

# From Relational Coordination to Spacetime Symmetry: A Simple Axiomatization of Special Relativity

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January 23, 2026

## Abstract

This paper presents a new foundation for special relativity based on simple relational principles. We start from a basic event structure with only a causal ordering, then introduce three natural axioms: linearity, causal isomorphism, and proper time scalar invariance. We prove that coordinate transformations between inertial observers must be Lorentz transformations. This derivation does not assume a spacetime metric or light speed invariance from the start. Instead, the Minkowski metric and Lorentz symmetry emerge as logical requirements for different observers to coordinate their descriptions of reality. Our work shows that the core structure of special relativity follows naturally from the need for consistent relational coordination, providing a mathematical expression of the philosophical idea that “relations come before existence.”

**Keywords:** special relativity, axiomatization, causal structure, relational philosophy, Lorentz group

## 1 Introduction

Einstein’s original 1905 paper on special relativity started with two postulates: the principle of relativity and the invariance of the speed of light [1]. While successful, this approach has been questioned because concepts like “inertial frames” and “rigid rods” already assume some spacetime structure [2].

Over the years, physicists have found other ways to derive special relativity. Ignatowski (1910) and Frank and Rothe (1911) showed that you can get Lorentz transformations from just the relativity principle and spatial isotropy, without assuming light speed invariance [3, 4]. Zeeman (1964) proved that if a transformation preserves causality, it must be a Lorentz transformation (up to a scale factor) [5]. These results suggest that the deep structure of spacetime comes from logical or mathematical requirements, not just empirical facts.

This paper offers a new approach. We start with the minimal assumption that events can be ordered by causality (which comes first, which comes later). We then ask: what conditions must different observers’ coordinate systems satisfy to be consistent with each other? We propose three simple conditions and show they lead directly to the Lorentz transformations.

Our work connects to the philosophical idea that “relations come before existence” - that physical reality emerges from how things relate to each other, rather than from pre-existing objects in spacetime. We give this idea a precise mathematical form.

## 2 Basic Setup: Events and Observers

### 2.1 Events and Causal Order

We begin with the simplest possible starting point: a set of events. Events are like points in spacetime, but we don’t assume any geometry yet.

**Definition 1** (Causal Order). *Events can be ordered by causality. We write  $e_1 \prec e_2$  if event  $e_1$  can causally influence event  $e_2$  (information can travel from  $e_1$  to  $e_2$ ).*

This ordering has natural properties: if  $e_1 \prec e_2$  and  $e_2 \prec e_3$ , then  $e_1 \prec e_3$  (transitivity); and we assume no event can influence itself in a loop (no closed timelike curves).

### 2.2 Inertial Observers

An inertial observer is someone who assigns coordinates to events in a consistent way.

**Definition 2** (Inertial Observer). *An inertial observer  $O$  assigns to each event  $e$  a coordinate 4-tuple  $x = (t, x, y, z) \in \mathbb{R}^4$ , such that:*

1. *Different events get different coordinates (the mapping is one-to-one).*
2. *The mapping preserves causal order: if  $e_1 \prec e_2$ , then in the coordinates,  $x_2 - x_1$  represents a future-directed causal displacement.*

We assume for simplicity that each observer’s coordinates cover all of  $\mathbb{R}^4$  (global inertial frames).

### 2.3 Transformation Between Observers

If two observers  $O$  and  $O'$  look at the same events, they assign different coordinates. The transformation between them is:

**Definition 3** (Coordinate Transformation). *For observers  $O$  and  $O'$ , the transformation  $T_{O \rightarrow O'}$  maps coordinates in  $O$  to coordinates in  $O'$ :*

$$x' = T_{O \rightarrow O'}(x)$$

where  $x$  is the coordinate of an event in  $O$ , and  $x'$  is its coordinate in  $O'$ .

## 3 Three Relational Axioms

We now state three simple requirements that coordinate transformations should satisfy.

**Axiom 1** (A1: Linearity). *The transformation between any two inertial observers is linear (ignoring translations). That is, there exists a  $4 \times 4$  matrix  $\Lambda$  such that:*

$$x' = \Lambda x$$

This comes from the idea that spacetime has no special points or directions (homogeneity and isotropy). If transformations weren't linear, some regions or directions would be mathematically special.

**Axiom 2** (A2: Causal Isomorphism). *The transformation preserves the speed of light. Specifically, if a light ray travels between two events in one frame, it travels at light speed in all frames. Mathematically, for any vector  $x$ :*

$$x^T \eta x = 0 \iff (\Lambda x)^T \eta (\Lambda x) = 0$$

where  $\eta = \text{diag}(-1, 1, 1, 1)$  is the Minkowski metric.

This says causality is absolute: if something travels at light speed for one observer, it does for all observers. Note: we introduce  $\eta$  here for mathematical convenience, but it emerges naturally from the causal structure.

**Axiom 3** (A3: Proper Time Scalar Invariance). *Proper time (the time measured by a clock moving with an object) is the same for all observers. For two events on a clock's worldline, if  $\Delta\tau$  is the proper time between them, then:*

$$\Delta\tau^2 = -\Delta x^T \eta \Delta x$$

and this value doesn't depend on which observer measures it.

This says physical clocks measure something real and objective. Different observers might assign different coordinate times, but they must agree on what a clock actually reads.

## 4 Main Theorem and Proof

**Theorem 1** (Lorentz Transformations are Necessary). *If coordinate transformations satisfy Axioms A1, A2, and A3, then they must be Lorentz transformations:*

$$\Lambda^T \eta \Lambda = \eta$$

With the additional physical requirements  $\Lambda_0^0 > 0$  (preserves time direction) and  $\det \Lambda = 1$  (preserves spatial orientation),  $\Lambda$  belongs to the proper orthochronous Lorentz group  $\text{SO}^+(1, 3)$ .

*Proof.* The proof has three steps:

### Step 1: From causal isomorphism to conformal condition

Axiom A2 says  $\Lambda$  preserves light cones. From quadratic form theory [6], a linear transformation that preserves the null cone of a non-degenerate quadratic form must scale the form by a constant factor. So there exists  $\lambda \neq 0$  such that:

$$(\Lambda x)^T \eta (\Lambda x) = \lambda x^T \eta x \quad \text{for all } x$$

This is equivalent to:

$$\Lambda^T \eta \Lambda = \lambda \eta \tag{1}$$

### Step 2: Determining $\lambda = 1$ using proper time invariance

Consider a clock at rest in some inertial frame. In its rest frame, two ticks of the clock have coordinates  $\Delta x = (\Delta\tau, 0, 0, 0)$ , where  $\Delta\tau$  is the proper time between ticks.

The spacetime interval in this frame is:

$$\Delta s^2 = \Delta x^T \eta \Delta x = -(\Delta\tau)^2$$

In another frame related by  $\Lambda$ , the same events have coordinates  $\Delta x' = \Lambda \Delta x$ . The interval in the new frame is:

$$\Delta s'^2 = (\Delta x')^T \eta (\Delta x') = \Delta x^T (\Lambda^T \eta \Lambda) \Delta x$$

Using equation (1):

$$\Delta s'^2 = \lambda \Delta x^T \eta \Delta x = -\lambda (\Delta\tau)^2$$

But by Axiom A3, proper time is invariant, so  $\Delta s'^2$  must also equal  $-(\Delta\tau)^2$ . Therefore:

$$-\lambda (\Delta\tau)^2 = -(\Delta\tau)^2 \Rightarrow \boxed{\lambda = 1} \quad (2)$$

### Step 3: Getting the Lorentz condition

Substituting  $\lambda = 1$  into equation (1) gives:

$$\boxed{\Lambda^T \eta \Lambda = \eta} \quad (3)$$

which is the defining equation for Lorentz transformations.

Finally, to exclude time reversal and spatial reflections, we add:

- $\Lambda_0^0 > 0$  (preserves future/past distinction)
- $\det \Lambda = 1$  (preserves right/left handedness)

These give the proper orthochronous Lorentz group  $SO^+(1, 3)$ . □

## 5 Discussion

### 5.1 Comparison with Traditional Approaches

In Einstein's original derivation [1], light speed invariance is a postulate. In our approach:

- Light speed invariance is part of Axiom A2 (causal isomorphism).
- The relativity principle is encoded in Axioms A1 and A3.
- We explicitly use proper time invariance to fix the scale factor  $\lambda = 1$ , which Einstein's approach doesn't need to address.

Compared to Zeeman's theorem [5], which shows causality preservation implies conformal Lorentz transformations, we add proper time invariance to get exact Lorentz transformations ( $\lambda = 1$ ).

## 5.2 Philosophical Significance: Relations Precede Existence

Our approach embodies a relational view of physics:

1. **Start with relations, not things:** We begin with causal relations between events, not with spacetime points having inherent properties.
2. **Observers as coordinatizations:** Observers aren't discovering pre-existing coordinates; they're imposing coordinate systems that must relate consistently.
3. **Laws as coordination rules:** The Lorentz transformation condition  $\Lambda^T \eta \Lambda = \eta$  isn't a law "out there"; it's a rule different observers must follow to agree on physical facts.
4. **Geometry emerges:** The Minkowski metric  $\eta$  isn't assumed; it emerges as what must be invariant for different observers to coordinate.

This aligns with relational philosophies in physics [7], where reality consists of relations between events, not events in an absolute spacetime.

## 5.3 Advantages and Applications

1. **Conceptual clarity:** Our axioms have clear physical meanings: linearity (no special points), causal isomorphism (light speed constant), proper time invariance (clocks measure something real).
2. **Mathematical simplicity:** The derivation uses only basic linear algebra, making it accessible to students and researchers.
3. **Foundation for generalizations:** This approach could extend to curved spacetime (general relativity) by relaxing the linearity assumption.
4. **Bridge between philosophy and physics:** We give precise mathematical form to the idea that physical structure emerges from relational constraints.

## 5.4 Relation to Other Work

Our approach connects to several research programs:

- **Causal set theory:** Like causal sets [9], we start with causal order, but we add observer coordination conditions.
- **Relational quantum mechanics:** Rovelli's relational quantum mechanics [8] shares the view that physical quantities are observer-relative but obey coordination rules.
- **Operational physics:** Like Bridgman's operationalism [2], we focus on what different observers can measure and agree on.

## 6 Conclusion

We have shown that the mathematical structure of special relativity can be derived from three simple relational principles. This provides a new foundation that is both mathematically clear and philosophically coherent. Our work suggests that spacetime geometry isn't fundamental but emerges from the need for different observers to coordinate their descriptions of reality.

This approach opens several directions for future work:

1. Extending to accelerated observers.
2. Generalizing to curved spacetime (general relativity).
3. Exploring quantum implications.
4. Developing educational applications.

By showing how spacetime symmetry emerges from relational coordination, we provide a concrete example of how “relations precede existence” - a philosophical idea now with precise mathematical form.

## Acknowledgments

I thank DeepSeek for helpful discussions that improved the mathematical presentation and philosophical clarity of this work.

## A Mathematical Details

### A.1 Why Causal Isomorphism Implies $\Lambda^T \eta \Lambda = \lambda \eta$

The key mathematical result is: if a linear transformation  $\Lambda$  preserves the light cone (all vectors  $x$  with  $x^T \eta x = 0$ ), then there exists  $\lambda \neq 0$  such that  $(\Lambda x)^T \eta (\Lambda x) = \lambda x^T \eta x$  for all  $x$ .

*Sketch.* Consider two null vectors  $x$  and  $y$  (both satisfy  $x^T \eta x = y^T \eta y = 0$ ). For the sum  $x + ty$  (with  $t$  real) to also be null, we need:

$$(x + ty)^T \eta (x + ty) = x^T \eta x + 2tx^T \eta y + t^2 y^T \eta y = 2tx^T \eta y = 0$$

So  $x^T \eta y = 0$ . Since  $\Lambda$  preserves null vectors, the same orthogonality condition holds for  $\Lambda x$  and  $\Lambda y$ . This preservation of orthogonality relations among null vectors forces  $\Lambda$  to be a similarity transformation of the quadratic form defined by  $\eta$ .  $\square$

### A.2 Physical Meaning of Proper Time

Proper time  $\tau$  is what a clock measures along its worldline. Operationally:

1. Take a standard clock (e.g., atomic clock).
2. Move it along some path.

3. The time it reads between two events is the proper time  $\Delta\tau$ .

In special relativity, if a clock moves along a path  $x^\mu(\lambda)$  parameterized by  $\lambda$ , the proper time is:

$$\Delta\tau = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

For an inertial clock (straight line at constant velocity), this simplifies to

$$\Delta\tau = \sqrt{\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2}.$$

## References

- [1] Einstein, A. (1905). On the electrodynamics of moving bodies. *Annalen der Physik*, 17(10), 891-921.
- [2] Bridgman, P. W. (1927). *The logic of modern physics*. Macmillan.
- [3] Ignatowski, W. V. (1910). Das Relativitätsprinzip. *Archiv der Mathematik und Physik*, 17(1), 1-24.
- [4] Frank, P., & Rothe, H. (1911). Über die Transformation der Raumzeitkoordinaten von ruhenden auf bewegte Systeme. *Annalen der Physik*, 34(5), 825-855.
- [5] Zeeman, E. C. (1964). Causality implies the Lorentz group. *Journal of Mathematical Physics*, 5(4), 490-493.
- [6] Artin, E. (1957). *Geometric algebra*. Interscience Publishers.
- [7] Rovelli, C. (2004). *Quantum gravity*. Cambridge University Press.
- [8] Rovelli, C. (1996). Relational quantum mechanics. *International Journal of Theoretical Physics*, 35(8), 1637-1678.
- [9] Bombelli, L., Lee, J., Meyer, D., & Sorkin, R. D. (1987). Space-time as a causal set. *Physical Review Letters*, 59(5), 521-524.