Lyapunov Orbits

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All code available at: `https://github.com/Meoli96/Trajectory`

Lyapunov Orbits

Lyapunov orbits are planar symmetric periodic orbits around the collinear points for the 3CRBP. In order to find such solutions, the Jacobian of the linearized dynamical model around the equilibrium point is considered, and through diagonalization, two real eigenvalues (one positive $\boldsymbol{E_u}$ and one negative $\boldsymbol{E_s}$) and one complex pair $\boldsymbol{E_{c1}}$, $\boldsymbol{E_{c2}}$ is found.

Periodic motion is possible only in the subspace spanned by the eigenvectors corresponding to the complex eigenvalues. Deriving the linearized equation of motion in this subspace, the state equation for a Lyapunov orbit is obtained as:

$$\boldsymbol{x} = \boldsymbol{x_L} + \alpha_0(\cos(\omega t)Re(\boldsymbol{E_{c1}}) - \sin(\omega t)Im(\boldsymbol{E_{c1}}) \tag{1}$$

with α_0 the amplitude of the periodic motion, ω the frequency of the motion, and $\boldsymbol{x_L}$ the equilibrium point. The linearization will eventually diverge, and the periodic motion will be accurate only for small amplitudes.

To improve the accuracy of the solution and make possible the computation of a family of periodic orbits, the problem is formulated as a two-point boundary value problem (TPBVP), thanks to the simmetries of the 3CRBP. A shooting method is then employed to solve the TPBVP, and a first nonlinear Lyapunov orbit is found from the linear guess.

Once a first solution is found, a continuation method is employed to find a family of periodic orbits. The Pseudo Archlength Continuation (PAC) is used to follow the family of solutions in the solution space. The following pseudocode illustrates the procedure to vary the solution step ds to achieve the continuation:

```
if norm(G) > tol:
    ds = ds/1.1
else:
    ...
    ds = ds*1.2
    if ds > ds_max:
        ds = ds_max
```

where G is the residual of the PAC, tol is the tolerance, and ds_{max} is the maximum step size. This method allows for accuracy on the computation by reducing the step size when the residual is too high, but also for efficiency by increasing the step size when the residual is low enough.

A family of nonlinear Lyapunov orbits is then found around L1 and L2:

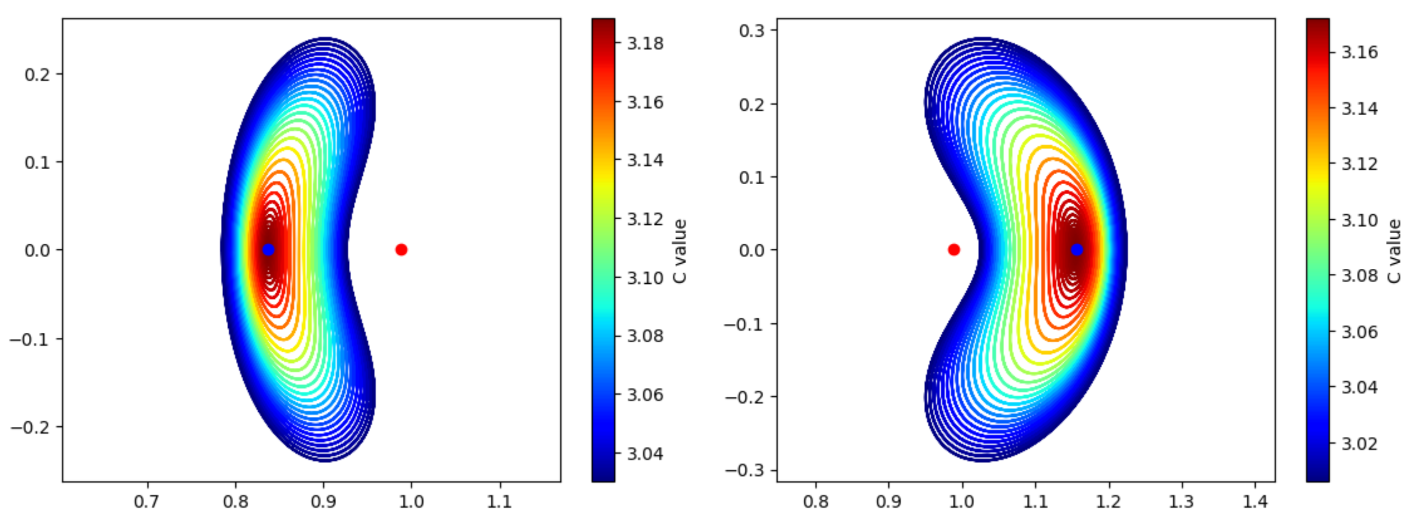

Figure 1: Lyapunov orbits around L1 and L2

Isoenergetic Lyapunov orbits

As per request, two pair of isoenergetic Lyapunov orbits is seeked. Two orbits in the Jacobi range $C_J = [3.1370, 3.1493]$ are found in the family around L1 from the previously computed family. A bisection method on the solution step ds is employed to find the two pairs. The following pseudocode illustrates the procedure:

```
ds_a = 0
ds_b = next_orb_ds
while abs(C_J - C_J_target) > tol:
    ... # Compute nonlinear Lyapunov orbit with ds_guess
    if C_J > C_J_target:
        ds_a = ds_guess
    else:
        ds_b = ds_guess
    ds_guess = (ds_a + ds_b)/2
```

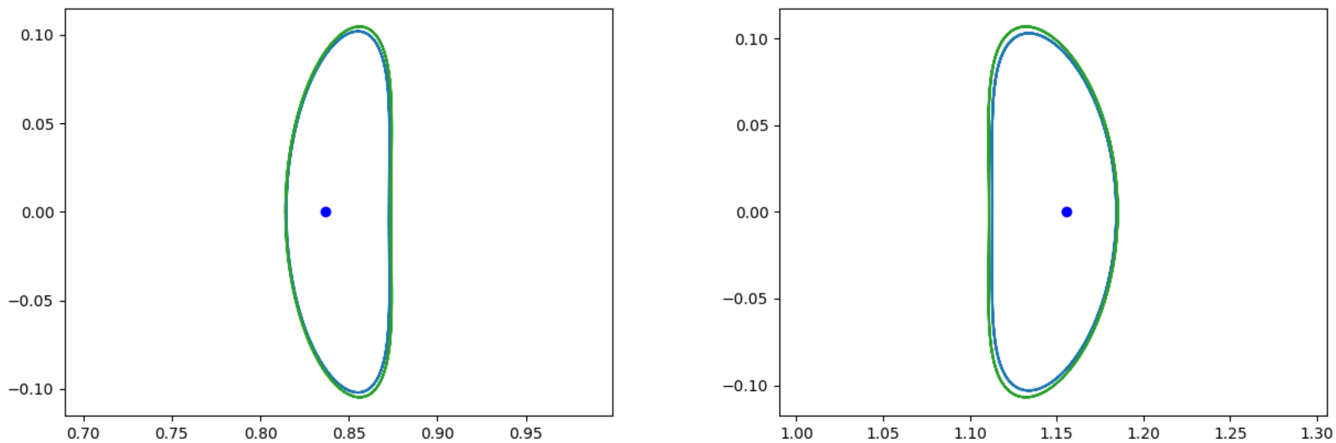

Figure 2: Isoenergetic Lyapunov orbits

In this way two pairs of isoenergetic orbits are found, with a Jacobi constant of $C_J = 3.1443$ for the first pair and $C_J = 3.1422$ for the second pair.

Manifolds

Manifolds can be computed by perturbing a generic point on the Lyapunov orbit in the stable or unstable subspace spanned by their respective eigenvector. Letting $\boldsymbol{p}$ be the generic point on the Lyapunov orbit, the perturbation is computed as:

$$\boldsymbol{x} = p \pm \epsilon \frac{\boldsymbol{E}}{||\boldsymbol{E}||} \tag{2}$$

with ϵ the perturbation, and $\boldsymbol{E}$ the stable or unstable eigenvector

Manifolds provide an useful insight on the movement of matter of the system, and can be used to compute low-DV transfers between Lyapunov orbits or regions in space. In particular, stable manifolds will asimptotically approach the Lyapunov orbit, while unstable manifolds will asimptotically diverge from it.

With the task to compute trajectories, three regions in space are identified:

- The Earth region **E**, inside the ZVC
- The Moon region **M**, inside the ZVC
- The outer space region **X**, outside the ZVC

With the use of Poincarè sections, state cuts can be computed to find the initial state to propagate the desired itinerary. Two of such sections are defined as:

$$\Sigma_1 : \{x = 1 - mu\} \qquad \Sigma_2 : \{y = 0\} \tag{3}$$

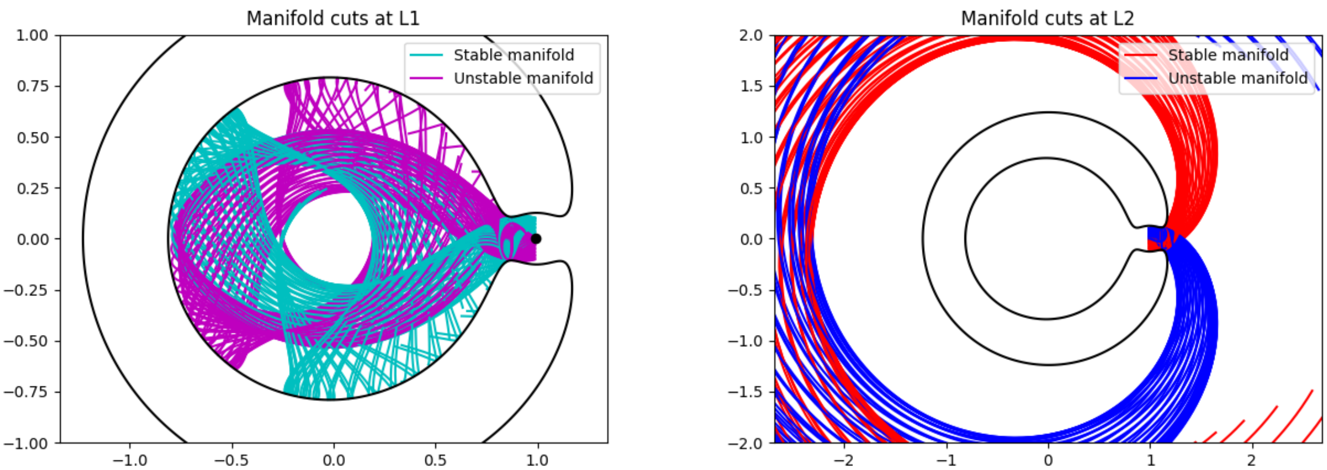

Figure 3: Manifolds around L1 and L2, stable (red) and unstable (blue)

[X M E] itinerary

In order to explore the [X M E] itinerary, the stable manifold of L1 and unstable manifold of L2 are computed, divided by the Poincarè section Σ_1, and the state value at the section are found. By choosing one initial state between the two in

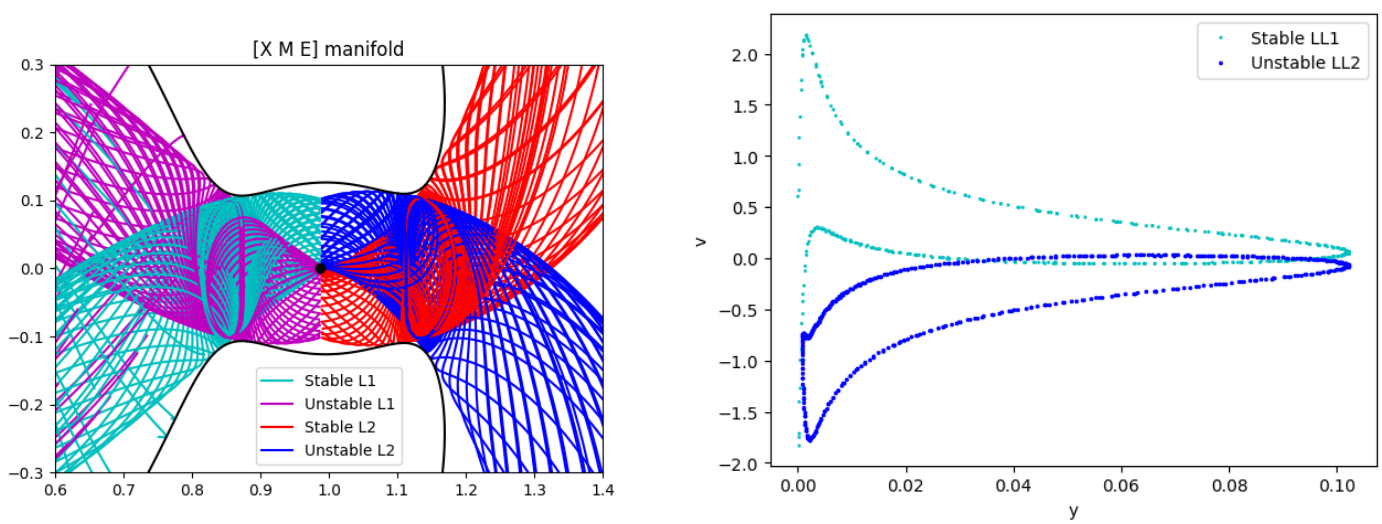

Figure 4: Stable L1 and unstable L2

the hodograph plane and propagate it backward and forward in time, a [X M E] itinerary is found. Due to the problem symmetry, an [X E M] itinerary is also found by inverting the y-u components of the same initial state.

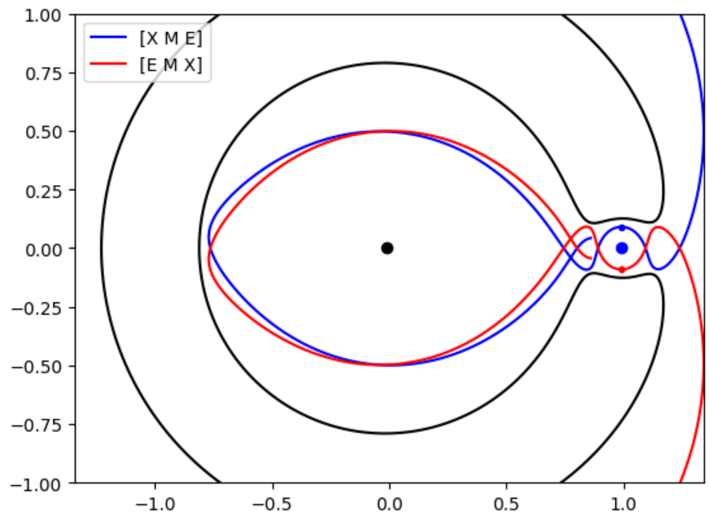

Figure 5: [X M E] and [E M X] itineraries

[E M E] itinerary

To plan an [E M E] itinerary the stable and unstable manifolds of L1 are computed, with the Poincarè section Σ_2.

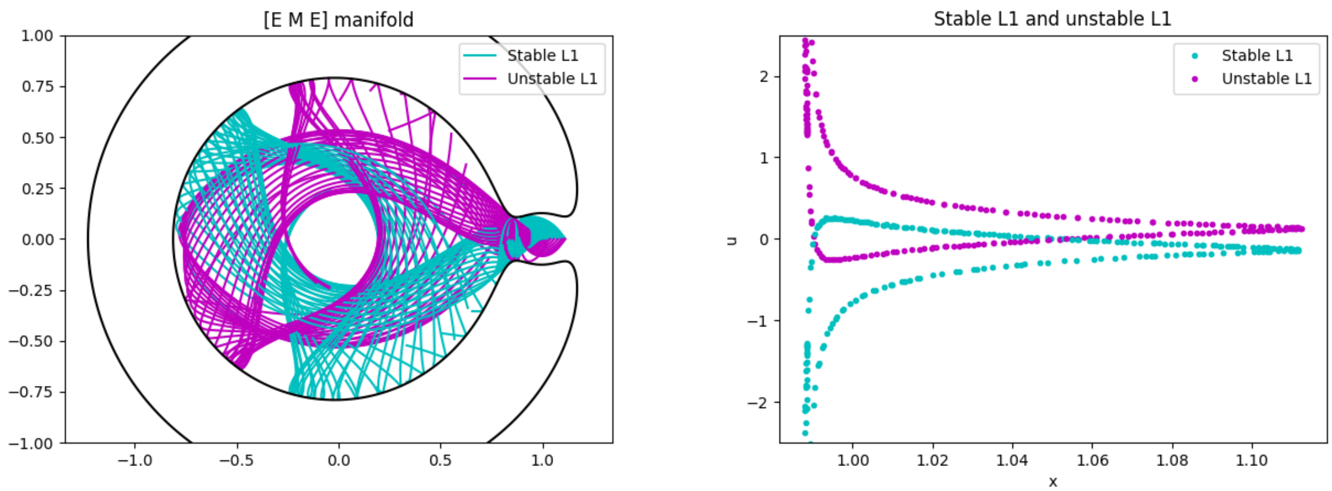

Figure 6: Stable and unstable L1

With the same methodology as before, we compute a trajectory that goes from the Earth to the Moon and back to the Earth.

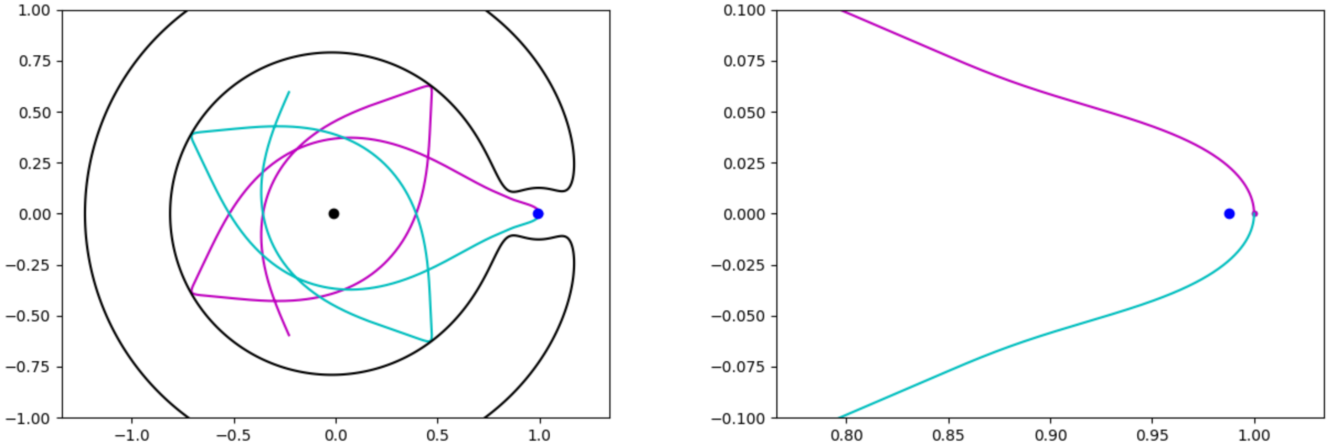

Figure 7: [E M E] itinerary

[X M X] itinerary

Finally, to plan an [X M X] itinerary, the stable and unstable manifolds of L2 are considered, with the Poincarè section Σ_2. The trajectory is computed as before, and the [X M X] itinerary is found.

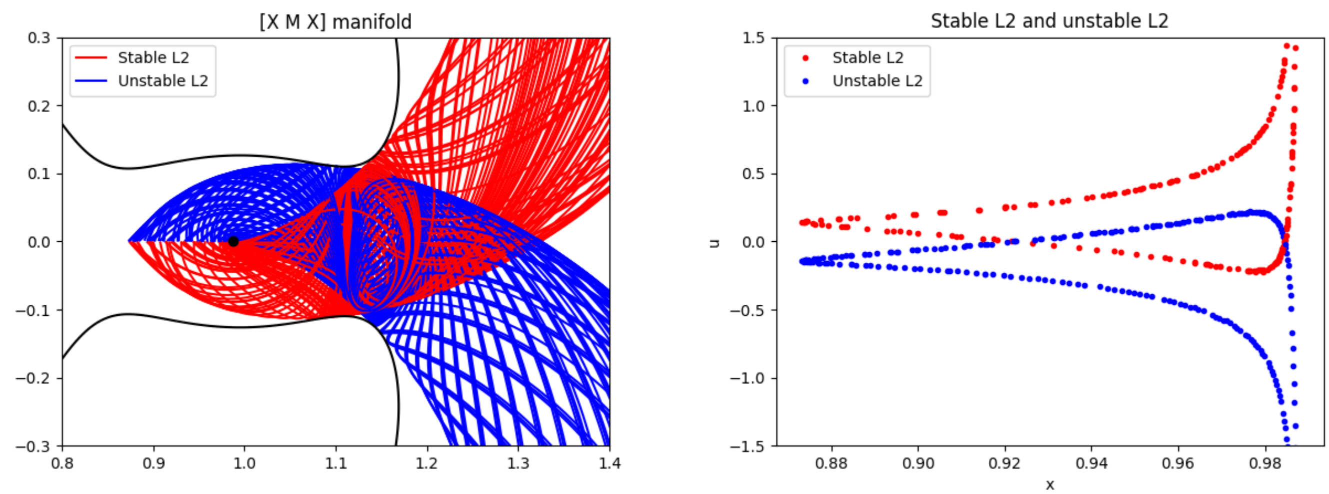

Figure 8: Stable and unstable L2

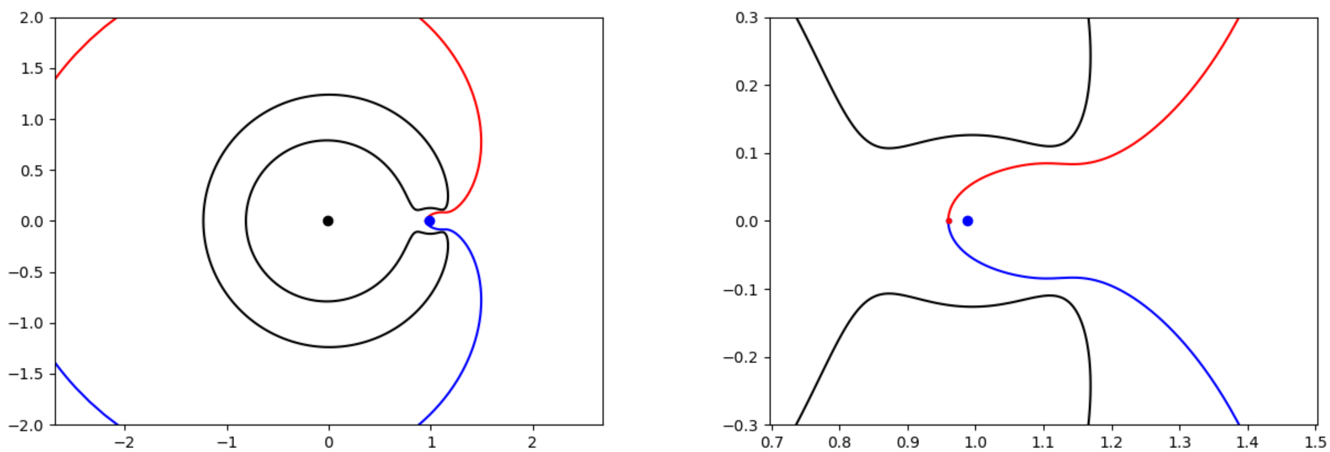

Figure 9: [X M X] itinerary