Lecture notes in Derived Algebraic Geometry
Session 1

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References

- Toën-Vezzosi *Homotopical Algebraic Geometry* ;
- Lurie *Derived Algebraic Geometry* ;
- Antieau *Derived Algebraic Geometry.*

Motivations for derived geometry

Let k be an algebraically closed field, and U, V be subvarieties (say, reduced, of finite type and maybe integral) of the affine space $\mathbf{A}^n_k$. The scheme-theoretic intersection of U and V is then the fibered product $U \times_{\mathbf{A}^n_k} V$. However, this thing is not a subvariety in general, for example it is not always reduced[1]. How can we properly understand intersections, especially when they are not transversal ?

Let W be an irreducible component of $U \cap V$ and A be the local ring of $\mathbf{A}^n$ at the generic point η_W. Serre's intersection formula gives the correct intersection multiplicity, if we take into account higher Tor groups :

$$\chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V) = \sum (-1)^i \ell_A(\mathrm{Tor}_i^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V))$$

where $\mathrm{Tor}_i^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V) = H_i(A/\mathfrak{p}_U \otimes^{\mathbb{L}}_A A/\mathfrak{p}_V)$ is the derived tensor product.

As a matter of fact, the higher Tor terms are useless for curves, since the usual tensor product already captures all the subtleties of the intersection. In higher dimension however, it is not enough and the higher correction terms are important.

Example 0.1. Even the simplest examples can already show that the usual tensor product does not retain enough information : the intersection between the union of two planes and another plane in four-dimensional space needs a correction term.

Let $U = V(x_1x_3, x_1x_4, x_2x_3, x_2x_4) = V(x_1, x_2) \cup V(x_3, x_4)$ and $V = V(x_1 - x_3, x_2 - x_4)$ inside $\mathbf{A}^4_k$. Then set-theoretically, the intersection $U \cap V$ consists of one rational point $P = (0, 0, 0, 0)$. Let us compute the correct intersection inside the Chow ring. Let $U_1 = V(x_1, x_2)$ and $U_2 = V(x_3, x_4)$ such that $U = U_1 \cup U_2$:

$$[V] \cdot [U] = [V] \cdot [U_1] + [V] \cdot [U_2] = 1 + 1 = 2$$

[1] take $U = V(y)$ and $V = V(y - x^2)$ inside $\mathbf{A}^2$ for example.

since the two intersections $V \cap U_1$ and $V \cap U_2$ are done transversely.

What does the tensor product see of this intersection ? Let us compute it :

$$\dim_k \frac{k[x_1, x_2, x_3, x_4]_{(P)}}{(x_1 - x_3, x_2 - x_4, x_1x_3, x_1x_4, x_2x_3, x_2x_4)} = \dim_k \frac{k[x, y]_{(0,0)}}{(x^2, xy, y^2)} = 3.$$

So we get the wrong intersection number if we don't account for the fact that :

$$\dim_k \mathrm{Tor}_1^{k[x_1,x_2,x_3,x_4]_{(P)}} \left(\frac{k[x_1, x_2, x_3, x_4]_{(P)}}{(x_1 - x_3, x_2 - x_4)}, \frac{k[x_1, x_2, x_3, x_4]_{(P)}}{(x_1x_3, x_1x_4, x_2x_3, x_2x_4)} \right) = 1.$$

With this and the vanishing of the higher Tor groups, the Serre intersection formula gives the correct intersection number of $3 - 1 = 2$.

What lesson can we learn from this formula and this example ? In essence, we understand that the tensor product $A/\mathfrak{p}_U \otimes_A A/\mathfrak{p}_V$ on its own does not know everything that happens at the intersection, and in particular that it is the wrong object to model the intersection. In particular, the fibered product $U \times_{\mathbf{A}^n} V$ is also the wrong geometric object.

The Serre formula really tells us that the correct object that should represent the intersection is the *derived* tensor product $A/\mathfrak{p}_U \otimes_A^{\mathbb{L}} A/\mathfrak{p}_V$. Since it is not a ring, it doesn't give us a scheme : we have to enlarge our framework.

Recall that a scheme X can be seen as a functor $X : \mathrm{CAlg} \to \mathrm{Set}$ on commutative rings, that locally looks like a representable functor $\mathrm{Spec}(A)$, that is a sheaf for the big Zariski site, and whose transition maps are sufficiently geometric.

To correctly understand intersection, one needs to enlarge CAlg to a suitable notion of *derived* rings, at least such that derived tensor products $A \otimes^{\mathbb{L}} B$ are such objects.

Something else has to be changed, and it is the target category Set (for multiple reasons). The kind of functors that geometers are usually interested in are of the following form, whose values are sets of isomorphism classes of families of geometric objects :

$$A \in \mathrm{CAlg}_k \mapsto \{[X_t \mid t \in \mathrm{Spec}(A)]\}$$

where X_t is a specified kind of geometric object defined over $k(t)$. However, those functors do not locally look like $\mathrm{Spec}(A)$, nor are they Zariski sheaves ! Most of the time, automorphisms come into play and make such a moduli functor locally look like a quotient of a scheme by a group action. This is absolutely not a scheme in general.

The way to correct this is to remember the geometric objects and not just their isomorphism classes, and also keep track of the action of the automorphism group. What we get is not a set anymore, but a groupoid.

Replacing Set by Grpd yields the notion of a stack. Why not replace it by Cat ? To keep the geometric intuition, the best idea is to instead use the ∞-category $\mathcal{S}\mathrm{pc}$ of *spaces*.

1 A brief introduction to ∞-categories

1.1 Simplicial sets

Let Δ be the category of finite ordered sets, together with order-preserving maps. Denoting by $[i]$ the object $0 < 1 < \cdots < i$, we then have :

$$\mathrm{Hom}_\Delta([i], [j]) = \{f : \{0, \ldots, i\} \to \{0, \ldots, j\} \mid f \text{ preserves the order}\}.$$

The category of simplicial sets is then defined as $\mathrm{sSet} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$. In other words, a simplicial set X is a collection of sets $X_n = X([n])$ for each $n \geq 0$, together with a transition map $\alpha^* : X_n \to X_m$ for every order-preserving map $\alpha : [m] \to [n]$.

Remark 1.1. Since such an α can be epi-mono factorized, it is enough to remember the action on X of the specific *face* maps $\delta_n^i : [n-1] \hookrightarrow [n]$ skipping i, and the *degeneracy* maps $\sigma_n^i : [n+1] \twoheadrightarrow [n]$ doubling i. In this regard, a simplicial set is usually depicted as a diagram :

$$\cdots \; \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \; X_1 \; \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \; X_0$$

or more cleanly, discarding the degeneracies :

$$\cdots \; \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \; X_2 \; \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \; X_1 \; \rightrightarrows \; X_0.$$

Let $\Delta^n = \mathrm{Hom}(-, [n]) \in \mathrm{sSet}$ be the representable *n-simplex.* Then by Yoneda, an element of X_n is equivalently a morphism of simplicial sets $\Delta^n \to X$.

The boundary $\partial\Delta^n \hookrightarrow \Delta^n$ is defined as the union of the faces $\partial^k\Delta^n$, images of $\delta_n^k \circ -$.

For every $k \leqslant n$, define the *k-th horn* $\Lambda_k^n \subset \Delta^n$ to be the union of all the $(n-1)$-faces except the k-th one :

$$\Lambda_k^n = \bigcup_{i \neq k} \partial^i \Delta^n.$$

If $C \in \mathrm{Cat}$ is a category, one can see it as a simplicial set via its *nerve* : $\mathrm{N}(C)$ is the simplicial set defined by $\mathrm{N}(C)_n = \mathrm{Fun}([n], C)$, where $[n]$ is seen as a category with its ordered structure. An n-simplex is then a string of n composable morphisms in C. The face maps compose successive morphisms, and the degeneracies add identity morphisms.

Remark 1.2. Every morphism of simplicial sets $\Lambda_1^2 \to \mathrm{N}(C)$ can be extended in a unique way to Δ^2, because the third face has to be the composition of the other two. However, this is not the case for Λ_0^2 and Λ_2^2, except when C is a groupoid.

On the other hands, simplicial sets coming from topological spaces (via the singular construction) have the property that *all* horns can be extended, albeit not in a unique way :

Definition 1.3. A simplicial set X is a *Kan complex* if for every $n > 0$ and $0 \leqslant i \leqslant n$, every map $\Lambda_i^n \to X$ can be extended to Δ^n.

Theorem 1.4 (Milnor)**.** *The* $\mathrm{Sing}(-)$ *construction gives an equivalence between the homotopy category of CW-complexes and the that of Kan complexes.*

In other words, there is a diagram

$$\begin{tikzcd} \mathrm{Grpd} \arrow[r, hook] \arrow[d, "\mathrm{N}"'] & \mathrm{Cat} \arrow[d] \\ \mathrm{Kan} \arrow[r] & ?? \end{tikzcd}$$

waiting to be completed to a square. The notion of ∞-category is the relaxation of both "all inner horns can be filled uniquely" and "all horns can be filled" into "all inner horns can be filled", so that both categories *and* Kan complexes (or CW-complexes) can be seen as the same kind of object.

Definition 1.5. A *weak Kan complex* is a simplicial set X such that for every $n \geq 2$ and $0 < k < n$, the restriction map

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \to \mathrm{Hom}_{\mathrm{sSet}}(\Lambda^n_k, X)$$

is surjective.

A theorem of Joyal states that X is a Kan complex if and only if it is an ∞-groupoid, i.e. every "morphism" (1-simplex) is an "isomorphism" (whatever this means). This is the good notion of spaces that will be used later.

Definition 1.6. An ∞-*category* is a weak Kan complex.

The reason to use two different terms to denote the same object is both historical and contextual. A good theory of higher categories (categories together with 2-morphisms between morphisms, 3-morphisms between 2-morphisms, and so on) had been sought for a long time, especially since the higher coherence axioms tend to take exponentially many pages to define. Multiple models for ∞-category theory were proposed, and today thanks to the work of Lurie (mainly), it is the weak Kan complex model that is being used everywhere. Also, the term *weak Kan complex* can be used to emphasize the simplicial nature of the object, whereas the term ∞-*category* is used as a way to forget the exact model in favor of the general higher categorical framework that is needed in practice.

Definition 1.7. An ∞-*functor* (or just *functor*) between ∞-categories is a morphism of simplicial sets.

Let X be an ∞-category and $x, y \in X$ be two objects (understand : let $x, y \in X_0$ be two 0-simplices). Then the *mapping space* $\mathrm{Map}_X(x, y)$ is defined as the following pullback in sSet :

$$\begin{tikzcd}
\mathrm{Map}_X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[d, "{(s,t)}"] \\
\Delta^0 \arrow[r, "{(x,y)}"'] & X \times X
\end{tikzcd}$$

where $\underline{\mathrm{Hom}}(Y, X)_n = \mathrm{Hom}_{\mathrm{sSet}}(Y \times \Delta^n, X)$ is the internal Hom inside sSet.

Fact $\mathrm{Map}_X(x, y)$ is always a Kan complex (whence the name of mapping *space*).

Also, if $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ (defined as the internal Hom object $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ in sSet) is also an ∞-category.

1.2 Homotopy category

The nerve functor $N : \mathrm{Cat} \to \mathrm{sSet}$ is fully faithful (so that there is no harm in seeing a category as an ∞-category), and the image is the collection of ∞-categories for which composition of morphisms is uniquely defined. This functor has a left adjoint

$$\mathrm{h} : \mathrm{sSet} \to \mathrm{Cat}$$

which is easy to describe. An object of $\mathrm{h}X$ is simply a 0-simplex of X (an element of X_0). Morphisms between objects in $\mathrm{h}X$ are homotopy classes of maps in X.

Definition 1.8. Let $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ be an ∞-category and $f : x \to y$ a morphism in $\mathcal{C}$. It is an *equivalence* if for every $n \geq 2$, there is a dotted extension arrow in every diagram of the form :

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, hook] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

A theorem of Joyal states that this is equivalent to $\mathrm{h}f$ being an isomorphism in $\mathrm{h}\mathcal{C}$.

Definition 1.9. A functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is an *equivalence* if :

- $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is essentially surjective ;
- $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(Fx, Fy)$ is an equivalence of Kan complexes (i.e. a weak homotopy equivalence between their geometric realizations).

1.3 How to build an ∞-category from a model category

Let $\mathscr{M} = (C, W, \mathrm{Cof}, \mathrm{Fib})$ be a model category. In particular, one should have two examples in mind :

- $C = \mathrm{Ch}(R)_{\geqslant 0}$ the category of bounded below chain complexes of modules over a ring R, with W the class of quasi-isomorphisms and Cof the class of split monomorphisms with projective cokernel ;
- $C = \mathrm{sSet}$ the category of simplicial sets, with W the class of weak homotopy equivalences and Cof the class of levelwise monomorphisms.

One can construct the subcategory $\mathscr{M}^\circ$ of fibrant-cofibrant objects (it works out much nicer when $\mathscr{M}$ is a simplicial or combinatorial model category). In the examples above, they are the bounded below complexes of projectives, or the Kan complexes. One can then construct the Dwyer-Kan localization $\mathrm{L}_{\mathrm{DK}}(\mathscr{M}^\circ, W)$ which is a simplicial category, and take its homotopy coherent nerve $[n] \mapsto \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathrm{L}_{\mathrm{DK}}(\mathscr{M}^\circ, W))$ to obtain an ∞-category $\infty(\mathscr{M})$.

We only care about one example here, and will never use this rather technical definition. Kan complexes are the fibrant (automatically cofibrant) objects in sSet with its model structure described above. Then define the ∞-*category of spaces* as $\mathcal{S}\mathrm{pc} = \infty(\mathrm{Kan})$[2].

1.4 Joins and slices

The purpose of this subsection is to make sense of the notion of limits and colimits inside ∞-categories.

Definition 1.10. Let C and D be (1-)categories. Define a new category $C * D$ as follows :

[2] The localization and coherent nerve processes do not fundamentally alter the objects, so that an object of $\mathcal{S}\mathrm{pc}$ is still just a Kan complex (aka an ∞-groupoid). The purpose of this whole construction is mainly to construct the higher simplices and the ∞-category $\mathcal{S}\mathrm{pc}$ as a whole.

- $\mathrm{Ob}(C * D) = \mathrm{Ob}(C) \amalg \mathrm{Ob}(D)$
- $\mathrm{Hom}_{C*D}(x,y) = \begin{cases} \mathrm{Hom}_C(x,y) \text{ if } x \in C, y \in C, \\ \mathrm{Hom}_D(x,y) \text{ if } x \in D, y \in D, \\ * \text{ if } x \in C, y \in D, \\ \emptyset \text{ if } x \in D, y \in C. \end{cases}$

For example, $C * \Delta^0$ formally adds a final object to C and $\Delta^0 * C$ formally adds an initial object.

The next definition mimicks the previous one for simplicial sets :

Definition 1.11. Let $X, Y \in \mathrm{sSet}$. For every finite linearly ordered set J, define :

$$\mathrm{Cut}(J) = \left\{ (J_1, J_2) \middle| \begin{array}{l} J = J_1 \amalg J_2, \\ x < y \text{ for all } x \in J_1, y \in J_2 \end{array} \right\}$$

and then

$$(X * Y)(J) = \coprod_{(J_1,J_2) \in \mathrm{Cut}(J)} X(J_1) \times Y(J_2).$$

For every $\alpha : J \to J'$, a cut of J' induces in the obvious way a cut of J so there is a well-defined transition map α^*.

Remark 1.12. This is compatible with the definition for 1-categories, as :

$$\mathrm{N}(C * D) = \mathrm{N}(C) * \mathrm{N}(D).$$

In particular, $\Delta^n * \Delta^m = \Delta^{n+m+1}$. Also, if $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories then $\mathcal{C} * \mathcal{D}$ is also an ∞-category.

Notice that $(X * Y)_n = X_n \amalg Y_n \amalg \coprod_{i+j=n-1} X_i * Y_j$, so that $X * Y$ receives a map both from X and from Y. Therefore there are functors :

$$X * - : \mathrm{sSet} \to \mathrm{sSet}_{X/}$$

and

$$- * Y : \mathrm{sSet} \to \mathrm{sSet}_{Y/}.$$

Fact For every simplicial set S, the functor $S * -$ admits a right adjoint $\mathrm{sSet}_{S/} \to \mathrm{sSet}$ defined by sending $p : S \to X$ to $X_{p/}$ with

$$(X_{p/})_n = \mathrm{Hom}_p(S * \Delta^n, X).$$

Similarly, $- * S$ admits a right adjoint sending p to $X_{/p}$, with

$$(X_{/p})_n = \mathrm{Hom}_p(\Delta^n * S, X).$$

In particular if $p : \Delta^n \to X$ classifies an n-simplex $\sigma \in X_n$, write $X_{\sigma/}$ and $X_{/\sigma}$ instead.

For a simplicial set K, also denote by $K^{\triangleright}$ the cocone $K * \Delta^0$ and by $K^{\triangleleft}$ the cone $\Delta^0 * K$.

Definition 1.13. If $F : \mathcal{J} \to \mathcal{C}$ is an ∞-functor, define the *cocovers of* $\mathcal{C}$ *under* F as the pullback[3] in $\mathcal{C}\mathrm{at}_\infty$:

$$\begin{tikzcd} \mathcal{C}_{F/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\mathcal{J}^{\triangleright}, \mathcal{C}) \arrow[d] \\ * \arrow[r] & \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \end{tikzcd}$$

Definition 1.14. A *colimit* for F is an initial object in $\mathcal{C}_{F/}$.

Remark 1.15. A colimit for F is an object of $\mathcal{C}_{F/}$, and can therefore be identified as a map $\overline{F} : \mathcal{J}^{\triangleright} \to \mathcal{C}$. In particular[4], the colimit *remembers* the whole diagram, not just the object - which is the value of $\overline{F}$ on the added point of $\mathcal{J}^{\triangleright}$, and which is also called the colimit.

One can of course dualize everything to get the notion of a limit.

Theorem 1.16 (Rectification theorem, Lurie)**.** *Let J be a fibrant simplicial category and $\mathscr{M}$ be a combinatorial simplicial model category. Let $F : J \to \mathscr{M}$ be a simplicial functor and $\eta : F \to C$ be a natural transformation, C being constant to some $c \in \mathscr{M}$. Then the following are equivalent :*

1. *η exhibits c as a homotopy colimit of F ;*
2. *The image of η in $\infty(\mathscr{M})$ is an ∞-categorical colimit for $\infty(F)$.*

1.5 Cartesian fibrations and the Grothendieck construction

This subsection is about how one can build new ∞-categories from old ones, and more importantly how to talk about sheaves of ∞-categories.

Example 1.17. How is $\mathrm{QCoh}(-)$ a sheaf, or even a functor of categories on schemes ?

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes. Then we have a pullback functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ and similarly for g. However, f^*g^* and $(gf)^*$ are not equal, there is merely a natural transformation between them.

Instead of looking at such a strict definition of functoriality, why don't we look at the category $\mathrm{QCoh}_{\mathrm{Sch}}$ whose objects are pairs $(X, \mathcal{F})$ with X a scheme and $\mathcal{F} \in \mathrm{QCoh}(X)$ and whose morphisms $(X', \mathcal{F}') \to (X, \mathcal{F})$ are morphisms $f : X' \to X$ together with $\phi : f^*\mathcal{F} \to \mathcal{F}'$ in $\mathrm{QCoh}(X')$, and its projection functor $(X, \mathcal{F}) \mapsto X$ to the category Sch of schemes ?

This is a much better behaved object, which knows enough to witness a form of functoriality for $\mathrm{QCoh}(-)$: it is a cartesian fibration.

Definition 1.18. Let $p : C \to D$ be a functor (between 1-categories), $\overline{f} : \overline{x} \to p(y)$ be a morphism in D. Suppose that there is a morphism $f : x \to y$ in C such that $p(x) = \overline{x}$ and $p(f) = \overline{f}$.

We say that f is *p-cartesian* if for every object $z \in C$, the canonical map

$$\mathrm{Hom}_C(z, x) \to \mathrm{Hom}_C(z, y) \times_{\mathrm{Hom}_D(p(z), p(y))} \mathrm{Hom}_D(p(z), \overline{x})$$

[3]this should be a homotopy pullback, but one can prove that the arrow on the right is a fibration for the Joyal model structure on simplicial sets, so that there is no replacement to be made.

[4]and this makes sense, even for 1-categories, since the legs of the (co)cone are maps that have to be considered in the data of the (co)limit

is a bijection.

Say that p is a *cartesian fibration* if every $\overline{f}$ has such a cartesian lift f.

For a category C, define $\mathrm{Cart}(C) = \mathrm{Cat}^{\mathrm{Cart}}_{/C}$ to be the $(2,1)$-category of cartesian fibrations over C.

The functoriality data hidden in the cartesian fibration can be uncovered by straightening it into a functor of categories.

Theorem 1.19 (Straightening/Unstraightening, Grothendieck)**.** *There is an equivalence between* $(2,1)$*-categories*

$$\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Cat}) \simeq \mathrm{Cart}(C).$$

Lecture notes in Derived Algebraic Geometry
Session 2

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10 February 2025

1 More on limits in an ∞-category

There is another way to define the notion of a (co)limit in an ∞-category, which looks a bit more like the 1-categorical notion. Let $F : I \to \mathcal{C}$ be an element of $\mathrm{Map}(I, \mathcal{C})$, with I a simplicial set and $\mathcal{C}$ an ∞-category.

Definition 1.1. A *cone* over F is a pair (y, η) where $y \in \mathcal{C}$ and $\eta : \mathcal{C}_y \to F$ is a natural transformation (i.e. a map $\eta : I \times \Delta^1 \to C$ which restricts to $C_y : I \to \Delta^0 \xrightarrow{y} \mathcal{C}$ on $\{0\}$ and to F on $\{1\}$).

Let (y, η) be a cone over F and $x \in \mathcal{C}$. Letting $c : \mathcal{C} \to \mathcal{C}^I$ be the functor taking an object $z \in \mathcal{C}$ to the constant diagram $c_z : I \to \mathcal{C}$ on z, we can define a map :

$$\mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{c} \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, c_y) \xrightarrow{\eta_*} \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, F)$$

up to a contractible choice for the composition with η.

Proposition 1.2. *A cone (y, η) over F is a limit cone for F if for all $x \in \mathcal{C}$, the map above is a homotopy equivalence.*

Example 1.3. If I is discrete, then $\mathrm{Fun}(I, \mathcal{C}) = \prod_I \mathcal{C}$. A diagram $F : I \to \mathcal{C}$ is then a collection of objects $\{y_i\}_{i \in I}$, and a cone over F is a collection of maps of the form $\{y \xrightarrow{\pi_i} y_i\}_{i \in I}$. It is a limit cone if and only if $\mathrm{Map}(x, y) \simeq \prod_{i \in I} \mathrm{Map}(x, y_i)$.

Lemma 1.4. *Any two limits (y, η) and (y', η') for F are equivalent.*

Proof. Since (y', η') is a limit, we have $\mathrm{Map}(c_y, F) \simeq \mathrm{Map}(y, y')$. The image of η gives a map $f : y \to y'$ such that $\eta' \circ f \simeq \eta$. Similarly, there is a map $g : y' \to y$ such that $\eta \circ g \simeq \eta'$. We then get

$$\eta \circ g \circ f \simeq \eta' \circ f \simeq \eta$$

so $g \circ f \simeq \mathrm{id}$. Similarly, $f \circ g \simeq \mathrm{id}$. □

Remark 1.5. One can show the (much) stronger statement that the ∞-category of limit cones over a given functor is either empty or trivial (equivalent to $\{*\}$). This means that when a diagram has a limit, then the “space” of limits is contractible, whereas the lemma only showed that it is connected.

Exercise 1.6. Let $I = \{0 \to 1 \leftarrow 0'\}$ be the "pullback" diagram. A functor $F : I \to \mathcal{C}$ is equivalently the data of two maps $b \xrightarrow{h} d \xleftarrow{k} c$ in $\mathcal{C}$. Show that up to equivalence, the datum of a cone $\eta : c_a \to F$ is equivalent to the data of two morphisms $b \xleftarrow{i} a \xrightarrow{j} c$ and of a choice of equivalence $h \circ i \simeq k \circ j$.

Proposition 1.7. *A cone (y, η) over $F : I \to \mathcal{S}\mathrm{pc}$ is a limit cone if and only if for all $x \in \mathcal{S}\mathrm{pc}$, the map*

$$[x, y]_{\mathcal{S}\mathrm{pc}} = \pi_0 \operatorname{Map}(x, y) \to [c_x, F]_{\mathcal{S}\mathrm{pc}^I} = \pi_0 \operatorname{Map}(c_x, F)$$

is an equivalence.

In other words, the fact that a cone *of spaces* is a limit can be checked in the homotopy category. (This does not mean that the limit can be computed in the homotopy category.)

Exercise 1.8. Check that the pullback of a diagram $b \to c \leftarrow d$ in $\mathcal{S}\mathrm{pc}$ can be computed as the ordinary limit (iterated pullback) of the following diagram in sSet :

$$\begin{tikzcd} b \arrow[dr] & & c^{\Delta^1} \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & & d \arrow[dl] \\ & c & & c & \end{tikzcd}$$

Proposition 1.9. *Let $\mathcal{C}$ be a Kan-enriched category, I a simplicial set and $F : I \to \mathrm{N}_\Delta(\mathcal{C})$ a map to the simplicial nerve. Consider the functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{sSet}$ and apply the simplicial nerve functor to get a functor*

$$\mathrm{N}_\Delta(\underline{\mathrm{Hom}}(x, -)) : \mathrm{N}_\Delta(\mathcal{C}) \to \mathcal{S}\mathrm{pc}.$$

Then a cone (y, η) is a limit cone for F if and only if the cone $(\underline{\mathrm{Hom}}(x, y), \mathrm{N}_\Delta(\underline{\mathrm{Hom}}(x, \eta)))$ is a limit in $\mathcal{S}\mathrm{pc}$.

In other words, "the Map functor preserves limits in the second argument" in the following way :

$$\operatorname{Map}_{\mathcal{C}}(x, \lim_I F) \simeq \lim_{i \in I} \operatorname{Map}(x, F(i)).$$

The limit on the left is performed in $\mathcal{C}$, whereas the one on the right is performed in the ∞-category of spaces $\mathcal{S}\mathrm{pc}$.

Remark 1.10. A cone for F is equivalent to the datum of a map

$$(I \times \Delta^1)/(I \times \Delta^0) \to \mathcal{C}$$

where the contracted $I \times \Delta^0$ is the copy of I that is located at $\{0\}$. This quotient is another model for $I^{\triangleleft}$.

Of course, one gets the notion of a colimit by dualizing this whole section.

Theorem 1.11. *Let $\mathcal{C}$ be an ∞-category. Then the following are equivalent :*

1. *$\mathcal{C}$ has an initial object and pushouts ;*

2. *$\mathcal{C}$ has finite coproducts and coequalizers ;*
3. *$\mathcal{C}$ has finite limits.*

Moreover, if $\mathcal{C}$ has arbitrary coproducts, then the following are equivalent :

1. *$\mathcal{C}$ has all colimits ;*
2. *$\mathcal{C}$ has pushouts ;*
3. *$\mathcal{C}$ has coequalizers ;*
4. *$\mathcal{C}$ has geometric realizations (colimits of diagrams of shape Δ^{op}).*

2 Cartesian fibrations

Let $p : \mathcal{X} \to \mathcal{Y}$ be a functor of ∞-categories modelled by an inner fibration of simplicial sets. It means that it has the right lifting property with respect to every inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$: every square of solid arrows of the following form has a dashed lift

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

for all $0 < k < n$. Let $f : \Delta^1 \to \mathcal{X}$ be a an arrow in $\mathcal{X}$.

Definition 2.1. The map f is *p-cartesian* if every diagram of solid arrows of the following form has a dashed lift :

$$\begin{tikzcd} \Delta^1 = \Delta^{\{n-1,n\}} \arrow[r, hook] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

Similarly, f is *p-cocartesian* if the same condition is true with $\Delta^{\{0,1\}} \hookrightarrow \Lambda^n_0$ instead.

Lemma 2.2. *f is p-cartesian if and only if the map*

$$\mathcal{X}_{/f} \to \mathcal{X}_{/y} \times_{\mathcal{Y}_{/p(y)}} \mathcal{Y}_{/p(f)}$$

is a trivial fibration of simplicial sets. (A trivial fibration is a map that has the right lifting property with respect to every boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$.)

Definition 2.3. A functor $p : \mathcal{X} \to \mathcal{Y}$ is a *Cartesian fibration* if it is an inner fibration such that every lifting problem

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, "1"'] & \mathcal{X} \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

has a solution $\Delta^1 \to \mathcal{X}$ which is p-cartesian. Dually, p is a *coCartesian fibration* if it is an inner fibration such that every lifting problem

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, "0"'] & \mathcal{X} \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

has a solution $\Delta^1 \to \mathcal{X}$ which is p-cocartesian.

A Cartesian (resp. coCartesian) fibration is a tractable way of constructing a contravariant (resp. covariant) functor with values in ∞-categories, via the following straightening construction.

Let $p : \mathcal{E} \to \mathcal{C}$ be a coCartesian fibration.

1. For all $x \in \mathcal{C}$, let $\mathcal{E}_x$ be the fiber $\mathcal{E} \times_{\mathcal{C}} \Delta^0$ over x. It is automatically a weak Kan complex since p is an inner fibration.

2. Let $f : x \to y$ be a morphism in $\mathcal{C}$, we want to construct an associated functor $f_! : \mathcal{E}_x \to \mathcal{E}_y$. Let $z \in \mathcal{E}_x$. Since $p(z) = x$, we have a morphism $f : p(z) \to y$ in $\mathcal{C}$, so a square

$$\begin{tikzcd} \Delta^0 \arrow[r, "z"] \arrow[d, "0"'] & \mathcal{E} \arrow[d, "p"] \\ \Delta^1 \arrow[r, "f"'] & \mathcal{C} \end{tikzcd}$$

   There *exists* a p-cocartesian lift $\Delta^1 \to \mathcal{E}$, seen as a map $z \to w$ with $w \in \mathcal{E}_y$. We want to define $f_!(z) = w$: how ambiguous is this construction ?

3. Given $z' \in \mathcal{E}_x$ and $\alpha : z \to z'$, we can choose another p-cocartesian lift $z' \to w' = f_!(z')$. By p-cocartesianity, the following diagram can be extended with a dashed arrow, giving the desired equivalence :

$$\begin{tikzcd} z \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & z' \arrow[d] \\ w \arrow[r, dashed] & w' \end{tikzcd}$$

Proposition 2.4. *Let $p : \mathcal{E} \to \mathcal{C}$ be a coCartesian fibration and $f : \Delta^1 \to \mathcal{C}$ be a map $x \to y$. Then there exists a functor $\mathcal{E}_x \times \Delta^1 \to \mathcal{E}$ such that*

1. *For every $z \in \mathcal{E}_x$, restricting to z gives an arrow $\{z\} \times \Delta^1 \to \mathcal{E}$ of the form $\beta : z \to z'$ which is a p-cocartesian lift of f ;*

2. *The restriction to $\mathcal{E}_x \times \{1\}$ yields the desired functor $f_! : \mathcal{E}_x \to \mathcal{E}_y$.*

Ideas of the proof. Let $\mathrm{Fun}_f(\Delta^1, \mathcal{E}) = \{g : \Delta^1 \to \mathcal{E} \mid p \circ g \simeq f\}$ and $\mathrm{Fun}_f^{\mathrm{cc}}(\Delta^1, \mathcal{E})$ be the subcategory on the p-cocartesian lifts. There is a functor $\mathrm{Fun}_f^{\mathrm{cc}}(\Delta^1, \mathcal{E}) \to \mathcal{E}_x$ taking the source, and it is a trivial fibration. Choose a section $\mathcal{E}_x \to \mathrm{Fun}_f^{\mathrm{cc}}(\Delta^1, \mathcal{E})$: by adjunction, it gives the functor announced in the statement. Composing this section with the target functor yields the desired functor :

$$f_! : \mathcal{E}_x \to \mathrm{Fun}^{\mathrm{cc}}_f(\Delta^1, \mathcal{E}) \xrightarrow{\text{target}} \mathcal{E}_y.$$

□

More generally, for every simplex $\sigma : \Delta^n \to \mathcal{C}$ one can define $\mathrm{Fun}^{\mathrm{cc}}_\sigma(\Delta^n, \mathcal{E})$, and then a functor

$$\vartheta(p) : \Delta^{\mathrm{op}}_{/\mathcal{C}} \to \mathrm{sSet}$$

associating to $\sigma : \Delta^n \to \mathcal{C}$ the ∞-category $\mathrm{Fun}^{\mathrm{cc}}_\sigma(\Delta^n, \mathcal{E})$. Denoting by $W_\mathcal{C}$ the set of morphisms $f : [n] \to [m]$ in Δ_C such that $f(0) = 0$, one can show that $\vartheta(p)$ sends its elements to Joyal equivalences.

Corollary 2.5. *For every coCartesian fibration $p : \mathcal{E} \to \mathcal{C}$, there is a functor*

$$\Theta(p) : \mathrm{N}(\Delta^{\mathrm{op}}_{/\mathcal{C}})[W_\mathcal{C}^{-1}] \to \mathcal{C}\mathrm{at}_\infty$$

and the map from the source category to $\mathcal{C}$ taking the initial vertex is a Joyal equivalence, whence a functor $\mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$.

This is the straightening of p. We finally obtain the un/straightening theorem that was needed to construct functors with values in ∞-categories.

Theorem 2.6. *For every ∞-category $\mathcal{C}$, there are equivalences of ∞-categories*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty) \simeq \mathrm{CoCart}(\mathcal{C})$$

and

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty) \simeq \mathrm{Cart}(\mathcal{C}).$$

Cartesian fibrations also give a way to compute limits of diagrams of ∞-categories.

Theorem 2.7. *Let $F : \mathcal{J} \to \mathcal{C}\mathrm{at}_\infty$ be an ∞-functor, i.e. a diagram of ∞-categories. Let $\int F \to \mathcal{J}^{\mathrm{op}}$ be the associated Cartesian fibration. Then the limit of F exists and can be computed as the ∞-category of cartesian sections (that is, functors that send edges to cartesian maps) :*

$$\lim_{\mathcal{J}} F \simeq \mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{J}^{\mathrm{op}}}(\mathcal{J}^{\mathrm{op}}, \int F).$$

3 Cocompletions

3.1 The 1-categorical story

Let C be an ordinary 1-category. There is a Yoneda embedding $C \to \mathcal{P}(C)^\heartsuit$ into the 1-category of presheaves of sets on C. It has the universal property that $\mathrm{Fun}'(\mathcal{P}(C)^\heartsuit, D) \simeq \mathrm{Fun}(C, D)$ for every cocomplete category D, the notation Fun' denoting the subcategory of functors that preserve colimits.

Exercise 3.1. Show that the Yoneda functor does not preserve coproducts.

This is a problem, but it can be corrected by considering sifted colimits.

Definition 3.2. A 1-category C is *1-sifted* if it is not empty and if C-indexed colimits in Set commute with finite products.

Exercise 3.3. Show that C is 1-sifted if and only if it is not empty and the diagonal map $\Delta : C \to C \times C$ is cofinal.

Example 3.4. The category $\Delta^{\mathrm{op}}_{\leqslant 1}$ modelling reflexive coequalizers is 1-sifted. Note that the category $\{[1] \rightrightarrows [0]\}$ modelling coequalizers is not 1-sifted.

Definition 3.5. Let C be a 1-category with finite coproducts. Define $\mathcal{P}_\Sigma(C)^\heartsuit$ as the full subcategory of $\mathcal{P}(C)^\heartsuit$ on functors preserving finite coproducts (since they are contravariant, it means that they map finite sums to products).

Proposition 3.6. *1. The inclusion $\mathcal{P}_\Sigma(C)^\heartsuit \hookrightarrow \mathcal{P}(C)^\heartsuit$ preserves 1-sifted colimits and has a left adjoint. In particular, $\mathcal{P}_\Sigma(C)^\heartsuit$ is a reflexive localization of $\mathcal{P}(C)^\heartsuit$, so it is presentable ;*

2. *The Yoneda embedding $j : C \to \mathcal{P}(C)^\heartsuit$ lands in $\mathcal{P}_\Sigma(C)^\heartsuit$;*

3. *$j : C \to \mathcal{P}_\Sigma(C)^\heartsuit$ preserves finite coproducts ;*

4. *A presheaf X is in $\mathcal{P}_\Sigma(C)^\heartsuit$ if and only if it can be written as a reflexive coequalizer of filtered colimits of representables.*

Denoting by Fun'' *the category of sifted-colimit preserving functors and by* $\mathrm{Fun}^{\amalg}$ *the category of coproduct-preserving functors, we also have the universal properties*

$$\mathrm{Fun}''(\mathcal{P}_\Sigma(C)^\heartsuit, D) \simeq \mathrm{Fun}(C, D)$$

and

$$\mathrm{Fun}^{\amalg}(C, D) \simeq \mathrm{Fun}'(\mathcal{P}_\Sigma(C)^\heartsuit, D)$$

for every category D with colimits.

How does the story extend to ∞-categories ?

3.2 The ∞-categorical version

Let $\mathcal{C}$ be a an ∞-category and $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{pc})$ be its presheaf ∞-category. Then $\mathcal{P}(\mathcal{C})$ has all colimits, and there is again an equivalence

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}'(\mathcal{P}(\mathcal{C}), \mathcal{D})$$

for cocomplete ∞-categories $\mathcal{D}$.

Definition 3.7. An ∞-category $K \in \mathcal{C}\mathrm{at}_\infty$ is *sifted* if K-indexed colimits in $\mathcal{S}\mathrm{pc}$ commute with finite products.

Example 3.8. Δ^{op} is sifted. Its diagonal is cofinal ; it is well-known that if $X_{\bullet,\bullet}$ is a bisimplicial set then its geometric realization can be computed as the colimit of the diagonal $X_{n,n}$. On the other hand, the 1-sifted category $\Delta^{\mathrm{op}}_{\leqslant 1}$ is not sifted.

Definition 3.9. Let $\mathcal{C}$ be an ∞-category with finite coproducts. Then $\mathcal{P}_\Sigma(\mathcal{C})$ is defined as the full subcategory of $\mathcal{P}(\mathcal{C})$ on functors that preserve products.

The same proposition as in the 1-categorical picture holds.

Proposition 3.10. *1. The inclusion $\mathcal{P}_\Sigma(\mathcal{C}) \hookrightarrow \mathcal{P}(\mathcal{C})$ preserves sifted colimits and has a left adjoint. In particular, $\mathcal{P}_\Sigma(\mathcal{C})$ is presentable, and it has limits and colimits ;*

2. The Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ lands in $\mathcal{P}_\Sigma(\mathcal{C})$;

3. $j : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$ preserves finite coproducts ;

4. A presheaf X is in $\mathcal{P}_\Sigma(\mathcal{C})$ if and only if it is the geometric realization of filtered colimits of representables.

Also, we have the universal properties

$$\mathrm{Fun}''(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

and

$$\mathrm{Fun}^{\amalg}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}'(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}).$$

In particular, any functor $\mathcal{C} \to \mathcal{D}$ can be extended to a sifted-colimit-preserving functor $\mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{D}$.

4 Animated rings

There are now several equivalent ways to define derived schemes. The classical picture modelled them on simplicial commutative rings, but here we will prefer animated rings.

4.1 Compact projective objects

Definition 4.1. Let $\mathcal{C}$ be an ∞-category. An object $x \in \mathcal{C}$ is *compact projective* (resp. *compact*) if the functor $\mathrm{Map}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathcal{S}\mathrm{pc}$ commutes with sifted (resp. filtered) colimits.

Since a filtered colimit is a particular case of a sifted colimit, compact projective objects are compact. Denote by $\mathcal{C}^\omega$ the ∞-category of compact objects in $\mathcal{C}$, and by $\mathcal{C}^{\omega\mathrm{proj}}$ the ∞-category of compact projective objects in $\mathcal{C}$.

Definition 4.2. An ∞-category $\mathcal{C}$ is *compactly generated* if the natural map $\mathrm{Ind}(\mathcal{C}^\omega) \to \mathcal{C}$ is an equivalence. It is *generated by compact projective objects* if the natural map $\mathcal{P}_\Sigma(\mathcal{C}^{\omega\mathrm{proj}}) \to \mathcal{C}$ is an equivalence.

Remark 4.3. One can also define *1-compact projective* objects in a 1-category C as those $x \in C$ such that $\mathrm{Hom}_C(x, -)$ preserves 1-sifted colimits. C is then said to be *generated by 1-compact projective objects* if the natural map $\mathcal{P}_\Sigma(C^{\omega 1-\mathrm{proj}}) \to C$ is an equivalence.

Definition 4.4. Let C be an 1-category generated by its 1-compact projective objects. Then define its *animation* $\mathrm{Ani}(C)$ as the ∞-category $\mathcal{P}_\Sigma(C^{\omega\mathrm{proj}}) \subset \mathrm{Fun}(\mathrm{N}(C)^{\omega\mathrm{proj},\mathrm{op}}, \mathcal{S}\mathrm{pc})$.

4.2 Animated algebras

Example 4.5. Let R be a commutative ring. Denote by $\mathrm{Ani}(R)$ or $\mathrm{CAlg}_R^{\mathrm{ani}}$ the animation of the category of R-algebras.

What are the 1-compact projective algebras ?

Lemma 4.6. *Let R be a commutative ring. Then R is 1-compact projective (aka strongly finitely presented) if and only if it is a retract of a finite polynomial algebra over* $\mathbf{Z}$.

Proof. The free $\dashv$ forgetful adjunction $\mathrm{Set} \rightleftarrows \mathrm{CAlg}_{\mathbf{Z}}$ gives a reflexive coequalizer diagram

$$\begin{tikzcd} \mathbf{Z}[\mathbf{Z}[\underline{R}]] \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbf{Z}[\underline{R}] \arrow[l] \arrow[r, "\varepsilon"] & R \end{tikzcd}$$

Letting $S_1 = \mathbf{Z}[\mathbf{Z}[\underline{R}]]$ and $S_0 = \mathbf{Z}[\underline{R}]$, one shows that the coequalizer $\mathrm{colim}(S_1 \rightrightarrows S_0)$ is isomorphic to R. If R is 1-compact projective, then we have

$$\begin{aligned} &\mathrm{colim}(\mathrm{Hom}(R, S_1) \rightrightarrows \mathrm{Hom}(R, S_0)) \\ &\simeq \mathrm{Hom}(R, \mathrm{colim}(S_1 \rightrightarrows S_0)) \\ &\simeq \mathrm{Hom}(R, R). \end{aligned}$$

The image of the identity gives a section of ε, so R is a retract of the polynomial algebra $\mathbf{Z}[\underline{R}]$. Also, $\mathbf{Z}[\underline{R}] = \mathrm{colim}_{F \subset \underline{R} \text{ finite}} \mathbf{Z}[F]$ and since R is compact it factors through a $\mathbf{Z}[F]$. In particular, it is a retract of a *finite* polynomial algebra. $\square$

Remark 4.7. Animated commutative algebras can therefore be described in a more concise way as :

$$\mathrm{CAlg}_R^{\mathrm{ani}} \simeq \mathcal{P}_\Sigma(\mathrm{Poly}_R)$$

where Poly_R is the category of finite polynomial algebras over R.

Remark 4.8. As explained in the previous subsection, any animated R-algebra is the geometric realization of a simplicial object $X_\bullet$, where each X_n is a filtered colimit of finitely generated polynomial algebras.

Definition 4.9. An animated algebra $S \in \mathrm{Ani}(R)$ is *n-truncated* if for every $X \in \mathrm{Poly}_R$, the space $\mathrm{Map}_{\mathrm{Ani}(R)}(X, S)$ is n-truncated. (Recall that being n-truncated means that the homotopy groups of order higher than n vanish.) This defines a full subcategory $\tau_{\leqslant n} \mathrm{Ani}(R) \hookrightarrow \mathrm{Ani}(R)$.

This full inclusion has a left adjoint $\tau_{\leqslant n}$, given by composition with the usual truncation functor on spaces :

$$\tau_{\leqslant n} S : \mathrm{Poly}_R^{\mathrm{op}} \xrightarrow{S} \mathcal{S}\mathrm{pc} \xrightarrow{\tau_{\leqslant n}} \mathcal{S}\mathrm{pc}_{\leqslant n} \hookrightarrow \mathcal{S}\mathrm{pc}.$$

This composition still preserves finite products, so it is still an animated algebra ; and the functor $\tau_{\leqslant n} : \mathrm{Ani}(R) \to \mathrm{Ani}(R)$ that we obtain indeed lands in $\tau_{\leqslant n} \mathrm{Ani}(R)$.

Definition 4.10. For an animated R-algebra S, define its *homotopy ring* as

$$\pi_*(S) = \pi_*(\mathrm{Map}_{\mathrm{Ani}(R)}(R[t], S)).$$

It is a graded commutative ring.

Definition 4.11. If $S \in \mathrm{Ani}(R)$, then the space $S(R[t]) = \mathrm{Map}_{\mathrm{Ani}(R)}(R[t], S) \in \mathcal{S}\mathrm{pc}$ is called the *underlying space* of S.

Remark 4.12. There is a fully faithful functor

$$\mathrm{CAlg}_R^{\heartsuit} \hookrightarrow \mathrm{Ani}(R)$$

which is right adjoint to the functor $\pi_0 : \mathrm{CAlg}_R^{\mathrm{ani}} \to \mathrm{CAlg}_R^{\heartsuit}$.

5 Spectra

Recall that the suspension and loop space functors $\Sigma, \Omega : \mathcal{S}\mathrm{pc}_* \to \mathcal{S}\mathrm{pc}_*$ defined as the following pushout and pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

are not inverse of each other, nor are they even equivalences. A spectrum is a sequence $(X_i)_{i \geqslant 0}$ of pointed spaces together with equivalences $X_i \simeq \Omega X_{i+1}$; this construction effectively formally inverts the suspension and loop space functors and make them inverse equivalences of each other.

It turns out that spectra are much easier to define as an ∞-category than as a (model) 1-category :

Definition 5.1. Let $\mathrm{Eq}(\mathcal{S}\mathrm{pc}_*)$ be the full subcategory of $\mathcal{S}\mathrm{pc}_*^{\Delta^1}$ on arrows which are equivalences. It bears two functors $s, t : \mathrm{Eq}(\mathcal{S}\mathrm{pc}_*) \to \mathcal{S}\mathrm{pc}_*$, namely the source and target functor. Define the ∞-category of *spectra* $\mathcal{S}\mathrm{p} = \mathrm{Eq}(\mathcal{S}\mathrm{pc}_*) \times_{\mathcal{S}\mathrm{pc}_*} \mathrm{Eq}(\mathcal{S}\mathrm{pc}_*) \times_{\mathcal{S}\mathrm{pc}_*} \dots$ as the limit (infinite iterated pullback) in $\mathcal{C}\mathrm{at}_\infty$ of the following diagram :

$$\begin{tikzcd}[column sep=small] \mathrm{Eq}(\mathcal{S}\mathrm{pc}_*) \arrow[dr, "t"] & & \mathrm{Eq}(\mathcal{S}\mathrm{pc}_*) \arrow[dl, "\Omega\circ s"'] \arrow[dr, "t"] & & \mathrm{Eq}(\mathcal{S}\mathrm{pc}_*) \arrow[dl, "\Omega\circ s"'] \arrow[dr, "t"] & & \dots \arrow[dl, "\Omega\circ s"'] \\ & \mathcal{S}\mathrm{pc}_* & & \mathcal{S}\mathrm{pc}_* & & \dots & \end{tikzcd}$$

In simpler words, objects of $\mathcal{S}\mathrm{p}$ are lists $(X_i)_{i \geq 0}$ of pointed spaces together with a list of equivalences $X_i \simeq \Omega X_{i+1}$, and maps $X \to Y$ are lists of pointed maps $f_i : X_i \to Y_i$ together with a list of chosen homotopies filling the following square :

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "\simeq"'] & Y_i \arrow[d, "\simeq"] \arrow[dl, Rightarrow, shorten=2mm] \\ \Omega X_{i+1} \arrow[r, "\Omega f_{i+1}"'] & \Omega Y_{i+1} \end{tikzcd}$$

Remark 5.2. When passing from 1-categories to ∞-categories, the category Set is replaced by the ∞-category $\mathcal{S}\mathrm{pc}$. Following this idea, the category Ab of abelian groups has to be replaced by the ∞-category $\mathcal{S}\mathrm{p}_{\geqslant 0}$ of connective spectra (those whose negative homotopy groups vanish). Indeed, Ab is the category of commutative monoids in Set which are grouplike ; and by May's recognition theorem, the grouplike commutative monoids in $\mathcal{S}\mathrm{pc}$ are exactly the connective spectra.

Lecture notes in Derived Algebraic Geometry
Session 3

Course by F. Binda, notes by E. Hecky

17 February 2025

1 A word about simplicial rings and $\mathbb{E}_\infty$-rings

Recall from the previous session that if C is a category with coproducts, then $\mathrm{Ani}(C) = \mathcal{P}_\Sigma(C^{1-\omega\mathrm{proj}})$ is its animation. In particular, $\mathrm{CAlg}^{\mathrm{ani}} = \mathrm{Fun}^\Pi(\mathrm{Poly}_{\mathbf{Z}}^{\mathrm{op}}, \mathcal{S}\mathrm{pc})$ is the ∞-category of animated rings, and it contains fully faithfully the usual category $\mathrm{CAlg}^\heartsuit$ as the subcategory of 0-truncated objects.

Remark 1.1. Denoting by SCR the simplicial category of simplicial commutative rings, there is an equivalence

$$\infty(\mathrm{N}_\Delta(\mathrm{SCR})) \simeq \mathrm{CAlg}^{\mathrm{ani}}$$

between animated rings and the ∞-category associated to the simplicial nerve of SCR. That is the main reason why the simplicial approach of Toën-Vezzosi gives the same theory in the end as the more modern approach using animated rings.

Another approach that is linked to animated rings is the theory of $\mathbb{E}_\infty$-rings. Recall that the ∞-category $\mathcal{S}\mathrm{p}$ of spectra admits a structure of a symmetric monoidal stable ∞-category. In particular, it is possible to consider commutative algebra objects, and we get the ∞-category $\mathrm{CAlg}(\mathcal{S}\mathrm{p})$ of $\mathbb{E}_\infty$-rings spectra.

One can associate to any polynomial algebra a connective ring spectrum, yielding a functor

$$\mathrm{Poly}_{\mathbf{Z}} \to \mathrm{CAlg}(\mathcal{S}\mathrm{p}_{\geqslant 0})$$

which then extends to

$$\mathrm{CAlg}^{\mathrm{ani}} \to \mathbb{E}_\infty\text{-rings}.$$

However, this functor is not essentially surjective. Indeed, if A is an animated ring, then $\pi_*(A)$ is a graded commutative ring, each $\pi_n(A)$ is a module over the classical ring $\pi_0(A)$. More importantly, the even homotopy groups $\pi_{2*}(A)$ have a structure of a divided power algebra[1].

This is not the case for every $\mathbb{E}_\infty$-ring spectrum. For example, the algebra $\mathbb{F}_p[X]$ with X of degree 2 does not admit a PD-structure, so it doesn't come from any animated ring.

That being said, this phenomenon purely comes from torsion, and if we consider rational algebras, we get an equivalence :

$$\mathrm{CAlg}^{\mathrm{ani}}_{\mathbf{Q}} \simeq \mathrm{CAlg}(\mathcal{S}\mathrm{p}_{\geqslant 0})_{\mathbf{Q}/}.$$

[1] if A is an algebra over the rational numbers $\mathbf{Q}$, then this structure is just the usual division by integers.

2 Base change

Lemma 2.1. *Let $\varphi : R \to R'$ be a morphism in* $\mathrm{CAlg}^{\heartsuit}$*. It induces a functor* $\mathrm{Poly}_R \to \mathrm{Poly}_{R'}$ *sending P to $P \otimes_R R'$, inducing an adjunction :*

$$\mathrm{CAlg}_R^{\mathrm{ani}} \rightleftarrows \mathrm{CAlg}_{R'}^{\mathrm{ani}}$$

between extension and restriction of scalars.

Proof. In the following diagram, the diagonal composition descends to a unique extension of scalars functor :

$$\begin{tikzcd} \mathrm{Poly}_R \arrow[r, "-\otimes_R R'"] \arrow[d, hook, "j"'] \arrow[dr] & \mathrm{Poly}_{R'} \arrow[d, hook, "j"] \\ \mathrm{CAlg}_R^{\mathrm{ani}} \arrow[r, dashed, "\exists!"'] & \mathrm{CAlg}_{R'}^{\mathrm{ani}} \end{tikzcd}$$

as the image of $j \circ (- \otimes_R R')$ by the equivalence

$$\mathrm{Fun}_{\Sigma}(\mathrm{Poly}_R, \mathrm{CAlg}_{R'}^{\mathrm{ani}}) \simeq \mathrm{Fun}'(\mathrm{CAlg}_R^{\mathrm{ani}}, \mathrm{CAlg}_{R'}^{\mathrm{ani}}).$$

The existence of a right adjoint comes from presentability and the adjoint functor theorem. $\square$

This construction yields the same result as the "naive" idea of taking $R \in \mathrm{CAlg}^{\heartsuit}$ and $S, T \in \mathrm{CAlg}_R^{\mathrm{ani}}$, and defining $S \otimes_R^{\mathbb{L}} T$ as the corresponding pushout of animated algebras. To see this, just notice that they have the exact same behavior on polynomial algebras, and both constructions preserve enough colimits to extend to the same functor.

Remark 2.2. There is an equivalence $\mathrm{Ani}(\mathrm{CAlg}_R^{\heartsuit}) \simeq \mathrm{CAlg}_{R/}^{\mathrm{ani}}$ between the animation of the category of (classical) R-algebras and the slice category of animated algebras under R.

This remark being true for static rings R motivates the following definition for all animated rings.

Definition 2.3. Let $A \in \mathrm{CAlg}^{\mathrm{ani}}$ be an animated ring. Then define the ∞-category of animated A-algebras as :

$$\mathrm{CAlg}_A^{\mathrm{ani}} = \mathrm{Ani}(\mathrm{CAlg}^{\heartsuit})_{A/}.$$

In fact this is even functorial in A, in the sense that a functor

$$\mathrm{CAlg}_{(-)}^{\mathrm{ani}} : \mathrm{Ani}(\mathrm{CAlg}^{\heartsuit}) \to \mathcal{C}\mathrm{at}_{\infty}$$

can be obtained by straightening the extension/restriction of scalars adjunction constructed above.

3 Animated modules

Let $R \in \mathrm{CAlg}^{\heartsuit}$ be a classical ring. Then the compact projective R-modules $(\mathrm{Mod}_R^{\heartsuit})^{\omega\mathrm{proj}}$ are the retracts of finite free modules. One then gets the ∞-category of animated R-modules by taking Σ-presheaves of spaces on those modules.

It is possible to promote this construction in two different ways :

- allow R to be an animated ring ;
- make this construction functorial in R.

This section's goal is to do both at once.

Let $\mathrm{CRMod}^\heartsuit$ be the following 1-category. Its objects are pairs (A, M) with $A \in \mathrm{CAlg}^\heartsuit$ a ring and $M \in \mathrm{Mod}_A^\heartsuit$ an A-module. A morphism $(A, M) \to (B, N)$ consists of a ring map $f : A \to B$, and of a B-linear map $\varphi : M \otimes_A B \to N$ (or equivalently by adjunction, an A-linear map from M to the restriction of N).

Proposition 3.1. *The compact projective objects of* $\mathrm{CRMod}^\heartsuit$ *are the (retracts of) finite free modules over (retracts of) finite polynomial algebras over* $\mathbf{Z}$*. In other words, up to retracts, every compact projective object in this category looks like* $(\mathbf{Z}[\underline{X}], \mathbf{Z}[\underline{X}]^{\oplus n})$ *with* $\underline{X}$ *a finite set of unknowns and* $n \geqslant 0$*.*

Ideas of the proof. Let C be the category which is claimed to be $(\mathrm{CRMod}^\heartsuit)^{\omega\mathrm{proj}}$ in the statement. Notice that every object in $\mathrm{CRMod}^\heartsuit$ is a 1-sifted colimit of objects of C. Also, C is the closure under finite coproducts of the category spanned by the two generating objects $(\mathbf{Z}[T], 0)$ and $(\mathbf{Z}, \mathbf{Z})$. Finally, both these objects are themselves compact projective. $\square$

Keeping this notation, the animation $\mathrm{Ani}(\mathrm{CRMod}^\heartsuit)$ is the ∞-category of presheaves of spaces on this C. We then get two projection functors :

$$\mathrm{CAlg}^{\mathrm{ani}} \xleftarrow{\mathrm{pr}_1} \mathrm{Ani}(\mathrm{CRMod}^\heartsuit) \xrightarrow{\mathrm{pr}_2} \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{ani}}.$$

Notice that both functors have sections.

Remark 3.2. For every static ring $R \in \mathrm{CAlg}^\heartsuit$, the forgetful functor

$$\mathrm{Ani}(\mathrm{CRMod}_R^\heartsuit) \to \mathrm{CAlg}_R^{\mathrm{ani}} \times \mathrm{Mod}_R^{\mathrm{ani}}$$

is conservative.

This ∞-category together with the inner fibration pr_1 gives the desired definition and functoriality of the ∞-categories of animated modules.

Definition 3.3. For every animated ring $A \in \mathrm{CAlg}^{\mathrm{ani}}$, let Mod_A be the fiber of pr_1 over A. In other words, it is the following pullback

$$\begin{tikzcd} \mathrm{Mod}_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ani}(\mathrm{CRMod}^\heartsuit) \arrow[d, "\mathrm{pr}_1"] \\ \Delta^0 \arrow[r, "A"'] & \mathrm{CAlg}^{\mathrm{ani}} \end{tikzcd}$$

and it is indeed an ∞-category because pr_1 is an inner fibration.

Actually, pr_1 is fortunately more than an inner fibration. It is both a Cartesian and a coCartesian fibration, whence a functoriality of $\mathrm{Mod}_{(-)}$ both covariantly and contravariantly.

Theorem 3.4. pr_1 *is a biCartesian fibration.*

Proof. Let us prove that it is coCartesian. Concretely, for every animated ring map $f : A \to B$, we need to find $\mathcal{A}, \mathcal{B} \in \mathrm{Ani}(\mathrm{CRMod}^\heartsuit)$ whose pr_1 are respectively A and B, and a map $\phi : \mathcal{A} \to \mathcal{B}$ such that for every $X \in \mathrm{Ani}(\mathrm{CRMod}^\heartsuit)$, the following diagram is homotopy-cartesian in $\mathcal{S}\mathrm{pc}$:

$$\begin{tikzcd} \mathrm{Map}(\mathcal{B}, X) \arrow[r, "\phi"] \arrow[d, "\mathrm{pr}_1"'] & \mathrm{Map}(\mathcal{A}, X) \arrow[d, "\mathrm{pr}_1"] \\ \mathrm{Map}(B, \mathrm{pr}_1(X)) \arrow[r, "f"'] & \mathrm{Map}(A, \mathrm{pr}_1(X)) \end{tikzcd}$$

The homotopy-cartesianity condition is equivalent to the fibers of the vertical arrows being homotopy equivalent.

Since every possible choice of $\mathcal{A}$ and $\mathcal{B}$ are sifted colimits of compact projective objects, and since pullbacks commute with cosifted limits, it will suffice to treat the case of A and B compact projective, and to look for $\mathcal{A}$ and $\mathcal{B}$ compact projective as well. By definition of compact projectivity, it also suffices to treat the case where X itself is compact projective.

Now the mapping spaces in the diagram have all become discrete sets ! Write $\mathcal{A} = (A, M)$, $\mathcal{B} = (B, N)$ and $X = (\mathrm{pr}_1(X), M_X)$. Take $\psi : B \to \mathrm{pr}_1(X)$ and compare $\mathrm{pr}_1^{-1}(\psi) \in \mathrm{Map}(\mathcal{B}, X)$ which we can describe as

$$\mathrm{pr}_1^{-1}(\psi) = \{\tilde{\psi} : (B, N) \to (\mathrm{pr}_1(X), M_X) \mid \mathrm{pr}_1(\tilde{\psi}) = \psi\} \simeq \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{pr}_1(X)}}(N \otimes_B \mathrm{pr}_1(X), M_X),$$

with $\mathrm{pr}_1^{-1}(\psi \circ f) \in \mathrm{Map}(\mathcal{A}, X)$ which can similarly be described as

$$\mathrm{pr}_1^{-1}(\psi \circ f) \simeq \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{pr}_1(X)}}((M \otimes_A B) \otimes_B \mathrm{pr}_1(X), M_X).$$

Of course, only $f : A \to B$ is given, and we are free to choose M and N. We just need a morphism $\mathrm{pr}_2(\phi) : M \otimes_A B \to N$ such that $\mathrm{pr}_2(\phi) \otimes \mathrm{pr}_1(X)$ is an equivalence for every choice of X.

One can take $M = A$ and $N = B$ (and the fibers are both M_X), or even more stupidly, one can take $M = 0$ and $N = 0$ (and the fibers are both singletons). $\square$

This is the exact theorem that was needed to show the functoriality of the ∞-category of animated modules. We get two functors :

- $\mathrm{Mod} \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ani,op}}, \mathrm{Pr}^{\mathrm{R}})$ sending $\phi : A \to A'$ to the ∞-functor $\mathrm{Mod}(\phi) : \mathrm{Mod}_{A'} \to \mathrm{Mod}_A$ given by (generalized) restriction of scalars ;
- $\mathrm{Mod}^* \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ani}}, \mathrm{Pr}^{\mathrm{L},\otimes})$ sending $A \to A'$ to the symmetric monoidal ∞-functor of (generalized) extension of scalars $- \otimes_A^{\mathbb{L}} A' : \mathrm{Mod}_A \to \mathrm{Mod}_{A'}$.

Remark 3.5. There is another possible construction for animated modules, which is to define finite free animated A-modules for $A \in \mathrm{CAlg}^{\mathrm{ani}}$ as $\mathrm{FFree}_A = A \otimes_{\mathbf{Z}}^{\mathbb{L}} \mathrm{FFree}_{\mathbf{Z}}$ and then animate this to get Mod_A.

4 Geometrical notions animated

This section defines flat and étale maps of animated modules, as well as how one can perform animated localizations and quotients.

4.1 Flatness

Definition 4.1. Let $A \in \mathrm{CAlg}^{\mathrm{ani}}$ be an animated ring and $M \in \mathrm{Mod}_A$ be a (connective) A-module. M is *flat* if one/all of the following equivalent conditions holds :

1. (Homological flatness) The functor $M \otimes^{\mathbb{L}} -$ is left exact (i.e. it preserves finite limits) ;
2. (No higher Tor) For every $N \in \mathrm{Mod}_A$ which is n-truncated, the tensor product $M \otimes^{\mathbb{L}} N$ is also n-truncated ;
3. (Fiber-flatness) The following conditions hold :
   (a) $\pi_0(M)$ is a flat $\pi_0(A)$-module,
   (b) for every $i \geq 0$, the map
   $$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(M) \to \pi_i(M)$$
   is an equivalence.

Definition 4.2. An animated ring map $f : A \to B$ is called *flat* if B is flat as an A-module. It is called *faithfully flat* if it is flat and $- \otimes^{\mathbb{L}}_A B$ is a conservative functor.

Theorem 4.3 (Lurie, classically Lazard)**.** *Let A be an animated ring and $M \in \mathrm{Mod}_A$. Then M is flat if and only if it is a filtered colimit of finite free A-modules.*

4.2 Localization

When localizing, one has to be careful : trying to invert an element in a positive homotopy group of a ring forces us to create elements in the negative homotopy groups, infinitely far down - so localizing with respect to a positive homotopy class completely destroys any connectivity. This subsection is only about how to localize with respect to an element in the π_0 of an animated ring.

Let A be an animated ring. Its underlying space $\mathrm{Map}(\mathbf{Z}[T], A)$ contains the underlying space of units $\mathrm{Map}(\mathbf{Z}[T^{\pm 1}], A)$, which we awkwardly denote by $A^\times$ here. The inclusion induces an identification $\pi_0(A^\times) = \pi_0(A)^\times$.

Proposition 4.4. *Let $f : A \to B$ be an animated ring map, and $a \in \pi_0(A)$ be such that $f(a) \in \pi_0(B)^\times$. The following are then equivalent :*

1. *For every animated ring R, we have $\mathrm{Map}(B, R) \simeq \mathrm{Map}'(A, R)$ where Map' denotes the maps which send a to an invertible element in R ;*
2. *For every n, there is an equivalence*
   $$\pi_n(B) \simeq \pi_n(A) \otimes_{\pi_0(A)} \pi_0(A)[1/a].$$

When this is the case, we say that B is a/the *localization of A with respect to a*. The module B is then automatically flat over A, as in classical algebraic geometry.

How can one construct such a localization ? It is not hard, one can follow the same process as in classical ring theory : just construct the localization as the derived tensor product

$$\begin{tikzcd} \mathbf{Z}[T] \arrow[r, "a"] \arrow[d] & A \arrow[d] \\ \mathbf{Z}[T^{\pm 1}] \arrow[r] & B = A \otimes^{\mathbb{L}}_{\mathbf{Z}[T]} \mathbf{Z}[T^{\pm 1}] \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Defining it this way automatically proves the proposition. All choices of B, coming from all the possible choices of $a : \mathbf{Z}[T] \to A$, are contractibly equivalent.

With the exact same construction, it is of course possible to localize modules.

4.3 Quotients

Let A be an animated ring and $f_1, \ldots, f_n \in \pi_0(A)$ be elements, which we gather as a map $\mathbf{Z}[T_1, \ldots, T_n] \to A$. Then one can define the quotient, following the same strategy as the localization :

$$A /\!\!/ (f_1, \ldots, f_n) = A \otimes^{\mathbb{L}}_{Z[T_1,\ldots,T_n]} \frac{\mathbf{Z}[T_1, \ldots, T_n]}{(T_1, \ldots, T_n)}.$$

Proposition 4.5. *The quotient behaves like the usual quotient in classical ring theory :*

1. $\pi_0(A /\!\!/ (f_1, \ldots, f_n)) \simeq \pi_0(A)/(f_1, \ldots, f_n)$ *;*
2. *If A is a static local ring then $A /\!\!/ (f_1, \ldots, f_n)$ is equivalent to $A/(f_1, \ldots, f_n)$* if and only if *$(f_1, \ldots, f_n)$ is a regular sequence*[2] *;*
3. *$A /\!\!/ (f)$ is the cofiber of $f : A \to A$ (it was not clear that this cofiber had a ring structure), and more generally $A /\!\!/ (f_1, \ldots, f_n)$ is the tensor product of the cofibers of the $f_i : A \to A$*[3].

Exercise 4.6. A particular element of $\pi_0(A)$ is 0. What is the quotient $A /\!\!/ (0)$?

Definition 4.7. A map of animated rings $f : A \to B$ is *(formally) étale* if the following conditions hold :

1. $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is (formally) étale ;
2. For every $n \geq 0$, the map

$$\pi_n(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_n(B)$$

is an equivalence.

Theorem 4.8 (Lurie)**.** *Let A be an animated ring. Then the functor*

$$\pi_0 : \mathrm{CAlg}_A^{\text{ét}} \to \left(\mathrm{CAlg}^{\heartsuit}_{\pi_0(A)}\right)^{\text{ét}}$$

is an equivalence.

[2] this was to be expected, since we only want derived intersections of classical things to stay classical if they behaved well in the first place.

[3] this is to be compared with the Koszul complex

Theorem 4.9 (Lurie/Matthew/Toën-Vezzosi)**.** *For every faithfully flat animated ring map $A \to B$, the following diagram is a limit diagram in $\mathcal{C}\mathrm{at}_\infty$:*

$$\mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B \rightrightarrows \mathrm{Mod}_{B\otimes B} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

This theorem is the animated version of Grothendieck's faithfully flat descent. In fact, the functor $\mathrm{Mod}^* \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ani}}, \mathrm{Pr}^{\mathrm{L}})$ constructed above is a sheaf for the flat topology.

5 Square-zero extensions and the cotangent complex

This last section aims to go towards the construction of the cotangent complex, which will be finished in the next session.

Recall that if $A \in \mathrm{CAlg}^\heartsuit$ is a static commutative ring and $M \in \mathrm{Mod}_A^\heartsuit$ is a static A-module, then one can construct $A \oplus M \in \mathrm{CAlg}^\heartsuit_{A//A}$ as an augmented A-algebra. The product in this algebra is given by the following formula :

$$(a, m) \cdot (a', m') = (aa', am' + a'm),$$

which is a square-zero multiplication.

Remark 5.1. By design, this augmented algebra corepresents ($\mathbf{Z}$-linear) derivations :

$$\mathrm{Der}(A, M) \simeq \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{/A}}(A, A \oplus M).$$

It sends a derivation D to the map $s : a \mapsto (a, D(a))$.

If we want to introduce derivations in an ∞-categorical context, there is a problem : how can one define the Leibniz rule $D(ab) = D(a)b + aD(b)$ without strict equality ? The answer is that thanks to $A \oplus M$, we don't care : it suffices to define $A \oplus M$ and to define derivations as the things corepresented by this augmented algebra.

It looks like we have just moved the problem, because we now have to define the correct multiplication on $A \oplus M$. There is a brutal model-categorical way to do this, but we don't have to.

It suffices to animate the functor $\mathrm{CRMod}^\heartsuit \to \mathrm{CAlg}$ sending the static (A, M) to $A \oplus M$ as defined above. We get a functor $F : \mathrm{Ani}(\mathrm{CRMod}^\heartsuit) \to \mathrm{CAlg}^{\mathrm{ani}}$, satisfying the following basic properties :

1. F preserves static objects ;
2. If $M \simeq 0$ in Mod_A, then $F(M) \simeq A$ in $\mathrm{CAlg}^{\mathrm{ani}}$;
3. $A \oplus M \in \mathrm{CAlg}^{\mathrm{ani}}_{A//A}$ is an augmented A-algebra.

The third point is really easy to understand once we have the second one : consider the stupid sequence $0 \to M \to 0$ in Mod_A, and simply apply F. Since $F(0) \simeq A$, we obtain the desired sequence $A \to A \oplus M \to A$.

Lecture notes in Derived Algebraic Geometry
Session 4

Course by F. Binda, notes by E. Hecky

21 February 2025

The aim of this last session is to finish the construction of the cotangent complex, and to discuss a bit about how to globalize the ring-theoretic construction that were introduced in the previous session.

1 Square-zero extensions and the cotangent complex

This section is actually the continuation of the last section of the previous lesson. Recall that we defined an augmented A-algebra $A \oplus M$ for every animated module M over an animated ring A, satisfying intuitive basic properties.

Lemma 1.1. *There exists a fiber sequence*

$$M \to A \oplus M \to A$$

where the second map is an effective epimorphism, and a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit}) \arrow[r] \arrow[d, "\oplus"'] & \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit}) \arrow[d, "\mathrm{pr}_1"] \\ \mathrm{CAlg}^{\mathrm{ani}} \arrow[r] & \mathrm{CAlg}^{\mathrm{ani}} \end{tikzcd} \qquad \begin{tikzcd} (A, M) \arrow[r, mapsto] \arrow[d, mapsto] & (A, 0) \arrow[d, mapsto] \\ A \oplus M \arrow[r, mapsto] & A \end{tikzcd}$$

Remember that classically, $\mathrm{Der}(A, M) \simeq \mathrm{Hom}_{/A}(A, A \oplus M)$. This motivates the definition of derivations in the animated context.

Definition 1.2. Let $(A, M) \in \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit})$. Then define

$$\mathrm{Der}(A, M) = \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}_{/A}}(A, A \oplus M).$$

Remark 1.3. $A \oplus M$ is a loop space ! Indeed, there is a pullback square of the form

$$\begin{tikzcd} A \oplus M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ A \arrow[r] & A \oplus M[1] \end{tikzcd}$$

Also, $\mathrm{Der}(A, M)$ is the fiber of the map

$$\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}}(A, A \oplus M) \to \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}}(A, A).$$

Proposition 1.4. *The functor*

$$\mathrm{Der}(A,-) : \mathrm{Mod}_A \to \mathcal{S}\mathrm{pc}$$

is accessible and preserves limits. Therefore, it has a left adjoint

$$G : \mathcal{S}\mathrm{pc} \to \mathrm{Mod}_A.$$

Proof. For the accessibility, it is enough to observe that $\mathrm{Map}(A,-)$ is accessible since $M \mapsto A \oplus M$ commutes with sifted colimits by design (it is an animation), so it is accessible.

To show that the functor preserves limits, notice that the forgetful functor $\mathrm{CAlg}^{\mathrm{ani}}_{/A} \to \mathrm{CAlg}^{\mathrm{ani}}$ detects limits, so it is enough to show that if $p : K \to \mathrm{Mod}_A$ is a diagram of A-modules, then

$$\mathrm{Map}_{/A}(A[\underline{T}], A \oplus \lim_K p) \simeq \lim_K \mathrm{Map}_{/A}(A[\underline{T}], A \oplus p).$$

Why is this the case ? Recall that there is an adjunction :

$$\mathbb{L}\mathrm{Sym}_{\mathbf{Z}} : \mathrm{Mod}^{\mathrm{ani}}_{\mathbf{Z}} \rightleftarrows \mathrm{CAlg}^{\mathrm{ani}} : \mathrm{for}$$

between the (derived) free symmetric algebra and forgetful functors. Apply this adjunction and recall that small limits in Mod_A commute with $\oplus$ (which is simply the product in Mod_A) to obtain :

$$\begin{aligned}
\mathrm{Map}_{\mathrm{CAlg}_{/A}}(A[\underline{T}], A \oplus \lim_K p) &\simeq \mathrm{Map}_{\mathrm{Mod}_A}(A^{\oplus n}, \mathrm{for}(A \oplus \lim_K p)) \\
&\simeq \lim_K \mathrm{Map}_{\mathrm{Mod}_A}(A^{\oplus n}, \mathrm{for}(A \oplus p)) \\
&\simeq \lim_K \mathrm{Map}_{\mathrm{CAlg}_{/A}}(\mathbb{L}\mathrm{Sym}(A^{\oplus n}) = A[\underline{T}], A \oplus p).
\end{aligned}$$

□

One can now define the cotangent complex.

Definition 1.5. The *absolute cotangent complex* $\mathbb{L}_A$ is the module $G(*)$.

Remark 1.6. There is another, spectral way to construct the cotangent complex. If $\mathcal{C}$ is a symmetric monoidal stable ∞-category, then one can define a functor $G : \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ sending an augmented algebra $f : A \to \mathbf{1}$ to its cofiber. There is a theorem stating that G[1] induces an equivalence

$$\partial G : \mathcal{S}\mathrm{p}(\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})) \simeq \mathcal{S}\mathrm{p}(\mathcal{C}) \simeq \mathcal{C}$$

which can be used to define the cotangent complex. The advantage of this approach is that it lets us defined $A \oplus M$ for any non-necessarily animated $\mathbb{E}_\infty$-ring spectrum A.

Theorem 1.7. *The construction $A \mapsto \mathbb{L}_A$ extends to a functor*

$$\mathrm{CAlg}^{\mathrm{ani}} \to \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit})$$

sending $f : A \to B$ to $B \otimes^{\mathbb{L}}_A \mathbb{L}_A \to \mathbb{L}_B$. If A is a polynomial algebra then $\mathbb{L}_A \simeq \Omega_{A/\mathbf{Z}}$.

[1] or more precisely, its Goodwillie derivative

Definition 1.8. For an animated ring map $f : A \to B$, its *relative cotangent complex* $\mathbb{L}_{B/A}$ is the cofiber of the map $\mathbb{L}_A \otimes_A^{\mathbb{L}} B \to \mathbb{L}_B$.

Proposition 1.9. *There is an equivalence*

$$\mathrm{Map}_{\mathrm{Mod}_B}(\mathbb{L}_{B/A}, N) \simeq \mathrm{Map}_{\mathrm{CAlg}_{A//B}}(B, B \oplus N).$$

Define the space $\mathrm{Der}_A(B, N)$ of A-linear derivations from B to N to be this space.

Proposition 1.10. *The cotangent complex satisfies the following basic properties :*

1. *(Base change) If $B \simeq B' \otimes_{A'} A$, then $\mathbb{L}_{B/A} \simeq \mathbb{L}_{B'/A'} \otimes B$;*
2. *If $A \to B \to C$ is a fiber sequence, then there is a fiber sequence*
$$C \otimes_B \mathbb{L}_{B/A} \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B} \ ;$$
3. *For every map $f : A \to B$, there is an associated map $\varepsilon(f) : B \otimes_A \mathrm{Cof}(f) \to \mathbb{L}_{B/A}$. We have the following connectivity estimate : if the fiber of f is connective, then the fiber of $\varepsilon(f)$ is 2-connective.*

An important "slogan" in derived algebraic geometry which gives even more importance to this object is the idea that all animated rings can be built from discrete rings with square-zero extensions.

Definition 1.11. A map $S' \to S$ in $\mathrm{CAlg}^{\mathrm{ani}}$ is a *square-zero extension* if there is an S-module M and a derivation $d \in \pi_0 \mathrm{Map}_{/S}(S, S \oplus M[1])$ such that the following square is cartesian :

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S \arrow[d, "d_0"] \\ S \arrow[r, "d"'] & S \oplus M[1] \end{tikzcd}$$

Here, d_0 denotes the trivial derivation $S \to S \oplus M[1]$ which is just the inclusion of the first factor.

Proposition 1.12 (Lurie, DAG)**.** *Let M be an animated module over S. If S' is a square-zero extension of S by M and if $R \to S$ is any ring map, then the space of lifts* $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}_{/S}}(R, S')$ *is a torsor under* $\mathrm{Der}(R, M)$.

Concretely, this means that to get a lift $R \to S'$ in the following diagram

$$\begin{tikzcd} R \arrow[r, dashed] \arrow[dr] & S' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "d_0"] \\ & S \arrow[r, "d"'] & S \oplus M[1] \end{tikzcd}$$

it is sufficient and necessary that the derivation $R \to S \xrightarrow{d} S \oplus M[1]$ in $\pi_0 \mathrm{Der}(R, M[1]) = \pi_0 \mathrm{Map}_{\mathrm{Mod}_R}(\mathbb{L}_R, M[1])$ is the trivial derivation.

Example 1.13. Postnikov towers give rise to square zero extensions ! Recall that for every $n \geq 0$, there is a functor

$$\tau_{\leqslant n} : \mathrm{CAlg}^{\mathrm{ani}} \to \mathrm{CAlg}^{\mathrm{ani}}.$$

Then $\tau_{\leqslant n} R$ is a square-zero extension of $\tau_{\leqslant n-1} R$ by $(\pi_n R)[n]$, i.e. there is a pullback square :

$$\begin{tikzcd}
\tau_{\leqslant n} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tau_{\leqslant n-1} R \arrow[d, "d_0"] \\
\tau_{\leqslant n-1} R \arrow[r, dashed, "\exists d"'] & \tau_{\leqslant n-1} R \oplus (\pi_n R)[n+1]
\end{tikzcd}$$

This explains the philosophy of derived algebraic geometry, that after passing from varieties to schemes by adding non-reduced points, we pass from schemes to derived schemes by adding finer infinitesimal information. This Postnikov construction allows for handy inductive arguments using the cotangent complex.

Sketch of why this is true One easy thing to see first is that $\tau_{\leqslant n} R$ should equalize the two maps to the tensor product :

$$\tau_{\leqslant n} R \to \tau_{\leqslant n-1} R \rightrightarrows \tau_{\leqslant n-1} R \otimes_{\tau_{\leqslant n} R} \tau_{\leqslant n-1} R$$

(even though this might not be an equalizer diagram). Apply the $\tau_{\leqslant n+1}$ functor to this to get

$$\tau_{\leqslant n} R \to \tau_{\leqslant n-1} R \rightrightarrows \underbrace{\tau_{\leqslant n+1}(\tau_{\leqslant n-1} R \otimes_{\tau_{\leqslant n} R} \tau_{\leqslant n-1} R)}_{\star}.$$

What is $\star$? We can compute its homotopy groups with the long exact sequence and check that $\pi_*(\star) \simeq \pi_*(\tau_{\leqslant n-1} R \oplus \pi_n(R)[n+1])$. This should be a good clue that $\star$ is homotopy equivalent to the bottom-right object of the square in the statement. This would in turn show that there is a map from $\tau_{\leqslant n} R$ to the pullback. Of course, this all depends on the existence of a suitable derivation d, which will not be discussed.

2 Another word on geometrical aspects

Definition 2.1. Let $k \to R$ be a map of animated rings. We say that R is *derived formally smooth over* k if for every square-zero extension of k-algebras $S' \to S$ and for every map $R \to S$, there exists a lift in the following diagram of k-algebras :

$$\begin{tikzcd}
R \arrow[r] \arrow[dr, dashed] & S \\
& S' \arrow[u]
\end{tikzcd}$$

Proposition 2.2. *With the same notation, being derived formally smooth is equivalent to the vanishing of* $\pi_0 \operatorname{Map}(\mathbb{L}_{R/k}, M[1])$ *for every* $M \in \mathrm{Mod}_R$.

Definition 2.3. With the same notation as above and in the derived formally smooth assumption, we say that R is *derived formally étale* if moreover the space of lifts is contractible.

Proposition 2.4. *Being derived formally étale is equivalent to being formally étale in the sense of the definition from the previous session (namely, that the π_0 is étale and that $\pi_i(R) \simeq \pi_i(k) \otimes_{\pi_0(k)} \pi_0(R)$).*

Exercise 2.5. Let R be an animated ring and $x_1, \ldots, x_c \in \pi_0 R$. Then prove that

$$\mathbb{L}_{(R/\!\!/(x_1,\ldots,x_c))/R} \simeq R /\!\!/ (x_1, \ldots, x_c)^{\oplus c}[1].$$

If $\mathcal{J}$ is the fiber of the map $R \to R /\!\!/ (x_1, \ldots, x_c)$, then

$$\mathbb{L} \simeq \mathcal{J} \otimes_R R /\!\!/ (x_1, \ldots, x_c)[1].$$

Concretely, the cotangent complex "contains" the conormal bundle. To be more precise, if $A \to B$ is a map of static rings in $\mathrm{CAlg}^{\heartsuit}$, then

$$\tau_{\leqslant 1}\mathbb{L}_{B/A} \simeq \left[\mathcal{I}/\mathcal{I}^2 \to \Omega_A \otimes_A B\right].$$

Theorem 2.6 (Quillen, Lurie)**.** *Let $f : A \to B$ be a map whose cofiber is n-connective. Then $\varepsilon(f)$ is $(n+2)$-connective. Moreover, f is an equivalence is and only if $\mathbb{L}_{B/A} \simeq 0$ and $\pi_0(A) \simeq \pi_0(B)$.*

Definition 2.7. Let R be an animated ring and $M \in \mathrm{Mod}_R$. We say that M has *Tor-amplitude in* $[a, b]$ if for every static module $N \in \mathrm{Mod}^{\heartsuit}_{\pi_0 R}$, we have $H_i(M \otimes^{\mathbb{L}}_R N) = 0$ whenever $i \notin [a, b]$.

The following theorem, conjectured by Quillen decades earlier, is striking : it explains that under mild assumptions, the cotangent complex is either extremely simple (concentrated in degrees 0 and 1) or extremely complex (unbounded).

Theorem 2.8 (Avramov)**.** *If $f : k \to R$ is a map of noetherian rings, then it is locally complete intersection if and only if R is locally of finite flat dimension and $\mathbb{L}_{R/k}$ has bounded Tor-amplitude.*

Proposition 2.9. *Let A be an animated ring and $B \in \mathrm{CAlg}^{\omega}_A$ be a compact A-algebra. Then $\mathbb{L}_{B/A}$ is also compact as a B-module. The converse is also true if $\pi_0(B)$ is of finite presentation over $\pi_0(A)$.*

3 Global derived geometry

This last section aims to give a glimpse on how one can globalize the ring-theoretic constructions that have been discussed up until now, in order to define derived schemes.

3.1 Prestacks and affine derived schemes

There are two approaches, the first using ringed topoi and looking like the "locally ringed space" approach to schemes. The second one, which will be followed here, is the "functor of points and descent conditions" approach. It is in a sense more natural, at least algebraically.

Definition 3.1. Let PreStack $=$ Fun(CAlg$^{\text{ani}}$, $\mathcal{S}$pc) be the category of functors on derived rings. By Yoneda, any animated ring A defines a representable prestack $\mathrm{Spec}(A) = \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}}(A, -)$. Let $\mathrm{dSch}^{\mathrm{aff}}$ be the full subcategory of PreStack spanned by these affine derived schemes $\mathrm{Spec}(A)$.

Recall that there is an adjunction between the inclusion $\mathrm{CAlg}^{\heartsuit} \hookrightarrow \mathrm{CAlg}^{\mathrm{ani}}$ and the π_0 functor : this yields an adjunction by further abstract nonsense :

$$\pi_0^* : \mathrm{PreStack}^{\mathrm{cl}} \rightleftarrows \mathrm{PreStack} : (-)^{\mathrm{cl}}$$

where $\mathrm{PreStack}^{\mathrm{cl}} = \mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \mathcal{S}\mathrm{pc})$.

Remark 3.2. For every animated ring A, we have $\mathrm{Spec}(A)^{\mathrm{cl}} = \mathrm{Spec}(\pi_0 A)$.

3.2 Grothendieck topologies

Let us define the Zariski and étale topologies on $\mathrm{dSch}^{\mathrm{aff}}$.

Definition 3.3. Let I be a set and $\{j_\alpha : \mathrm{Spec}(B_\alpha) \to \mathrm{Spec}(A)\}_{\alpha \in I}$ be a family of maps in $\mathrm{dSch}^{\mathrm{aff}}$. Such a family is a *Zariski covering* (resp. *étale covering*) if the following conditions hold.

1. The functor $\mathrm{Mod}_A \to \prod_\alpha \mathrm{Mod}_{B_\alpha}$ is conservative ;

2. Each map $A \to B_\alpha$ is Zariski, i.e. it is an open immersion at the π_0 level and $\pi_i(B_\alpha) \simeq \pi_i(A) \otimes_{\pi_0(A)} \pi_0(B_\alpha)$ (resp. each map is (formally) étale in the sense already defined earlier).

A prestack F is then a sheaf for the topology τ (here, τ is Zar or ét) if for every such covering family, $F(A)$ is the limit of the corresponding Čech diagram $F(B^{\otimes \bullet})$.

One can similarly define a hypersheaf by imposing a descent condition for hypercovers.

Definition 3.4. A *derived stack* is a prestack that is an étale hypersheaf.

Remark 3.5. To define modules on a prestack $\mathfrak{X}$ instead of just an animated ring A, one can construct the right Kan extension of $\mathrm{Mod}^* : \mathrm{CAlg}^{\mathrm{ani,op}} \to \mathrm{Pr}^{\mathrm{L},\otimes}$ to get a new functor

$$\mathrm{QCoh} : \mathrm{PreStack} \to \mathrm{Pr}^{\mathrm{L},\otimes}.$$

By definition,

$$\mathrm{QCoh}(\mathfrak{X}) \simeq \lim_{\mathrm{Spec}(B) \in \mathrm{dSch}^{\mathrm{aff}}_{/\mathfrak{X}}} \mathrm{QCoh}(\mathrm{Spec}(B))$$

where $\mathrm{QCoh}(\mathrm{Spec}(B)) = \mathrm{Mod}_B$ is simply the ∞-category of animated B-modules.

Definition 3.6. A Zariski stack $\mathfrak{X}$ is a *derived scheme* if there exists a cover by affine derived schemes : there exists a family $\mathcal{U} = (\mathcal{U}_\alpha)$ with $\mathcal{U}_\alpha \to \mathfrak{X}$ open immersions from affine derived schemes $\mathcal{U}_\alpha \simeq \mathrm{Spec}(A_\alpha)$, such that $\coprod_\alpha \mathcal{U}_\alpha \to \mathfrak{X}$ is an effective epimorphism.

In this definition, "open immersion" refers to a morphism $\mathfrak{Y} \to \mathfrak{X}$ such that $\mathfrak{Y} \times_{\mathfrak{X}} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is an open immersion for every animated ring A. More generally, one can define representability in the following way.

Definition 3.7. A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is *(-1)-geometric* or *representable* if for every map $\mathrm{Spec}(A) \to \mathfrak{Y}$, the pullback $\mathfrak{X} \times_{\mathfrak{Y}} \mathrm{Spec}(A)$ is an affine derived scheme $\mathrm{Spec}(B)$.

The higher levels of geometricity are defined recursively.

Definition 3.8. If $\mathfrak{X}$ is a derived stack, we say that it is *(-1)-geometric* if it is affine. For every $n \geqslant 0$, we say that $\mathfrak{X}$ is *n-geometric* if the following conditions hold :

1. The diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is $(n-1)$-geometric ;
2. There is an affine covering $\coprod \mathrm{Spec}(B_i) \twoheadrightarrow \mathfrak{X}$ that is $(n-1)$-geometric and étale.

A morphism of derived stacks $f : \mathfrak{X} \to \mathfrak{Y}$ is *m-geometric* for $m \geqslant 0$ if for every map $\mathrm{Spec}(A) \to \mathfrak{Y}$, the pullback P in the following square

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

is m-geometric.

Along with the different levels of geometricity, there is also a hierarchy of levels of *stack-ity.*

Definition 3.9. A derived stack $\mathfrak{X}$ is an *n-stack* if for every static ring $A \in \mathrm{CAlg}^{\heartsuit}$, the space $\mathfrak{X}(A)$ is n-truncated.

In other words, higher stacks have "higher coherence data".

Remark 3.10. A classical Artin stack is a 1-geometric 1-stack with these definitions. A *higher Deligne-Mumford stack* would be a geometric stack with "smooth" replacing the word *étale* in the second condition of Definition 3.8.

Proposition 3.11. *The functor* $\mathrm{Perf} : \mathrm{CAlg}^{\mathrm{ani}} \to \mathcal{S}\mathrm{pc}$ *sending an animated ring A to the space of perfect complexes* $\left(\mathrm{Mod}_A^{\mathrm{perf}}\right)^{\simeq}$ *is a locally geometric stack : it is a filtered colimit of geometric stacks by monomorphic transition maps.*

Ideas of the proof. This filtered colimit is none other than $\mathrm{Perf} = \bigcup_{a \leqslant b} \mathrm{Perf}_{[a,b]}$ where $\mathrm{Perf}_{[a,b]}$ denotes the perfect complexes with Tor-amplitude in $[a, b]$.

One can show that $\mathrm{Perf}_{[a,b]}$ is $(b-a+1)$-geometric, by induction on $n = b-a+1$. The case $n = 1$ is already interesting, since $\mathrm{Perf}_{[a,a]} \simeq \mathrm{Vect} \simeq \coprod_m \mathrm{BGL}_m$. This is indeed 1-geometric, each $\mathrm{BGL}_m = [*/\mathrm{GL}_m]$ being a geometric realization. □