Lecture notes in Derived Algebraic Geometry
Session 4

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The aim of this last session is to finish the construction of the cotangent complex, and to discuss a bit about how to globalize the ring-theoretic construction that were introduced in the previous session.

1 Square-zero extensions and the cotangent complex

This section is actually the continuation of the last section of the previous lesson. Recall that we defined an augmented A-algebra $A \oplus M$ for every animated module M over an animated ring A, satisfying intuitive basic properties.

Lemma 1.1. *There exists a fiber sequence*

$$M \to A \oplus M \to A$$

where the second map is an effective epimorphism, and a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit}) \arrow[r] \arrow[d, "\oplus"'] & \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit}) \arrow[d, "\mathrm{pr}_1"] \\ \mathrm{CAlg}^{\mathrm{ani}} \arrow[r] & \mathrm{CAlg}^{\mathrm{ani}} \end{tikzcd} \qquad \begin{tikzcd} (A,M) \arrow[r, mapsto] \arrow[d, mapsto] & (A,0) \arrow[d, mapsto] \\ A \oplus M \arrow[r, mapsto] & A \end{tikzcd}$$

Remember that classically, $\mathrm{Der}(A,M) \simeq \mathrm{Hom}_{/A}(A, A\oplus M)$. This motivates the definition of derivations in the animated context.

Definition 1.2. Let $(A,M) \in \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit})$. Then define

$$\mathrm{Der}(A,M) = \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}_{/A}}(A, A \oplus M).$$

Remark 1.3. $A \oplus M$ is a loop space ! Indeed, there is a pullback square of the form

$$\begin{tikzcd} A \oplus M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ A \arrow[r] & A \oplus M[1] \end{tikzcd}$$

Also, $\mathrm{Der}(A,M)$ is the fiber of the map

$$\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}}(A, A\oplus M) \to \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}}(A,A).$$

Proposition 1.4. *The functor*

$$\mathrm{Der}(A,-) : \mathrm{Mod}_A \to \mathcal{S}\mathrm{pc}$$

is accessible and preserves limits. Therefore, it has a left adjoint

$$G : \mathcal{S}\mathrm{pc} \to \mathrm{Mod}_A.$$

Proof. For the accessibility, it is enough to observe that $\mathrm{Map}(A,-)$ is accessible since $M \mapsto A \oplus M$ commutes with sifted colimits by design (it is an animation), so it is accessible.

To show that the functor preserves limits, notice that the forgetful functor $\mathrm{CAlg}^{\mathrm{ani}}_{/A} \to \mathrm{CAlg}^{\mathrm{ani}}$ detects limits, so it is enough to show that if $p : K \to \mathrm{Mod}_A$ is a diagram of A-modules, then

$$\mathrm{Map}_{/A}(A[\underline{T}], A \oplus \lim_K p) \simeq \lim_K \mathrm{Map}_{/A}(A[\underline{T}], A \oplus p).$$

Why is this the case ? Recall that there is an adjunction :

$$\mathbb{L}\mathrm{Sym}_{\mathbf{Z}} : \mathrm{Mod}^{\mathrm{ani}}_{\mathbf{Z}} \rightleftarrows \mathrm{CAlg}^{\mathrm{ani}} : \mathrm{for}$$

between the (derived) free symmetric algebra and forgetful functors. Apply this adjunction and recall that small limits in Mod_A commute with $\oplus$ (which is simply the product in Mod_A) to obtain :

$$\begin{aligned}
\mathrm{Map}_{\mathrm{CAlg}_{/A}}(A[\underline{T}], A \oplus \lim_K p) &\simeq \mathrm{Map}_{\mathrm{Mod}_A}(A^{\oplus n}, \mathrm{for}(A \oplus \lim_K p)) \\
&\simeq \lim_K \mathrm{Map}_{\mathrm{Mod}_A}(A^{\oplus n}, \mathrm{for}(A \oplus p)) \\
&\simeq \lim_K \mathrm{Map}_{\mathrm{CAlg}_{/A}}(\mathbb{L}\mathrm{Sym}(A^{\oplus n}) = A[\underline{T}], A \oplus p).
\end{aligned}$$

□

One can now define the cotangent complex.

Definition 1.5. The *absolute cotangent complex* $\mathbb{L}_A$ is the module $G(*)$.

Remark 1.6. There is another, spectral way to construct the cotangent complex. If $\mathcal{C}$ is a symmetric monoidal stable ∞-category, then one can define a functor $G : \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ sending an augmented algebra $f : A \to \mathbf{1}$ to its cofiber. There is a theorem stating that G[1] induces an equivalence

$$\partial G : \mathcal{S}\mathrm{p}(\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})) \simeq \mathcal{S}\mathrm{p}(\mathcal{C}) \simeq \mathcal{C}$$

which can be used to define the cotangent complex. The advantage of this approach is that it lets us defined $A \oplus M$ for any non-necessarily animated $\mathbb{E}_\infty$-ring spectrum A.

Theorem 1.7. *The construction $A \mapsto \mathbb{L}_A$ extends to a functor*

$$\mathrm{CAlg}^{\mathrm{ani}} \to \mathrm{Ani}(\mathrm{CRMod}^{\heartsuit})$$

sending $f : A \to B$ to $B \otimes^{\mathbb{L}}_A \mathbb{L}_A \to \mathbb{L}_B$. If A is a polynomial algebra then $\mathbb{L}_A \simeq \Omega_{A/\mathbf{Z}}$.

[1] or more precisely, its Goodwillie derivative

Definition 1.8. For an animated ring map $f : A \to B$, its *relative cotangent complex* $\mathbb{L}_{B/A}$ is the cofiber of the map $\mathbb{L}_A \otimes_A^{\mathbb{L}} B \to \mathbb{L}_B$.

Proposition 1.9. *There is an equivalence*

$$\mathrm{Map}_{\mathrm{Mod}_B}(\mathbb{L}_{B/A}, N) \simeq \mathrm{Map}_{\mathrm{CAlg}_{A//B}}(B, B \oplus N).$$

Define the space $\mathrm{Der}_A(B, N)$ of A-linear derivations from B to N to be this space.

Proposition 1.10. *The cotangent complex satisfies the following basic properties :*

1. *(Base change) If $B \simeq B' \otimes_{A'} A$, then $\mathbb{L}_{B/A} \simeq \mathbb{L}_{B'/A'} \otimes B$;*
2. *If $A \to B \to C$ is a fiber sequence, then there is a fiber sequence*
$$C \otimes_B \mathbb{L}_{B/A} \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B}\ ;$$
3. *For every map $f : A \to B$, there is an associated map $\varepsilon(f) : B \otimes_A \mathrm{Cof}(f) \to \mathbb{L}_{B/A}$. We have the following connectivity estimate : if the fiber of f is connective, then the fiber of $\varepsilon(f)$ is 2-connective.*

An important "slogan" in derived algebraic geometry which gives even more importance to this object is the idea that all animated rings can be built from discrete rings with square-zero extensions.

Definition 1.11. A map $S' \to S$ in $\mathrm{CAlg}^{\mathrm{ani}}$ is a *square-zero extension* if there is an S-module M and a derivation $d \in \pi_0 \mathrm{Map}_{/S}(S, S \oplus M[1])$ such that the following square is cartesian :

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S \arrow[d, "d_0"] \\ S \arrow[r, "d"'] & S \oplus M[1] \end{tikzcd}$$

Here, d_0 denotes the trivial derivation $S \to S \oplus M[1]$ which is just the inclusion of the first factor.

Proposition 1.12 (Lurie, DAG)**.** *Let M be an animated module over S. If S' is a square-zero extension of S by M and if $R \to S$ is any ring map, then the space of lifts* $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ani}}_{/S}}(R, S')$ *is a torsor under* $\mathrm{Der}(R, M)$.

Concretely, this means that to get a lift $R \to S'$ in the following diagram

$$\begin{tikzcd} R \arrow[r, dashed] \arrow[dr] & S' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "d_0"] \\ & S \arrow[r, "d"'] & S \oplus M[1] \end{tikzcd}$$

it is sufficient and necessary that the derivation $R \to S \xrightarrow{d} S \oplus M[1]$ in $\pi_0 \mathrm{Der}(R, M[1]) = \pi_0 \mathrm{Map}_{\mathrm{Mod}_R}(\mathbb{L}_R, M[1])$ is the trivial derivation.

Example 1.13. Postnikov towers give rise to square zero extensions ! Recall that for every $n \geq 0$, there is a functor

$$\tau_{\leqslant n} : \mathrm{CAlg}^{\mathrm{ani}} \to \mathrm{CAlg}^{\mathrm{ani}}.$$

Then $\tau_{\leqslant n} R$ is a square-zero extension of $\tau_{\leqslant n-1} R$ by $(\pi_n R)[n]$, i.e. there is a pullback square :

$$\begin{tikzcd}
\tau_{\leqslant n} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tau_{\leqslant n-1} R \arrow[d, "d_0"] \\
\tau_{\leqslant n-1} R \arrow[r, dashed, "\exists d"'] & \tau_{\leqslant n-1} R \oplus (\pi_n R)[n+1]
\end{tikzcd}$$

This explains the philosophy of derived algebraic geometry, that after passing from varieties to schemes by adding non-reduced points, we pass from schemes to derived schemes by adding finer infinitesimal information. This Postnikov construction allows for handy inductive arguments using the cotangent complex.