Lecture notes in Derived Algebraic Geometry
Session 4

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The aim of this last session is to finish the construction of the cotangent complex, and to discuss a bit about how to globalize the ring-theoretic construction that were introduced in the previous session.

1 Square-zero extensions and the cotangent complex

This section is actually the continuation of the last section of the previous lesson. Recall that we defined an augmented A-algebra $A \oplus M$ for every animated module M over an animated ring A, satisfying intuitive basic properties.

Lemma 1.1. *There exists a fiber sequence*

$$M \to A \oplus M \to A$$

where the second map is an effective epimorphism, and a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit}) \arrow[r] \arrow[d, "\oplus"'] & \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit}) \arrow[d, "\mathrm{pr}_1"] \\ \mathrm{CAlg}^{\mathrm{ani}} \arrow[r] & \mathrm{CAlg}^{\mathrm{ani}} \end{tikzcd} \qquad \begin{tikzcd} (A,M) \arrow[r, mapsto] \arrow[d, mapsto] & (A,0) \arrow[d, mapsto] \\ A \oplus M \arrow[r, mapsto] & A \end{tikzcd}$$

Remember that classically, $\mathrm{Der}(A,M) \simeq \mathrm{Hom}_{/A}(A, A \oplus M)$. This motivates the definition of derivations in the animated context.

Definition 1.2. Let $(A,M) \in \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit})$. Then define

$$\mathrm{Der}(A,M) = \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}_{/A}}(A, A \oplus M).$$

Remark 1.3. $A \oplus M$ is a loop space ! Indeed, there is a pullback square of the form

$$\begin{tikzcd} A \oplus M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ A \arrow[r] & A \oplus M[1] \end{tikzcd}$$

Also, $\mathrm{Der}(A,M)$ is the fiber of the map

$$\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}}(A, A \oplus M) \to \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}}(A, A).$$

Proposition 1.4. *The functor*

$$\mathrm{Der}(A,-) : \mathrm{Mod}_A \to \mathcal{S}\mathrm{pc}$$

is accessible and preserves limits. Therefore, it has a left adjoint

$$G : \mathcal{S}\mathrm{pc} \to \mathrm{Mod}_A.$$

Proof. For the accessibility, it is enough to observe that $\mathrm{Map}(A,-)$ is accessible since $M \mapsto A \oplus M$ commutes with sifted colimits by design (it is an animation), so it is accessible.

To show that the functor preserves limits, notice that the forgetful functor $\mathrm{CAlg}^{\mathrm{ani}}_{/A} \to \mathrm{CAlg}^{\mathrm{ani}}$ detects limits, so it is enough to show that if $p : K \to \mathrm{Mod}_A$ is a diagram of A-modules, then

$$\mathrm{Map}_{/A}(A[\underline{T}], A \oplus \lim_K p) \simeq \lim_K \mathrm{Map}_{/A}(A[\underline{T}], A \oplus p).$$

Why is this the case ? Recall that there is an adjunction :

$$\mathbb{L}\mathrm{Sym}_{\mathbf{Z}} : \mathrm{Mod}^{\mathrm{ani}}_{\mathbf{Z}} \rightleftarrows \mathrm{CAlg}^{\mathrm{ani}} : \mathrm{for}$$

between the (derived) free symmetric algebra and forgetful functors. Apply this adjunction and recall that small limits in Mod_A commute with $\oplus$ (which is simply the product in Mod_A) to obtain :

$$\begin{aligned}
\mathrm{Map}_{\mathrm{CAlg}_{/A}}(A[\underline{T}], A \oplus \lim_K p) &\simeq \mathrm{Map}_{\mathrm{Mod}_A}(A^{\oplus n}, \mathrm{for}(A \oplus \lim_K p)) \\
&\simeq \lim_K \mathrm{Map}_{\mathrm{Mod}_A}(A^{\oplus n}, \mathrm{for}(A \oplus p)) \\
&\simeq \lim_K \mathrm{Map}_{\mathrm{CAlg}_{/A}}(\mathbb{L}\mathrm{Sym}(A^{\oplus n}) = A[\underline{T}], A \oplus p).
\end{aligned}$$

□

One can now define the cotangent complex.

Definition 1.5. The *absolute cotangent complex* $\mathbb{L}_A$ is the module $G(*)$.

Remark 1.6. There is another, spectral way to construct the cotangent complex. If $\mathcal{C}$ is a symmetric monoidal stable ∞-category, then one can define a functor $G : \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ sending an augmented algebra $f : A \to \mathbf{1}$ to its cofiber. There is a theorem stating that G[1] induces an equivalence

$$\partial G : \mathcal{S}\mathrm{p}(\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})) \simeq \mathcal{S}\mathrm{p}(\mathcal{C}) \simeq \mathcal{C}$$

which can be used to define the cotangent complex. The advantage of this approach is that it lets us defined $A \oplus M$ for any non-necessarily animated $\mathbb{E}_\infty$-ring spectrum A.

Theorem 1.7. *The construction $A \mapsto \mathbb{L}_A$ extends to a functor*

$$\mathrm{CAlg}^{\mathrm{ani}} \to \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit})$$

sending $f : A \to B$ to $B \otimes^{\mathbb{L}}_A \mathbb{L}_A \to \mathbb{L}_B$. If A is a polynomial algebra then $\mathbb{L}_A \simeq \Omega_{A/\mathbf{Z}}$.

[1] or more precisely, its Goodwillie derivative

Definition 1.8. For an animated ring map $f : A \to B$, its *relative cotangent complex* $\mathbb{L}_{B/A}$ is the cofiber of the map $\mathbb{L}_A \otimes_A^{\mathbb{L}} B \to \mathbb{L}_B$.

Proposition 1.9. *There is an equivalence*

$$\mathrm{Map}_{\mathrm{Mod}_B}(\mathbb{L}_{B/A}, N) \simeq \mathrm{Map}_{\mathrm{CAlg}_{A//B}}(B, B \oplus N).$$

Define the space $\mathrm{Der}_A(B, N)$ of A-linear derivations from B to N to be this space.

Proposition 1.10. *The cotangent complex satisfies the following basic properties :*

1. *(Base change) If $B \simeq B' \otimes_{A'} A$, then $\mathbb{L}_{B/A} \simeq \mathbb{L}_{B'/A'} \otimes B$;*
2. *If $A \to B \to C$ is a fiber sequence, then there is a fiber sequence*
$$C \otimes_B \mathbb{L}_{B/A} \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B} \ ;$$
3. *For every map $f : A \to B$, there is an associated map $\varepsilon(f) : B \otimes_A \mathrm{Cof}(f) \to \mathbb{L}_{B/A}$. We have the following connectivity estimate : if the fiber of f is connective, then the fiber of $\varepsilon(f)$ is 2-connective.*

An important "slogan" in derived algebraic geometry which gives even more importance to this object is the idea that all animated rings can be built from discrete rings with square-zero extensions.

Definition 1.11. A map $S' \to S$ in $\mathrm{CAlg}^{\mathrm{ani}}$ is a *square-zero extension* if there is an S-module M and a derivation $d \in \pi_0 \mathrm{Map}_{/S}(S, S \oplus M[1])$ such that the following square is cartesian :

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S \arrow[d, "d_0"] \\ S \arrow[r, "d"'] & S \oplus M[1] \end{tikzcd}$$

Here, d_0 denotes the trivial derivation $S \to S \oplus M[1]$ which is just the inclusion of the first factor.

Proposition 1.12 (Lurie, DAG)**.** *Let M be an animated module over S. If S' is a square-zero extension of S by M and if $R \to S$ is any ring map, then the space of lifts* $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}_{/S}}(R, S')$ *is a torsor under* $\mathrm{Der}(R, M)$.

Concretely, this means that to get a lift $R \to S'$ in the following diagram

$$\begin{tikzcd} R \arrow[r, dashed] \arrow[dr] & S' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "d_0"] \\ & S \arrow[r, "d"'] & S \oplus M[1] \end{tikzcd}$$

it is sufficient and necessary that the derivation $R \to S \xrightarrow{d} S \oplus M[1]$ in $\pi_0 \mathrm{Der}(R, M[1]) = \pi_0 \mathrm{Map}_{\mathrm{Mod}_R}(\mathbb{L}_R, M[1])$ is the trivial derivation.

Example 1.13. Postnikov towers give rise to square zero extensions ! Recall that for every $n \geq 0$, there is a functor

$$\tau_{\leqslant n} : \mathrm{CAlg}^{\mathrm{ani}} \to \mathrm{CAlg}^{\mathrm{ani}}.$$

Then $\tau_{\leqslant n} R$ is a square-zero extension of $\tau_{\leqslant n-1} R$ by $(\pi_n R)[n]$, i.e. there is a pullback square :

$$\begin{tikzcd}
\tau_{\leqslant n} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tau_{\leqslant n-1} R \arrow[d, "d_0"] \\
\tau_{\leqslant n-1} R \arrow[r, dashed, "\exists d"'] & \tau_{\leqslant n-1} R \oplus (\pi_n R)[n+1]
\end{tikzcd}$$

This explains the philosophy of derived algebraic geometry, that after passing from varieties to schemes by adding non-reduced points, we pass from schemes to derived schemes by adding finer infinitesimal information. This Postnikov construction allows for handy inductive arguments using the cotangent complex.

Sketch of why this is true One easy thing to see first is that $\tau_{\leqslant n} R$ should equalize the two maps to the tensor product :

$$\tau_{\leqslant n} R \to \tau_{\leqslant n-1} R \rightrightarrows \tau_{\leqslant n-1} R \otimes_{\tau_{\leqslant n} R} \tau_{\leqslant n-1} R$$

(even though this might not be an equalizer diagram). Apply the $\tau_{\leqslant n+1}$ functor to this to get

$$\tau_{\leqslant n} R \to \tau_{\leqslant n-1} R \rightrightarrows \underbrace{\tau_{\leqslant n+1}(\tau_{\leqslant n-1} R \otimes_{\tau_{\leqslant n} R} \tau_{\leqslant n-1} R)}_{\star}.$$

What is $\star$? We can compute its homotopy groups with the long exact sequence and check that $\pi_*(\star) \simeq \pi_*(\tau_{\leqslant n-1} R \oplus \pi_n(R)[n+1])$. This should be a good clue that $\star$ is homotopy equivalent to the bottom-right object of the square in the statement. This would in turn show that there is a map from $\tau_{\leqslant n} R$ to the pullback. Of course, this all depends on the existence of a suitable derivation d, which will not be discussed.

2 Another word on geometrical aspects

Definition 2.1. Let $k \to R$ be a map of animated rings. We say that R is *derived formally smooth over* k if for every square-zero extension of k-algebras $S' \to S$ and for every map $R \to S$, there exists a lift in the following diagram of k-algebras :

$$\begin{tikzcd}
R \arrow[r] \arrow[dr, dashed] & S \\
& S' \arrow[u]
\end{tikzcd}$$

Proposition 2.2. *With the same notation, being derived formally smooth is equivalent to the vanishing of* $\pi_0 \operatorname{Map}(\mathbb{L}_{R/k}, M[1])$ *for every* $M \in \mathrm{Mod}_R$.

Definition 2.3. With the same notation as above and in the derived formally smooth assumption, we say that R is *derived formally étale* if moreover the space of lifts is contractible.

Proposition 2.4. *Being derived formally étale is equivalent to being formally étale in the sense of the definition from the previous session (namely, that the π_0 is étale and that $\pi_i(R) \simeq \pi_i(k) \otimes_{\pi_0(k)} \pi_0(R)$).*

Exercise 2.5. Let R be an animated ring and $x_1, \dots, x_c \in \pi_0 R$. Then prove that

$$\mathbb{L}_{(R /\!\!/ (x_1,\dots,x_c))/R} \simeq R /\!\!/ (x_1, \dots, x_c)^{\oplus c}[1].$$

If $\mathcal{J}$ is the fiber of the map $R \to R /\!\!/ (x_1, \dots, x_c)$, then

$$\mathbb{L} \simeq \mathcal{J} \otimes_R R /\!\!/ (x_1, \dots, x_c)[1].$$

Concretely, the cotangent complex "contains" the conormal bundle. To be more precise, if $A \to B$ is a map of static rings in $\mathrm{CAlg}^{\heartsuit}$, then

$$\tau_{\leqslant 1}\mathbb{L}_{B/A} \simeq \left[\mathcal{I}/\mathcal{I}^2 \to \Omega_A \otimes_A B\right].$$

Theorem 2.6 (Quillen, Lurie)**.** *Let $f : A \to B$ be a map whose cofiber is n-connective. Then $\varepsilon(f)$ is $(n+2)$-connective. Moreover, f is an equivalence is and only if $\mathbb{L}_{B/A} \simeq 0$ and $\pi_0(A) \simeq \pi_0(B)$.*

Definition 2.7. Let R be an animated ring and $M \in \mathrm{Mod}_R$. We say that M has *Tor-amplitude in* $[a,b]$ if for every static module $N \in \mathrm{Mod}^{\heartsuit}_{\pi_0 R}$, we have $H_i(M \otimes^{\mathbb{L}}_R N) = 0$ whenever $i \notin [a,b]$.

The following theorem, conjectured by Quillen decades earlier, is striking : it explains that under mild assumptions, the cotangent complex is either extremely simple (concentrated in degrees 0 and 1) or extremely complex (unbounded).

Theorem 2.8 (Avramov)**.** *If $f : k \to R$ is a map of noetherian rings, then it is locally complete intersection if and only if R is locally of finite flat dimension and $\mathbb{L}_{R/k}$ has bounded Tor-amplitude.*

Proposition 2.9. *Let A be an animated ring and $B \in \mathrm{CAlg}^{\omega}_A$ be a compact A-algebra. Then $\mathbb{L}_{B/A}$ is also compact as a B-module. The converse is also true if $\pi_0(B)$ is of finite presentation over $\pi_0(A)$.*

3 Global derived geometry

This last section aims to give a glimpse on how one can globalize the ring-theoretic constructions that have been discussed up until now, in order to define derived schemes.

3.1 Prestacks and affine derived schemes

There are two approaches, the first using ringed topoi and looking like the "locally ringed space" approach to schemes. The second one, which will be followed here, is the "functor of points and descent conditions" approach. It is in a sense more natural, at least algebraically.

Definition 3.1. Let PreStack $= \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ani}}, \mathcal{S}\mathrm{pc})$ be the category of functors on derived rings. By Yoneda, any animated ring A defines a representable prestack $\mathrm{Spec}(A) = \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}}(A, -)$. Let $\mathrm{dSch}^{\mathrm{aff}}$ be the full subcategory of PreStack spanned by these affine derived schemes $\mathrm{Spec}(A)$.

Recall that there is an adjunction between the inclusion $\mathrm{CAlg}^{\heartsuit} \hookrightarrow \mathrm{CAlg}^{\mathrm{ani}}$ and the π_0 functor : this yields an adjunction by further abstract nonsense :

$$\pi_0^* : \mathrm{PreStack}^{\mathrm{cl}} \rightleftarrows \mathrm{PreStack} : (-)^{\mathrm{cl}}$$

where $\mathrm{PreStack}^{\mathrm{cl}} = \mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \mathcal{S}\mathrm{pc})$.

Remark 3.2. For every animated ring A, we have $\mathrm{Spec}(A)^{\mathrm{cl}} = \mathrm{Spec}(\pi_0 A)$.

3.2 Grothendieck topologies

Let us define the Zariski and étale topologies on $\mathrm{dSch}^{\mathrm{aff}}$.

Definition 3.3. Let I be a set and $\{j_\alpha : \mathrm{Spec}(B_\alpha) \to \mathrm{Spec}(A)\}_{\alpha \in I}$ be a family of maps in $\mathrm{dSch}^{\mathrm{aff}}$. Such a family is a *Zariski covering* (resp. *étale covering*) if the following conditions hold.

1. The functor $\mathrm{Mod}_A \to \prod_\alpha \mathrm{Mod}_{B_\alpha}$ is conservative ;
2. Each map $A \to B_\alpha$ is Zariski, i.e. it is an open immersion at the π_0 level and $\pi_i(B_\alpha) \simeq \pi_i(A) \otimes_{\pi_0(A)} \pi_0(B_\alpha)$ (resp. each map is (formally) étale in the sense already defined earlier).

A prestack F is then a sheaf for the topology τ (here, τ is Zar or ét) if for every such covering family, $F(A)$ is the limit of the corresponding Čech diagram $F(B^{\otimes \bullet})$.

One can similarly define a hypersheaf by imposing a descent condition for hypercovers.

Definition 3.4. A *derived stack* is a prestack that is an étale hypersheaf.

Remark 3.5. To define modules on a prestack $\mathfrak{X}$ instead of just an animated ring A, one can construct the right Kan extension of $\mathrm{Mod}^* : \mathrm{CAlg}^{\mathrm{ani,op}} \to \mathrm{Pr}^{\mathrm{L},\otimes}$ to get a new functor

$$\mathrm{QCoh} : \mathrm{PreStack} \to \mathrm{Pr}^{\mathrm{L},\otimes}.$$

By definition,

$$\mathrm{QCoh}(\mathfrak{X}) \simeq \lim_{\mathrm{Spec}(B) \in \mathrm{dSch}^{\mathrm{aff}}_{/\mathfrak{X}}} \mathrm{QCoh}(\mathrm{Spec}(B))$$

where $\mathrm{QCoh}(\mathrm{Spec}(B)) = \mathrm{Mod}_B$ is simply the ∞-category of animated B-modules.

Definition 3.6. A Zariski stack $\mathfrak{X}$ is a *derived scheme* if there exists a cover by affine derived schemes : there exists a family $\mathcal{U} = (\mathcal{U}_\alpha)$ with $\mathcal{U}_\alpha \to \mathfrak{X}$ open immersions from affine derived schemes $\mathcal{U}_\alpha \simeq \mathrm{Spec}(A_\alpha)$, such that $\coprod_\alpha \mathcal{U}_\alpha \to \mathfrak{X}$ is an effective epimorphism.

In this definition, "open immersion" refers to a morphism $\mathfrak{Y} \to \mathfrak{X}$ such that $\mathfrak{Y} \times_{\mathfrak{X}} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is an open immersion for every animated ring A. More generally, one can define representability in the following way.

Definition 3.7. A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is *(-1)-geometric* or *representable* if for every map $\operatorname{Spec}(A) \to \mathfrak{Y}$, the pullback $\mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Spec}(A)$ is an affine derived scheme $\operatorname{Spec}(B)$.

The higher levels of geometricity are defined recursively.

Definition 3.8. If $\mathfrak{X}$ is a derived stack, we say that it is *(-1)-geometric* if it is affine. For every $n \geqslant 0$, we say that $\mathfrak{X}$ is *n-geometric* if the following conditions hold :

1. The diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is $(n-1)$-geometric ;

2. There is an affine covering $\coprod \operatorname{Spec}(B_i) \twoheadrightarrow \mathfrak{X}$ that is $(n-1)$-geometric and étale.

A morphism of derived stacks $f : \mathfrak{X} \to \mathfrak{Y}$ is *m-geometric* for $m \geqslant 0$ if for every map $\operatorname{Spec}(A) \to \mathfrak{Y}$, the pullback P in the following square

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{X} \arrow[d, "f"] \\ \operatorname{Spec}(A) \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

is m-geometric.

Along with the different levels of geometricity, there is also a hierarchy of levels of *stack-ity.*

Definition 3.9. A derived stack $\mathfrak{X}$ is an *n-stack* if for every static ring $A \in \mathrm{CAlg}^{\heartsuit}$, the space $\mathfrak{X}(A)$ is n-truncated.

In other words, higher stacks have "higher coherence data".

Remark 3.10. A classical Artin stack is a 1-geometric 1-stack with these definitions. A *higher Deligne-Mumford stack* would be a geometric stack with "smooth" replacing the word *étale* in the second condition of Definition 3.8.

Proposition 3.11. *The functor* $\mathrm{Perf} : \mathrm{CAlg}^{\mathrm{ani}} \to \mathcal{S}\mathrm{pc}$ *sending an animated ring A to the space of perfect complexes* $\left(\mathrm{Mod}_A^{\mathrm{perf}}\right)^{\simeq}$ *is a locally geometric stack : it is a filtered colimit of geometric stacks by monomorphic transition maps.*

Ideas of the proof. This filtered colimit is none other than $\mathrm{Perf} = \bigcup_{a \leqslant b} \mathrm{Perf}_{[a,b]}$ where $\mathrm{Perf}_{[a,b]}$ denotes the perfect complexes with Tor-amplitude in $[a, b]$.

One can show that $\mathrm{Perf}_{[a,b]}$ is $(b-a+1)$-geometric, by induction on $n = b-a+1$. The case $n = 1$ is already interesting, since $\mathrm{Perf}_{[a,a]} \simeq \mathrm{Vect} \simeq \coprod_m \mathrm{BGL}_m$. This is indeed 1-geometric, each $\mathrm{BGL}_m = [*/\mathrm{GL}_m]$ being a geometric realization. $\square$