Iran Lemma

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This is well known in Olympiad Geometry and is often used to solve some configuration based problems. However, it can be tricky sometimes. This problem is highly instructive because to solve such problems(in the synthetic method) we can find many configurations which are very useful in Olympiad lemma. First we will look at the lemmas and solve the original problems and see a couple of examples.

1 The Problem

Example 1.1: Iran TST 2009/9

In triangle ABC, D, E and F are the points of tangency of incircle with the center of I to BC, CA and AB respectively. Let M be the foot of the perpendicular from D to EF. P is on DM such that $DP = MP$. If H is the Orthocenter of BIC, prove that PH bisects EF.

Even though it looks like a random problem, it has some lemmas that are quite common in olympiad to this date.

2 Lemmas

There is a basic lemma for warm up,

Lemma 2.1

Let H be the orthocenter of $\triangle ABC$. D, E, F are the foot from A, B, and C respectively. Then prove that A is the D-excenter of $\triangle DEF$.

Solution—
This requires angle chasing. Note that $AEHF$, $BHDF$, $CHDE$, $BCEF$, $CAFD$, $ABDE$ are all cyclic. Now that gives, $\angle AFE = \angle ACB$. Moreover,

$$90^\circ - \angle ACB = \angle EBC = \angle HBC = \angle HBD = \angle HFD$$

Similarly, $\angle EFH = 90^\circ - \angle ACB$.

$\therefore \angle AFE = \angle ACB = \dfrac{1}{2}(180^\circ - \angle DFE)$, and similarly $\angle AEF = \dfrac{1}{2}(180^\circ - \angle DEF)$. And the rest follows. $\square$

Lemma 2.2

In triangle $\triangle ABC$ D, E, F are the touchpoints of BC, CA, and AB respectively. Line CI and EF intersects at C_1. Similarly, Line BI and EF intersects at B_1. Then, C_1FIB, B_1EIC are cyclic quadrilaterals and ID as their radical axis.

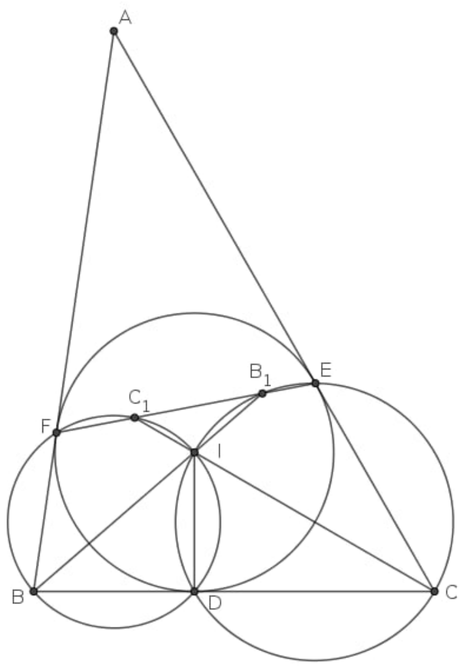
Solution—
Notice that, $IF \perp AB$, $ID \perp BC$. Thus, $IFBD$ is cyclic. And,

$$\angle C_1IB = \angle ICB + \angle IBC = \frac{1}{2}B + \frac{1}{2}C = 90^\circ - \frac{1}{2}A$$

But,

$$\angle AFE = 90^\circ - \frac{1}{2}A$$

$\therefore C_1FBD$ is cyclic. Consequently, C_1FBDI is cyclic.



Similarly, B_1FBDI is also cyclic. So we are done! $\square$

Note that this lemma also gives BI, CI are the diameters of the circles mentioned in the lemma. And, since $IF \perp AB$ and $IE \perp AC$ thus, BC_1B_1C is cyclic with diameter BC.

Lemma 2.3

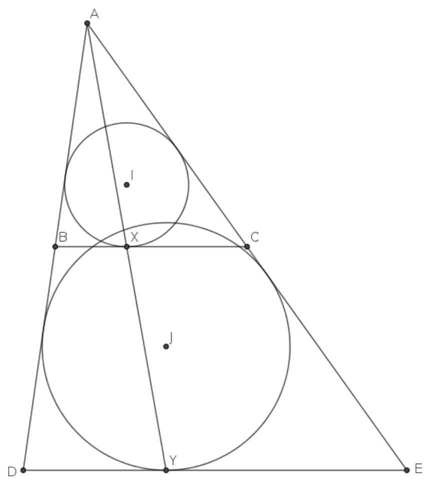
For $\triangle ABC$ let D be the touchpoint on BC and incircle. X be the touchpoint of BC and A-excircle of $\triangle ABC$. AX and the A-excircle intersects for the second time at Z. Prove that $A - X - Z$ (A, X, and Z are collinear.).

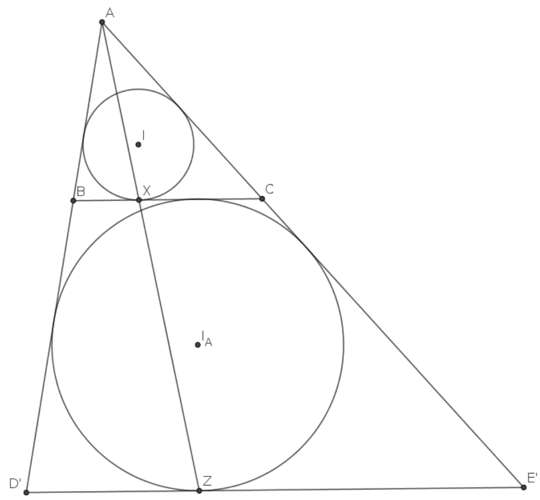
Solution—

First, forget about the excircles even if it's good prcatice to consider every information, you sometimes want to forget if it makes thing *simple*. Just consider a *random* line parallel to BC. Which we shall call DE such that $D \in AB$, $E \in AC$. Let I, J be the incicle of $\triangle ABC$ and ADE respectively. Those incircles touches BC and DE at X and Y respectively.

Claim— A, X, Y are collinear.

Proof—
This must hold as $\triangle ABC \sim \triangle ADE \implies \triangle ABX \sim \triangle ADY$. Which gives us $\angle XAB = \angle YAD$. Thus our claim is proved. $\square$





Now, this must hold if the incircle of $\triangle ADE$ is the A-excircle of $\triangle ABC$. Thus we are done!

Lemma 2.4

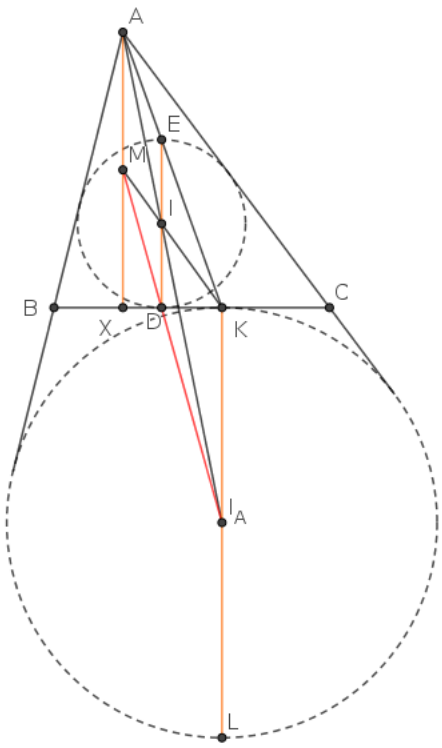
AX is the foot of $\triangle ABC$. I_A is the A-excenter of $\triangle ABC$. Let M be the midpoint of AX, and D is the touchpoint of the incircle of ABC and BC. Then MI_A passes through D.

Solution—

Let E be the antipode of D with respect to the incircle, and let the A-excircle touch BC at K, and L be the antipode of K with respect to the A-excircle. Then by previous lemma we have $A - E - K$ and $A - D - L$, which implies that $K - I - M$. Now, denote r, and r_a as the radius of the incircle and A-excircle respectively. Which gives us,

$$\frac{AE}{AK} = \frac{r}{r_a} = \frac{MI}{MK} = \frac{DI}{KI_A}$$

$\therefore \triangle MID \sim \triangle MKI_A$. And the rest follows $\square$



3 Now, the solution

Example 3.1

In triangle ABC, D, E and F are the points of tangency of incircle with the center of I to BC, CA and AB respectively. Let M be the foot of the perpendicular from D to EF. P is on DM such that $DP = MP$. If H is the Orthocenter of BIC, prove that PH bisects EF.

Solution—

Let $C_1 = EF \cap CI$, $B_1 = EF \cap CI$, $PH \cap EF = X$
Then, by Lemma 2.2 we have (C_1FBDI) and $(B_1EIDC$ are cyclic with diameter BI, CI respectively. Which gives us, (B_1C_1BC) is cyclic with BC as its diameter. Thus, $B - C_1 - H$ and $C - B_1 - H$ are collinear.

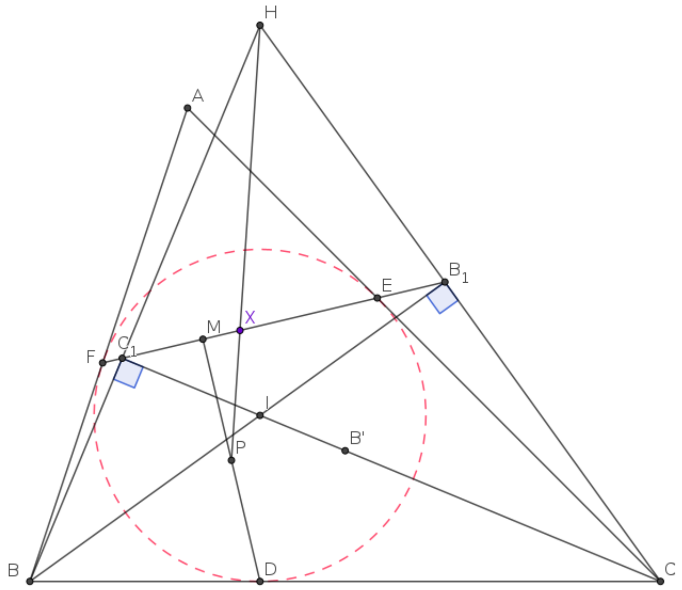


Figure: Iran TST 2009

Note that H is then the D-excenter of $\triangle DC_1B_1$, by lemma 2.1. And as DC_1B_1 is the orthic triangle of $\triangle HBC$, I is the incenter of DC_1B_1. Thus, PH is passes through the touchpoint of EF and the incircle of DB_1C_1.
Thus, $IX \perp B_1C_1 \implies IX \perp EF$. Since, $IE = IF$, thus X is the midpoint of EF, and we are done!