Lecture notes in Derived Algebraic Geometry
Session 2

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1 More on limits in an ∞-category

There is another way to define the notion of a (co)limit in an ∞-category, which looks a bit more like the 1-categorical notion. Let $F : I \to \mathcal{C}$ be an element of $\mathrm{Map}(I, \mathcal{C})$, with I a simplicial set and $\mathcal{C}$ an ∞-category.

Definition 1. A *cone* over F is a pair (y, η) where $y \in \mathcal{C}$ and $\eta : \mathcal{C}_y \to F$ is a natural transformation (i.e. a map $\eta : I \times \Delta^1 \to C$ which restricts to $C_y : I \to \Delta^0 \xrightarrow{y} \mathcal{C}$ on $\{0\}$ and to F on $\{1\}$).

Let (y, η) be a cone over F and $x \in \mathcal{C}$. Letting $c : \mathcal{C} \to \mathcal{C}^I$ be the functor taking an object $z \in \mathcal{C}$ to the constant diagram $c_z : I \to \mathcal{C}$ on z, we can define a map :

$$\mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{c} \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, c_y) \xrightarrow{\eta_*} \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, F)$$

up to a contractible choice for the composition with η.

Proposition 1. *A cone (y, η) over F is a limit cone for F if for all $x \in \mathcal{C}$, the map above is a homotopy equivalence.*

Example 1. If I is discrete, then $\mathrm{Fun}(I, \mathcal{C}) = \prod_I \mathcal{C}$. A diagram $F : I \to \mathcal{C}$ is then a collection of objects $\{y_i\}_{i \in I}$, and a cone over F is a collection of maps of the form $\{y \xrightarrow{\pi_i} y_i\}_{i \in I}$. It is a limit cone if and only if $\mathrm{Map}(x, y) \simeq \prod_{i \in I} \mathrm{Map}(x, y_i)$.

Lemma 1. *Any two limits (y, η) and (y', η') for F are equivalent.*

Proof. Since (y', η') is a limit, we have $\mathrm{Map}(c_y, F) \simeq \mathrm{Map}(y, y')$. The image of η gives a map $f : y \to y'$ such that $\eta' \circ f \simeq \eta$. Similarly, there is a map $g : y' \to y$ such that $\eta \circ g \simeq \eta'$. We then get

$$\eta \circ g \circ f \simeq \eta' \circ f \simeq \eta$$

so $g \circ f \simeq \mathrm{id}$. Similarly, $f \circ g \simeq \mathrm{id}$. □

Remark 1. One can show the (much) stronger statement that the ∞-category of limit cones over a given functor is either empty or trivial (equivalent to $\{*\}$). This means that when a diagram has a limit, then the "space" of limits is contractible, whereas the lemma only showed that it is connected.

Exercise 1. Let $I = \{0 \to 1 \leftarrow 0'\}$ be the "pullback" diagram. A functor $F : I \to \mathcal{C}$ is equivalently the data of two maps $b \xrightarrow{h} d \xleftarrow{k} c$ in $\mathcal{C}$. Show that up to equivalence, the datum of a cone $\eta : c_a \to F$ is equivalent to the data of two morphisms $b \xleftarrow{i} a \xrightarrow{j} c$ and of a choice of equivalence $h \circ i \simeq k \circ j$.

Proposition 2. *A cone (y, η) over $F : I \to \mathcal{S}\mathrm{pc}$ is a limit cone if and only if for all $x \in \mathcal{S}\mathrm{pc}$, the map*

$$[x, y]_{\mathcal{S}\mathrm{pc}} = \pi_0 \operatorname{Map}(x, y) \to [c_x, F]_{\mathcal{S}\mathrm{pc}^I} = \pi_0 \operatorname{Map}(c_x, F)$$

is an equivalence.

In other words, the fact that a cone *of spaces* is a limit can be checked in the homotopy category. (This does not mean that the limit can be computed in the homotopy category.)

Exercise 2. Check that the pullback of a diagram $b \to c \leftarrow d$ in $\mathcal{S}\mathrm{pc}$ can be computed as the ordinary limit (iterated pullback) of the following diagram in sSet :

$$\begin{tikzcd}
b \arrow[dr] & & c^{\Delta^1} \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & & d \arrow[dl] \\
& c & & c &
\end{tikzcd}$$

Proposition 3. *Let $\mathcal{C}$ be a Kan-enriched category, I a simplicial set and $F : I \to \mathrm{N}_\Delta(\mathcal{C})$ a map to the simplicial nerve. Consider the functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{sSet}$ and apply the simplicial nerve functor to get a functor*

$$\mathrm{N}_\Delta(\underline{\mathrm{Hom}}(x, -)) : \mathrm{N}_\Delta(\mathcal{C}) \to \mathcal{S}\mathrm{pc}.$$

Then a cone (y, η) is a limit cone for F if and only if the cone $(\underline{\mathrm{Hom}}(x, y), \mathrm{N}_\Delta(\underline{\mathrm{Hom}}(x, \eta)))$ is a limit in $\mathcal{S}\mathrm{pc}$.

In other words, "the Map functor preserves limits in the second argument" in the following way :

$$\operatorname{Map}_{\mathcal{C}}(x, \lim_I F) \simeq \lim_{i \in I} \operatorname{Map}(x, F(i)).$$

The limit on the left is performed in $\mathcal{C}$, whereas the one on the right is performed in the ∞-category of spaces $\mathcal{S}\mathrm{pc}$.

Remark 2. A cone for F is equivalent to the datum of a map

$$(I \times \Delta^1)/(I \times \Delta^0) \to \mathcal{C}$$

where the contracted $I \times \Delta^0$ is the copy of I that is located at $\{0\}$. This quotient is another model for $I^{\triangleleft}$.

Of course, one gets the notion of a colimit by dualizing this whole section.

Theorem 1. *Let $\mathcal{C}$ be an ∞-category. Then the following are equivalent :*

1. *$\mathcal{C}$ has an initial object and pushouts ;*

2. *$\mathcal{C}$ has finite coproducts and coequalizers ;*

3. *$\mathcal{C}$ has finite limits.*

Moreover, if $\mathcal{C}$ has arbitrary coproducts, then the following are equivalent :

1. *$\mathcal{C}$ has all colimits ;*

2. *$\mathcal{C}$ has pushouts ;*

3. *$\mathcal{C}$ has coequalizers ;*

4. *$\mathcal{C}$ has geometric realizations (colimits of diagrams of shape Δ^{op}).*

2 Cartesian fibrations

Let $p : \mathcal{X} \to \mathcal{Y}$ be a functor of ∞-categories modelled by an inner fibration of simplicial sets. It means that it has the right lifting property with respect to every inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$: every square of solid arrows of the following form has a dashed lift

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

for all $0 < k < n$. Let $f : \Delta^1 \to \mathcal{X}$ be a an arrow in $\mathcal{X}$.

Definition 2. The map f is *p-cartesian* if every diagram of solid arrows of the following form has a dashed lift :

$$\begin{tikzcd} \Delta^1 = \Delta^{\{n-1,n\}} \arrow[r, hook] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

Similarly, f is *p-cocartesian* if the same condition is true with $\Delta^{\{0,1\}} \hookrightarrow \Lambda^n_0$ instead.

Lemma 2. *f is p-cartesian if and only if the map*

$$\mathcal{X}_{/f} \to \mathcal{X}_{/y} \times_{\mathcal{Y}_{/p(y)}} \mathcal{Y}_{/p(f)}$$

is a trivial fibration of simplicial sets. (A trivial fibration is a map that has the right lifting property with respect to every boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$.)

Definition 3. A functor $p : \mathcal{X} \to \mathcal{Y}$ is a *Cartesian fibration* if it is an inner fibration such that every lifting problem

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, "1"'] & \mathcal{X} \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

has a solution $\Delta^1 \to \mathcal{X}$ which is p-cartesian. Dually, p is a *coCartesian fibration* if it is an inner fibration such that every lifting problem

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, "0"'] & \mathcal{X} \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

has a solution $\Delta^1 \to \mathcal{X}$ which is p-cocartesian.

A Cartesian (resp. coCartesian) fibration is a tractable way of constructing a contravariant (resp. covariant) functor with values in ∞-categories, via the following straightening construction.

Let $p : \mathcal{E} \to \mathcal{C}$ be a coCartesian fibration.

1. For all $x \in \mathcal{C}$, let $\mathcal{E}_x$ be the fiber $\mathcal{E} \times_{\mathcal{C}} \Delta^0$ over x. It is automatically a weak Kan complex since p is an inner fibration.

2. Let $f : x \to y$ be a morphism in $\mathcal{C}$, we want to construct an associated functor $f_! : \mathcal{E}_x \to \mathcal{E}_y$. Let $z \in \mathcal{E}_x$. Since $p(z) = x$, we have a morphism $f : p(z) \to y$ in $\mathcal{C}$, so a square

$$\begin{tikzcd} \Delta^0 \arrow[r, "z"] \arrow[d, "0"'] & \mathcal{E} \arrow[d, "p"] \\ \Delta^1 \arrow[r, "f"'] & \mathcal{C} \end{tikzcd}$$

   There *exists* a p-cocartesian lift $\Delta^1 \to \mathcal{E}$, seen as a map $z \to w$ with $w \in \mathcal{E}_y$. We want to define $f_!(z) = w$: how ambiguous is this construction ?

3. Given $z' \in \mathcal{E}_x$ and $\alpha : z \to z'$, we can choose another p-cocartesian lift $z' \to w' = f_!(z')$. By p-cocartesianity, the following diagram can be extended with a dashed arrow, giving the desired equivalence :

$$\begin{tikzcd} z \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & z' \arrow[d] \\ w \arrow[r, dashed] & w' \end{tikzcd}$$

Proposition 4. *Let $p : \mathcal{E} \to \mathcal{C}$ be a coCartesian fibration and $f : \Delta^1 \to \mathcal{C}$ be a map $x \to y$. Then there exists a functor $\mathcal{E}_x \times \Delta^1 \to \mathcal{E}$ such that*

1. *For every $z \in \mathcal{E}_x$, restricting to z gives an arrow $\{z\} \times \Delta^1 \to \mathcal{E}$ of the form $\beta : z \to z'$ which is a p-cocartesian lift of f ;*

2. *The restriction to $\mathcal{E}_x \times \{1\}$ yields the desired functor $f_! : \mathcal{E}_x \to \mathcal{E}_y$.*

Ideas of the proof. Let $\mathrm{Fun}_f(\Delta^1, \mathcal{E}) = \{g : \Delta^1 \to \mathcal{E} \mid p \circ g \simeq f\}$ and $\mathrm{Fun}_f^{\mathrm{cc}}(\Delta^1, \mathcal{E})$ be the subcategory on the p-cocartesian lifts. There is a functor $\mathrm{Fun}_f^{\mathrm{cc}}(\Delta^1, \mathcal{E}) \to \mathcal{E}_x$ taking the source, and it is a trivial fibration. Choose a section $\mathcal{E}_x \to \mathrm{Fun}_f^{\mathrm{cc}}(\Delta^1, \mathcal{E})$: by adjunction, it gives the functor announced in the statement. Composing this section with the target functor yields the desired functor :

$$f_! : \mathcal{E}_x \to \mathrm{Fun}^{\mathrm{cc}}_f(\Delta^1, \mathcal{E}) \xrightarrow{\text{target}} \mathcal{E}_y.$$

□

More generally, for every simplex $\sigma : \Delta^n \to \mathcal{C}$ one can define $\mathrm{Fun}^{\mathrm{cc}}_\sigma(\Delta^n, \mathcal{E})$, and then a functor

$$\vartheta(p) : \Delta^{\mathrm{op}}_{/\mathcal{C}} \to \mathrm{sSet}$$

associating to $\sigma : \Delta^n \to \mathcal{C}$ the ∞-category $\mathrm{Fun}^{\mathrm{cc}}_\sigma(\Delta^n, \mathcal{E})$. Denoting by $W_\mathcal{C}$ the set of morphisms $f : [n] \to [m]$ in Δ_C such that $f(0) = 0$, one can show that $\vartheta(p)$ sends its elements to Joyal equivalences.

Corollary 1. *For every coCartesian fibration $p : \mathcal{E} \to \mathcal{C}$, there is a functor*

$$\Theta(p) : \mathrm{N}(\Delta^{\mathrm{op}}_{/\mathcal{C}})[W^{-1}_\mathcal{C}] \to \mathcal{C}\mathrm{at}_\infty$$

and the map from the source category to $\mathcal{C}$ taking the initial vertex is a Joyal equivalence, whence a functor $\mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$.

This is the straightening of p. We finally obtain the un/straightening theorem that was needed to construct functors with values in ∞-categories.

Theorem 2. *For every ∞-category $\mathcal{C}$, there are equivalences of ∞-categories*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty) \simeq \mathrm{CoCart}(\mathcal{C})$$

and

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty) \simeq \mathrm{Cart}(\mathcal{C}).$$

Cartesian fibrations also give a way to compute limits of diagrams of ∞-categories.

Theorem 3. *Let $F : \mathcal{J} \to \mathcal{C}\mathrm{at}_\infty$ be an ∞-functor, i.e. a diagram of ∞-categories. Let $\int F \to \mathcal{J}^{\mathrm{op}}$ be the associated Cartesian fibration. Then the limit of F exists and can be computed as the ∞-category of cartesian sections (that is, functors that send edges to cartesian maps) :*

$$\lim_{\mathcal{J}} F \simeq \mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{J}^{\mathrm{op}}}(\mathcal{J}^{\mathrm{op}}, \int F).$$

3 Cocompletions

3.1 The 1-categorical story

Let C be an ordinary 1-category. There is a Yoneda embedding $C \to \mathcal{P}(C)^\heartsuit$ into the 1-category of presheaves of sets on C. It has the universal property that $\mathrm{Fun}'(\mathcal{P}(C)^\heartsuit, D) \simeq \mathrm{Fun}(C, D)$ for every cocomplete category D, the notation Fun' denoting the subcategory of functors that preserve colimits.

Exercise 3. Show that the Yoneda functor does not preserve coproducts.

This is a problem, but it can be corrected by considering sifted colimits.

Definition 4. A 1-category C is *1-sifted* if it is not empty and if C-indexed colimits in Set commute with finite products.

Exercise 4. Show that C is 1-sifted if and only if it is not empty and the diagonal map $\Delta : C \to C \times C$ is cofinal.

Example 2. The category $\Delta^{\mathrm{op}}_{\leqslant 1}$ modelling reflexive coequalizers is 1-sifted. Note that the category $\{[1] \rightrightarrows [0]\}$ modelling coequalizers is not 1-sifted.

Definition 5. Let C be a 1-category with finite coproducts. Define $\mathcal{P}_\Sigma(C)^\heartsuit$ as the full subcategory of $\mathcal{P}(C)^\heartsuit$ on functors preserving finite coproducts (since they are contravariant, it means that they map finite sums to products).

Proposition 5.

1. *The inclusion $\mathcal{P}_\Sigma(C)^\heartsuit \hookrightarrow \mathcal{P}(C)^\heartsuit$ preserves 1-sifted colimits and has a left adjoint. In particular, $\mathcal{P}_\Sigma(C)^\heartsuit$ is a reflexive localization of $\mathcal{P}(C)^\heartsuit$, so it is presentable ;*
2. *The Yoneda embedding $j : C \to \mathcal{P}(C)^\heartsuit$ lands in $\mathcal{P}_\Sigma(C)^\heartsuit$;*
3. *$j : C \to \mathcal{P}_\Sigma(C)^\heartsuit$ preserves finite coproducts ;*
4. *A presheaf X is in $\mathcal{P}_\Sigma(C)^\heartsuit$ if and only if it can be written as a reflexive coequalizer of filtered colimits of representables.*

Denoting by Fun'' *the category of sifted-colimit preserving functors and by* $\mathrm{Fun}^{\amalg}$ *the category of coproduct-preserving functors, we also have the universal properties*

$$\mathrm{Fun}''(\mathcal{P}_\Sigma(C)^\heartsuit, D) \simeq \mathrm{Fun}(C, D)$$

and

$$\mathrm{Fun}^{\amalg}(C, D) \simeq \mathrm{Fun}'(\mathcal{P}_\Sigma(C)^\heartsuit, D)$$

for every category D with colimits.

How does the story extend to ∞-categories ?

3.2 The ∞-categorical version

Let $\mathcal{C}$ be a an ∞-category and $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{pc})$ be its presheaf ∞-category. Then $\mathcal{P}(\mathcal{C})$ has all colimits, and there is again an equivalence

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}'(\mathcal{P}(\mathcal{C}), \mathcal{D})$$

for cocomplete ∞-categories $\mathcal{D}$.

Definition 6. An ∞-category $K \in \mathcal{C}\mathrm{at}_\infty$ is *sifted* if K-indexed colimits in $\mathcal{S}\mathrm{pc}$ commute with finite products.

Example 3. Δ^{op} is sifted. Its diagonal is cofinal ; it is well-known that if $X_{\bullet,\bullet}$ is a bisimplicial set then its geometric realization can be computed as the colimit of the diagonal $X_{n,n}$. On the other hand, the 1-sifted category $\Delta^{\mathrm{op}}_{\leqslant 1}$ is not sifted.

Definition 7. Let $\mathcal{C}$ be an ∞-category with finite coproducts. Then $\mathcal{P}_\Sigma(\mathcal{C})$ is defined as the full subcategory of $\mathcal{P}(\mathcal{C})$ on functors that preserve products.

The same proposition as in the 1-categorical picture holds.

Proposition 6.

1. *The inclusion $\mathcal{P}_\Sigma(\mathcal{C}) \hookrightarrow \mathcal{P}(\mathcal{C})$ preserves sifted colimits and has a left adjoint. In particular, $\mathcal{P}_\Sigma(\mathcal{C})$ is presentable, and it has limits and colimits ;*
2. *The Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ lands in $\mathcal{P}_\Sigma(\mathcal{C})$;*
3. *$j : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$ preserves finite coproducts ;*
4. *A presheaf X is in $\mathcal{P}_\Sigma(\mathcal{C})$ if and only if it is the geometric realization of filtered colimits of representables.*

Also, we have the universal properties

$$\mathrm{Fun}''(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

and

$$\mathrm{Fun}^{\amalg}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}'(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}).$$

In particular, any functor $\mathcal{C} \to \mathcal{D}$ can be extended to a sifted-colimit-preserving functor $\mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{D}$.