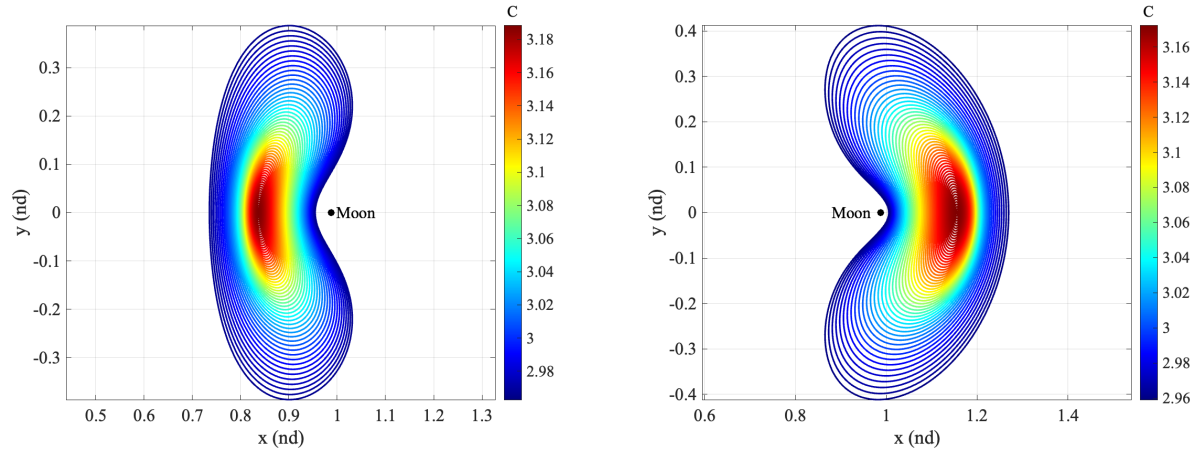
Homework 1: computation of Lyapunov orbits in the PCR3BP

1 Introduction

Lyapunov orbits are planar symmetric periodic solution of the CR3BP located around the three collinear points. They are the smallest planar periodic orbit near the libration points but can be extended in amplitude composing a whole continuous family for each libration point. They can be easily computed and, thanks to their instability, give useful insight on the natural transport of matter inside binary planetary systems that can be approximated with the CR3BP model.

We will focus in the followings on the planar version of the CR3BP, i.e, the PCR3BP, and the Earth-Moon system, but the same results hold for other planetary systems such as Sun-Earth and Sun-Mars to name some, as long as the mass parameter is such that the collinear points are unstable. In particular the phase space near the collinear libration points is of the type *saddle* × *center* × *center*, i.e., two real eigenvalues (one positive and one negative) and four purely imaginary ones in 3D, and *saddle* × *center* in the planar case.



(a) **(b)**

Figure 1: Earth-Moon $\mathcal{L}_1$ and $\mathcal{L}_2$ Lyapunov orbits.

2 State transition matrix

Before delving into the computation of the Lyapunov families we introduce the state transition matrix (STM). Let's suppose we have a n-dimensional autonomous dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \tag{1}$$

and an initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Then we can define the trajectory $\mathbf{x}(t)$ with the flow function $\boldsymbol{\phi}$ of the dynamical model $\mathbf{f}$ below

$$\boldsymbol{\phi} : \mathbf{x}(0) \to \mathbf{x}(t), \quad \mathbf{x}(t) = \boldsymbol{\phi}(\mathbf{x}_0, t) = \mathbf{x}_0 + \int_0^t \mathbf{f}(\mathbf{x}(\tau))\, d\tau \tag{2}$$

A trajectory $\mathbf{x}(t)^*$ displaced from $\mathbf{x}(t)$ starting from the initial point, i.e., $\mathbf{x}^*(t) = \mathbf{x}(t) + \delta\mathbf{x}(t)$, will satisfy in any instant the equation of motion, so

$$\dot{\mathbf{x}}^*(t) = \dot{\mathbf{x}}(t) + \delta\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t) + \delta\dot{\mathbf{x}}(t)) \simeq \mathbf{f}(\mathbf{x}(t)) + \frac{\partial \mathbf{f}(\mathbf{x}(t))}{\partial \mathbf{x}}\delta\mathbf{x}(t) \tag{3}$$

that leads to

$$\delta\dot{\mathbf{x}}(t) = \frac{\partial \mathbf{f}(\mathbf{x}(t))}{\partial \mathbf{x}}\delta\mathbf{x}(t) \tag{4}$$

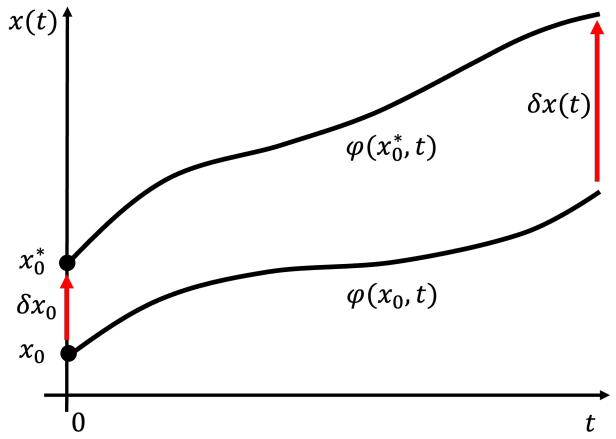


Figure 2

From Eq. 2 written for the displaced trajectory one can retrieve the displacement over time

$$\mathbf{x}(t) + \delta\mathbf{x}(t) = \boldsymbol{\phi}(\mathbf{x}_0 + \delta\mathbf{x}_0, t) \simeq \boldsymbol{\phi}(\mathbf{x}_0, t) + \frac{\partial \boldsymbol{\phi}(\mathbf{x}_0, t)}{\partial \mathbf{x}_0}\delta\mathbf{x}_0 \Longrightarrow \delta\mathbf{x}(t) = \frac{\partial \boldsymbol{\phi}(\mathbf{x}_0, t)}{\partial \mathbf{x}_0}\delta\mathbf{x}_0 \tag{5}$$

We now define $\dfrac{\partial \boldsymbol{\phi}(\mathbf{x}_0, t)}{\partial \mathbf{x}_0}$ as the matrix $\boldsymbol{\Phi}(t, 0)$ that takes the name of state transition matrix, and indicate the jacobian of the dynamical model $\dfrac{\partial \mathbf{f}(\mathbf{x}(t))}{\partial \mathbf{x}}$ with $\mathbf{A}(\mathbf{x}(t))$. Note that just like $\mathbf{A}$ also $\boldsymbol{\Phi}$ depends on the trajectory but we omit it for simplicity.

Let's combine Eq. 4 and Eq. 5 and derive in time the latter

$$\begin{cases} \delta\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t))\boldsymbol{\Phi}(t, 0)\delta\mathbf{x}_0 \\ \delta\dot{\mathbf{x}}(t) = \dot{\boldsymbol{\Phi}}(t, 0)\delta\mathbf{x}_0 \end{cases} \tag{6}$$

by comparison one obtains the differential equation for the STM, also know as variational equation of the dynamical model

$$\dot{\boldsymbol{\Phi}} = \mathbf{A}(\mathbf{x}(t))\boldsymbol{\Phi}(t, 0) \tag{7}$$

Finally, since $\delta\mathbf{x}(t) = \mathbf{\Phi}(t,0)\delta\mathbf{x}_0$ and $\delta\mathbf{x}(0) = \mathbf{\Phi}(t,0)\delta\mathbf{x}_0$, the initial condition for Eq. 7 is $\mathbf{\Phi}(0,0) = \mathcal{I}_d$ (the identity matrix).

As final remark note that the same results hold true also in the case of a non-autonomous dynamical system. In this case one simply has to substitute $\mathbf{f}(\mathbf{x}(t))$ with $\mathbf{f}(\mathbf{x}(t),t)$ and $\mathbf{A}(\mathbf{x}(t))$ with $\mathbf{A}(\mathbf{x}(t),t)$. In addition the flow function now depends also on the initial time t_0 as we cannot simply set it null.

3 Linearized motion near the collinear equilibrium points

Let's start considering the full nonlinear equations of motion (EOM) of the PCR3BP in Eq. 8 written in non-dimensional form

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = 2\begin{bmatrix}\dot{y}\\ -\dot{x}\end{bmatrix} + \begin{bmatrix}U_x\\ U_y\end{bmatrix} \tag{8}$$

where U_x and U_y are the partial derivatives of the total potential U given in Eq. 9 with respect to x and y:

$$U = \frac{x^2+y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{9}$$

r_1 and r_2 being the distances of the third body from to the Earth and the Moon, whose positions are $-\mu$ and $1-\mu$ respectively. Let's rewrite the second order system of Eq. 8 as a first order system of four differential equations and use u and v instead of $\dot{x}$ and $\dot{y}$:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{u}\\ \dot{v}\end{bmatrix} = \begin{bmatrix}u\\ v\\ 2v+U_x\\ -2u+U_y\end{bmatrix} \tag{10}$$

which is conveniently written in compact form as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x}$ is the state vector and $\mathbf{f}(\mathbf{x})$ is the dynamical model.

We now proceed on linearizing the nonlinear EOM in the neighborhood of one of the three collinear Lagrangian points. Let's define the state of the generic Lagrangian point ($\mathcal{L}$) as $\mathbf{x_L}$, then the generic state can be expressed as $\mathbf{x} = \mathbf{x_L} + \delta\mathbf{x}$ and the linear equations of motion write as:

$$\delta\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x_L})\delta\mathbf{x}, \quad \mathbf{A}(\mathbf{x}) = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ U_{xx} & U_{xy} & 0 & 2\\ U_{xy} & U_{yy} & -2 & 0\end{bmatrix}(\mathbf{x}) \tag{11}$$

where $\mathbf{A}(\mathbf{x_L})$ is the jacobian of the dinamical model $\mathbf{f}(\mathbf{x})$ computed at the $\mathbf{x_L}$, and so is a constant matrix.

Note that the differential equations in Eq. 11 are linear but are also coupled and little if any information can be extracted, so the next step is to try to diagonalize the linear system. We compute first the four eigenvalues (λ) and eigenvectors ($\mathbf{E}$) of $\mathbf{A}(\mathbf{x_L})$. For the three collinear points the eigenvalues are of the form:

$$\lambda_u > 0, \quad \lambda_s = -\lambda_u (< 0), \quad \lambda_{c1}, \lambda_{c2} = \pm j\omega$$

λ_u and its eigenvector $\mathbf{E_u}$ are called unstable eigenvalue and eigenvector respectively, similarly λ_s and $\mathbf{E_s}$ are the stable eigenvalue and eigenvector, while $\lambda_{c1}, \lambda_{c2}$ and $\mathbf{E_{c1}}, \mathbf{E_{c2}}$ are

the center eigenvalues and eigenvectors. According to the spectral analysis of A, the phase space near $\mathcal{L}$ allows both asymptotic motion approaching and departing from $\mathcal{L}$, and both periodic motion around it, and so the definition of *saddle* $\times$ *center*. The convergent motion happens in the portion of the 4D phase space spanned by $\mathbf{E_s}$, the divergent motion happens in the span of $\mathbf{E_u}$, while periodic motion is possible only in the center subspace, i.e., the set spanned by $\mathbf{E_{c1}}, \mathbf{E_{c2}}$.

Let's go back to the diagonalization of the linear EOM (Eq. 11). We use as base vector for the eigenspace representation the matrix given by $\mathbf{Z} = \left[\mathbf{E_u} \quad \mathbf{E_s} \quad Re\left(\mathbf{E_{c1}}\right) \quad Im\left(\mathbf{E_{c1}}\right)\right]$, and define the associated coordinates $\mathbf{e} = \left[\xi \quad \eta \quad \alpha \quad \beta\right]^T$. The transformation from the eigenspace to the cartesian space is $\delta\mathbf{x} = \mathbf{Ze}$, which yields

$$\dot{\mathbf{e}} = \mathbf{Z}^{-1}\mathbf{AZe} = \mathbf{De} \tag{12}$$

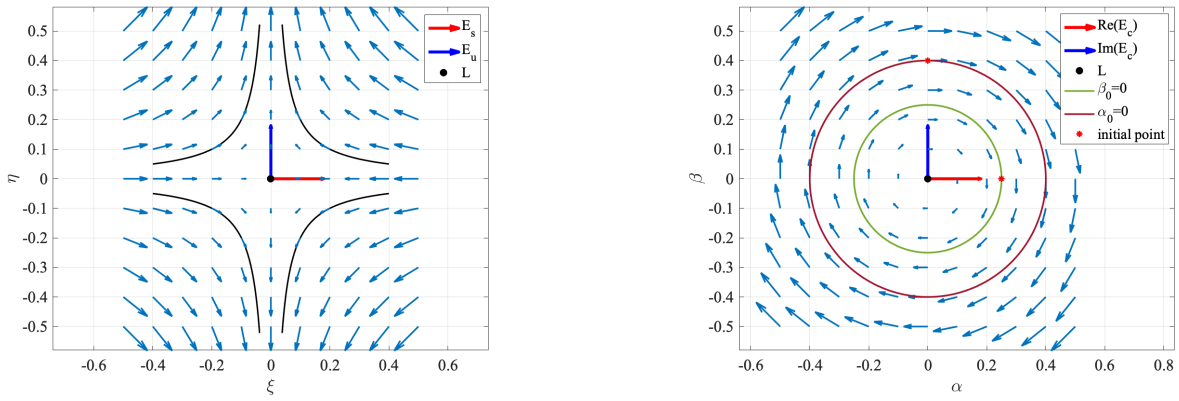
Due to the presence of a couple of imaginary eigenvalues $\mathbf{A}$ cannot be fully diagonalized, instead one has the Jordan form

$$\mathbf{D} = \begin{bmatrix} \lambda_u & 0 & 0 & 0 \\ 0 & \lambda_s & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}$$

The solution of Eq. 12 is now much easier. Selecting the initial conditions $\mathbf{e_0} = [\xi_0 \quad \eta_0 \quad \alpha_0 \quad \beta_0]^T$ one has

$$\begin{cases} \xi(t) = \xi_0 \exp \lambda_u t \\ \eta(t) = \eta_0 \exp \lambda_s t \\ \alpha(t) = \alpha_0 \cos \omega t + \beta_0 \sin \omega t \\ \beta(t) = \beta_0 \cos \omega t - \alpha_0 \sin \omega t \end{cases} \tag{13}$$

Note that the periodic motion has always the same period $T = 2\pi/\omega$, regardless of the choice of the initial condition. From Fig. 3 one can see the flow in the stable-unstable subspace and in the center subspace near $\mathcal{L}_1$ of the Earth-Moon system.



(a) **(b)**

Figure 3: Flow near the lunar $\mathcal{L}_1$ in the stable/unstable (a) and center (b) subspaces.

Eq. 13 can be rewritten in matrix form introducing the eigen-representation of the state

transition matrix (STM) $\Phi_e(t)$

$$\mathbf{e}(t) = \mathbf{\Phi_e}(t)\mathbf{e_0}, \quad \mathbf{\Phi_e}(t) = \begin{bmatrix} \exp\lambda_u t & 0 & 0 & 0 \\ 0 & \exp\lambda_s t & 0 & 0 \\ 0 & 0 & \cos\omega t & \sin\omega t \\ 0 & 0 & -\sin\omega t & \cos\omega t \end{bmatrix}$$

which yields the cartensian representation of the flow to be:

$$\delta\mathbf{x}(t) = \mathbf{Z}\mathbf{\Phi_e}(t)\mathbf{e_0}$$

In order to build a periodic motion we set $\mathbf{e_0} = [0 \quad 0 \quad \alpha_0 \quad 0]^T$, and get the linearized equation of a Lyapunov orbit:

$$\mathbf{x}(t) = \mathbf{x_L} + \alpha_0(\cos(\omega t)\ Re(\mathbf{E_{c1}}) - \sin(\omega t)\ Im(\mathbf{E_{c1}})) \tag{14}$$

4 Nonlinear motion and TPBVP formulation

With Eq. 14 we obtained a linear approximation of a small-amplitude Lyapunov orbits near $\mathcal{L}$, however, if we were to propagate the orbit in the nonlinear problem, it would eventually diverge. Furthermore the linear approximation doesn't describe Lyapunov orbits with greater amplitude. We are now concerned in computing Lyapunov orbits in the full PCR3BP and start by noting a symmetry in the dynamical model: if we apply the transformation $(y, t) \rightarrow (-y, -t)$ we get the same equation of motion. This implies that, if we have a trajectory $\boldsymbol{\phi}(\mathbf{x_0}^+, t)$ starting in $\mathbf{x_0}^+ = \begin{bmatrix} x_0 & y_0 & u_0 & v_0 \end{bmatrix}$, the trajectory $\boldsymbol{\phi}(\mathbf{x_0}^-, -t)$ with $\mathbf{x_0}^- = \begin{bmatrix} x_0 & -y_0 & -u_0 & v_0 \end{bmatrix}$ will experience the same EOM and so it will be the mirror image of $\boldsymbol{\phi}(\mathbf{x_0}^+, t)$ with respect to the xz-plane as shown in Fig. 4. In other words in the PCR3BP (but also in the CR3BP) trajectories are symmetrical with respect to time and the xz-plane.

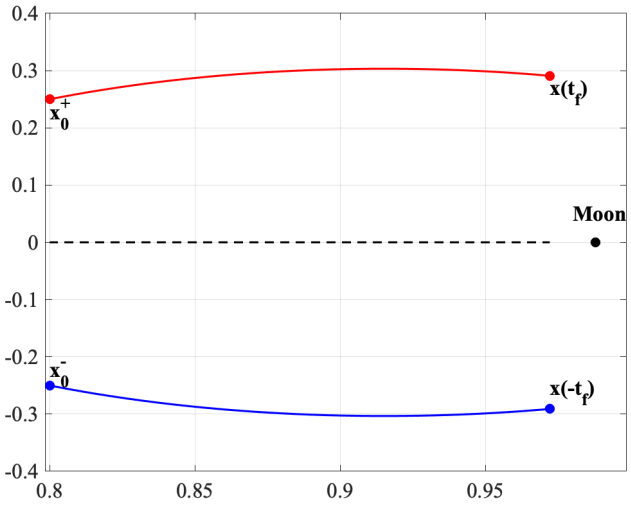


Figure 4: symmetric flow in the PCR3BP: (red) a trajectory starting at $\mathbf{x_0}^+ = \begin{bmatrix} x_0 & y_0 & u_0 & v_0 \end{bmatrix}$ with $t : 0 \rightarrow t_f$, and (blue) a trajectory starting at $\mathbf{x_0}^- = \begin{bmatrix} x_0 & -y_0 & -u_0 & v_0 \end{bmatrix}$ with $t : 0 \rightarrow -t_f$.

Hence a sufficient condition for periodicity in the PCR3BP is to have two perpendicular crossings of the x-axis and Lyapunov orbits fulfill this condition (to see this one must simply note that the first two components of $Re(\mathbf{E_{c1}})$ and $Im(\mathbf{E_{c1}})$ are parallel to the x and y axes).

One can exploit this property to compute Lyapunov orbits with a single-shooting scheme. The idea is to initialize the orbit at one crossing and target the other one by gradually adjusting the initial conditions. Let's define our two point boundary value problem (TPBVP). Since the orbit is initialized on the x-axis perpendicularly we just need to compute the coordinates x_0 and v_0 and the periods T. At $T/2$ the orbits must cross again perpendicularly the x-axis, hence it is desired that $y(T/2)$ and $u(T/2)$ are zero, the other state components are free instead. Then the design variables are $\mathbf{X} = [x_0,\ v_0,\ T]^T$ and the constraint vector is $\mathbf{F}(\mathbf{X}) = [y(T/2),\ u(T/2)]^T = \mathbf{0}$.

The time domain for the periodic orbit is $t \in [0, T]$, but T is unknown. We can introduce it in the dynamical model by re-scaling the time as $t = T\tau$, in this way each periodic orbit must be integrated on the fixed time span $\tau \in [0, 1]$. The scaled dynamical model is

$$\mathbf{x}' = T\mathbf{f}(\mathbf{x}(\tau)) \tag{15}$$

$$\mathbf{\Phi}' = T\mathbf{A}(\mathbf{x}(\tau))\mathbf{\Phi} \tag{16}$$

where the superscript (') indicates a derivative in τ.

Let's now suppose that the actual, unknown, solution for our BVP, $\mathbf{X}^*$, is not far from our best knowledge of it, $\mathbf{X}$, hence $\mathbf{X}^* = \mathbf{X} + \delta\mathbf{X}$ with $\delta\mathbf{X}$ small. By substitution in the constraint vector one obtains:

$$\mathbf{0} = \mathbf{F}(\mathbf{X}^*) \simeq \mathbf{F}(\mathbf{X}) + \mathbf{DF}(\mathbf{X})\delta\mathbf{X} \tag{17}$$

where $\mathbf{DF}(\mathbf{X})$ is the Jacobian of $\mathbf{F}$ with respect to $\mathbf{X}$, computed on $\mathbf{X}$. The linear system above is under-determined and so we will use the pseudo-inverse of $\mathbf{DF}$ to compute the correction $\delta\mathbf{X}$:

$$\delta\mathbf{X} = -(\mathbf{DF}^T\mathbf{DF})^{-1}\mathbf{DF}^T\mathbf{F}(\mathbf{X}) \tag{18}$$

Since with Eq. 17 we linearized in the neighborhood of $\mathbf{X}$, we must repeat this process until some stopping conditions are met. One can set a tolerance on the norm of $\mathbf{F}$, e.g. $1 \cdot 10^{-12}$, or iterate until $|\delta\mathbf{X}_k| > |\delta\mathbf{X}_{k-1}|$ (in this case no further improvement is possible).

Lastly, one must explicit $\mathbf{DF}$. We define the periodic orbit as the flow $\boldsymbol{\phi}(\tau, \mathbf{x_0}, T)$ and compute its partials with respect to the initial condition $\dfrac{\partial\boldsymbol{\phi}}{\partial\mathbf{x_0}}$ and the period $\dfrac{\partial\boldsymbol{\phi}}{\partial T}$. The former is simply the STM $\mathbf{\Phi}(\tau)$, the latter requires a little more computation. Recall that $\boldsymbol{\phi}$ solves the EOM, then

$$\boldsymbol{\phi}(\tau)' = T\mathbf{f}(\boldsymbol{\phi}(\tau, \mathbf{x_0}, T)) \tag{19}$$

which once differentiated in T leads to

$$\frac{\partial\boldsymbol{\phi}'}{\partial T} = \mathbf{f}(\boldsymbol{\phi}(\tau, \mathbf{x_0}, T)) + T\frac{\partial\mathbf{f}}{\partial\boldsymbol{\phi}}\frac{\partial\boldsymbol{\phi}}{\partial T} \tag{20}$$

Let now define $\mathbf{w}(\tau) = \dfrac{\partial\boldsymbol{\phi}}{\partial T}$ and note that $\dfrac{\partial\mathbf{f}}{\partial\boldsymbol{\phi}}$ is just the Jacobian of the dynamical model $\mathbf{A}$, the above equation becomes

$$\mathbf{w}(\tau)' = \mathbf{f} + T\mathbf{A}(\tau)\mathbf{w}(\tau) \tag{21}$$

whose solution is given by $\mathbf{w}(\tau) = \mathbf{w_o}(\tau) + \mathbf{w}_p(\tau)$, where $\mathbf{w_o}(\tau)$ solves the homogeneous system $\mathbf{w}(\tau)' = T\mathbf{A}\mathbf{w}(\tau)$ and $\mathbf{w_p}(\tau)$ is the particular integral. The homogeneous system $\mathbf{w}' = T\mathbf{A}\mathbf{w}$ is a linear system with varying coefficients whose solution is known and is $\mathbf{w_o}(\tau) = \mathbf{\Phi}(\tau)\mathbf{w_o}(0)$. However the initial condition $\mathbf{w_o}(0)$ is zero so $\mathbf{w_o}$ doesn't play any role (the idea is that at $\tau = 0$ a variation on T doesn't affect the evolution of the orbits, matter

of fact at $\tau = 0$ we have no flow but just a point in phase space). For the particular integral one can use the generalized variation of constant formula and the final solution is then:

$$\mathbf{w}(\tau) = \Phi(\tau) \int_0^\tau \Phi(s)^{-1} \mathbf{f}(\boldsymbol{\phi}(s, \mathbf{x_0}, T))\, ds = \Phi(\tau) \int_0^\tau \mathbf{f}(\boldsymbol{\phi}(0, \mathbf{x_0}, T))\, ds = \tau \Phi(\tau) \mathbf{f}(\mathbf{x_0}) \tag{22}$$

We have now all the elements to compose the Jacobian:

- $\dfrac{\partial \mathbf{F}}{\partial [x_0, v_0]}$ is a 2x2 matrix given by the second and third rows and the first and fourth columns of $\mathbf{\Phi}(1/2)$,
- $\dfrac{\partial \mathbf{F}}{\partial T}$ is a 2x1 vector given by the second and third components of $\mathbf{w}(1/2)$.

If we explicit $\mathbf{\Phi}$ as

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{xx} & \phi_{xy} & \phi_{xu} & \phi_{xv} \\ \phi_{yx} & \phi_{yy} & \phi_{yu} & \phi_{yv} \\ \phi_{ux} & \phi_{uy} & \phi_{uu} & \phi_{uv} \\ \phi_{vx} & \phi_{vy} & \phi_{vu} & \phi_{vv} \end{bmatrix}$$

then $\mathbf{DF}$ is

$$\mathbf{DF} = \begin{bmatrix} \phi_{yx} & \phi_{yv} & \frac{1}{2}\mathbf{\Phi}(2,:)\mathbf{f}(\mathbf{x_0}) \\ \phi_{ux} & \phi_{uv} & \frac{1}{2}\mathbf{\Phi}(3,:)\mathbf{f}(\mathbf{x_0}) \end{bmatrix} \tag{23}$$

Finally, to compute a small amplitude Lyapunov orbit one must retrieve from Eq. 14 the state associated to one of the two crossings, then propagate forward in time from $\tau = 0$ and $\tau = 1/2$, evaluate $\mathbf{F}$ and $\mathbf{DF}$ and the correct the initial guess. The process must be repeated until convergence.

5 Continuation schemes

Here we will see how to compute the whole family of orbits once a converged solution has been obtained. The underlying idea is that multiple solutions of the BVP exist and they belong to a smooth curve (or branch of solutions). In other words the design vector $\mathbf{X}$ is not an isolated point instead it can be parameterized in a convenient way. Once a point of the solution branch has been obtained, it can be used to compute a new one nearby with some numerical continuation scheme. We will see to methodologies: the natural continuation (NC) and the pseudo-arclength continuation (PAC).

5.1 Natural continuation

NC consists of varying in a controlled manner just one component, said the natural parameter, of the design vector. In this case one is assuming implicitly that all the other components can be described as a function of the natural parameter. In our case we could choose for example x_0 as the natural parameter and consequently consider $v_0 = v_0(x_0)$ and $T = T(x_0)$.

Let's suppose that we have computed our small-amplitude Lyapunov orbit $\mathbf{X_1}$, we will look for a new orbit $\mathbf{X_2}$ at distance δx_0 from the previous, i.e., $x_{02} = x_{01} + \delta x_0$. This means that x_0 is now a known fixed quantity which has to be then removed from $\mathbf{X}$ as well as the corresponding column of $\mathbf{DF}$. By choosing δx_0 small enough the orbit with initial guess $x_{02} = x_{01} + \delta x_0$, $v_{02} = v_{01}$ and $T_2 = T_1$ lies in the basin of attraction of the Newton

method which will eventually converge to the actual Lyapunov orbit at x_{02} (if it exists). In other words, we are stepping along the solution branch by varying x_0 and using the previous solution as initial guess for the new one.

NC is a simple continuation scheme that works when the design vector is a monotonous function of the natural parameter. Lyapunov orbit belong to this case.

OSS (1): one should note that the use of v_{01} and T_1 to approximate v_{02} and T_2 was mandatory since there is no way to evaluate the shape of the functions $v_0 = v_0(x_0)$ and $T = T(x_0)$ at x_{01}.

OSS (2): Lyapunov orbits are not the only planar periodic and symmetric solutions near the equilibrium points, and so a step δx_0 too large could yield the shooting scheme to converge to another family.

5.2 Pseudo-arclength continuation

PAC is a continuation scheme slightly more complicated then NC but way more powerful. As its name suggests, it makes use of an arclength parameter to parameterize the solution branch. Let $\mathbf{X}(s)$ be the solution branch, s be the arclength parameter and $\mathbf{X}_1 = \mathbf{X}(s_1)$ a converged solution of $\mathbf{F}(\mathbf{X}) = \mathbf{0}$. The idea is to increment s in order to step along $\mathbf{X}(s)$ starting from $\mathbf{X}_1$ while exploiting the vector $\boldsymbol{\tau}$ tangent to the curve. So let assume $\mathbf{X}(s_2) = \mathbf{X}(s_1 + \delta s)$ and substitute inside the constraint vector:

$$\mathbf{F}(\mathbf{X}(s_2)) \simeq \mathbf{F}(\mathbf{X}_1) + \frac{\partial \mathbf{F}}{\partial \mathbf{X}} \frac{\partial \mathbf{X}}{\partial s}(s_1) = \mathbf{0} + \mathbf{DF}(\mathbf{X}_1)\boldsymbol{\tau}(s_1) \tag{24}$$

We want the left hand side to be zero which happens only if $\mathbf{DF}(\mathbf{X}_1)\boldsymbol{\tau}(s_1) = \mathbf{0}$, i.e. only if $\boldsymbol{\tau}(s_1)$ belongs to the null-space of $\mathbf{DF}(\mathbf{X}_1)$. This suggest to look for the next solution $\mathbf{X}_2$ at a distance Δs from $\mathbf{X}_1$.

To implement PAC one must first solve the TPBVP for the small amplitude Lyapunov (our $\mathbf{X}_1$) and compute the Jacobian $\mathbf{DF}(\mathbf{X}_1)$ and its null-space. Then set Δs and use $\mathbf{X}_2 = \mathbf{X}_1 + \Delta s\, \boldsymbol{\tau}_1$ as initial guess for the next orbit to solve again the TPBVP. However we must make sure that $\mathbf{X}_2$ remains along $\boldsymbol{\tau}_1$ during the shooting scheme, so a further constraint must be added to the problem. With PAC one has to solve

$$\mathbf{G}(\mathbf{X}) = \begin{bmatrix} \mathbf{F}(\mathbf{X}) \\ \boldsymbol{\tau}_1^T(\mathbf{X} - \mathbf{X}_1) - \Delta s \end{bmatrix} = \mathbf{0} \tag{25}$$

where the Jacobian is $\mathbf{DG} = \begin{bmatrix} \mathbf{DF} \\ \boldsymbol{\tau}_1^T \end{bmatrix}$. Once $\mathbf{X}_2$ is computed, one must compute $\boldsymbol{\tau}_2$ from $\mathbf{DF}(\mathbf{X}_2)$ and repeat the process for $\mathbf{X}_3$ and so on.

OSS (1): MATLAB has the built-in function *null* to compute the null-space, however the sign of the tangent vector is unknown. One must make sure that $\boldsymbol{\tau}_k$ and $\boldsymbol{\tau}_{k-1}$ are not opposite otherwise the shooting scheme will converge to a previous solution.

OSS (2): The dimension of the null-space can be greater than one. This means that multiple solution branches, so multiple families of orbit (as many as the dimension of the null-space), intersect at the current orbit. In this case it is sufficiently to select the $\boldsymbol{\tau}_k$ closest

in orientation to $\boldsymbol{\tau}_{k-1}$ to remain on the current solution branch.

OSS (3): The arclength parameter doesn't have a straightforward physical interpretation as the natural parameter in NC. It is convenient to start with a small Δs like $1 \cdot 10^{-3}$ and adjust it via trial and error.

6 Homework 1

Compute the Lyapunov families of $\mathcal{L}_1$ and $\mathcal{L}_2$ in the Earth-Moon system. Starting from a small amplitude orbit, choose a numerical continuation scheme (either NC or PAC) and compute the whole family. Then, using the bisection methods, compute the orbit pairwise, in the range of Jacobi constant common to both $\mathcal{L}_1$ and $\mathcal{L}_2$ families. The goal is to have pairs of $\mathcal{L}_1$ and $\mathcal{L}_2$ orbits with the same Jacobi constant.

Some remarks:

1. make sure you have at least two/three pairs of orbit within the range $[3.1370, 3.1493]$ of the Jacobi constant;
2. the Jacobi constant (C) used here is $C = 2U - (u^2 + v^2)$.

Hints for the implementation:

- use *ode113* for the numerical integrator;
- when you have to solve a linear system $\mathbf{Ax} = \mathbf{b}$, use the matlab command "\" ($\mathbf{x} = \mathbf{A}\backslash\mathbf{b}$), it will take care of the pseudo-inverse if $\mathbf{A}$ isn't squared;
- first compute all the Lyapunov families and then do their pairwise re-computation;
- the bisection method must be applied to the natural parameter or to the arclength. In either cases at each bisection step one must use the shooting to recompute the orbit;
- if you choose PAC, than while computing the Lyapunov families you need to store also the tangent vectors and the step-sizes to later apply bisection to the Δss. Or you can apply bisection to x_0.

Name	Value
μ_{Earth}	$3.98600436 \cdot 10^5\ km^3/s^2$
μ_{Moon}	$4.902799 \cdot 10^3\ km^3/s^2$
Earth-Moon distance	$384400\ km$
ode113 *RelTol*	$3 \cdot 10^{-14}$
ode113 *AbsTol*	$1 \cdot 10^{-14}$

Table 1: Parameters setting.

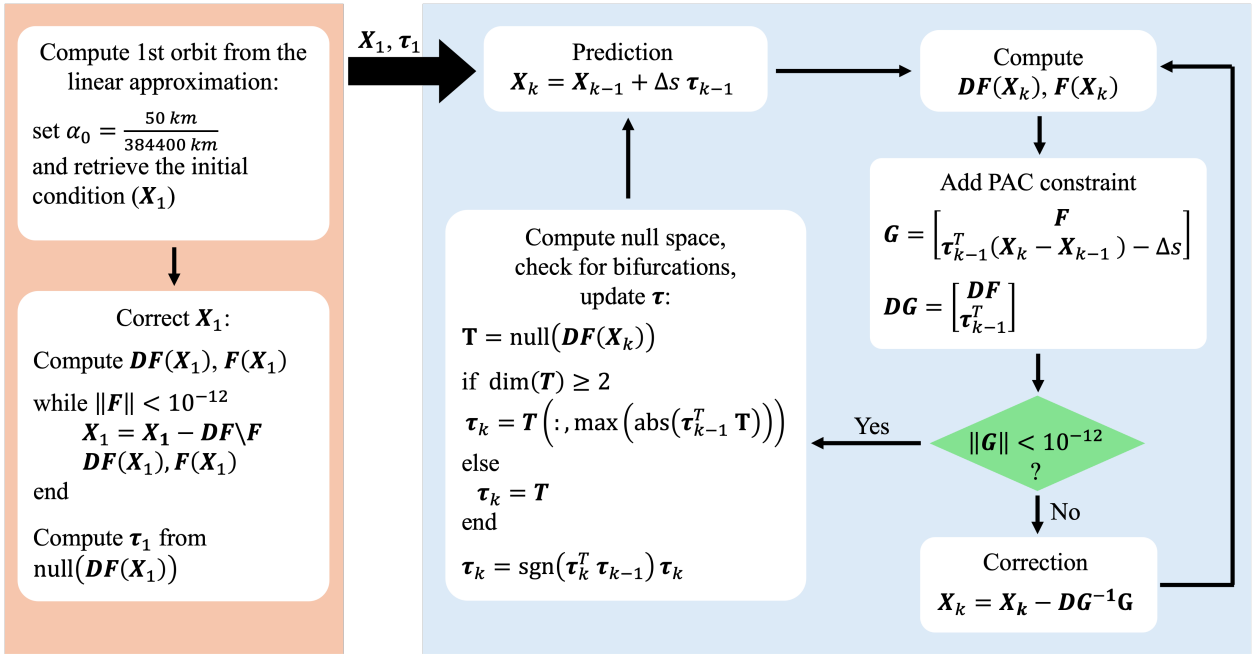


Figure 5: Pseudo code for the PAC scheme.