

Metalogica: Assignment 2

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4 Exercise 4

4.a

To proof Show that $(P \rightarrow (\neg Q \vee R))$ is a sentence of L_{PL} .

Proof To be a sentence of L_{PL} a sentence must be well-formed. We use the rules of Definition 2.1.1 to discover whether a sentence is well-formed and thus an element of L_{PL} .

For $(P \rightarrow (\neg Q \vee R))$ to be a sentence of L_{PL} subsentences P and $\neg Q \vee R$ need to be well-formed too (*iii*). P is well-formed because it is an atomic sentence (*i*). $\neg Q \vee R$ is well-formed if both $\neg Q$ and R are well formed (*iii*). $\neg Q$ is well formed if Q is well-formed (*ii*). Q is well formed because it is an atomic sentence (*i*). R is well formed because it is an atomic sentence (*i*). Because all the subsentences of $(P \rightarrow (\neg Q \vee R))$ are well formed according to Definition 2.1.1, $(P \rightarrow (\neg Q \vee R))$ is well-formed and a sentence of L_{PL} . \square

4.b

To proof Show that $(P \vee (Q \vee R \wedge S))$ is not a sentence of L_{PL} .

Proof To proof $(P \vee (Q \vee R \wedge S))$ is not a sentence of L_{PL} we must find a subsentence of $(P \vee (Q \vee R \wedge S))$ that is not an element of L_{PL} . Let's assume $(P \vee (Q \vee R \wedge S))$ is well-formed. Then subsentences P and $(Q \vee R \wedge S)$ must be well-formed too. P is well formed because it is an atomic sentence (*i*). Because $(Q \vee R \wedge S)$ is not well-formed according to either (*i*), (*ii*) or (*iii*) it must be a case of *iv*: it is not a sentence of L_{PL} . Because we found a subsentence of $(P \vee (Q \vee R \wedge S))$ that is not an element of L_{PL} , $(P \vee (Q \vee R \wedge S))$ is not an element of L_{PL} . \square

5 Exercise 5

Because $(\varphi \wedge \psi) \rightarrow \chi$ of L_{PL} is not a contradiction there must be an interpretation where $(\varphi \wedge \psi) \rightarrow \chi$ holds.

Because $(\varphi \wedge \psi) \rightarrow \chi$ of L_{PL} is not a tautology there must be an interpretation of $(\varphi \wedge \psi) \rightarrow \chi$ that is a counter example, so where the premise is true and the conclusion is false and therefore $(\varphi \wedge \psi) \not\vdash \chi$. Because of the fact that $(\varphi \wedge \psi) \rightarrow \chi$ has an interpretation that holds and because there is an interpretation that is a counterexample χ cannot be a logical consequence of $(\varphi \wedge \psi)$. So $\varphi \wedge \psi \not\models \chi$

6 Exercise 6

6.a

The image shows a handwritten logical derivation on lined paper. The derivation is structured as follows:

- Left side (Proving $P \rightarrow (Q \vee P)$):**
 - Assume P (labeled $[P]^1$).
 - From P , derive $Q \vee P$ using $\vee I$.
 - Discharge the assumption P to get $P \rightarrow (Q \vee P)$ using $\rightarrow I_1$.
- Right side (Proving $(Q \vee P) \rightarrow P$):**
 - Assume $Q \vee P$ (labeled $[Q \vee P]^3$).
 - Assume Q (labeled $[Q]^2$).
 - From Q , derive $Q \rightarrow P$ using $\rightarrow E$.
 - From $Q \rightarrow P$ and Q , derive P using $\rightarrow E$.
 - Discharge the assumption Q to get $Q \rightarrow P$ using $\rightarrow I$.
 - From $Q \rightarrow P$ and $Q \vee P$, derive P using $\vee E_2$.
 - Discharge the assumption $Q \vee P$ to get $(Q \vee P) \rightarrow P$ using $\rightarrow I^3$.
- Conclusion:**
 - Combine the two results using $\leftrightarrow I$ to get $P \leftrightarrow (Q \vee P)$.

□

6.b

$$\begin{array}{c}
 \frac{E \wedge (E \rightarrow D) \quad \wedge E}{E} \quad \frac{E \wedge (E \rightarrow D) \quad \wedge E}{E \rightarrow D} \rightarrow E \\
 \frac{[C]^1 \quad D}{C \wedge D} \wedge I \\
 \frac{(A \wedge B) \leftrightarrow (C \wedge D) \quad C \wedge D}{A \wedge B} \leftrightarrow E \\
 \frac{A \wedge B \quad \wedge E}{A} \quad \frac{A}{C \rightarrow A} \rightarrow I^1
 \end{array}$$

□

6.c

$$\begin{array}{c}
 \frac{[H]^2 \rightarrow I \quad [G]^1 \rightarrow I}{G \leftrightarrow H} \leftrightarrow I \\
 \frac{G \leftrightarrow H \quad \neg(G \leftrightarrow H)}{\perp} \neg I^1 \\
 \frac{\perp}{\neg G} \neg I^2 \\
 \frac{\neg G}{H \rightarrow \neg G} \rightarrow I^3 \\
 \frac{H \rightarrow \neg G}{\neg G \leftrightarrow H} \leftrightarrow I \\
 \frac{[G]^6 \rightarrow I \quad [H]^5 \rightarrow I}{\neg H \rightarrow \neg G \quad \neg G \rightarrow \neg H} \leftrightarrow I \\
 \frac{\neg H \rightarrow \neg G \quad \neg G \rightarrow \neg H}{\neg G \leftrightarrow \neg H} \leftrightarrow I \\
 \frac{\neg G \leftrightarrow \neg H \quad [G]^3 \leftrightarrow E}{\neg G} \leftrightarrow E \\
 \frac{\neg G}{G \rightarrow H} \rightarrow I^4 \\
 \frac{G \rightarrow H \quad H \rightarrow G}{G \leftrightarrow H} \leftrightarrow I^5 \\
 \frac{G \leftrightarrow H \quad \neg(G \leftrightarrow H)}{\perp} \neg I^6 \\
 \frac{\perp}{\neg G \rightarrow H} \neg I^6 \\
 \frac{\neg G \rightarrow H}{\neg G \leftrightarrow H} \leftrightarrow I
 \end{array}$$

□

7 Exercise 7

7.a

$$\begin{array}{c}
 \frac{[Hy \wedge Gy]^1}{Gy} \wedge E \quad \frac{\forall x (Gx \rightarrow \neg Fx) \quad \forall E}{Gy \rightarrow \neg Fy} \rightarrow E \quad \frac{[Hy \wedge Gy]^1}{Hy} \wedge E \\
 \hline
 \neg Fy \quad \frac{Hy \wedge \neg Fy}{\exists x (Hx \wedge Gx)} \exists E^1 \\
 \hline
 \frac{Hy \wedge \neg Fy}{\exists x (Hx \wedge \neg Fx)} \exists I
 \end{array}$$

□

7.b

$$\begin{array}{c}
 \frac{[\forall x (Ga \rightarrow Fx)]^3}{[Ga]^1} \forall E \quad \frac{Ga \rightarrow Fy}{Fy} \rightarrow E \\
 \hline
 Fy \quad \forall I \\
 \frac{\forall x Fx}{Ga \rightarrow \forall x Fx} \rightarrow I^1 \\
 \hline
 \forall x (Ga \rightarrow Fx) \leftrightarrow (Ga \rightarrow \forall x Fx)
 \end{array}
 \quad
 \begin{array}{c}
 \frac{[Ga \rightarrow \forall x Fx]^3}{[Ga]^2} \rightarrow E \\
 \hline
 \forall x Fx \quad \forall E \\
 \frac{Fy}{Ga \rightarrow Fy} \rightarrow I^2 \\
 \frac{Ga \rightarrow Fy}{\forall x (Ga \rightarrow Fx)} \forall I \\
 \hline
 \forall x (Ga \rightarrow Fx) \leftrightarrow (Ga \rightarrow \forall x Fx) \leftrightarrow I^3
 \end{array}$$

□

7.c

$$\begin{array}{c}
 \frac{\frac{\frac{[\forall x (Fx \rightarrow Ga)]^6}{\forall E} \quad \frac{Fy \rightarrow Ga \quad [Fy]^1}{\rightarrow E}}{Ga \quad [\exists x Fx]^2}{\exists E^1} \quad \frac{Ga}{\rightarrow I^2}}{\exists x Fx \rightarrow Ga} \quad \text{---} \\
 \forall x (Fx \rightarrow Ga) \rightarrow (\exists x Fx \rightarrow Ga)
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\frac{[Fy]^5 \quad [\exists x Fx]^4}{[Fy] \quad [\exists x Fx \rightarrow Ga]^6}{\exists E^4} \quad \frac{Ga}{\rightarrow I^5}}{Fy \rightarrow Ga} \quad \frac{\forall I}{\forall x (Fx \rightarrow Ga)} \quad \frac{\leftrightarrow I^6}{\leftrightarrow I^6}
 \end{array}$$

□

Van Dalen exercises

5 Exercise 5

5.a

To proof: $\Gamma \vdash \varphi \Rightarrow \Gamma \cup \Delta \vdash \varphi$.

Proof Suppose $\Gamma \vdash \varphi$. Then there exists a derivation D from Γ to conclusion φ . Take the set of all hypothesis on D needed to deduce φ from Γ and call this set H . Then $H \subseteq \Gamma$. Because $\Gamma \subseteq \Gamma \cup \Delta$ this means $H \subseteq \Gamma \cup \Delta$. Because the conclusion of a derivation D stays the same even when premises are added and because the conclusion of D is φ it follows that $\Gamma \cup \Delta \vdash \varphi$. \square

5.b

To proof: $\Gamma \vdash \varphi; \Delta; \varphi \vdash \psi \Rightarrow \Gamma \cup \Delta \vdash \psi$.

Proof Suppose $\Gamma \vdash \varphi$, Δ and $\varphi \vdash \psi$. Then there exist a derivation D from Γ to φ and a derivation D' from φ to ψ . If we concatenate derivations D and D' we have a derivation D'' from Γ to ψ , so $\Gamma \vdash \psi$. Because of the answer in 5.a it follows that $\Gamma \cup \Delta \vdash \psi$. \square

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a $I(\Gamma \vdash \varphi \text{ if } \varphi \in \Gamma)$

Because there is a derivation D from Γ to conclusion φ such that there is a finite Δ in Γ such that $\Gamma \vdash \varphi$

b $\Gamma \vdash \varphi, \Gamma' \vdash \psi \Rightarrow \Gamma \cup \Gamma' \vdash \varphi \wedge \psi$

Because there is a derivation D from Γ to conclusion φ and a derivation D' from Γ' to conclusion ψ , there must be a finite Δ in Γ and a finite Δ' in Γ' such that φ and ψ are derived of Δ and Δ' . Because of the \wedge -rule we can put branches Δ and Δ' in parallel and conclude $\Gamma \cup \Gamma' \vdash \varphi \wedge \psi$.

c $\Gamma \vdash \varphi \wedge \psi \Rightarrow \Gamma \vdash \varphi$ and $\Gamma \vdash \psi$,

d $\Gamma \cup \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \rightarrow \psi$,

$$\text{e } \Gamma \vdash \varphi, \Gamma' \vdash \varphi \rightarrow \psi \Rightarrow \Gamma \cup \Gamma' \vdash \psi,$$

$$\text{f } \Gamma \vdash \perp \Rightarrow \Gamma \vdash \varphi,$$

$$\text{g } \Gamma \cup \not\vdash \vdash \perp \Rightarrow \Gamma \vdash \varphi.$$