Metalogica: Assignment 2

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4 Exercise 4

4.a

To proof Show that $(P \to (\neg Q \vee R))$ is a sentence of L_{PL}.
Proof To be a sentence of L_{PL} a sentence must be well-formed. We use the rules of Definition 2.1.1 to discover whether a sentence is well-formed and thus an element of L_{PL}.
For $(P \to (\neg Q \to R))$ to be a sentence of L_{PL} subsentences P and $\neg Q \vee R$ need to be well-formed too *(iii)*. P is well-formed because it is an atomic sentence *(i)*. $\neg Q \vee R$ is well-formed if both $\neg Q$ and R are well formed *(iii)*. $\neg Q$ is well formed if Q is well-formed *(ii)*. Q is well formed because it is an atomic sentence *(i)*. R is well formed because it is an atomic sentence *(i)*. Because all the subsentences of $(P \to (\neg Q \vee R))$ are well formed according to Definition 2.1.1, $(P \to (\neg Q \vee R))$ is well-formed and a sentence of L_{PL}. $\square$

4.b

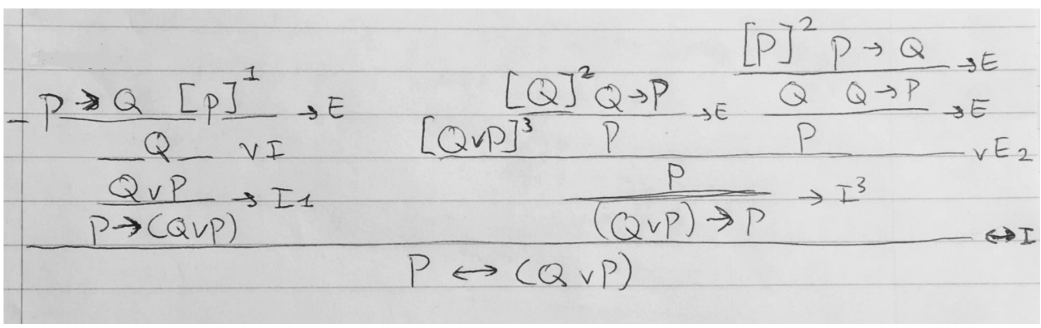
To proof Show that $(P \vee (Q \vee R \wedge S))$ is not a sentence of L_{PL}.
Proof To proof $(P \vee (Q \vee R \wedge S))$ is not a sentence of L_{PL} we must find a subsentence of $(P \vee (Q \vee R \wedge S))$ that is not an element of L_{PL}. Let's assume $(P \vee (Q \vee R \wedge S))$ is well-formed. Than subsentences P and $(Q \vee R \wedge S)$ must be well-formed too. P is well formed because it is an atomic sentence *(i)*. Because $(Q \vee R \wedge S)$ is not well-formed according to either *(i)*, *(ii)* or *(iii)* it must be a case of *iv*: it is not a sentence of L_{PL}. Because we found a subsentence of $(P \vee (Q \vee R \wedge S))$ that is not an element of L_{PL}, $(P \vee (Q \vee R \wedge S))$ is not an element of L_{PL}. $\square$

5 Exercise 5

Because $(\varphi \wedge \psi) \to \chi$ of L_{PL} is not a contradiction there must be an interpretation where $(\varphi \wedge \psi) \to \chi$ holds.
Because $(\varphi \wedge \psi) \to \chi$ of L_{PL} is not a tautology there must be an interpretation of $(\varphi \wedge \psi) \to \chi$ that is a counter example, so where the premise is true and the conclusion is false and therefore $(\varphi \wedge \psi) \not\to \chi$. Because of the fact that $(\varphi \wedge \psi) \to \chi$ has an interpretation that holds and because there is an interpretation that is a counterexample χ cannot be a logical consequence of $(\varphi \wedge \psi)$. So $\varphi \wedge \psi \not\models \chi$

6 Exercise 6

6.a



$\square$

6.b

$$
\begin{prooftree}
\AxiomC{$(A \wedge B) \leftrightarrow (C \wedge D)$}
\AxiomC{$[C]^1$}
\AxiomC{$E \wedge (E \to D)$}
\RightLabel{$\wedge E$}
\UnaryInfC{E}
\AxiomC{$E \wedge (E \to D)$}
\RightLabel{$\wedge E$}
\UnaryInfC{$E \to D$}
\RightLabel{$\to E$}
\BinaryInfC{D}
\RightLabel{$\wedge I$}
\BinaryInfC{$C \wedge D$}
\RightLabel{$\leftrightarrow E$}
\BinaryInfC{$A \wedge B$}
\RightLabel{$\wedge E$}
\UnaryInfC{A}
\RightLabel{$\to I^1$}
\UnaryInfC{$C \to A$}
\end{prooftree}
$$

□

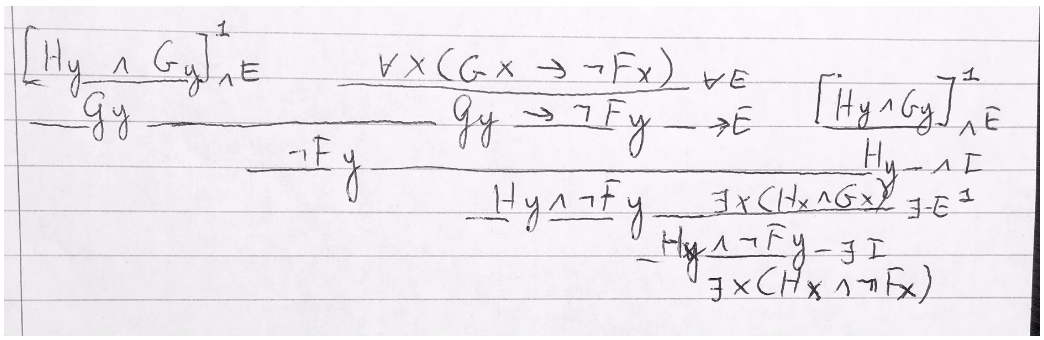
6.c

$$
\begin{prooftree}
\AxiomC{$[H]^2$}
\RightLabel{$\to I$}
\UnaryInfC{$G \to H$}
\AxiomC{$[G]^1$}
\RightLabel{$\to I$}
\UnaryInfC{$H \to G$}
\BinaryInfC{$G \leftrightarrow H$}
\AxiomC{$\neg(G \leftrightarrow H)$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg G$}
\RightLabel{$\to I^2$}
\UnaryInfC{$H \to \neg G$}
\AxiomC{$\neg(G \leftrightarrow H)$}
\AxiomC{$[\neg G]^6$}
\RightLabel{$\to I$}
\UnaryInfC{$\neg H \to \neg G$}
\AxiomC{$[\neg H]^5$}
\RightLabel{$\to I$}
\UnaryInfC{$\neg G \to \neg H$}
\RightLabel{$\leftrightarrow I$}
\BinaryInfC{$\neg G \leftrightarrow \neg H$}
\AxiomC{$[G]^3$}
\RightLabel{$\leftrightarrow E$}
\BinaryInfC{H}
\RightLabel{$\to I^3$}
\UnaryInfC{$G \to H$}
\AxiomC{$[\neg G]^6$}
\UnaryInfC{$\neg H \to \neg G$}
\AxiomC{$[\neg H]^5$}
\UnaryInfC{$\neg G \to \neg H$}
\RightLabel{$\leftrightarrow I$}
\BinaryInfC{$\neg G \leftrightarrow \neg H$}
\AxiomC{$[H]^4$}
\RightLabel{$\leftrightarrow E$}
\BinaryInfC{G}
\RightLabel{$\to I^4$}
\UnaryInfC{$H \to G$}
\BinaryInfC{$G \leftrightarrow H$}
\RightLabel{$\bot^5$}
\BinaryInfC{H}
\RightLabel{$\to I^6$}
\UnaryInfC{$\neg G \to H$}
\RightLabel{$\leftrightarrow I$}
\BinaryInfC{$\neg G \leftrightarrow H$}
\end{prooftree}
$$

□

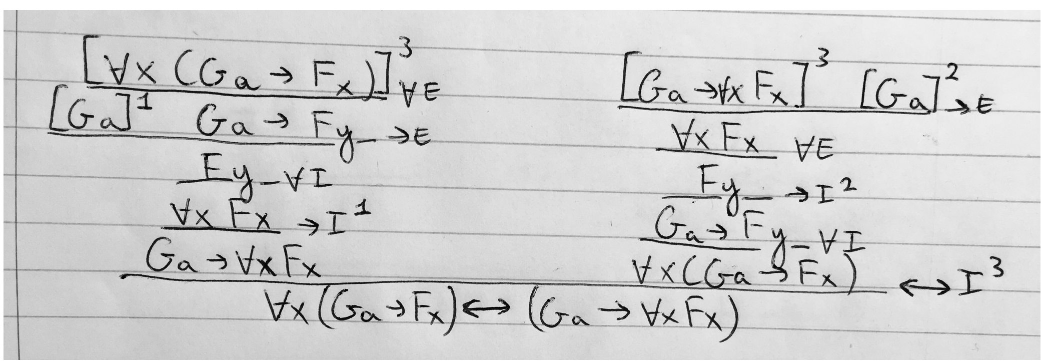
7 Exercise 7

7.a



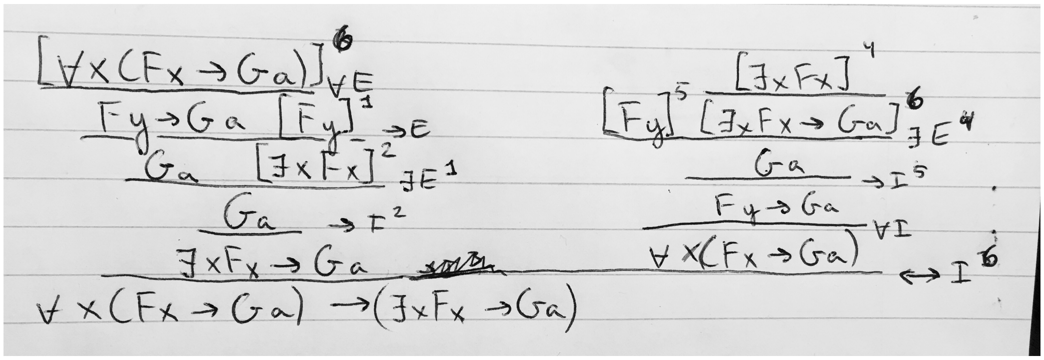
□

7.b



□

7.c



[∀x(Fx → Ga)]⁶ ∀E
Fy → Ga [Fy]¹ →E
Ga [∃xFx]² ∃E¹
Ga →I²
∃xFx → Ga
∀x(Fx → Ga) → (∃xFx → Ga)
[∃xFx]⁴
[Fy]⁵ [∃xFx → Ga]⁶ ∃E⁴
Ga →I⁵
Fy → Ga ∀I
∀x(Fx → Ga)
↔I⁶

□

Van Dalen exercises

5 Exercise 5

5.a

To proof: $\Gamma \vdash \varphi \Rightarrow \Gamma \cup \Delta \vdash \varphi$.
Proof Suppose $\Gamma \vdash \varphi$. Then there exists a derivation D form Γ to conclusion φ. Take the set of all hypothesis on D needed to deduce φ from Γ and call this set H. Then $H \subseteq \Gamma$. Because $\Gamma \subseteq \Gamma \cup \Delta$ this means $H \subseteq \Gamma \cup \Delta$. Because the conclusion of a derivation D stays the same even when premises are added and because the conclusion of D is φ it follows that $\Gamma \cup \Delta \vdash \varphi$. □

5.b

To proof: $\Gamma \vdash \varphi; \Delta; \varphi \vdash \psi \Rightarrow \Gamma \cup \Delta \vdash \psi$.
Proof Suppose $\Gamma \vdash \varphi$, Δ and $\varphi \vdash \psi$. Then there exist a derivation D from Γ to φ and a derivation D' from φ to ψ. If we concatenate derivations D and D' we have a derivation D'' from Γ to ψ, so $\Gamma \vdash \psi$. Because of the answer in 5.a it follows that $\Gamma \cup \Delta \vdash \psi$. □

10

a I($\Gamma \vdash \varphi$ if $\varphi \in \Gamma$)
Because there is a derivation D from Γ to conclusion φ such that there is a finite Δ in Γ such that $\Gamma \vdash \varphi$

b $\Gamma \vdash \varphi$, $\Gamma' \vdash \psi \Rightarrow \Gamma \cup \Gamma' \vdash \varphi \wedge \psi$
Because there is a derivation D from Γ to conclusion φ and a derivation D' from Γ' to conclusion ψ, there must be a finite Δ in Γ and a finite Δ' in Γ' such that φ and ψ are derived of Δ and Δ'. Because of the $\wedge$-rule we van put branches Δ and Δ' in parallel and conclude $\Gamma \cup \Gamma' \vdash \varphi \wedge \psi$.

c $\Gamma \vdash \varphi \wedge \psi \Rightarrow \Gamma \vdash \varphi$ and $\Gamma \vdash \psi$,

d $\Gamma \cup \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \rightarrow \psi$, e $\Gamma \vdash \varphi, \Gamma' \vdash \varphi \to \psi \Rightarrow \Gamma \cup \Gamma' \vdash \psi$,

f $\Gamma \vdash \bot \Rightarrow \Gamma \vdash \varphi$,

g $\Gamma \cup \not\varphi \vdash \bot \Rightarrow \Gamma \vdash \varphi$.