In this second preliminary chapter, we present the basic tools of the theory of pure motives: algebraic cycles, adequate equivalences and Weil cohomologies.

0.1 Algebraic cycles and adequate relations

0.1.1 Let k be a base field and denote $\mathcal{P}(k)$ the *category of smooth projective schemes over* k, also sometimes called smooth projective k-varieties from now on.

For each $X \in \mathcal{P}(k)$ let $\mathcal{Z}^*(X)$ be the graded group of algebraic cycles on X, that is, the free abelian group generated by the integral closed subschemes Z of X and graded by codimension[1]. Denote by $[Z]$ the image of the subscheme Z in $\mathcal{Z}^*(X)$ (and in its quotients).

For each commutative ring F, the elements of

$$\mathcal{Z}^r(X)_F = \mathcal{Z}^r(X) \otimes_{\mathbf{Z}} F$$

are called the *algebraic cycles of codimension*[2] r *with coefficients in* F.

Intersection theory constructs a partially defined bilinear composition law on algebraic cycles: the intersection product, counting multiplicites, of two cycles whose components intersect properly (see [**?**]). To extend this definition to all cycles, the classical approach consists in using an adequate equivalence relation, see [**?**].

0.1.1.1 Definition. An equivalence relation $\sim$ on algebraic cycles is said to be *adequate* if it satisfies the following conditions for each $X, Y \in \mathcal{P}(k)$:

1) $\sim$ is compatible with the F-linear structure and with the grading,

2) for each $\alpha, \beta \in \mathcal{Z}^*(X)_F$ there exists an $\alpha' \sim \alpha$ such that α' and β intersect properly (so that the intersection product $\alpha' \cdot \beta$ is well-defined),

3) for each $\alpha \in \mathcal{Z}^*(X)_F$ and each $\gamma \in \mathcal{Z}^*(X \times Y)_F$ intersecting properly $(\mathrm{pr}_X^{XY})^{-1}(\alpha)$, we have $\alpha = 0 \implies \gamma_*(\alpha) \sim 0$ where

$$\gamma_*(\alpha) = \mathrm{pr}_Y^{XY}(\gamma \cdot (\mathrm{pr}_X^{XY})^{-1}(\alpha)).$$

0.1.1.2 Exercise. Show that these conditions ensure that an "external" product $\alpha \times \beta$ (on $X \times Y$) is ~ 0 as soon as $\alpha \sim 0$ or $\beta \sim 0$.

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[1] we do not ask of X that it is connected or even equidimensional, the codimension of an integral subscheme is still well-defined.

[2] we avoid the word "degree" to prevent any confusion with the degree of 0-cycles defined in **??** or with the degree of the cycle class in cohomology, which is twice the codimension. Furthermore, it is also useful to consider the grading defined by dimension instead of codimension.

0.1.2 These conditions ensure that the intersection product which is partially defined on $\mathcal{Z}^*(X)_F$ factors through the quotient by $\sim$, and yields a well-defined composition law on

$$\mathcal{Z}^*_\sim(X)_F = \mathcal{Z}^*(X)_F/\sim$$

endowing it with the structure of a *commutative graded F-algebra* (and not graded-commutative)[3].

N.B. One shall be careful not to confuse $\mathcal{Z}^*_\sim(X)\otimes_{\mathbf{Z}} F$ with its quotient $\mathcal{Z}^*_\sim(X)_F$.

The third condition defining adequate relations implies that constructing the algebra $\mathcal{Z}^*_\sim(X)_F$ is *contravariant* in X (let γ be the transpose of the graph Γ_f of the given morphism f, obtained by exchanging in Γ_f the factors $X\times Y\to Y\times X$). We also obtain a homomorphism of F-modules *in the other way* $f_*:\mathcal{Z}^*_\sim(X)_F\to\mathcal{Z}^*_\sim(Y)_F$ shifting the grading by (−the generic relative dimension of f), if the latter is constant on the components of X (this time, take $\gamma=\Gamma_f$). This induced homomorphism f_* is not compatible with intersection products, however we have the so-called *projection formula*

$$f_*(\alpha\cdot f^*(\beta)) = f_*(\alpha)\cdot\beta.$$

In addition, for each cartesian square in $\mathcal{P}(k)$

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

we have the formula $f^*p_* = q_*f'^*$ (see [**?**]).

0.1.3 The elements of

$$\mathcal{Z}^{\dim X+r}(X\times Y)_F,\quad \text{resp.}\quad \mathcal{Z}^{\dim X+r}_\sim(X\times Y)_F$$

are called *algebraic correspondences of degree r with coefficients in F* (resp. modulo $\sim$) *from X to Y* (or between X and Y). When X is not equidimensional, one shall consider $\dim X$ as a locally constant function on X.

For example, the transpose of the graph of a morphism $Y\to X$ is a correspondence of degree 0 from X to Y.

The formula

$$g\circ f = \mathrm{pr}^{XYZ}_{XZ*}\left(\mathrm{pr}^{XYZ*}_{XY}(f)\cdot\mathrm{pr}^{XYZ*}_{YZ}(g)\right)$$

defines an associative composition law[4] for correspondences modulo $\sim$ (see [**?**])

$$\mathcal{Z}^{\dim Y+r}_\sim(X\times Y)_F\otimes_F\mathcal{Z}^{\dim Z+s}_\sim(Y\times Z)_F\to\mathcal{Z}^{\dim Z+r+s}_\sim(X\times Z)_F,$$

[3] other notations often used are $A^*_\sim(X)_F$, $C^*_\sim(X)_F$, $A^*_\sim(X,F)$, ...

[4] that this formula defines $g\circ f$ and not $f\circ g$ is compatible with the above definition of the degree of a correspondence, and is consistent with the traditionally contravariant point of view on motives.

adding the degrees and endowing $\mathcal{Z}_\sim^{\dim X}(X \times X)_F$ with the structure of a (non necessarily commutative) F-algebra, the algebra of correspondences of degree 0. The unit is the class of the diagonal $\Delta_X \subset X \times X$. This algebra is also endowed with an (anti-)involution given by transposition ${}^\top$.

0.1.4 For $f \in \mathcal{Z}_\sim(X \times Y)_F$, $\alpha \in \mathcal{Z}_\sim(X)$ and $\beta \in \mathcal{Z}_\sim(Y)$, we define generally

$$\begin{aligned} f_*(\alpha) &= \mathrm{pr}^{XY}_{Y*}(f \cdot \mathrm{pr}^{XY*}_X(\alpha)) \in \mathcal{Z}_\sim(Y), \\ \text{and } f^*(\beta) &= \mathrm{pr}^{XY}_{X*}(f \cdot \mathrm{pr}^{XY*}_Y(\beta)) \in \mathcal{Z}_\sim(X). \end{aligned}$$

For $\gamma \in \mathcal{Z}_\sim^{\dim X+r}(X \times Y)_F$, $a \in \mathcal{Z}_\sim^{\dim X'+p}(X \times X')_F$ and $b \in \mathcal{Z}_\sim^{\dim Y+q}(Y \times Y')_F$, we have the useful formula:

$$(a, b)^*(\gamma) = b \circ \gamma \circ a^\top \in \mathcal{Z}_\sim^{\dim X'+p+q+r}(X' \times Y')_F.$$

0.1.4.1 Exercise. Let k'/k be a field extension. Start with an adequate equivalence relation $\sim$ on algebraic cycles on objects of $\mathcal{P}(k')$. By “restriction”, it induces an adequate equivalence relation (still denoted $\sim$) on algebraic cycles α on objects of $\mathcal{P}(k)$: $\alpha \sim 0$ if and only if $\alpha_{k'} \sim 0$. Whence there is a canonical injective homomorphism

$$\mathcal{Z}_\sim^*(X)_F \hookrightarrow \mathcal{Z}_\sim^*(X_{k'})_F.$$

When $k' = \overline{k}$, the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ acts naturally on $\mathcal{Z}_\sim^*(X_{\overline{k}})_F$, and the previous homomorphism induces

$$\mathcal{Z}_\sim^*(X)_F \hookrightarrow \mathcal{Z}_\sim^*(X_{\overline{k}})_F^{\mathrm{Gal}(\overline{k}/k)}.$$

Show that this homomorphism is bijective if F is a $\mathbf{Q}$-algebra, but not surjective in general if $F = \mathbf{Z}$.

0.2 Review of the usual adequate relations

0.2.1 They are the:

- rational equivalence $\sim_{\mathrm{rat}}$,
- algebraic equivalence $\sim_{\mathrm{alg}}$,
- homological equivalences $\sim_{\mathrm{hom}}$,
- numerical equivalence $\sim_{\mathrm{num}}$,

to which we add the interesting equivalence $\sim_{\otimes\mathrm{nil}}$ of "smash-nilpotence" (or $\otimes$-nilpotence) appearing in a work of V. Voevodsky [**?**][5].

To compare them, say that $\sim$ is finer than $\approx$, and write $\sim\succ\approx$, if $\alpha\sim 0$ implies $\alpha\approx 0$. We then have

$$\sim_{\mathrm{rat}}\succ\sim_{\mathrm{alg}}\succ\sim_{\mathrm{hom}}\succ\sim_{\mathrm{num}},\ \text{and}$$

$$\sim_{\mathrm{rat}}\succ\sim_{\mathrm{alg}}\succ\sim_{\otimes\mathrm{nil}}\succ\sim_{\mathrm{hom}}\succ\sim_{\mathrm{num}}\ \text{ if } F\supset\mathbf{Q}.$$

As a foreshadowing, let us mention that we conjecture that the last three equivalences of this sequence coincide exactly (Grothendieck, Voevodsky).

0.2.2 A cycle $\alpha\in\mathcal{Z}^*(X)_F$ is *rationally equivalent to* 0 (that is, $\alpha\sim_{\mathrm{rat}}0$) if there exists $\beta\in\mathcal{Z}^*(X\times\mathbb{P}^1)_F$ such that $\beta(0)$ and $\beta(\infty)$ are well-defined[6] and such that $\alpha=\beta(0)-\beta(\infty)$.

The fact that $\sim_{\mathrm{rat}}$ is adequate is among the foundations of intersection theory: the difficult condition is the second one, which is W. Chow's "moving lemma", see [**?**].

The graded rings $\mathcal{Z}^*(X)/\sim_{\mathrm{rat}}$ are called the *Chow rings* and are customarily denoted $\mathrm{CH}^*(X)$. We have $\mathrm{CH}^*(X)\otimes F=\mathcal{Z}^*_{\mathrm{rat}}(X)_F$, also denoted $\mathrm{CH}^*(X)_F$.

0.2.2.1 Lemma. $\sim_{\mathrm{rat}}$ is the finest adequate relation.

Indeed, let $\sim$ be an adequate relation on algebraic cycles with coefficients in F. By the third condition of adequateness, it suffices to show that $[0]\sim[\infty]$ on $\mathbb{P}^1$. By the second condition, there exists a cycle $\sum n_i[x_i]\sim[1]$ with $n_i\in F$, such that $\sum n_i[x_i]\cdot[1]$ is well-defined (that is, $x_i\neq 1$). Apply the third condition with γ the graph of the polynomial $1-\prod\left(\frac{x-x_i}{1-x_i}\right)^{m_i}$ (with $m_i>0$) and $\alpha=\sum n_i[x_i]-[1]$. We obtain that $mn[1]\sim m[0]$ where $m=\sum m_i$ and $n=\sum n_i$. Since the m_i can still be chosen freely, we conclude that $n[1]\sim[0]$. Now, apply the automorphism $x\mapsto 1/x$ (and the third condition again) to obtain $n[1]\sim[\infty]$, whence $[0]\sim[\infty]$ as required.

In virtue of this lemma, we can identify any adequate relation on algebraic cycles with coefficients in F with the data of a homogeneous ideal $I^*_\sim(X)=\{\alpha\in\mathrm{CH}(X)_F,\alpha\sim 0\}$ of the Chow ring $\otimes F$ of every $X\in\mathcal{P}(k)$, satisfying a compatibility condition with the bifunctoriality of Chow groups.

0.2.2.2 Exercises.

1) Let $x\in\mathbb{P}^1(k)$. Show that the diagonal of $\mathbb{P}^1\times\mathbb{P}^1$ decomposes modulo $\sim$ as the sum of the idempotent correspondences $[x]\times\mathbb{P}^1$ and $\mathbb{P}^1\times[x]$, and that this decomposition is independent of x.

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[5] when these symbols appear as subscripts or superscripts, we will shorten $\sim_{\mathrm{rat}}$ as rat, and so on.

[6] more precisely, such that each component Z of β is dominant over $\mathbb{P}^1$, $\beta(0)$ (resp. $\beta(\infty)$) being then the sum of the cycles associated to the closed subschemes $Z(0)$ (resp. $Z(\infty)$) of X, see [**?**].

2) More generally, let $t \in \mathcal{Z}^1_\sim(\mathbb{P}^n)_F$ be the class of a hyperplane in $\mathbb{P}^n$, and let $[\Delta] \in \mathcal{Z}^n_\sim(\mathbb{P}^n \times \mathbb{P}^n)_F$ be the class of the diagonal. Show that $[\Delta] = \sum t^i \times t^{n-i}$.

   Deduce that $\mathcal{Z}^*_\sim(X \times \mathbb{P}^n)_F \cong \mathcal{Z}^*_\sim(X)_F[t]/(t^{n+1})$ as graded F-algebras, t being of degree 1. Explain how this generalizes Bézout's theorem.

0.2.3 The definition of $\sim_{\mathrm{alg}}$ is analogous to that of $\sim_{\mathrm{rat}}$. The projective line $\mathbb{P}^1$ is replaced by an arbitrary smooth projective curve (or equivalently[7] any smooth connected projective k-scheme, that is how one can prove that the condition $\sim_{\mathrm{alg}} 0$ is stable under addition) and 0 and ∞ are replaced by two k-rational points. It is easy to see that $\mathcal{Z}^*_{\mathrm{alg}}(X)_F = \mathcal{Z}^*_{\mathrm{alg}}(X) \otimes_{\mathbf{Z}} F$.

0.2.4 According to [**?**], an element $\alpha \in \mathcal{Z}^*(X)_F$ is "smash-nilpotent" or $\otimes$-nilpotent if there exists $N > 0$ such that $\alpha \times \alpha \times \cdots \times \alpha$ is rationally equivalent to 0 on X^N. One checks without difficulty that this is indeed an adequate relation[8], and that $\mathcal{Z}^*_{\otimes\mathrm{nil}}(X)_F = \mathcal{Z}^*_{\otimes\mathrm{nil}}(X) \otimes_{\mathbf{Z}} F$.

0.2.4.1 Proposition ([?]). If F is a $\mathbf{Q}$-algebra, then $\sim_{\mathrm{alg}}$ is finer than $\sim_{\otimes\mathrm{nil}}$.

Proof. Let $\alpha \in \mathrm{CH}^r(X)_F$ be an element sent to 0 in $\mathcal{Z}^r_{\mathrm{alg}}(X)_F$. There exists a genus g smooth connected projective curve T, two points t_0 and t_1 and $\beta \in \mathrm{CH}^r(X \times T)_F$, such that $\alpha = \beta(t_0) - \beta(t_1)$. We have $\alpha^{\otimes N} = \beta^{\otimes N}([t_0] - [t_1])^{\otimes N}$, so that it suffices to prove that $([t_0] - [t_1])^{\otimes N} = 0$ in $\mathrm{CH}^N(T^N)_{\mathbf{Q}}$ for N large enough (in fact, $N = 2g$ suffices if $g \geq 1$, which we can assume). This relies on the fact that the canonical morphism $S^n(T) \to J(T)$ from the n-th symmetric power of T to the Jacobian given by $x_1 + \cdots + x_n \mapsto x_1 + \cdots + x_n - nt_0$ identifies $S^n(T)$ with a projective bundle on $J(T)$, as soon as $n \geq 2g-1$ [**?**]. Knowing the structure of the Chow groups of a projective bundle [**?**], we deduce that the inclusion $\iota : S^{2g-1}(T) \hookrightarrow S^{2g}(T)$ given by $x_1 + \cdots + x_{2g-1} \mapsto x_1 + \cdots + x_{2g-1} + t_0$ induces an isomorphism $\iota^* : \mathrm{CH}^{2g}(S^{2g}(T)) \cong \mathrm{CH}^{2g-1}(S^{2g-1}(T))$. Furthermore, $\mathrm{CH}^{2g}(S^{2g}(T))_{\mathbf{Q}}$ is identified with the symmetric elements in $\mathrm{CH}^{2g}(T^{2g})_{\mathbf{Q}}$ [**?**]. This lets us see $([t_0] - [t_1])^{\otimes 2g}$ as a cycle on $S^{2g}(T)$, and it is easy to see that $\iota^*(([t_0] - [t_1])^{\otimes 2g}) = 0$. □

0.2.4.2 Remarks.

1) The bound $2g$ for the nilpotence exponent is not optimal. One can show that the optimal bound is $g + 1$, see point 2) of **??**.

2) Let A be an elliptic curve, and x_0 and x_1 two distinct points in $A(k)$. The above proof shows that the cycle $([x_0] - [x_1]) \times ([x_0] - [x_1])$ on $A \times A$ is zero modulo

[7] any two points can be joined by a smooth connected projective curve: cut the variety by hypersurfaces general enough and of high enough degree passing through the two points, so that the section is a smooth connected projective curve (they even exist if k is finite, see [**?**]).

[8] this is even obvious when using the adequateness criterion given in the next chapter.

rational equivalence, while $[x_0] - [x_1]$ on A is not itself zero. This disproves the converse of exercise **??**.

A remarkable aspect of this result is that even though $\sim_{\mathrm{rat}}$ and $\sim_{\otimes\mathrm{nil}}$ seem very similar by definition, the groups of cycles modulo $\sim_{\mathrm{rat}}$ and modulo $\sim_{\otimes\mathrm{nil}}$ have wildly different properties when *k is algebraically closed.* On the one hand, Chow groups are "continuous" invariants, in general "enormous" (and varying with k) as shown by D. Mumford. On the other hand, the groups $\mathcal{Z}^*_{\mathrm{alg}}(X)$ and *a fortiori* $\mathcal{Z}^*_{\sim}(X)$ for every adequate equivalence coarser that $\sim_{\mathrm{alg}}$ (in particular $\sim_{\otimes\mathrm{nil}}$) are "discrete" invariants, countable and *invariant by extensions of k* in virtue of the theory of Chow forms [**?**].

0.2.5 Homological equivalence relies on the notion (and the choice) of a Weil cohomology, which will be explained below. It is the only usual adequate equivalence for which it is unclear *a priori* that $\mathcal{Z}^*_{\sim}(X) \otimes_{\mathbf{Z}} F = \mathcal{Z}^*_{\sim}(X)_F$ for each F contained in the ring of coefficients of the chosen cohomology[9].

0.2.6 We customarily call 0-*cycle* a cycle of dimension 0, that is a linear combination $\sum n_i[P_i]$ of closed points. Its degree is defined as

$$\sum n_i[k(P_i) : k].$$

It only depends on the class of the 0-cycle modulo algebraic equivalence[10] (this is a rephrasing of Poncelet's "conservation of number principle" [**?**]).

0.2.7 An element $\alpha \in \mathcal{Z}^r(X)_F$ is *numerically equivalent to* 0 if for each cycle β of dimension r, the 0-cycle $\alpha \cdot \beta$ (well-defined in $\mathrm{CH}^d(X)$) is of degree $\langle \alpha, \beta \rangle$ equal to zero.

It is known that $\mathcal{Z}^r_{\mathrm{num}}(X)_{\mathbf{Q}} = \mathcal{Z}^r_{\mathrm{alg}}(X)_{\mathbf{Q}}$ when $r \leq 1$ (Matsusaka [**?**], reducing the problem to the case of a surface) but the kernel of the quotient map $\mathcal{Z}^r_{\mathrm{alg}}(X)_{\mathbf{Q}} \to \mathcal{Z}^r_{\mathrm{num}}(X)_{\mathbf{Q}}$ is infinite-dimensional in general if $r \geq 2$ [**?**].

0.2.7.1 Proposition. If F is an integral domain of characteristic zero, then the F-module $\mathcal{Z}^r_{\mathrm{num}}(X)_F$ is free of finite type and $\mathcal{Z}^r_{\mathrm{num}}(X)_F = \mathcal{Z}^r_{\mathrm{num}}(X) \otimes_{\mathbf{Z}} F$. What is more, if $F \supset \mathbf{Q}$ and X has pure dimension d, then the "degree of the intersection 0-cycle" pairing

$$\mathcal{Z}^r_{\mathrm{num}}(X)_F \times \mathcal{Z}^{d-r}_{\mathrm{num}}(X)_F \to F, \quad (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle$$

is a perfect pairing.

Since the proof relies on the existence of Weil cohomologies, we postpone it (to **??**).

0.2.7.2 Exercise. Suppose that F is a field. Show that $\sim_{\mathrm{num}}$ is the coarsest non-trivial adequate equivalence relation.

[9] as such, it is also unclear that $\mathcal{Z}^*_{\sim}(X)$ has finite rank, whereas this is obviously the case for $\mathcal{Z}^*_{\sim}(X)_K$ where K is the coefficient ring.

[10] and of course, it only depends on its class modulo numerical equivalence, by definition.

0.3 Weil cohomologies

0.3.1 Taking inspiration from [**?**], we axiomatize the properties of the cohomologies mentioned in **??**.

The product $\times_k$ endows the category $\mathcal{P}(k)$ of smooth projective k-schemes with the structure of a (non additive) symmetric monoidal category. The unit is the point $\operatorname{Spec} k$ and there are canonical isomorphisms

$$X \times Y \cong Y \times X, \quad (X \times Y) \times Z \cong X \times (Y \times Z).$$

The diagonal $\Delta_X : X \hookrightarrow X \times X$ endows each object $X \in \mathcal{P}(k)$ with the structure of a counitary cocommutative coalgebra with respect to this monoidal structure.

Recall that $VecGr_K$ denotes the rigid $\otimes$-category of finite-dimensional $\mathbf{Z}$-graded vector spaces over a field K (with commutativity given by the Koszul sign rule). Let $VecGr_K^{\geq 0}$ be its full (not rigid) sub-$\otimes$-category spanned by objects concentrated in nonnegative degrees.

0.3.1.1 Definition. A (pure) Weil cohomology[11] H^* is a functor

$$H^* : \mathcal{P}(k)^{\mathrm{op}} \to VecGr_K^{\geq 0}$$

compatible with the monoidal structure, satisfying

$*$) $\dim_K H^2(\mathbb{P}^1) = 1$,

and endowed with the additional data 1) and 2) given below[12].

Notice that the comultiplication of each object X of $\mathcal{P}(k)$ induces *ipso facto* a multiplication (the cup-product) on $H^*(X)$, making $H^*(X)$ a (unitary) graded-commutative $\mathbf{N}$-graded K-algebra (that is, a unitary commutative algebra object in $VecGr_K^{\geq 0}$).

For each $d \in \mathbf{Z}$, denote (r) the "Tate twist" operation $V^* \mapsto V^* \otimes H^2(\mathbb{P}^1)^{\otimes(-r)}$ in $VecGr_K$ (considering that the $*$) condition implies the $\otimes$-invertibility of $H^2(\mathbb{P}^1)$ in $VecGr_K$).

The additional data in **??** are:

1) (trace, Poincaré duality) for each $X \in \mathcal{P}(k)$ with pure dimension d, a K-linear map $H^{2d}(X)(d) \xrightarrow{\mathrm{Tr}_X} K$, which is an isomorphism when X is geometrically connected, satisfying $\mathrm{Tr}_{X\times Y} = \mathrm{Tr}_X \mathrm{Tr}_Y$ (after the obvious identifications that are needed) and such that the product of $H^*(X)$ induces for each i a duality pairing

$$\langle\ ,\ \rangle : H^i(X) \times H^{2d-i}(X)(d) \to H^{2d}(X)(d) \xrightarrow{\mathrm{Tr}_X} K.$$

[11] in the literature it is often asked that K is of characteristic zero, however the general case, even that of a product of fields, is useful.

[12] several authors also require H^* to satisfy the "weak and hard Lefschetz theorems". We avoid it, on the one hand because we will have to consider generalizations of the notion of Weil cohomologies to the mixed case (**??**) where the "Lefschetz theorems" lose their relevance, and on the other hand because we will construct in the pure case *a priori* non-classical Weil cohomologies for which the "Lefschetz theorems" are not yet known to hold (**??**).

2) (cycle classes) for each $X \in \mathcal{P}(k)$, abelian group homomorphisms

$$\gamma_X^r = \gamma_{X,H}^r : \mathrm{CH}^r(X) \to H^{2r}(X)(r)$$

that are

- contravariant in X,
- compatible with the "external product" $\gamma_{X\times Y}^{r+s}(\alpha \times \beta) = \gamma_X^r(\alpha) \otimes \gamma_Y^s(\beta)$,
- "normalized" such that γ^d composed with the trace Tr_X coincides with the 0-cycles degree map considered in **??** when X has pure dimension d.

Compatibility with the monoidal structure means that for each pair (X, Y) there exists a canonical Künneth isomorphism between graded-commutative graded algebras:

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

functorially in X and Y (and compatible with the associativity and commutativity constraints). The projections $\pi_X^i = \pi_{X,H}^i : H^*(X) \to H^i(X) \hookrightarrow H^*(X)$ on homogeneous components are called the *Künneth projectors* of X.

0.3.2 Some formal consequences of the definition

See [?], [?].

- For each $r \in \mathbf{Z}$, $K(r)$ is isomorphic (not canonically in general) to K concentrated in degree $-2r$. The double Tate torsion $(r)(s)$ is the same as the simple torsion $(r+s)$.

- $H^*(\mathrm{Spec}\, k) = H^0(\mathrm{Spec}\, k) \overset{\mathrm{Tr}_{\mathrm{Spec}\, k}}{=} K$.

- $H^i(X) = 0$ for $i > 2\dim X$.

- Poincaré duality can also be written in the following form, for each $r \in \mathbf{Z}$

$$\langle\ ,\ \rangle : H^i(X)(r) \times H^{2d_X - i}(X)(d-r) \to H^{2d_X}(X)(d) \xrightarrow{\mathrm{Tr}_X} K,$$

  and we have $\langle x, x'\rangle = (-1)^i \langle x', x\rangle$ for each $x \in H^i(X)(r)$ and $x' \in H^{2d-i}(X)(d-r)$.

- $\langle \gamma_r(\alpha), \gamma_{d-r}(\beta)\rangle = \langle \alpha, \beta\rangle$.

- Let $X, Y \in \mathcal{P}(k)$ with dimensions d_X and d_Y respectively, and $f : X \to Y$ be a morphism. To $f^* = H^*(f)$, Poincaré duality assigns the "Gysin morphism" $f_* : H^*(X)(d_X) \to H^{*-2(d_X-d_Y)}(Y)(d_Y)$, adjoint to f^* and satisfying the *projection formula* $f_*(f^*(y)\cdot x) = y \cdot f_*(x)$. One also abusively denotes f_* the homomorphisms obtained by the Gysin morphism after Tate twists.

- The trace map Tr_X is p^X_*, p^X being the structural morphism $X \to \operatorname{Spec} k$. For each morphism $f : X \to Y$ as before, we have $\mathrm{Tr}_X = \mathrm{Tr}_Y \circ f_*$. From $p^{X\times Y} = p^X \times p^Y$ and the Künneth formula, we deduce $\mathrm{Tr}_{X\times Y} = \mathrm{Tr}_X \otimes \mathrm{Tr}_Y$.

- By the Künneth formula and duality, there are non graded canonical isomorphisms

$$H(X \times Y)(d_X) \cong H(X)(d_X) \otimes H(Y) \cong \mathrm{Hom}_K(H(X), H(Y))$$

given by $x \otimes y \mapsto (z \mapsto \langle z, x\rangle y)$, and more precisely there are canonical isomorphisms $u \mapsto \underline{u} = (z \mapsto \mathrm{pr}^{XY}_{Y*}(\mathrm{pr}^{XY*}_X(z) \cdot u))$:

$$\begin{aligned} H^{2d_X+i}(X \times Y)(d_X) &\cong \bigoplus_{j\geq 0} H^{2d_X-j}(X)(d_X) \otimes H^{j+i}(Y) \\ &\cong \bigoplus_{j\geq 0} \mathrm{Hom}_K(H^j(X), H^{j+i}(Y)) \end{aligned}$$

Moreover, for each $v \in H^{2d_Y+h}(Y \times Z)(d_Y)$ the element

$$w = \mathrm{pr}^{XYZ}_{XZ*}(\mathrm{pr}^{XYZ*}_{XY}(u) \cdot \mathrm{pr}^{XYZ*}_{YZ}(v))$$

satisfies $\underline{w} = \underline{v} \circ \underline{u} \in \bigoplus_{j\geq 0} \mathrm{Hom}_K(H^j(X), H^{j+i+h}(Z))$.

- Let $x \in H^i(X)$, $y \in H^j(Y)$ with $i = j \bmod 2$. We have

$$\langle y', \underline{x \otimes y}(x')\rangle = \langle x', \underline{y \otimes x}(y')\rangle,$$

so that we can identify $(-1)^i \underline{y \otimes x}$ with the Tate-twisted transpose of $\underline{x \otimes y}$. It follows that denoting c_{XY} the switching map $X \times Y \to Y \times X$, we have

$$\forall u \in H^{2d_X+2r}(X \times Y)(d_X), \quad \underline{c^*_{XY}(u)} = \underline{u}^\top.$$

- Let $x \in H^{\mathrm{even}}(X \times W)(d_X)$, $y \in H^{\mathrm{even}}(Y \times Z)(d_Y)$, $v \in H(X \times Y)(d_X)$ and $w = \underline{c^*_{WY}(x \otimes y))}(v) \in H(W \times Z)(d_W)$. Then $\underline{w} = \underline{y} \circ \underline{v} \circ \underline{x}^\top$.

- $\gamma^0_X([X])$ is the unit of the algebra $H^*(X)$.

- Functoriality and compatibility of cycle classes with the "external" product imply compatibility with the "internal" product (intersection product in $\mathrm{CH}(X)$, cup-product in $H^*(X)$):

$$\gamma^{r+s}_X(\alpha \cdot \beta) = \gamma^{r+s}_X(\Delta^*(\alpha \times \beta)) = \Delta^*(\gamma^r_X(\alpha) \otimes \gamma^s_X(\beta)) = \gamma^r_X(\alpha) \cdot \gamma^s_X(\beta).$$

- Let $g : Y \to X$ be a morphism in $\mathcal{P}(k)$ and suppose that X has pure dimension d_X. Then with

$$u = \gamma_{X\times Y}(\Gamma^\top_g) \in H^{2d_X}(X \times Y)(d_X),$$

we have $\underline{u} = g^* \in \mathrm{Hom}_K(H^*(X), H^*(Y))$. If Y has pure dimension d_Y then with

$$u^\top = \gamma_{X\times Y}(\Gamma_g) \in H^{2d_Y}(X\times Y)(d_X),$$

we also have

$$\underline{u^\top} = g_* \in \mathrm{Hom}_K(H^*(Y), H^{*-2(d_X-d_Y)}(X)(d_X-d_Y)).$$

- Let $\beta \in \mathrm{CH}(X\times Y)$. Reusing the notation $\theta \in \mathrm{CH}(X) \mapsto \beta_*(\theta) = \mathrm{pr}^{XY}_{Y*}(\beta \cdot \mathrm{pr}^{XY*}_X(\theta)) \in \mathrm{CH}(Y)$ already introduced in **??**, we have $\underline{\gamma_{X\times Y}(\beta)} \circ \gamma_X = \gamma_Y \circ \beta_*$. When $\beta = \Gamma_f$ is the graph of a morphism $f : X \to Y$, we have $\underline{\gamma_{X\times Y}(\Gamma_f)} = f_*$, whence $f_*\gamma_X = \gamma_Y f_*$.
- $H^*(X \amalg Y)$ is canonically isomorphic to $H^*(X)\oplus H^*(Y)$. This lets us circumvent any equidimensionality assumption by letting

$$H^{2d_X+i}(X\times Y)(d_X) = \bigoplus_j H^{2d_j+i}(X_j\times Y)(d_j)$$

where X_j is the component of X having pure dimension d_j.

0.3.3 Lefschetz trace formula. Let $V = V^+\oplus V^-$ and $W = W^+\oplus W^-$ be two (finite-dimensional) super-vector spaces and let $\tilde V$ and $\tilde W$ be their respective K-duals. Recall that the commutativity constraint c_{VW} is the interchange of factors with the Koszul sign rule

$$V\otimes W \cong W\otimes V, \quad c_{VW}(v\otimes w) = (-1)^{\delta v\cdot\delta w} w\otimes v,$$

where δv (resp. δw) denotes the parity of a homogeneous element v (resp. w).

We identify $V\otimes\tilde W$ with the dual of $\tilde V\otimes W$ by the formula $\langle \tilde v\otimes w, v\otimes\tilde w\rangle = (-1)^{\delta v\cdot\delta w}\langle\tilde v, v\rangle\langle w,\tilde w\rangle$ (also we can assume that $\delta v = \delta\tilde v$ and $\delta w = \delta\tilde w$).

Moreover, we identify $\tilde V\otimes W$ with $\mathrm{Hom}(V,W)$ by $\tilde v\otimes w \mapsto (v'\mapsto\langle\tilde v, v'\rangle w)$ and similarly $\tilde W\otimes V$ is identified with $\mathrm{Hom}(W,V)$.

As such, the endomorphism $(\tilde w\otimes v)\circ(\tilde v\otimes w)$ of V is given by $v'\mapsto\langle w,\tilde w\rangle\langle v',\tilde v\rangle v$, that is, $(\tilde w\otimes v)\circ(\tilde v\otimes w) = \langle w,\tilde w\rangle\tilde v\otimes v$. The supertrace $s\mathrm{tr}(\tilde v\otimes v)$ is $\langle\tilde v, v\rangle$.

We deduce that $\langle\tilde v\otimes w, c_{\tilde W V}(\tilde w\otimes v)\rangle = s\mathrm{tr}((\tilde w\otimes v)\circ(\tilde v\otimes w))$, and by linearity that

$$\langle\phi, c_{\tilde W V}(\psi)\rangle = s\mathrm{tr}(\psi\circ\phi)$$

for each $\phi\in\mathrm{Hom}(V,W)$ and $\psi\in\mathrm{Hom}(W,V)$ of matching parities.

Apply this super-linear algebra formula to

$$V^+ = H^+(X), \quad V^- = H^-(X), \quad W^+ = H^+(Y), \quad W^- = H^-(Y),$$

and

$$\phi \in H^{2d_Y+i}(X\times Y)(d_Y+n), \quad \psi\in H^{2d_X-i}(Y\times X)(d_X-n),$$

where H^+ and H^- respectively denote the even and odd parts of cohomology. We also have $c_{\tilde{W}V}(\psi) = (c_{YX})_*(\psi) = (c_{XY})^*(\psi) = \psi^\top$.

We then obtain the *Lefschetz trace formula*:

$$\boxed{\langle \phi, \psi^\top \rangle = \sum_{j=0}^{2d_X} (-1)^j \mathrm{tr}(\psi \circ \phi | H^j(X)).}$$

In the case where $X = Y$, $i = n = 0$ and $\psi = \pi_X^j$, we have $(\pi_X^j)^\top = \pi_X^{2d_X - j}$, whence

$$\boxed{\mathrm{tr}(\phi | H^j(X)) = (-1)^j \langle \phi, \pi_X^{2d_X - j} \rangle.}$$

0.3.4 Homological equivalence. Let H^* be a Weil cohomology with coefficients in K and F be a subring of K. An element $\alpha \in \mathcal{Z}^r(X)_F$ is *homologically equivalent to* 0 if $H^*(\alpha) = \gamma^r(\alpha) = 0$. It follows from the above formula $\gamma_X^{r+s}(\alpha \cdot \beta) = \gamma_X^r(\alpha) \cdot \gamma_X^s(\beta)$ that this is an adequate equivalence relation, denoted $\sim_{\mathrm{hom}}$ or $\sim_H$.

Via γ_X^r, $\mathcal{Z}^r_{\mathrm{hom}}(X)_F$ is identified with an F-submodule (necessarily without torsion) of $H^{2r}(X)(r)$. It thus has finite rank if $F = K$, but in general it is not known whether the quotient map $\mathcal{Z}^r_{\mathrm{hom}}(X)_F \otimes_{\mathbf{Z}} K \to \mathcal{Z}^r_{\mathrm{hom}}(X)_K$ is injective[13], and even whether $\mathcal{Z}^r_{\mathrm{hom}}(X)$ has finite rank.

It is immediate that $\sim_{\otimes\mathrm{nil}}$ is finer than $\sim_{\mathrm{hom}}$. On the other hand, $\sim_{\mathrm{hom}}$ is finer than $\sim_{\mathrm{num}}$ (by multiplicativity of cycle classes, or using exercise **??**).

0.3.4.1 Exercise. Show that $\sim_{\mathrm{hom}}$ is coarser than $\sim_{\mathrm{alg}}$ without using **??**.

[13]if K has positive characteristic p, this is false. There are counter-examples with X a supersingular abelian surface over a field k of characteristic p and $r = 1$.