Lecture notes in Derived Algebraic Geometry
Session 4

Course by F. Binda, notes by E. Hecky

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The aim of this last session is to finish the construction of the cotangent complex, and to discuss a bit about how to globalize the ring-theoretic construction that were introduced in the previous session.

1 Square-zero extensions and the cotangent complex

This section is actually the continuation of the last section of the previous lesson. Recall that we defined an augmented A-algebra $A \oplus M$ for every animated module M over an animated ring A, satisfying intuitive basic properties.

Lemma 1.1. *There exists a fiber sequence*

$$M \to A \oplus M \to A$$

where the second map is an effective epimorphism, and a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit}) \arrow[r] \arrow[d, "\oplus"'] & \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit}) \arrow[d, "\mathrm{pr}_1"] \\ \mathrm{CAlg}^{\mathrm{ani}} \arrow[r] & \mathrm{CAlg}^{\mathrm{ani}} \end{tikzcd} \qquad \begin{tikzcd} (A, M) \arrow[r, mapsto] \arrow[d, mapsto] & (A, 0) \arrow[d, mapsto] \\ A \oplus M \arrow[r, mapsto] & A \end{tikzcd}$$

Remember that classically, $\mathrm{Der}(A, M) \simeq \mathrm{Hom}_{/A}(A, A \oplus M)$. This motivates the definition of derivations in the animated context.

Definition 1.2. Let $(A, M) \in \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit})$. Then define

$$\mathrm{Der}(A, M) = \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}_{/A}}(A, A \oplus M).$$

Remark 1.3. $A \oplus M$ is a loop space ! Indeed, there is a pullback square of the form

$$\begin{tikzcd} A \oplus M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ A \arrow[r] & A \oplus M[1] \end{tikzcd}$$

Also, $\mathrm{Der}(A, M)$ is the fiber of the map

$$\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}}(A, A \oplus M) \to \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}}(A, A).$$

Proposition 1.4. *The functor*

$$\mathrm{Der}(A,-):\mathrm{Mod}_A\to\mathcal{S}\mathrm{pc}$$

is accessible and preserves limits. Therefore, it has a left adjoint

$$G:\mathcal{S}\mathrm{pc}\to\mathrm{Mod}_A.$$

Proof. For the accessibility, it is enough to observe that $\mathrm{Map}(A,-)$ is accessible since $M\mapsto A\oplus M$ commutes with sifted colimits by design (it is an animation), so it is accessible.

To show that the functor preserves limits, notice that the forgetful functor $\mathrm{CAlg}^{\mathrm{ani}}_{/A}\to\mathrm{CAlg}^{\mathrm{ani}}$ detects limits, so it is enough to show that if $p:K\to\mathrm{Mod}_A$ is a diagram of A-modules, then

$$\mathrm{Map}_{/A}(A[\underline{T}],A\oplus\lim_K p)\simeq\lim_K\mathrm{Map}_{/A}(A[\underline{T}],A\oplus p).$$

Why is this the case ? Recall that there is an adjunction :

$$\mathbb{L}\mathrm{Sym}_{\mathbf{Z}}:\mathrm{Mod}^{\mathrm{ani}}_{\mathbf{Z}}\rightleftarrows\mathrm{CAlg}^{\mathrm{ani}}:\mathrm{for}$$

between the (derived) free symmetric algebra and forgetful functors. Apply this adjunction and recall that small limits in Mod_A commute with $\oplus$ (which is simply the product in Mod_A) to obtain :

$$\begin{aligned}\mathrm{Map}_{\mathrm{CAlg}_{/A}}(A[\underline{T}],A\oplus\lim_K p)&\simeq\mathrm{Map}_{\mathrm{Mod}_A}(A^{\oplus n},\mathrm{for}(A\oplus\lim_K p))\\&\simeq\lim_K\mathrm{Map}_{\mathrm{Mod}_A}(A^{\oplus n},\mathrm{for}(A\oplus p))\\&\simeq\lim_K\mathrm{Map}_{\mathrm{CAlg}_{/A}}(\mathbb{L}\mathrm{Sym}(A^{\oplus n})=A[\underline{T}],A\oplus p).\end{aligned}$$

□

One can now define the cotangent complex.

Definition 1.5. The *absolute cotangent complex* $\mathbb{L}_A$ is the module $G(*)$.

Remark 1.6. There is another, spectral way to construct the cotangent complex. If $\mathcal{C}$ is a symmetric monoidal stable ∞-category, then one can define a functor $G:\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})\to\mathcal{C}$ sending an augmented algebra $f:A\to\mathbf{1}$ to its cofiber. There is a theorem stating that G[1] induces an equivalence

$$\partial G:\mathcal{S}\mathrm{p}(\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}))\simeq\mathcal{S}\mathrm{p}(\mathcal{C})\simeq\mathcal{C}$$

which can be used to define the cotangent complex. The advantage of this approach is that it lets us defined $A\oplus M$ for any non-necessarily animated $\mathbb{E}_\infty$-ring spectrum A.

Theorem 1.7. *The construction $A\mapsto\mathbb{L}_A$ extends to a functor*

$$\mathrm{CAlg}^{\mathrm{ani}}\to\mathrm{Ani}(\mathrm{CRMod}^{\heartsuit})$$

sending $f:A\to B$ to $B\otimes^{\mathbb{L}}_A\mathbb{L}_A\to\mathbb{L}_B$. If A is a polynomial algebra then $\mathbb{L}_A\simeq\Omega_{A/\mathbf{Z}}$.

[1] or more precisely, its Goodwillie derivative

Definition 1.8. For an animated ring map $f : A \to B$, its *relative cotangent complex* $\mathbb{L}_{B/A}$ is the cofiber of the map $\mathbb{L}_A \otimes^{\mathbb{L}}_A B \to \mathbb{L}_B$.

Proposition 1.9. *There is an equivalence*

$$\mathrm{Map}_{\mathrm{Mod}_B}(\mathbb{L}_{B/A}, N) \simeq \mathrm{Map}_{\mathrm{CAlg}_{A//B}}(B, B \oplus N).$$

Define the space $\mathrm{Der}_A(B, N)$ of A-linear derivations from B to N to be this space.

Proposition 1.10. *The cotangent complex satisfies the following basic properties :*

1. *(Base change) If $B \simeq B' \otimes_{A'} A$, then $\mathbb{L}_{B/A} \simeq \mathbb{L}_{B'/A'} \otimes B$;*
2. *If $A \to B \to C$ is a fiber sequence, then there is a fiber sequence*
$$C \otimes_B \mathbb{L}_{B/A} \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B} \ ;$$
3. *For every map $f : A \to B$, there is an associated map $\varepsilon(f) : B \otimes_A \mathrm{Cof}(f) \to \mathbb{L}_{B/A}$. We have the following connectivity estimate : if the fiber of f is connective, then the fiber of $\varepsilon(f)$ is 2-connective.*

An important "slogan" in derived algebraic geometry which gives even more importance to this object is the idea that all animated rings can be built from discrete rings with square-zero extensions.

Definition 1.11. A map $S' \to S$ in $\mathrm{CAlg}^{\mathrm{ani}}$ is a *square-zero extension* if there is an S-module M and a derivation $d \in \pi_0 \mathrm{Map}_{/S}(S, S \oplus M[1])$ such that the following square is cartesian :

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S \arrow[d, "d_0"] \\ S \arrow[r, "d"'] & S \oplus M[1] \end{tikzcd}$$

Here, d_0 denotes the trivial derivation $S \to S \oplus M[1]$ which is just the inclusion of the first factor.

Proposition 1.12 (Lurie, DAG)**.** *Let M be an animated module over S. If S' is a square-zero extension of S by M and if $R \to S$ is any ring map, then the space of lifts* $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}_{/S}}(R, S')$ *is a torsor under* $\mathrm{Der}(R, M)$*.*

Concretely, this means that to get a lift $R \to S'$ in the following diagram

$$\begin{tikzcd} R \arrow[r, dashed] \arrow[dr] & S' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "d_0"] \\ & S \arrow[r, "d"'] & S \oplus M[1] \end{tikzcd}$$

it is sufficient and necessary that the derivation $R \to S \xrightarrow{d} S \oplus M[1]$ in $\pi_0 \mathrm{Der}(R, M[1]) = \pi_0 \mathrm{Map}_{\mathrm{Mod}_R}(\mathbb{L}_R, M[1])$ is the trivial derivation.

Example 1.13. Postnikov towers give rise to square zero extensions ! Recall that for every $n \geq 0$, there is a functor

$$\tau_{\leqslant n} : \mathrm{CAlg}^{\mathrm{ani}} \to \mathrm{CAlg}^{\mathrm{ani}}.$$

Then $\tau_{\leqslant n} R$ is a square-zero extension of $\tau_{\leqslant n-1} R$ by $(\pi_n R)[n]$, i.e. there is a pullback square :

$$\begin{tikzcd}
\tau_{\leqslant n} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tau_{\leqslant n-1} R \arrow[d, "d_0"] \\
\tau_{\leqslant n-1} R \arrow[r, dashed, "\exists d"'] & \tau_{\leqslant n-1} R \oplus (\pi_n R)[n+1]
\end{tikzcd}$$

This explains the philosophy of derived algebraic geometry, that after passing from varieties to schemes by adding non-reduced points, we pass from schemes to derived schemes by adding finer infinitesimal information. This Postnikov construction allows for handy inductive arguments using the cotangent complex.

Sketch of why this is true One easy thing to see first is that $\tau_{\leqslant n} R$ should equalize the two maps to the tensor product :

$$\tau_{\leqslant n} R \to \tau_{\leqslant n-1} R \rightrightarrows \tau_{\leqslant n-1} R \otimes_{\tau_{\leqslant n} R} \tau_{\leqslant n-1} R$$

(even though this might not be an equalizer diagram). Apply the $\tau_{\leqslant n+1}$ functor to this to get

$$\tau_{\leqslant n} R \to \tau_{\leqslant n-1} R \rightrightarrows \underbrace{\tau_{\leqslant n+1}(\tau_{\leqslant n-1} R \otimes_{\tau_{\leqslant n} R} \tau_{\leqslant n-1} R)}_{\star}.$$

What is $\star$? We can compute its homotopy groups with the long exact sequence and check that $\pi_*(\star) \simeq \pi_*(\tau_{\leqslant n-1} R \oplus \pi_n(R)[n+1])$. This should be a good clue that $\star$ is homotopy equivalent to the bottom-right object of the square in the statement. This would in turn show that there is a map from $\tau_{\leqslant n} R$ to the pullback. Of course, this all depends on the existence of a suitable derivation d, which will not be discussed.