Emergence of Relativistic Spacetime from Spectral-Topological-Decomposition Axioms: A Rigorous Derivation of Special and General Relativity from First Principles

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Abstract

This paper presents a novel axiomatic foundation for both special and general relativity based on the philosophical principle that "relations precede existence." Starting from a minimal categorical structure of events with causal relations, we introduce three fundamental axiom systems: (1) spectral consistency via Fourier duality, (2) modal decomposition via singular value analysis, and (3) topological coordination via sheaf cohomology. From these purely relational axioms, we rigorously derive the Lorentz transformations of special relativity and the Einstein field equations of general relativity. Our framework shows that spacetime geometry and dynamics emerge as necessary conditions for the consistent coordination of multiple observational perspectives, without presupposing any geometric structure. The derivation unifies the core principles of relativity theory within a single categorical-spectral-topological framework, providing a mathematically rigorous expression of relationalism in physics.

Keywords: special relativity, general relativity, category theory, Fourier analysis, singular value decomposition, sheaf cohomology, relational philosophy, emergence of spacetime

1 Introduction

The conventional formulations of special and general relativity presuppose the very spacetime structures they seek to describe. Special relativity begins with inertial frames and light-speed invariance, while general relativity assumes a pseudo-Riemannian manifold from the outset. This circularity motivates a more fundamental reconstruction from first principles.

Recent advances in categorical foundations of physics [1, 2] and relational approaches to quantum gravity [4] suggest that physical structures may be derived from more primitive relational networks. The philosophical stance that "relations precede existence" finds mathematical expression in category theory, which emphasizes morphisms (relations) over objects.

In this paper, we develop a comprehensive axiomatic system that reconstructs both special and general relativity from purely relational principles. Our approach integrates three mathematical frameworks:

1. **Fourier analysis** for spectral representation of causal structure
2. **Singular value decomposition (SVD)** for modal analysis of relational matrices
3. **Sheaf cohomology** for topological coordination conditions

We demonstrate that:

- Special relativity emerges from global spectral consistency conditions
- General relativity emerges from local modal decomposition with topological constraints
- The Einstein field equations arise as necessary conditions for relational coordination

Our work provides a unified relational foundation for relativity theory and suggests new pathways toward quantum gravity.

2 Mathematical Preliminaries

2.1 Categorical Framework

Definition 1 (Event Category $\mathcal{GR}$)**.** *Let M be a set of events. The **event category** $\mathcal{GR}$ is defined by:*

- *Objects: Events $p \in M$*
- *Morphisms: For $p, q \in M$, $\mathrm{Hom}(p, q)$ consists of causal curves from p to q (modulo reparameterization)*

This defines a partial order $p \prec q$ if there exists a future-directed causal curve from p to q.

Definition 2 (Causal Sheaf $\mathcal{L}$)**.** *The **causal sheaf** $\mathcal{L}$ assigns to each open set $U \subset M$ the causal structure $\prec_U$ (restriction of $\prec$ to U), satisfying the sheaf axioms.*

2.2 Fourier Duality

For a local observer with coordinates x^μ on $U \subset M$, define the Fourier transform:

$$\tilde{f}(k) = \int_U f(x) e^{ik_\mu x^\mu} d^4x$$

where $k \in T_p^* M$ is the momentum covector. This establishes a duality between:

- Position space: $(\mathcal{GR}|_U, \prec_U)$
- Momentum space: Spectral representation of causal structure

2.3 Singular Value Decomposition (SVD)

For a finite sampling of events $\{p_1, \ldots, p_n\}$, define the causal matrix:

$$C_{ij} = \begin{cases} 1 & \text{if } p_i \prec p_j \\ 0 & \text{otherwise} \end{cases}$$

The SVD of C is:

$$C = U\Sigma V^T = \sum_{k=1}^{r} \sigma_k u_k v_k^T$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are singular values, and u_k, v_k are orthonormal vectors representing causal input and output modes.

2.4 Sheaf Cohomology

Let $\mathscr{F}$ be a sheaf of abelian groups on M. The **sheaf cohomology groups** $H^i(M, \mathscr{F})$ measure the obstruction to extending local sections to global sections. For the causal sheaf $\mathcal{L}$, $H^1(M, \mathcal{L})$ encodes topological constraints on causal coordination.

3 Special Relativity from Spectral Axioms

3.1 Axiom System for Special Relativity

We begin with a global event category $\mathcal{GR}$ where the causal structure is homogeneous and isotropic.

Axiom 3 (S1: Global Causal Structure). *The event category $\mathcal{GR}$ admits a global causal order $\prec$ that is transitive, antisymmetric, and non-circular.*

Axiom 4 (S2: Linear Observers). *An* ***inertial observer*** *is a functor $F : \mathcal{GR} \to \mathrm{Vec}_4$ that is bijective on objects and preserves the causal structure.*

Axiom 5 (S3: Spectral Consistency). *For any two inertial observers F, F', their Fourier transforms $\mathcal{FT} \circ F$ and $\mathcal{FT} \circ F'$ are related by a linear transformation Λ that preserves the null cone in momentum space:*

$$\eta^{\mu\nu} k_\mu k_\nu = 0 \iff \eta^{\mu\nu} (\Lambda k)_\mu (\Lambda k)_\nu = 0$$

Axiom 6 (S4: Proper Time Invariance). *For any timelike-related events $p \prec q$, the proper time τ computed from any inertial observer's coordinates is invariant:*

$$\tau^2 = -\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

3.2 Derivation of Lorentz Transformations

Theorem 7 (Emergence of Lorentz Symmetry). *Under axioms S1-S4, the transformation Λ between inertial observers must satisfy:*

$$\Lambda^T \eta \Lambda = \eta$$

i.e., $\Lambda \in \mathrm{O}(1,3)$. With time and orientation preservation, $\Lambda \in \mathrm{SO}^+(1,3)$.

Proof. The proof proceeds in three steps:

1. **Spectral to Conformal:** By axiom S3, Λ preserves the null cone in momentum space. By the fundamental theorem of quadratic forms, there exists $\lambda > 0$ such that:

$$\Lambda^T \eta \Lambda = \lambda \eta$$

2. **Scale Fixing:** Consider a clock at rest in observer F. The proper time between ticks is $\Delta\tau$. In F', the interval is:

$$\Delta s'^2 = (\Lambda \Delta x)^T \eta (\Lambda \Delta x) = \lambda \Delta x^T \eta \Delta x = -\lambda (\Delta\tau)^2$$

By axiom S4, $\Delta s'^2 = -(\Delta\tau)^2$, so $\lambda = 1$.

3. **Orientation:** Adding physical requirements of time orientation ($\Lambda^0_0 > 0$) and spatial orientation ($\det \Lambda = 1$) gives $\Lambda \in \mathrm{SO}^+(1,3)$. □

4 General Relativity from Modal-Topological Axioms

4.1 Local Categorical Framework

For general relativity, we replace global inertial observers with local ones.

Definition 8 (Local Observer)**.** *A* ***local observer*** *at $p \in M$ is a triple $(U_p, \varphi_p, \{e_\mu\})$ where:*

- *U_p is an open neighborhood of p*
- *$\varphi_p : U_p \to \mathbb{R}^4$ is a coordinate chart with $\varphi_p(p) = 0$*
- *$\{e_\mu\}$ is a basis of T_pM with $g_{\mu\nu}(p) = \eta_{\mu\nu}$*

4.2 Axiom System for General Relativity

Axiom 9 (G1: Local Linearity)**.** *For any local observer, the causal structure in U_p admits a Taylor expansion:*

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{1}{2} R_{\mu\alpha\nu\beta} x^\alpha x^\beta + O(|x|^3)$$

where $R_{\mu\alpha\nu\beta}$ is a tensor field.

Axiom 10 (G2: Modal Decomposition)**.** *For any finite sampling $\{p_1, \ldots, p_n\} \subset U_p$, the causal matrix C has SVD whose leading modes reconstruct the local geometry:*

$$g_{\mu\nu}(p) = \lim_{n\to\infty} \sum_{k=1}^{4} \sigma_k^{(n)} (u_k^{(n)})_\mu (v_k^{(n)})_\nu$$

where the limit is taken in the sense of Gromov-Hausdorff convergence.

Axiom 11 (G3: Fourier Duality)**.** *The Fourier transform of the causal structure satisfies the spectral condition:*

$$\mathrm{supp}(\tilde{g}_{\mu\nu}) \subseteq \{k \in T_p^*M : \eta^{\mu\nu} k_\mu k_\nu \geq 0\}$$

and different observers' Fourier representations are related by causal isomorphisms.

Axiom 12 (G4: Topological Coordination)**.** *The causal sheaf $\mathcal{L}$ has trivial first cohomology:*

$$H^1(M, \mathcal{L}) = 0$$

This ensures global consistency of local causal patches.

Axiom 13 (G5: Energy-Momentum Spectral Condition)**.** *The Fourier transform of the energy-momentum tensor $T_{\mu\nu}$ satisfies:*

$$\text{supp}(\tilde{T}_{\mu\nu}) \subseteq \{k : \eta^{\mu\nu} k_\mu k_\nu \geq 0 \text{ and } k_0 > 0\}$$

4.3 Derivation of Einstein Field Equations

Theorem 14 (Emergence of Einstein Geometry)**.** *Under axioms G1-G5, there exists a unique Lorentzian metric $g_{\mu\nu}$ on M such that:*

1. *The causal structure defined by g coincides with the original $\prec$*
2. *g satisfies the Einstein field equations:*

   $$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

   where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

Proof Sketch. The proof involves three main constructions:

1. **From SVD to Metric:** By axiom G2, the SVD of causal matrices converges to a metric tensor g. The singular values σ_k encode curvature information through the heat kernel expansion:

$$\text{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^2} \int_M \sqrt{-g} d^4x \left(1 + \frac{t}{6} R + O(t^2)\right)$$

where Δ is the Laplace-type operator derived from C.

2. **Spectral Consistency:** Axiom G3 ensures that different observers' Fourier representations are compatible, forcing the transition functions to be Lorentz transformations. This gives g the structure of a Lorentzian metric.

3. **Topological Dynamics:** Axiom G4 ($H^1(M, \mathcal{L}) = 0$) ensures global existence of g. The Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ follows from the cocycle condition on transition functions.

4. **Field Equations:** Axiom G5 (energy-momentum spectral condition) combined with the modal decomposition of g yields:

$$\delta S_{\text{relation}} / \delta g^{\mu\nu} = 0$$

where S_{relation} is an action functional constructed from the singular values σ_k and Fourier modes. Variation gives $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. □

4.4 Example: Schwarzschild Metric from Relational Data

Consider a spherically symmetric causal matrix C_{ij} with singular values σ_k. In the continuum limit, the SVD modes u_k, v_k become spherical harmonics. The reconstructed metric takes the form:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

The mass parameter M emerges from the asymptotic decay of singular values: $\sigma_k \sim 1/k^{3/2}$.

5 Unification and Physical Interpretation

5.1 The Relational Paradigm

Our framework realizes the philosophical principle "relations precede existence" through:

1. **Primitive Structure:** Only events and causal relations (category $\mathcal{GR}$)
2. **Emergent Geometry:** Spacetime metric g from SVD of causal matrices
3. **Emergent Dynamics:** Einstein equations from spectral-topological consistency
4. **Emergent Observers:** Local coordinate systems as representation functors

5.2 Comparison with Existing Approaches

- **Zeeman's Theorem:** Our approach extends Zeeman's causality-preserving maps [3] to the nonlinear regime via local linearization.
- **Causal Set Theory:** Similar discrete starting point, but we add spectral and topological axioms to recover continuum geometry.
- **Emergent Gravity:** Unlike condensed matter analogs, emergence here is from pure relational data without presupposing any background.
- **Category Theory in Physics:** Goes beyond applications to quantum theory [1] to reconstruct classical spacetime.

5.3 Predictions and Novel Features

1. **Discrete-Continuum Correspondence:** The scaling of singular values σ_k with k predicts the dimension and curvature of emergent spacetime.
2. **Quantum Gravity Pathway:** Quantizing the causal matrix C rather than g suggests a natural route to quantum gravity.
3. **Modified Gravity:** Relaxing axiom G4 allows for topological defects corresponding to extended gravity theories.
4. **Testable Signatures:** Discrete remnants at Planck scale could affect high-energy astrophysical observations.

6 Conclusions and Outlook

We have presented a complete axiomatic reconstruction of special and general relativity from purely relational principles. By integrating categorical, spectral, and topological methods, we have shown that:

- Spacetime geometry emerges from the singular value decomposition of causal matrices
- Lorentz symmetry emerges from spectral consistency conditions

- Einstein field equations emerge from topological coordination requirements

This work provides a rigorous mathematical foundation for relational physics and suggests several promising directions for future research:

6.1 Quantum Gravity

Quantizing the causal matrix C leads naturally to a theory of quantum spacetime. The singular values σ_k become quantum operators, and the Einstein equations emerge in the classical limit.

6.2 Unification with Quantum Theory

Extending the categorical framework to include quantum events (superpositions of causal relations) may unify general relativity with quantum mechanics in a relational setting.

6.3 Mathematical Extensions

- Higher categorical structures for higher-dimensional gravity
- Non-commutative Fourier transforms for non-commutative geometries
- Derived categories for quantum gravitational corrections

6.4 Empirical Tests

- Planck-scale discreteness in high-energy astrophysical data
- Modified dispersion relations from spectral axioms
- Topological constraints on cosmic structure

Our framework demonstrates that physics may fundamentally concern not what exists, but how relations are coordinated. Spacetime and its laws emerge as the necessary language for consistent communication between all possible observers.

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A Mathematical Details

A.1 SVD Convergence to Continuum Geometry

Theorem 15 (SVD Convergence)**.** *Let $\{p_i^{(n)}\}_{i=1}^n$ be a sequence of event sets in M becoming dense as $n \to \infty$. Let $C^{(n)}$ be the corresponding causal matrices with SVD*

$C^{(n)} = U^{(n)}\Sigma^{(n)}(V^{(n)})^T$. *Then there exists a Lorentzian metric g on M such that:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{4} \sigma_k^{(n)} u_k^{(n)} (v_k^{(n)})^T = g$$

in the sense of Gromov-Hausdorff convergence of metric spaces.

A.2 Sheaf Cohomology and Einstein Equations

Proposition 16 (Cohomological Interpretation)**.** *The Einstein field equations are equivalent to the condition that the Čech cocycle $\{\psi_{ij}\}$ defined by coordinate transitions satisfies:*

$$[\psi] = 0 \in H^1(M, \mathrm{SO}^+(1,3))$$

where $\mathrm{SO}^+(1,3)$ is the sheaf of local Lorentz transformations.

A.3 Fourier Spectral Conditions

Lemma 17 (Spectral Support)**.** *The Fourier transform of a causal structure satisfies:*

$$\mathrm{supp}(\tilde{g}_{\mu\nu}) \subseteq \{(k,x) : \eta^{\mu\nu} k_\mu k_\nu \geq 0\}$$

if and only if the original structure satisfies causality (no superluminal propagation).

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