In this second preliminary chapter, we present the basic tools of the theory of pure motives : algebraic cycles, adequate equivalences and Weil cohomologies.

0.1 Algebraic cycles and adequate relations

0.1.1 Let k be a base field and denote $\mathcal{P}(k)$ the *category of smooth projective schemes over k*, also sometimes called smooth projective k-varieties from now on.

For each $X \in \mathcal{P}(k)$ let $\mathcal{Z}^*(X)$ be the graded group of algebraic cycles on X, that is, the free abelian group generated by the integral closed subschemes Z of X and graded by codimension[1]. Denote by $[Z]$ the image of the subscheme Z in $\mathcal{Z}^*(X)$ (and in its quotients).

For each commutative ring F, the elements of

$$\mathcal{Z}^r(X)_F = \mathcal{Z}^r(X) \otimes_{\mathbf{Z}} F$$

are called the *algebraic cycles of codimension*[2] *r with coefficients in F*.

Intersection theory constructs a partially defined bilinear composition law on algebraic cycles: the intersection product, counting multiplicites, of two cycles whose components intersect properly (see [**?**]). To extend this definition to all cycles, the classical approach consists in using an adequate equivalence relation, see [**?**].

0.1.1.1 Definition. An equivalence relation $\sim$ on algebraic cycles is said to be *adequate* if it satisfies the following conditions for each $X, Y \in \mathcal{P}(k)$:

1) $\sim$ is compatible with the F-linear structure and with the grading,

2) for each $\alpha, \beta \in \mathcal{Z}^*(X)_F$ there exists an $\alpha' \sim \alpha$ such that α' and β intersect properly (so that the intersection product $\alpha' \cdot \beta$ is well-defined),

3) for each $\alpha \in \mathcal{Z}^*(X)_F$ and each $\gamma \in \mathcal{Z}^*(X \times Y)_F$ intersecting properly $(\mathrm{pr}_X^{XY})^{-1}(\alpha)$, we have $\alpha = 0 \implies \gamma_*(\alpha) \sim 0$ where

$$\gamma_*(\alpha) = \mathrm{pr}_Y^{XY}(\gamma \cdot (\mathrm{pr}_X^{XY})^{-1}(\alpha)).$$

0.1.1.2 Exercise. Show that these conditions ensure that an "external" product $\alpha \times \beta$ (on $X \times Y$) is ~ 0 as soon as $\alpha \sim 0$ or $\beta \sim 0$.

[1] we do not ask of X that it is connected or even equidimensional, the codimension of an integral subscheme is still well-defined.

[2] we avoid the word "degree" to prevent any confusion with the degree of 0-cycles defined in **??** or with the degree of the cycle class in cohomology, which is twice the codimension. Furthermore, it is also useful to consider the grading defined by dimension instead of codimension.