AN INTRODUCTION TO MOTIVES (PURE MOTIVES, MIXED MOTIVES, PERIODS)

Yves André
Translated to English by E. Hecky

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Abstract *Motives* have been introduced 40 years ago by A. Grothendieck as "a systematic theory of arithmetic properties of algebraic varieties as embodied in their groups of classes of cycles". This text provides an exposition of the geometric foundations of the theory (pure and mixed), and a panorama of major developments which have occurred in the last 15 years. The last part is devoted to a study of *periods* of motives, with emphasis on examples (polyzeta numbers, notably).

Foreword Many disciplines have adopted in their technical vocabulary the term *motive*, with one of its standard meanings: that of "reason" (which has a subjective connotation) and that of "constitutive element".

By introducing this term in Algebraic Geometry 40 years ago, A. Grothendieck was playing with both meanings at once. The goal was to find the common *motive* in different cohomological phenomena (for example, the transcendental and arithmetic notions of weight), or even the *motive* explaining certain mysterious relations between the integrals of algebraic functions, and so on. The aim was also to decompose algebraic varieties, from a cohomological point of view, in simple *motives* prone to recombination.

The fundamental intuition of Grothendieck was that this chemistry of motives was ruled by the theory of algebraic correspondences. Motives should form some sort of purely algebraic universal cohomology, wherefrom all the other cohomology theories could be deduced as different "realizations". They should also give birth to a generalization of Galois theory to systems of multivariate polynomials.

One can distinguish three eras in the evolution of the theory. Up until the end of the 1970s, it was basically reduced to the conjectural grothendieckian corpus of pure motives, as it is exposed at the end of Saavedra's book [**?**]. This piece of "sci-fi mathematics" was only referred to as a source of inspiration.

In the 1980s the theory has seen two major developments. The first is the finding of non-conjectural results, even if it required to sacrifice the very notion of motive and to replace it with realizations systems. The second is the establishment of a wonderful program towards a general theory of mixed motives, and the key idea of motivic cohomology. The Deligne, Beilinson and Block-Kato conjectures on the relations between the values of L-functions and regulators have played a major role in making the theory popular amongst the arithmeticians. These developments are exposed in the collective work *Motives* (Seattle Conference 1991, AMS).

With the focus shifting back to algebraic correspondences, the next years have seen genuine breakthroughs giving non-conjectural foundations to the theory, both in pure motives with the proof of the conjecture on the semi-simplicity of numerical motives, and in mixed motives with a mature theory of motivic cohomology[1].

Our aim is to make these developments accessible to non-specialists, while giving them in the first two parts a unitary vision of the theory[2].

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[1]enough to initiate work on the Milnor-Bloch-Kato conjectures.

[2]restricting ourselves to the geometric and categorical aspects: there will be no treatment of L-

functions and nothing will be said about “Langlands philosophy” [**?**]

Part I

PURE MOTIVES

Chapter 1

ORIGINS : ENUMERATIVE GEOMETRY, COHOMOLOGY, GALOIS THEORY

In this "heuristical" chapter, we will outline freely and without trying to be exhaustive, some common threads leading to motives. It is not a historical reconstruction of the birth of the theory, and the notions will be reviewed in greater details in the upcoming chapters.

1.1 Enumerative geometry

1.1.1 One of the oldest and most elementary theorems in enumerative projective geometry is that of Bézout[1] on the number of points of intersection between two plane curves without common component. If the curves are defined as the vanishing loci of polynomials $P(x, y)$ and $Q(x, y)$ respectively, then they intersect at $\deg P \cdot \deg Q$ points, without forgetting multiplicities and points at infinity. Poncelet[2] suggested a *dynamical* approach to this theorem, deforming Q into a product $\prod(a_i x + b_i y)$ of linear components in general position and using his "conservation of number principle". It is then clear that each of the $\deg Q$ lines $a_i x + b_i y$ intersects $\{P = 0\}$ in exactly $\deg P$ points, and the theorem follows.

A modern formulation of this principle is the fact that the algebraic equivalence of algebraic cycles, and *a fortiori* their rational equivalence, is finer than the numerical equivalence. Let us briefly recall their definitions.

An *algebraic cycle* on a smooth projective variety X is a formal linear combination $Z = \sum n_i Z_i$ of irreducible subvarieties Z_i of X, with integers coefficients n_i (or rational, depending on the context). Two cycles Z and Z' are said to be *rationally equivalent* (denoted $Z \sim_{\mathrm{rat}} Z'$) if they may be transformed into one another by a succession

[1] 1779, but stated by MacLaurin in 1720.

[2] Traité des propriétés projectives des figures (1822).

of rational deformations, parametrized by the projective line $\mathbb{P}^1$. They are said to be *algebraically equivalent* (denoted $Z \sim_{\mathrm{alg}} Z'$) if they can be transformed into one another by algebraic deformations. They are said to be *numerically equivalent* (denoted $Z \sim_{\mathrm{num}} Z'$) if the intersection numbers $Z \cdot Y = \sum n_i Z_i \cdot Y$ and $Z' \cdot Y = \sum n'_i Z'_i \cdot Y$ agree for every subvariety Y of X of complementary codimension.

Obviously, $\sim_{\mathrm{rat}} \implies \sim_{\mathrm{alg}}$, and the "conservation of number principle" of Poncelet states that $\sim_{\mathrm{alg}} \implies \sim_{\mathrm{num}}$.

The theory of these equivalences on algebraic cycles had already developed around 1925, especially thanks to efforts by Van der Waerden and the Italian School (independently).

Basic examples

1. A *divisor* is an algebraic cycle $\sum n_i Z_i$ where each Z_i is of codimension one (and each n_i is an integer). One can rationally deform such a divisor to one with nonnegative coefficients, in its own linear system as long as the latter is nonempty.

2. Let $Z = \sum n_i Z_i$ be a divisor on a curve X. Then $Z \sim_{\mathrm{num}} 0 \iff Z \sim_{\mathrm{alg}} 0 \iff \sum n_i = 0$. If X is an elliptic curve, then $Z \sim_{\mathrm{rat}} 0$ if and only if $\sum n_i = 0$ *and* $\sum_X n_i Z_i = 0_X$, where $\sum_X$ is the group law on the elliptic curve X. In higher genus, a similar criterion holds, substituting X for its associated abelian "jacobian" variety.

1.1.2 Enumerative geometry traditionally deals with counting geometric objects in a given family, having prescribed relations with a given configuration (Chasles, Schubert[3], Zeuthen[4], ...). It proceeds by rephrasing the problem in terms of intersection numbers of subvarieties of grassmannians parametrizing the configurations. These intersection numbers are the central subject of Schubert calculus.

Basic examples

1. Counting the number of lines in $\mathbb{P}^3$ intersecting four given lines in general position. We find two: the lines intersecting a given line in $\mathbb{P}^3$ are parametrized by a divisor σ_1 in the 4-dimensional grassmannian $\mathbb{G}\mathrm{r}(1,3)$. According to Schubert calculus, the self-intersection number σ_1^4 is 2. In fact, this is another instance of the "conservation of number principle"[5]: with deformations, one reduces the problem to the case where the first two (resp. last two) lines lie in a plane P (resp. Q), intersect in a point A (resp. B), and are in general position for those constraints. The two lines are then AB and $P \cap Q$.

[3]Kalkül der abzählenden Geometrie (1879).
[4]Lehrbuch der abzählenden Methoden der Geometrie (1914).
[5]as J.-B. Bost pointed out.

2. Another classical counting problem in algebraic geometry is that of counting solutions to polynomial equations in finite fields (Gauss, Artin, Schmidt, Hasse, Weil, ...). Let ν_n be the number of points in $\mathbb{F}_{q^n}$ of a given smooth projective variety X defined over $\mathbb{F}_q$. We encode these numbers in the zeta function Z_X, defined by the series
$$\log Z_X(t) = \sum \nu_n \frac{t^n}{n}.$$
One can compute them as intersection numbers on $X \times X$:
$$\nu_n = \langle \Delta, \Gamma_{\mathrm{Fr}_X^n} \rangle$$
where Δ is the diagonal subvariety in $X \times X$, and $\Gamma_{\mathrm{Fr}_X^n}$ is the graph of the n-fold iterated Frobenius map of X (locally, Fr_X is induced by $x \mapsto x^q$). As for many other enumerative problems on curves, one reduces it to studying algebraic correspondences modulo numerical equivalence between curves.

1.1.3 All of these enumerative problems live naturally in the "enumerative category of smooth projective varieties" over a field k.

This $\mathbf{Q}$-linear category, temporarily denoted $\mathcal{E}(k)$, has as objects the smooth projective varieties on k. The collection of morphisms is given by the degree 0 algebraic correspondences modulo numerical equivalence
$$\mathcal{E}(k)(X,Y) = \{\text{algebraic cycles with rational coefficients} \\ \text{of codimension } \dim X \text{ on } X \times Y\}/\sim_{\mathrm{num}}.$$
The composition of morphisms is given by the well-known rule from intersection theory :
$$g \circ f = (\mathrm{pr}^{XYZ}_{XZ})_* \left((\mathrm{pr}^{XYZ}_{XY})^* f \cdot (\mathrm{pr}^{XYZ}_{YZ})^* g \right)$$
involving the three "3-to-2" projections from $X \times Y \times Z$.

One could say that enumerative geometry in the broadest sense is the study of the category $\mathcal{E}(k)$. It seems that A. Grothendieck was the first one to consider this category in and of itself, and to point out its remarkable properties.

These properties are even more apparent if one takes the *pseudo-abelian envelope* of $\mathcal{E}(k)$, obtained by formally introducing kernels of idempotent endomorphisms. In Grothendieck's lingo, this pseudo-abelian envelope denoted $\mathrm{NM}^{\mathrm{eff}}(k)_{\mathbf{Q}}$ is called the *category of effective numerical motives*. Grothendieck had conjectured the following "miraculous" property :
$$\mathrm{NM}^{\mathrm{eff}}(k)_{\mathbf{Q}} \text{ is a semi-simple abelian category}$$
which was later proved by U. Jannsen in 1991 [?]. The category $\mathrm{NM}^{\mathrm{eff}}(k)_{\mathbf{Q}}$ is moreover endowed with a monoidal structure induced by the cartesian product of varieties.

1.1.3.1 Recall. An additive category $\mathcal{C}$ is said to be *pseudo-abelian* if for each endomorphism $e \in \mathcal{C}(A, A)$ satisfying $e^2 = e$, one can write A as a direct sum $A_1 \oplus A_2$ such that e is the projection $A \to A_1$ followed by the inclusion $A_1 \to A$. We then say that A_1 is the image of e and denote it $e(A)$ or eA. We observe that A_2 is then the image of the idempotent endomorphism $1 - e$.

Let $\mathcal{C}$ be an additive category. Let $\mathcal{C}^\natural$ be the category whose objects are the pairs (A, e) with A an object of $\mathcal{C}$ and $e \in \mathcal{C}(A, A)$ an idempotent endomorphism. The abelian group of morphisms from (A, e) to (A', e') is defined as $e' \circ \mathcal{C}(A, A') \circ e$, namely the subgroup of $\mathcal{C}(A, A')$ spanned by morphisms of the form $e' \circ f \circ e$. One checks without effort that $\mathcal{C}^\natural$ is pseudo-abelian. It is the *pseudo-abelian envelope* of $\mathcal{C}$. The obvious functor $\mathcal{C} \to \mathcal{C}^\natural$ is fully faithful.

1.2 Cohomology of algebraic varieties

1.2.1 When the base field is the field of complex numbers $\mathbf{C}$, the use of topological methods - especially cohomology theories - in algebraic geometry dates back to Poincaré, Picard, and then Lefschetz[6].

Through various ways (simplicial methods, differential forms, ...), we assign to each smooth projective variety X over a field k a graded k-algebra $H^*(X)$. It satisfies the "Künneth formula": $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$. Moreover, we assign to each algebraic cycle of codimension r over X its fundamental class, which lives in $H^{2r}(X)$. The "cup-product" of $H^*(X)$ is compatible with the intersection product of cycles.

Two cycles Z and Z' are said to be *homologically equivalent* (denoted $Z \sim_{\text{hom}} Z'$) if the fundamental class of their difference is zero. This equivalence lies between the algebraic and the numerical ones.

Basic example Back to Bézout's theorem. We have $H^2(\mathbb{P}^2, \mathbf{Q}) = H^4(\mathbb{P}^2, \mathbf{Q}) = \mathbf{Q}$ and the cup-product $H^2(\mathbb{P}^2, \mathbf{Q}) \times H^2(\mathbb{P}^2, \mathbf{Q}) \to H^4(\mathbb{P}^2, \mathbf{Q})$ is multiplication. The fundamental class of a curve $C \subset \mathbb{P}^2$ is given by its degree and the fundamental class of a zero-dimensional cycle $\sum n_i P_i$ is its degree $\sum n_i$. The number of points at the intersection of two curves C and C' intersecting properly is then given by the class of $C \cdot C'$, namely $\deg C \cdot \deg C'$.

The Lefschetz trace formula provides a powerful cohomological tool to compute the number of (isolated) fixed points of an endomorphism of X :

$$|\mathrm{Fix}(\alpha)| = \sum_{0}^{2\dim X} (-1)^i \mathrm{tr}(\alpha^* | H^i(X)).$$

1.2.2 It is A. Weil who thought that such methods could yield results about varieties defined over other fields, especially over a finite field $k = \mathbb{F}_q$. A formal application of

[6] the applications to the study of complex algebraic varieties was a major incentive for those authors.

the Lefschetz trace formula to iterated the Frobenius map :

$$\nu_n = \sum_{0}^{2\dim X} (-1)^i \mathrm{tr}((\mathrm{Fr}_X^n)^* | H^i(X))$$

would lead to an expression of the zeta function of X as a rational function :

$$Z_X(t) = \frac{\det(1 - t\mathrm{Fr}_X^* | H^-(X))}{\det(1 - t\mathrm{Fr}_X^* | H^+(X))}.$$

Proving such a formula was one of the main goals of the construction of étale cohomology. This goal was attained in 1963 by Grothendieck and M. Artin.

In fact, there is an étale cohomology with coefficients in each field $\mathbf{Q}_\ell$ of ℓ-adic numbers for $\ell \neq p$ where p is the characteristic of k. This plethora makes the situation more obscure: in addition to the awkwardly arbitrary choice of ℓ, we don't really know how to relate these ℓ-adic cohomologies.

Ideally, we would want a *universal cohomology theory* with rational coefficients satisfying the Künneth formula. However, such a theory with values in $\mathbf{Q}$-vector spaces cannot exist if $p > 0$, because of functoriality considerations about supersingular elliptic curves[7]. There is still hope to find such a "generalized" cohomology theory with values in a suitable semi-simple $\mathbf{Q}$-linear monoidal category in lieu of $\mathbf{Q}$-vector spaces.

1.2.3 Grothendieck suggested to take $\mathrm{NM}^{\mathrm{eff}}(k)_{\mathbf{Q}}$ as such a category[8]. There is an obvious contravariant monoidal functor

$$\mathfrak{h} : \{\text{smooth projective } k\text{-varieties}\} \to \mathrm{NM}^{\mathrm{eff}}(k)_{\mathbf{Q}}$$

mapping a variety to itself ($\mathfrak{h}(X)$ is called the motive of X), and mapping a morphism to the transpose of its graph modulo numerical equivalence. We expect that $\mathfrak{h}$ provides a universal cohomology theory for smooth projective varieties, and in particular that each ℓ-adic étale cohomology as well as each reasonable "Weil" cohomology like crystalline

[7] this is well-known work by J.-P. Serre, see below **??**.

[8] "We had the clear impression that in a sense, which remained very vague at first, all these theories should 'be the same', that they 'gave the same results'. It is to express this intuitive idea of 'kinship' between different cohomology theories that I introduced the notion of 'motive' associated to an algebraic variety. By this term I intend to suggest that it is the 'common motive' (or the common reason) underlying this multitude of cohomological invariants associated to the variety [...] Thus, the motive associated to an algebraic variety would constitute the 'ultimate' cohomological invariant, from which all the others (associated to the different possible cohomology theories) would be deduced, as different 'musical incarnations', or different 'realizations'..." [**?**, A. Grothendieck, Récoltes et semailles, § 16]

cohomology would factor through $\mathfrak{h}$:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathfrak{h}(X) \arrow[d, dashed] \\ & H(X) = H_{\mathrm{t}}(X \otimes \overline{k}, \mathbf{Q}_\ell) \end{tikzcd} \qquad (\ell\text{-adic realization})$$

This expected factorization property means exactly that for each reasonable[9] "Weil cohomology", *the homological and the numerical equivalences agree* :

$$\sim_{\mathrm{hom}} \Longleftrightarrow \sim_{\mathrm{num}} ?$$

It is a deep conjecture in algebraic cycles theory (one of Grothendieck's standard conjectures, see **??** below).

Accepting this conjecture, one can show that the category $\mathrm{NM}^{\mathrm{eff}}(k)_{\mathbf{Q}}$ is graded in a natural way, and one can give a geometric meaning to the decomposition $H(X) = \bigoplus H^i(X)$ and also to the factorization of the zeta function when $k = \mathbb{F}_q$:

$$Z_X(t) = \frac{\prod_{i \text{ odd}} \det(1 - t\mathrm{Fr}_X^* | H^i(X))}{\prod_{i \text{ even}} \det(1 - t\mathrm{Fr}_X^* | H^i(X))}.$$

Namely, it is a reflection of a decomposition $\mathfrak{h}(X) = \bigoplus \mathfrak{h}^i(X)$ in $\mathrm{NM}^{\mathrm{eff}}(k)_{\mathbf{Q}}$.

Basic examples

1. The elementary formula
$$|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + \cdots + q^n$$
expresses the fact that the motive of $\mathbb{P}^n$ decomposes as
$$\mathfrak{h}(\mathbb{P}^n) = \mathbf{1} \oplus \mathbf{1}(-1) \oplus \cdots \oplus \mathbf{1}(-n),$$
where $\mathbf{1} = \mathfrak{h}(\operatorname{Spec} k)$ and $\mathbf{1}(-r) = (\mathfrak{h}^2(\mathbb{P}^1))^{\otimes r}$.

2. The motive of a smooth projective curve X decomposes as
$$\mathfrak{h}(X) = \mathbf{1} \oplus \mathfrak{h}^1(X) \oplus \mathbf{1}(-1).$$
The $\mathfrak{h}^1(X)$ component is a substitute for the jacobian $J(X)$ of X up to isogeny, in the sense where for any two curves X and X',
$$\mathrm{NM}^{\mathrm{eff}}(k)_{\mathbf{Q}}(\mathfrak{h}^1(X), \mathfrak{h}^1(X')) \cong \mathrm{Hom}(J(X), J'(X)) \otimes \mathbf{Q}.$$
Understanding and generalizing Weil's idea of using the jacobian as a geometric substitute for the cohomology $H^1(X, \mathbf{Q})$ has been incidentally one of the first clues leading Grothendieck to the notion of motive.

[9] "reasonable" might even be over-cautiousness.

The rôle of motives in the *factorization of zeta functions* of varieties over finite fields or number fields is a captivating aspect of the theory, reminiscent of Artin's formalism on factorizations of Dedekind's zeta functions in L-functions. In fact, the "arithmetical" part of the theory of motives provides a vast generalization (partly conjectural) in higher dimension of Artin's theory.

1.3 Galois theory

1.3.1 Let k be a field, $\overline{k}$ a separable closure, and $\mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k. Let k' be a finite-dimensional commutative k-algebra. Recall that the finite k-scheme $Z = \operatorname{Spec} k'$ is said to be *étale* if the following equivalent properties are satisfied :

(i) $k' \otimes \overline{k} \cong \overline{k}^{[k':k]}$,

(ii) $k' \cong \prod k_\alpha$ where k_α/k are finite separable extensions,

(iii) $|Z(\overline{k})| = [k' : k]$.

If moreover k is perfect, these properties are also equivalent to

(iv) k' is reduced.

The functor

$$\{\text{finite étale } k\text{-schemes}\} \to \{\text{finite continuous } \mathrm{Gal}(\overline{k}/k)\text{-sets}\}$$

given by $Z \mapsto Z(\overline{k})$ is an equivalence of categories known as the "Galois-Grothendieck correspondence".

1.3.2 It happens more often that we have to deal with linear representations of $\mathrm{Gal}(\overline{k}/k)$ than with finite $\mathrm{Gal}(\overline{k}/k)$-sets. As an example, the finite extensions of k already provide representations of $\mathrm{Gal}(\overline{k}/k)$, but there are also Frobenius characters, Artin L-series, ...

This suggests to "linearize" the Galois-Grothendieck correspondence, that is to replace it with an equivalence of the form

$$\{\ ?\ \} \xrightarrow{\sim} \left\{ \begin{matrix} \text{finite-dimensional } \mathbf{Q}\text{-vector spaces with} \\ \text{a continuous linear action of } \mathrm{Gal}(\overline{k}/k) \end{matrix} \right\}$$

the mysterious category $\{\ ?\ \}$ being semi-simple monoidal $\mathbf{Q}$-linear and linked to finite étale k-schemes. As the latter are simply the 0-dimensional smooth projective varieties, one idea would be to take the subcategory of $\mathrm{NM}^{\mathrm{eff}}(k)_{\mathbf{Q}}$ spanned by the 0-dimensional varieties. These objects are called *Artin motives.*

It is not difficult to check that the category $\mathrm{AM}(k)_{\mathbf{Q}}$ of Artin motives provides an answer to this linearization problem. More precisely, there is a commutative diagram

$$\begin{tikzcd}
\{\text{finite étale } k\text{-schemes}\} \arrow[r, "\sim"] \arrow[d, "\mathfrak{h}"'] & \{\text{finite continuous } \mathrm{Gal}(\overline{k}/k)\text{-sets}\} \arrow[d, "\mathfrak{l}"] \\
\mathrm{AM}(k)_{\mathbf{Q}} \arrow[r, "\sim"] & \left\{\begin{matrix}\text{finite-dimensional } \mathbf{Q}\text{-vector spaces with} \\ \text{a continuous linear action of } \mathrm{Gal}(\overline{k}/k)\end{matrix}\right\}
\end{tikzcd}$$

where $\mathfrak{l}$ is the contravariant linearization functor given by $S \mapsto \mathbf{Q}^S$, an element $g \in \mathrm{Gal}(\overline{k}/k)$ acting on $f \in \mathbf{Q}^S$ by $(g(f))(s) = f(g^{-1}(s))$.

Moreover, the tensor product in $\mathrm{AM}(k)_{\mathbf{Q}}$ corresponding to the cartesian product of k-varieties of dimension 0 is compatible with the tensor product of representations of $\mathrm{Gal}(\overline{k}/k)$.

1.3.3 Changing the point of view, all of this suggests to replace $\mathrm{AM}(k)_{\mathbf{Q}}$ by the more general $\mathrm{NM}^{\mathrm{eff}}(k)_{\mathbf{Q}}$ and to gain a *higher dimensional Galois theory*, namely a Galois theory for systems of multivariate polynomials. It is the root of Grothendieck's idea of *motivic Galois groups*, which will be outlined in chapter **??**. In this theory, the (pro)finite groups are replaced by (pro)algebraic groups.

Just as usual Galois theory translates problems about étale algebras into questions about finite or profinite groups, motivic Galois theory aims to reduce a whole class of geometric problems down to questions in the classical theory of representations of reductive groups and their Lie algebras. This philosophy has inspired, either directly or indirectly, numerous research projects in the last thirty years.

Chapter 2

RIGID $\otimes$-CATEGORIES, TANNAKIAN CATEGORIES

In this preliminary chapter, we present an overview of the theory of rigid $\otimes$-categories and of Tannakian categories, underlying the Galois theory of motives.

Tannakian theory is a powerful and versatile tool which plays for motives the rôle that usual Galois theory plays in arithmetic, or that fundamental groups play in topology. The theory is itself inspired by these two situations.

2.1 Introduction

Start from ordinary Galois theory: above (**??**) we have considered the equivalent categories

$$\{\text{finite étale } k\text{-schemes}\} \cong \{\text{finite continuous } \operatorname{Gal}(\overline{k}/k)\text{-sets}\}.$$

How to reconstruct the group $\operatorname{Gal}(\overline{k}/k)$ from these categories ? The answer is well-known and leads to Grothendieck's point of view on fundamental groups in algebraic geometry: $\operatorname{Gal}(\overline{k}/k)$ is the automorphism group of the forgetful functor

$$\{\text{finite continuous } \operatorname{Gal}(\overline{k}/k)\text{-sets}\} \to \{\text{finite sets}\}.$$

After $\mathbf{Q}$-linearization, we obtained the following equivalent categories

$$AM(k)_{\mathbf{Q}} \xrightarrow{\sim} \left\{ \begin{matrix} \text{finite-dimensional } \mathbf{Q}\text{-vector spaces with} \\ \text{a continuous linear action of } \operatorname{Gal}(\overline{k}/k) \end{matrix} \right\}$$

where the category on the left is that of Artin motives over k, and the same problem of reconstructing $\operatorname{Gal}(\overline{k}/k)$ from these categories arises. In this context, classical Tannaka-Krein duality for locally compact groups suggests to take into account the tensor product, namely the monoidal structure on these categories. Indeed, we recover the group $\mathrm{Gal}(\bar{k}/k)$ as the automorphism group of the forgetful ⊗-functor

$$\left\{\begin{matrix}\text{finite-dimensional } \mathbf{Q}\text{-vector spaces with} \\ \text{a continuous linear action of } \mathrm{Gal}(\bar{k}/k)\end{matrix}\right\} \to \{\text{finite-dimensional } \mathbf{Q}\text{-vector spaces}\}.$$

Suppose for now that $k \subset \mathbf{C}$. In this case, the space $\mathbf{Q}^{Z(\bar{k})}$ attached to each finite étale k-scheme (or more generally, to each Artin motive) is identified to its Betti cohomology $H_B(Z) = H^0(Z(\mathbf{C}), \mathbf{Q})$ in such a way that $\mathrm{Gal}(\bar{k}/k) \cong \mathrm{Aut}^{\otimes} H_{B|\,\mathrm{AM}(k)_{\mathbf{Q}}}$. This suggests a definition of the absolute pure motivic Galois group $G_{\mathrm{mot},k}$ in higher dimension by replacing $\mathrm{AM}(k)_{\mathbf{Q}}$ with the ⊗-category $\mathrm{NM}^{\mathrm{eff}}(k)_{\mathbf{Q}}$ of effective numerical motives (see **??**). Setting up a correct definition of $G_{\mathrm{mot},k}$ is the origin of Tannakian category theory, initiated by Grothendieck (and developed by N. Saavedra [**?**] and P. Deligne [**?**]).

2.2 Rigid ⊗-categories

2.2.1 In this theory the primordial object of study is the category $Rep_F G$ of finite-dimensional representations over a field F of an affine F-group scheme G. The algebra of functions on G inherits a coalgebra structure, making it a commutative Hopf algebra A_G, so that $G = \mathrm{Spec}\, A_G$.

The starting point is that the F-linear abelian category $Rep_F G$ is equivalent to the category of comodules of finite F-dimension over the *coalgebra* A_G. The *algebra* structure on A_G is hidden in the monoidal structure $\otimes : Rep_F G \times Rep_F G \to Rep_F G$.

2.2.2 We call ⊗-*category*[1] over a commutative ring F any F-linear category $\mathcal{T}$ endowed with a ⊗-structure, that is :

i) a bilinear bifunctor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$,

ii) a unit object $\mathbf{1}$,

iii) natural isomorphisms

$$\begin{aligned} a_{LMN} &: L \otimes (M \otimes N) \xrightarrow{\sim} (L \otimes M) \otimes N \\ c_{MN} &: M \otimes N \xrightarrow{\sim} N \otimes M \quad \text{such that } c_{NM} = c_{MN}^{-1} \\ u_M &: M \otimes \mathbf{1} \xrightarrow{\sim} M, \quad u'_M : \mathbf{1} \otimes M \xrightarrow{\sim} M \end{aligned}$$

with coherence axioms described by several evident commutative diagrams[2].

[1] unitary symmetric monoidal F-linear category in [**?**].

[2] a triangle with vertices $(M \otimes \mathbf{1}) \otimes N$, $M \otimes (1 \otimes N)$, $M \otimes N$, a pentagon whose vertices are different ways of parenthesizing $K \otimes L \otimes M \otimes N$, and a hexagon connecting the cyclic permutations of $L \otimes M \otimes N$, see [**?**]

A *rigid*[3] $\otimes$-category over F is a $\otimes$-category $\mathcal{T}$ over F endowed with

iv) an autoduality ${}^\vee : \mathcal{T} \to \mathcal{T}^{\mathrm{op}}$ such that for each object M, the functor $? \otimes M^\vee$ is left adjoint to $? \otimes M$, and $M^\vee \otimes ?$ is right adjoint to $M \otimes ?$.

Notice that this fourth datum provides natural transformations $\varepsilon : M \otimes M^\vee \to \mathbf{1}$ and $\eta : \mathbf{1} \to M^\vee \otimes M$ called respectively evaluation and coevaluation. They satisfy the relations $\varepsilon \otimes \mathrm{id}_M \circ \mathrm{id}_M \otimes \eta = \mathrm{id}_M$ and $\mathrm{id}_{M^\vee} \otimes \varepsilon \circ \eta \otimes \mathrm{id}_{M^\vee} = \mathrm{id}_{M^\vee}$. For a morphism f, we also write ${}^t f$ instead of $f^\vee$.

In a rigid $\otimes$-category, each endomorphism f has a *trace* defined as the following element of the commutative F-algebra $\mathrm{End}(\mathbf{1})$, obtained as the composition

$$\mathbf{1} \xrightarrow{f} M^\vee \otimes M \xrightarrow{c_{M^\vee M}} M \otimes M^\vee \xrightarrow{\varepsilon} \mathbf{1}.$$

The *rank* or *dimension* of the object M is the trace of the identity id_M, that is the element $\varepsilon \circ c_{M^\vee M} \circ \eta \in \mathrm{End}(\mathbf{1})$.

2.2.2.1 Examples.

1) For a field F and $\mathcal{T} = Rep_F G$, the unit $\mathbf{1}$ is the trivial representation F and the notion of rank agrees with the usual one.

2) Let $VecGr_K$ (resp. $sVec_K$) the category of finite-dimensional $\mathbf{Z}$-graded (resp. $\mathbf{Z}/2$-graded) vector spaces over a field K. The tensor product $\otimes_K$ makes it a rigid $\otimes$-category, the commutativity constraint c_{MN} swapping factors with the Koszul sign rule[4]. The unit is K concentrated in degree 0. The rank of an object V^* is its super-dimension $\dim V^+ - \dim V^-$, where $V^+ = \bigoplus V^{2i}$ and $V^- = \bigoplus V^{2i+1}$.

3) Combining the previous two examples, we obtain the rigid $\otimes$-category of super-representations of G. In practice, we rather consider the sub-rigid $\otimes$-category $Rep_F(G, \varepsilon)$ defined in the following way. Since ε is in the center of $G(F)$ and of order 1 or 2, $Rep_F(G, \varepsilon)$ is spanned by the super-representations whose even part is exactly where ε acts trivially.

4) Vector bundles over a given F-scheme X form a rigid $\otimes$-category with $\mathrm{End}(\mathbf{1}) = F$ when X is proper and geometrically connected. The rank agrees with the usual notion.

2.2.2.2 Remark. In a pseudo-abelian rigid $\otimes$-category over a field F of characteristic zero, we can define symmetric and exterior powers of each object M :

$$S^n M = s_n(M), \quad \bigwedge^n M = \lambda_n(M),$$

where $s_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma$ (resp. $\lambda_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma)\sigma$) denotes the symmetrization (resp. antisymmetrization) projection.

[3] also known as *autonomous*.

[4] in other words, there is a minus sign when swapping two odd factors.

2.2.3 A $\otimes$-*functor* between $\otimes$-categories is the data of an F-linear functor

$$\omega : \mathcal{T} \to \mathcal{T}'$$

and of a collection of natural isomorphisms

$$o_{MN} : \omega(M \otimes N) \xrightarrow{\sim} \omega(M) \otimes \omega(N)$$
$$o_{\mathbf{1}} : \omega(\mathbf{1}) \xrightarrow{\sim} \mathbf{1}'$$

compatible with the constraints a, c, u and u'.

If the $\otimes$-categories are rigid, then such a functor is automatically compatible with the duality $^\vee$ (up to natural isomorphism). Moreover, thanks to this duality, every natural transformation between $\otimes$-functors is then a natural isomorphism (see [**?**]).

2.2.3.1 Exercise. Construct the free rigid $\otimes$-category generated by one object M.

2.3 Tannakian categories

2.3.1 Let F be a field and $\mathcal{T}$ be an abelian rigid $\otimes$-category over F such that $\mathrm{End}(\mathbf{1}) = F$. A *fiber functor* on $\mathcal{T}$ is a faithful exact $\otimes$-functor

$$\omega : \mathcal{T} \to Vec_K$$

whose target is the rigid $\otimes$-category of finite-dimensional vector spaces over a certain field extension K of F. If such a functor exists, then we say that $\mathcal{T}$ is *Tannakian*[5].

We can then define the affine K-group scheme $G = \underline{\mathrm{Aut}}^{\otimes} \omega$ called the *Tannakian group* of $\mathcal{T}$ associated with ω. For each extension K'/K, $(\underline{\mathrm{Aut}}^{\otimes} \omega)(K')$ is the automorphism group of the extended $\otimes$-functor $\omega_{K'} : \mathcal{T} \to Vec_{K'}$.

2.3.1.1 Theorem (Deligne [?], 7). Let $\mathcal{T}$ be a rigid $\otimes$-category over a field F of characteristic zero, which is abelian and such that $\mathrm{End}(\mathbf{1}) = F$. Then $\mathcal{T}$ is Tannakian if and only if the rank of every object is a natural number, if and only if for each object M, we have $\bigwedge^n M = 0$ for n large enough.

2.3.2 If there exists a fiber functor ω with $K = F$, we say that $\mathcal{T}$ is *neutral Tannakian.*

In this case, ω becomes an *equivalence* of rigid $\otimes$-categories

$$\omega : \mathcal{T} \to Rep_F G$$

[5]in reference to classical Tannaka-Krein theory for compact groups. It deserves as well, if not more, to be called a Kreinian category, but usage decided otherwise, see [**?**]. The Tannakian part of the theory provides a left inverse to $G \mapsto Rep_F G$ and the Kreinian part tells us that it is also a right inverse.

where $Rep_F G$ denotes the rigid $\otimes$-category of finite-dimensional F-representations of $G = \underline{\mathrm{Aut}}^{\otimes}\,\omega$. Moreover, the category of fiber functors ω over F is equivalent to the groupoid of G-torsors.

This establishes a correspondence between $\otimes$-categorical properties and properties of the associated groups. Here are some examples :

- G is an algebraic group (that is, G is of finite type over F) if and only if $\mathcal{T}$ admits a $\otimes$-generator. This means that there is an object M such that every object of $\mathcal{T}$ is a subquotient of a finite sum $\bigoplus(M^{\otimes m_i} \otimes M^{\vee,\otimes n_i})$. In this case, we write $\mathcal{T} = \langle M\rangle^{\otimes}$ and G is identified with a Zariski closed algebraic subgroup of $GL(\omega(M))$. A subspace of $\bigoplus(\omega(M)^{\otimes m_i} \otimes (\omega(M)^{\vee})^{\otimes n_i})$ is stable under the action of G if and only if it is the image by ω of a subobject of $\bigoplus(M^{\otimes m_i} \otimes M^{\vee,\otimes n_i})$.

- (if the characteristic of F is zero) G is a pro-reductive[6] if and only if $\mathcal{T}$ is semi-simple.

2.3.3 Let $\phi : \mathcal{T}' \to \mathcal{T}$ be an exact $\otimes$-functor between neutral Tannakian categories. Let $\omega : \mathcal{T} \to Vec_F$ be a fiber functor. Then $\omega' = \omega \circ \phi : \mathcal{T}' \to Vec_F$ is also a fiber functor, and there is a homomorphism

$$f = \phi^* : G = \underline{\mathrm{Aut}}^{\otimes}\,\omega \to G' = \underline{\mathrm{Aut}}^{\otimes}\,\omega'$$

between the associated Tannakian groups.

Conversely, each group homomorphism $f : G \to G'$ yields a $\otimes$-functor $\phi = f^* : Rep_F G' \to Rep_F G$. We have that :

- f is a monomorphism (a closed immersion) if and only if each object M of $\mathcal{T}$ is a subquotient of the image by ϕ of an object N' of $\mathcal{T}'$,

- f is an epimorphism (faithfully flat) if and only if ϕ is fully faithful and for each object M' of $\mathcal{T}'$, each subobject of $\phi(M')$ is the image by ϕ of a subobject of M' (this last condition is automatically satisfied if $\mathcal{T}$ is semi-simple).

- ϕ identifies $\mathcal{T}'$ with the category of objects of $\mathcal{T}$ endowed with an action of G' factoring the action of G.

2.3.4 The Tannakian group G is not uniquely determined by $\mathcal{T}$ (for example, two F-groups which are inner forms of each other have equivalent $\otimes$-categories of representations), the datum ω is essential.

In a more intrisic and general manner, we can attach to each Tannakian category[7] $\mathcal{T}$ over F a commutative Hopf algebra $\mathcal{O}(\pi(\mathcal{T}))$ in the category of ind-objects of $\mathcal{T}$,

[6] An affine scheme group G over a field F of characteristic zero is pro-reductive if its unipotent radical is trivial. This is equivalent to G being a inverse limit of reductive algebraic F-groups (non necessarily connected).

[7] or even more generally, any abelian rigid $\otimes$-category such that $\mathrm{End}(\mathbf{1}) = F$ perfect, whose objects and morphism F-spaces are of finite length.

such that for every fiber functor $\omega : \mathcal{T} \to Vec_K$, $\omega(\mathcal{O}(\pi(\mathcal{T})))$ is the algebra of functions of the K-group scheme $\underline{\mathrm{Aut}}^{\otimes}\,\omega$.

Denote by $\pi(\mathcal{T})$ the same object in the opposite category (that of pro-objects of $\mathcal{T}^{\mathrm{op}}$) seen as an "affine $\mathcal{T}$-group scheme". As such, $\omega(\pi(\mathcal{T}))$ is identified with the usual affine K-group scheme $\underline{\mathrm{Aut}}^{\otimes}\,\omega$. If $\mathcal{O}(\pi(\mathcal{T}))$ is cocommutative, then $\pi(\mathcal{T})$ itself can be seen as an affine K-group scheme in the usual sense.

Each exact $\otimes$-functor $\phi : \mathcal{T}' \to \mathcal{T}$ between Tannakian categories over F yields a homomorphism $\pi(\mathcal{T}) \to \phi\pi(\mathcal{T}')$ and identifies $\mathcal{T}'$ with the category of objects of $\mathcal{T}$ endowed with an action of $\phi\pi(\mathcal{T}')$ factoring the action of $\pi(\mathcal{T})$, see [**?**].

2.3.5 A *Tannakian subcategory* of a Tannakian category $\mathcal{T}$ is a full subcategory $\mathcal{T}'$ of $\mathcal{T}$ stable by $\otimes$ and $^\vee$ such that each subobject and quotient in $\mathcal{T}$ of an object of $\mathcal{T}'$ lies in $\mathcal{T}'$[8]. The corresponding homomorphim $\mathcal{O}(\pi(\mathcal{T}')) \to \mathcal{O}(\pi(\mathcal{T}))$ is then faithfully flat, and for every fiber functor on $\mathcal{T}$ we get a faithfully flat homomorphism between the Tannakian groups.

[8] as a counter-example, the fully faithful restriction $\otimes$-functor $Rep_F GL_n \to Rep_F B_n$, where B_n is the subgroup of triangular matrices, does not make $Rep_F GL_n$ a Tannakian subcategory.

Chapter 3

ALGEBRAIC CYCLES AND COHOMOLOGIES (CASE OF SMOOTH PROJECTIVE VARIETIES)

In this second preliminary chapter, we present the basic tools of the theory of pure motives: algebraic cycles, adequate equivalences and Weil cohomologies.

3.1 Algebraic cycles and adequate relations

3.1.1 Let k be a base field and denote $\mathcal{P}(k)$ the *category of smooth projective schemes over* k, also sometimes called smooth projective k-varieties from now on.

For each $X \in \mathcal{P}(k)$ let $\mathcal{Z}^*(X)$ be the graded group of algebraic cycles on X, that is, the free abelian group generated by the integral closed subschemes Z of X and graded by codimension[1]. Denote by $[Z]$ the image of the subscheme Z in $\mathcal{Z}^*(X)$ (and in its quotients).

For each commutative ring F, the elements of

$$\mathcal{Z}^r(X)_F = \mathcal{Z}^r(X) \otimes_{\mathbf{Z}} F$$

are called the *algebraic cycles of codimension*[2] r *with coefficients in* F.

Intersection theory constructs a partially defined bilinear composition law on algebraic cycles: the intersection product, counting multiplicites, of two cycles whose components intersect properly (see [**?**]). To extend this definition to all cycles, the classical approach consists in using an adequate equivalence relation, see [**?**].

[1]we do not ask of X that it is connected or even equidimensional, the codimension of an integral subscheme is still well-defined.

[2]we avoid the word "degree" to prevent any confusion with the degree of 0-cycles defined in **??** or with the degree of the cycle class in cohomology, which is twice the codimension. Furthermore, it is also useful to consider the grading defined by dimension instead of codimension.

3.1.1.1 Definition. An equivalence relation $\sim$ on algebraic cycles is said to be *adequate* if it satisfies the following conditions for each $X, Y \in \mathcal{P}(k)$:

1) $\sim$ is compatible with the F-linear structure and with the grading,

2) for each $\alpha, \beta \in \mathcal{Z}^*(X)_F$ there exists an $\alpha' \sim \alpha$ such that α' and β intersect properly (so that the intersection product $\alpha' \cdot \beta$ is well-defined),

3) for each $\alpha \in \mathcal{Z}^*(X)_F$ and each $\gamma \in \mathcal{Z}^*(X \times Y)_F$ intersecting properly $(\mathrm{pr}_X^{XY})^{-1}(\alpha)$, we have $\alpha = 0 \implies \gamma_*(\alpha) \sim 0$ where
$$\gamma_*(\alpha) = \mathrm{pr}_Y^{XY}(\gamma \cdot (\mathrm{pr}_X^{XY})^{-1}(\alpha)).$$

3.1.1.2 Exercise. Show that these conditions ensure that an "external" product $\alpha \times \beta$ (on $X \times Y$) is ~ 0 as soon as $\alpha \sim 0$ or $\beta \sim 0$.

3.1.2 These conditions ensure that the intersection product which is partially defined on $\mathcal{Z}^*(X)_F$ factors through the quotient by $\sim$, and yields a well-defined composition law on
$$\mathcal{Z}^*_\sim(X)_F = \mathcal{Z}^*(X)_F / \sim$$
endowing it with the structure of a *commutative graded F-algebra* (and not graded-commutative)[3].

N.B. One shall be careful not to confuse $\mathcal{Z}^*_\sim(X) \otimes_{\mathbf{Z}} F$ with its quotient $\mathcal{Z}^*_\sim(X)_F$.

The third condition defining adequate relations implies that constructing the algebra $\mathcal{Z}^*_\sim(X)_F$ is *contravariant* in X (let γ be the transpose of the graph Γ_f of the given morphism f, obtained by exchanging in Γ_f the factors $X \times Y \to Y \times X$). We also obtain a homomorphism of F-modules *in the other way* $f_* : \mathcal{Z}^*_\sim(X)_F \to \mathcal{Z}^*_\sim(Y)_F$ shifting the grading by ($-$the generic relative dimension of f), if the latter is constant on the components of X (this time, take $\gamma = \Gamma_f$). This induced homomorphism f_* is not compatible with intersection products, however we have the so-called *projection formula*
$$f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta.$$
In addition, for each cartesian square in $\mathcal{P}(k)$
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "q"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
we have the formula $f^* p_* = q_* f'^*$ (see [?]).

[3] other notations often used are $A^*_\sim(X)_F$, $C^*_\sim(X)_F$, $A^*_\sim(X, F)$, ...

3.1.3 The elements of

$$\mathcal{Z}^{\dim X+r}(X\times Y)_F,\quad\text{resp.}\quad \mathcal{Z}^{\dim X+r}_\sim(X\times Y)_F$$

are called *algebraic correspondences of degree r with coefficients in F* (resp. modulo $\sim$) *from X to Y* (or between X and Y). When X is not equidimensional, one shall consider $\dim X$ as a locally constant function on X.

For example, the transpose of the graph of a morphism $Y\to X$ is a correspondence of degree 0 from X to Y.

The formula

$$g\circ f=\mathrm{pr}^{XYZ}_{XZ*}\left(\mathrm{pr}^{XYZ*}_{XY}(f)\cdot\mathrm{pr}^{XYZ*}_{YZ}(g)\right)$$

defines an associative composition law[4] for correspondences modulo $\sim$ (see [**?**])

$$\mathcal{Z}^{\dim Y+r}_\sim(X\times Y)_F\otimes_F\mathcal{Z}^{\dim Z+s}_\sim(Y\times Z)_F\to\mathcal{Z}^{\dim Z+r+s}_\sim(X\times Z)_F,$$

adding the degrees and endowing $\mathcal{Z}^{\dim X}_\sim(X\times X)_F$ with the structure of a (non necessarily commutative) F-algebra, the algebra of correspondences of degree 0. The unit is the class of the diagonal $\Delta_X\subset X\times X$. This algebra is also endowed with an (anti-)involution given by transposition $^\top$.

3.1.4 For $f\in\mathcal{Z}_\sim(X\times Y)_F$, $\alpha\in\mathcal{Z}_\sim(X)$ and $\beta\in\mathcal{Z}_\sim(Y)$, we define generally

$$f_*(\alpha)=\mathrm{pr}^{XY}_{Y*}(f\cdot\mathrm{pr}^{XY*}_X(\alpha))\in\mathcal{Z}_\sim(Y),$$
$$\text{and }f^*(\beta)=\mathrm{pr}^{XY}_{X*}(f\cdot\mathrm{pr}^{XY*}_Y(\beta))\in\mathcal{Z}_\sim(X).$$

For $\gamma\in\mathcal{Z}^{\dim X+r}_\sim(X\times Y)_F$, $a\in\mathcal{Z}^{\dim X'+p}_\sim(X\times X')_F$ and $b\in\mathcal{Z}^{\dim Y+q}_\sim(Y\times Y')_F$, we have the useful formula:

$$(a,b)^*(\gamma)=b\circ\gamma\circ a^\top\in\mathcal{Z}^{\dim X'+p+q+r}_\sim(X'\times Y')_F.$$

3.1.4.1 Exercise. Let k'/k be a field extension. Start with an adequate equivalence relation $\sim$ on algebraic cycles on objects of $\mathcal{P}(k')$. By "restriction", it induces an adequate equivalence relation (still denoted $\sim$) on algebraic cycles α on objects of $\mathcal{P}(k)$: $\alpha\sim 0$ if and only if $\alpha_{k'}\sim 0$. Whence there is a canonical injective homomorphism

$$\mathcal{Z}^*_\sim(X)_F\hookrightarrow\mathcal{Z}^*_\sim(X_{k'})_F.$$

When $k'=\overline{k}$, the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ acts naturally on $\mathcal{Z}^*_\sim(X_{\overline{k}})_F$, and the previous homomorphism induces

$$\mathcal{Z}^*_\sim(X)_F\hookrightarrow\mathcal{Z}^*_\sim(X_{\overline{k}})_F^{\mathrm{Gal}(\overline{k}/k)}.$$

Show that this homomorphism is bijective if F is a $\mathbf{Q}$-algebra, but not surjective in general if $F=\mathbf{Z}$.

[4]that this formula defines $g\circ f$ and not $f\circ g$ is compatible with the above definition of the degree of a correspondence, and is consistent with the traditionally contravariant point of view on motives.

3.2 Review of the usual adequate relations

3.2.1 They are the:

- rational equivalence $\sim_{\mathrm{rat}}$,
- algebraic equivalence $\sim_{\mathrm{alg}}$,
- homological equivalences $\sim_{\mathrm{hom}}$,
- numerical equivalence $\sim_{\mathrm{num}}$,

to which we add the interesting equivalence $\sim_{\otimes\mathrm{nil}}$ of "smash-nilpotence" (or $\otimes$-nilpotence) appearing in a work of V. Voevodsky [**?**][5].

To compare them, say that $\sim$ is finer than $\approx$, and write $\sim\succ\approx$, if $\alpha\sim 0$ implies $\alpha\approx 0$. We then have

$$\sim_{\mathrm{rat}}\succ\sim_{\mathrm{alg}}\succ\sim_{\mathrm{hom}}\succ\sim_{\mathrm{num}},\ \text{and}$$

$$\sim_{\mathrm{rat}}\succ\sim_{\mathrm{alg}}\succ\sim_{\otimes\mathrm{nil}}\succ\sim_{\mathrm{hom}}\succ\sim_{\mathrm{num}}\ \text{ if } F\supset\mathbf{Q}.$$

As a foreshadowing, let us mention that we conjecture that the last three equivalences of this sequence coincide exactly (Grothendieck, Voevodsky).

3.2.2 A cycle $\alpha\in\mathcal{Z}^*(X)_F$ is *rationally equivalent to* 0 (that is, $\alpha\sim_{\mathrm{rat}}0$) if there exists $\beta\in\mathcal{Z}^*(X\times\mathbb{P}^1)_F$ such that $\beta(0)$ and $\beta(\infty)$ are well-defined[6] and such that $\alpha=\beta(0)-\beta(\infty)$.

The fact that $\sim_{\mathrm{rat}}$ is adequate is among the foundations of intersection theory: the difficult condition is the second one, which is W. Chow's "moving lemma", see [**?**].

The graded rings $\mathcal{Z}^*(X)/\sim_{\mathrm{rat}}$ are called the *Chow rings* and are customarily denoted $\mathrm{CH}^*(X)$. We have $\mathrm{CH}^*(X)\otimes F=\mathcal{Z}^*_{\mathrm{rat}}(X)_F$, also denoted $\mathrm{CH}^*(X)_F$.

3.2.2.1 Lemma. $\sim_{\mathrm{rat}}$ is the finest adequate relation.

Indeed, let $\sim$ be an adequate relation on algebraic cycles with coefficients in F. By the third condition of adequateness, it suffices to show that $[0]\sim[\infty]$ on $\mathbb{P}^1$. By the second condition, there exists a cycle $\sum n_i[x_i]\sim[1]$ with $n_i\in F$, such that $\sum n_i[x_i]\cdot[1]$ is well-defined (that is, $x_i\neq 1$). Apply the third condition with γ the graph of the polynomial $1-\prod\left(\frac{x-x_i}{1-x_i}\right)^{m_i}$ (with $m_i>0$) and $\alpha=\sum n_i[x_i]-[1]$. We obtain that $mn[1]\sim m[0]$ where $m=\sum m_i$ and $n=\sum n_i$. Since the m_i can still be chosen freely, we conclude that $n[1]\sim[0]$. Now, apply the automorphism $x\mapsto 1/x$ (and the third condition again) to obtain $n[1]\sim[\infty]$, whence $[0]\sim[\infty]$ as required.

In virtue of this lemma, we can identify any adequate relation on algebraic cycles with coefficients in F with the data of a homogeneous ideal $I^*_\sim(X)=\{\alpha\in\mathrm{CH}(X)_F,\alpha\sim$

[5] when these symbols appear as subscripts or superscripts, we will shorten $\sim_{\mathrm{rat}}$ as rat, and so on.

[6] more precisely, such that each component Z of β is dominant over $\mathbb{P}^1$, $\beta(0)$ (resp. $\beta(\infty)$) being then the sum of the cycles associated to the closed subschemes $Z(0)$ (resp. $Z(\infty)$) of X, see [**?**].

$0\}$ of the Chow ring $\otimes F$ of every $X \in \mathcal{P}(k)$, satisfying a compatibility condition with the bifunctoriality of Chow groups.

3.2.2.2 Exercises.

1) Let $x \in \mathbb{P}^1(k)$. Show that the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ decomposes modulo $\sim$ as the sum of the idempotent correspondences $[x] \times \mathbb{P}^1$ and $\mathbb{P}^1 \times [x]$, and that this decomposition is independent of x.

2) More generally, let $t \in \mathcal{Z}^1_\sim(\mathbb{P}^n)_F$ be the class of a hyperplane in $\mathbb{P}^n$, and let $[\Delta] \in \mathcal{Z}^n_\sim(\mathbb{P}^n \times \mathbb{P}^n)_F$ be the class of the diagonal. Show that $[\Delta] = \sum t^i \times t^{n-i}$.

   Deduce that $\mathcal{Z}^*_\sim(X \times \mathbb{P}^n)_F \cong \mathcal{Z}^*_\sim(X)_F[t]/(t^{n+1})$ as graded F-algebras, t being of degree 1. Explain how this generalizes Bézout's theorem.

3.2.3 The definition of $\sim_{\mathrm{alg}}$ is analogous to that of $\sim_{\mathrm{rat}}$. The projective line $\mathbb{P}^1$ is replaced by an arbitrary smooth projective curve (or equivalently[7] any smooth connected projective k-scheme, that is how one can prove that the condition $\sim_{\mathrm{alg}} 0$ is stable under addition) and 0 and ∞ are replaced by two k-rational points. It is easy to see that $\mathcal{Z}^*_{\mathrm{alg}}(X)_F = \mathcal{Z}^*_{\mathrm{alg}}(X) \otimes_{\mathbf{Z}} F$.

3.2.4 According to [**?**], an element $\alpha \in \mathcal{Z}^*(X)_F$ is "smash-nilpotent" or $\otimes$-nilpotent if there exists $N > 0$ such that $\alpha \times \alpha \times \cdots \times \alpha$ is rationally equivalent to 0 on X^N. One checks without difficulty that this is indeed an adequate relation[8], and that $\mathcal{Z}^*_{\otimes\mathrm{nil}}(X)_F = \mathcal{Z}^*_{\otimes\mathrm{nil}}(X) \otimes_{\mathbf{Z}} F$.

3.2.4.1 Proposition ([?]). If F is a $\mathbf{Q}$-algebra, then $\sim_{\mathrm{alg}}$ is finer than $\sim_{\otimes\mathrm{nil}}$.

Proof. Let $\alpha \in \mathrm{CH}^r(X)_F$ be an element sent to 0 in $\mathcal{Z}^r_{\mathrm{alg}}(X)_F$. There exists a genus g smooth connected projective curve T, two points t_0 and t_1 and $\beta \in \mathrm{CH}^r(X \times T)_F$, such that $\alpha = \beta(t_0) - \beta(t_1)$. We have $\alpha^{\otimes N} = \beta^{\otimes N}([t_0] - [t_1])^{\otimes N}$, so that it suffices to prove that $([t_0] - [t_1])^{\otimes N} = 0$ in $\mathrm{CH}^N(T^N)_{\mathbf{Q}}$ for N large enough (in fact, $N = 2g$ suffices if $g \geq 1$, which we can assume). This relies on the fact that the canonical morphism $S^n(T) \to J(T)$ from the n-th symmetric power of T to the Jacobian given by $x_1 + \cdots + x_n \mapsto x_1 + \cdots + x_n - nt_0$ identifies $S^n(T)$ with a projective bundle on $J(T)$, as soon as $n \geq 2g-1$ [**?**]. Knowing the structure of the Chow groups of a projective bundle [**?**], we deduce that the inclusion $\iota : S^{2g-1}(T) \hookrightarrow S^{2g}(T)$ given by $x_1 + \cdots + x_{2g-1} \mapsto x_1 + \cdots + x_{2g-1} + t_0$ induces an isomorphism $\iota^* : \mathrm{CH}^{2g}(S^{2g}(T)) \cong \mathrm{CH}^{2g-1}(S^{2g-1}(T))$. Furthermore, $\mathrm{CH}^{2g}(S^{2g}(T))_{\mathbf{Q}}$ is identified with the symmetric elements in $\mathrm{CH}^{2g}(T^{2g})_{\mathbf{Q}}$ [**?**]. This lets us see $([t_0] - [t_1])^{\otimes 2g}$ as a cycle on $S^{2g}(T)$, and it is easy to see that $\iota^*(([t_0] - [t_1])^{\otimes 2g}) = 0$. $\square$

[7]any two points can be joined by a smooth connected projective curve: cut the variety by hypersurfaces general enough and of high enough degree passing through the two points, so that the section is a smooth connected projective curve (they even exist if k is finite, see [**?**]).

[8]this is even obvious when using the adequateness criterion given in the next chapter.

3.2.4.2 Remarks.

1) The bound $2g$ for the nilpotence exponent is not optimal. One can show that the optimal bound is $g+1$, see point 2) of **??**.

2) Let A be an elliptic curve, and x_0 and x_1 two distinct points in $A(k)$. The above proof shows that the cycle $([x_0]-[x_1]) \times ([x_0]-[x_1])$ on $A \times A$ is zero modulo rational equivalence, while $[x_0]-[x_1]$ on A is not itself zero. This disproves the converse of exercise **??**.

A remarkable aspect of this result is that even though $\sim_{\mathrm{rat}}$ and $\sim_{\otimes \mathrm{nil}}$ seem very similar by definition, the groups of cycles modulo $\sim_{\mathrm{rat}}$ and modulo $\sim_{\otimes \mathrm{nil}}$ have wildly different properties when *k is algebraically closed*. On the one hand, Chow groups are "continuous" invariants, in general "enormous" (and varying with k) as shown by D. Mumford. On the other hand, the groups $\mathcal{Z}^*_{\mathrm{alg}}(X)$ and *a fortiori* $\mathcal{Z}^*_{\sim}(X)$ for every adequate equivalence coarser that $\sim_{\mathrm{alg}}$ (in particular $\sim_{\otimes \mathrm{nil}}$) are "discrete" invariants, countable and *invariant by extensions of k* in virtue of the theory of Chow forms [**?**].

3.2.5 Homological equivalence relies on the notion (and the choice) of a Weil cohomology, which will be explained below. It is the only usual adequate equivalence for which it is unclear *a priori* that $\mathcal{Z}^*_{\sim}(X) \otimes_{\mathbf{Z}} F = \mathcal{Z}^*_{\sim}(X)_F$ for each F contained in the ring of coefficients of the chosen cohomology[9].

3.2.6 We customarily call 0-*cycle* a cycle of dimension 0, that is a linear combination $\sum n_i [P_i]$ of closed points. Its degree is defined as

$$\sum n_i [k(P_i) : k].$$

It only depends on the class of the 0-cycle modulo algebraic equivalence[10] (this is a rephrasing of Poncelet's "conservation of number principle" [**?**]).

3.2.7 An element $\alpha \in \mathcal{Z}^r(X)_F$ is *numerically equivalent to* 0 if for each cycle β of dimension r, the 0-cycle $\alpha \cdot \beta$ (well-defined in $\mathrm{CH}^d(X)$) is of degree $\langle \alpha, \beta \rangle$ equal to zero.

It is known that $\mathcal{Z}^r_{\mathrm{num}}(X)_{\mathbf{Q}} = \mathcal{Z}^r_{\mathrm{alg}}(X)_{\mathbf{Q}}$ when $r \leq 1$ (Matsusaka [**?**], reducing the problem to the case of a surface) but the kernel of the quotient map $\mathcal{Z}^r_{\mathrm{alg}}(X)_{\mathbf{Q}} \to \mathcal{Z}^r_{\mathrm{num}}(X)_{\mathbf{Q}}$ is infinite-dimensional in general if $r \geq 2$ [**?**].

[9] as such, it is also unclear that $\mathcal{Z}^*_{\sim}(X)$ has finite rank, whereas this is obviously the case for $\mathcal{Z}^*_{\sim}(X)_K$ where K is the coefficient ring.

[10] and of course, it only depends on its class modulo numerical equivalence, by definition.

3.2.7.1 Proposition. If F is an integral domain of characteristic zero, then the F-module $\mathcal{Z}^r_{\mathrm{num}}(X)_F$ is free of finite type and $\mathcal{Z}^r_{\mathrm{num}}(X)_F = \mathcal{Z}^r_{\mathrm{num}}(X) \otimes_{\mathbf{Z}} F$. What is more, if $F \supset \mathbf{Q}$ and X has pure dimension d, then the "degree of the intersection 0-cycle" pairing

$$\mathcal{Z}^r_{\mathrm{num}}(X)_F \times \mathcal{Z}^{d-r}_{\mathrm{num}}(X)_F \to F, \quad (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle$$

is a perfect pairing.

Since the proof relies on the existence of Weil cohomologies, we postpone it (to **??**).

3.2.7.2 Exercise. Suppose that F is a field. Show that $\sim_{\mathrm{num}}$ is the coarsest non-trivial adequate equivalence relation.

3.3 Weil cohomologies

3.3.1 Taking inspiration from [**?**], we axiomatize the properties of the cohomologies mentioned in **??**.

The product $\times_k$ endows the category $\mathcal{P}(k)$ of smooth projective k-schemes with the structure of a (non additive) symmetric monoidal category. The unit is the point $\operatorname{Spec} k$ and there are canonical isomorphisms

$$X \times Y \cong Y \times X, \quad (X \times Y) \times Z \cong X \times (Y \times Z).$$

The diagonal $\Delta_X : X \hookrightarrow X \times X$ endows each object $X \in \mathcal{P}(k)$ with the structure of a counitary cocommutative coalgebra with respect to this monoidal structure.

Recall that $VecGr_K$ denotes the rigid $\otimes$-category of finite-dimensional $\mathbf{Z}$-graded vector spaces over a field K (with commutativity given by the Koszul sign rule). Let $VecGr_K^{\geq 0}$ be its full (not rigid) sub-$\otimes$-category spanned by objects concentrated in nonnegative degrees.

3.3.1.1 Definition. A (pure) Weil cohomology[11] H^* is a functor

$$H^* : \mathcal{P}(k)^{\mathrm{op}} \to VecGr_K^{\geq 0}$$

compatible with the monoidal structure, satisfying

∗) $\dim_K H^2(\mathbb{P}^1) = 1$,

and endowed with the additional data 1) and 2) given below[12].

[11] in the literature it is often asked that K is of characteristic zero, however the general case, even that of a product of fields, is useful.

[12] several authors also require H^* to satisfy the "weak and hard Lefschetz theorems". We avoid it, on the one hand because we will have to consider generalizations of the notion of Weil cohomologies to the mixed case (**??**) where the "Lefschetz theorems" lose their relevance, and on the other hand because we will construct in the pure case *a priori* non-classical Weil cohomologies for which the "Lefschetz theorems" are not yet known to hold (**??**).

Notice that the comultiplication of each object X of $\mathcal{P}(k)$ induces *ipso facto* a multiplication (the cup-product) on $H^*(X)$, making $H^*(X)$ a (unitary) graded-commutative $\mathbf{N}$-graded K-algebra (that is, a unitary commutative algebra object in $VecGr_K^{\geq 0}$).

For each $d \in \mathbf{Z}$, denote (r) the "Tate twist" operation $V^* \mapsto V^* \otimes H^2(\mathbb{P}^1)^{\otimes(-r)}$ in $VecGr_K$ (considering that the $*$) condition implies the $\otimes$-invertibility of $H^2(\mathbb{P}^1)$ in $VecGr_K$).

The additional data in **??** are:

1) (trace, Poincaré duality) for each $X \in \mathcal{P}(k)$ with pure dimension d, a K-linear map $H^{2d}(X)(d) \xrightarrow{\mathrm{Tr}_X} K$, which is an isomorphism when X is geometrically connected, satisfying $\mathrm{Tr}_{X\times Y} = \mathrm{Tr}_X \mathrm{Tr}_Y$ (after the obvious identifications that are needed) and such that the product of $H^*(X)$ induces for each i a duality pairing

$$\langle\ ,\ \rangle : H^i(X) \times H^{2d-i}(X)(d) \to H^{2d}(X)(d) \xrightarrow{\mathrm{Tr}_X} K.$$

2) (cycle classes) for each $X \in \mathcal{P}(k)$, abelian group homomorphisms

$$\gamma_X^r = \gamma_{X,H}^r : \mathrm{CH}^r(X) \to H^{2r}(X)(r)$$

that are

- contravariant in X,
- compatible with the "external product" $\gamma_{X\times Y}^{r+s}(\alpha \times \beta) = \gamma_X^r(\alpha) \otimes \gamma_Y^s(\beta)$,
- "normalized" such that γ^d composed with the trace Tr_X coincides with the 0-cycles degree map considered in **??** when X has pure dimension d.

Compatibility with the monoidal structure means that for each pair (X, Y) there exists a canonical Künneth isomorphism between graded-commutative graded algebras:

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

functorially in X and Y (and compatible with the associativity and commutativity constraints). The projections $\pi_X^i = \pi_{X,H}^i : H^*(X) \to H^i(X) \hookrightarrow H^*(X)$ on homogeneous components are called the *Künneth projectors* of X.

3.3.2 Some formal consequences of the definition

See [?], [?].

- For each $r \in \mathbf{Z}$, $K(r)$ is isomorphic (not canonically in general) to K concentrated in degree $-2r$. The double Tate torsion $(r)(s)$ is the same as the simple torsion $(r+s)$.

- $H^*(\mathrm{Spec}\, k) = H^0(\mathrm{Spec}\, k) \overset{\mathrm{Tr}_{\mathrm{Spec}\, k}}{=} K$.

- $H^i(X) = 0$ for $i > 2\dim X$.

- Poincaré duality can also be written in the following form, for each $r \in \mathbf{Z}$

$$\langle\ ,\ \rangle : H^i(X)(r) \times H^{2d_X-i}(X)(d-r) \to H^{2d_X}(X)(d) \xrightarrow{\mathrm{Tr}_X} K,$$

and we have $\langle x, x'\rangle = (-1)^i\langle x', x\rangle$ for each $x \in H^i(X)(r)$ and $x' \in H^{2d-i}(X)(d-r)$.

- $\langle\gamma_r(\alpha), \gamma_{d-r}(\beta)\rangle = \langle\alpha, \beta\rangle$.

- Let $X, Y \in \mathcal{P}(k)$ with dimensions d_X and d_Y respectively, and $f : X \to Y$ be a morphism. To $f^* = H^*(f)$, Poincaré duality assigns the "Gysin morphism" $f_* : H^*(X)(d_X) \to H^{*-2(d_X-d_Y)}(Y)(d_Y)$, adjoint to f^* and satisfying the *projection formula* $f_*(f^*(y)\cdot x) = y\cdot f_*(x)$. One also abusively denotes f_* the homomorphisms obtained by the Gysin morphism after Tate twists.

- The trace map Tr_X is p^X_*, p^X being the structural morphism $X \to \operatorname{Spec} k$. For each morphism $f : X \to Y$ as before, we have $\mathrm{Tr}_X = \mathrm{Tr}_Y \circ f_*$. From $p^{X\times Y} = p^X \times p^Y$ and the Künneth formula, we deduce $\mathrm{Tr}_{X\times Y} = \mathrm{Tr}_X \otimes \mathrm{Tr}_Y$.

- By the Künneth formula and duality, there are non graded canonical isomorphisms

$$H(X \times Y)(d_X) \cong H(X)(d_X) \otimes H(Y) \cong \mathrm{Hom}_K(H(X), H(Y))$$

given by $x \otimes y \mapsto (z \mapsto \langle z, x\rangle y)$, and more precisely there are canonical isomorphisms $u \mapsto \underline{u} = (z \mapsto \mathrm{pr}^{XY}_{Y*}(\mathrm{pr}^{XY*}_X(z) \cdot u))$:

$$\begin{aligned} H^{2d_X+i}(X \times Y)(d_X) &\cong \bigoplus_{j\geq 0} H^{2d_X-j}(X)(d_X) \otimes H^{j+i}(Y) \\ &\cong \bigoplus_{j\geq 0} \mathrm{Hom}_K(H^j(X), H^{j+i}(Y)) \end{aligned}$$

Moreover, for each $v \in H^{2d_Y+h}(Y \times Z)(d_Y)$ the element

$$w = \mathrm{pr}^{XYZ}_{XZ*}(\mathrm{pr}^{XYZ*}_{XY}(u) \cdot \mathrm{pr}^{XYZ*}_{YZ}(v))$$

satisfies $\underline{w} = \underline{v} \circ \underline{u} \in \bigoplus_{j\geq 0} \mathrm{Hom}_K(H^j(X), H^{j+i+h}(Z))$.

- Let $x \in H^i(X)$, $y \in H^j(Y)$ with $i = j \bmod 2$. We have

$$\langle y', \underline{x \otimes y}(x')\rangle = \langle x', \underline{y \otimes x}(y')\rangle,$$

so that we can identify $(-1)^i\underline{y \otimes x}$ with the Tate-twisted transpose of $\underline{x \otimes y}$. It follows that denoting c_{XY} the switching map $X \times Y \to Y \times X$, we have

$$\forall u \in H^{2d_X+2r}(X \times Y)(d_X), \quad \underline{c^*_{XY}(u)} = \underline{u}^\top.$$

- Let $x \in H^{\mathrm{even}}(X \times W)(d_X)$, $y \in H^{\mathrm{even}}(Y \times Z)(d_Y)$, $v \in H(X \times Y)(d_X)$ and $w = \underline{c^*_{WY}(x \otimes y))}(v) \in H(W \times Z)(d_W)$. Then $\underline{w} = \underline{y} \circ \underline{v} \circ \underline{x}^\top$.
- $\gamma^0_X([X])$ is the unit of the algebra $H^*(X)$.
- Functoriality and compatibility of cycle classes with the "external" product imply compatibility with the "internal" product (intersection product in $\mathrm{CH}(X)$, cup-product in $H^*(X)$):

$$\gamma_X^{r+s}(\alpha \cdot \beta) = \gamma_X^{r+s}(\Delta^*(\alpha \times \beta)) = \Delta^*(\gamma^r_X(\alpha) \otimes \gamma^s_X(\beta)) = \gamma^r_X(\alpha) \cdot \gamma^s_X(\beta).$$

- Let $g : Y \to X$ be a morphism in $\mathcal{P}(k)$ and suppose that X has pure dimension d_X. Then with
$$u = \gamma_{X \times Y}(\Gamma_g^\top) \in H^{2d_X}(X \times Y)(d_X),$$
we have $\underline{u} = g^* \in \mathrm{Hom}_K(H^*(X), H^*(Y))$. If Y has pure dimension d_Y then with
$$u^\top = \gamma_{X \times Y}(\Gamma_g) \in H^{2d_Y}(X \times Y)(d_X),$$
we also have
$$\underline{u^\top} = g_* \in \mathrm{Hom}_K(H^*(Y), H^{*-2(d_X - d_Y)}(X)(d_X - d_Y)).$$
- Let $\beta \in \mathrm{CH}(X \times Y)$. Reusing the notation $\theta \in \mathrm{CH}(X) \mapsto \beta_*(\theta) = \mathrm{pr}^{XY}_{Y*}(\beta \cdot \mathrm{pr}^{XY*}_X(\theta)) \in \mathrm{CH}(Y)$ already introduced in **??**, we have $\underline{\gamma_{X \times Y}(\beta)} \circ \gamma_X = \gamma_Y \circ \beta_*$. When $\beta = \Gamma_f$ is the graph of a morphism $f : X \to Y$, we have $\underline{\gamma_{X \times Y}(\Gamma_f)} = f_*$, whence $f_* \gamma_X = \gamma_Y f_*$.
- $H^*(X \amalg Y)$ is canonically isomorphic to $H^*(X) \oplus H^*(Y)$. This lets us circumvent any equidimensionality assumption by letting
$$H^{2d_X+i}(X \times Y)(d_X) = \bigoplus_j H^{2d_j+i}(X_j \times Y)(d_j)$$
where X_j is the component of X having pure dimension d_j.

3.3.3 Lefschetz trace formula. Let $V = V^+ \oplus V^-$ and $W = W^+ \oplus W^-$ be two (finite-dimensional) super-vector spaces and let $\tilde{V}$ and $\tilde{W}$ be their respective K-duals. Recall that the commutativity constraint c_{VW} is the interchange of factors with the Koszul sign rule

$$V \otimes W \cong W \otimes V, \quad c_{VW}(v \otimes w) = (-1)^{\delta v \cdot \delta w} w \otimes v,$$

where δv (resp. δw) denotes the parity of a homogeneous element v (resp. w).

We identify $V \otimes \tilde{W}$ with the dual of $\tilde{V} \otimes W$ by the formula $\langle \tilde{v} \otimes w, v \otimes \tilde{w} \rangle = (-1)^{\delta v \cdot \delta w} \langle \tilde{v}, v \rangle \langle w, \tilde{w} \rangle$ (also we can assume that $\delta v = \delta \tilde{v}$ and $\delta w = \delta \tilde{w}$).

Moreover, we identify $\tilde{V} \otimes W$ with $\mathrm{Hom}(V, W)$ by $\tilde{v} \otimes w \mapsto (v' \mapsto \langle \tilde{v}, v' \rangle w)$ and similarly $\tilde{W} \otimes V$ is identified with $\mathrm{Hom}(W, V)$.

As such, the endomorphism $(\tilde{w} \otimes v) \circ (\tilde{v} \otimes w)$ of V is given by $v' \mapsto \langle w, \tilde{w} \rangle \langle v', \tilde{v} \rangle v$, that is, $(\tilde{w} \otimes v) \circ (\tilde{v} \otimes w) = \langle w, \tilde{w} \rangle \tilde{v} \otimes v$. The supertrace $s\mathrm{tr}(\tilde{v} \otimes v)$ is $\langle \tilde{v}, v \rangle$.

We deduce that $\langle \tilde{v} \otimes w, c_{\tilde{W}V}(\tilde{w} \otimes v) \rangle = s\mathrm{tr}((\tilde{w} \otimes v) \circ (\tilde{v} \otimes w))$, and by linearity that

$$\langle \phi, c_{\tilde{W}V}(\psi) \rangle = s\mathrm{tr}(\psi \circ \phi)$$

for each $\phi \in \mathrm{Hom}(V, W)$ and $\psi \in \mathrm{Hom}(W, V)$ of matching parities.

Apply this super-linear algebra formula to

$$V^+ = H^+(X), \quad V^- = H^-(X), \quad W^+ = H^+(Y), \quad W^- = H^-(Y),$$

and

$$\phi \in H^{2d_Y+i}(X \times Y)(d_Y + n), \quad \psi \in H^{2d_X-i}(Y \times X)(d_X - n),$$

where H^+ and H^- respectively denote the even and odd parts of cohomology. We also have $c_{\tilde{W}V}(\psi) = (c_{YX})_*(\psi) = (c_{XY})^*(\psi) = \psi^\top$.

We then obtain the *Lefschetz trace formula*:

$$\boxed{\langle \phi, \psi^\top \rangle = \sum_{j=0}^{2d_X} (-1)^j \mathrm{tr}(\psi \circ \phi | H^j(X)).}$$

In the case where $X = Y$, $i = n = 0$ and $\psi = \pi_X^j$, we have $(\pi_X^j)^\top = \pi_X^{2d_X-j}$, whence

$$\boxed{\mathrm{tr}(\phi | H^j(X)) = (-1)^j \langle \phi, \pi_X^{2d_X-j} \rangle.}$$

3.3.4 Homological equivalence. Let H^* be a Weil cohomology with coefficients in K and F be a subring of K. An element $\alpha \in \mathcal{Z}^r(X)_F$ is *homologically equivalent to* 0 if $H^*(\alpha) = \gamma^r(\alpha) = 0$. It follows from the above formula $\gamma_X^{r+s}(\alpha \cdot \beta) = \gamma_X^r(\alpha) \cdot \gamma_X^s(\beta)$ that this is an adequate equivalence relation, denoted $\sim_{\mathrm{hom}}$ or $\sim_H$.

Via γ_X^r, $\mathcal{Z}^r_{\mathrm{hom}}(X)_F$ is identified with an F-submodule (necessarily without torsion) of $H^{2r}(X)(r)$. It thus has finite rank if $F = K$, but in general it is not known whether the quotient map $\mathcal{Z}^r_{\mathrm{hom}}(X)_F \otimes_{\mathbf{Z}} K \to \mathcal{Z}^r_{\mathrm{hom}}(X)_K$ is injective[13], and even whether $\mathcal{Z}^r_{\mathrm{hom}}(X)$ has finite rank.

It is immediate that $\sim_{\otimes\mathrm{nil}}$ is finer than $\sim_{\mathrm{hom}}$. On the other hand, $\sim_{\mathrm{hom}}$ is finer than $\sim_{\mathrm{num}}$ (by multiplicativity of cycle classes, or using exercise **??**).

3.3.4.1 Exercise. Show that $\sim_{\mathrm{hom}}$ is coarser than $\sim_{\mathrm{alg}}$ without using **??**.

[13]if K has positive characteristic p, this is false. There are counter-examples with X a supersingular abelian surface over a field k of characteristic p and $r = 1$.