $v, w \in V$. Then*

1. $v^+, v^- \in V^+$;
2. $v = v^+ - v^-$;
3. $v^+ \perp v^-$ *(i.e. $v^+ \wedge v^- = 0$).*

Furthermore,

4. *if $p, q \in V$ satisfy 1 and 2, i.e. $p, q \in V^+$ and $v = p - q$, then $p \geq v^+$ and $q \geq v^-$; we may say $v = v^+ - v^-$ is the minimal such decomposition;*
5. *the elements v^+, v^- are uniquely determined by properties 2 and 3.*

Also

6. $(-v)^- = v^+$ *and* $(-v)^+ = v^-$;
7. *if $\alpha \geq 0$, then $(\alpha v)^+ = \alpha v^+$ and $(\alpha v)^- = \alpha v^-$;*
8. $-v^- \leq v \leq v^+$;
9. $v \leq w$ if and only if $v^+ \leq w^+$ *and* $v^- \geq w^-$.

Proof. (1) Evident from definitions.
(2) We calculate $v^+ - v = (v \vee 0) - v = (v - v) \vee (0 - v) = 0 \vee (-v) = v^-$.
(3) We calculate $0 = v^- - v^- = v^- + (v \wedge 0) = (v^- + v) \wedge (0 + v^-) = v^+ \wedge v^-$.
(4) From $v \leq p$ and $0 \leq p$, we get $v^+ = v \vee 0 \leq p$. Then we also have $v^- = v^+ - v \leq p - v = q$.
(5) Assume $p, q \in V$ satisfy (2) and (3), then (1) automatically follows from (3). Using X.345, we calculate

$$0 = p \wedge q = p - (p-q)^+ = p - v^+.$$

So $p = v^+$ and $q = p - v = v^+ - v = v^-$.
(6) It is evident that $(-v)^- = (--v) \vee 0 = v \vee 0$.
(7) We calculate $\alpha v^+ = \alpha(v \vee 0) = (\alpha v) \vee 0 = (\alpha v)^+$; the calculation for αv^- is similar.
(8) This is clear from $-v^- = v \wedge 0 \ \leq \ v \ \leq \ v \vee 0 = v^+$.
(9) $v \leq w$ implies $v^+ = v \vee 0 \leq w \vee 0 = w^+$ and $-v^- = v \wedge 0 \leq w \wedge 0 = -w^-$.
Conversely, we have $v = v^+ - v^- \leq w^+ - w^- = w$. □

Proposition X.347. *Let V be a Riesz space and $v, w \in V$. Then*

1. $(v+w)^+ \leq v^+ + w^+$;
2. $(v+w)^- \leq v^- + w^-$.

Proof. From X.346 we get $v \leq v^+$ and $w \leq w^+$, so $v + w \leq v^+ + w^+$. Also $0 \leq v^+ + w^+$. So

$$(v+w)^+ = (v+w) \vee 0 \leq v^+ + w^+.$$

Then we also have

$$(v+w)^- = (-v-w)^+ \leq (-v)^+ + (-w)^+ = v^- + w^-.$$

□

10.3.1.2 Absolute value

Let V be a Riesz space and $v \in V$. Then the absolute value of v is

$$|v| := v \vee (-v) = -(v \wedge (-v)).$$

If the Riesz space is a real function space with pointwise order, then $|f| = |\cdot| \circ f$ as usual, where $|\cdot| : \mathbb{R} \to \mathbb{R}$ is the usual absolute value function.

Lemma X.348. *Let V be a Riesz space, $v, w \in V$ and $\alpha \in \mathbb{R}$. Then*

1. $|v| = v^+ + v^-$;
2. $|v| \in V^+$;
3. *if* $v \in V^+$, *then* $|v| = v$;
4. $|v| = |-v|$;
5. $|\alpha v| = |\alpha| \cdot |v|$;
6. $\big||v|\big| = |v|$;
7. $0 \leq v^+ \leq |v|$ *and* $0 \leq v^- \leq |v|$;
8. $|v| = 0$ if and only if $v = 0$.

Proof. We prove (1):

$$|v| = v \vee (-v) = (2v) \vee 0 - v = 2v^+ - v = 2v^+ - (v^+ - v^-) = v^+ - v^-$$

and (6), using the absorption law:

$$\big||v|\big| = |v| \vee (-|v|) = v \vee (-v) \vee \big(v \wedge (-v)\big) = v \vee (-v) = |v|.$$

The rest are immediate consequences, using the results of X.346. □

Lemma X.349. *Let V be a Riesz space and $v, w \in V$, then*

1. $(v+w) \vee (v-w) = v + |w|$;

2. $(v+w)\wedge(v-w) = v-|w|$*;*

or, equivalently,

3. $v\vee w = \frac{1}{2}\big(v+w+|v-w|\big)$*;*

4. $v\wedge w = \frac{1}{2}\big(v+w-|v-w|\big)$*.*

Proof. We calculate, using X.340

$$(v+w)\vee(v-w) = v+w\vee(-w) = v+|w| \quad\text{and}\quad (v+w)\wedge(v-w) = v+w\wedge(-w) = v-|w|.$$

The next two equalities follow by the substitutions $v+w\leftrightarrow v$ and $v-w\leftrightarrow w$. □

Corollary X.349.1. *Let* V *be a Riesz space and* $v,w\in V$*, then*

$$|v-w| = (v\vee w)-(v\wedge w).$$

Corollary X.349.2. *Let* V *be a Riesz space and* $v,w\in V$*, then*

1. $|v|\vee|w| = \frac{1}{2}\Big(|v|+|w|+\big||v|-|w|\big|\Big)$*;*

2. $|v|\wedge|w| = \frac{1}{2}\Big(|v|+|w|-\big||v|-|w|\big|\Big)$*.*

Proof. Substitute $v\to|v|$ and $w\to|w|$. □

Corollary X.349.3. *Let* V *be a Riesz space and* $u,v,w\in V$*, then*

1. $|u\vee v-u\vee w|+|u\wedge v-u\wedge w| = |v-w|$*;*

2. $|v^+-w^+|\leq|v-w|$ *and* $|v^--w^-|\leq|v-w|$*.*

Proof. (1) Using the proposition, we get

$$|u\vee v-u\vee w|+|u\wedge v-u\wedge w| = (u\vee v)\vee(u\vee w)-(u\vee v)\wedge(u\vee w)+(u\wedge v)\vee(u\wedge w)-(u\wedge v)\wedge(u\wedge w).$$

Using distributivity this simplifies to $(v\vee w)-(v\wedge w) = |v-w|$.
(2) Using (1) we have

$$|v-w| = |0\vee v-0\vee w|+|0\wedge v-0\wedge w| = |v^+-w^+|+|v^--w^-| \geq \begin{cases}|v^+-w^+|\\ |v^--w^-|.\end{cases}$$

□

Proposition X.350. *Let* V *be a Riesz space and* $v,w\in V$*, then*

1. $|v|\vee|w| = \frac{1}{2}\Big(|v+w|+|v-w|\Big)$*;*

2. $|v|\wedge|w| = \frac{1}{2}\Big||v+w|-|v-w|\Big|$*;*

or, equivalently,

3. $|v+w|\vee|v-w| = |v|+|w|$*;*

4. $|v+w|\wedge|v-w| = \big||v|-|w|\big|$*.*

Also

5. $|v| + |w| = |v + w| + |v - w| - \big||v| - |w|\big|$;
6. $|v + w| + |v - w| = 2|v| + 2|w| - \big||v + w| - |v - w|\big|$;

and

7. $|v| + |w| = \big||v| - |w|\big| + \big||v + w| - |v - w|\big|$.

The only real trick is in the proof of (1). All the other results follow from elementary substitutions.

Proof. (3,4,6) Are equivalent to (1,2,5) by the replacements $v \leftrightarrow v + w$ and $w \leftrightarrow v - w$.
(1) We calculate

$$\begin{aligned}
|v| \vee |w| &= v \vee (-v) \vee w \vee (-w) = \big(v \vee (-w)\big) \vee \big((-v) \vee w\big) \\
&= \frac{1}{2}\Big((v - w) + |v + w|\Big) \vee \frac{1}{2}\Big((-v + w) + |-v - w|\Big) \\
&= \frac{1}{2}|v + w| + \frac{1}{2}\big((v - w) \vee (-v + w)\big) = \frac{1}{2}\Big(|v + w| + |v - w|\Big).
\end{aligned}$$

(5) Using X.342, (1) and X.349.2 we get

$$\begin{aligned}
|v| + |w| &= |v| \vee |w| + |v| \wedge |w| \\
&= \frac{1}{2}\Big(|v + w| + |v - w|\Big) + \frac{1}{2}\Big(|v| + |w| - \big||v| - |w|\big|\Big).
\end{aligned}$$

This simplifies to the required equation.
(7) Follows from substituting (6) into (5).
(2) Follows from (7) and X.349.2. ☐

Corollary X.350.1. *Let V be a Riesz space and $v, w \in V$, then the following are equivalent:*

1. $|v| \wedge |w| = 0$;
2. $|v + w| = |v - w|$;
3. $|v| \vee |w| = |v + w|$.

The absolute value also satisfies the triangle inequality.

Proposition X.351 (Triangle and reverse triangle inequality in Riesz spaces)**.** *Let V be a Riesz space and $v, w \in V$, then*

$$|v| + |w| \geq \big|v + w\big| \geq \big||v| - |w|\big|.$$

Proof. The first inequality is the triangle inequality. It follows straight from X.347.
The second inequality is the reverse triangle inequality and follows from the triangle inequality as in XIII.15. ☐

10.3.2 Subsets

Let V be a Riesz space. A subset E is called

- a Riesz subspace if it is both a subspace and a sublattice;
- solid if for all $v \in E$ the interval $[-|v|, |v|]$ is a subset of E;
- a band if for all subsets $S \subseteq E$ we have $\sup(S) \subset E$.

Lemma X.352. *Let V be a Riesz space and $v, w \in V$, then*

$$|w| \leq |v| \iff -|v| \leq w \leq |v|.$$

Proof. We have $w \leq |w|$ and $|w| \leq |v|$, so $w \leq |v|$. Also $-w \leq |-w| = |w|$, so $-|v| \leq -|w| \leq w$. Conversely, $-|v| \leq w$ implies $-w \leq |v|$. So $|w| = w \vee (-w) \leq |v|$. $\square$

Corollary X.352.1. *Let V be a Riesz space and $E \subseteq V$ a subset. Then E is solid* if and only if

$$\forall v \in E : \forall w \in V : \; |w| \leq |v| \implies w \in E.$$

Lemma X.353. *Let V be a Riesz space and $E \subseteq V$ a subset. Then E is an (order) ideal* if and only if *it is a solid Riesz subspace.*

Proof. TODO $\square$

10.3.3 Disjointness

10.3.4 Archimedean

Chapter 11

Some results and applications

11.1 Rotations

Rodrigues' rotation formula
eigenvectors and eigenvalues of rotation.

11.2 Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

All have eigenvalues ±1. The eigenspaces are spanned by

$$v_{x+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_{x-} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_{y+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad v_{y-} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad v_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\mathrm{Tr}[\sigma_i \sigma_j] = \delta_{ij}$$

Part XI

Analysis

file:///C:/Users/user/Downloads/0-8176-4442-3.pdf https://link.springer.com/content/pdf/10.1007%2F978-0-387-84895-2.pdf https://zr9558.files.wordpress.com/2014/08/a-guide-to-distribution-theory-and-fourier-transforms.pdf file:///C:/Users/user/Downloads/978-0-8176-4675-2.pdf
Integral mean value theorem https://en.wikipedia.org/wiki/Mean_value_theorem#Mean_value_theorems_for_definite_integrals
TODO: Hölder, Monkowski, Lyapounov

Chapter 1

Limits

TODO: squeeze theorem!

1.1 Bachmann-Landau notation

1.1.1 Asymptotic bounds: O, Θ, Ω

Let $(X, \mathcal{T})$ be a topological space and $(V, \|\cdot\|)$ a normed space. Let $x_0 \in X$ and $f, g : X \setminus \{x_0\} \to V$ be functions. The statement

- "$f(x) = O(g(x))$ as $x \to x_0$" means there exists a neighbourhood S of x_0 and a constant $M \in \mathbb{R}$ such that
$$\forall x \in S : \; \|f(x)\| \leq M\|g(x)\|;$$
- "$f(x) = \Omega(g(x))$ as $x \to x_0$" means there exists a neighbourhood S of x_0 and a constant $M \in \mathbb{R}$ such that
$$\forall x \in S : \; \|f(x)\| \geq M\|g(x)\|;$$
- "$f(x) = \Theta(g(x))$ as $x \to x_0$" means there exists a neighbourhood S of x_0 and constants $M_1, M_2 \in \mathbb{R}$ such that
$$\forall x \in S : \; M_1\|g(x)\| \leq \|f(x)\| \leq M_2\|g(x)\|.$$

We may add the word "uniformly" to these statements to mean we can take $S = X$.
We may suppress the x dependence for legibility and write e.g $f = O(g)$ instead.

Lemma XI.1. *Let $(X, \mathcal{T})$ be a topological space and $(V, \|\cdot\|)$ a normed space. Let $x_0 \in X$ and $f, g : X \setminus \{x_0\} \to V$ be functions. Then*

1. $f = O(g)$ if and only if $g = \Omega(f)$ *as* $x \to x_0$*;*
2. $f = \Theta(g)$ if and only if $f = O(g)$ *and* $f = \Omega(g)$ *as* $x \to x_0$.

Lemma XI.2. *Let $(X, \mathcal{T})$ be a topological space and $(V, \|\cdot\|)$ a normed space. Let $x_0 \in X$ and $f : X \setminus \{x_0\} \to V$ be functions. Then "being $\Theta(f)$ as $x \to x_0$" is an equivalence relation.*

Lemma XI.3. *Let $(X, \mathcal{T})$ be a topological space and $(V, \|\cdot\|)$ a normed space. Let $x_0 \in X$ and $f, g : X \setminus \{x_0\} \to V$ be functions. Then*

1. *$f(x) = O(g(x))$ as $x \to x_0$* if and only if *there exists a neighbourhood S of x_0 such that $\|f(x)\|/\|g(x)\|$ is bounded on S;*

2. *$f(x) = \Omega(g(x))$ as $x \to x_0$* if and only if *there exists a neighbourhood S of x_0 such that $\|f(x)\|/\|g(x)\|$ is bounded below on S by a strictly positive constant.*

1.1.2 Asymptotic domination and equality: $o, \sim, \omega$

Let $(X, \mathcal{T})$ be a topological space and $(V, \|\cdot\|)$ a normed space. Let $x_0 \in X$ and $f, g : X \setminus \{x_0\} \to V$ be functions. The statement

- "$f(x) = o(g(x))$ as $x \to x_0$" means $\lim_{x \to x_0} \frac{\|f(x)\|}{\|g(x)\|} = 0$;
- "$f(x) \sim_{x_0} g(x)$" means $\lim_{x \to x_0} \frac{\|f(x)\|}{\|g(x)\|} = 1$;
- "$f(x) = \omega(g(x))$ as $x \to x_0$" means $\lim_{x \to x_0} \frac{\|f(x)\|}{\|g(x)\|} = \infty$.

Lemma XI.4. *Let $(X, \mathcal{T})$ be a topological space and $(V, \|\cdot\|)$ a normed space. Let $x_0 \in X$ and $f, g : X \setminus \{x_0\} \to V$ be functions.*
Then $f = o(g)$ if and only if *$g = \omega(f)$ as $x \to x_0$.*

Lemma XI.5. *Let $(X, \mathcal{T})$ be a topological space and $(V, \|\cdot\|)$ a normed space. Let $x_0 \in X$ and $f, g : X \setminus \{x_0\} \to V$ be functions. Then*

1. *$f \sim_{x_0} g \iff (f - g) \in o(g)$ as $x \to x_0$;*

2. *$\sim_{x_0}$ is an equivalence relation;*

3. *$f \sim_{x_0} g \implies f = \Theta(g)$ as $x \to x_0$.*

Lemma XI.6. *Let $(X, \mathcal{T})$ be a topological space and $(V, \|\cdot\|)$ a normed space. Let $x_0 \in X$ and $f, g, h, k : X \setminus \{x_0\} \to V$ be functions. Then*

1. *if $f = o(h)$ and $g = O(k)$, then $fg = o(hk)$ as $x \to x_0$;*

2. *if $f = O(h)$ and $g = O(k)$, then $fg = O(hk)$ as $x \to x_0$;*

3. *if $f = o(h)$ and $h = O(k)$, then $f = o(k)$ as $x \to x_0$.*

Chapter 2

Differentiation

`file:///C:/Users/user/Downloads/978-1-4614-3894-6.pdf` `file:///C:/Users/user/Downloads/2011_Bookmatter_TheRicciFlowInRiemannianGeomet.pdf`

2.1 Derivatives of functions between normed groups

Let G, H be normed groups, $f : G \to H$ a function and $x_0 \in G$. We call f differentiable if there exists a continuous homomorphism A_{x_0} such that

$$\lim_{x\to 1} \frac{\left\| f(xx_0)f(x_0)^{-1}A_{x_0}(x)^{-1} \right\|}{\|x\|} = 0.$$

We call A_{x_0} a derivative of f at x_0.

Proposition XI.7. *Let G, H be normed groups, $f : G \to H$ a function and $x_0 \in G$. There exists at most one derivative of f at x_0.*

Proof. TODO ☐

2.1.1 Fréchet derivatives on normed vector spaces

2.2 Directional derivatives

2.3 For real normed vector spaces

TODO: directional / Gateaux derivative for locally convex TVSs?

2.3.1 Directional derivatives

Let V, W be normed vector spaces and $f : U \subseteq V \to W$ a function defined on an open subset U. For $a, u \in V$, we call

$$\partial_u f|_a := \lim_{t\to 0} \frac{f(a+tu) - f(a)}{t}$$

the directional derivative of f at a in the direction u,if it exists.

- If $V = \mathbb{R}^n$, then we define $\frac{\partial f}{\partial x^i} f \coloneqq \partial_{\mathbf{e}_i} f$, where $\mathcal{E} = \langle \mathbf{e}_i \rangle_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. These directional derivatives are called the partial derivatives w.r.t. the basis $\mathcal{E}$.
- If $V = \mathbb{R}$, then there is, up to scalar multiplication, only one direction u. We denote the directional derivative $f'(a) \coloneqq \partial_u f|_a$.

For a given function $f : V \to W$, the directional derivative is a partial function of both a direction and a point:

$$(V \times V) \not\rightarrow W : \quad (u, a) \mapsto \partial_u f(a)$$

Partial application in the first argument gives a function

$$\partial_u f : \ V \not\rightarrow W : \ a \mapsto \partial_u f(a) \coloneqq \partial_u f|_a$$

that is also referred to as the directional derivative of f in the direction u.

Lemma XI.8. *Let $f, g : V \to W$, $u \in V$ and $\lambda \in \mathbb{F}$, then*

1. $\partial_u(f + g) = \partial_u f + \partial_u g$;
2. $\partial_u(fg) = (\partial_u f)g + f(\partial_u g)$;
3. $\partial_u(\lambda f) = \lambda \partial_u f$.

Proposition XI.9. *Let $B : V_1 \oplus V_2 \to W$ be a bilinear form. Then, for $(x, y), (a, b) \in V_1 \oplus V_2$*

$$\partial_{(x,y)} B|_{(a,b)} = B(x, b) + B(a, y).$$

Proof. We calculate

$$\begin{aligned}
\partial_{(x,y)} B|_{(a,b)} &= \lim_{t\to 0} \frac{B(a + tx, b + ty) - B(a, b)}{t} \\
&= \lim_{t\to 0} \frac{1}{t}(B(a, b) + tB(a, y) + tB(x, b) + t^2 B(x, y) - B(a, b)) \\
&= B(x, b) + B(a, y) + \lim_{t\to 0} tB(x, y) \\
&= B(x, b) + B(a, y).
\end{aligned}$$

□

2.3.1.1 Partial derivatives

TODO notation D^α for multiindex α. Also $|\alpha| = \sum_i \alpha_i$.

2.3.1.2 Gateaux derivative

Partial application of the directional derivative in the second argument gives a function

$$\mathrm{d}_a f : V \not\rightarrow W : u \mapsto \mathrm{d}_a f(u) \coloneqq \partial_u f|_a = \lim_{t\to 0} \frac{f(a + tu) - f(a)}{t}$$

that is referred to as the Gateaux differential of f at the point a.

If $\mathrm{d}_a f : V \nrightarrow W$ is a bounded linear map, we will refer to it as the Gateaux derivative.

The Gateaux differential is homogeneous even if it is not linear:

Lemma XI.10. *Let $f : V \to W$ be a function between normed spaces and $a, u \in V$. If $\partial_u f$ is defined at a, then*

$$\mathrm{d}_a f(\lambda u) = \partial_{\lambda u} f(a) = \lambda \partial_u f(a) = \lambda\, \mathrm{d}_a f(u) \qquad \forall \lambda \in \mathbb{F}.$$

Proof. $\partial_{\lambda u} f(a) = \lim_{t\to 0} \frac{f(a+t\lambda u)-f(a)}{t} = \lim_{t\lambda \to 0} \frac{f(a+t\lambda u)-f(a)}{t\lambda/\lambda} = \lambda \partial_u f(a)$. ☐

TODO mean value theorem?

2.3.2 Hadamard derivative

2.3.3 Fréchet derivative

If a function has a (bounded linear) Gateaux derivative at a and the limit in the definition of the derivative

$$\mathrm{d}_a f : V \nrightarrow W : u \mapsto \mathrm{d}_a f(u) \coloneqq \partial_u f|_a = \lim_{t\to 0} \frac{f(a+tu)-f(a)}{t}$$

is uniform in all u on the $S(\mathbf{0},1)$, then we say the function is (Fréchet) differentiable at a and has Fréchet derivative $\mathrm{d}_a f$.
We may also write $\mathrm{d}f$, leaving the a implicit.

Proposition XI.11. *Let V, W be normed vector spaces and $f : U \subseteq V \to W$ a function defined on an open subset U. Let $a \in V$.*
Then f is Fréchet differentiable at a if and only if *there exists a bounded linear map $A : V \to W$ such that $f(a+x)$ can be written as*

$$f(a+x) = f(a) + A(x) + o(x) \qquad \text{as} \qquad x \to 0.$$

In this case $A = \mathrm{d}_a f$.

Proof. First assume f is Fréchet differentiable at a. Then

$$\forall \varepsilon > 0 : \exists \delta > 0 : \; \forall u \in S(\mathbf{0},1) : \forall t \in \mathbb{R} : \; t < \delta \implies \varepsilon > \left\| \frac{f(a+tu)-f(a)}{t} - \mathrm{d}_a f(u) \right\| = \frac{\|f(a+tu)-f(a)-\mathrm{d}_a f(tu)\|}{|t|} = \frac{\|f(a+tu)-f(a)-\mathrm{d}_a f(tu)\|}{\|tu\|}.$$

Now each vector x in V can be written as tu for some $t \in \mathbb{R}$ and $u \in S(\mathbf{0},1)$, so this can be written as

$$\forall \varepsilon > 0 : \exists \delta > 0 : \; \forall x \in V : \; \|x\| < \delta \implies \varepsilon > \frac{\|f(a+x)-f(a)-\mathrm{d}_a f(x)\|}{\|x\|}$$

which is exactly the statement $f(a+x) = f(a) + \mathrm{d}_a f(x) + o(x)$ as $x \to 0$.
The logic can be reversed to obtain the equivalence. ☐

Proposition XI.12. *If a function is Fréchet differentiable at a point a, then it is continuous at a.*

Proof. Assume f is has Fréchet derivative A. Then

$$0 = \lim_{x \to a} \|f(x) - f(a) - \mathrm{d}_a f(x-a)\| = \left\| \lim_{x \to a} f(x) - f(a) - \mathrm{d}_a f(\lim_{x \to a} x - a) \right\| = \left\| \lim_{x \to a} f(x) - f(a) \right\|.$$

□

Lemma XI.13. *The Fréchet derivative is the same for equivalent norms.*

2.3.3.1 Link with Gateaux derivative

`https://link.springer.com/content/pdf/bbm%3A978-3-642-16286-2%2F1.pdf`
`http://www.m-hikari.com/ams/ams-password-2008/ams-password17-20-2008/behmardiAMS17-20-2008.pdf`

Proposition XI.14. *If a function between subsets of normed spaces is Fréchet differentiable, it is also Gateaux differentiable and the Fréchet derivative is equal to the Gateaux derivative.*

Proof. Let A be the Fréchet derivative of $f : U \subseteq V \to W$. Then for all $u \in V$

$$\begin{aligned} 0 &= \lim_{t \to 0} \frac{\|f(a+tu) - f(a) - A(tu)\|}{\|tu\|} = \lim_{t \to 0} \frac{\|(f(a+tu) - f(a))/t - A(u)\|}{\|u\|} \\ &= \frac{\|\lim_{t \to 0}(f(a+tu) - f(a))/t - A(u)\|}{\|u\|} = \frac{\|\mathrm{d}_a f(u) - A(u)\|}{\|u\|}. \end{aligned}$$

□

For this reason we will also denote the Fréchet derivative of f at a as $\mathrm{d}_a f$. We will sometimes also write $f'(a)$.

Example

TODO!
There are functions that have a Gateaux derivative, but not a Fréchet derivative at certain points. For example

$$f : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

which has $\partial_{\mathbf{u}} f(\mathbf{0}) = 0$ for all $\mathbf{u} \in \mathbb{R}^2$ and thus the Gateaux derivative at zero is $\mathrm{d}f = 0$. Composing f with $t \mapsto (t, t^2)$ yields the function $t \mapsto \begin{cases} t^{-2} & t \neq 0 \\ 0 & t = 0 \end{cases}$, which is not continuous at 0. So f is not continuous at zero and a fortiori is not Fréchet differentiable.

Proposition XI.15. *If there exists a basis β of V such that the partial derivatives of $f : U \subseteq V \to W$ w.r.t. β exist and are continuous in $a \in V$, then f is Fréchet differentiable in a.*

Proof. □

TODO for finite dimensions! Expand to criterion for Gateaux to Fréchet.

Example

2.3.3.2 The Jacobian

Let $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be a function. Then $A_{\mathrm{d}f}$ is a matrix with

$$[A_{\mathrm{d}f}]_{ij} = [\mathrm{d}f\mathbf{e}_j]_i = \left[\frac{\partial f}{\partial x^j}\right]_i .$$

This matrix is called the Jacobian J_f.

2.3.4 Differentiation of a normed algebra

Proposition XI.16 (Leibniz rule)**.** *Let A be a normed algebra and $a, b \in (\mathbb{R} \to A)$ elements that have derivatives. Then*

$$(ab)' = a'b + b'a.$$

Proof. We calculate

$$\begin{aligned}
0 = 0 \cdot a'(t)b'(t) &= \lim_{\epsilon \to 0} \epsilon a'(t)b'(t) \\
&= \lim_{\epsilon \to 0} \epsilon \frac{a(t+\epsilon) - a(t)}{\epsilon} \frac{b(t+\epsilon) - b(t)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{a(t+\epsilon)b(t+\epsilon) - a(t+\epsilon)b(t) - a(t)b(t+\epsilon) + a(t)b(t)}{\epsilon} + \frac{a(t)b(t)}{\epsilon} - \frac{a(t)b(t)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{a(t+\epsilon)b(t+\epsilon) - a(t)b(t)}{\epsilon} - \frac{a(t+\epsilon) - a(t)}{\epsilon} b(t) - a(t)\frac{b(t+\epsilon) - b(t)}{\epsilon} \\
&= (ab)' - a'b - ab'.
\end{aligned}$$

□

Proposition XI.17. *Let A be an algebra and $p \in A$ such that $p^2 = p$ and p' exists. Then*

1. $p' = pp' + p'p$;
2. $pp'p = 0$;
3. $(p')^2 = p'pp' + p(p')^2p$;
4. $p'' = 2(p')^2 + pp'' + p''p$;
5. $pp''p = -2p(p')^2p$.

Proof. (1) We calculate $p' = (p^2)' = pp' + p'p$.
(2) Multiply (1) by p on the left and right.
(3) We calculate $(p')^2 = (pp' + p'p)(pp' + p'p) = pp'pp' + pp'p'p + p'ppp' + p'pp'p = 0p + pp'p'p + p'pp' + p'0$.
(4) Take derivative of (1).
(5) Multiply (3) by p on the left and right. □

Corollary XI.17.1. *Let* Tr *be a trace functional on A and $p \in A$ as before. Then* $\mathrm{Tr}(p') = 0$.

Proof. $\mathrm{Tr}(p') = \mathrm{Tr}(pp' + p'p) = \mathrm{Tr}(p^2p') + \mathrm{Tr}(p'p^2) = \mathrm{Tr}(pp'p) + \mathrm{Tr}(pp'p) = 2Tr(0) = 0$. □

Proposition XI.18. *Let p_0, p_1 be differentiable idempotents such that $p_0p_1 = 0 = p_1p_0$. Then $p_0'p_1 = -p_0p_1'$.*

Proof. We have $0 = p_0p_1$, so $0 = 0' = p_0'p_1 + p_0p_1'$. □

Corollary XI.18.1. *Let p_0, p_1 be differentiable idempotents such that $p_0p_1 = 0 = p_1p_0$. Then*

1. $p_1p_0'p_0 = -p_1p_1'p_0$;
2. $p_1p_0'p_1 = 0$;
3. $p_1(p_0')^2p_1 = p_1p_1'p_0p_1'p_1$;
4. $p_0(p_0')^2p_1 = 0$;
5. $p_1p_0''p_1 = 2p_1(p_0')^2p_1$.

If in addition p_2 is a differentiable idempotent such that $p_0p_2 = 0p_2p_0$ and $p_1p_2 = 0p_2p_1$, then

6. $p_1p_0'p_2 = 0$.

Proof. (1) We have

$$p_1p_0'p_0 = p_1(p_1p_0')p_0 = -p_1(p_1'p_0)p_0 = -p_1p_1'p_0.$$

(2) We have $p_1p_0'p_1 = -(p_1'p_0)p_1 = -p_1'(p_0p_1) = 0$.
(3) We have $p_1(p_0')^2p_1 = (p_1p_1p_0')(p_0'p_1p_1) = (p_1p_1'p_0)(p_0p_1'p_1)$.
(4) We have $p_0(p_0')^2p_1 = (p_0p_0')(p_0'p_1) = -(p_0p_0')(p_0p_1') = -(p_0p_0'p_0)p_1' = 0$.
(5) We have, using XI.17, $p_1p_0''p_1 = p_1(2(p_0')^2 + p_0p_0'' + p_0''p_0)p_1 = 2p_1(p_0')^2p_1$.
(6) We have $(p_1p_0')p_2 = -(p_1'p_0)p_2 = -p_1'(p_0p_2) = 0$. □

2.4 Taylor expansion

Radius of convergence

2.5 Classification of spaces

Let X, Y be subsets of normed vector spaces and X be open. We call a function $f : X \to Y$

- smooth at $x_0 \in V$ if all derivatives of f at x_0 exist;
- analytic at $x_0 \in V$ if the Taylor series of f at x_0 exists and has non-zero radius of convergence.

Lemma XI.19. *Let $f : X \to Y$ be a smooth function. Then all derivatives are continuous.*

Let X, Y be subsets of normed vector spaces and X be open.

- $\mathcal{C}^r(X, Y)$ is the space of functions in $(X \to Y)$ whose first r derivatives exist and are continuous;
- $\mathcal{C}^\infty(X, Y)$ is the space of functions in $(X \to Y)$ that are smooth at all points in X;
- $\mathcal{C}^\omega(X, Y)$ is the space of functions in $(X \to Y)$ that are analytic at all points in X.

If $Y = \mathbb{C}$, we write $\mathcal{C}^r(X), \mathcal{C}^\infty(X)$ and $\mathcal{C}^\omega(X)$. We can also use subscripts $_0$ and $_c$ to denote the extra conditions of vanishing at infinity and having compact support.

Chapter 3

Non-standard analysis

`http://www.lightandmatter.com/calc/`

Proposition XI.20. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Then f is continuous at $x \in \mathbb{R}$* if and only if *for all infinitesimal δ there exists an infinitesimal ϵ such that*

$$f(x+\delta) = f(x) + \epsilon.$$

Alternatively we can state this as

$$f(x+\delta) \approx f(x)$$

for all infinitesimal δ.

Clearly continuity is a requirement for differentiability: if $f(x+\delta) - f(x)$ is not infinitesimal, then $\frac{f(x+\delta)-f(x)}{\delta}$ will not be finite.

Lemma XI.21. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and $y : \mathbb{R} \to \mathbb{R}$ a continuous function. Assume f differentiable at $y_0 \in \operatorname{im}(y)$. Consider f as depending on y and y as depending on x. Then*

$$\left.\frac{\mathrm{d}f}{\mathrm{d}y}\right|_{y_0} = \operatorname{st}\left(\frac{\Delta_y f}{\Delta y}\right) = \operatorname{st}\left(\frac{\Delta_x f \circ y}{\Delta_x y}\right).$$

Proof. We calculate, setting $y_0 = y(x_0)$ and using continuity of y,

$$\begin{aligned}\operatorname{st}\left(\frac{\Delta_x f \circ y}{\Delta_x y}\right) &= \operatorname{st}\left(\frac{f(y(x_0+\Delta x)) - f(y(x_0))}{y(x_0+\Delta x) - y(x_0)}\right) = \operatorname{st}\left(\frac{f(y(x_0)+\delta) - f(y(x_0))}{y(x_0)+\delta - y(x_0)}\right) \\ &= \operatorname{st}\left(\frac{f(y(x_0)+\delta) - f(y(x_0))}{\delta}\right) = \left.\frac{\mathrm{d}f}{\mathrm{d}y}\right|_{y_0}.\end{aligned}$$

□

Proposition XI.22 (Chain rule)**.** *Let y, f be real functions, differentiable at points x_0 and $y_0 = y(x_0)$, respectively. Then*

$$\left.\frac{\mathrm{d}f}{\mathrm{d}y}\right|_{y_0} \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_0} = \left.\frac{\mathrm{d}f \circ y}{\mathrm{d}x}\right|_{x_0}.$$

Proof. We calculate, using XI.21,

$$\left.\frac{\mathrm{d}f}{\mathrm{d}y}\right|_{y_0} \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_0} = \operatorname{st}\left(\frac{\Delta_y f}{\Delta y}\frac{\Delta_x y}{\Delta x}\right) = \operatorname{st}\left(\frac{\Delta_x f \circ y}{\Delta_x y}\frac{\Delta_x y}{\Delta x}\right) = \operatorname{st}\left(\frac{\Delta_x f \circ y}{\Delta x}\right) = \left.\frac{\mathrm{d}f \circ y}{\mathrm{d}x}\right|_{x_0}.$$

□

Chapter 4

Series and sequences

TODO MacLaurin, propto, other expansions (multipolar, binomial etc) Taylor polynomials, big O, possible big O types

4.1 Sequences

4.1.1 Convergence

Applying the definition of convergent sequences to sequences in $\mathbb{R}$ gives the so-called $\varepsilon - n_0$ criterion for convergence:

Proposition XI.23. *Let (x_n) be a real sequence. Then (x_n) converges to L* if and only if

$$\forall\varepsilon > 0 : \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq n_0 \implies |x_n - L| < \varepsilon.$$

The definition for divergence to ±∞ is identical.
TODO Bolzano-Weierstrass

4.1.1.1 Examples of sequences

Proposition XI.24. *Let $p \in \mathbb{R}$. Then*

$$\lim_{n\to\infty} n^p = \begin{cases} +\infty & (p > 0) \\ 1 & (p = 0) \\ 0 & (p < 0) \end{cases}.$$

Proposition XI.25. *Let $r \in \mathbb{R}$. Then*

$$\lim_{n\to\infty} r^n = \begin{cases} +\infty & (r > 1) \\ 1 & (r = 1) \\ 0 & (-1 < r < 1) \\ \textit{does not exist} & (r \leq -1) \end{cases}.$$

4.1.2 Difference calculus

TODO see section under series
TODO $\Delta^+_\epsilon, \Delta^-_\epsilon$ for real functions! Also central difference Δ.

4.1.2.1 Difference operators

Let $\langle x_n \rangle$ be a sequence. We define

- the forward difference operator $\Delta^+ : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by $\Delta^+ x_n \coloneqq (\Delta^+ x)_n \coloneqq x_{n+1} - x_n$; and
- the backward difference operator $\Delta^- : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by $\Delta^- x_n \coloneqq (\Delta^- x)_n \coloneqq x_n - x_{n-1}$.

4.2 Series

Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a sequence. The series generated by this sequence is the sequence

$$\left\langle \sum_{i=0}^{n} a_i \right\rangle_{n\in\mathbb{N}}.$$

The n^{th} element of this sequence, $\sum_{i=0}^{n} a_n$, is called the n^{th} partial sum.
If the sequence of partial sums converges, we call the series convergent otherwise it is divergent.
The expression

$$\sum_{i=0}^{\infty} a_n$$

may be used to denote the limit of the series or the series itself.

- If the series $\sum_{i=0}^{\infty} a_n$ converges to $+\infty$ as a sequence in $\overline{\mathbb{R}}$, we say it diverges to $+\infty$ and write $\sum_{i=0}^{\infty} a_n = +\infty$.
- If it converges to $-\infty$ in $\overline{\mathbb{R}}$, we say it diverges to $-\infty$ and write $\sum_{i=0}^{\infty} a_n = -\infty$.

4.2.1 Difference calculus

TODO see section under sequences

Proposition XI.26 (Summation by parts)**.** *Let $\langle x_k \rangle$ and $\langle y_k \rangle$ be sequences in some field and $m, n \in \mathbb{N}$. Then*

$$\begin{aligned}\sum_{k=m}^{n} x_k \Delta^+ y_k &= (x_n y_{n+1} - x_{m-1} y_m) - \sum_{k=m}^{n} y_k \Delta^- x_k \\ &= (x_n y_{n+1} - x_m y_m) - \sum_{k=m}^{n-1} y_{k+1} \Delta^+ x_k.\end{aligned}$$

4.2.2 Series of positive real numbers

`https://en.wikipedia.org/wiki/Convergence_tests`

4.2.2.1 Ratio test hierarchy

`https://en.wikipedia.org/wiki/Ratio_test`

4.2.2.2 Root test hierarchy

`https://en.wikipedia.org/wiki/Root_test#Root_tests_hierarchy`

Proposition XI.27 (Cauchy's criterion)**.** *Let* $\sum_{n=0}^{\infty} a_n$ *be a positive series. Define the number*

$$C = \limsup_{n\to\infty} \sqrt[n]{a_n}.$$

Then

- *if* $C < 1$*, the series converges;*
- *if* $C > 1$*, the series diverges;*
- *if* $C = 1$*, the test is inconclusive.*

4.2.3 Series in normed abelian groups

Let $\sum_{i=0}^{\infty} a_i$ be a convergent series. We call the series

- absolutely convergent if the series $\sum_{i=0}^{\infty} \|a_i\|$ is convergent;
- unconditionally convergent if for all bijections $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{i=0}^{\infty} a_{\sigma(i)}$ is convergent and conditionally convergent otherwise.

Proposition XI.28. *Let* $\sum_{i=0}^{\infty} a_i$ *be a series in a complete abelian normed group. If* $\sum_{i=0}^{\infty} a_i$ *is absolutely convergent, it is convergent.*

Proof. Reverse triangle inequality TODO! ☐

Proposition XI.29. *`https://en.wikipedia.org/wiki/L%C3%A9vy%E2%80%93Steinitz_theorem`*

Corollary XI.29.1 (Riemann series theorem)**.** *Let* $\sum_{i=0}^{\infty} a_i$ *be a convergent series that does not converge absolutely and* $L \in \overline{\mathbb{R}}$*. There exists a bijection* $\sigma : \mathbb{N} \to \mathbb{N}$ *such that*

$$\sum_{i=0}^{\infty} a_{\sigma(i)} = L.$$

There also exists a bijection $\tau : \mathbb{N} \to \mathbb{N}$ *such that* $\sum_{i=0}^{\infty} a_{\tau(i)}$ *fails to approach any limit, finite or infinite.*

Proof. `https://en.wikipedia.org/wiki/Riemann_series_theorem` ☐

Corollary XI.29.2. *A convergent series is unconditionally convergent* if and only if *it is absolutely convergent.*

4.2.4 Convergence

Theorem XI.30 (Tannery's theorem)**.** *Let* $s_n = \sum_{k=0}^{\infty} a_{n,k}$ *be the limit of a convergent series for each* n *such that* $\lim_{n\to\infty} a_{n,k}$ *converges to* a_k *for all* n*. If there exists a sequence* M_k *such that* $|a_{n,k}| \leq M_k$ *for all* n, k *and* $\sum_{k=0}^{\infty} M_k < \infty$*, then*

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = \sum_{k=0}^{\infty} \lim_{n\to\infty} a_{n,k} = \sum_{k=0}^{\infty} a_k.$$

Proof. Choose an arbitrary $\varepsilon > 0$. For any $n, N \in \mathbb{N}$ we can write

$$\left| s_n - \sum_{k=0}^{\infty} a_k \right| \leq \sum_{k=0}^{\infty} |a_{n,k} - a_k| \leq \sum_{k=0}^{N} |a_{n,k} - a_k| + 2\sum_{k>N} M_k \leq N \max_{k<N} |a_{n,k} - a_k| + 2\sum_{k>N} M_k.$$

So we aim to find some N_0 such that $2\sum_{k>N} M_k \leq \varepsilon/2$ for all $N \geq N_0$, which of course we can. Then we choose an n_0, in function of this N_0 and ε, such that $\max_{k<N_0} |a_{n,k} - a_k| \leq \varepsilon/(2N_0)$ for all $n \geq n_0$. It is clear we can do so for each k separately, but there are only finitely many ks so we take the largest n_0. Then

$$\left| s_n - \sum_{k=0}^{\infty} a_k \right| \leq \varepsilon \qquad \text{for all } n \geq n_0\text{, implying the limit is zero.}$$

□

`https://www.coloradomesa.edu/math-stat/documents/JohnGillresearchnoteTanneryTheorem.pdf`

4.2.5 Examples of series

4.2.5.1 Geometric series

4.2.5.2 Harmonic series

4.3 Functions defined by series

4.3.1 Power series

A power series is a partial function of the form

$$f : \mathbb{C} \not\to \mathbb{C} : z \mapsto \sum_{i=0}^{\infty} a_i (z - z_0)^i,$$

where $z_0 \in \mathbb{C}$ and $\langle a_n \rangle$ is a sequence of complex numbers.

TODO more general algebras.

Proposition XI.31 (Cauchy-Hadamard)**.** *Let* $f : \mathbb{C} \not\to \mathbb{C} : z \mapsto \sum_{i=0}^{\infty} a_i (z - z_0)^i$ *be a power series. Define the real number* R *by*

$$\frac{1}{R} \coloneqq \limsup_{n\to\infty} \left(|a_n|^{1/n} \right).$$

For all $z \in \mathbb{C}$ *such that* $|z - z_0| < R$*, the value* $f(z)$ *is well-defined.*

If $\limsup_{n\to\infty}\left(|a_n|^{1/n}\right)\to\infty$, we consider R to be zero.

Proof. XI.27 TODO. ☐

The R defined in XI.31 is called the radius of convergence of the power series.

4.3.1.1 Taylor and MacLaurin

4.3.2 Laurent series

A Laurent series is a partial function of the form

$$f : \mathbb{C} \not\rightarrow \mathbb{C} : z \mapsto \sum_{i=-\infty}^{\infty} a_i(z-z_0)^i := \sum_{i=0}^{\infty} a_i(z-z_0)^i + a_{-i}(z-z_0)^{-i},$$

where $z_0 \in \mathbb{C}$ and $\langle a_n \rangle, \langle a_{-n} \rangle$ are sequences of complex numbers.
The series $\sum_{i=1}^{\infty} a_{-i}(z-z_0)^{-i}$ is called the principal part of the Laurent series.

Proposition XI.32. *Let $f : \mathbb{C} \not\rightarrow \mathbb{C} : z \mapsto \sum_{i=-\infty}^{\infty} a_i(z-z_0)^i$ be a Laurent series. Define the real numbers r and R by*

$$r := \limsup_{n\to\infty}\left(|a_{-n}|^{1/n}\right); \frac{1}{R} \qquad := \limsup_{n\to\infty}\left(|a_n|^{1/n}\right).$$

For all $z \in \mathbb{C}$ such that $r < |z-z_0| < R$, the value $f(z)$ is well-defined.

So the domain of convergence of a Laurent series is an annulus around z_0.

4.3.3 Puiseux series

`https://en.wikipedia.org/wiki/Puiseux_series`

4.4 Sequences and series in normed structures

4.5 Matrix exponential

The matrix exponential is quite simply defined as the MacLaurin series of the normal exponential applied to square matrices.

Let X be an $n \times n$ matrix. The exponential of X is given by the power series

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}.$$

It's nice to know that for any $n\times n$ real or complex matrix X, this series does actually converge.
The matrix exponential is also a continuous function of X.
Here are also some elementary properties of the matrix exponential, that may be useful for somebody somewhere.

Let X be an arbitrary $n \times n$ matrix. Let C be invertible.

1. $e^0 = \mathbb{1}_n$.
2. $\left(e^X\right)^\dagger = e^{X^\dagger}$.
3. e^X is invertible and $\left(e^X\right)^{-1} = e^{-X}$.
4. $\left(e^X\right)^* = e^{A^*}$
5. $\left(e^X\right)^\dagger = e^{A^\dagger}$
6. $\left(e^X\right)^\intercal = e^{A^\intercal}$
7. $e^{CXC^{-1}} = Ce^XC^{-1}$

It is in general **not** true that $e^{X+Y} = e^X e^Y$; this is only true if X and Y commute.
Here are a number of properties of the matrix exponential that will be useful later. We will assume X, Y are complex $n \times n$ matrix.

The map $\mathbb{R} \to \mathbb{C}^{n\times n} : t \mapsto e^{tX}$ is a smooth curve in $\mathbb{C}^{n\times n}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{tX} = Xe^{tX} = e^{tX}X.$$

In particular,

$$\tfrac{\mathrm{d}}{\mathrm{d}t} e^{tX}\Big|_{t=0} = X.$$

Lie product formula

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m$$

Finally for the determinant we have:

$$\det\left(e^X\right) = e^{\mathrm{Tr}(X)}$$

4.6 Binomial theorem and binomial series

Chapter 5

Real functions

5.1 Exponentiation

Exponentiation is usually introduced as repeated multiplication, e.g $3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$. We call the number being repeatedly multiplied the base (3 in this case) and the number of times it is multiplied (5 in this case) we call the exponent or power. Taking a real number as the base is not too difficult to do: $3.2564^3 = 3.2564 \times 3.2564 \times 3.2564 = 34.531324622144\ldots$
Now what happens if we put a real number in the exponent? What is $3^{2.5}$? Because 2.5 is a rational number, we can still solve this using standard exponentiation and a square root ($3^{2.5} = \sqrt{3^5}$). Something like 3^π gets even more tricky.
We resolve this situation if the following way: TODO

5.1.1 The square of a number

TODO !! square of real nonnegative

5.1.2 $n^{\mathbf{th}}$ roots

TODO in particular of one.

5.1.3 Exponential functions

Corresponds to power for rational numbers.
Assume $a > 0$ and $b > 0$, and x and y are any real numbers, then the exponential function has the following properties:

1. $a^0 = 1$
2. $a^{x+y} = a^x a^y$
3. $a^{-x} = \frac{1}{a^x}$
4. $(a^x)^y = a^{xy}$
5. $(ab)^x = a^x b^x$

Limits:

1. If $a > 1$, then $\lim_{x \to -\infty} a^x = 0$ and $\lim_{x \to \infty} a^x = \infty$
2. If $0 < a < 1$, then $\lim_{x \to -\infty} a^x = \infty$ and $\lim_{x \to \infty} a^x = 0$

5.2 Logarithms

The logarithm, denoted

$$\log_a : [0, \infty[\to [-\infty, \infty]$$

is defined as the inverse of the exponential function with base a. Thus

$$\log_a(a^x) = x \quad \forall x \in \mathbb{R} \qquad \text{and} \qquad a^{\log_a(x)} = x \forall x > 0$$

If $x > 0, y > 0, a > 0, b > 0$ and $a \neq 1, b \neq 1$, then

1. $\log_a 1 = 0$
2. $\log_a(xy) = \log_a x + \log_a y$
3. $\log_a(\frac{1}{x}) = -\log_a x$
4. $\log_a(x^y) = y \log_a x$
5. $\log_a x = \frac{\log_b x}{\log_b a}$

5.3 Polynomial and rational functions

The polynomial functions are a very important class of functions. They can be specified by equations of the form

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

where the numbers a_i specified by the index i are called the coefficients corresponding to the i^{th} power of x. We can assume that a_n is not zero (if it is we reduce n until it isn't). That way n gives the degree of the polynomial expression.
Polynomial functions with a degree of one are called linear.
TODO def rational functions

5.3.1 Linear functions

5.3.2 Quadratic functions

focus, directrix, vertex, axis
root formula

5.3.3 Fundamental theorem of algebra

5.4 Absolute value

TODO + standard definition of distance

5.5 Transformations

TODO + transforming the graph (translation x/y, scaling x/y)

Chapter 6

Real Analysis

Dual numbers
Dini theorem
TODO: all versions of homogeneity.
The objects of study in real analysis are real functions, and more generally functions between Euclidean spaces, i.e. real, finite-dimensional, normed vector spaces.

6.1 Limits

In this section infinities can appear because we assume $\mathbb{N}$ and $\mathbb{R}$ are embedded in their Dedekind-MacNeille completions $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.
`https://en.wikipedia.org/wiki/Interchange_of_limiting_operations`

6.1.1 Real functions

Applying the definition of limits to functions in $\mathbb{R}$ gives the so-called $\varepsilon - \delta$ definition of limits:

Proposition XI.33. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a real function and p a limit point o A. Then $L \in \mathbb{R}$ is the limit of $f(x)$ as x approaches p* if and only if

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : |x - p| < \delta \implies |f(x) - L| < \varepsilon.$$

TODO: criteria for limits involving infinities.
In addition we can define some special types of limits.

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and p a limit point of A. Then

- the left limit of $f(x)$ as x approaches p is the limit of $f|_{A \cap]-\infty, p]}$ as $x \to p$, denoted

$$\lim_{x \xrightarrow{<} p} f(x) \qquad \text{or} \qquad \lim_{x \to p^-} f(x);$$

- the right limit of $f(x)$ as x approaches p is the limit of $f|_{A \cap [p, +\infty[}$ as $x \to p$, denoted

$$\lim_{x \xrightarrow{>} p} f(x) \qquad \text{or} \qquad \lim_{x \to p^+} f(x).$$

Lemma XI.34. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and p a limit point of both $A \cap]-\infty, p]$ and $A \cap [p, +\infty[$. Then $\lim_{x\to p} f(x)$ exists if the left and right limit exist and are equal to each other. In this case*

$$\lim_{x\to p} f(x) = \lim_{x\to p^-} f(x) = \lim_{x\to p^+} f(x).$$

6.1.2 Properties of limits

TODO: for functions $X \to \mathbb{R}$.
We enumerate some of the properties for limits here. These properties are valid for all types of limit of real functions, so long as all limits in the equation are limits to the same point (or infinity, TODO elaborate!). In fact the properties also hold true for sequences of real numbers.

- Taking the limit is a linear operation:

$$\lim(f(x) + g(x)) = \lim f(x) + \lim g(x)$$

$$\lim(a \cdot f(x)) = a \cdot \lim f(x) \qquad \forall a \in \mathbb{R}$$

- The limit of the product:

$$\lim(f(x) \cdot g(x)) = (\lim f(x)) \cdot (\lim g(x))$$

- The limit of the quotient (assuming $\lim g(x) \neq 0$):

$$\lim \left(\frac{f(x)}{g(x)}\right) = \frac{\lim f(x)}{\lim g(x)}$$

- The limit of a power (with m an integer and n a positive integer):

$$\lim[f(x)]^{m/n} = (\lim f(x))^{m/n}$$

- Partial order is preserved. Assume $f(x) \leq g(x)$ on some interval containing x_0. Then

$$\lim f(x) \leq \lim g(x)$$

  The same is not true for the strict order $<$!

6.1.2.1 The squeeze theorem

6.2 Continuity

Proposition XI.35. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $p \in A \cap A'$, where A' is the set of limit points of A. Then*

$$f \text{ is continuous at } p \quad \iff \quad \lim_{x\to p} f(x) = f(x).$$

6.2.1 Left and right continuity

TODO

6.2.2 Discontinuities

If a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is not continuous at p, then p is a limit point of A by VIII.136 and VIII.130.

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $p \in A$ such that f is not continuous at p. Then

- if $\lim_{x\to p} f(x)$ exists and is finite, we call p a removable discontinuity;
- if both $\lim_{x\to p^-} f(x)$ and $\lim_{x\to p^+} f(x)$ exist and are finite, but are different, we call p a jump discontinuity;
- if either $\lim_{x\to p^-} f(x)$ or $\lim_{x\to p^+} f(x)$ do not exist, we call p an essential discontinuity.

Removable and jump discontinuities are also called discontinuities of the first kind. Essential discontinuities are also called discontinuities of the second kind.

TODO: should discontinuities of type 1/x be considered essential?

6.2.2.1 Jump discontinuities

Proposition XI.36. *Let f be a monotone real-valued function on an interval I. Then all discontinuities are jump discontinuities.*

Theorem XI.37 (Darboux-Froda)**.** *Let f be a monotone real-valued function on an interval I. Then the set of discontinuities is at most countable.*

TODO: intervals must be closed / open?? In XI.36 and XI.37.

Let f be a real function. We define the difference operator

$$\Delta_{x_0} f := \lim_{x\to x_0+} f(x) - \lim_{x\to x_0-} f(x)$$

6.3 Functions compact subsets

6.3.1 Min-max theorem

$f(p) \leq f(x) \leq f(q)$

6.3.2 Intermediate value theorem

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6.4 Derivatives

6.5 Stone-Weierstrass

Theorem XI.38 (Stone-Weierstrass)**.** *Let X be a compact Hausdorff space. Let $A \subseteq \mathcal{C}(X)$ be a unital $*$-subalgebra. Suppose that A separates points, i.e. for all $x \neq y$ in X there exists $f \in A$ with $f(x) \neq f(y)$. Then A is dense in $\mathcal{X}$ with respect to $\|\cdot\|_\infty$.*

Chapter 7

Measure theory

TODO (bounded) finitely additive signed measures form Riesz spaces.

7.1 Pre-measures

Let $\mathcal{S}$ be a semi-ring on a set Ω and $(M, +, 0)$ a commutative monoid. An M-valued pre-measure on $\mathcal{S}$ is a map $\mu : \mathcal{S} \to M$ satisfying

- $\mu(\emptyset) = 0$;
- μ is (finitely) additive: if $A, B \in \mathcal{S}$ are disjoint, then
$$\mu(A \cup B) = \mu(A) + \mu(B).$$

If $M = \overline{\mathbb{R}^+}$, then we call the pre-measure positive.

Proposition XI.39. *Let $\mathcal{S}$ be a semi-ring on Ω. Every pre-measure $\mu_{\mathcal{S}}$ on $\mathcal{S}$ extends uniquely to a pre-measure μ on $\mathfrak{R}\{\mathcal{S}\}$, the ring generated by $\mathcal{S}$.*

Lemma XI.40. *If there exists an $A \in \mathcal{S}$ such that $\mu(A)$ is cancellative, then the requirement $\mu(\emptyset) = 0$ is redundant.*

Proof. We calculate $\mu(A) = \mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset)$, so $\mu(\emptyset) = 0$. □

For positive pre-measures this is requiement is $\mu(A) < \infty$.

7.1.1 Outer measures

Let Ω be a non-empty set. An outer measure on Ω is a map $\nu : \mathcal{P}(\Omega) \to [0, \infty]$ satisfying

- $\nu(\emptyset) = 0$;
- if $E \subseteq F \subseteq \Omega$, then $\nu(E) \leq \nu(F)$;
- ν is σ-subadditive: for every sequence (E_n) of pairwise disjoint subsets of Ω, we

have

$$\nu\left(\biguplus_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \nu(E_n).$$

It is important to note that outer measures are not in general measures or pre-measures.

Proposition XI.41. *Let $\mathcal{R}$ be a ring with universe set Ω and μ a measure on $\mathcal{R}$. Then*

$$\mu^* : \mathcal{R} \to [0,\infty] : E \mapsto \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) \;\middle|\; (E_n)_{n\in\mathbb{N}} \subseteq \mathcal{R} \text{ with } E \subseteq \bigcup_{n\in\mathbb{N}} E_n\right\}$$

with the convention that $\inf\emptyset = \infty$*, defines an outer measure on* Ω*.*

Let ν be an outer measure on a set Ω. We say that a set $E \subseteq \Omega$ is ν-measurable, if

$$\forall A \subseteq \Omega: \ \nu(A) = \nu(A\cap E) + \nu(A \setminus E).$$

7.2 σ-algebras and measurable spaces

Let Ω be a non-empty set. A σ-algebra $\mathcal{A}$ on Ω is a subset of $\mathcal{P}(\Omega)$ such that

- $\emptyset \in \mathcal{A}$;
- if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$;
- for any sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$, one has $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{A}$.

The elements of a σ-algebra are called events.
A pair $(\Omega, \mathcal{A})$ of a set and a σ-algebra on the set is called a measurable space.

Example

For any non-empty set Ω, the following are σ-algebras:

- $\{\emptyset, \Omega\}$;
- $\mathcal{P}(\Omega)$.

Lemma XI.42. *A σ-algebra is closed w.r.t. all countable operations involving complements, unions, intersections and differences.*

Proof. All can be expressed in terms of unions and complements:

$$E \cap F = (E^c \cup F^c)^c \qquad E \setminus F = (E^c \cup F)^c.$$

□

Lemma XI.43. *Let Ω be a non-empty set. Let $\{\mathcal{A}_i\}_{i\in I}$ be an arbitrary family of σ-algebras. Then the intersection $\bigcap_{i\in I}\mathcal{A}_i$ is a σ-algebra.*

Corollary XI.43.1. *Let $\mathcal{S} \subset \mathcal{P}(\Omega)$. Then there exists a smallest σ-algebra containing $\mathcal{S}$.*

This σ-algebra is called the σ-algebra generated by $\mathcal{S}$.

7.2.1 Properties of measure spaces

7.2.1.1 Separated measure spaces

Let $(\Omega, \mathcal{A})$ be a measurable space and $x, y \in \Omega$. Then x, y are called separated or distinguishable if there exists $A \in \mathcal{A}$ such that $x \in A$ and $y \notin A$.
The measurable space $(\Omega, \mathcal{A})$ is called separated if all distinct $x, y \in \Omega$ are separated.

Lemma XI.44. *The relation that relates indistinguishable points is an equivalence relation. This relation is the identity relation* if and only if *the measurable space is separated.*

7.2.2 Measurable functions

Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces.
A function $f : \Omega_1 \to \Omega_2$ is called measurable if

$$\forall E \in \mathcal{A}_2 : f^{-1}[E] \in \mathcal{A}_1.$$

We may also say f is $\mathcal{A}_1/\mathcal{A}_2$-measurable to emphasise which σ-algebras are being used.
We denote the set of $\mathcal{A}_1/\mathcal{A}_2$-measurable functions by $\mathcal{M}(\mathcal{A}_1, \mathcal{A}_2)$.

Suppose we are moving around in Ω_1 and tracking the output of the function in Ω_2. We would like to be able to explore the contents of an event in Ω_2 from within an event in Ω_1.

Lemma XI.45. *Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces and $\mathcal{A}_2$ is generated by $\mathcal{S}$. Then $f : \Omega_1 \to \Omega_2$ is measurable* if and only if $\forall S \in \mathcal{S} : f^{-1}[S] \in \mathcal{A}_1$.

7.2.2.1 σ-algebras generated by functions

Let Ω_1 be a set and $(\Omega_2, \mathcal{A})$ a measurable space. Let $f : \Omega_1 \to \Omega_2$ be a function. The pull-back σ-algebra or the σ-algebra generated by f is

$$\sigma(f) \coloneqq \left\{ f^{-1}[A] \;\middle|\; A \in \mathcal{A} \right\}.$$

The pull-back σ-algebra is a σ-algebra.

Lemma XI.46. *Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces and $f : \Omega_1 \to \Omega_2$ a function. Then f is $\mathcal{A}_1/\mathcal{A}_2$-measurable* if and only if $\sigma(f) \subseteq \mathcal{A}_1$.

Lemma XI.47. *Let Ω_1 be a set, $(\Omega_2, \mathcal{A})$ a measurable space and $f : \Omega_1 \to \Omega_2$ a function. Then $x, y \in (\Omega_1, \sigma(f))$ are indistinguishable* if and only if $f(x), f(y) \in (\Omega_2, \mathcal{A})$ *are indistinguishable.*

Proof. TODO For all $A \in \mathcal{A}$, we have $f(x) \in A \iff x \in f^{-1}[A]$ and $y \in f^{-1}[A] \iff f(y) \in A$. The indistinguishability of x, y means the left-hand sides are equivalent. The indistinguishability of $f(x), f(y)$ means the right-hand sides are equivalent. ☐

Corollary XI.47.1. *Let Ω_1 be a set, $(\Omega_2, \mathcal{A})$ a measurable space and $f : \Omega_1 \to \Omega_2$ a function. Then*

1. $\ker(f)$ *is a subset of the indistinguishability relation of $\sigma(f)$;*
2. *if $(\Omega_2, \mathcal{A})$ is separated, then $\ker(f)$ equals the indistinguishability relation of $\sigma(f)$.*

7.2.2.2 The Doob-Dynkin property

A measurable set $(\Omega, \mathcal{A})$ is said to have the Doob-Dynkin property if for any set Ω_1, measurable space $(\Omega_2, \mathcal{A}_2)$ and function $f : \Omega_1 \to \Omega_2$, a function $g : \Omega_1 \to \Omega$ is $\sigma\{f\}/\mathcal{A}$-measurable if and only if it factors through f, i.e.

$$\exists h \in \mathcal{M}(\mathcal{A}_2 \to \mathcal{A}) : \quad g = h \circ f.$$

Example

The measurable space $(\Omega, \mathcal{A}) = (\{0, 1\}, \{\emptyset, \Omega\})$ does not have the Doob-Dynkin property. Take, for example, any sets Ω_1, Ω_2 and equip Ω_2 with the σ-algebra $\{\emptyset, \Omega_2\}$. Then for any function $f : \Omega_1 \to \Omega_2$, we have $\sigma(f) = \{\emptyset, \Omega_1\}$ and thus any function $g : \Omega_1 \to \Omega$ is measurable. Clearly it is not true that any function between arbitrary sets factors through any other function.

Proposition XI.48. *A measurable set* $(\Omega, \mathcal{A})$ *has the Doob-Dynkin property* if and only if $\mathcal{A}$ *is*

- *separated; and*
- *for any measurable space* $(\Omega_1, \mathcal{A}_1)$*, subset* $D \subseteq \Omega_1$ *and measurable function* $f : (D, \mathcal{A}_1|_D) \to (\Omega, \mathcal{A})$*,* f *can be extended to a measurable function on* $(\Omega_1, \mathcal{A}_1)$*.*

Proof. TODO functionsLeftRightRelations
http://www.numdam.org/article/SPS_1990__24__46_0.pdf https://mathoverflow.net/questions/263863/does-the-doob-dynkin-lemma-hold-for-any-measurable-space-that-se
https://math.stackexchange.com/questions/2193181/proof-of-doob-dynkin-lemma-when-x-is
□

7.2.3 Borel-σ-algebras

Let $(X, \mathcal{T})$ be a topological space. The σ-algebra on X generated by $\mathcal{T}$ is called the Borel-σ-algebra of $(X, \mathcal{T})$. The measurable space consisting of X equipped with the Borel-σ-algebra is called the Borel-measurable space.

Lemma XI.49. *Let* $(X, \mathcal{T})$ *be a topological space. If* $\mathcal{T}$ *has a countable basis* $\mathcal{B}$*, then the Borel-σ-algebra is generated by the basis* $\mathcal{B}$*.*

TODO:

$$\begin{aligned}\mu(B) &= \sup\{\mu(C) \mid C \subseteq B,\ C \text{ compact}\} \\ &= \inf\{\mu(O) \mid B \subseteq O,\ O \text{ open}\}.\end{aligned}$$

cfr compact open topology??

Lemma XI.50. *Every continuous function between topological spaces is a measurable function between Borel-measurable spaces.*

Proposition XI.51. *Let* $(\Omega, \mathcal{A})$ *be a measurable space and* (Y, d) *a metric space. We equip* Y *with the Borel-σ-algebra associated to the metric topology.*
Suppose that a sequence of measurable functions $f_n : \Omega \to Y$ *converges pointwise to a function* $f : \Omega \to Y$*. Then* f *is measurable.*

Proof. By XI.45 it is enough to show that for all closed sets $C \subset Y$, the inverse image $f^{-1}[C]$ is in $\mathcal{A}_1$.
We now use that in a metric space the closure of any space can be written as the countable intersection of a sequence of open sets, by TODOref. So

$$C = \overline{C} = \bigcap_{k \in \mathbb{N}} O_k.$$

Now we combine this with the fact that $f_n(x) \to f(x)$ for every $x \in \Omega$, and that every metric space is sequential, to obtain the equivalences

$$\begin{aligned} x \in f^{-1}[C] &\iff f(x) \in C \\ &\iff \forall k \in \mathbb{N} : f(x) \in O_k \\ &\iff \forall k \in \mathbb{N} : \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : f_n(x) \in O_n \\ &\iff x \in \bigcap_{k \in \mathbb{N}} \bigcup_{n_0 \in \mathbb{N}} \bigcap_{n \geq n_0} f_n^{-1}[O_n] \end{aligned}$$

As all O_n are open, they are in $\mathcal{A}_2$ and thus $f_n^{-1}[O_n] \in \mathcal{A}_1$. Finally σ-algebras are closed under countable unions and intersections. □

TODO restate proof: prove that liminf and limsup are measurable.
`https://math.stackexchange.com/questions/1343860/limit-of-measurable-functions-is-mea`

Proposition XI.52 (Doob-Dynkin lemma)**.** *TODO:* $[0,1]$ *and* $[-\infty, \infty]$ *have the Doob-Dynkin property.*

7.3 Measure spaces

Let $(\Omega, \mathcal{A})$ be a measurable space and $(M, +, 0, \xi)$ a commutative convergence monoid. An M-valued measure on $(\Omega, \mathcal{A})$ is a map $\mu : \mathcal{A} \to [0, \infty]$ satisfying

- $\mu(\emptyset) = 0$;
- μ is σ-additive: for every sequence (E_n) in $\mathcal{A}$ of pairwise disjoint sets, we have

$$\mu\left(\biguplus_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} \mu(E_n).$$

The triple $(\Omega, \mathcal{A}, \mu)$ is called a measure space. If

- $\mu(\Omega) < \infty$, we call μ a finite measure;
- $M = [0,1]$ and $\mu(\Omega) = 1$, we call μ a probability measure and we call $(\Omega, \mathcal{A}, \mu)$ a probability space;
- there exists a sequence (E_n) in $\mathcal{A}$ with $\Omega = \bigcup_{n \in \mathbb{N}} E_n$ and $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$, then we call μ σ-finite.

We typically consider $\mathcal{A}$ as ordered by inclusion.

Lemma XI.53. *Finite additivity is implied by σ-additivity.*

Proof. Let $E_1, E_2 \in \mathcal{A}$ be disjoint sets. Then

$$\begin{aligned}\mu(E_1 \uplus E_2) &= \mu(E_1 \uplus E_2 \uplus \emptyset \uplus \emptyset \uplus \ldots) \\ &= \mu(E_1) + \mu(E_2) + \mu(\emptyset) + \mu(\emptyset) + \ldots \\ &= \mu(E_1) + \mu(E_2).\end{aligned}$$

□

As with pre-measures, if there exists an event E such that $\mu(E) < \infty$, then $\mu(\emptyset) = 0$ is redundant. See XI.40.

Lemma XI.54. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $E \in \mathcal{E}$. Then $\mathcal{A}' = \mathcal{P}(E) \cap \mathcal{A}$ is a σ-algebra on E and $(E, \mathcal{A}', \mu|_{\mathcal{A}'})$is a measure space.*

Proof. We need to show that $\mathcal{A}'$ is a σ-algebra. Now complements are w.r.t. E. These are still in the σ-algebra because $E \setminus A = \Omega \setminus ((\Omega \setminus E) \cup A)$. □

Lemma XI.55. *Let $(\Omega, \mathcal{A})$ be a measurable space. Positive linear combinations of measures are measures: let μ_1, μ_2 be measures on $(\Omega, \mathcal{A})$ and $0 \leq c \in \mathbb{R}$. Then $c\mu_1 + \mu_2$ is a measure on $(\Omega, \mathcal{A})$.*

Example

- For any non-empty set Ω define

$$\mu : \mathcal{P}(\Omega) \to [0, \infty] : E \mapsto \begin{cases} \#(E) & (E \text{ is finite}) \\ \infty & (E \text{ is infinite}) \end{cases}.$$

  Then $(\Omega, \mathcal{P}(\Omega), \mu)$ is a measure space and μ is called the counting measure.

- Let $(\Omega, \mathcal{A})$ be a measurable space and $x \in \Omega$ and define

$$\delta_x : \mathcal{A} \to [0, \infty] : E \mapsto \begin{cases} 1 & (x \in E) \\ 0 & (x \notin E) \end{cases}.$$

  Then $(\Omega, \mathcal{A}, \delta_x)$ is a measure space and δ_x is called a Dirac measure.

Proposition XI.56 (Pushforward measure)**.** *Let $(\Omega_1, \mathcal{A}, \mu)$ be a measure space and $(\Omega_2, \mathcal{B})$ a measurable space. Let $f : \Omega_1 \to \Omega_2$ be a measurable function. Then*

$$\nu = \mu \circ f^{-1}|_{\mathcal{B}} : \mathcal{B} \to [0, \infty] : B \mapsto \mu(f^{-1}[B])$$

is a measure on $(\Omega_2, \mathcal{B})$.

Proof. Clearly this is well-defined due to f being measurable. We have

$$\nu(\emptyset) = \mu(f^{-1}[\emptyset]) = \mu(\emptyset) = 0$$

for every sequence (E_n) in $\mathcal{B}$ of pairwise disjoint sets, we have

$$\nu\left(\biguplus_{n \in \mathbb{N}} E_n\right) = \mu\left(f^{-1}\left[\biguplus_{n \in \mathbb{N}} E_n\right]\right) = \mu \biguplus_{n \in \mathbb{N}} f^{-1}[E_n] = \sum_{n \in \mathbb{N}} \mu(f^{-1}[E_n]) = \sum_{n \in \mathbb{N}} \nu(E_n).$$

□

Proposition XI.57. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Then*

1. *μ is order-preserving (if $\mathcal{A} \subset \mathcal{P}(\Omega)$ is ordered by inclusion);*
2. *μ is σ-subadditive, i.e. for any sequence (E_n) in $\mathcal{A}$, we have*
$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \mu(E_n);$$
3. *if (E_n) is a converging sequence in $\mathcal{A}$, then $(\mu(E_n))$ also converges with*
$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right).$$

Proof. TODO + upper and lower convergence + non-sequential. □

Corollary XI.57.1. *Let $\langle \Omega, \mathcal{A}, \mu \rangle$ be a measure space. Then*

1. *if (E_n) is an increasing sequence in $\mathcal{A}$, then $\mu(E_n)$ is also increasing and*
$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sup_{n\in\mathbb{N}} \mu(E_n) = \lim_{n\to\infty} \mu(E_n);$$
2. *if (E_n) is a decreasing sequence in $\mathcal{A}$ and $\mu(E_1) < \infty$, then $\mu(E_n)$ is also decreasing and*
$$\mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \inf_{n\in\mathbb{N}} \mu(E_n) = \lim_{n\to\infty} \mu(E_n).$$

Corollary XI.57.2. *Let μ, ν be measures defined on the same measure space $\langle \Omega, \mathcal{A} \rangle$. Assume $\mathcal{A} = \sigma\{\mathcal{F}\}$ for some π-system $\mathcal{F}$ which contains Ω. Then $\mu = \nu$* if and only if *$\mu(A) = \nu(A)$ for all $A \in \mathcal{F}$.*

Proof. Define
$$\mathcal{E} = \{A \in \mathcal{A} \mid \mu(A) = \nu(A)\}.$$
By the $\pi - \lambda$ theorem III.163.1 it is enough to show that $\mathcal{E}$ is a Dynkin system.

- By assumption $\Omega \in \mathcal{E}$.
- Assume $A \subset B$ are sets in $\mathcal{E}$. Then
$$\mu(B) = \mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A)$$
which implies
$$\mu(B \setminus A) = \mu(B) - \mu(A) = \nu(B) - \nu(A) = \nu(B \setminus A).$$
- Let $\langle A_i \rangle$ be a monotonically increasing family of sets in $\mathcal{E}$. Then
$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sup_{i\in\mathbb{N}} \mu(A_i) = \sup_{i\in\mathbb{N}} \nu(A_i) = \nu\left(\bigcup_{i\in\mathbb{N}} A_i\right).$$

□

7.3.1 Null sets and completeness

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. A set $A \subseteq \Omega$ is a null set if there exists a measurable set $B \in \mathcal{A}$ such that $A \subseteq B$ and $\mu(B) = 0$.
A measure space is called complete if very null set is measurable.
A proposition $P(x)$ referencing some $x \in \Omega$ is said to be true almost everywhere (or a.e.) if $\{x \in \Omega \mid P(x) \text{ is false}\}$ is a null set.

Lemma XI.58. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. If A is a measurable null set, then $\mu(A) = 0$.*

Proof. Let $B \supseteq A$ be a measurable set with $\mu(B) = 0$. Then $\mu(A) \leq \mu(B) = 0$. ☐

Lemma XI.59. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\langle A_i \rangle$ a sequence of measurable null sets. Then*

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = 0.$$

Proof. This follows by σ-sub-additivity:

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i) = 0.$$

☐

Proposition XI.60. *Every measure space can be completed (TODO)*

7.3.2 Convergence on measure spaces

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and (Y, ξ) a convergence space. Let F be a filter in $\mathcal{FP}(\Omega \to Y))$ and $f \in (\Omega \to Y)$.

- The almost everywhere convergence (a.e.) on $((\Omega \to Y))$ is defined by

$$f \in \lim_{a.e.} F \qquad \iff \qquad \exists \text{ null set } A : \forall x \in A^c : \; f(x) \in \lim_{\xi} \operatorname{ev}_x(F)$$

TODO initial convergence

7.3.3 Decompositions of measures

7.4 Carathéodory's construction of measures

Chapter 8

Integration theory

8.1 Riemann integration

See also Reed/Simon and XIII.79.

8.1.1 Riemann-Stieltjes

8.2 Lebesgue integration

`https://math.stackexchange.com/questions/2218114/theoretical-advantages-of-lebesgue-i`
`https://math.stackexchange.com/questions/3202630/what-are-the-advantages-of-the-riema`
We want to define an integral functional $I(f)$.
Riemann: $I(f) = \lim_n I_n(f)$
Lebesgue $I(f) = \lim_n I(f_n)$.

8.2.1 Simple functions

TODO order on function spaces

Let Ω be a set. A simple function (or step function) on Ω is a function with a range of finite cardinality.
Let B be a set. We denote the subset of simple functions in $(\Omega \to B)$ by $\mathrm{SF}(\Omega, B)$

Lemma XI.61. *Let $s : \Omega \to Y$ be a simple function into a vector space. Then s can be written as*

$$s(x) = \sum_{\lambda \in s[\Omega]} \lambda \cdot \chi_{s^{-1}[\lambda]}(x).$$

Additionally for some $k \in \mathbb{N}$ we can write $s[\Omega] = \bigcup_{i=1}^{k}\{\lambda_i\}$ and $A_i = s^{-1}[\lambda_i]$. Then the A_i form a partition of Ω and

$$s(x) = \sum_{i=1}^{k} \lambda_i \cdot \chi_{A_i}(x).$$

This is known as the canonical form of s. Conversely every function of this form is a simple function.

Lemma XI.62. *Let $(\Omega, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces such that $\mathcal{B}$ contains all singleton sets. Let $s : \Omega \to Y$ be a simple function.*
Then s is measurable if and only if *$s^{-1}[\lambda] \in \mathcal{A}$ for all $\lambda \in s[\Omega]$.*
This is equivalent to saying the partition $\{A_i\}_{i=1}^k$ is a subset of $\mathcal{A}$.

Proof. The direction $\boxed{\Rightarrow}$ is clear, since $\mathcal{B}$ is assumed to contain all singleton sets.
For the $\boxed{\Leftarrow}$ direction, let $B \in \mathcal{B}$. Then

$$s^{-1}[B] = \bigcup_{\lambda \in s[\Omega]} s^{-1}[B \cap \{\lambda\}],$$

which is a finite union of measurable sets. ☐

Lemma XI.63. *Let Ω be a set, G an abelian group and $s, t \in \mathrm{SF}(\Omega, G)$. If s and t have canonical forms*

$$s = \sum_{i=1}^{k} a_i \cdot \chi_{A_i} \qquad \text{and} \qquad t = \sum_{j=1}^{l} b_j \cdot \chi_{B_j},$$

then $s + t$ has canonical form

$$s + t = \sum_{i=1}^{k} \sum_{j=1}^{l} (a_i + b_j) \chi_{A_i} \chi_{B_j} = \sum_{i,j \in (1:k)\times(1:l)} (a_i + b_j) \chi_{A_i \cap B_j}.$$

Proposition XI.64. *Let $(\Omega, \mathcal{A})$ be a measurable set. Every measurable function $f : \Omega \to \mathbb{R}$ is the pointwise limit of a sequence of simple functions.*

TODO: generalise??

Proof. `https://proofwiki.org/wiki/Measurable_Function_is_Pointwise_Limit_of_Simple_Functions` ☐

8.2.1.1 Integration of simple functions

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and Y a vector space. Let

$$s : \Omega \to Y : x \mapsto \sum_{\lambda \in s[\omega]} \lambda \cdot \chi_{s^{-1}[\lambda]}(x) = \sum_{i=1}^{k} \lambda_i \cdot \chi_{A_i}(x)$$

be a measurable simple function. We define the integral of s over Ω w.r.t. μ as

$$\int_\Omega s \,\mathrm{d}\mu := \sum_{\lambda \in s[\omega]} \lambda \cdot \mu(s^{-1}[\lambda]) = \sum_{i=1}^{k} \mu(A_i) \cdot \lambda_j.$$

Using the convention that $0 \times \infty = 0$. (TODO: clarify)
We call s integrable if the integral is finite (TODO clarify + below).

The integral can be seen as a map $\mathrm{SF}(\Omega, Y) \to \overline{Y}$, where $\overline{Y}$ is the Dedekind-MacNeille completion (TODO!).

For any $E \in \mathcal{A}$ we also define the integral over E as the map

$$\int_E \mathrm{d}\mu : \mathrm{SF}(\Omega, Y) \to \overline{Y} : s \mapsto \int_E s|_E \,\mathrm{d}\mu|_E.$$

Where the last integral is taken over the measure space $(E, \mathcal{A}', \mu|_{\mathcal{A}'})$ as defined in XI.54.

Proposition XI.65. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and Y a vector space over $\mathbb{F}$. Then*

1. *the integral is linear: $\forall c \in \mathbb{F}$ and $\forall s, t \in \mathrm{SF}(\Omega, Y)$:*

$$\int_\Omega (c \cdot s + t) \,\mathrm{d}\mu = c \cdot \int_\Omega s \,\mathrm{d}\mu + \int_\Omega t \,\mathrm{d}\mu.$$

If Y is a normed space, then

2. *for all $s \in \mathrm{SF}(\Omega, Y)$:*

$$\int_\Omega \|s\| \,\mathrm{d}\mu \leq \left\| \int_\Omega s \,\mathrm{d}\mu \right\|;$$

3. *if $E_1 \subseteq E_2$ are events in $\mathcal{A}$, then*

$$\int_{E_1} \|s\| \,\mathrm{d}\mu \leq \int_{E_2} \|s\| \,\mathrm{d}\mu.$$

If Y is an ordered space, then

1. *if $s(x) \leq t(x)$ for all $x \in \Omega$, then*

$$\int_\Omega s \,\mathrm{d}\mu \leq \int_\Omega t \,\mathrm{d}\mu.$$

Proof. (1) Let s, t have canonical forms

$$s = \sum_{i=1}^{k} a_i \cdot \chi_{A_i} \qquad \text{and} \qquad t = \sum_{i=1}^{k} b_i \cdot \chi_{B_i}.$$

Then we calculate

$$\begin{aligned}
\int_\Omega (c \cdot s + t) \,\mathrm{d}\mu &= \sum_{i=1}^{k} \sum_{j=1}^{l} \mu(A_i \cap B_j) \cdot (ca_i + b_j) \\
&= c \sum_{i=1}^{k} \sum_{j=1}^{l} \mu(A_i \cap B_j) \cdot a_i + \sum_{i=1}^{k} \sum_{j=1}^{l} \mu(A_i \cap B_j) \cdot b_j \\
&= c \sum_{i=1}^{k} \mu(A_i) \cdot a_i + \sum_{j=1}^{l} \mu(B_j) \cdot b_j \\
&= c \int_\Omega s \,\mathrm{d}\mu + \int_\Omega t \,\mathrm{d}\mu.
\end{aligned}$$

The property (2) is just the triangle inequality. The other properties can be proven in a similar fashion to (1). $\square$

Proposition XI.66. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $s : \Omega \to [0, +\infty[$ a measurable simple function. The map*

$$\nu : \mathcal{A} \to [0, +\infty] : E \mapsto \int_E s \, \mathrm{d}\mu = \int_\Omega s \cdot \chi_E \, \mathrm{d}\mu$$

defines a measure.

Proof. From the positive linearity of both measures and integration (XI.55, XI.65) it is enough to consider $s = \chi_A$ for some $A \in \mathcal{A}$. In this case

$$\nu(E) = \int_\Omega \chi_A \chi_B \, \mathrm{d}\mu = \int_\Omega \chi_{A\cap B} \, \mathrm{d}\mu = \mu(A \cap E).$$

It is clear that $\nu(\emptyset) = 0$. For σ-additivity, let (E_n) be a sequence of disjoint sets in $\mathcal{A}$ and calculate

$$\nu\left(\biguplus_{n\in\mathbb{N}} E_n\right) = \mu\left(A \cap \biguplus_{n\in\mathbb{N}} E_n\right) = \mu\left(\biguplus_{n\in\mathbb{N}} (A \cap E_n)\right) = \sum_{n\in\mathbb{N}} \mu(A\cap E_n) = \sum_{n\in\mathbb{N}} \nu(E_n).$$

□

Corollary XI.66.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $s : \Omega \to [0, +\infty]$a measurable simple function and (E_n) a converging sequence in $\mathcal{A}$. Then*

$$\lim_{n\to\infty} \int_{E_n} s \, \mathrm{d}\mu = \int_{\lim_{n\to\infty} E_n} s \, \mathrm{d}\mu.$$

Proof. Define the measure $\nu : E \mapsto \int_E s \, \mathrm{d}\mu$. Then by XI.57

$$\lim_{n\to\infty} \int_{E_n} s \, \mathrm{d}\mu = \lim_{n\to\infty} \nu(E_n) = \nu(\lim_{n\to\infty} E_n) = \int_{\lim_{n\to\infty} E_n} s \, \mathrm{d}\mu.$$

□

8.2.2 Positive real functions

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f : \Omega \to [0, +\infty]$ be a positive measurable function. Define
We define the Lebesgue integral of f on Ω w.r.t. μ as

$$\int_\Omega f \, \mathrm{d}\mu \coloneqq \sup\left\{ \int_\Omega s \, \mathrm{d}\mu \;\middle|\; s \in \mathrm{SF}(\Omega, [0, +\infty[) \;\wedge\; s \leq f \right\}.$$

We call f integrable when $\int_\Omega f \, \mathrm{d}\mu < \infty$.

For simple functions this definition corresponds to the previous one by XI.65.
This definition a priori makes sense even when f is not assumed to be measurable. However the integral has undesirable properties in this case, such as not being additive.

Example

If δ_x is the Dirac measure associated to a point $x \in \Omega$ in a measurable space, then

$$\int_\Omega f \, \mathrm{d}\delta_x = f(x).$$

Lemma XI.67. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $E \in \mathcal{A}$ and $f : \Omega \to [0, +\infty]$ be a measurable function. Then*

$$\int_E f \, \mathrm{d}\mu = \int_\Omega f \cdot \chi_E \, \mathrm{d}\mu \qquad \text{and} \qquad \int_\Omega f \, \mathrm{d}\mu = \int_{\Omega \setminus E} f \, \mathrm{d}\mu + \int_E f \, \mathrm{d}\mu.$$

Proposition XI.68. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f : \Omega \to [0, +\infty]$ be a positive function.*
If f is measurable, then there exists an increasing sequence of positive measurable step functions (s_n) that converges point-wise to f.

The converse is also true and is given by XI.51.

Proof. Assume f measurable. If we can find an increasing sequence of positive measurable step functions (t_n) that converges point-wise to $\mathrm{id} : [0, +\infty] \to [0, +\infty]$, then

$$f = \mathrm{id} \circ f = \lim_{n \to \infty} t_n \circ f = \sup_{n \in \mathbb{N}} (t_n \circ f)$$

and so $s_n = t_n \circ f$ gives the sequence we are looking for. And we can find such a sequence (t_n). For example

$$t_n = n \chi_{[n, +\infty[} + \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{[\frac{k-1}{2^n}, \frac{k}{2^n}[}.$$

□

Proposition XI.69. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f, g : \Omega \to [0, +\infty]$ be positive measurable functions. Then*

1. *if $f \leq g$, then $\int_\Omega f \, \mathrm{d}\mu \leq \int_\Omega g \, \mathrm{d}\mu$;*
2. *if $E_1 \subseteq E_2$ are events in $\mathcal{A}$, then $\int_{E_1} f \, \mathrm{d}\mu \leq \int_{E_2} f \, \mathrm{d}\mu$;*
3. (Beppo Levi's lemma) *if (f_n) is an increasing sequence of positive functions that converges to f point-wise, then $(\int_\Omega f_n \, \mathrm{d}\mu)_n$ is an increasing sequence and*

$$\lim_{n \to \infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega \lim_{n \to \infty} f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu;$$

4. *the integral is positive linear: $\forall c \geq 0$:*

$$\int_\Omega (cf + g) \, \mathrm{d}\mu = c \int_\Omega f \, \mathrm{d}\mu + \int_\Omega g \, \mathrm{d}\mu.$$

Proof. (1) For all $s \in \mathrm{SF}(\Omega, [0, +\infty[)$ we have that $s \leq f$ implies $s \leq g$.
(2) This follows from $f \cdot \chi_{E_1} \leq f \cdot \chi_{E_2}$ and XI.67.
(3) That the sequence $(\int_\Omega f_n \,\mathrm{d}\mu)_n$ is increasing follows from point 1. For increasing sequences the limits are suprema, by monotone convergence VIII.194. Also, for all $m \in \mathbb{N}$, we have $f_m \leq \sup_{n\in\mathbb{N}} f_n$, which implies, by point 1., that $\int_\Omega f_m \,\mathrm{d}\mu \leq \int_\Omega \sup_{n\in\mathbb{N}} f_n \,\mathrm{d}\mu$. So

$$\lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int_\Omega f_n \,\mathrm{d}\mu \leq \int_\Omega \sup_{n\in\mathbb{N}} f_n \,\mathrm{d}\mu = \int_\Omega \lim_{n\to\infty} f_n \,\mathrm{d}\mu.$$

For the other inequality, it is enough to prove that $c\int_\Omega s \,\mathrm{d}\mu \leq \lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu$ for all $0 < c < 1$ and $s \in \mathrm{SF}(\Omega, [0, +\infty[)$ such that $s \leq f$. Fix such a c and $s = \sum_{i=1}^k \lambda_i \chi_{A_i}$. Consider the sets

$$E_n = \{x \in \Omega \mid cs(x) \leq f_n(x)\} = \bigcup_{i=1}^k \left(f_n^{-1}[\,[c\lambda_i, +\infty]\,] \cap A_i\right).$$

Then (E_n) is an increasing sequence in $\mathcal{A}$ with $\Omega = \bigcup_{n\in\mathbb{N}} E_n$ and

$$c\int_\Omega s \,\mathrm{d}\mu = \int_{\bigcup_n E_n} cs \,\mathrm{d}\mu = \lim_{n\to\infty} \int_{E_n} cs \,\mathrm{d}\mu$$

by XI.66.1. Also

$$\int_{E_n} cs \,\mathrm{d}\mu \leq \int_{E_n} f_n \,\mathrm{d}\mu \leq \int_\Omega f_n \,\mathrm{d}\mu$$

by the previous points and so the result follows from the fact that limits preserve inequalities, VIII.190.
(4) Take sequences (s_n) and (t_n) of positive measurable step functions that increase pointwise to f and g, respectively. Then $cs_n + t_n$ converges pointwise to $cf + g$ by the linearity of the limit and

$$\begin{aligned}\int_\Omega (cf + g) \,\mathrm{d}\mu &= \lim_{n\to\infty} \int_\Omega (cs_n + t_n) \,\mathrm{d}\mu \\ &= \lim_{n\to\infty} \left(c\int_\Omega s_n \,\mathrm{d}\mu + \int_\Omega t_n \,\mathrm{d}\mu\right) \\ &= c\lim_{n\to\infty} \int_\Omega s_n \,\mathrm{d}\mu + \lim_{n\to\infty} \int_\Omega t_n \,\mathrm{d}\mu \\ &= c\int_\Omega f \,\mathrm{d}\mu + \int_\Omega g \,\mathrm{d}\mu.\end{aligned}$$

□

Proposition XI.70 (Fatou's lemma)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and (h_n) any sequence of positive measurable functions in $(\Omega \to [0, +\infty])$. Then*

$$f : \Omega \to [0, +\infty] : x \mapsto \liminf_{n\to\infty} h_n(x)$$

is measurable and

$$\int_\Omega \liminf_{n\to\infty} h_n \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_\Omega h_n \,\mathrm{d}\mu.$$

Proof. Consider the sequence (f_n) defined by $f_n(x) = \inf_{k\geq n} h_k(x)$. This is an increasing sequence of positive functions and $f_n \leq h_n$ for all $n \in \mathbb{N}$. Each f_n is measurable because

$$f_n^{-1}[\,[t,+\infty]\,] = \bigcap_{k=n}^{\infty} h_k^{-1}[\,[t,+\infty]\,].$$

This shows that f is measurable by XI.51. Then we can use Beppo Levi's lemma XI.69 to obtain

$$\int_\Omega \liminf_{n\to\infty} h_n \,\mathrm{d}\mu = \lim_{n\to\infty}\int_\Omega f_n \,\mathrm{d}\mu = \liminf_{n\to\infty}\int_\Omega f_n \,\mathrm{d}\mu \leq \liminf_{n\to\infty}\int_\Omega h_n \,\mathrm{d}\mu$$

using monotonicity of the integral XI.69 and liminf XI.69 for the last inequality. □

Proposition XI.71. *Let $\langle\Omega, \mathcal{A}, \mu\rangle$ be a measure space and $f : \Omega \to [0,+\infty]$ a positive measurable function. Then*

1. $\int_\Omega f \,\mathrm{d}\mu = 0$ if and only if $f(x) = 0$ *a.e.;*
2. *if f is integrable, then $f(x) < +\infty$ a.e.*

Proof. (1) Set $E = \{x \in \mathbb{R} \mid f(x) \neq 0\}$. Then $f(x) = 0$ a.e. is equivalent to $\mu(E) = 0$.
Assume $\mu(E) = 0$, then

$$\int_\Omega f \,\mathrm{d}\mu = \int_{\Omega\setminus E} f \,\mathrm{d}\mu + \int_E f \,\mathrm{d}\mu = 0 + \int_E f \,\mathrm{d}\mu \leq \sup_{x\in E}(f(x)).$$

Now for all $s \in \mathrm{SF}(\Omega \to [0,+\infty[) \cap {\downarrow} f$ we have

$$0 \leq \int_E s \,\mathrm{d}\mu \leq \max_x(s(x))\mu(E) = 0$$

and so the supremum $\int_E f \,\mathrm{d}\mu$ is zero as well.
Now assume $\int_\Omega f \,\mathrm{d}\mu = 0$. Consider the sets $E_n = f^{-1}[\,]\frac{1}{n},+\infty]\,]$. Then $\frac{1}{n}\chi_{E_n} \in {\downarrow} f$ is simple. Hence $\frac{1}{n}\mu(E_n) \leq \int_\Omega f \,\mathrm{d}\mu = 0$, meaning $\mu(E_n) = 0$ for all n. So $\mu(E) = \sup_{n\in\mathbb{N}} \mu(E_n) = 0$.
(2) Towards contraposition, assume the set $E = \{x \in \mathbb{R} \mid f(x) = +\infty\}$ has non-zero measure. Then $a/\mu(E)\chi_E \in \mathrm{SF}(\Omega, [0,+\infty[) \cap {\downarrow} f$ for all real $a > 0$ and

$$\int_\Omega f \,\mathrm{d}\mu \geq \int_\Omega \frac{a\chi_E}{\mu(E)} \,\mathrm{d}\mu = a$$

so f is not integrable. □

8.2.2.1 Product measures

Fubini!

8.2.3 Real functions

Let Ω be a set and $f : \Omega \to \mathbb{C}$ a function. Then we can uniquely decompose f into $f = u + iv$ where $u, v : \Omega \to \mathbb{R}$. We can further decompose

$$\begin{cases} u = u^+ - u^- & \text{such that } u^+u^- = 0 \\ v = v^+ - v^- & \text{such that } v^+v^- = 0. \end{cases}$$

If Ω carries a σ-algebra such that f is measurable, then u^+, u^-, v^+, v^- are also measurable.

Let $\langle \Omega, \mathcal{A}, \mu \rangle$ be a measure space. We say a measurable function $f : \Omega \to \mathbb{C}$ is integrable, if $|f| : \Omega \to [0, \infty[$ is integrable. In this case we define the integral of f as

$$\int_\Omega f \, \mathrm{d}\mu = \int_\Omega u^+ \, \mathrm{d}\mu - \int_\Omega u^- \, \mathrm{d}\mu + i \int_\Omega v^+ \, \mathrm{d}\mu - i \int_\Omega v^- \, \mathrm{d}\mu.$$

The set of all integrable functions in $(\Omega \to \mathbb{C})$ is denoted $\mathcal{L}^1(\Omega, \mathcal{A}, \mu)$ or $\mathcal{L}^1(\mu)$.

TODO

Proposition XI.72 (Reverse Fatou lemma). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and (h_n) any sequence of measurable functions that is dominated by a positive integrable function g (i.e. $h_n \leq g$ for all $n \in \mathbb{N}$). Then*

$$f : \Omega \to [0, +\infty] : x \mapsto \limsup_{n\to\infty} h_n(x)$$

is measurable and

$$\int_\Omega f \, \mathrm{d}\mu \geq \limsup_{n\to\infty} \int_\Omega h_n \, \mathrm{d}\mu.$$

Proof. By the corollary $g(x) < +\infty$ a.e. and so also $g - h_n < +\infty$ a.e. Applying Fatou's lemma XI.70 gives

$$\int \liminf g - h_n \leq \liminf \int g - h_n$$

□

8.2.4 Integration of vector-valued functions

TODO: define integration by

$$\left\langle u, \int T(t) \, \mathrm{d}t v \right\rangle := \int \langle u, T(t)v \rangle \, \mathrm{d}t$$

?? This case Riemann?? Or only finite dim??

8.2.4.1 Weak and strong measurability

TODO is Bochner measurable measurable with Y given Borel σ-algebra?
A function $f : X \to B$ is called Bochner-measurable if it is equal μ-almost everywhere to a function g taking values in a separable subspace B_0 of B, and such that the inverse image $g^{-1}[U]$ of every open set U in B belongs to Σ. Equivalently, f is limit μ-almost everywhere of a sequence of simple functions.

Theorem XI.73 (Pettis measurability theorem). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and Y a normed vector space. Then f is strongly measurable* if and only if *f is weakly measurable and almost surely separably valued.*

8.2.4.2 Bochner integration

TODO: the Bochner integral is the unique extension of the integral of simple functions to the set of Bochner measurable functions???? (i.e. simple functions dense in Bochner space, with L^1 metric)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and Y a normed vector space. Then a Bochner measurable function $f : \Omega \to Y$ is called Bochner integrable if there exists a sequence of integrable simple functions $\langle s_n \rangle \subset \mathrm{SF}(\Omega, Y)$ such that

$$\lim_{n\to\infty} \int_\Omega \|f - s_n\| \,\mathrm{d}\mu = 0.$$

Take such a sequence $\langle s_n \rangle$. The Bochner integral of f on Ω w.r.t. μ is defined as

$$\int_\Omega f \,\mathrm{d}\mu \coloneqq \lim_{n\to\infty} \int_\Omega s_n \,\mathrm{d}\mu.$$

Lemma XI.74. *The Bochner integral is well-defined: let* $\langle s_n \rangle, \langle t_n \rangle \in {}^{\mathbb{N}}\mathrm{SF}(\Omega, Y)$ *be sequences such that*

$$\lim_{n\to\infty} \int_\Omega \|f - s_n\| \,\mathrm{d}\mu = 0 = \lim_{n\to\infty} \int_\Omega \|f - t_n\| \,\mathrm{d}\mu.$$

Then

1. *the limits* $\lim_{n\to\infty} \int_\Omega s_n \,\mathrm{d}\mu$ *and* $\lim_{n\to\infty} \int_\Omega t_n \,\mathrm{d}\mu$ *exist;*
2. $\lim_{n\to\infty} \int_\Omega s_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega t_n \,\mathrm{d}\mu$.

Proof. TODO □

Proposition XI.75 (Bochner integrability criterion)**.** *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space and* Y *a normed vector space.*
A Bochner measurable function f *is Bochner integrable* if and only if

$$\int_\Omega \|f\| \,\mathrm{d}\mu < \infty.$$

Proposition XI.76. *Linearity and monotonicity.*

Proposition XI.77. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space* Y *a normed vector space and* T *a closed operator on* Y*. If* $T \circ f$ *is integrable, then*

$$\int_\Omega (T \circ f) \,\mathrm{d}\mu = T\left(\int_\Omega f \,\mathrm{d}\mu\right).$$

Proof. TODO □

Corollary XI.77.1. *If* T *is bounded, then* $T \circ f$ *is integrable and*

$$\int_\Omega (T \circ f) \,\mathrm{d}\mu = T\left(\int_\Omega f \,\mathrm{d}\mu\right).$$

TODO Dominated convergence.

8.2.4.3 Pettis integration

8.3 Further topics

TODO rename!

8.3.1 Absolute continuity and mutual singularity

Let μ, ν be measures on the measurable space $(\Omega, \mathcal{A})$. We say

- ν is absolutely continuous w.r.t. μ if $\mu(A) = 0 \implies \nu(A) = 0$ for all $A \in \mathcal{A}$;
- μ and ν are mutually singular if there exists a set $A \in \mathcal{A}$ with $\mu(A) = 0$ and $\nu(A^c) = 0$.

Theorem XI.78 (Radon-Nikodym)**.** *Let μ, ν be measures on the measurable space $(\Omega, \mathcal{A})$. Then ν is absolutely continuous w.r.t. μ* if and only if *there exists a measurable function $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$?) such that*

$$\nu(A) = \int_\Omega f \cdot \chi_A \, \mathrm{d}\mu \qquad \forall A \in \mathcal{A}.$$

The function f is uniquely determined a.e. (w.r.t. μ).

8.3.2 Lebesgue decomposition

Theorem XI.79 (Lebesgue decomposition theorem)**.** *Let μ, ν be two measures on a measurable space $(\Omega, \mathcal{A})$. Then ν can be written uniquely as*

$$\nu = \nu_{ac} + \nu_{sing}$$

where μ and ν_{sing} are mutually singular and ν_{ac} is absolutely continuous w.r.t. μ.

8.3.3 Convolution

TODO Young's convolution inequality

8.4 Duality in integration

Distributions with kernels will be example.

Chapter 9

Complex analysis

A complex function is a function in $(U \subseteq \mathbb{C} \to \mathbb{C})$.

9.1 The complex numbers as a real algebra

The set of complex numbers $\mathbb{C}$ is isomorphic to the algebra $\mathbb{R}^2$ where multiplication is defined by

$$(a,b)\cdot(x,y) \coloneqq (ax-by, ay+bx).$$

Thus the regular representation $\rho_{(a,b)}$ has matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

9.2 Holomorphic functions

Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be a complex function. We say

1. f is holomorphic at $z \in U$ if the limit $\lim_{h\to 0} \dfrac{f(z+h)-f(z)}{h}$ exists;
2. f is holomorphic in $S \subset U$ if it is holomorphic at every point in S;
3. f is holomorphic if it is holomorphic at every point in U;
4. f is entire if $U = \mathbb{C}$ and it is holomorphic at every point in $\mathbb{C}$.

Lemma XI.80. *Holomorphic functions are continuous.*

Lemma XI.81. *Let f, g be holomorphic. Then*

1. *$f+g$ is holomorphic and $(f+g)' = f' + g'$;*
2. *fg is holomorphic and $(fg)' = f'g + fg'$;*

3. if $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and

$$(f/g)' = \frac{f'g - fg'}{g^2};$$

4. the chain rule holds.

9.2.1 Cauchy-Riemann equations

The space of complex numbers $\mathbb{C}$ is a real 2-dimensional vector space.

Lemma XI.82. *Let $f : U \subseteq \mathbb{C} \to C$ be a complex function. Then f is holomorphic* if and only if *it is Fréchet differentiable as a function $F : V \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathrm{d}F$ is the regular representation of some $z \in \mathbb{C}$.*

Corollary XI.82.1 (Cauchy-Riemann equations)**.** *Let $f : U \subseteq \mathbb{C} \to C$ be a complex function. Then f is complex differentiable at $z_0 = a + bi$* if and only if *the real derivatives*

$$\frac{\partial\,\Re\mathrm{e}(f(a+ib))}{\partial a}, \frac{\partial\,\Re\mathrm{e}(f(a+ib))}{\partial b}, \frac{\partial\,\Im\mathrm{m}(f(a+ib))}{\partial a} \quad \text{and} \quad \frac{\partial\,\Im\mathrm{m}(f(a+ib))}{\partial b}.$$

exist, are continuous and the equations

$$\frac{\partial\,\Re\mathrm{e}(f(a+ib))}{\partial a} = \frac{\partial\,\Im\mathrm{m}(f(a+ib))}{\partial b} \quad \text{and} \quad \frac{\partial\,\Re\mathrm{e}(f(a+ib))}{\partial b} = -\frac{\partial\,\Im\mathrm{m}(f(a+ib))}{\partial a}$$

hold. In particular

$$|f'(z_0)|^2 = \det J_f(a,b)$$

where $\det J_f(a,b)$ is the Jacobian determinant of f as a function $V \subseteq \mathbb{R}^2 \to \mathbb{R}^2$.

Proof. We need the Jacobian

$$\begin{pmatrix} \frac{\partial\,\Re\mathrm{e}(f(a+ib))}{\partial a} & \frac{\partial\,\Re\mathrm{e}(f(a+ib))}{\partial b} \\ \frac{\partial\,\Im\mathrm{m}(f(a+ib))}{\partial a} & \frac{\partial\,\Im\mathrm{m}(f(a+ib))}{\partial b} \end{pmatrix}$$

to be of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ in order for it to be a regular representation of some $f' = \frac{\partial\,\Re\mathrm{e}(f(a+ib))}{\partial a} + i\frac{\partial\,\Im\mathrm{m}(f(a+ib))}{\partial a} \in \mathbb{C}$. TODO ref requirement continuously differentiable.
The Jacobian determinant is a simple calculation:

$$\begin{aligned} \det J_f(a,b) &= \frac{\partial\,\Re\mathrm{e}(f(a+ib))}{\partial a}\frac{\partial\,\Im\mathrm{m}(f(a+ib))}{\partial b} - \frac{\partial\,\Im\mathrm{m}(f(a+ib))}{\partial a}\frac{\partial\,\Re\mathrm{e}(f(a+ib))}{\partial b} \\ &= \frac{\partial\,\Re\mathrm{e}(f(a+ib))}{\partial a}^2 + \frac{\partial\,\Im\mathrm{m}(f(a+ib))}{\partial a}^2 = |f'|^2. \end{aligned}$$

□

9.2.2 Cauchy's theorem, Morera's theorem and the integral formula

Theorem XI.83 (Cauchy's theorem)**.** *Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic complex function and γ a simple closed Jordan curve whose interior lies in U. Then*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

TODO: looser requirements for γ

Proof. TODO: generalised Stokes + Cauchy-Riemann! ☐

Corollary XI.83.1. *Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic complex function. Then there exists a primitive $F : U \to \mathbb{C}$ such that $\frac{\mathrm{d}F}{\mathrm{d}z} = f$.*

Proof. The integral $\int_\gamma f(z)\,\mathrm{d}z$ depends only on the endpoints of γ. ☐

Theorem XI.84 (Morera's theorem)**.** *Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be a complex function on an open set U such that*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0$$

for every closed, piecewise C^1 curve γ in U, then f is holomorphic.

Proof. $\int f(z)\,\mathrm{d}z$ is a primitive, so f is complex differentiable. ☐

TODO: triangles enough!

Theorem XI.85 (Cauchy's integral formula)**.** *Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic complex function and γ a simple closed Jordan curve whose interior lies in U. Then*

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

for any point z in the interior of γ.

Proof. Let $C_{z,\epsilon}$ be a small circle inside γ around z of radius ϵ and opposite orientation (TODO explicate!). Then $\frac{f(\zeta)}{\zeta - z}$ is holomorphic in the region between γ and $C_{z,\epsilon}$, so

$$\begin{aligned}
\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta &= -\oint_{C_{z,\epsilon}} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta && (9.1)\\
&= -\oint_{C_{z,\epsilon}} \frac{f(\zeta) - f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z}\,\mathrm{d}\zeta && (9.2)\\
&= -\oint_{C_{z,\epsilon}} \frac{f(\zeta) - f(z)}{\zeta - z}\,\mathrm{d}\zeta - \oint_{C_{z,\epsilon}} \frac{f(z)}{\zeta - z}\,\mathrm{d}\zeta. && (9.3)
\end{aligned}$$

In the limit $\epsilon \to 0$, the first part is bounded by

$$|\oint_{C_{z,\epsilon}} \frac{f(\zeta) - f(z)}{\zeta - z}\,\mathrm{d}\zeta| \leq \sup_\zeta \left|\frac{f(\zeta) - f(z)}{\zeta - z}\right| \cdot |2\pi\epsilon| \to 0$$

because $\frac{f(\zeta)-f(z)}{\zeta - z}$ remains bounded. For the second part, we have

$$\begin{aligned}
-\oint_{C_{z,\epsilon}} \frac{f(z)}{\zeta - z}\,\mathrm{d}\zeta &= f(z)\oint_{C_{z,\epsilon}} \frac{\mathrm{d}\zeta}{\zeta - z}\\
&= f(z)\int_0^{2\pi} \frac{\epsilon i e^{-it}}{\epsilon e^{-it}}\,\mathrm{d}t\\
&= f(z)2\pi i.
\end{aligned}$$

☐

Corollary XI.85.1. *Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic complex function. Then f has infinitely many complex derivatives and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta$$

for all z in the interior of γ.

Proof. TODO ☐

Proposition XI.86. *Let f be holomorphic in an open set Ω and D a disc centered at z_0 whose closure is contained in Ω. Then f has a power series expansion at z_0*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$. The coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad \textit{for all } n \geq 0.$$

Proof. TODO ☐

Corollary XI.86.1. *If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then*

$$|f^{(n)}(z_0)| \leq \frac{n! \|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z \in C} |f(z)|$.

Proof. TODO ☐

Thus the distance from z_0 to the nearest singular point is the radius of convergence of the power series.

Corollary XI.86.2 (Liouville's theorem)**.** *If f is entire and bounded, then f is contant.*

Proof. It suffices to show that $f'(z_0) = 0$. This can be seen by taking $R \to \infty$ in the previous inequality. ☐

Corollary XI.86.3 (Fundamental theorem of algebra)**.** *Let $P(z) = a_n z^n + \ldots + a_0$ be a polynomial of degree $n \geq 1$ with complex coefficients. Then $P(z)$ has precisely n roots. If these roots are denoted $w_1, \ldots, w_n$, then we can write*

$$P(z = a_n (z - w_1)(z - w_2) \ldots (z - w_n).$$

Proposition XI.87. *Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a complex function that is holomorphic in a region $U \subseteq \Omega$ that contains a closed annulus $\{z \in \mathbb{C} \mid r \leq |z - z_0| \leq R\}$ for some $z_0 \in \mathbb{C}$ and $r, R \in \mathbb{R}$. Then f has a Laurent series expansion that converges in the interior of the annulus:*

$$f(z) = \sum_{i=-\infty}^{\infty} a_i (z - z_0)^i$$

for all $z \in \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$.

Proof. TODO ☐

TODO uniqueness ??? `https://en.wikipedia.org/wiki/Laurent_series#Uniqueness`

9.2.3 Analytic continuation

Proposition XI.88. *Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic function on a connected open set Ω. Suppose there exists a sequence of distinct points with limit point in Ω on which f vanishes. Then $f = \underline{0}$.*

Proof. TODO □

Corollary XI.88.1. *Let $f, g : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be holomorphic functions on a connected open set Ω. Suppose there exists a sequence of distinct points $\langle z_n \rangle$ with limit point in Ω such that $f(z_n) = g(z_n)$ for all n. Then $f = g$.*

In particular this holds if f and g agree on some non-empty open subset of Ω.

Corollary XI.88.2. *Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic function on an open set Ω and $\Omega' \supseteq \Omega$ a connected open subset of $\mathbb{C}$. Then there exists at most one holomorphic function f' on Ω' such that $f'|_\Omega = f$.*

In this case the function f' is called an analytic continuation of f into Ω'.

Proposition XI.89 (Symmetry principle)**.** *Let Ω be an open subset of $\mathbb{C}$ such that $\overline{\Omega} = \Omega$. Denote by Ω^+ the part of Ω that lies in the upper half plane and by Ω^- the part lying in the lower half plane. Set $I = \mathbb{R} \cap \Omega$.*
Let $f^+ : \Omega^+ \to \mathbb{C}$ and $f^- : \Omega^- \to \mathbb{C}$ be holomorphic functions that extend continuously to I and

$$\forall x \in I : \; f^+(x) = f^-(x).$$

Then the compound function

$$f : \Omega \to \mathbb{C} : z \mapsto f(z) = \begin{cases} f^+(z) & z \in \Omega^+ \\ f^+(z) = f^-(z) & z \in I \\ f^-(z) & z \in \Omega^- \end{cases}$$

is holomorphic on all of Ω.

Proposition XI.90 (Schwarz reflection principle)**.** *Let $\Omega, \Omega^+, \Omega^-$ and I be as in XI.89 and $f : \Omega^+ \to \mathbb{C}$ a holomorphic function that extends continuously to I. Then there exists a holomorphic function $g : \Omega \to \mathbb{C}$ such that $g|_{\Omega^+} = f$.*

Proof. Idea: define $g|_{\Omega^-}(z) = \overline{f(\overline{z})}$. TODO details. □

9.2.3.1 Analytic continuation along a curve

Theorem XI.91 (Monodromy theorem)**.** *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic function and Ω the set of all points that admit an analytic continuation.*
If two curves are homotopic in Ω, then they give the same analytic continuation.

9.2.4 Limits and holomorphic functions

Proposition XI.92. *Let $\langle f_n \rangle$ be a sequence of holomorphic functions on Ω that converges uniformly to a function f in every compact subset of Ω. Then*

1. *f is holomorphic in Ω;*

2. *$\langle f_n' \rangle$ converges uniformly to f' on every compact subset of Ω;*

3. *for all $k \in \mathbb{N}$, the sequence $\left\langle f_n^{(k)} \right\rangle$ converges uniformly to $f^{(k)}$ on every compact subset of Ω.*

Proof. (1) TODO (+ `https://www.math.wustl.edu/~sk/limits.pdf`)
(2) TODO
(3) By induction on k. □

TODO: this is part of the motivation for the compact-open topology (use this terminology?).

Proposition XI.93. *Let Ω be a open set in $\mathbb{C}$ and $F : \Omega \times [0,1] \to \mathbb{C}$ a function such that*

- *$F(\cdot, s)$ is holomorphic (in the first variable) for all $s \in [0,1]$;*
- *F is continuous.*

Then the function

$$f : \Omega \to \mathbb{C} : z \mapsto f(z) = \int_0^1 F(z,s)\,\mathrm{d}s$$

is holomorphic.

Proposition XI.94 (Runge's approximation theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact.*

1. *Any function holomorphic in a neighbourhood of K can be approximated uniformly on K by rational functions whose singularities are in K^c.*
2. *If K^c is connected, then the approximating functions can be taken to be polynomials.*
3. *If K^c is not connected, then there exists a function f holomorphic on a neighbourhood of K that cannot be approximated uniformly by polynomials on K.*

If inner and outer part of annulus then Laurent series (i.e. particular form of approximating rationals)

9.3 Singularities

9.3.1 Laurent series

Proposition XI.95. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and let Ω contain two concentric circles with center z_0 and radii $0 < r < R$. Then f has a Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

that converges on the annulus between the concentric circles.

Proof. Call the outer circle C_1 and the inner C_2. Then we can use the integral formula to write

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

TODO □

9.3.2 Isolated singularities

9.3.3 Non-isolated singularities

9.4 Meromorphic functions

9.4.1 Zeros and poles

Lemma XI.96. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on an open set Ω that has a zero at $z_0 \in \Omega$ and does not vanish identically on Ω. Then there exists an open neighbourhood $U \subseteq \Omega$ of z_0 such that*

$$f|_U(z) = (z - z_0)^n g(z)$$

for all $z \in U$, some holomorphic $g : U \to \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$.
Additionally, the n is uniquely determined by f and z_0; it is independent of U.

Proof. TODO ☐

Let $f : \Omega \to \mathbb{C}$ be a function on an open set Ω.

- If f is holomorphic with a zero at z_0, the n in XI.96 is called the multiplicity or order of the zero z_0.
- If $\lim_{z\to z_0} 1/f(z) = 0$ and is $1/f$ holomorphic in a neighbourhood of z_0, then z_0 is called a pole of f. The n in the expansion XI.96 of $1/f$ is called the multiplicity or order of the pole z_0.

If $n = 1$, we call the zero or pole simple.

Poles lie in $\overline{\Omega} \setminus \Omega$. TODO sort out definition meromorphic function.

Lemma XI.97. *Let $f : \Omega \to \mathbb{C}$ be a function on an open set Ω that has a pole at $z_0 \in \overline{\Omega}$. Then there exists a neighbourhood U of z_0 such that f is holomorphic on $U \setminus \{z_0\}$.*

This means a pole in necessarily an isolated singularity (TODO ref later)

Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a function. We call f meromorphic if either f or $1/f$ is holomorphic at each point $z \in \Omega$.

Lemma XI.98. *Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a function. Then f is meromorphic* if and only if *for each point $z \in \Omega$, either f is holomorphic or z is a pole.*

9.4.2 Laurent series and residues

Proposition XI.99. *Let $f : \Omega \to \mathbb{C}$ be a function on an open set Ω that has a pole of order n at z_0, then the Laurent series of f at z_0 is of the form*

$$f(z) = \sum_{i=-n}^{\infty} a_i (z - z_0)^i$$

and converges on a punctured disk $B(z_0, \epsilon) \setminus \{z_0\}$.

Proof. By XI.96 we have that

$$(1/f)|_{B(z_0,\epsilon)}(z) = (z-z_0)^n g(z)$$

for some holomorphic, non-vanishing g. Thus $f|_{B(z_0,\epsilon)}(z) = (z-z_0)^{-n}1/g(z)$. Now $1/g$ is holomorphic, so we can expand it as a power series. This makes the product a Laurent series starting at $-n$.
To conclude, we remark that the inner radius of the annulus on which the Laurent series converges is zero: by XI.32

$$r = \limsup_{i\to\infty}\left(|a_{-i}|^{1/i}\right) = 0.$$

□

TODO: pole iff Laurent series has finite principal part!

9.4.2.1 Partial fraction decomposition

Proposition XI.100 (Partial fraction decomposition)**.** *Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a meromorphic function with finitely many poles $z_0, \ldots z_n$. Let $P_{z_k}(z)$ be the principal part of the Laurent expansion around z_k. Then we have the decomposition*

$$f(z) = \sum_{k=0}^{n} P_k(z) + F(z)$$

where $F(z)$ is a holomorphic function.

Proof. The function $F(z) = f(z) - \sum_{k=0}^{n} P_k(z)$ has no poles and thus is holomorphic. Indeed, for all $z \in \Omega \setminus \{z_0, \ldots z_n\}$, both $f(z)$ and $\sum_{k=0}^{n} P_k(z)$ are holomorphic, meaning that $F(z)$ is too.
Take some pole z_k. Then the Laurent series of $F(z)$ around z_k is of the form

$$F(z) = P_k(z) + \sum_{i=0}^{\infty} a_{k,i}(z-z_k)^i - \sum_{j=0}^{n} P_j(z) = \sum_{i=0}^{\infty} a_{k,i}(z-z_k)^i - \sum_{\substack{j=0\\ j\neq k}}^{n} P_j(z),$$

which has no singularity at z_k. □

TODO link with integrals of fractional functions.

9.4.2.2 Residues

Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ have a pole at z_0. Given the Laurent series expansion

$$f(z) = \sum_{i=-n}^{\infty} a_i(z-z_0)^i,$$

the coefficient a_{-1} is called the residue at the pole z_0. We write $\mathrm{Res}_{z_0} f \coloneqq a_{-1}$.

Proposition XI.101. *Let f have a pole of order n at z_0, then*

$$\mathrm{Res}_{z_0} f = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1}(z-z_0)f(z).$$

Proof. This follows straight from the series expansion

$$(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \ldots + a_{-1}(z-z_0)^{n-1} + a_0(z-z_0)^n + \ldots$$

☐

Corollary XI.101.1. *If f has a simple pole at z_0, then* $\operatorname{Res}_{z_0} f = \lim_{z\to z_0}(z-z_0)f(z)$.

Proposition XI.102. *Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a meromorphic function that has one pole at z_0. Let γ be a simple curve such that $z_0 \in \int \gamma \subseteq \Omega$. Then*

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}_{z_0} f.$$

Proof. By truncated Laurent expansion. TODO ☐

Corollary XI.102.1 (Residue formula)**.** *Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a meromorphic function and γ a simple curve that encompasses N poles $z_1, \ldots, z_N$. Then*

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \operatorname{Res}_{z_k} f.$$

9.4.3 The argument principle

Proposition XI.103 (Argument principle)**.** *Let f be a meromorphic function in an open set containing a simple curve γ and its interior. Assume f has no poles or zeros on γ. Then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \textit{number of zeros of } f \textit{ inside } \gamma \;-\; \textit{number of poles of } f \textit{ inside } \gamma$$

where the poles and zeros are counted with their multiplicities.

Proof. TODO ☐

Theorem XI.104 (Rouché's theorem)**.** *Let f and g be holomorphic functions on an open set that contains a closed simple curve γ and its interior. If*

$$|f(z)| > |g(z)| \qquad \forall z \in \gamma,$$

then f and $f+g$ have the same number of zeros inside γ.

Proof. For $t \in [0,1]$ we define

$$f_t(z) = f(z) + tg(z).$$

This means $f_0 = f$ and $f_1 = f + g$. The condition $|f(z)| > |g(z)|$ ensures that f_t has no zeros on γ. Also f_t is holomorphic and thus has no poles. So the number of zeros n_t of f_t inside γ is given by

$$n_t = \frac{1}{2\pi i}\oint_\gamma \frac{f_t'(z)}{f_t(z)}\,\mathrm{d}z.$$

It is then enough to observe that n_t is continuous as a real function of t. ☐

Corollary XI.104.1 (Open mapping theorem)**.** *If f is holomorphic and non-constant, then it is an open map.*

Proof. TODO ☐

Proposition XI.105 (Maximum modulus principle)**.** *If f is a non-constant holomorphic function in a simply connected region Ω, then $|f(z)|$ cannot attain a maximum in Ω.*

Proof. TODO ☐

Corollary XI.105.1. *Let Ω be a simply connected open set with compact closure $\overline{\Omega}$. If f is holomorphic on Ω and continuous on $\overline{\Omega}$, then*

$$\sup_{z\in\Omega}|f(z)| \leq \sup_{z\in\overline{\Omega}\setminus\Omega}|f(z)|.$$

9.4.4 The ring of polynomials over $\mathcal{M}_\Omega$

Let $\Omega \subseteq \mathbb{C}$ be an open set. Let $\mathcal{M}_\Omega$ be the set of meromorphic functions in $(Omega \to \mathbb{C})$.

Proposition XI.106. *For any $\Omega \subseteq \mathbb{C}$, the set $\mathcal{M}_\Omega$ is a field. With*

- *as zero the constant function $\underline{0}$;*
- *as identity the constant function $\underline{1}$.*

9.4.4.1 Poles and roots

We say a polynomial in $\mathcal{M}_\Omega[X]$ has a pole at $z \in \Omega$, if one of its coefficients has a pole at z.

Lemma XI.107. *Let $p, q \in \mathcal{M}_\Omega[X]$ be monic and $r = pq$. If either p or q has a pole at $z_0 \in \Omega$, then r has a pole at z_0 as well.*

Proof. Let h be the maximal order of z_0 as a pole among the coefficients of p and k the maximal order of z_0 as a pole among the coefficients of q. Then both $\lim_{z\to z_0}(z-z_0)^h p(z)$ and $\lim_{z\to z_0}(z-z_0)^k q(z)$ are non-zero polynomials (due to being monic) with constant coefficients and $h+k \geq 1$.
Now assume $r(z)$ is holomorphic at z_0, then

$$0 = \lim_{z\to z_0}(z-z_0)^{h+k}r(z) = \Big(\lim_{z\to z_0}(z-z_0)^h p(z)\Big)\Big(\lim_{z\to z_0}(z-z_0)^k q(z)\Big) \neq 0,$$

which is a contradiction. ☐

Corollary XI.107.1. *Let $p \in \mathcal{M}_\Omega[x]$ be monic and have a prime decomposition $p = \prod_{k=1}^n q_k^{m_k}$. Then $z_0 \in \Omega$ is a pole of $p(z)$* if and only if *it is a pole of one of the $q_k(z)$.*

Proposition XI.108. *Let $p, q \in \mathcal{M}_\Omega[x]$ be monic and relatively prime, then $p(z)$ and $q(z)$ are relatively prime as polynomials in $\mathbb{C}[X]$ for all $z \in \Omega \setminus (S_p \cup S_q \cup H)$, where*

- *S_p is the set of poles of p;*
- *S_q is the set of poles of q;*
- *H is isolated and closed in Ω.*

Proof. By Bézout's identity, we can write $\alpha[x](z)p[x](z)+\beta[x](z)q[x](z) = f(z)$ for some monic $\alpha, \beta \in \mathcal{M}_\Omega[x]$. Let H be the set of poles and zeros of $f(z)$ (which contains $S_p \cup S_q$ by XI.107). For all $z_0 \notin H$ we have

$$\left(\frac{\alpha[x](z_0)}{f(z_0)}\right) p[x](z_0) + \left(\frac{\beta[x](z_0)}{f(z_0)}\right) q[x](z_0) = 1,$$

meaning $p[x](z_0)$ and $q[x](z_0)$ are relatively prime (TODO ref). □

Corollary XI.108.1. *Let $p \in \mathcal{M}_\Omega[x]$ be monic and irreducible. Then $p(z)$ has simple roots for all $z \in \Omega \setminus H$ for some isolated and closed set $H \subseteq \Omega$.*

Proof. If the degree of p is 1, then the result is immediate. Assume the degree of p is greater than 1.
For any $z \in \Omega$, any non simple root of $p[x](z)$ must also be a root of $\frac{\partial p[x](z)}{\partial x}$. As $p[x]$ is irreducible, $p[x]$ and $\frac{\partial p[x]}{\partial x}$ are relatively prime. By the proposition $p[x](z_0)$ and $\frac{\partial p[x](z_0)}{\partial x}$ are relatively prime for all $z_0 \in \Omega \setminus H$, which means they do not share roots. □

9.4.4.2 Algebroid functions

9.4.4.3 Puiseux series

9.5 Conformal mappings

Chapter 10

Calculus

10.1 Exploring the concept of change

TODO: diffeomorphism

In physics how things change is quite important. Much of physics is concerned with the question of, given a particular system at a particular time, how that system will evolve.

We have not yet really introduced a mathematical construct that expresses an idea of change. We will do so here.

In particular we will consider ways to express how the output of a function changes if we (slightly) change its input.

To motivate the discussion below, consider the function represented by the graph in figure TODO.

Locally at any one point the rate of change of the function can be described using the slope at that point. That makes intuitive sense; when walking up a mountain the slope is a measure for how quickly the altitude changes.

The slope between two points can be calculated by dividing the vertical distance by the horizontal distance. This definition of slope obviously depends on two points. We would quite like to be able to talk about the slope at a single point (the way we would intuitively when walking up a hill). To do that we can just bring both points very close together.

As can be seen on the picture this procedure gives the slope of the tangent line at that point (straight lines have a constant slope).

10.1.1 Speed

At this point we can give an important physical motivating example, namely the speed of an object. Say we throw an apple straight up into the air. Its vertical movement is plotted in figure TODO.

We may want to know its speed at different times. We can calculate speed by taking the displacement and dividing it by the time it takes traverse that distance. We can now make an important distinction between average speed (the slope between two distinct points) and instantaneous speed (the limit when we bring both points together).

10.2 The derivative

As motivated above, the rate of change of a (real) function is the difference in output divided by the difference in input of two points:

$$\frac{f(y)-f(x)}{y-x}.$$

We conventionally call $h = y - x$. We can then write the above quantity (which is called the Newton quotient) as

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}.$$

The derivative of f at x is then just the limit of the Newton quotient with h going to zero. This limit does not always exist. If the limit exists for all x, the function is called differentiable. A function may also be differentiable in some points and not in others.

We can now use the definition and properties of limits to calculate derivatives, such as in the following example. This process is slow and laborious even for relatively simple functions. Luckily the derivative has some important properties that lets us calculate the derivative of many functions with relative ease.

We can define a (real) function that, for any input, calculates the derivative of a particular fixed function f at that point and gives that as its output. This new function is often called the derivative of the function f.

There are many ways to write the derivative of f:

$$\lim_{h\to 0} \frac{f(x+h)-f(h)}{h} \equiv f'(x) \equiv \frac{\mathrm{d}f}{\mathrm{d}x} \equiv \frac{\mathrm{d}f(x)}{\mathrm{d}x}$$

A mathematician would want me to emphasize that the expression $\frac{\mathrm{d}f}{\mathrm{d}x}$ should be read as a whole and is technically not a division, but a physicist would say that (in some situations) it can be viewed as such, where $\div f$ and $\div x$ are (the in this context relevant) infinitesimal variations of f and x. Do not tell any mathematicians I said this.

> Example
>
> TODO derivative of polynomial function using limits.

In certain situations a dot is used to indicate a derivative with respect to time (i.e. the derivative of a quantity in function of time). So we might for example use $x(t)$ to denote the position in function of time (here x is not used to refer to a variable but to a function, the notation is standard and usually it clear from the context what x refers to). We can then use the notation

$$x'(t) \equiv \dot{x}(t).$$

In fact $\dot{x}(t)$, is just the speed.

When using the notation $\frac{\mathrm{d}f}{\mathrm{d}x}$, this usually refers to the function that is the derivative of f. If we want to evaluate this function in a particular point (say x_0), we can write something like this

$$\left.\tfrac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0}.$$

10.2.1 Slope of a curve

$$\text{slope of the normal} = \frac{-1}{\text{slope of the tangent}}$$

10.2.2 Properties of the derivative

Here we give some properties of the derivative:

- The derivative is a linear operation:

  $$(f+g)'(x) = f'(x) + g'(x)$$

  and

  $$(c \cdot f)'(x) = c \cdot f'(x) \qquad \forall c \in \mathbb{R}$$

- Product rule

  $$(f \cdot g)'(x) = f(x) \cdot g'(x) + f'(x)g(x)$$

- The derivative of $\frac{1}{f(x)}$, assuming $f(x) \neq 0$:

  $$\left(\frac{1}{f(x)}\right)' = \frac{f'(x)}{f(x)^2}.$$

- Combining the previous two properties, we get the quotient rule (assuming $g(x) \neq 0$)

  $$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

- Finally we have the very important chain rule. This tells us how to take the derivative of composite functions:

  $$(f \circ g)'(x) = f'(g(x))g'(x).$$

  We can also write this as

  $$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}.$$

  TODO example

- Derivative of an inverse

  $$\frac{\mathrm{d}f^{-1}(x)}{\mathrm{d}x} = \frac{1}{f'(f^{-1}(x))}$$

TODO Faà di Bruno

10.2.3 Derivatives of some common functions

Using the results below together with the properties above we can calculate the derivative of a large number of functions.

- Let n be an integer larger than or equal to 1 and let $f(x) = x^n$. Then

  $$f'(x) = nx^{n-1}.$$

  Using this result together with the property of linearity, we can easily calculate the derivative of any polynomial function. TODO: general exponent

- The derivatives of the trigonometric functions can be derived from

  $$\sin'(x) = \cos \qquad \text{and} \qquad \cos'(x) = -\sin(x)$$

  TODO: list

- TODO cyclometric

- TODO hyperbolic

10.2.3.1 The exponential and logarithm

Define natural logarithm ln and *the* exponential function exp.

$$\frac{\mathrm{d}\ln x}{\mathrm{d}x} = \frac{1}{x}$$

$$a^x = e^{x \ln a} \qquad (a > 0, x \in \mathbb{R})$$

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

and growth.

10.2.4 Applications of differentiation

10.2.4.1 Extreme values

Link increasing, decreasing and derivatives. + derivative zero everywhere = constant. critical points. singulat points. concavity and inflections

10.2.4.2 Rolle's lemma

10.2.4.3 Mean-value theorem

10.2.4.4 L'Hôpital's rules

10.2.5 Higher order derivatives

When we take the derivative of a function, we we get a new function. We can now take the derivative of this new function. This is called taking the second order derivative. This process can be repeated for as long as the derivatives exist. We write the n-th order derivative as

$$f^{(n)}(x) = \frac{\mathrm{d}^n f}{\mathrm{d}n^n}.$$

So for example $f''(x) = f^{(2)}(x)$.

10.2.6 Implicit differentiation

10.2.7 Partial derivatives

10.2.7.1 Definition

10.2.7.2 Geometric interpretation

10.2.8 Meaning of the differential $\div$

TODO conventional use + examples with nabla
TODO: put series here!

10.2.9 Generalisations and types of derivatives

TODO: Liebnitz rule!!! + linear.

10.3 Integration

TODO intuition, solving strategies, solving intelligently SEE: The electric field (first write all quantities, then)

10.3.1 Areas as limits of sums

10.3.1.1 Sums and sigma notation

10.3.1.2 Trapezoid rule

10.3.1.3 Midpoint rule

10.3.1.4 Simpson's rule

10.3.2 The definite integral

10.3.3 Computing different areas and volumes

10.3.3.1 Rotation bodies

10.3.3.2 Surface bounded by function of polar coordinate θ

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} [f(\theta)]^2 \div \theta$$

10.3.4 The fundamental theorem of calculus

10.3.4.1 Indefinite integrals

anti-derivative $+C$

10.3.4.2 Some elementary integrals

10.3.5 Properties of integrals

10.3.5.1 Linearity

10.3.5.2 Mean-value theorem

10.3.5.3 Integrals of piece-wise continuous functions

10.3.6 Techniques of integration

10.3.6.1 Integrals of rational functions

10.3.6.2 Substitutions

+ inverse substitutions

10.3.6.3 Integration by parts

10.3.7 Improper integrals

10.3.8 Different types of integrals

10.3.8.1 Riemann

10.3.8.2 Lebesgue

10.3.8.3 Stieltjes

10.3.8.4 Cauchy

10.3.9 From infinite sum to integral

Using measure

10.4 Complex analysis

holomorphic functions, residue theorem

10.4.1 Complex integration and analyticity

10.4.2 Laurent series and isolated singularities

10.4.3 Residue calculus

10.4.4 Conformal mapping

TODO Solving intelligently (later using physics): Green functions, method of mirrors (+ cfr. general section on equations) charge distributions
separation of variables (Legendre polynomials)
going from discrete sum to integral (also opposite with dirac delta). volume int using $\mathcal{V}$ and surface $\mathcal{S}$
surface int goes to zero at infinity.

10.5 Dirac delta

10.5.1 In one dimension

10.5.2 In three dimensions

10.5.3 Properties

Composition of the Dirac δ with a smooth, continuously differentiable function g follows from the following relation

$$\int_{\mathbb{R}} \delta(g(x))f(g(x))|g'(x)| \div x = \int_{g(\mathbb{R})} \delta(u)f(u) \div u$$

Thus we say that

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$$

Where x_i are the simple roots of g.

10.6 Silly integrals

$$\int x^{\mathrm{d}x} - 1 = x \ln(x) - x + c$$

Chapter 11

Distributions

11.1 The space of test functions

11.1.1 Canonical LF topology

11.1.1.1 Convergence

11.1.2 The space of test functions

Let X be an open subset of a normed vector space. The space of test functions on X is $\mathcal{C}_c^\infty(X)$ equipped with the canonical LF topology. This space is also commonly denoted $\mathcal{D}(X)$.

Example

The bump function

$$\phi : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} e^{1/(x^2-1)} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

is a test function in $\mathcal{D}(\mathbb{R})$.

Lemma XI.109. *Every test function is bounded.*

Proof. TODO □

11.2 The module of distributions

Let X be an open subset of a normed vector space. A linear map $T : \mathcal{D}(X) \to \mathbb{C}$ is a distribution on X is an element of the topological dual of $\mathcal{D}(X)$ equipped with the strong dual topology. It is denoted $\mathcal{D}'(X)$.

Proposition XI.110. *The space of distributions $\mathcal{D}'(X)$ is a module over the ring $\mathcal{C}^\infty(X)$. If $T \in \mathcal{D}'(X)$ and $a \in \mathcal{C}^\infty(X)$, then the multiplication is defined by*

$$(aT)(\phi) = T(a\phi) \qquad \forall \phi \in \mathcal{D}(X).$$

11.2.1 Types of distributions

11.2.1.1 Regular distributions

Lemma XI.111. *Let X be an open subset of a normed vector space and $f : X \to \mathbb{C}$ a locally integrable function in $L^1_{loc}(X)$. Then*

$$T_f : \mathcal{D}(X) \to \mathbb{C} : \phi \mapsto \int_X f(x)\phi(x)\,\mathrm{d}x$$

is a distribution.

Proof. TODO We just need to show continuity (from boundedness) ☐

Distributions of the form T_f for some $f \in L^1_{\text{loc}}(X)$ are called regular distributions.

Lemma XI.112. *Let $f, g : X \to \mathbb{C}$ be locally, absolutely integrable functions. If $T_f = T_g$, then $f(x) = g(x)$ a.e.*

11.2.1.2 Dirac delta distribution

Let X be an open subset of a normed vector space and $x_0 \in X$. The Dirac delta distribution at x_0 is the distribution

$$\delta_{x_0} : \mathcal{D}(X) \to \mathbb{C} : \phi \mapsto \phi(x_0).$$

We write $\delta \coloneqq \delta_0$.

Proposition XI.113. *Let X be an open subset of a normed vector space containing 0. Suppose $\langle f_n : X \to \mathbb{C}\rangle$ is a sequence of functions such that*

1. $\int_X f_n(x)\,\mathrm{d}x = 1$ *for all n;*
2. *there exists a constant C such that $\int_X |f_n(x)|\,\mathrm{d}x \leq C$ for all n;*
3. $\lim_{n\to\infty} \int_{|x|>r} |f_n(x)|\,\mathrm{d}x = 0$ *for all $r > 0$.*

If ϕ is bounded on X and continuous at 0, then

$$\lim_{n\to\infty} \int_X f_n(x)\phi(x)\,\mathrm{d}x = \phi(0)$$

and in particular $f_n \to \delta$ in $\mathcal{D}'(X)$.

Notice that the condition of ϕ being bounded on X and continuous at 0 is much less strong than $\phi \in \mathcal{D}$.

Proof. TODO + iff? Then we could define delta sequences as sequences that converge to δ. TODO link with (and formulation of integral mean value theorem) ☐

Sequences $\langle f_n\rangle$ satisfying the assumptions of the proposition are called delta sequences.

Lemma XI.114. *If f_n is such that $\int f_n(x)\,\mathrm{d}x = 1$ and the support shrinks to zero, then f_n is a delta sequence.*

Lemma XI.115. *Let $f : \mathbb{R}^N \to \mathbb{C}$ be an integrable function with $\int_{\mathbb{R}^N} f(x)\,\mathrm{d}x = 1$. Then $\left\langle n^N f(nx)\right\rangle_{n\in\mathbb{N}}$ is a delta sequence.*

11.2.2 The derivative of a distribution

Let X be an open subset of a normed vector space V and let $T \in \mathcal{D}'(X)$. For $u \in V$ we define the directional derivative of T as

$$\partial_u(T) \coloneqq -T \circ \partial_u.$$

In particular, if $V = \mathbb{R}$, we define

$$T' \coloneqq \frac{\mathrm{d}}{\mathrm{d}x}(T) \coloneqq -T \circ \frac{\mathrm{d}}{\mathrm{d}x}.$$

Lemma XI.116. *Let T_f be an integral distribution such that $\partial_u(f)$ is well-defined on X, then*

$$\partial_u(T_f) = T_{\partial_u(f)}.$$

Proof. TODO by partial integration. ☐

Lemma XI.117. *Let X be an open subset of $\mathbb{R}^N$ and let $T \in \mathcal{D}'(X)$. Then*

$$D^\alpha T = (-1)^{|\alpha|} T \circ D^\alpha.$$

Proposition XI.118. *Let X be an open subset of a normed vector space V, $T \in \mathcal{D}'(X)$ and $u \in V$. Then*

1. *$\partial_u T \in \mathcal{D}'(X)$;*
2. *∂_u is linear on the module $\mathcal{D}'(X)$.*

Proposition XI.119. *Let θ be the Heaviside function. Then*

$$(T_\theta)' = \delta.$$

11.2.2.1 Jump discontinuities in integral distributions

Proposition XI.120. *Let f be a function that is $\mathcal{C}^1$ on $]-\infty, x_0[$ and $]x_0, +\infty[$ for some $x_0 \in \mathbb{R}$. Then the derivative of f as distribution is given by*

$$f' = (f|_{\mathbb{R}\setminus\{x_0\}})' + (\Delta_{x_0} f)\delta_{x_0}.$$

Note that if f is continuous at x_0, this gives $f' = (f|_{\mathbb{R}\setminus\{x_0\}})'$ as distributions.

Proof. Let $\phi \in \mathcal{D}(\mathbb{R})$. Then we calculate

$$\begin{aligned}
f'(\phi) &= \int_{-\infty}^{\infty} f(x)\phi'(x)\,\mathrm{d}x = \int_{-\infty}^{x_0} f(x)\phi'(x)\,\mathrm{d}x + \int_{x_0}^{\infty} f(x)\phi'(x)\,\mathrm{d}x \\
&= \Big[f(x)\phi(x)\Big]_{-\infty}^{x_0} - \int_{-\infty}^{x_0} f'(x)\phi(x)\,\mathrm{d}x + \Big[f(x)\phi(x)\Big]_{x_0}^{+\infty} - \int_{x_0}^{+\infty} f'(x)\phi(x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}\setminus\{x_0\}} f'(x)\phi(x)\,\mathrm{d}x + \lim_{x\to x_0+} f(x)\phi(x) - \lim_{x\to x_0-} f(x)\phi(x) \\
&= \int_{\mathbb{R}} f|_{\mathbb{R}\setminus\{x_0\}}\phi'(x)\,\mathrm{d}x + \phi(x_0)\lim_{x\to x_0+} f(x) - \phi(x_0)\lim_{x\to x_0-} f(x) = \int_{\mathbb{R}} (f|_{\mathbb{R}\setminus\{x_0\}})'\phi(x)\,\mathrm{d}x + (\Delta_{x_0} f)\delta_{x_0}.
\end{aligned}$$

☐

Corollary XI.120.1. *Let f be a function that is $\mathcal{C}^1$ on $]-\infty, x_0[$ and $]x_0, +\infty[$ for some $x_0 \in \mathbb{R}$. Then the k^{th} derivative of f as distribution is given by*

$$f^{(k)} = (f|_{\mathbb{R}\setminus\{x_0\}})^{(k)} + \sum_{j=0}^{k-1} (\Delta_{x_0} f^{(j)}) \delta_{x_0}^{(k-1-j)}.$$

11.2.3 Convolution

11.3 Sobolev spaces

TODO: after L^p spaces.

11.3.1 Weak and strong derivatives

Let X be an open subset of a normed vector space, $f \in L^p(X)$ and D^α a derivative on X. If there exists $g \in L^q(X)$ such that $D^\alpha T_f = T_g$, then g is the weak α derivative of f in $L^q(X)$.

The weak α derivative is unique if it exists, due to XI.112.
The definition of weak derivative translates to the requirement

$$\int_X f D^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_X g\phi \, \mathrm{d}x \qquad \forall \phi \in \mathcal{D}(X).$$

Let X be an open subset of a normed vector space, $f \in L^p(X)$ and D^α a derivative on X. We call $g \in L^q(X)$ a strong α derivative if there exists a sequence $\langle f_n \rangle \subset \mathcal{C}^\infty(X)$ such that

- $f_n \to f$ in $L^p(X)$; and
- $D^\alpha f_n \to g$ in $L^q(X)$.

Theorem XI.121. *Let $f \in L^p(X)$. Then $g \in L^q(X)$ is a weak α derivative* if and only if *it is a strong α derivative.*

11.3.2 Sobolev spaces

Let X be an open subset of a normed vector space, $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. Then the Sobolev space $W^{k,p}(X)$ is defined as

$$W^{k,p}(X) \coloneqq \{T \in \mathcal{D}'(X) \mid T \text{ has a weak } \alpha \text{ derivative in } L^p(X) \text{ for all } |\alpha| \leq k\}.$$

In particular each distribution in $W^{k,p}(X)$ is an integral distribution T_f for some f in $L^p(X)$.

Lemma XI.122. *Let X be an open subset of a normed vector space, $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. Then*

$$\mathcal{D}(X) \subset W^{k,p}(X) \subset L^p(X),$$

so that $W^{p,k}(X)$ is a dense subspace of $L^p(X)$.

Proposition XI.123. *A Sobolev space $W^{k,p}(X, \mathrm{d}\mu)$ is a Banach space with norm*

$$\|f\|_{W^{k,p}(X,\mathrm{d}\mu)} = \begin{cases} \left(\sum_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(X,\mathrm{d}\mu)}\right)^{1/p} & 1 \leq p < \infty \\ \max_{|\alpha|\leq k} \|D^\alpha f\|_{L^\infty(X,\mathrm{d}\mu)} & p = \infty. \end{cases}$$

If the measure is clear, we may also write $W^{k,p}(X)$.
In particular $W^{k,2}(X, \mathrm{d}\mu)$ is a Hilbert space with inner product

$$\langle f, g\rangle \sum_{|\alpha|\leq k} \int_X D^\alpha f(x) \overline{D^\alpha g(x)} \, \mathrm{d}\mu(x).$$

The Hilbert space $W^{k,2}(X, \mathrm{d}\mu)$ is more commonly denoted $H^k(X, \mathrm{d}\mu)$.

TODO: alternative definition: use ordinary derivative and take completion???

Proposition XI.124. *Let X be an open subset of a normed vector space, $1 \leq p < \infty$ and $k \in \mathbb{N}$. Then $W^{k,p}(X)$ coincides with the closure of $\mathcal{C}^\infty(X) \cap W^{k,p}(X)$ in the $W^{k,p}(X)$-norm.*

Let X be an open subset of a normed vector space, $1 \leq p < \infty$ and $k \in \mathbb{N}$. We define $W_0^{k,p}(X)$ to be the closure of $\mathcal{C}_c^\infty(X)$ in the $W^{k,p}(X)$-norm.

Part XII

Operator algebras

Chapter 1

Banach algebras

In this part we set $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Usually operator algebras are assumed to be complex. We will attempt to give results for real algebras where possible.

A normed algebra is an associative algebra A over $\mathbb{F}$ with norm $\|\cdot\|$ such that $(\mathbb{F}, A, +, \|\cdot\|)$ is a normed space and

$$\forall x, y \in A : \quad \|xy\| \leq \|x\|\|y\|.$$

We say A is unital if there exists a unit element $\mathbf{1} \in A$ such that

$$\forall x \in A : \mathbf{1} \cdot x = x = x \cdot \mathbf{1} \qquad \text{and} \qquad \|\mathbf{1}\| = 1.$$

A Banach algebra is a normed algebra that is also a Banach space.

TODO: which results also hold for normed algebras?

Lemma XII.1. *Let A be a Banach algebra. The multiplication map $\cdot : A \times A \to A : (x, y) \mapsto xy$ is continuous.*

Proof. Because $A \times A$ is a metric space, we can combine VIII.182 and VIII.148.1 to conclude that the multiplication map is continuous iff $x_n y_n \to xy$ whenever $x_n \to x$ and $y_n \to y$. Assume $x_n \to x$ and $y_n \to y$. Then

$$\begin{aligned} \|x_n y_n - xy\| &= \|x_n y_n - x y_n + x y_n - xy\| \leq \|(x_n - x) y_n\| + \|x(y_n - y)\| \\ &\leq \|x_n - x\| \cdot \|y_n\| + \|x\| \cdot \|y_n - y\| = \|x_n - x\| \cdot \|y_n - y + y\| + \|x\| \cdot \|y_n - y\| \\ &\leq \|x_n - x\| \cdot (\|y_n - y\| + \|y\|) + \|x\| \cdot \|y_n - y\| \to 0 \end{aligned}$$

□

As a consequence multiplication by a fixed factor, $x \mapsto cx$ or $x \mapsto xc$ for some c, is also continuous, by VIII.151.1. This is also immediate from the boundedness of multiplication $\|xy\| \leq \|x\|\|y\|$ and X.72.

Lemma XII.2. *Let A be a Banach algebra and $D \subset A$ a subset. Suppose $a \in A$ commutes with all elements of D, then a commutes with the closure $\overline{D}$.*

Proof. Take an arbitrary element $d \in \overline{D}$. Take an arbitrary $\epsilon > 0$. Then we can find an $x \in D$ such that $\|x - d\| \leq \epsilon$. Then, using that a and x commute,

$$\begin{aligned}\|ad - da\| &= \|a(d + x - x) - (d + x - x)\| \\ &= \|a(d - x) - (d - x)a\| \leq 2\epsilon\|a\|.\end{aligned}$$

Because we can choose ϵ arbitrarily small, $\|ad - da\|$ must be zero. □

Proposition XII.3. *Let A be a Banach algebra and $S \subset A$ a subset. Then*

$$\mathcal{B}(S) \coloneqq \overline{\operatorname{span}}\{s_1 \cdot s_2 \cdot \ldots \cdot s_k \mid k \geq 1, s_1, \ldots, s_k \in S\}$$

is the smallest Banach subalgebra in A that contains S.

Proposition XII.4. *Let A be a Banach algebra and $J \subset A$ an ideal. Then A/J is a Banach algebra with the quotient norm*

$$\|x + J\|_J = \inf_{j \in J}\|x - j\|.$$

1.1 ∗-algebras

A ∗-algebra is a ∗-r(i)ng $(A, +, \cdot, *)$, with involution $*$, that is an associative algebra over a commutative ∗-ring $(R, +, \cdot, ')$, with involution $'$, such that

$$\forall r \in R, x \in A : \quad (rx)^* = r'x^*.$$

A complex ∗-algebra is a ∗-algebra where the ∗-ring R is $\mathbb{C}$ with complex conjugation as the involution $'$.
A real ∗-algebra is a ∗-algebra where the ∗-ring R is $\mathbb{R}$ with the identity map as the involution $'$.
If a ∗-algebra is also a Banach algebra and for all elements $\|x^*\| = \|x\|$, then it is called a Banach-∗-algebra.

TODO: drop condition $\|x^*\| = \|x\|$? Not required for C^* (already implied).

Lemma XII.5. *Let A be a ∗-algebra. The unitisation $A^\dagger = A \oplus \mathbb{F}$ can also be seen as a ∗-algebra with the involution defined by*

$$(a, \lambda)^* = (a^*, \overline{\lambda}) \qquad \forall a \in A, \lambda \in \mathbb{F}.$$

Lemma XII.6. *Let A be a unital ∗-algebra. Then*

1. $\mathbf{1}^* = \mathbf{1}$;
2. *if x is invertible, then x^* is invertible with $(x^*)^{-1} = (x^{-1})^*$;*
3. $\sigma(x^*) = \left\{\overline{\lambda} \mid \lambda \in \sigma(x)\right\}$.

Proof. Take some $x \in A$.

1. $\mathbf{1}^*x = (x^* \cdot \mathbf{1})^* = x^{**} = x$. Similarly $x\mathbf{1}^* = x$.
2. $x^* \cdot (x^{-1})^* = (x^{-1}x)^* = \mathbf{1}^* = \mathbf{1}$. Similarly $(x^{-1})^* \cdot x^* = \mathbf{1}$.

□

Proposition XII.7. *Let A be a Banach-$*$-algebra and $S \subset A$ a subset. Then*

$$\mathcal{B}^*(S) := \mathcal{B}(S \cup S^*)$$

is the smallest Banach-$*$-subalgebra in A that contains S, where $\mathcal{B}$ is defined as in XII.3 and $S^* = \{s^* \in A \mid s \in S\}$.

Let A be a $*$-algebra and $x \in A$. We say that x is

1. normal, if $x^*x = xx^*$;
2. self-adjoint, if $x = x^*$;
3. unitary, if $x^*x = xx^* = \mathbf{1}$ (assuming A unital);
4. a projection, if $x = x^* = x^2$.

The set of all

1. normal elements in A is denoted $\mathcal{N}(A)$;
2. self-adjoint elements in A is denoted $\mathcal{SA}(A)$;
3. unitaries in A is denoted $\mathcal{U}(A)$;
4. projections in A is denoted $\mathcal{P}(A)$.

Lemma XII.8. *We have the following implications:*

$$\textit{projection} \Rightarrow \textit{self-adjoint} \Rightarrow \textit{normal} \Leftarrow \textit{unitary.}$$

Lemma XII.9. *Let A be a unital $*$-algebra and $p \in \mathcal{P}(A)$. Then $\mathbf{1} - p$ is a projection.*

Proof. We simply calculate

$$(\mathbf{1} - p)^2 = (\mathbf{1} - p)(\mathbf{1} - p) = \mathbf{1} - p - p + p = \mathbf{1} - p = (\mathbf{1} - p)^*.$$

□

Lemma XII.10. *Let A be a $*$-algebra and $x \in A$. Then there are unique self-adjoint elements $x_1, x_2 \in A$ such that $x = x_1 + i \cdot x_2$. They are given by*

$$x_1 = \frac{x + x^*}{2} \qquad \textit{and} \qquad x_2 = \frac{x - x^*}{2i}.$$

We call x_1 and x_2 the real part and imaginary part of x, respectively.

1.1.1 $*$-homomorphisms

Let A, B be $*$-algebras. A $*$-homomorphism is a linear, multiplicative, $*$-preserving map $\Psi : A \to B$.
If A, B are unital and $\Psi(\mathbf{1}_A) = \mathbf{1}_B$, then we say Ψ is unital.

Lemma XII.11. *Let A be a $*$-algebra, then $*$-homomorphisms map*

1. *normal elements to normal elements;*
2. *self-adjoints to self-adjoints;*
3. *projections to projections;*
4. *unitaries to unitaries, if the $*$-homomorphism is unital.*

1.1.2 $*$-matrix algebras

TODO define matrix algebra.

Let A be a $*$-algebra. Then the matrix algebra $A^{n\times n}$ is considered a $*$-algebra with the star operation given defined by

$$[a^*]_{i,j} \coloneqq [a]^*_{j,i}.$$

for all components of $a \in A^{n\times n}$.

Notice that the $*$-operation acts as the element-wise $*$-operation composed with the transpose.

1.2 Unitisation

Let A be a Banach algebra. Then the unitisation of A is the algebra $A^\dagger = A \oplus \mathbb{F}$ with multiplication

$$(x,\lambda)\cdot(y,\mu) = (xy + \lambda y + \mu x, \lambda\mu)$$

and a norm that extends the norm $\|\cdot\|$ on A to a norm on $A^\dagger$. In other words, there is an isometric embedding

$$A \hookrightarrow A^\dagger : x \mapsto (x, 0).$$

TODO: is $A^\dagger$ necessarily complete?

Lemma XII.12. *For any Banach algebra A, $A^\dagger$ is a unital Banach algebra with unit $\mathbf{1} = (0,1)$.*

Proof. TODO: is $A^\dagger$ necessarily complete? $\square$

It is possible to use multiple norms for the unitisation.

Proposition XII.13. *Let A be a Banach algebra. Of the possible norms for $A^\dagger$, the 1-norm*

$$\|(x,\lambda)\|_1 = \|x\| + |\lambda|$$

is minimal and the operator norm

$$\|(x,\lambda)\|_{op} = \sup\{\|xa + \lambda a\| \mid a \in A \wedge \|a\| \leq 1\}$$

is maximal. All possible norms are equivalent.

Proof. TODO: prove the operator norm is actually a norm and isometric. $\square$

We set

$$\tilde{A} = \begin{cases} A & \text{if } A \text{ unital} \\ A^\dagger & \text{if } A \text{ non-unital.} \end{cases}$$

If a Banach algebra A is unital, we can identify $\mathbb{F}$ with $\mathbb{F} \cdot \mathbf{1} \subseteq A$.
Alternatively we could define $\tilde{A}$ as the smallest unital Banach algebra containing A.

Lemma XII.14. *Let A be a Banach algebra. Then A is an ideal of $A^\dagger$.*

Lemma XII.15. *Let A be a Banach algebra. We have the split exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "\iota"] & A^\dagger \arrow[r, shift left, "\pi_2"] & \mathbb{F} \arrow[l, hook, shift left, "\lambda"] \arrow[r] & 0. \end{tikzcd}$$

Lemma XII.16. *Let A, B be Banach algebras. Every algebra homomorphism $\Psi : A \to B$ extends uniquely to a unital homomorphism $\Psi^\dagger : A^\dagger \to B^\dagger$:*

$$\Psi^\dagger : A^\dagger \to B^\dagger : (a, \lambda) \mapsto (\Psi(a), \lambda).$$

Proof. We want $\Psi^\dagger((a,0)) = (\Psi(a), 0)$ for all $a \in A$. Because Ψ is unital, we have $\Psi^\dagger((\mathbf{0}, 1)) = (\mathbf{0}, 1)$. So

$$\Psi^\dagger((a,\lambda)) = \Psi^\dagger((a,0)) + \lambda\Psi^\dagger((\mathbf{0},1)) = (\Psi(a),0) + \lambda(\mathbf{0},1) = (\Psi(a),\lambda).$$

□

Corollary XII.16.1. *Let $\pi_1 : A^\dagger \to A$ be the projection on the first component: $\pi_1(a,\alpha) = a$. The unital extension $\Psi^\dagger$ commutes with π_2:*

$$\pi_2 \circ \Psi^\dagger = \Psi^\dagger \circ \pi_2 = \Psi \circ \pi_2.$$

Restricted to A, this is equal to Ψ.

As before we set, for $\Psi : A \to B$ an algebra homomorphism

$$\tilde{\Psi} = \begin{cases} \Psi & \text{if } A \text{ unital} \\ \Psi^\dagger & \text{if } A \text{ non-unital.} \end{cases}$$

Thus $\tilde{\Psi}$ is a function on $\tilde{A}$.

Lemma XII.17. *Let A, B be Banach algebras and $\Psi : A \to B$ and algebra homomorphism. Then*

1. $\operatorname{im}(\Psi^\dagger) = (\operatorname{im}\Psi)^\dagger$;
2. $\ker(\Psi^\dagger) = \ker(\Psi) \oplus \{0\}$;
3. $\Psi^\dagger$ *is injective* if and only if Ψ *is injective;*
4. $\Psi^\dagger$ *is surjective* if and only if Ψ *is surjective;*
5. $\|\Psi^\dagger\| = \max\{\|\Psi\|, 1\}$;
6. $\Psi^\dagger$ *is isometric* if and only if Ψ *is isometric.*

Proof. The third point follows from the second and X.23. □

Let A be a Banach algebra. We define the scalar mapping to be

$$s = \lambda \circ \pi : A^\dagger \to A^\dagger : (a, \lambda) \mapsto (0, \lambda).$$

Notice that $\pi \circ s = \pi$.

1.2.1 Approximate units

Let A be a Banach algebra. A net $(e_\lambda)_{\lambda \in \Lambda}$ is an approximate unit if

1. $\|e_\lambda\| \leq 1$ for all λ;
2. $a = \lim_{\lambda \to \infty} e_\lambda \cdot a = \lim_{\lambda \to \infty} a \cdot e_\lambda$.

We call $(e)_\lambda$ is an increasing approximate unit if $\lambda_0 \leq \lambda_1$ implies $0 \leq e_{\lambda_0} \leq e_{\lambda_1}$.

Lemma XII.18. *If A is unital, any approximate unit in A converges to $\mathbf{1}$.*

TODO: usually increasing. When not?

1.3 Algebras of real and complex functions

1.4 Complexification

1.5 Complex analysis on Banach algebras

There is a theory of holomorphic (analytic) functions from open sets in $\mathbb{C}$ taking values in a Banach space, which is nearly identical to the usual complex-valued theory. In particular, most of the standard theorems of complex analysis, such as the Cauchy Integral Formula, Liouville's Theorem,and the existence and radius of convergence of Taylor and Laurent expansions,have exact analogs in this setting.

Lemma XII.19. *TODO: holomorphic?? And check sign.*

$$\frac{\mathrm{d}}{\mathrm{d}z} R_x(z) = (x - z)^{-2}.$$

1.6 Series in Banach algebras

1.6.1 Neumann series

Proposition XII.20 (Neumann series)**.** *Let A be a unital Banach algebra and $x \in A$. If $\|x\| < 1$, then $\mathbf{1} - x$ is invertible with inverse*

$$(\mathbf{1} - x)^{-1} = \sum_{n=0}^{\infty} x^n.$$

Equivalently, if $\|\mathbf{1} - x\| < 1$, then x is invertible with inverse

$$x^{-1} = \sum_{n=0}^{\infty} (\mathbf{1} - x)^n.$$

Proof. Since $\|x^n\| \leq \|x\|^n$ for all $n \geq 1$ and $\sum\|x\|^n$ is a convergent geometric series, the series $\sum x^n$ is convergent by XIII.73. ☐

Corollary XII.20.1. *Let* $x \in \mathrm{GL}(A)$*, then* $B(x, \|x^{-1}\|^{-1}) \subset \mathrm{GL}(A)$*. The invertible elements* $\mathrm{GL}(A)$ *form an open subset of* A*.*

Proof. The second assertion follows from the first by VIII.115. Let $y \in B(x, \|x^{-1}\|^{-1})$. Then

$$\|\mathbf{1} - x^{-1}y\| \leq \|x^{-1}\| \cdot \|x - y\| < \|x^{-1}\| \cdot \|x^{-1}\|^{-1} = 1$$

and, by the proposition, $x^{-1}y$ is invertible. Similarly yx^{-1} is invertible and thus y is invertible by II.115. ☐

Corollary XII.20.2. *The map* ${}^{-1} : \mathrm{GL}(A) \to \mathrm{GL}(A) : x \mapsto x^{-1}$ *is continuous.*

Proof. Take a convergent sequence $(x_n) \subset \mathrm{GL}(A)$ with limit x. We wish to prove (x_n^{-1}) converges to x^{-1}, because then the map is continuous by VIII.182. We can choose an n_0 such that $\forall n \geq n_0 : x_n \in B(x, \|x^{-1}\|^{-1})$. From now on we consider only the tails $(x_n)_{n=n_0}^\infty$ and $(x_n^{-1})_{n=n_0}^\infty$, which have the same limits by TODO ref. Then

$$\|x^{-1}\| \cdot \|x - x_n\| < \|x^{-1}\| \cdot \|x^{-1}\|^{-1} = 1.$$

Also

$$\|\mathbf{1} - x^{-1}x_n\| = \|x^{-1}(x - x_n)\| \leq \|x^{-1}\| \cdot \|x - x_n\| < 1.$$

We calculate, using the inequalities to apply the Neumann series formula and geometric series formula:

$$\begin{aligned}
\|x_n^{-1} - x^{-1}\| &= \|(x_n^{-1}x - \mathbf{1})x^{-1}\| = \|((x^{-1}x_n)^{-1} - \mathbf{1})x^{-1}\| \\
&= \left\|\left(\sum_{k=0}^{\infty}[\mathbf{1} - x^{-1}x_n]^k - \mathbf{1}\right)x^{-1}\right\| = \left\|\left(\sum_{k=1}^{\infty}[\mathbf{1} - x^{-1}x_n]^k\right)x^{-1}\right\| \\
&\leq \sum_{k=1}^{\infty}\|\mathbf{1} - x^{-1}x_n\|^k \cdot \|x^{-1}\| = \sum_{k=1}^{\infty}\|x^{-1}(x - x_n)\|^k \cdot \|x^{-1}\| \\
&\leq \|x^{-1}\| \sum_{k=1}^{\infty}\|x - x_n\|^k \cdot \|x^{-1}\|^k = \|x^{-1}\| \sum_{k=0}^{\infty}\|x - x_n\|^k \cdot \|x^{-1}\|^k - \|x^{-1}\| \\
&= \frac{\|x^{-1}\|}{1 - \|x - x_n\| \cdot \|x^{-1}\|} - \|x^{-1}\| = \frac{\|x - x_n\| \cdot \|x^{-1}\|^2}{1 - \|x - x_n\| \cdot \|x^{-1}\|}.
\end{aligned}$$

As the right-hand side converges to 0, so must the left-hand side. Thus (x_n^{-1}) converges to x^{-1}. ☐

1.6.2 The exponential

Proposition XII.21. *Let* A *be a Banach algebra and* $a \in A$*. Then the series*

$$\sum_{i=1}^{\infty}\frac{a^i}{i!}$$

converges. We denote its limit $\exp(a) - 1$ *or* $e^a - 1$*.*

Proof. By

$$\left\|\sum_{i=1}^{N}\frac{a^i}{i!}\right\| \leq \sum_{i=1}^{N}\frac{\|a\|^i}{i!}$$

it is absolutely convergent and thus convergent, by XIII.73. □

The function $a \mapsto \exp(a) = \mathbf{1} + \sum_{i=1}^{\infty}\frac{a^i}{i!}$ is the exponential mapping.

Lemma XII.22. *The exponential mapping is continuous.*

Proof. TODO - see Coleman □

Proposition XII.23. *Let A be a unital Banach algebra and $a, b \in A$. If a and b commute, then*

$$\exp(a+b) = \exp(a)\exp(b).$$

Proof. TODO - Coleman □

Corollary XII.23.1. *The image of the exponential function is contained in* $\mathrm{GL}(A)$*.*

Proof. As a and $-a$ commute, we have $\exp(-a)\exp(a) = \exp(0) = \mathbf{1}$. □

Lemma XII.24. *Let A be a Banach algebra and $a \in A$. Then*

$$\exp(a) = \lim_{n\to\infty}\left(\mathbf{1} + \frac{a}{n}\right)^n.$$

Proof. TODO □

1.7 Finite elements

elements of the socle. `https://link.springer.com/content/pdf/10.1023/A:1009717500980.pdf`
have finite spectrum
`http://matwbn.icm.edu.pl/ksiazki/sm/sm104/sm10431.pdf`

1.8 The spectrum

TODO: remove unital requirement.

Let A be a complex Banach algebra. The spectrum of an element $x \in A$ is defined as

$$\sigma(x) = \sigma_A(x) \coloneqq \left\{\lambda \in \mathbb{C} \;\middle|\; x - \lambda\cdot\mathbf{1} \in \tilde{A} \text{ is not invertible}\right\}.$$

If A is a real Banach algebra, then the spectrum of $x \in A$ is defined as

$$\sigma_A(x) \coloneqq \sigma_{A_\mathbb{C}}(x) = \left\{\lambda \in \mathbb{C} \;\middle|\; x - \lambda\cdot\mathbf{1} \in \widetilde{A_\mathbb{C}} \text{ is not invertible}\right\}.$$

The resolvent set of an element $x \in A$ is

$$\rho(x) = \mathbb{C} \setminus \sigma(x)$$

and its resolvent map is

$$R_x : \rho(x) \to A : z \mapsto (z - x)^{-1}.$$

The spectral radius of $x \in A$ is

$$\mathrm{spr}(x) = \max\{|\lambda| \mid \lambda \in \sigma(x)\}.$$

TODO redefine spr with sup. Max is then proposition (spectrum compact).

Lemma XII.25. *Let A be a non-unital Banach algebra. Then $0 \in \sigma_A(a)$ for all $a \in A$.*

Proof. Because $A \subset A^\dagger$ as an ideal, $(a, 0) \in A^\dagger$ is not invertible. □

Lemma XII.26. *Let A be a real Banach algebra. Then for all $a \in A$ and $\mu_1, \mu_2 \in \mathbb{R}$:*

1. *$\sigma(a) = \overline{\sigma(a)}$;*
2. *$\mu_1 + \mu_2 i \in \sigma(a)$* if and only if *$(a - \mu_1)^2 + \mu_2^2$ is not invertible in $\tilde{A}$.*

Proof. (1) Assume $\lambda \notin \sigma(a)$, so $(a - \lambda)^{-1}$ exists. Then

$$\mathbf{1} = \overline{\mathbf{1}} = \overline{(a-\lambda)(a-\lambda)^{-1}} = (a - \overline{\lambda})\overline{(a-\lambda)^{-1}},$$

so $a - \overline{\lambda}$ is invertible and $\overline{\lambda} \notin \sigma(a)$. The converse is identical, using $\overline{\lambda}$.
(2) By (1), $a - (\mu_1 + \mu_2 i)$ is invertible if and only if $a - (\mu_1 - \mu_2 i)$ is invertible. Because $a - (\mu_1 + \mu_2 i)$ and $a - (\mu_1 - \mu_2 i)$ commute, this is equivalent to saying

$$(a - (\mu_1 + \mu_2 i))(a - (\mu_1 - \mu_2 i)) = (a - \mu_1)^2 + \mu_2^2$$

is invertible in $\widetilde{A_{\mathbb{C}}}$, by II.115 and thus also in A, by TODO ref. □

Proposition XII.27. *Let B be a complex Banach algebra and $A = B_{\mathbb{R}}$, then for all $a \in A$*

$$\sigma_A(a) = \sigma_B(a) \cup \overline{\sigma_B(a)}.$$

Proof. TODO □

Proposition XII.28. *For any $x \in A$, the spectrum $\sigma(x)$ is a compact subset of $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|x\|\}$. In particular, $\mathrm{spr}(x) \leq \|x\|$.*

Proof. Let $\lambda \in \mathbb{C}$ be such that $|\lambda| > \|x\|$, then

$$1 > \frac{\|x\|}{|\lambda|} = \frac{\lambda - (\lambda - \|x\|)}{|\lambda|} = \left\|\mathbf{1} - \left(\mathbf{1} - \frac{x}{\lambda}\right)\right\|.$$

By XII.20, $\mathbf{1} - x/\lambda$ is invertible and thus so is $x - \lambda$.
It is then enough to show that $\sigma(x)$ is closed. By XII.20.1, $\mathrm{GL}(A)$ is open and the set of non-invertibles $A \setminus \mathrm{GL}(A)$ is closed. Consider $f : \mathbb{C} \to A : \lambda \mapsto x - \lambda$. Then $\sigma(x) = f^{-1}[A \setminus \mathrm{GL}(A)]$ is the preimage of a closed set under a continuous map, and hence is closed. □

Proposition XII.29 (Spectral radius formula)**.** *Let A be a Banach algebra and $x \in A$. Then*

$$\mathrm{spr}(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|x^n\|^{1/n}.$$

Proof. TODO, with complex analysis. Does it imply XII.28? □

Theorem XII.30. *Let A be a unital Banach algebra. For every $x \in A$, the spectrum $\sigma(x)$ is non-empty.*

Proof. TODO, uses complex analysis (Liouville's theorem). Also move higher? □

Corollary XII.30.1 (Gelfand-Mazur)**.** *Let A be a unital complex Banach algebra. If every non-zero element is invertible, then $A = \mathbb{C} \cdot \mathbf{1}$.*

Proof. Suppose $x \in A \setminus (\mathbb{C} \cdot \mathbf{1})$. Then $\sigma(x) = \emptyset$, contradicting the theorem. □

In other words, $\mathbb{C}$ is the only normed complex division algebra.

Proposition XII.31. *Let A be a unital Banach algebra and $x, y \in A$. Then $\mathbf{1} - xy$ is invertible* if and only if *$\mathbf{1} - yx$ is invertible.*

Proof. Assume $\mathbf{1} - xy$ invertible. Then the inverse of $\mathbf{1} - yx$ is

$$y(\mathbf{1} - xy)^{-1}x + \mathbf{1}.$$

□

Corollary XII.31.1. *Let A be a unital Banach algebra and $x, y \in A$. Then*

$$\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}.$$

Proof. Assuming $\lambda \neq 0$, we have $\lambda \in \sigma(xy) \iff \frac{xy}{\lambda} - \mathbf{1}$ is invertible. □

It is important to include 0: there are cases when $0 \in \sigma(xy)$, but $0 \notin \sigma(yx)$.

Lemma XII.32. *Let A, B be unital Banach algebras and $\Psi : A \to B$ a unital algebra homomorphism. Then for all $x \in A$: $\sigma(\Psi(x)) \subseteq \sigma(x)$ and hence $\operatorname{spr}(\Psi(x)) \leq \operatorname{spr}(x)$.*

Proof. By contraposition: Assume $\lambda \notin \sigma(x)$, then $x - \lambda$ has an inverse, call it a. Then $(\Psi(x) - \lambda)$ has an inverse by

$$(\Psi(x) - \lambda)\Psi(a) = \Psi(x - \lambda)\Psi(a) = \Psi((x - \lambda)a) = \Psi(\mathbf{1}) = \mathbf{1},$$

meaning $\lambda \notin \sigma(\Psi(x))$. □

In general if B is a subalgebra of a Banach algebra A, then for any $x \in B$, $\sigma_B(x) \supseteq \sigma_A(x)$.

Proposition XII.33. *Let A be a unital Banach algebra and suppose that $S \subset A$ is a set of pairwise commuting elements. Then there exists a unital commutative Banach subalgebra C such that $S \subset C \subset A$ and*

$$\sigma_A(s) = \sigma_C(s) \qquad \text{for all } s \in S.$$

Proof. TODO □

1.8.1 Quasinilpotent operators

Let A be a Banach algebra and $x \in A$. If $\sigma(x) = \{0\}$, then x is called quasinilpotent

https://www.jstor.org/stable/2042882?seq=1 https://www.researchgate.net/profile/Zbigniew-Slodkowski/publication/265547661_A_note_on_quasinilpotent_elements_of_a_Banach_algebra/links/5e7e8f94458515efa0b0fe83/A-note-on-quasinilpotent-
pdf?origin=publication_detail

https://www.cambridge.org/core/services/aop-cambridge-core/content/view/AC3CBD3000D16515D0BD83C07B703186/S0013091500015352a.pdf/finite_dimensionality_nilpotents_and_quasinilpotents_in_banach_algebras.pdf
https://www.cambridge.org/core/services/aop-cambridge-core/content/view/C8F26DDF45A29D689C726A29D8F0BC2A/S0017089500008429a.pdf/algebraic-ideals-of-semiprime-
pdf

Proposition XII.34. *Every nilpotent element is quasinilpotent. The converse holds for finite elements in semisimple Banach algebras (TODO correct version of finite).*

Proof. Let x be a nilpotent element in a (unital) Banach algebra A. By the spectral radius formula XII.29, we have

$$\operatorname{spr}(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = 0.$$

This implies $\sigma(x) = \{0\}$.
Now assume x a finite quasimilpotent element. Then $\dim(xAx) = n$ and so $x^2, \ldots x^{n+2}$ a linearly dependent and thus there exists a polynomial p of degree at most $n+2$ such that $p(x) = 0$. We can factorise $p(x) = x^k q(x)$ where q is some polynomial such that $q(0) \neq 0$ and $k \in \mathbb{N}$. Then by spectral mapping, we have that $\sigma(q(x)) = q(\sigma(x)) = q(\{0\}) \neq \{0\}$. Thus $q(x)$ is invertible and we have

$$x^k = x^k q(x) q(x)^{-1} = 0 \cdot q(x)^{-1} = 0.$$

This means x is nilpotent. □

1.9 Characters

> Let A be a Banach algebra. A character on A is a non-zero algebra homomorphism $A \to \mathbb{C}$.

In other words, a character on A is a non-zero multiplicative linear functional $A \to \mathbb{C}$. In particular, if A is a real algebra, a character is still complex valued, but now an $\mathbb{R}$-linear functional on A.

Lemma XII.35. *Let A be a real Banach algebra. Let φ be a character on A. Then*

$$\varphi \in A' \iff \varphi = \overline{\varphi} \iff \varphi[A] \subset \mathbb{R}.$$

Proposition XII.36. *Let A be a Banach algebra and φ a character on A, then φ is continuous and*

1. $\|\varphi\| \leq 1$;
2. $\|\varphi\| = 1$ *if A contains an approximate unit;*
3. $\|\varphi\| = 1 = \varphi(\mathbf{1})$ *if A is unital.*

Proof. We first prove that φ is unital if A is unital: As $\varphi(x) = \varphi(x \cdot \mathbf{1}) = \varphi(x)\varphi(\mathbf{1})$ for all $x \in A$ and $\varphi \neq 0$, it follows that $\varphi(\mathbf{1}) = 1$.
Next $\|\varphi\| \leq 1$ follows from $\|\tilde{\varphi}\| \leq 1$ by XII.17. To this end suppose that for some $x \in \tilde{A}$, $|\tilde{\varphi}(x)| > \|x\|$. Then $x - \tilde{\varphi}(x)$ is invertible by corollary XII.28. Thus

$$1 = \tilde{\varphi}(\mathbf{1}) = \tilde{\varphi}((x-\tilde{\varphi}(x))^{-1})\tilde{\varphi}(x-\tilde{\varphi}(x)) = \tilde{\varphi}((x-\tilde{\varphi}(x))^{-1})[\tilde{\varphi}(x)-\tilde{\varphi}(x)] = \tilde{\varphi}((x-\tilde{\varphi}(x))^{-1})\cdot 0 = 0$$

which is a contradiction. Then $\|\tilde{\varphi}\| \leq 1$ and thus $\|\varphi\| \leq 1$. By X.72, φ is continuous.
Then it just remains to be shown that $\|\varphi\| = 1$ if A contains an approximate unit. TODO □

1.10 Commutative Banach algebras

1.10.1 The (Gelfand) spectrum

Let A be a commutative Banach algebra. The (Gelfand) spectrum (or character space) $\hat{A}$ is the set of characters on A.

If we equip $\hat{A}$ with the weak-$*$ topology, the Banach-Alaoglu theorem (TODO ref) implies $\hat{A}$ is a locally compact Hausdorff space.
TODO $\hat{A}$ compact if and only if A unital.
All elements of the Gelfand spectrum are unital and continuous, by XII.36 and XII.36.
If A is a real algebra, then

$$\hat{A} = \left\{ \varphi|_A \;\middle|\; \varphi \in \widehat{A_{\mathbb{C}}} \right\}.$$

Proposition XII.37. *Let A be a unital Banach algebra, and suppose that $S \subseteq A$ is a subset of pairwise commuting elements. Then there exists a unital commutative Banach subalgebra $C \subseteq A$ with $S \subseteq C$ such that*

$$\forall s \in S: \quad \sigma_A(s) = \sigma_C(s).$$

Proof. TODO ☐

Proposition XII.38. *Let A be a unital Banach algebra and $\mathcal{J}$ is a maximal ideal. Then*

1. *$\mathcal{J}$ is closed;*
2. *if A is complex and commutative, then $A/\mathcal{J} \cong \mathbb{C}$;*
3. *if A is real and commutative, then $A/\mathcal{J} \cong \mathbb{R}$ or $A/\mathcal{J} \cong \mathbb{C}$.*

Proof. If $\mathcal{J}$ is not closed, then $\overline{\mathcal{J}} = A$ by maximality. So by proposition VIII.114 every open set must intersect $\mathcal{J}$, in particular the set $\mathrm{GL}(A)$, open by XII.20.1. Because $\mathcal{J}$ is an ideal containing an invertible element, $\mathcal{J} = A$ by TODO ref. A contradiction.
If A is commutative, $A/\mathcal{J}$ is a field by TODO ref. It is also a unital Banach algebra (TODO), so $A/\mathcal{J} \cong \mathbb{C}$ by the Gelfand-Mazur theorem, XII.30.1. ☐

Proposition XII.39. *Let A be a complex unital commutative Banach algebra. Then we have a bijection*

$$\ker : \hat{A} \rightarrowtail\!\!\!\!\rightarrow \{\textit{maximal ideals in } A\} : \varphi \mapsto \ker(\varphi).$$

Proof. First we verify that for each character φ the kernel is a maximal ideal. Indeed applying X.59 to φ we get an isomorphism $A/\ker\varphi \cong \operatorname{im}\varphi = \mathbb{C}$, meaning $\ker(\varphi)$ has codimension 1 and thus is a maximal proper subspace. By II.119, $\ker(\varphi)$ is an ideal.
To prove ker is injective: let $\ker(\varphi) = \ker(\psi)$. Take some $a \in A$, which we can uniquely write as $\lambda + n$, with $n \in \ker(\varphi) = \ker(\psi)$. TODO ref + extract lemma! Then $\varphi(a) = \lambda = \psi(a)$, so $\varphi = \psi$.
For surjectivity, take a maximal ideal $\mathcal{J}$. By XII.38, $A/\mathcal{J} \cong \mathbb{C}$, so the quotient map $A \to A/\mathcal{J} \cong \mathbb{C}$ can be seen as a character with kernel $\mathcal{J}$. ☐

TODO characterMaximalIdealsReal!

Proposition XII.40. *Let A be a unital commutative Banach algebra and $x \in A$. Then*

$$\sigma(x) = \left\{ \varphi(x) \;\middle|\; \varphi \in \hat{A} \right\} = \hat{x}[\hat{A}].$$

TODO: introduce notation $\hat{x}$ earlier?

Proof. Suppose $\lambda = \varphi(x)$ for some $\varphi \in \hat{A}$. Then $x - \lambda \in \ker\varphi$, which is a proper ideal. So $x - \lambda \notin \mathrm{GL}(A)$ and $\lambda \in \sigma(x)$.
Suppose $\lambda \in \sigma(x)$. Because $x - \lambda$ is non-invertible, the ideal generated by it is proper, by II.117, and $x - \lambda$ lies in a maximal ideal, by II.116. By XII.39, this means $x - \lambda \in \ker\varphi$ for some $\varphi \in \hat{A}$ and then $\lambda = \varphi(x)$. ☐

1.10.2 The Gelfand transform

Let A be a unital commutative Banach algebra. The Gelfand transform of A is the map

$$\wedge : A \to C(\hat{A}) : x \mapsto \hat{x}$$

where $\hat{x}(\varphi) = \varphi(x)$ for all $x \in A, \varphi \in \hat{A}$.

Lemma XII.41. *Let A be a unital commutative Banach algebra with Gelfand transform $\wedge$: $A \to C(\hat{A})$. Then*

1. *$\wedge$ is well-defined, in the sense that $\hat{x} \in C(\hat{A})$;*
2. *$\wedge$ is linear and multiplicative;*
3. *$\|\hat{x}\| = \mathrm{spr}(x) \leq \|x\|$.*

Thus the Gelfand transform is a unital, norm-contractive Banach algebra homomorphism.

Proof. We prove in turn:

1. We have equipped $\hat{A}$ with the weak-$*$ topology. Then proposition XIII.44 says each $\hat{x}$ is continuous.
2. $\hat{x}(\lambda\varphi + \psi) = \lambda\varphi(x) + \psi(x) = \lambda\hat{x}(\varphi) + \hat{x}(\psi)$ and $\hat{x}(\varphi\psi) = \varphi(x)\psi(x) = \hat{x}(\varphi)\hat{x}(\psi)$.
3. For all $x \in A$:
$$\|\hat{x}\| = \sup_{\varphi\in\hat{A}}\left(\frac{|\varphi(x)|}{\|\varphi\|}\right) = \sup_{\varphi\in\hat{A}}(|\varphi(x)|) = \mathrm{spr}(x) \leq \|x\|$$
where we have used that $\|\varphi\| = 1$ by XII.36, $\sigma(x) = \left\{|\varphi(x)| \;\middle|\; \varphi \in \hat{A}\right\}$ by XII.40 and the inequality is from XII.28.

☐

Chapter 2

C^*-algebras

2.1 C^*-algebras

A (complex) C^*-algebra is a complex Banach-$*$-algebra A such that

$$\forall x \in A : \quad \|x^*x\| = \|x\|^2.$$

This identity is known as the C^*-identity, the C^*-property, the C^*-condition or the C^*-axiom.

A real C^*-algebra is a real Banach-$*$-algebra such that

Example

TODO: Concrete C^*-algebras.
$\mathcal{C}(X)$ for some compact X (need Hausdorff?). TODO: norm well defined (i.e. bounded) and for $g \in \mathcal{C}(X)$, $\sigma(g) = g[X]$.

Proposition XII.42. *The C^*-identity is equivalent to*

$$\forall x \in A : \quad \|x^*x\| = \|x^*\| \cdot \|x\|.$$

Proof. TODO. Highly non-trivial. TODO: move later. □

Lemma XII.43. *The C^*-identity is equivalent to*

$$\forall x \in A : \quad \|x^*x\| \geq \|x\|^2.$$

Proof. Let $x \in A$, then $\|x\|^2 \leq \|x^*x\| \leq \|x\| \cdot \|x^*\|$ and so $\|x\| \leq \|x^*\|$. By replacing x with x^* and using $x^{**} = x$ we also get $\|x\| \geq \|x^*\|$. Then $\|x^*x\| \leq \|x^*\| \cdot \|x\| = \|x\|^2$. Together with the original inequality this implies the C^*-identity. □

Let A be a C^*-algebra and D a subset of A. The C^*-algebra generated by D, $C^*(D)$, is the smallest C^*-subalgebra of A containing D. TODO refine def.

Lemma XII.44. *$C^*(\mathbf{1}, a)$ is commutative.*

Lemma XII.45. *Let A be a C^*-algebra. The C^*-identity implies*

1. *$\|\mathbf{1}\| = 1$.*
2. *the involution $*$ is isometric: $\|x^*\| = \|x\|$;*
3. *the involution $*$ is continuous.*

Lemma XII.46. *Let A be a C^*-algebra. Then the sets $\mathcal{N}(A), \mathcal{SA}(A), \mathcal{U}(A)$ and $\mathcal{P}(A)$ are closed in A.*

Proof. This follows from the continuity of the multiplication and the involution $*$. ☐

Proposition XII.47. *Let A be a C^*-algebra and $x \in A$ a normal element. Then $r(x) = \|x\|$.*

Proof. We compute

$$\|x\|^4 = \|x^*x\|^2 = \|x^*xx^*x\| = \left\|(x^*)^2x^2\right\| = \left\|x^2\right\|^2$$

where we have repeatedly applied the C^*-identity and used normality once. We conclude that $\left\|x^2\right\| = \|x\|^2$. Inductively we obtain $\left\|x^{(2^n)}\right\| = \|x\|^{2^n}$. By the spectral radius formula, XII.29, we get

$$\operatorname{spr}(x) = \lim_{n\to\infty} \left\|x^{(2^n)}\right\|^{1/2^n} = \lim_{n\to\infty} \|x\| = \|x\|.$$

☐

Corollary XII.47.1. *Let A be a $*$-algebra. There exists at most one norm on A turning it into a C^*-algebra. If there is such a norm, it is given by $\|x\| = \sqrt{\operatorname{spr}(x^*x)}$.*

Proof. By the C^*-identity $\|x\| = \sqrt{\|x^*x\|}$ and x^*x is normal, so we can apply the proposition. ☐

It is important to note that the spectral radius is a purely algebraic property and is independent of the norm.

2.1.1 C^*-homomorphisms

TODO drop unital

Proposition XII.48. *Let A, B be unital C^*-algebras and $\Psi : A \to B$ a unital $*$-homomorphism. Then Ψ is bounded (and thus continuous) with $\|\Psi\| = 1$.*

Proof. Because x^*x and $\Psi(x^*x)$ are normal, we calculate using XII.47

$$\|x\|^2 = \|x^*x\| = \operatorname{spr}(x^*x) \geq \operatorname{spr}(\Psi(x^*x)) = \|\Psi(x^*x)\| = \|\Psi(x)^*\Psi(x)\| = \|\Psi(x)\|^2,$$

where the inequality follows from an application of lemma XII.32 to x^*x. Hence $\|\Psi\| \leq 1$. Equality follows from $\Psi(\mathbf{1}) = 1$. ☐

Corollary XII.48.1. *The kernel of Ψ is a closed $*$-ideal.*

Proof. TODO ref. ☐

Lemma XII.49. *A surjective $*$-homomorphism from a unital C^*-algebra is unital.*

Proof. Let $\Psi : A \to B$ be a surjective $*$-homomorphism with A unital. For all $a \in A$:

$$\Psi(a)\Psi(\mathbf{1}) = \Psi(a\mathbf{1}) = \Psi(a) \qquad \text{and} \qquad \Psi(\mathbf{1})\Psi(a) = \Psi(\mathbf{1}a) = \Psi(a).$$

As all elements of B are of the form $\Psi(a)$, $\Psi(\mathbf{1})$ is a multiplicative identity for B. ☐

2.1.1.1 Lifts

TODO: move to $*$-algebra homomorphisms?

Proposition XII.50. *Let $\Psi : A \to B$ be a surjective $*$-homomorphism between C^*-algebras. Then*

1. *every self-adjoint element $b \in B$ has a self-adjoint lift $a \in A$, such that $\|a\| = \|b\|$;*
2. *every positive element $b \in B$ has a positive lift $a \in A$, such that $\|a\| = \|b\|$;*
3. *every element $b \in B$ has a lift $a \in A$ such that $\|a\| = \|b\|$.*

In general normal elements, unitaries and projections do not lift to normal elements, unitaries and projections, unless Ψ is injective.

Lemma XII.51. *Let $\Psi : A \to B$ be an injective $*$-homomorphism between C^*-algebras.*

1. *if $\Psi(a)$ is a normal element, then a is a normal element;*
2. *if $\Psi(p)$ is a projection, then p is a projection;*
3. *if $\Psi(u)$ is a unitary element, then u is a unitary element.*

Proof. If $\Psi(a)\Psi(a)^* = \Psi(a)^*\Psi(a)$, then $\Psi(a^*a) = \Psi(aa^*)$ and $a^*a = aa^*$ by injectivity. The other conditions are verified similarly. □

2.2 Direct sums of C^*-algebras

TODO!

2.2.1 Unitisation of C^*-algebras

For Banach-$*$-algebras we may have a choice of norms to put on the unitisation. For C^*-algebras there is exactly one.
TODO: of course C^*-algebras use supremum norms. They are fundamentally operators after all!

Proposition XII.52. *Let A be a C^*-algebra. Then there exists a unique norm on $A^\dagger$ that turns it into a C^*-algebra: the operator norm*

$$\|(a,\lambda)\| \coloneqq \sup\{\|ax + \lambda x\| \mid x \in A \land \|x\| \leq 1\}.$$

Proof. There is at most one such norm, by XII.47.1. Because the operator norm is a suitable norm for $A^\dagger$ by XII.13, we just need to verify the C^*-identity:

$$\begin{aligned}\|(a,\lambda)^*(a,\lambda)\| &= \sup_{\|x\|\leq 1} \|(a,\lambda)^*(a,\lambda)x\| \\ &\geq \sup_{\|x\|\leq 1} \|x^*\| \cdot \|(a,\lambda)^*(a,\lambda)x\| \geq \sup_{\|x\|\leq 1} \|x^*(a,\lambda)^*(a,\lambda)x\| \\ &= \sup_{\|x\|\leq 1} \|((a,\lambda)x)^*(a,\lambda)x\| = \sup_{\|x\|\leq 1} \|(a,\lambda)x\|^2 = \|(a,\lambda)\|^2,\end{aligned}$$

where we have used the C^*-identity in A because $(a,\lambda)x = ax + \lambda x \in A$. □

2.3 Functionals and spectrum

Lemma XII.53. *Let A be a unital C^*-algebra and $x \in A$ a self-adjoint element. Then*

$$\forall t \in \mathbb{R} : \quad \|x + it\|^2 = \left\|x^2 + t^2\right\| \leq \|x\|^2 + t^2.$$

Proposition XII.54. *Let A be a unital C^*-algebra and $\varphi : A \to \mathbb{C}$ a linear functional satisfying $\|\varphi\| = \varphi(\mathbf{1})$. If $a \in A$ is self-adjoint, then $\varphi(a) \in \mathbb{R}$.*

Proof. We may assume $\varphi \neq 0$ and $\varphi(\mathbf{1}) = 1$. Using XII.53, we calculate

$$|\varphi(x) + it|^2 = |\varphi(x + it)|^2 \leq \|x + it\|^2 \leq \|x\|^2 + t^2.$$

By IX.12 this means $\varphi(x) \in \mathbb{R}$. □

TODO: alternate proof in Fillmore using exponential map.

Corollary XII.54.1. *Let $x \in A$ be self-adjoint. Then $\sigma(x) \subseteq \mathbb{R}$.*

Proof. By XII.37, and the fact that x is self-adjoint (TODO ref) we may assume A commutative. Let $\lambda \in \sigma(x)$. By XII.40 there is a $\varphi \in \hat{A}$ such that $\lambda = \varphi(x)$. By XII.36, $\|\varphi\| = \varphi(\mathbf{1}) = 1$. Then by the proposition $\lambda = \varphi(x) \in \mathbb{R}$. □

Corollary XII.54.2. *Let $\varphi : A \to \mathbb{C}$ be a <u>linear</u> functional satisfying $\|\varphi\| = \varphi(\mathbf{1})$. Then φ is $*$-preserving, i.e. for all $x \in A$*

$$\varphi(x^*) = \overline{\varphi(x)}.$$

Proof. By XII.10 we can write $x = x_1 + ix_2$. Then $\varphi(x^*) = \varphi(x_1 - ix_2) = \varphi(x_1) - i\varphi(x_2)$. By the proposition $\varphi(x_1), \varphi(x_2) \in \mathbb{R}$. □

Corollary XII.54.3. *Every character on A is $*$-preserving.*

Proposition XII.55. *Let A be a unital C^*-algebra, $B \subseteq A$ a unital C^*-subalgebra and $x \in B$. Then x is invertible in B* if and only if *x is invertible in A.*

Proof. If an inverse exists in B, said inverse will also be in A.
Conversely, suppose x not invertible in B. Then either x^*x or xx^* is not invertible in B by XII.6 and II.115. Let y be one of the two that is not invertible. Because y is self-adjoint, $\sigma_B(y) \subset \mathbb{R}$, by XII.54.1. Thus $(y_n) = (y + \frac{i}{n})$ is a sequence of invertibles converging to y. By XII.20.1, $\|y_n - y\| \geq \left\|y_n^{-1}\right\|^{-1}$ must hold for all n, otherwise y would be invertible. Thus $\left\|y_n^{-1}\right\|^{-1}$ must converge to zero and $\left\|y_n^{-1}\right\|$ must diverge (TODO ref). Since the inversion map is continuous on $\mathrm{GL}(A)$, by XII.20.2, it follows that y cannot be invertible in A. Since x is invertible if and only if x^* is invertible, by XII.6, y being non-invertible implies x is not invertible, by II.115. □

Corollary XII.55.1. *Let A be a C^*-algebra, $B \subseteq A$ a C^*-subalgebra and $x \in B$.*
The spectrum of x is independent of the surrounding algebra:

$$\sigma_B(x) = \sigma_A(x).$$

Proof. Apply the proposition to $\tilde{A}$ and $\tilde{B}$. Note that if A, B are non-unital, the unit of $B^\dagger$ is the same as that of $A^\dagger$. □

This does not hold in general for Banach-$*$-algebras!

Proposition XII.56. *Let A be a unital C^*-algebra and $x \in A$ a normal element. Then the map*

$$\hat{x}|_{\widehat{C^*(x,\mathbf{1})}} : \widehat{C^*(x,\mathbf{1})} \to \sigma(x) : \varphi \mapsto \varphi(x)$$

is a homeomorphism.

Proof. By XII.55.1 the map is independent of the surrounding algebra (TODO: $C^*(x,\mathbf{1})$ also independent?). So without WLOG we take $A = C^*(x,\mathbf{1})$. Because x is normal, $C^*(x,\mathbf{1})$ is commutative (TODO:ref). By XII.40 the map is surjective and well-defined, in that it maps into the codomain $\sigma(x)$. It is also continuous by XIII.44.
To show injectivity, suppose $\varphi(x) = \psi(x)$ for some $\varphi, \psi \in \hat{A}$. Because (TODO ref)

$$A = \overline{\operatorname{span}}\,\{x^n(x^*)^m \mid n, m \geq 0\}$$

and characters are continuous homomorphisms, XII.36, we see that $\varphi = \psi$.
Finally we need to show the map is open. Because $\sigma(x)$ is compact, this follows from VIII.173. □

2.4 Commutative C^*-algebras

`https://math.stackexchange.com/questions/4401500/how-to-see-that-pure-states-are-mult`
`rq=1` TODO

2.4.1 The Gelfand-Naimark theorem

TODO: non-unital case!

Theorem XII.57 (Gelfand-Naimark)**.** *Let A be a unital commutative C^*-algebra. Then the Gelfand transform*

$$\wedge : A \to \mathcal{C}(\hat{A}) : x \mapsto \hat{x}$$

is an isometric $*$-isomorphism.

Here we view $\mathcal{C}(\hat{A})$ as a C^*-algebra with involution $f^*(\varphi) = \overline{f(\varphi)}$ for all $\varphi \in \hat{A}$. TODO: more in exercises.

Proof. We already know the Gelfand transform is a homomorphism by XII.41.
We first prove it is a $*$-homomorphism: $\forall x \in A : \hat{x}^* = (x^*)^\wedge$. To that end, take some $\varphi \in \hat{A}$. Then

$$(\hat{x})^*(\varphi) = \overline{\hat{x}(\varphi)} = \overline{\varphi(x)} = \varphi(x^*) = (x^*)^\wedge(\varphi)$$

where the third equality is due to XII.54.3.
For isometry, notice that every element is normal due to commutativity. By XII.47 and XII.41, we have

$$\|x\| = \operatorname{spr}(x) = \|\hat{x}\|.$$

Then we just need to show the Gelfand transform is bijective (TODO ref). Injectivity follows from isometry, VIII.211. For surjectivity we want to apply the Stone-Weierstrass theorem XI.38. To show the image of A separates points, take $\varphi \neq \psi$ in $\hat{A}$. Then $\varphi(x) \neq \psi(x)$ for some $x \in A$, meaning $\hat{x}(\varphi) \neq \hat{x}(\psi)$.
By the Stone-Weierstrass theorem XI.38 the image of A is dense in $\hat{\mathcal{A}}$. By VIII.213 the image of an isometry from a complete space is closed. Thus the image of A is all of $\hat{\mathcal{A}}$. □

Corollary XII.57.1. *Every commutative C^*-algebra is isomorphic to $C(X)$ for some compact Hausdorff space X.*
This space X is unique up to isomorphism.

Proof. The first part follows directly from the Gelfand-Naimark theorem. For the second part, we know that $\widehat{C(X)} \cong X$ by TODO ref. And if $X \cong Y$, then $C(X) \cong C(Y)$?TODO ref? $\square$

Proposition XII.58. *If A (commutative) generated by one element a, then A is isomorphic to the C^*-algebra of continuous functions on the spectrum of a which vanish at 0.*

Proposition XII.59. *Any injective $*$-homomorphism of C^*-algebras is an isometry.*

Proof. Let $\Psi : A \to B$ be an injective $*$-homomorphism of C^*-algebras. It is enough to show $\|\Psi(a)\| = \|a\|$ for self-adjoint $a \in A$. Then for arbitrary $x \in A$, we have

$$\|\Psi(x)\| = \sqrt{\|\Psi(x)^*\Psi(x)\|} = \sqrt{\|\Psi(x^*x)\|} = \sqrt{\|x^*x\|} = \|x\|.$$

We can assume A, B unital by passing to $\Psi^\dagger$ using XII.17.
We can also assume A, B are commutative by restricting Ψ to $\Psi' : C^*(a, \mathbf{1}) \to C^*(\Psi(a), \mathbf{1})$.
By the Gelfand-Naimark theorem XII.57 we can suppose we have is a unital injection $\Psi : \mathcal{C}(X) \to \mathcal{C}(Y)$ for some compact Hausdorff spaces X, Y. There is then (TODO ref) some continuous surjection $\alpha : Y \to X$ such that $\forall f \in \mathcal{C}(X) : \Psi(f) = f \circ \alpha$. Then $\|\Psi(f)\| = \|f\|$ because both functions have the same range. $\square$

TODO non-unital Gelfand-Naimark!

2.5 Continuous functional calculus

TODO: relocate.

Lemma XII.60. *Let $D \subseteq \mathbb{C}$. Then*

$$\mathcal{C}(D) = C^*(I_D, \mathbf{1}_D).$$

Proof. TODO ref to Stone-Weierstrass. $\square$

Theorem XII.61 (Continuous functional calculus)**.** *Let A be a unital C^*-algebra and $x \in A$ a normal element. There exists a unique $*$-homomorphism*

$$\Phi_x : \mathcal{C}(\sigma(x)) \to A : f \mapsto f(x)$$

such that $\mathrm{id}_{\sigma(x)}(x) = x$ *and* $\mathbf{1}_{\sigma(x)}(x) = \mathbf{1}_A$. *Moreover*

$$\operatorname{im} \Phi_x = C^*(x, \mathbf{1}).$$

Proof. Unicity from XII.60 TODO.
For existence, let $B = C^*(x, \mathbf{1})$, which is commutative because x is normal. Then $\hat{x} : \hat{B} \to \sigma(x) : \varphi \mapsto \varphi(x)$ is a homeomorphism by XII.56 and $\hat{x}^t : \mathcal{C}(\sigma(x)) \to \mathcal{C}(\hat{B})$ is an isometric isomorphism (TODO!). The we have the isometric $*$-homomorphism

$$\Phi_x = \iota \circ \wedge^{-1} \circ \hat{x}^t : \ \mathcal{C}(\sigma(x)) \xrightarrow{\hat{x}^t} \mathcal{C}(\hat{B}) \xrightarrow{\wedge^{-1}} B \xhookrightarrow{\iota} A$$

where the inverse Gelfand transform $\wedge^{-1}$ exists and is isometric by the Gelfand-Naimark theorem XII.57, because B is commutative.
We verify

$$\mathrm{id}_{\sigma(x)}(x) = \Phi_x(\mathrm{id}_{\sigma(x)}) = (\iota \circ \wedge^{-1} \circ \hat{x}^t)(\mathrm{id}_{\sigma(x)}) = (\iota \circ \wedge^{-1})(\mathrm{id}_{\sigma(x)} \circ \hat{x}) = (\iota \circ \wedge^{-1})(\hat{x}) = \iota(x) = x$$

using the definition of the transpose, cancellation of $I_\sigma(x)$ and inverse of Gelfand transform.
We also verify

$$\mathbf{1}_{\sigma(x)}(x) = \Phi_x(\mathbf{1}_{\sigma(x)}) = (\iota \circ \wedge^{-1} \circ \hat{x}^t)(\mathbf{1}_{\sigma(x)}) = (\iota \circ \wedge^{-1})(\mathbf{1}_{\sigma(x)} \circ \hat{x}) = (\iota \circ \wedge^{-1})(\mathbf{1}_{\hat{B}}) = \iota(\mathbf{1}) = \mathbf{1}$$

using the fact that the Gelfand transform of $\mathbf{1}$ is $\varphi \mapsto \varphi(\mathbf{1})$, which is $\mathbf{1}_{\hat{B}}$ by XII.36. ☐

For polynomials in $z, \overline{z}$, this functional calculus works as expected, because it is a $*$-homomorphism. In fact we can apply functional calculus to any continuous defined on a superset of the spectrum: restricting to the spectrum still yields a continuous function by VIII.140.

Proposition XII.62. *Let A be a unital C^*-algebra and $x \in A$ be a normal element. Let f be a continuous function on the spectrum of x and let $y \in A$ commute with x. Then $f(x)$ commutes with y.*

TODO proof + restructure + define joint functional calculus? (but we still need case where y is not necessarily normal)

Lemma XII.63. *Let X be a compact space and view $\mathcal{C}(X)$ as a unital commutative C^*-algebra. Fix $g \in \mathcal{C}(X)$. The functional calculus is then given simply by composition:*

$$\mathcal{C}(g[X]) \to \mathcal{C}(X) : f \mapsto f(g) = f \circ g.$$

Proof. Composition is a unital $*$-homomorphism with the right properties. The claim follows from uniqueness of the functional calculus. ☐

Proposition XII.64. *Let A be a unital C^*-algebra and $x \in A$ be a normal element with functional calculus Φ_x.*

1. *If B is a unital C^*-algebra and $\Psi : A \to B$ a unital $*$-homomorphism, then*

$$\Psi \circ \Phi_x = \Phi_{\Psi(x)},$$

   which means

$$\forall f \in \mathcal{C}(\sigma(x)) : \quad \Psi(f(x)) = f(\Psi(x)).$$

2. *For any $f \in \mathcal{C}(\sigma(x))$:*

$$\sigma(f(x)) = f(\sigma(x)).$$

   This is the spectral mapping theorem.

3. *For any $f \in \mathcal{C}(\sigma(x))$ and $g \in \mathcal{C}(\sigma(f(x)))$:*

$$(g \circ f)(x) = \Phi_x(g \circ f) = \Phi_{f(x)}(g) = g(f(x)).$$

Proof.

1. The claim is well-defined because $\sigma(\Psi(x)) \subseteq \sigma(x)$, by XII.32. It is easy to check both sides are unital $*$-homomorphisms from $\mathcal{C}(\sigma(x))$ to B, sending the identity function to $\Psi(x)$. The claim then follows from uniqueness of the functional calculus.

2. Let $B = C^*(x, \mathbf{1})$, which is commutative because x is normal. We calculate

$$\sigma(f(x)) = \left\{ \varphi(f(x)) \;\middle|\; \varphi \in \hat{B} \right\} = \left\{ f(\varphi(x)) \;\middle|\; \varphi \in \hat{B} \right\} = f(\sigma(x)).$$

   using XII.40 and the previous point.

3. For all $\varphi \in \hat{B}$:

$$\varphi((g \circ f)(x)) = g(f(\varphi(x))) = g(\varphi(f(x)) = \varphi(g(f(x)))$$

   Using the first point. Hence $(g \circ f)(x) = g(f(x))$ by TODO ref.

□

Proposition XII.65. *Let $K \subset \mathbb{R}$ be non-empty and compact; $f : K \to \mathbb{C}$ a continuous function; A a unital C^*-algebra and Ω_K the set of self-adjoint elements in A with spectrum contained in K. The function*

$$f : \Omega_K \subset A \to A : a \mapsto f(a)$$

is continuous.

Proof. The map $A \to A$ given by $a \mapsto a^n$ is continuous for every $n \geq 0$, because multiplication is continuous. Then every polynomial f induces a continuous map $A \to A$. Then $\epsilon/3$ by Stone-Weierstrass. □

TODO: also for non-compact K and $K \subseteq \mathbb{C}$. Then Ω_K set of normal elements.

Proposition XII.66. *Let A be a unital C^*-algebra, $x \in A$ a normal element and $f \in \mathcal{C}(\sigma(x))$. Then*

$$\|f(x)\| = \sup_{\lambda \in \sigma(x)} |f(\lambda)|.$$

Proof. We calculate

$$\|f(x)\| = \operatorname{spr}(f(x)) = \sup |\sigma(f(x))| = \sup |f(\sigma(x))| = \sup_{\lambda \in \sigma(x)} |f(\lambda)|.$$

□

Corollary XII.66.1. *Let x be an normal element in a unital C^*-algebra and $\lambda_0 \in \rho(x)$. Then $d(\lambda_0, \sigma(x)) = \|R_x(\lambda_0)\|^{-1}$.*

Proof. We calculate

$$\|R_x(\lambda_0)\| = \left\| \frac{1}{x - \lambda_0 \cdot \mathbf{1}} \right\| = \sup_{\lambda \in \sigma(x)} \left| \frac{1}{\lambda - \lambda_0} \right| = \frac{1}{\inf_{\lambda \in \sigma(x)} |\lambda - \lambda_0|} = \frac{1}{d(\lambda_0, \sigma(x))}.$$

□

TODO: for general closed operators $d(\lambda_0, \sigma(x)) \geq \|R_x(\lambda_0)\|^{-1}$??

Lemma XII.67. *If two polynomials in $z, \overline{z}$ agree on the spectrum of a normal element, they give an equation the element obeys.*

The proof is the unicity of the functional calculus.

Corollary XII.67.1. *Let A be a unital C^*-algebra and $x \in A$ a normal element. Then*

1. *x is self-adjoint* if and only if $\sigma(x) \subseteq \mathbb{R}$;
2. *x is unitary* if and only if $\sigma(x) \subseteq \mathbb{T}$;
3. *x is a projection* if and only if $\sigma(x) \subseteq \{0, 1\}$.

Proof. TODO spectral mapping

1. By XII.54.1 we have that self-adjoint implies real spectrum. The converse follows from the lemma applied to $z = \overline{z}$.
2. Assume x unitary. By the C^*-identity $\|x\| = \sqrt{\|x^*x\|} = \sqrt{\|\mathbf{1}\|} = 1$, by XII.45. By XII.47, $\operatorname{spr}(x) = \|x\| = 1$. Also $\left\|x^{-1}\right\|^{-1} = 1$. So $\sigma(x) \subseteq \mathbb{T}$ by XII.20.1. The converse follows from the lemma applied to $1 = z\overline{z} = \overline{z}z$.
3. Assume x a projection. Then $x - \lambda$ has an inverse given by $-\lambda^{-1} + (1-\lambda)^{-1}\lambda^{-1}x$ if $\lambda \notin \{0, 1\}$:

$$(x - \lambda)\left(-\frac{1}{\lambda} + \frac{x}{(1-\lambda)\lambda}\right) = \mathbf{1} - \frac{x}{\lambda} + \frac{x - x\lambda}{(1-\lambda)\lambda} = \mathbf{1} - \frac{x}{\lambda} + \frac{x}{\lambda} = \mathbf{1}.$$

   The converse follows from the lemma applied to $z = \overline{z} = z^2$.

□

Proposition XII.68. *Let A be a unital C^*-algebra. Then every element in A can be written as a linear combination of at most four unitaries.*

Proof. Let $x \in A$. By XII.10 we can write $x = x_1 + ix_2$ for some self-adjoint x_1, x_2. TODO □

2.6 Positivity

`https://link.springer.com/content/pdf/10.1023/A:1009717500980.pdf`

2.6.1 Positive elements

Let A be a C^*-algebra. An element $a \in A$ is positive if it is normal and $\sigma(a) \subset [0, +\infty[$.
We write $a \geq 0$.
The set of all positive elements of A is the positive cone of A

$$A^+ \coloneqq \{a \in A \mid a \geq 0\}.$$

By XII.67.1 every projection is positive and every positive element is in fact self-adjoint.
By XIII.169 we have for all positive $a \in A$:

$$\|a\| = \sup \sigma(a) = \max \sigma(a).$$

Proposition XII.69. *Let A be a unital C^*-algebra and $a \in A$ a self-adjoint element. Then*

$$a \in A^+ \qquad \iff \qquad \exists b \in \mathcal{SA}(A): \quad a = b^2.$$

If we further require b to be positive, then it is unique.

Proof. Let $a \in A^+$. Then define $b = \sqrt{a}$ by spectral calculus. Then we have

$$b^2 = (\Phi_a(\sqrt{\ }))^2 = \Phi_a(\sqrt{\ }) \cdot \Phi_a(\sqrt{\ }) = \Phi_a(\sqrt{\ }^2) = \Phi_a(I_{\sigma(a)}) = a.$$

Converse by spectral mapping XII.64. $\square$

Lemma XII.70. *Let A be a unital C^*-algebra and $a \in A$ self-adjoint. Then the following are equivalent:*

1. *a is positive;*
2. $\left\|\frac{1}{2}\mathbf{1} - a/\|a\|\right\| \leq \frac{1}{2}$;
3. $\|r\mathbf{1} - a/\|a\|\| \leq r$ *for all* $r \geq 1/2$.
4. $\|r\mathbf{1} - a/\|a\|\| \leq r$ *for some* $r \geq 1/2$.

Proof. The proof is cyclic:
(1. $\Rightarrow$ 2.) Assume a positive. Then $\sigma(a) \subseteq [0, \|a\|]$. By spectral mapping, XII.64, we have $\sigma(\frac{1}{2}\mathbf{1} - a/\|a\|) \subseteq [-1/2, 1/2]$ and thus $\left\|\frac{1}{2}\mathbf{1} - a/\|a\|\right\| \leq \frac{1}{2}$, by XII.47.
(2. $\Rightarrow$ 3.) Write $r = 1/2 + r'$, so $r' \geq 0$. Then

$$\left\|r\mathbf{1} - \frac{a}{\|a\|}\right\| = \left\|\frac{1}{2}\mathbf{1} + r'\mathbf{1} - \frac{a}{\|a\|}\right\| \leq \|r'\mathbf{1}\| + \left\|\frac{1}{2}\mathbf{1} - \frac{a}{\|a\|}\right\| \leq r' + \frac{1}{2} = r.$$

(3. $\Rightarrow$ 4.) Clear.
(4. $\Rightarrow$ 1.) By XII.47, $\sigma(r\mathbf{1} - a/\|a\|) \subseteq [-r, r]$. By spectral mapping, this means $\sigma(a) \subseteq [0, 2r\|a\|]$ and thus a is positive. $\square$

Proposition XII.71. *Let A be a C^*-algebra and $a \in A$. Then a is positive* if and only if *$a = b^*b$ for some $b \in A$.*

TODO: link with $\sqrt{a}$?

Corollary XII.71.1. *Let A be a concrete C^*-algebra of bounded operators on some Hilbert space $\mathcal{H}$ and $a \in A$. Then a is positive as an element of the C^* algebra* if and only if *a is positive as an operator on $\mathcal{H}$. for all $x \in \mathcal{H} : \langle x, ax\rangle \geq 0$.*

Proof. If a is positive, then $a = b^*b$ and thus

$$\forall x \in \mathcal{H} : \langle x, ax\rangle = \langle x, b^*bx\rangle = \langle bx, bx\rangle = \|bx\|^2 \geq 0,$$

meaning a is a positive operator.
Conversely, the spectrum is contained in the closure of the numerical range (TODO ref), which is a subset of $[0, \infty[$. $\square$

Consequently if $\langle x, ax\rangle \geq 0$ for all $x \in \mathcal{H}$, then a is self-adjoint.

2.6.2 Partial order on self-adjoint elements

Proposition XII.72. *Let A be a C^*-algebra. The set A^+ is a salient pointed convex cone.*

Proof. That A^+ is a cone follows from spectral mapping (XII.64), as does the salience of A^+. For convexity, we verify additive closure (see X.54) Take $a, b \in A$ and set $p = \frac{\|a\|+\|b\|}{\|a+b\|} \geq 1 \geq 1/2$. Then

$$\left\| p\mathbf{1} - \frac{a+b}{\|a+b\|} \right\| \leq \frac{1}{\|a+b\|} \left(\| \|a\| - a\| + \| \|b\| - b\| \right) \leq \frac{\|a\|+\|b\|}{\|a+b\|} = p$$

using the triangle inequality and point 3. of XII.70 with $r = 1$. $\square$

By X.338, we have:

Corollary XII.72.1. *The relation $\leq$ defined by*

$$a \leq b \qquad \iff \qquad b - a \in A^+$$

is a vector partial order on A.

TODO in general $(A, \leq)$ is not a Riesz space. (See e.g. the absolute value)

Lemma XII.73. *If $\alpha, \beta \in \mathbb{R}$ and $\alpha \leq \beta$, then $\alpha\mathbf{1} \leq \beta\mathbf{1}$.*

Lemma XII.74. *If $0 \leq a \leq b$ and a invertible, then b invertible and $0 \leq a^{-1} \leq b^{-1}$.*

Lemma XII.75. *Let A be a unital C^*-algebra and v an arbitrary element $v \in A$. If v^*v is positive, then $v = 0$.*

Proposition XII.76. *Let A be a unital C^*-algebra and $a \in A$ a self-adjoint element. Then $a \leq \|a\| \cdot \mathbf{1}$.*

Proof. This follows from

$$\sigma(\|a\| - a) = \{\|a\| - \lambda \mid \lambda \in \sigma(a)\}$$

using the fact that the spectrum of a is real, XII.54.1. $\square$

Corollary XII.76.1. *If $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.*

Proof. TODO $\square$

2.6.2.1 Lattice properties of self-adjoint operators

`https://www.ams.org/journals/proc/1951-002-03/S0002-9939-1951-0042064-2/S0002-9939-1951-0042064-2.pdf`

Proposition XII.77. *Let A be a C^*-algebra. The real vector space of self-adjoint operators $\mathcal{SA}(A)$ is a Riesz space* if and only if *A is commutative.*

Proof. TODO $\square$

Corollary XII.77.1. *Let A be a unital C^*-algebra, and $a \in A$ a self-adjoint element. Then there exists a unique decomposition $a = a^+ - a^-$ where $a^+, a^- \in A^+$ and $a^+a^- = 0$.*

Proof. TODO $\square$

The corollary XII.77.1 can also be proved using functional calculus, by setting $a^+ = f^+(a)$ and $a^- = f^-(a)$ where

$$f^+ : x \mapsto \begin{cases} x & (x \geq 0) \\ 0 & (x < 0) \end{cases} \qquad \text{and} \qquad f^- : x \mapsto \begin{cases} 0 & (x \geq 0) \\ -x & (x < 0) \end{cases}$$

Corollary XII.77.2 (Cartesian decomposition)**.** *Let A be a C^*-algebra and $a \in A$. Then we can decompose a as*

$$(p_1 - p_2) + i(p_3 - p_4)$$

where p_1, p_2, p_3, p_4 are positive.

Proof. By XII.10. ☐

Thus $\operatorname{span}_{\mathbb{C}}(A^+) = A$.

Proposition XII.78. *The set of all bounded self-adjoint operators on a Hilbert space is an anti-lattice.*

2.6.2.2 Operator monotonicity

Delicate!

Proposition XII.79. *Assume $0 \leq a \leq b$. Then*

1. *$\sqrt{a} \leq \sqrt{b}$;*
2. *$0 \leq ab$ if a, b commute.*

It is not true that $a^2 \leq b^2$ or that ab is positive in general!

2.6.3 Absolute value

Let A be a unital C^*-algebra. For each $a \in A$, the absolute value of a is $|a| = (a^*a)^{1/2}$.

The square root is well defined using functional calculus on the self-adjoint element a^*a.

Lemma XII.80. *Let A be a unital C^*-algebra.*

1. *Let $u \in \mathcal{U}$, then $|u| = 1$.*
2. *Let $a \in A$, then $|a|$ is positive and thus self-adjoint.*
3. *The map $a \mapsto |a|$ is continuous.*

Proof. For the first point, $|u| = (u^*u)^{1/2} = \mathbf{1}^{1/2} = \mathbf{1}$.
The second point follows from spectral mapping XII.64 using the fact that $z \mapsto \overline{z}z$ has positive image in $\mathbb{C}$.
For the third point, $a \mapsto a^*a$ is continuous by XII.1 and XII.45. Then $a \mapsto |a|$ is continuous by XII.65. ☐

Lemma XII.81. *Let T be a bounded operator on a Hilbert space $\mathcal{H}$. Then $|T|$ is the only positive operator A in $\mathcal{B}(\mathcal{H})$ such that $\|Ax\| = \|Tx\|$ for all $x \in \mathcal{H}$.*

Proof. We have for all $x \in \mathcal{H}$,

$$\langle Ax, Ax\rangle = \langle Tx, Tx\rangle \implies \langle (A^*A - T^*T)x, x\rangle = 0,$$

which implies $T^*T = A^*A$. Now A is positive, so $A^*A = A^2$ and taking the squared root give $A = \sqrt{T^*T} = |T|$. □

We do <u>not</u>, in general, have a triangle inequality $|a+b| \leq |a| + |b|$.

<u>Example</u>

Consider the C^* algebra $\mathbb{C}^{2\times 2}$ with

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = |A| \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

Then

$$A + B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1-\sqrt{2} & 1=\sqrt{2} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1-\sqrt{2} & 1=\sqrt{2} \\ 1 & 1 \end{pmatrix}^{-1}$$

So

$$|A+B| = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \qquad \text{and} \qquad |A| + |B| = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}.$$

Finally $|A| + |B| - |A+B| = \begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & 3-\sqrt{2} \end{pmatrix}$ is not positive:

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & 3-\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 - \sqrt{2} < 0.$$

2.6.3.1 Polar decomposition

Proposition XII.82 (Polar decomposition)**.** *Let A be a unital C^*-algebra and $a \in \mathrm{GL}(A)$ an invertible element. Then there exists a unique decomposition*

$$a = u(a)|a|$$

such that $u(a)$ is unitary. The map $u : \mathrm{GL}(A) \to \mathcal{U}(A)$ is continuous.

Proof. If a is invertible, then so are a^* and $|a|$ by spectral mapping XII.64 ($0 \notin \sigma(a^*)$ and $0 \notin \sigma(|a|)$). Put $u(a) = a|a|^{-1}$. Clearly $a = u(a)|a|$ and $u(a)$ is unitary because it is invertible and

$$u(a)^*u(a) = |a|^{-1}a^*a|a|^{-1} = |a|^{-1}|a|^2|a|^{-1} = \mathbf{1}.$$

The continuity of u: $a \mapsto a|a|^{-1}$ follows from the continuity of multiplication, XII.1, the continuity of the absolute value, XII.80 and the continuity of the inverse, XII.20.2. □

TODO: polar decomposition for non-invertible elements. Then u is a partial isometry.

2.6.4 Positive maps

Let A, B be C^*-algebras. Then $f : A \to B$ is a positive map if

$$\forall x \in A : \quad x \geq 0 \implies f(x) \geq 0.$$

By XII.71, this is equivalent to the condition that $f(x^*x) \geq 0$ for all $x \in A$.
TODO: `https://www-m5.ma.tum.de/foswiki/pub/M5/CQC/Masterarbeit.pdf` `https://iopscience.iop.org/article/10.1088/0305-4470/34/29/308`

2.6.4.1 Positive functionals and states

Let A be a C^*-algebra. A linear functional ρ on A is positive, written $\rho \geq 0$, if

$$\forall x \in A : \quad x \geq 0 \implies \rho(x) \geq 0.$$

A state on a C^*-algebra A is a positive linear functional of norm 1. The set $\mathcal{S}(A)$ of all states on A is called the state space of A.

Not necessarily multiplicative!

Example

Let A be a concrete C^*-algebra of operators acting non-degenerately on $\mathcal{H}$ and $\xi \in \mathcal{H}$. Define

$$\rho_\xi : A \to \mathbb{C} : x \mapsto \langle \xi, x\xi \rangle\,,$$

then ρ_ξ is a positive linear functional on A of norm $\|\xi\|^2$, so ρ_ξ is a state if $\|\xi\| = 1$. Such a state is called a vector state of A.

Proposition XII.83. *Let A be a C^*-algebra and ρ a positive linear functional on A. Then*

$$A \times A \to \mathbb{C} : (a, b) \mapsto \rho(a^*b)$$

is a positive Hermitian form.

Corollary XII.83.1. *This form obeys the Cauchy-Schwarz inequality, X.108:*

$$\forall a, b \in A : \quad |\rho(a^*b)|^2 \leq \rho(a^*a)\rho(b^*b).$$

Or

$$\forall a, b \in A : \quad |\rho(ab)|^2 \leq \rho(aa^*)\rho(b^*b).$$

Proposition XII.84. *Let ω be a linear functional over a C^*-algebra A. The following are equivalent:*

1. *ω is positive;*
2. *ω is continuous and $\|\omega\| = \lim_\lambda \omega(e_\lambda^2)$ for some approximate unit $\{e_\lambda\}$.*

Proposition XII.85. *Let ω be a positive linear functional over a C^*-algebra A and $a, b \in A$, then*

1. *$\omega(a^*) = \overline{\omega(a)}$;*
2. *$|\omega(a)|^2 \leq \omega(a^*a)\|\omega\|$;*
3. *$|\omega(a^*ba)| \leq \omega(a^*a)\|b\|$;*
4. *$\|\omega\| = \sup\{\omega(a^*a) \mid \|a\| = 1\}$*
5. *for any approximate unit $\{e_\lambda\}$, $\|\omega\| = \lim_\lambda \omega(e_\lambda^2)$*

Proof. By XII.10 and XII.77.1 we can write $a \in A$ as

$$a = x_{1,+} - x_{1,-} + i(x_{2,+} - x_{2,-}).$$

Where $x_{1,+}, x_{1,-}, x_{2,+}, x_{2,-}$ are self-adjoint. Then

$$\rho(a^*) = \rho(x_{1,+} - x_{1,-} - i(x_{2,+} - x_{2,-})) = \rho(x_{1,+}) - \rho(x_{1,-}) - i(\rho(x_{2,+}) - \rho(x_{2,-})) = \overline{\rho(a)}.$$

Where the last equality follows because ρ takes real values on self-adjoint elements. (TODO!) □

Corollary XII.85.1. *Let ω_1 and ω_2 be positive linear functionals over a C^*-algebra A. Then $\omega_1 + \omega_2$ is a positive linear functional and*

$$\|\omega_1 + \omega_2\| = \|\omega_1\| + \|\omega_2\|.$$

Thus the state space is a convex subset of the dual of A.

Corollary XII.85.2. *Let X be a compact Hausdorff space. Let ω be a positive linear functional on $\mathcal{C}(X)$. Then ω is continuous and $\|\omega\| = 1$.*

TODO: move up for Riesz-Markov?

Proposition XII.86. *If A is commutative, the pure states are exactly the characters.*

2.6.5 Comparison of projectors

Lattice

2.6.6 General comparison theory

2.7 Matrix C^*-algebras

Lemma XII.87. *Let A be a C^*-algebra. There exists a norm that makes $A^{n\times n}$ a C^*-algebra. TODO expression*

2.7.1 Completely positive maps

> Let A, B be C^*-algebras and $f : A \to B$ a linear function. We call f completely positive if for all $n \in \mathbb{N}$ the pointwise extension of f in $(A^{n\times n} \to B^{n\times n})$ is positive.

2.7.2 Completely bounded maps

Let A, B be C^*-algebras and $f : A \to B$ a linear function. We call f completely bounded if for all $n \in \mathbb{N}$ the pointwise extension of f in $(A^{n\times n} \to B^{n\times n})$ is bounded.

Chapter 3

Representations and states

3.1 Representations

TODO: link to general definition!

A representation of a C^*-algebra A on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$.
A subrepresentation of π is the restriction of π to a closed invariant subspace of $\mathcal{H}$.
We say a representation $\pi : A \to \mathcal{B}(\mathcal{H})$ is

1. faithful if it is injective;
2. non-degenerate if $\overline{\pi(A)\mathcal{H}} = \mathcal{H}$;
3. cyclic w.r.t. a unit vector $\xi \in \mathcal{H}$ if $\overline{\pi(A)\xi} = \mathcal{H}$.

Lemma XII.88. *Let A be a C^*-algebra and $\pi : A \to \mathcal{B}(\mathcal{H})$ a representation of A. Then π is a faithful representation of $A/\ker\pi$.*

Proposition XII.89. *Let $\pi : A \to \mathcal{H}$ be a representation of a C^*-algebra, then π being faithful is equivalent to any of the following:*

1. $\ker\pi = \{0\}$;
2. $\|\pi(a)\| = \|a\|$ *for all* $a \in A$;
3. $\pi(a) > 0$ *for all* $a > 0$.

Lemma XII.90. *Let $\pi : A \to \mathcal{H}$ be a representation and P_1 be a projector with closed range $\mathcal{H}_1$. Then $\pi|_{\mathcal{H}_1}$ is a subrepresentation* if and only if

$$\forall a \in A : \quad \pi(a)P_1 = P_1\pi(a).$$

Proof. Assume $P_1\pi(a) = \pi(a)P_1$ for all $a \in A$. Then multiplying by P_1 gives

$$P_1\pi(a)P_1 = \pi(a)P_1 \quad \forall a \in A$$

which expresses invariance. Conversely, assume this invariance condition. Then

$$\pi(a)P_1 = P_1\pi(a)P_1 = (P_1\pi(a)P_1)^{**} = (P_1\pi(a^*)P_1)^* = (\pi(a^*)P_1)^* = P_1\pi(a).$$

□

Lemma XII.91. *Let $\pi : A \to \mathcal{H}$ be a representation of a C^*-algebra A. Define*

$$\mathcal{H}_0 \coloneqq \{x \in \mathcal{H} \mid \forall a \in A : \pi(a)x = 0\}.$$

Then π is non-degenerate if and only if $\mathcal{H}_0 = \{0\}$.

Proof. It is enough to prove that

$$(\pi(A)\mathcal{H})^\perp = \mathcal{H}_0.$$

This implies $\overline{\pi(A)\mathcal{H}} = \mathcal{H}_0^\perp$ by XIII.138.2. The claim then follows from X.115.
The proof then rests on the equality

$$\forall x, y \in \mathcal{H}, a \in A : \quad \langle x, \pi(a)y \rangle = \langle \pi(a^*)x, y \rangle .$$

An $x \in \mathcal{H}$ is an element of $(\pi(A)\mathcal{H})^\perp$ iff the left side is zero for all $y \in \mathcal{H}, a \in A$. An $x \in \mathcal{H}$ is an element of $\mathcal{H}_0$ iff $\pi(a^*)x = 0$ for all $a^* \in A$. This is equivalent to saying the right side is zero for all $y \in \mathcal{H}, a \in A$ by the non-degeneracy of the inner product X.102. ☐

Proposition XII.92. *Let $\pi : A \to \mathcal{H}$ be a non-degenerate representation. Then π is the direct sum of a family of cyclic representations.*

Two representations π, ρ of A on Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively are (unitarily) equivalent if there is a unitary operator $U \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that

$$\forall x \in A : \quad U\pi(x)U^* = \rho(x).$$

3.1.1 Irreducible representations

TODO connect to more general definitions.

A set D of bounded operators on a Hilbert space $\mathcal{H}$ is called algebraically irreducible if the only subspaces of $\mathcal{H}$ invariant under the action of D are the trivial subspaces $\{0\}$ and $\mathcal{H}$.
The set D is called topologically irreducible if the only closed invariant subspaces of $\mathcal{H}$ are the trivial subspaces.
A representation $\pi : A \to \mathcal{H}$ is called irreducible if $\pi[A]$ is irreducible.

Proposition XII.93. *For C^*-algebras the notions of algebraic and topological irreducibility are equivalent.*

Proposition XII.94. *Let D be a self-adjoint set of bounded operators on the Hilbert space $\mathcal{H}$. The following are equivalent:*

1. *D is irreducible;*
2. *the commutant D' consists of multiples of the identity operator;*
3. *every non-zero vector $x \in \mathcal{H}$ is cyclic for D in $\mathcal{H}$, or $D = 0$ and $\mathcal{H} = \mathbb{C}$.*

3.2 The GNS construction

TODO: move (much) higher, at least before positivity!

Lemma XII.95. *Let ω be a state on a C^*-algebra A. Then the map*

$$A \times A \to \mathbb{C} : (x, y) \mapsto \omega(x^* y)$$

is a pre-inner product on A.

Lemma XII.96. *Let ω be a state on a C^*-algebra A. Then*

$$N_\omega \coloneqq \{x \in A \mid \omega(x^* x) = 0\}$$

is a closed left ideal in A.

Proof. TODO: left ideal criterion?
Let $x, y \in N_\omega$, Then

$$\omega((x + y)^*(x + y)) = \omega(x^* x) + \omega(y^* x) + \omega(x^* y) + \omega(y^* y) = 0$$

where $\omega(y^* x), \omega(x^* y)$ are zero by corollary X.108.2 to the Cauchy-Schwarz inequality. □

Lemma XII.97. *Let ω be a state on a C^*-algebra A. Then $H^0_\omega = A/N_\omega$ is an inner product space with inner product*

$$\langle [x], [y] \rangle_\omega = \omega(x^* y)$$

where $[x] = x + N_\omega$ for $x \in A$.
We also define $\mathcal{H}_\omega$ as the Hilbert space completion of H^0_ω.

Proof. We verify this inner product is well-defined: take $x, x' \in [x]$ and $y, y' \in [y]$. Then $(x' - x) \in N_\omega$ and $(y' - y) \in N_\omega$. Again using corollary X.108.2 to the Cauchy-Schwarz inequality, we see

$$\omega(x^* y) = \omega(x^* y) + \omega(x^*(y' - y)) = \omega(x^* y') = \omega(x^* y') + \omega((x' - x)^* y') = \omega(x'^* y').$$

□

Lemma XII.98. *Let ω be a state on a C^*-algebra A and write $[x] = x + N_\omega$. Then for any increasing approximate unit $(e_n)_n$,*

$$\xi_\omega = \lim_{n \to \infty} [e_n]$$

is a unit vector in $\mathcal{H}_\omega$ which does not depend on the choice of approximate unit.

Theorem XII.99 (Gelfand-Naimark-Segal)**.** *Let ω be a state on a C^*-algebra A. The map $\pi_\omega : A \to \mathcal{B}(H^0_\omega)$ defined by*

$$\pi_\omega(x) : H^0_\omega \to H^0_\omega : [y] \mapsto [xy]$$

is a well-defined representation of A on H^0_ω. This can be extended to a representation of A on $\mathcal{H}_\omega$. Also

1. *π_ω is cyclic w.r.t. ξ_ω;*
2. *for all $a \in A :\ \omega(a) = \langle \xi_\omega, \pi_\omega(a)\xi_\omega \rangle_\omega$.*

This is the unique representation with these properties, up to unitary equivalence.

Corollary XII.99.1. *Let ω be a state over the C^*-algebra A and $\tau : A \to A$ a $*$-automorphism that leaves ω invariant:*

$$\forall a \in A : \quad \omega(\tau(a)) = \omega(a).$$

Then there exists a unique unitary operator U on $\mathcal{H}_\omega$ such that

$$\forall a \in A : \quad U\pi_\omega(a)U^* = \pi_\omega(\tau(a))$$

and $U\xi_\omega = \xi_\omega$.

Let $\omega : A \to \mathbb{C}$ be a state. We call the cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega)$ constructed in the GNS theorem is called the canonical cyclic representation of A associated with ω.

Lemma XII.100. *Let ω be a state over a C^*-algebra A and $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega)$ the associated cyclic representation. There is a bijective correspondence*

$$\omega_T(a) = \langle T\xi_\omega, \pi_\omega(a)\xi_\omega \rangle$$

between positive functionals ω_T over A majorised by ω and positive operators T in the commutant π_ω' with $\|T\| \leq 1$.

Proposition XII.101. *Let ω be a state over a C^*-algebra A and $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega)$ the associated cyclic representation. The following are equivalent:*

1. *$(\mathcal{H}_\omega, \pi_\omega)$;*
2. *ω is a pure state;*
3. *ω is an extremal point of the state space $\mathcal{S}(A)$.*

Theorem XII.102. *Let A be a C^*-algebra. Then A is isomorphic to a norm-closed self-adjoint algebra of bounded operators on a Hilbert space.*

3.3 Multiplier algebras

3.3.1 Essential ideals

TODO move to section about ideals

Let J be an ideal of a C^*-algebra A. Then J is called essential if for all $a \in A$ we have that $aJ = \{0\}$ implies $a = 0$.

Lemma XII.103. *Let A be a C^*-algebra and $I \subset A$ an ideal. Then $I^2 = I$.*

Proof. Let $a \in I^+$. Then $a = (a^{1/2})^2 \in I^2$. As I and I^2 are C^*-algebras, they are spanned by their positive elements. So $I^2 \subset I \subset I^2$. □

Lemma XII.104. *Let A be a C^*-algebra and $I, J \subset A$ ideals. Then $IJ = I \cap J$.*

Proof. We calculate $I \cap J = (I \cap J)^2 \subset IJ \subset I \cap J$ using XII.103. □

Proposition XII.105. *Let J be an ideal of a C^*-algebra A. Then the following are equivalent:*

1. *J is essential;*

2. $\forall a \in A: \; Ja = \{0\}$ *implies* $a = 0$*;*

3. *every other non-zero ideal in* A *has a non-zero intersection with* J*.*

Proof. Assume (3) and let $a \in A$ such that $aJ = \{0\}$. Let $I = \overline{AaA}$ be the ideal generated by a. ☐

3.3.2 Multiplier algebras

Proposition XII.106. *Let* A *be a* C^**-algebra and* π_1, π_2 *faithful, non-degenerate representations. Then the idealisers* $I(\pi_1[A]), I(\pi_2[A])$ *of* $\pi_1[A]$ *and* $\pi_2[A]$ *are isomorphic to each other.*

Proof. We need to show

$$I(\pi_1[A]) = \{T \in \mathcal{B}(\mathcal{H}_1) \mid T\pi_1[A] \cup \pi_1[A]T \subseteq \pi_1[A]\} \cong \{T \in \mathcal{B}(\mathcal{H}_2) \mid T\pi_2[A] \cup \pi_2[A]T \subseteq \pi_2[A]\} = I(\pi_2[A]).$$

We first show $I(\pi[A])$ contains $\pi[A]$ as an essential ideal. That it contains A as an ideal is obvious. Non-degeneracy of the representation means that the only $x \in \mathcal{H}$ that is mapped to 0 by all $\pi[A]$ is 0, by XII.91.
The idealiser is the largest subalgebra of $\mathcal{B}(\mathcal{H})$ that contains A as an ideal. The ideal is necessarily ☐

Let A be a C^*-algebra. The multiplier algebra $M(A)$ of A is the largest C^*-algebra that contains A as an essential ideal.

The multiplier algebra is the non-commutative analogue of Stone–Čech compactification: if A is commutative, then $A \cong C(X)$ and

$$M(A) \cong C_b(X) \cong C(\beta(X)),$$

where $\beta(X)$ denotes the Stone–Čech compactification of X.

Lemma XII.107. *If* A *is a unital* C^**-algebra, then* $M(A) = A$*.*

If we view the C^*-algebra A as a Hilbert A-module, then $M(A)$ is the set of adjointable operators on A.

Proposition XII.108. *Let* A *be a* C^**-algebra and* π_1, π_2 *faithful, non-degenerate representations. Then the idealisers* $I(\pi_1[A]), I(\pi_2[A])$ *of* $\pi_1[A]$ *and* $\pi_2[A]$ *are isomorphic to each other and to the multiplier algebra* $M(A)$*.*
$M(A)$ *can be realised as the idealiser*

$$M(A) \cong I(\pi[A]) = \{T \in \mathcal{B}(\mathcal{H}) \mid T\pi[A] \cup \pi[A]T \subseteq \pi[A]\}$$

of A *in* $\mathcal{B}(\mathcal{H})$*.*

Proof. We need to show

$$I(\pi_1[A]) = \{T \in \mathcal{B}(\mathcal{H}_1) \mid T\pi_1[A] \cup \pi_1[A]T \subseteq \pi_1[A]\} \cong \{T \in \mathcal{B}(\mathcal{H}_2) \mid T\pi_2[A] \cup \pi_2[A]T \subseteq \pi_2[A]\} = I(\pi_2[A]).$$

We first show $I(\pi[A])$ contains $\pi[A]$ as an essential ideal. That it contains A as an ideal is obvious. Non-degeneracy of the representation means that the only $x \in \mathcal{H}$ that is mapped to 0 by all $\pi[A]$ is 0, by XII.91.
The idealiser is the largest subalgebra of $\mathcal{B}(\mathcal{H})$ that contains A as an ideal. The ideal is necessarily ☐

For example, let $\mathcal{H}$ be a Hilbert space. Then $M(\mathcal{K}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ is the algebra of compact operators on $\mathcal{H}$.
We write $\mathcal{U}M(A)$ to mean the unitary elements of the multiplier algebra.
Let $\pi : A \to M(B)$ be a $*$-homomorphism. If $\overline{\text{span}}(\pi(A)B) = B$, then π can be uniquely extended to $\overline{\pi} : M(A) \to M(B)$.

3.4 Universal C^*-algebras

Let $\mathcal{X}$ be a non-empty set. We formally write $\mathcal{X}^* = \{x^* \mid x \in \mathcal{X}\}$ and view it as a set disjoint from $\mathcal{X}$. A noncommutative-$*$-polynomial with variables in $\mathcal{X}$ is a formal expression of the form

$$\sum_{k=1}^{m} \lambda_k x_{k,1} x_{k,2} \dots x_{k,n_k}$$

where $m, n_k \in \mathbb{N}$, $x_{k,n} \in \mathcal{X} \cup \mathcal{X}^*$ and $\lambda_k \in \mathbb{C}$.
a polynomal relation $\mathcal{R}$ on $\mathcal{X}$ is a collection of formal statements of the form

$$\|p_j(\mathcal{X})\| \leq r_j$$

indexed by some index set J where $r_j \in \mathbb{R}^{\geq 0}$ and p_j is a noncommutative-$*$-polynomial with variables in $\mathcal{X}$.

Let $\mathcal{X}$ be a non-empty set and $\mathcal{R}$ a set of polynomial relations on $\mathcal{X}$.
A representation of $(\mathcal{X} \mid \mathcal{R})$ is a C^*-algebra A together with a map $\pi : \mathcal{X} \to A$ such that $\mathcal{R}$ becomes true in the image of π.
A representation $\pi_u : \mathcal{X} \to B$ of $(\mathcal{X} \mid \mathcal{R})$ is called universal if for any other representation $\pi : \mathcal{X} \to A$ of $(\mathcal{X} \mid \mathcal{R})$, there exists a unique $*$-homomorphism $\varphi : B \to A$ such that $\varphi \circ \pi_u = \pi$.
In this case we call B the universal C^*-algebra generated by $(\mathcal{X} \mid \mathcal{R})$ and write $B = C^*(\mathcal{X} \mid \mathcal{R})$.

A polynomial relation $\mathcal{R}$ on $\mathcal{X}$ is said to be bounded, if for every $x \in \mathcal{X}$, we have

$$\sup \{\|\pi(x)\| \mid \pi : \mathcal{X} \to A \text{ is a representation of } (\mathcal{X} \mid \mathcal{R})\} < \infty.$$

Example

- The relation $(\mathcal{X} \mid \mathcal{R}) = (\{a\} \mid \{\|a - a^*\| \leq 0\})$ is not bounded.
- The relation $(\mathcal{X} \mid \mathcal{R}) = (\{x, y\} \mid \{\|\mathbf{1} - x^*x - y^*y\| \leq 0\})$ is bounded: writing x for $\pi(x)$, the spectrum of $x^*x = 1 - y^*y$ is positive and bounded by 1 according to the spectral mapping theorem XII.64. Now $\|\pi(x)\| = \sqrt{\text{spr}(\pi(x)^*\pi(x))} \leq 1$ by XII.47 and so

$$\sup \{\|\pi(x)\| \mid \pi : \mathcal{X} \to A \text{ is a representation of } (\mathcal{X} \mid \mathcal{R})\} \leq 1 < \infty.$$

- The relation $(\mathcal{X} \mid \mathcal{R}) = (\mathcal{X} \mid \bigcup_{x \in \mathcal{X}} \{\|x - x^*\| \leq 0, \left\|x - x^2\right\| \leq 0\})$ is bounded. It gives rise to a universal C^*-algebra generated by projections.

Proposition XII.109. *Let $\mathcal{X}$ be a non-empty set and $\mathcal{R}$ a polynomial relation on $\mathcal{X}$. Then $(\mathcal{X} \mid \mathcal{R})$ is bounded if and only if $C^*(\mathcal{X} \mid \mathcal{R})$ exists.*

Proof. TODO ☐

3.5 Direct limits

3.5.1 AF

3.5.2 UHF

3.5.3 Stable algebras

`http://web.math.ku.dk/~rordam/manus/encyc.pdf`

Chapter 4

Von Neumann Algebras

TODO: definitions of SOT and WOT!

A concrete C^*-algebra $A \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if it is closed in the SOT.

4.1 von Neumann bicommutant theorem

Proposition XII.110. *Let $S \subset \mathcal{B}(\mathcal{H})$ be a set for some Hilbert space $\mathcal{H}$. Then*

1. *S' is a Banach algebra;*
2. *S' is a C^*-algebra if $S = S^*$;*
3. *S' $\subseteq \mathcal{B}(\mathcal{H})$ is WOT-closed.*

Chapter 5

Group C^*-algebras and crossed products

I also like the name “Automorphism groups”.

5.1 Group C^*-algebras

5.1.1 Discrete groups

Let G be a finite group and R a r(i)ng. The group ring RG is the set of functions $(G \to R)$ with pointwise addition and the convolution product

$$(x \star y)(g) = \sum_h x(h)y(h^{-1}g) = \sum_{g=hk} x(h)y(k)$$

for all $x, y \in RG$ and $g \in G$.

The a group ring can be seen as a free module generated by G. (TODO: this as definition?) The group algebra $\mathbb{C}G$ has an involution:

$$x^*(g) = \overline{x(g^{-1})} \qquad \text{for all } x \in \mathbb{C}G.$$

And it admits a norm making it a C^*-algebra.

5.1.2 Locally compact Hausdorff groups

For topological groups we are not restricted to finite sums. For locally compact Hasdorff groups we can in fact define multiplication, involution and a norm on the space $C_c(G)$ of complex-valued continuous functions of compact support.

Lemma XII.111. *Let G be a locally compact Hausdorff group. Convolution is a bilinear operation that maps $C_c(G) \times C_c(G) \to C_c(G)$ defined by*

$$(f \star g)(t) \coloneqq \int_G f(s)g(s^{-1}t) \,\mathrm{d}\mu(s).$$

Proof. Continuity follows from the dominated convergence theorem. Also

$$\operatorname{supp}(f \star g) \subseteq \operatorname{supp}(f) \cdot \operatorname{supp}(g)$$

where $\cdot$ is the group multiplication. $\square$

Lemma XII.112. *The algebra $C_c(G)$ has an involutive anti-linear anti-automorphism*

$$* : f \mapsto f^* = (s \mapsto \overline{f(s^{-1})}\Delta(s^{-1}))$$

where Δ is the modular function on G. This means $C_c(G)$ is a $*$-algebra with $*$ as involution.

5.2 C^*-dynamical systems

A C^*-dynamical system is a triple (G, α, A) consisting of a locally compact group G, a C^*-algebra A and a homomorphism α of G into $\operatorname{Aut}(A)$, such that $g \mapsto a_g(a)$ is continuous for all $a \in A$.

5.2.1 Covariant homomorphisms and representations

Let (G, α, A) be a C^*-dynamical system. A covariant homomorphism into the multiplier algebra $M(D)$ of some C^*-algebra D is a pair (ρ, U) where

- $\rho : A \to M(D)$ is a $*$-homomorphism and
- $U : G \to \mathcal{U}\, M(D)$ is a strictly continuous homomorphism between groups

satisfying

$$\rho(\alpha_g(a)) = U_g \rho(a) U_g^* \qquad \text{for all } g \in G.$$

We say (ρ, U) is non-degenerate if ρ is.

5.2.1.1 Integrated forms

Given a covariant homomorphism (ρ, U) on a C^*-dynamical system (G, α, A) into $M(D)$ we can parcel these two functions into one function $C_c(G, A) \to M(D)$, called the integrated form

$$(\rho \rtimes U)(f) := \int_G \rho(f(r)) U_r \,\mathrm{d}\mu(r)$$

where μ is the left Haar measure.

Lemma XII.113. *Let (ρ, U) be a covariant homomorphism. Then $\rho \rtimes U$ is a $*$-homomorphism.*

5.2.1.2 Induced covariant morphisms

Given a $*$-homomorphism $\rho : A \to M(D)$ we can extend it naturally to a covariant homomorphism.

Let (G,α,A) be a C^*-dynamical system and $\rho : A \to M(D)$ a $*$-homomorphism. Then the covariant homomorphism induced from ρ $\operatorname{Ind}\rho$ is the covariant homomorphism $(\widetilde{\rho}, 1\otimes\lambda)$ of (G,α,A) into $M(D\otimes\mathcal{K}(L^2(G))$ where

- $\lambda : G \to \mathcal{U}(L^2(G))$ is the left regular representation of G given by $(\lambda_s\xi)(t) = \xi(s^{-1}t)$;
- ρ is the composition

$$A \xrightarrow{\widetilde{\alpha}} C_b(G,A) \hookrightarrow M(A\otimes C_0(G)) \xrightarrow{\rho\otimes M} M(D\otimes\mathcal{K}(L^2(G)))$$

where $\widetilde{\alpha} : A \to C_b(G,A)$ is defined by $\widetilde{\alpha}(a)(s) = \alpha_{s^{-1}}(a)$ and

$$M : C_0(G) \to \mathcal{B}(L^2(G)) = M(\mathcal{K}(L^2(G)))$$

denotes the representation by multiplication operators.

The regular representation of (G,α,A) is $\Lambda^G_A \coloneqq \operatorname{Ind}(\mathrm{id}_A)$.

Lemma XII.114. *Let $\rho : A \to M(D)$ be a $*$-homomorphism. Then*

$$\operatorname{Ind}\rho = ()$$

5.2.1.3 Covariant representations

A (covariant) representation of a C^*-dynamical system (G,α,A) on a Hilbert space $\mathcal{H}$ is a covariant homomorphism (π,U) into $M(\mathcal{K}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$.

Covariant representations (π,U) on $\mathcal{H}$ and (π',U') on $\mathcal{H}'$ are unitarily equivalent if there is a unitary operator $W : \mathcal{H} \to \mathcal{H}'$ such that

$$\pi'(a) = W\pi(a)W^* \qquad \text{and} \qquad U'_g = WU_gW^*$$

for all $a\in A, g\in G$.

Suppose (π,U) and (ρ,V) are covariant representations on $\mathcal{H}$ and $\mathcal{V}$ respectively. Their direct sum $(\pi,U)\oplus(\rho,V)$ is the covariant representation $(\pi\oplus\rho, U\oplus V)$ on $\mathcal{H}\oplus\mathcal{V}$ given by $(\pi\oplus\rho)(a) \coloneqq \pi(a)\oplus\rho(a)$ and $(U\oplus V)_s \coloneqq U_s\oplus V_s$.

5.2.2 Crossed products

The crossed product $A\rtimes_\alpha G$ will be defined as the completion of $C_c(G,A)$, viewed as a $*$-algebra in a certain way, with respect to a certain norm.
First the algebra: the set of functions $G\to A$ with compact support naturally comes equipped with scalar multiplication and vectorial addition. We define the multiplication as

$$f\star g : G \to \mathbb{C} : x \mapsto \int_G f(s)\alpha_s(g(s^{-1}x))\,\mathrm{d}\mu(s)$$

and the involution $*$ by

$$f^* : x \mapsto \Delta(x^{-1})\alpha_x(f(x^{-1})^*).$$

Notice the appearance of α in the definitions.
Next we define a norm. This will be done using integrated forms. Let (π, U) be a covariant representation of a C^*-dynamical system (A, G, α) on $\mathcal{H}$. Then

$$(\pi \rtimes U)(f) \coloneqq \int_G \pi(f(r))U_r \, \mathrm{d}\mu(r)$$

defines a $*$-representation of the $*$-algebra $C_c(G, A)$ on $\mathcal{H}$, called the integrated form.
Then we can define a norm, called the universal norm, on $C_c(G, A)$ by

$$\|f\| \coloneqq \sup\{\|(\pi \rtimes U)(f)\| \mid (\pi, U) \text{ is a covariant representation of } (A, G, \alpha)\}$$

The supremum[1] is finite because $\|f\| \leq \|f\|_1$.
The completion of $C_c(G, A)$ with respect to the universal norm is called the crossed product $A \rtimes_\alpha G$.
In fact, when evaluating the supremum for the universal norm, we do not need to consider all representations of (A, G, α): Let (π, U) be a covariant representation of (A, G, α) on $\mathcal{H}$. Let

$$\mathcal{E} \coloneqq \overline{\operatorname{span}}\,\{\pi(a)h \mid a \in A; h \in \mathcal{H}\}$$

be the essential subspace of π. We call the corresponding subrepresentation $\operatorname{ess}\pi$. Because

$$U_s h = \pi(\alpha_{s^{-1}}(a))U_s\pi(a)h \qquad \forall a \in A, h \in \mathcal{H},$$

it is clear $\mathcal{E}$ is invariant under U as well. Call U' the restriction of U to $\mathcal{E}$.
Then $\|(\operatorname{ess}\pi \rtimes U')(f)\| = \|(\pi \rtimes U)(f)\|$ and so

$$\|f\| = \sup\{\|(\pi \rtimes U)(f)\| \mid (\pi, U) \text{ is a non-degenerate covariant representation of } (A, G, \alpha)\}$$

Proposition XII.115. *If (A, G, α) is a dynamical system, then the map sending a covariant pair (π, U) to its integrated form $\pi \rtimes U$ is a one-to-one correspondence between non-degenerate covariant representations of (A, G, α) and non-degenerate representations of $A \rtimes_\alpha G$. This correspondence preserves direct sums, irreducibility and equivalence.*

5.2.2.1 Universal property

In general the crossed product $A \rtimes_\alpha G$ does not contain a copy of either A or G. The multiplier algebra $M(A \rtimes_\alpha G)$ does however: There exist injective homomorphisms

$$i_A : A \to M(A \rtimes_\alpha G) \qquad i_G : G \to \mathcal{U}M(A \rtimes_\alpha G)$$

satisfying

1. $i_A(\alpha_r(a)) = i_G(r)i_A(a)i_G(r)^*$ for all $a \in A, r \in G$;
2. $A \rtimes_\alpha G = \overline{\operatorname{span}}\left\{i_A(a)\int_G f(s)i_G(s)\,\mathrm{d}\mu(s) \;\middle|\; a \in A, f \in C_c(G)\right\}$;
3. if (π, U) is a covariant representation of (G, A, α), then

$$\pi = (\pi \rtimes U) \circ i_A \qquad \text{and} \qquad U = (\pi \rtimes U) \circ i_G.$$

---

[1] One may worry we are taking the supremum over a class and not a set (the covariant representations do not form a set). Luckily the class is a subclass of the real numbers and thus a set.

This is a universal property. Suppose another C^*-algebra B and maps

$$j_A : A \to M(B) \qquad \text{and} \qquad j_G : G \to \mathcal{U}M(B)$$

satisfy these conditions, then there exists an isomorphism $\Psi : A \rtimes_\alpha G \to B$ such that

$$\Psi \circ i_A = j_A \qquad \text{and} \qquad \Psi \circ i_G = j_G.$$

The existence of such homomorphisms i_A, i_G is proved by explicitly giving them. They are defined by

$$\begin{aligned} &(i_A(a)f)(t) = af(t) \qquad && (i_G(s)f)(t) = \alpha_s(f(s^{-1}t) \\ &(fi_A(a))(t) = f(t)\alpha_t(a) \qquad && (fi_G(s))(t) = f(t^{-1}s)\Delta(s^{-1}) \end{aligned}$$

for all $f \in C_c(G, A), a \in A, t \in G$.
We can use the existence of these embeddings to prove the proposition. We need to show the existence of an inverse of the map from non-degenerate covariant representations to integrated forms.
Let ω be a non-degenerate covariant representation of $A \rtimes_\alpha G$. Because it is non-degenerate, we can extend it uniquely to a representation of $M(A \rtimes_\alpha G)$. Using i_A, i_G we can restrict this representation to a representation of A and G. Together they form a covariant representation and one can check it is exactly the original representation ω.

5.2.2.2 Induced representations

Part XIII

Functional Analysis

Chapter 1

Vector space convergence

1.1 Vector space convergence

Let $(\mathbb{F}, V, +)$ be a vector space and ξ a convergence on V. Then $(\mathbb{F}, V, +, \xi)$ is a convergence vector space if

- vector addition $+ : V \times V \to V$ is continuous;
- scalar multiplication $\cdot : \mathbb{F} \times V \to V$ is continuous.

Lemma XIII.1. *If $(\mathbb{F}, V, +, \xi)$ is a convergence vector space, then $(V, +, 0, \xi)$ is a convergence group.*

Proof. We just need to show that $v \mapsto -v$ is continuous, but this scalar multiplication and thus continuous by assumption. $\square$

Lemma XIII.2. *If $(\mathbb{F}, V, +, \xi)$ is a convergence vector space, then*

1. *the function $\mathbb{F} \to \operatorname{span}\{v\} : \lambda \mapsto \lambda \cdot v$ is a homeomorphism for all $v \in V$;*
2. *the function $V \to V : v \mapsto \lambda \cdot v$ is a homeomorphism (?) for all $\lambda \in \mathbb{F}$.*

Proof. The functions $\lambda \mapsto (\lambda, v)$ and $v \mapsto (\lambda, v)$ are continuous by VIII.63.1. Composition with the continuous scalar product gives the result by continuity of composition (VIII.43).
They are both clearly invertible (for the first, note that the kernel is $\{0\}$). The inverse of the second is of the same form and thus immediately continuous.
For the inverse of the first TODO!?! $\square$

Proposition XIII.3. *Let V be a vector space, $\{V_i\}_{i \in I}$ a set of convergence vector spaces and $\{L_i : V \to V_i\}_{i \in I}$ a set of linear maps. Then the initial convergence on V w.r.t. $\{L_i : V \to V_i\}_{i \in I}$ makes V a convergence vector space.*

Proof. Continuity of vector addition follows from VIII.239.
We verify continuity of scalar multiplication $m : \mathbb{F} \times V \to V : (\lambda, v) \mapsto \lambda v$. Using VIII.58.1, we need to verify that $L_i \circ m$ is continuous for all $i \in I$. Because the L_i are linear, we have

$$L_i(\lambda v) = \lambda L_i(v)$$

for all $\lambda \in \mathbb{F}, v \in V$. This means that $L_i \circ m = m_i \circ L_i$, where m_i is scalar multiplication in V_i, and thus continuous. So $L_i \circ m$ is continuous. $\square$

Proposition XIII.4. *Let V be a vector space over a field $\mathbb{F}$. And $\mathcal{F} \subseteq \mathcal{FP}(V)$ a family of filters. There exists a vector space convergence ξ on V such that $\mathcal{F} = \lim_\xi^{-1}(0)$* if and only if

1. *if $F \in \mathcal{F}$ and $G \supseteq F$, then $G \in \mathcal{F}$;*
2. *if $F, G \in \mathcal{F}$, then $F + G \in \mathcal{F}$;*
3. *if $F \in \mathcal{F}$, then $\mathcal{V}_\mathbb{F}(0) \cdot F \in \mathcal{F}$;*
4. *if $v \in V$, then $\mathcal{V}_\mathbb{F}(0) \cdot v \in \mathcal{F}$;*
5. *if $F \in \mathcal{F}$ and $\lambda \in \mathbb{F}$, then $\lambda \cdot F \in \mathcal{F}$.*

Note the similarity with VIII.232 for convergence groups. A group convergence is completely determined by $\lim_\xi^{-1}(0)$ due to the translation homeomorphisms VIII.231.

Proof. Assume first that $\mathcal{F} = \lim_\xi^{-1}(0)$ for some vector space convergence ξ.

1. This is just the monotonicity of the convergence.
2. If $F, G \to 0$, then $F \otimes G \to (0, 0)$ by VIII.62. By continuity of addition we have $F + G \to 0$.
3. The convergence on the scalar field is pretopological, so $\mathcal{V}_\mathbb{F}(0) \to 0$. By VIII.62 $\mathcal{V}_\mathbb{F}(0) \otimes F \to (0, 0)$ and by continuity of the scalar multiplication $\mathcal{V}_\mathbb{F}(0) \cdot F \to 0$.
4. By XIII.2.
5. By XIII.2.

Now assume the five points hold. Define the convergence ξ by $F \to v$ iff $F - v \in \mathcal{F}$. We need to show that this is a convergence and that it makes both the vector addition and scalar multiplication continuous.

Monotonicity is guaranteed by (1). To show the convergence is centered, note that $\mathcal{F} \neq \emptyset$ by (4), so long as $V \neq \emptyset$. Then for any $F \in \mathcal{F}$, $\big\{\{0\}\big\} = 0 \cdot F \in \mathcal{F}$ by (5).

To show that the vector addition is continuous, take $F \to (v_1, v_2)$. Then $p_1^{\downarrow\downarrow}(F) = F_1 \to v_1$ and $p_2^{\downarrow\downarrow}(F) = F_2 \to v_2$, i.e. $F_1 - v_1 \in \mathcal{F}$ and $F_2 - v_2 \in \mathcal{F}$. By (1), $(F_1 - v_1) + (F_2 - v_2) = (F_1 + F_2) - (v_1 + v_2) \in \mathcal{F}$, so $F_1 + F_2 \to v_1 + v_2$. Thus by VIII.61, $F_1 + F_2 = +^{\downarrow\downarrow}[F_1 \otimes F_2] \subseteq +^{\downarrow\downarrow}[F] \to v_1 + v_2$ and the addition is continuous.

Let $G \to (\lambda, v)$. Then $G_1 = p_1^{\downarrow\downarrow}(G) \to \lambda$ and $G_2 = p_2^{\downarrow\downarrow}(G) \to v$, so $G_1 \supseteq \mathcal{V}_\mathbb{F}(\lambda)$. We have

$$\begin{aligned}
\cdot^{\downarrow\downarrow}[G] - \lambda \cdot v &\supseteq \cdot^{\downarrow\downarrow}[G_1 \otimes G_2] - \lambda \cdot v = G_1 \cdot G_2 - \lambda \cdot v \\
&\supseteq \mathcal{V}_\mathbb{F}(\lambda) \cdot G_2 - \lambda \cdot v \\
&= (\mathcal{V}_\mathbb{F}(0) + \lambda) \cdot ((G_2 - v) + v) - \lambda \cdot v \\
&\supseteq \lambda \cdot (G_2 - v) + \mathcal{V}_\mathbb{F}(0) \cdot (G_2 - v) + \lambda \cdot v + \mathcal{V}_\mathbb{F}(0) \cdot v - \lambda \cdot v \\
&= \lambda \cdot (G_2 - v) + \mathcal{V}_\mathbb{F}(0) \cdot (G_2 - v) + \mathcal{V}_\mathbb{F}(0) \cdot v \in \mathcal{F}.
\end{aligned}$$

So $\cdot^{\downarrow\downarrow}[G] \to \lambda \cdot v$, making the scalar multiplication continuous. Note the last inclusion is not an equality because we go from one instance of G_2 and $\mathcal{V}_\mathbb{F}(0)$ to two! □

Proposition XIII.5. *Let initial convergence w.r.t. a set of linear functions is a vector space convergence.*

Proposition XIII.6. *Let (V, ξ) be a vector space convergence and $A \subseteq V$. Then*

1. *if A is balanced, then* $\mathrm{adh}_\xi(A)$ *is balanced;*
2. *if A is convex, then* $\mathrm{adh}_\xi(A)$ *is convex;*
3. *if A is a subspace, then* $\mathrm{adh}_\xi(A)$ *is a subspace.*

Proof. (1) We use VIII.60.2 and VIII.45 to compute

$$\begin{aligned}\overline{\mathrm{B}}(0,1)\cdot\mathrm{adh}_\xi(A) &= \cdot^{\downarrow}[\mathrm{adh}_{\mathbb{F}}(\overline{\mathrm{B}}(0,1))\times\mathrm{adh}_\xi(A)] = \cdot^{\downarrow}[\mathrm{adh}_{\mathbb{F}\otimes\xi}(\overline{\mathrm{B}}(0,1)\times A)] \\ &\subseteq \mathrm{adh}_\xi\left(\cdot^{\downarrow}\left[\overline{\mathrm{B}}(0,1)\times A\right]\right) = \mathrm{adh}_\xi(\overline{\mathrm{B}}(0,1)\cdot A) = \mathrm{adh}_\xi(A).\end{aligned}$$

Alternative proof: Take $|\lambda|\leq 1$ and $v\in\mathrm{adh}_\xi(A)$, then we need to show that $\lambda v\in\mathrm{adh}_\xi(A)$. We have $A\in\mathcal{V}_\xi(v)^{\#}$ and $A\subseteq\lambda^{-1}A$. So for all $B\in\mathcal{V}_\xi(v)$:

$$A\,\#\,B \quad\Longrightarrow\quad \lambda^{-1}A\,\#\,B \quad\Longrightarrow\quad A\,\#\,\lambda B.$$

Thus $A\in\mathcal{V}_\xi(\lambda v)^{\#}$, which is what we needed to show by VIII.20.
(2) Take $0\leq r\leq 1$ and $v,w\in\mathrm{adh}_\xi(A)$, then we need to show that $rv+(1-r)w\in\mathrm{adh}_\xi(A)$. To that end, take some arbitrary $B\in\mathcal{V}_\xi(rv+(1-r)w)$. Now $B-(rv+(1-r)w)\in\mathcal{V}_\xi(0)$, so by VIII.234 we can find a $U\in\mathcal{V}_\xi(0)$ such that $U+U\subseteq B-(rv+(1-r)w)$, which means $r\big(r^{-1}U+v\big)+(1-r)\big((1-r)^{-1}U+w\big)\subseteq B$. Now $r^{-1}U,(1-r)^{-1}U\in\mathcal{V}_\xi(0)$ because $v\mapsto\lambda\cdot v$ is a homeomorphism (XIII.2) and VIII.51, so $r^{-1}U+v\,\#\,A$ and $(1-r)^{-1}U+w\,\#\,A$. We take $v'\in r^{-1}U+v\cap A$ and $w'\in(1-r)^{-1}U+w\cap A$. Then $rv'+(1-r)w'$ is in A by convexity and in B by construction, so $A\,\#\,B$.
(3) Clearly $\mathrm{adh}_\xi(A)$ is not empty. It is then enough to verify that $\mathrm{adh}_\xi(A)+\mathrm{adh}_\xi(A)\subseteq\mathrm{adh}_\xi(A)$ and $\mathbb{F}\cdot\mathrm{adh}_\xi(A)\subseteq\mathrm{adh}_\xi(A)$. We use VIII.60.2 and VIII.45 to compute

$$\begin{aligned}\mathrm{adh}_\xi(A)+\mathrm{adh}_\xi(A) &= +^{\downarrow}[\mathrm{adh}_\xi(A)\times\mathrm{adh}_\xi(A)] = +^{\downarrow}[\mathrm{adh}_{\xi\otimes\xi}(A\times A)] \\ &\subseteq \mathrm{adh}_\xi\left(+^{\downarrow}\left[A\times A\right]\right) = \mathrm{adh}_\xi(A+A) = \mathrm{adh}_\xi(A)\end{aligned}$$

and

$$\begin{aligned}\mathbb{F}\cdot\mathrm{adh}_\xi(A) &= \cdot^{\downarrow}[\mathrm{adh}_{\mathbb{F}}(\mathbb{F})\times\mathrm{adh}_\xi(A)] = \cdot^{\downarrow}[\mathrm{adh}_{\mathbb{F}\otimes\xi}(\mathbb{F}\times A)] \\ &\subseteq \mathrm{adh}_\xi\left(\cdot^{\downarrow}\left[\mathbb{F}\times A\right]\right) = \mathrm{adh}_\xi(\mathbb{F}\cdot A) = \mathrm{adh}_\xi(A).\end{aligned}$$

□

Corollary XIII.6.1. *A hyperplane in a convergence vector space is either closed or dense.*

Proof. Let H be a hyperplane in a vector space V. Then $H\subseteq\mathrm{adh}(H)$ and $\mathrm{adh}(H)$ is a subspace. Because H is a coatom, we have either $\mathrm{adh}(H)=H$ or $\mathrm{adh}(H)=V$. In the first case H is closed, in the second dense. □

1.1.1 Algebraic convergence

Let V be a vector space over a field $\mathbb{F}$. The algebraic convergence on V is the strongest vector space convergence on V. We denote this convergence $\mathfrak{a}$ and thus write $F\xrightarrow{\mathfrak{a}}x$ and $x\in\lim_{\mathfrak{a}}F$.

Note that the algebraic convergence is not just the discrete convergence because, for example, $\langle n^{-1}v\rangle\to 0$ for all $v\in V$ by continuity of the scalar product and the fact that $\dot{v}\to v$.
We need to show that the definition makes sense.

Proposition XIII.7. *Let V be a vector space over a field $\mathbb{F}$. There exists a strongest vector space convergence on V and this convergence is defined by*

$$\forall F \in \mathcal{FP}(V): \qquad F \in \lim{}^{-1}(0) \iff \exists v \in V:\ F \supseteq \mathcal{V}_{\mathbb{F}}(0) \cdot v.$$

Proof. Using XIII.4 it is easy to see that this convergence is a vector space convergence. Conversely, this is the strongest possible convergence by point (4) of XIII.4. ☐

Note that the algebraic convergence of a non-trivial vector space is not topological!

Lemma XIII.8. *Let V be a vector space over a field $\mathbb{F}$, $v \in V$ and $A \subseteq V$ a subset. Then*

1. $$\begin{aligned} \mathcal{V}_{\mathfrak{a}}(0) &= \bigcap_{v \in V} \uparrow \mathcal{V}_{\mathbb{F}}(0) \cdot v \\ &= \{B \in \mathcal{P}(V) \mid \forall v \in V : \exists \Gamma_v \in \mathcal{V}_{\mathbb{F}}(0):\ \Gamma_v \cdot v \subseteq B\} \\ &= \left\{ \bigcup_{v \in V} \Gamma_v \cdot V \;\middle|\; \forall v \in V:\ \Gamma_v \in \mathcal{V}_{\mathbb{F}}(0) \right\}; \end{aligned}$$
2. $\operatorname{inh}_{\mathfrak{a}}(A) = \{x \in V \mid \forall v \in V : \exists \Gamma_v \in \mathcal{V}_{\mathbb{F}}(0):\ x + \Gamma_v \cdot v \subseteq A\}$;
3. $\operatorname{adh}_{\mathfrak{a}}(A) = \{x \in V \mid \forall v \in V : \forall \Gamma \in \mathcal{V}_{\mathbb{F}}(0):\ (x + \Gamma \cdot v) \mathrel{\#} A\}$.

Proof. (1) The first equality follows straight from XIII.7.
(2) We have $\operatorname{inh}_{\mathfrak{a}}(A) = \{x \mid A \in \mathcal{V}_{\mathfrak{a}}(x)\} = \{x \mid A - x \in \mathcal{V}_{\mathfrak{a}}(0)\}$. From (1) we get

$$\begin{aligned} \operatorname{inh}_{\mathfrak{a}}(A) &= \{x \mid \forall v \in V : \exists \Gamma_v \in \mathcal{V}_{\mathbb{F}}(0) : \Gamma_v \cdot v \subseteq A - x\} \\ &= \{x \mid \forall v \in V : \exists \Gamma_v \in \mathcal{V}_{\mathbb{F}}(0) : x + \Gamma_v \cdot v \subseteq A\}. \end{aligned}$$

(3) TODO ☐

1.1.1.1 The algebraic interior or core

Let V be a vector space and $A \subseteq V$ a subset. Then algebraic inherence $\operatorname{inh}_{\mathfrak{a}}(A)$ is also called the algebraic interior or core of A.

Proposition XIII.9. *Let V be a vector space and $A \subseteq V$ a subset. Then*

1. *A is absorbing* if and only if *$0 \in \operatorname{inh}_{\mathfrak{a}}(A)$;*
2. *if A is convex, then $\operatorname{inh}_{\mathfrak{a}}(A)$ is convex and open.*

Proof. (1) We have that

$$\begin{aligned} A \text{ is absorbing} &\iff \forall v \in V : \exists \epsilon > 0 :\ \mathrm{B}(0, \epsilon) \cdot v \subseteq A \\ &\iff \forall v \in V : \exists \Gamma \in \mathcal{V}_{\mathbb{F}}(0) :\ \Gamma \cdot v \subseteq A \\ &\iff 0 \in \operatorname{inh}_{\mathfrak{a}}(A). \end{aligned}$$

(2) We first show convexity: take $x, y \in \operatorname{inh}_{\mathfrak{a}}(A)$. Then there exist relevant v, w, Γ_v, Γ_w such that $x + \Gamma_v \cdot v \subseteq A$ and $y + \Gamma_w \cdot w \subseteq A$. By X.48 we have $\lambda\big(x + \Gamma_v \cdot v\big) + (1-\lambda)\big(y + \Gamma_w \cdot w\big) \subseteq A$ for all $0 \leq \lambda \leq 1$, so

$$\lambda x + (1-\lambda)y + (\Gamma_v \cap \Gamma_w)\big(\lambda v + (1-\lambda)w\big) \subseteq \lambda x + (1-\lambda)y + \Gamma_v \cdot \lambda v + \Gamma_w \cdot (1-\lambda)w \subseteq A.$$

To show $\mathrm{inh}_{\mathfrak{a}}(A)$ is open, we use VIII.32. Take $x \in \mathrm{inh}_{\mathfrak{a}}(A)$. Then for all $v \in V$ we can find a $\Gamma_v \in \mathcal{V}_{\mathbb{F}}(0)$ such that $x + \Gamma_v \cdot v \subseteq A$. This means $x + \bigcup_{v\in V} \Gamma_v \cdot v \subseteq A$. Because the convergence on $\mathbb{F}$ is topological, we may take the Γ_v open. To conclude with VIII.32 it is enough to show that $x + \bigcup_{v\in V} \Gamma_v \cdot v \subseteq \mathrm{inh}_{\mathfrak{a}}(A)$.
Pick some $y = x + c_w w \in x + \bigcup_{v\in V} \Gamma_v \cdot v \subseteq A$, meaning $c_w \in \Gamma_w$. We can find an $0 < \epsilon_w < |c_w|$ such that $c_w + \mathrm{B}(0, \epsilon_w) \subseteq \Gamma_w$ by VIII.32. Then for all $1 - \frac{\epsilon_w}{|c_w|} < \delta < 1$ we have $|c_w - \delta c_w| = (1-\delta)|c_w| < \frac{\epsilon_w}{|c_w|}|c_w| = \epsilon_w$ and so $x + \delta c_w w \in A$.
Now pick an arbitrary $u \in V$. We have $x + \Gamma_u \cdot u \subseteq A$. By convexity we have

$$\delta^{-1}(x + \delta c_w w) + (1-\delta^{-1})\big(x + \Gamma_u \cdot u\big) = x + c_w w + (1-\delta^{-1})\Gamma_u \cdot u \subseteq A.$$

This means that for all $v \in V$ we have $y + (1-\delta^{-1})\Gamma_v \cdot v \subseteq A$ and thus $y \in \mathrm{inh}_{\mathfrak{a}}(A)$. □

Proposition XIII.10. *Let V be a vector space and $A \subseteq V$ an algebraically open subset. Then*

1. *$A + U$ is algebraically open for any subspace $U \subseteq V$;*

Proof. TODO □

1.2 Functionals

Let V be a vector space over a field $\mathbb{F}$.

1. A functional on V is a map $V \to \mathbb{F}$;
2. A linear functional on V is a linear map from V to $\mathbb{F}$;
3. A real functional on V is a map $V \to \mathbb{R}$.

Lemma XIII.11. *Let V be a vector space and $U \subseteq V$ a subspace. Then U is a hyperplane* if and only if *it is the kernel of a functional.*

Lemma XIII.12. *Let $f : V \to \mathbb{F}$ be a linear functional and $x \notin \ker(f)$. Let $A \subseteq V$ be a balanced set. Then* $(x + A) \perp \ker(f)$ if and only if $A \subseteq f^{-\downarrow}(\mathrm{B}(0, |f(x)|))$.

Proof. Suppose $A \subseteq f^{-\downarrow}(\mathrm{B}(0, |f(x)|))$. Then for all $a \in A$: $f(x + a) = f(x) + f(a) \neq 0$.
Conversely, suppose $A \not\subseteq f^{-\downarrow}(\mathrm{B}(0, |f(x)|))$, i.e. there exists $a \in A$ such that $|f(a)| \geq |f(x)|$. Then $v = -\frac{f(x)}{f(a)}a \in A$, because A is balanced and so $f(x + v) = f(x) - \frac{f(x)}{f(a)}f(a) = 0$ and so $(x + A) \between \ker(f)$. □

1.2.1 Real functionals

Let V be a real or complex vector space. Let $f : V \to \mathbb{R}$ be a real functional. We say

- f is subadditive or satisfies the triangle inequality if $\forall x, y \in V : f(x + y) \leq f(x) + f(y)$;
- f is convex if $\forall x, y \in V, \lambda \in [0, 1] : f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$;
- f is positively homogeneous if $\forall x \in V, \lambda \geq 0 : f(\lambda x) = \lambda f(x)$;
- f is absolutely homogeneous if $\forall x \in V, \lambda \in \mathbb{F} : f(\lambda x) = |\lambda| f(x)$;

- f separates points if $\forall v \in V : f(v) = 0 \implies v = 0$.

We call f

- sublinear if it is subadditive and positively homogeneous;
- a seminorm if it is subadditive and absolutely homogeneous;
- a norm if it is a point-separating seminorm.

TODO general valued fields.

Lemma XIII.13. *Let V be a real or complex vector space and $f : V \to \mathbb{R}$ be a real functional. Then*

1. *absolute homogeneity $\implies$ positive homogeneity;*
2. *subadditivity+positive homogeneity $\implies$ convexity $\implies$ subadditivity.*

Thus norms and seminorms are sublinear.

Lemma XIII.14. *A subadditive, absolutely homogenous function $f : V \to \mathbb{R}$ is non-negative:*

$$f : V \to \mathbb{R}_{\geq 0}.$$

Thus norms and seminorms are functions $V \to \mathbb{R}_{\geq 0}$.

Proof. For all $v \in V$ we have $0 = f(v - v) \leq f(v) + f(-v) = 2f(v)$, so $f(v) \geq 0$. □

Proposition XIII.15 (Reverse triangle inequality)**.** *Let V be a vector space and $\|\cdot\| : V \to \mathbb{R}$ a function that satisfies the triangle inequality and has $\|-v\| = \|v\|$ for all $v \in V$. Then $\forall v, w \in V$:*

1. *$|\|v\| - \|w\|| \leq \|v - w\|$;*
2. *$|\|v\| - \|w\|| \leq \|v + w\|$.*

In particular this holds if $\|\cdot\|$ is a norm or seminorm.

Proof. We calculate $\|v\| = \|v - w + w\| \leq \|v - w\| + \|w\|$, so $\|v\| - \|w\| \leq \|v - w\|$. By swapping $v \leftrightarrow w$ we also get $-\|v\| + \|w\| \leq \|w - v\| = \|v - w\|$ and thus the first inequality is established.

For the second inequality, set $w \to -w$ and use $\|-w\| = \|w\|$. □

1.2.1.1 Epigraphs

Let V be a vector space and $f : V \to \mathbb{R}$ a real functional on V. Then epigraph of f is defined as

$$\operatorname{epi}(f) \coloneqq \{(v, r) \in V \times \mathbb{R} \mid f(v) \leq r\}.$$

Lemma XIII.16. *Let V be a vector space and $f : V \to \mathbb{R}$ a real functional on V. Then for all $v \in V$:*

$$f(v) = \inf \{r \mid (v, r) \in \operatorname{epi}(f)\}.$$

Proposition XIII.17. *Let V be a real vector space and $f : V \to \mathbb{R}$ a functional. Then*

1. *f is convex* if and only if $\mathrm{epi}(f)$ *is a convex subset of* $V \oplus \mathbb{R}$*;*
2. *f is positively homogeneous* if and only if $\mathrm{epi}(f)$ *is a cone in* $V \oplus \mathbb{R}$.

Proof. (1) First assume f convex and pick $(v,s),(w,t) \in \mathrm{epi}(f)$ and $\lambda \in [0,1]$. Then we need to show that $(\lambda v+(1-\lambda)w, \lambda s+(1-\lambda)t) \in \mathrm{epi}(f)$. This is equivalent to saying $f(\lambda v+(1-\lambda)w) \leq \lambda s + (1-\lambda)t$. Indeed we have $f(\lambda v + (1-\lambda)w) \leq \lambda f(v) + (1-\lambda)f(w) \leq \lambda s + (1-\lambda)t$ by the convexity of f.
Conversely, assume $\mathrm{epi}(f)$ convex. Then $(v, f(v)), (w, f(w)) \in \mathrm{epi}(f)$, $(\lambda v + (1-\lambda)w, \lambda f(v) + (1-\lambda)f(w)) \in \mathrm{epi}(f)$ for all $\lambda \in [0,1]$. This implies $f(\lambda v + (1-\lambda)w) \leq \lambda f(v) + (1-\lambda)f(w)$.
(2) First assume f is positively homogeneous, take $(v,s) \in \mathrm{epi}(f)$ and $r > 0$. Then we need to show that $r(v,s) = (rv, rs) \in \mathrm{epi}(f)$. This follows because of the implications $f(v) \leq s \implies rf(v) \leq rs \implies f(rv) \leq rs$.
Conversely, assume that $\mathrm{epi}(f)$ is a cone. Then $\lambda \cdot \mathrm{epi}(f) = \mathrm{epi}(f)$ for all $\lambda > 0$ by X.53. We then calculate using XIII.16:

$$\begin{aligned}
f(\lambda v) &= \inf\{r \mid (\lambda v, r) \in \mathrm{epi}(f)\} \\
&= \inf\{r \mid (\lambda v, r) \in \lambda \cdot \mathrm{epi}(f)\} \\
&= \inf\left\{r \;\middle|\; \lambda(v, \lambda^{-1}r) \in \lambda \cdot \mathrm{epi}(f)\right\} \\
&= \inf\left\{r \;\middle|\; (v, \lambda^{-1}r) \in \mathrm{epi}(f)\right\} \\
&= \inf\{\lambda r \mid (v, r) \in \mathrm{epi}(f)\} = \lambda f(v).
\end{aligned}$$

□

Corollary XIII.17.1. *A functional on a real vector space is sublinear* if and only if *its epigraph is a convex cone.*

1.2.1.2 Convex functionals

Proposition XIII.18. *Let $p : V \to \mathbb{R}$ be convex functional. Then*

$$P : V \to \mathbb{R} : x \mapsto \inf_{t>0} t^{-1}p(tx)$$

is sublinear and $P(x) \leq p(x)$.
Also, if $f : V \to \mathbb{R}$ is a linear functional, then $f \leq p \iff f \leq P$.

Proof. For sublinearity: let $x, y \in V$, then for all $s, t > 0$

$$P(x+y) \leq \frac{s+t}{st} p\left(\frac{st}{s+t}(x+y)\right) = \frac{s+t}{st} p\left(\frac{s}{s+t}(tx) + \frac{t}{s+t}(sy)\right) \leq t^{-1}p(tx) + s^{-1}p(sy).$$

This implies that $P(x+y) \leq P(x) + P(y)$.
For positive homogeneity: let $x \in V, \lambda \geq 0$

$$P(\lambda x) = \inf_{t>0} t^{-1}p(t\lambda x) = \inf_{t\lambda>0} \lambda(t\lambda)^{-1}p(t\lambda x) = \inf_{t>0} \lambda(t)^{-1}p(tx) = \lambda P(x).$$

Finally we prove that $f \leq p \implies f \leq P$ for linear functionals f. For all $t > 0$ we have $f(tx) \leq p(tx)$, which implies $f(x) = t^{-1}f(tx) \leq t^{-1}p(tx) \leq P(x)$. So $f \leq P$. □

1.2.1.3 Seminorms

Lemma XIII.19. *The kernel of a seminorm is a vector space.*

Note this does not follow from X.22 because seminorms are not linear.

Proof. Let $p : V \to \mathbb{R}$ be a seminorm. We verify the subspace criterion X.2. First $0 \in \ker(p)$ because $p(0) = p(0 \cdot 0) = |0|p(0) = 0$.
Now take $v, w \in \ker(p)$ and $\lambda \in \mathbb{F}$. Then $0 \leq p(v + \lambda w) \leq p(v) + |\lambda|p(w) = 0$, so $v + \lambda w \in \ker(p)$. ☐

Proposition XIII.20. *Let V be a vector space, $p : V \to \mathbb{R}$ a seminorm and $\lambda \in \mathbb{F}$. Then*

$$\{v \in V \mid p(v) < \lambda\} \qquad \textit{and} \qquad \{v \in V \mid p(v) \leq \lambda\}$$

are absolutely convex and absorbent.

Proof. Take $|\mu| + |\nu| \leq 1$ and $v, w \in \{v \in V \mid p(v) \leq \lambda\}$. Then $p(|\mu|v + |\nu|w) \leq |\mu|p(v) + |\nu|p(w) \leq (|\mu| + |\nu|)\lambda \leq \lambda$.
For absorbence, take $v \in V$. Then $\frac{\lambda}{2p(v)}v \in \{v \in V \mid p(v) < \lambda\}$. ☐

1.2.1.4 Gauges

Let V be a vector space and $A \subseteq V$ an absorbent subset. The function

$$p_A : V \to \mathbb{R}^{\geq 0} : v \mapsto \inf\left\{\lambda \in \mathbb{R}^{\geq 0} \;\middle|\; v \in \lambda A\right\}$$

is called the gauge or Minkowski functional of A.

The function p_A is well-defined (i.e. $p_A(v)$ is finite for all $v \in V$) because A is absorbent.

Lemma XIII.21. *Let V be a vector space, $A \subseteq V$ an absorbent subset and $\lambda \in \mathbb{R}^{>0}$. Then*

1. *$\lambda > p_A(v) \implies \lambda^{-1}v \in A$;*
2. *$\lambda^{-1}v \in A \implies \lambda \geq p_A(v)$.*

In particular, we have

$$p_A^{-\downarrow}[\mathrm{B}(0,1)] = \{v \in V \mid p_A(v) < 1\} \subseteq A \subseteq \{v \in V \mid p_A(v) \leq 1\} = p_A^{-\downarrow}[\overline{\mathrm{B}}(0,1)].$$

Proposition XIII.22. *Let V be a vector space and $A \subseteq V$ an absorbent subset. Then*

1. *$p_A^{-\downarrow}[\mathrm{B}(0,1)] = \mathrm{inh}_{\mathfrak{a}}(A)$;*
2. *$p_A^{-\downarrow}[\overline{\mathrm{B}}(0,1)] = \mathrm{adh}_{\mathfrak{a}}(A)$.*

Proof. TODO ☐

We can summarise XIII.21 and XIII.22 as

$$\mathrm{inh}_{\mathfrak{a}}(A) = \{v \in V \mid p_A(v) < 1\} \subseteq A \subseteq \{v \in V \mid p_A(v) \leq 1\} = \mathrm{adh}_{\mathfrak{a}}(A).$$

Proposition XIII.23. *Let V be a vector space and $f : V \to \mathbb{R}^{\geq 0}$ a function. Then the following are equivalent:*

1. *f is positively homogenous;*
2. *for all $A \subseteq V$ such that $f^{-\downarrow}(\mathrm{B}(0,1)) \subseteq A \subseteq f^{-\downarrow}(\overline{\mathrm{B}}(0,1))$, A is absorbent and $f = p_A$;*
3. *$f = p_A$ for some absorbent subset A.*

Note that positive homogeneity is equivalent to strictly positive homogeneity.

Proof. (1) ⇒ (2) To show absorbence, take some $v \in V$. Then for any $\epsilon > 0$, $f\big((f(v)+\epsilon)^{-1}v\big) = \frac{f(v)}{f(v)+\epsilon} < 1$, so $(f(v)+\epsilon)^{-1}v \in A$.
Now take some A and fix some arbitrary $v \in V$. We have $p_A(v) = \inf\{\lambda \in \mathbb{R}^{\geq 0} \mid v \in \lambda A\}$, so

$$\begin{aligned} p_A(v) &\leq \inf\{\lambda \in \mathbb{R}^{\geq 0} \mid v \in \lambda f^{-\downarrow}(\mathrm{B}(0,1))\} \\ &= \inf\{\lambda \in \mathbb{R}^{\geq 0} \mid v \in f^{-\downarrow}(\mathrm{B}(0,\lambda))\} \\ &= \inf\{\lambda \in \mathbb{R}^{\geq 0} \mid f(v) < \lambda\} = f(v) \end{aligned} \quad \text{and} \quad \begin{aligned} p_A(v) &\geq \inf\{\lambda \in \mathbb{R}^{\geq 0} \mid v \in \lambda f^{-\downarrow}(\overline{\mathrm{B}}(0,1))\} \\ &= \inf\{\lambda \in \mathbb{R}^{\geq 0} \mid v \in f^{-\downarrow}(\overline{\mathrm{B}}(0,\lambda))\} \\ &= \inf\{\lambda \in \mathbb{R}^{\geq 0} \mid f(v) \leq \lambda\} = f(v). \end{aligned}$$

We conclude that $f(v) = p_A(v)$.
(2) ⇒ (3) Immediate.
(3) ⇒ (1) We calculate for $t \geq 0$

$$\begin{aligned} f(tv) &= \inf\{\lambda \in \mathbb{R}^{\geq 0} \mid tv \in \lambda A\} = \inf\{\lambda \in \mathbb{R}^{\geq 0} \mid v \in t^{-1}\lambda A\} \\ &= \inf\{t\lambda \in \mathbb{R}^{\geq 0} \mid v \in \lambda A\} = t\inf\{\lambda \in \mathbb{R}^{\geq 0} \mid v \in \lambda A\} = tf(v). \end{aligned}$$

□

Lemma XIII.24. *Let V be a vector space, $A \subseteq V$ an absorbent subset and $a \in A$. If there exists a subspace $U \subseteq A$ such that $a \in U$, then $p_A(a) = 0$.*

Proof. For all $\epsilon > 0$, $\epsilon^{-1}a \in A$, so $a \in \epsilon A$. □

Proposition XIII.25. *Let V be a vector space and $A \subseteq V$ an absorbent subset. Then*

1. *p_A is absolutely homogenous if A is balanced;*
2. *p_A is subadditive if A is convex;*
3. *p_A is point-separating if A is balanced and contains only the trivial subspace.*

Proof. (1) By X.45 we have $\mu A = |\mu| A$ and thus

$$\begin{aligned} p_A(\mu \cdot v) &= \inf\{\lambda \in \mathbb{R}^{\geq 0} \mid \mu \cdot v \in \lambda A\} = \inf\left\{\lambda \in \mathbb{R}^{\geq 0} \;\middle|\; v \in \frac{\lambda}{\mu} A\right\} \\ &= \inf\left\{\lambda \in \mathbb{R}^{\geq 0} \;\middle|\; v \in \frac{\lambda}{|\mu|} A\right\} = \inf\{|\mu|\lambda \in \mathbb{R}^{\geq 0} \mid v \in \lambda A\} = |\mu| \cdot p_A(v). \end{aligned}$$

(2) Take $v, w \in V$. Now take arbitrary $\epsilon > 0$, so $(p_A(v)+\epsilon)^{-1}v \in A$ and $(p_A(w)+\epsilon)^{-1}w \in A$ by XIII.21. By convexity of A, we have

$$\frac{v+w}{p_A(v)+p_A(w)+2\epsilon} = \frac{p_A(v)+\epsilon}{p_A(v)+p_A(w)+2\epsilon}(p_A(v)+\epsilon)^{-1}v + \frac{p_A(w)+\epsilon}{p_A(v)+p_A(w)+2\epsilon}(p_A(w)+\epsilon)^{-1}w \in A.$$

By XIII.21 this means $p_A(v)+p_A(w)+2\epsilon \geq p_A(v+w)$ and because ϵ was arbitrary, we conclude that $p_A(v+w) \leq p_A(v)+p_A(w)$.
(3) Assume A contains only the trivial subspace. Then for all $v \in V$ there exists some $\lambda \in \mathbb{F}$ such that $\lambda \cdot v \notin A$. Now for all $|c| \geq |\lambda|$, $c \cdot v \notin A$ because A is balanced. Then $p_A(2\lambda \cdot v) \neq 0$ and because p_A is absolutely homogeneous we have $p_A(v) = (2\lambda)^{-1}p_A(2\lambda \cdot v) \neq 0$. □

Corollary XIII.25.1. *The gauge of an absolutely convex and absorbent subset is a seminorm. If the subset contains only the trivial subspace, then the gauge is a norm.*

Proposition XIII.26. *Let V be a vector space, $A, B \subseteq V$ absolutely convex and absorbent subsets and $\mathcal{E}$ a set of absolutely convex and absorbent subsets.*

1. *For all $\lambda \in \mathbb{F} \setminus \{0\}$, the gauge of λA is $|\lambda|^{-1} p_A$.*
2. *The gauge of $\bigcap \mathcal{E}$ is $\sup \{p_K \mid K \in \mathcal{E}\}$.*
3. *If $A \subseteq B$, then $p_B \leq p_A$.*

1.2.2 Hahn-Banach extension theorems

Theorem XIII.27 (Hahn-Banach majorised by convex functionals)**.** *Let V be a real vector space, $U \subset V$ a subspace and p a convex functional on V. Let $f : U \to \mathbb{R}$ be a linear functional that is bounded by p:*

$$\forall u \in U : \quad f(u) \leq p(u).$$

Then f has an extension $\tilde{f} : V \to \mathbb{R}$ such that $\tilde{f}$ is a linear functional on V bounded by p:

$$\forall v \in V : \tilde{f}(v) \leq p(v) \qquad \textit{and} \qquad \forall u \in U : \tilde{f}(u) = f(u).$$

Proof. As a first step, we want to extend f to a functional g on a space that is one dimension larger than U. This means g is of the form

$$g : U \oplus \operatorname{span}\{v_1\} \to \mathbb{R} : v + \alpha v_1 \mapsto f(v) + \alpha c$$

for some $v_1 \in V \setminus U$.
If we want g to be majorised by p, then we need to find a c such that

$$\forall v \in U : \forall \alpha \in \mathbb{R} : \ g(\alpha v_1 + v) = \alpha c + f(v) \leq p(\alpha v_1 + v)$$

this means that we need

$$\forall v \in U : \forall \alpha \in \mathbb{R} : \ \frac{-p(v - |\alpha| v_1) + f(v)}{|\alpha|} \leq c \leq \frac{p(v + |\alpha| v_1) - f(v)}{|\alpha|}$$

and we can find such a c if and only if

$$\forall v \in U : \forall \alpha \in \mathbb{R} : \ -p(v - |\alpha| v_1) + f(v) \leq p(v + |\alpha| v_1) - f(v),$$

which is equivalent to $2f(v) \leq p(v + |\alpha| v_1) + p(v - |\alpha| v_1)$. This follows from

$$\begin{aligned} f(v) \leq p(v) &= p(\tfrac{1}{2}(v + |\alpha| v_1) + \tfrac{1}{2}(v - |\alpha| v_1)) \\ &\leq \tfrac{1}{2} p(v + |\alpha| v_1) + \tfrac{1}{2} p(v - |\alpha| v_1). \end{aligned}$$

So we can extend the domain of f by one dimension such that it is still majorised by p.
An extension by multiple dimensions is determined by a subset of $V \times \mathbb{R}$. Consider the family of all such subsets that determine a majorised extension of f. This is a family of finite character. We apply the Teichmüller-Tukey lemma, I.201, to obtain a maximal element.
This maximal element has domain V, because if it did not, it could be extended and was not a maximal element. ☐

Clearly if V has a well-ordered Hamel basis, we do not need choice as we can just take successive vs in the basis and find cs constructively.

Corollary XIII.27.1 (Hahn-Banach majorised by sublinear functionals)**.** *Any majorant p that is sublinear is also convex and can be used in the Hahn-Banach theorem.*

Corollary XIII.27.2 (Hahn-Banach majorised by seminorms)**.** *Let $(\mathbb{F}, V, +)$ be a real or complex vector space, $U \subset V$ a subspace and p a seminorm on V. Let $f : U \to \mathbb{F}$ be a linear functional that is bounded by p:*

$$\forall u \in U : \quad |f(u)| \leq p(u).$$

Then f has an extension $\tilde{f} : V \to \mathbb{R}$ such that $\tilde{f}$ is a linear functional on V bounded by p:

$$\forall v \in V : |\tilde{f}(v)| \leq p(v) \qquad \textit{and} \qquad \forall u \in U : \tilde{f}(u) = f(u).$$

Proof. For <u>real</u> vector fields, we notice that every seminorm is a sublinear function, so we can use XIII.27.1 to find an extension $\tilde{f}$. We then just need to check it satisfies $\forall v \in V : |\tilde{f}(v)| \leq p(v)$. From XIII.27.1 we know $\forall v \in V : \tilde{f}(v) \leq p(v)$. To prove $-\tilde{f}(v) \leq p(v)$, we calculate

$$-\tilde{f}(v) = \tilde{f}(-v) \leq p(-v) = |-1|p(v) = p(v).$$

For <u>complex</u> vector fields, we can write $f = f_1 + if_2$ with f_1, f_2 real functionals on U, which can also be seen as a real vector space. First take f_1. Now $\forall u \in U f_1(u) \leq |f(x)| \leq p(x)$, so we can extend f_1 to $\tilde{f}_1$ by XIII.27.1.
Now by complex linearity, $if(u) = f(iu)$ so

$$i[f_1(u) + if_2(u)] = -f_2(u) + if_1(u) = f_1(iu) + if_2(iu) \implies f_2(u) = -if_1(iu).$$

So we set $\tilde{f}(v) = \tilde{f}_1(v) - i\tilde{f}_1(iv)$. It is easy to show $\tilde{f}$ is $\mathbb{C}$-linear. For boundedness, write $\tilde{f}(v) = |\tilde{f}(v)|e^{i\theta}$ then

$$|\tilde{f}(v)| = e^{-i\theta}\tilde{f}(v) = \tilde{f}(e^{-i\theta}v) = \tilde{f}_1(e^{-i\theta}v) \leq p(e^{-i\theta}v) = |e^{-i\theta}|p(v) = p(v).$$

□

Corollary XIII.27.3. *Let X be a normed space and $Z \subset X$ a subspace. Any bounded linear functional in Z' can be extended to a bounded linear functional in X' with the same norm.*

Proof. Let $f : Z \to \mathbb{F}$ be such a functional. Extend f by the previous theorem, XIII.27.2, using $p(x) = \|f\|_Z \|x\|$. □

Corollary XIII.27.4. *Let X be a normed space and $x_0 \neq 0$ an element of X. Then there exists a bounded linear functional ω_{x_0} such that*

$$\|\omega_{x_0}\| = 1 \qquad \textit{and} \qquad \omega_{x_0}(x_0) = \|x_0\|.$$

Proof. Extend the functional $f : \operatorname{span}\{x_0\} \to \mathbb{F}$ defined by

$$f(x) = f(ax_0) = a\|x_0\|.$$

□

Corollary XIII.27.5. *Let X be a normed space. Then $\forall x \in X$:*

$$\|x\| = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|}.$$

Proof. We calculate

$$\|x\| \geq \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|} \geq \frac{|\omega_x(x)|}{\|\omega_x\|} = \frac{\|x\|}{1} = \|x\|$$

where the first inequality follows from $|f(x)| \leq \|f\|\|x\|$ for all $f \in X', x \in X$. □

1.2.2.1 Hahn-Banach separation

Lemma XIII.28. *Let V be a real vector space, A an absorbent set and $x_0 \notin A$. Consider the functional $f_{x_0} : \operatorname{span}\{x_0\} \to \mathbb{F} : tx_0 \mapsto t$. Then $f_{x_0}(x) \leq p_A(x)$ for all $x \in \operatorname{span}\{x_0\}$.*

Proof. Let $x = tx_0$. If $t \leq 0$, then the inequality is immediate. Suppose $t > 0$. Because $p_A(x_0) \geq 1$ (by the converse of XIII.21), we have

$$f_{x_0}(x) = f_{x_0}(tx_0) = t \leq tp_A(x_0) = p_A(tx_0) = p_A(x)$$

using positive homogeneity (XIII.23). □

Proposition XIII.29. *Let V be a real or complex vector space and A an algebraically open and absolutely convex set. If U is a subspace such that $A \perp U$, then there exists a hyperplane $H \supseteq U$ such that $A \perp H$.*

Proof. Take $a \in A$. Then $0 \in a - A = \operatorname{inh}_{\mathfrak{a}}(a - A)$, so $a - A$ is absorbing by XIII.9 and $a - A + U$ is also absorbing.
Then we have

$$U \perp A \implies 0 \notin U - A \implies a \notin a - A + U.$$

Consider the functional f_a of XIII.28, which is majorised by the gauge p_{a-A+U}. Then f_a can be extended to all V by the Hahn-Banach extension theorem XIII.27.2, because p_{a-A+U} is a seminorm.
We note that $U \subseteq \ker(f_a)$, because $p_{a-A+U}(u) = 0$ by XIII.24.
In order to conclude with XIII.12, we need to show that $A - a \subseteq f_a^{-\downarrow}(\mathrm{B}(0, |f_a(a)|)) = f_a^{-\downarrow}(\mathrm{B}(0, 1))$.
Indeed $A - a \subseteq U + A - a = \operatorname{inh}_{\mathfrak{a}}(U + A - a) = p_{U+A-a}^{-\downarrow}[\mathrm{B}(0, 1)] \subseteq f_a^{-\downarrow}[\mathrm{B}(0, 1)]$ by XIII.10 XIII.22. □

Proposition XIII.30. *Let V be a real or complex vector space, $a \in V$ and A a subset such that $A - a$ is absolutely convex and absorbent. If U is a subspace such that $A \perp U$, then there exists a hyperplane $H \supseteq U$ such that $A \perp H$.*

Proof. Consider the set $U + A - a$. This is absolutely convex and absorbent.
Then we have

$$U \perp A \implies 0 \notin U + A \implies -a \notin U + A - a \implies a \notin U + A - a.$$

Consider the functional f_a of XIII.28, which is majorised by the gauge p_{U+A-a}. Then f_a can be extended to all V by the Hahn-Banach extension theorem XIII.27.2.
We note that $U \subseteq \ker(f_a)$, because $p_{U+A-a}(u) = 0$ by XIII.24.
In order to conclude with XIII.12, we need to show that $A - a \subseteq f_a^{-\downarrow}(\mathrm{B}(0, |f_a(a)|)) = f_a^{-\downarrow}(\mathrm{B}(0, 1))$.
Indeed $A - a \subseteq U + A - a \subseteq p_{U+A-a}^{-\downarrow}[\overline{\mathrm{B}}(0, 1)] \subseteq f_a^{-\downarrow}[\overline{\mathrm{B}}(0, 1)]$. □

Proposition XIII.31. *Let A be a open convex subset of a locally convex TVS and M a vector subspace such that $A \perp M$. Then there exists a closed hyperplane $H \supseteq M$ such that $A \perp H$.*

1.2.2.2 Banach limits

Proposition XIII.32. *There exists a linear map $L : l^\infty(\mathbb{N}) \to \mathbb{C}$ satisfying*

1. $L(x) = \lim_{n\to\infty} x_n$ *if the limit exists;*
2. $L((x_{n+1})_{n\in\mathbb{N}}) = L((x_n)_{n\in\mathbb{N}})$*;*
3. *if* $\forall n \in \mathbb{N} : x_n \geq 0$*, then* $L(x) \geq 0$*;*
4. $\|L\| = 1$.

Such a linear map is called a Banach limit.

Proof. TODO, after Cesàro means. ☐

1.3 Topological vector spaces

A topological vector space (or TVS) is a convergence vector space that is topological.

As with convergence groups, any pretopological vector space convergence is topological, see VIII.236.

1.3.1 Neighbourhoods and base

Proposition XIII.33. *Let V be a vector space and $N \in \mathcal{FP}(V)$. Then $N = \mathcal{N}_\xi(0)$ for some topological convergence on V* if and only if

1. *for all $A \in N$ and $\lambda \in \mathbb{F}$: $\lambda A \in N$;*
2. *for all $A \in N$, there exists some $B \in N$ such that $B + B \subseteq A$;*
3. *each $A \in N$ is absorbent;*
4. *N has a balanced base.*

Proof. We adapt XIII.4 to the present situation.
First assume N has a balanced and absorbent base. We check the five conditions for $\mathcal{F} = \dot{N}$.

1. Immediate because $\mathcal{F} = \dot{N}$.
2. Take $F, G \in \dot{N}$. We need to show that $\uparrow(F + G) \supseteq N$, which means that for all $A \in N$ there exist $B \in F$ and $C \in G$ such that $B + C \subseteq A$. We can take $B = C$ equal to the B of point (2).
3. Take $F \in \dot{N}$. We need to show that $\uparrow(\mathcal{V}_\mathbb{F}(0) \cdot F) \supseteq N$, which means that for all $A \in N$ there exists a $\Gamma \in \mathcal{V}_\mathbb{F}(0)$ and $B \in F$ such that $\Gamma \cdot B \subseteq A$. We can take $B = \text{balcore}(A) \in N \subseteq F$ and $\Gamma = B(0, 1)$.

4. Take $v \in V$. We need to show that all $A \in N$ contain $\Gamma \cdot v$ for some $\Gamma \in \mathcal{V}_{\mathbb{F}}(0)$. Because A is absorbent, there exists an $r > 0$ such that $v \in cA$ for all $|c| \geq r$. Conversely $c^{-1}v \in A$ for all $|c^{-1}| \leq r^{-1}$. So $\mathrm{B}(0, r^{-1}) \cdot v \subseteq A$ and $B(0, r^{-1}) \in \mathcal{V}_{\mathbb{F}}(0)$.
5. Take $F \in \dot{N}$ and $\lambda \in \mathbb{F}$. We need to show that for all $A \in N$ there exists a $B \in F$ such that $\lambda \cdot B \subseteq A$. We can take $B = \lambda^{-1}A \in N \subseteq F$.

Now assume ξ is a topological vector space convergence and $N = \mathcal{N}_\xi(0)$.

1. By point (5) of XIII.4, we have that for all $\lambda \in \mathbb{F}$, $\lambda \cdot N \supseteq N$, so for all $A \in N$, there exists a $B \in N$ such that $A = \lambda B$. This means $\lambda^{-1}A \in N$ for all $\lambda \in \mathbb{F}, A \in N$.
2. By VIII.234;
3. For absorbence, take $A \in N$ and $v \in V$. Then there exists a $\Gamma \in \mathcal{V}_{\mathbb{F}}(0)$ such that $\Gamma \cdot v \subseteq A$. Now we can find a $r > 0$ such that $\mathrm{B}(0, r) \subseteq \Gamma$, so for all $|c| \geq r^{-1}$ we have $v \in cA$.
4. By point (3) of XIII.4, $\mathcal{V}_{\mathbb{F}}(0) \cdot N \supseteq N$. Take $A \in N$. Then there exists a $\Gamma \in \mathcal{V}_{\mathbb{F}}(0)$ and $B \in N$ such that $\Gamma \cdot B \subseteq A$. We can find sume ball $\mathrm{B}(0, \epsilon) \subseteq \Gamma$, so $\mathrm{B}(0, 1) \cdot \epsilon^{-1}B \subseteq \epsilon^{-1}B \subseteq A$. Thus $\epsilon^{-1}B$ is balanced and a neighbourhood by point(1). So every $A \in N$ contains a balanced set in N.

□

1.3.2 Continuity

Lemma XIII.34. *Let V be a TVS and W a normed space. A linear function $f : V \to W$ is continuous* if and only if *there exists a neighbourhood $U \in \mathcal{N}_V(0)$ such that f is bounded on U.*

Proof. Assume f continuous, then $\mathrm{B}(0, 1) \in \mathcal{N}_W(0) = \mathcal{N}_W(f(0))$ implies $U = f^{-\downarrow}(\mathrm{B}(0, 1)) \in \mathcal{N}_V(0)$ by VIII.46 and f is bounded on U by construction.
Now assume there exists a neighbourhood $U \in \mathcal{N}_V(0)$ such that f is bounded on U. Then $f^{\downarrow}(U) \subseteq \mathrm{B}_W(0, C)$. It is then enough to show that for all $\mathrm{B}_W(0, \epsilon)$, $f^{-\downarrow}(\mathrm{B}_W(0, \epsilon)) \in \mathcal{N}_V(0)$. Indeed

$$f^{-\downarrow}(\mathrm{B}_W(0, \epsilon)) = \frac{\epsilon}{C} f^{-\downarrow}(\mathrm{B}_W(0, C)) = \frac{\epsilon}{C} U \in \mathcal{N}_V(0).$$

□

Proposition XIII.35. *Let V be a TVS and $f : V \to \mathbb{F}$ a linear functional on V. Then the following are equivalent:*

1. *f is continuous;*
2. *there exists a neighbourhood $U \in \mathcal{N}_V(0)$ such that f is bounded on U*
3. $\ker(f)$ *is closed;*
4. $\ker(f)$ *is not dense.*

Proof. (1) ⇔ (2) By XIII.34.
(1) ⇒ (3) Because $\ker(f) = f^{-\downarrow}(\{0\})$, it is closed by VIII.138.
(3) ⇒ (1) Now assume $\ker(f)$ closed. If $\ker(f) = V$, then f is constant and thus continuous by VIII.48. If $\ker(f) \neq V$, we can find some some $x \in \ker(f)^c$, which is open. Thus $\ker(f)^c - x$ is a neighbourhood of the origin, meaning we can take a balanced subset A by XIII.33. Now

$(x+A) \perp \ker(f)$ by construction, so f is bounded on A by XIII.12 and thus f is continuous by XIII.34.
(2) $\Leftrightarrow$ (3) By XIII.11 $\ker(f)$ is a hyperplane and by XIII.6.1 this hyperplane is either closed or dense. ☐

TODO???

Proposition XIII.36. *Let V and W be TVSs and $f : V \to W$ a linear function.*

1. *If f is continuous and W is Hausdorff, then $\ker(f)$ is closed.*
2. *If f has closed kernel and finite-dimensional image, then f is continuous.*

Proof. (1) Because W is Hausdorff, it is also T_1 and thus $\{0\}$ is closed by VIII.75. Then $\ker(f) = f^{-\downarrow}(\{0\})$ is closed by VIII.138.
(2) ☐

??

1.3.3 Locally convex convergence

> Let (V, ξ) be a convergence vector space. If ξ is based in the convex sets, then ξ is called locally convex.

Lemma XIII.37. *TODO? Every locally convex convergence vector space is topological.*

Proof. Cfr. XIII.9 ☐

Lemma XIII.38. *Let (V, ξ) be a TVS. Then the following are equivalent:*

1. *ξ is locally convex;*
2. *$\mathcal{N}_\xi(0)$ is based in the convex sets;*
3. *$\mathcal{N}_\xi(0)$ is based in the absolutely convex sets.*

Proof. (1) $\Leftrightarrow$ (2) One direction is immediate, for the other it is enough to note that if U is convex, then so is the translated set $x + U$ for all $x \in V$.
(2) $\Leftrightarrow$ (3) One direction is immediate, the other follows because the balanced core of a convex set is convex by X.46. ☐

Proposition XIII.39. *Let V be a vector space and $N \in \mathcal{FP}(V)$. Then $N = \mathcal{N}_\xi(0)$ for some topological convergence on V* if and only if

1. *for all $A \in N$ and $\lambda \in \mathbb{F}$: $\lambda A \in N$;*
2. *each $A \in N$ is absorbent;*
3. *N has an absolutely convex base.*

Proof. This almost completely follows from XIII.33 and XIII.38. We just need to show that for all $A \in N$, there exists some $B \in N$ such that $B + B \subseteq A$. We may take $B = \frac{1}{2}A'$, where A' is a convex subset of A, because for all $v, w \in A'$ we have $\frac{1}{2}v + \frac{1}{2}w \in A'$ by convexity. ☐

1.3.3.1 Seminorms and gauges

Proposition XIII.40. *Let (V, ξ) be a TVS and $p_K : V \to \mathbb{R}^{\geq 0}$ the gauge of some absorbent set K. Then the following are equivalent:*

1. *p_K is continuous;*
2. *p_K is continuous at 0;*
3. *$K \in \mathcal{N}_\xi(0)$.*

Proof. (1) ⇒ (2) Immediate.
(2) ⇒ (3) We have $p_K^{-\downarrow}[\mathrm{B}(0,1)] \subseteq K$ by XIII.23. If p_K is continuous at 0, then K is a neighbourhood of 0 by VIII.46.
(3) ⇒ (1) Assume K a neighbourhood of 0 and take some neighbourhood Γ of 0 in $\mathbb{R}$. Then Γ contains a ball $\mathrm{B}(0,\epsilon)$. By VIII.46, it is enough to show that $p_K^{-\downarrow}[\mathrm{B}(0,\epsilon)]$ is a neighbourhood. Indeed

$$p_K^{-\downarrow}[\mathrm{B}(0,\epsilon)] \supseteq p_K^{-\downarrow}\left[\overline{\mathrm{B}}\left(0, \frac{\epsilon}{2}\right)\right] = p_K^{-\downarrow}\left[\frac{\epsilon}{2}\,\overline{\mathrm{B}}(0,1)\right] = \frac{\epsilon}{2} p_K^{-\downarrow}[\overline{\mathrm{B}}(0,1)] \supseteq \frac{\epsilon}{2} K,$$

and the result follows because $\frac{\epsilon}{2}K$ is a neighbourhood of 0 by XIII.33. □

Proposition XIII.41. *Let V be a vector space. A topology on V is locally convex* if and only if *it is the initial topology w.r.t. some set S of seminorms on V.*

Proof. Assume V is a locally convex TVS. Let $\mathcal{B}$ be a locally convex base of $\mathcal{N}(0)$. Define $S = \{p_K \mid K \in \mathcal{B}\}$. By XIII.40 all seminorms in S are continuous.
By VIII.59 and XIII.40 each W in the neighbourhood of the origin in the initial topology w.r.t. S contains a scaled version of some K in $\mathcal{B}$. So the topology on V is coarser than the initial topology, meaning they are equal.
TODO converse. □

Note: metrisable is not equivalent to normable!

1.4 General duality theory

1.4.1 Paired spaces

A pairing is a triple (V, W, b) where V, W are vector spaces over $\mathbb{F}$ and $b : V \times W \to \mathbb{F}$ is a bilinear form. Often we will write the pairing as just (V, W).
We say W distinguishes points of V or is separating on V if

$$\forall v \in V : \exists w \in W : b(v, w) \neq 0.$$

A dual system, dual pair or duality over a field $\mathbb{F}$ is a pairing (V, W, b) such that V distinguishes points of W and W distinguishes points of V.

Lemma XIII.42. *Let (V, W, b) be a pairing. The curried map $w \mapsto b(\cdot, w)$ is injective* if and only if *V distinguishes points of W.*

In this case W is isomorphic with a space of linear functionals (the image of the curried function), so we can also say a dual system is a pair (V, W) where W is a space of linear functionals on V that distinguishes points of V.

Example

- Let V be a vector space. Then (V, V^*, b) with $b : V \times V^* : (v, f) \mapsto f(v)$ is a dual pair.
- Let V be a locally convex Hausdorff space. Hahn-Banach implies (V, V') is a dual pair.

1.4.2 Weak topologies

Let (X, Y, b) be paired vector spaces. Then for each $y \in Y$, the map

$$p_y : X \to \mathbb{R}_{\geq 0} x \mapsto |b(x, y)|$$

determines a seminorm on X.
The weakest topology on X for which the seminorms $\{p_y \mid y \in Y\}$ are continuous is called the weak topology $\sigma(X, Y)$ on X for the pair (X, Y).

Proposition XIII.43. *Let (X, Y, b) be a pairing. The following are equivalent:*

1. *X distinguishes points of Y;*
2. *the map $Y \to X^* : y \mapsto y^*$ is injective, where y^* is defined by*

$$y^* : X \to \mathbb{F} : x \mapsto b(x, y);$$

3. *$\sigma(Y, X)$ is Hausdorff.*

Proof. TODO ☐

1.4.2.1 Weak-$*$ topology

Proposition XIII.44. *Let X be a Banach space and let X' have the weak-$*$ topology. Then a linear functional $\theta : X' \to \mathbb{C}$ is continuous* if and only if

$$\exists x \in X : \forall \omega \in X' : \quad \theta(\omega) = \omega(x).$$

Proof. TODO 9.2 in lecture notes. ☐

1.4.3 Mackey topology

Theorem XIII.45 (Mackey-Arens)**.**

1.5 Operators on topological vector spaces

1.5.1 Continuous operators

1.5.1.1 Closed graph theorem

Theorem XIII.46 (Closed graph theorem)**.** *Let $f : X \to Y$ be a map from a topological space X into a Hausdorff space Y.*

1. *if f is continuous, then f has closed graph;*
2. *if X is compact, then the converse also holds.*

Proof. TODO ☐

TODO: for Banach spaces X complete enough!!!!!!

1.5.2 Compact operators

A linear operator $T : X \to Y$ between TVSs is compact if it maps a neighbourhood of the origin to a precompact set, i.e.

$$\exists U \in \mathcal{N}(0) : \overline{T[U]} \text{ is compact.}$$

The set of compact linear operators in $(X \to Y)$ is denoted $\mathcal{K}(X, Y)$.

TODO: doesn't the neighbourhood need to be bounded in some way?????

Proposition XIII.47. *Let X be a normed space and Y a TVS and $T : X \to Y$ a linear operator. Then the following are equivalent:*

1. *T is a compact operator;*
2. *there exists a neighbourhood $U \subset X$ of the origin and a compact set $V \subset Y$ such that $T[U] \subset V$;*
3. *the image of the unit ball of X, $T[B(\mathbf{0}, 1)]$, is precompact in Y;*
4. *the image of any bounded set in X is precompact in Y.*

If Y is a normed space, these are also equivalent to

5. *for any bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, the sequence $(Tx_n)_{n\in\mathbb{N}}$ contains a converging subsequence.*

Proof. TODO ☐

Lemma XIII.48. *Let X, Y be TVSs.*

1. *Then $\mathcal{K}(X, Y)$ is a vector space.*
2. *If X, Y are normed spaces, then $\mathcal{K}(X, Y)$ is a subspace of $\mathcal{B}(X, Y)$.*

Proof. (1) Let $K, K' : X \to Y$ be compact operators. Then, by VIII.230 (TODO opposite inclusion!),

$$\overline{K[B(0,1)] + K'[B(0,1)]} \subseteq \overline{K[B(0,1)]} + \overline{K'[B(0,1)]}, \qquad \overline{K[\lambda B(0,1)]} = \lambda \overline{K[B(0,1)]}.$$

(2) Let $K \in \mathcal{K}(X, Y)$. Then the image of the unit ball is precompact, meaning it is bounded. So K is bounded by X.87. ☐

Lemma XIII.49. *Let $T : V \to W$ be a bounded operator. If W has the Heine-Borel property, then T is compact.*

Proof. The set $T[B(\mathbf{0},1)]$ is bounded because T is. By the Heine-Borel (TODO ref) property of W, $\overline{B(\mathbf{0},1)}$ is compact. □

Corollary XIII.49.1. *Bounded operators with as image a finite dimensional normed space are compact.*

Corollary XIII.49.2. *The identity on a normed space X is compact* if and only if *X is finite-dimensional.*

Proof. TODO ref. □

Proposition XIII.50. *Compact operators map weakly convergent sequences to strongly convergent sequences. TODO! + remove from Hilbert section.*

Corollary XIII.50.1. *Let V be an inner product space and $\langle e_n \rangle$ a sequence of orthonormal vectors in V. If K is a compact operator, then $\lim_{n\to\infty} Ke_n = 0$.*

Proof. Any sequence of orthonormal vectors $\langle e_n \rangle$ converges weakly to 0. Because K is compact, $\langle Ke_n \rangle$ converges strongly to zero. TODO ref. □

Corollary XIII.50.2. *If V is infinite-dimensional and K is invertible, then its inverse is unbounded.*

Proof. Due to $\lim_{n\to\infty} Ke_n = 0$ the operator K cannot be bounded below, so K^{-1} is not bounded by X.91. □

1.6 Continuity

`https://en.wikipedia.org/wiki/Bilinear_map#Continuity_and_separate_continuity`

Chapter 2

Functionals on vector spaces

Theorem XIII.51 (Riesz-Markov-Kakutani representation theorem)**.** *Let X be a locally compact Hausdorff space. For any positive linear functional ψ on $C_c(X)$, there is a unique Radon measure μ on X such that*

$$\forall f \in C_c(X) : \quad \psi(f) = \int_X f(x)\,\mathrm{d}\mu(x).$$

2.1 Algebraic duality

2.1.1 Linear functionals

Let V be a vector space over a field $\mathbb{F}$.
The (algebraic) dual of V, denoted V^*, is the vector space of all linear functionals on V.

$$V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}).$$

Proposition XIII.52. *Let V be a vector space. Then $\dim V^* \geq \dim V$ and*

$$\dim V^* = \dim V \iff V \text{ is finite-dimensional.}$$

If V is finite-dimensional with a basis $v_1, \ldots, v_n$, then the dual basis $\varphi_1, \ldots, \varphi_n$ is the set of linear functionals on V such that

$$\varphi_j(v_k) = \begin{cases} 1 & (k = j), \\ 0 & (k \neq j) \end{cases}.$$

This dual basis is indeed a basis of V^*.

Proof. We first assume V is finite-dimensional and prove the dual basis is a basis, which proves $\dim V^* = \dim V$. We then assume V is infinite-dimensional and prove $\dim V^* \neq \dim V$.[1]

1. Assume V is finite-dimensional. To show the dual basis spans V^*, take a linear functional φ. Now define $a_i = \varphi(v_i)$. It is clear that $\varphi = \sum_{i=1}^n a_i\varphi_i$. To show linear independence, take a combination

   $$b_1\varphi_1 + \ldots + b_n\varphi_n = 0.$$

   Filling in all basis vectors v_i in turn, gives $b_i = 0$ for all i.

---

[1] Reference: `https://mathoverflow.net/questions/13322/slick-proof-a-vector-space-has-the-same-dimension-as-`

2. Assume V is infinite-dimensional. At first let us assume $\dim_{\mathbb{F}} V \geq |\mathbb{F}|$. Then we can apply lemma X.12 to obtain $\dim_{\mathbb{F}} V = |V|$. Let β be a basis for V. The elements of V^* correspond bijectively to functions from β to $\mathbb{F}$. Thus

$$|V^*| = |\mathbb{F}^\beta| = |\mathbb{F}|^{|\beta|} > |\beta| = |V|.$$

Now we relax the condition $\dim_{\mathbb{F}} V \geq |\mathbb{F}|$. We first note that every field contains a subfield that is at most denumerable. Take such a field $K \subset \mathbb{F}$. We introduce the new vector space $W = \operatorname{span}_K(\beta)$. Every functional from W to K extends to a functional from V to $\mathbb{F}$. Hence

$$\dim_{\mathbb{F}} V = \dim_K W < \dim_K W^* \leq \dim_{\mathbb{F}} V^*$$

using $\dim_K W \geq |K| \geq \aleph_0$.

□

Proposition XIII.53. *Let* $f \in \operatorname{Hom}(V,W)$ *and* $\mathcal{V}, \mathcal{W}$ *bases of* V, W. *The*

$$(f^*)^{\mathcal{V}^*}_{\mathcal{W}^*} = ((f)^{\mathcal{W}}_{\mathcal{V}})^{\mathrm{T}}.$$

2.1.1.1 Annihilator subspace

Let $U \subset V$ be a subspace. The <u>annihilator</u> of U, denoted U^0, is the set of functionals that are identically zero on U:

$$U^0 = \{\varphi \in V^* \mid \forall u \in U : \varphi(u) = 0\}.$$

Proposition XIII.54. *Let* $U \subset V$ *be a subspace and* $T \in \operatorname{Hom}(V,W)$.

1. U^0 *is a subspace of* V^*;
2. $\dim U^* + \dim U^0 = \dim V^*$;
3. $\ker T^t = (\operatorname{im} T)^0$
4. T *is surjective* if and only if T^t *is injective.*

Proof.

1. Elementary application of subspace criterion, proposition X.2.
2. Consider the inclusion $\iota : U \hookrightarrow V$. Then the dimension theorem X.25.2 applied to ι' gives

$$\dim \operatorname{im} \iota' + \dim \ker \iota' = \dim V^*.$$

Now $\dim \ker \iota'$ are $\varphi \in V^*$ such that $\varphi \circ \iota = 0$. These are exactly the elements of the annihilator. Any functional on U can be extended to a functional on V, so ι' is surjective and $\dim \operatorname{im} \iota' = \dim U^*$.

3. There are two inclusions. First assume $\varphi \in \ker T'$, so $\forall v \in V$

$$0 = (\varphi \circ T)(v) = \varphi(Tv).$$

Thus $\varphi \in (\operatorname{im} T)^0$. The other inclusion uses the same equality.

4. $T \in \operatorname{Hom}(V,W)$ is surjective iff $\operatorname{im} T = W$ iff $(\operatorname{im} T)^0 = \{0\}$ iff $\ker T' = \{0\}$ iff T' is injective.

□

2.1.2 The transpose of a map

Let $f : V \to W \in \mathrm{Hom}_{\mathbb{F}}(V, W)$. The dual map[a] or transpose f^t is the linear map

$$f^t : W^* \to V^* : l \mapsto f^t(l) = l \circ f.$$

---

[a] The dual map f^t is often denoted f^* or f'. We avoid this because it clashes with the notation of the Hilbert adjoint.

Lemma XIII.55. *Let $f \in \mathrm{Hom}(U, V)$ and $g \in \mathrm{Hom}(V, W)$.*

- $(g \circ f)^t = f^t \circ g^t$*;*
- $\mathrm{id}_V^t = \mathrm{id}_{V^*}$*;*
- f *is an isomorphism* if and only if f^t *is an isomorphism;*
- $(f^t)^{-1} = (f^{-1})^t$

TODO: merge

Lemma XIII.56. *Let $S, T \in \mathrm{Hom}(V, W)$ and $\alpha \in \mathbb{F}$. Then*

1. $(S + T)^t = S^t + T^t$*;*
2. $(\alpha T)^t = \alpha T^t$
3. *if T is invertible, then T^t is invertible and*
$$(T^t)^{-1} = (T^{-1})^t.$$

Proposition XIII.57. *Let $U \subset V$ be a subspace and $T \in \mathrm{Hom}(V, W)$, where V, W are finite-dimensional.*

1. $\dim \ker T^t = \dim \ker T + \dim W - \dim V$*;*
2. $\dim \mathrm{im}\, T^t = \dim \mathrm{im}\, T$*;*
3. $\mathrm{im}\, T^t = (\ker T)^0$
4. T *is injective* if and only if T^t *is surjective.*

Proof.

1. Using $\dim V^* = \dim V$, we have
$$\begin{aligned}\dim \ker T^t &= \dim(\mathrm{im}\, T)^0 = \dim W - \dim \mathrm{im}\, T \\ &= \dim W - (\dim V - \dim \ker T) = \dim \ker T + \dim W - \dim V\end{aligned}$$
where the equalities come from proposition XIII.54 and the dimension theorem for linear maps, theorem X.25.2.
2. Still using these results, we can calculate
$$\begin{aligned}\dim \mathrm{im}\, T^t &= \dim W^* - \dim \ker T^t = \dim W^* - \dim(\mathrm{im}\, T)^0 \\ &= \dim\,(\mathrm{im}\, T)^* = \dim \mathrm{im}\, T.\end{aligned}$$

3. Take $\varphi = T^t(\psi) \in \operatorname{im} T^t$ where $\psi \in W^*$. If $v \in \ker T$, then
$$\varphi(v) = \left(T^t(\psi)\right) v = (\psi \circ T)(v) = \psi(Tv) = \psi(0) = 0.$$
Hence $\varphi \in (\ker T)^0$ and $\operatorname{im} T^t \subset (\ker T)^0$. We prove the equality by showing the dimensions are the same. Indeed:
$$\dim \operatorname{im} T^t = \dim \operatorname{im} T = \dim V - \dim \ker T = \dim(\ker T)^0.$$
4. $T \in \operatorname{Hom}(V, W)$ is injective iff $\ker T = \{0\}$ iff $(\ker T)^0 = V^*$ iff $\operatorname{im} T^t = V^*$ iff T^t is surjective.

□

2.1.3 Bidual spaces

Let V be a vector space. The bidual space is the dual of the dual $V^{**} = (V^*)^*$.

Let V be a vector space over $\mathbb{F}$ and $v \in V$. The evaluation map $\operatorname{ev} : V \to V^{**} : v \mapsto \operatorname{ev}_v$ is given by
$$\operatorname{ev}_v : V^* \to \mathbb{F} : l \mapsto l(v).$$

Lemma XIII.58. *Let V be a vector space. The evaluation map $\operatorname{ev} : V \to V^{**} : v \mapsto \operatorname{ev}_v$ is linear:*
$$\forall v, w \in V, a \in \mathbb{F} : \quad \operatorname{ev}_{av+w} = a \operatorname{ev}_v + \operatorname{ev}_w .$$

Lemma XIII.59. *Let V be vector space over $\mathbb{F}$. The evaluation map is injective.*

Proof. Assume $\operatorname{ev}_v = \operatorname{ev}_w$ for some $v, w \in V$. Then
$$0 = \operatorname{ev}_v - \operatorname{ev}_w = \operatorname{ev}_{v-w} .$$
So $\forall l \in V^* : \operatorname{ev}_{v-w}(l) = l(v-w) = 0$. Now define the sublinear functional by
$$p(x) = \begin{cases} \alpha & x = \alpha(v-w) \\ 0 & \text{else.} \end{cases}$$
Then the functional f defined on $\operatorname{span}\{v-w\}$ by $f(\alpha(v-w)) = \alpha$ is bounded by p and can be extended to a functional on all V by the Hahn-Banach theorem XIII.27.1 if $v - w \neq 0$. Then $f(v-w) \neq 0$, which contradicts our assumptions. Thus $v = w$. □

Proposition XIII.60. *The mapping $\operatorname{ev} : V \to V^{**} : v \mapsto \operatorname{ev}_v$ is an isomorphism* if and only if *V is finite-dimensional.*

Proof. Assume V finite dimensional. As the evaluation map is injective, it is an isomorphism by X.34. The other direction is a dimensional argument by proposition XIII.52. □

2.1.4 Considerations of naturality

TODO Dual basis not natural, determined by inner product.
V canonically embeddable in V^{**}.

2.2 Topological duality

Let X be a normed space. Define

$$X' = \{\omega \mid \omega : X \to \mathbb{F} \text{ is a continuous map}\}$$

with the norm

$$\|\omega\| = \sup\{|\omega(x)| \mid x \in X, \|x\| \leq 1\}.$$

Then $(X', \|\cdot\|)$ is a normed space. We call X' the continuous dual or topological dual of X.

Lemma XIII.61. *The topological dual X' is a subspace of the algebraic dual X^*. If X is finite-dimensional, then $X' = X^*$.*

Lemma XIII.62. *Let X be a normed space and let $x \in X$ $\omega \in X'$ be a bounded linear functional. Then*

$$\begin{aligned}\|\omega\| &= \sup\{|\omega(v)| \mid \|v\| = 1\} \\ &= \sup\left\{\frac{|\omega(v)|}{\|v\|} \;\middle|\; v \neq 0\right\} \\ &= \inf\{c > 0 \mid |\omega(v)| \leq c\|v\| \forall v \in X\}\end{aligned}$$

and

$$\begin{aligned}\|x\| &= \sup\{|\varphi(x)| \mid \|\varphi\| = 1\} \\ &= \sup\left\{\frac{|\varphi(x)|}{\|\varphi\|} \;\middle|\; \varphi \neq 0\right\}.\end{aligned}$$

Proof. We prove the third equality. Let α be the infimum. Let $\epsilon > 0$, then by the definition $|\omega[(\|x\| + \epsilon)^{-1}x]| \leq \|\omega\|$. Hence $|\omega(x)| \leq \|\omega\|(\|x\| + \epsilon)$. Letting $\epsilon \to 0$ gives $|\omega(x)| \leq \|\omega\|\|x\|$ for all x. So $\alpha \leq \|\omega\|$. On the other hand, $|\omega(x)| \leq c$ for all x with $\|x\| = 1$. Hence $\|\omega\| \leq \alpha$. □

Proposition XIII.63. *The continuous dual of $l^p(J)$ is $l^q(J)$ where $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q}$. Also, the continuous dual of l^1 is l^∞.*

2.2.1 The (topological) transpose of a map

Let $T \in \mathcal{B}(V, W)$. The dual map $T^t : W' \to V'$ is called the adjoint or the transpose of T.

The notation T^t is consistent for maps on both the algebraic and topological duals: if T is bounded, $T^t : W^* \to V^*$ restricts to $T^t|_{W'} = T^t : W' \to V'$.

Proposition XIII.64. *Let $T \in \mathcal{B}(V, W)$. Then the transpose T^t is a bounded operator in $\mathcal{B}(W, V)$ with $\|T^t\| = \|T\|$.*

Proof. The operator T^t is linear since $\forall f_1, f_2 \in W', \forall a \in \mathbb{F}, \forall x \in V$:

$$(T^t(af_1 + f_2))(x) = (af_1 + f_2)(Tx) = af_1(Tx) + f_2(Tx) = a(T^t f_1)(x) + (T^t f_2)(x).$$

For the equality of norms, we prove two inequalities. First $\forall x \in V, f \in W'$

$$|f(Tx)| \leq \|f\|\|Tx\| \leq \|f\|\|x\|\|T\| \implies \frac{|f(Tx)|}{\|x\|} \leq \|f\|\|T\|.$$

taking the supremum over $x \in V$, we get $\|T^t f\| = \|f \circ T\| \leq \|f\|\|T\|$ and taking the supremum over $f \in W'$ gives $\|T^t\| \leq \|T\|$. This shows that T^t is bounded.
For the other inequality, we use corollary XIII.27.4 to the Hahn-Banach theorem: for every $x \in V$, there exists a bounded functional ω_x such that $\|\omega_x\| = 1$ and $\omega_x(x) = \|x\|$. Then we can calculate:

$$\|Tx\| = \omega_{Tx}(Tx) = (T^t\omega_{Tx})(x) \leq \left\|T^t\omega_{Tx}\right\|\|x\| \leq \left\|T^t\right\|\|\omega_{Tx}\|\|x\| = \left\|T^t\right\|\|x\|$$

So $\|T\| \leq \|T^t\|$. Combining gives $\|T^t\| = \|T\|$. □

Corollary XIII.64.1. *The map $T \mapsto T^t$ is an isometric isomorphism in $(\mathcal{B}(X,Y) \to \mathcal{B}(Y',X'))$.*

Lemma XIII.65. *Let $S, T \in \mathcal{B}(V,W)$ and $\alpha \in \mathbb{F}$. Then*

1. $(S+T)^t = S^t + T^t$;
2. $(\alpha T)^t = \alpha T^t$
3. *if T is invertible, then T^t is invertible and*

$$(T^t)^{-1} = (T^{-1})^t.$$

Let $T \in \mathcal{B}(U,V)$ and $S \in \mathcal{B}(V,W)$. Then

4. $(ST)^t = T^t S^t$

2.2.2 Bidual spaces

Just like for algebraic duality, we can define a topological bidual space (or second dual space) V''.

Proposition XIII.66. *Let V be a normed space. For each $v \in V$*

$$\text{ev}_v : V' \to \mathbb{F} : \omega \mapsto \omega(v)$$

is bounded and thus an element of V''.
The evaluation map $\text{ev} : V \to V''$ *is*

1. *isometric (and thus injective):* $\|\text{ev}_v\| = \|v\|$;
2. *bounded with norm* $\|\text{ev}\| = 1$.

Proof. Let $v \in V$. Then

$$\|\text{ev}_v\| = \sup\{\|\text{ev}_v(\omega)\| \mid \|\omega\| = 1\} = \sup\{\|\omega(v)\| \mid \|\omega\| = 1\} \leq \sup\{\|v\|\,\|\omega\| \mid \|\omega\| = 1\} = \|v\|.$$

(1) Setting $\omega = \langle v/\|v\|, \cdot\rangle$, we get

$$\|\text{ev}_v\| \leq |\,\text{ev}_v(\omega)| = |\,\langle v/\|v\|, v\rangle\,| = \|v\|.$$

Together with the calculation above, this gives $\|\text{ev}_v\| = \|v\|$.
(2) $\|\text{ev}\| = \sup\{\|\text{ev}_v\| \mid \|v\| = 1\} = \sup\{\|v\| \mid \|v\| = 1\} = 1$. □

Lemma XIII.67. *Let V be normed space over $\mathbb{F}$ and $v \in V$. For each $v \in V$*

$$\text{ev}_v : V' \to \mathbb{F} : \omega \mapsto \omega(v)$$

is bounded with norm $\|v\|$ and thus $\text{ev} \in V''$ *with* $\|\text{ev}\| = 1$.

2.2.2.1 Reflexive spaces

A normed space V is reflexive if the evaluation map $\mathrm{ev} : V \to V''$ is surjective:

$$\operatorname{im} \mathrm{ev} = V''.$$

If V is reflexive, then V'' is isometrically isomorphic to V. The converse is not necessarily true.

Lemma XIII.68. *Every finite-dimensional space is reflexive.*

Proposition XIII.69. *A separable normed space X with a non-separable dual space X' cannot be reflexive.*

Proof. TODO ☐

Thus l^1 is not reflexive.

Proposition XIII.70. *If the dual space X' of a normed space X is separable, then X itself is separable.*

Proof. TODO ☐

Chapter 3

Banach spaces

- A Banach space is a normed vector space that is complete as a metric space.
- A Hilbert space is an inner product space that is complete as a metric space.

A finite-dimensional normed / inner product space is automatically a Banach / Hilbert space by proposition X.80.

Every proper subspace U of a normed vector space V has empty interior. A nice consequence of this is that any closed proper subspace is necessarily nowhere dense. So if V is a Banach space, the Baire category theorem implies that V cannot be a countable union of closed proper subspaces. In particular, an infinite dimensional Banach space cannot be a countable union of finite dimensional subspaces. This means, for example, that a vector space of countable dimension (e.g the space of polynomials) cannot be equipped with a complete norm.

The space $\mathcal{B}(V, W)$ is a Banach space.

TODO: quotient of Banach spaces.

Complemented subspace problem: `https://arxiv.org/pdf/math/0501048v1.pdf`

3.1 Function spaces

3.1.1 The spaces $\mathcal{L}^p(X, \mathrm{d}\mu)$

3.1.2 The spaces $L^p(X, \mathrm{d}\mu)$

Theorem XIII.71 (Riesz-Fisher)**.** *The space $L^p(X, \mathrm{d}\mu)$ is complete.*

For L^∞: essential supremum.

3.1.2.1 Locally integrable spaces

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. The locally L^p space is the space

$$L^p_{\text{loc}}(\Omega) \coloneqq \{f \in (\Omega \to \mathbb{C}) \mid f \in L^p(K) \text{ for all compact } K \subset \Omega\}.$$

The functions in $L^1_{\text{loc}}(\Omega)$ are called locally integrable on Ω.

TODO: deal with equivalence classes??

3.1.3 Sequence spaces

TODO: $L^p(A,\mu)$ with μ counting measure.
Let J be a countable index set and $x : J \to \mathbb{F}$ a sequence indexed by J. We define

$$\|x\|_p := \left(\sum_{j\in J} |x(j)|^p\right)^{1/p} \qquad \text{and} \qquad \|x\|_\infty = \sup_{j\in J} |x(j)|.$$

So $\|\cdot\|_1$ is the standard norm on $\mathbb{F}^n$. For general sequences there is no guarantee that these norms do not diverge.

Let J be an index set, D a directed set and $p \geq 1$,

$$\begin{aligned}
\ell^p(J) &= \left\{x : J \to \mathbb{F} \;\middle|\; \|x\|_p < +\infty\right\},\\
\ell^\infty(J) &= \{x : J \to \mathbb{F} \mid \|x\|_\infty < +\infty\},\\
c_0(D) &= \left\{x : D \to \mathbb{F} \;\middle|\; \lim_{n\to\infty} |x(n)| = 0\right\},\\
c_{00}(D) &= \{x : D \to \mathbb{F} \mid \{n \in D \mid x(n) \neq 0\} \text{ has finite cardinality}\}.
\end{aligned}$$

unless specified we equip c_0 and c_{00} with the norm $\|\cdot\|_\infty$.

Lemma XIII.72. *c_{00} is dense in ℓ^p if it is equipped with the norm $\|\cdot\|_p$ and dense in c_0 if it is equipped with the norm $\|\cdot\|_\infty$.*

Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q}$. We have the inequalities

$$\begin{aligned}
\|xy\|_1 &\leq \|x\|_p\|y\|_q && \text{(Hölder inequality)}\\
\|x+y\|_p &\leq \|x\|_p + \|y\|_p && \text{(Minkowski inequality)}
\end{aligned}$$

which follow from the general cases (TODO ref) by applying the counting measure.

3.1.3.1 Operators on sequence spaces

TODO Gribanov's theorems
3.7.1, 3.7.2 of Hanson / Yakovlev.

3.2 Series in Banach spaces

TODO `https://link.springer.com/content/pdf/10.1007%2F978-0-8176-4687-5_3.pdf`

Let $\langle x_n \rangle$ be a sequence in a Banach space X. As for series of scalars, we say a series $\sum_{n=1}^\infty x_n$ is

- unconditionally convergent if $\sum_{n=1}^\infty x_{\sigma(n)}$ converges for every permutation σ of $\mathbb{N}$;
- absolutely convergent if $\sum_{n=1}^\infty \|x_n\| < \infty$.

Proposition XIII.73. *Let $\langle x_n \rangle$ be a sequence in a Banach space X. If $\sum_{n=1}^{\infty}$ converges absolutely, then it converges unconditionally.*

Proof. Assume absolute convergence, so $\sum \|x_i\| < \infty$. Then (for $m < n$)

$$\left\| \sum_{i=1}^{n} x_i - \sum_{i=1}^{m} x_i \right\| = \left\| \sum_{i=m+1}^{n} x_i \right\| \leq \sum_{i=m+1}^{n} \|x_i\| = \sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{m} \|x_i\|,$$

and because $\sum \|x_i\|$ converges, it is a Cauchy sequence and by the inequality so is $\sum x_i$. By completeness this sequence is convergent.

By (TODO ref) $\sum \left\| x_{\sigma(i)} \right\|$ converges for any permutation σ of $\mathbb{N}$. We can then repeat the argument to show $\sum x_{\sigma(i)}$ is also convergent and thus unconditionally convergent. ☐

3.2.1 Fourier series

TODO Sacks 7.1

3.3 Completions and constructions

Proposition XIII.74. *The completions of a space with respect to two different norms are isomorphic* if and only if *the norms are equivalent.*

TODO move down

3.3.1 Tensor products

TODO Ryan `https://math.stackexchange.com/questions/2712906/does-mathcalb-mathcalh-math` `https://math.stackexchange.com/questions/35191/operator-norm-and-tensor-norms?noredirect=1&lq=1`

3.3.2 Direct sums

For arbitrary direct sums we can generalise: now that we have a concept of limits, we can relax the requirement that all but finitely many terms be zero. Instead we require that the sequence of norms is bounded in some way. This gives a whole family of related concepts of direct sum, named for which sequence space the sequence of norms belongs to.

Let $\{V_i\}_{i\in I}$ be an arbitrary family of Banach spaces over a field $\mathbb{F}$ and let $\ell(I, \mathbb{F})$ be a space of sequences in $\mathbb{F}$ indexed by I. Then the ℓ-direct sum is the vector space with as field

$$\bigoplus_{i\in I}^{\ell} V_i = \left\{ (v_i)_{i\in I} \;\middle|\; \forall i \in I : v_i \in V_i \quad \text{and} \quad (\|v_i\|_{V_i})_{i\in I} \in \ell(I, \mathbb{F}) \right\}.$$

In particular we have, for all $1 \leq p < \infty$, the ℓ^p-direct sum

$$\bigoplus_{i\in I}^{p} V_i \coloneqq \left\{ (v_i)_{i\in I} \;\middle|\; \forall i \in I : v_i \in V_i \quad \text{and} \quad \sqrt[p]{\sum_{i\in I} \|v_i\|_{V_i}^{p}} < \infty \right\}$$

and the ℓ^∞-direct sum

$$\bigoplus_{i\in I}^{\infty} V_i := \left\{ (v_i)_{i\in I} \;\middle|\; \forall i \in I : v_i \in V_i \quad \text{and} \quad \sup_{i\in I} \|v_i\|_{V_i} < \infty \right\}.$$

Proposition XIII.75. *For any sequence space that is a Banach space the direct sum is a Banach space. TODO: in particular algebraic direct sum as c_{00}? (one possible norm)? and finite direct sums?*

3.3.2.1 Direct sum of identical spaces

Proposition XIII.76. *Let V be a Banach space over $\mathbb{F}$, I an arbitrary index set and $\ell(I, \mathbb{F})$ a banach sequence space.*

$$\bigoplus_{i\in I}^{\ell} V \cong \ell \otimes V$$

3.4 Operators on Banach spaces

3.4.1 Closed operators

Proposition XIII.77. *Let X, Y be Banach spaces and $S, T \in \mathcal{L}(X, Y)$ with $\operatorname{dom}(S) = \operatorname{dom}(T)$. If S is a closed operator and there exist $\alpha, \beta, \gamma \in \mathbb{R}^+$ such that $0 < \gamma \leq 1$ and $\beta < 1/\gamma$ and*

$$\|(S - T)u\| \leq \alpha\|u\| + \beta\|Su\|^\gamma u^{1-\gamma} \qquad \textit{for all } u \in \operatorname{dom}(S) = \operatorname{dom}(T),$$

then T is also closed.

Proof. TODO Jeribi. □

3.5 Bounded operators

Proposition XIII.78. *Let V, W be normed spaces. The vector space $\mathcal{B}(V, W)$ with the operator norm is a Banach space* if and only if *W is a Banach space.*

Corollary XIII.78.1. *Let V be a normed space. The continuous dual X' is a Banach space.*

Corollary XIII.78.2. *Topologically reflexive spaces are Banach spaces.*

Proposition XIII.79 (Bounded linear extension)**.** *Let $T : \operatorname{dom}(T) \subseteq X \to Y$ be a bounded operator between normed spaces. Then T has a unique extension*

$$\widetilde{T} : \overline{\operatorname{dom}(T)} \to Y$$

where $\widetilde{T}$ is a bounded operator with $\|\widetilde{T}\| = \|T\|$.

Proof. Normed vector spaces have the unique extension property because they are Hausdorff, VIII.164. We just need to show the norm stays the same:
Clearly $\|\tilde{T}\| \geq \|T\|$. For the converse take any $x \in X$. As $\overline{\operatorname{dom}(T)} = X$, there exists a sequence $\langle x_i \rangle \subset \operatorname{dom}(T)$ that converges to x. Then

$$\|\tilde{T}(x)\|_Y = \left\| T\left(\lim_{i\to\infty} x_i\right)\right\|_Y = \lim_{i\to\infty} \|T(x_i)\|_Y \leq \lim_{i\to\infty} \|T\|\,\|x_i\|_X = \|T\|\,\|x\|_X.$$

□

3.5.1 Contractions

A linear operator T on a normed space is a contraction if and only if it is bounded and $\|T\| < 1$.

3.5.1.1 Neumann series

Lemma XIII.80. *Let T be a bounded linear operator on a normed space X with $\|T\| < 1$. Then the series $\sum_{i=1}^{\infty} T^i(b)$ converges for all $b \in X$ and is the unique fixed point of $F(x) = T(x) + b$.*

Proof. The function F is a contraction if and only if $\|T\| < 1$. So it has a unique fixed point. Starting the fixed point iteration at b yields the series:

$$\begin{aligned} F(b) &= Tb + b \\ F(Tb + b) &= T^2 b + Tb + b \\ &\dots. \end{aligned}$$

Alternatively we could have used the inequality $\|T^n b\| \leq \|T\|^n \|b\|$, the convergence of the geometric series and XIII.73 to prove convergence. Proving it is a fixed point is then elementary. □

Corollary XIII.80.1 (Neumann series)**.** *Let T be a bounded linear operator with $\|T\| < 1$. Then*

$$(\mathrm{id} - T)^{-1} = \sum_{i=1}^{\infty} T^i$$

with uniform convergence. Also

$$\left\|(\mathrm{id} - T)^{-1}\right\| \leq \frac{1}{1 - \|T\|}.$$

Proof. Let $x \in X$. Then set $(\mathrm{id} - T)^{-1} x = y$. This is equivalent to $x = y - Ty$ and means y is the fixed point of $y \mapsto Ty + x$. So $y = \sum_{i=1}^{\infty} T^i x$.
The convergence is uniform by TODO ref.
Finally we have

$$\left\|(\mathrm{id} - T)^{-1}\right\| = \left\| \sum_{i=1}^{\infty} T^i \right\| \leq \sum_{i=1}^{\infty} \|T^i\| = \frac{1}{1 - \|T\|}$$

by the geometric series. □

TODO ref: uniform convergence if $\sum_i \|T_i\| < \infty$??

3.5.2 The uniform boundedness principle

TODO: if a family of bounded operators on a Banach space is pointwise bounded, then it is uniformly bounded.

Theorem XIII.81 (Uniform boundedness principle)**.** *Let $\mathcal{F} \subset \mathcal{B}(X, Y)$ be a family of bounded operators where X is a Banach space and Y a normed space, such that*

$$\sup\{\|Tx\| \mid T \in \mathcal{F}\} < \infty \qquad \text{for all } x \in X.$$

Then $\sup\{\|T\| \mid T \in \mathcal{F}\} < \infty$.

Proof. The proof is an application of the Baire category theorem. Define the closed subsets K_n as

$$K_n = \{x \in X \mid \forall T \in \mathcal{F} : \|Tx\| \leq n\}.$$

These are closed because the functional $f_T : X \to \mathbb{R} : x \mapsto \|Tx\|$ is bounded and

$$K_n = \bigcap_{T \in \mathcal{F}} f_T^{-1}[\,[0,n]\,].$$

By assumption, $X = \bigcup_{n\in\mathbb{N}} K_n$. As X is a Banach space, and thus a complete metric space, we can apply the Baire category theorem, VIII.168, to conclude that there is a K_n with non-empty interior (by contraposition of the Baire condition). Take $x_0 \in K_n^\circ$, then $-x_0 + K_n^\circ \subset K_{n2}$. So $\mathbf{0} \in (K_{2n})^\circ$ and we can find a ρ such that $B(\mathbf{0}, \rho) \subset K_{2n}$. By proposition X.87 we have $\|T\| \leq 2n/\rho$ for all $T \in \mathcal{F}$. □

Corollary XIII.81.1 (Banach-Steinhaus)**.** *Let X be a Banach space and Y a normed space. Let $T_n : X \to Y$ be a sequence of bounded operators. If T_n converges pointwise to $T : X \to Y$: $Tx = \lim_n T_n x$, then $\sup_n \|T_n\| < \infty$ and thus T is bounded.*

Proof. Any convergent sequence in a normed space is bounded, so we can apply the uniform boundedness principle. □

3.5.3 Open mapping and closed graph theorems

Proposition XIII.82. *Let X, Y be Banach spaces and $T : X \to Y$ a surjective bounded operator. Then the image of the open unit ball $B(\mathbf{0}, 1) \subset X$ contains an open ball about $\mathbf{0} \in Y$.*

Proof. We first prove $0 \in \overline{T[B(\mathbf{0}, r)]}^\circ$ for every $r > 0$: (TODO: make computations lemma.)

- Using $X = \bigcup_{n=1}^\infty B(\mathbf{0}, n)$, we see by surjectivity

  $$Y = T[X] = T\left[\bigcup_{n=1}^\infty B(\mathbf{0}, n)\right] = \bigcup_{n=1}^\infty T[B(\mathbf{0}, n)].$$

  Because Y has the Baire property (theorem VIII.168) and Y is both open and non-empty, it may not be meagre, by lemma VIII.166. So for some $n \in \mathbb{N}$, $T[B(\mathbf{0}, n)]$ is non-rare, meaning that $\overline{T[B(\mathbf{0}, n)]}$ has non-empty interior.

- Because

  $$\overline{T[B(\mathbf{0}, n)]} = \overline{2nT[B(\mathbf{0}, 1/2)]} = 2n\overline{T[B(\mathbf{0}, 1/2)]},$$

  $\overline{T[B(\mathbf{0}, 1/2)]}$ must have non-empty interior. Let $B(y_0, \epsilon) \subset \overline{T[B(\mathbf{0}, 1/2)]}$.

- Note $B(0, \epsilon) = y_0 - B(y_0, \epsilon) \subset \overline{T[B(\mathbf{0}, 1)]}$ and thus $B(0, r\epsilon) \subset \overline{T[B(\mathbf{0}, r)]}$.

We then prove $\overline{T[B(\mathbf{0}, 1/2)]} \subset T[B(\mathbf{0}, 1)]$, proving the proposition.

- Choose some $y_0 \in \overline{T[B(\mathbf{0}, 1/2)]}$. Then every neighbourhood $B(y_0, \epsilon/4)$ intersects $T[B(\mathbf{0}, 1/2)]$.

- Then

  $$B(y_0, \epsilon/4) = y_0 - B(\mathbf{0}, \epsilon/4) \subset y_0 - \overline{T[B(\mathbf{0}, 1/4)]},$$

  so $y_0 - \overline{T[B(\mathbf{0}, 1/4)]}$ intersects $T[B(\mathbf{0}, 1/2)]$. Take a $y_1 \in \overline{T[B(\mathbf{0}, 1/4)]}$ such that $y_0 - y_1$ is in this intersection. Then we have an $x_0 \in B(\mathbf{0}, 1/2)$ such that $T(x_0) = y_0 - y_1$.

- We can continue recursively choosing $y_{n+1} \in \overline{T[B(\mathbf{0}, 2^{-(n+1)})]}$ and $x_n \in B(\mathbf{0}, 2^{-n})$ such that $y_n - y_{n+1} = T(x_n)$.
- Consider the sequence $\sum_{k=0}^{n} x_k$. It is a Cauchy sequence in X. Call its limit x. Then $x \in B(\mathbf{0}, 1)$.
- Because $\|y_n\| \leq 2^{-n}\|T\|$, (y_n) converges to zero. Then

  $$\left(T\left(\sum_{k=1}^{n} x_k\right)\right)_{n\in\mathbb{N}} = (y_0 - y_{n+1})_{n\in\mathbb{N}}$$

  converges to y_0. Thus $T(x) = y_0 \in T[B(\mathbf{0}, 1)]$.

□

Proposition XIII.83. *Let X, Y be normed spaces and $T : X \to Y$ a linear map. If $\mathbf{0}$ lies in the interior of $T[B(\mathbf{0}, r)]$ for some $r > 0$, then T is open.*

Proof. TODO: make computations lemma. Given the assumption, 0 lies in the interior of $T[B(\mathbf{0}, \epsilon)]$ for all $\epsilon > 0$. Because $T[B(x, \epsilon)] = T(x) + T[B(\mathbf{0}, \epsilon)]$, $T(x)$ lies in the interior of $T[B(x, \epsilon)]$, for all $x \in X$. Thus for all neighbourhoods $U(x) \subset X$, $T(x) \subset T[U]^\circ$ and so $T[U] \subset T[U]^\circ$, so $T[U]$ is open. □

Theorem XIII.84 (Open mapping)**.** *Let X, Y be Banach spaces and $T : X \to Y$ a surjective bounded operator. Then T is an open map.*

Proof. This is the consequence of propositions XIII.82 and XIII.83. □

Corollary XIII.84.1 (Bounded inverse theorem)**.** *Let X, Y be Banach spaces. If $T : X \to Y$ is is continuous, linear and bijective, then T is a homeomorphism.*

Proposition XIII.85. *Let $T : \mathrm{dom}(T) \subset X \to Y$ be a bounded linear operator. Then*

1. *if $\mathrm{dom}(T)$ is a closed subset of X, then T has closed graph;*
2. *if T has closed graph and Y is complete, then $\mathrm{dom}(T)$ is a closed subset of X.*

Proof. We use proposition X.92 twice: First assume (x_n) and (Tx_n) converge to x and y, respectively. Then $x \in \mathrm{dom}(T)$ by closure and $y = Tx$ by continuity.
Now assume T has closed graph and Y is complete. Take $x \in \overline{\mathrm{dom}(T)}$ and $(x_n) \subset \mathrm{dom}(T)$ converging to x. Since T is bounded:

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\|\|x_n - x_m\|,$$

so (Tx_n) is Cauchy by VIII.216 and thus by completeness has a limit, say y. Then $Tx = y$ by continuity. Since T has closed graph, $x \in \mathrm{dom}(T)$. So $\overline{\mathrm{dom}(T)} \subseteq \mathrm{dom}(T)$ and $\mathrm{dom}(T)$ is closed. □

For closed graph theorem, see TVS.

3.5.4 Compact operators

Proposition XIII.86. *Let $L \in \mathrm{Hom}(V, W)$ with V, W Banach spaces. Then L is compact* if and only if *the image of any bounded subset of V under L is totally bounded in W.*

TODO proof

3.5.4.1 Calkin algebra

Proposition XIII.87. *Let X be a Banach space. Then $\mathcal{K}(X)$ is a closed two-sided ideal in $\mathcal{B}(X)$.*

Proof. TODO + $*$-ideal for Hilbert spaces. $\square$

Let X be a Banach space. The Calkin algebra is the quotient $\mathcal{B}(X)/\mathcal{K}(X)$.

TODO: quotient algebra ($[A][B] = [AB]$)

Proposition XIII.88. *Let $[T] \in \mathcal{B}(X)/\mathcal{K}(X)$. Then the following are equivalent:*

1. *$[T]$ is invertible in the Calkin algebra;*
2. *$\exists S \in \mathcal{B}(X)$: both $\mathbf{1} - TS$ and $\mathbf{1} - ST$ are compact;*
3. *T has closed range and finite-dimensional kernel and cokernel.*

Proof. Point 1. and 2. are easily equivalent: $[S]$ is an inverse of $[T]$ if and only if $[\mathbf{1}] = [S][T] = [ST]$ and $[\mathbf{1}] = [T][S] = [TS]$. Then

$$[\mathbf{1}] = [ST] \iff [ST - \mathbf{1}] = [0] \qquad [\mathbf{1}] = [TS] \iff [TS - \mathbf{1}] = [0]$$

and $[F] = [0]$ if and only if F is compact.
TODO $\square$

3.6 Unbounded operators

Chapter 4

Spectral theory and functional calculus

4.1 Invariant subspaces

Let $L \in \operatorname{Hom}(V)$ be an endomorphism. A subspace U of V is invariant under L if $T|_U$ is an endomorphism on U. In other words, $u \in U$ implies $Tu \in U$.

Clearly this definition only works for endomorphisms, not for linear maps in general. This is true for the rest of the theory about eigenvalues and eigenvectors.

Example

Let $L \in \operatorname{Hom}(V)$. The following are invariant under L:

- $\{0\}$;
- $\ker L$;
- $\operatorname{im} L$.

4.2 The spectrum

TODO: eigenvalue problem $Lx = \lambda x$
generalised eigenvalue problem $Lx = \lambda Tx$
nonstandard eigenvalue problem $A(\gamma)x = 0$.
TODO: consistency $\lambda \operatorname{id} - L$, not $L - \lambda \operatorname{id}$. TODO: everything is now in $\mathbb{C}$.

Let $L : \operatorname{dom}(L) \subset V \to V$ be an operator on a complex normed vector space V. For $\lambda \in \mathbb{C}$ the resolvent $R_L(\lambda)$ is defined as

$$R_L(\lambda) \coloneqq (\lambda \operatorname{id}_V - L)^{-1} : \operatorname{im}(\lambda \operatorname{id}_V - L) \to \operatorname{dom}(L),$$

if this inverse exists (i.e. if $\lambda \operatorname{id}_V - L$ is injective).

- The resolvent set $\rho(L)$ is the set

$$\begin{aligned}\rho(L) &:= \{\lambda \in \mathbb{C} \mid R_L(\lambda) \in \mathcal{B}(V)\} \\ &= \{\lambda \in \mathbb{C} \mid R_L(\lambda) \text{ exists, has domain } V \text{ and is bounded}\}.\end{aligned}$$

- The spectrum of L is the complement of the resolvent set: $\sigma(L) := \mathbb{C} \setminus \rho(L)$.
- The spectral radius $\operatorname{spr}(L)$ is $\sup_{\lambda\in\sigma(L)} |\lambda|$.

Lemma XIII.89. *Let T be an operator on a normed vector space V. Then $\lambda \in \rho(T)$* if and only if *$\lambda \operatorname{id}_V - T$ is surjective and bounded from below.*

Proof. By X.91, $\lambda \operatorname{id}_V - T$ has a bounded inverse $(\lambda \operatorname{id}_V - T)^{-1} : \operatorname{im}(\lambda \operatorname{id}_V - T) \to V$ if and only if it is bounded below. In order for λ to be in the resolvent set, we need $(\lambda \operatorname{id}_V - T)^{-1}$ to be defined everywhere, i.e. $\operatorname{im}(\lambda \operatorname{id}_V - T) = V$. □

4.2.1 The three-way classification of the spectrum

Let $L : \operatorname{dom}(L) \subset V \to V$ be an operator on a complex vector space V.

- The point spectrum or discrete spectrum $\sigma_{\mathrm{p}}(L)$ contains the values of λ where $\lambda \operatorname{id}_V - L$ fails to be injective, so the resolvent fails to exist. These values are called the eigenvalues of L.

  We call

  - $\ker(\lambda \operatorname{id}_V - L)$ the multiplicity space or geometric eigenspace of λ; and
  - $\dim \ker(\lambda \operatorname{id}_V - L)$ the (geometric) multiplicity of λ.
- The continuous spectrum $\sigma_{\mathrm{c}}(L)$ is the set of all values of $\lambda \in \sigma(L)$ such that the resolvent $R_L(\lambda)$ exists and is densely defined.
- The residual spectrum $\sigma_{\mathrm{r}}(L)$ is the set of all values of $\lambda \in \sigma(L)$ such that the resolvent $R_L(\lambda)$ exists, but is not densely defined.

  We call

  - $\operatorname{im}(\lambda \operatorname{id}_V - L)^{\perp}$ the deficiency subspace of λ; and
  - $\dim(\operatorname{im}(\lambda \operatorname{id}_V - L)^{\perp})$ the deficiency of λ.

The sets $\sigma_{\mathrm{p}}(T), \sigma_{\mathrm{c}}(T)$ and $\sigma_{\mathrm{r}}(T)$ are disjoint.

In finite dimensions we know that

$$\lambda \operatorname{id}_V - L \text{ is surjective} \qquad\iff\qquad \lambda \operatorname{id}_V - L \text{ is injective}$$

and all linear operator are bounded. So in this case there can only ever be a point spectrum.

Proposition XIII.90. *If T is an operator on a Banach space that is not closed, then $\sigma(T) = \mathbb{C}$.*

Proof. We can find a sequence $x_n \to x$ such that $Tx_n \to y$, but $Tx \neq y$. Then for all $\lambda \in \mathbb{C}$ we have $z_n = (\lambda \operatorname{id} - T)x_n \to \lambda x - y$. If $R_T(\lambda)$ was a bounded inverse of $(\lambda \operatorname{id} - T)$, then

$R_T(\lambda) \circ (\lambda \operatorname{id} - T)x_n \to R_T(\lambda)(\lambda x - y)$. We need to show that $R_T(\lambda)(\lambda x - y) \neq x$. Indeed

$$\begin{aligned} R_T(\lambda)(\lambda x - y) &= R_T(\lambda)(\lambda x - Tx + Tx - y) \\ &= R_T(\lambda)(\lambda x - Tx) + R_T(\lambda)(Tx - y) \\ &= x + R_T(\lambda)(Tx - y), \end{aligned}$$

and $R_T(\lambda)(Tx - y) \neq 0$, because $Tx - y \neq 0$ and the kernel of $R_T(\lambda)$ is trivial because it is injective. ☐

Example

Closed operators may also have empty resolvent set. `https://math.stackexchange.com/questions/3262168/closed-operator-with-trivial-resolvent-set`

So spectral theory is only interesting for closed operators. In this case the three-way classification exhausts the possibilities.

Proposition XIII.91. *Let X be a Banach space and T a closed linear operator on X. Then $\lambda \in \sigma(T)$ if and only if $\lambda \operatorname{id}_X - T : \operatorname{dom}(T) \to V$ is not bijective.*

Proof. If $\lambda \operatorname{id}_X - T$ is not bijective, then clearly $\lambda \in \sigma(T)$.
Conversely, assume $\lambda \operatorname{id}_X - T$ is bijective. Then $(\lambda \operatorname{id}_X - T)^{-1} : X \to \operatorname{dom}(T)$ is closed by X.93 and has as domain a Banach space, so it is bounded by the closed graph theorem XIII.46. ☐

Corollary XIII.91.1. *Let T a closed operator on a Banach space. Then*

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

Proposition XIII.92. *Let $T : X \to X$ be an operator on a Banach space and $\lambda \in \sigma_c$, then $R_\lambda(T)$ is unbounded.*

Proof. If $R_\lambda(T)$ is bounded, $\lambda \operatorname{id}_V - T$ then is bounded below by lemma X.91 and has closed range by proposition X.95. Then because $\operatorname{im}(\lambda \operatorname{id}_V - T)$ is dense, this means T is surjective, which is a contradiction because then $\lambda \in \rho(T)$. ☐

4.2.2 Resolvents

Proposition XIII.93 (Resolvent identity)**.** *Let T be a linear operator and $\lambda, \mu \in \mathbb{C}$ such that $R_T(\lambda), R_T(\mu)$ exist. Then $R_T(\lambda)$ and $R_T(\mu)$ commute and*

$$\frac{R_T(\lambda) - R_T(\mu)}{\lambda - \mu} = -R_T(\lambda)R_T(\mu).$$

Proof. The commutativity of the resolvents follows from I.129.
We calculate

$$\begin{aligned} R_T(\lambda) - R_T(\mu) &= R_T(\lambda)R_T(\mu)(\mu \operatorname{id} - T) - R_T(\mu)R_T(\lambda)(\lambda \operatorname{id} - T) \\ &= \mu R_T(\lambda)R_T(\mu) - R_T(\lambda)R_T(\mu)T - \lambda R_T(\mu)R_T(\lambda) + R_T(\mu)R_T(\lambda)T \\ &= (\mu - \lambda)R_T(\lambda)R_T(\mu). \end{aligned}$$

☐

Corollary XIII.93.1. *Let T ba a linear operator. Then*

1. *$R_T(\lambda)$ is holomorphic in $\rho(T)$;*
2. *$R'_T(\lambda) = -R_T(\lambda)^2$;*
3. *$R_T^{(n)}(\lambda) = n!(-1)^n R_T(\lambda)^{n+1}$ for all $n \in \mathbb{N}$.*

Corollary XIII.93.2. *Let T ba a linear operator and $\lambda_0 \in \rho(T)$. Then the Taylor expansion of $R_T(\lambda)$ around λ_0 is*

$$\begin{aligned}R_T(\lambda) &= \sum_{n=0}^{\infty}(\lambda-\lambda_0)^n(-1)^n R_T(\lambda_0)^{n+1}\\ &= R_T(\lambda_0)\sum_{n=0}^{\infty}\Big((\lambda_0-\lambda)R_T(\lambda_0)\Big)^n\\ &= \frac{R_T(\lambda_0)}{1+(\lambda-\lambda_0)R_T(\lambda_0)},\end{aligned}$$

with convergence radius $1/\|R_T(\lambda_0)\|$.

Proof. If $|(\lambda_0-\lambda)| \leq 1/\|R_T(\lambda_0)\|$, then $(\lambda_0-\lambda)R_T(\lambda_0)$ is a contraction and we can use the Neumann series expansion XIII.80.1, which gives the last equality. ☐

Corollary XIII.93.3. *For all $\lambda \in \rho(T)$, we have $d(\lambda, \sigma(T)) \geq \|R_T(\lambda)\|^{-1}$.*

Corollary XIII.93.4. *The resolvent set $\rho(T)$ is open.*

Proposition XIII.94. *Let $S : \operatorname{dom}(S) \subseteq X \to X$ be a closed, bijective linear operator on a Banach space X. Let $T \in \mathcal{B}(X)$ be a bounded operator with $\|T\| < \left\|S^{-1}\right\|^{-1}$, then $S-T$ is invertible and*

$$(S-T)^{-1} = S^{-1}\sum_{n=0}^{\infty}\left(S^{-1}T\right)^n.$$

Also $\operatorname{dom}((S-T)^{-1}) = X$.

The assumptions are enough to guarantee $S^{-1} \in \mathcal{B}(X)$, so $\left\|S^{-1}\right\|$ makes sense.

Proof. By injectivity, we can define S^{-1}. It is closed by X.93 and has closed domain, so it is bounded by the closed graph theorem XIII.46. Since $\left\|S^{-1}T\right\| \leq \left\|S^{-1}\right\| \|T\| < 1$, it follows that by XIII.80.1 that $\operatorname{id} - S^{-1}T$ has bounded inverse and that we can expand its bounded inverse as a Neumann power series. So we can calculate

$$(S-T)^{-1} = S^{-1}(\operatorname{id} - S^{-1}T)^{-1} = S^{-1}\sum_{n=0}^{\infty}\left(S^{-1}T\right)^n.$$

☐

Corollary XIII.94.1. *Let T be a bounded operator on a Banach space X. For $|\lambda| > \|T\|$ the resolvent $R_T(\lambda)$ is bounded and given by*

$$R_T(\lambda) = \sum_{n=0}^{\infty}\frac{T^n}{\lambda^{n+1}}$$

with uniform convergence. The norm is bounded by

$$\|R_T(\lambda)\| \leq \frac{1}{|\lambda| - \|T\|}.$$

Proof. We use the proposition with $S = \lambda \operatorname{id}$. The norm bound follows from the Neumann series expansion XIII.80.1. $\square$

Corollary XIII.94.2. *Let T be a bounded linear operator on a Banach space. Then*

1. $\sigma(T) \subset [-\|T\|, \|T\|]$;
2. $\sigma(T)$ *is compact.*

For the point spectrum a simpler argument also leads to $\sigma_{\mathrm{p}}(T) \subset [-\|T\|, \|T\|]$: let λ be an eigenvalue with eigenvector x. Then

$$|\lambda|\,\|x\| = \|\lambda x\| = \|Tx\| \leq \|T\|\,\|x\|.$$

Proposition XIII.95. *Let T be an injective operator with dense range. Then for all $\lambda \neq 0$*

$$R_{T^{-1}}(\lambda^{-1}) = -\lambda T R_T(\lambda) = \lambda - \lambda^2 R_T(\lambda).$$

Proof. This is a reformulation of the calculation

$$\frac{1}{\lambda^{-1} - T^{-1}} = \frac{\lambda T}{\lambda T}\frac{1}{\lambda^{-1} - T^{-1}} = \frac{\lambda T}{T - \lambda} = \frac{\lambda T - \lambda^2 + \lambda^2}{T - \lambda} = \frac{\lambda\cancel{(T-\lambda)}}{\cancel{T-\lambda}} + \frac{\lambda^2}{T-\lambda} = \lambda - \lambda^2 R_T(\lambda).$$

TODO: make rigourous!! $\square$

Corollary XIII.95.1. *Let T be an injective operator with dense range. Then for all $\lambda \neq 0$*

1. $\sigma(T^{-1}) \setminus \{0\} = (\sigma(T) \setminus \{0\})^{-1}$;
2. $\sigma_p(T^{-1}) \setminus \{0\} = (\sigma_p(T) \setminus \{0\})^{-1}$.

4.2.3 Approximate spectrum and Weyl sequences

The set of all λ such that $T - \lambda \operatorname{id}_V$ is not bounded from below is called the approximate point spectrum σ_{ap}.
If $\lambda \in \sigma_{\mathrm{ap}}(T)$, then λ is an approximate eigenvalue of T.

Proposition XIII.96. *Let T be an operator. Then*

1. $\sigma_{ap}(T) \subset \sigma(T)$;
2. *if T is closed, then* $\sigma_p(T) \cup \sigma_c(T) \subset \sigma_{ap}(T)$.

Proof. (1) Assume $\lambda \notin \sigma(T)$. Then $(T - \lambda \operatorname{id}_V)^{-1}$ is bounded, so its inverse $T - \lambda \operatorname{id}_V$ is bounded below by X.91 and $\lambda \in \sigma_{\mathrm{ap}}(T)$.
(2) Assume $\lambda \notin \sigma_{\mathrm{ap}}(T)$, so $T - \lambda \operatorname{id}_V$ is bounded below. Then $T - \lambda \operatorname{id}_V$ is injective by X.91 and $\lambda \notin \sigma_{\mathrm{p}}(T)$. By proposition X.95 the range $\operatorname{im}(T - \lambda \operatorname{id}_V)$ is closed, so it cannot be a proper dense subset of X and $\lambda \notin \sigma_{\mathrm{c}}(T)$. $\square$

Proposition XIII.97 (Weyl sequences)**.** *Let T be an operator on a normed vector space V. Then $\lambda \in \sigma_{ap}(T)$* if and only if *there exists a sequence of unit vectors $(e_n)_{n\in\mathbb{N}}$ for which*

$$\lim_{n\to\infty} \|\lambda e_n - T e_n\| = 0.$$

Proof. Assume there is such a sequence $(e_n)_{n\in\mathbb{N}}$. Then for all $\epsilon > 0$, we can find a unit vector e_k such that $\|(\lambda \operatorname{id}_V 0T)e_n\| \leq \epsilon = \epsilon\|e_n\|$. This is clearly not bounded below.
This other direction is just an inversion of this argument. □

A sequence as described in XIII.97 is called a Weyl sequence for λ. This gives meaning to the name "approximate eigenvalue".

Corollary XIII.97.1. *Let T be an operator. Then $\sigma(T) \cap \overline{\rho(T)} \subseteq \sigma_{ap}(T)$.*

Proof. Let $\lambda \in \sigma(T) \cap \overline{\rho(T)}$. We show there is a Weyl sequence for λ.
We can find a sequence $\langle \lambda_n \rangle \subseteq \rho(T)$ such that $\lambda_n \to \lambda$. Now $d(\lambda_n, \sigma(T)) \to 0$, so by XIII.93.3, we can find a sequence of unit vectors $\langle x_n \rangle$ such that $\|R_T(\lambda_n)x_n\| \to \infty$. Now we can rescale $\langle x_n \rangle$ such that $\|R_T(\lambda_n)x_n\| = 1$.
Then $\|x_n\| \to 0$, and hence

$$\begin{aligned}
\|(\lambda \operatorname{id} - T)R_T(\lambda_n)x_n\| &= \left\| \frac{\lambda \operatorname{id} - T}{\lambda_n \operatorname{id} - T} x_n \right\| \\
&= \left\| \frac{(\lambda \operatorname{id} - T) + (\lambda_n \operatorname{id} - T) - (\lambda_n \operatorname{id} - T)}{\lambda_n \operatorname{id} - T} x_n \right\| \\
&= \left\| \left(\operatorname{id} + \frac{(\lambda \operatorname{id} - T) - (\lambda_n \operatorname{id} - T)}{\lambda_n \operatorname{id} - T} \right) x_n \right\| \\
&= \left\| \left(\operatorname{id} + \frac{\lambda \operatorname{id} - \lambda_n \operatorname{id}}{\lambda_n \operatorname{id} - T} \right) x_n \right\| \\
&= \|x_n + (\lambda \operatorname{id} - \lambda_n \operatorname{id})R_T(\lambda_n)x_n\| \\
&\leq \|x_n\| + |\lambda - \lambda_n| \, \|R_T(\lambda_n)x_n\| \to 0.
\end{aligned}$$

Thus $\langle R_T(\lambda_n)x_n \rangle$ is the kind of sequence we were looking for. □

4.2.4 Parts of the spectrum

Example

Operator with empty spectrum. TODO `https://math.stackexchange.com/questions/1344287/example-operator-with-empty-spectrum`.

4.2.4.1 The point spectrum: eigenvalue and eigenvectors

In this section we study invariant subspaces with dimension 1, i.e. subspaces $U = \operatorname{span}\{v\}$ such that

$$Lv = \lambda v.$$

Suppose $L \in \operatorname{Hom}_{\mathbb{F}}(V)$.

- A scalar $\lambda \in \mathbb{F}$ is called an eigenvalue of L if there exists a $v \in V$ such that $v \neq 0$ and $Lv = \lambda v$.
- Such a vector v is called an eigenvector.
- The set of all eigenvectors associated with an eigenvalue λ is called the eigenspace $E_\lambda(L)$. Because

$$E_\lambda(L) = \ker(L - \lambda \operatorname{id}_V)$$

it is indeed a vector space.

The dimension of $E_\lambda(L)$ is the geometric multiplicity of λ.

Proposition XIII.98. *Let* $L \in \operatorname{Hom}_{\mathbb{F}}(V)$ *and* $\lambda \in \mathbb{F}$*, then*

$$\lambda \text{ is an eigenvalue of } L \qquad \iff \qquad \lambda \text{ is in the point spectrum } \sigma_p(L).$$

Proof. The equation $Lv = \lambda v$ is equivalent to $(L - \lambda \operatorname{id}_V)v = 0$. ☐

Proposition XIII.99. *Let* $L \in \operatorname{Hom}(V)$ *be an operator on some vector space. Suppose* $\lambda_1, \ldots, \lambda_m$ *are distinct eigenvalues of* L *and* $v_1, \ldots, v_m$ *are corresponding eigenvectors. Then* $\{v_1, \ldots, v_m\}$ *is linearly independent.*

Proof. The proof goes by contradiction. Assume $\{v_1, \ldots, v_m\}$ is linearly dependent. Let k be the smallest positive integer such that

$$v_k \in \operatorname{span}\{v_1, \ldots, v_{k-1}\}.$$

So there exists a nontrivial linear combination

$$v_k = a_1 v_1 + \ldots + a_{k-1} v_{k-1}.$$

Applying L to both sides gives

$$\lambda_k v_k = a_1 \lambda_k v_1 + \ldots + a_{k-1} \lambda_k v_{k-1}.$$

Multipliying the previous combination by λ_k and subtracting both equations gives

$$0 = a_1(\lambda_k - \lambda_1)v_1 + \ldots + a_{k-1}(\lambda_k - \lambda_{k-1})v_{k-1}.$$

By assumption of linear independence of $\{v_1, \ldots, v_{k-1}\}$ this combination must be trivial, however none of the $(\lambda_k - \lambda_i)$ can be zero, so all the a_i must be zero. This is a contradiction with the assumption of linear dependence. ☐

Corollary XIII.99.1. *For each operator on* V*, the set of distinct eigenvalues has at most cardinality* $\dim V$.

Corollary XIII.99.2. *Let* $L \in \operatorname{Hom}(V)$*. Suppose* $\lambda_1, \ldots, \lambda_m$ *are distinct eigenvalues of* L*. Then*

$$E_{\lambda_1}(L) \oplus \ldots \oplus E_{\lambda_m}(L)$$

is a direct sum. Furthermore, the sum of geometric multiplicities is less than or equal to the dimension of V*:*

$$\dim E_{\lambda_1}(L) + \ldots + \dim E_{\lambda_m}(L) \leq \dim V.$$

4.2.4.2 Residual spectrum

Proposition XIII.100. *Let* L *be a densely defined linear operator on a Hilbert space. If* λ *is in the residual spectrum of* L *with deficiency* m*, then* $\overline{\lambda}$ *is in the point spectrum of* L^* *with multiplicity* m*.*

Proof. By XIII.158 we have

$$\operatorname{im}(\lambda \operatorname{id} - L)^\perp = \ker(\lambda \operatorname{id} - L)^* = \ker(\overline{\lambda} \operatorname{id} - L^*).$$

☐

4.2.4.3 Compression spectrum

The set of λ for which $T - \lambda I$ does not have dense range is the compression spectrum $\sigma_{\mathrm{cp}}(T)$ of T.

Then $\sigma_{\mathrm{r}}(T) = \sigma_{\mathrm{cp}}(T) \setminus \sigma_{\mathrm{p}}(T)$.

4.2.4.4 The essential spectrum

TODO `https://en.wikipedia.org/wiki/Spectrum_(functional_analysis)#Classification_of_points_in_the_spectrum`

4.2.5 The spectral radius

The spectral radius $\mathrm{spr}(T)$ of a operator T is given by

$$\mathrm{spr}(T) \coloneqq \sup_{\lambda \in \sigma(T)} |\lambda|.$$

4.2.6 The spectrum of operators on Hilbert spaces

Proposition XIII.101. *Let $T \in \mathcal{B}(H)$ for some Hilbert space H. Then*

1. $\sigma(T) \neq \emptyset$*;*
2. $\rho(T) = \overline{\rho(T^*)}$*, where the bar denotes complex conjugation.*

Proof. (1) Let $x, y \in H$ and define

$$f(\lambda) = \langle x, R_\lambda(T) y \rangle .$$

If $\sigma(T) = \emptyset$, then f is an entire function. Now

$$\|R_\lambda(T)\| \leq \frac{1}{|\lambda| - \|T\|} \to 0 \quad \text{as} \quad |\lambda| \to \infty.$$

By Liouville's theorem (TODO ref) we must have $f \equiv 0$. Because the x, y we arbitrary we must have $R_\lambda(T)y = 0$ for all $y \in H$, such that $R_\lambda(T)$ is not injective, which is impossible as it is an inver

Proposition XIII.102. *Let K be a compact operator on a Banach space. Then*

$$\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}.$$

Proof. For all $\lambda \neq 0$, we have that $\lambda \,\mathrm{id} - K$ is Fredholm with index zero (and thus bounded). Then by the Fredholm alternative XIII.228 $\lambda \,\mathrm{id} - K$ is either bijective or neither injective nor surjective, meaning λ is either in $\rho(T)$ or in $\sigma_{\mathrm{p}}(T)$. □

Proposition XIII.103. *Let K be a compact operator on a Banach space X. Then*

1. *for all $\lambda \in \sigma(K) \setminus \{0\}$ there exists a least m such that* $\ker(\lambda \,\mathrm{id} - K)^m = \ker(\lambda \,\mathrm{id} - K)^{m+1}$*. This space is finite dimensional and reducing for K;*
2. *for $\alpha > 0$ the number of eigenvalues λ such that $|\lambda| \geq \alpha$ is finite;*

3. 0 *is the only accumulation point; if* X *is infinite dimensional, then* $0 \in \sigma(K)$*;*

4. $\sigma(K)$ *is at most countably infinite;*

5. every $\lambda \in \sigma(K) \setminus \{0\}$ *is a pole of the resolvent* R_K*.*

Proof. `https://en.wikipedia.org/wiki/Spectral_theory_of_compact_operators` □

TODO: if K is a self-adjoint compact operator on a Hilbert space H, then H has an orthonormal basis of eigenvectors of K. se.
(2) Take $\lambda \in \rho(T)$. Then

$$((\lambda \operatorname{id} - A)^{-1})^* = (\overline{\lambda} \operatorname{id} - A^*)^{-1}$$

so $\overline{\lambda} \in \rho(T^*)$ iff $((\lambda \operatorname{id} - A)^{-1})^*$ is bounded iff $(\lambda \operatorname{id} - A)^{-1}$ is bounded iff $\lambda \in \rho(T)$. □

Lemma XIII.104. *Let* L *be a densely defined operator on a Hilbert space* H*. Take* $\lambda \in \sigma_p(L)$ *and* $\mu \in \sigma_p(L^*)$*. If* $\lambda \neq \overline{\mu}$*, then*

$$\ker(\lambda \operatorname{id} - L) \perp \ker(\mu \operatorname{id} - L^*).$$

Proof. Take non-zero eigenvectors x, y such that $Ax = \lambda x$ and $A^* y = \mu y$. Then

$$\lambda \langle y, x\rangle = \langle y, \lambda x\rangle = \langle y, Ax\rangle = \langle A^* y, x\rangle = \langle \mu y, x\rangle = \overline{\mu} \langle y, x\rangle .$$

So we have $(\lambda - \overline{\mu}) \langle y, x\rangle = 0$. □

Proposition XIII.105. *Let* L *be a densely defined operator on a Hilbert space* H*. Then the following are equivalent:*

1. the residual spectrum of L *is empty;*

2. $\overline{\sigma_p(L^*)} \subseteq \sigma_p(L)$*;*

as are the following:

1. the residual spectrum of L^* *is empty;*

2. $\sigma_p(L) \subseteq \overline{\sigma_p(L^*)}$*.*

In particular all these statements hold if L *is normal.*

Proof. Consider, for all $x \in \operatorname{dom}(L), y \in \operatorname{dom}(L^*)$, the equality

$$\langle (\lambda \operatorname{id} - L)x, y\rangle = \left\langle x, (\overline{\lambda} \operatorname{id} - L^*)y\right\rangle .$$

We can make the following inferences:

- If $\lambda \in \overline{\sigma_{\mathrm{p}}(L^*)}$, then the equality holds in particular for all eigenvectors y. This implies $\langle (\lambda \operatorname{id} - L)x, y\rangle = 0$. By X.118.2 $\operatorname{im}(\lambda \operatorname{id} - L)$ may then not be dense, so it cannot be injective because the residual spectrum of L is empty.
- Assume $\lambda \operatorname{id} - L$ injective and take $y \perp \operatorname{im}(\lambda \operatorname{id} - L)$. Then by the equality $\left\langle x, (\overline{\lambda} \operatorname{id} - L^*)y\right\rangle = 0$ for all $x \in \operatorname{dom}(L)$, which is dense. So $(\overline{\lambda} \operatorname{id} - L^*)y = 0$ by X.118.2. Now $\lambda \notin \sigma_{\mathrm{p}}(L)$, so $\overline{\lambda} \notin \sigma_{\mathrm{p}}(L^*)$. Thus $y = 0$ and $\operatorname{im}(\lambda \operatorname{id} - L)^\perp = \{0\}$, meaning $\operatorname{im}(\lambda \operatorname{id} - L)$ is dense.

The arguments for the second set of statements are similar.
If L is normal, then $\ker(\lambda \operatorname{id} - L) = \ker \overline{\lambda} \operatorname{id} - L^*$ by XIII.167.1, so $\sigma_{\mathrm{p}}(L) = \overline{\sigma_{\mathrm{p}}(L^*)}$. ☐

Proposition XIII.106. *Let T be a closed, densly defined operator on a Hilbert space.*

1. *If $\lambda \in \rho(T)$, then $\overline{\lambda} \in \rho(T^*)$.*
2. *If $\lambda \in \sigma_r(T)$, then $\overline{\lambda} \in \sigma_p(T^*)$.*
3. *If $\lambda \in \sigma_p(T)$, then $\overline{\lambda} \in \sigma_r(T^*) \cup \sigma_p(T^*)$.*

Proof. TODO Compare with XIII.105. CLosure necessary? ☐

Proposition XIII.107. *Let T be a unitary operator. Then*

1. *$\sigma_r(T) = \emptyset$;*
2. *$\sigma(T) \subset \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.*

TODO: move to more general place??

Lemma XIII.108. *The eigenvalues of a bounded dissipative linear operator lie in the half-plane $\mathfrak{Im}\,\lambda \geq 0$.*

4.2.6.1 Rayleigh quotient

Lemma XIII.109. *Let L be an operator on a Hilbert space. If x is an eigenvector with eigenvalue λ, then*

$$J_L(x) = \lambda.$$

Proof. Let x be an eigenvector with eigenvalue λ, then

$$J_L(x) = \frac{\langle x, Lx \rangle}{\langle x, x \rangle} = \lambda \frac{\langle x, x \rangle}{\langle x, x \rangle} = \lambda.$$

☐

Proposition XIII.110. *If U is unitary, then $\sigma(U) \subset \mathbb{T}$.*

4.3 Spectral theory for types of operators

4.3.1 Compact operators

Proposition XIII.111. *Let K be a compact operator on a Banach space. Then*

$$\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}.$$

Proof. For all $\lambda \neq 0$, we have that $\lambda \operatorname{id} - K$ is Fredholm with index zero (and thus bounded). Then by the Fredholm alternative XIII.228 $\lambda \operatorname{id} - K$ is either bijective or neither injective nor surjective, meaning λ is either in $\rho(T)$ or in $\sigma_{\mathrm{p}}(T)$. ☐

Proposition XIII.112. *Let K be a compact operator on a Banach space X. Then*

1. *for all $\lambda \in \sigma(K) \setminus \{0\}$ there exists a least m such that $\ker(\lambda \operatorname{id} - K)^m = \ker(\lambda \operatorname{id} - K)^{m+1}$. This space is finite dimensional and reducing for K;*

2. *for $\alpha > 0$ the number of eigenvalues λ such that $|\lambda| \geq \alpha$ is finite;*
3. *0 is the only accumulation point; if X is infinite dimensional, then $0 \in \sigma(K)$;*
4. *$\sigma(K)$ is at most countably infinite;*
5. *every $\lambda \in \sigma(K) \setminus \{0\}$ is a pole of the resolvent R_K.*

Proof. `https://en.wikipedia.org/wiki/Spectral_theory_of_compact_operators` □

TODO: if K is a self-adjoint compact operator on a Hilbert space H, then H has an orthonormal basis of eigenvectors of K.

4.3.2 Multiplication operators

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. A multiplication operator is an operator of the form

$$T : L^p(\Omega, \mu) \to L^p(\Omega, \mu) : u(x) \mapsto a(x)u(x)$$

for some $a \in L^\infty(\Omega, \mu)$

Proposition XIII.113. *Let $T : L^p(\Omega, \mu) \to L^p(\Omega, \mu) : u \mapsto a \cdot u$ be a multiplication operator. Then*

$$\|T\| = \|a\|_{L^\infty}.$$

Proof. From the inequality $\|Tu\|_{L^p} \leq \|a\|_{L^\infty} \|u\|_{L^p}$ we get $\|T\| \leq \|a\|_{L^\infty}$.
TODO □

Lemma XIII.114. *Let $T : L^2(\Omega, \mu) \to L^2(\Omega, \mu) : u \mapsto a \cdot u$ be a multiplication operator with $a \in L^\infty(\Omega, \mu)$. Then T^* is the multiplication operator*

$$T^* : L^2(\Omega, \mu) \to L^2(\Omega, \mu) : u \mapsto \overline{a} \cdot u.$$

Proof. From

$$\langle Tu, v\rangle = \int_\Omega a \cdot u \cdot \overline{v} \, \mathrm{d}\mu = \int_\Omega u \cdot \overline{\overline{a} \cdot v} \, \mathrm{d}\mu$$

it follows that $T^*v = \overline{a} \cdot v$. □

Corollary XIII.114.1. *Then*

1. *T is self-adjoint if a is real-valued;*
2. *T is skew-adjoint if a is purely imaginary;*
3. *T is unitary if $|a(x)| \equiv 1$.*

Let E_λ be the level set

$$E_\lambda = \{x \in \Omega \mid a(x) = \lambda\}$$

Proposition XIII.115. *Let $T : L^2(\Omega, \mu) \to L^2(\Omega, \mu) : u \mapsto a \cdot u$ be a multiplication operator with $a \in \mathcal{C}(\Omega)$. Then*

1. *$\sigma_p(T) = \{\lambda \in \mathrm{im}(a) \mid \mu(E_\lambda) > 0\}$;*

2. $\sigma_c(T) = \left\{\lambda \in \overline{\mathrm{im}(a)} \;\middle|\; \mu(E_\lambda) = 0\right\}$;
3. $\sigma_r(T) = \emptyset$;
4. $\rho(T) = \mathbb{C} \setminus \overline{\mathrm{im}(T)}$.

Proof. TODO ☐

4.4 The spectral theorem

`https://link.springer.com/content/pdf/10.1007%2F978-1-4614-7116-5.pdf`
`http://individual.utoronto.ca/jordanbell/notes/SVD.pdf` `https://digitalcommons.mtu.edu/cgi/viewcontent.cgi?article=2133&context=etdr`
`https://web.ma.utexas.edu/mp_arc/c/09/09-32.pdf`

4.5 Functional calculus

4.5.1 Holomorphic functional calculus

Theorem XIII.116 (Holomorphic functional calculus)**.** *Let A be a Banach algebra and $x \in A$. Consider the function*

$$\Phi_x : \mathcal{C}^\infty(\sigma(x), \mathbb{C}) \to A : f \mapsto f(x) \coloneqq \oint_\Gamma f(z) R_x(z) \,\mathrm{d}z.$$

Here Γ is any simple Jordan curve that contains $\sigma(x)$ such that f is holomorphic in a region that contains Γ and its interior. Then

1. *Φ_x is well-defined: it does not depend on the particular curve Γ;*
2. *Φ_x is a homomorphism;*
3. *for any polynomial $p \in \mathbb{C}[X]$, we have $\Phi_x(p) = p(x)$;*
   in particular $\Phi_x(\mathrm{id}_\mathbb{C}) = x$ and $\Phi_x(\underline{1}) = \mathrm{id}_A$;
4. *$\sigma(\Phi_x(f)) = f[\sigma(x)]$;*
5. *Φ_x is continuous if $\mathcal{C}^\infty(\sigma(x), \mathbb{C})$ is equipped with continuous convergence (?).*

TODO: $\mathcal{C}^\infty(\sigma(x))$ should be the space of functions that are analytic in some neighbourhood of $\sigma(x)$. Is it??

Proof. TODO ☐

TODO unbounded operators

4.5.1.1 Riesz eigenprojections

Holomorphic functional calculus applied to

$$\chi_{S,\delta} : A \to \{0,1\} : x \mapsto \begin{cases} 1 & d(x,S) \leq \delta \\ 0 & \text{otherwise.} \end{cases}$$

TODO: spectral measure with only disconnected parts in σ-algebra??
TODO: P_Δ and $E_\Delta \coloneqq \operatorname{im} P_\Delta$.

Lemma XIII.117. $\sigma(T|_{E_\Delta}) = \sigma(T) \cap \Delta$.

We call $\dim E_\lambda$ the algebraic multiplicity of λ.

4.5.1.2 Frobenius covariants

TODO P_λ is a Frobenius covariant. `https://en.wikipedia.org/wiki/Frobenius_covariant`
TODO cfr. Lagrange polynomial??

4.6 Jordan decomposition

4.6.1 Eigennilpotent

Let a be a finite element in a semisimple Banach algebra and $\lambda \in \sigma(a)$. The eigennilpotent operator of a at λ is defined as

$$D_\lambda \coloneqq (a - \lambda)P_\lambda.$$

This definition works because we can find a $\delta < d(\lambda, \sigma(a) \setminus \{\lambda\})$.

Lemma XIII.118. *Let a be a finite element in a semisimple Banach algebra and $\lambda \in \sigma(a)$. The eigennilpotent operator D_λ is nilpotent.*

Proof. By spectral mapping XIII.116, D_λ is quasinilpotent. Because a is finite, it is nilpotent by XII.34. □

4.6.2 Jordan vectors

Let V be a finite dimensional vector space and T an operator on V. A Jordan vector of T belonging to the eigenvalue λ is a vector $x \in V$ such that

$$(\lambda \operatorname{id}_V - T)^k x = 0$$

for some $k \in \mathbb{N}$. The least such k is called the degree of x and is denoted $\deg_J(x)$.

Eigenvectors are Jordan vectors of degree 1.

Proposition XIII.119. *Let V be a finite dimensional vector space, T an operator on V $\lambda \in \sigma(T)$ and $x \in V$. Then x is a Jordan vector of T belonging to the eigenvalue λ* if and only if *$x \in E_\lambda$.*

Proof. Let $x \in E_\lambda$. Then $x = P_\lambda x$ and thus

$$(\lambda \operatorname{id}_V - T)^k x = (\lambda \operatorname{id}_V - T)^k P_\lambda x = \big((\lambda \operatorname{id}_V - T)P_\lambda\big)^k x = D_\lambda^k x,$$

which is zero for some k because D_λ is nilpotent.
Conversely, assume x is a Jordan vector of T belonging to the eigenvalue λ. We can write $x = x_1 + x_2 \in E_\lambda \oplus E_{\mathbb{C}\setminus\{\lambda\}}$. Then (because E_λ is reducing for $T - \lambda \operatorname{id}_V$)

$$0 = (\lambda \operatorname{id}_V - T)^k x = (\lambda \operatorname{id}_V - T)^k x_1 + (\lambda \operatorname{id}_V - T)^k x_2 \in E_\lambda \oplus E_{\mathbb{C}\setminus\{\lambda\}}$$

Thus we have $(\lambda \operatorname{id}_V - T)^k x_1 = 0$ and $(\lambda \operatorname{id}_V - T)^k x_2 = 0$ separately. Now $T - \lambda \operatorname{id}_V$ is invertible on $E_{\mathbb{C}\setminus\{\lambda\}}$, so $x_2 = 0$ (TODO ref). This means that $x = x_1 \in E_\lambda$. ☐

Let $m = \deg_N(D_\lambda)$. Then we have

$$\{0\} \subsetneq \ker(\lambda \operatorname{id}_V - T) \subsetneq \ker(\lambda \operatorname{id}_V - T)^2 \subsetneq \ldots \subsetneq \ker(\lambda \operatorname{id}_V - T)^{m-1} \subsetneq \ker(\lambda \operatorname{id}_V - T)^m = V.$$

We define $E_\lambda^k \coloneqq \ker(\lambda \operatorname{id}_V - T)^k$. In particular

- E_λ^1 is the geometric eigenspace;
- E_λ^{m-1} is the algebraic eigenspace.

Lemma XIII.120. *Let V be a finite dimensional vector space, T an operator on V, $\lambda \in \sigma(T)$ and $x \in E_\lambda$. Then*

1. $1 \leq \dim \ker(\lambda \operatorname{id}_V - T) \leq \dim E_\lambda$;
2. $1 \leq \deg_J(x) \leq \dim_E \lambda$.

The lemma says the geometric multiplicity is smaller than the algebraic multiplicity.

Proof. Every eigenvector is a Jordan vector, so $\ker(\lambda \operatorname{id}_V - T) \subseteq E_\lambda$.
For all $k \in \mathbb{N}$ smaller then the degree of x, $(\lambda \operatorname{id}_V - T)^k x$ is a Jordan vector and thus in E_λ. TODO all $(\lambda \operatorname{id}_V - T)^k x$ are linearly independent (like in XII.34) ☐

Let V be a finite dimensional vector space, T an operator on V and $\lambda \in \sigma(T)$. The eigenvalue λ is called

- simple if the algebraic multiplicity is 1;
- semisimple if every Jordan vector in E_λ has degree 1;
- prime if the geometric multiplicity is 1.

If all eigenvalues of T are semisimple, then T is called a diagonal operator.

Lemma XIII.121. *An operator T is diagonal iff T is of the form $\sum_j a_j P_j$, where $a_j \in \mathbb{F}$ and P_j are projectors that commute pairwise.*

4.6.3 Characteristic polynomial and equation

Let V be a finite dimensional vector space and T an operator on V. The characteristic polynomial $p_T(x)$ of T is the polynomial

$$p_T(x) \coloneqq \det(x \operatorname{id}_V - T).$$

Proposition XIII.122. *Let V be a finite dimensional vector space, T an operator on V and $\sigma(T) = \{\lambda_j\}_{j=1}^r$. Then*

$$p_T(x) = \prod_{j=1}^{r}(x - \lambda_j)^{\dim E_{\lambda_j}}.$$

Proof. TODO ☐

Corollary XIII.122.1. *A number $\lambda \in \mathbb{C}$ is an eigenvalue of T* if and only if *it is a root of $p_T(x)$.*

The equation $p_T(x) = 0$ is the characteristic equation of T.

4.6.4 Spectral representation

Proposition XIII.123. *Let V be a finite dimensional complex vector space and T an operator on V. There exists a unique decomposition $T = S + D$ such that*

- *S is diagonal;*
- *D is nilpotent;*
- *$SD = DS$.*

If $\sigma(T) = \{\lambda_j\}_{j=1}^r$, this decomposition is given by

$$T = \sum_{j=1}^{r} \lambda_r P_{\lambda_r} + \sum_{j=1}^{r} D_{\lambda_r}.$$

4.6.5 Partial fraction decomposition of the resolvent

For any operator T on a vector space V with eigenvalue λ_0, the resolvent $R_T(\lambda)$ has a pole at λ_0.

Proposition XIII.124. *Let T be an operator on a finite dimensional vector space V and $\lambda_0 \in \sigma(T)$. Then the Laurent expansion of $R_T(\lambda)$ around λ_0 is of the form*

$$R_T(\lambda) = \frac{P_0}{\lambda - \lambda_0} + \sum_{n=1}^{\deg_N(D_0)-1} \frac{D_0^n}{(\lambda - \lambda_0)^{n+1}} + \sum_{n=0}^{\infty}(-1)^n S_0^{n+1}(\lambda - \lambda_0)^n,$$

where $P_0 \coloneqq P_{\lambda_0}, D_0 \coloneqq D_{\lambda_0}$ and S_0 is some fixed operator.

Proof. TODO ☐

The holomorphic part of the Laurent expansion of $R_T(\lambda)$ at λ_0 is called the reduced resolvent of T w.r.t. λ_0:

$$S_{T,\lambda_0}(\lambda) := \sum_{n=0}^{\infty}(-1)^n S_0^{n+1}(\lambda-\lambda_0)^n = R_T(\lambda) - \left(\frac{P_0}{\lambda-\lambda_0} + \sum_{n=1}^{\deg_N(D_0)-1}\frac{D_0^n}{(\lambda-\lambda_0)^{n+1}}\right).$$

Proposition XIII.125. *Let T be an operator on a finite dimensional vector space V and $\lambda_0 \in \sigma(T)$. Then*

$$R_{T|_{(\mathrm{id}_V - P_0)}}(\lambda) = S_{T,\lambda_0}|_{\mathrm{id}_V - P_0}(\lambda).$$

Proposition XIII.126. *Let T be an operator on a finite dimensional vector space V with $\sigma(T) = \{\lambda_j\}_{j=1}^r$. The partial fraction decomposition of $R_T(\lambda)$ is given by*

$$R_T(\lambda) = \sum_{j=1}^{r}\left(\frac{P_{\lambda_j}}{\lambda-\lambda_j} + \sum_{n=1}^{\deg_N(D_{\lambda_j})-1}\frac{D_{\lambda_j}^n}{(\lambda-\lambda_j)^{n+1}}\right).$$

The partial fraction decomposition of $S_{T,\lambda_k}(\lambda)$ is given by

$$S_{T,\lambda_k}(\lambda) = \sum_{\substack{j=1\\ j\neq k}}^{r}\left(\frac{P_{\lambda_j}}{\lambda-\lambda_j} + \sum_{n=1}^{\deg_N(D_{\lambda_j})-1}\frac{D_{\lambda_j}^n}{(\lambda-\lambda_j)^{n+1}}\right).$$

Proof. The poles of $R_T(\lambda)$ are exactly the eigenvalues of T. There are finitely many of them, so we can use partial fraction decomposition, XI.100. We just need to show that the holomorphic part is zero. For that we note that $\lim_{\lambda\to\infty} R_T(\lambda) = 0$ and all principal parts tend to 0 at infinity as well. Thus the holomorphic part also tends to 0, making it bounded. By Liouville's theorem, XI.86.2, we get that it is identically zero. □

Corollary XIII.126.1 (Sylvester-Lagrange formula)**.** *Let f be a holomorphic function on an open set that contains $\sigma(T)$. Then*

$$f(T) = \sum_{j=1}^{r}\left(f(\lambda_j)P_{\lambda_j} + \sum_{n=1}^{\deg_N(D_{\lambda_j})-1}\frac{f^{(n)}(\lambda_j)D_{\lambda_j}^n}{n!}\right).$$

Proof. We have

$$f(T) = \oint_\Gamma f(\lambda)R_T(\lambda)\,\mathrm{d}\lambda = 2\pi i\sum_{j=1}^{r}\mathrm{Res}_{\lambda_j} f(\lambda)R_T(\lambda)$$

by the residue theorem (TODO ref for operators). □

Corollary XIII.126.2 (Cayley-Hamilton)**.** *Let $p_T(x)$ be the characteristic polynomial of T. Then $p_T(T) = 0$.*

Proof. Since $p_T(x) = \prod_{j=1}^r (x-\lambda_j)^{\dim E_{\lambda_j}}$ and $\dim E_{\lambda_j} \geq \deg_N(D_{\lambda_j})$, we see that $p_T(\lambda)R_T(\lambda)$ has no poles and is holomorphic, meaning that $oint_\Gamma f(\lambda)R_T(\lambda)\,\mathrm{d}\lambda = 0$ by Cauchy's theorem (TODO ref for operators). □

4.6.6 Normal operators

Proposition XIII.127. *If T is a normal operator, then $P_\lambda = P_\lambda^*$ and $D_\lambda = D_\lambda^* = 0$.*

This means normal operators are diagonalisable.

Proof. TODO ☐

Corollary XIII.127.1. *Let V be a finite dimensional complex vector space and T a normal operator on V with $\sigma(T) = \{\lambda_j\}_{j=1}^r$.*

1. *We have the spectral decompositions*

$$T = \sum_{j=1}^{r} \lambda_r P_{\lambda_r} \qquad \text{and} \qquad T^* = \sum_{j=1}^{r} \overline{\lambda_r} P_{\lambda_r}.$$

2. *We have*

$$R_T(\lambda) = \sum_{j=1}^{r} \frac{P_{\lambda_j}}{\lambda - \lambda_j} \qquad \text{and} \qquad S_{T,\lambda_k}(\lambda) = \sum_{\substack{j=1 \\ j\neq k}}^{r} \frac{P_{\lambda_j}}{\lambda - \lambda_j}$$

4.6.7 Jordan decomposition

TODO matrix representation + matrix representation of Lagrange-Sylvester. See Baumgärtel

Chapter 5

Hilbert spaces

5.1 Examples

5.1.1 The ℓ^2 spaces

Sequence spaces ℓ^p Hilbert iff $p = 2$. (TODO: other sequence spaces?)

5.1.2 Direct sum

Let $(V_i)_{i\in I}$ be a family of Hilbert spaces. By considering them as Banach spaces we can take the ℓ^2-direct sum. (TODO: other sequence spaces?)

Proposition XIII.128. *Let $(V_i)_{i\in I}$ be a family of Hilbert spaces. The ℓ^2-direct sum is a Hilbert space.*

This gives the conventional interpretation of the Hilbert space direct sum: it is the ℓ^2-direct sum of the summands as Banach spaces.

5.2 Strong and weak convergence

5.2.1 Weak convergence of vectors

Let $\mathcal{H}$ be a Hilbert space. The weak convergence on $\mathcal{H}$ is the initial convergence w.r.t.

$$\{\langle x, -\rangle : \mathcal{H} \to \mathbb{F} \mid x \in \mathcal{H}\}.$$

Thus $F \xrightarrow{w} v$ if and only if $\langle x, F\rangle \longrightarrow \langle x, v\rangle$ for all $x \in \mathcal{H}$.
We denote the weak convergence $\xrightarrow{w}$ or $\lim^w$ and write $\mathcal{H}^w$ to denote the convergence space $(\mathcal{H}, \lim^w)$.

By XIII.5 we have that the weak convergence is a vector space convergence.

Lemma XIII.129. *Norm convergence is finer than weak convergence.*

Proof. The functions in $\{\langle x, -\rangle : \mathcal{H} \to \mathbb{F} \mid x \in \mathcal{H}\}$ are also continuous in the norm convergence. $\square$

Example

Let $\langle e_n \rangle$ be an orthonormal basis. Then the Riemann-Lebesgue lemma, X.126.4, gives $e_n \xrightarrow{w} 0$, even though clearly $e_n \not\to 0$.

So norm convergence is strictly finer.

Proposition XIII.130. *Let $\mathcal{H}$ be a Hilbert space, $x \in \mathcal{H}$ and $\langle x_n \rangle \subseteq \mathcal{H}$. Then*

1. $x_n \xrightarrow{w} x$ *implies* $\|x\| \leq \liminf \|x_n\|$;
2. $x_n \longrightarrow x$ if and only if $x_n \xrightarrow{w} x$ *and* $\|x_n\| \to \|x\|$.

Proof. (1) We have

$$\|x\|^2 = \langle x, x \rangle = \lim \langle x, x_n \rangle = \liminf \langle x, x_n \rangle \leq \|x\| \liminf \|x_n\|,$$

where the limit is in $\mathbb{R}$. We use that for convergent sequences, $\lim = \liminf$.
(2) The direction $\Rightarrow$ is clear form XIII.129 and the continuity of the norm.
For the converse, we have

$$\|x - x_n\|^2 = \|x\| - 2\,\Re\mathrm{e}(\langle x, x_n \rangle) + \|x_n\| \to 0,$$

because $\|x_n\| \to \|x\|$ and $\langle x, x_n \rangle \to \langle x, x \rangle = \|x\|^2$. □

`https://math.stackexchange.com/questions/1461363/closed-unit-ball-of-hilbert-space-se`

5.2.1.1 Weak Cauchy filters

Cauchy filters in $\mathcal{H}^w$ are called weak Cauchy filters.

Proposition XIII.131. *Let $\mathcal{H}$ be a real or complex Hilbert space. A filter F in $\mathcal{FP}(\mathcal{H})$ is a weak Cauchy filter* if and only if *$\langle x, F \rangle$ converges in $\mathbb{F}$ for all x.*

Proof. The following steps are equivalent:

- F is weak Cauchy;
- $F - F \xrightarrow{w} 0$;
- $\langle x, F \rangle - \langle x, F \rangle \to 0$ for all $x \in \mathcal{H}$;
- $\langle x, F \rangle$ is Cauchy in the scalar field for all $x \in \mathcal{H}$;
- $\langle x, F \rangle$ converges for all $x \in \mathcal{H}$.

The last equivalence follows because $\mathbb{R}$ and $\mathbb{C}$ are complete. □

Corollary XIII.131.1. *Every weak Cauchy filter is bounded in norm.*

Proof. Consider the family of bounded linear operator $\langle F, - \rangle$. Then $\langle F, - \rangle$ converges pointwise, by the proposition. By Banach-Steinhaus, XIII.81.1, we have $\sup_{y \in F} \|\langle y, - \rangle\| = \sup_{y \in F} \|y\| < \infty$. □

Note that the previous proposition does not mean a weak Cauchy filter is necessarily weakly convergent, as there may not exist a vector v such that $\langle x, F\rangle$ converges to $\langle x, v\rangle$ for all $x \in \mathcal{H}$. However, we have the following proposition:

Proposition XIII.132. *Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{H}^w$ is Cauchy complete.*

Proof. Let F be a weak Cauchy filter. Let $\langle e_i\rangle_{i\in I}$ be an orthonormal basis of $\mathcal{H}$. Let $c_i = \lim \langle e_i, F\rangle$. Then $F \xrightarrow{w} \sum_{i\in I} c_i e_i$.
First we check this sum is well-defined. TODO!! □

TODO: $\langle F, F\rangle$ weak convergent?????

5.2.2 Strong and weak convergence of operators

Let $\mathcal{H}$ be a Hilbert space. Then the

- strong operator topology is the topology of pointwise convergence on $\mathcal{L}(\mathcal{H})$;
- weak operator topology is the topology of pointwise convergence on $\mathcal{L}(\mathcal{H}^w)$.

We write $F \xrightarrow{SOT} A$ and $F \xrightarrow{WOT} A$ for the convergence of F to A in the strong and weak operator topologies.

TODO introduce shifts earlier.

Example

Consider the left and right shifts S_l and S_r on $\ell^2(\mathbb{N})$. Then

- $S_l \xrightarrow{SOT} 0$, but $S_l \overset{norm}{\not\longrightarrow} 0$;
- $S_r \xrightarrow{WOT} 0$, but $S_r \overset{SOT}{\not\longrightarrow} 0$.

In general taking adjoints is not continuous w.r.t. strong operator convergence, because $S_l \xrightarrow{SOT} 0$, but $S_r = S_l^* \overset{SOT}{\not\longrightarrow} 0$.

`https://math.stackexchange.com/questions/1054288/the-set-of-all-normal-operators-on-a`
Note normal operators not SOT-closed!

Proposition XIII.133. *If $\langle A_n\rangle$ is a sequence of normal operators that converges to a normal operator A in the strong operator topology, then $A_n^* \xrightarrow{SOT} A^*$.*

Proposition XIII.134. *Let $\langle A_n\rangle$ be a sequence of bounded operators on a Hilbert space and $A \in \mathcal{L}(\mathcal{H})$. Then*

1. *if $A_n \xrightarrow{SOT} A$, then $\|A\| \leq \liminf \|A_n\|$;*
2. *If $A_n x \longrightarrow Ax$ for all x in a dense subset of $\mathcal{A}$ and $\langle A_n\rangle$ is a bounded sequence, then $A_n \xrightarrow{SOT} A$.*

Proof. (1) For all unit vectors x we have

$$\|Ax\| = \left\| \lim_{SOT} A_n x \right\| = \lim_{n\to\infty} \|A_n x\| \leq$$

□

TODO: same for WOT.
TODO: Cauchy sequences are bounded. Does this follow form general Banach theory?

5.3 Tools to study operators

5.3.1 Spectrum

5.3.2 Numerical range

Lemma XIII.135. *Let T be an operator on a Hilbert space. If $\lambda \in \sigma_c(T)$, then there exists a Weyl sequence for λ.*

Proof. If $\lambda \in \sigma_\mathrm{c}(T)$, then $R_T(\lambda)$ is densely defined. □

Proposition XIII.136 (Spectral inclusion property of numerical range)**.** *Let T be an operator on a Hilbert space. Then*

1. $\sigma_p(T) \subseteq \mathrm{W}(T)$;
2. $\sigma_r(T) \subseteq \mathrm{W}(T)$;
3. $\sigma_{ap}(T) \subseteq \overline{\mathrm{W}(T)}$.

In particular $\sigma_c(T) \subseteq \overline{\mathrm{W}(T)}$.

Proof. (1) Let $\lambda \in \sigma_\mathrm{p}(T)$. Then $Tx = \lambda x$ for some unit vector x and so

$$\langle x, Tx\rangle = \langle x, \lambda x\rangle = \lambda \langle x, x\rangle = \lambda \|x\|^2 = \lambda,$$

which means that $\lambda \in \mathrm{W}(A)$.
(2) Let $\lambda \in \sigma_\mathrm{r}(T)$. Then there exists a unit vector $x \in \operatorname{im}(\lambda \operatorname{id} - T)^\perp$, so

$$0 = \langle x, (\lambda \operatorname{id} - T)x\rangle = \lambda \langle x\rangle^2 - \langle x, Tx\rangle = \lambda - \langle x, Tx\rangle,$$

which means that $\lambda \in \mathrm{W}(A)$.
(3) Let $\lambda \in \sigma_\mathrm{ap}(T)$. Then there exists a Weyl sequence $\langle e_n\rangle$ for λ by XIII.97. Then

$$\|(\lambda \operatorname{id} - T)e_n\| = \|e_n\| \, \|(\lambda \operatorname{id} - T)e_n\| \geq |\langle e_n, (\lambda \operatorname{id} - T)e_n\rangle| = |\lambda - \langle e_n, Te_n\rangle| \to 0.$$

Thus $\lambda \in \overline{\mathrm{W}(T)}$.
Finally we have $\sigma_\mathrm{c}(T) \subseteq \sigma_\mathrm{ap}(T)$ by XIII.96 and so $\sigma_\mathrm{c}(T) \subseteq \overline{\mathrm{W}(T)}$. □

Corollary XIII.136.1. *If T is a closed operator on a Hilbert space, then $\sigma(T) \subseteq \overline{\mathrm{W}(T)}$.*

5.4 Projectors and minimisation problems

Every subspace is a convex, non-empty subset.

Theorem XIII.137 (Hilbert projection theorem)**.** *Let $\mathcal{H}$ be a Hilbert space, K a closed, convex, non-empty subset of $\mathcal{H}$.*

1. *There exists a unique element of K of least norm. i.e. there exists a unique $k_0 \in K$ such that*
$$\|k_0\| = \inf\{\|k\| \mid k \in K\}.$$
i.e. $\min\{\|k\| \mid k \in K\}$ *exists.*
2. *For any $h \in \mathcal{H}$ there exists a unique point k_0 in K such that*
$$\|h - k_0\| = \inf\{\|h - k\| \mid k \in K\}.$$
We use this to define the distance $d(h, K) \coloneqq \|h - k_0\|$.
3. *If K is a (closed) subspace, then k_0 is also the unique point in K such that $(h - k_0) \perp K$.*

The idea for the first part of the proof is to take a sequence $\langle\|k_i\|\rangle \to \inf\{\|k\| \mid k \in K\}$. By the parallelogram law $\langle k_i\rangle$ is Cauchy and by completeness it has a limit k_0.

Proof. (1) We can find a sequence $\langle k_i\rangle$ in K such that $\|k_i\|$ converges to $d = \inf\{\|k\| \mid k \in K\}$ by VIII.192. By the parallelogram law

$$\begin{aligned}\|k_i - k_j\|^2 &= 2\|k_i\|^2 + 2\|k_j\|^2 - 4\left\|\frac{1}{2}(k_i + k_j)\right\|^2 \\ &\leq 2\|k_i\|^2 + 2\|k_j\|^2 - 4d^2\end{aligned}$$

the sequence $\langle k_i\rangle$ is Cauchy. So it converges to some k_0 in K because K is a closed subset of a complete space.
To prove uniqueness, take another $k_0' \in K$ such that $\|k_0'\| = d$. By convexity $\frac{1}{2}(k_0 + k_0') \in K$, hence
$$d \leq \left\|\tfrac{1}{2}(k_0 + k_0')\right\| \leq \tfrac{1}{2}(\|k_0\| + \|k_0'\|) = d.$$
So $\left\|\frac{1}{2}(k_0 + k_0')\right\| = d$. The parallelogram law gives

$$d^2 = \left\|\frac{k_0 + k_0'}{2}\right\|^2 = d^2 - \left\|\frac{k_0 - k_0'}{2}\right\|^2;$$

hence $\|k_0 - k_0'\|^2 = 0$ and thus $h_0 = k_0$.
(2) The element k_0 considered in point 1. is the point closest to a particular choice for h, namely $h = 0$. For other h consider the set $K - h$, which is again closed and convex.
(3) For all $k \in K$ and $a \in \mathbb{F}$, we have

$$\|h - k_0\| \leq \|h - k_0 + ak\|$$

and thus, by lemma X.125, $(h - k_0) \perp k$, meaning $(h - k_0) \perp K$.
For the converse (i.e. uniqueness), suppose $f_0 \in K$ such that $(h - f_0) \perp K$. Then for all $f \in K$ we have $(h - f_0) \perp (f_0 - f)$ so that

$$\begin{aligned}\|h - f\|^2 &= \|(h - f_0) + (f_0 - f)\|^2 \\ &= \|h - f_0\|^2 + \|f_0 - f\|^2 \geq \|h - f_0\|^2.\end{aligned}$$

So $\|h - f_0\| = \inf\{\|h - k\| \mid k \in K\} = d(h, K)$ and thus $f_0 = k_0$. □

Corollary XIII.137.1. *Let $\mathcal{H}$ be a Hilbert space and K a closed vector subspace. Then $\mathcal{H} = K^\perp \oplus K$.*

Proof. We need to prove every vector $x \in \mathcal{H}$ has a unique decomposition of the form

$$x = y + z \qquad y \in K,\ z \in K^\perp.$$

Such a decomposition exists: we can take $y = k_0$ and $z = x - k_0$. We have already proved uniqueness. We can also give another argument for uniqueness: assume another such decomposition $x = y' + z'$. Then $y - y' = z - z'$ where the left side is in K and the right in $K^\perp$. The only element in $K \cap K^\perp$ is 0, so $y = y'$ and $z = z'$. ☐

The ability to make such decompositions in general is unique to Hilbert spaces, see theorem XIII.140.

5.4.1 Orthogonal projection and decomposition

Let $\mathcal{H}$ be a Hilbert space. Given a subspace K and an element $x \in \mathcal{H}$, we call the unique element $y \in K$ of the decomposition $K \oplus K^\perp$ the orthogonal projection of x on K. It is denoted $P_K(x)$. This defines a function $P_K : \mathcal{H} \to K$ called the orthogonal projection on K.

Proposition XIII.138. *Let P be the orthogonal projection on a closed subspace K. Then*

1. *P is a linear operator on $\mathcal{H}$;*
2. *$\|Px\| \leq \|x\|$ for all $x \in \mathcal{H}$;*
3. *$P^2 = P$;*
4. *$\ker P = K^\perp$ and $\operatorname{im} P = K$;*
5. *$\operatorname{id}_{\mathcal{H}} - P$ is the orthogonal projection of $\mathcal{H}$ onto $K^\perp$.*

Proof. These are mostly direct results of the decomposition. In particular 5. follows if we know $K^\perp$ is closed, which it is by proposition X.118. ☐

Corollary XIII.138.1. *Let $\mathcal{H}$ ba a Hilbert space and K a closed subspace, then $\mathcal{H} = K \oplus K^\perp$.*

Proof. Let P be the orthogonal projection on K. Then by X.36

$$\mathcal{H} = \operatorname{im} P \oplus \ker P = K \oplus K^\perp.$$

☐

Corollary XIII.138.2. *Let $\mathcal{H}$ be a Hilbert space.*

1. *If K is a subspace, then $(K^\perp)^\perp = \overline{K}$ is the closure of K.*
2. *If A is a subset, then $(A^\perp)^\perp$ is the closed linear span of A.*

Proof. (1) Assume K is closed. Then using $0 = (I - P_K)x \;\Leftrightarrow\; x = P_K x$, we see

$$(K^\perp)^\perp = \ker(I - P_K) = \operatorname{im} P_K = K.$$

Then, if K is not closed, $(K^\perp)^\perp = (\overline{K}^\perp)^\perp = \overline{K}$, by proposition X.118.
(2) Using X.115 we calculate $(A^\perp)^\perp = (\operatorname{span}(A)^\perp)^\perp = \overline{\operatorname{span}(A)}$. ☐

Corollary XIII.138.3. *Let A be a subset of a Hilbert space $\mathcal{H}$. Then* $\operatorname{span}(A)$ *is dense in $\mathcal{H}$* if and only if $A^\perp = \{0\}$.

Proof. The subspace $\operatorname{span}(A)$ is dense in $\mathcal{H}$ iff $\overline{\operatorname{span}(A)} = \mathcal{H}$ iff $(\operatorname{span}(A)^\perp)^\perp = (A^\perp)^\perp = \mathcal{H}$ iff $A^\perp = \{0\}$.
In the last step we have used that $A^\perp$ is closed so that $((A^\perp)^\perp)^\perp = \overline{A^\perp} = A^\perp$, see X.118. ☐

5.4.1.1 Existence of orthonormal bases

Corollary XIII.138.4. *Let D be an orthonormal subset of a Hilbert space $\mathcal{H}$, then D is an ortonormal basis* if and only if *it is maximal.*

Proof. This is a restatement of the previous corollary in the language of X.129. ☐

Corollary XIII.138.5. *Every Hilbert space has an orthonormal basis.*

Proof. Every inner product space has a maximal orthonormal set by X.127. This maximal orthonormal set is an orthonormal set by the proposition. ☐

Corollary XIII.138.6. *An orthonormal subset of a Hilbert space is an orthonormal basis* if and only if *it is maximal.*

Lemma XIII.139. *Let $\mathcal{H}$ be a Hilbert space and K a closed subspace. Let $\{e_i\}_{i\in I}$ be an orthonormal basis of K. Then*

$$P_K(x) = \sum_{i\in I} \langle e_i, x\rangle\, e_i.$$

Proof. We can extend $\{e_i\}_{i\in I}$ to an orthonormal basis $\{e_i\}_{i\in J}$ of $\mathcal{H}$. Then

$$x = \sum_{i\in J} \langle e_i, x\rangle\, e_i = \sum_{i\in I} \langle e_i, x\rangle\, e_i + \sum_{i\notin I} \langle e_i, x\rangle\, e_i,$$

which is clearly a decomposition in $K \oplus K^\perp$. This is unique, so we have found $P_K(x)$. ☐

5.4.1.2 When are inner product spaces complete?

Notice that some of the results obtained for Hilbert spaces have one direction that is generally true for inner product spaces: in any inner product space we have

- $\overline{K} \subset (K^\perp)^\perp$;
- $\operatorname{span}(A)$ dense in $\mathcal{H}$ implies $A^\perp = \{0\}$;
- if D is an orthonormal basis, then it is maximal.

See X.118, X.118.1 and X.129.
The converses are only true for Hilbert spaces.

Theorem XIII.140. *Let H be an inner product space. If any of the following hold, H is a Hilbert space:*

1. *For any orthonormal set D,*

$$D \text{ is maximal} \quad\Longrightarrow\quad D \text{ is an orthonormal basis.}$$

2. *For any subset A, $A^{\perp} = \{0\}$ implies* $\operatorname{span}(A)$ *is dense in H.*
3. *For any subspace K, we have $(K^{\perp})^{\perp} = \overline{K}$.*
4. *For all closed subspaces K we can decompose $H = K \oplus K^{\perp}$.*

Proof. We prove the first statement implies H is a Hilbert space. The other three imply the first and thus that H is a Hilbert space.

1. We prove the contrapositive: assume H is not complete, we wish to show that 1. does not hold, i.e. there exists a maximal orthonormal subset of H that is not an orthonormal basis.

   Let $\mathcal{H}$ be the completion of H and take a unit vector $v \in \mathcal{H} \setminus H$. Now working in the completion, we have the decomposition $\operatorname{span}\{v\} \oplus \operatorname{span}\{v\}^{\perp}$. Consider the subspace $\operatorname{span}\{v\} + H = \operatorname{span}\{v\} \oplus (H \cap \operatorname{span}\{v\}^{\perp})$. We can extend $\{v\}$ to a maximal orthonormal set $\{v\} \cup D$ by X.127.

   We claim D is the orthonormal set we want:

   Firstly it is maximal. Assume, towards a contradiction, that D is not maximal in H, so there exists an orthonormal set $D' \supsetneq D$. Take $w \in D' \setminus D$ and let w' be the normalisation of $w - \langle v, w \rangle v$. Then $w' \perp v$ and $w' \perp D$, so $\{v\} \cup D \cup \{w'\}$ is an orthonormal set in $\operatorname{span}\{v\} + H$, which contradicts the maximality of $\{v\} \cup D$.

   Secondly it cannot be total. Indeed if $\operatorname{Cl}_H(\operatorname{span}(D)) = H \cap \overline{\operatorname{span}(D)}$ were equal to H, then $H \subseteq \overline{\operatorname{span}(D)}$ and thus $\mathcal{H} = \overline{H} \subseteq \overline{\operatorname{span}(D)} \subseteq \mathcal{H}$, meaning $\overline{\operatorname{span}(D)} = \mathcal{H}$. But $v \notin \overline{\operatorname{span}(D)}$, so $\overline{\operatorname{span}(D)} \neq \mathcal{H}$.

2. 2. clearly implies 1. We can also adapt the proof above to show 2. implies H is a Hilbert space: Assume H is not complete and let $\mathcal{H}$ be the completion of H. There exists a $v \in \mathcal{H} \setminus H$. All orthogonal complements are taken in the completion. The set

   $$U \coloneqq H \cap \{v\}^{\perp}$$

   is not dense in $\mathcal{H}$ for the same reason D was not total above. We claim that the orthogonal complement of U in H is $\{0\}$:

   $$U^{\perp} \cap H = \{0\}.$$

   First we claim U is dense in $\{v\}^{\perp}$: take a $w \in \{v\}^{\perp}$ and let $(x_n)_{n \in \mathbb{N}} \subseteq H$ converge to w (this is possible because $w \in \mathcal{H}$ and H is dense in $\mathcal{H}$). Fix some $x \in H$ such that $\langle x, v \rangle \neq 0$, then we have the following sequence in U that converges to w:

   $$n \mapsto x_n - \langle x_n, v \rangle \frac{x}{\langle x, v \rangle}.$$

   Then because U is dense in $\{v\}^{\perp}$,

   $$U^{\perp} \cap H = \overline{U}^{\perp} \cap H = (\{v\}^{\perp})^{\perp} \cap H = \operatorname{span}\{v\} \cap H = \{0\}.$$

3. Assume 3. Let D be a maximal orthonormal set. Then

   $$\overline{\operatorname{span}(D)} = (\operatorname{span}(D)^{\perp})^{\perp} = (D^{\perp})^{\perp} = \{0\}^{\perp} = H,$$

   so D is an orthonormal basis.

4. Assume 4. Let D be a maximal orthonormal set. Then $D^{\perp}$ is a closed subspace, so

   $$H = D^{\perp} \oplus (D^{\perp})^{\perp} = \{0\} \oplus (D^{\perp})^{\perp} = (\operatorname{span}(D)^{\perp})^{\perp} = \overline{\operatorname{span}(D)}.$$

□

5.4.1.3 Orthogonal decomposition

Theorem XIII.141. *A Banach space such all of its closed subspaces are complemented is isomorphic to a Hilbert space.*

Proof. TODO Lindestrauss and Tzafriri in 1971. Only real?? ☐

Proposition XIII.142. *Let $\mathcal{H}$ be a Hilbert space and let $\{V_i\}_{i\in I}$ be a family of closed, (pairwise) orthogonal subspaces. Then*

$$\bigoplus_{i\in I} V_i \qquad \text{is a closed subspace of } \mathcal{H}.$$

Proof. Let (v_n) be a Cauchy sequence in $\bigoplus_{i\in I} V_i$ which converges to w. Let $v_{i,n}$ be the component of v_n in V_i. By orthogonality we have

$$\|v_n - v_m\|^2 = \sum_{i\in I} \|v_{i,n} - v_{i,m}\|^2.$$

Then

$$\|v_{i,n} - v_{i,m}\| \leq \|v_n - v_m\|$$

which implies $(v_{i,n})_n$ is a Cauchy sequence in the closed space V_i which therefore converges to $w_i \in V_i$. Now there are only a finite number of i for which there exist non-zero $v_{i,n}$ (TODO proof!!!!). So then

$$\lim_n v_n = \lim_n \sum_{i\in I} v_{i,n} = \sum_{i\in I} w_i \in \bigoplus_{i\in I} V_i$$

where the interchange of limits and last equality follow because the sums are finite. ☐

Lemma XIII.143. *Let $\mathcal{H}$ be a Hilbert space and $A \supseteq B \supseteq C$ subspaces with B closed. Then*

$$(A \ominus B) \oplus (B \ominus C) = A \ominus C.$$

Proof. Take $v \in (A\ominus B) \oplus (B\ominus C)$. Then either $\{v\} \perp C$ or $\{v\} \perp B$, but this implies $\{v\} \perp C$, so $v \in A \ominus C$.
Take $v \in A \ominus C$. We can uniquely write $v = v_1 + v_2 \in (A \ominus B) \oplus B = A$. We just need to show that $v_2 \in B \ominus C$. Indeed assume $\langle c, v_2\rangle \neq 0$ for some $c \in C$. Then

$$\langle c, v\rangle = \langle c, v_1 + v_2\rangle = \langle c, v_1\rangle + \langle c, v_2\rangle = \langle c, v_2\rangle \neq 0,$$

so $v \notin A \ominus C$, a contradiction. ☐

5.4.2 Projection and minimisation in finite-dimensional spaces

Lemma XIII.144. *Let K be a subspace of $\mathbb{F}^n$ spanned by the orthonormal basis $\{\mathbf{u}_i\}_{i=1}^k$. Then*

$$P_K = QQ^* \qquad \text{where} \qquad Q = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_k\end{bmatrix}.$$

Proof. $P_K(\mathbf{x}) = \sum_{i=1}^k \mathbf{u}_i \langle \mathbf{u}_i, \mathbf{x}\rangle = \sum_{i=1}^k \mathbf{u}_i\mathbf{u}_i^*\mathbf{x} = \left(\sum_{i=1}^k \mathbf{u}_i\mathbf{u}_i^*\right)\mathbf{x} = QQ^*\mathbf{x}$. ☐

Corollary XIII.144.1. *For any matrix A with QR factorisation $A = QR$, we have*

$$P_{\text{col}(A)} = QQ^*.$$

In general $P_{\operatorname{col}(A)} = A(A^*A)^{-1}A^*$.

Proposition XIII.145 (Normal equations)**.** *Let $\{\mathbf{v}_i\}_{i=1}^k$ be linearly independent set of vectors in $\mathbb{F}^n$. Set $K = \operatorname{span}\{\mathbf{v}_i\}_{i=1}^k$. Then for all $\mathbf{x} \in \mathbb{F}^n$*

$$P_K(\mathbf{x}) = \sum_{i=1}^k c_i \mathbf{v}_i,$$

where $\begin{bmatrix} c_1 & c_2 & \ldots & c_k \end{bmatrix}^{\mathrm{T}}$ is the solution of

$$\begin{bmatrix} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_1, \mathbf{v}_k \rangle \\ \langle \mathbf{v}_2, \mathbf{v}_1 \rangle & \langle \mathbf{v}_2, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_2, \mathbf{v}_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{v}_k, \mathbf{v}_1 \rangle & \langle \mathbf{v}_k, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_k, \mathbf{v}_k \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} \langle \mathbf{v}_1, \mathbf{x} \rangle \\ \langle \mathbf{v}_2, \mathbf{x} \rangle \\ \vdots \\ \langle \mathbf{v}_k, \mathbf{x} \rangle \end{bmatrix}.$$

This system of linear equations is consistent, yielding a unique solution.

The equations in this proposition are known as normal equations and the matrix

$$G(\mathbf{v}_1, \ldots, \mathbf{v}_k) \coloneqq \begin{bmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \\ \vdots \\ \mathbf{v}_k^* \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_k \end{bmatrix} = \begin{bmatrix} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_1, \mathbf{v}_k \rangle \\ \langle \mathbf{v}_2, \mathbf{v}_1 \rangle & \langle \mathbf{v}_2, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_2, \mathbf{v}_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{v}_k, \mathbf{v}_1 \rangle & \langle \mathbf{v}_k, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_k, \mathbf{v}_k \rangle \end{bmatrix}$$

is known as the Gram matrix or Grammian.

Proof. TODO ☐

Proposition XIII.146. *Let $A \in \mathbb{F}^{m \times n}$, $\mathbf{b} \in \mathbb{F}^m$ and $\mathbf{x}_0 \in \mathbb{F}^n$. Then*

$$\min_{\mathbf{x} \in \mathbb{F}^n} \|\mathbf{b} - A\mathbf{x}\| = \|\mathbf{b} - A\mathbf{x}_0\|$$

if and only if

$$A^*A\mathbf{x}_0 = A^*\mathbf{b}.$$

We regard $\mathbf{x}_0$ as the "best approximate solution" to the (not necessarily consistent) system $A\mathbf{x} = \mathbf{b}$.

Proof. TODO ☐

5.4.3 Riesz representation

Theorem XIII.147 (Riesz-Fréchet representation theorem)**.** *Let $\mathcal{H}$ be a Hilbert space. For every continuous linear functional $\omega \in \mathcal{H}'$, there exists a unique $v_\omega \in \mathcal{H}$ such that*

$$\omega(x) = \langle v_\omega, x \rangle \qquad \forall x \in \mathcal{H}.$$

Moreover, $\|v_\omega\|_{\mathcal{H}} = \|\omega\|_{\mathcal{H}'}$.

The idea of the proof is as follows: consider $\mathcal{H} \cong \ker\omega \oplus \operatorname{im}\omega$. So we can find a subspace $U \subseteq \mathcal{H}$ such that $\mathcal{H} = \ker\omega \oplus U$. Clearly $\dim U = \dim\operatorname{im}\omega = \dim\mathbb{F} = 1$. Between 1-dimensional spaces there can only be one linear map, up to rescaling. This map is given by $x \mapsto \langle v, x\rangle$ for some $v \in U$, where the scaling determines the v. So we choose v such that $\omega|_U = x \mapsto \langle v, x\rangle$.
Now we want extend this form of $\omega|_U$ to the whole of $\mathcal{H}$. This works exactly if $v \in (\ker\omega)^\perp$. So we need $U = (\ker\omega)^\perp$ which is true if and only if $\mathcal{H} = \ker\omega \oplus U = \ker\omega \oplus (\ker\omega)^\perp$, which only works in general if $\ker\omega$ is closed and $\mathcal{H}$ is a Hilbert space. Now $\ker\omega$ is closed if and only if it is continuous, by X.86.
With this idea we give a full proof:

Proof. If $\ker\omega = \mathcal{H}$, we can take $v_\omega = 0$.
Assume $\ker\omega \neq \mathcal{H}$, then $(\ker\omega)^\perp \neq \{0\}$ by XIII.138.3, because $\ker\omega$ is closed (X.86). So we can take a non-zero $u \in (\ker\omega)^\perp$. We can choose it such that $\omega(u) = 1$, by rescaling. Now let $h \in \mathcal{H}$. We can write $h = (h - \omega(h)u) + \omega(h)u \in \ker\omega \oplus (\ker\omega)^\perp$, because $\omega(h - \omega(h)u) = 0$. So

$$0 = \langle u, h - \omega(h)u\rangle = \langle u, h\rangle - \omega(u)\|u\|^2.$$

If $v_\omega = \|u\|^{-2}u$, then $\omega(h) = \langle v_\omega, h\rangle$ for all $h \in \mathcal{H}$.
For uniqueness: assume we can find two vectors v_ω, v'_ω such that for all $h \in \mathcal{H}$ we have $\omega(h) = \langle v_\omega, h\rangle = \langle v'_\omega, h\rangle$. Then $v_\omega - v'_\omega \perp \mathcal{H}$, so $v_\omega - v'_\omega = 0$. ☐

Together with lemma X.108.1 this gives:

Corollary XIII.147.1. *The map $C_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}' : v \mapsto \langle v, \cdot\rangle$ is a bijective anti-linear isometry.*

Corollary XIII.147.2. *Every Hilbert space is reflexive.*

Proof. TODO ☐

Corollary XIII.147.3. *Every bounded functional defined on a closed subspace of $\mathcal{H}$ can be extended to a functional on $\mathcal{H}$ with the same norm.*

Proof. The functional on the closed subspace, say K, can be represented as $x \mapsto \langle v, x\rangle_K$ for some $v \in K$. The extended functional is then simply given by $x \mapsto \langle v, x\rangle_{\mathcal{H}}$. ☐

Proposition XIII.148 (Representation of sesquilinear forms)**.** *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces over $\mathbb{F}$ and $h : \mathcal{H}_1, \mathcal{H}_2 \to \mathbb{F}$ a bounded sesquilinear form. Then there exists a unique bounded operator $S : \mathcal{H}_1 \to \mathcal{H}_2$ such that*

$$h(x, y) = \langle Sx, y\rangle .$$

This operator has the property $\|S\| = \|h\|$.

Proof. For fixed x, $y \mapsto h(x, y)$ is a bounded linear functional, so by the Riesz representation theorem XIII.147 this can be represented by a unique v_x. Let S be the function $x \mapsto v_x$. Then $h(x, y) = \langle Sx, y\rangle$.
To prove this function S is linear, take arbitrary $x_1, x_2 \in \mathcal{H}_1; y \in \mathcal{H}_2$ and $\lambda \in \mathbb{F}$. Then

$$\begin{aligned}\langle S(\lambda x_1 + x_2), y\rangle &= h(\lambda x_1 + x_2, y) = \overline{\lambda} h(x_1, y) + h(v, y_2) \\ &= \overline{\lambda}\langle Sx_1, y\rangle + \langle Sx_2, y\rangle = \langle \lambda Sx_1 + Sx_2, y\rangle ,\end{aligned}$$

so S is linear by lemma X.103.

The equality of norms follows from

$$\begin{aligned}\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|\langle Sx, y\rangle|}{\|x\|\|y\|} &\geq \sup_{\substack{x\neq 0\\ Sx\neq 0}} \frac{|\langle Sx, Sx\rangle|}{\|x\|\|Sx\|} = \sup_{x\neq 0} \frac{\|Sx\|}{\|x\|} = \|S\| \\ &\leq \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{\|Sx\|\|y\|}{\|x\|\|y\|} = \sup_{x\neq 0} \frac{\|Sx\|}{\|x\|} = \|S\|\end{aligned}$$

where the second inequality is Cauchy-Schwarz. □

5.5 Orthonormal bases

Hamel basis / Schauder basis / Hilbert basis
Every Hilbert basis is Schauder basis if V is separable.
Hamel basis too big in Banach space??
Necessity of completeness for existence of complete orthonormal system, i.e. orthonormal system $\{a_i\}_{i\in I}$ (so $a_i \cdot a_j = \delta_{ij}$) with

$$v = \sum_{i\in I} (a_i \cdot v) a_i$$

for all v. This is equivalent with

$$v \cdot w = \sum_{i\in I} (v \cdot a_i)(a_i \cdot w)$$

for all v, w.

Theorem XIII.149 (Riesz-Fischer). *Let $\{e_i\}_{i\in I}$ be an orthonormal basis of a Hilbert space H and $\alpha : I \to \mathbb{C}$ a net. Then*

$$\sum_{i\in I} \alpha_i e_i$$

converges if and only if $\sum_{i\in I} |\alpha_i|^2 < \infty$.

Proof. If $\sum_{i\in I} \alpha_i e_i$ converges, then $\sum_{i\in I} |\alpha_i|^2$ is bounded by the Bessel inequality X.126.2. By monotone convergence, $\sum_{i\in I} |\alpha_i|^2 < \infty$ is equivalent to saying the sum converges. By (ref TODO) α has finite support. So $\sum_{i\in I} \alpha_i e_i$ can be expressed as the series

$$\sum_{k\in\mathbb{N}} \alpha_{i_k} e_{i_k}.$$

By completeness it is enough to show that $\langle s_n \rangle = \langle \sum_{k=0}^n \alpha_{i_k} e_{i_k} \rangle$ is Cauchy. Let $n < m$, then

$$\|s_n - s_m\|^2 = \left\| \sum_{k=m+1}^n \alpha_{i_k} e_{i_k} \right\|^2 = \sum_{k=m+1}^n \|\alpha_{i_k} e_{i_k}\|^2 = \sum_{k=m+1}^n |\alpha_{i_k}|^2 = \sum_{k=0}^n |\alpha_{i_k}|^2 - \sum_{k=0}^m |\alpha_{i_k}|^2.$$

Since $\left\langle \sum_{k=0}^n |\alpha_{i_k}|^2 \right\rangle$ is convergent, it is Cauchy and thus so is $\langle s_n \rangle$. □

Corollary XIII.149.1. *Let $\mathcal{H}$ be a Hilbert space and D be an orthonormal basis of $\mathcal{H}$. Then $\mathcal{H}$ is isometrically isomorphic to $\ell^2(D)$.*

Corollary XIII.149.2. *Hilbert spaces whose orthonormal bases have the same cardinality are isometrically isomorphic.*

??

Lemma XIII.150. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Then $L^2(\Omega, \mu)$ is separable* if and only if *μ is σ-finite.*

Lemma XIII.151. *Let $\{\phi_n(x)\}_{n=0}^{\infty}$ be an orthonormal basis of $L^2(\Omega, \mu)$ and $\{\psi_n(x)\}_{n=0}^{\infty}$ be an orthonormal basis of $L^2(\Lambda, \nu)$, then $\{\phi_n(x)\psi_m(y)\}_{n,m=0}^{\infty}$ is an orthonormal basis of $L^2(\Omega \times \Lambda, \mu \times \nu)$.*

Proof. The set $\{\phi_n(x)\psi_m(y)\}_{n,m=0}^{\infty}$ is orthonormal:

$$\iint_{\Omega\times\Lambda} \phi_n(x)\psi_m(y)\overline{\phi_{n'}(x)\psi_{m'}(y)}\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) = \int_\Omega \phi_n(x)\overline{\phi_{n'}(x)}\,\mathrm{d}\mu(x)\cdot\int_\Lambda \psi_m(y)\overline{\psi_{m'}(y)}\,\mathrm{d}\nu(y) = \delta_{n,n'}\delta_{m,m'},$$

using Fubini's theorem and the Hölder inequality (TODO refs).
To show $D = \{\phi_n(x)\psi_m(y)\}_{n,m=0}^{\infty}$ is an orthonormal basis, we verify point 5. of X.130: if $f \perp D$, then $f = 0$.
If $f \perp D$, then for all $m, n \in \mathbb{N}$

$$0 = \langle f, \phi_n\psi_m\rangle = \iint_{\Omega\times\Lambda} f(x,y)\overline{\phi_n(x)\psi_m(y)}\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) = \int_\Omega \left(\int_\Lambda f(x,y)\overline{\psi_m(y)}\,\mathrm{d}\nu(y)\right)\overline{\phi_n(x)}\,\mathrm{d}\mu(x).$$

Using point 5. of X.130 in $L^2(\Omega, \mu)$, we see that for all m the function $x \mapsto \int_\Lambda f(x,y)\overline{\psi_m(y)}\,\mathrm{d}\nu(y)$ is 0 as an element of $L^2(\Omega, \mu)$, i.e. it is 0 a.e. as a function of x. Let

$$E_m = \left\{ x \in \Omega \;\middle|\; \int_\Lambda f(x,y)\overline{\psi_m(y)}\,\mathrm{d}\nu(y) \neq 0 \right\}$$

and set $E = \bigcup_{m\in\mathbb{N}} E_m$. Then

$$\mu(E) = \mu\left(\bigcup_{m\in\mathbb{N}} E_m\right) \leq \sum_{m\in\mathbb{N}} \mu(E_m) = 0.$$

For $x \notin E$, we have $\int_\Lambda f(x,y)\overline{\psi_m(y)}\,\mathrm{d}\nu(y) = 0$, so by the same logic $f(x,y) = 0$ for almost all y.
Now $|f|^2$ is integrable and

$$\iint_{\Omega\times\Lambda} |f(x,y)|^2\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) = \int_{\Omega\setminus E}\int_\Lambda |f(x,y)|^2\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) = 0,$$

so $f = 0$ in $L^2(\Omega \times \Lambda, \mu \times \nu)$. □

5.6 Adjoints of operators

Let H, K be Hilbert spaces and $T : H \not\to K$ an operator. An adjoint of T is an operator

$S : K \not\to H$ such that

$$\langle w, Tv\rangle_K = \langle Sw, v\rangle_H \quad \forall v \in \operatorname{dom}(T),\ \forall w \in \operatorname{dom}(S).$$

Theorem XIII.152 (Hellinger-Toeplitz). *Let $T : H \to K$ be an operator between Hilbert spaces (which is defined everywhere), then T has an adjoint that is defined everywhere* if and only if *it is bounded.*

Proof. The "if" will be shown below by explicit construction. For the "only if", take such an operator T.
First we show T has closed graph, by using proposition X.92: assume (x_n) converges to x and (Tx_n) converges to y. Then

$$\langle z, Tx\rangle = \langle Sz, x\rangle = \lim_n \langle Sz, x_n\rangle = \lim_n \langle z, Tx_n\rangle = \langle z, y\rangle$$

where we have used the boundedness of $x \mapsto \langle z, x\rangle$. By the non-degeneracy of the inner product, $Tx = y$. So the graph of T is closed. Similarly the graph of S is closed. Applying the closed graph theorem XIII.46, yields the boundedness of T and S. □

Corollary XIII.152.1. *Everywhere-defined symmetric operators are bounded.*

Example

The adjoint of the left-shift operator

$$S_L : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) : (x_n)_{n\in\mathbb{N}} \mapsto (x_{n+1})_{n\in\mathbb{N}}$$

is the right-shift operator

$$S_R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) : (x_n)_{n\in\mathbb{N}} \mapsto \left(\begin{cases} x_{n-1} & (n \geq 1) \\ 0 & (n = 0)\end{cases}\right)_{n\in\mathbb{N}}.$$

5.6.1 The adjoint as a relation

Lemma XIII.153. *Let $T : H \not\to K$ be an operator between Hilbert spaces. Let S_1, S_2 be adjoints of T then for all $x \in \operatorname{dom}(S_1) \cap \operatorname{dom}(S_2)$ we have $S_1(x) - S_2(x) \in \operatorname{dom}(T)^\perp$.*
Conversely, let S be an adjoint of T and $x \in \operatorname{dom}(S)$. Then for all $v \in \operatorname{dom}(T)^\perp$ there exists an adjoint S' such that $S'(x) = S(x) + v$.

Proof. For all $u \in \operatorname{dom}(T)$ we have

$$\langle S_1(x) - S_2(x), u\rangle_H = \langle S_1(x), u\rangle_H - \langle S_2(x), u\rangle_H = \langle x, Tu\rangle_K - \langle x, Tu\rangle_K = 0.$$

So $\{S_1(x) - S_2(x)\} \in \operatorname{dom}(T)^\perp$.
For the converse, set $S' = S + \frac{\langle x, \cdot\rangle_K}{\langle x, x\rangle_K} v$. This is an adjoint: for all $a \in \operatorname{dom}(T), b \in \operatorname{dom}(S') = \operatorname{dom}(S)$ we have

$$\langle S'b, a\rangle_H = \langle Sb, a\rangle_H + \frac{\langle x, b\rangle_K}{\langle x, x\rangle_K} \langle v, a\rangle_H = \langle Sb, a\rangle_H = \langle b, Ta\rangle_K .$$

□

Corollary XIII.153.1. *Let $T : H \nrightarrow K$ be a densely defined operator between Hilbert spaces. Let S_1, S_2 be adjoints of T then for all $x \in \operatorname{dom}(S_1) \cap \operatorname{dom}(S_2)$ we have $S_1(x) = S_2(x)$.*

Proof. We have $\operatorname{dom}(T)^\perp = \overline{\operatorname{dom}(T)}^\perp = H^\perp = \{0\}$. So $S_1(x) - S_2(x) = 0$. ☐

Corollary XIII.153.2. *Let $T : H \nrightarrow K$ be an operator between Hilbert spaces. Then*

$$\bigcup \{\operatorname{graph}(S) \mid S \in (K \nrightarrow H) \text{ is an adjoint of } T\}$$

is the graph of an operator if and only if *T is densely defined.*

Let $T : H \nrightarrow K$ be an operator between Hilbert spaces. We define the adjoint T^* as the relation on (H, K) with graph

$$\operatorname{graph}(T^*) \coloneqq \bigcup \{\operatorname{graph}(S) \mid S \in (K \nrightarrow H) \text{ is an adjoint of } T\}.$$

- If $T^* = T$, we say T is self-adjoint.
- If $T^* = -T$, we say T is skew-adjoint.

We denote the set of self-adjoint operators on H by $\mathcal{SA}(H)$.

Note that, by XIII.153.2, the adjoint is a function if and only if T is densely defined.

Lemma XIII.154. *Let $T : H \nrightarrow K$ be a densely defined operator between Hilbert spaces. If S is an adjoint of T that is defined everywhere, then $T^* = S$.*

Corollary XIII.154.1. *Let H be a Hilbert space. Then $\operatorname{id}_H^* = \operatorname{id}_H$.*

Proposition XIII.155. *Let $T : H \nrightarrow K$ be an operator between Hilbert spaces. Then*

$$\operatorname{dom}(T^*) = \{x \in K \mid \operatorname{dom}(T) \to \mathbb{F} : u \mapsto \langle x, Tu \rangle \text{ is a bounded functional}\}.$$

Proof. $\boxed{\subseteq}$ If $\omega_x : u \mapsto \langle x, Tu \rangle$ is bounded, then its domain can be extended by linearity to all of H and it has a Riesz vector x^* such that $\omega_x = u \mapsto \langle x^*, u \rangle$. The linear operator with domain $\operatorname{span}\{x\}$ that maps x to x^* is then an adjoint.
$\boxed{\supseteq}$ If $x \in \operatorname{dom}(T^*)$, then, using the Cauchy-Schwarz inequality,

$$|\langle x, Tu \rangle| = |\langle T^*x, u \rangle| \leq \|T^*x\| \, \|u\|,$$

so the functional $u \mapsto \langle x, Tu \rangle$ is bounded. ☐

Corollary XIII.155.1. *The domain $\operatorname{dom}(T^*)$ is a vector space and in particular contains 0.*

Corollary XIII.155.2. *Let $S, T : H \nrightarrow K$ be operators between Hilbert spaces such that $S \subseteq T$. Then $T^* \subseteq S^*$.*

Corollary XIII.155.3. *Let H, K be Hilbert spaces. Then $(*, *)$ is a Galois connection between $((H \nrightarrow K), \subseteq)$ and $((K \nrightarrow H), \subseteq)$.*

Proof. TODO ☐

Proposition XIII.156 (Algebraic properties of the adjoint)**.** *Let T, S be compatible operators between Hilbert spaces and $\lambda \in \mathbb{C}$. Then*

1. $S^* + T^* \subseteq (S+T)^*$*;*

2. $S^*T^* \subseteq (TS)^*$*;*

3. $\begin{pmatrix} \mathrm{id} & 0 \\ 0 & \overline{\lambda}\,\mathrm{id} \end{pmatrix} \operatorname{graph}(T^*) \subseteq (\lambda T)^*$*;*

4. if $\lambda \neq 0$*, then* $\begin{pmatrix} \mathrm{id} & 0 \\ 0 & \overline{\lambda}\,\mathrm{id} \end{pmatrix} \operatorname{graph}(T^*) = (\lambda T)^*$*;*

5. $(T + \lambda\,\mathrm{id})^* = T^* + \overline{\lambda}\,\mathrm{id}$.

Proof. (1) Let f be an adjoint of S and g an adjoint of T. It is enough to see that $f+g$ is an adjoint of $S+T$. Indeed $\forall w \in \operatorname{dom}(f+g), v \in \operatorname{dom}(S+T)$

$$\langle (f+g)(w), v\rangle = \langle f(w), v\rangle + \langle g(w), Tv\rangle = \langle w, Sv\rangle + \langle w, Tv\rangle = \langle w, (S+T)v\rangle .$$

(2) Let f be an adjoint of T and g an adjoint of S. It is enough to see that gf is an adjoint of TS. Indeed

$$\langle g\circ f(w), v\rangle = \langle f(w), Sv\rangle = \langle w, TSv\rangle \qquad \forall w \in \operatorname{dom}(g\circ f), v \in \operatorname{dom}(TS).$$

(3) Let f be an adjoint of T. It is enough to see that $\overline{\lambda}f$ is an adjoint of λT. Indeed

$$\left\langle \overline{\lambda} f(w), v\right\rangle = \lambda \langle f(w), v\rangle = \lambda \langle w, Tv\rangle = \langle w, \lambda Tv\rangle \qquad \forall w \in \operatorname{dom}(f), v \in \operatorname{dom}(T).$$

(4) One inclusion has already been proved. For the other inclusion, let f be an adjoint of λT. It is enough to see that $\overline{\lambda^{-1}}f$ is an adjoint of T, because then $f = \overline{\lambda}\cdot\overline{\lambda^{-1}}f \subseteq \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \overline{\lambda}\,\mathrm{id} \end{pmatrix} \operatorname{graph}(T^*)$. Indeed

$$\left\langle \overline{\lambda^{-1}} f(w), v\right\rangle = \lambda^{-1} \langle f(w), v\rangle = \left\langle w, \lambda^{-1}\lambda Tv\right\rangle = \langle w, Tv\rangle \quad \forall w \in \operatorname{dom}(f), v \in \operatorname{dom}(T).$$

(5) From (1) and (3), we have $T^* + \overline{\lambda}\,\mathrm{id} \subseteq (T+\lambda\,\mathrm{id})^*$. Conversely, let f be an adjoint of $(T+\lambda\,\mathrm{id})$. It is enough to see that $f - \overline{\lambda}\,\mathrm{id}$ is an adjoint of T. Indeed, $\forall w \in \operatorname{dom}(f - \overline{\lambda}\,\mathrm{id}), v \in \operatorname{dom}(T)$,

$$\begin{aligned}\left\langle (f - \overline{\lambda}\,\mathrm{id})(w), v\right\rangle &= \langle f(w), v\rangle - \lambda \langle w, v\rangle \\ &= \langle w, (T+\lambda\,\mathrm{id})(v)\rangle - \lambda\langle w, v\rangle \\ &= \langle w, Tv\rangle - \lambda\langle w, v\rangle + \lambda\langle w, v\rangle = \langle w, Tv\rangle .\end{aligned}$$

□

Proposition XIII.157. *Let* $T : H \not\to K$ *be an operator between Hilbert spaces. Then*

$$\operatorname{graph}(T^*) = \left(\begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \operatorname{graph}(T)\right)^{\perp} = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \operatorname{graph}(T)^{\perp}.$$

If T *is densely defined, then* T^* *is a closed operator.*

Proof. We have

$$\operatorname{graph}(T^*) = \bigcup \{\operatorname{graph}(S) \mid S \in (K \not\to H) \text{ is an adjoint of } T\}.$$

Take an adjoint S and (w, Sw) in $\operatorname{graph}(S)$. Then for all $v \in \operatorname{dom}(T)$:

$$0 = \langle w, Tv\rangle_K - \langle Sw, v\rangle_H = \langle w, Tv\rangle_K + \langle Sw, -v\rangle_H = \langle (w, Sw), (Tv, -v)\rangle_{K\oplus H}.$$

So $(Tv, -v) = \begin{pmatrix} 0 & -\operatorname{id} \\ \operatorname{id} & 0 \end{pmatrix}(v, Tv) \in \operatorname{graph}(S)^\perp$.

The final equality follows from X.117, using the fact that $\begin{pmatrix} 0 & -\operatorname{id} \\ \operatorname{id} & 0 \end{pmatrix}$ is a surjective isometry.
If T is densely defined, then T^* is a function by XIII.153.2. It is closed by X.118. ☐

Corollary XIII.157.1. *Let $T : H \not\to K$ be a densely defined operator between Hilbert spaces. Then T^* is densely defined* if and only if *T is closable. In this case $\overline{T} = T^{**}$.*

Proof. From the proposition we have

$$\begin{aligned}\operatorname{graph}(T^{**}) &= \begin{pmatrix} 0 & -\operatorname{id} \\ \operatorname{id} & 0 \end{pmatrix}\operatorname{graph}(T^*)^\perp = \begin{pmatrix} 0 & -\operatorname{id} \\ \operatorname{id} & 0 \end{pmatrix}\left(\begin{pmatrix} 0 & -\operatorname{id} \\ \operatorname{id} & 0 \end{pmatrix}\operatorname{graph}(T)^\perp\right)^\perp \\ &= \begin{pmatrix} 0 & -\operatorname{id} \\ \operatorname{id} & 0 \end{pmatrix}^2\operatorname{graph}(T)^{\perp\perp} = -\operatorname{graph}(T)^{\perp\perp} = \overline{\operatorname{graph}(T)} = \operatorname{graph}(\overline{T}).\end{aligned}$$

The right-hand side is the graph of an operator iff T is closable and the left-hand side is the graph of an operator iff T^* is densely defined, by XIII.153.2.
By definition, the left-hand side is equal to $\operatorname{graph}(T^{**})$, so $T^{**} = \overline{T}$. ☐

Proposition XIII.158. *Let $T : H \not\to K$ be a densely defined operator between Hilbert spaces. Then*

1. $\ker(T^*) = \operatorname{im}(T)^\perp$;
2. $\ker(T) = \operatorname{im}(T^*)^\perp$.

Proof. First take $v \in \ker(T^*)$, then $T^*(v) = 0$ which implies

$$\forall x \in \operatorname{dom}(T) : \langle T^*(v), x\rangle = 0 \implies \forall x \in \operatorname{dom}(T) : \langle v, T(x)\rangle = 0 \implies v \perp \operatorname{im}(T).$$

Next take $v \perp \operatorname{im}(T)$ TODO complete. ☐

TODO: link with previous? + Drop densely defined.

Proposition XIII.159. *Let $S : K \not\to H$ and $T : H \not\to K$ be linear operators between Hilbert spaces. If*

$$\operatorname{im}(S \cap T^*) = H \qquad \textit{and} \qquad \operatorname{im}(T \cap S^*) = K,$$

then S and T are densely defined with $S^* = T$ and $T^* = S$.

Proof. Notice that $S \cap T^*$ and $T \cap S^*$ are linear operators that are adjoints of each other.
We claim that they are densely defined: take $x \in \operatorname{dom}(S\cap T^*)^\perp$. Then there exists some $y \in H$ such that $x = (T \cap S^*)y$ because of surjectivity. Now for all $z \in \operatorname{dom}(S \cap T^*)$

$$0 = \langle z, x\rangle = \langle z, (T \cap S^*)y\rangle = \langle (S \cap T^*)z, y\rangle,$$

so $\langle z', y\rangle = 0$ for all $z' \in H$, by surjectivity. This means, by X.103, that $y = 0$ and thus also $x = (T\cap S^*)y = 0$. We conclude that $\operatorname{dom}(S\cap T^*)^\perp = \{0\}$, meaning $(S\cap T^*)$ is densely defined. The argument for $(T \cap S^*)$ is similar.

It follows that S and T must be densely defined. We have, by XIII.158,

$$\ker(S) = \operatorname{im}(S^*)^\perp \subseteq \operatorname{im}(T \cap S^*)^\perp = \{0\}.$$

Similarly $\ker(T) = \ker(S^*) = \ker(T^*) = \{0\}$.
So we have $\ker(S) = \ker(T^*)$, $\operatorname{im}(S) \subseteq \operatorname{im}(S \cap T^*)$ and $\operatorname{im}(T^*) \subseteq \operatorname{im}(S \cap T^*)$. The equality $S = T^*$ follows from I.132. The equality $T = S^*$ is similar. □

Proposition XIII.160. *Let $T : H \not\to K$ be a densely defined operator between Hilbert spaces. Then*

1. $\operatorname{im}(T)$ *is dense in K* if and only if T^* *is injective;*
2. *if T and T^* are injective, then $(T^*)^{-1} = (T^{-1})^*$;*
3. *if T is closable and $\overline{T}$ is injective, then $\overline{T}^{-1} = \overline{T^{-1}}$.*

Proof. (1) This is immediate from XIII.158 and X.23:

$$\operatorname{im}(T) \text{ is dense} \qquad \iff \qquad \{0\} = \operatorname{im}(T)^\perp = \ker(T^*).$$

(2) We have $\operatorname{graph}(T^{-1}) = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \operatorname{graph}(T)$. Also note that $\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$ commute. Then we compute using XIII.157:

$$\begin{aligned} \operatorname{graph}((T^*)^{-1}) &= \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \operatorname{graph}(T)^\perp \\ &= \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \operatorname{graph}(T)^\perp \\ &= \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \left(\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \operatorname{graph}(T) \right)^\perp = \operatorname{graph}((T^{-1})^*). \end{aligned}$$

The penultimate equality follows from X.117, using the fact that $\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$ is a surjective isometry. □

`https://arxiv.org/pdf/1507.08418.pdf` `https://link.springer.com/article/10.1007/s43036-020-00068-4`

5.6.2 Bounded operators

Proposition XIII.161. *Let $T : H \to K$ be a closed densely defined operator between Hilbert spaces. Then*

1. $T \in \mathcal{B}(H, K)$ if and only if $T^* \in \mathcal{B}(K, H)$.

In this case

2. $\|T\| = \|T^*\|$;
3. $T^* = C_H^{-1} T^t C_K$*, where C_K is the Riesz isometry from XIII.147.1.*

Proof. First assume $T \in \mathcal{B}(H,K)$. Then $u \mapsto \langle x, Tu \rangle$ is a bounded functional for all $x \in K$, so $\operatorname{dom}(T^*) = K$ by XIII.155. Also T^* is closed by XIII.157, so it is bounded by the closed graph theorem XIII.46.
Now assume $T^* \in \mathcal{B}(K,H)$. By the previous argument $T = \overline{T} = T^{**} \in \mathcal{B}(H,K)$.
The function $(x,u) \mapsto \langle x, Tu \rangle$ is a bounded sesquilinear form. By proposition XIII.148, T^* must be the unique S from the proposition, which has norm $\|T\|$.
Finally we note that $C_H^{-1} T^t C_K$ is an adjoint with domain K and conclude by XIII.154. ☐

Lemma XIII.162. *The adjoint defines a map $* : \mathcal{B}(H,K) \to \mathcal{B}(K,H)$ that is anti-linear and continuous in the weak and uniform operator topologies. It is continuous in the strong operator topology* if and only if *finite dimensional.*

Proof. By the proposition the adjoint map is anti-linear. It is also bounded with norm 1. Then by corollary X.73 it must be bounded.
TODO ☐

Lemma XIII.163. *Let $S, T \in \mathcal{B}(H,K)$ and $\lambda \in \mathbb{F}$.*

1. $(T^*)^* = T$;
2. $(S+T)^* = S^* + T^*$;
3. $(\lambda T)^* = \bar{\lambda} T^*$;
4. $\mathrm{id}_V^* = \mathrm{id}_V$.

Let $T \in \mathcal{B}(H_1, H_2), S \in \mathcal{B}(H_2, H_3)$

5. $(ST)^* = T^* S^*$.

Proof. These follow straight from XIII.156 and the fact that the operators and adjoints are everywhere defined. ☐

> Useful exercise: The identities of XIII.163 can also be proven by elementary manipulations. For example, to prove 1., we take arbitrary $v \in H$ and $w \in K$, Then
>
> $$\langle w, Tv \rangle = \langle T^* w, v \rangle = \overline{\langle v, T^* w \rangle} = \overline{\langle (T^*)^* v, w \rangle} = \langle w, (T^*)^* v \rangle .$$
>
> By lemma X.103 we have $Tv = (T^*)^* v$ for all $v \in V$.

Proposition XIII.164. *Let H, K be Hilbert spaces and $T : H \to K$ a bijective bounded linear operator with bounded inverse. Then $(T^*)^{-1}$ exists and*

$$(T^*)^{-1} = (T^{-1})^*.$$

Proof. We prove $(T^{-1})^*$ is both a left- and a right-inverse of T^*: $\forall x \in H, y \in K$

$$\begin{aligned} \left\langle T^*(T^{-1})^* x, y \right\rangle &= \left\langle x, T^{-1} T y \right\rangle = \langle x, y \rangle \\ \left\langle x, (T^{-1})^* T^* y \right\rangle &= \left\langle T T^{-1} x, y \right\rangle = \langle x, y \rangle \end{aligned}$$

So, by lemma X.103, $T^*(T^{-1})^* = \mathrm{id}_H$ and $(T^{-1})^* T^* = \mathrm{id}_K$. ☐

Proposition XIII.165. *Let $T \in \mathcal{B}(H,K)$. Then*

$$\ker T = (\operatorname{im} T^*)^\perp \qquad \textit{and thus} \qquad \overline{\operatorname{im}(T)} \subseteq \ker(T^*)^\perp.$$

Proof.

$$x \in \ker T \iff Tx = 0 \iff \forall y \in K : \langle y, Tx \rangle = 0 \iff \forall y \in K : \langle T^* y, x \rangle = 0 \iff x \perp T^*[K].$$

☐

In particular $\operatorname{im}(T)$ is closed iff it is equal to $\ker(T^*)^\perp$. This is sometimes known as the closed range theorem. This is, e.g the case when T is bounded below, see X.95.

Proposition XIII.166. *Let $T \in \mathcal{B}(H, K)$ with H, K Hilbert spaces. Then*

$$\|T^*T\| = \|T\|^2 = \|TT^*\|.$$

Proof. For $\|T^*T\| = \|T\|^2$ first observe that

$$\|T^*T\| \leq \|T^*\| \cdot \|T\| = \|T\|^2.$$

Conversely, $\forall x \in H$:

$$\|T(x)\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle \leq \|T^*Tx\| \cdot \|x\| \leq \|T^*T\| \cdot \|x\|^2.$$

The other equality follows by applying the first to T^* and using $\|T^*\| = \|T\|$. ☐

5.6.3 Normal operators

A densely defined linear operator T on a Hilbert space H is normal if it is closed and $TT^* = T^*T$.

Self-adjoint and unitary operators are normal.
TODO 3.10 Self-Adjoint, Unitary and Normal Operators from Kreyszig.

Proposition XIII.167. *Let $T : H \not\rightarrow H$ be a densely defined operator. Then T is normal* if and only if $\operatorname{dom}(T) = \operatorname{dom}(T^*)$ *and* $\forall x \in H : \|Tx\| = \|T^*x\|$.

Proof. First, assume T normal. Then

$$\|Tx\|^2 = |\langle Tx, Tx \rangle| = |\langle T^*Tx, x \rangle| = |\langle TT^*x, x \rangle| = |\langle T^*x, T^*x \rangle| = \|T^*x\|^2.$$

For the converse, we have $\langle Tx, Ty \rangle = \langle T^*x, T^*y \rangle$ for all $x, y \in H$ by polarisation. From this we have $\langle T^*Tx, y \rangle = \langle TT^*x, y \rangle$ and normality follows from X.136.
TODO question of domain. `https://www.math.drexel.edu/faculty/mjz55/wp-content/uploads/sites/8/2017/01/normalnotes.pdf`. ☐

Corollary XIII.167.1. *If T is a normal operator, then* $\ker T = \ker T^*$.

Proof. We have $x \in \ker(T) \iff \|Tx\| = 0 \iff \|T^*x\| = 0 \iff x \in \ker(T^*)$. ☐

Corollary XIII.167.2. *If T is a normal operatorm then*

1. $\sigma_r(T) = \emptyset$;
2. $\sigma(T) = \sigma_{ap}(T)$.

Proof. If T is normal, then so is $\lambda \operatorname{id} - T$. Now $\lambda \in \sigma_{\mathrm{r}}(T)$ iff $\ker(\lambda \operatorname{id} - T) = \{0\}$ and $\operatorname{im}(\lambda \operatorname{id} - T)^\perp \neq \{0\}$, but $\operatorname{im}(\lambda \operatorname{id} - T)^\perp = \ker(\lambda \operatorname{id} - T)^* = \ker(\lambda \operatorname{id} - T)$. By XIII.158 and the previous corollary. This is a contradiction.
(2) then follows straight from (1). □

Theorem XIII.168. *The closure of the numerical range of a normal operator is the convex hull of its spectrum.*

Proof. Normal operators T are by definition closed, so $\sigma(T) \subseteq \overline{\mathrm{W}(T)}$ by XIII.136. TODO □

Lemma XIII.169. *For normal elements the spectral radius equals the norm.*

Lemma XIII.170. *A normal operator on a Hilbert space is invertible* if and only if *it is bounded below.*

5.6.4 Symmetric and self-adjoint operators

5.6.4.1 Domain related matters

A symmetric operator A is self-adjoint if and only if $\operatorname{dom}(A) = \operatorname{dom}(A^*)$.

Lemma XIII.171. *Let A be a symmetric densely defined operator on a Hilbert space H. Then*

1. $\operatorname{dom}(A) \subseteq \operatorname{dom}(A^*)$*;*
2. $A = A^*|_{\operatorname{dom}(A)}$*;*
3. *A is closable and $\overline{A} = A^{**}$.*

Proof. (1, 2) This is immediate from symmetry: $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in \operatorname{dom}(A)$.
(3) From (1) we see that A^* is densely defined, because the superset of a dense set is dense. The result follows by XIII.157.1. □

Corollary XIII.171.1. *A closed and densely defined symmetric operator is self-adjoint* if and only if *A^* is symmetric.*

Proof. If A^* is not symmetric, A can clearly not be self-adjoint.
Assume A^* is symmetric. Then $\operatorname{dom}(A) \subseteq \operatorname{dom}(A^*) \subseteq \operatorname{dom}(\overline{A}) = \operatorname{dom}(A)$. □

Proposition XIII.172. *A self-adjoint operator cannot have a proper symmetric extension.*

Proof. Assume A self-adjoint and $A \subseteq B$ for some symmetric operator B. Then

$$A \subseteq B \subseteq B^* \subseteq A^* = A,$$

so $A = B$. We have used XIII.171 and XIII.155.2. □

Corollary XIII.172.1. *Let A be a densely defined symmetric operator. If $\overline{A}$ is self-adjoint, then it is the unique self-adjoint extension of A.*

Note that $\overline{A}$ is always an operator by XIII.171.

Proof. Let B be a self-adjoint extension of A. Then $\overline{A} = A^{**} \subseteq B^{**} = B$, by XIII.155.2. This means that B is symmetric extension of the self-adjoint operator $\overline{A}$, which, by the proposition, implies $B = \overline{A}$. □

In general it is possible for an unbounded, symmetric operator to not have a self-adjoint extension or have multiple self-adjoint extensions, even if it is densely defined. (TODO example)

Let A be a densely defined symmetric operator whose closure is self-adjoint. We call A

- essentially self-adjoint;
- a core for A.

Example

Consider the operator

$$A : L^2(a,b) \to L^2(a,b) : f \mapsto i\frac{\mathrm{d}f}{\mathrm{d}x}$$

with domain

$$\operatorname{dom}(A) = \left\{ f \in L^2(a,b) \;\middle|\; \tfrac{\mathrm{d}f}{\mathrm{d}x} \in L^2(a,b),\ f(a) = 0 = f(b) \right\}.$$

Then

$$\begin{aligned}\langle g, Af\rangle &= \int_a^b \overline{g(x)} i \tfrac{\mathrm{d}f(x)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \overline{g(b)}f(b) - \overline{g(a)}f(a) - \int_a^b \left(i\tfrac{\mathrm{d}}{\mathrm{d}x}\overline{g(x)}\right) f(x)\,\mathrm{d}x \\ &= \int_a^b \overline{i\tfrac{\mathrm{d}g(x)}{\mathrm{d}x}} f(x)\,\mathrm{d}x = \langle Ag, f\rangle\,.\end{aligned}$$

So A is symmetric and $\operatorname{dom}(A^*) = \left\{ f \in L^2(a,b) \;\middle|\; \tfrac{\mathrm{d}f}{\mathrm{d}x} \in L^2(a,b) \right\}$. We cannot extend $\operatorname{dom}(A)$ while keeping $\operatorname{dom}(A^*)$ the same, because A would no longer be symmetric due to boundary terms.
There are, however, multiple ways we can extend A to a self-adjoint operator (in each case $\operatorname{dom}(A^*)$ must shrink).
Let A_α, for $\alpha \in \mathbb{R}$, be the operator A with domain

$$\operatorname{dom}(A_\alpha) = \left\{ f \in L^2(a,b) \;\middle|\; \tfrac{\mathrm{d}f}{\mathrm{d}x} \in L^2(a,b),\ f(b) = e^{i\alpha} f(b) \right\}.$$

We must have $\forall f \in \operatorname{dom}(A_\alpha)$ and $g \in \operatorname{dom}(A_\alpha^*)$ that

$$\overline{g(b)}f(b) - \overline{g(a)}f(a) = f(a)\left(e^{i\alpha}\overline{g(b)} - \overline{g(a)}\right) = 0,$$

so we have $e^{-i\alpha}g(b) = g(a)$ and thus $g(b) = e^{i\alpha}g(a)$, which means $\operatorname{dom}(A_\alpha^*) = \operatorname{dom}(A_\alpha)$.
So A_α is a self-adjoint extension of A for all $\alpha \in \mathbb{R}$.
TODO: compare Aharonov-Bohm

Notice that the operator

$$T : L^2(a,b) \to L^2(a,b) : f \mapsto i\frac{\mathrm{d}f}{\mathrm{d}x}$$

with domain

$$\operatorname{dom}(T) = \left\{ f \in L^2(a,b) \;\middle|\; \tfrac{\mathrm{d}f}{\mathrm{d}x} \in L^2(a,b) \right\}$$

is not symmetric. In this case

$$\mathrm{dom}(T^*) = \left\{ f \in L^2(a,b) \;\middle|\; \tfrac{\mathrm{d}f}{\mathrm{d}x} \in L^2(a,b),\ f(a) = 0 = f(b) \right\},$$

so $\mathrm{dom}(T^*) \subseteq \mathrm{dom}(T)$.

5.6.4.2 Spectrum and related criteria

Lemma XIII.173. *Let A be a symmetric operator on a complex Hilbert space H. If $\exists z \in \mathbb{C} \setminus \mathbb{R} :\ \mathrm{im}(A + z\,\mathrm{id}) = H$, then A is densely defined.*

Proof. Let $A + z\,\mathrm{id}$ be surjective and suppose, towards a contradiction that there exists an $y \perp \mathrm{dom}(A)$. Then $y = (A + z\,\mathrm{id})x$ for some $x \in \mathrm{dom}(A)$ by surjectivity. Then

$$0 = \Im\mathrm{m}\,\langle x, y\rangle = \Im\mathrm{m}\,\langle x, (A + z\,\mathrm{id})x\rangle = \cancel{\Im\mathrm{m}\,\langle x, Ax\rangle} + \Im\mathrm{m}\,\langle x, zx\rangle = \Im\mathrm{m}(z)\|x\|^2.$$

By assumption, $\Im\mathrm{m}(z) \neq 0$, so $x = 0$, meaning $y = (A+z\,\mathrm{id})x = 0$ and thus $\mathrm{dom}(A)^\perp = \{0\}$. ☐

Proposition XIII.174. *Let A be a symmetric operator on a complex Hilbert space H. Then $A + z\,\mathrm{id}_H$ is bounded below by $|\,\Im\mathrm{m}\,\lambda|$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. We first calculate, $\forall x \in H$:

$$\Im\mathrm{m}\,\langle x, (A + z\,\mathrm{id}_H)x\rangle = \cancel{\Im\mathrm{m}\,\langle x, Ax\rangle} + \Im\mathrm{m}\,z\|x\|^2.$$

Thus

$$|\,\Im\mathrm{m}\,\lambda|\;\|x\|^2 = |\,\Im\mathrm{m}\,\langle x, (A + z\,\mathrm{id}_H)x\rangle\,| \leq |\,\langle x, (A + z\,\mathrm{id}_H)x\rangle\,| \leq \|x\|\;\|(A + z\,\mathrm{id}_H)x\|,$$

which means that $\|(A + z\,\mathrm{id}_H)x\| \geq |\,\Im\mathrm{m}\,\lambda|\;\|x\|$, so $A + z\,\mathrm{id}_H$ is bounded below by $|\,\Im\mathrm{m}\,\lambda|$. ☐

Corollary XIII.174.1. *Let A be a symmetric operator on a complex Hilbert space H. Then $\sigma_{ap}(A) \subseteq \mathbb{R}$.*

Corollary XIII.174.2. *The eigenvalues of a symmetric operator are real.*

Proof. This is immediate using $\sigma_{\mathrm{p}}(A) \subseteq \sigma_{\mathrm{ap}}(A)$. We can also give a direct calculation: Assume there exists an $x \in \ker(\lambda\,\mathrm{id}_H - A) \setminus \{0\}$. Then $Ax = \lambda x$ and thus

$$\lambda\|x\|^2 = \lambda\,\langle x, x\rangle = \langle x, \lambda x\rangle = \langle x, Ax\rangle = \langle Ax, x\rangle = \langle \lambda x, x\rangle = \overline{\lambda}\,\langle x, x\rangle = \overline{\lambda}\|x\|^2.$$

Because $\|x\|^2 \neq 0$, we have $\lambda = \overline{\lambda}$, meaning λ is real. ☐

Corollary XIII.174.3. *Let A be a symmetric operator on a complex Hilbert space H. Then for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the resolvent $R_A(\lambda)$ well-defined and bounded by $\|R_A(\lambda)\| \leq 1/|\,\Im\mathrm{m}\,\lambda|$.*

Note this does not mean $\mathbb{C} \setminus \mathbb{R} \subseteq \rho(A)$, as $\mathrm{dom}(R_A(\lambda))$ may not be all of H.

Proof. This is an application of X.91. ☐

Proposition XIII.175. *Let A be a symmetric operator on a Hilbert space H. The following are equivalent:*

1. $\exists z \in \mathbb{C} :\ \mathrm{im}(A + z\,\mathrm{id}) = H = \mathrm{im}(A + \overline{z}\,\mathrm{id})$;

2. A is self-adjoint;

3. $\sigma_r(A) = \emptyset$.

In this case $\sigma(A) = \sigma_{ap}(A)$.

Notice that in (1) we include $\mathbb{R}$ and in (3) we exclude $\mathbb{R}$.

Proof. (1) ⇒ (2) From XIII.171, we have $A \subseteq A^*$ and thus $A + z\,\mathrm{id} = (A^* + z\,\mathrm{id}) \cap (A + z\,\mathrm{id})$. From point (5) of XIII.156, we have $A + z\,\mathrm{id} = (A^* + z\,\mathrm{id}) \cap (A + z\,\mathrm{id}) = (A + \overline{z}\,\mathrm{id})^* \cap (A + z\,\mathrm{id})$. We then use XIII.159 with $S = A + z\,\mathrm{id}$ and $T = A + \overline{z}\,\mathrm{id}$ to obtain $(A + z\,\mathrm{id})^* = A + \overline{z}\,\mathrm{id}$. Subtracting $\overline{z}\,\mathrm{id}$ from each side yields the result.
(2) ⇒ (3) Fix some $z \in \sigma(A) \setminus \sigma_{\mathrm{p}}(A)$ we need to show that $\overline{\mathrm{im}(\lambda\,\mathrm{id} - A)} = H$. Indeed

$$\overline{\mathrm{im}(A + z\,\mathrm{id})} = \ker(A^* + \overline{z}\,\mathrm{id})^{\perp} = \ker(A + \overline{z}\,\mathrm{id})^{\perp} = \{0\}^{\perp} = H.$$

(3) ⇒ (1) We have $\sigma(A) = \sigma_{\mathrm{ap}}(A)$. Because $\sigma_{\mathrm{ap}} \subseteq \mathbb{R}$, by XIII.174.1, we have that $A + z\,\mathrm{id}$ is surjective for all $\mathbb{C} \setminus \sigma(A) = \mathbb{C} \setminus \sigma_{\mathrm{ap}}(A) \supseteq \mathbb{C} \setminus \mathbb{R}$. □

Corollary XIII.175.1. *Every surjective symmetric operator is self-adjoint.*

Proof. Take $z = 0$ in point (1). □

Proposition XIII.176. *Let A be a closed symmetric operator. Then one of the following cases holds:*

- *A is self-adjoint, in which case* $\sigma(A) \subseteq \mathbb{R}$*;*
- $\sigma(A) = \overline{\mathbb{C}^{\uparrow}}$*;*
- $\sigma(A) = \overline{\mathbb{C}^{\downarrow}}$*;*
- $\sigma(A) = \mathbb{C}$.

If A is not densely-defined, then the last case holds.

We have denoted the closed upper half plane $\overline{\mathbb{C}^{\uparrow}}$ and the closed lower half plane $\overline{\mathbb{C}^{\downarrow}}$.

Proof. First assume A self-adjoint, then $\sigma(A) \subseteq \mathbb{R}$ by a combination of XIII.174.1 and XIII.175. Now note that if there exists a real $\lambda \in \mathbb{R}$ such that $\lambda \in \rho(A)$, then in particular $\lambda\,\mathrm{id} - A$ is surjective, so A is self-adjoint by XIII.175.
Now assume A not self-adjoint and pick a $\lambda \in \mathbb{C}^{\uparrow}$. From XIII.175 we must have either $\lambda \in \sigma(A)$ or $\overline{\lambda} \in \sigma(A)$ (or both).
If $\lambda \in \rho(A)$, then $\mathbb{C}^{\uparrow} \subseteq \rho(A)$ and if $\overline{\lambda} \in \rho(A)$, then $\mathbb{C}^{\downarrow} \subseteq \rho(A)$.
Indeed take some $\mu \in \mathbb{C}$. By XIII.174.3 we only need to check surjectivity of $\mu\,\mathrm{id} - A$. Now note that

$$\begin{aligned}(\mu\,\mathrm{id} - A)R_A(\lambda) &= (\mu\,\mathrm{id} - \lambda\,\mathrm{id} + \lambda\,\mathrm{id} - A)R_A(\lambda)\\ &= (\mu - \lambda)R_A(\lambda) + (\lambda\,\mathrm{id} - A)R_A(\lambda)\\ &= (\mu - \lambda)R_A(\lambda) + \mathrm{id},\end{aligned}$$

which we can consider as a bounded perturbation of id. Thus by XIII.94, $(\mu\,\mathrm{id} - A)R_A(\lambda)$ is surjective if $\|(\mu - \lambda)R_A(\lambda)\| < 1$, i.e. $|\mu - \lambda| < |\mathfrak{Im}(\lambda)|$ using XIII.174.3.
We can iterate this construction to cover the whole of $\mathbb{C}^{\uparrow}$. The argument for $\overline{\lambda}$ is similar. □

Example

Spectrum half plane TODO `https://math.stackexchange.com/questions/893899/spectrum-of-symmetric-non-selfadjoint-operator-on-hilbert-space` `https://math.stackexchange.com/questions/925097/spectrum-of-self-adjoint-operator-on-hilbert-space-real`

Proposition XIII.177. *Let A be a closed symmetric operator on a Hilbert space. Then A is positive* if and only if $\sigma(A) \subseteq [0, \infty[$.

Proof. XIII.168 ☐

Proposition XIII.178. *Let A be a self-adjoint operator. Then*

1. $\inf \sigma(A) = \inf \mathrm{W}(A)$*;*
2. $\sup \sigma(A) = \sup \mathrm{W}(A)$.

Proof. XIII.168 ☐

Proposition XIII.179. *Let T be a densely defined self-adjoint operator. Then*

1. $\sigma_r(T) = \emptyset$*;*
2. *let $\lambda_1, \lambda_2 \in \sigma_p(T)$ and $\lambda_1 \neq \lambda_2$, then*

$$\mathrm{null}(\lambda_1 \,\mathrm{id} - T) \perp \mathrm{null}(\lambda_2 \,\mathrm{id} - T).$$

Proof. TODO ☐

Proposition XIII.180. *Let T be a symmetric operator on a Hilbert space H. Then*

1. *the eigenvalues of T are real;*
2. *the eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof. This is an application of XIII.104 and XIII.105. ☐

5.6.4.3 Compact self-adjoint operators

Proposition XIII.181. *Every compact self-adjoint operator L on a nontrivial Hilbert space has an eigenvalue λ with $|\lambda| = \|L\|$.*

Proposition XIII.182. *Let A be a compact self-adjoint operator. Then the only possible accumulation point of $\sigma(A)$ is 0.*

TODO self-adjoint not necessary? See XIII.112?

Proof. Assume $\sigma(A)$ is infinite. Then take $\langle \lambda_n \rangle \subset \sigma(A)$. Any associated sequence $\langle x_n \rangle$ of eigenvectors is orthogonal. We can take it to be orthonormal. By XIII.50.1 we have

$$0 = \lim_{n\to\infty} \|Ax_n\|^2 = \lim_{n\to\infty} \langle Ax_n, Ax_n \rangle = \lim_{n\to\infty} \lambda_n^2 \langle x_n, x_n \rangle = \lim_{n\to\infty} \lambda_n^2,$$

so $\langle \lambda_n \rangle$ converges to 0. ☐

Theorem XIII.183. *Every spectral value $\lambda \neq 0$ of a compact self-adjoint linear operator $A : H \to H$ is an eigenvalue of finite multiplicity that can only accumulate at $\lambda = 0$. Conversely, a self-adjoint operator having these properties is compact.*

Proof. TODO See XIII.112 ☐

5.6.4.4 Self-adjoint extensions of symmetric operators

Cayley transform Consider the Möbius transform

$$\mathbb{C} \setminus \{\overline{\lambda}\} \to \mathbb{C} : x \mapsto \frac{x-\lambda}{x-\overline{\lambda}} \qquad \text{for some } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

This transform maps

- the real line to $\mathbb{T} \setminus \{1\}$;
- the half-plane above / below the real line containing λ to the interior of the unit disk;
- the half plane containing $\overline{\lambda}$ to the exterior of the unit disk;
- in particular $\lambda \mapsto 0$ and $\overline{\lambda} \mapsto \infty$.

Conventional choice: $\lambda = i$.

Defect indices Or deficiency(?) `https://link-springer-com.ezproxy.ulb.ac.be/content/pdf/10.1007/978-94-007-4753-1.pdf`
Cfr. dilation theory through Cayley transform.

5.6.4.5 Positive self-adjoint extensions of symmetric operators

Theorem XIII.184 (Friedrich's extension)**.** *Let A be a positive symmetric operator on a Hilbert space H. Then A has a unique positive self-adjoint extension $\widetilde{A}$ with domain* $\operatorname{dom}(\widetilde{A}) \subseteq \operatorname{Cl}_{\|\cdot\|_{A+\mathrm{id}}}(\operatorname{dom}(A))$.

Proof. Set $H_A \coloneqq \operatorname{Cl}_{\|\cdot\|_{A+\mathrm{id}}}(\operatorname{dom}(A))$. By X.153, we have $H_A \subseteq \operatorname{Cl}_{\|\cdot\|}(\operatorname{dom}(A))$.
For existence, we can construct the operator $\widetilde{A}$ as follows:

$$\begin{aligned} \operatorname{dom}(\widetilde{A}) &\coloneqq \big\{x \in H_A \;\big|\; \exists x' \in H : \forall y \in H_A : \; \langle y, x\rangle_{A+\mathrm{id}} = \langle y, x'\rangle \big\} \\ \widetilde{A}x &\coloneqq x' - x. \end{aligned}$$

Now $\widetilde{A}$ is an extension of A, because for all $x \in \operatorname{dom}(A)$, we can take $x' = Ax + x$. So $\widetilde{A}x = Ax$. But $\operatorname{dom}(\widetilde{A})$ may be larger than $\operatorname{dom}(A)$, because we can extended $\langle y, x\rangle_{A+\mathrm{id}}$ to be defined on all of H_A by continuity.
By construction $\operatorname{dom}(\widetilde{A}) \subseteq \operatorname{Cl}_{\|\cdot\|_{A+\mathrm{id}}}(\operatorname{dom}(A))$.
Now we claim $\operatorname{im}(\widetilde{A} + \mathrm{id}) = H$. Indeed for any $x' \in H$, the functional $H_A \to H_A : y \mapsto \langle y, x'\rangle$ is bounded. By Riesz representiation XIII.147, we can find an $x \in H_A$ such that $\langle y, x\rangle_{A+\mathrm{id}} = \langle y, x'\rangle$. Thus $(\widetilde{A} + \mathrm{id})x = x'$.
By XIII.175 we conclude that $\widetilde{A}$ is self-adjoint.
For uniqueness, assume there exists a second such extension $\widehat{A}$. For all $y \in \operatorname{dom}(A)$ and $x \in \operatorname{dom}(\widehat{A})$, we have

$$\left\langle y, (\widehat{A} + \mathrm{id})x \right\rangle = \left\langle (\widehat{A} + \mathrm{id})y, x \right\rangle = \langle (A + \mathrm{id})y, x\rangle = \overline{\langle x, (A + \mathrm{id})y\rangle} = \overline{\langle x, y\rangle_{A+\mathrm{id}}} = \langle y, x\rangle_{A+\mathrm{id}}\,.$$

By continuity this holds for all $y \in H_A$. And thus by definition $\widetilde{A}x = \widehat{A}x$ for all $x \in \operatorname{dom}(\widetilde{A})$. Thus $\widetilde{A} \subseteq \widehat{A}$, but self-adjoint operators are maximal by XIII.172, so $\widetilde{A} = \widehat{A}$. □

5.6.4.6 Bounded self-adjoint operators

Lemma XIII.185. *Let $A, B \in \mathcal{B}(H)$. Then*

1. *A^*A, AA^* and $A + A^*$ are self-adjoint;*
2. *if A, B are self-adjoint, then AB is self-adjoint* if and only if *A, B commute.*

Corollary XIII.185.1. *Let $A \in \mathcal{B}(H)$. Then there exist unique self-adjoint operators S, T such that*

$$A = S + iT \qquad A^* = S - iT.$$

Proof. Indeed $S = (A + A^*)/2$ and $T = (A - A^*)/2i$ are self-adjoint. □

Corollary XIII.185.2. *The operator A is normal* if and only if *S, T commute.*

Proof. We calculate the commutator

$$[S, T] = \left[\frac{A + A^*}{2}, \frac{A - A^*}{2i}\right] = \frac{A^*A - AA^*}{2i} = \frac{1}{2i}[A^*, A].$$

□

Proposition XIII.186. *The set of bounded self-adjoint operators forms an anti-lattice.*

Proof. TODO + generalised to self-adjoint operators?? □

5.6.5 Orthogonal projections

`https://planetmath.org/latticeofprojections`
`https://zfn.mpdl.mpg.de/data/Reihe_A/35/ZNA-1980-35a-0437.pdf`
We denote the set op projections on a Hilberts space $\mathcal{H}$ by $\mathcal{P}(\mathcal{H})$.
TODO: $\operatorname{im}(P) = \ker P^{*\perp}$ shows that we need $P = P^*$ for orthogonality.

Proposition XIII.187. *Let P be a bounded operator P on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

1. *P is an orthogonal projection onto a closed subspace of $\mathcal{H}$;*
2. *$P^2 = P$ and $P = P^*$;*
3. *$P^2 = P$ and $\|P\| \in \{0, 1\}$;*
4. *$P^2 = P$ and $\|P\| \leq 1$;*

Proof. $\boxed{(1) \Rightarrow (2)}$ Suppose first that P is the orthogonal projection operator onto a closed subspace K. Clearly $P^2 = P$. Let $x, y \in \mathcal{H}$ and write $x = x_1 + x_2, y = y_1 + y_2$ where $x_1, y_1 \in K$ and $x_2, y_2 \in K^\perp$. Then

$$\langle Px, y\rangle = \langle x_1, y_1 + y_2\rangle = \langle x_1, y_1\rangle + \langle x_1, y_2\rangle = \langle x_1, y_1\rangle = \langle x_1 + x_2, y_2\rangle = \langle x, Py\rangle .$$

So $P = P^*$.
$\boxed{(2) \Rightarrow (3)}$ We calculate $\|P\| = \left\|P^2\right\| = \|P^*P\| = \|P\|^2$ using XIII.166. The solutions to this equation are $\{0, 1\}$.
$\boxed{(3) \Rightarrow (4)}$ This is clear.

$\boxed{(4) \Rightarrow (1)}$ Define $K = \operatorname{im} P$, then K is closed because $x \in K$ iff $Px = x$ and thus for any converging sequence $(x_n)_n \subset K$: $\lim x_n = \lim Px_n = P(\lim x_n)$, so the limit is in K.
We just need to show orthogonality: $Px \perp x - Px$. For this we use X.125: for all $a \in \mathbb{F}$

$$\|Px\| = \|Px + aPx - aPx\| = \|P(Px + a(x - Px))\| \leq \|P\| \cdot \|Px + a(x - Px)\| \leq \|Px + a(x - Px)\|.$$

We conclude $Px \perp x - Px$. □

Proposition XIII.188. *Let $\mathcal{H}$ be a Hilbert space and let P be an orthogonal projector on a closed subspace K. Then* $\operatorname{id} - P$ *is the orthogonal projector on $K^\perp$.*

Proof. Any $x \in \mathcal{H}$ can be uniquely decomposed as $x_1 + x_2 \in K \oplus K^\perp$. If $Px = x_1$, then $(\operatorname{id} - P)x = x_1 + x_2 - x_1 = x_2$. □

Corollary XIII.188.1. *The set of projectors $\mathcal{P}(\mathcal{H})$ is a subset of* $[0, \operatorname{id}]$.

Proof. Let $P \in \mathcal{P}(\mathcal{H})$. Then $P \geq 0$ follows from $P = P^2 = P^* P$. □

Proposition XIII.189. *Let $\mathcal{H}$ be a Hilbert space and P, Q be projections. The following are equivalent:*

1. *$PQ = QP$;*
2. *PQ is a projection;*
3. *QP is a projection;*
4. *$P + Q - PQ$ is a projection;*
5. $\operatorname{im}(PQP) = \operatorname{im}(P) \cap \operatorname{im}(Q)$*;*
6. *$PQP = QP$;*
7. $\mathcal{H} = \big(\operatorname{im}(P) \cap \operatorname{im}(Q)\big) \oplus \big(\operatorname{im}(P) \cap \operatorname{im}(Q)^\perp\big) \oplus \big(\operatorname{im}(P)^\perp \cap \operatorname{im}(Q)\big) \oplus \big(\operatorname{im}(P)^\perp \cap \operatorname{im}(Q)^\perp\big)$.

Proof. Points (1), (2), (3) are equivalent by the equation $(PQ)^* = Q^* P^* = QP$, and the fact that (1) implies $(PQ)^2 = PQPQ = PPQQ = PQ$.
(4) If P, Q commute, then

$$\begin{aligned}(P + Q - PQ)^* &= P + Q - (PQ)^* = P + Q - Q^* P^* = P + Q - QP = P + Q - PQ \\ (P + Q - PQ)^2 &= P^2 + PQ - P^2 Q + QP + Q^2 - QPQ - PQP - PQP + PQPQ \\ &= P + Q + 3PQ - 4PQ = P + Q - PQ.\end{aligned}$$

Assume (4), then $(P + Q - PQ)^* = P + Q - QP = P + Q - PQ$. This implies $PQ = QP$.
$\boxed{(1) \Rightarrow (5)}$ Clearly $\operatorname{im}(PQP) \subseteq \operatorname{im}(P) \cap \operatorname{im}(Q)$. For the inverse inequality, take $x \in im(P) \cap \operatorname{im}(Q)$. Then $PQP(x) = PQ(x) = P(x) = x$, so $x \in \operatorname{im}(PQP)$.
$\boxed{(5) \Rightarrow (6)}$ We decompose $\mathcal{H} = \operatorname{im}(PQP) \oplus \ker(PQP)$ and show that the operators are the same on both parts. For all $x \in \mathcal{H}$ we have

$$x \in \ker(PQP) \iff \langle x, PQPx \rangle = 0 \iff \langle QPx, QPx \rangle = 0 \iff \|QPx\| = 0 \iff x \in \ker QP.$$

Now let $x \in \operatorname{im}(PQP) = \operatorname{im}(P) \cap \operatorname{im}(Q)$. Then $QPx = Qx = x = PQPx$.
$\boxed{(6) \Rightarrow (3)}$ PQP is always a projection.

$\boxed{(6) \Rightarrow (7)}$ Take some $x \in \mathcal{H}$. Then we can uniquely decompose $x = P(x) + (x - P(x)) = x_P + x_{P^\perp} \in \operatorname{im}(P) \oplus \operatorname{im}(P)^\perp$. We can then further decompose $x_P = x_{P,Q} + x_{P,Q^\perp}$ and $x_{P^\perp} = x_{P^\perp,Q} + x_{P^\perp,Q^\perp}$. In order to have the decomposition of the proposition, we need to show that $x_{P,Q}, x_{P,Q^\perp} \in \operatorname{im}(P)$ and $x_{P^\perp,Q}, x_{P^\perp,Q^\perp} \in \operatorname{im}(P)^\perp$.
First take $x_{P,Q} = QPx$. From (6) we have $P(QPx) = PQPx = QPx$, so $x_{P,Q} \in \operatorname{im}(P)$. For the others we have similar calculations (also using the identity $PQP = PQ$):

$$\begin{aligned}
P(x_{P,Q^\perp}) &= P\big((\mathrm{id}-Q)P\big)x = Px - PQPx = Px - QPx = (\mathrm{id}-Q)Px = x_{P,Q^\perp} \\
(\mathrm{id}-P)(x_{P^\perp,Q}) &= (\mathrm{id}-P)\big(Q(\mathrm{id}-P)\big)x = (Q - QP - PQ + PQP)x = (Q - QP)x = Q(\mathrm{id}-P)x = x_{P^\perp,Q} \\
(\mathrm{id}-P)(x_{P^\perp,Q^\perp}) &= (\mathrm{id}-P)\big((\mathrm{id}-Q)(\mathrm{id}-P)\big)x = (\mathrm{id}-P-Q+QP-P+P+PQ-PQP)x \\
&= (\mathrm{id}-Q-P+QP)x = (\mathrm{id}-Q)(\mathrm{id}-P)x = x_{P^\perp,Q^\perp}.
\end{aligned}$$

$\boxed{(7) \Rightarrow (1)}$ Take $x \in \mathcal{H}$ and decompose it as $x_{P,Q} + x_{P,Q^\perp} + x_{P^\perp,Q} + x_{P^\perp,Q^\perp}$. Then $PQx = P(x_{P,Q} + x_{P^\perp,Q}) = x_{P,Q}$ and $QP = Q(x_{P,Q} + x_{P,Q^\perp}) = x_{P,Q}$, so $PQ = QP$. ☐

Proposition XIII.190. *Let P, Q be orthogonal projections onto subspaces $\operatorname{im}(P)$ and $\operatorname{im}(Q)$ of $\mathcal{H}$.*

1. *The following are equivalent to $\operatorname{im}(P) \perp \operatorname{im}(Q)$:*
   - *(a)* $QP = 0$*;*
   - *(b)* $PQ = 0$*;*
   - *(c)* $Q + P$ *is an orthogonal projection.*
2. *The following are equivalent to $\operatorname{im}(P) \subseteq \operatorname{im}(Q)$:*
   - *(a)* $QP = P$*;*
   - *(b)* $PQ = P$*;*
   - *(c)* $Q - P$ *is an orthogonal projection;*
   - *(d)* $P \leq Q$*;*
   - *(e)* $\|Px\| \leq \|Qx\|$ *for all* $x \in \mathcal{H}$*.*

Proof. (1) We have:
$\boxed{(a) \Leftrightarrow (b) \Leftrightarrow \operatorname{im}(P) \perp \operatorname{im}(Q)}$ By XIII.189.
$\boxed{(a,b) \Leftrightarrow (c)}$ We know $(P+Q)^* = P^* + Q^* = P + Q$ and we can write

$$(P+Q)^2 = P^2 + Q^2 + PQ + QP = P + Q + PQ + QP,$$

So clearly (a) or (b) imply (c). Conversely, assume $PQ + QP = 0$, implying $PQ = -QP$. By left- and right-multiplication by P this implies both

$$PPQ = PQ = -PQP \qquad \text{and} \qquad PQP = -QPP = -QP.$$

So $PQ = -PQP = QP$, meaning $PQ = 1/2(PQ + QP) = 0$.
(2) We prove the following:
$\boxed{(a) \Leftrightarrow (b) \Leftrightarrow \operatorname{im}(P) \subseteq \operatorname{im}(Q)}$ By XIII.189.

$\boxed{(a,b) \Rightarrow (c)}$ Obviously $(Q-P)^* = Q-P$. Also

$$(Q-P)^2 = Q+P-PQ-QP = Q+P-2P = Q-P.$$

$\boxed{(c) \Rightarrow (a,b)}$ Now from

$$Q-P = (Q-P)^2 = Q+P-PQ-QP$$

we obtain $2P = PQ+QP$. The result then follows if we can show that $PQ = QP$. This follows by multiplying the equality on the left and on the right by P to obtain $QP = 2P - PQP$ and $PQ = 2P - PQP$, respectively.

$\boxed{(c) \Rightarrow (d)}$ This follows because all projections are positive.

$\boxed{(d) \Rightarrow (a,b)}$ Assume, towards a contradiction, that $\operatorname{im}(P) \nsubseteq \operatorname{im}(Q)$. Then we can take $v \in \operatorname{im}(P) \setminus \operatorname{im}(Q)$. Then

$$\langle v, (Q-P)v\rangle = \langle v, Qv\rangle - \langle v, v\rangle = \langle Qv, Qv\rangle - \langle Qv, Qv\rangle - \langle v - Qv, v - Qv\rangle = -\|v - Qv\|^2.$$

Because $v \notin \operatorname{im}(Q)$, $\|v - Qv\|$ is not zero and thus $Q-P$ is not positive.

$\boxed{(d) \Leftrightarrow (e)}$ By the equivalence

$$\|Px\| \leq \|Qx\| \iff \langle Px, Px\rangle \leq \langle Qx, Qx\rangle \iff \langle Px, x\rangle \leq \langle Qx, x\rangle \iff \langle (Q-P)x, x\rangle \geq 0.$$

□

We can generalise part 2(d) of the previous proposition to a slightly larger class of operators.

Lemma XIII.191. *Let $P \in \mathcal{P}(\mathcal{H})$ and $T \in [0, \mathrm{id}]$, then the following are equivalent:*

1. $\operatorname{im}(T) \subseteq \operatorname{im}(P)$;
2. $T \leq P$.

Proof. As T is self-adjoint, we have $\|T\| = \operatorname{nr}(T) \leq 1$ by X.161.
Assume (1) so that for all $x \in \mathcal{H}$ we get

$$\langle x, Tx\rangle = \langle x, PTx\rangle = \langle Px, PTx\rangle \leq \|Px\|^2 \operatorname{nr}(T) \leq \|Px\|^2 = \langle Px, Px\rangle = \langle x, Px\rangle.$$

So $\langle x, (P-T)x\rangle \geq 0$ and thus $T \leq P$.
Assume (2). The energy form associated with T is a pre-inner product by X.149. The Cauchy-Schwarz inequality X.108 gives

$$|\langle v, Tw\rangle|^2 \leq \langle v, Tv\rangle \langle w, Tw\rangle \leq \langle v, Pv\rangle \langle w, Pw\rangle.$$

So if $v \in \operatorname{im}(P)^\perp$, then $\langle v, Tw\rangle = 0$ for all $w \in \mathcal{H}$. So $\operatorname{im}(T) \perp \operatorname{im}(P)^\perp$, implying $\operatorname{im}(T) \subseteq \operatorname{im}(P)^{\perp\perp} = \operatorname{im}(P)$. □

Proposition XIII.192. *Let $\mathcal{H}$ be a Hilbert space. Let $\{P_i\}_{i\in I}$ be an arbitrary subset of $\mathcal{P}(\mathcal{H})$ and let $K_i = \operatorname{im}(P_i)$ for all $i \in I$. Then, as a subset of $[0, \mathrm{id}]$,*

1. $\inf\{P_i\}_{i\in I} = P_M$ *where* $M = \bigcap_{i\in I} K_i$;
2. $\sup\{P_i\}_{i\in I} = P_N$ *where* $N = \bigcap\{K \subseteq \mathcal{H} \mid K \text{ is closed} \land \forall i \in I : K_i \subseteq K\}$.

The set of projections on $\mathcal{H}$ is thus a complete lattice as a subset of $[0, \mathrm{id}]$.
If I is finite, then $N = \operatorname{span}(\bigcup_{i\in I} K_i)$. TODO: always closure of this N????

In particular this means $\mathcal{P}(\mathcal{H})$ is a complete lattice as itself, with the same suprema and infima. It is not a lattice as a subset of $\mathcal{SA}(\mathcal{H})$ (TODO + example ??).

Proof. (1) By XIII.190 P_M is a lower bound of $\{P_i\}_{i\in I}$ in $[0, \mathrm{id}]$. Let T be a lower bound of $\{P_i\}_{i\in I}$ in $[0, \mathrm{id}]$. By XIII.191 $\operatorname{im}(T) \subseteq K_i$ for all $i \in I$, so $\operatorname{im}(T) \subseteq M$ and thus $T \leq P$ again by XIII.191. This means P is the greatest lower bound.
(2) The mapping $T \mapsto \mathrm{id} - T$ keeps $[0, \mathrm{id}]$ invariant and inverts the order. Then $\inf\{\mathrm{id} - P_i\}_{i\in I}$ is a projection due to the previous point and so $\sup\{P_i\}_{i\in I}$ is also a projection. The expression for N is clear from XIII.190. ☐

5.6.5.1 Sets of pairwise disjoint projections

TODO!

5.6.5.2 Derivatives of orthogonal projections

Proposition XIII.193. *Let $\{P_i\}_{i\in I}$ be a set of pairwise disjoint orthogonal projectors which have derivatives and take $i \neq j$ in I. Then*

1. $P_i'P_j = -P_iP_j'$;
2. *if* $\mathrm{id} \in \uparrow\{P_i\}_{i\in I}$, *then*

$$P_iP_i' = \sum_{j\neq i} P_i'P_j \qquad \textit{and} \qquad P_i'P_i = \sum_{j\neq i} P_jP_i'.$$

Proof. (1) We have $P_iP_j = 0$, so $0 = P_i'P_j + P_iP_j'$.
(2) We calculate, using $\mathrm{id} = \sum_{j\in I} P_j$ and XI.17:

$$P_iP_i' = P_iP_i'\left(\sum_{j\in I} P_j\right) = P_iP_i'P_i + \sum_{j\neq i} P_iP_i'P_j = 0 - \sum_{j\neq i} P_iP_iP_j' = -\sum_{j\neq i} P_iP_j' = \sum_{j\neq i} P_i'P_j.$$

☐

Corollary XIII.193.1. *Let P_1, P_2 be orthogonal projections such that $P_1 + P_2 = \mathrm{id}$. Then*

$$P_1P_1' = P_1'P_2 \qquad \textit{and} \qquad P_1'P_1 = P_2P_1'.$$

5.6.6 Isometries

We recall that isometries are injective and continuous. On Hilbert spaces they are also closed. See VIII.211, VIII.212 and VIII.213.

Proposition XIII.194. *Let $T \in \mathcal{B}(H, K)$ with H, K Hilbert spaces. Then*

1. *T is an isometry* if and only if $T^*T = \mathrm{id}_H$;
2. *T is unitary* if and only if $T^*T = \mathrm{id}_H$ *and* $TT^* = \mathrm{id}_K$, *i.e.* $T^{-1} = T^*$.

Proof. (1) For all $v, w \in H$ we have

$$\langle Tv, Tw \rangle = \langle T^*Tv, w \rangle .$$

The left-hand side is equal to $\langle v, w \rangle$ iff T is an isometry. The right-hand side is equal to $\langle v, w \rangle$ iff $T^*T = \mathrm{id}_H$, by X.136.
(2) If T is invertible, it must have a left and right inverse. By lemma I.142 they must be the same. ☐

Corollary XIII.194.1. *An isometry $T \in \mathcal{B}(H)$ is unitary* if and only if *it is normal.*

Lemma XIII.195. *Let T be an isometry between Hilbert spaces H and K. Then TT^* is an orthogonal projection.*

Proof. Clearly $(TT^*)^* = TT^*$. Also $(TT^*)^2 = T(T^*T)T^* = T\,\mathrm{id}_H\,T^* = TT^*$. ☐

5.6.6.1 Wandering spaces and unilateral shifts

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{V} \subseteq \mathcal{H}$ a closed subspace and $T : \mathcal{H} \to \mathcal{H}$ a linear map. Then $\mathcal{V}$ is called a wandering space for T if $T^p[\mathcal{V}] \perp T^q[\mathcal{V}]$ for every $p \neq q \in \mathbb{N}$.

Lemma XIII.196. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{V} \subseteq \mathcal{H}$ a closed subspace and $T : \mathcal{H} \to \mathcal{H}$ a linear isometry.*

1. *$\mathcal{V}$ is a wandering space for T* if and only if *$T^n[\mathcal{V}] \perp \mathcal{V}$ for all $n \in \mathbb{N}$;*
2. *$T[\mathcal{H}]^\perp$ is a wandering subspace for T;*
3. *if $\mathcal{V}$ is a wandering space for T, then $T^n[\mathcal{V}] \cong \mathcal{V}$ for all $n \in N$.*

Proof. (1) The direction $\Rightarrow$ is clear. For the converse, assume $T^n[\mathcal{V}] \perp \mathcal{V}$ for all $n \in \mathbb{N}$. We need to show that $T^p[\mathcal{V}] \perp T^q[\mathcal{V}]$ for every $p \neq q \in \mathbb{N}$. WLOG we may assume $p \leq q$. Take arbitrary $x \in T^p[\mathcal{V}]$ and $y \in T^q[\mathcal{V}]$. Then

$$\langle x, y \rangle = \langle T^p(u), T^q(v) \rangle = \left\langle u, T^{q-p}(v) \right\rangle = 0$$

because $\mathcal{V} \perp T^{q-p}[\mathcal{V}]$.
(2) For all $n \geq 1$ we have

$$T^n\left[T[\mathcal{H}]^\perp\right] \subset T^n[\mathcal{H}] = T\left[T^{n-1}[\mathcal{H}]\right] \subset T[\mathcal{H}] \perp T[\mathcal{H}]^\perp.$$

(3) For all $n \in \mathbb{N}$ the operator T^n is an isometry. It is injective by VIII.211, and thus maps its domain bijectively to its image. ☐

An isometry T on a Hilbert space $\mathcal{H}$ is called a unilateral shift if there is a closed subspace $\mathcal{V} \subseteq \mathcal{H}$ that is wandering for T such that

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} T^n[\mathcal{V}].$$

We call the subspace $\mathcal{V}$ generating for T and $\dim(\mathcal{V})$ the multiplicity of T.

By XIII.196, we see that any isometry $T : \mathcal{H} \to \mathcal{H}$ is a unilateral shift when restricted to $\bigoplus_{n=0}^{\infty} T^n\left[T[\mathcal{H}]^\perp\right]$.

Lemma XIII.197. *Let T be an isometry on $\mathcal{H}$. If T is a unilateral shift, then it is generated by $T[\mathcal{H}]^\perp$.*

Proof. Let $\mathcal{V}$ be the generating subspace of the unilateral shift T. We calculate

$$T[\mathcal{H}] = T\left[\bigoplus_{n=0}^{\infty} T^n[\mathcal{V}]\right] = \bigoplus_{n=1}^{\infty} T^n[\mathcal{V}] = \bigoplus_{n=0}^{\infty} T^n[\mathcal{V}] \ominus \mathcal{V} = \mathcal{H} \ominus \mathcal{V} = \mathcal{V}^\perp,$$

so $\mathcal{V} = T[\mathcal{H}]^\perp$. ☐

A unilateral shift is determined up to unitary equivalence by its multiplicity:

Lemma XIII.198. *Let $T : \mathcal{H} \to \mathcal{H}$ and $T' : \mathcal{H}' \to \mathcal{H}'$ be unilateral shifts generated by $\mathcal{V}$ and $\mathcal{V}'$ such that* $\dim(\mathcal{V}) = \dim(\mathcal{V}')$*. Then there exists an unitary $U : \mathcal{H}' \to \mathcal{H}$ such that*

$$T' = U^* T U$$

Proof. Choose an isometric isomorphism $u : \mathcal{V}' \to \mathcal{V}$. Then any $x \in \mathcal{H}'$ can be written as $x = \sum_{n=0}^{\infty} T^n(x_n)$. Then define

$$Ux = \sum_{n=0}^{\infty} T^n(ux_n).$$

☐

Theorem XIII.199 (Wold decomposition)**.** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ an isometry. Then $\mathcal{H}$ decomposes into an orthogonal sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$such that $\mathcal{H}_0, \mathcal{H}_1$ reduce T and*

$$T|_{\mathcal{H}_0} \text{ is unitary} \quad \text{and} \quad T|_{\mathcal{H}_1} \text{ is a unilateral shift.}$$

This decomposition is uniquely determined and given by

$$\mathcal{H}_0 = \bigcap_{n=0}^{\infty} T^n[\mathcal{H}] \qquad \text{and} \qquad \mathcal{H}_1 = \bigoplus_{n=0}^{\infty} T^n[\mathcal{V}] \qquad \text{where} \qquad \mathcal{V} = T[\mathcal{H}]^\perp.$$

Proof. The subspace $\mathcal{V} = T[\mathcal{H}]^\perp$ is wandering by XIII.196. Then T is a unilateral shift in the subspace

$$\mathcal{H}_1 = \bigoplus_{n=0}^{\infty} T^n[\mathcal{V}].$$

Now $v \in \mathcal{H}_0 = \mathcal{H}_1^\perp$ if and only if it is perpendicular to $\bigoplus_{i=0}^{n} T^i[\mathcal{V}]$ for all n and we have

$$\begin{aligned}\bigoplus_{i=0}^{n} T^i[\mathcal{V}] &= \bigoplus_{i=0}^{n} T^i[\mathcal{H} \ominus T[\mathcal{H}]] = \bigoplus_{i=0}^{n} T^i[\mathcal{H}] \ominus T^{i+1}[\mathcal{H}] \\ &= (\mathcal{H} \ominus T[\mathcal{H}]) \oplus (T[\mathcal{H}] \ominus T^2[\mathcal{H}]) \oplus \ldots \oplus (T^n[\mathcal{H}] \ominus T^{n+1}[\mathcal{H}]) = \mathcal{H} \ominus T^{n+1}[\mathcal{H}]\end{aligned}$$

using X.117 and XIII.143, which is applicable because $T^i[\mathcal{V}]$ is closed by VIII.213. So $\mathcal{H}_0 \subseteq T^n[\mathcal{H}]$ for all n.

Finally $T|_{\mathcal{H}_0}$ is unitary because it is an isometry and surjective on $\mathcal{H}_0$. ☐

5.6.6.2 Left and right shifts on ℓ^2

Consider the space $\ell^2(\mathbb{N})$ with o.n. basis $\langle e_i \rangle$. Then

- the right shift operator S_r is the operator that maps $e_i \mapsto e_{i+1}$;
- the left shift operator S_l is the operator that maps $e_i \mapsto \begin{cases} e_{i-1} & i \geq 1 \\ 0 & i = 0 \end{cases}$.

Lemma XIII.200. *S_r is a unilateral shift*

Proposition XIII.201. *$S_r = S_l^*$ (also converse?)*

5.6.6.3 Partial isometries

An operator $T \in \mathcal{L}(H, H')$ is called a partial isometry if there is a closed subspace $K \subseteq H$ such that

- $T|_K$ is an isometry;
- $T|_{K^\perp} = 0$.

Clearly every partial isometry is bounded.

Lemma XIII.202. *An operator $T \in \mathcal{L}(H, H')$ is a partial isometry* if and only if *$T|_{\ker(T)^\perp}$ is an isometry.*

Proposition XIII.203. *Let $T \in \mathcal{B}(H, H')$. The following are equivalent:*

1. *T is a partial isometry;*
2. *$T^*TT^* = T^*$;*
3. *$TT^*T = T$;*
4. *$TT^* : H' \to H'$ is a projection;*
5. *$T^*T : H \to H$ is a projection;*
6. *T^* is a partial isometry.*

Moreover,

1. *T^*T is the projection onto $\ker(T)^\perp$;*
2. *$\operatorname{im}(T)$ is closed and TT^* is the projection onto $\operatorname{im}(T)$.*

Proof. $\boxed{(1) \Rightarrow (2)}$ By X.103 it is enough to show that $\langle T^*TT^*x, y\rangle = \langle T^*x, y\rangle$ for all $x \in H', y \in H$. Take such x, y. We decompose $y = y_1 \oplus y_2 \ker(T) \oplus \ker(T)^\perp$. Then

$$\langle T^*TT^*x, y_1\rangle = \langle TT^*x, Ty\rangle = 0 = \langle x, Ty_1\rangle = \langle T^*x, y_1\rangle$$

and

$$\langle T^*TT^*x, y_2\rangle = \langle TT^*x, Ty_2\rangle = \langle T^*x, y_2\rangle,$$

where we have used the fact that both y_2 and T^*x are elements of $\ker(T)^\perp = \overline{\operatorname{im}(T^*)}$, and T is an isometry on this space. In conclusion, we have

$$\langle T^*TT^*x, y\rangle = \langle T^*TT^*x, y_1\rangle + \langle T^*TT^*x, y_2\rangle = \langle T^*x, y_1\rangle + \langle T^*x, y_2\rangle = \langle T^*x, y\rangle$$

for all $x \in H', y \in H$, so $T^*TT^* = T^*$.
$\boxed{(2) \Leftrightarrow (3)}$ By taking adjoints: $(TT^*T)^* = T^*TT^*$.
$\boxed{(2) \Rightarrow (4,5)}$ Clearly T^*T and TT^* are self-adjoint. We just need to show idempotency:

$$(T^*T)^2 = (T^*T)(T^*T) = (T^*TT^*)T = T^*T \qquad (TT^*)^2 = (TT^*)(TT^*) = T(T^*TT^*) = TT^*.$$

$\boxed{(4) \Rightarrow (1)}$ Assume TT^* a projection. Let $v \in \ker(T)^\perp = \overline{\operatorname{im}(T^*)}$. Then there exists a sequence $\langle v_n\rangle \in H'^{\mathbb{N}}$ such that $\lim_{n\to\infty} T^*v_n = v$. Then

$$\begin{aligned}\|Tv\|^2 &= \lim_{n\to\infty} \|TT^*v_n\|^2 = \lim_{n\to\infty} \langle TT^*v_n, TT^*v_n\rangle \\ &= \lim_{n\to\infty} \left\langle (TT^*)^2 v_n, v_n\right\rangle = \lim_{n\to\infty} \langle TT^*v_n, v_n\rangle \\ &= \lim_{n\to\infty} \langle T^*v_n, T^*v_n\rangle = \lim_{n\to\infty} \|T^*v_n\|^2 = \|v\|^2,\end{aligned}$$

so T is a partial isometry.
$\boxed{(5,6)}$ Applying the proposition to T^* instead of T yields the equivalences with $T = TT^*T$, and thus with the rest of the statements.
TODO + $\operatorname{im}(T^*) = \ker(T)^\perp$ means support and range are exchanged between T and T^*. ◻

Let T be a partial isometry. We call

- T^*T the support projection or initial projection of T;
- TT^* the range projection or final projection of T.

Proposition XIII.204. *Let H, H' be Hilbert spaces with $K \subseteq H$ and $L \subseteq H'$ closed subspaces. Then the following are equivalent:*

1. *T is a partial isometry with support K and range L;*
2. *(T, T^*) is a Galois connection between $(H, \perp_K)$ and $(H', \perp_L)$.*

Here $\perp_K$ is defined by

$$x \perp_K y \qquad \Leftrightarrow_{def} \qquad P_K(x) \perp P_K(y).$$

Proof. The direction (2) $\Rightarrow$ (1) is immediate from XIII.203, because T, T^* are generalised inverses.
For the other direction, we first prove T preserves the relational structure. Take arbitrary $x = x_1 + x_2$ and $y = y_1 + y_2$ in $K \oplus K^\perp$ such that $x \perp_K y$. Then

$$\langle T(x), T(y)\rangle = \langle T(x_1), T(y_1)\rangle = \langle x_1, y_1\rangle = 0.$$

So $T(x) \perp T(y)$ and, because $T(x), T(y) \in L$, we have $T(x) \perp_L T(y)$. The argument for T^* is similar.

For the Galois condition, we need to show that $T^*T(x) \perp_K y \implies x \perp_K y$. Indeed

$$\begin{aligned} T^*T(x) \perp_K y &\iff T^*T(x_1) \perp y_1 \\ &\iff 0 = \langle T^*T(x_1), y_1\rangle = \langle T(x_1), T(y_1)\rangle = \langle x_1, y_1\rangle \\ &\iff P_K(x) \perp P_K(y). \end{aligned}$$

□

Corollary XIII.204.1. *Let $T : H \to H'$ be a partial isometry with support K and range L. Then*

$$T(x) \perp P_L(y) \iff P_K(x) \perp T^*(y)$$

for all $x \in H, y \in H'$.

Proof. This is the Galois identity II.50, although the direct proof is also very simple. □

5.6.6.4 Unitaries

Bilateral shifts

5.7 Dirac notation

`https://core.ac.uk/download/pdf/25263496.pdf` `https://michael-herbst.com/talks/2014.07.22_Mathematical_Concept_Dirac_Notation.pdf` `http://galaxy.cs.lamar.edu/~rafaelm/webdis.pdf` `https://plato.stanford.edu/entries/qt-nvd/` `file:///C:/Users/user/Downloads/Abdus%20Salam,%20E.P.%20Wigner%20(Ed.)%20-%20Aspects%20of%20Quantum%20Theory%20-%20Dedicated%20to%20Dirac%E2%80%99s%2070th%20Birthday-Cambridge%20University%20Press%20(1972).pdf` `https://aip.scitation.org/doi/pdf/10.1063/1.1705001`

Lemma XIII.205.
1. $T|\varphi\rangle\langle\psi| = |T\varphi\rangle\langle\psi| = |\varphi\rangle\langle\psi|T = |\varphi\rangle\langle T^*\psi|$;
2. $|\varphi\rangle\langle\psi||\xi\rangle\langle\eta| = \langle\psi, \xi\rangle\, |\varphi\rangle\langle\eta|$;
3. $(|\varphi\rangle\langle\psi|)^* = |\psi\rangle\langle\varphi|$.

Lemma XIII.206. *Let H be a Hilbert space and $\langle e_i\rangle_{i\in I}$ a basis for H. Then*

$$\mathrm{id}_H = \sum_{i\in I} |e_i\rangle\langle e_i| \qquad \textit{in the strong limit.}$$

Proof. TODO!! □

Lemma XIII.207. *Let H be a Hilbert space, $\langle e_i\rangle_{i\in I}$ a basis for H and T an operator on H. Then*

$$T = \sum_{i,j\in I} \langle e_i|T|e_j\rangle\, |e_i\rangle\langle e_j|.$$

in the strong limit.

Proof. TODO!! Tannery. □

5.8 Hilbert space ideals

5.8.1 Finite-rank operators

Remember that finite-rank operators are bounded by definition (this is not automatic, cfr. X.86).

Proposition XIII.208 (Finite rank singular value decomposition)**.** *Let V be an inner product space and $T \in \operatorname{Hom}(V)$. Then T is a finite-rank operator* if and only if *T can be written in the form*

$$T = \sum_{i=1}^{N} \lambda_i |v_i\rangle\langle w_i|,$$

where $(v_i)_{i=1}^N$ and $(w_i)_{i=1}^N$ are finite sets of vectors and $(\lambda_i)_{i=1}^N$ are positive (non-zero) numbers. The numbers $(\lambda_i)_{i=1}^N$ in this decomposition are uniquely determined by the operator.

The numbers $(\lambda_i)_{i=1}^N$ are called the singular values of the operator.

Proof. Because $\operatorname{im}(T)$ is finite-dimensional, we can find an orthonormal basis $(v_i)_{i=1}^N$ for it. Then we can write

$$Tx = \sum_{i=1}^{N} |v_i\rangle \langle v_i|Tx\rangle = \sum_{i=1}^{N} |v_i\rangle \langle T^* v_i|Tx\rangle = \sum_{i=1}^{N} |v_i\rangle \langle \lambda_i w_i|Tx\rangle = \sum_{i=1}^{N} \lambda_i |v_i\rangle \langle w_i|Tx\rangle$$

where $\lambda_i = \|T^* v_i\|$ and $w_i = \frac{T^* v_i}{\lambda_i}$.
We just need to show that the λ_i are independent of the chosen basis $(v_i)_{i=1}^N$. TODO!!!! ☐

Corollary XIII.208.1. *Every finite-rank operator on a Hilbert space is a finite sum of rank-1 operators.*

Lemma XIII.209. *Let H be Hilbert space. The set of finite rank operators on H is a $*$-ideal in H.*

5.8.2 Compact operators

Proposition XIII.210. *Let $T \in \mathcal{B}(H)$. Then the following are equivalent:*

1. *T is compact;*
2. *T^* is compact;*
3. *there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of finite rank operators such that $\|T - T_n\| \to 0$.*

This is false in Banach spaces. (TODO Enflo, approximation property, goose problem)

Proof. TODO ☐

Corollary XIII.210.1 (Canonical expansion)**.** *Any compact operator T on a Hilbert space $\mathcal{H}$ can be written in the form*

$$T = \sum_{i=1}^{\infty} \lambda_i |v_i\rangle\langle w_i|,$$

where $(v_i)_{i=1}^\infty$ and $(w_i)_{i=1}^\infty$ are orthonormal sets and $(\lambda_i)_{i=1}^\infty$ is a monotonically decreasing sequence of positive numbers with $\lim_{i\to\infty} \lambda_i = 0$.

As in XIII.208 for finite-rank operators we call $(\lambda_i)_{i=1}^{\infty}$ the singular values of T. They are uniquely determined by the operator.

Proof. TODO (one way is with polar decomposition and spectral theorem. Are there others?) ☐

Compare with XIII.207.

Lemma XIII.211. *Let H be a Hilbert space. Then the set of compact operators on H, $\mathcal{K}(H)$ is a two-sided $*$-ideal of H.*

Proposition XIII.212. *Let H be a Hilbert space with orthonormal basis $(e_i)_{i\in I}$. If $T \in \mathcal{B}(H)$ and*

$$\sum_{i\in I}\|Te_i\|^2 < \infty,$$

then T is a compact operator. + Converse??

Proof. TODO + weaken $T \in \mathcal{B}(H)$? ☐

Corollary XIII.212.1. *An integral operator defined by a square integrable kernel $K \in L^2(A\times A, \mu)$ is compact.*

Proposition XIII.213. *Let T be an operator on a Hilbert space. Then the following are equivalent:*

1. *T is compact;*
2. *for all sequences $\langle x_n\rangle$, weak convergence $x_n \overset{w}{\to} x$ implies the strong convergence $Ax_n \to Ax$;*
3. *for any two weakly convergent sequences $x_n \overset{w}{\to} x$ and $y_n \overset{w}{\to} y$ the energy form is continuous in both arguments:*

$$\lim_{n\to\infty}\langle x_n, y_n\rangle_T = \lim_{n\to\infty}\langle x_n, Ty_n\rangle = \langle x, Ty\rangle = \langle x, y\rangle_T\,.$$

Lemma XIII.214. *Let H be a Hilbert space and $P \in \mathcal{P}(H)$. If P is compact, then P has finite rank.*

5.8.3 Positive operators

5.8.3.1 Polar decomposition

Proposition XIII.215. *Let H be a Hilbert space and $T \in \mathcal{B}(H)$. There exists a unique partial isometry V such that $T = V|T|$ and $\ker(V) = \operatorname{supp}(T)$.*

TODO: should this be $\ker(V) = \ker(T)$??

Proof. TODO ☐

Lemma XIII.216. *There is only one positive operator A such that $T = VA$ for some partial isometry.*

Proof. TODO uniqueness positive squared root. ☐

`https://encyclopediaofmath.org/wiki/Polar_decomposition`

5.8.4 Trace class operators

TODO Simon

5.9 Dilation theory

5.9.1 Dilations, N-dilations and power dilations

Let $\mathcal{H} \subseteq \mathcal{H}'$ be Hilbert spaces and let $P_\mathcal{H}$ be the projection on $\mathcal{H}$. If a pair of linear maps $S : \mathcal{H}' \to \mathcal{H}'$ and $T : \mathcal{H} \to \mathcal{H}$ satisfy the relation

$$T = P_\mathcal{H} S|_\mathcal{H}$$

then T is called a compression of S and S a dilation of T. This is abbreviated $T \prec U$.

- Let $N \in \mathbb{N}$. If $T^k = P_\mathcal{H} S^k|_\mathcal{H}$ for all $k \leq N$, then S is called an N-dilation.
- If this holds for all $k \in \mathbb{N}$, then S is called a power dilation.
- If $T^* = P_\mathcal{H} S^*|_\mathcal{H}$, we call TODO??

We call $\mathcal{H}'$ minimal if the only reducing subspace for S that contains $\mathcal{H}$ is $\mathcal{H}'$.

If S is a dilation of T, then we clearly have $T = P_\mathcal{H} S P_\mathcal{H}|_\mathcal{H}$.

Lemma XIII.217. *Let $S : \mathcal{H}' \to \mathcal{H}'$ be an N-dilation of $T : \mathcal{H} \to \mathcal{H}$ and p a polynomial of degree at most N. Then*

$$p(T) = P_\mathcal{H} p(S)|_\mathcal{H}.$$

Let $\mathcal{H}$ be a Hilbert space. We call $T \in \mathcal{B}(\mathcal{H})$ a contraction if $\|T\| \leq 1$.

Proposition XIII.218. *Let $\mathcal{H} \cong \mathcal{H} \oplus \{0\} \subseteq \mathcal{H} \oplus \mathcal{H} = \mathcal{H}^2$ be a Hilbert space. Every contraction T on $\mathcal{H}$ has a unitary dilation U on $\mathcal{H}^2$.*

Proof. From $\|T\| \leq 1$ (and the fact that T^*T is normal), we have that $\mathbf{1} - T^*T \geq 0$ by spectral mapping. We can define $D_T = \sqrt{\mathbf{1} - T^*T}$. Then

$$U = \begin{pmatrix} T & D_{T^*} \\ D_T & -T^* \end{pmatrix}$$

is a dilation of T and it is unitary:

$$\begin{aligned} UU^* &= \begin{pmatrix} TT^* + D_{T^*}^2 & TD_T^* - D_{T^*}T \\ D_T T^* - T^* D_{T^*}^* & D_T^2 + T^*T \end{pmatrix} = \begin{pmatrix} \mathbf{1} & TD_T - D_{T^*}T \\ D_T T^* - T^* D_{T^*} & \mathbf{1} \end{pmatrix} \\ U^*U &= \begin{pmatrix} T^*T + D_T^2 & T^* D_{T^*} - D_T^* T^* \\ D_{T^*}^* T - TD_T & D_{T^*}^2 + TT^* \end{pmatrix} = \begin{pmatrix} \mathbf{1} & T^* D_{T^*} - D_T T^* \\ D_{T^*} T - TD_T & \mathbf{1}. \end{pmatrix} \end{aligned}$$

We have used that D_T is self-adjoint for all contractions T. We just need to show that $TD_T = D_{T^*}T$. Clearly we have

$$T(D_T)^2 = T(\mathbf{1} - T^*T) = T - TT^*T = (\mathbf{1} - TT^*)T = (D_{T^*}T)^2 T.$$

By functional-like calculus (TODO!!) we have $TD_T = D_{T^*}T$. $\square$

The operator D_T in the previous proof is sometimes called the defect operator of T. It measures in some sense how far T is from being a unitary operator. If T is unitary, then $D_T = 0 = D_{T^*}$. If T is an isometry, then $D_T = 0$ (by XIII.195) and D_{T^*} is a projector (TT^* is a projector by XIII.194, so $\mathbf{1} - TT^*$ is too by XIII.188 and $D_{T^*} = \sqrt{\mathbf{1} - TT^*} = \sqrt{(\mathbf{1} - TT^*)^2} = \mathbf{1} - TT^*$).

Proposition XIII.219. *Let $\mathcal{H} \cong \mathcal{H} \oplus \{0\} \subseteq \mathcal{H} \oplus \mathcal{H} = \mathcal{H}^2$ be a Hilbert space. Every isometry T on $\mathcal{H}$ has a unitary power dilation U on $\mathcal{H}^2$.*

Proof. Consider the unitary dilation of XIII.218. When T is an isometry this reduces to

$$U = \begin{pmatrix} T & D_{T^*} \\ 0 & -T^* \end{pmatrix} = \begin{pmatrix} T & \mathbf{1} - TT^* \\ 0 & -T^* \end{pmatrix},$$

where we have used that $D_{T^*} = \sqrt{\mathbf{1} - TT^*} = \sqrt{(\mathbf{1} - TT^*)^2} = \mathbf{1} - TT^*$ is a projector.
Now for all $n \in \mathbb{N}$ we have $U^n = \begin{pmatrix} T^n & * \\ 0 & (-T^*)^n \end{pmatrix}$, so in particular $P_{\mathcal{H}} U^n|_{\mathcal{H}} = T^n$, meaning U is a power dilation of T. □

Lemma XIII.220. *Let T a contraction on a Hilbert space $\mathcal{H}$. Then $V_T : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H} : x \mapsto (Tx, D_T x)$ is an isometry.*

Proof. For all $x \in \mathcal{H}$ we have

$$\|V_T x\| = \sqrt{\|Tx\|^2 + \|D_T x\|^2} = \sqrt{\langle Tx, Tx\rangle + \langle D_T x, D_T x\rangle} = \sqrt{\langle T^*Tx, x\rangle + \langle D_T^2 x, x\rangle} = \sqrt{\langle x, x\rangle} = \|x\|.$$

□

Proposition XIII.221. *Let $\mathcal{H} \cong \mathcal{H} \oplus \{0\}^N \subseteq \mathcal{H}^{N+1}$ be a Hilbert space. Every contraction T on $\mathcal{H}$ has a unitary N-dilation U on $\mathcal{H}^{N+1}$.*

Proof. Let U' be a unitary dilation of T on $\mathcal{H}^2$. Let $C_1 = U'_{-,1}$ and $C_2 = U'_{-,2}$ denote the columns. Then

$$U = \begin{pmatrix} C_1 & \mathbb{0}^{2\times N-1} & C_2 \\ \mathbb{0}^{N-1\times 1} & \mathbb{1}^{N-1\times N-1} & \mathbb{0}^{N-1\times 1} \end{pmatrix}$$

is unitary by

$$\begin{pmatrix} C_1^* & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \\ C_2^* & \mathbb{0} \end{pmatrix} \begin{pmatrix} C_1 & \mathbb{0} & C_2 \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \end{pmatrix} = \begin{pmatrix} C_1^*C_1 & \mathbb{0} & C_1^*C_2 \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \\ C_2^*C_1 & \mathbb{0} & C_2^*C_2 \end{pmatrix} = \mathbb{1}^{N+1\times N+1}.$$

We just need to show that the (1,1)-component of U^k is T^k for all $k \in 1 : N$. In order to perform the multiplication, we rewrite U such that the row and column partitions are the same, i.e. $(2|(N-3)|2) \times (2|(N-3)|2)$:

$$U = \begin{pmatrix} \begin{bmatrix} T & 0 \\ D_T & 0 \end{bmatrix} & \mathbb{0} & \begin{bmatrix} 0 & D_{T^*} \\ 0 & -T^* \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ \mathbb{0} & \mathbb{0} \end{bmatrix} & \begin{bmatrix} \mathbb{0} & 0 \\ \mathbb{1} & \mathbb{0} \end{bmatrix} & \mathbb{0} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \mathbb{0} & 1 \\ \mathbb{0} & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{pmatrix}$$

TODO □

Proposition XIII.222 (von Neumann's inequality)**.** *Let T be a contraction on some Hilbert space $\mathcal{H}$. Then, for every polynomial $p \in \mathbb{C}[z]$,*

$$\|p(T)\| \leq \sup_{|z|=1} |p(z)|.$$

Proof. Suppose the degree of p is N. Let U be a unitary N-dilation of T. Then

$$\|p(T)\| = \|P_{\mathcal{H}} p(U)|_{\mathcal{H}}\| \leq \|p(U)\| = \sup_{z\in\sigma(U)} |p(z)| \leq \sup_{|z|=1} |p(z)|$$

since the spectrum of U is contained in the unit circle. □

Theorem XIII.223 (Sz.-Nagy's dilation theorem)**.** *Let $\mathcal{H} \subseteq \ell^2(\mathbb{N}) \otimes \mathcal{H}$ be Hilbert spaces. Every contraction on $\mathcal{H}$ has a unitary power dilation on $\ell^2(\mathbb{N}) \otimes \mathcal{H}$.*

5.10 Constructions

5.10.1 Direct sum

5.10.2 Tensor product

`https://web.ma.utexas.edu/mp_arc/c/14/14-2.pdf`

Chapter 6

Types of operators

6.1 Fredholm operators

An operator $T \in \mathcal{B}(X,Y)$ between Banach spaces is called a Fredholm operator if T has a finite-dimensional kernel and cokernel.
The Fredholm index of T is defined as

$$\operatorname{idx} T \coloneqq \dim\ker T - \dim\operatorname{coker} T.$$

We denote the space of Fredholm operators from X to Y as $\mathcal{F}(X,Y)$. If $X = Y$, we write $\mathcal{F}(X)$.

Example

1. If $X = Y$ is finite-dimensional, then all operators are Fredholm with index 0.
2. The left shift $S_l : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) : (x_n)_n \mapsto (x_{n+1})_n$ has index 1.
3. The right shift $S_r = S_l^*$ has index -1.

Lemma XIII.224. *A Fredholm operator has closed range.*

Lemma XIII.225. *Let $T \in \mathcal{B}(H)$ be a bounded operator on a Hilbert space. Then $\dim\operatorname{coker} T = \dim\ker T^*$.*

Proof. TODO (is it correct?) $\ker(T^*) = \operatorname{im}(T)^\perp$. □

Proposition XIII.226. *Let $S, T \in \mathcal{F}(X)$, $\lambda \in \mathbb{F}$ and $K \in \mathcal{K}(X)$. Then*

1. $\operatorname{idx}(ST) = \operatorname{idx}(S) + \operatorname{idx}(T)$;
2. $\operatorname{idx}(T + K) = \operatorname{idx}(T)$;
3. $\operatorname{idx}(\lambda T) = \operatorname{idx}(T)$, *if* $\lambda \neq 0$;
4. $\operatorname{idx}(T) = 0$ if and only if $T = K' + L$ *for some compact* K' *and invertible* L.

Let $T \in \mathcal{F}(H)$ for some Hilbert space H. Then

5. $\operatorname{idx}(T^*) = -\operatorname{idx}(T)$.

TODO: integrate with corollary??

Lemma XIII.227. *Let the commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "T"] & Y \arrow[r] \arrow[d, "S"] & Z \arrow[r] \arrow[d, "R"] & 0 \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

have short exact rows. If any two of T, S, R are Fredholm, then so is the third and

$$\operatorname{idx} S = \operatorname{idx} T + \operatorname{idx} R.$$

Proof. TODO snake lemma to obtain long exact

$$0 \to \ker T \to \ker S \to \ker R \to \operatorname{coker} T \to \operatorname{coker} S \to \operatorname{coker} R \to 0.$$

□

Corollary XIII.227.1.

1. *Let $T \in \mathcal{F}(X)$ and $S \in \mathcal{F}(Y)$ be Fredholm, then so is $T \oplus S$ with*
$$\operatorname{idx}(T \oplus S) = \operatorname{idx}(T) + \operatorname{idx}(S).$$
2. *Let $T \in \mathcal{F}(X, Y)$ and $S \in \mathcal{F}(Y, Z)$ be Fredholm, then so is ST with*
$$\operatorname{idx}(ST) = \operatorname{idx}(T) + \operatorname{idx}(S).$$
3. *Let $K \in \mathcal{K}(X)$ be compact, then $\operatorname{id}_X + K$ is Fredholm with*
$$\operatorname{idx}(\operatorname{id}_X + K) = 0.$$

Lemma XIII.228 (Fredholm alternative)**.** *Let T be a Fredholm operator of index zero. Then either T is bijective, or it is neither injective nor surjective.*

Proof. The operator T is injective iff $\dim \ker(T) = 0$ and surjective iff $\dim \operatorname{coker}(T) = 0$. □

6.2 Integral operators and transforms

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Then an integral operator or integral transform is a map of the form

$$T : U \subset (\Omega \to \mathbb{C}) \to (\Omega \to \mathbb{C}) : f \mapsto \int_\Omega K(x, y) f(y) \, \mathrm{d}\mu(y)$$

where $K \in (\Omega \times \Omega \to \mathbb{C})$ is the kernel or nucleus of T.
The kernel is called

- symmetric if $K(x, y) = \overline{K(y, x)}$;

- Volterra if $\Omega = \mathbb{R}$ and $K(x,y) = 0$ for $y > x$;
- convolutional if Ω is a group and $K(x,y) = F(x-y)$ for some function F;
- Hilbert-Schmidt if $K \in L^2(\Omega \times \Omega)$, i.e.

  $$\int_{\Omega\times\Omega} |K(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y < \infty;$$

- singular if $K(x,y)$ is unbounded on $\Omega \times \Omega$.

Lemma XIII.229. *Hilbert-Schmidt integral operators are compact operators on $L^2(\Omega\times\Omega)$.*

Proof. A Hilbert-Schmidt integral operator T maps $L^2(\Omega)$ to $L^2(\Omega)$ functions:

$$\begin{aligned}\|Tu\|_{L^2}^2 &= \int_\Omega \left|\int_\Omega K(x,y)u(y)\,\mathrm{d}\mu(y)\right|^2 \mathrm{d}\mu(x) \\ &\leq \int_\Omega \left(\int_\Omega |K(x,y)|^2\,\mathrm{d}\mu(y)\right)\left(|u(y)|^2\,\mathrm{d}\mu(y)\right)\mathrm{d}\mu(x) \\ &= \left(\int_\Omega\int_\Omega |K(x,y)|^2\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\right)\left(|u(y)|^2\,\mathrm{d}\mu(y)\right) < \infty\end{aligned}$$

where we have used the Cauchy-Schwarz inequality. This also immediately shows Hilbert-Schmidt integral operators are bounded.
TODO Compact ☐

Proposition XIII.230. *Let T be an integral operator with kernel $K(x,y)$, then T^* is the integral operator with kernel $\overline{K(y,x)}$.*

Proof. TODO ☐

Proposition XIII.231. *Let A be a Borel set and $K : A \times A \to \mathbb{C}$ a measurable function such that the integral operator with kernel K is bounded. Then the adjoint of the integral operator is again an integral operator with kernel $K^*(x,y) = \overline{K(y,x)}$.*

Proposition XIII.232. *Let T be a Volterra integral operator. Then $\sigma(T) = \sigma_c(T) = \{0\}$.*

Proof. TODO ☐

6.2.1 Integral equations

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. An integral equation is an equation containing an unknown function on Ω and an integral over Ω.
An integral equation is

- of the first kind if it is of the form

  $$\int_\Omega K(x,y)u(y)\,\mathrm{d}\mu(y) = f(x) \qquad x \in \Omega$$

  where f is a given function and u is the unknown function;

- of the second kind if it is of the form
$$\lambda u(x) - \int_\Omega K(x,y)u(y)\,\mathrm{d}\mu(y) = f(x) \qquad x \in \Omega$$
where f is a given function, λ is a scalar and u is the unknown function.

Proposition XIII.233. *Let*
$$\lambda u(x) - \int_\Omega K(x,y)u(y)\,\mathrm{d}\mu(y) = f(x)$$
be an integral equation of the second kind. This integral equation has a unique solution u if
$$|\lambda| > \sup_{x\in\Omega}\int_\Omega |K(x,y)|\,\mathrm{d}\mu(y).$$

Proof. Let the map T be defined by
$$T(u) = x \mapsto \frac{1}{\lambda}\left(\int_\Omega K(x,y)u(y)\,\mathrm{d}\mu(y) + f(x)\right)$$
so that solutions of the integral equation are exactly the fixed points of T. Then
$$\|Tu - Tv\|_\infty = \sup_{x\in\Omega}\frac{1}{|\lambda|}\left|\int_\Omega K(x,y)(u(y)-v(y))\,\mathrm{d}\mu(y)\right| \leq \frac{1}{|\lambda|}\sup_{x\in\Omega}\int_\Omega |K(x,y)|\,\mathrm{d}\mu(y)\cdot\|u-v\|_\infty.$$
So T is a contraction if $|\lambda| > \sup_{x\in\Omega}\int_\Omega |K(x,y)|\,\mathrm{d}\mu(y)$. The result follows from VIII.224. □

6.3 Convolution operators

Chapter 7

Fourier transforms

7.1 Types of Fourier transform

7.1.1 Discrete Fourier transform

Then N-dimensional discrete Fourier transform (DFT) is the linear transformation $\mathbb{C}^N \to \mathbb{C}^N$ defined by the matrix DFT_N with components

$$[DFT_N]_{j,k} = \frac{1}{\sqrt{N}} \omega_N^{(j-1)(k-1)},$$

where ω_N is the N^{th} root of unity.

Lemma XIII.234.

1. *The DFT_N matrix is the Vandermonde matrix of the roots of unity, up to the normalisation factor $1/\sqrt{N}$.*
2. *The DFT_N matrix is unitary.*

Proof. (1) Just an observation.
(2) We calculate

$$[DFT_N \cdot DFT_N]_{j,l} = \frac{1}{N}\sum_{k=1}^{N} \omega_N^{jk}\overline{\omega_N}^{kl} = \frac{1}{N}\sum_{k=1}^{N} \omega_N^{k(j-l)} = \delta_{j,l}.$$

□

Part XIV

Operator equations

Chapter 1

Semigroups and evolution operators

1.1 Semigroups of linear operators

Let X be a Banach space and $(I, +, 0, \leq)$ a partially ordered group with positive cone I^+. We call a function $T : I^+ \to \mathcal{L}(X)$ a semigroup of linear operators (or just semigroup) if

- $T(0) = \mathrm{id}_X$;
- $T(s+t) = T(s)T(t)$ for all $s, t \in I^+$.

A function $T : I \to \mathcal{L}(X)$ satisfying the same conditions is called a group of linear operators.
A semigroup is called strongly continuous if it is continuous when I is equipped with the order convergence and X with the strong operator topology.
We call T bounded if $T(t)$ is bounded for all $t \in I^+$.

Lemma XIV.1. *A semigroup of linear operators $T : I^+ \to \mathcal{L}(X)$ is bounded* if and only if *$T[S]$ is a set of bounded operators for some $S \subseteq I+$ that generates I^+ as a monoid.*

Proposition XIV.2. *Let $T : I^+ \to \mathcal{L}(X)$ be a semigroup of operators on a Banach space. Then the following are equivalent:*

1. *T is strongly continuous;*
2. *T is strongly continuous at 0;*
3. *$T(t)$ is uniformly bounded on some neighbourhood of 0 and there exists a dense subset $D \subseteq X$ such that $\lim_{t\to 0} T(t)x = x$ for all $x \in D$.*

Proof. (1) ⇒ (2) If a semigroup is strongly continuous, then it is obviously strongly continuous at 0.
(2) ⇒ (3) Because $T(0) = \mathrm{id}_X$, strong continuity at 0 means that $\lim_{t\to 0} T(t)x = x$ for all $x \in X$. Thus this also holds for all x in any (dense) subset of X.
For the uniform bound we use the uniform boundedness principle XIII.81. It is then enough to show that there exists a neighbourhood U of 0 such that $\sup\{\|T(t)x\| \mid t \in U\} < \infty$ for all $x \in X$.

TODO
(3) $\Rightarrow$ (1)
Conversely, assume a semigroup $T : I^+ \to \mathcal{L}(X)$ is continuous at 0. Take $t_0 \in I^+$. By VIII.38 it is enough to check that T is left and right continuous at t_0.
First let F be a filter in $\mathcal{FP}(\uparrow t_0)$ that converges to t_0. Then $F - t_0 \in \mathcal{FP}(I^+)$ and $F - t_0 \to 0$. Thus $T[F] = T[F - t_0 + t_0] = T[F - t_0]T[t_0] \to T[0]T[t_0] = T[t_0]$, meaning that T is right continuous.
TODO left continuity. ☐

TODO cfr SOT.

1.1.1 Growth bounds

Proposition XIV.3. *Let $T : \mathbb{R}^+ \to \mathcal{L}(X)$ be a strongly continuous semigroup of bounded linear operators. Then there exist contstants $w \in \mathbb{R}$ and $M \geq 1$ such that*

$$\forall t \in \mathbb{R}^+ : \quad \|T(t)\| \leq Me^{wt}.$$

Proof. Choose M such that for all $0 \leq s \leq 1$, $\|T(s)\| \leq M$. ☐

1.1.2 C_0-semigroups

A C_0-semigroup

Chapter 2

Operator equations

Linear Operator Equations: Approximation and Regularization

2.1 Terminology

Let $T : X \nrightarrow Y$ be a (non-)linear operator between normed spaces. An operator equation is an equation of the form

$$T(u) = f \qquad f \in Y \ u \text{ unknown.}$$

A solution to is operator equation is a vector $a \in X$ such that $T(a) = f$.

2.1.1 Well- and ill-posed problems

Some natural questions associated to such problems are:

1. Whether a solution exists for a given $f \in Y$.
2. Whether this solution is unique.
3. Whether this solution depends continuously on f, i.e. whether the solution is stable under perturbation.

The later questions depend on the affirmative answers of the former. A problem is called well-posed if the answer to all three questions is positive and ill-posed if not. The terminology is due to Jacques Hadamard (TODO ref Hadamard, Jacques (1902). Sur les problèmes aux dérivées partielles et leur signification physique. Princeton University Bulletin. pp. 49–52.)

Proposition XIV.4. *Let $T(u) = f$ be an operator equation where T is a linear operator. Then the problem of solving this operator equation is well-posed for all $f \in \operatorname{im}(T)$ if and only if T is bounded below.*

Proof. By X.91. □

Proposition XIV.5. *Let T be a linear operator and $\lambda \in \mathbb{C}$. Consider the operator equation*

$$Tu = \lambda u + f.$$

This problem is well-posed if and only if $\lambda \in \rho(T)$.
In particular,

1. *if $\lambda \in \sigma_p(T)$, then uniqueness fails.*
2. *if $\lambda \in \sigma_r(T)$, then existence fails for some f,*
3. *if $\lambda \in \sigma_c(T)$, then the solution does not depend continuously on f. TODO: verify with (re)definition of continuous spectrum.*

2.1.2 Equations of the first and second kind

- An operator equation of the form $T(u) = f$ is said to be of the first kind
- An operator equation of the form $\lambda u - T(u) = f$ for some non-zero scalar λ is said to be of the second kind.

2.2 Equations on function spaces

2.2.1 Relevant spaces

2.2.2 Solutions

classical, weak, distributional.

2.2.2.1 Green's functions

2.2.3 Boundary conditions

It is often useful to identify a subset of functions using boundary conditions.

Let X be a topological space and Ω a closed subset. A boundary condition is an operator equation of the form

$$\big[Tu\big]_{\partial\Omega} = f$$

where f is a function on $\partial\Omega$. The operator $B := u \mapsto \big[Tu\big]_{\partial\Omega}$ is called a boundary operator.
The boundary condition is called

- Dirichlet or first-type if $T = \mathrm{id}$;
- Neumann or second-type if Ω is a Riemannian manifold and $T = \partial_{\mathbf{n}}$ where $\mathbf{n}$ is the vector normal to Ω; (TODO: correct setting??)
- Robin or third-type if $T = \mathrm{id} + \alpha\partial_{\mathbf{n}}$ for some non-zero constant α.

The boundary condition is called homogenous if $f \equiv 0$.
If the boundary condition is specified on a subset of $\partial\Omega$, then we have a partial boundary condition.

- If $\partial\Omega$ is partitioned with a partial boundary condition specified on each partition, we call this a mixed boundary condition.

- A Cauchy boundary condition consists of both a partial Dirichlet and a partial Neumann boundary condition.

TODO: An Introduction to the Finite Element Method - Reddy + TODO generalised functions?

Lemma XIV.6. *A boundary condition is of the first, second or third type* if and only if *it is of the general form*

$$\Big[(\alpha \operatorname{id} + \beta\partial_{\mathbf{n}})u\Big]_{\partial\Omega} = f$$

where α, β are constants that are not both zero.
In particular, the boundary condition is

1. *Dirichlet if $\beta = 0$;*
2. *Neumann if $\alpha = 0$;*
3. *Robin if $\alpha \neq 0 \neq \beta$.*

Lemma XIV.7. *Functions obeying a homogeneous boundary condition with linear boundary operator form a subspace.*

Proof. These functions are exactly the functions in the kernel of the boundary operator. □

2.2.4 Periodic boundary conditions

TODO

2.3 Linear dynamical systems

2.4 Approximating well-posed problems

2.5 Regularising ill-posed problems

2.6 Regularised approximation methods

Chapter 3

Ordinary differential equations

`https://www.mat.univie.ac.at/~gerald/ftp/book-ode/ode.pdf file:///C:/Users/user/Downloads/978-3-030-47849-0.pdf file:///C:/Users/user/Downloads/Polyanin%20A.,%20D.,%20Zaitsev%20V.%20F.,%20Handbook%20of%20exact%20solutions%20for%20ordinary%20differential%20equations.pdf`

3.1 Classification

An n^{th} order (ordinary) differential equation (or ODE) is an equation of the form

$$F(t, u, u', u'', \dots, u^{(n)}) \equiv 0.$$

We call a real function $u : [a, b] \to \mathbb{R}$ a solution of this differential equation on $[a, b]$ if it has at least n continuous derivatives such that $F(t, u(t), u'(t), u''(t), \dots, u^{(n)}(t))$ is zero for all $t \in [a, b]$. The set of all solutions is called the general solution.
We call the differential equation

1. linear if $\frac{\partial^2 F}{\partial u^{(i)} \partial u^{(j)}} \equiv 0$ for all $i, j \in [0, n]$;
2. homogenous if $F(t, 0, \dots, 0) = 0$.

Unless explicitly stated, we will always assume that F can be solved for the highest derivative, such that the ODE can be written as

$$u^{(n)} \equiv f(t, u, u', \dots, u^{(n-1)}).$$

A linear n^{th} order differential equation can be written in the form

$$\sum_{i=0}^{n} a_i(t) u^{(i)}(t) - g(t) = 0$$

where $a_i, g \in (]a, b[\to \mathbb{R})$.

Let $\sum_{i=0}^{n} a_i(t) u^{(i)}(t) \equiv g(t)$ be a linear differential equation. We call the functions a_i the coefficients and we say the ODE has constant coefficients if all the a_i are constants.

In the linear case we can introduce the linear operator

$$L \coloneqq \sum_{i=0}^{n} a_i \left(\frac{\mathrm{d}}{\mathrm{d}t}\right).$$

Then the differential equation can be written as $Lu = g$.
The differential equation is homogenous if and only if $g = 0$.

3.1.1 Differential problems

3.1.1.1 Initial value problems

An initial value problem (IVP) is an n^{th} order ODE together with initial conditions

$$u^{(i)}(t_0) = c_i \qquad i \in 0 : n-1$$

where t_0 and the c_i are real numbers.

TODO: replace derivatives with Lipschitz continuity.

Theorem XIV.8. *Let* $u^{(n)} \equiv f(t, u, u', \ldots, u^{(n-1)})$ *be an* n^{th} *order ODE with initial conditions*

$$u^{(i)}(t_0) = c_i \qquad i \in 0 : n-1.$$

Assume

$$f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial u'}, \ldots, \frac{\partial f}{\partial u^{(n-1)}}$$

are defined and continuous on a neighbourhood of $(t_0, c_0, \ldots, c_{n-1}) \in \mathbb{R}^{n+1}$.
Then there exists $\epsilon > 0$ *such that the IVP has a unique solution on the interval* $]t_0 - \epsilon, t_0 + \epsilon[$.

Proof. TODO □

It is possible for a solution u to be defined outside the interval $]t_0 - \epsilon, t_0 + \epsilon[$, but not be a solution to the IVP.

Example

Consider the IVP

$$u' = u^2 \qquad u(0) = c.$$

The function

$$u : \mathbb{R} \setminus \{1/c\} \to \mathbb{R} : t \mapsto \frac{c}{1 - ct}$$

is the solution only for $t < 1/c$.

3.1.1.2 Boundary value problems

A boundary value problem (BVP) is an n^{th} order ODE together with boundary conditions.

The questions of existence and uniqueness are less clear than for IVPs.

Example

Consider the ODE

$$u''(t) + u(t) = 0 \qquad 0 < t < \pi.$$

The general solution is $u(t) = c_1 \sin t + c_2 \cos t$. All solutions have $u(0) = u(\pi)$. So the BVP with boundary conditions

$$u(0) = 0 \qquad u(\pi) = 1$$

has no solution and the BVP with boundary conditions

$$u(0) = 0 \qquad u(\pi) = 0$$

has infinitely many solutions.

3.1.2 Systems of differential equations

3.2 Existence and uniqueness

3.2.1 Existence

Generally solutions to IVPs exist if f (TODO ref) is continuous. Such existence results are generally only local.

Example

Consider $u' = u^2$ with condition $u(1) = -1$. A solution is given by $u(t) = -t^{-1}$. This solution does not exist at 0, even though $u \mapsto u^2$ is continuous.

3.2.1.1 Picard-Lindelöf theorem

Theorem XIV.9 (Picard-Lindelöf)**.** *Consider $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and the differential problem of finding $y : \mathbb{R} \to \mathbb{R}^d$ such that*

$$y' = f(t, y) \qquad y(t_0) = y_0 \qquad (t_0 \in \mathbb{R}, y_0 \in \mathbb{R}^d).$$

Let R be the rectangle $[t_0, t_0 + a] \times B(y_0, b)$ for some $a, b \in \mathbb{R}$. If f is continuous on R and uniformly Lipschitz continuous w.r.t. y, then the differential problem has a unique solution $y(t)$ on $[t_0, t_0 + \alpha]$, where $\alpha = \min(a, b/M)$ and M is a bound for $|f(t, y)|$ on the rectangle R.

Any norm on $\mathbb{R}^d$ can be used to define $B(y_0, b)$, as they are all equivalent.

Proof. For any potential solution $y(t)$, the function $t \mapsto f(t, y(t))$ is integrable (TODO ref). So the solution must satisfy

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) \, \mathrm{d}s.$$

We use this the construct a sequence of approximate solutions. Set $y_0 : t \mapsto y_0$ and

$$y_{n+1} : [t_0, t_0 + a] \to \mathbb{R}^d : t \mapsto y_0 + \int_{t_0}^{t} f(s, y_n(s)) \, \mathrm{d}s.$$

These integrals can be taken because the graph of each y_n lies in the rectangle R:

$$|y_n(t) - y_0| \leq \int_{t_0}^{t} |f(s, y_{n-1}(s))| \, \mathrm{d}s \leq M\alpha \leq b.$$

TODO ☐

3.2.1.2 Peano's existence theorem

Theorem XIV.10 (Peano's existence theorem). *Consider $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and the differential problem of finding $y : \mathbb{R} \to \mathbb{R}^d$ such that*

$$y' = f(t, y) \qquad y(t_0) = y_0 \qquad (t_0 \in \mathbb{R}, y_0 \in \mathbb{R}^d).$$

Let R be the rectangle $[t_0, t_0 + a] \times B(y_0, b)$ for some $a, b \in \mathbb{R}$.
If f is continuous on R and $|f(t, y)|$ is bounded on R with bound M, then the differential problem has at least one solution $y(t)$ on $[t_0, t_0 + \alpha]$.

Any norm on $\mathbb{R}^d$ can be used to define $B(y_0, b)$, as they are all equivalent.

Proof. TODO ☐

3.2.2 Differential inequalities

3.2.3 Dependence on initial conditions and parameters

3.3 First order differential equations

3.3.1 Existence and uniqueness

3.3.2 Qualitative properties of solutions

3.3.3 A miscellany of solutions for different types of equations

3.3.3.1 Separable equations

The logistic equation.

3.3.3.2 Exact equations

3.3.3.3 Linear first order differential equations

3.3.3.4 Homogeneous equations

3.3.3.5 Bernoulli equations

3.4 Systems of equations and higher order equations

3.4.1 Problem statement and notation

Equivalence systems and higher order

3.4.2 Existence and uniqueness

3.4.3 Second order equations

3.4.4 Higher order linear equations

3.4.5 Systems of first order equations

3.5 Qualitative analysis

3.6 Solutions by infinite series and Bessel functions

3.7 Second order differential equations

3.7.1 Solutions with Green's functions

Proposition XIV.11. *Consider a second order linear differential equation on an interval* $[a, b]$*, which is of the general form*

$$Lu = a_2u'' + a_1u' + a_0u = f$$

where $a_2, a_1, a_0, f \in \mathcal{C}([a, b])$*. Consider mixed homogenous boundary conditions of first, second or third type, i.e. of the form*

$$B_au = c_1u(a) + c_2u'(a) = 0$$
$$B_bu = c_3u(b) + c_4u'(b) = 0$$

where at least one of c_1, c_2 *and* c_3, c_4 *is non-zero.*
Assume that $a_2(x) \neq 0$ *for all* $x \in [a, b]$*,* $f \in L^2([a, b])$ *and the kernel of* L *is trivial.*
Then there exists a unique solution of the form

$$u(x) = \int_a^b G(x, y)f(y)\,\mathrm{d}y$$

where G *is a bounded function in* $([a, b] \times [a, b] \to \mathbb{C})$*.*

Proof. We want to find a kernel G such that

$$L(G(\cdot, y)) = \delta_y$$

as distributions for all fixed $y \in [a, b]$ and

$$B_aG(\cdot, y) = 0 = B_bG(\cdot, y).$$

TODO □

3.7.2 Sturm-Liouville theory

3.7.2.1 Strum-Liouville problems and operators

A Sturm-Liouville equation is a real second-order ODE of the form

$$-(pu')' + qu = \lambda\omega u$$

where $\lambda \in \mathbb{R}$ and $p, p', q, \omega \in \mathcal{C}([a,b])$ for some $a, b \in \mathbb{R}$. Also p is assumed strictly positive on $]a,b[$ and ω strictly positive on $[a,b]$.
A Sturm-Liouville operator is a linear operator on the Sobolev space $W^{2,2}([a,b])$ of the form

$$L : u \mapsto \frac{1}{\omega}\Big(-(pu')' + qu \Big)$$

where p, q, ω are as above.
A Sturm-Liouville problem is a Sturm-Liouville equation with mixed homogenous boundary conditions of first, second or third type, i.e. of the form

$$c_1 u(t_b) + c_2 u'(t_b) = 0$$

where t_b is a or b and at least one of c_1, c_2 is non-zero.
We call the Sturm-Liouville problem

- regular if p is strictly positive on $[a,b]$ and boundary conditions are specified at a and b;
- singular if any of the following hold:
  - $p(a) = 0$ and there is no boundary condition at a;
  - $p(b) = 0$ and there is no boundary condition at a;
  - $p(a) = 0 = p(b)$ and there are no boundary conditions; or
  - $[a,b]$ is infinite.

For finite $[a,b]$, all $u \in \mathcal{C}([a,b])$ are square integrable: u is necessarily bounded by VIII.171.1 and the integral is bounded by this bound times $|b-a|$. If $[a,b]$ is infinite we still require solutions to be square integrable.
We may consider $\mathcal{C}([a,b]) \subset (\mathbb{R} \to \mathbb{C})$, in which case the Sturm-Liouville operator maps real functions to real functions.

Proposition XIV.12. *Consider a Sturm-Liouville problem. Take the Sobolev space $W^{2,2}([a,b], \omega(x)\,\mathrm{d}x)$ and let $\mathcal{H}$ be the subspace of functions that obey the boundary conditions. Then the Sturm-Liouville operator L restricted to $\mathcal{H}$ is self-adjoint.*

We take Lu to be defined if $Lu \in \mathcal{C}([a,b])$. If $[a,b]$ is infinite, we

Proof. □

Chapter 4

Partial differential equations

Transport equation, Laplace's equation, Heat equation, Wave equation

4.1 Classification

An n^{th} order partial differential equation (or PDE) is an equation of the form

$$F(x, \{D^\alpha u\}_{|\alpha|\leq m}) \equiv 0.$$

We call a function $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ a solution of this differential equation on Ω if $D^\alpha u$ exists and is continuous for $|\alpha| \leq m$ and $F(x, \{D^\alpha u(x)\}_{|\alpha|\leq m})$ is zero for all $x \in \Omega$.
The set of all solutions is called the general solution.
We call the differential equation

1. linear if $\frac{\partial^2 F}{\partial(D^\alpha u)\partial(D^\beta u)} \equiv 0$ for $|\alpha|, |\beta| \leq m$;
2. homogenous if $F(x, 0, \ldots, 0) = 0$.

Lemma XIV.13. *A PDE is linear* if and only if *it can be written in the form*

$$Lu(x) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u(x) = g(x).$$

A linear PDE is homogeneous if and only if $g = 0$.

4.1.1 Elliptic, Hyperbolic and Parabolic PDEs

Part XV

Probability theory

`file:///C:/Users/user/Downloads/Gut2005_Book_ProbabilityAGraduateCourse.pdf` `https://services.math.duke.edu/~rtd/PTE/PTE5_011119.pdf` TODO: Kolmogorov 0-1 law Gut p21

Chapter 1

Probability spaces

1.1 Kolmogorov axioms

A measure space $\langle \Omega, \mathcal{A}, P\rangle$ is called a probability space if the measure P is normalised: $P(\Omega) = 1$.

Lemma XV.1. *Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space and $A, B \subseteq \Omega$ measurable sets. Then*

1. $P(A^c) = 1 - P(A)$;
2. $P(A) = P(A \setminus B) + P(A \cap B)$
3. $P(A \cup B) + P(A \cap B) = P(A) + P(B)$;

Proof. (1) $\Omega = A \uplus A^c$ is a disjoint union.
(2) $B = (B \setminus A) \uplus (A \cap B)$ is a disjoint union.
(3) Using (2) we get

$$A \cup B = \Big(A \setminus (A \cap B)\Big) \uplus \Big(B \setminus (A \cap B)\Big) \uplus (A \cap B) = P(A) - P(A \cap B) + P(B) - P(A \cap B) + P(A \cap B).$$

$\square$

Corollary XV.1.1.

1. $P(A \cup B) \leq P(A) + P(B)$;
2. $A \subset B$ *implies* $P(B) = P(A) + P(B \setminus A)$;
3. $A \subset B$ *implies* $P(A) \leq P(B)$.

Theorem XV.2 (The inclusion-exclusion formula)**.** *Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space and $\langle A_k \rangle$ a sequence of events. Then*

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k) \; - \sum_{1 \leq i < j \neq n} P(A_i \cap A_j) \; + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) \; - \; \ldots \\ + \; (-1)^{n+1} P(A_1 \cap A_2 \cap \ldots \cap A_n).$$

This can also be written as

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{S\subset 1:n} (-1)^{\#(S)+1} P\left(\bigcap_{i\in S} A_i\right).$$

Proof. Set $A = \bigcup_{k=1}^{n} A_k$ and consider the function

$$f = \prod_{k=1}^{n} (\chi_A - \chi_{A_k})$$

in $(\Omega \to \{0,1\})$. This function is identically zero. Expanding $f = 0$ yields the equation

$$\chi_A = \sum_{k=1}^{n} \chi_{A_k} - \sum_{1\leq i<j\neq n} \chi_{A_i}\cdot\chi_{A_j} + \sum_{1\leq i<j<k\leq n} \chi_{A_i}\cdot\chi_{A_j}\cdot\chi_{A_k} - \ldots + (-1)^{n+1}\chi_{A_1}\cdot\chi_{A_2}\cdot\ldots\cdot\chi_{A_n}.$$

Integrating both sides of the equation over the measure P gives the result. □

Corollary XV.2.1 (Bonferroni inequalities)**.**

$$\begin{aligned}
P\left(\bigcup_{k=1}^{n} A_k\right) &\leq \sum_{k=1}^{n} P(A_k)\\
P\left(\bigcup_{k=1}^{n} A_k\right) &\geq \sum_{k=1}^{n} P(A_k) - \sum_{1\leq i<j\neq n} P(A_i\cap A_j)\\
P\left(\bigcup_{k=1}^{n} A_k\right) &\leq \sum_{k=1}^{n} P(A_k) - \sum_{1\leq i<j\neq n} P(A_i\cap A_j) + \sum_{1\leq i<j<k\leq n} P(A_i\cap A_j\cap A_k)
\end{aligned}$$

Proof. TODO `https://planetmath.org/proofofbonferroniinequalities` □

Corollary XV.2.2. *If the events of the sequence* $\langle A_k\rangle$ *are independent, then*

$$P\left(\bigcup_{k=1}^{n} A_k\right) = 1 - \prod_{k=1}^{n}(1 - P(A_k)).$$

Also

$$P\left(\bigcup_{k=1}^{n} A_k\right) \geq 1 - \exp\left(-\sum_{k=1}^{n} P(A_k)\right).$$

Proposition XV.3. *Let* $\langle \Omega, \mathcal{A}, \mu\rangle$ *be a probability space and* $\mathcal{F}$ *an algebra that generates the* σ*-algebra* $\mathcal{A} = \sigma\{\mathcal{F}\}$*. For any* $A \in \mathcal{A}$ *and* $\varepsilon > 0$ *there exists a set* $A_\varepsilon \in \mathcal{F}$ *such that*

$$\mu(A \,\Delta\, A_\varepsilon) \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$ and define

$$\mathcal{E} = \{A \in \mathcal{A} \mid \mu(A \,\Delta\, A_\varepsilon) \leq \varepsilon \text{ for some } A_\varepsilon \in \mathcal{F}\}.$$

Clearly $\mathcal{F} \subseteq \mathcal{E} \subseteq \mathcal{A}$. So if $\mathcal{E}$ is a σ-algebra, then it is equal to $\mathcal{A}$ and the proposition is proven.

- $\Omega \in \mathcal{A}$ because $\Omega \in \mathcal{F}$.

- Let $A \in \mathcal{E}$. Then $A^c \,\Delta\, (A_\varepsilon)^c = A \,\Delta\, A_\varepsilon < \varepsilon$, so $A^c \in \mathcal{E}$.
- Let $\langle A_i \rangle$ be a sequence of sets in $\mathcal{E}$ and set $A = \bigcup_{i=0}^\infty A_i$. Then

  $$\lim_{n\to\infty} P\left(\bigcup_{i=0}^n A_i\right) = P\left(\lim_{n\to\infty}\bigcup_{i=0}^n A_i\right) = P(A)$$

  and there exists an $n_0 \in \mathbb{N}$ such that

  $$\varepsilon/2 > P(A) - P\left(\bigcup_{i=0}^{n_0} A_i\right) = P\left(A \setminus \bigcup_{i=0}^{n_0} A_i\right).$$

  TODO

$\square$

1.2 Independence

Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space. The events in a set $\{A_i\}$ are called independent if for all finite $F \subset \{A_i\}$ we have

$$P\left(\bigcap_{A_i\in F} A_i\right) = \prod_{A_i\in F} P(A_i).$$

Lemma XV.4. *Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space and A, B independent events. Then $\{A, B^c\}, \{A^c, B\}$ and $\{A^c, B^c\}$ are also independent.*

Lemma XV.5. *Null sets are independent of any event, in particular of themselves.*

Proof. Let A be a null set and B any event. Then

$$0 \leq P(A\cap B) \leq P(A\setminus B) + P(A\cap B) = P(A) = 0,$$

so

$$P(A\cap B) = 0 = P(A)P(B).$$

$\square$

1.2.1 Independent collections of events

Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space. Let $\{\mathcal{A}_i\}_{i\in I}$ be a countable family of sets of events. The sets of events in this family are called independent if (the image of) every choice function on $\{\mathcal{A}_i\}_{i\in I}$ is independent.

Proposition XV.6. *Let $\{\mathcal{A}_i\}_{i\in I}$ be a countable family of independent sets of events. Then*

1. *$\{\mathfrak{D}\{\mathcal{A}_i\}\}$ are independent sets of events;*
2. *if the $\mathcal{A}_i$ are π-systems, then $\{\sigma\{\mathcal{A}_i\}\}$ are independent sets of events.*

Proof. The second part follows from the first by III.163.

We prove the first part by induction on the cardinality of I. For $\#(I) = 1$ any section contains only one set, which is necessarily independent.

For the induction step, let $s : I \to \bigcup\{\mathfrak{D}\{\mathcal{A}_i\}\}$ be a section. Take $i_0 \in I$. We need to show that $s[I \setminus \{i_0\}] \cup \{A\}$ is independent for all $A \in \mathfrak{D}\{\mathcal{A}_{i_0}\}$. By the induction hypothesis we may assume that $s[I \setminus \{i_0\}] \cup \{A\}$ is independent for all $A \in \mathcal{A}_{i_0}$.

Let $B \in s[I]$ and define

$$\mathcal{E}_B = \{A \in \mathfrak{D}\{\mathcal{A}_{i_0}\} \mid P(A \cap B) = P(A)P(B)\} .$$

TODO □

1.2.2 Pair-wise independence

Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space. The events in a set $\{A_k\}_{k\in I}$ are called pair-wise independent if for all $i \neq j \in I$ we have

$$P(A_i \cap A_j) = P(A_i) \cdot P(A_j).$$

Clearly independence implies pair-wise independence. The converse is not true.

Example

Let $\Omega = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$, $\mathcal{A} = \mathcal{P}(\Omega)$ and $P = A \mapsto 1/4 \cdot \#(A)$.
Set $A_k = \{\text{the } k^{\text{th}} \text{ coordinate equals } 1\}$ for $k = 1,2,3$. Then

$$\begin{aligned} P(A_k) &= \frac{1}{2} && \forall k \in \{1,2,3\} \\ P(A_i \cap A_j) &= \frac{1}{4} && \forall i \neq j \in \{1,2,3\} \\ P(A_i)P(A_j) &= \frac{1}{4} && \forall i \neq j \in \{1,2,3\} \\ P(A_1 \cap A_2 \cap A_3) &= \frac{1}{4} \\ P(A_1)P(A_2)P(A_3) &= \frac{1}{8}. \end{aligned}$$

The sets A_1, A_2, A_3 are pair-wise independent, but not independent.

1.3 Conditional probability

Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space, A and B be two events, and suppose that $P(A) > 0$. The conditional probability of B given A is defined as

$$P(B|A) := \frac{P(A \cap B)}{P(A)}.$$

TODO Radon-Nikodym for $P(A) = 0$!

Lemma XV.7. *Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space and A an event with non-zero probability. Then*

$$P(\cdot|A) : B \mapsto P(B|A)$$

is a probability measure on the measurable space $\langle \Omega, \mathcal{A}\rangle$.

Lemma XV.8. *Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space, $A, B \in \mathcal{A}$ and $P(A) > 0$. The events A and B are independent* if and only if *$P(B|A) = P(B)$.*

Proof. The equations $P(B|A) = \frac{P(A\cap B)}{P(A)} = P(B)$ and $P(A \cap B) = P(A)P(B)$ are equivalent by multiplying / dividing by $P(A)$. □

1.3.1 Chain rule and law of total probability

Theorem XV.9 (Law of total probability)**.** *Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space and $\langle H_i\rangle_{i\in I}$ a (countable) partition of Ω. Then, for any event $A \in \mathcal{A}$*

$$P(A) = \sum_{i\in I} P(A|H_i) \cdot P(H_i).$$

Proof. $A = A \cap \Omega = \biguplus_{i\in I}(A \cap H_i)$ is a disjoint union. So

$$P(A) = P\left(\biguplus_{i\in I}(A \cap H_i)\right) = \sum_{i\in I} P(A \cap H_i) = \sum_{i\in I} P(A|H_i)P(H_i).$$

□

1.3.2 Bayes' formula

Theorem XV.10 (Bayes' formula)**.** *Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space and $\langle H_i\rangle_{i\in I}$ a (countable) partition of Ω. Then, for any event A of non-zero probability,*

$$P(H_k|A) = \frac{P(A|H_k) \cdot P(H_k)}{P(A)} = \frac{P(A|H_k) \cdot P(H_k)}{\sum_{i\in I} P(A|H_i) \cdot P(H_i)}.$$

Proof. We calculate

$$P(H_k|A) = \frac{P(H_k \cap A)}{P(A)} = \frac{P(A|H_k) \cdot P(H_k)}{P(A)}.$$

□

1.4 Sequences of events

1.4.1 Infinitely often events

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\langle A_n\rangle$ be a sequence of events. We say $\omega \in A_n$ infinitely often (or i.o.) if $\omega \in \limsup_{n\to\infty} A_n$.
The event $\limsup_{n\to\infty} A_n$ is also denoted $\{A_n \text{i.o.}\}$.

1.4.1.1 Borel-Cantelli lemma

Proposition XV.11 (Borel-Cantelli lemma)**.** *Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space.*

1. *If $\langle A_n \rangle$ is an arbitrary sequence of events, then*

$$\sum_{n=1}^{\infty} P(A_n) < \infty \implies P(A_n i.o.) = 0;$$

2. *If $\langle A_n \rangle$ is a sequence of pair-wise independent events, then*

$$\sum_{n=1}^{\infty} P(A_n) = \infty \implies P(A_n i.o.) = 1.$$

Proof. (1) We have, for all k,

$$P(A_n \text{i.o.}) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right) \leq P\left(\bigcup_{m=k}^{\infty} A_m\right) \leq \sum_{m=k}^{\infty} P(A_m),$$

and $\sum_{m=k}^{\infty} P(A_m) \to 0$ as $k \to \infty$.
(2) TODO ☐

Corollary XV.11.1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\langle A_n \rangle$ be a sequence of pair-wise independent events. Then*

$$P(A_n i.o.) = \begin{cases} 0 & \sum_{n=1}^{\infty} P(A_n) < \infty \\ 1 & \sum_{n=1}^{\infty} P(A_n) = \infty. \end{cases}$$

Chapter 2

Random variables, random vectors and random elements

Let $(\Omega, \mathcal{A}, P)$ be a probability space and (S, d) a metric space which we consider as a measurable space $(X, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra.

- A measurable function $X : \Omega \to S$ is called a random element.
- If S is a normed vector space, then X is called a random vector.
- If $S = \mathbb{R}$, then X is called a random variable (or r.v.).
- If $S = \overline{\mathbb{R}} = [-\infty, +\infty]$, then X is called an extended random variable.

Lemma XV.12. *Let $X : (\Omega, \mathcal{A}, P) \to (S, d)$ be a random element. Then the pushforward measure*

$$\mathbb{P}_X : \mathcal{B} \to [0, +\infty] : B \mapsto P(X^{-1}(B)) = P(\{\omega \in \Omega \mid X(\omega) \in B\})$$

is a probability measure on $(V, \mathcal{B})$.

Proof. The pushforward measure is a measure by XI.56. It is normalised because $X^{-1}[V] = \Omega$, so $\mathbb{P}(V) = P(\Omega) = 1$. $\square$

The pushforward probability measure $\mathbb{P}_X$ is called the induced probability measure or the probability measure induced by X. The probability space $(S, \mathcal{B}, \mathbb{P}_X)$ is the induced probability space.

We will often write $P(X \in B)$ for $\mathbb{P}_X(B) = P(X^{-1}(B)) = P(\{\omega \in \Omega \mid X(\omega) \in B\})$

2.1 Equivalence relations on random vectors

2.1.1 Almost sure equivalence

Let $X, Y : (\Omega, \mathcal{A}, P) \to (S, \mathcal{B})$ be random elements. We say X and Y are almost surely (a.s.) equal, denoted $X \sim Y$, if they differ on at most a null set:

$$X \sim Y \iff P(\{\omega \in \Omega \mid X(\omega) \neq Y(\omega)\}) = 0 \iff P(\{\omega \in \Omega \mid X(\omega) = Y(\omega)\}) = 1.$$

We also say X and Y are equivalent random vectors.

2.1.2 Equivalence in distribution

Let $X, Y : (\Omega, \mathcal{A}, P) \to (S, \mathcal{B})$ be random elements. We say X and Y are equal in distribution, denoted $X \overset{d}{=} Y$, if they assign the same probability the each event in $\mathcal{B}$

$$X \overset{d}{=} Y \iff \forall B \in \mathcal{B} : \; P(X \in B) = P(Y \in B).$$

Lemma XV.13. *Let $X, Y : (\Omega, \mathcal{A}, P) \to (S, \mathcal{B})$ be random elements. If X and Y are a.s. equal, then they are equal in distribution.*

Proof. Set $C = \{\omega \in \mathcal{A} \mid X(\omega) \in B\}$, $D = \{\omega \in \mathcal{A} \mid Y(\omega) \in B\}$ and $E = \{\omega \in \Omega \mid X(\omega) \neq Y(\omega)\}$. Clearly $C \cap E$ and $D \cap E$ are measurable null sets, so $P(C \cap E) = 0 = P(D \cap E)$. Also $C \setminus E = D \setminus E$.
We calculate

$$P(X \in B) = P(C) = P(C \setminus E) + P(C \cap E) = P(D \setminus E) + P(D \cap E) = P(D) = P(Y \in B).$$

□

The converse of this lemma is not true.

Example

Toss a fair coin. The universe set is $\Omega = \{\text{heads}, \text{tails}\}$. Consider the random vectors

$$X : \omega \mapsto \begin{cases} 1 & \omega = \text{heads} \\ 0 & \omega = \text{tails} \end{cases} \qquad \text{and} \qquad Y : \omega \mapsto \begin{cases} 0 & \omega = \text{heads} \\ 1 & \omega = \text{tails.} \end{cases}$$

Clearly $P(X = 1) = P(X = 0) = P(Y = 1) = P(Y = 0) = 1/2$. Thus we see that $X \overset{d}{=} Y$. But clearly X and Y are not almost surely equal. In fact they are surely unequal.

2.2 Distribution functions

Let X be a random variable on a probability space $(\Omega, \mathcal{A}, P)$. The distribution function of X is the function

$$F_X : \mathbb{R} \to [0, 1] : x \mapsto P(X \leq x) = P(\{\omega \in \Omega \mid X(\omega) \leq x\}).$$

TODO: this is the Riemann-Stieltjes function. Generalise.

Proposition XV.14. *Let X be a random variable on a probability space $(\Omega, \mathcal{A}, P)$ and F the distribution function of X. Then*

1. *F is monotonically increasing;*
2. *$\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$;*
3. *F is right-continuous at every point.*

Conversely, any function $F : \mathbb{R} \to \mathbb{R}$ that satisfies these properties is the distribution function of some random variable.

Proof. (1) Let $x \leq y$. Then $\{\omega \in \Omega \mid X(\omega) \leq x\} \subseteq \{\omega \in \Omega \mid X(\omega) \leq y\}$, so

$$F(x) = P(\{\omega \in \Omega \mid X(\omega) \leq x\}) \leq P(\{\omega \in \Omega \mid X(\omega) \leq y\}) = F(y).$$

(2) Follows from XI.57.
(3) TODO
(Converse) TODO □

Corollary XV.14.1. *Every discontinuity is a jump discontinuity, so all left limits exist. Also there are at most countably many discontinuities.*

Proof. By XI.36 and XI.37. □

Proposition XV.15. *Let X_1, X_2 be random variables. Then*

$$F_{X_1+X_2}(u) = \int_{-\infty}^{\infty} F_{X_1}(u-y)\,\mathrm{d}F_{X_2}(y).$$

If both distributions are absolutely continuous, then

$$f_{X_1+X_2}(u) = \int_{-\infty}^{\infty} f_{X_1}(u-y) f_{X_2}(y)\,\mathrm{d}y.$$

Proof. Fubini TODO □

Lemma XV.16. *Let X be a random variable. Then $P\{X = x\} = \Delta F_X(x)$.*

2.2.1 Probability density functions

2.2.2 Transformed random variables

Proposition XV.17. *Let $X : \Omega \to \mathbb{R}$ be a random variable and $g : \mathbb{R} \to \mathbb{R}$ a Borel measurable function.*

1. *If (g, h) forms a Galois connection, then $F_{g(X)}(y) = F_X(h(y))$.*
2. *If (g, h) forms an antitone Galois connection, then $F_{g(X)}(y) = 1 - F_X(h(y)) + \Delta F_X(h(y))$.*
3. *If g is either strictly increasing or strictly decreasing, then*

$$f_{g(X)}(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y} \right| & y \in \operatorname{im}(g) \\ 0 & y \notin \operatorname{im}(g). \end{cases}$$

In particular we can find a Galois connection (g, h) if g is right continuous and increasing.

Proof. (1) We calculate

$$F_{g(X)}(y) = P\{g(X) \leq y\} = P\{X \leq h(y)\} = F_X(h(y)).$$

(2) Similarly,

$$F_{g(X)}(y) = P\{g(X) \leq y\} = P\{X \geq h(y)\} = 1 - P\{X < h(y)\} = 1 - P\{X \leq h(y)\} + P\{X = h(y)\} = 1 - F_X(h(y)) + \Delta F_X$$

(3) TODO ☐

Corollary XV.17.1. *Let $X : \Omega \to \mathbb{R}$ be a random variable and $a, b \in \mathbb{R}$. Set $Y = aX + b$.*

1. *If $a > 0$, then*

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right) \qquad f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right) \qquad f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

2.2.3 Conditional distributions

Let X, Y be a random variables on a single probability space $(\Omega, \mathcal{A}, P)$. The conditional distribution function of X given Y is the function

$$F_{X|Y} : \mathbb{R}^2 \to [0,1] : (x, y) \mapsto P\{X \leq x | Y = y\} = \frac{P\{X \leq x \wedge Y = y\}}{P\{Y = y\}}.$$

2.3 Convergence

Let $\langle X_n \rangle$ be a sequence of random elements in $((\Omega, \mathcal{A}, P) \to (S, d))$ and X a random element in the same set. We say

- $\langle X_n \rangle$ converges almost surely to X if

  $$P(\{\omega \in \Omega \mid X_n(\omega) \to X(\omega) \text{ as } n \to \infty\}) = 1.$$

  We write $X_n \xrightarrow{a.s.} X$.

Lemma XV.18. *Suppose Y, X, X_n are random vectors such that $X_n \xrightarrow{a.s.} X$, $E[Y] < \infty$ and $|X_n| \leq Y$ for all n. Then $E[|X_n - X|] \to 0$ as $n \to \infty$.*

Proof. This is just the Lebesgue dominated convergence theorem TODO ref. ☐

2.4 Expected value

Let X be a random vector on a probability space $(\Omega, \mathcal{A}, P)$. We define the exprectated value of X as

$$\operatorname{E} X \coloneqq \int_\Omega X(\omega) \, \mathrm{d}P(\omega).$$

assuming X is integrable.

2.4.1 Moments

2.4.1.1 Raw and central moments

2.4.1.2 Moment generating function

2.4.1.3 Normalised moments

2.4.1.4 Examples of moments

Expected value

Variance and standard deviation

Skewness

Kurtosis

2.4.2 Mean and variance

Proposition XV.19. *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space and* $X, Y : \Omega \to \mathbb{R}$ *random variables with finite means* μ_X, μ_Y *and standard deviations* σ_X*,* σ_Y*. Then*

$$|\,\mathrm{E}[XY] - \mu_X\mu_Y| \leq \sigma_X\sigma_Y.$$

Proof. This follows from the CSB inequality, X.108:

$$\begin{aligned}\sigma_X\sigma_Y &= \sqrt{\mathrm{E}[(X-\mu_X)^2]}\sqrt{\mathrm{E}[(Y-\mu_Y)^2]} \\ &\geq |\,\mathrm{E}[(X-\mu_X)(Y-\mu_Y)]| \\ &= |\,\mathrm{E}[XY] + \mu_X\mu_Y - \mu_X\mu_Y - \mu_X\mu_Y| = |\,\mathrm{E}[XY] - \mu_X\mu_Y|.\end{aligned}$$

□

2.4.3 Cumulants

2.4.4 Conditional expectation

2.5 Joint distributions

2.5.1 Marginal distributions

2.6 Independence of random elements

A set of random elements $\{X_i\}_{i\in 1:n}$ is called independent if for all sets $\{A_i\}_{i\in 1:n}$ of n Borel measurable sets we have that $\{X_i^{-1}[A_i]\}_{i\in 1:n}$ is a set of independent events. i.e.

$$P\left(\bigcap_{i=1}^{n} X_i^{-1}[A_i]\right) = \prod_{i=1}^{n} P(X_i^{-1}[A_i]).$$

Proposition XV.20. *The random variables* $X_1, \dots X_n$ *are independent* if and only if

$$F_{X_1,\dots,X_n}(x_1,\dots,x_n) = \prod_{i=1}^n F_{X_i}(x_i).$$

Proposition XV.21. *Let* $X_1, \dots X_n$ *be independent random elements and* $f_1, \dots, f_n$ *measurable functions. Then* $f_1 \circ X_1, \dots, f_n \circ X_n$ *are independent.*

Proof. Let $A_1, \dots, A_n$ be Borel measurable sets. Since the $f_i^{-1}[A_i]$ are Borel measurable, we have

$$P\left(\bigcap_{i=1}^n (f_i \circ X_i)^{-1}[A_i]\right) = P\left(\bigcap_{i=1}^n X_i^{-1}[f_i^{-1}[A_i]]\right) = \prod_{i=1}^n P(X_i^{-1}[f_i^{-1}[A_i]]) = \prod_{i=1}^n P((f_i \circ X_i)^{-1}[A_i]).$$

□

2.7 Derived random elements

Proposition XV.22. *Let* $\langle X_i \rangle$ *be a sequence of i.i.d. random vectors. Let* N *be a positive integer-valued r.v. that is independent of each* X_i*. Then*

$$\mathrm{E}\left[\sum_{i=1}^N X_i\right] = \mathrm{E}[N]\,\mathrm{E}[X_1].$$

Proof. We calculate

$$\begin{aligned}
\mathrm{E}\left[\sum_{i=1}^N X_i\right] &= \sum_{n=0}^\infty \mathrm{E}\left[\sum_{i=1}^n X_i | N = n\right] P\{N=n\} \\
&= \sum_{n=0}^\infty \sum_{i=1}^n \mathrm{E}\left[X_i | N=n\right] P\{N=n\} \\
&= \sum_{n=0}^\infty \sum_{i=1}^n \mathrm{E}\left[X_i\right] P\{N=n\} \\
&= \sum_{n=0}^\infty n\,\mathrm{E}\left[X_1\right] P\{N=n\} \\
&= \mathrm{E}\left[X_1\right] \sum_{n=0}^\infty nP\{N=n\} \\
&= \mathrm{E}\left[X_1\right] \mathrm{E}[N].
\end{aligned}$$

□

Chapter 3

Distributions

TODO: uniform distribution

3.1 Distributions of discrete random variables

3.1.1 Bernoulli trials

3.1.1.1 Bernoulli distribution

3.1.1.2 Geometric distribution

3.1.2 Iterated Bernoulli trials

3.1.2.1 Binomial distribution

3.1.2.2 Negative binomial or Pascal distribution

3.1.2.3 Hypergeometric distribution

3.1.2.4 Negative hypergeometric distribution

3.1.3 Poisson distribution

Let $\lambda > 0$ be a positive real number. The Poisson distribution with parameter λ is the distribution of a random variable $X : (\Omega, \mathcal{A}, P) \to \mathbb{N}$ with probability density

$$f_\lambda(k) = \begin{cases} \frac{\lambda^k e^{-\lambda}}{k!} & k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

We write $X \sim \text{Poisson}(\lambda)$.

3.2 Distributions of continuous random variables

3.2.1 Gamma distribution and subfamilies

3.2.1.1 Gamma distribution

Let α, β be positive real numbers. The gamma distribution with parameters α, β is the distribution of a random variable $X : (\Omega, \mathcal{A}, P) \to \mathbb{R}_+$ with probability density

$$f_{\alpha,\beta}(x) = \frac{x^{\alpha-1}e^{-\beta x}\beta^{\alpha}}{\Gamma(\alpha)},$$

where Γ is the Gamma function.

- We call α the shape parameter.
- We call β the rate parameter.
- We call $1/\beta$ the scale parameter.

We write $X \sim \mathrm{Gamma}(\alpha, \beta)$.

TODO lower incomplete gamma function.

Lemma XV.23. *Let α, β, k be positive real numbers. Let $X \sim \mathrm{Gamma}(\alpha, \beta)$ be a random variable. Then $kX \sim \mathrm{Gamma}(\alpha, \beta/k)$.*

3.2.1.2 Erlang distribution

An Erlang distribution is a gamma distibution where the shape parameter α is an integer n. We wite $\mathrm{Erlang}(n, \beta) \coloneqq \mathrm{Gamma}(n, \beta)$.

Proposition XV.24. *Let $X \sim \mathrm{Erlang}(n, \beta)$. Then the cumulative distribution function of X is given by*

$$F_{n,\beta}(x) = 1 = \sum_{k=0}^{n-1} \frac{(\beta x)^k}{k!} e^{-\beta x}.$$

3.2.1.3 Exponential distribution

An exponential distribution is an Erlang distribution with shape parameter $n = 1$. Thus $\mathrm{Exp}(\beta) \coloneqq \mathrm{Erlang}(1, \beta) = \mathrm{Gamma}(1, \beta)$.

3.2.1.4 χ^2-distribution

Let $k \in \mathbb{N}^{\times}$ be strictly positive integer. Then we call a gamma distribution with shape $k/2$ and rate 1 a χ^2-distribution with k degrees of freedom. We write $\chi^2_k = \mathrm{Gamma}(k/2, 1)$.

3.3 Distributions of random vectors

Chapter 4

Convergence

4.1 Types of convergence

Chapter 5

Stochastic processes

`https://math.stackexchange.com/questions/1309853/proving-galmarinos-test/1596012`
`https://link.springer.com/content/pdf/10.1007%2F978-3-319-78768-8.pdf`
`https://people.math.harvard.edu/~knill/books/KnillProbability.pdf`
`file:///C:/Users/user/Downloads/(Advances%20in%20applied%20mathematics)%20Kirkwood,%20James%20R%20-%20Markov%20Processes-CRC%20Press%20(2015).pdf` `file:///C:/Users/user/Downloads/(De%20Gruyter%20Studies%20in%20Mathematics)%20Kolokoltsov%20V.N.%20-%20Markov%20processes,%20semigroups%20and%20generators-De%20Gruyter%20(2011).pdf`

5.1 Processes

Let $(\Omega, \mathcal{A}, P)$ be a probability space, (S, d) a metric space and $(I, \leq)$ a partially ordered index set. A stochastic process X is a function

$$I \times \Omega \to S : (t, \omega) \mapsto X_t(\omega)$$

such that the partial application X_t is a random element for all $t \in I$.

- A sample path, trajectory or realisation is a partial application

$$X_{-}(\omega) : t \mapsto X_t(\omega).$$

- If $I \subseteq \mathbb{N}$, we call the stochastic process a stochastic sequence.
- If I is a subinterval of $\mathbb{R}$, we call the stochastic process a continuous process.
- If $I \subseteq \mathbb{R}^+$, we call the index time and the process a continuous time process.
- If $I \subseteq \mathbb{R}^k$, we call the stochastic process a k-parameter or multiparameter process.

Lemma XV.25. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, (S, d) a metric space, $(I, \leq)$ a partially ordered index set and X a function $X : I \times \Omega \to S : (t, \omega) \mapsto X_t(\omega)$. The following are equivalent:*

1. *X is a stochastic process;*

2. *X is measurable w.r.t. the σ-algebra generated by $\bigcup_{\substack{t\in I\\ A\in\mathcal{A}}}\{\{t\}\}\times A$;*

3.

Then X is a stochastic process if and only if *X is measurable w.r.t. the σ-algebra $\mathcal{P}(I)\otimes\mathcal{A}$.*

Proof. □

Lemma XV.26. *Let $(\Omega,\mathcal{A},P)$ be a probability space, (S,d) a metric space, $(I,\leq)$ a partially ordered index set and X a function $X : I\times\Omega\to S : (t,\omega)\mapsto X_t(\omega)$.*
Then X is a stochastic process if and only if curry$(X) : \Omega\to(I\to S)$ *is measurable, where $(I\to S)$ is equipped with the σ-algebra $\bigotimes_{t\in I}\mathcal{B}$ and $\mathcal{B}$ is the Borel-σ-algebra on S.*

Proof. TODO!! (is it true??) □

5.1.1 Equivalent stochastic processes

Let $(\Omega,\mathcal{A},P)$ be a probability space, (S,d) a metric space and $(I,\leq)$ a partially ordered index set. Let X,Y be stochastic processes. We call X and Y

- equivalent or indistinguishable if
$$P(X_- = Y_-) = 1.$$
- stochastically equivalent or modifications of each other if
$$\forall t\in I : P(X_t = Y_t) = 1.$$

Lemma XV.27. *Equivalence and stochastic equivalence are equivalence relations.*

Lemma XV.28. *Let $(\Omega,\mathcal{A},P)$ be a probability space, (S,d) a metric space, $(I,\leq)$ a partially ordered index set and X,Y stochastic processes. Then*

1. *X and Y are equivalent if and only if*
$$\exists \text{ null set } A\subseteq\Omega : \forall t\in I : \forall\omega\in A^c :\; X_t(\omega) = Y_t(\omega);$$
2. *X and Y are stochastically equivalent if and only if*
$$\forall t\in I : \exists \text{ null set } A\subseteq\Omega : \forall\omega\in A^c :\; X_t(\omega) = Y_t(\omega).$$

In particular equivalence implies stochastic equivalence.

The opposite implication does not hold:

Example

Let T be a random variable that is uniformly distributed over the interval $[0,1]$. Define
$$X_t = \chi_{\{t=T\}} \qquad \text{and} \qquad Y_t = \omega\mapsto 0.$$
So the sample paths of Y are identically zero and the sample paths of X are zero everywhere except at one point, where it is one.

Now X and Y are stochastically equivalent: for all t we have

$$P(X_t = Y_t) = P(X_t = 0) = P(T \neq t) = 1.$$

But X and Y are not equivalent: for all ω, the sample paths differ by exactly one point, so the probability of them coinciding is zero.

Proposition XV.29. *If the index set I is countable, then stochastic equivalence implies equivalence.*

Proof. Let X, Y be stochastically equivalent processes on a countable index set I. We calculate

$$\begin{aligned} P(X_- = Y_-) &= P\left(\bigcap_{i\in I}\{X_i = Y_i\}\right) = 1 - P\left(\bigcup_{i\in I}\{X_i \neq Y_i\}\right) \\ &\geq 1 - \sum_{i\in I} P(\{X_i \neq Y_i\}) = 1. \end{aligned}$$

□

So for countable index sets stochastic equivalence and equivalence are equivalent.

Proposition XV.30. *Let X be a stochastic process such that all sample paths of X lie in a separable ...*

Proof. TODO □

Corollary XV.30.1. *Let X and Y be stochastic processes whose sample paths are almost surely right-continuous (resp. left-continuous). Then X and Y are equivalent* if and only if *they are stochastically equivalent.*

5.1.2 Filtration

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(I, \leq)$ an ordered index set. A filtration is an order-preserving function from I to the lattice of sub-σ-algebras of $\mathcal{A}$.
If i is mapped to $\mathcal{A}_i$, we call $(\Omega, \mathcal{A}, \{\mathcal{A}_i\}_{i\in I}, P)$ a filtered probability space.
Let $t \in I$. Then we define

- the left limit at t as $\mathcal{A}_{t-} \coloneqq \bigvee_{s<t} \mathcal{A}_s$;
- the right limit at t as $\mathcal{A}_{t+} \coloneqq \bigwedge_{t<s} \mathcal{A}_s$;
- the limit at infinity as $\mathcal{A}_\infty \coloneqq \bigvee_{s\in I} \mathcal{A}_s$.

The suprema and infima are taken in the lattice of σ-algebras on Ω. By III.19 we have

$$\bigwedge \mathcal{E} = \bigcap \mathcal{E} \qquad \text{and} \qquad \bigvee \mathcal{E} = \sigma\left\{\bigcup \mathcal{E}\right\},$$

where $\mathcal{E}$ is any set of σ-algebras on Ω.

Lemma XV.31. *Let $(\Omega, \mathcal{A}, \{\mathcal{A}_i\}_{i\in I}, P)$ be a filtered probability space. Then for all $t \in I$*

1. $\mathcal{A}_{t-} \subseteq \mathcal{A}_t \subseteq \mathcal{A}_{t+}$;
2. $\mathcal{A}_t \subseteq \mathcal{A}_\infty$.

Proof. Using the fact that $t \mapsto \mathcal{A}_t$ is order-preserving and III.54, we have

$$\mathcal{A}_{t-} = \bigvee \mathcal{A}_{\downarrow t\setminus\{t\}} \subseteq \mathcal{A}_{\bigvee \downarrow t\setminus\{t\}} \subseteq \mathcal{A}_{\bigvee \downarrow t} = \mathcal{A}_t = \mathcal{A}_{\bigwedge \uparrow t} \subseteq \mathcal{A}_{\bigwedge \uparrow t\setminus\{t\}} \subseteq \bigwedge \mathcal{A}_{\uparrow t\setminus\{t\}} = \mathcal{A}_{t+}.$$

□

5.1.2.1 Adapted processes and natural filtrations

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(I, \leq)$ an ordered index set and (S, d) a metric space.

- Given a filtration $\{\mathcal{A}_t\}_{t\in I}$, a stochastic process $X : I \times \Omega \to S$ is adapted to the filtration if for all $t \in I$ X_t is measurable w.r.t. $\mathcal{A}_t$.
- Given a stochastic process $X : I \times \Omega \to S$, the natural filtration $\mathrm{NF}(X)$ is the filtration generated by X:

$$\mathcal{A}_t \coloneqq \sigma \left\{ X_s^{-1}[B] \;\middle|\; s \leq t, B \in \mathcal{B}(S) \right\}.$$

Let $\mathrm{NF}^S_{I,\Omega}$ be the set of natural filtrations generated by stochastic processes in $(I \times \Omega \to S)$.

Lemma XV.32. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(I, \leq)$ an ordered index set and (S, d) a metric space. Let $X : I \times \Omega \to S$ be a stochastic process and $\{\mathcal{A}_t\}_{t\in I}$ a filtration. Then*

$$X \text{ is adapted to } \{\mathcal{A}_t\}_{t\in I} \iff \mathrm{NF}(X) \leq \{\mathcal{A}_t\}_{t\in I}.$$

Lemma XV.33. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(I, \leq)$ an ordered index set, (S, d) a metric space and $X : I \times \Omega \to S$ a stochastic process.*
For all $t \in I$ we define $\mathcal{A}_i$ as the set of all subsets of Ω of the form

$$\bigcap_{j\in J} X_{t_j}^{-1}[B_j]$$

for some (countable) sequence $\langle t_j \rangle$ in $\downarrow t$ and $\langle B_j \rangle$ in $\mathcal{B}(S)$.
Then $\{\mathcal{A}_t\}_{t\in I}$ is the natural filtration $\mathrm{NF}(X)$.

Proof. Clearly $\left\{ X_s^{-1}[B] \;\middle|\; s \leq t, B \in \mathcal{B}(S) \right\}_{t\in I} \subseteq \{\mathcal{A}_t\}_{t\in I} \subseteq \mathrm{NF}(X)$. So it is enough to prove that $\mathcal{A}_t$ is a σ-algebra for all $t \in I$. By the monotone class theorem, III.162, it is in fact enough to show that $\mathcal{A}_t$ is a monotone class for all $t \in I$. Closure under countable (monotone) intersections is clear.

□

Proposition XV.34. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(I, \leq)$ an ordered index set and (S, d) a metric space.*
The set $\mathrm{NF}^S_{I,\Omega}$ of natural filtrations forms a sublattice of the lattice of filtrations of $\mathcal{A}$ on I.

Proof. □

5.1.2.2 Usual and natural conditions

Usual conditions, natural conditions

5.1.3 Stopping times

Let $(\Omega, \mathcal{A}, \{\mathcal{A}_i\}_{i\in I}, P)$ be a filtered probability space, $(I, \leq)$ an ordered index set and $\overline{I}$ its Dedekind-MacNeille completion. A stopping time is a function $\tau : \Omega \to I$ such that $\{\tau \leq t\} \in \mathcal{A}_t$ for all $t \in I$.

We equip the set of stopping times with pointwise order.

Lemma XV.35. *Let $(\Omega, \mathcal{A}, \{\mathcal{A}_i\}_{i\in I}, P)$ be a filtered probability space and $(I, \leq)$ an ordered index set. Then*

1. *the constant function $\underline{t}$ is a stopping time for all $t \in I$;*
2. *the set of stopping times is a sublattice of $(\Omega \to I)$.*

Proof. (1) For all $i \in I$: we have

$$\{\underline{t} \leq i\} = \begin{cases} \Omega & (t \leq i) \\ \emptyset & (t > i) \end{cases}$$

In both cases $\{\underline{t} \leq i\}$ is an element of each σ-algebra.
(2) □

Theorem XV.36 (Galmarino's test)**.** *Let $(\Omega, \mathcal{A}, \{\mathcal{A}_i\}_{i\in I}, P)$ be a filtered probability space and $(I, \leq)$ an ordered index set.*
A function $\tau : \Omega \to I$ is a stopping time if and only if *for all $\omega, \omega' \in \Omega$:*

$$\tau(\omega) = t \wedge \Big(\forall s \in {\downarrow} t : X_s(\omega) = X_s(\omega')\Big) \implies \tau(\omega') = t.$$

5.1.3.1 Stopped processes

Let $X : I \times \Omega \to S$ be a stochastic process and $\tau : \Omega \to I$ a stopping time. Then the stopped process X^τ is defined by

5.1.4 Measurability requirements

5.1.4.1 Joint measurability

Let $(\Omega, \mathcal{A}, P)$ be a probability space, (S, d) a metric space, $(I, \mathcal{T}, \leq)$ a topological poset and X a stochastic process. We call $X : \Omega \times I \to S$ jointly measurable if it is measureable w.r.t. $\mathcal{A} \otimes \mathcal{B}(I)$.

TODO: order algebra???????

Lemma XV.37. *All right-continuous and left-continuous processes are jointly measurable.*

Proposition XV.38. *If X is a jointly measurable stochastic process and $\tau : \Omega \to I$ a random time, then*

$$X_\tau : \Omega \to S : \omega \mapsto X(\omega)_\omega$$

is measurable and thus a random element.

Proof. □

5.1.4.2 Predictable and optional processes

5.1.4.3 Progressively measurable processes

5.2 Martingales

5.3 Properties and classes of processes

5.3.1 Processes generated by transition probabilities

`https://arxiv.org/pdf/1603.00251.pdf`

5.3.1.1 Markov processes

5.3.1.2 Feller processes

“C_0-semigroup”??

5.3.1.3 Lévy processes

5.3.2 Processes by distribution

5.3.2.1 Wiener processes

5.3.2.2 Bessel processes

5.3.3 Integer-valued processes

5.3.3.1 Birth-death processes

An integer-valued Markov process is called a birth-death process.

5.3.3.2 Counting processes and birth processes

A counting process is a stochastic process

$$N : (I, \leq) \times (\Omega, \mathcal{A}, P) \to \mathbb{N}$$

for some probability space Ω and ordered set I such that for all $s, t \in I$:

$$\forall s, t \in I : \quad s \leq t \implies N(s) \leq N(t).$$

If the counting process N is a Markov process, then it is called a birth process.

5.3.3.3 Renewal processes

`https://en.wikipedia.org/wiki/Renewal_theory`

5.3.3.4 Poisson processes

TODO: renewal Markov

A stochastic process
$$N : \mathbb{R}_+ \times (\Omega, \mathcal{A}, P) \to \mathbb{Z}$$
is called a Poisson process if it has independent increments and there exists a continuous, increasing function $\Lambda : \mathbb{R} \to \mathbb{R}$ such that
$$N_t - N_s \sim \text{Poisson}(\Lambda(t) - \Lambda(s))$$
for all $s \leq t$.
The function Λ is called the cumulative rate. If $\frac{\mathrm{d}\Lambda(s)}{\mathrm{d}s}$ exists, it is called the (instantaneous) rate.
If Λ is of the form $t \mapsto t\lambda$ for some $\lambda \in \mathbb{R}_+$, then we say N is a homogenous Poisson process.

Lemma XV.39. *Let N_t be a Poisson process with cumulative rate Λ.*

1. *Let $\theta : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function. Then $N_{\theta(t)}$ is a Poisson process with cumulative rate $\Lambda \circ \theta$, if $\Lambda \circ \theta$ is increasing.*
2. *Let $c \in \mathbb{R}$. Then $\Lambda + \underline{c}$ is also a cumulative rate function for N_t.*
3. *Let K be a random, integer-valued element. Then $N_t + K$ is also a Poisson process with cumulative rate Λ.*

From now on we assume, WLOG, that $\Lambda(0) = 0$.

Lemma XV.40. *Let N be a Poisson process with cumulative rate Λ such that $N_0 = 0$ and $\Lambda(0) = 0$. Then for each $t \in \mathbb{R}_+$, we have $N_t \sim \text{Poisson}(\Lambda(t))$.*

Proof. This follows directly from $N_t = N_t - N_0 \sim \text{Poisson}(\Lambda(t) - \Lambda(0)) = \text{Poisson}(\Lambda(t))$. □

Proposition XV.41. *For any continuous, increasing function $\Lambda : \mathbb{R}_+ \to \mathbb{R}$, there exists a Poisson processes with cumulative rate Λ.*

Proof. Construct homogenous Poisson process N_t with rate 1 (TODO ref). Then $N_{\Lambda(t)}$ is a Poisson process with cumulative rate Λ by XV.39. □

Proposition XV.42. *Let N be a Poisson process. Then*

1. *N is a Markov process;*
2. *if N is a homogenous Poisson process, then it is a Lévy process.*

Proof. TODO □

Proposition XV.43. *Let N be a Poisson process. For each $n \in \mathbb{N}$, consider the stopping time*
$$\tau_n : (\Omega, \mathcal{A}, P) \to \mathbb{R}_+ : \omega \mapsto \inf \{t \mid N_t(\omega) - N_0(\omega) \geq n\}.$$

1. *The distribution of* τ_n *is given by*

$$F_{\tau_n}(t) = 1 - \sum_{k=0}^{n-1} \frac{\Lambda(t)^k}{k!} e^{-\Lambda(t)} = F_{\mathrm{Erlang}(n,1)}(\Lambda(t)).$$

2. *If* N *is a homogenous Poisson process with rate* λ*, then* $\tau_n \sim \mathrm{Erlang}(n, \lambda)$*. In particular* $\tau_1 \sim \mathrm{Exp}(\lambda)$*.*

3. *If* Λ *is bijective, then* $N_{\Lambda^{-1}(t)}$ *is a homogenous Poisson process with rate* 1 *and stopping times* $\Lambda(\tau_n)$*.*

Proof. (1) We have

$$\begin{aligned} F_{\tau_n}(t) &= P\{\tau_n \le t\} = P\{N_t \ge n\} \\ &= \sum_{k=n}^{\infty} P\{N_t = k\} = \sum_{k=n}^{\infty} \frac{\Lambda(t)^k}{k!} e^{-\Lambda(t)} \\ &= (e^{\Lambda(t)} - \sum_{k=0}^{n-1} \frac{\Lambda(t)^k}{k!}) e^{-\Lambda(t)} = 1 - \sum_{k=0}^{n-1} \frac{\Lambda(t)^k}{k!} e^{-\Lambda(t)}. \end{aligned}$$

We see this is equal to $F_{\mathrm{Erlang}(n,1)}(\Lambda(t))$ by comparing with XV.24.
(2) This follows by the substitution $\Lambda(t) = \lambda t$ and comparison with XV.24.
(3) We calculate

$$\inf\left\{t \;\middle|\; N_{\Lambda^{-1}(t)}(\omega) - N_0(\omega) \ge n\right\} = \inf\left\{\Lambda(u) \mid N_u(\omega) - N_0(\omega) \ge n\right\} = \Lambda(\tau_n),$$

using the substitution $u = \Lambda^{-1}(t)$ and TODO ref inf preserving. □

Proposition XV.44. *Let* N *be a homogenous Poisson process with rate* λ*. Then the interarrival times* $S_n = \tau_n - \tau_{n-1}$ *are i.i.d. random variables with distribution* $\mathrm{Exp}(\lambda)$*.*

Proof. TODO □

5.4 Stochastic integration

5.4.1 Stochastic differential equations

Part XVI

Statistics

TODO: comparison of two runs of a simulation as measure of accuracy and variance??

Chapter 1

Descriptive statistics

TODO: Odds!

1.1 Statistics

1.2 Random samples

1.3 Cox's theorem

Chapter 2

Some important distributions

2.1 Discrete distributions

2.2 Continuous distributions

Chapter 3

Multivariate statistics

Chapter 4

Convergence and limits

Chapter 5

Statistical models and parametric point estimation

5.1 Point estimators

5.2 Confidence regions

Chapter 6

Estimation methods and estimation theory

Chapter 7

Hypothesis testing

Chapter 8

Bayesian statistical inference and nonparametric statistical inference

Chapter 9

Experimental methods

ML / LS curve fitting error propagation

Part XVII

Geometry

Chapter 1

Introduction

Space is obviously quite important for physics. Finding a mathematical model for space presents some challenges. Many branches of mathematics have tried to model essential characteristics of space in different ways. This has lead to many different mathematical meanings, one of which we have already seen: the vector space.

> SPACE. The word came into English—from Old French from Latin—around 1300. The OED entry distinguishes many meanings. In one sense (under heading 6b) it has room as a synonym. This word derives from the Old English and is related to the modern German Raum. Under heading 17 the OED defines "a space" as "an instance of any of various mathematical concepts, usually regarded as a set of points having some specified structure." Among the quotations is a nice one from 1932: "The word 'space' has gradually acquired a mathematical significance so broad that it is virtually equivalent to the word 'class', as used in logic." (M. H. Stone Linear Transformations in Hilbert Space p. 1.) The space age was well under way by 1914 when Hausdorff's Grundzüge der Mengenlehre (Fundamentals of Set Theory) gave axioms for a METRIC SPACE (metrischer Raum) and for a TOPOLOGICAL SPACE (topologischer Raum).[1]

The main branch of mathematics that is relevant for the modeling of space is obviously geometry. Just like the word "space", the label geometry is applied to many parts of mathematical reasoning.

1.1 Erlangen programm

In 1872 Felix Klein proposed his Erlangen programm (named after the University Erlangen-Nürnberg, where Klein worked) which attempted to classify different geometries based on symmetries. In particular it allowed geometry to be viewed as the study of properties of figures that remain invariant under a certain group of transformations. So for example, in Euclidean geometry may be viewed as the study of properties that remain invariant under isometry transformations (these can roughly be viewed as the translations and rotations). Two shapes are called congruent if there is an isometry that maps one onto the other. If we take a rectangle, its corners will always be 90° angles no matter how we rotate or translate it. Angles are in general invariant under such transformations. Other invariants include

---

[1] From *Earliest Known Uses of Some of the Words of Mathematics*, `http://jeff560.tripod.com/s.html`

- Distances;
- Areas;
- Volumes;
- Whether lines are parallel, or not;
- Whether points are or other shapes, or not;
- Whether points are collinear, or not;

Another important aspect to the Erlangen program is its hierarchical nature: if we take a larger group of transformations, then fewer aspects will invariants of all the transformations. Conversely, if we restrict the group of transformations, more aspects will be invariants. For example, the isometry transformations are part of a larger group of transformations called affine transformations. In particular they are the affine transformations that preserve distance. Of the bulleted list of invariants above, the last three are affine invariants, but the first three are **not**.

In this way projective geometry may be seen as the underlying, unifying frame. Restricting it a bit we get affine geometry; restricting it a bit more we get Euclidean geometry.

1.2 Analytic–synthetic distinction

TODO

Chapter 2

Euclidean and related geometry

2.1 Axiomatic (or synthetic) Euclidean geometry

Historically it was easy: geometry was the study of shapes, either on a flat plane or in space (or what was somewhat naively thought to be space). By the third century BC Euclid had published his *Elements*. In it he gave gave the postulates of what is now called Euclidean geometry. Euclid held his postulates to be self-evidently applicable the the actual, physical space in the real world.

2.1.1 Euclid's *Elements*

There are 13 books[1] in the *Elements*.

- Books I to IV and VI discuss planar geometry.

Book I lays out the fundamentals of planar geometry involving straight lines.

Book II contains propositions to do with squares and rectangles. This has a strong link with geometric algebra due to the link with the squaring of a number.

Book III lays out the fundamentals of planar geometry involving circles.

Book IV deals with the intersection of circles and rectilinear figures.

Book VI discusses similar figures.

- Books V and VII-X deal with number theory, with numbers treated geometrically as lengths of line segments or areas of regions.
- Books XI-XIII concern solid geometry.
- There are apocryphal books XIV and XV, probably written by Hypsicles and Isidore of Miletus, respectively. These are not usually included.

What follows is the first part of book I[2], which contains some definitions and Euclid's postulates.

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[1] It must be remembered that in ancient times book binding technology was not as advanced and works were published in smaller subunits called *books*. Each book was only a few dozen pages long, in today's pages, but published on scrolls.

[2] Translation by TODO ref

BOOK I

DEFINITIONS

1. A **point** is that which has no part.
2. A **line** is breadthless length.
3. The extremities of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A **plane surface** is a surface which lies evenly with the straight lines on itself.
8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called **rectilineal**.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.
11. An **obtuse angle** is an angle greater than a right angle.
12. An acute angle is an angle less than a right angle.
13. A **boundary** is that which is an extremity of anything.
14. A **figure** is that which is contained by any boundary or boundaries.
15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;
16. And the point is called the **centre** of the circle.
17. A **diameter** of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.
18. A **semicircle** is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.
19. **Rectilineal figures** are those which are contained by straight lines, **trilateral** figures being those contained by three, **quadrilateral** those contained by four, and **multilateral** those contained by more than four straight lines.
20. Of trilateral figures, an **equilateral triangle** is that which has its three sides equal, an **isosceles triangle** that which has two of its sides alone equal, and a **scalene triangle** that which has its three sides unequal.
21. Further, of trilateral figures, a **right-angled triangle** is that which has a right angle, an **obtuse-angled triangle** that which has an obtuse angle, and an **acute-angled triangle** that which has its three angles acute.
22. Of quadrilateral figures, a **square** is that which is both equilateral and right-angled; an **oblong** that which is right-angled but not equilateral; a **rhombus** that which is equilateral but not right-angled; and a **rhomboid** that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called **trapezia**.

23. **Parallel** straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

POSTULATES

Let the following be postulated :

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

COMMON NOTIONS

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

The rest of book I contains propositions. The other books contain definitions and propositions. When Euclid uses "equal", he means equal in magnitude. This is effectively what is now called congruence. This equals is obviously an equivalence relation (the transitivity and reflexivity are asserted as "common notions" 1. and 4.). Common notions 2. and 3. assert that the operations of addition and subtraction are compatible with the equality relation.

There is some debate as to whether the common notions were (in part or at all) included by Euclid[3].

In the postulates, Euclid only explicitly asserts the *existence* of the constructed objects, in his reasoning *uniqueness* is implicitly assumed.

2.1.1.1 The fifth postulate.

Reading through the postulates, it may be obvious that the fifth is of quite a different nature than the rest. It looks like is should be a proposition and for centuries mathematicians tried, unsuccessfully, to prove it as such.

Euclid himself, it seems, mistrusted the postulate and proved the first 28 propositions without using it.

Many equivalent formulations of the fifth postulate exist, the most famous probably being Playfair's axiom:

> In a plane, through a point not on a given straight line, at most one line can be drawn that never meets the given line.

---

[3] See ref TODO for much insightful commentary

In fact there is always exactly one. It can be shown (and in fact follows from proposition 27) that even in absolute geometry we can always find at least one parallel line.
Only in the beginning of the 19^{th} century did mathematicians start what Euclidean geometry would look like without the fifth postulate. Leaving out this postulate, one obtains what is known as absolute geometry.
If the fifth postulate were provable as a theorem, absolute geometry would be the same as Euclidean geometry. It turns out that this is not the case and including the negation of the fifth postulate leads to a consistent set of axioms, describing a non-Euclidean geometry.

2.1.2 Other axiomatic systems

Euclid's axioms do not actually provide the complete, logical foundation he thought they did. Many authors have offered their own sets of axioms that meet modern standards or rigour. Moritz Pasch was the first to accomplish this task in 1882.

2.1.2.1 David Hilbert

proposed a set of axioms in 1899 that did not depart too greatly in spirit from Euclid's, but was complete. The result was an axiom system constructed with six primitive notions (point, line, plane, betweenness, congruence and containment) and twenty axioms devided into 5 classes (incidence, order, congruence, parallels and continuity).

2.1.2.2 George Birkhoff

created a set of axioms[4] for planar geometry that uses real numbers in 1932. Leveraging the mathematics of real numbers, only four axioms were needed. Birkhoff's reason for introducing them was that they may be readily verified in the real world using a ruler and a protractor. Their simplicity also allowed them to be used in high-school books.
The primitive terms are:

(a) **Points**, designated by $A, B, C, \ldots$

(b) Particular sets of points called **lines**, designated by $l, m, \ldots$

(c) The symmetric relation **distance** between any two points A, B, designated by $d(A, B)$

(d) The **angle** determined by three ordered points A, O, B ($A \neq O, B \neq O$), designated $\angle AOB$, is a relation between two lines, AO and BO.

Then the axioms may be stated as follows:

Postulate I *Postulate of Line Measure.* The points $A, B, \ldots$ of any line can be put into 1:1 correspondence with the real numbers x so that $|x_B - x_A| = d(A, B)$ for all points A and B.

Postulate II *Point-Line Postulate.* There is one and only one straight line, l, that contains any two given distinct points P and Q.

> Before continuing with the rest of the postulates, we must interject with a few definitions.

---

[4] TODO ref.

- For any three points A, B, C on the same line (i.e. collinear points), point B is said to be between A and C if

$$d(A, C) = d(A, B) + d(B, C)$$

- A line segment AC is the set of points on the line through A and C that are between A and C. Equivalently, it is the set of points P such that

$$x_A \leq x_P \leq x_C \quad \text{or} \quad x_C \leq x_P \leq x_A$$

- A ray or half-line l' with end-point O is defined by two distinct points O, A on line l as the set of all points A' on l such that O is not between A and A'. TODO fig.
- A broken line $ABC \ldots KL$ consists of a collection of segments $AB, BC, CD, \ldots, KL$. The points $A, B, \ldots, L$ are the vertices of the broken line.
- If the initial point A and the terminal point L coincide, the broken line is called a polygon.
- A polygon with three distinct vertices is called a triangle.

Postulate III *Postulate of Angle Measure.* The rays $l, m, n, \ldots$ through any point O can be put into 1:1 correspondence with the real numbers $a \mod 2\pi$ so that if A and B are points (not equal to O) of l and m, respectively, the difference $(a_m - a_l) \mod 2\pi$ of the numbers associated with the lines l and m is the angle $\angle AOB$. Furthermore, if the point B on m varies continuously in a line r not containing the vertex O, the number a_m varies continuously also.

Postulate IV *Postulate of Similarity*[5]. If in two triangles $\triangle ABC$ and $\triangle A'B'C'$ and for some constant $k > 0$,

$$d(A', B') = k \cdot d(A, B), \quad d(A', C') = k \cdot d(A, C) \quad \text{and} \quad \angle B'A'C' = \pm\angle BAC,$$

then

$$d(B', C') = k \cdot d(B, C), \quad \angle C'B'A' = \pm\angle CBA, \quad \text{and} \quad \angle A'C'B' = \pm\angle ACB.$$

Some more definitions:

- As a consequence of postulate II, two distinct lines have either one point in common, or none. In the first case they are said to intersect in their common point; in the second case, they are said to be parallel.
- Two figures are called similar if all corresponding distances are in proportion and all corresponding angles are equal or all negatives of each other.
- Two figures are called congruent if they are similar with a ratio of proportionality equal to one.

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[5] This postulate rules out non-Euclidean geometries

Modifications to allow for higher dimensions

2.2 Analytic Euclidean geometry

In the 17^{th} century René Descartes and Pierre de Fermat departed from this purely axiomatic (or synthetic) approach and introduced the analytic approach, explicitly using a coordinate system. This was an important reason for developing linear algebra (as lines and planes can be represented by linear equations) and one of the reasons we talk about vector *spaces*. In fact n-dimensional Euclidean space can be modeled using an n-dimensional vector space with the standard inner product (that supplies the notions of distance and angle).

For the rest of the geometries mentioned here, we will focus on the analytic side of things, as that approach is more useful for the practicing physicist.

2.2.1 Introducing the model

All this talk of axioms may be frustrating for an engineer of physicist who just wants to be able to calculate. We now introduce a model (in fact a class of models) that can easily be used to calculate with. It is of course important to verify that these models do in fact describe Euclidean geometries. We will do that by verifying that they satisfy Birkhoff's postulates.

The models are based on real vector spaces equipped with the dot product, together with definitions for the primitive terms point, line, distance and angle.

TODO:
position and displacement vector
Zero
dimension
V
Using bold $\mathbf{v}$ for vectors

Taking a look at the axiom for angle measurement, it is obvious that we need a way to determine the angle between two vectors where the angle can be anything from 0 to 2π. The problem with this is that the dot product only gives us the cosine of the angle $\cos\theta$. Inverting that, we get a number between 0 and π. In other words, always the smallest angle between two vectors (TODO fig). What we need to do in order to get an number between 0 and 2π is fix an **orientation**. To do that we need a basis (TODO need?). We can then multiply the result of the arc cosine by -1 and take the angle mod 2π if the orientation of the vectors is negative.

Point A point A is modeled by a vector $\mathbf{v}_A \in V$.

Line A line is any set of vectors of the following form

$$l = \{\mathbf{v}_A + \lambda(\mathbf{v}_B - \mathbf{v}_A) \mid \lambda \in \mathbb{R}\}$$

where $\mathbf{v}_A$ and $\mathbf{v}_B$ are distinct vectors in V. Conventionally this is denoted

$$l \;\leftrightarrow\; \mathbf{v}_A + \lambda(\mathbf{v}_B - \mathbf{v}_A).$$

Distance The distance between points $\mathbf{v}_A$ and $\mathbf{v}_B$ is given by

$$d(A,B) = \|\mathbf{v}_B - \mathbf{v}_A\| = \sqrt{(\mathbf{v}_B - \mathbf{v}_A)\cdot(\mathbf{v}_B - \mathbf{v}_A)}$$

Angle The angle $\angle AOB$ is given by

$$\angle AOB = \pm\cos^{-1}\left(\frac{(\mathbf{v}_A - \mathbf{v}_O)\cdot(\mathbf{v}_B - \mathbf{v}_O)}{\|\mathbf{v}_A - \mathbf{v}_O\|\cdot\|\mathbf{v}_B - \mathbf{v}_O\|}\right)$$

The angle is negative if $((\mathbf{v}_A - \mathbf{v}_O), (\mathbf{v}_B - \mathbf{v}_O))$ has a negative orientation.

2.2.2 Compatibility with Birkhoff's postulates

TODO: lots of figures
It turns out it's easiest to consider the postulates in a different order, so that is what we will do.

Postulate II This proof contains two parts:

(a) *Existence*: for any two points a straight line can be found that contains both points.

(b) *Uniqueness*: only one such line can be found. We need to show that any such line we can construct is equivalent.

The proof is as follows:

(a) Existence is easy. Take two arbitrary points P, Q. Consider the line

$$l \;\leftrightarrow\; \mathbf{v}_P + \lambda(\mathbf{v}_Q - \mathbf{v}_P)$$

setting $\lambda = 0$ we see that the line contains $\mathbf{v}_P$; setting $\lambda = 1$ we see that the line contains $\mathbf{v}_Q$. So this line is a good line.

(b) Say we have another straight line m such that m contains $\mathbf{v}_P$ and $\mathbf{v}_Q$. The line m can be written as

$$m \;\leftrightarrow\; \mathbf{v}_A + \mu(\mathbf{v}_B - \mathbf{v}_A)$$

for some $\mathbf{v}_A$ and $\mathbf{v}_B$. We must show that $m = l$. We split this into two parts: $l \subset m$ and $m \subset l$.

$\boxed{l \subset m}$ Because $\mathbf{v}_P, \mathbf{v}_Q \in m$ there must exist $\mu_P, \mu_Q \in \mathbb{R}$ such that

$$\begin{cases}\mathbf{v}_A + \mu_P(\mathbf{v}_B - \mathbf{v}_A) = \mathbf{v}_P \\ \mathbf{v}_A + \mu_Q(\mathbf{v}_B - \mathbf{v}_A) = \mathbf{v}_Q.\end{cases} \tag{2.1}$$

These expressions for $\mathbf{v}_P$ and $\mathbf{v}_Q$ can be filled in in the expression for the line l:

$$l \leftrightarrow \mathbf{v}_P + \lambda(\mathbf{v}_Q - \mathbf{v}_P) \tag{2.2}$$

$$\mathbf{v}_A + \mu_P(\mathbf{v}_B - \mathbf{v}_A) + \lambda((\mathbf{v}_A + \mu_Q(\mathbf{v}_B - \mathbf{v}_A)) - (\mathbf{v}_A + \mu_P(\mathbf{v}_B - \mathbf{v}_A))) \tag{2.3}$$

$$\mathbf{v}_A + \mu_P(\mathbf{v}_B - \mathbf{v}_A) + \lambda(\mu_Q(\mathbf{v}_B - \mathbf{v}_A) - \mu_P(\mathbf{v}_B - \mathbf{v}_A)) \tag{2.4}$$

$$\mathbf{v}_A + [\mu_P + \lambda(\mu_Q - \mu_P)](\mathbf{v}_B - \mathbf{v}_A). \tag{2.5}$$

For every $\lambda \in \mathbb{R}$, the expression $(\mu_P + \lambda(\mu_Q - \mu_P))$ is a real number and thus a value μ can take. This means that every point of l is also a point of m and thus $l \subset m$.

$\boxed{m \subset l}$ Because $\mathbf{v}_P$ and $\mathbf{v}_Q$ are distinct (and thus $\mu_P \neq \mu_Q$), equations (2.1) can be inverted to obtain

$$\begin{cases} \mathbf{v}_A = \mathbf{v}_P + \frac{-\mu_P}{\mu_P - \mu_Q}(\mathbf{v}_Q - \mathbf{v}_P) \\ \mathbf{v}_B = \mathbf{v}_P + \frac{\mu_P - 1}{\mu_P - \mu_Q}(\mathbf{v}_Q - \mathbf{v}_P) \end{cases}$$

With a very similar line of reasoning, we can see that $m \subset l$.

This concludes the proof.

Postulate I Many such bijections can be found, each corresponding with a different placement and orientation of the ruler used. Assume that the points A, B, C, D are on the line l and are distinct. We elect to place the beginning of our ruler at point $C$and consider the half of the line on which D lies as being in the positive direction. Consider the function

$$x : l \to \mathbb{R} : A \mapsto x_A = \pm \|\mathbf{v}_A - \mathbf{v}_C\|$$

where the expression for x_A is positive if C is not between A and D. We now need to prove two things

(a) The proposed mapping x is a 1:1 correspondence (i.e. a bijection) and

(b) The distance $d(A, B) = \|\mathbf{v}_B - \mathbf{v}_A\|$ is equal to $|x_B - x_A|$.

We proceed as follows:

(a) We first introduce the special unit vector

$$\hat{v}_l \equiv \frac{\mathbf{v}_D - \mathbf{v}_C}{\|\mathbf{v}_D - \mathbf{v}_C\|}$$

Because D and C lie on the line and taking into account the second postulate, we see that we can write the line l as

$$l \;\leftrightarrow\; \mathbf{v}_C + \lambda \hat{v}_l.$$

- Now to prove surjectivity, we need to prove that for any real number y. There is a point $\mathbf{v}_E$ on the line such that $x_E = y$. Take an arbitrary real number y. The claim is now that the relevant point is given by $\mathbf{v}_E = \mathbf{v}_C + y\hat{v}_l$. Indeed

$$x_E = \pm\|(\mathbf{v}_C + y\hat{v}_l) - \mathbf{v}_C\| = \pm\|y\hat{v}_l\| = \pm|y|\|\hat{v}_l\| = \pm|y|.$$

  Now this is positive if $\mathbf{v}_E$ is on the D side of $\mathbf{C}$, which is exactly the case if y is positive, so $x_E = y$.

- Injectivity states that if we have two distinct points $A, B \in l$, then $x_A \neq x_B$. To prove this, write A and B as

$$\begin{cases} \mathbf{v}_A = \mathbf{v}_C + y_A \hat{v}_l \\ \mathbf{v}_B = \mathbf{v}_C + y_B \hat{v}_l \end{cases}$$

  which must necessarily be possible for some $y_A, y_B \in \mathbb{R}$ with $y_A \neq y_B$. Reasoning as before we obtain

$$\begin{cases} x_A = y_A \\ x_B = y_B. \end{cases}$$

  Thus $x_A \neq x_B$.

(b) As before we write

$$\begin{cases} \mathbf{v}_A = \mathbf{v}_C + y_A \hat{v}_l \\ \mathbf{v}_B = \mathbf{v}_C + y_B \hat{v}_l \end{cases}$$

Consequently

$$\begin{aligned} d(A,B) &= \|\mathbf{v}_B - \mathbf{v}_A\| && (2.6) \\ &= \|(\mathbf{v}_C + y_B \hat{v}_l) - (\mathbf{v}_C + y_A \hat{v}_l)\| && (2.7) \\ &= \|y_B \hat{v}_l - y_A \hat{v}_l\| && (2.8) \\ &= |y_B - y_A| \cdot \|\hat{v}_l\| && (2.9) \\ &= |y_B - y_A| && (2.10) \end{aligned}$$

and

$$|x_B - x_A| = |y_B - y_A|. \tag{2.11}$$

This concludes the proof.

Postulate III The proof that our model satisfies this axiom is similar to the last one. We will again propose a mapping that we will show to be a bijection with the requisite properties. We choose a ray n to act as our reference. For any ray l with a point A on it, we define the unit vector along the ray $\hat{e}_l$ as

$$\hat{e}_l = \frac{\mathbf{v}_A - \mathbf{v}_O}{\|\mathbf{v}_A - \mathbf{v}_O\|}.$$

As a consequence of the second postulate these unit vectors are unique. Then we define the function a on the rays through O as follows:

$$a_l = \pm \cos^{-1}(\hat{e}_l \cdot \hat{e}_n) \mod 2\pi$$

where the minus sign appears if $(\hat{e}_l, \hat{e}_n)$ has a negative orientation.

We also define $\hat{e}_t$ as the unique unit vector that makes $(\hat{e}_n, \hat{e}_t)$ a positively oriented orthonormal basis (TODO ?).

Now we need to show that

(a) The mapping a is a bijection.

(b) If A and B are points (not equal to O) of rays l and m through O, then

$$\angle AOB = (a_m - a_l) \mod 2\pi$$

(c) If B varies continuously, then a_m varies continuously also.

We proceed as follows:

(a) We first prove injectivity and then surjectivity.

- To prove injectivity we take two rays l and m. Assuming that $a_l = a_m$, we need to show that $l = m$. Clearly $a_l = a_m$ implies that

$$\hat{e}_l \cdot \hat{e}_n = \hat{e}_m \cdot \hat{e}_n$$

and that $(\hat{e}_l, \hat{e}_n)$ and $(\hat{e}_m, \hat{e}_n)$ have the same orientation. The unit vector $\hat{e}_l$ has the following orthonormal decomposition:

$$\hat{e}_l = (\hat{e}_l \cdot \hat{e}_n)\hat{e}_n + (\hat{e}_l \cdot \hat{e}_t)\hat{e}_t$$

Using the fact that it is a unit vector, we get

$$\hat{e}_l^2 = (\hat{e}_l \cdot \hat{e}_n)^2 + (\hat{e}_l \cdot \hat{e}_t)^2 = 1$$

Thus $\hat{e}_l \cdot \hat{e}_t = \pm\sqrt{1 - (\hat{e}_l \cdot \hat{e}_n)^2}$. This shows that $\hat{e}_l$ is one of two vectors. For one of those the orientation of $(\hat{e}_l, \hat{e}_n)$ is positive, for it is negative (TODO: show). Same for $\hat{e}_m$. Thus because the orientation of $(\hat{e}_l, \hat{e}_n)$ and $(\hat{e}_m, \hat{e}_n)$ is the same, $\hat{e}_l$ and $\hat{e}_m$ are the same vector. Then considering the rays through $\mathbf{v}_O$ and $\mathbf{v}_O + \hat{e}_l$, and through $\mathbf{v}_O$ and $\mathbf{v}_O + \hat{e}_m$, postulate II gives $l = m$ and the sought-after injectivity.

- For surjectivity we must find a ray l for every angle in $[0, 2\pi[$ such that a_l equals that angle. Take an arbitrary angle $\theta \in [0, 2\pi[$. TODO

(b) TODO

(c) TODO

This concludes the proof.

Postulate IV TODO

2.2.3 Towards categoricity

TODO $\mathbb{E}$

2.2.4 Spatial analytic geometry

2.3 Projective geometry

Also in the 17$^{\text{th}}$ mathematicians were trying to see what happened if certain axioms or concepts were left out of Euclids axiomatic system. In particular Girard Desargues started the systematic study of projective geometry, which does not have any concept of distance or parallel lines. The original motivation for this was an attempt to understand perspective.

There are several axiomatisations of projective geometry (such as those by Whitehead, Coxeter, Hilbert & Cohn-Vossen and Greenberg). We will not be considering those.

From an analytic point of view, the n-dimensional projective space over an arbitrary field K can be constructed as follows:

- Take an n-dimensional vector space V over the field K. We define K_0 and V_0 as resp. the sets K and V without the neutral element for the addition, 0. i.e.

$$V_0 \equiv V \setminus \{0\} \qquad K_0 \equiv K \setminus \{0\}$$

- We define the equivalence relation $\sim$ on V_0:

$$v \sim w \;\Leftrightarrow\; \exists \lambda \in K_0 : v = \lambda w.$$

It should be clear that this is indeed an equivalence relation.

- We define $[v]$ as the equivalence class that contains v. Explicitly this is given by

  $$[v] = \{\lambda v \mid \lambda \in K_0\}.$$

  This gives a partition of V_0 (i.e. $[v] = [w] \Leftrightarrow v \sim w$ and $v \nsim w \Rightarrow [v] \cap [w] = \emptyset$).

- The projective space $P(V)$ associated to V can now be defined as the set of equivalence classes. In other words, it is a quotient space:

  $$P(V) \;=\; V_0/\sim \;=\; \{[v] \mid v \in V_0\}$$

  Each equivalence class is called a *point* of the projective space. It may be strange to call a set a point (TODO point about models and isomorphism)

- In the construction above we have collapsed whole lines (containing e.g the points λv for all $\lambda \in K$) into single points. It therefore makes sense to *define* the dimension of $P(V)$ to be one less than the dimension of V, if V has a finite dimension that is.

In order to get the full projective geometry, according to the Erlanger program, we now also need a group of transformations. For the construction proposed above, the projective transformations $\phi : P(V) \to P(W)$ can be constructed as follows.

- Take the vector space $\mathrm{GL}(V, W)$ of the bijective linear maps from V to W. We would like to define the projective transformations as the transformations of the form

  $$P(V) \to P(W) : [v] \mapsto [f(v)]$$

  where f is an element of $\mathrm{GL}(V, W)$. It is not immediately clear that this well defined. The problem is that $[v]$ is a set of which v is only one element. If we take a different element of $u \in [v]$, how do we know that $f(u) \in [f(v)]$? In other words how do we know that projective transformation does not depend on the (arbitrarily chosen) representative v of the projective point $[v]$?

- Luckily we can use the result that for all $v \in V_0$, $[f(v)] = [g(v)]$ if and only if $[f] = [g]$. The notation $[f]$ makes sense because $\mathrm{GL}(V, W)$ is itself a vector space.

2.4 Affine geometry

Tangent space and bundle

2.5 Back to Euclidean geometry

2.6 Non-Euclidean geometry

The development of non-Euclidean geometries (with different sets of axioms) was another exciting enrichment of the field. These axiomatic systems roughly describe shapes in space that is not flat. In 2D this translates to doing geometry on surface that is not flat, like a sphere. From an analytic viewpoint, vector spaces (embodying linearity) are obviously no longer
Tangent bundle!!

Chapter 3

Manifolds

3.1 Definition

The basic idea is that a manifold is an object M that looks locally like a piece of $\mathbb{R}^n$, i.e. around any point we can find an area small enough that it looks flat.

A topological space M is locally Euclidean of dimension n is every point $p \in M$ has a neighbourhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call

- the pair $(U, \phi : U \to \mathbb{R}^n)$ a chart;
- U a coordinate neighbourhood or a coordinate open set;
- ϕ a coordinate map or a coordinate system on U:

$$\phi : U \to \mathbb{R}^n : p \mapsto (x^1(p), \ldots, x^n(p)).$$

A chart (U, ϕ) is centred at $p \in U$ if $\phi(p) = \mathbf{0}$. A chart about p is a chart (U, ϕ) such that $p \in U$.

TODO: for well defined: open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ cannot be homeomorphic if $n \neq m$.

An n-dimensional (topological) manifold M is a second-countable, Hausdorff topological space that is locally Euclidean of dimension n.

[Picture with manifold and overlapping patches mapping to Rm phi1 phi2 + mappings between R's .]
By requiring the topology to be second-countable (C2) and Hausdorff, we immediately guaranty our manifold has whole load of nice properties. An important one is the ability to embed manifolds in higher-dimensional Euclidean spaces (cfr. Whitney's embedding theorem). Other properties that second-countable Hausdorff spaces have include being metrizable, completely normal and paracompact.
Also note that subspaces of a Hausdorff (resp. C2) space are automatically Hausdorff (resp. C2).

Proposition XVII.1. *Every discrete space is a 0-dimensional manifold.*

Lemma XVII.2. *Every manifold is locally path-connected.*

This follows from the homeomorphisms with the path-connected space $\mathbb{R}^n$. Thus for manifolds the notions of connectedness and path-connectedness coincide.

Let $(U, \phi), (V, \psi)$ be two charts such that $U \cap V \neq \emptyset$. The maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \qquad \text{and} \qquad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are called transition functions or change of coordinate maps.

Transition functions are compositions of homeomorphisms and thus homeomorphisms.

An atlas on a manifold M is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ that covers M, i.e. $M = \bigcup_\alpha U_\alpha$.

3.2 Types of manifolds

Manifolds are very useful in many areas of physics and mathematics. Consequently there are many extensions and types of manifolds.

3.2.1 Topological manifolds

Topological manifolds are manifolds with no additional structure. The moniker topological is redundant, but can be used to emphasise that there is no additional structure, or, if there was previously additional structure, that one should forget about that additional structure.

3.2.2 Differential manifolds

Differential manifolds have an atlas that allows differential calculus to be used on the manifold. Each chart allows the use of calculus using the standard differential structure on linear space. (TODO!) The only difficulty is with the transition maps.

Two charts are called C^k-compatible if the transition functions are in C^k.

A C^k atlas on a manifold M is a collection $\{(U_\alpha, \phi_\alpha)\}$ of C^k-compatible charts that covers M, i.e. $M = \bigcup_\alpha U_\alpha$.

Lemma XVII.3. *Let $\{(U_\alpha, \phi_\alpha\}$ be an atlas. If two charts (V, ψ) and (W, σ) are compatible with the atlas, they are compatible with each other.*

Two atlases are C^k-equivalent if the union of their sets of charts forms a C^k-atlas.

Lemma XVII.4. *The C^k-equivalence of atlases is an equivalence relation. Each equivalence class is called a distinct C^k differential structure of the manifold.*

A maximal atlas is an atlas that is not contained in a larger atlas.

Lemma XVII.5. *The union of a C^k-equivalence class is a maximal atlas. Conversely every maximal atlas is the union of a C^k-equivalence class.*

A differential structure is sometimes defined as a maximal atlas. By this lemma this is equivalent.

Corollary XVII.5.1. *Any atlas is contained in a unique maximal atlas.*

A differential manifold is a manifold with a C^k differential structure.

In fact we can even recover the manifold from the differential structure
It is however not meaningful to talk of a C^k-manifold as any manifold with a C^k-atlas with $k > 0$ can be given a C^∞-atlas. In fact every C^k-structure is uniquely *smoothable* to a C^∞-structure.
The differential structure allows the definition of the globally differentiable tangent space. This process is discussed in the next section.
A manifold may also be defined as an equivalence class of atlases (TODO).
In dimensions smaller than 4 every topological manifold has a unique differentiable structure and in dimensions larger than 4 every compact topological manifold hasa finite number of differentiable structures. Dimension 4 is a mystery.
There are topological manifolds with no differentiable structure.

3.2.3 Smooth manifolds

Smooth manifolds are differentiable manifolds for which all the transition maps are smooth (i.e. infinitely differentiable). All concepts in the previous section relating to differential manifolds apply if C^k is replaced by C^∞ or "smooth".
We will see later that:

Proposition XVII.6. *Every differential manifold is uniquely smoothable. i.e. every C^k differential structure has a unique C^∞ structure that is C^k equivalent.*

Consequently there is no real difference between differential and smooth manifolds. We will mainly study the latter.

3.2.4 Analytic manifolds

Analytic manifolds are smooth manifolds with the additional condition that each transition map is analytic or C^ω: the Taylor expansion is absolutely convergent and equals the function on some open ball.

3.3 Manifolds with boundaries

3.4 Submanifolds

Chapter 4

Smooth manifolds

In this chapter all manifolds are assumed smooth, i.e. C^∞.

4.1 The tangent space

4.1.1 Functions on manifolds

Let M, N be manifolds of dimension m, n. A map $F : M \to N$ is said to be smooth, or C^∞, at a point $p \in M$ if there are charts (U, ϕ) about $p \in M$ and (V, ψ) about $F(p) \in N$ such that

$$\psi \circ F \circ \phi^{-1} : \phi[F^{-1}[V] \cap U] \subset \mathbb{R}^m \to \mathbb{R}^n$$

is smooth at $\phi(p)$.
The function $F : M \to N$ is said to be smooth if it is smooth are every point $p \in M$.

Notice that the smoothness of a function is independent of the charts chosen:

Lemma XVII.7. *Let M, N be manifolds of dimension m, n. If a function $F : M \to N$ is smooth at $p \in M$, then for any charts (U', ϕ') about $p \in M$ and (V', ψ') about $F(p) \in N$,*

$$\psi' \circ F \circ (\phi')^{-1} : \phi'[F^{-1}[V'] \cap U'] \subset \mathbb{R}^m \to \mathbb{R}^n$$

is C^∞ at $\phi'(p)$.

Proof. Let $F : M \to N$ be smooth and $p \in M$, $(U, \phi), (V, \psi)$ be as in the definition. Then

$$\psi' F \circ (\phi')^{-1} = (\psi' \circ \psi) \circ (\psi F \circ \phi^{-1}) \circ (\phi \circ (\phi')^{-1})$$

is smooth at $\phi'(p)$, because it is a composition of smooth maps. □

Lemma XVII.8. *Let $F : M \to N$ and $G : N \to P$ be smooth maps of manifolds, then $G \circ F$ is smooth.*

A diffeomorphism of manifolds is a bijective C^∞ maps $F : M \to N$ such that F^{-1} is also C^∞.

Lemma XVII.9. *If (U, ϕ) is a chart on a manifold M, then the coordinate map ϕ is a diffeomorphism.*

Proof. By definition $\phi : U \subset M \to \phi(U) \subset \mathbb{R}^n$ is a homeomorphism, so we just need to check ϕ and ϕ^{-1} are smooth. Let $(\mathbb{R}^n, I_{\mathbb{R}^n})$ be a chart of $\mathbb{R}^n$. Then

$$I_{\mathbb{R}^n} \circ \phi \circ \phi^{-1} \qquad \text{and} \qquad \phi \circ \phi^{-1} \circ I_{\mathbb{R}^n}$$

are both the identity map and thus smooth, so both ϕ and ϕ^{-1} are C^∞. □

Lemma XVII.10. *Let U be an open subset of a manifold M of dimension n. If $F : U \to \mathbb{R}^n$ is a diffeomorphism into an open subset of $\mathbb{R}^n$, the (U, F) is a chart in the differentiable structure of M.*

$C^\infty(M)$; $C^\infty(M)_p$ algebra of germs of functions in $C^\infty(M)$ at p. TODO Pullback? Previous chapter?

4.1.2 Derivatives of functions on manifolds

Let $r^1, \ldots, r^n$ denote the standard coordinates on $\mathbb{R}^n$. Then $x^i = r^i \circ \phi$.

Let $p \in U$. We define the partial derivative $\frac{\partial f}{\partial x^i}$ at the point p as

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)).$$

Alternatively, as a function on $\phi(U)$, we write

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}.$$

Proposition XVII.11. *Let $(U, (x^1, \ldots, x^n))$ be a chart on a manifold. Then*

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j.$$

Proof. By direct computation:

$$\frac{\partial x^i}{\partial x^j}(p) = \frac{\partial (x^i \circ \phi^{-1})}{\partial r^j}(\phi(p)) = \frac{\partial (r^i \circ \phi \circ \phi^{-1})}{\partial r^j}(\phi(p)) = \frac{\partial r^i}{\partial r^j}(\phi(p)) = \delta^i_j.$$

□

TODO inverse function theorem?

4.1.3 Derivations of functions on manifolds

A derivation at a point p on a manifold M (or a point-derivation of $C^\infty_p(M)$) is a map $D : C^\infty_p(M) \to \mathbb{R}$ that

- is linear;
- satisfies the Leibniz identity:

$$\forall f, g \in C^\infty_p(M) : \quad D(fg) = (Df)g(p) + f(p)(Dg).$$

A tangent vector at a point p in a manifold M is a derivation at p. The tangent space at p is the vector space of tangent vectors, denoted T_pM.

Lemma XVII.12. *Let M be a manifold and $p \in M$. The tangent space at p is a vector space.*

Proof. The space of linear maps $D : C_p^\infty(M) \to \mathbb{R}$ is a vector space. We check the criterion X.2:

- The zero map satisfies the Leibniz identity.
- Let D_1, D_2 be derivations. Then $\forall f, g \in C_p^\infty(M)$ and $\lambda \in \mathbb{R}$:

$$\begin{aligned}(\lambda D_1 + D_2)(fg) &= \lambda D_1(fg) + D_2(fg) \\ &= \lambda(D_1 f)g(p) + \lambda f(p)(D_1 g) + (D_2 f)g(p) + f(p)(D_2 g) \\ &= (\lambda(D_1 f) + (D_2 f))g(p) + f(p)(\lambda(D_1 g) + (D_2 g)) \\ &= (\lambda D_1 + D_2 f)(f)g(p) + f(p)(\lambda D_1 + D_2)(g).\end{aligned}$$

□

If U is an open subset of M containing p, then $C_p^\infty(U) = C_p^\infty(M)$ and thus $T_pU = T_pM$.

4.1.3.1 Curves and derivations

4.1.4 The differential of a map between manifolds

4.1.4.1 Computations in coordinates

$F^j = y^j \circ F$

$$\begin{aligned}\mathrm{d}F_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f &= \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) = \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) \\ &= \left(\frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}\right) f.\end{aligned}$$

4.1.4.2 Computation using curves

4.1.4.3 Critical and regular points

4.1.5 Bases for the tangent space at a point

4.2 Submanifolds

4.2.1 Rank theorems

Theorem XVII.13 (Global rank theorem)**.** *Let $F : M \to N$ be a smooth map of constant rank between smooth manifolds. Then*

1. *if F is surjective, it is a smooth submersion;*
2. *if F is injective, it is a smooth immersion;*
3. *if F is bijective, it is a diffeomorphism;*

4.3 Tangent and cotangent spaces

As mentioned above, if a manifold $\mathcal{S}$ is a submanifold of some Euclidean space $\mathbb{R}^n$, the tangent space at a point p is the set of vectors that can be expressed as $v = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0}$, where $\gamma(t)$ is a smooth curve lying in $\mathcal{S}$ and satisfying $\gamma(0) = p$. Unfortunately this definition of a tangent space only works if the manifold is embedded in a Euclidean space.

We obtain a more abstract definition by generalising the notion of a directional derivative. If we still assume our manifold $\mathcal{S}$ is embedded in $\mathbb{R}^n$ and f is a smooth function on $\mathcal{S}$, we define the directional derivative of f at the point p and in the direction v to be

$$(D_v f)(p) = \tfrac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\big|_{t=0},$$

where γ is any smooth curve lying in $\mathcal{S}$ with $\gamma(0) = 0$ and $\frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0} = v$. This directional derivative associates a number to each smooth function f and satisfies the usual product rule for derivatives.

We now define the tangent space at p to an arbitrary manifold $\mathcal{M}$, denoted $T_p(\mathcal{M})$, as the set of all linear maps X from $C^\infty(\mathcal{M})$ into $\mathbb{R}$ satisfying the product rule

$$X(fg) = X(f)g(p) + f(p)X(g)$$

for all f and g in $C^\infty(\mathcal{M})$, and localisation which means that if f equals g in a neighbourhood of p, then $X(f) = X(g)$. Obviously $T_p(\mathcal{M})$ is a real vector space and an element of $T_p(\mathcal{M})$ is called a tangent vector at p.

If $x_1, \ldots, x_n$ is a local coordinate system, then one can prove that each tangent vector X at m can be expressed uniquely as

$$X(f) = \sum_{k=1}^{n} a_k \frac{\partial f}{\partial x_k}(p) = \sum_{k=1}^{n} a_k \partial_k(p)$$

for some real constants $a_1, \ldots, a_n$. This means that the tangent space has the same dimension as the manifold at every point.

This particular basis is called the coordinate basis and is in general not orthonormal.

====================

Now that we have defined our manifold, we can start adding structures and concepts on top, starting with vectors.

The tangent space T_p can be identified with the space of directional derivative operators along curves through p. These operators act on smooth functions on the manifold M (i.e. on C^∞ maps $f : M \to \mathbb{R}$).

This is a vector space. Linearity is definitely ok. We need to check that vector addition closes. To do that we verify the Leibniz (product) rule.

So we want to construct the tangent space using only things intrinsic to the manifold (no embedding). Tangent vectors to curves through P? What does that mean?

Directional derivatives. Basis: partial derivatives ∂_μ (i.e. directional derivatives along curve with constant x^ν for all $\nu \neq \mu$, parametrized by x^μ).

Actually derivative? OK

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \partial_\mu f$$

So the partial derivatives $\{\partial_\mu\}$ represent a good basis for the directional derivatives. (Coordinate basis)

4.3.1 Transformation under coordinate transformations

Change of basis immediate through chain rule. New coordinate system $x^{\mu'}$:

$$\partial_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu}$$

Thus also transformation rules for vectors: $\mathbf{v} = v^{\mu} \partial_{\mu}$.

$$\begin{aligned} v^{\mu} \partial_{\mu} &= v^{\mu'} \partial_{\mu'} && (4.1) \\ &= v^{\mu} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu} && (4.2) \end{aligned}$$

So

$$v^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} v^{\mu}$$

for all changes of coordinates (not just linear transformations)

4.3.2 Cotangent space

Gradient is one-form:

$$\nabla f \left(\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}f}{\mathrm{d}\lambda}$$

f itself is not a one-form: one-forms exist only in one point and to get the derivative of f we need a neighbourhood.

Basis for one-forms given by the gradients of the coordinate functions:

$$\mathrm{d}x^{\mu}(\partial_{\nu}) = \frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}$$

with transformation rule

$$\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \mathrm{d}x^{\mu} \qquad \omega_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \omega_{\mu}$$

where $\omega = \omega_{\mu} \, \mathrm{d}x^{\mu}$ is a one-form.

!! Partial derivative of tensor of higher rank than a scalar is not a tensor. (Show) We will introduce several alternatives.

4.3.3 Vector fields

Vector field: maps smooth functions to smooth functions all over the manifold by taking derivative at each point. We can define commutator by its action of a function $f(x^{\mu})$

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)).$$

This is a vector field with components

$$[X, Y]^{\mu} = X^{\lambda} \partial_{\lambda} Y^{\mu} - Y^{\lambda} \partial_{\lambda} X^{\mu}$$

4.3.4 Tensors and tensor bundles

$$T = T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \, \partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_k} \otimes \mathrm{d}x^{\nu_1} \otimes \mathrm{d}x^{\nu_l}$$

$$T^{\mu'_1 \cdots \mu'_k}{}_{\nu'_1 \ldots \nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l}$$

Chapter 5

(Pseudo-)Riemannian differential geometry

TODO tangent space of regular level set (see wiskunde I).

We model spacetime as a pseudo-Riemannian manifold. This is a differentiable manifold equipped with a smooth, symmetric metric tensor that is non-degenerate everywhere. In this section we will see what all these words mean and define some more mathematics that will prove invaluable in the study of general relativity.

5.1 Constructions on the manifold

5.1.1 The tangent space

5.1.2 The metric

We refer to the components of the $(0,2)$-tensor as $g_{\mu\nu}$ (while $\eta_{\mu\nu}$ is reserved specifically for the Minkowski metric). The inverse metric can be defined via

$$g^{\mu\sigma}g_{\sigma\nu} = g_{\lambda\nu}g_{\lambda\mu} = \delta^{\mu}_{\nu}$$

The inverse metric is also symmetric.

We do not require the metric to be positive-definite (this is what makes pseudo-Riemannian geometry pseudo). This metric cannot serve as a metric in the topological sense, but has many other uses.

As a symmetric 4×4 matrix, it has 10 independent components. The form of $g_{\alpha\beta}(x)$ will be different in different coordinate systems for the same geometry. Since there are 4 arbitrary functions involved in transforming 4 coordinates, there are really only 6 independent functions associated with a metric. (TODO explain + reword)

5.1.2.1 Line element

For example the line element of flat, Euclidean space in Cartesian coordinates is given by

$$\mathrm{d}s = \left[(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2\right]^{1/2}$$

Conventionally the line element is written as a quadratic relation for $\mathrm{d}s^2$. The line element is then given by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

The line element of two dimensional Euclidean space in polar coordinates is given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + (r\,\mathrm{d}\phi)^2.$$

The line element on a sphere is

$$\mathrm{d}s^2 = a^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

where a is the radius.
We can write the line element in the following form (TODO why linear)

$$\mathrm{d}s^2 = g_{\alpha\beta}(x)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$$

where $g_{\alpha\beta}(x)$ is the metric.

5.1.2.2 Canonical form

A line element specifies a geometry, but many different line elements describe the same spacetime geometry.
A metric $g_{\mu\nu}$ in its canonical form has components:

$$g_{\mu\nu} = \mathrm{diag}(-1,\ldots,-1,+1,\ldots,+1,0,\ldots,0)$$

The signature of the metric is the number of positive and negative eigenvalues. For example the metric $\eta_{\mu\nu}$ has a signature "minus-plus-plus-plus". If any of the eigenvalues are zero, the metric is degenerate. If the metric contains one minus and all the rest plus (or all minus and one plus), it is called Lorentzian.
In special relativity, the metric is the Minkowski metric $\eta_{\alpha\beta}$. The (Einstein) equivalence principle requires that it be possible at each point P to change to a new set of coordinates $\tilde{x}_P$ such that

$$\tilde{g}_{\alpha\beta}(\tilde{x}_P) = \eta_{\alpha\beta}$$

In fact we can even make the first derivatives vanish. Second derivatives cannot be made to vanish, they express curvature. In other words we can find a coordinate transformation $x^\mu \to x^{\hat{\mu}}$ such that

$$g_{\hat{\mu}\hat{\nu}}(P) = \eta_{\hat{\mu}\hat{\nu}} \qquad \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(P) = 0.$$

Such coordinates are known as locally inertial coordinates, and the associated basis vectors constitute a local Lorentz frame.
Value of locally inertial coordinates: perform calculations and express answer in coordinate independent form. Often just knowing that locally inertial coordinates exist is useful knowing the exact transformations necessary to obtain them. We can perform calculations and express answer in coordinate independent form, so that it is not modified when we transform back.

TODO: derivation

5.1.3 Tensor densities

Tensor densities are objects that do not transform like tensors, but like tensors multiplied by a power (called the weight) of the Jacobian. A prototypical example is given by the Levi-Civita symbol.
The transformation of the Levi-Civita symbol can be inferred by noting that for any $n \times n$ matrix M, the determinant is given by

$$\epsilon_{\mu'_1\mu'_2\dots\mu'_n}|M| = \epsilon_{\mu_1\mu_2\dots\mu_n} M^{\mu_1}{}_{\mu'_1} M^{\mu_2}{}_{\mu'_2} \dots M^{\mu_n}{}_{\mu'_n}$$

(TODO ref linear algebra; after transform is this OK?) By setting $M^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}$ we have

$$\epsilon_{\mu'_1\mu'_2\dots\mu'_n} = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| \epsilon_{\mu_1\mu_2\dots\mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}} \frac{\partial x^{\mu_2}}{\partial x^{\mu'_2}} \dots \frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}}$$

So the Levi-Civita symbol has weight 1.
Another example of tensor densities is given by the determinant of the metric $g = |g_{\mu\nu}|$. The transformation law

$$g(x^{\mu'}) = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right|^{-2} g(x^{\mu})$$

is easily derived by taking the the determinant of the transformation law for $g_{\mu\nu}$. Here the weight is -2.

5.1.3.1 Tensors from tensor densities

A tensor density with weight w can be turned into a tensity by multiplying it with $|g|^{w/2}$. The absolute value is necessary because we have not required $g$to be positive definite, and in fact for Lorentzian metrics $|g| = -g$. This factor cancels the Jacobian factor.
For example we can define the Levi-Civita tensor

$$\varepsilon_{\mu_1\mu_2\dots\mu_n} \equiv \sqrt{|g|}\epsilon_{\mu_1\mu_2\dots\mu_n}$$

This can also be defined with upper indices:

$$\varepsilon^{\mu_1\mu_2\dots\mu_n} \equiv \frac{1}{\sqrt{|g|}}\epsilon^{\mu_1\mu_2\dots\mu_n}$$

The indices of the Levi-Civita tensor can be contracted as follows:

$$\varepsilon^{\mu_1\mu_2\dots\mu_p\alpha_1\alpha_2\dots\alpha_{n-p}}\varepsilon_{\mu_1\mu_2\dots\mu_p\beta_1\beta_2\dots\beta_{n-p}} = (-1)^s p!(n-p)!\delta^{[\alpha_1}_{\beta_1}\dots\delta^{\alpha_{n-p}]}_{\beta_{n-p}}$$

TODO: also definable as pseudo-tensor. Then transition SR to GR: $\epsilon \to \varepsilon$?

5.1.3.2 Variational calculus for tensor densities

TODO g

5.1.4 Differential forms

Differential forms are interesting (and called as such) because they can be differentiated and integrated without additional geometric structure.

5.1.4.1 Exterior derivative

The exterior derivative d maps $(0,p)$-forms to $(0,p+1)$-forms and is defined as

$$\mathrm{d} : \Lambda^p \to \Lambda^{p+1} : \omega \mapsto \mathrm{d}\omega = \frac{1}{p!}\partial_\nu \omega_{\mu_1,\ldots,\mu_p} \, \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}.$$

This can also be written in components as

$$(\mathrm{d}\omega)_{\mu_1 \ldots \mu_{p+1}} = (p+1)\partial_{[\mu_1}\omega_{\mu_2 \ldots \mu_{p+1}]}$$

In particular we can take the exterior derivative of a scalar (0-form), which gives the gradient.

$$(\mathrm{d}f)_\mu = \partial_\mu f \qquad \text{so} \qquad \mathrm{d}f = \mathrm{d}x^\mu \partial_\mu f$$

The exterior derivative has some interesting properties:

1. A modified version of the Leibniz rule applies. Say ω is a p-form an η a q-form, then

   $$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{d}\eta)$$

2. If we try applying the exterior derivative twice (TODO factor), we get zero because the simple derivative commutes.

   $$\begin{aligned} \mathrm{d}(\mathrm{d}\omega) &= \frac{1}{p!}\partial_\rho \partial_\nu \omega_{\mu_1,\ldots,\mu_p} \, \mathrm{d}x^\rho \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} && (5.1) \\ &= 0 && (5.2) \end{aligned}$$

3. It *is* a tensor. (TODO)

We say a p-form ω is

- closed if $\mathrm{d}\omega = 0$, and
- exact if $\omega = \mathrm{d}\eta$ for some η.

Exact forms are closed, but the inverse is not necessarily true.

5.1.4.2 Cohomology classes

5.1.4.3 Hodge duality

The idea for Hodge duality stems from the dimensionality of spaces of p-forms. On an n-dimensional manifold M the space of p-forms has dimension

$$\dim(\Lambda^p(M)) = \frac{n!}{p!(n-p)!}$$

For example, if $n = 4$, then

$$\begin{aligned} \Lambda^0(M) &= C^\infty && \dim 1 && (5.3) \\ \mathrm{d}x^\mu A_\mu \in \Lambda^1(M) &= T^*(M) && \dim 4 && (5.4) \\ \frac{1}{2}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \omega_{\mu\nu} &\in \Lambda^2(M) && \dim 6 && (5.5) \\ \frac{1}{3!}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \alpha_{\mu\nu\rho} &\in \Lambda^3(M) && \dim 4 && (5.6) \\ \frac{1}{4!}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \beta_{\mu\nu\rho\sigma} &\in \Lambda^4(M) && \dim 1 && (5.7) \\ & && && (5.8) \end{aligned}$$

In general we have

$$\dim(\Lambda^p) = \dim(\Lambda^{n-p}),$$

which suggests some idea of duality (in the sense of transforming twice gives the same result). This duality is called Hodge duality. It is embodied by the Hodge star operator $*$, which is defined on an n-dimensional manifold as a map from p-forms to $(n-p)$-forms.

$$* : \Lambda^p \to \Lambda^{n-p} : A \mapsto *A$$

where

$$\begin{aligned}(*A)_{\mu_1\dots\mu_{n-p}} &= \frac{\sqrt{|g|}}{p!} g^{\mu_1\nu_1}\dots g^{\mu_{n-p}\nu_{n-p}} \epsilon_{\nu_1\dots\nu_p\mu_1\dots\mu_{n-p}} A_{\nu_1\dots\nu_p} &(5.9)\\ &= \frac{1}{p!}\varepsilon^{\nu_1\dots\nu_p}{}_{\mu_1\dots\mu_{n-p}} A_{\nu_1\dots\nu_p} &(5.10)\end{aligned}$$

Because the Levi-Civita tensor is a proper tensor, its indices can be raised and lowered using the metric (TODO!!). This is why the second expression makes sense and is equal to the first. Applying the Hodge star twice returns ± the original form.

$$**A = (-1)^{s+p(n-p)}$$

where s is the number of minus signs in the signature of the metric.
For example, in three dimensional Euclidean space the Levi-Civita tensor is equal to the Levi-Civita symbol ($\varepsilon = \epsilon$). The Hodge dual of the wedge product of two 1-forms gives another 1-form

$$*(U \wedge V)_i = \epsilon_i^{ij} U_i V_k$$

which is exactly the cross product.

In electrodynamics Maxwells equations can be written as

$$\begin{cases} \mathrm{d}F = 0 \\ \mathrm{d}{*F} = *J \end{cases}$$

In Minkowski space all closed forms are exact, so we can find an A such that

$$F = \mathrm{d}A.$$

This is the vector potential.

Strongly and weakly coupled theories. TODO

5.1.4.4 Integration of differential forms & volumes

TODO redo post calculus.
On an n-dimensional manifold M, integration over a region $\Sigma \subset M$ may be seen as a map from an n-form field ω, i.e. the integrand, to the real numbers

$$\int_\Sigma : \omega \to \mathbb{R}.$$

(TODO example of line integral?)
To properly motivate this however, we need to show that

1. the integrand is a $(0, n)$ tensor and
2. the integrand is antisymmetric.

The first point can intuitively be motivated by viewing the $\mathrm{d}^n x$ as taking an infinitesimal (parallelipedal) area *defined by three vectors* and outputting its (infinitesimal) volume. See fig TODO. This mapping is linear. The second second point follows because the volume is oriented. This leads us to propose the identification

$$\mathrm{d}^n x \equiv \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{n-1} \tag{5.11}$$

This is actually a tensor density (TODO: why + as opposed to $\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_n}$)
This is consistent with the appearance of the Jacobian under change of coordinates (in the calculations in Euclidean space we have done so far)

$$\mathrm{d}^n x' = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| \mathrm{d}^n x.$$

To see that this is the case, we note that

$$\mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^{n-1} = \frac{1}{n!} \epsilon_{\mu_1 \ldots \mu_n} \, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_n}$$

The factor $1/n!$ takes care of the overcounting by summing over the permutations of the indices. The Levi-Civita symbol ϵ does not change under coordinate transformations, so

$$\begin{align} \epsilon_{\mu_1 \ldots \mu_n} \, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_n} &= \epsilon_{\mu_1 \ldots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \, \mathrm{d}x^{\mu_1'} \wedge \ldots \wedge \mathrm{d}x^{\mu_n'} \tag{5.12} \\ &= \left| \frac{\partial x^{\mu}}{\partial x^{\mu}} \right| \epsilon_{\mu_1' \ldots \mu_n'} \, \mathrm{d}x^{\mu_1'} \wedge \ldots \wedge \mathrm{d}x^{\mu_n'} \tag{5.13} \end{align}$$

This clearly implies equation 5.11.
Recognising its nature as a tensor density, we can construct the invariant volume element by multiplying by $\sqrt{|g|}$:

$$\sqrt{|g'|} \, \mathrm{d}^n x' = \sqrt{|g'|} \, \mathrm{d}x^{0'} \wedge \ldots \wedge \mathrm{d}x^{(n-1)'} = \sqrt{|g|} \, \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^{n-1} = \sqrt{|g|} \, \mathrm{d}^n x$$

The invariant volume element can in fact be identified with the Levi-Civita tensor ε:

$$\begin{align} \varepsilon &= \varepsilon_{\mu_1 \ldots \mu_n} \, \mathrm{d}x_1^{\mu} \otimes \ldots \otimes \mathrm{d}x_n^{\mu} \tag{5.14} \\ &= \frac{1}{n!} \varepsilon_{\mu_1 \ldots \mu_n} \, \mathrm{d}x_1^{\mu} \wedge \ldots \wedge \mathrm{d}x_n^{\mu} \tag{5.15} \\ &= \frac{1}{n!} \sqrt{|g|} \epsilon_{\mu_1 \ldots \mu_n} \, \mathrm{d}x_1^{\mu} \wedge \ldots \wedge \mathrm{d}x_n^{\mu} \tag{5.16} \\ &= \sqrt{|g|} \, \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^{n-1} \tag{5.17} \\ &= \sqrt{|g|} \, \mathrm{d}^n x \tag{5.18} \end{align}$$

Finally the main result is that the integral I of a scalar function $\phi(x)$ over a region Σ of an n-manifold is written as

$$\boxed{I = \int_{\Sigma} \phi(x) \sqrt{|g|} \, \mathrm{d}^n x.}$$

This can be evaluated with the usual rules of calculus.
In a more abstract notation, we can also write

$$I = \int_\Sigma \phi(x)\epsilon.$$

One problem with this conception of integrals is that it is not well suited to integral of vectors. And as such things like Stokes' theorem are hard to formulate.

5.1.5 Vielbeins

$$e^a(x) = \mathrm{d}x^\mu e_\mu^{\ a}(x) \tag{5.19}$$

$$e^a(\tilde{x}) = \mathrm{d}\tilde{x}^\mu e_\mu^{\ a}(\tilde{x}) \tag{5.20}$$

$$e^a(x) = \Lambda^a_{\ b}(x) e^b(x) \qquad \text{where} \Lambda^\intercal \eta \Lambda = \eta \tag{5.21}$$

$$\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \mathrm{d}t \wedge \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi r^2 \sin\theta$$

$$\begin{cases} x^0 = t \\ x^1 = r\cos\theta \\ x^2 = r\sin\theta\cos\phi \\ x^3 = r\sin\theta\sin\phi \end{cases}$$

$$\frac{1}{4!} e^a \wedge e^b \wedge e^c \wedge e^d \epsilon_{abcd} = \frac{1}{4!}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\sigma \wedge \mathrm{d}x^\rho\, \underbrace{e_\mu^{\ a} e_\nu^{\ b} e_\rho^{\ c} e_\sigma^{\ d} \epsilon_{abcd}}_{\det e \epsilon \mu\nu\rho\sigma} = \mathrm{d}^4x\sqrt{-g} = \frac{1}{4!}\sqrt{-g}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \epsilon_{\mu\nu\rho\sigma}$$

$$g_{\mu\nu} = e_\mu^{\ a} e_\nu^{\ b} \eta_{ab}$$

$$\det g = (\det e)^2 \det\eta$$

$$\det e = \sqrt{-\det g}$$

5.2 Curvature

TODO how to characterise curvature. TODO: why connection through derivative.

5.2.1 Connection

$$V \in TM, \qquad V = V^\mu(x)\frac{\partial}{\partial x^\mu} = \tilde{V}^\mu \frac{\partial}{\partial \tilde{x}^\mu}$$

With

$$\tilde{V}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} V^\nu$$

If we change coordinates

$$\partial_\mu V^\nu = \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\nu}{\partial \tilde{x}^\rho}\tilde{V}^\rho\right) \tag{5.22}$$

$$= \frac{\partial \tilde{x}^\sigma}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\sigma}\left(\frac{\partial x^\nu}{\partial \tilde{x}^\rho}\tilde{V}^\rho\right) \tag{5.23}$$

$$= \frac{\partial \tilde{x}^\sigma}{\partial x^\mu}\frac{\partial x^\nu}{\partial \tilde{x}^\rho}\tilde{\partial}_\sigma \tilde{V}^\rho + \frac{\partial x^\sigma}{\partial x^\mu}\frac{\partial x^\nu}{\partial \tilde{x}^\sigma}\tilde{x}^\rho \tilde{V}^\rho \tag{5.24}$$

We want to introduce a "new derivative" that transforms as a tensor.

$$(\partial_\mu \;\rightarrow\; \nabla_\mu)$$

5.2.1.1 Covariant derivative

In general the covariant derivative is a map from (p, q)-tensors to $(p, q + 1)$-tensors with the following properties

1. It is linear:
$$\nabla(aT + bS) = a\nabla T + b\nabla S$$
where $a, b \in \mathbb{R}$ and T, S are tensors.

2. Liebnitz:
$$\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$$

These properties mean that ∇ needs to take the form (TODO: why + notes about indices and placement)

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

The objects $\Gamma^\nu_{\mu\lambda}$ are called connection coefficients.
The behaviour of ∇ under coordinate transformations depends on how the connection coefficients transform. They need to transform in a precisely non-tensorial way to negate the non-tensorial behaviour of ∂_μ.
In particular the covariant derivative needs to transform as follows:

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu$$

The left side of this equation gives

$$\nabla_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'} \tag{5.25}$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda \tag{5.26}$$

The right side gives

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda$$

These two equations are the same for any V^ν if the connection coefficients transform as

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\mu 2}} x^\lambda$$

which still leaves plenty of room for choosing a specific form of the connection.
So far in this calculation we have calculated the covariant derivative of vectors. What happens when computing the covariant derivative of tensors in general? This matter can be settled by looking at the covariant derivative acting on a one-form ω_ν. Following the same reasoning as before, we get

$$\nabla_\mu \omega_n u = \partial_\mu \omega_n u + \tilde{\Gamma}^\lambda_{\mu\nu} \omega_\lambda$$

where $\tilde{\Gamma}^\lambda_{\mu\nu}$ has the same transformation properties as $\Gamma^\lambda_{\mu\nu}$, but does not generally have to share any other similarity.
The two connections can be related if we require the covariant derivative to have some more useful properties:

3. Commutes with contractions

$$\nabla_\mu (T^\lambda{}_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$$

4. Reduces to the partial derivative on scalars

$$\nabla_\mu \phi = \partial_\mu \phi$$

To see the effect of these new properties, we can take the covariant derivative of the scalar field $\omega_\lambda V^\lambda$:

$$\nabla_\mu(\omega_\lambda V^\lambda) = (\nabla_\mu \omega_\lambda) V^\lambda + \omega_\lambda (\nabla_\mu V^\lambda) \tag{5.27}$$

$$= (\partial_\mu \omega_\lambda) V^\lambda + \tilde{\Gamma}^\sigma_{\mu\lambda} \omega_\sigma V^\lambda + \omega_\lambda (\partial_\mu V^\lambda) + \omega_\lambda \Gamma^\lambda_{\mu\rho} V^\rho \tag{5.28}$$

From property 4., this covariant derivative must just be partial derivative, meaning that the connection coefficients must cancel. In general

$$\tilde{\Gamma}^\sigma_{\mu\lambda} = -\Gamma^\sigma_{\mu\lambda}$$

because both ω_σ and V^λ were completely arbitrary. In conclusion

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda$$

This leads us to the following formula for an arbitrary tensor T:

$$\nabla_\sigma T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l} = \partial_\sigma T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l} \tag{5.29}$$

$$+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\sigma\mu_2\dots\mu_k}{}_{\nu_1\nu_2\dots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\sigma\dots\mu_k}{}_{\nu_1\nu_2\dots\nu_l} + \dots + \Gamma^{\mu_k}_{\sigma\lambda} T^{\mu_1\mu_2\dots\sigma}{}_{\nu_1\nu_2\dots\nu_l} \tag{5.30}$$

$$- \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1\mu_2\dots\mu_k}{}_{\lambda\nu_2\dots\nu_l} - \Gamma^\lambda_{\sigma\nu_2} T^{\mu_1\mu_2\dots\mu_k}{}_{\nu_1\lambda\dots\nu_l} - \dots - \Gamma^\lambda_{\sigma\nu_l} T^{\mu_1\mu_2\dots\mu_k}{}_{\nu_1\nu_2\dots\lambda} \tag{5.31}$$

5.2.1.2 Christoffel connection

There are still many possible connection coefficients that satisfy the above requirements. There is however a unique one that is defined by the metric.

The first thing to note is that the difference between two connection coefficients is a $(1,2)$-tensor. This is because the non-tensorial part in the transformation rule cancels. Say ∇_μ and $\hat{\nabla}_\mu$ are two different covariant derivatives with connection coefficients $\Gamma^\lambda_{\mu\nu}$ and $\hat{\Gamma}^\lambda_{\mu\nu}$. Then the difference

$$S^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \hat{\Gamma}^\lambda_{\mu\nu}$$

is the $(1,2)$-tensor.

From any given connection $\Gamma^\lambda_{\mu\nu}$, a new one can be formed by permuting the lower indices. This new object still satisfies the transformation requirement and is thus a good connection. The difference between these two is a tensor known as the torsion tensor $T^\lambda_{\mu\nu}$.

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 2\Gamma^\lambda_{[\mu\nu]}$$

Now a unique connection can be defined on a manifold with a metric $g_{\mu\nu}$ by introducing two additional properties:

- The connection is torsion free, meaning
  $$\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 0 \qquad \text{or, equivalently} \qquad T^{\lambda}{}_{\mu\nu} = 0$$
  The lower indices of a torsion free connection commute.
- The connection is metric compatible. This means that the covariant derivative of the metric is zero everywhere.
  $$\nabla_\rho g_{\mu\nu} = 0$$

Such a covariant derivative has a number of nice properties:

- The covariant derivative of the inverse metric is zero:
  $$\nabla_\rho g^{\mu\nu} = 0$$
- As a consequence the covariant derivative commutes with the raising and lowering of indices, e.g
  $$g_{\mu\lambda}\nabla_\rho V^\lambda = \nabla_\rho (g_{\mu\lambda}V^\lambda) = \nabla_\rho V_\mu$$
- The covariant derivative of the Levi-Civita tensor is also zero:
  $$\nabla_\lambda \varepsilon_{\mu\nu\rho\sigma} = 0$$

To see that the stated properties define a unique connection, we proceed as follows: The covariant derivative of the metric (which is zero) can be written out as

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^{\sigma}_{\mu\nu} g_{\sigma\rho} - \Gamma^{\sigma}_{\mu\rho} g_{\nu\sigma} = 0$$

because it has two lower indices. Then we consider the sum

$$\nabla_\mu g_{\nu\rho} + \nabla_\nu g_{\mu\rho} - \nabla_\rho g_{\mu\nu} = 0$$

Expanding this and using the fact that that the lower indices commute, gives

$$0 = \partial_\mu g_{\nu\rho} - \Gamma^{\sigma}_{\mu\nu} g_{\sigma\rho} - \cancel{\Gamma^{\sigma}_{\mu\rho} g_{\nu\sigma}} \tag{5.32}$$

$$+\partial_\nu g_{\mu\rho} - \Gamma^{\sigma}_{\nu\mu} g_{\sigma\rho} - \cancel{\Gamma^{\sigma}_{\nu\rho} g_{\mu\sigma}} \tag{5.33}$$

$$-\partial_\rho g_{\mu\nu} + \cancel{\Gamma^{\sigma}_{\rho\mu} g_{\sigma\nu}} + \cancel{\Gamma^{\sigma}_{\rho\nu} g_{\mu\sigma}} \tag{5.34}$$

multiplying with -1, this becomes

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + \Gamma^{\sigma}_{\mu\nu} g_{\sigma\rho} = 0$$

This can be solved for the connection by multiplying by $g^{\lambda\rho}$:

$$\boxed{\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right)}$$

This connection is so important it is known by several different names: Christoffel connection, Riemannian connection and sometimes Levi-Civita connection. The associated coefficients are known as the Christoffel symbols.

When considering cartesian coordinates in flat space, the connection coefficients vanish. This is not the case for curvilinear coordinates.

Because it is possible to make the metric vanish in a single point even in curved space (as explained before), the Christoffel symbols can be made to vanish in a point by a change of coordinates. In a neighbourhood this is in general not possible.

Divergence and Stokes' theorem. The covariant divergence of a vector V^μ is given by

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda$$

This contraction of the Christoffel symbol can be calculated as follows:

$$\begin{aligned}
\Gamma^\mu_{\mu\lambda} &= \frac{1}{2} g^{\mu\rho} \left(\partial_\mu g_{\lambda\rho} + \partial_\lambda g_{\rho\mu} - \partial_\rho g_{\mu\lambda} \right) && (5.35)\\
&= \frac{1}{2} g^{\mu\rho} \partial_\lambda g_{\rho\mu} && (5.36)\\
&= \frac{1}{2} \operatorname{Tr}(g^{-1} \partial_\lambda g) && (5.37)\\
&= \frac{1}{2} \operatorname{Tr}(\partial_\lambda \ln g) && (5.38)\\
&= \frac{1}{2} \partial_\lambda \operatorname{Tr}(\ln g) && (5.39)\\
&= \frac{1}{2} \partial_\lambda \ln |g| && (5.40)\\
&= \partial_\lambda \ln |g|^{1/2} && (5.41)\\
&= \frac{1}{\sqrt{|g|}} \partial_\lambda \sqrt{|g|} && (5.42)
\end{aligned}$$

where the first and third terms of the first equality cancel because the metric is symmetric and we have used the matrix identity $\ln \det M = \operatorname{Tr} \ln M$. (TODO transfer to log requires positive definiteness?). Thus the covariant divergence can be written as

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} V^\mu \right).$$

TODO stokes' theorem.

Relation to other derivatives. If ∇_μ is a torsion free covariant derivative, ω_μ is a one-form, and X^μ and Y^μ are vector fields, then

$$(\mathrm{d}\omega)_{\mu\nu} = \partial_{[\mu} \omega_{\nu]} = \nabla_{[\mu} \omega_{\nu]}$$

and

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu = X^\lambda \nabla_\lambda Y^\mu - Y^\lambda \nabla_\lambda X^\mu.$$

5.2.2 Parallel transport

In flat space the derivative along a curve is given by

$$\frac{\mathrm{d}V^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \partial_\mu V^\mu$$

For parallel transport this derivative should vanish. In curved space the partial derivative is replaced by a covariant one. This gives the coraviant directional derivative

$$\frac{D}{d\tau} \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \nabla_\mu.$$

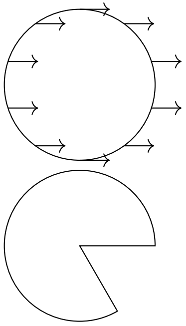
For the parallel transport of a vector, it is applied to a vector and set to zero:

$$\frac{DV^\mu}{d\tau} = \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\nabla_\nu V^\mu = \frac{\mathrm{d}V^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}V^\rho = 0$$

or

$$\boxed{\tfrac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\nu V^\mu + \Gamma^\mu_{\nu\rho}\tfrac{\mathrm{d}x^\nu}{\mathrm{d}\tau}V^\rho = 0}$$

This is the **equation of parallel transport**.



If the connection is metric compatible, the metric is parallel transported. This means parallel transport preserves properties like norm, orthogonality etc.

5.2.3 Geodesics

Two way of seeing it:

1. Parallel transports it's own tangent vector
2. Shortest distance between two points.

5.2.3.1 As a curve that parallel transports it's tangent vector

Setting directional covariant derivative of $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ to zero gives

$$\frac{D}{d\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \boxed{\tfrac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\sigma}\tfrac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\tfrac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0}$$

This is the **geodesic equation**. In flat space using Cartesian coordinates, this reduces to the equation of straight lines.
Actually this procedure constrains the parametrisation of the curve. In general a curve may be thought of as a geodesic if it parallel transports the unit tangent vector. The in formula above the norm of the tangent vector was also forced to be constant. This constrains the parametrisation to be an affine parameter, which is a parameter of the form

$$\lambda = a\tau + b$$

with τ the arc length (i.e. proper time) and a and b constants.
Now say α is an arbitrary parametrisation and $v(\alpha) = \left|\frac{\mathrm{d}x^\mu}{\mathrm{d}\alpha}\right|$ the norm of the tangent vector $\frac{\mathrm{d}x^\mu}{\mathrm{d}\alpha}$. The unit tangent vector is then $v^{-1}\frac{\mathrm{d}x^\mu}{\mathrm{d}\alpha}$. Requiring that this be parallel transported gives

$$\begin{aligned} \frac{D}{d\alpha}\left(v^{-1}\tfrac{\mathrm{d}x^\mu}{\mathrm{d}\alpha}\right) &= \tfrac{\mathrm{d}x^\rho}{\mathrm{d}\alpha}\nabla_\rho\left[v^{-1}\tfrac{\mathrm{d}x^\mu}{\mathrm{d}\alpha}\right] && (5.43)\\ &= \tfrac{\mathrm{d}x^\rho}{\mathrm{d}\alpha}\left[\partial_\rho\left(v^{-1}\tfrac{\mathrm{d}x^\mu}{\mathrm{d}\alpha}\right) + \Gamma^\mu_{\rho\sigma}v^{-1}\tfrac{\mathrm{d}x^\sigma}{\mathrm{d}\alpha}\right] && (5.44)\\ &= \tfrac{\mathrm{d}x^\rho}{\mathrm{d}\alpha}\left[\tfrac{\mathrm{d}x^\mu}{\mathrm{d}\alpha}\partial_\rho v^{-1} + v^{-1}\partial_\rho\tfrac{\mathrm{d}x^\mu}{\mathrm{d}\alpha} + \Gamma^\mu_{\rho\sigma}v^{-1}\tfrac{\mathrm{d}x^\sigma}{\mathrm{d}\alpha}\right] = 0. && (5.45) \end{aligned}$$

Multiplying both sides by v gives

$$0 = \frac{\mathrm{d}x^\mu}{\mathrm{d}\alpha} v \frac{\mathrm{d}x^\rho}{\mathrm{d}\alpha} \partial_\rho v^{-1} + \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\alpha^2} + \Gamma^\mu_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\alpha} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\alpha}$$

which is the geodesic equation derived above plus an extra term of the form $f(\alpha)\frac{\mathrm{d}x^\mu}{\mathrm{d}\alpha}$, with

$$\begin{aligned} f(\alpha) &= v\tfrac{\mathrm{d}v^{-1}}{\mathrm{d}\alpha} && (5.46)\\ &= -v^{-1}\tfrac{\mathrm{d}v}{\mathrm{d}\alpha} && (5.47)\\ &= -\left(\tfrac{\mathrm{d}^2\tau}{\mathrm{d}\alpha^2}\right)\left(\tfrac{\mathrm{d}\tau}{\mathrm{d}\alpha}\right)^{-1} && (5.48) \end{aligned}$$

using $v = \frac{\mathrm{d}\tau}{\mathrm{d}\alpha}$, because the proper time is the arc length (TODO rephrase?). The factor $f(\alpha)$ is obviously zero for affine parameters.
For timelike paths we can write the geodesic equation in terms of the four-velocity $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$:

$$u^\lambda \nabla_\lambda u^\mu = 0$$

For massive particles the four-momentum p^μ is mu^μ, making this equivalent to

$$p^\lambda \nabla_\lambda p^\mu = 0$$

5.2.3.2 As the shortest distance between two points

For timelike paths. Consider the proper time functional for a timelike path parametrised by λ:

$$\tau_{AB} = \int \left(-g_{\mu\nu}\tfrac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\tfrac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)^{1/2} \mathrm{d}\lambda$$

We want to find the stationary paths, with $\delta\tau = 0$. Computing the variation and setting $f = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}$ gives

$$\begin{aligned} \delta\tau &= \int \delta\sqrt{-f}\,\mathrm{d}\lambda && (5.49)\\ &= -\int \frac{1}{2}(-f)^{-1/2}\delta f\,\mathrm{d}\lambda. && (5.50) \end{aligned}$$

Now we can reparametrise the path, taking the proper time as the new parameter. This means the tangent vector is the four-velocity u^μ, which fixes f

$$f = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = g_{\mu\nu}u^\mu u^\nu = -1,$$

so

$$\delta\tau = -\frac{1}{2}\int \delta f\,\mathrm{d}\tau$$

This means stationary paths of the proper time functional are also stationary paths of the simpler integral

$$I = \frac{1}{2}\int f\,\mathrm{d}\tau = \frac{1}{2}\int g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

and vice versa.

We can explicitly vary this integral with (TODO why)

$$\begin{cases} x^\mu \to x^\mu_\delta x^\mu \\ g_{\mu\nu} \to g_{\mu\nu} + (\partial_\sigma g_{\mu\nu})\delta x^\sigma. \end{cases}$$

Plugging this into the expression for I and keeping only terms that are first order in δx^μ, we get

$$\delta I = \frac{1}{2}\int \left[\partial_\sigma g_{\mu\nu} \tfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tfrac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \delta x^\sigma + g_{\mu\nu} \tfrac{\mathrm{d}(\delta x^\mu)}{\mathrm{d}\tau} \tfrac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{\mu\nu} \tfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tfrac{\mathrm{d}(\delta x^\nu)}{\mathrm{d}\tau}\right] \mathrm{d}\tau$$

The last two terms can be integrated by parts; for example,

$$\begin{align} \int \left[g_{\mu\nu} \tfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tfrac{\mathrm{d}(\delta x^\nu)}{\mathrm{d}\tau}\right] \mathrm{d}\tau &= g_{\mu\nu} \tfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \delta x^\nu \big|_{\text{at boundary}} - \int \tfrac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu} \tfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right) \delta x^\nu \, \mathrm{d}\tau \tag{5.51} \\ &= -\int \left[g_{\mu\nu} \tfrac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \tfrac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}\tau} \tfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right] \delta x^\nu \, \mathrm{d}\tau \tag{5.52} \\ &= -\int \left[g_{\mu\nu} \tfrac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \partial_\sigma g_{\mu\nu} \tfrac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \tfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right] \delta x^\nu \, \mathrm{d}\tau \tag{5.53} \end{align}$$

where we have used that δx^ν vanishes at the boundary.
The total variation is then

$$\delta I = -\int \left[g_{\mu\sigma} \tfrac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \frac{1}{2}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tfrac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right] \delta x^\sigma \, \mathrm{d}\tau.$$

Since we are searching for stationary points, we want δI to vanish for any variation δx^σ. This implies that the expression inside the square brackets must vanish. Multiplying it with the inverse metric $g^{\rho\sigma}$ yields

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} + \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0$$

Which is exactly the geodesic equation with the Christoffel symbols as the connection.
This procedure provides a convenient way to calculate the Christoffel symbols for a given metric: by explicitly varying the integral I with the metric of interest plugged in.

Null geodesics. The geodesic formula was found using a very specific parametrisation, which is not a problem because all regular curves can be arc parametrised. Unfortunately we also restricted ourselves to timelike paths, because for null paths $\tau = 0$.
Now the geodesic equation derived above is still perfectly valid, even if τ can no longer be considered a valid parameter. An affine parameter is now any parameter such that the geodesic equation is satisfied, but now there is no special one. They are all related by the fact that if λ is an affine parameter, any parameter of the form $a\lambda + b$ is as well.
It is often convenient to normalise the affine parameter λ along a null geodesic such that

$$p^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$$

Timelike geodesics are maxima. Locally that is. We can see that this is true because any timelike path can be arbitrarily well approximated by a null curve.

5.2.3.3 Exponential map

TODO
Geodesically incomplete
Riemann normal coordinates

5.2.4 Riemann curvature tensor

We now want an object that embodies our idea of curvature. Seeing as we are going to define a new object to fit out intuition, we need to flesh it out a bit first. We begin by naming some properties of flat spacetime:

- Parallel transport around a closed loop leaves vectors unchanged;
- Covariant derivatives of tensors commute;
- Initially parallel geodesics remain parallel.

TODO motivation. Riemann tensor ${\mathcal{R}^{\rho}}_{\sigma\mu\nu}$ is a $(1,3)$-tensor. It is antisymmetric in the last two indices.

$$[\nabla_\mu, \nabla_\nu]V^\rho = \nabla_\mu\nabla_\nu V^\rho - \nabla_\nu\nabla_\mu V^\rho \tag{5.54}$$

$$= \left(\partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu}\nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma}\nabla_\nu V^\rho\right) - \left(\partial_\nu(\nabla_\mu V^\rho) - \Gamma^\lambda_{\nu\mu}\nabla_\lambda V^\rho + \Gamma^\rho_{\nu\sigma}\nabla_\mu V^\rho\right) \tag{5.55}$$

$$= 2\left(\partial_{[\mu}(\nabla_{\nu]}V^\rho) - \Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho + \Gamma^\rho_{[\mu|\sigma|}\nabla_{\nu]}V^\rho\right) \tag{5.56}$$

$$= 2\left(\partial_{[\mu}\partial_{\nu]}V^\rho + \partial_{[\mu}(\Gamma^\rho_{\nu]\sigma}V^\sigma) - \Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho + \Gamma^\rho_{[\mu|\sigma|}\partial_{\nu]}V^\sigma + \Gamma^\rho_{[\mu|\sigma|}\Gamma^\rho_{\nu]\sigma}V^\sigma\right) \tag{5.57}$$

$$= \cancel{[\partial_\mu, \partial_\nu]V^\rho} + 2\left(\partial_{[\mu}(\Gamma^\rho_{\nu]\sigma}V^\sigma) - \Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho + \Gamma^\rho_{[\mu|\sigma|}\partial_{\nu]}V^\sigma + \Gamma^\rho_{[\mu|\sigma|}\Gamma^\rho_{\nu]\sigma}V^\sigma\right) \tag{5.58}$$

$$= 2\left(\partial_{[\mu}(\Gamma^\rho_{\nu]\sigma})V^\sigma + \cancel{\partial_{[\mu}(V^\sigma)\Gamma^\rho_{\nu]\sigma}} - \Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho + \cancel{\Gamma^\rho_{[\mu|\sigma|}\partial_{\nu]}V^\sigma} + \Gamma^\rho_{[\mu|\sigma|}\Gamma^\rho_{\nu]\sigma}V^\sigma\right) \tag{5.59}$$

$$= 2\left(\partial_{[\mu}\Gamma^\rho_{\nu]\sigma} + \Gamma^\rho_{[\mu|\sigma|}\Gamma^\rho_{\nu]\sigma}\right)V^\sigma - 2\Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho \tag{5.60}$$

$$= \left(\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\sigma}\Gamma^\rho_{\nu\sigma} - \Gamma^\rho_{\nu\sigma}\Gamma^\rho_{\mu\sigma}\right)V^\sigma - {T^\lambda}_{\mu\nu}\nabla_\lambda V^\rho \tag{5.61}$$

$$\equiv {\mathcal{R}^\rho}_{\mu\nu\sigma}V^\sigma - {T^\lambda}_{\mu\nu}\nabla_\lambda V^\rho \tag{5.62}$$

Where [,] is the antisymmetrisation, $[\mu|\sigma|\nu]$ means that only μ and ν are antisymmetrised and T is the torsion tensor. So the Riemann tensor is identified as

$$\boxed{{\mathcal{R}^\rho}_{\mu\nu\sigma} \equiv \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\sigma}\Gamma^\rho_{\nu\sigma} - \Gamma^\rho_{\nu\sigma}\Gamma^\rho_{\mu\sigma}}$$

5.2.4.1 Properties of the curvature tensor

To investigate the properties of the Riemann tensor, the upper index is lowered

$$\mathcal{R}_{\rho\sigma\mu\nu} = g_{\rho\lambda}{\mathcal{R}^\lambda}_{\sigma\mu\nu}$$

and an explicit expression is obtained in locally inertial coordinates. In locally inertial coordinates the metric and its first derivatives vanish, so the Christoffel symbols do as well, but not their derivatives.

$$\begin{aligned}
\mathcal{R}_{\hat\rho\hat\sigma\hat\mu\hat\nu} &= g_{\hat\rho\hat\lambda}\left(\partial_{\hat\mu}\Gamma^{\hat\lambda}_{\hat\nu\hat\sigma} - \partial_{\hat\nu}\Gamma^{\hat\lambda}_{\hat\mu\hat\sigma}\right) && (5.63)\\
&= \frac{1}{2} g_{\hat\rho\hat\lambda} g^{\hat\lambda\hat\tau}\left[(\partial_{\hat\mu}\partial_{\hat\nu} g_{\hat\sigma\hat\tau} + \partial_{\hat\mu}\partial_{\hat\sigma} g_{\hat\tau\hat\nu} - \partial_{\hat\mu}\partial_{\hat\tau} g_{\hat\nu\hat\sigma}) - (\partial_{\hat\nu}\partial_{\hat\mu} g_{\hat\sigma\hat\tau} + \partial_{\hat\nu}\partial_{\hat\sigma} g_{\hat\tau\hat\mu} - \partial_{\hat\nu}\partial_{\hat\tau} g_{\hat\mu\hat\sigma})\right] && (5.64)\\
&= \frac{1}{2}\left[\partial_{\hat\mu}\partial_{\hat\sigma} g_{\hat\rho\hat\nu} - \partial_{\hat\mu}\partial_{\hat\rho} g_{\hat\nu\hat\sigma} - \partial_{\hat\nu}\partial_{\hat\sigma} g_{\hat\rho\hat\mu} + \partial_{\hat\nu}\partial_{\hat\rho} g_{\hat\mu\hat\sigma}\right] && (5.65)
\end{aligned}$$

This derivation was done in a special coordinate system, but all tensorial equations that follow from it must be true in any coordinate system. A few such equations are now listed:

1. The Riemann tensor is antisymmetric in its first two indices.

$$\boxed{\mathcal{R}_{\rho\sigma\mu\nu} = -\mathcal{R}_{\sigma\rho\mu\nu}}$$

2. The Riemann tensor is antisymmetric in its last two indices.

$$\boxed{\mathcal{R}_{\rho\sigma\mu\nu} = -\mathcal{R}_{\rho\sigma\nu\mu}}$$

3. The Riemann tensor is invariant under exchange of the first and last pair of indices.

$$\boxed{\mathcal{R}_{\rho\sigma\mu\nu} = \mathcal{R}_{\mu\nu\rho\sigma}}$$

4. Thus sum of cyclic permutations of the last three indices vanishes.

$$\mathcal{R}_{\rho\sigma\mu\nu} + \mathcal{R}_{\rho\mu\nu\sigma} + \mathcal{R}_{\rho\nu\sigma\mu} = 0$$

   This is equivalent to the vanishing of the antisymmetric part of the last three indices.

$$\boxed{\mathcal{R}_{\rho[\sigma\mu\nu]} = 0}$$

Number of parameters. TODO

Bianchi identity. TODO

$$\nabla_{[\lambda}\mathcal{R}_{\rho\sigma]\mu\nu} = 0$$

5.2.4.2 Derived quantities

Tricks for decomposition: taking contractions and taking (anti)symmetric parts.

Ricci tensor. This Ricci tensor is defined as

$$\mathcal{R}_{\mu\nu} \equiv \mathcal{R}^{\lambda}{}_{\mu\lambda\nu}.$$

The Ricci tensor associated with the Christoffel connection is automatically symmetric:

$$\begin{aligned}
\mathcal{R}_{\mu\nu} &= g^{\rho\lambda}\mathcal{R}_{\rho\mu\lambda\nu} = g^{\rho\lambda}\mathcal{R}_{\lambda\nu\rho\mu} = \mathcal{R}^{\rho}{}_{\nu\rho\mu} && (5.66)\\
&= \mathcal{R}_{\nu\mu} && (5.67)
\end{aligned}$$

The trace of the Ricci tensor is called the Ricci scalar (or curvature scalar):

$$R \equiv \mathcal{R}^{\mu}{}_{\mu} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$$

Now a useful form of the Bianchi identity can be obtained by multiplying it by $3g^{\nu\sigma}g^{\mu\lambda}$:

$$\begin{aligned} 0 &= 3g^{\nu\sigma}g^{\mu\lambda}\nabla_{[\lambda}\mathcal{R}_{\rho\sigma]\mu\nu} && (5.68)\\ &= \frac{3g^{\nu\sigma}g^{\mu\lambda}}{6!}\left[(\nabla_\lambda\mathcal{R}_{\rho\sigma\mu\nu} + \nabla_\rho\mathcal{R}_{\sigma\lambda\mu\nu} + \nabla_\sigma\mathcal{R}_{\lambda\rho\mu\nu}) - (\nabla_\lambda\mathcal{R}_{\sigma\rho\mu\nu} + \nabla_\rho\mathcal{R}_{\lambda\sigma\mu\nu} + \nabla_\sigma\mathcal{R}_{\rho\lambda\mu\nu})\right] && (5.69)\\ &= g^{\nu\sigma}g^{\mu\lambda}\left[\nabla_\lambda\mathcal{R}_{\rho\sigma\mu\nu} + \nabla_\rho\mathcal{R}_{\sigma\lambda\mu\nu} + \nabla_\sigma\mathcal{R}_{\lambda\rho\mu\nu}\right] && (5.70)\\ &= \nabla^\mu\mathcal{R}_{\rho\mu} - \nabla_\rho R + \nabla^\nu\mathcal{R}_{\rho\nu} && (5.71) \end{aligned}$$

or

$$\boxed{\nabla^\mu\mathcal{R}_{\rho\mu} = \frac{1}{2}\nabla_\rho R.}$$

Weyl tensor. TODO Why. In n dimensions the Weyl tensor is given by

$$C_{\rho\sigma\mu\nu} \equiv \mathcal{R}_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g\rho[\mu\mathcal{R}_{\nu]\sigma} - g\sigma[\mu\mathcal{R}_{\nu]\rho}\right) + \frac{2}{(n-1)(n-2)}g\rho[\mu\mathcal{R}_{\nu]\sigma}$$

TODO properties.

Einstein tensor. The Einstein tensor is defined as

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}.$$

Now the Bianchi identity reduces to

$$\boxed{\nabla^\mu G_{\mu\nu} = 0}$$

5.3 Isometries

5.3.1 About isometries

TODO post geometry.

$$g'_{\mu\nu}(y) = g_{\mu\nu}(y)$$

Symmetries of arbitrary tensor fields.

5.3.2 Lie derivatives

In general $g_{\mu\nu}(x_p)$ is different from $g_{\mu\nu}(x_q)$. Are there directions we can move in on the manifold so that the metric (or any other function on the manifold) doesn't change. So we want

$$g_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial\tilde{x}^\mu}\frac{\partial x^\sigma}{\partial\tilde{x}^\nu}g_{\rho\sigma}(x)$$

to equal the original metric.

We consider an infinitessimal transformation:

$$\tilde{x}^\mu = x^\mu + \epsilon V^\mu$$

$$\delta y^\mu = \frac{\partial y^\mu}{\partial x^\nu}\delta x^\nu$$

5.3.2.1 Lie derivative on a scalar

We are comparing

$$\begin{cases} \phi(\tilde{x}) = \phi(x+\epsilon V) = \phi(x) + \epsilon V^\mu \partial_\mu \phi(x) + \mathcal{O}(\epsilon^2) \\ \tilde{\phi}(\tilde{x}) = \phi(x) \end{cases}$$

$$\begin{aligned} L_V \phi &\equiv \lim_{\epsilon\to 0} \frac{\phi(\tilde{x}) - \tilde{\phi}(\tilde{x})}{\epsilon} \tag{5.72} \\ &= \lim_{\epsilon\to 0} \frac{\phi(x+\epsilon V^\mu) - \phi(x)}{\epsilon} \tag{5.73} \\ &= \lim_{\epsilon\to 0} \frac{\cancel{\phi(x)} + \epsilon V^\mu \partial_\mu \phi(x) + \mathcal{O}(\epsilon^2) - \cancel{\phi(x)}}{\epsilon} \tag{5.74} \\ &= V^\mu \partial_\mu \phi(x) \tag{5.75} \end{aligned}$$

Ordinary directional derivative.
In general:

$$L_V : (p,q)\text{forms} \to (p,q)\text{forms}$$

5.3.2.2 Lie derivative of a vector field

Again we define

$$L_V W^\mu \equiv \lim_{\epsilon\to 0} \frac{W^\mu(\tilde{x}) - \tilde{W}^\mu(\tilde{x})}{\epsilon}$$

Again we are comparing two quantities

$$\begin{aligned} W(\tilde{x}) &= W(x+\epsilon V) = W(x) + \epsilon V^\mu \partial_\mu W(x) + \mathcal{O}(\epsilon^2) \tag{5.76} \\ \tilde{W}(\tilde{x}) &= \frac{\partial \tilde{x}^\mu}{\partial x^\nu} W^\nu(x) \tag{5.77} \\ &= \frac{\partial(x^\mu + \epsilon V^\mu)}{\partial x^\nu} W^\nu(x) \tag{5.78} \\ &= \left(\delta^\mu_\nu + \epsilon \frac{\partial V^\mu}{\partial x^\nu}\right) W^\nu(x) \tag{5.79} \\ &= W^\mu(x) + \epsilon W^\nu(x) \partial_\nu V^\mu \tag{5.80} \end{aligned}$$

Putting everything together, we get

$$L_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$

Normal derivative not covariant, but here same as covariant derivative (extra bits cancel).
Not defined with respect to any particular metric.
Some properties:

- Partial derivatives can be replaced by covariant ones.
- The Lie derivative is antisymmetric in V and W and defines a commutator

  $$[V,W]^\mu \equiv L_V W^\mu = -L_W V^\mu$$

  This satisfies the Jacobi identity

  $$[V,[W,X]]^\mu + [X,[V,W]]^\mu + [W,[X,V]]^\mu$$

  and thus is a Lie bracket. The Jacobi identity is equivalent to

  $$L_V[W,X]^\mu = [L_V W, X]^\mu + [W, L_V X]^\mu$$

5.3.2.3 Lie derivative of other tensor fields

The definitions above are readily generalised

$$\tilde{x}^\mu(x) = x^\mu + \epsilon V^\mu(x)$$

$$L_V T = \lim_{\epsilon \to 0} \frac{T(\tilde{x}) - \tilde{T}(\tilde{x})}{\epsilon}$$

where we need

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho} = \delta^\mu_\rho + \epsilon \partial_\rho V^\mu + \mathcal{O}(\epsilon^2) \qquad \text{and} \qquad \frac{\partial x^\mu}{\partial \tilde{x}^\rho} = \delta^\mu_\rho - \epsilon \partial_\rho V^\mu + \mathcal{O}(\epsilon^2)$$

Again partial derivatives can be replaced by covariant derivatives. We illustrate with a $(0,2)$-tensor $T_{\mu\nu}$.

$$\begin{cases} T_{\mu\nu}(\tilde{x}) = T_{\mu\nu}(x) + \epsilon V^\rho \partial_\rho T_{\mu\nu} + \mathcal{O}(\epsilon^2) \\ \tilde{T}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\mu}{\partial \tilde{x}^\mu} \frac{\partial x^\nu}{\partial \tilde{x}^\nu} T_{\mu\nu} = T_{\mu\nu} - \epsilon \partial_\mu V^\rho T_{\rho\nu}(x) - \epsilon \partial_\nu V^\sigma T_{\mu\sigma} + \mathcal{O}(\epsilon^2) \end{cases}$$

Filling this in gives

$$L_V T_{\mu\nu} = \lim_{\epsilon \to 0} \frac{T_{\mu\nu}(\tilde{x}) - \tilde{T}_{\mu\nu}(\tilde{x})}{\epsilon} \tag{5.81}$$

$$= \lim_{\epsilon \to 0} \frac{\cancel{T_{\mu\nu}(x)} + \epsilon V^\rho \partial_\rho T_{\mu\nu} + \mathcal{O}(\epsilon^2) - \cancel{T_{\mu\nu}} + \epsilon T_{\rho\nu} \partial_\mu V^\rho + \epsilon T_{\mu\sigma} \partial_\nu V^\sigma + \mathcal{O}(\epsilon^2)}{\epsilon} \tag{5.82}$$

$$= V^\rho \partial_\rho T_{\mu\nu} + T_{\rho\nu} \partial_\mu V^\rho + T_{\mu\rho} \partial_\nu V^\rho \tag{5.83}$$

$$= V^\rho \left(\nabla_\rho T + \Gamma^\lambda_{\rho\mu} T_{\lambda\nu} + \Gamma^\lambda_{\rho\nu} T_{\mu\lambda} \right) + T_{\rho\nu} \left(\nabla_\mu V^\rho - \Gamma^\rho_{\mu\lambda} V^\lambda \right) + T_{\mu\rho} \left(\nabla_\nu V^\rho - \Gamma^\rho_{\nu\lambda} V^\lambda \right) \tag{5.84}$$

$$= V^\rho \nabla_\rho T + T_{\rho\nu} \nabla_\mu V^\rho + T_{\mu\rho} \nabla_\nu V^\rho + \left(\Gamma^\lambda_{\rho\mu} V^\rho T_{\lambda\nu} - \Gamma^\rho_{\mu\lambda} V^\lambda T_{\rho\nu} \right) + \left(\Gamma^\lambda_{\rho\nu} V^\rho T_{\mu\lambda} - \Gamma^\rho_{\nu\lambda} V^\lambda T_{\mu\rho} \right) \tag{5.85}$$

$$= V^\rho \nabla_\rho T + T_{\rho\nu} \nabla_\mu V^\rho + T_{\mu\rho} \nabla_\nu V^\rho \tag{5.86}$$

Isometries from an algebra

$$[L_V, L_W] = L_{[V,W]}$$

Enough to verify scalars and vectors.

5.3.2.4 Lie derivative of tensor densities

TODO

5.3.3 Killing vectors

The Lie derivative of the metric tensor. This is $(0,2)$-tensor, so the formula is the one given above. Due to metric compatibility, the first term is zero

$$L_V g_{\mu\nu} = V^\rho \nabla_\rho g_{\mu\nu} + g_{\lambda\nu} \nabla_\mu V^\lambda + g_{\mu\lambda} \nabla_\nu V^\lambda \tag{5.87}$$

$$= g_{\lambda\nu} \nabla_\mu V^\lambda + g_{\mu\lambda} \nabla_\nu V^\lambda \tag{5.88}$$

$$= \nabla_\mu V_\nu + \nabla_\nu V_\nu \tag{5.89}$$

$$\tag{5.90}$$

An infinitesimal coordinate transformation is a symmetry of the metric if $L_V g_{\mu\nu} = 0$, which is equivalent to requiring V to satisfy the equations

$$\nabla_\mu V_\nu + \nabla_\nu V_\nu = 0 = \nabla_{(\mu} V_{\nu)}.$$

Such vectors are called Killing vectors. These equations are equivalent to

$$\nabla_\mu V_\nu = \nabla_{[\mu} V_{\nu]}$$

Properties:

1. Killing vectors form a Lie algebra. If V and W are Killing vectors, i.e. $L_V g_{\mu\nu} = L_W g_{\mu\nu} = 0$, then $[V, W]$ is a Killing vector because

$$L_{[V,W]} g_{\mu\nu} = L_V L_W g_{\mu\nu} - L_W L_V g_{\mu\nu} = 0$$

2. If all the components of the metric are independent of a particular coordinate, say y

$$\partial_y g_{\mu\nu} \qquad \forall \mu, \nu$$

   Then $V = \partial_y$ is a Killing vector. A coordinate system in which a Killing vector is a partial derivative is said to be *adapted* to the Killing vector (or isometry) in question. TODO derive Killing equations from this.

3. Two Killing vectors commute if and only if there is a coordinate system that is adapted to both of them.

You can use the equations in 2 ways:

- Impose symmetries on the metric.
- Find the Killing vectors for a given metric, which gives the symmetries

Example

Algebra of Killing vectors in Minkowski and two-sphere.

5.3.4 Conserved quantities

5.3.4.1 Conserved charges along geodesics

Let K^μ be a Killing vector field and $x^\mu(\tau)$ a geodesic with four-velocity x^μ. Then the quantity

$$Q_K = K_\mu u^\mu$$

is constant along the geodesic. Indeed,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}\tau} Q_K = \tfrac{\mathrm{d} K_\mu u^\mu}{\mathrm{d}\tau} &= u^\mu \tfrac{\mathrm{d} K_\mu}{\mathrm{d}\tau} + K_\mu \tfrac{\mathrm{d}}{\mathrm{d}\tau} u^\mu && (5.91) \\ &= u^\mu u^\nu \nabla_\nu K_\mu + 0 && (5.92) \\ &= \frac{1}{2} \left(\nabla_\nu K_\mu + \nabla_\mu K_\nu \right) u^\mu u^\nu = 0 && (5.93) \end{aligned}$$

where the last equality is due to the Killing equations.

5.3.4.2 Conserved currents from the energy-momentum tensor

Let K^μ be a Killing vector field and $T^{\mu\nu}$ the covariantly conserved symmetric energy-momentum tensor ($\nabla_\mu T^{\mu\nu} = 0$). Then the current

$$J^\mu_K = T^{\mu\nu} K_\nu$$

is covariantly conserved. Indeed,

$$\begin{aligned} \nabla_\mu J^\mu_K &= (\nabla_\mu T^{\mu\nu}) K_\nu + T^{\mu\nu} \nabla_\mu K_\nu && (5.94) \\ &= 0 + \frac{1}{2} T^{\mu\nu} \left(\nabla_\mu K_\nu + \nabla_\nu K_\mu \right) = 0 && (5.95) \end{aligned}$$

5.3.4.3 Komar currents

The Einstein tensor is symmetric and conserved (from the Bianchi identity), so we have the conserved current

$$J^\mu_1 = G^\mu{}_\nu K^\nu =$$

5.3.5 Killing tensors

5.3.5.1 Killing(-Stäckel) tensors

A Killing tensor $K_{\beta_1 \ldots \beta_n}$ is a totally symmetric tensor satisfying

$$\nabla_{(\alpha} K_{\beta_1 \ldots \beta_n)} = 0.$$

The charge

$$Q_K = K_{\beta_1 \ldots \beta_n} u^{\beta_1} \ldots u^{\beta_n}$$

is constant along the geodesic.

5.3.5.2 Killing-Yano tensors

A Killing-Yano tensor $Y_{\beta_1 \ldots \beta_n}$ is a totally anti-symmetric tensor satisfying

$$\nabla_{(\alpha} Y_{\beta_1) \ldots \beta_n} = 0 \qquad \text{or, equivalenty} \qquad \nabla_\alpha Y_{\beta_1 \ldots \beta_n} = \nabla_{[\alpha} Y_{\beta_1 \ldots \beta_n]}$$

The tensorial charges

$$Z_{\beta_1 \ldots \beta_{n-1}} = u^\beta Y_{\beta \beta_1 \ldots \beta_{n-1}}$$

are conserved along geodesics.

5.3.5.3 Symmetries and conserved charges (Komar integrals)

5.3.5.4 Conservation laws

Conservation laws

- for geodesics
- for spacetime

$$Q = V^\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} g_{\mu\nu}$$

$\frac{\mathrm{d}}{\mathrm{d}\tau}Q$ if V is Killing and $\dot{x}^\mu$ is a geodesic.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Q = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(V_\mu \dot{x}^\mu\right) = \left(\frac{DV_\mu}{D\tau}\right)\dot{x}^\mu + V_\mu \frac{D\dot{x}^\mu}{\tau} \tag{5.96}$$

$$= \underbrace{\dot{x}^\rho D_\rho V_\mu \dot{x}^\mu}_{=0\text{because Killing}} + \underbrace{V_\mu \frac{D\dot{x}^\mu}{D\tau}}_{=0\text{geodesic}} \tag{5.97}$$

5.3.6 Maximally symmetric spaces

In D=4, maximally 10 symmetries. Minkowski maximally symmetric, but not uniquely so. (Depends on number of killing vectors (which also form an algebra and can commute or not))

$$K^\mu(x)\text{Killing} \quad \Leftrightarrow \quad D_{[\mu}K_{\nu]} = 0$$

$$K_\mu(x) = K_\mu(x^*) + \partial_\nu K_\mu(x^*)(x^\nu - x^{\nu*}) + \frac{1}{2}\partial_\rho\partial_\nu K_\mu(x^*)(x^\nu - x^{\nu*}(x^\rho - x^{\rho*}) + \dots$$

$$D_\mu K_\nu = \partial_\mu K_\nu - \Gamma^\rho_{\mu\nu}K_\rho$$

$$\partial_\nu K_\mu(x^*) = D_\nu K_\mu(x^*) + \Gamma_{\nu\mu}^{\ \ \rho} K_\rho(x^*)$$

With

$$D_\nu K_\mu(x^*) = \underbrace{\cancel{D_{(\nu}K_{\mu)}}}_{0\text{because Killing}} + D_{[\nu}K_{\mu]}(x^*)$$

So

$$\frac{D(D-1)}{2} \qquad \text{for} D = 4 \quad \Rightarrow \quad 6\text{coeff.}$$

Maximally symmetric:

- Minkowski: 4 translations, 6 Lorentz ($\Lambda = 0$)

$$[P_\mu, P_\nu] = 0, \qquad [M_{\mu\nu}, P_\rho] = 2\eta_{\rho[\mu}P_{\nu]}, \qquad [M_{\mu\nu}, M^{\rho\sigma}] = 4\delta_{[\mu}^{\ [\rho} M_{\nu]}^{\ \ \sigma]}$$

$$M^{\mu\nu} = x^\mu\partial^\nu - x^\nu\partial^\mu \qquad G = \mathbb{R}^4 \rtimes \mathrm{SO}(1,3)$$

- de Sitter $G = \mathrm{SO}(1,4)$ ($\Lambda > 0$)
- Anti-de Sitter $G = \mathrm{SO}(2,3)$

On a side note

$$dS_4 \equiv \frac{\mathrm{SO}(1,4)}{\mathrm{SO}(1,3)} \qquad AdS_4 \equiv \frac{\mathrm{SO}(2,3)}{\mathrm{SO}(1,3)}$$

Confer:

$$S^p = \frac{\mathrm{SO}(p+1)}{\mathrm{SO}(p)}$$

riemann normal coordinates + freely falling frames

5.4 Conformal transformations

5.4.1 Conformal Killing vectors

$$L_C g_{\mu\nu} = \nabla_\mu C_\nu + \nabla_\nu C_\mu = 2\omega(x) g_{\mu\nu}$$

A Killing vector for a metric is at least a conformal Killing vector for any conformally rescaled metric.

5.4.1.1 Conserved charges along null geodesics

$$Q_C = C_\mu u^\mu$$

5.4.1.2 Conserved currents from the energy-momentum tensor

energy-momentum tensor is traceless

$$J^\mu_C = T^{\mu\nu} C_\nu$$

5.4.2 Conformal Killing(-Yano) tensors

Chapter 6

Lie groups and algebras

6.1 Lie Group

A Lie group is a topological group that is also a differential manifold. This means we can apply differentials, which is of course very important. So important in fact that Sophus Lie called Lie groups infinitesimal groups when he first introduced them. Not only that, but it means we can consider tangent spaces, which will also be important later. TODO better justification
Bearing in mind the link between the topology and group properties explored in the section on topological groups, we quite naturally arrive at the following definition:

> A Lie group is a smooth manifold G which is also a group and such that both the group product $G \times G \to G$ and the inverse map $G \to G$ are smooth.

There is a particular type of Lie group that will be of particular importance to us, namely the matrix Lie group. In fact we will almost exclusively consider matrix Lie groups.

6.1.1 Matrix Lie group

For matrix groups there is a simpler condition to see whether it is a Lie group or not:

> All closed subgroups of $\mathrm{GL}(n, \mathbb{C})$ are matrix Lie groups.

The condition that it be closed means that for every sequence in the Lie group the limit needs to be in the Lie group as well, if there is one. (Or you can say every Cauchy sequence in the Lie group has to have a limit in the Lie group). This is a technicality and is satisfied for most of the interesting subgroups of $\mathrm{GL}(n, \mathbb{C})$.
We have already seen that all subgroups of $\mathrm{GL}(n, \mathbb{C})$ are topological groups. To prove the assertion then we must only verify that it is a smooth manifold. Because $\mathbb{C}^{n\times n}$ is a manifold and a matrix Lie group is a subset of $\mathbb{C}^{n\times n}$, the matrix Lie group inherits Hausdorffness and second-countability from $\mathbb{C}^{n\times n}$. To show it is smooth and locally homeomorphic to $\mathbb{R}^m$ in every point, we will explicitly construct such homeomorphisms using the matrix exponential.

6.1.1.1 Exponential maps

The homeomorphisms will be constructed based on the exponential map.

$$\exp : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C}) : X \mapsto e^X$$

This map is not a bijection, however if we restrict it to a neighbourhood of $\mathbb{0}$, it is locally a bijection. In fact it maps that neighbourhood to a neighbourhood of $\mathbb{1}$. More formally

> There exists a neighbourhood U of $\mathbb{0}$ and a neighbourhood V of $\mathbb{1}$ such that the exponential mapping takes U homeomorphically onto V.

This result should not be surprising. For X close to $\mathbb{0}$ we have the approximation $e^X \approx \mathbb{1} + X + \mathcal{O}(X^2)$. So for matrices in a small neighbourhood U around $\mathbb{0}$ the exponential mapping can be seen as approximately linear, which is injective. In order to get surjectivity, we restrict the codomain of the mapping to the image of U under the exponential mapping. This is a neighbourhood of $\mathbb{1}$ because $e^0 = \mathbb{1}$.

We have obtained a bijection and because the matrix exponential is continuous, this restriction of it is also continuous. We would now like to show that the map maps open sets in our matrix Lie group, which we shall now call G, to open sets of $\mathbb{R}^m$. Unfortunately it doesn't. There is no reason why $\mathbb{0}$ or any matrices in U should be elements of G. (Remember that the relevant group operation for matrix groups is the matrix multiplication, for which the neutral element is $\mathbb{1}$; the matrix $\mathbb{0}$ is of no particular importance in this context.) Being a group, the matrix Lie group must contain $\mathbb{1}$; being a topological group, it must contain a neighbourhood of $\mathbb{1}$; being a subspace of $\mathrm{GL}(n, \mathbb{C})$ endowed with the subspace topology, the intersection of V with that neighbourhood is an open set in G which we will call V'.

So if we invert the restricted matrix exponential, we get a homeomorphism from one neighbourhood of G to $\mathrm{GL}(n, \mathbb{C})$, which can then be composed with a homeomorphism to $\mathbb{R}^m$.

> The inverse map $\exp^{-1} : V' \to U$ is called the logarithm.

From this we can construct a homeomorphism from a neighbourhood of any element A of G. By multiplying each element of V' with A we get a neighbourhood V_A of A. We define the following homeomorphism on V_A: multiply by A^{-1} (this is bijective due to associativity of the group operation and continuous due to the definition of topological groups) and then send through the inverted, restricted matrix exponential. This composition of homeomorphisms is a homeomorphism. So for each element $A \in G$ we can find a neighbourhood V_A that is homeomorphic to $\mathbb{R}^m$ thanks to this homeomorphism.

> Example
>
> TODO Finite Lie group

6.1.1.2 Lie algebra of a matrix Lie group

TODO: justification

> Let G be a matrix Lie group. The Lie algebra of G, denoted $\mathfrak{g}$, is the set of all matrices X_t such that e^{itX_t} is in G for all **real** numbers t. We call the matrices X_t generators of the group.

Now here we have a complication. There are actually two conventions. The definition above is the convention most often used in physics. In the mathematics literature the Lie algebra in usually defined using e^{tX_t}, not e^{itX_t}. The physics convention gives rise to Hermitian generators in the algebras of $\mathrm{U}(n)$ and $\mathrm{SU}(n)$. This is useful because we are often interested in turning them into quantum operators, which correspond to observables only if they are Hermitian. The downside of this convention however is that it makes our life much more difficult in other places, and it even means that some definitions don't make any sense. In what follows we will generally be using the physics convention. We will however make use of the mathematics convention when the need arises. Also if there are interesting differences in the mathematics definition, we will mention those as well.

To try to grasp why the definition given above is useful, we introduce the notion of parametrization of group elements.

6.1.1.3 Parametrization of group elements.

When first introducing the matrix groups, we pointed out how the elements could be written in function of real parameters. We now make this notion more concrete and begin by defining a one-parameter subgroup.

A function $A : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ is called a one-parameter subgroup of $\mathrm{GL}(n, \mathbb{C})$ if

1. A is continuous,
2. $A(0) = \mathbb{1}_n$,
3. $A(t + s) = A(t)A(s)$ for all $t, s \in \mathbb{R}$.

If A is a one-parameter subgroup of $\mathrm{GL}(n, \mathbb{C})$, then it has the following property:

There exists a unique $n \times n$ complex matrix X such that

$$A(t) = e^{tX}$$

So X_t is in $\mathfrak{g}$ if and only if the one-parameter subgroup generated by X_t lies in G. Conversely for any one-parameter subgroup that is a subgroup of G, there exists a generator and that generator is by definition part of the algebra.
Before continuing we shall consider some examples of algebras of matrix Lie groups. In general we shall call the algebra of a Lie group the lowercase version of the name of the Lie group. e.g the Lie algebra of $\mathrm{GL}(n, \mathbb{C})$ is $\mathfrak{gl}(n, \mathbb{C})$.

Example

1. If X is any $n \times n$ complex matrix, then e^{itX} is invertible. Thus the Lie algebra, $\mathfrak{gl}(n, \mathbb{C})$, of the invertible matrices, $\mathrm{GL}(n, \mathbb{C})$, is the space of all complex $n \times n$ matrices.

2. If we use the mathematical convention, then the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$ is the space of all real $n \times n$ matrices, denoted $\mathfrak{gl}(n, \mathbb{R})$. To prove this we first remark that is X is any real $n \times n$ matrix, then e^{tX} will be invertible and real. Conversely, if e^{tX} is

real for all real t, then $X = \frac{\mathrm{d}}{\mathrm{d}t} e^{tX}\Big|_{t=0}$ will also be real. Obviously in the physics convention the above no longer holds true.

3. The Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ of $\mathrm{SL}(n,\mathbb{C})$ is the space of all complex $n \times n$ matrices with zero trace. To prove this we use that

$$\det(e^X) = e^{\mathrm{Tr}(X)}.$$

If $\mathrm{Tr}(X) = 0$, then $\det(e^{itX}) = 1$ for all real numbers t. On the other hand, if X is any $n \times n$ matrix such that $\det(e^{itX}) = 1$ for all t, then $e^{it\,\mathrm{Tr}(X)} = 1$ for all t. This means that $it\,\mathrm{Tr}(X)$ is an integer multiple of $2\pi i$ for all t, which is only possible if $\mathrm{Tr}(X) = 0$.

4. Lie algebra of $\mathrm{U}(N)$. If X is to be a generator in our algebra, we need e^{itX} to be unitary. So

$$\left(e^{itX}\right)^\dagger = \left(e^{itX}\right)^{-1} = e^{-itX}.$$

We also have that

$$\left(e^{itX}\right)^\dagger = e^{-itX^\dagger}.$$

Which gives us

$$e^{-itX} = e^{-itX^\dagger}.$$

Differentiating at $t = 0$ we see that the generators have to be Hermitian ($X = X^\dagger$).

We can also prove this by writing out the definition of the matrix exponential.

$$U(N) \ni U = e^{it_i X_i}$$

$$\mathbb{1} = U^\dagger U = (\mathbb{1} - it_i X_i^\dagger + \ldots)(\mathbb{1} + it_i X_i + \ldots) \tag{6.1}$$

$$= \mathbb{1} + it_i(X_i^\dagger - X_i) + \ldots = \mathbb{1} \tag{6.2}$$

So we require the generators to be Hermitian matrices ($X_i^\dagger = X_i$). We have N^2 independent X_i that are Hermitian.

$$\mathfrak{u}(N) = \{H \in \mathrm{GL}(N,\mathbb{C}), H^\dagger = H\}$$

In the mathematics convention this condition becomes that the generators have to be skew-Hermitian, i.e. $X_i^\dagger = -X_i$.

5. Lie algebra of $\mathrm{SU}(N)$. Combining the arguments for the algebras of the unitary and special linear group, we see that the generators must be unitary and of trace zero. In other words the algebra is given by

$$\mathfrak{su}(N) = \{H \in \mathfrak{u}(N), \mathrm{Tr}[H] = 0\}$$

and has dimension $N^2 - 1$.

For $N = 2$ we have:

$$\begin{cases} \mathfrak{su}(2) = \{\sigma_1, \sigma_2, \sigma_3\} \\ \mathfrak{u}(2) = \{\sigma_1, \sigma_2, \sigma_3, \mathbb{1}\} \end{cases}$$

Where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

6. Lie algebra of $\mathrm{O}(N)$. As explained above, if we want the algebra to be real, we need to make use of the mathematical convention. So $O = e^{t_i X_i}$.

$$\mathbb{1} = O^\intercal O = e^{t_i X_i^\intercal} e^{t_i X_i} = (\mathbb{1} + t_i X_i^\intercal + \ldots)(\mathbb{1} + t_i X_i + \ldots) \tag{6.3}$$

$$= \mathbb{1} + t_i(X_i^\intercal + X_i) + \ldots \tag{6.4}$$

So we require the generators to be antisymmetric matrices ($X_i^\intercal = -X_i$).

$$\mathfrak{o}(N) = \{X \in \mathrm{GL}(N, \mathbb{R}), X^\intercal = -X\} = \mathfrak{so}(N)$$

The dimension of $\mathfrak{o}(N)$ is $\frac{N(N-1)}{2}$.

The Lie algebra as defined above is in some way prototypical. i.e. when we make this notion more abstract, we want the abstract notion to behave in a similar fashion and have many of the same properties. Of course to do that we first need an idea of what properties these Lie algebras actually have. This is what we will be exploring next.

If G is a *connected* matrix Lie group, then every $A \in G$ can be written in the form

$$A = e^{X_1} e^{X_2} \ldots e^{X_m}$$

for some $X_1, X_2, \ldots, X_m$ in $\mathfrak{g}$

Every continuous homomorphism between two matrix Lie groups is smooth.

A matrix X is in $\mathfrak{g}$ if and only if there exists a smooth curve γ in $\mathbb{C}^{n \times n}$ such that

1. $\gamma(t)$ lies in G for all t;
2. $\gamma(0) = \mathbb{1}$;
3. $\left.\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|_{t=0} = X$

Thus $\mathfrak{g}$ is the tangent space at the identity to G.

- If we assume $X \in \mathfrak{g}$, we can take $\gamma(t) = \exp(tX)$. This $\gamma(t)$ satisfies the points of the proposition above.
- We now assume $\gamma(t)$ is a smooth curve in G with $\gamma(0) = \mathbb{1}$.

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{\gamma(t + \delta t) - \gamma(t)}{\delta t} = \gamma(t) \left(\lim_{\delta t \to 0} \frac{\gamma(\delta t) - \gamma(0)}{\delta t} \right) \tag{6.5}$$

$$= \gamma(t) \left.\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|_{t=0} = \gamma(t) X \tag{6.6}$$

From which we get that

$$\gamma(t) = \exp tX$$

Now this is interesting, so interesting in fact that we use this last proposition to construct a general definition of a Lie algebra associated to a Lie group.
We can now also reintroduce the physics convention. We just divide all elements of any algebra by the imaginary unit i. The elements of this algebra may not be closed under the bracket operation, but that does not matter as we have a different definition to work from now: they are elements of the tangent space at identity to G, rescaled with a fractor $-i$.
We have already seen that in the mathematics convention the commutator belongs to the algebra (remembering to sum according to Einstein notation):

$$[X_i, X_j] = f_{ij}^k X_k$$

Where we call f_{ij}^k a structure constant (with respect to the chosen basis of course). In the physics convention, we obviously need to deal with the factor i:

$$[X_i, X_j] = i f_{ij}^k X_k$$

> From the antisymmetry of the bracket we get:
>
> $$f_{ij}^k + f_{ji}^k = 0$$
>
> From the Jacobi identity we get:
>
> $$f_{ie}^m f_{jk}^e + f_{je}^m f_{ki}^e + f_{ke}^m f_{ij}^e = 0$$

And lastly a final property of Lie algebras of matrix Lie groups follows straight from the Baker-Campbell-Hausdorff formula:

$$e^A e^B = e^C \qquad \text{with} \qquad C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]]) + \dots$$

> To know the (local) structure of a Lie group close to the identity one only needs to know the commutator of the generators $[X_i, X_j]$

6.2 Lie Algebra

Again we will start by restricting our attention to Lie algebra's of matrix Lie groups. That way we can give some examples that will (hopefully) aid in the understanding of the general case.

6.2.1 Definition

> Let G be a matrix Lie group, with Lie algebra (g). Let X and Y be elements of $\mathfrak{g}$. Then
>
> 1. $sX \in \mathfrak{g}$, for all real numbers s,
> 2. $X + Y \in \mathfrak{g}$,
> 3. $-i(XY - YX) \in \mathfrak{g}$.

The first two points mean that the Lie algebra is actually a vector space over the real numbers. This is important and serves as the crux of our first generalisation of Lie algebras, so we will have a quick look at the proofs of the statements above.

1. This first point is fairly straightforward, since $e^{t(sX)} = e^{(ts)X}$, which must be in G if X is in $\mathfrak{g}$.

2. If X and Y commute, this is again immediate. If they don't however we need to do a little more work. We start from the Lie product formula:

   $$e^{t(X+Y)} = \lim_{m\to\infty}\left(e^{\frac{tX}{m}}e^{\frac{tY}{m}}\right)^m$$

   Clearly if $X, Y \in \mathfrak{g}$, for every m, $e^{\frac{tX}{m}}$ and $e^{\frac{tY}{m}}$ are elements of G. Since G is a group, $\left(e^{\frac{tX}{m}}e^{\frac{tY}{m}}\right)^m$ is in G. Now because G is a matrix Lie group, and thus *closed* in $\mathrm{GL}(n,\mathbb{C})$, the limit must also be in G. (If that is the limit is in $\mathrm{GL}(n,\mathbb{C})$, which it is because $e^{t(X+Y)}$ is invertible). This shows that $X+Y$ is in $\mathfrak{g}$.

3. The third point follows from the product rule of the differential operator. Alternatively we can use the Baker-Campbell-Hausdorff formula:

   $$e^{tX}e^{sY} = e^{tX+sY+\frac{ts}{2}[X,Y]+\ldots}$$

   This together with the first two points shows the third point.

We have shown that the Lie algebra is a vector space over the real numbers, but crucially a Lie algebra is in general not a vector space over complex numbers, even if it consists of matrices with complex entries. For an example we consider the algebra $\mathfrak{su}(n)$, which consists of Hermitian matrices with zero trace. Assume X is such a matrix. Now because $(iX)^\dagger = -iX^\dagger = -iX$, iX is not Hermitian. As a consequence it cannot be an element of $\mathfrak{su}(n)$ and thus $\mathfrak{su}(n)$ is not a complex vector space.

If we follow the mathematical definition, the third point becomes $XY - YX \in \mathfrak{g}$. This will be important later. In fact it's so important we will give it name.

> Given two $n \times n$ matrices A and B, the bracket (or commutator) of A and B, denoted $[A, B]$ is defined to be
>
> $$[A, B] = AB - BA$$

Using the mathematical convention, the Lie algebra of any matrix Lie group is closed under brackets. This is in general not the case using the physics convention. Take for example the algebra $\mathfrak{su}(2)$, generated by $\{\sigma_1, \sigma_2, \sigma_3\}$. Then

$$[\sigma_1, \sigma_2] = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

which is not Hermitian and thus not an element of $\mathfrak{su}(2)$! This also means that $\mathfrak{su}(n)$ (with $n > 1$) is not an algebra in according to the definition we are about to give.

Despite the problems with the conventions, these properties seem nice. We would like to study things that exhibit these properties in general. So based on this we define a Lie algebra in general in the following way.

> A (finite-dimensional) real or complex Lie algebra $\mathfrak{g}$ is an n-dim (real or complex) vector space with the following map:
>
> $$[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g} : (X, Y) \mapsto [X, Y]$$

that has the following properties

1. Bilinear: $\forall X, Y, Z \in \mathfrak{g}, \qquad a, b \in \mathbb{R} \quad (\text{or}\mathbb{C})$:
$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$
2. Antisymmetric: $\forall X, Y \in \mathfrak{g}$
$$[X, Y] = -[Y, X]$$
3. Satisfies the Jacobi identity: $\forall X, Y, Z \in \mathfrak{g}$
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

The only surprising thing in this definition is the appearance of the Jacobi identity. It can be thought of as a condition that takes the place of associativity, but is weaker. In fact every Lie algebra can be embedded in into some associative algebra so that the bracket corresponds to the operation $XY - YX$.

If we follow the mathematical convention, the Lie algebra of a matrix Lie group is a real Lie algebra in the sense of the above definition. Unfortunately this is not true for the physics convention.

As noted above, this notably means $\mathfrak{su}(n)$ (with $n > 1$) is not an algebra in according this definition. There are several ways to solve this problem. The obvious one would be to redefine the bracket operator when using the physics convention (i.e. say that $[A, B] = -i\,(AB - BA)$). This is usually not done. We could also extend $\mathfrak{su}(n)$ to include iX for every $X \in \mathfrak{su}(n)$ (this is called the complexification of $\mathfrak{su}(n)$), which would mean that $\mathfrak{su}(n)$ is actually $\mathfrak{sl}(n)$ (i.e. we drop the condition that the elements of $\mathfrak{su}(n)$ have to be Hermitian). This is apparently actually done sometimes in the physics literature. Or finally we can do what we will do in these notes, namely forget about this definition, use the definition we will motivate in the next section and write the extra i whenever it pops up.

Furthermore for every finite-dimensional real or complex vector space V, let $\mathfrak{gl}(V)$ denote the space of linear maps of V into itself. Then $\mathfrak{gl}(V)$ is a real or complex Lie algebra with the bracket operation $[A, B] = AB - BA$.

6.2.2 Lie algebra of a Lie group

We finally define the Lie algebra:

The Lie algebra of a Lie group G is the tangent space at the identity with the bracket operation defined by
$$[v, w] = [X^v, X^w]_e.$$

6.3 Representations of Lie algebras

TODO: representations of Lie groups: Representation vs linear group action. Continuous groups must be represented on the physical Hilbert space by unitary operators $U(T(\theta))$.

We start with some definitions.

A homomorphism between two algebras $\mathfrak{g}_1, \mathfrak{g}_2$ is a map that preserves $[,]$:

$$\phi : \mathfrak{g}_1 \to \mathfrak{g}_2 : [X_1, X_2] \mapsto \phi([X_1, X_2]) = [\phi(X_1), \phi(X_2)]$$

If the map is invertible, it is called an isomorphism.

Every Lie group homomorphism gives rise to a Lie algebra homomorphism.

Let G and H be matrix Lie groups, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Suppose that $\Phi : G \to H$ is a Lie group homomorphism. Then there exists a unique real linear homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\Phi\left(e^X\right) = e^{\phi(X)}$$

for all $X \in \mathfrak{g}$. The map ϕ has the following additional properties:

1. $\phi\left(AXA^{-1}\right) = \Phi(A)\phi(X)\Phi(A)^{-1}$, for all $X \in \mathfrak{g}, A \in G$
2. $\phi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi\left(e^{tX}\right)\Big|_{t=0}$, for all $X \in \mathfrak{g}$

An algebra representations is a homomorphism between an abstract algebra and the space of linear operators.

$$D : \mathfrak{g} \to \mathrm{GL}(n, \mathbb{R}) \text{ or } \mathrm{GL}(n, \mathbb{C}) : X \mapsto D(X)$$

A representation is said to be faithful if it is injective.

Ado's theorem: Any finite dimensional Lie algebra admits a faithful matrix representation.

This nontrivial theorem means that every Lie algebra can be viewed as a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$, and thus as an algebra of a matrix Lie group.

6.3.1 Adjoint representation

Let G be a matrix Lie group, with Lie algebra (g). Let X be an element of $\mathfrak{g}$ and A an element of G.

$$AXA^{-1} \in \mathfrak{g}$$

This means that the following definition makes sense:

Let G be a matrix Lie group with algebra $\mathfrak{g}$. Then for each $A \in G$ we define the linear map $\mathrm{Ad}_A : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$\mathrm{Ad}_A(X) = AXA^{-1}$$

- $\mathrm{Ad}_A^{-1} = \mathrm{Ad}_{A^{-1}}$.
- The map $A \to \mathrm{Ad}_A$ is a group homomorphism of G into $\mathrm{GL}(\mathfrak{g})$.
- $\mathrm{Ad}_A([X,Y]) = [\mathrm{Ad}_A(X), \mathrm{Ad}_A(Y)] \qquad \forall A \in G, X, Y \in \mathfrak{g}$.

Because $A \to \mathrm{Ad}_A$ is a group homomorphism, we have an associated algebra homomorphism, $X \mapsto \mathrm{ad}_X$.

The associated Lie algebra map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by

$$\mathrm{ad}_X(Y) = [X, Y]$$

This last property generalises well and we can use it to define the adjunct map for a Lie algebra in general.
The maps Ad and ad give the adjoint representations of G and $\mathfrak{g}$.
The adjoint representation ad_{X_i} is linear, and thus can be represented as a matrix. So for every X_i in the basis, we have a T_i that maps the coordinates of a $Y \in \mathfrak{g}$ to $[X_i, Y]$. If we write $Y = c_1X_1 + c_2X_2 + c_3X_3 + \ldots$, then

$$T_i \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} if^1_{11}c_1 + if^1_{12}c_2 + \ldots \\ if^2_{11}c_1 + if^2_{12}c_2 + \ldots \\ \vdots \end{pmatrix}$$

So

$$T_i = \begin{pmatrix} if^1_{11} & if^1_{12} & \ldots \\ if^2_{11} & if^2_{12} & \ldots \\ \vdots & & \end{pmatrix} \qquad \text{or} \qquad (T_i)^k_j = if^k_{ij}$$

These matrices have the following property (derived from the Jacobi identity):

$$[T_i, T_j] = -if^k_{ij}T_k$$

The Cartan-Killing form

$$\begin{aligned} g_{ij} &\equiv \mathrm{Tr}[T_i \cdot T_j] && (6.7) \\ &= -f^e_{ik}f^k_{je} && (6.8) \end{aligned}$$

The quadratic Casimir in a given representation of an algebra is given by

$$C_2 = g^{ij}X_iX_j$$

This is an invariant for a specific representation.

The quadratic Casimir commutes with any element X of the algebra:

$$[C_2, X] = 0$$

In general $C_2 \notin \mathfrak{g}$

A Casimir is an operator that commutes with all generators.

Example

The angular momentum operators have to structure of $\mathfrak{su}(2)$

$$[L_i, L_j] = i\epsilon_{ijk}L_k \qquad (L_i \text{ generators}) \tag{6.9}$$

$$[L^2, L_i] = 0 \tag{6.10}$$

Example

Find the Casimir operator of the fundamental representation of $\mathfrak{su}(2)$.
We call $\tau_i = \frac{\sigma_i}{2}$, so that

$$[\tau_i, \tau_j] = i\epsilon_{ijk}\tau_k$$

We then compute

$$C_2 = \sum_{i,j} g^{ij}\tau_i\tau_j = \frac{1}{2}\sum_{i,j}\delta_{ij}\tau_i\tau_j = \frac{3}{8}\mathbb{1} \tag{6.11}$$

$$= \frac{1}{2}s(s+1)\mathbb{1} \qquad \Rightarrow \qquad s = \tfrac{1}{2} \tag{6.12}$$

Where we used that $g^{ij} = (g_{ij})^{-1}$ and $g_{ij} = \epsilon_{ike}\epsilon_{jke} = 2\delta_{ij}$

6.3.2 Representations of $\mathfrak{su}(2)$

6.3.2.1 The algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$

are isomorphic

$$\mathfrak{so}(3) = \{X \in \mathrm{GL}(3, \mathbb{R}), X^\intercal = -X\}$$

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example

Show that $[X_i, X_j] = -\epsilon_{ijk}X_k$

Let J_i be

$$J_i = -iX_i$$

so that $[J_i, J_j] = i\epsilon_{ijk}J_k$. Then J_i are generators of $\mathfrak{su}(2)$.

The groups have the same algebra, which means they are the same around identity

6.3.2.2 Building the $\mathfrak{su}(2)$ representation

we run into the problem that the J_i cannot be diagonalised simultaneously, i.e. they don't commute. So we choose a basis in which J_3 is diagonal, then we define

$$J_\pm \equiv J_1 \pm iJ_2$$

With the following properties:

$$\begin{cases} [J_3, J_\pm] = [J_3, J_1] \pm i[J_3, J_2] = iJ_2 \pm J_1 = \pm J_\pm \\ [J_+, J_-] = i[J_2, J_1] - i[J_1, J_2] = 2J_3 \end{cases}$$

We now notate a basis of states V with $|j,m\rangle$

$$J_3 |j,m\rangle = m |j,m\rangle$$

where m is an eigenvalue of J_3 and j is the biggest eigenvalue ($m \leq j$). We can find enough eigenvectors to make the basis because J_3 is diagonal.
From the relation

$$\begin{aligned} J_3 (J_\pm |j,m\rangle) &= (J_\pm J_3 + [J_3, J_\pm]) |j,m\rangle && (6.13) \\ &= J_\pm m |j,m\rangle \pm J_\pm |j,m\rangle && (6.14) \\ &= (m \pm 1) (J_\pm |j,m\rangle) && (6.15) \end{aligned}$$

we get the following

$$\begin{cases} J_+ |j,j\rangle = 0 \qquad (j > 0) \\ J_- |j,j_-\rangle = 0 \qquad (j_- \text{ smallest eigenvalue of } J_3) \end{cases}$$

We also see that the eigenvalues are spaced an integer apart, from j_- to j.
Because the trace of a commutator is zero (as the trace is cyclic), we also have that

$$\mathrm{Tr}[J_3] = \frac{1}{2}\mathrm{Tr}([J_+, J_-]) = 0 = \sum_{j_-}^{j} m$$

which means that

$$0 = j + (j-1) + \ldots + (j_- + 1) + j_- \quad \Rightarrow \quad j + j_- = 0 \quad \Rightarrow \quad j_- = -j$$

So the dimension of V is $2j+1$, which must be an integer, meaning that j must be half-integer.
We also impose the following normalisation:

$$\langle j,m| =\rangle 1 \qquad \langle j,j| =\rangle 1$$

For a generic state $|j,m\rangle$ we get the following:

$$\begin{aligned} J_3 |j,m\rangle &= m |j,m\rangle && (6.16) \\ J_+ |j,m\rangle &= [(j+1+m)(j+m)]^{1/2} |j,m+1\rangle && (6.17) \\ J_- |j,m\rangle &= [(j+1-m)(j+m)]^{1/2} |j,m-1\rangle && (6.18) \end{aligned}$$

Example

Fundamental representation of $\mathfrak{su}(2)$ (i.e. of dimension 2).

$$j = 1/2 \qquad \begin{cases} |1/2, +1/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |1/2, -1/2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

$$J_3 = -\frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\sigma_3}{2}$$

$$\begin{cases} J_+ \, |1/2, 1/2\rangle = 0 \\ J_+ \, |1/2, -1/2\rangle = |1/2, 1/2\rangle \end{cases} \qquad \begin{cases} J_- \, |1/2, 1/2\rangle = |1/2, -1/2\rangle \\ J_- \, |1/2, -1/2\rangle = 0 \end{cases}$$

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\sigma_2}{2} \qquad J_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\sigma_2}{2}$$

Example

The $j = 1$ representation of $\mathfrak{su}(2)$

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{cases} J_+ \, |1,1\rangle = 0 \\ J_+ \, |1,0\rangle = \sqrt{2}\, |1,1\rangle \\ J_+ \, |1,-1\rangle = \sqrt{2}\, |1,0\rangle \end{cases} \qquad \begin{cases} J_- \, |1,1\rangle = \sqrt{2}\, |1,0\rangle \\ J_- \, |1,0\rangle = \sqrt{2}\, |1,-1\rangle \\ J_- \, |1,-1\rangle = 0 \end{cases}$$

$$J_+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad J_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Example

Exercise: calculate C_2

$$C_2 = \frac{1}{2}l(l+1) \qquad (l = 1)$$

Chapter 7

Riemannian manifolds

7.1 Riemannian metrics

Let M be a smooth manifold. A Riemannian metric on M is a smooth covariant 2-tensor field $g \in T^*M \otimes T^*M$ whose value g_p at each point $p \in M$ is an inner product on T_pM. We often use the notation

$$\langle v, w \rangle_g := g_p(v, w)$$

where $p \in M$ and $v, w \in T_pM$. Similarly we write $\|v\|_g := \sqrt{\langle v, v \rangle_g}$.

A Riemannian manifold is a pair (M, g) where M is a smooth manifold and g is a Riemannian metric on M.

Most things work with boundary as well.

Lemma XVII.14. *Every smooth manifold admits a Riemannian metric.*

Proof. TODO with partition of unity. ☐

Example

The Euclidean metric is the Riemannian metric g_E on the manifold $\mathbb{R}^n$ whose value at each $x \in \mathbb{R}^n$ is the standard inner product on $T_x\mathbb{R}^n$.

7.1.1 Isometries

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. An isometry from (M_1, g_1) to (M_2, g_2) is a diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^* g_2 = g_1$.

We say $\varphi : M_1 \to M_2$ is a local isometry if for each point $p \in M_1$ their is a neighbourhood $U(p)$ such that $\varphi|_U$ is an isometry onto an open subset of M_2.

A Riemannian n-manifold is called flat if it is locally isometric to a Euclidean space.

An isometry from (M, g) to itself is called an isometry of (M, g). The set of isometries of (M, g) is a group under composition, the isometry group of (M, g), denoted $\mathrm{Iso}(M, g)$.

Lemma XVII.15. *All Riemannian 1-manifolds are flat.*

Lemma XVII.16. *A mapping $\varphi : M \to M'$ between smooth manifolds is an isometry* if and only if *φ is a smooth bijection and each differential $\mathrm{d}\varphi_p : T_pM \to T_{\varphi(p)}M'$ is a linear isometry.*

Proof. The only part to prove is that φ is automatically a diffeomorphism if it is a smooth bijection. This follows from the global rank theorem XVII.13 because F has contant rank (equal to the dimension of M and M'). ☐

7.1.2 Local representations for metrics

Let $(x^1, \ldots, x^n)$ be smooth local coordinates on the neighbourhood $U \subseteq M$. Then $g|_U$ can be written as

$$g|_U = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$$

The definition of inner product translates to the requirement that $[g(p)]_{ij}$ be a symmetric, non-singular matrix. Using symmetry we get the symmetric product

$$g|_U = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

Example

The Euclidean metric can be expressed as

$$g_E = \sum_i \mathrm{d}x^i\,\mathrm{d}x^i = \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^i$$

so $g_{ij} = \delta_{ij}$.

Proposition XVII.17. *Given a smooth local frame for TM we can construct a smooth orthonormal frame with the same span.*

Proof. Gram-Schmidt. ☐

7.1.3 Constructing Riemannian metrics

7.1.4 Riemannian immersions

Proposition XVII.18. *Let (M, g) be a Riemannian manifold, M' a smooth manifold and $F : M' \to M$ a smooth map. Then $g' = F^*g$ is a Riemannian metric on M* if and only if *F is an immersion.*

Proof. The only reason $g' = F^*g$ may fail to be a metric is if it is not definite. First assume F is not an immersion. Then there exist $p \in M'$ and $v, w \in T_pM'$ such that $\mathrm{d}F_p(v) = \mathrm{d}F_p(w)$ and $v \neq w$. Then $v - w \neq 0$, but

$$\langle v - w, v - w\rangle_{g'} = \langle \mathrm{d}F(v - w), \mathrm{d}F(v - w)\rangle_g = \langle 0, 0\rangle_g = 0.$$

Conversely, assume g' not definite. Then there exists a $v \neq 0$ such that $0 = \|v\|_{g'} = \|\mathrm{d}F(v)\|_g$, implying $\mathrm{d}F(v) = 0$. Thus the kernel of $\mathrm{d}F$ is not $\{0\}$, meaning it is not injective by X.23 and thus F is not an immersion by definition. ☐

The metric $g' = F^*g$ of the proposition is called the metric induced by F.
An immersion (resp. embedding) $F : (M, g) \to (M', g')$ is called an isometric immersion (resp. isometric embedding) if $g' = F^*g$.

Lemma XVII.19. *Existence of adapted orthonormal frames.*

Let (M, g) be a Riemannian manifold and $M' \subseteq M$ a smooth submanifold. A vector $v \in T_pM$, for some $p \in M'$, is called normal to M' if $\langle v, w \rangle_g = 0$ for every $w \in T_pM'$. The space of all vectors normal to M' at $p \in M'$ is called the normal space N_pM' at p.

Clearly $N_pM' = (T_pM')^\perp$ and

$$T_pM = T_pM' \oplus N_pM'.$$

Proposition XVII.20 (Normal bundle)**.** *Let (M, g) be a Riemannian m-manifold without boundary and $M' \subseteq M$ a an immersed n-submanifold. The set*

$$NM' = \bigsqcup_{p \in M'} N_pM'$$

is a smooth subbundle of $TM|_{M'}$ of rank $(m - n)$.

The vector bundle NM' is called the normal bundle of M'.
A section of the normal bundle NM' is called a normal vector field along M'.
The tangential projection $\pi^\top : TM|_{M'} \to TM'$ and the normal projection $\pi^\perp : TM|_{M'} \to NM'$ are the maps that for each $p \in M'$ restrict to the orthogonal projections $T_pM \to T_pM'$ and $T_pM \to N_pM'$.

Lemma XVII.21. *The tangential and normal projections are smooth bundle homomorphisms*

7.1.5 Riemannian products

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. The product manifold $M_1 \times M_2$ has a natural Riemannian metric $g = g_1 \oplus g_2$ called the product metric defined by

$$g_{p_1,p_2} : (T_{p_1}M_1 \oplus T_{p_2}M_2)^2 \to \mathbb{R} : (v_1 + v_2, w_1 + w_2) \mapsto g_1|_{p_1}(v_1, w_1) + g_2|_{p_2}(v_2, w_2)$$

where we have identified $T_{(p_1,p_2)}(M_1 \times M_2)$ with $T_{p_1}M_1 \oplus T_{p_2}M_2$.

7.1.6 Riemannian submersions

7.1.6.1 Horizontal and vertical tangent spaces

Suppose M, M' are smooth manifolds, $\pi : M \to M'$a smooth submersion and g a Riemannian metric on M.
TODO: we can view M as a fibre bundle with as fibres the properly embedded smooth manifolds $M_y = \pi^{-1}(y)$.
At each point $x \in M$ we can split T_xM into two subspaces $V_x \oplus H_x$, the horizontal and vertical tangent spaces at x, defined by

$$V_x \coloneqq \ker \mathrm{d}\pi_x = T_x(M_{\pi(x)}) \qquad \text{and} \qquad H_x = (V_x)^\perp.$$

Where the equality $\ker \mathrm{d}\pi_x = T_x(M_{\pi(x)})$ is due to (TODO tangent space to a submanifold).
Notice that the definition of V_x does not depend on the metric, but the definition of H_x does.
A horizontal vector field on M consists of vectors in the horizontal tangent space and a vectical vector field on M consists of vectors in the vertical tangent tangent space on M.
A vector field X on M is a horizontal lift of a vector field X' on M' if X is horizontal and π-related to X, which means that

$$\forall x \in M : \; \mathrm{d}\pi_x(X_x) = X'_{\pi(x)}.$$

Proposition XVII.22. *Let M, M' be smooth manifolds, $\pi : M \to M'$ a smooth submersion and g a Riemannian metric on M.*

1. *Every smooth vector field W on M can uniquely be expressed as the sum of a smooth horizontal and a smooth vertical vector field:*

$$W = W^H + W^V.$$

2. *Every smooth vector field on M' has a unique smooth horizontal lift to M.*
3. *For every $x \in M$ and $v \in H_x$, there is a vector field $X' \in \mathfrak{X}(M')$ whose horizontal lift X satisfies $X_x = v$.*

The last part of the previous proposition says that any horizontal vector can be extended to a horizontal lift on all of M.
Importantly, it is not true that every horizontal vector field on M is a horizontal lift.

> Example
>
> Take $\pi : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$. Let W be the smooth vector field $y\partial_x$ on $\mathbb{R}^2$. At any point $V_p = \text{span}\{\partial_y\}$ and $H_p = \text{span}\{\partial_x\}$, so W is horizontal. But there is no vector field on $\mathbb{R}$ whose horizontal lift is W. Indeed $\mathrm{d}\pi_p(W) = y\partial_x$ is not constant on $\pi^{-1}(p)$ because it depends on y.

7.1.6.2 Riemannian submersions

> Let $\pi : (M, g) \to (M', g')$ be a smooth submersion between Riemannian manifolds. Then π is a Riemannian submersion if $\mathrm{d}\pi_x|_{H_x} : H_x \to T_{\pi(x)}M'$ is a (bijective) linear isometry for all $x \in M$.

Equivalently, the submersion π is a Riemannian submersion if the metrics satisfy

$$\forall x \in M : \ \forall v, w \in H_x : \ g_x(v, w) = g'_{\pi(x)}(\mathrm{d}\pi_x(v), \mathrm{d}\pi_x(w)).$$

7.1.6.3 Riemannian coverings

7.1.7 Basic constructions derived from the metric

7.1.7.1 Raising and lowering indices

Let M be a smooth manifold. Given a Riemannian metric g in M, we define a bundle homomorphism

$$\hat{g} : TM \to T^*M : v \mapsto g_p(v, \cdot).$$

In other words we have $\hat{g}(v)(w) = g_p(v, w)$ for all $p \in M$ and $v, w \in T_pM$.
Musical isomorphisms

7.1.7.2 Inner products of tensors

We define $\langle \omega, \eta \rangle_g \coloneqq \langle \omega^\sharp, \eta^\sharp \rangle$. Then

$$\langle \omega, \eta \rangle = g_{kl}(g^{ki}\omega_i)(g^{lj}\eta_j) = \delta^i_l g^{lj}\omega_i\eta_j = g^{ij}\omega_i\eta_j.$$

7.2 Connections

7.2.1 Affine connection

Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M and let $\Gamma(E)$ denote the space of sections of E. A connection in E is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) : (X, Y) \mapsto \nabla_X Y$$

satisfying the following properties:

1. $\nabla_X Y$

7.3 Geodesics

7.4 Curvature

Chapter 8

Algebraic geometry

Chapter 9

Geometric topology

9.1 Vector bundles

9.1.1 Definition

9.1.2 Operations on vector bundles

Part XVIII

Number theory

Chapter 1

Primes

Chapter 2

Irrational numbers

Chapter 3

Congruences and modular arithmetic

Part XIX

K-theory

Chapter 1

K-theory for additive categories

Let C be an additive category. Consider the isomorphism classes $[E]$ of objects E in C with an addition operation given by

$$[E] + [F] = [E \oplus F] \qquad E, F \in \mathsf{C}.$$

Lemma XIX.1. *The isomorphism classes of* C *form an abelian monoid* $M(\mathsf{C})$ *under this addition operation:*

$$E \oplus (F \oplus G) \cong (E \oplus F) \oplus G, \qquad E \oplus F \cong F \oplus E, \qquad \textit{and} \qquad E \oplus 0 \cong E.$$

Note that it is necessary to use isomorphism classes, because in general

$$E \oplus (F \oplus G) \neq (E \oplus F) \oplus G \qquad \text{even though} \qquad E \oplus (F \oplus G) \cong (E \oplus F) \oplus G.$$

The K functor is in this case just the Grothendieck functor G applied after making the category a monoid:

$$K(-) \coloneqq G(M(-)).$$

Chapter 2

Topological K-theory

2.1 The group $K(X)$

Let X be a compact topological space. The category $\mathsf{Vect}(X)$ of vector bundles over X with the direct sum is an additive category. We define the K group of X as

$$K(X) \coloneqq K(\mathsf{Vect}(X)).$$

Let $\mathcal{E}_n$ be the trivial bundle of rank n over a compact space X.

Proposition XIX.2. *Every element x of $K(X)$ can be written as $[E] - [\mathcal{E}_n]$ for some n and some vector bundle E over X.*
Moreover, $[E] - [\mathcal{E}_p] = [F] - [\mathcal{E}_q]$ if and only if *there exists an integer n such that $E \oplus \mathcal{E}_{q+n} \cong F \oplus \mathcal{E}_{p+n}$.*

Proof. Immediate from Grothendieck construction and the fact that for all vector bundles E there exists a vector bundle F such that $E \oplus F \cong \mathcal{E}_n$. ☐

Corollary XIX.2.1. *Let E, F be vector bundles over X. Then $[E] = [F]$ in $K(X)$* if and only if *$E \oplus \mathcal{E}_n \cong F \oplus \mathcal{E}_n$ for some n.*

Proposition XIX.3. *For topological spaces K is a contravariant functor on the category of compact spaces.*

2.2 The group $\widetilde{K}(X)$ for pointed spaces

Let (X, x_0) be a pointed compact space. The projection $\pi : X \to \{x_0\}$ induces a homomorphism $K(\{x_0\}) \cong \mathbb{Z} \to K(X)$. The reduced K-theory $\widetilde{K}(X)$ of X is the cokernel of this homomorphism:

$$0 \longrightarrow \mathbb{Z} \longrightarrow K(X) \longrightarrow \widetilde{K}(X) \longrightarrow 0.$$

Lemma XIX.4. *There is a canonical splitting generated by the inclusion $\{x_0\} \hookrightarrow X$ so that*

$$K(X) \cong \mathbb{Z} \oplus \widetilde{K}(X) \qquad \text{and} \qquad \widetilde{K}(X) \cong \ker(K(\{x_0\} \hookrightarrow X)).$$

Proposition XIX.5. *The composition* $\gamma : M(\mathsf{Vect}(X)) \to K(X) \to \widetilde{K}(X)$ *is a surjective homomorphism. Moreover,* $\gamma([E]) = \gamma([F])$ if and only if $E \oplus \mathcal{E}_p \cong F \oplus \mathcal{E}_q$ *for some* p, q.

This gives a more direct definition of $\widetilde{K}(X)$ as the quotient of $M(\mathsf{Vect}(X))$ by the equivalence relation

$$[E] \sim [F] \qquad \Longleftrightarrow \qquad \exists p, q \in \mathbb{N} : \; E \oplus \mathcal{E}_p \cong F \oplus \mathcal{E}_q.$$

2.3 The relative K-group $K(X, Y)$

Let X be a compact topological space and Y a closed subspace. Consider the triples (E, F, α) where E, F are vector bundles over X and α is an isomorphism $E_Y \to F_Y$ where E_Y and F_Y are the vector bundles E, F restricted to Y.
We define the sum of two triples to be

$$(E, F, \alpha) + (E', F', \alpha') = (E \oplus E', F \oplus F', \alpha \oplus \alpha').$$

We call two triples $(E, F, \alpha), (E', F', \alpha')$ isomorphic if there exist isomorphisms $f : E \to E'$ and $g : F \to F'$ such that the diagram

$$\begin{tikzcd} E|_Y \arrow[r, "\alpha"] \arrow[d, "f|_Y"'] & F|_Y \arrow[d, "g|_Y"] \\ E'|_Y \arrow[r, "\alpha'"] & F'|_Y \end{tikzcd} \qquad \text{commutes.}$$

We consider the equivalence relation of "stable isomorphism" on these triples, that is two triples $(E, F, \alpha), (E', F', \alpha')$ are equivalent if and only if there exist triples (G, G, I_{G_Y}) and $(G', G', I_{G'_Y})$ such that

$$\begin{cases} (E, F, \alpha) + (G, G, I_{G_Y}) = (E \oplus G, F \oplus G, \alpha \oplus I_{G_Y}) & \text{and} \\ (E', F', \alpha') + (G', G', I_{G'_Y}) = (E' \oplus G', F' \oplus G', \alpha' \oplus I_{G'_Y}) \end{cases}$$

are isomorphic.
Then $K(X, Y)$ is the set of equivalence classes of such triples. We denote the equivalence class of a triple (E, F, α) by $d(E, F, \alpha)$.

Proposition XIX.6. *Let X be a compact space and Y a closed subspace. Then*

1. *$K(X, Y)$ is an abelian group with as neutral element*

   $$0 = d(G, G, I_{G_Y})$$

   and

   $$d(E, F, \alpha) + d(F, E, \alpha^{-1}) = 0;$$

2. *$K(X) \cong K(X, \emptyset)$;*
3. *$d(E, F, \alpha) + d(F, G, \beta) = d(E, G, \beta \circ \alpha)$.*

Proof. □

Proposition XIX.7. *Let i be the homomorphism*

$$i : K(X,Y) \to K(X) : d(E, F\alpha) \mapsto [E] - [F]$$

and j the homomorphism

$$i : K(X) \to K(Y) : [E] - [F] \mapsto [E|_Y] - [F|_Y].$$

Then we have the exact sequence

$$K(X,Y) \xrightarrow{i} K(X) \xrightarrow{j} K(Y).$$

Moreover, if Y is a retract of X (i.e. the inclusion $Y \hookrightarrow X$ admits a left-inverse), then we have the split exact sequence

$$0 \longrightarrow K(X,Y) \longrightarrow K(X) \longrightarrow K(Y) \longrightarrow 0.$$

Corollary XIX.7.1. *Let (X, x_0) be a pointed space. Then $\{x_0\}$ is a retract of X and thus*

$$K(X, \{x_0\}) \cong \ker(K(X) \to K(\{x_0\})) \cong \widetilde{K}(X)$$

Proposition XIX.8. *The projection $\pi : X \to X/Y$ induces an isomorphism $K(X/Y, \{y\}) \to K(X,Y)$.*

Proposition XIX.9. *Let Y be a closed subspace of a compact space X. Then we have the exact sequence*

$$\widetilde{K}(X/Y) \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}(Y).$$

Theorem XIX.10 (Atiyah-Jänich). *Let X be a compact Hausdorff space and H a Hilbert space. Then*

$$[X, \mathcal{F}(H)] \cong K(X).$$

2.4 Clifford modules and the functor $K^{p,q}$

Proposition XIX.11. *Let A, B be $\mathbb{R}$-algebras.*

$$(\mathsf{C}^A)^B \cong \mathsf{C}^{A \otimes_{\mathbb{R}} B}$$

$$\mathsf{C}^{A \oplus B} \simeq \mathsf{C}^A \times \mathsf{C}^B.$$

Let $\mathsf{Vect}(X)^{p,q}$ be the category of $\mathrm{Cl}^{p,q}$-modules W such that W is a vector bundle over X.
We define $K^{p,q}(X)$ as the Grothendieck group of the functor

$$\mathsf{Vect}(X)^{p,q+1} \to \mathsf{Vect}(X)^{p,q}$$

Theorem XIX.12. *Let X be a compact space. Then $K^{0,0}$ and $K(0,1)$ are canonically isomorphic to $K(X)$ and $K^{-1}(X)$.*

2.4.1 Description via gradings

Let $E \in \mathsf{Vect}(X)^{p,q}$. A grading of E is an endomorphism η of E regarded as an object of $\mathsf{Vect}(X)$ such that

1. $\eta^2 = I$;

2. $\eta\rho(e_i) = -\rho(e_i)\eta$.

Equivalently, a grading on E is a $\mathrm{Cl}^{p,q+1}$-structure on E extending the $\mathrm{Cl}^{p,q}$-structure where $\eta = \rho(e_{p+q+1})$.

Chapter 3

K-theory for C^*-algebras

3.1 Homotopy equivalence of unitaries

TODO ref on homotopy + $\sim_h$.

Lemma XIX.13. *If* $u_1 \sim_h v_1$ *and* $u_2 \sim_h v_2$*, then* $u_1 u_2 \sim_h v_1 v_2$*.*

Let A be a unital C^*-algebra. We let $\mathcal{U}_0(A) \subseteq \mathcal{U}(A)$ denote the set of all unitaries homotopic with $\mathbf{1}$ in $\mathcal{U}(A)$.

Lemma XIX.14. *Let* A *be a unital* C^**-algebra.*

1. *For each self-adjoint element* $h \in A$*,* $\exp(ih) \in \mathcal{U}_0(A)$*.*
2. *If* $u \in \mathcal{U}(A)$ *with* $\sigma(u) \neq \mathbb{T}$*, then* $u \in \mathcal{U}_0(A)$*.*
3. *If* $u, v \in \mathcal{U}(A)$ *with* $\|u - v\| < 2$*, then* $u \sim_h v$*.*

Proof.

1. By spectral mapping, XII.64 , and XII.67.1 we see that $\exp(ih)$ is unitary. The homotopy is given by $t \mapsto \exp(ith)$.
2. If $\sigma(u) \neq \mathbb{T}$, then for some real θ, $\exp(i\theta) \notin \sigma(u)$. This means the exponential has a well defined inverse $f : \sigma(u) \to]\theta, \theta + 2\pi[: \exp(it) \mapsto t$. By spectral mapping, XII.64 , and XII.67.1 we see that $f(u)$ is self-adjoint. It follows that $u = \exp(if(u))$, so we can conclude using (1).
3. Assume $\|u - v\| < 2$. Then

   $$2 > \|u - v\| = \|v^*\|\|u - v\| \geq \|v^* u - 1\|$$

   so $-2 \notin \sigma(v^* u - 1)$ and $-1 \notin \sigma(v^* u)$ by spectral mapping. By (2) $v^* u \sim_h \mathbf{1}$ and hence $u \sim_h v$ be XIX.13.

□

Corollary XIX.14.1. *The unitary group* $\mathrm{U}(\mathbb{C}^{n \times n})$ *is connected for all* $n \in \mathbb{N}$*.*

Proof. Every element in $\mathbb{C}^{n\times n}$ has finite spectrum (the eigenvalues). So we conclude by (2) of XIX.14. $\square$

Because $\|u-v\| \leq \|u\| + \|v\| = 2$ for all $u, v \in \mathcal{U}(A)$, two unitaries are only not homotopic if they lie at a distance of exactly 2.
TODO generalise to GL:

Lemma XIX.15. *Let A be a unital C^*-algebra. Then*

1. *$\mathcal{U}_0(A)$ is a normal subgroup of $\mathcal{U}(A)$;*
2. *$\mathcal{U}_0(A)$ is open and closed relative to $\mathcal{U}(A)$;*
3. *an element $u \in \mathcal{U}(A)$ belongs to $\mathcal{U}_0(A)$* if and only if *for some self-adjoint elements $h_1, \ldots, h_n \in A$*
$$u = \exp(ih_1) \cdot \ldots \cdot \exp(ih_n).$$

Proof. TODO ref.

1. First note $\mathcal{U}_0(A)$ is closed under multiplication by XIX.13. Let u_t be a continuous path from $\mathbf{1}$ to u. Then u_t^{-1} and (for all $v \in \mathcal{U}(A)$) v^*u_tv are continuous paths from $\mathbf{1}$ to u^{-1} and v^*uv, respectively.
2. TODO.

$\square$

Lemma XIX.16 (Whitehead)**.** *Let A be a unital C^*-algebra and $u, v \in \mathcal{U}(A)$. Then*

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & \mathbf{1} \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & \mathbf{1} \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} \qquad \text{in } \mathcal{U}(A^{2\times 2}).$$

It follows in particular that
$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

Proof. First
$$\sigma_A \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \sigma_{\mathbb{C}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \{1\}, \qquad \text{so} \qquad \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \sim_h \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$
by (2) of XIX.14. Hence
$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & \mathbf{1} \end{pmatrix}\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & \mathbf{1} \end{pmatrix}\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \sim_h \begin{pmatrix} u & 0 \\ 0 & \mathbf{1} \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} uv & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

The other claims follow in a similar way. $\square$

Lemma XIX.17. *Let A, B be unital C^*-algebras and let $\Psi : A \to B$ be a surjective $*$-homomorphism. Then*

1. *$\Psi(\mathcal{U}_0(A)) = \mathcal{U}_0(B)$;*
2. *if $u \in \mathcal{U}(B)$, and if $u \sim_h \Psi(v)$ for some $v \in \mathcal{U}(A)$, then u lifts to a unitary element in A.*

Proof. TODO $\square$

Proposition XIX.18. *Let A be a unital C^*-algebra and for all $a \in \mathrm{GL}(A)$, let $u(a)|a|$ be the polar decomposition of a. Then*

1. *$a \sim_h u(a)$ in $\mathrm{GL}(A)$;*
2. *for all $v_1, v_2 \in \mathcal{U}(A)$*

$$v_1 \sim_h v_2 \quad \textit{in} \quad \mathrm{GL}(A) \qquad \iff \qquad v_1 \sim_h v_2 \quad \textit{in} \quad \mathcal{U}(A).$$

Proof. TODO ☐

Thus the polar decomposition gives a deformation retract of $\mathrm{GL}(A)$ onto $\mathcal{U}(A)$. TODO also for matrices!!

3.2 Projections

Let A be a C^*-algebra. Two projections $p, q \in \mathcal{P}(A)$ are orthogonal if $pq = 0$. We write $p \perp q$.

TODO: $pq = 0$ iff $qp = 0$.

Lemma XIX.19. *Let A be a C^*-algebra and $p, q \in \mathcal{P}(A)$. Then the following are equivalent:*

1. *$p + q \in \mathcal{P}(A)$;*
2. *p and q are orthogonal;*
3. *$p + q \leq 1$.*

Proof. TODO ☐

Every "almost-idempotent" can be approximated by a projection: TODO

3.2.1 Partial isometries

An element $v \in A$ is a partial isometry if v^*v and vv^* are projections. We call v^*v the support projection and vv^* the range projection.

Lemma XIX.20. *Let A be a C^*-algebra and $v \in A$. If either v^*v or vv^* is a projection, then*

1. *$v = vv^*v$ and $v^* = v^*vv^*$;*
2. *the other is also a projection and v is a partial isometry.*

Proof. First assume v^*v is a projection. Put $z = (\mathbf{1} - vv^*)v$. Then

$$z^*z = v^*(\mathbf{1}-vv^*)(\mathbf{1}-vv^*)v = v^*v - v^*vv*v - v^*vv^*v + v^*vv^*vv^*vv^*v = v^*v - v^*v - v^*v + v^*v = 0.$$

Now by the C^*-identity $\|z\| = \sqrt{\|z^*z\|} = 0$, so $0 = z = v - vv^*v$. Multiplying this on the right by v^* gives $vv^* = (vv^*)^2$. Because vv^* is self-adjoint, this shows us it is a projection.
If we assume vv^* is a projection, we put $z = (\mathbf{1} - v^*v)v^*$ instead and again $z^*z = 0$. ☐

Lemma XIX.21. *Let $v \in \mathcal{B}(\mathcal{H})$ be a partial isometry. Then*

$$\ker v = (I - v^*v)\mathcal{H} \qquad \ker v^* = (I - vv^*)\mathcal{H}$$

and

$$v\mathcal{H} = vv^*\mathcal{H} \qquad v^*\mathcal{H} = v^*v\mathcal{H}.$$

Proof. First assume $x \in \ker v$. Then $(I - vv^*)x = Ix = x$, so $x \in (I - vv^*)\mathcal{H}$. Conversely, $v(I - v^*v)\mathcal{H} = (v - vv^*v)\mathcal{H} = 0\mathcal{H} = \{0\}$ by XIX.20.
Also by $XIX.20$, we have

$$v\mathcal{H} \subseteq vv^*\mathcal{H} \subseteq vv^*v\mathcal{H} = v\mathcal{H}.$$

□

3.2.2 Equivalence of projections

Let A be a C^*-algebra and $p, q \in A$. We write

1. $p \sim q$ if there exists $v \in A$ such that $p = v^*v$ and $q = vv^*$. This is (Murray-von Neumann) equivalence.
2. $p \sim_u q$ if there exists $u \in \mathcal{U}(\tilde{A})$ such that $q = upu^*$. This is unitary equivalence.

TODO: We can also take $A^\dagger$ ipv $\tilde{A}$.

Lemma XIX.22. *Both Murray-von Neumann equivalence and unitary equivalence are equivalence relations.*

Proof. TODO transitivity. □

Lemma XIX.23. *Let A be a unital C^*-algebra and $p, q \in \mathcal{P}(A)$. Then*

$$p \sim_u q \qquad \iff p \sim q \quad \textit{and} \quad \mathbf{1} - p \sim \mathbf{1} - q.$$

Proof. Assume $p \sim_u q$, so we can write $q = upu^*$ for some $u \in \mathcal{U}(A)$. Put $v = up$ and $w = u(\mathbf{1} - p)$. Then

$$\begin{aligned} v^*v &= p^*u^*up = p, & vv^* &= upp^*u^* = upu^* = q \\ w^*w &= (\mathbf{1} - p)u^*u(\mathbf{1} - p) = \mathbf{1} - p & ww^* &= u(\mathbf{1} - p)(\mathbf{1} - p)u^* = u(\mathbf{1} - p)u^* = \mathbf{1} - q. \end{aligned}$$

Assume the converse. TODO □

Lemma XIX.24. *Let $p, q \in \mathcal{P}(A)$. If $\exists z \in \mathrm{GL}(\tilde{A})$ such that $q = zpz^{-1}$, then $p \sim_u q$.*

Proof. We have $zp = qz$ and $pz^* = z^*q$, so p commutes with z^*z:

$$pz^*z = z^*qz = z^*zp.$$

Now put $u = z|z|^{-1}$ and calculate

$$upu^* = z|z|^{-1}p|z|^{-1}z^* = zp|z|^{-2}z^* = qz(z^*z)^{-1}z^* = q.$$

TODO: clarify rules of calculation. □

Proposition XIX.25. *Let $p, q \in \mathcal{P}(A)$. If $\|p - q\| < 1$, then $p \sim_h q$.*

Proposition XIX.26. *Let A be a C^*-algebra and $p, q \in \mathcal{P}(A)$. Then*

1. *if $p \sim_h q$, then $p \sim_u q$;*
2. *if $p \sim_u q$, then $p \sim q$;*

and

3. *if $p \sim q$, then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $A^{2\times 2}$;*
4. *if $p \sim_u q$, then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $A^{2\times 2}$.*

Proof. TODO ☐

3.2.2.1 Decomposition into matrix algebras

Proposition XIX.27. *Let A be a unital C^*-algebra. Let $p_1, \ldots, p_n$ be pairwise orthogonal and Murray-von Neumann equivalent projections for which $p_1 + \ldots + p_n = \mathbf{1}$. Then $A \cong (p_1 A p_1)^{n\times n}$.*

Proof. TODO ☐

3.2.3 Semigroups of projections

Let A be a C^*-algebra. We define $\mathcal{P}_\infty(A)$ as

$$\mathcal{P}_n(A) = \mathcal{P}(A^{n\times n}) \qquad \mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A)$$

and equip it with the binary operation $\oplus$:

$$\forall p, q \in \mathcal{P}_\infty(A) : \quad p \oplus q = \operatorname{diag}(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

The involution on $\mathcal{P}_\infty(A)$ is the transposed pointwise application of $*$.

Lemma XIX.28. *Let A be a C^*-algebra.*

1. *If $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, then $p \oplus q \in \mathcal{P}_{n+m}(A)$.*
2. *The operation $\oplus$ is associative, making $\mathcal{P}_\infty(A)$ a semigroup.*
3. *If $p, q, r, p + q \in \mathcal{P}_\infty(A)$, then*

$$(p + q) \oplus r = p \oplus r + q \oplus r \qquad \text{and} \qquad r \oplus (p + q) = r \oplus p + r \oplus q.$$

Proof. The first point follows from

$$(p \oplus q)^* = \begin{pmatrix} p^* & 0 \\ 0 & q^* \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = p \oplus q$$

and

$$(p \oplus q)^2 = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} p^2 & 0 \\ 0 & q^2 \end{pmatrix} = p \oplus q.$$

The others are easy. ☐

We define a relation $\sim_0$ on $\mathcal{P}_\infty(A)$ as follows: for $p \in \mathcal{P}_n(A)$ and $p \in \mathcal{P}_m(A)$,

$$p \sim_0 q \quad \Leftrightarrow_{\text{def}} \quad \exists v \in A^{m\times n}: \quad p = v^*v \ \wedge \ q = vv^*.$$

If $m = n$, then $\sim_0$ equivalence is Murray-von Neumann equivalence.

Lemma XIX.29. *The relation $\sim_0$ is an equivalence relation on $\mathcal{P}_\infty(A)$.*

Lemma XIX.30. *Let A be a C^*-algebra and $p, q, r, p', q' \in \mathcal{P}_\infty(A)$. Then*

1. *$p \sim_0 p \oplus 0^{n\times n}$; in particular, $0 \sim_0 0^{n\times n}$;*
2. *if $p \sim_0 p'$ and $q \sim_0 q'$, then $p \oplus q \sim_0 p' \oplus q'$;*
3. *$p \oplus q \sim_0 q \oplus p$;*
4. *if $p, q \in \mathcal{P}_n(A)$ such that $pq = 0$, then $p + q \in \mathcal{P}_n(A)$ and $p + q \sim_0 p \oplus q$.*

Proof. TODO ☐

We set

$$\mathcal{V}(A) = \mathcal{P}_\infty(A)/\sim_0$$

and let $[p]_\mathcal{V}$ denote the equivalence class containing p. We define addition on $\mathcal{V}(A)$ by

$$[p]_\mathcal{V} + [q]_\mathcal{V} = [p \oplus q]_\mathcal{V} \qquad \forall p, q \in \mathcal{P}_\infty(A).$$

Clearly $\mathcal{V}(A)$ is a commutative monoid with identity $[0]_0$. The $\mathcal{V}$ comes from "vector bundle". The addition $[p]_\mathcal{V} + [q]_\mathcal{V}$ is well-defined for all projections p, q. If $p \perp q$, then

$$[p]_\mathcal{V} + [q]_\mathcal{V} = [p + q]_\mathcal{V}$$

by XIX.30. In general this does not work, which is essentially the reason we work with matrices in $\mathcal{P}_\infty(A)$, not just with projections.

Lemma XIX.31. *Then $\mathcal{V}(-) : \mathsf{C^*alg} \to \mathsf{CMon}$ is a functor that sends morphisms $f : A \to B$ in $\mathsf{C^*alg}$, i.e. $*$-homomorphisms, to*

$$\mathcal{V}(f) : \mathcal{V}(A) \to \mathcal{V}(B) : [p]_\mathcal{V} \mapsto [f(p)]_\mathcal{V}.$$

Proof. We need to check the mapping of morphisms is well-defined. Then the functorial properties are immediate.
First we note that $*$-homomorphisms map projections to projections.
Let $[p]_\mathcal{V} = [q]_\mathcal{V}$. Then $p \sim_0 q$ and thus $\exists v \in A^{m\times n}$ such that $p = v^*v$ and $q = vv^*$. Thus

$$f(p) = f(v^*v) = f(v)^*f(v) \qquad \text{and} \qquad f(q) = f(vv^*) = f(v)f(v)^*.$$

So $f(p) \sim_0 f(q)$ and thus $[f(p)]_\mathcal{V} = [f(q)]_\mathcal{V}$, meaning the mapping of morphisms is well-defined. ☐

Define a relation $\sim_s$ on $\mathcal{P}_\infty(A)$ as follows: for $p, q \in \mathcal{P}_\infty(A)$,

$$p \sim_s q \quad \Leftrightarrow_{\text{def}} \quad \exists r \in \mathcal{P}_\infty : \quad p \oplus r \sim_0 q \oplus r.$$

The relation $\sim_s$ is called stable equivalence.

Lemma XIX.32. *Let A be unital. Then for all $p, q \in \mathcal{P}_\infty(A)$*

$$p \sim_s q \iff p \oplus \mathbf{1}_n \sim_0 q \oplus \mathbf{1}_n$$

for some integer n.

Proof. Assume $p \sim_s q$, so $p \oplus r \sim_0 q \oplus r$ for some $r \in \mathcal{P}_n(A)$. By XII.9, we know $(\mathbf{1} - r)$ is a projection. By the second point of XIX.30, we have $p \oplus r \oplus (\mathbf{1}_n - r) \sim_0 q \oplus r \oplus (\mathbf{1}_n - r)$. Now $r(\mathbf{1}_n - r) = r - r = 0$, so $r \oplus (\mathbf{1}_- r) \sim_0 r \oplus (\mathbf{1}_n - r)$ by the fourth point of XIX.30. By the second point

$$p \oplus \mathbf{1}_n \sim_0 p \oplus r \oplus (\mathbf{1}_n - r) \sim_0 q \oplus r \oplus (\mathbf{1}_n - r) \sim_0 q \oplus \mathbf{1}_n.$$

The converse is immediate. □

3.3 The K_{00} functor

Let A be a C^*-algebra. Then we define the functor K_{00} as the composition of two functors

$$K_{00} = G \circ \mathcal{V} : \mathsf{C^*alg} \to \mathsf{Ab}$$

and let $[\cdot]_0$ be the mapping

$$\mathcal{P}_\infty(A) \xrightarrow{[\cdot]_\mathcal{V}} \mathcal{V}(A) \xrightarrow{g_0} K_{00}(A)$$

where g_0 is the Grothendieck map $x \mapsto (x, 0)$.

Note that $[\cdot]_0$ is not surjective and g_0 is not necessarily injective.

Proposition XIX.33 (The standard picture of K_{00})**.** *Let A be a C^*-algebra, then*

$$\begin{aligned} K_{00}(A) &= \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_\infty(A)\} \\ &= \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_n(A), n \in \mathbb{N}\}. \end{aligned}$$

Moreover,

1. $[p \oplus q]_0 = [p]_0 + [q]_0$ *for all projections $p, q \in \mathcal{P}_\infty(A)$;*
2. $[0_A]_0 = 0$*;*
3. *if $p, q \in \mathcal{P}_n(A)$ and $p \sim_h q \in \mathcal{P}_n(A)$, then $[p]_0 = [q]_0$.*

and

4. *if $p, q \in \mathcal{P}_n(A)$ and $pq = 0$, then $[p + q]_0 = [p]_0 + [q]_0$;*
5. *for all $p, q \in \mathcal{P}_\infty(A)$, $[p]_0 = [q]_0 \iff p \sim_s q$.*

For $*$-homomorphisms $f, g : K_{00}(A) \to K_{00}(B)$:

1. $K_{00}(f)([p]_0) = [f(p)]_0$*;*

2. $\forall p \in \mathcal{P}_\infty(A) : f([p]_0) = g([p]_0) \implies f = g$*;*

3. If f, g *are orthogonal, i.e. for all* $a \in A$*:* $f(a) \cdot g(a) = 0$*, then*

$$K_{00}(f + g) = K_{00}(f) + K_{00}(g).$$

Proof. The first equality is a property of the Grothendieck map. For the second equality, take $p \in \mathcal{P}_m(A)$ and $p \in \mathcal{P}_n(A)$, then $[p]_0 - [q]_0 = [p \oplus 0^{n\times n}]_0 - [q \oplus^{m\times m}]_0$. Then

1. This follows because $[p \oplus q]_\mathcal{V} = [p]_\mathcal{V} + [q]_\mathcal{V}$ and the Grothendieck map is a homomorphism. TODO refs.
2. $[0_A]_0 + [0_A]_0 = [0_A \oplus 0_A]_0 = [0_A]_0$, since $0_A \oplus 0_A \sim_0 0_A$.
3. By XIX.26,

$$p \sim_h q \implies p \sim q \implies p \sim_0 q \implies [p]_\mathcal{V} = [q]_\mathcal{V} \implies [p]_0 = [q]_0.$$

and

4. This is just point (4) of XIX.30 combined $[p \oplus q]_0 = [p]_0 + [q]_0$.
5. If $p \sim_s q$, then $p \oplus r \sim_0 q \oplus r$, so $[p]_0 + [r]_0 = [q]_0 + [r]_0$ and $[p]_0 = [q]_0$ because $K_{00}(A)$ is a group.

   Conversely, if $[p]_0 = [q]_0$, then there is an $[r]_\mathcal{V}$ such that $[p]_\mathcal{V} + [r]_\mathcal{V} = [q]_\mathcal{V} + [r]_\mathcal{V}$. Hence

$$[p \oplus r]_\mathcal{V} = [q \oplus r]_\mathcal{V} \implies p \oplus r \sim_0 q \oplus r \implies p \sim_s q.$$

   Note that we cannot conclude $p \sim_0 q$ from $[p]_0 = [q]_0$, because the Grothedieck map is not necessarily injective.

For the morphisms,

1. $K_{00}(f)([p]_0) = G(\mathcal{V}(f))([p]_0) = \mathcal{V}(f)([p]_0) = [f(p)]_0$.
2. Assume $\forall p \in \mathcal{P}_\infty(A) : f([p]_0) = g([p]_0)$. Take an arbitrary $[p]_0 - [q]_0 \in K_{00}(A)$, then

$$f([p]_0 - [q]_0) = f([p]_0) - f([q]_0) = g([p]_0) - g([q]_0) = g([p]_0 - [q]_0).$$

3. For all $p \in \mathcal{P}_\infty(A)$:

$$K_{00}(f + g)([p]_0) = K_{00}([f(p) + g(p)]_0) = K_{00}([f(p)]_0 + [g(p)]_0) = (K_{00}(f) + K_{00}(g))([p]_0)$$

   where we have used point (4) of XIX.30.

□

Proposition XIX.34 (Universal property of K_{00})**.** *Let* A *be a* C^**-algebra and* G *an Abelian group. Suppose* $\nu : \mathcal{P}_\infty(A) \to G$ *is a function that satisfies*

1. $\nu(p \oplus q) = \nu(p) + \nu(q)$ *for all projections* $p, q \in \mathcal{P}_\infty(A)$*;*

2. $\nu(0_A) = 0$*;*

3. if $p, q \in \mathcal{P}_n(A)$ *and* $p \sim_h q \in \mathcal{P}_n(A)$*, then* $\nu(p) = \nu(q)$*.*

Then there is a unique group homomorphism $\alpha : K_{00}(A) \to G$ which makes the diagram

$$\begin{tikzcd} \mathcal{P}_\infty(A) \arrow[d, "{[\cdot]_0}"] \arrow[dr, "\nu"] & \\ K_{00}(A) \arrow[r, dashed, "\exists!\alpha"'] & G \end{tikzcd}$$

commute.

In the third point we can also use $\sim_u, \sim_0$ or $\sim_s$:

Proposition XIX.35. *Let A be a C^*-algebra, G an Abelian group, and $\nu : \mathcal{P}_\infty(A) \to G$ a function that satisfies $\nu(0_A) = 0$ and $\nu(p \oplus q) = \nu(p) + \nu(q)$ for all projections $p, q \in \mathcal{P}_\infty(A)$. Then the following are equivalent:*

3. *for all n and all $p, q \in \mathcal{P}_n(A)$: if $p \sim_h q$, then $\nu(p) = \nu(q)$;*

3′. *for all n and all $p, q \in \mathcal{P}_n(A)$: if $p \sim_u q$, then $\nu(p) = \nu(q)$;*

3″. *for all $p, q \in \mathcal{P}_\infty(A)$: if $p \sim_0 q$, then $\nu(p) = \nu(q)$;*

3‴. *for all $p, q \in \mathcal{P}_\infty(A)$: if $p \sim_s q$, then $\nu(p) = \nu(q)$;*

Proof. We cyclically prove $(3''') \Rightarrow (3'') \Rightarrow (3') \Rightarrow (3) \Rightarrow (3''')$:

$\boxed{(3''') \Rightarrow (3'')}$ Assume $(3''')$ and take arbitrary $p, q \in \mathcal{P}_\infty(A)$ such that $p \sim_0 q$. Then $p \oplus 0 \sim_0 q \oplus 0$, so $p \sim_s q$ and $\nu(p) = \nu(q)$ by $(3''')$.

$\boxed{(3'') \Rightarrow (3')}$ Assume $(3'')$ and take arbitrary $p, q \in \mathcal{P}_n(A)$ for some n such that $p \sim_u q$. Then $p \sim q$ by XIX.26 and thus $p \sim_0 q$, so $\nu(p) = \nu(q)$ by $(3'')$.

$\boxed{(3') \Rightarrow (3)}$ Assume $(3')$ and take arbitrary $p, q \in \mathcal{P}_n(A)$ for some n such that $p \sim_h q$. Then $p \sim_u q$ by XIX.26, so $\nu(p) = \nu(q)$ by $(3')$.

$\boxed{(3) \Rightarrow (3''')}$ Assume (3) and take arbitrary $p, q \in \mathcal{P}_\infty(A)$ such that $p \sim_s q$. Then there exists an $r \in \mathcal{P}_\infty(A)$ such that $p \oplus r \sim_0 q \oplus r$. If $p \oplus r \in \mathcal{P}_m$ and $q \oplus r \in \mathcal{P}_n$, then

$$p \oplus r \oplus 0^{n\times n} \sim_0 p \oplus r \sim_0 q \oplus r \sim_0 q \oplus r \oplus 0^{m\times m}.$$

And since both sides are in $\mathcal{P}_{n+m}(A)$, we have $p \oplus r \oplus 0^{n\times n} \sim q \oplus r \oplus 0^{m\times m}$. By XIX.26 this implies

$$p \oplus r \oplus 0^{n\times n} \oplus 0^{(m+n)\times(m+n)} \sim_h q \oplus r \oplus 0^{m\times m} \oplus 0^{(m+n)\times(m+n)}.$$

Using (3) this gives

$$\nu(p)+\nu(r) = \nu(p\oplus r\oplus 0^{n\times n}\oplus 0^{(m+n)\times(m+n)}) = \nu(q\oplus r\oplus 0^{m\times m}\oplus 0^{(m+n)\times(m+n)}) = \nu(q)+\nu(r)$$

which implies $\nu(p) = \nu(q)$.

□

Proposition XIX.36 (Homotopy invariance of K_{00})**.** *Let A, B be C^*-algebras. If $\varphi, \psi : A \to B$ are homotopic $*$-homomorphisms, then*

$$K_{00}(\varphi) = K_{00}(\psi).$$

Proof. For every $p \in \mathcal{P}_\infty(A)$, $\varphi(p) \sim_h \psi(p)$. So

$$K_{00}(\varphi)([p]_0) = [\varphi(p)]_0 = [\psi(p)]_0 = K_{00}(\psi)([p]_0),$$

meaning $K_{00}(\varphi) = K_{00}(\psi)$ $\square$

Corollary XIX.36.1. *If A and B are homotopy equivalent, then $K_{00}(A) \cong K_{00}(B)$.*

Proof. If $\psi \circ \varphi \sim_h I_A$, then

$$K_{00}(\psi) \circ K_{00}(\varphi) = K_{00}(I_A) = I_{K_{00}(A)}.$$

Similarly $\varphi \circ \psi \sim_h I_B$ implies

$$K_{00}(\varphi) \circ K_{00}(\psi) = K_{00}(I_B) = I_{K_{00}(B)}.$$

So $K_{00}(\varphi) : A \to B$ is invertible with inverse $K_{00}(\psi)$. $\square$

Proposition XIX.37. *The functor K_{00} is not half exact for non-unital C^*-algebras.*

This is essentially the motivation to work with K_0, not K_{00}.

3.4 The K_0 functor

Let A be a C^*-algebra and $\pi : A^\dagger \to \mathbb{C}$ the projection of the second component of $A^\dagger$onto $\mathbb{C}$. Then we define $K_0(A)$ as the kernel of $K_{00}(\pi) : K_{00}(A^\dagger) \to K_{00}(\mathbb{C})$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "\iota"] & A^\dagger \arrow[r, shift left, "\pi"] & \mathbb{C} \arrow[l, shift left, hook', "\lambda"] \arrow[r] & 0 \end{tikzcd}$$

Proposition XIX.38. *Let A be a C^*-algebra, then $K_{00}(\iota)$ in an embedding $K_{00}(\iota) : K_{00}(A) \hookrightarrow K_0(A)$.*

Proof. To show injectivity, it is enough to note that ι is split monic. Then $K_{00}(\iota)$ is also split monic in the category Ab and thus injective.
To show the image is a subset of $K_0(A)$, take some arbitrary $[p]_0 - [q]_0 \in K_{00}(A)$ with $p, q \in \mathcal{P}_\infty(A)$. Then we claim

$$K_{00}(\iota)([p]_0 - [q]_0) = K_{00}(\iota)([p]_0) - K_{00}(\iota)([q]_0) = [\iota(p)]_0 - [\iota(q)]_0$$

maps to zero under $K_{00}(\pi)$. Indeed:

$$K_{00}(\pi)([\iota(p)]_0 - [\iota(q)]_0) = K_{00}(\pi)([\iota(p)]_0) - K_{00}(\pi)([\iota(q)]_0) = [\pi(\iota(p))]_0 - [\pi(\iota(q))]_0 = 0$$

using that fact that $\pi \circ \iota = 0$. So $K_{00}(\iota)([p]_0 - [q]_0) \in \ker(K_{00}(\pi)) = K_0(A)$. $\square$

So we can identify $K_{00}(A) \cong \operatorname{im} K_{00}(\iota) \subseteq K_0(A)$ and we can naturally extend $[\cdot]_0$ to a function

$$\mathcal{P}_\infty(A) \to K_{00}(A) \hookrightarrow K_0(A).$$

Proposition XIX.39. *Let A be a unital C^*-algebra. Then $K_{00}(A) \cong K_0(A)$.*

Proof. By XIX.38 we have $K_{00}(\iota) : K_{00}(A) \hookrightarrow K_0(A)$. We just need to show $K_{00}(\iota)$ is surjective. To do this we are going to decompose $I_{A^\dagger}$ into the form

$$I_{A^\dagger} = \iota \circ \mu + \mu' \circ \pi$$

with the property that $\iota \circ \mu \cdot \mu' \circ \pi$ is the zero map, i.e. $\iota \circ \mu$ and $\mu' \circ \pi$ are orthogonal. Such a decomposition is given by

$$\mu : A^\dagger \to A : (a, \alpha) \mapsto a + \alpha \qquad \text{and} \qquad \mu' : \mathbb{C} \to A^\dagger : \alpha \mapsto (-\alpha, \alpha),$$

which works because A is unital.
We verify

$$\begin{aligned} &(\iota \circ \mu)(a, \alpha) + (\mu' \circ \pi)(a, \alpha) = (a + \alpha, 0) + (-\alpha, a\alpha) = (a, \alpha) \\ &(\iota \circ \mu)(a, \alpha) \cdot (\mu' \circ \pi)(a, \alpha) = (a + \alpha, 0) \cdot (-\alpha, \alpha) = (-\alpha(a + \alpha) + \alpha(a + \alpha) + 0, 0) = (0, 0). \end{aligned}$$

Then take some $s \in K_0(A) = \ker(K_{00}(\pi))$. We calculate

$$\begin{aligned} s &= I_{K_{00}(A^\dagger)}(s) = K_{00}(I_{A^\dagger})(s) = K_{00}(\iota \circ \mu + \mu' \circ \pi)(s) \\ &= K_{00}(\iota \circ \mu)(s) + K_{00}(\mu' \circ \pi)(s) = (K_{00}(\iota) \circ K_{00}(\mu))(s) + (K_{00}(\mu') \circ K_{00}(\pi))(s) \\ &= K_{00}(\iota) \circ K_{00}(\mu))(s) + 0 \in \operatorname{im} K_{00}(\iota). \end{aligned}$$

□

For this reason $K_0(A)$ is often defined as $K_{00}(A)$ for unital algebras.

Lemma XIX.40. *Let A, B be C^*-algebras. For every morphism $f : A \to B$, we can view $K_{00}(f)$ as a morphism on a subgroup of $K_0(A)$ by identifying it with*

$$K_{00}(\iota) K_{00}(f) K_{00}(\iota)^{-1}.$$

Then $K_{00}(f)$ can uniquely be extended to a morphism $K_0(f) : K_0(A) \to K_0(B)$.
This makes K_0 a functor.

Proof. Consider the diagram

$$\begin{tikzcd} K_{00}(A) \arrow[r, "K_{00}(\iota_A)"] \arrow[d, "K_{00}(f)"] & K_0(A) \arrow[r, "\subseteq"] \arrow[d, dashed, "K_0(f)"] & K_{00}(A^\dagger) \arrow[r, "K_{00}(\pi_A)"] \arrow[d, "K_{00}(f^\dagger)"] & K_{00}(\mathbb{C}) \arrow[d, equal] \\ K_{00}(B) \arrow[r, "K_{00}(\iota_B)"'] & K_0(B) \arrow[r, "\subseteq"'] & K_{00}(B^\dagger) \arrow[r, "K_{00}(\pi_B)"'] & K_{00}(\mathbb{C}) \end{tikzcd}$$

which is commutative because functors preserve commutative diagrams. Uniqueness is immediate from the commutativity of the middle square. To show existence, we must show that

$$K_{00}(f^\dagger)[K_0(A)] \subseteq K_0(B).$$

Take $r \in K_0(A) = \ker K_{00}(\pi_A)$. Then using the fact that $f^\dagger$ commutes with π, i.e.

$$\pi_B \circ f^\dagger = f^\dagger \circ \pi_A,$$

we get

$$(K_{00}(\pi_B) \circ K_{00}(f^\dagger))(r) = (K_{00}(f^\dagger) \circ K_{00}(\pi_A))(r) = K_{00}(f^\dagger)(0) = 0.$$

So $K_{00}(f^\dagger)(r) \in \ker K_{00}(\pi_B) = K_0(B)$. □

Proposition XIX.41 (The standard picture of K_0)**.** *Let A be a C^*-algebra, then*

$$K_0(A) = \left\{[p]_0 - [s(p)]_0 \;\middle|\; p \in \mathcal{P}_\infty(A^\dagger)\right\}.$$

Moreover, for all $p, q \in \mathcal{P}_\infty(A^\dagger)$, the following are equivalent:

1. $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$,
2. $p \oplus \mathbf{1}_k \sim_0 q \oplus \mathbf{1}_l$ *in $\mathcal{P}_\infty(A^\dagger)$ for some $k, l \in \mathbb{N}$,*
3. *there exist scalar projections r_1, r_2 such that $p \oplus r_1 \sim_0 q \oplus r_2$ in $\mathcal{P}_\infty(A^\dagger)$.*

If $\varphi : A \to B$ is a $*$-homomorphism, then

$$K_0(\varphi)([p]_0 - [s(p)]_0) = [\varphi^\dagger(p)]_0 - [s(\varphi^\dagger(p))]_0$$

for all $p \in \mathcal{P}_\infty(A^\dagger)$.

Proof. For all $p \in \mathcal{P}_\infty(A^\dagger)$, $[p]_0 - [s(p)]_0$ is in $K_0(A) = \ker K_{00}(\pi)$:

$$K_{00}(\pi)([p]_0 - [s(p)]_0) = [\pi(p)]_0 - [(\pi \circ s)(p)]_0 = [\pi(p)]_0 - [\pi(p)]_0 = 0.$$

Conversely, let $g \in K_0(A)$, then by the standard picture of $K_{00}(A^\dagger)$ there are $e, f \in \mathcal{P}((A^\dagger)^{n\times n})$ such that $g = [e]_0 - [f]_0$. Put

$$p = \begin{pmatrix} e & 0 \\ 0 & \mathbf{1}_n - f \end{pmatrix}, \qquad q = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}.$$

Then $(\mathbf{1}_n - f)$ and f are orthogonal,

$$(\mathbf{1}_n - f)f = f - f^2 = f - f = 0.$$

So, by XIX.33, $[\mathbf{1}_n]_0 = [\mathbf{1}_n - f + f]_0 = [\mathbf{1}_n - f]_0 + [f]_0$ and we get

$$[p]_0 - [q]_0 = [e]_0 + [\mathbf{1}_n - f]_0 - [\mathbf{1}_n]_0 = [e]_0 - [f]_0 = g.$$

Using $q = s(q)$ and $K_{00}(\pi)(g) = 0$, we get

$$[s(p)]_0 - [q]_0 = [s(p)]_0 - [s(q)]_0 = K_{00}(s)(g) = (K_{00}(\lambda) \circ K_{00}(\pi))(g) = 0.$$

So $[s(p)]_0 = [q]_0$ and we get $g = [p]_0 - [s(p)]_0$.
We prove the equivalent statements cyclically:

$\boxed{(1) \Rightarrow (3)}$ Suppose $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$ for some $p, q \in \mathcal{P}_\infty(A^\dagger)$ this implies

$$[p \oplus s(q)]_0 = [q \oplus s(p)]_0 \implies p \oplus s(q) \sim_s q \oplus s(p) \qquad \text{in } \mathcal{P}_\infty(A^\dagger)$$

by XIX.33. By XIX.32 this implies $p \oplus s(q) \oplus \mathbf{1}_n \sim_0 q \oplus s(p) \oplus \mathbf{1}_n$. Putting $r_1 = s(q) \oplus \mathbf{1}_n$ and $r_2 = s(p) \oplus \mathbf{1}_n$, which are scalar projections, this is exactly (3).

$\boxed{(3) \Rightarrow (2)}$ If r_1 is a scalar projection in $\mathcal{P}_k(A)$ and r_2 in $\mathcal{P}_l(A)$, then $r_1 \sim_0 \mathbf{1}_k$ and $r_2 \sim_0 \mathbf{1}_l$, by TODO ref. Hence $p \oplus \mathbf{1}_k \sim_0 q \oplus \mathbf{1}_l$.

$\boxed{(2) \Rightarrow (1)}$

□

Proposition XIX.42 (Half exactness of K_0)**.** *Every short exact sequence of C^*-algebras*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\varphi"] & A \arrow[r, "\psi"] & B \arrow[r] & 0 \end{tikzcd}$$

induces an exact sequence of Abelian groups

$$\begin{tikzcd} K_0(I) \arrow[r, "K_0(\varphi)"] & K_0(A) \arrow[r, "K_0(\psi)"] & K_0(B). \end{tikzcd}$$

Proposition XIX.43. *Every split exact sequence of C^*-algebras*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\varphi"] & A \arrow[r, shift left, "\psi"] & B \arrow[l, shift left, "\lambda"] \arrow[r] & 0 \end{tikzcd}$$

induces a split exact sequence of Abelian groups

$$\begin{tikzcd} 0 \arrow[r] & K_0(I) \arrow[r, "K_0(\varphi)"] & K_0(A) \arrow[r, shift left, "K_0(\psi)"] & K_0(B) \arrow[l, shift left, "K_0(\lambda)"] \arrow[r] & 0 \end{tikzcd}$$

Proposition XIX.44. *For every pair A, B of C^*-algebras,*

$$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B).$$

3.4.1 Homotopy, suspensions and cones

Proposition XIX.45 (Homotopy invariance of K_0)**.** *Let A, B be C^*-algebras. If $\varphi, \psi : A \to B$ are homotopic $*$-homomorphisms, then*

$$K_0(\varphi) = K_0(\psi).$$

If A and B are homotopy equivalent, then $K_0(A) \cong K_0(B)$.

Proof. If φ is homotopic to ψ, then $\varphi^\dagger$ is homotopic to $\psi^\dagger$ and thus $K_{00}(\varphi^\dagger) = K_{00}(\psi^\dagger)$, by XIX.36. Restricting to $K_0(A)$ yields the result. □

In particular, $K_0(A) = 0$ for every contractible C^*-algebra A.

Let A be a C^*-algebra. The cone over A is

$$CA \coloneqq \{ f \in C([0,1], A) \mid f(0) = 0 \}.$$

The suspension of A is

$$SA \coloneqq \{ f \in C([0,1], A) \mid f(0) = f(1) = 0 \}.$$

Lemma XIX.46. *If the operations are pointwise and the norm the supremum norm, then CA and SA are C^*-algebras.*

Lemma XIX.47. *Let A be a C^*-algebra. The cone CA is contractible. The suspension SA is contractible if A is contractible.*

Proof. Let $\gamma_t : CA \to CA$ be defined by $\gamma_t(f)(s) = f(st)$ for all $t \in [0,1]$. Then γ defines a contraction of CA.
For any contraction $\beta_t : A \to A$ of A, $\gamma_t : SA \to SA : f \mapsto \beta_t \circ f$ is a contraction of SA. ☐

Lemma XIX.48. *Let A be a C^*-algebra. Then SA is a closed ideal of CA and $A \cong CA/SA$. We have the short exact sequence*

$$0 \longrightarrow SA \xhookrightarrow{\iota} CA \longrightarrow A \longrightarrow 0.$$

Proof. Consider the map

$$CA \to A : f \mapsto f(1).$$

This is a surjective morphism with kernel SA. So SA is a closed ideal by ref TODO. ☐

Let A, B be C^*-algebras and $\alpha : A \to B$ a morphism. The mapping cone for α is

$$C_\alpha \coloneqq \{(a,f) \in A \oplus CB \mid f(1) = \alpha(a)\}.$$

The mapping cone is a C^*-algebra.

Lemma XIX.49. *The mapping cone α is related to A and B in the short exact sequence*

$$0 \longrightarrow SB \xrightarrow{\iota: f \mapsto (0,f)} C_\alpha \xrightarrow{\pi:(a,f)\mapsto a} A \longrightarrow 0.$$

Moreover, the sequence

$$K_0(C_\alpha) \xrightarrow{\pi_*} K_0(A) \xrightarrow{\alpha_*} K_0(B)$$

is exact.

Proof. TODO ☐

3.5 The K_1 functor

As with the projections, we define, for some unital C^*-algebra,

$$\mathcal{U}_n(A) = \mathcal{U}(A^{n\times n}), \qquad \mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A)$$

as well as the binary operations $\oplus$ on $\mathcal{U}_\infty(A)$

$$\forall u, v \in \mathcal{U}_\infty(A) : \quad u \oplus v = \operatorname{diag}(u,v) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

Lemma XIX.50. *Let A be a unital C^*-algebra. If $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, then $u \oplus v \in \mathcal{U}_{n+m}(A)$.*

3.5.1 Normalised matrices

The group of normalised invertible matrices is

$$\mathrm{GL}_n^\dagger(A) \coloneqq \left\{a \in \mathrm{GL}_n(A^\dagger) \;\middle|\; \pi(a) = \mathbf{1}_n\right\}$$

and the group of normalised unitary matrices is

$$\mathcal{U}_n^\dagger(A) \coloneqq \left\{u \in \mathcal{U}_n(A^\dagger) \;\middle|\; \pi(u) = \mathbf{1}_n\right\}.$$

The group operation for both is multiplication. We also define

$$\mathrm{GL}_\infty^\dagger(A) \coloneqq \bigcup_{n=1}^\infty \mathrm{GL}_n^\dagger(A) \qquad \text{and} \qquad \mathcal{U}_\infty^\dagger(A) \coloneqq \bigcup_{n=1}^\infty \mathcal{U}_n^\dagger(A).$$

Proposition XIX.51. *Let A be a C^*-algebra and $n \in \mathbb{N} \cup \{\infty\}$. Then the groups*

$$\begin{array}{rr} \mathrm{GL}_n^\dagger(A)/\mathrm{GL}_n^\dagger(A)_0, & \qquad \mathcal{U}_n^\dagger(A)/\mathcal{U}_n^\dagger(A)_0 \\ \mathrm{GL}_n(A^\dagger)/\mathrm{GL}_n(A^\dagger)_0, & \qquad \mathcal{U}_n(A^\dagger)/\mathcal{U}_n(A^\dagger)_0 \end{array}$$

are pairwise isomorphic. If A is unital, then

$$\mathrm{GL}_n^\dagger(A) \cong \mathrm{GL}_n(A) \qquad \textit{and} \qquad \mathcal{U}_n^\dagger(A) \cong \mathcal{U}_n(A).$$

Proof. First note that by XIX.15, all quotients are normal subgroups.
Consider the continuous map

$$\phi : \mathrm{GL}_n^\dagger(A) \to \mathcal{U}_n^\dagger(A) : z \mapsto z|z|^{-1}.$$

The induced map

$$\psi : \mathrm{GL}_n^\dagger(A)/\mathrm{GL}_n^\dagger(A)_0 \to \mathcal{U}_n^\dagger(A)/\mathcal{U}_n^\dagger(A)_0 : [z] \mapsto [\phi(z)]$$

is a group homomorphism by (TODO ref) and is bijective: it is clearly surjective. For injectivity, we prove the kernel is trivial. Indeed let $[\phi(z)] = \mathbf{1}$, then $z|z|^{-1} \sim_h \mathbf{1}_n$. By XIX.18, $z|z|^{-1} \sim_h z$, so $z \sim_h \mathbf{1}_n$ by transitivity.
To prove $\mathrm{GL}_n(A^\dagger)/\mathrm{GL}_n(A^\dagger)_0 \cong \mathrm{GL}_n^\dagger(A)/\mathrm{GL}_n^\dagger(A)_0$, consider the isomorphism $[z] \mapsto [z\pi(z^{-1})]$.
Restricting to unitaries gives the last isomorphism. ☐

3.5.2 Equivalence of unitaries

We define a relation $\sim_1$ on $\mathcal{U}_\infty(A)$ as follows: for $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$,

$$u \sim_1 v \quad \Leftrightarrow_{\text{def}} \quad \exists k \geq \max\{m,n\} : \quad u \oplus \mathbf{1}_{k-n} \sim_h v \oplus \mathbf{1}_{k-m} \quad \text{in } \mathcal{U}_k(A).$$

With the convention that $w \oplus 1_0 = w$ for all $w \in \mathcal{U}_\infty(A)$.

Lemma XIX.52. *Let A be a unital C^*-algebra. Then for all $u, v, u', v' \in \mathcal{U}(A)$*

1. *$\sim_1$ is an equivalence relation on $\mathcal{U}_\infty(A)$;*
2. *$u \sim_1 u \oplus \mathbf{1}_k$*

3. $u \oplus v \sim_1 v \oplus u$;

4. if $u \sim_1 u'$ and $v \sim_1 v'$, then $u \oplus v \sim_1 u' \oplus v'$;

5. if $u, v \in \mathcal{U}_n(A)$, then $uv \sim_1 vu \sim_1 u \oplus v$.

3.5.3 The K_1 functor

For each C^*-algebra A, we define

$$K_1(A) = \mathcal{U}_\infty(A^\dagger)/\sim_1 .$$

Define the binary operation $+$ on $K_1(A)$ by $[u]_1 + [v]_1 = [u \oplus v]_1$.

Because

$$[u]_1 + [v]_1 = [u \oplus v]_1 = [uv]_1,$$

the $K_1(A)$ group is naturally a multiplicative group that we are writing in additive notation for uniformity with other K groups. Notice in particular that

$$[\mathbf{1}]_1 = 0.$$

In fact we could have defined

$$[u]_1 + [v]_1 = [uv]_1,$$

so that there is less dependence on matrices. This is different than for projections where the definition

$$[p]_0 + [q]_0 = [p + q]_0$$

did not work because $p + q$ was not necessarily a projection.

Proposition XIX.53. *Let A be a C^*-algebra. Then $K_1(A)$ is isomorphic to any of the following:*

$$\begin{aligned} &\mathrm{GL}^\dagger_\infty(A)/\,\mathrm{GL}^\dagger_\infty(A)_0, \qquad && \mathcal{U}^\dagger_\infty(A)/\mathcal{U}^\dagger_\infty(A)_0 \\ &\mathrm{GL}_\infty(A^\dagger)/\,\mathrm{GL}_\infty(A^\dagger)_0, && \mathcal{U}_\infty(A^\dagger)/\mathcal{U}_\infty(A^\dagger)_0. \end{aligned}$$

Proof. The four groups are isomorphic by XIX.51. Consider $\mathcal{U}_\infty(A^\dagger)/\mathcal{U}_\infty(A^\dagger)_0$. We just need to see that $\forall u \in \mathcal{U}_\infty(A^\dagger)$

$$u \sim_1 \mathbf{1} \iff u \in \mathcal{U}_\infty(A^\dagger)_0$$

by TODO ref. This is clear. □

Proposition XIX.54 (Universal property of K_1)**.** *Let A be a C^*-algebra and G an Abelian group. Suppose $\nu : \mathcal{U}_\infty(A^\dagger) \to G$ is a function that satisfies*

1. $\nu(u \oplus v) = \nu(u) + \nu(v)$ *for all unitaries* $u, v \in \mathcal{U}_\infty(A^\dagger)$;

2. $\nu(\mathbf{1}_A) = 0$;

3. *if* $u, v \in \mathcal{U}_n(A^\dagger)$ *and* $u \sim_h v \in \mathcal{U}_n(A^\dagger)$, *then* $\nu(p) = \nu(q)$.

Then there is a unique group homomorphism $\alpha : K_1(A) \to G$ which makes the diagram

$$\begin{tikzcd}
\mathcal{U}_\infty(A^\dagger) \arrow[d, "{[\cdot]_1}"] \arrow[dr, "\nu"] & \\
K_1(A) \arrow[r, dashed, "\exists!\alpha"'] & G
\end{tikzcd} \qquad \textit{commute.}$$

The functor K_1 is homotopy invariant, half exact, split exact and respects direct sums.

3.6 Exact sequences of K-groups

3.6.1 Suspensions

Lemma XIX.55. *Let A be a C^*-algebra. We have the isomorphisms*

$$\begin{aligned} SA &\cong A \otimes C_0(\mathbb{R}) \\ &\cong C_0(\mathbb{R}, A) \\ &\cong C_0(]0,1[, A) \\ &\cong \{f \in C(\mathbb{T}, A) \mid f(1) = 0\}. \end{aligned}$$

Lemma XIX.56. *Suspension is a functor $S : \mathsf{C^*alg} \to \mathsf{C^*alg}$.*

Proof. We know the suspension maps C^*-algebras to C^*-algebras. Let $f : A \to B$ be a $*$-homomorphism. Then $f_* : SA \to SB$ is well-defined and a $*$-homomorphism. The functorial properties are clearly satisfied. ☐

Theorem XIX.57. *The functors K_1 and $K_0 \circ S$ are naturally isomorphic.*

Proof. For every C^*-algebra A we define ☐

3.6.2 The index map

Suppose a short exact sequence of C^*-algebras

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\varphi"] & A \arrow[r, "\psi"] & B \arrow[r] & 0 \end{tikzcd}.$$

Then we want to define a map $\delta_1 : K_1(B) \to K_0(I)$, called the index map, such that the sequence

$$\begin{tikzcd}
K_1(I) \arrow[r, "K_1(\varphi)"] & K_1(A) \arrow[r, "K_1(\psi)"] & K_1(B) \arrow[d, "\delta_1"] \\
K_0(B) & K_0(A) \arrow[l, "K_0(\psi)"] & K_0(I) \arrow[l, "K_0(\varphi)"]
\end{tikzcd}$$

is exact.

3.6.2.1 Constructing the index map

Take an element $[u] \in K_1(B) = \mathcal{U}_\infty(B^\dagger)/\mathcal{U}_\infty(B^\dagger)_0$. Now the elements of $u \cdot \mathcal{U}_\infty(B^\dagger)_0$ do not in general lift to unitaries in $A^\dagger$.
We wish to measure to what degree this lifting is not possible. We expect such a map to be well-defined, because the elements of $\mathcal{U}_\infty(B^\dagger)_0$ should not impede the lifting, by XIX.17.
To do this, we define a function $\nu' : \mathcal{U}_\infty(B^\dagger) \to \mathcal{P}(I^\dagger)$ such that $\nu := [\cdot]_0 \circ \nu' : \mathcal{U}_\infty(B^\dagger) \to K_0(I)$ satisfies the universal property of the K_1 functor, XIX.54, meaning it uniquely factors through $K_1(B)$, giving a group homomorphism $\delta_1 : K_1(B) \to K_0(I)$ satisfying $\delta_1([u]_1) = \nu(u)$ for each $u \in \mathcal{U}_\infty(B^\dagger)$.
We define the map $\nu' : \mathcal{U}_\infty(B^\dagger) \to \mathcal{P}(I^\dagger)$ as follows:
Take $u \in \mathcal{U}_\infty(B^\dagger)$. First we would like to lift this to a unitary

Lemma XIX.58. *Suppose a short exact sequence of C^*-algebras*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\varphi"] & A \arrow[r, "\psi"] & B \arrow[r] & 0 \end{tikzcd}$$

and let $u \in \mathcal{U}_n(B^\dagger)$.

1. *There exists a unitary $v \in \mathcal{U}_{2n}(A^\dagger)$ and a projection $p \in \mathcal{P}_{2n}(I^\dagger)$ such that*
$$\psi^\dagger(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \qquad \varphi^\dagger(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*, \qquad s(p) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix}.$$
2. *Given these v, p, if $w \in \mathcal{U}_{2n}(A^\dagger)$ and $q \in \mathcal{P}_{2n}(I^\dagger)$ satisfy*
$$\psi^\dagger(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \qquad \varphi^\dagger(q) = w \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w^*,$$
then $s(q) = \operatorname{diag}(\mathbf{1}_n, 0_n)$ and $p \sim_u q$ in $\mathcal{P}_{2n}(I^\dagger)$.

Proof. (1). Because $\operatorname{diag}(u, u^*) \sim_h \operatorname{diag}(1, 1)$, we can use the first point of XIX.17 to see that $\operatorname{diag}(u, u^*)$ lifts to a unitary $v \in (A^\dagger)^{2n \times 2n}$, giving the first equation. Also

$$\psi^\dagger(v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*) = \psi^\dagger(v) \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \psi^\dagger(v^*) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix}.$$

Letting $\pi_1 : A^\dagger \to A$ be the projection as in XII.16.1, this means

$$\pi_1(\psi^\dagger(v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*)) = \psi(\pi_1(v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*)) = 0.$$

By the exactness of the sequence,

$$\pi_1(v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*) \in \operatorname{im}(\varphi),$$

meaning there is a $p \in (I^\dagger)^{2n \times 2n}$ such that $\varphi^\dagger(p) = v \operatorname{diag}(\mathbf{1}_n, 0) v^*$ and p is a projection by XII.51. For the third equality, $s(p) = \psi^\dagger(\varphi^\dagger(p)) = \operatorname{diag}(\mathbf{1}_n, 0)$.
(2). That $s(q) = \operatorname{diag}(\mathbf{1}_n, 0)$ follows from $\psi^\dagger(\varphi^\dagger(p)) = \operatorname{diag}(\mathbf{1}_n, 0)$ as before.
Then $\psi^\dagger(wv^*) = \mathbf{1}_{2n}$, so $\psi(\pi_1(wv^*)) = 0$ and $\pi_1(wv^*) \in \operatorname{im} \varphi$ by exactness. So we can find a $z \in (I^\dagger)^{2n \times 2n}$ such that $\varphi^\dagger(z) = wv^*$ and z is unitary by XII.51. From $\varphi^\dagger(zpz^*) = \varphi^\dagger(q)$ and the injectivity of $\varphi^\dagger$, we get $q = zpz^*$, meaning $p \sim_u q$ in $\mathcal{P}_{2n}(I^\dagger)$. ☐

We use this lemma to define a function

$$\nu : \mathcal{U}_\infty(B^\dagger) \to K_0(I)$$

which maps $u \in \mathcal{U}_\infty(B^\dagger)$ to $\nu(u) = [p]_0 - [s(p)]_0$ where $p \in \mathcal{P}_{2n}(I^\dagger)$ is as in the lemma. This map is well-defined by the lemma.

Lemma XIX.59. *The map $\nu : \mathcal{U}_\infty(B^\dagger) \to K_0(I)$ satisfies the universal property of $K_1(B)$:*

1. *$\nu(u_1 \oplus u_2) = \nu(u_1) + \nu(u_2)$ for all unitaries $u_1, u_2 \in \mathcal{U}_\infty(B^\dagger)$;*
2. *$\nu(\mathbf{1}) = 0$;*
3. *if $u_1, u_2 \in \mathcal{U}_n(B^\dagger)$ and $u_1 \sim_h u_2 \in \mathcal{U}_n(B^\dagger)$, then $\nu(u_1) = \nu(u_2)$.*

Proof. (1). For $j = 1, 2$, let u_j be given. Choose $v_j \in \mathcal{U}_{2n_j}(A^\dagger)$ and $p_j \in \mathcal{P}_{2n_j}(I^\dagger)$ as in the definition of the index map, i.e. $\nu(u_j) = [p_j]_0 - [s(p_j)]_0$. Then introduce

$$y = \begin{pmatrix} \mathbf{1}_{n_1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n_2} & 0 \\ 0 & \mathbf{1}_{n_1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n_2} \end{pmatrix} \in \mathcal{U}_{2(n_1+n_2)}(\mathbb{C})$$

□

Because the map ν satisfies the universal property of the K_1 functor, XIX.54, it uniquely factors throught $K_1(B)$, giving a group homomorphism $\delta_1 : K_1(B) \to K_0(I)$ satisfying $\delta_1([u]_1) = \nu(u)$ for each $u \in \mathcal{U}_\infty(B^\dagger)$.
The map δ_1 is called the index map associated with the short exact sequence.

3.6.2.2 Properties of the index map

With the index map we have the exact sequence:

$$\begin{tikzcd} K_1(I) \arrow[r, "K_1(\varphi)"] & K_1(A) \arrow[r, "K_1(\psi)"] & K_1(B) \arrow[d, "\delta_1"] \\ K_0(B) & K_0(A) \arrow[l, "K_0(\psi)"'] & K_0(I) \arrow[l, "K_0(\varphi)"'] \end{tikzcd}$$

Proposition XIX.60 (Naturality of the index map)**.** *Let*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\varphi"] \arrow[d, "\gamma"] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & I' \arrow[r, "\varphi'"] & A' \arrow[r, "\psi'"] & B' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of short exact rows of C^*-algebras. Let

$$\delta_1 : K_1(B) \to K_0(I) \qquad \text{and} \qquad \delta_1' : K_1(B') \to K_0(I')$$

be the index maps associated with both rows. Then the diagram

$$\begin{tikzcd} K_1(B) \arrow[r, "\delta_1"] \arrow[d, "K_1(\beta)"] & K_0(I) \arrow[d, "K_0(\gamma)"] \\ K_1(B') \arrow[r, "\delta_1'"] & K_0(I') \end{tikzcd} \qquad \textit{commutes.}$$

3.6.3 Higher K-groups

Proposition XIX.61. *There is a natural isomorphism between $K_1(A)$ and $K_0(SA)$.*

Proof. From the short exact sequence, XIX.48

$$0 \longrightarrow SA \longrightarrow CA \longrightarrow A \longrightarrow 0.$$

we get the long exact sequence, ref TODO.

$$\begin{tikzcd} K_1(SA) \arrow[r] & K_1(CA) \arrow[r] & K_1(A) \arrow[r] & K_0(SA) \arrow[r] & K_0(CA) \arrow[r] & K_0(A) \end{tikzcd} .$$

Because CA is contractible, we have

$$\begin{tikzcd} K_1(SA) \arrow[r] & 0 \arrow[r] & K_1(A) \arrow[r] & K_0(SA) \arrow[r] & 0 \arrow[r] & K_0(A) \end{tikzcd} .$$

By exactness this gives $K_1(A) \cong K_0(SA)$. The naturality is given by the naturality of the index map, XIX.60. □

3.6.4 Bott periodicity

Theorem XIX.62 (Bott periodicity)**.** *The functors K_0 and $K_1 \circ S$ are naturally isomorphic.*

3.6.5 The six-term exact sequence

Chapter 4

K-theory for graded C^*-algebras

4.1 Van Daele's picture

4.2 Karoubi's picture

Let A be a graded C^*-algebra.

Chapter 5

K-theory for group C^*-algebras

Part XX

Applied mathematics

Chapter 1

Optimisation

1.1 Lagrange multipliers

Chapter 2

Reformulating the problem: transforms

2.1 What are transforms?

2.2 Integral transforms

2.2.1 Some general theory

2.2.2 Fourier transform

tilde

2.2.2.1 Fourier series

2.2.2.2 Conjugated quantities and uncertainty

Time and frequency

Position and momentum space

2.2.2.3 Some important transforms

Dirac delta

2.2.3 Laplace transform

2.3 Coordinate transformations

Jacobian Scalar

2.3.1 Common coordinate transformations

2.4 Legendre transform

`https://www.lpsm.paris/pageperso/lecomte/references2014/making-sense-of-legendre-tran`
`pdf`

Chapter 3

Curves and surfaces in Euclidean space

3.1 Curves and parametrisations in Euclidean space

3.1.1 Definition of curves in $\mathbb{E}^n$

A curve can be describes with a mapping of the form

$$\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n : t \mapsto \gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$$

where I is an interval and each of the components $\gamma_i(t)$ are real functions.
We call the curve described by γ differentiable if each component is infinitely differentiable.
We call t the parameter of the curve and γ is called the parametrisation of the curve. The parametrisation contains information not only about the shape of the curve, but also about how it is traversed. We will often simply use the word curve when we mean parametrisation.

Example

- A line is a type of curve and can written as

  $$\gamma : \mathbb{R} \to \mathbb{E}^n : t \mapsto p + tv$$

  where $p \in \mathbb{E}^n$ is a point and $v \in \mathbb{R}^n$ is a vector.

- A circle is a curve and can be written as

  $$\gamma : [0, 2\pi] \to \mathbb{E}^2 : t \mapsto (m_1 + R\cos t, m_2 + R\sin t)$$

  where $m = (m_1, m_2) \in \mathbb{E}^2$ is a point and R is a positive number called the radius.

- A helix is a curve and can be written as

  $$\gamma : \mathbb{R} \to \mathbb{E}^3 : t \mapsto (a\cos t, a\sin t, bt)$$

  where a and b are real numbers.

Let $\gamma : I \to \mathbb{E}^n$ be a curve. A vector field along γ is a map of the form

$$Y : I \to T\mathbb{E}^n : t \mapsto Y(t) \in T_{\gamma(t)}\mathbb{E}^n$$

TODO specify components after geometry!!!!

3.1.2 Velocity and arc length

We define the velocity vector field along the curve as the map

$$\boldsymbol{\gamma'} : I \to T\mathbb{E}^n : t \mapsto \boldsymbol{\gamma'(t)} \equiv (\gamma_1'(t), \ldots, \gamma_n'(t))_{\gamma(t)} \in T_{\gamma(t)}\mathbb{E}^n$$

where $\boldsymbol{\gamma_i'}(t)$ means the derivative of $\gamma_i(t)$.
The speed of γ can then be defined as the function

$$v : I \to \mathbb{R} : t \mapsto v(t) \equiv \|\boldsymbol{\gamma'}(t)\|$$

and for $a, b \in I$ with $a \leq b$, we call

$$\int_a^b v(t)\,\mathrm{d}t = \int_a^b \|\boldsymbol{\gamma'}(t)\|\mathrm{d}t$$

the length of the stretch of γ between $\gamma(a)$ and $\gamma(b)$. This corresponds to our intuitive notion of length of a curve.

Another way to define the length of a curve, is by dividing the interval $[a, b]$ into k sections, each of length Δ. Thus we can write

$$a = t_0 < t_1 < \ldots < t_{k-1} < t_k = b \qquad \text{with } t_i - t_{i-1} = \Delta.$$

This defines a broken line with length

$$\sum_{i=1}^{k} \|\gamma(t_i) - \gamma(t_{i-1})\| = \sum_{i=0}^{k-1} \|(\gamma(t_i + \Delta) - \gamma(t_i))/\Delta\|\Delta.$$

We could define the length of the curve as the length of such a broken line in the limit of $k \to \infty$ (which also means that Δ goes to 0). So

$$\begin{aligned} \text{length} &= \lim_{k\to\infty, \Delta\to 0} \sum_{i=0}^{k-1} \|(\gamma(t_i + \Delta) - \gamma(t_i))/\Delta\|\Delta \\ &= \lim_{k\to\infty, \Delta\to 0} \sum_{i=0}^{k-1} \|(\gamma'(t_i))\|\Delta \\ &= \int_a^b \|\gamma'(u)\|\mathrm{d}u \end{aligned}$$

which is the definition we gave before.

We can also define the arc length as the function

$$s : I \to \mathbb{R} : t \mapsto s(t) \equiv \int_a^t v(u)\,\mathrm{d}u = \int_a^t \|\boldsymbol{\gamma'}(u)\|\mathrm{d}u$$

for a given $a \in I$. This is quite simply the length of the curve between $\gamma(a)$ and $\gamma(t)$, if $t \geq a$ and minus the length otherwise.

Lemma XX.1. *Let $I \to \mathbb{E}^n$ be a curve and $t, t' \in I$. Then*

$$\|\gamma(t') - \gamma(t)\| \leq s(t') - s(t).$$

Proof. Triangle inequality. TODO ☐

3.1.3 Tangent vectors

A tangent line to the curve γ in t_0 is a line through the point $\gamma(t_0)$ in the direction of $\boldsymbol{\gamma'}(t_0)$. This is obviously only defined if $\boldsymbol{\gamma'}(t_0) \neq 0$.
Any multiple of $\boldsymbol{\gamma'}(t_0)$ in $T_{\gamma(t_0)}\mathbb{E}^n$ is called a tangent vector in $\gamma(t_0)$. In particular a tangent vector with norm one is called a unit tangent vector.

With the right assumptions of differentiability etc. it is possible to write down a Taylor expansion of the curve γ.

$$\gamma(t) = \gamma(t_0) + (t - t_0)\boldsymbol{\gamma'}(t_0) + \ldots + \frac{1}{k!}(t - t_0)^k \boldsymbol{\gamma^{(k)}}(t_0) + (t - t_0)^{k+1} R_{k+1}(t)$$

We recognise the expression for the tangent line as the first order approximation

$$\gamma(t) \approx \gamma(t_0) + (t - t_0)\boldsymbol{\gamma'}(t_0).$$

3.1.4 Reparametrisations and arc length parametrisations

Two different parametrisations may look the same when drawn in space.

Let $I, \tilde{I} \subseteq \mathbb{R}$ be intervals and $\gamma : I \to \mathbb{E}^n$ a curve. If $h : \tilde{I} \to I$ is a diffeomorphism, then

$$\beta \equiv \gamma \circ h : \tilde{I} \to \mathbb{E}^n$$

is a curve *with the same image as* γ (i.e. it looks the same in space). We call β a reparametrisation of γ.

Because

$$\boldsymbol{\beta'}(t) = (\gamma \circ h)'(t) = \boldsymbol{\gamma}'(h(t))h'(t),$$

$\boldsymbol{\beta'}(t)$ and $\boldsymbol{\gamma'}(h(t))$ are proportional to each other and thus the tangent lines are the same. Also

$$v_\beta = |h'|(v_\gamma \circ h).$$

Because the arc length is also a geometric quantity, we would expect it to be the same for both parametrisations. Actually it turns out to be the same up to the sign, because the new parametrisation may traverse the curve in the opposite direction.

$$s_\beta(t) = \pm s_\gamma(h(t))$$

3.1.4.1 Arc length parametrisation

We would now like to find a reparametrisation such that the speed is always unity (i.e. one). This turns out to always be possible is the curve is regular.

A curve is called regular if $v(t) > 0$ for all t.

If the curve is regular, the the arc length is a diffeomorfism, as is it's inverse. Using the inverse in the place of the diffeomorphism h, we get exactly the reparametrisation we were looking for. It turns out that this reparametrisation is relatively unique: If β_1 and β_2 are reparametrisations of the same curve, both with speed 1, then $\beta_1(t) = \beta_2(\pm t + c)$, for a constant $c \in \mathbb{R}$. In other words, if we want a reparametrisation with speed 1 everywhere, then that reparametrisation is unique once we have chosen a direction and origin.

We call a curve with speed 1 everywhere an arc length parametrisation.

Let β be an arc length parametrisation, then the arc length is

$$s_\beta(t) = \int_a^t \mathrm{d}u = t - a$$

3.2 Curves on a sphere

Proposition XX.2. *Let $\gamma : I \to S^n$ be a curve on the unit sphere. Then for all t, δ such that $t, t+\delta \in I$:*

$$\begin{aligned} |\langle \gamma(t)|\gamma(t+\delta)\rangle|^2 &= 1 - \left\|P_{\gamma(t)^\perp}\gamma(t+\delta)\right\|^2 \\ &\leq 1 - \|\gamma(t+\delta) - \gamma(t)\|^2 \\ &\leq 1 - (s(t+\delta) - s(t))^2, \end{aligned}$$

where $P_{\gamma(t)^\perp}$ is the orthogonal projection on the subspace perpendicular to $\operatorname{span}(\gamma(t))$ *and $s(t)$ is the path length.*

Proof. Once we have proven the equality, the first inequality follows because $P_{\gamma(t)^\perp}\gamma(t+\delta) = P_{\gamma(t)^\perp}\gamma(t+\delta) - P_{\gamma(t)^\perp}\gamma(t)$ and $\left\|P_{\gamma(t)^\perp}\right\| \leq 1$. The second follows by XX.1.
Now we prove the equality. Set $\Delta\gamma = \gamma(t+\delta) - \gamma(t)$. Using the fact that $\|\gamma(t)\| = 1 = \|\gamma(t+\delta)\|$, we have

$$1 = \|\gamma(t+\delta)\| = \langle \gamma(t) + \Delta\gamma|\gamma(t) + \Delta\gamma\rangle = 1 + \langle\gamma(t)|\Delta\gamma\rangle + \langle\Delta\gamma|\gamma(t)\rangle + \langle\Delta\gamma|\Delta\gamma\rangle\,,$$

which implies that $\langle\gamma(t)|\Delta\gamma\rangle + \langle\Delta\gamma|\gamma(t)\rangle = -\langle\Delta\gamma|\Delta\gamma\rangle$. Using this, we can expand

$$\begin{aligned} |\langle \gamma(t)|\gamma(t+\delta)\rangle|^2 &= \langle\gamma(t)|\gamma(t)+\Delta\gamma\rangle\langle\gamma(t)+\Delta\gamma|\gamma(t)\rangle \\ &= 1 + \langle\gamma(t)|\Delta\gamma\rangle + \langle\Delta\gamma|\gamma(t)\rangle + \langle\Delta\gamma|\gamma(t)\rangle\langle\gamma(t)|\Delta\gamma\rangle \\ &= 1 - \langle\Delta\gamma|\Delta\gamma\rangle + \langle\Delta\gamma|\gamma(t)\rangle\langle\gamma(t)|\Delta\gamma\rangle \\ &= 1 - \langle\Delta\gamma|\left(\mathrm{id} - |\gamma(t)\rangle\langle\gamma(t)|\right)|\Delta\gamma\rangle = 1 - \left\|P_{\gamma(t)^\perp}\Delta\gamma\right\|^2. \end{aligned}$$

TODO ref diad projector lemma. ☐

3.3 Curves in flat Euclidean space

3.3.1 Frenet frame for regular curves

Given a curve $\gamma : I \to \mathbb{E}^2$, we wish to introduce a useful basis for $T_{\gamma(t)}\mathbb{E}^2$. By useful, we mean that it can serve as a natural reference frame for a particle traveling along the curve. The

Frenet frame also leads to a natural definition of the curvature of a curve (as well as torsion in 3 dimensional space).
A first obvious vector to introduce in our basis is the unit tangent vector, which we will call $\mathbf{T}$. Then there is only one way to extend this to a positively oriented orthonormal basis, which we call the normal unit vector $\mathbf{N}$. If the unit tangent vector is given by $\mathbf{T} = (T_1, T_2)$, then[1]

$$\mathbf{N} = (-T_2, T_1)$$

In two dimensions, the Frenet frame, for a curve $\gamma(t)$ at a point t_0, is given by $(\mathbf{T}, \mathbf{N})$ where

- $\mathbf{T}$ is the unit tangent vector to $\gamma(t)$ at t_0;

$$\mathbf{T} = \frac{\boldsymbol{\gamma'}}{\|\boldsymbol{\gamma'}\|}$$

- $\mathbf{N}$ is the unique vector such that $(\mathbf{T}, \mathbf{N})$ is a positively oriented orthonormal basis, called the normal unit vector.

From $(\mathbf{T} \cdot \mathbf{T}) = 1$, we see that $(\mathbf{T} \cdot \mathbf{T})' = 0$. Using the product rule, we see that it is also equal to $(\mathbf{T} \cdot \mathbf{T})' = \mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 2(\mathbf{T} \cdot \mathbf{T}')$. Thus we see that $\mathbf{T} \cdot \mathbf{T}' = 0$, meaning that the derivative of $\mathbf{T}$ must be perpendicular to $\mathbf{T}$. This is true for any unit vector. The unit normal vector is also perpendicular to $\mathbf{T}$, so we can find a $k \in \mathbb{R}$ such that $\mathbf{T}' = k\mathbf{N}$. Because $\mathbf{N}$ is a unit vector, k is given by $k = \mathbf{T}' \cdot \mathbf{N}$.
Following a similar line of reasoning, we see that $\mathbf{N}'$ must be proportional to $\mathbf{T}$, with a factor of proportionality equal to $\mathbf{T} \cdot \mathbf{N}'$. From

$$(\mathbf{T} \cdot \mathbf{N})' = \mathbf{T}' \cdot \mathbf{N} + \mathbf{T} \cdot \mathbf{N}' = 0$$

we see that the factor must equals $-k$.
The quantity k correspond to our intuitive notion of curvature (as we will see), but **only if the curve is arc length parametrised**. We also want the curvature to be a purely geometric quantity that does not depend on which parametrisation we choose (as k does). So, for regular curves, it makes sense to define the curvature κ as the factor $\mathbf{T}' \cdot \mathbf{N}$ for an arc length parametrisation of the curve. This uniquely determines the curvature κ of the curve up to the sign, which depends on the direction of traversal.
Now for a general regular curve with arc length $s(t)$, we denote the quantities associated with an arc length parametrisation with a tilde:

$$\begin{cases} \mathbf{T}(t) = \tilde{\mathbf{T}}(s(t)) \\ \mathbf{N}(t) = \tilde{\mathbf{N}}(s(t)) \\ \kappa(t) = \tilde{k}(s(t)) \end{cases}$$

We can calculate

$$\mathbf{T}'(t) = (\tilde{\mathbf{T}}(s(t)))' = \tilde{\mathbf{T}}'(s(t))s'(t) = v(t)\tilde{k}(s(t))\tilde{\mathbf{N}}(s(t)) = v(t)\kappa(t)N(t)$$

We now have a general expression for κ:

$$\kappa = \frac{\mathbf{T}' \cdot \mathbf{N}}{v}$$

---

[1] Effectively we are applying the two-dimensional complex structure to $\mathbf{T}$. Where a linear complex structure is a linear transformation that squares to minus identity.

Summarising

The Frenet-Serret formulae are

$$\begin{cases} \mathbf{T}' = \kappa v \mathbf{N} \\ \mathbf{N}' = -\kappa v \mathbf{T} \end{cases}$$

where $\kappa = \frac{\mathbf{T}' \cdot \mathbf{N}}{v}$ is called the (oriented) curvature and v is the speed at which the curve is traversed in that point; $v = 1$ for arc parametrised curves.

The curvature κ of any regular curve γ can be calculated directly from the first and second derivatives of the curve:

$$\kappa = \frac{\|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''\|}{\|\boldsymbol{\gamma}'\|^3}$$

Or

$$\kappa = \frac{|\boldsymbol{\gamma}' \quad \boldsymbol{\gamma}''|}{\|\boldsymbol{\gamma}'\|^3}$$

where $|\boldsymbol{\gamma}' \quad \boldsymbol{\gamma}''|$ is the determinant with $\boldsymbol{\gamma}', \boldsymbol{\gamma}''$ seen as column vectors.
The associated collection $\mathbf{T}, \mathbf{N}, \kappa$ is called the Frenet-Serret apparatus. In three dimensions this also includes the binormal unit vector $\mathbf{B}$ and torsion τ.

3.3.2 Curvature

First some examples to support the idea that our definition of curvature makes sense.

Example

- Let γ be a straight line with arc length parametrisation

$$\gamma : \mathbb{R} \to \mathbb{E}^2 : s \mapsto p + s\mathbf{s}$$

where $\|\mathbf{v}\| = 1$. Then $\mathbf{T}(s) = \mathbf{v}$ for all s and consequently $\mathbf{T}' = 0$. Thus a straight line has zero curvature.

- It can be proven that any curve with zero curvature is a straight line.
- Let γ be a circle centered at $m = (m_1, m_2)$ with radius R and arc length parametrisation

$$\gamma : \mathbb{R} \to \mathbb{E}^2 : s \mapsto \left(m_1 + R\cos\left(\frac{s}{R}\right), m_2 + R\sin\left(\frac{s}{R}\right) \right).$$

Then the Frenet frame, for all $s \in \mathbb{R}$, is given by

$$\begin{cases} \mathbf{T}(s) = \left(-\sin\left(\frac{s}{R}\right), \cos\left(\frac{s}{R}\right)\right) \\ \mathbf{N}(s) = \left(-\cos\left(\frac{s}{R}\right), -\sin\left(\frac{s}{R}\right)\right) \end{cases}$$

Thus

$$\mathbf{T}'(s) = \left(-\frac{1}{R}\cos\left(\frac{s}{R}\right), -\frac{1}{R}\sin\left(\frac{s}{R}\right) \right) = \frac{1}{R}\mathbf{N}(s),$$

meaning that $\kappa(s) = \frac{1}{R}$ for all $s \in \mathbb{R}$. A circle with radius R has a constant curvature $1/R$.

- Every curve with constant, non-zero, curvature is a (part of a) circle with radius $1/|\kappa|$.

3.3.2.1 Osculating parabola

Let β be an arc length parametrised curve. Then we can write the Taylor expansion:

$$\begin{aligned}\beta(s) &= \beta(s_0) + (s-s_0)\boldsymbol{\beta'}(s_0) + \frac{1}{2}(s-s_0)^2\boldsymbol{\beta''}(s_0) + (s-s_0)^3 R_3(s)\\ &= \beta(s_0) + (s-s_0)\mathbf{T}(s_0) + \frac{1}{2}(s-s_0)^2\kappa(s_0)\mathbf{N}(s_0) + \dots\end{aligned}$$

The second order approximation is a parabola, called the osculating parabola (which comes from the latin word osculans meaning kissing). Intuitively it may be thought of as the parabola with it's top in $\beta(s_0)$ that most closely matches the curve.
TODO Figure.
If κ is positive, β curves towards $\mathbf{N}$ and β locally lies on the same side of $\mathbf{T}$ as $\mathbf{N}$. If κ is negative, β curves away from $\mathbf{N}$ and β locally lies on the opposite side of $\mathbf{T}$ from $\mathbf{N}$.

3.3.2.2 Osculating circle

Again let β be an arc length parametrised curve.

- We call $1/|\kappa(s_0)|$ the radius of curvature of the curve at s_0.
- We call the point
  $$m \equiv \beta(s_0) + (1/\kappa(s_0))\mathbf{N}(s_0)$$
  the centre of curvature of the curve at s_0.
- The circle which has as its centre in the centre of curvature and a radius that is the same as the radius of curvature, is called the osculating circle of the curve at s_0.

The osculating circle may be parametrised as

$$c(s) = m + R\cos\left(\frac{s-s_0}{R}\right)(-\mathbf{N}(s_0)) + R\sin\left(\frac{s-s_0}{R}\right)\mathbf{T}(s_0)$$

where m is the centre of curvature and $R = \frac{1}{|\kappa(s_0)|}$ is the radius of curvature.
We can easily calculate that

$$\begin{cases} c(s_0) = \beta(s_0)\\ \mathbf{c'}(s_0) = \boldsymbol{\beta'}(s_0)\\ \mathbf{c''}(s_0) = \boldsymbol{\beta''}(s_0).\end{cases}$$

We say that the osculating circle approximates the curve to second order. It is the only circle that does that.
The osculating circle gives us quite a useful intuitive interpretation of the curvature: it is the inverse of the radius of the "best fitting" circle.

3.3.3 Intrinsic equations

An intrinsic equation of a curve is an equation that defines the curve using a relation between geometrical properties that are intrinsic to the curve and do not depend on the exact parametrisation.
Examples of such intrinsic quantities are: arc length s, tangential angle θ and curvature κ.

3.3.3.1 Tangential angle

The tangential angle θ is the angle of the unit tangent vector $\mathbf{T}$ with the line through points $(0,0)$ and $(0,1)$ in $\mathbb{E}^2$ (i.e. the "x-axis"). It is a function $\theta : I \to \mathbb{R}$ such that

$$\mathbf{T}(s) = (\cos(\theta(s)), \sin(\theta(s))).$$

The normal unit vector is then given by

$$\mathbf{N}(s) = (-\sin(\theta(s)), \cos(\theta(s))).$$

From $\mathbf{T}' = (-\sin(\theta)\theta', \cos(\theta)\theta') = \theta'\mathbf{N}$, we see that

$$\kappa(s) = \theta'(s).$$

3.3.3.2 Whewell equations

Suppose we have an equation for the tangential angle of a curve in function of the arc length ($\theta(s) = \ldots$). We would now like to find a parametrisation for that curve.
Consider the following curve:

$$\beta(s) = \left(\int_{s_0}^{s} \cos\theta(u)\,\mathrm{d}u, \int_{s_0}^{s} \sin\theta(u)\,\mathrm{d}u\right)$$

Then $\boldsymbol{\beta'}(s) = (\cos(\theta(s)), \sin(\theta(s)))$ for all $s \in I$, so β is arc length parametrised and tangential angle θ at all points. So β is exactly the parametrisation we were looking for.

Example

- Straight lines are determined by $\theta = c$ for some constant $c \in \mathbb{R}$.
- Circles are determined by $\theta(s) = \frac{s}{R}$ where $R \in \mathbb{R}$ is the radius.
- Catenary curves are determined by $\theta = \arctan\left(\frac{s}{R}\right)$.

3.3.3.3 Cesàro equations

Now suppose we have an equation for the curvature of a curve in function of the arc length ($\kappa(s) = \ldots$).
Because $\theta' = \kappa$, the Cesàro equation of a curve can be obtained from the Whewell equation by differentiating it.
The parametrisation is then given by

$$\beta(s) = \left(\int_{s_0}^{s} \cos\left(\int_{s_0}^{u} \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u, \int_{s_0}^{s} \sin\left(\int_{s_0}^{u} \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u\right)$$

Example

- Line: $\kappa = 0$
- Circle: $\kappa = 1/R$
- Logarithmic spiral: $\kappa = C/s$

- Circle involute: $\kappa = C/\sqrt{s}$
- Cornu spiral (or clothoid): $\kappa = Cs$
- Catenary: $\kappa = \frac{a}{s^2+a^2}$

3.3.4 Global properties of flat curves

So far we have mainly looked at *local* properties of curves, like curvature. Now we take a look at some global properties.

We call a curve $\gamma : \mathbb{R} \to \mathbb{E}^n$ closed if there is a strictly positive number $\omega \in R_0^+$ such that $\gamma(t+\omega) = \gamma(t)$ for all $t \in \mathbb{R}$. We call ω the period of γ.
If ω is a period of a curve, then any multiple of ω is also a period. We call the smallest period the real period ω.

A simple curve is a curve that does not cross itself.

A closed curve γ with real period ω is said to be simple if the restriction $\gamma|_{[0,\omega[} : [0,\omega[\to \mathbb{E}^n$ is injective.

Let $\beta : \mathbb{R} \to \mathbb{E}^2$ be an arc parametrised, closed curve with period L.

- We call
$$\int_0^L \kappa(s)\,\mathrm{d}s$$
the total curvature of β.
- We define the rotation index i_β of β as
$$i_\beta \equiv \frac{1}{2\pi}\left(\theta(L) - \theta(0)\right)$$

The rotation index must always be an integer, because $\mathbf{T}(0)$ must be the same as $\mathbf{T}(L)$. So the angle tangential angles must be the same, modulus 2π.

The total curvature is related to the rotation index:
$$\int_0^L \kappa(s)\,\mathrm{d}s = 2\pi i_\beta$$

Finally we formulate three theorems for simple, closed, flat curves (also called Jordan curves).

- *Umlaufsatz.* The rotation index of a simple closed curve is 1 or -1.
- *Jordan's theorem.* A simple closed curve divides the plain onto two parts: a bounded interior and an unbounded exterior.

- *Isoperimetric inequality.* The surface area of the bounded interior A satisfies the inequality
$$L^2 \geq 4\pi A$$
where L is a period. This is an equality only if the curve is a circle (and L the real period).

3.4 Curves in three dimensional Euclidean space

3.4.1 The Frenet frame

Let γ be a regular curve. Again we define $\mathbf{T}$ as the unit tangent vector. For the same reason as before, $\mathbf{T'}$ is perpendicular to $\mathbf{T}$, so we can use that to define $\mathbf{N}$. In three dimensions we need a third basis vector. There is only one vector that can be added to the orthonormal vectors $\mathbf{T}$ and $\mathbf{N}$ to make positively oriented orthonormal basis of $T_{\gamma(t)}\mathbb{E}^3$, namely $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

In three dimensions, the Frenet frame, for a curve $\gamma(t)$ at a point t_0, is given by $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ where

- $\mathbf{T}$ is the tangent unit vector to $\gamma(t)$ at t_0
$$\mathbf{T} \equiv \frac{\boldsymbol{\gamma'}}{\|\boldsymbol{\gamma'}\|}$$
- $\mathbf{N}$ is the normal unit vector
$$\mathbf{N} \equiv \frac{\mathbf{T'}}{\|\mathbf{T'}\|}$$
- $\mathbf{B}$ is the binormal unit vector
$$\mathbf{B} \equiv \mathbf{T} \times \mathbf{N}$$

From this definition it is obvious that $\mathbf{T}'$ is a multiple of $\mathbf{N}$. As in the one dimensional case, the factor connecting ($k = \|\mathbf{T'}\|$) them has geometric significance, so long as the speed is fixed. The big difference is that now the factor k is always positive.

Again we define the curvature κ of a curve as the factor k for an arc length parametrisation of the curve. As in the two dimensional case (and following the same reasoning), we have for general regular curves
$$\mathbf{T}' = k\mathbf{N} = v\kappa\mathbf{N}$$

We now prove that $\mathbf{B'} = l\mathbf{N}$ for some factor $l(t) \in \mathbb{R}$. First we write an orthonormal expansion of $\mathbf{B}'$
$$\mathbf{B}' = (\mathbf{B}' \cdot \mathbf{T})\mathbf{T} + (\mathbf{B}' \cdot \mathbf{N})\mathbf{N} + (\mathbf{B}' \cdot \mathbf{B})\mathbf{B}.$$

Now $\mathbf{B}' \cdot \mathbf{B} = 0$, which we have already shown to be true in the previous section because $\mathbf{B}$ is a unit vector (like $\mathbf{T}$). If we derive $\mathbf{B} \cdot \mathbf{T} = 0$, we get $\mathbf{B}' \cdot \mathbf{T} + \mathbf{B} \cdot \mathbf{T}' = \mathbf{B}' \cdot \mathbf{T} + k\mathbf{B} \cdot \mathbf{N} = \mathbf{B}' \cdot \mathbf{T} = 0$. Thus
$$\mathbf{B}' = (\mathbf{B}' \cdot \mathbf{N})\mathbf{N} = l\mathbf{N}$$

Again, to give $l = \mathbf{B'} \cdot \mathbf{N}$ geometric significance, we define the torsion τ as $-l$ for arc length parametrisations. The minus sign is a classical convention. The torsion *can* be negative. Again, following the same reasoning as we have twice before, we get for general regular curves

$$\mathbf{B}' = l\mathbf{N} = -v\tau\mathbf{N}$$

The Frenet-Serret formulae are

$$\begin{aligned}\mathbf{T}' &= && v\kappa\mathbf{N} \\ \mathbf{N}' &= -v\kappa\mathbf{T} && && +v\tau\mathbf{B} \\ \mathbf{B}' &= && -v\tau\mathbf{N}\end{aligned}$$

where $v = \|\boldsymbol{\gamma}'\|$ is the speed at which the curve is traversed in that point ($v = 1$ for arc parametrised curves) and

- $\kappa = \frac{\mathbf{T}'\cdot\mathbf{N}}{v}$ is called the curvature
- $\tau = \frac{-\mathbf{B}'\cdot\mathbf{N}}{v}$ is called the torsion

The only formula we have not yet proven is the second one. It can easily be seen to be correct if we take the orthonormal expansion $\mathbf{N}' = (\mathbf{N}' \cdot \mathbf{T})\mathbf{T} + (\mathbf{N}' \cdot \mathbf{N})\mathbf{N} + (\mathbf{N}' \cdot \mathbf{B})\mathbf{B}$ and calculate

$$\begin{aligned}&\mathbf{N}' \cdot \mathbf{T} = -\mathbf{N} \cdot \mathbf{T}' = -v\kappa\mathbf{N} \cdot \mathbf{N} = -v\kappa \\ &\mathbf{N}' \cdot \mathbf{N} = 0 \\ &\mathbf{N}' \cdot \mathbf{B} = -\mathbf{N} \cdot \mathbf{B}' = v\tau\mathbf{N} \cdot \mathbf{N} = v\tau\end{aligned}$$

The curvature κ and torsion τ of any regular curve γ can be calculated directly from the first, second and third derivatives of the curve. As in two dimensions, we have

$$\kappa = \frac{\|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''\|}{\|\boldsymbol{\gamma}'\|^3}$$

For the torsion we have

$$\tau = \frac{\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'' \cdot \boldsymbol{\gamma}'''}{\|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''\|^2} \qquad \text{or} \qquad \frac{|\boldsymbol{\gamma}' \quad \boldsymbol{\gamma}'' \quad \boldsymbol{\gamma}'''|}{\|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''\|^2}$$

where $|\boldsymbol{\gamma}' \quad \boldsymbol{\gamma}'' \quad \boldsymbol{\gamma}'''|$ is the determinant with $\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}'''$ seen as column vectors.
The associated collection $\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau$ is called the Frenet-Serret apparatus.

3.4.1.1 Osculating plane

We define the osculating plane in each point as the plane that contains **T** and **N**.
Because the Taylor expansion of an arc length parametrised curve β is still given by

$$\begin{aligned}\beta(s) &= \beta(s_0) + (s - s_0)\boldsymbol{\beta}'(s_0) + \frac{1}{2}(s - s_0)^2\boldsymbol{\beta}''(s_0) + (s - s_0)^3 R_3(s) \\ &= \beta(s_0) + (s - s_0)\mathbf{T}(s_0) + \frac{1}{2}(s - s_0)^2\kappa(s_0)\mathbf{N}(s_0) + \ldots\end{aligned}$$

the osculating plane can be seen as the tangent plane and it contains the osculating parabola.
If β lies in a plane, then all osculating planes are equal to this plane. That means that **B** does not change, so $\mathbf{B}' = 0$ and $\tau = 0$. In fact we can state that for any regular curve γ with curvature $\kappa > 0$, γ lies in a plane if and only if $\tau = 0$.
If a space curve γ lies in a plane, then the curvature κ of the curve is the absolute value of the curvature of the curve seen as a planar curve in two dimensions.

3.5 Surfaces in Euclidean space

In this section we introduce some of the key concepts

Chapter 4

Vector and tensor calculus

TODO analysis in vector notation. See Stat inf AQFT

4.1 Fields

4.1.1 What is a field?

People mean different things when they say field. We have already encountered the algebraic structure (e.g the fields $\mathbb{R}$ and $\mathbb{Q}$). In physics the term field usually means we associate a value or an object to each point in space. The vector field along a curve that we have already seen can be seen as a field in one dimensional space. In this section we will restrict our attention to Euclidean space. Details for other geometries will follow. In modern high energy physics the term field often refers specifically to fields of operators.

A field F associates an element of a set X to each point in space.

$$F : \mathbb{E}^n \to X$$

Where n is typically 2 or 3. Depending on X we call the field differently:

- For $X \subset \mathbb{R}$ we call F a scalar field.
- For X a three-dimensional vector field, we call F a vector field.

Examples of scalar fields include temperature in space and pressure distribution in a fluid. The flow of a fluid can be modeled using a vector field.

4.2 Differential calculus

There are two important cases: a field may be two or three dimensional. Because two dimensional fields can be seen as a special case of three dimensional ones, we will assume $n = 3$ for the rest of this section. This means that the field is essentially a function with three variables:

$$F(x, y, z)$$

This means there are three differential operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. We also fix and orthonormal basis for $\mathbb{E}^3$ with basisvectors

$$\hat{\mathbf{x}},\ \hat{\mathbf{y}},\ \hat{\mathbf{z}}$$

in that order.

4.2.1 Nabla, the vector differential operator

The main operations we can do in scalar and vector fields (taking the gradient, divergence and curl) can be expressed in terms of the nabla operator $\boldsymbol{\nabla}$ (also known as the del operator), which can be seen as a *vector* operator with three components:

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}$$

Formally this means it is an operator that when acting on a number produces a vector. More intuitively it means that is also makes sense to use it in conjunction with the dot and cross products. It must however be remembered that $\boldsymbol{\nabla}$ is not really a vector and we cannot just use vector identities with it (even if they do sometimes turn out to be correct). When in doubt, write out the components.

4.2.1.1 Gradient

Say F is a scalar field. A first, obvious, question for calculus to solve is: how fast does F vary? TODO after Taylor expansion. TODO directional derivative + intuition

$$\mathrm{d}F = (\nabla F) \cdot (\mathrm{d}\mathbf{l})$$

where

$$\nabla F = \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right) F = \frac{\partial F}{\partial x}\hat{\mathbf{x}} + \frac{\partial F}{\partial y}\hat{\mathbf{y}} + \frac{\partial F}{\partial z}\hat{\mathbf{z}}$$

Now

$$\mathrm{d}F = \nabla F \cdot \mathrm{d}\mathbf{l} = |\,\nabla F||\,\mathrm{d}\mathbf{l}|\cos\theta.$$

From this it is clear that $\mathrm{d}F$ is largest if $\theta = 0$ and smallest (i.e. zero) if $\theta = \pi$. Fixing $\theta = 0$ and viewing F as a one dimensional function (with variable $l = |\mathbf{l}|$) along this line, we see

$$\frac{\mathrm{d}F}{\mathrm{d}l} = |\,\nabla F|.$$

This leads us to a geometrical interpretation of the gradient:

- The gradient ∇F points in the direction of maximum increase of the function F.
- Locally the field does not vary perpendicular to ∇F.
- The magnitude $|\,\nabla F|$ gives the slope (rate of increase) along this maximal direction.

We call a point (x, y, z) a stationary point if $\nabla F = 0$ at (x, y, z). As for single variable calculus, local maxima and minima are stationary points.

4.2.1.2 Divergence

Assume we have a vector field

$$\mathbf{v} : \mathbb{E}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto \mathbf{v}(x, y, z) = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$$

we can then define the divergence as

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\end{aligned}$$

The divergence of a vector function is a *scalar*.
Intuitively the divergence can be thought of as the amount the vector field spreads out (diverges) from the point in question. If we think of the vector field as modeling the flow of a fluid, then a point of positive divergence is a source and a point of negative divergence is a drain.
TODO figure (like 18 in electro)

4.2.1.3 Curl

For a vector field $\mathbf{v}$, the curl can be defined as follows:

$$\begin{aligned}\nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}.\end{aligned}$$

The curl of a vector function is a *vector*.
Intuitively the curl is a measure of how much the vector swirls around the point in question. Again viewing the vector field as the flow of some liquid, the curl indicates how much a paddle wheel fixed at that point would rotate (TODO fig, like 19 in electro + paddle wheel).

4.2.2 Properties of vector derivatives

We now assume

- $k \in \mathbb{R}$ is a constant.
- f and g are scalar fields.
- $\mathbf{A}$ and $\mathbf{B}$ are vector fields.

All the vector derivatives are linear:

$$\begin{cases} \nabla(kf + g) = k\,\nabla f + \nabla g \\ \nabla \cdot (k\mathbf{A} + \mathbf{B}) = k\nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \\ \nabla \times (k\mathbf{A} + \mathbf{B}) = k\,\nabla \times \mathbf{A} + \nabla \times \mathbf{B} \end{cases}$$

4.2.2.1 Product rules

There are several relevant products to consider: scalar times scalar (fg), scalar times vector ($f\mathbf{A}$), dot product ($\mathbf{A}\cdot\mathbf{B}$) and cross product ($\mathbf{A}\times\mathbf{B}$). Accordingly, there are six product rules. Each can easily be verified by writing out the components and using the standard product rule from single variable calculus.

- For gradients
  (a) $\nabla(fg) = f\,\nabla g + g\,\nabla f$
  (b) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}$
- For divergences
  (c) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$
  (d) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$
- For curls
  (e) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$
  (f) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

One strange feature of these product rules is the occurrence of terms of the form $(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}$. This is clearer (at least to my mind) when written out in components (in three dimensions)

$$(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} = \left(A_x\frac{\partial B_x}{\partial x}\right)\hat{\mathbf{x}} + \left(A_y\frac{\partial B_y}{\partial y}\right)\hat{\mathbf{y}} + \left(A_z\frac{\partial B_z}{\partial z}\right)\hat{\mathbf{z}}.$$

4.2.2.2 Quotient rules

These can be easily obtained from the product rules.

$$\begin{aligned}\nabla\left(\frac{f}{g}\right) &= \frac{g\,\nabla f - f\,\nabla g}{g^2}\\ \nabla\cdot\left(\frac{\mathbf{A}}{g}\right) &= \frac{g(\nabla\cdot\mathbf{A}) - \mathbf{A}\cdot(\nabla g)}{g^2}\\ \nabla\times\left(\frac{\mathbf{A}}{g}\right) &= \frac{g(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla g)}{g^2}\end{aligned}$$

4.2.3 Second derivatives

4.2.3.1 The Laplacian

We introduce a second order operator, the Laplacian ∇^2:

$$\begin{aligned}\nabla^2 &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\\ &= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\cdot\left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\\ &= \frac{\partial^2}{\partial x^2}\hat{\mathbf{x}} + \frac{\partial^2}{\partial y^2}\hat{\mathbf{y}} + \frac{\partial^2}{\partial z^2}\hat{\mathbf{z}}\end{aligned}$$

This can be applied to a scalar field:

$$\nabla^2 F \equiv \frac{\partial^2 F}{\partial x^2}\hat{\mathbf{x}} + \frac{\partial^2 F}{\partial y^2}\hat{\mathbf{y}} + \frac{\partial^2 F}{\partial z^2}\hat{\mathbf{z}}$$

Or to a vector field by applying the Laplacian to each component individually:

$$\nabla^2 \mathbf{v} \equiv (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}$$

4.2.3.2 Constructing second derivatives from first order derivatives

There are five ways we can make second derivative operators by mixing gradient, divergence and curl:

1. Divergence of gradient:

$$\begin{aligned}\nabla \cdot (\nabla F) &= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right) \cdot \left(\frac{\partial F}{\partial x}\hat{\mathbf{x}} + \frac{\partial F}{\partial y}\hat{\mathbf{y}} + \frac{\partial F}{\partial z}\hat{\mathbf{z}}\right)\\ &= \frac{\partial^2 F}{\partial x^2}\hat{\mathbf{x}} + \frac{\partial^2 F}{\partial y^2}\hat{\mathbf{y}} + \frac{\partial^2 F}{\partial z^2}\hat{\mathbf{z}} = \nabla^2 F\end{aligned}$$

   So this is just the Laplacian.

2. The curl of a gradient:

$$\begin{aligned}\nabla \times (\nabla F) &= \left(\frac{\partial}{\partial y}\frac{\partial F}{\partial z} - \frac{\partial}{\partial z}\frac{\partial F}{\partial y}\right)\hat{\mathbf{x}} + \left(\frac{\partial}{\partial z}\frac{\partial F}{\partial x} - \frac{\partial}{\partial x}\frac{\partial F}{\partial z}\right)\hat{\mathbf{y}} + \left(\frac{\partial}{\partial x}\frac{\partial F}{\partial y} - \frac{\partial}{\partial y}\frac{\partial F}{\partial x}\right)\hat{\mathbf{z}}\\ &= 0\end{aligned}$$

   This is an important fact that hinges on the fact that cross derivatives commute.

3. The gradient of the divergence $\nabla(\nabla \cdot \mathbf{v})$ is *not* the same as the Laplacian of a vector. It does not have a special name.

4. The divergence of a curl:

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{v}) &= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\\ &= \frac{\partial}{\partial y}\frac{\partial v_x}{\partial z} - \frac{\partial}{\partial z}\frac{\partial v_x}{\partial y} + \frac{\partial}{\partial x}\frac{\partial v_z}{\partial y} - \frac{\partial}{\partial z}\frac{\partial v_y}{\partial x} + \frac{\partial}{\partial x}\frac{\partial v_z}{\partial y} - \frac{\partial}{\partial z}\frac{\partial v_z}{\partial x} \qquad = 0\end{aligned}$$

   Again this hinges on the fact the cross derivatives commute.

5. The curl of a curl gives nothing new:

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

We repeat two important facts for future reference:

- The curl of a gradient is always **zero**:

$$\nabla \times (\nabla F) = 0$$

- The divergence of a curl is always **zero**:

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

4.2.4 With respect to which coordinates?

Sometimes we will deal with maps that look like fields (they depend on x, y and z coordinate), but also depend on other variables, like time. We can still use all the results from this section. All derivatives are partial, so we just calculate as if the other variables were constant.
Sometimes we will deal with maps that depend on two or more sets of spatial coordinates. For example, the electric field in a point may depend on the locations of various charged particles. In this case we can still use the notation and result from this section, we just need to specify with respect to which set of coordinates we are applying the derivative.
For example, using the compact notation $\mathbf{r_1} = (x_1, y_1, z_1)$ and $\mathbf{r_2} = (x_2, y_2, z_2)$, we may have a quantity $T(\mathbf{r_1}, \mathbf{r_2}) = T(x_1, y_1, z_1, x_2, y_2, z_2)$. We can now write $\boldsymbol{\nabla}_{\mathbf{r_1}} T$ to mean

$$\boldsymbol{\nabla}_{\mathbf{r_1}} T = \frac{\partial T}{\partial x_1}\hat{\mathbf{x}} + \frac{\partial T}{\partial y_1}\hat{\mathbf{y}} + \frac{\partial T}{\partial z_1}\hat{\mathbf{z}}$$

and $\boldsymbol{\nabla}_{\mathbf{r_2}} T$ to mean

$$\boldsymbol{\nabla}_{\mathbf{r_2}} T = \frac{\partial T}{\partial x_2}\hat{\mathbf{x}} + \frac{\partial T}{\partial y_2}\hat{\mathbf{y}} + \frac{\partial T}{\partial z_2}\hat{\mathbf{z}}$$

4.2.5 Miscellaneous identities

$$\mathbf{a} \times (\nabla \times \mathbf{a}) = \nabla \left(\frac{a^2}{2}\right) - (\mathbf{a} \cdot \nabla)\mathbf{a}$$

4.2.6 Tensor derivatives

4.3 Integral calculus

4.3.1 Line integrals

Given a curve γ we define the line integral between $\gamma(a)$ and $\gamma(b)$ as

- The line integral of a scalar field F is given by

$$\int_a^b F[\gamma(t)]\, |\boldsymbol{\gamma'}(\mathbf{t})|\, \mathrm{d}t$$

- The line integral of a vector field $\mathbf{v}$ is given by

$$\int_a^b \mathbf{v}[\gamma(x)] \cdot \boldsymbol{\gamma'}(x)\, \mathrm{d}x.$$

This is usually written in the following way:

$$\int_{\mathbf{a}}^{(b)} \mathbf{v} \cdot \mathrm{d}\mathbf{l} \qquad \text{or} \qquad \int_{\gamma} \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

where $\mathbf{a} = \gamma(a)$ and $\mathbf{b} = \gamma(b)$

If γ is a closed loop (so $\mathbf{a} = \mathbf{b}$) we write

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

In general the value of the line integral depends on the curve γ. There exist some vector fields such that line integrals only depend on the endpoints, not on the path. Such vector fields are called conservative. For any conservative field $\mathbf{u}$:

$$\oint \mathbf{u} \cdot \mathrm{d}\mathbf{l} = 0.$$

4.3.2 Surface integrals

For a given surface $\mathcal{S}$ and vector field $\mathbf{v}$, we define the surface integral TODO

$$\iint_{\mathcal{S}} \mathbf{v} \cdot \mathrm{d}\mathbf{a}$$

$$\oiint \mathbf{v} \cdot \mathrm{d}\mathbf{a}$$

4.3.3 Volume integrals

TODO

$$\iiint_{\mathcal{V}} F \, \mathrm{d}\tau$$

4.4 Fundamental theorems of vector calculus

In this section we will state three very important results that are analoguous to the fundamental theorem of calculus:

$$\int_a^b \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) \mathrm{d}x = f(b) - f(a)$$

That is we will be inversing the operations of gradient divergence and curl using integrals.

4.4.1 Fundamental theorem for gradient

The fundamental theorem for gradients states that for any scalar field F

$$\boxed{\int_{\mathbf{a}}^{\mathbf{b}} (\nabla F) \cdot \mathrm{d}\mathbf{l} = F(\mathbf{b}) - F(\mathbf{a})}$$

An intuitive explanation can be given as follows: TODO figure.
This fundamental theorem is valid for any curve. Thus for any scalar field, the vector field ∇F is conservative.

4.4.2 Fundamental theorem for divergence

The fundamental theorem for divergences is also known as **Gauss's theorem**, **Green's theorem** or simply the **divergence theorem**.

$$\boxed{\iiint_{\mathcal{V}} (\nabla \cdot \mathbf{v}\, \mathrm{d}\tau) = \oiint_{\mathcal{S}} \mathbf{v} \cdot \mathrm{d}\mathbf{a}}$$

Where $\mathcal{S}$ is the surface of the volume $\mathcal{V}$.
If we view the vector field as the flow of a fluid, the divergence theorem can be interpreted as

$$\iiint (\text{sources within the volume}) = \oiint (\text{flow out through the surface}).$$

4.4.3 Fundamental theorem for curl

The fundamental theorem for curls is known as **Stokes' theorem**.

$$\boxed{\iint_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a} = \oint_{\mathcal{P} \mathbf{v} \cdot \mathrm{d}\mathbf{l}}}$$

Where $\mathcal{P}$ is the perimeter of the path $\mathcal{S}$.
Two comments

- The expressions on both sides of the equals sign have a sign ambiguity: The surface integral changes sign if you orient the surface differently and the line integral changes sign if you change direction of traversal.
- The surface integral does not depend on the exact surface chosen, only on its perimeter. This also means that
$$\oiint (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a} = 0.$$

4.5 Integrating by parts

Integration by parts in vector calculus is analogous to the one dimensional case: Use the product rule, integrate both sides and invoke the fundamental theorem. For vector calculus we have more product rules to exploit. Assume f is a scalar field and $\mathbf{A}$ and $\mathbf{B}$ are vector fields.

$$\iiint_{\mathcal{V}} f(\nabla \cdot \mathbf{A})\, \mathrm{d}\tau = -\iiint_{\mathcal{V}} \mathbf{A} \cdot (\nabla f)\, \mathrm{d}\tau + \oiint_{\mathcal{S}} f \cdot (\mathbf{A} \cdot \mathrm{d}\mathbf{a})$$

$$\iint_{\mathcal{S}} f(\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{a} = \iint_{\mathcal{S}} [\mathbf{A} \times (\nabla f)] \cdot \mathrm{d}\mathbf{a} + \oint_{\mathcal{P}} f \cdot (\mathbf{A} \cdot \mathrm{d}\mathbf{l})$$

$$\iiint_{\mathcal{V}} \mathbf{B} \cdot (\nabla \times \mathbf{A})\, \mathrm{d}\tau = \iiint_{\mathcal{V}} \mathbf{A} \cdot (\nabla \times \mathbf{B})\, \mathrm{d}\tau + \oiint_{\mathcal{S}} (\mathbf{A} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{a}$$

4.6 Other coordinate systems

4.6.1 General coordinate transformations

TODO
The formula for $\nabla \cdot X$ is incorrect. The notation with the 'usual' dot product is misleading. Properly it is for a diagonal metric:

$$\nabla \cdot F = \frac{1}{\rho}\frac{\partial(\rho F^i)}{\partial x^i}$$

where $\rho = \sqrt{\det g}$ is the coefficient of the differential volume element $dV = \rho\, dx^1 \wedge \ldots \wedge dx^n$, meaning ρ is also the Jacobian determinant, and where F^i are the components of F with respect to an unnormalized basis.
In polar coordinates we have $\rho = \sqrt{\det g} = r$, and:

$$\nabla \cdot X = \frac{1}{r}\frac{\partial(rX^r)}{\partial r} + \frac{1}{r}\frac{\partial(rX^\theta)}{\partial \theta}$$

In the usual normalized coordinates $X = \hat{\mathbf{X}}^r \frac{\partial}{\partial r} + \hat{\mathbf{X}}^\theta \frac{1}{r}\frac{\partial}{\partial \theta}$ this becomes:

$$\nabla \cdot X = \frac{1}{r}\frac{\partial(r\hat{\mathbf{X}}^r)}{\partial r} + \frac{1}{r}\frac{\partial \hat{\mathbf{X}}^\theta}{\partial \theta}$$

which agrees with the usual formula given in calculus books.

4.6.2 Spherical coordinates

- *Gradient*

$$\nabla F = \frac{\partial F}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\hat{\boldsymbol{\phi}}$$

- *Divergence*

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \sin\theta v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

- *Curl*

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left(\frac{\partial \sin\theta v_\theta}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial r v_\phi}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)\hat{\boldsymbol{\phi}}$$

- *Laplacian*

$$\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial F}{\partial \theta}\right) + \frac{1}{r^2\sin^2}\theta\frac{\partial^2 F}{\partial z^2}$$

4.6.3 Cylindrical coordinates

- *Gradient*

$$\nabla F = \frac{\partial F}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial F}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial F}{\partial z}\hat{\mathbf{z}}$$

- *Divergence*

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial s v_s}{\partial s} + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

- *Curl*
$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left(\frac{\partial s v_\phi}{\partial s} - \frac{\partial v_s}{\partial \phi}\right)\hat{\mathbf{z}}$$
- *Laplacian*
$$\nabla^2 F = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial F}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial z^2}$$

4.6.4 Polar coordinates

This is just like for cylindrical coordinates, except nothing depends on the z coordinate and vectors of a vector field do not have a component in the z direction. Thus we may set v_z, and anything that is operated on by $\frac{\partial}{\partial z}$, to zero.

- *Gradient*
$$\nabla F = \frac{\partial F}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial F}{\partial \phi}\hat{\boldsymbol{\phi}}$$
- *Divergence*
$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial s v_s}{\partial s} + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi}$$
- *Curl*
$$\nabla \times \mathbf{v} = \frac{1}{s}\left(\frac{\partial s v_\phi}{\partial s} - \frac{\partial v_s}{\partial \phi}\right)\hat{\mathbf{z}}$$
- *Laplacian*
$$\nabla^2 F = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial F}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 F}{\partial \phi^2}$$

4.7 Potentials

4.7.1 Irrotational fields

Irrotational fields are fields where the curl vanishes everywhere.

> For a vector field **F** the following conditions are equivalent:
>
> (a) $\nabla \times \mathbf{F} = 0$ everywhere.
>
> (b) $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot \mathrm{d}\mathbf{l}$ is independent of the path, for any given end points.
>
> (c) $\oint \mathbf{F} \cdot \mathrm{d}\mathbf{l} = 0$ for any closed loop.
>
> (d) **F** is the gradient of some scalar function, called the potential
>
> $$\mathbf{F} = -\nabla V$$

The potential is not unique, any constant can be added to V without changing ∇V. The minus sign in the definition is conventional.

4.7.2 Solenoidal fields

Solenoidal fields are fields where the divergence vanishes everywhere.

For a vector field **F** the following conditions are equivalent:

(a) $\nabla \cdot \mathbf{F} = 0$ everywhere.

(b) $\iint \mathbf{F} \cdot \mathrm{d}\mathbf{a}$ is independent of the exact path, for a given perimeter.

(c) $\oiint \mathbf{F} \cdot \mathrm{d}\mathbf{a} = 0$ for any closed surface.

(d) **F** is the curl of some vector function, called the vector potential

$$\mathbf{F} = \nabla \times \mathbf{A}$$

4.7.3 Helmholtz theorem

4.7.3.1 Decomposition in irrotational and solenoidal field

An arbitrary vector field **F** can always be written as the sum of an irrotational and a solenoidal field, and thus also as the sum of the gradient of a scalar and the curl of a vector.

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}$$

This is sometimes known as the **Helmholtz decomposition**.

4.7.3.2 Fields with prescribed divergence and curl

Say we have a scalar field D and a solenoidal vector fields **C**. Can we find a vector field **F** such that the divergence and curl are given by D and **C** respectively?

$$\begin{cases} \nabla \cdot \mathbf{F} = D \\ \nabla \times \mathbf{F} = \mathbf{C} \end{cases}$$

The answer is *yes* if $\mathbf{C}(\mathbf{r})$ and $D(\mathbf{r})$ go to zero at infinity faster than $\frac{1}{r^2}$. They even define **F** uniquely if **F** goes to zero at infinity.
The vector field field can be constructed as follows:

$$\mathbf{F} = -\nabla \cdot U + \nabla \times \mathbf{W}$$

where

$$\begin{cases} U(\mathbf{r}) = \frac{1}{2\pi} \int \frac{D(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|} \,\mathrm{d}\tau' \\ \mathbf{W}(\mathbf{r}) = \frac{1}{2\pi} \int \frac{\mathbf{C}(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|} \,\mathrm{d}\tau' \end{cases}$$

4.8 Laplace's equation

TODO intro + see electro

4.8.1 Uniqueness theorems

4.8.2 Method of images

4.8.3 Seperation of variables

Appendix A

Symbols

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