Lecture notes in Derived Algebraic Geometry
Session 2

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1 More on limits in an ∞-category

There is another way to define the notion of a (co)limit in an ∞-category, which looks a bit more like the 1-categorical notion. Let $F : I \to \mathcal{C}$ be an element of $\mathrm{Map}(I, \mathcal{C})$, with I a simplicial set and $\mathcal{C}$ an ∞-category.

Definition 1. A *cone* over F is a pair (y, η) where $y \in \mathcal{C}$ and $\eta : \mathcal{C}_y \to F$ is a natural transformation (i.e. a map $\eta : I \times \Delta^1 \to C$ which restricts to $C_y : I \to \Delta^0 \xrightarrow{y} \mathcal{C}$ on $\{0\}$ and to F on $\{1\}$).

Let (y, η) be a cone over F and $x \in \mathcal{C}$. Letting $c : \mathcal{C} \to \mathcal{C}^I$ be the functor taking an object $z \in \mathcal{C}$ to the constant diagram $c_z : I \to \mathcal{C}$ on z, we can define a map :

$$\mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{c} \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, c_y) \xrightarrow{\eta_*} \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, F)$$

up to a contractible choice for the composition with η.

Proposition 1. *A cone (y, η) over F is a limit cone for F if for all $x \in \mathcal{C}$, the map above is a homotopy equivalence.*

Example 1. If I is discrete, then $\mathrm{Fun}(I, \mathcal{C}) = \prod_I \mathcal{C}$. A diagram $F : I \to \mathcal{C}$ is then a collection of objects $\{y_i\}_{i \in I}$, and a cone over F is a collection of maps of the form $\{y \xrightarrow{\pi_i} y_i\}_{i \in I}$. It is a limit cone if and only if $\mathrm{Map}(x, y) \simeq \prod_{i \in I} \mathrm{Map}(x, y_i)$.

Lemma 1. *Any two limits (y, η) and (y', η') for F are equivalent.*

Proof. Since (y', η') is a limit, we have $\mathrm{Map}(c_y, F) \simeq \mathrm{Map}(y, y')$. The image of η gives a map $f : y \to y'$ such that $\eta' \circ f \simeq \eta$. Similarly, there is a map $g : y' \to y$ such that $\eta \circ g \simeq \eta'$. We then get

$$\eta \circ g \circ f \simeq \eta' \circ f \simeq \eta$$

so $g \circ f \simeq \mathrm{id}$. Similarly, $f \circ g \simeq \mathrm{id}$. □

Remark 1. One can show the (much) stronger statement that the ∞-category of limit cones over a given functor is either empty or trivial (equivalent to $\{*\}$). This means that when a diagram has a limit, then the "space" of limits is contractible, whereas the lemma only showed that it is connected.

Exercise 1. Let $I = \{0 \to 1 \leftarrow 0'\}$ be the "pullback" diagram. A functor $F : I \to \mathcal{C}$ is equivalently the data of two maps $b \xrightarrow{h} d \xleftarrow{k} c$ in $\mathcal{C}$. Show that up to equivalence, the datum of a cone $\eta : c_a \to F$ is equivalent to the data of two morphisms $b \xleftarrow{i} a \xrightarrow{j} c$ and of a choice of equivalence $h \circ i \simeq k \circ j$.

Proposition 2. *A cone (y, η) over $F : I \to \mathcal{S}\mathrm{pc}$ is a limit cone if and only if for all $x \in \mathcal{S}\mathrm{pc}$, the map*

$$[x, y]_{\mathcal{S}\mathrm{pc}} = \pi_0 \operatorname{Map}(x, y) \to [c_x, F]_{\mathcal{S}\mathrm{pc}^I} = \pi_0 \operatorname{Map}(c_x, F)$$

is an equivalence.

In other words, the fact that a cone *of spaces* is a limit can be checked in the homotopy category. (This does not mean that the limit can be computed in the homotopy category.)

Exercise 2. Check that the pullback of a diagram $b \to c \leftarrow d$ in $\mathcal{S}\mathrm{pc}$ can be computed as the ordinary limit (iterated pullback) of the following diagram in sSet :

$$\begin{tikzcd}
b \arrow[dr] & & c^{\Delta^1} \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & & d \arrow[dl] \\
& c & & c &
\end{tikzcd}$$

Proposition 3. *Let $\mathcal{C}$ be a Kan-enriched category, I a simplicial set and $F : I \to \mathrm{N}_\Delta(\mathcal{C})$ a map to the simplicial nerve. Consider the functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{sSet}$ and apply the simplicial nerve functor to get a functor*

$$\mathrm{N}_\Delta(\underline{\mathrm{Hom}}(x, -)) : \mathrm{N}_\Delta(\mathcal{C}) \to \mathcal{S}\mathrm{pc}.$$

Then a cone (y, η) is a limit cone for F if and only if the cone $(\underline{\mathrm{Hom}}(x, y), \mathrm{N}_\Delta(\underline{\mathrm{Hom}}(x, \eta)))$ is a limit in $\mathcal{S}\mathrm{pc}$.

In other words, "the Map functor preserves limits in the second argument" in the following way :

$$\operatorname{Map}_{\mathcal{C}}(x, \lim_I F) \simeq \lim_{i \in I} \operatorname{Map}(x, F(i)).$$

The limit on the left is performed in $\mathcal{C}$, whereas the one on the right is performed in the ∞-category of spaces $\mathcal{S}\mathrm{pc}$.

Remark 2. A cone for F is equivalent to the datum of a map

$$(I \times \Delta^1)/(I \times \Delta^0) \to \mathcal{C}$$

where the contracted $I \times \Delta^0$ is the copy of I that is located at $\{0\}$. This quotient is another model for $I^{\triangleleft}$.

Of course, one gets the notion of a colimit by dualizing this whole section.

Theorem 1. *Let $\mathcal{C}$ be an ∞-category. Then the following are equivalent :*

1. *$\mathcal{C}$ has an initial object and pushouts ;*

2. *$\mathcal{C}$ has finite coproducts and coequalizers ;*

3. *$\mathcal{C}$ has finite limits.*

Moreover, if $\mathcal{C}$ has arbitrary coproducts, then the following are equivalent :

1. *$\mathcal{C}$ has all colimits ;*

2. *$\mathcal{C}$ has pushouts ;*

3. *$\mathcal{C}$ has coequalizers ;*

4. *$\mathcal{C}$ has geometric realizations (colimits of diagrams of shape Δ^{op}).*