1 Syntax

Types	$A, B, C ::= \top \mid \mathbb{B} \mid A \to B \mid A \,\&\, B$
Expressions	$e ::= () \mid b \mid x \mid \mathbf{fix}\, x : A.\ e \mid \lambda x : A.\ e : B \mid e_1\ e_2 \mid e_1\ ,\ e_2 \mid e : A$
Values	$v ::= () \mid b \mid \lambda x : A.\ e : B \mid v_1\ ,\ v_2$

2 Subtyping

$\boxed{\rceil A \lceil}$ *(Top-like types)*

$$\textsc{TL-top}\ \frac{}{\rceil \top \lceil} \qquad \textsc{TL-and}\ \frac{\rceil A \lceil \qquad \rceil B \lceil}{\rceil A \,\&\, B \lceil} \qquad \textsc{TL-arrow}\ \frac{\rceil B \lceil}{\rceil A \to B \lceil}$$

$\boxed{A^\circ}$ *(Ordinary types)*

$$\textsc{O-top}\ \frac{}{\top^\circ} \qquad \textsc{O-base}\ \frac{}{\mathbb{B}^\circ} \qquad \textsc{O-arrow}\ \frac{}{(A \to B)^\circ}$$

$\boxed{A <: B}$ *(Subtyping)*

$$\textsc{S-top}\ \frac{\rceil B \lceil}{A <: B} \qquad \textsc{S-base}\ \frac{}{\mathbb{B} <: \mathbb{B}} \qquad \textsc{S-arrow}\ \frac{B_1 <: A_1 \qquad A_2 <: B_2}{A_1 \to A_2 <: B_1 \to B_2}$$

$$\textsc{S-andLL}\ \frac{A <: C}{A \,\&\, B <: C} \qquad \textsc{S-andLR}\ \frac{B <: C}{A \,\&\, B <: C} \qquad \textsc{S-andR}\ \frac{A <: B \qquad A <: C}{A <: B \,\&\, C}$$

3 Typing

$\boxed{\Gamma \vdash e \Leftrightarrow A}$ *(Bidirectional typing)*

$$\textsc{Typ-top}\ \frac{}{\Gamma \vdash () \Rightarrow \top} \qquad \textsc{Typ-base}\ \frac{}{\Gamma \vdash b \Rightarrow \mathbb{B}} \qquad \textsc{Typ-var}\ \frac{x : A \in \Gamma}{\Gamma \vdash x \Rightarrow A} \qquad \textsc{Typ-fix}\ \frac{\Gamma, x : A \vdash e \Leftarrow A}{\Gamma \vdash \mathbf{fix}\, x : A.\ e \Rightarrow A}$$

$$\textsc{Typ-abs}\ \frac{\Gamma, x : A \vdash e \Leftarrow B}{\Gamma \vdash \lambda x : A.\ e : B \Rightarrow A \to B} \qquad \textsc{Typ-app}\ \frac{\Gamma \vdash e_1 \Rightarrow A \to B \qquad \Gamma \vdash e_2 \Leftarrow A}{\Gamma \vdash e_1\ e_2 \Rightarrow B} \qquad \textsc{Typ-appTop}\ \frac{\Gamma \vdash e_1 \Rightarrow \top}{\Gamma \vdash e_1\ e_2 \Rightarrow \top}$$

$$\textsc{Typ-merge}\ \frac{\Gamma \vdash e_1 \Rightarrow A \qquad \Gamma \vdash e_2 \Rightarrow B \qquad A * B}{\Gamma \vdash e_1\ ,\ e_2 \Rightarrow A \,\&\, B}$$

$$\textsc{Typ-mergeV}\ \frac{\cdot \vdash v_1 \Rightarrow A \qquad \cdot \vdash v_2 \Rightarrow B \qquad v_1 \approx v_2}{\Gamma \vdash v_1\ ,\ v_2 \Rightarrow A \,\&\, B} \qquad \textsc{Typ-anno}\ \frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash e : A \Rightarrow A} \qquad \textsc{Typ-sub}\ \frac{\Gamma \vdash e \Rightarrow A \qquad A <: B}{\Gamma \vdash e \Leftarrow B}$$

4 Semantics

$\boxed{v \;\hookrightarrow_A\; v'}$ *(Type casting)*

$$\textsc{Cast-top}\ \frac{A^\circ \qquad \rceil A \lceil}{v \;\hookrightarrow_A\; ()} \qquad \textsc{Cast-base}\ \frac{}{b \;\hookrightarrow_{\mathbb{B}}\; b} \qquad \textsc{Cast-arrow}\ \frac{\neg\rceil A_2 \lceil \qquad B_1 <: A_1 \qquad A_2 <: B_2}{\lambda x : A_1.\ e : A_2 \;\hookrightarrow_{B_1 \to B_2}\; \lambda x : A_1.\ e : B_2} \qquad \textsc{Cast-mergeL}\ \frac{A^\circ \qquad v_1 \;\hookrightarrow_A\; v_1'}{v_1\,,\ v_2 \;\hookrightarrow_A\; v_1'}$$

$$\textsc{Cast-mergeR}\ \frac{A^\circ \qquad v_2 \;\hookrightarrow_A\; v_2'}{v_1\,,\ v_2 \;\hookrightarrow_A\; v_2'} \qquad \textsc{Cast-and}\ \frac{v \;\hookrightarrow_A\; v_1 \qquad v \;\hookrightarrow_B\; v_2}{v \;\hookrightarrow_{A\,\&\,B}\; v_1\,,\ v_2}$$

$\boxed{e \;\hookrightarrow\; e'}$ *(Small-step operational semantics)*

$$\textsc{Step-fix}\ \frac{}{\mathbf{fix}\, x : A.\ e \;\hookrightarrow\; e[x \mapsto \mathbf{fix}\, x : A.\ e] : A} \qquad \textsc{Step-beta}\ \frac{v \;\hookrightarrow_A\; v'}{(\lambda x : A.\ e : B)\, v \;\hookrightarrow\; e[x \mapsto v'] : B} \qquad \textsc{Step-betaTop}\ \frac{}{()\, v \;\hookrightarrow\; ()}$$

$$\textsc{Step-cast}\ \frac{v \;\hookrightarrow_A\; v'}{v : A \;\hookrightarrow\; v'} \qquad \textsc{Step-appL}\ \frac{e_1 \;\hookrightarrow\; e_1'}{e_1\, e_2 \;\hookrightarrow\; e_1'\, e_2} \qquad \textsc{Step-appR}\ \frac{e_2 \;\hookrightarrow\; e_2'}{v_1\, e_2 \;\hookrightarrow\; v_1\, e_2'} \qquad \textsc{Step-mergeL}\ \frac{e_1 \;\hookrightarrow\; e_1'}{e_1\,,\ e_2 \;\hookrightarrow\; e_1'\,,\ e_2} \qquad \textsc{Step-mergeR}\ \frac{e_2 \;\hookrightarrow\; e_2'}{v_1\,,\ e_2 \;\hookrightarrow\; v_1\,,\ e_2'}$$

$$\textsc{Step-anno}\ \frac{e \;\hookrightarrow\; e'}{e : A \;\hookrightarrow\; e' : A}$$