The Study on the Velocity Ratio of Binary Black Holes with Galactic Potential

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1.1 Background & Motivation

Background:

1. Supermassive black holes are typically found at galaxy centers and can give rise to binary black holes during galaxy mergers.
2. Double-peaked $[O_{III}]$ profiles in AGNs suggest dual AGN existence, yet a reliable diagnostic for their identification is currently unavailable.

1.1 Background & Motivation

Motivation:

1. Wang et al.[1] found a significant correlation between the velocity ratio and mass ratio of dual AGN, explained by Keplerian diagram.
2. Jiang and Yeh[2] proposed a theoretical framework for a supermassive binary black hole in the galactic center, using the Kandrup potential model[3].

[1] Wang et al., "Active galactic nuclei with double-peaked narrow lines: are they dual active galactic nuclei?"

[2] Jiang and Yeh, "Galaxies with supermassive binary black holes:(I) a possible model for the centers of core galaxies".

[3] Kandrup et al., "Supermassive black hole binaries as galactic blenders".

1.1 Background & Motivation

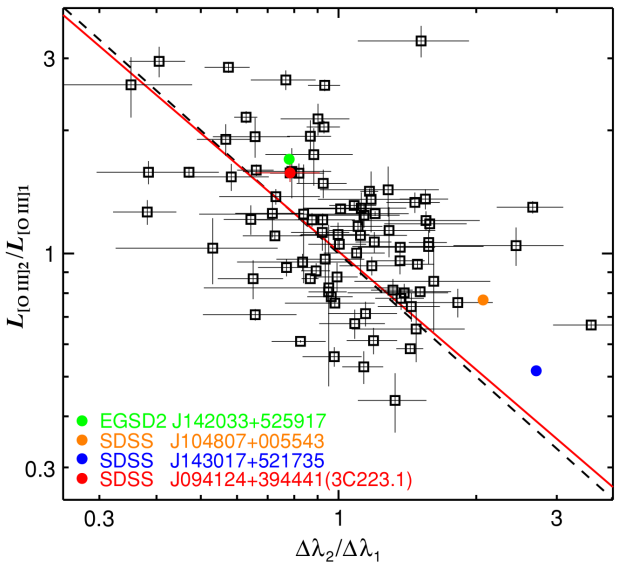


Figure: The black squares are samples of Wang et al. The red solid line is the best-fit linear correlation given in Eq.(1). The dashed line is $L_{[O_{III}]_1}/L_{[O_{III}]_2} = \epsilon_1/\epsilon_2 \cdot \Delta\lambda_2/\Delta\lambda_1$ by setting $\epsilon_1/\epsilon_2 = 1$. ϵ_1 and ϵ_2 are Eddington ratios.

$$\log\left(\frac{L_{[O_{III}]_2}}{L_{[O_{III}]_1}}\right) = (0.005 \pm 0.010) - (0.960 \pm 0.11)\log\left(\frac{\Delta\lambda_2}{\Delta\lambda_1}\right) \quad (1)$$

1.2 Objective & Procedure

The specific steps are as follows:

1. **Explore the relationship between velocity ratio and mass ratio.**
2. **Reflect the impact of galactic potential by modifying the initial conditions.**

2.1 Circular Orbit

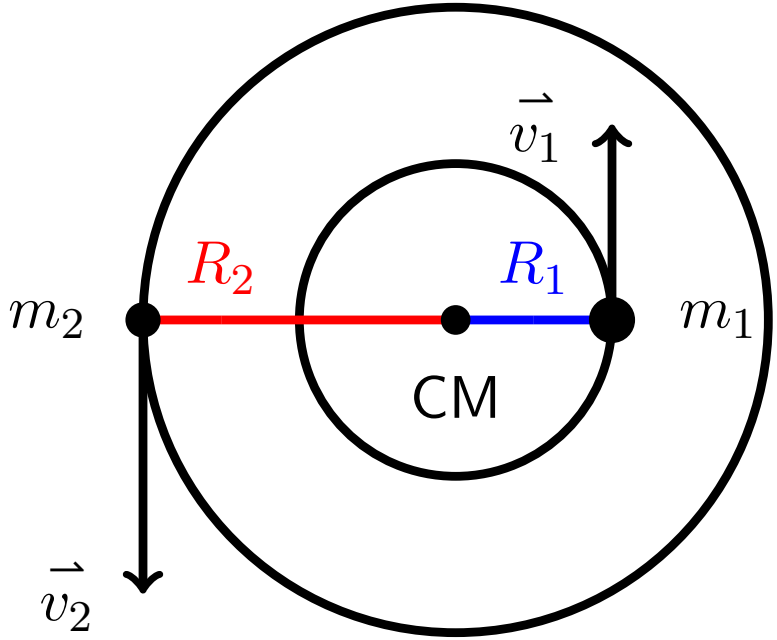


Figure: Binary system with circular orbits. Suppose that there are two particles P_1, P_2, of masses, m_1, m_2, orbiting their common center of mass at distances R_1, R_2 with velocities $\vec{v_1}$ and $\vec{v_2}$ respectively.

2.1 Circular Orbit

Theorem 2.1

From definition of center of mass (CM), we obtain

$$m_1 R_1 = m_2 R_2.$$

Definition 2.2 (Newton's law of gravity)

Let $R = R_1 + R_2$.

$$|\mathbf{F}_1| = |\mathbf{F}_2| = \frac{G m_1 m_2}{R^2},$$

where G is the universal gravitational constant.

2.1 Circular Orbit

Let $R = R_1 + R_2$. From Theorem 2.1, we get

$$R_1 = \frac{m_2}{m_1 + m_2} R, \quad \text{and} \quad R_2 = \frac{m_1}{m_1 + m_2} R.$$

Moreover, we obtain

$$m_1 = \frac{R_2}{R}(m_1 + m_2) \quad \text{and} \quad m_2 = \frac{R_1}{R}(m_1 + m_2). \tag{2}$$

2.1 Circular Orbit

From Definition 2.2 and Eq.(2), we obtain

$$|\mathbf{F}_1| = \frac{Gm_1m_2}{R^2} = m_1 \cdot \frac{G((m_1+m_2)R_1/R)}{R^2} = m_1 \cdot R_1 \frac{G(m_1+m_2)}{R^3}, \tag{3}$$

and

$$|\mathbf{F}_2| = \frac{Gm_1m_2}{R^2} = m_2 \cdot \frac{G((m_1+m_2)R_2/R)}{R^2} = m_2 \cdot R_2 \frac{G(m_1+m_2)}{R^3}. \tag{4}$$

Since $|\mathbf{F}_1| = |\mathbf{F}_2|$, we can get

$$\frac{m_1}{m_2} = \frac{R_2}{R_1}. \tag{5}$$

2.1 Circular Orbit

From Eq.(3) and Eq.(4), the angular velocity of the binary system ω, is

$$\omega = \sqrt{\frac{G(m_1 + m_2)}{R^3}}.$$

Then, let $\vec{v}$ be the velocity of the binary system using the relation $|\vec{v}| = \omega R$. Then the period T of binary system is

$$T = \frac{2\pi R}{|\vec{v}|} = \frac{2\pi R}{\omega R} = \frac{2\pi}{\omega}. \tag{6}$$

2.1 Circular Orbit

From Eq.(6), the circumferences of P_1 and P_2 are

$$2\pi R_1 = |\vec{v_1}|T, \quad \text{and} \quad 2\pi R_2 = |\vec{v_2}|T.$$

Then we obtain

$$|\vec{v_1}| = \frac{2\pi R_1}{T}, \quad \text{and} \quad |\vec{v_2}| = \frac{2\pi R_2}{T}. \tag{7}$$

2.1 Circular Orbit

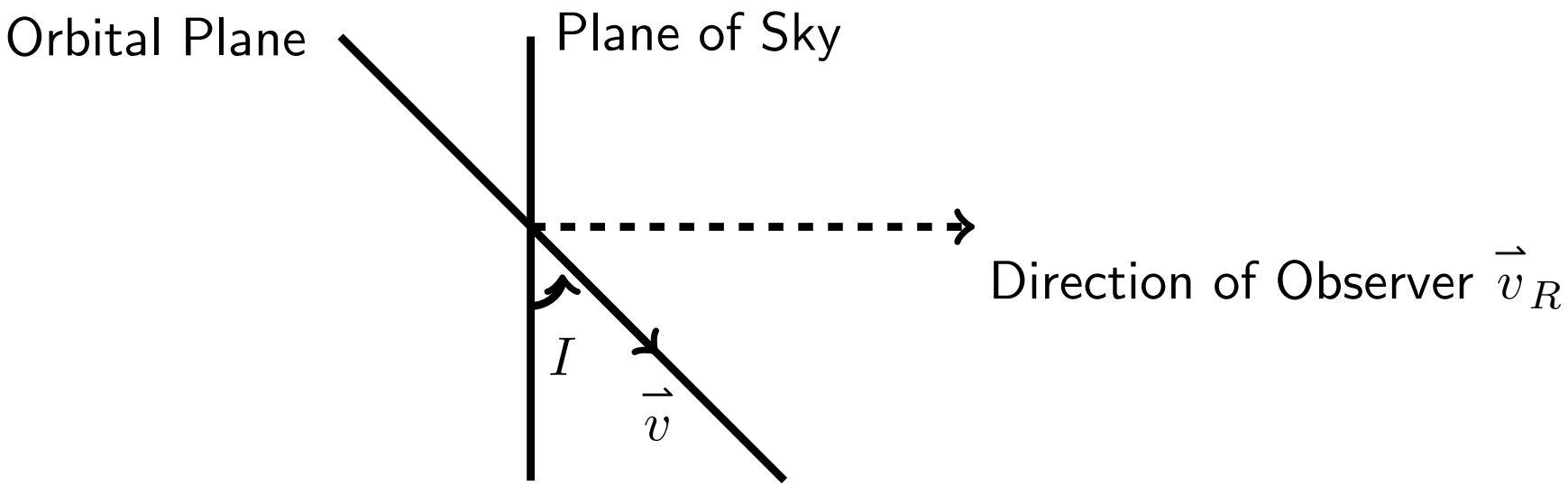


Figure: Inclination of an orbit.

From the Figure,

$$\vec{v}_R = \vec{v}\cos(90^o - I) = \vec{v}\sin(I).$$

2.1 Circular Orbit

Hence, the line of direction of observes of $\vec{v}_1$ and $\vec{v}_2$ are $\vec{v}_{R_1}$ and $\vec{v}_{R_2}$, is below :

$$\vec{v}_{R_1} = \vec{v}_1 \sin(I), \quad \text{and} \quad \vec{v}_{R_2} = \vec{v}_2 \sin(I). \tag{8}$$

Therefore, from Eq.(5), Eq.(7), and Eq.(8), we get

$$\frac{|\vec{v}_{R_1}|}{|\vec{v}_{R_2}|} = \frac{|\vec{v}_1 \sin(I)|}{|\vec{v}_2 \sin(I)|} = \frac{2\pi R_1/T}{2\pi R_2/T} = \frac{R_1}{R_2} = \frac{m_2}{m_1},$$

which could know that the relation between the observe of orbital velocity ratio and the mass ratio of binary system, is

$$\frac{|\vec{v}_{R_1}|}{|\vec{v}_{R_2}|} = \frac{m_2}{m_1}.$$

2.2 Elliptical Orbit

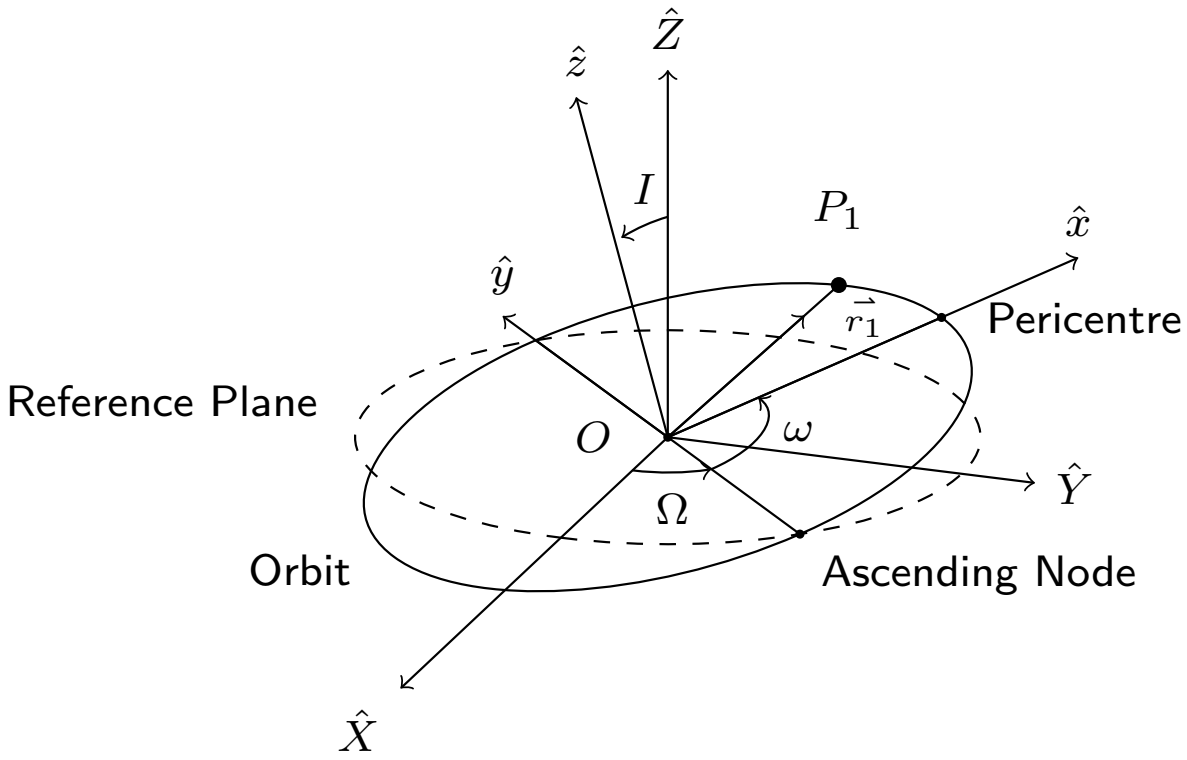


Figure: Orbital motion and reference plane in three-dimensional space.

2.2 Elliptical Orbit

Suppose that $(\hat{x}, \hat{y}, \hat{z})$ is the orbital plane system and $(\hat{X}, \hat{Y}, \hat{Z})$ is the general reference plane system. Let the reference line in the reference plane be $\hat{X}$-axis.

Assumption of angles:

1. Inclination of the orbit: $I \in [0, \pi]$.
2. Longitude of ascending node: Ω.
3. Argument of pericentre: ω.

Let the longitude of pericentre be ϖ, i.e.

$$\varpi = \Omega + \omega, \quad \text{as} \quad I \to 0.$$

2.2 Elliptical Orbit

Theorem 2.3 (Kepler's 1st Law)

Let star-planet radius vector denote $\vec{r}$. Then, in polar coordinate system,

$$|\vec{r}| = \frac{p}{1 + e\cos(\theta - \varpi)},$$

where e, ϖ, and p are arbitrary constants.

As $I \to 0$ and the orbit is elliptical, we get

$$|\vec{r_1}| = \frac{a(1 - e^2)}{1 + e\cos(\theta - \varpi)}, \tag{9}$$

where e is the eccentricity and θ is called the true longitude, which is the angle between the reference line and the radius vector.

2.2 Elliptical Orbit

Suppose that there are two particles P_1, P_2 with mass m_1, m_2, and with position vectors $\vec{r}_1$, $\vec{r}_2$ referred to some origin O fixed in inertial space. Furthermore, let orbital radius vector $\vec{r} = \vec{r}_2 - \vec{r}_1$ denotes the relative position of the particle P_2 with respect to P_1.

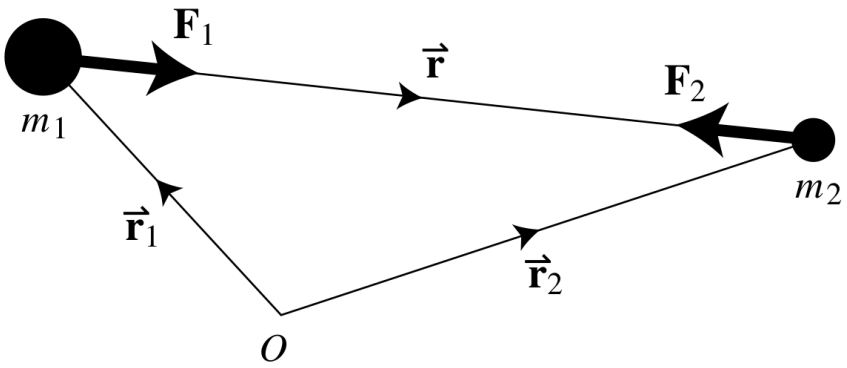


Figure: The vector diagram for P_1 and P_2.

2.2 Elliptical Orbit

Let $f = \theta - \varpi$. Then

$$r = |\vec{r}| = \frac{a(1-e^2)}{1+e\cos(f)}. \tag{10}$$

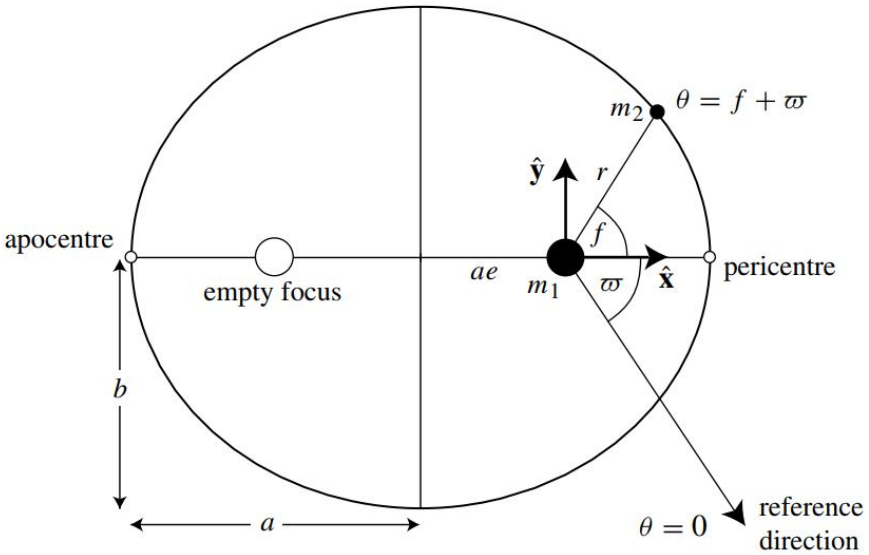


Figure: Orbital motion as $I \to 0$.

2.2 Elliptical Orbit

We can obtain the value of angular momentum of $\vec{r}$ that

$$|\vec{h}| = |\vec{r}|^2\dot{\theta}.$$

Moreover, since $\dot{f} = \dot{\theta}$,

$$|\vec{h}| = |\vec{r}|^2\dot{f}. \tag{11}$$

From proof of Kepler's 1st Law, we know that $p = |\vec{h}|^2/\mu$ and from Eq.(9) , we have $p = a(1 - e^2)$. Then we get

$$|\vec{h}|^2 = p\mu = a(1 - e^2)\mu, \tag{12}$$

where $\mu = G(m_1 + m_2)$.

2.2 Elliptical Orbit

Theorem 2.4 (Kepler's 3rd Law)

Consider a elliptical orbital planet motion and its orbital period is T. If a is the semi-major axis of the elliptical orbit, then

$$T^2 = \frac{4\pi^2}{\mu} a^3,$$

where $\mu = G(m_1 + m_2)$.

We define average angular velocity $n = 2\pi/T$. Then

$$\mu = \frac{4\pi^2}{T^2} a^3 = n^2 a^3. \tag{13}$$

2.2 Elliptical Orbit

From Eq.(13), we rewrite Eq.(12) to become

$$|\vec{h}|^2 = a(1-e^2)\mu = n^2a^4(1-e^2).$$

Thus, we have

$$|\vec{h}| = na^2\sqrt{1-e^2}. \tag{14}$$

From Eq.(10), Eq.(11), and Eq.(14), we derive that

$$\begin{aligned} |\vec{r}|\dot{f} &= \frac{|\vec{h}|}{|\vec{r}|} = na^2\sqrt{1-e^2}\cdot\frac{1}{|\vec{r}|} \\ &= na^2\sqrt{1-e^2}\cdot\frac{1+e\cos(f)}{a(1-e^2)} = \frac{na(1+e\cos(f))}{\sqrt{1-e^2}}. \end{aligned} \tag{15}$$

2.2 Elliptical Orbit

We differentiate Eq.(10) with respect to the time t, then

$$|\dot{\vec{r}}| = \frac{-a(1-e^2)(-e\sin(f))}{(1+e\cos(f))^2}\dot{f} = |\vec{r}|\dot{f}\frac{e\sin(f)}{1+e\cos(f)}. \tag{16}$$

Moreover, since the result of Eq.(15), we rewrite Eq.(16) to become

$$\begin{aligned} |\dot{\vec{r}}| &= |\vec{r}|\dot{f}\frac{e\sin(f)}{1+e\cos(f)} \\ &= \frac{na(1+e\cos(f))}{\sqrt{1-e^2}}\cdot\frac{e\sin(f)}{1+e\cos(f)} = \frac{nae\sin(f)}{\sqrt{1-e^2}}. \end{aligned} \tag{17}$$

2.2 Elliptical Orbit

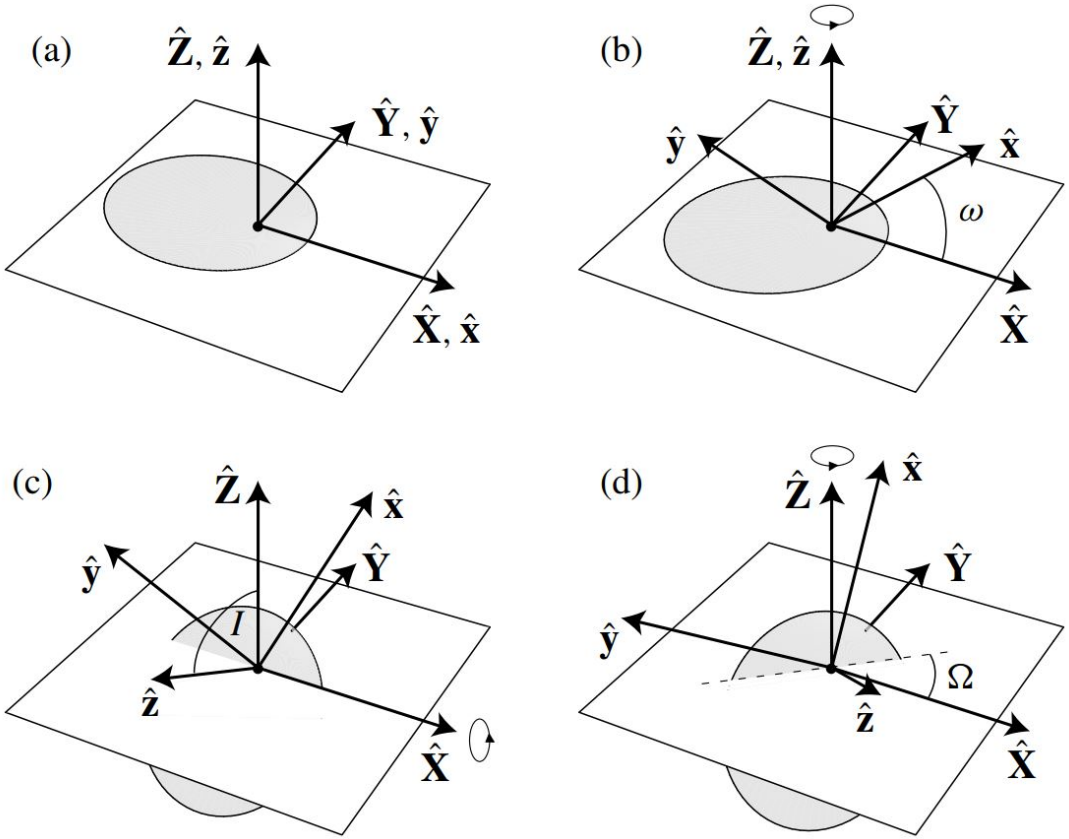


Figure: The rotation process.

2.2 Elliptical Orbit

(b) :

$$\mathbf{P}_1 = \begin{bmatrix} \cos(\omega) & -\sin(\omega) & 0 \\ \sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) :

$$\mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(I) & -\sin(I) \\ 0 & \sin(I) & \cos(I) \end{bmatrix}$$

(d) :

$$\mathbf{P}_3 = \begin{bmatrix} \cos(\Omega) & -\sin(\Omega) & 0 \\ \sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.2 Elliptical Orbit

Now, we consider the particle P_2 position at orbital plane in Cartesian system, $(\vec{x}, \vec{y}, \vec{z}) = (|\vec{r}|\cos(f), |\vec{r}|\sin(f), 0)$, then the position of P_2 at reference plane is

$$\begin{aligned}\begin{pmatrix}\vec{X}\\ \vec{Y}\\ \vec{Z}\end{pmatrix} &= \mathbf{P}_3\mathbf{P}_2\mathbf{P}_1\begin{pmatrix}\vec{x}\\ \vec{y}\\ \vec{z}\end{pmatrix} = \mathbf{P}_3\mathbf{P}_2\mathbf{P}_1\begin{pmatrix}\vec{r}\cos(f)\\ \vec{r}\sin(f)\\ 0\end{pmatrix}\\ &= \vec{r}\begin{pmatrix}\cos(\Omega)\cos(\omega+f)-\sin(\Omega)\sin(\omega+f)\cos(I)\\ \sin(\Omega)\cos(\omega+f)+\cos(\Omega)\sin(\omega+f)\cos(I)\\ \sin(\omega+f)\sin(I)\end{pmatrix}.\end{aligned} \tag{18}$$

2.2 Elliptical Orbit

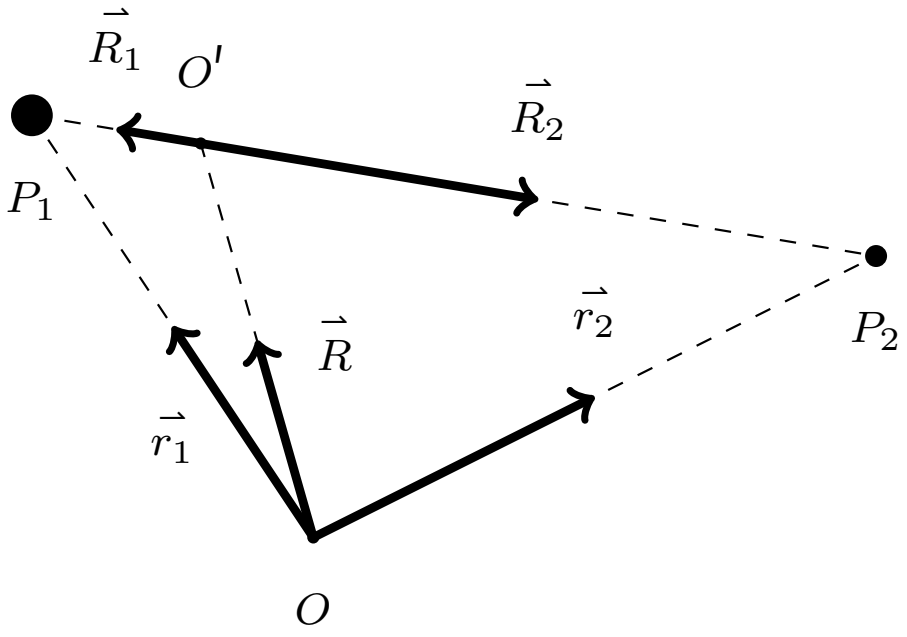


Figure: The position vector of the two particles P_1, P_2 with mass m_1, m_2 respectively, with respect to the origin, O, and with respect to the centre of mass, O'.

2.2 Elliptical Orbit

Let $\vec{R}$ denote the position vector of the centre of mass O', referred to the fixed origin O. Then, the relation of the vector $\vec{R}$ could be written as

$$m_1\vec{r_1} + m_2\vec{r_2} - (m_1 + m_2)\vec{R} = 0, \tag{19}$$

and we define

$$\vec{R_1} = \vec{r_1} - \vec{R}, \quad \text{and} \quad \vec{R_2} = \vec{r_2} - \vec{R}. \tag{20}$$

From Eq.(19) and Eq.(20), we get

$$m_1\vec{R_1} + m_2\vec{R_2} = 0. \tag{21}$$

2.2 Elliptical Orbit

Assume $r = |\vec{R_1}| + |\vec{R_2}|$ and from Eq.(19)-(21), we obtain

$$\begin{aligned} |\vec{R_1}| &= \frac{m_2}{m_1}|\vec{R_2}| = \frac{m_2}{m_1}|\vec{r_2} - \vec{R}| = \frac{m_2}{m_1}|\vec{r_2} - \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}| \\ &= \frac{m_2}{m_1} \cdot \frac{m_1}{m_1 + m_2}|\vec{r_2} - \vec{r_1}| = \frac{m_2}{m_1 + m_2}r. \end{aligned} \tag{22}$$

Similarly,

$$|\vec{R_2}| = \frac{m_1}{m_2}|\vec{R_1}| = \frac{m_1}{m_2} \cdot \frac{m_2}{m_1 + m_2}r = \frac{m_1}{m_1 + m_2}r. \tag{23}$$

Thus, from Eq.(22) and Eq.(23), we could have

$$\vec{R_1} = -\frac{m_2}{m_1 + m_2}\vec{r}, \quad \text{and} \quad \vec{R_2} = +\frac{m_1}{m_1 + m_2}\vec{r}. \tag{24}$$

2.2 Elliptical Orbit

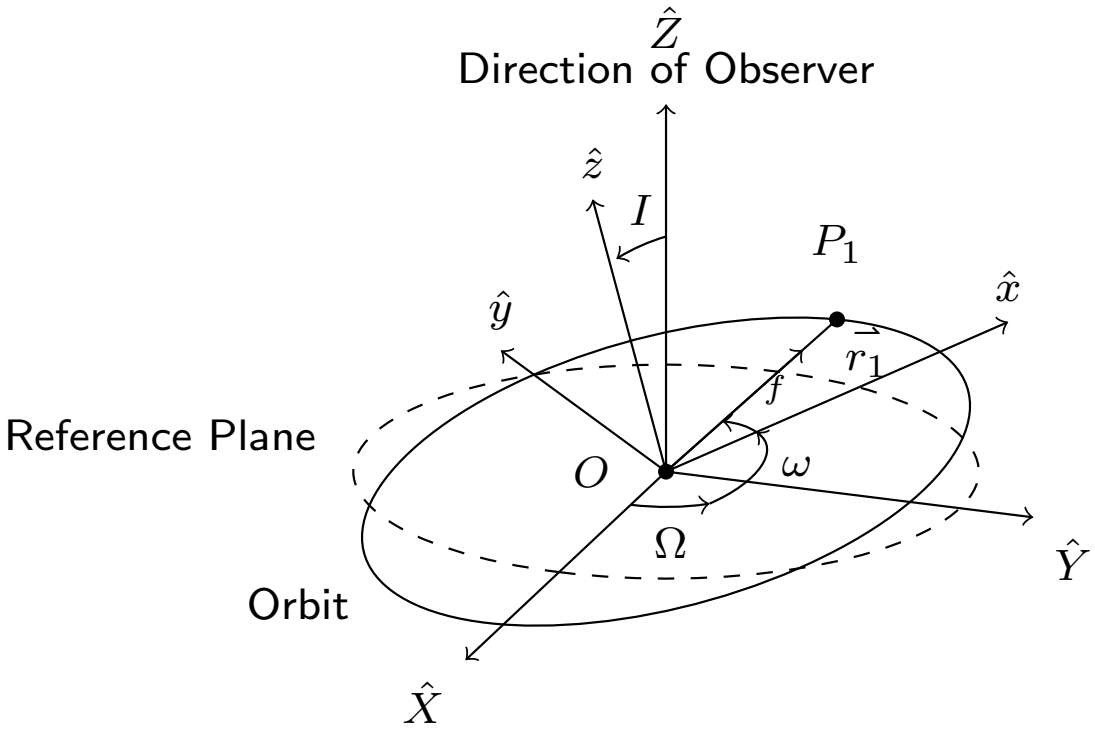


Figure: CM coordinate system of P_1 in three-dimensional space.

2.2 Elliptical Orbit

We consider the observer take the reference plane $(\hat{X}, \hat{Y})$ to be the plane of the sky perpendicular to the line of sight, the $\hat{Z}$-axis oriented towards the observer.

The velocity of particle P_1 projection on the $\hat{Z}$-axis, $\vec{v}_{1,\hat{z}}$, is presented[4] as below,

$$\vec{v}_{1,\hat{z}} = \frac{d}{dt}(\vec{r_1} \cdot \hat{Z}) = \dot{\vec{r_1}} \cdot \hat{Z}. \tag{25}$$

[4] Murray and Correia, "Keplerian Orbits and Dynamics of Exoplanets".

2.2 Elliptical Orbit

From Eq.(20), we rewrite Eq.(25) to be

$$\vec{v}_{1,\hat{z}} = (\dot{\vec{R}} + \dot{\vec{R}}_1) \cdot \hat{Z}. \tag{26}$$

Let $V_{\hat{Z}} = \dot{\vec{R}} \cdot \hat{Z}$. Then, from Eq.(26),

$$\vec{v}_{1,\hat{z}} = V_{\hat{Z}} + (\dot{\vec{R}}_1 \cdot \hat{Z}). \tag{27}$$

From Eq.(18) and Eq.(24),

$$\begin{aligned} \dot{\vec{R}}_1 \cdot \hat{Z} &= -\frac{d}{dt}\left(\frac{m_2}{m_1 + m_2}\vec{r} \cdot \hat{Z}\right) \\ &= -\frac{d}{dt}\left(\frac{m_2}{m_1 + m_2} r \sin(\omega + f) \sin(I)\right). \end{aligned} \tag{28}$$

2.2 Elliptical Orbit

Then, using the result of Eq.(28), Eq.(27) could be

$$\begin{aligned}\vec{v}_{1,\hat{z}} &= V_{\hat{Z}} - \frac{d}{dt}\left(\frac{m_2}{m_1+m_2} r \sin(\omega+f)\sin(I)\right) \\ &= V_{\hat{Z}} - \frac{m_2 \sin(I)}{m_1+m_2}\left(\dot{r}\sin(\omega+f) + r\dot{f}\cos(\omega+f)\right).\end{aligned} \tag{29}$$

2.2 Elliptical Orbit

Hence, from Eq.(15), Eq.(17), and Eq.(29),

$$\begin{aligned}\vec{v}_{1,\hat{z}} &= V_{\hat{Z}} - \frac{m_2 \sin(I)}{m_1 + m_2}(\dot{r}\sin(\omega + f) + r\dot{f}\cos(\omega + f)) \\ &= V_{\hat{Z}} - \frac{m_2 \sin(I)}{m_1 + m_2}\frac{na}{\sqrt{1-e^2}}(e\sin(f)\sin(\omega + f) \\ &\quad + (1 + e\cos(f))\cos(\omega + f)) \\ &= V_{\hat{Z}} - \frac{m_2 \sin(I)}{m_1 + m_2}C,\end{aligned} \tag{30}$$

where

$$C = \frac{na}{\sqrt{1-e^2}}(e\cos(\omega) + \cos(\omega + f)).$$

2.2 Elliptical Orbit

Similarly, we have the velocity of particle P_2, and denoted $\vec{v}_{2,\hat{z}}$, that is,

$$\vec{v}_{2,\hat{z}} = V_{\hat{Z}} + \frac{m_1 \sin(I)}{m_1 + m_2} C, \tag{31}$$

where

$$C = \frac{na}{\sqrt{1-e^2}} (e\cos(\omega) + \cos(\omega + f)).$$

2.2 Elliptical Orbit

Suppose that the proper motion of the barycentric velocity is zero, that is, $V_{\hat{Z}} = 0$. Then, from Eq.(30) and Eq.(31), the value of the radial velocity ratio is

$$\frac{|\vec{v}_{1,\hat{z}}|}{|\vec{v}_{2,\hat{z}}|} = \frac{|V_{\hat{Z}} - \frac{m_2 \sin(I)}{m_1+m_2} C|}{|V_{\hat{Z}} + \frac{m_1 \sin(I)}{m_1+m_2} C|} = \frac{|-\frac{m_2 \sin(I)}{m_1+m_2}|}{|\frac{m_1 \sin(I)}{m_1+m_2}|} = \frac{m_2}{m_1}. \tag{32}$$

Therefore, from Eq.(32), the relation between the observe of orbital radial velocity ratio and the mass ratio of binary system in space is

$$\frac{|\vec{v}_{1,\hat{z}}|}{|\vec{v}_{2,\hat{z}}|} = \frac{m_2}{m_1}. \tag{33}$$

3.1 Equations of Motion

Remark 3.1 (Galaxical Potential)

The form of galaxical potential[5] is

$$V(r) = -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')r'^2 dr' + \int_r^\infty \rho(r')r' dr'\right], \tag{34}$$

where G is the universal gravitational constant, r is a radius of the galaxy, and ρ is a density distribution function.

[5] Jiang and Yeh, "Galaxies with supermassive binary black holes:(I) a possible model for the centers of core galaxies".

3.1 Equations of Motion

Remark 3.2 (Nuker law)

We use the Nuker law[6] as the density distribution function[7]. The Nuker law is expressed below

$$\rho(r) = \rho_c r^{-\gamma}(1 + r^{\alpha})^{(\gamma-\beta)/\alpha}, \tag{35}$$

where ρ_c, α, β, and γ are constants.

[6] Lauer et al., "The centers of early-type galaxies with HST. I. An observational survey".

[7] Jiang and Yeh, "Galaxies with supermassive binary black holes:(I) a possible model for the centers of core galaxies".

3.1 Equations of Motion

From Eq.(35), Kandrup potential model[8] was chosing $\alpha = 2$, $\beta = 4$, and $\gamma = 0$.

Then the total mass of the galaxy upto a radius r is

$$M_g = 4\pi \int_0^\infty r'^2 \rho(r') dr' = \pi^2 \rho_c \tag{36}$$

[8] Kandrup et al., "Supermassive black hole binaries as galactic blenders".

3.1 Equations of Motion

From Eq.(36), we obtain

$$\rho_c = \frac{M_g}{\pi^2}. \tag{37}$$

From Eq.(34)-(35) and Eq.(37), we could get the potential

$$\begin{aligned} V(r) &= -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')r'^2 dr' + \int_r^\infty \rho(r')r' dr'\right] \\ &= -c\frac{\arctan r}{r}, \end{aligned} \tag{38}$$

where $c = \frac{2}{\pi}GM_g$.

3.1 Equations of Motion

From the galaxical potential, we get the galaxical force as

$$F_g(r) = -\frac{dV}{dr} = c\left[\frac{1}{r(1+r^2)} - \frac{\arctan(r)}{r^2}\right]. \tag{39}$$

3.1 Equations of Motion

Suppose that supermassive binary black hole system with a separate distance $\bar{r}$. Let the location of black holes be $\vec{r_1}$, $\vec{r_2}$, and let $\bar{r}_i = |\vec{r}_i| = \sqrt{\bar{x}_i^2 + \bar{y}_i^2 + \bar{z}_i^2}$, $i = 1, 2$.

Let $\bar{r}_{12} = |\vec{r}_2 - \vec{r}_1| = \sqrt{(\bar{x}_2 - \bar{x}_1)^2 + (\bar{y}_2 - \bar{y}_1)^2 + (\bar{z}_2 - \bar{z}_1)^2}$ in three-dimensional space.

Let the mass of two black holes be $\bar{m}_1$ and $\bar{m}_2$.

3.1 Equations of Motion

After we do non-dimensionalization for our system, it can be represented as :

$$\begin{cases} \frac{dx_i}{dt} = u_i \\ \frac{dy_i}{dt} = v_i \\ \frac{dz_i}{dt} = w_i \\ \frac{du_i}{dt} = -\frac{(m_1+m_2)x_i}{r_{12}^3} + \frac{cx_i}{r_i}\left[\frac{1}{r_i(1+r_i^2)} - \frac{\arctan(r_i)}{r_i^2}\right] \\ \frac{dv_i}{dt} = -\frac{(m_1+m_2)y_i}{r_{12}^3} + \frac{cy_i}{r_i}\left[\frac{1}{r_i(1+r_i^2)} - \frac{\arctan(r_i)}{r_i^2}\right] \\ \frac{dw_i}{dt} = -\frac{(m_1+m_2)z_i}{r_{12}^3} + \frac{cz_i}{r_i}\left[\frac{1}{r_i(1+r_i^2)} - \frac{\arctan(r_i)}{r_i^2}\right]. \end{cases} \tag{40}$$

3.1 Equations of Motion

From System (40), let

$$E_i = \frac{1}{2}(u_i^2 + v_i^2 + w_i^2) - \frac{m_j^3}{r_i(m_1 + m_2)^2} - c\frac{\arctan(r_i)}{r_i} = \frac{1}{2}C_J, \quad (41)$$

where $i \neq j$, and $i, j \in \{1, 2\}$.

Then we define the total energy E_{total} to be

$$E_{total} = E_1 + E_2. \quad (42)$$

3.1 Equations of Motion

From Eq.(41), we obtain

$$(u_i^2 + v_i^2 + w_i^2) = 2E_i + \frac{2m_j^3}{r_i(m_1+m_2)^2} + c\frac{2\arctan(r_i)}{r_i} \geq 0. \quad (43)$$

Suppose that $U_i = \frac{2m_j^3}{r_i(m_1+m_2)^2} + c\frac{2\arctan(r_i)}{r_i}$, where $i \neq j$, and $i, j \in \{1, 2\}$. Then we get

$$(u_i^2 + v_i^2 + w_i^2) = 2E_i + U_i \geq 0. \quad (44)$$

Hence,

$$E_i \geq -\frac{U_i}{2}. \quad (45)$$

3.2 $c = 0$ Case with a circular orbit

Let

- $m_1 = m_2 = 0.5$.
- $\vec{r_1}(0) = (1, 0, 0)$, $\vec{r_2}(0) = (-1, 0, 0)$.
- $\omega = \frac{\pi}{6}$, $I = \frac{\pi}{4}$, and $\Omega = \frac{\pi}{6}$.

We let $\mathbf{c} = \mathbf{0}$ and the initial velocity

- $(u_1(0), v_1(0), w_1(0)) = (0, \sqrt{\frac{Gm_2}{r_1}}, 0) = (0, \sqrt{0.5}, 0)$,
- $(u_2(0), v_2(0), w_2(0)) = (0, -\sqrt{\frac{Gm_1}{r_2}}, 0) = (0, -\sqrt{0.5}, 0)$.

After rotation, the initial positions of P_1 and P_2 are

- $\vec{r_1}(0) = (0.5732, 0.7392, 0.3563)$,
- $\vec{r_2}(0) = (-0.5732, -0.7392, -0.3563)$.

3.2 $c = 0$ Case with a circular orbit

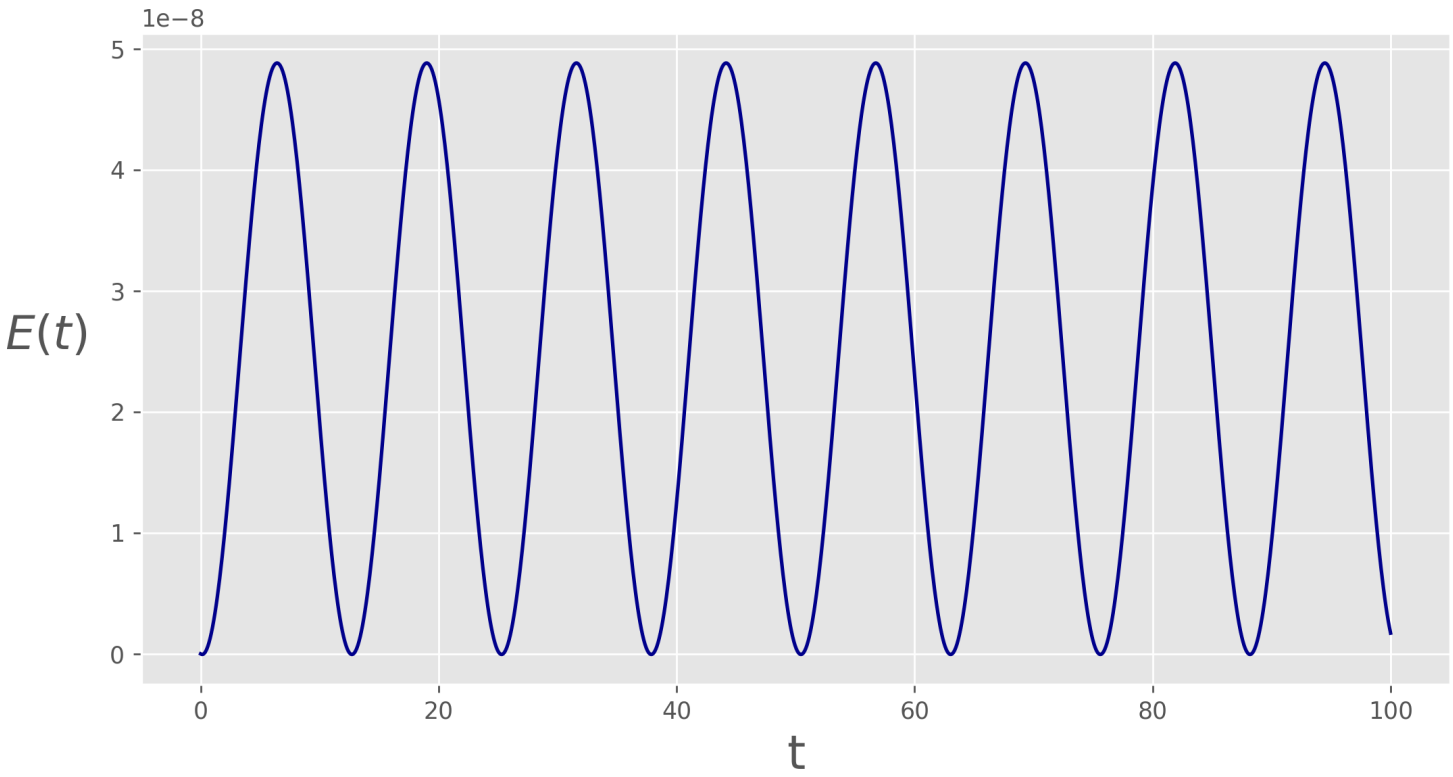
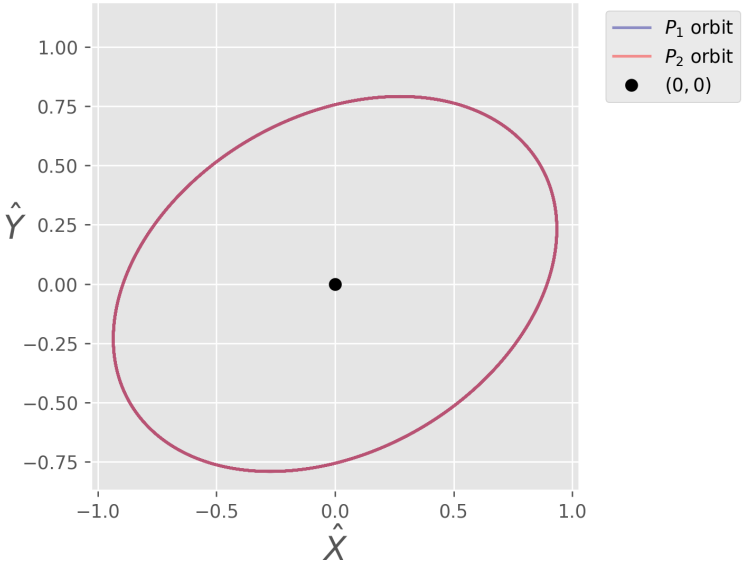
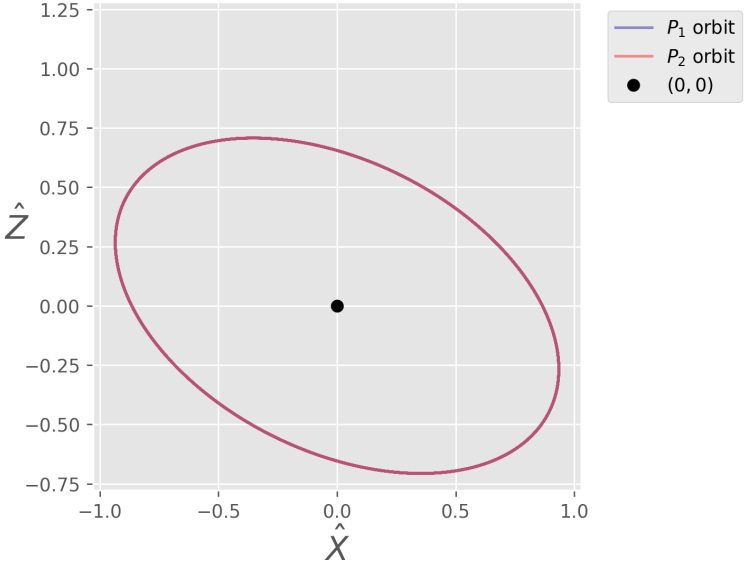


Figure: The total energy from $t = 0$ to $t = 100$.

3.2 $c = 0$ Case with a circular orbit



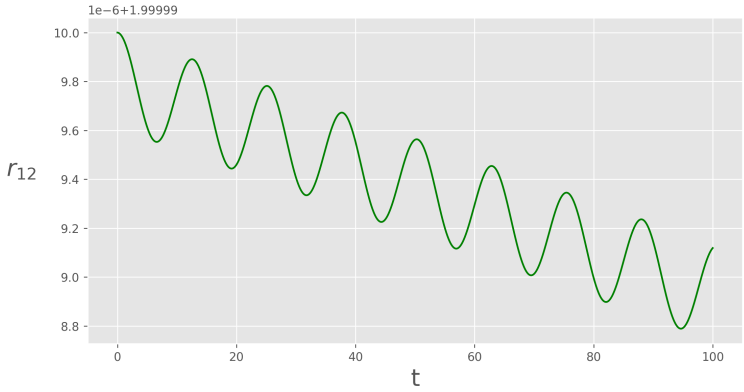
(a) The orbit on $\hat{X}\hat{Y}$-plane.



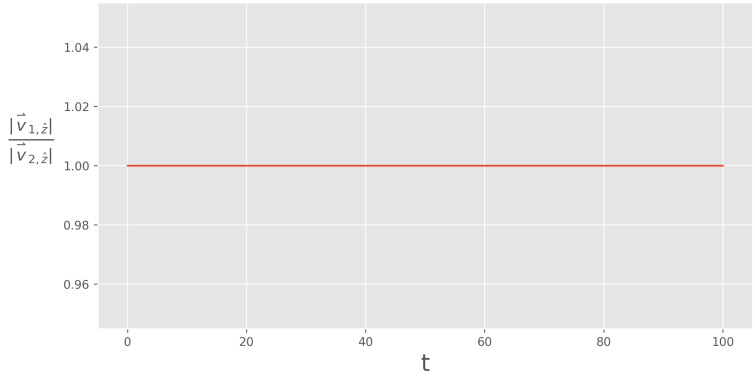
(b) The orbit on $\hat{X}\hat{Z}$-plane.

Figure: Plane orbits from $t = 0$ to $t = 100$.

3.2 $c = 0$ Case with a circular orbit



(a) The length of r_{12} from $t = 0$ to $t = 100$.



(b) The orbital radial velocity ratio from $t = 0$ to $t = 100$.

3.3 $c \neq 0$ Case with a elliptical orbit

Let $\mathbf{c} = \frac{2}{\pi}\mathbf{M_g}$ and the total mass of the galaxy[9] be

$$M_g = \frac{(m_1 + m_2)}{2} \cdot 100.$$

From Eq.(44), we let $E^c_{total} = U_1 + U_2$ with $m_1 = m_2 = 0.5$, i.e.

$$E^c_{total} = U_1 + U_2 = 0.5 + 4c \arctan(1). \tag{46}$$

Moreover, by Eq.(44), the initial velocity,

$$V_i^2(0) = u_i^2(0) + v_i^2(0) + w_i^2(0) = -\frac{1}{2}E_0.$$

[9] Milosavljević and Merritt, "Formation of galactic nuclei"

Equal mass case

Let $\mathbf{m_1} = \mathbf{m_2} = \mathbf{0.5}$ for Model A1 and Model A2.

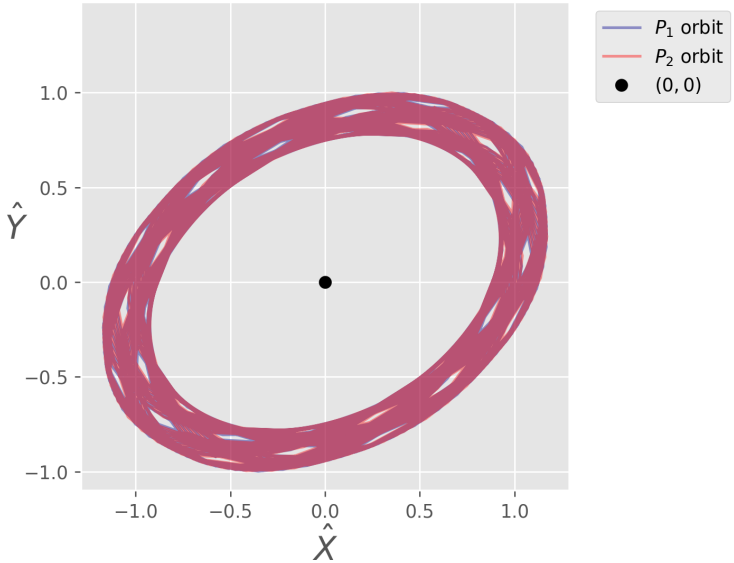
After rotation, the initial positions of P_1 and P_2 are

- $\vec{r_1}(0) = (0.5732, 0.7392, 0.3563)$,
- $\vec{r_2}(0) = (-0.5732, -0.7392, -0.3563)$.

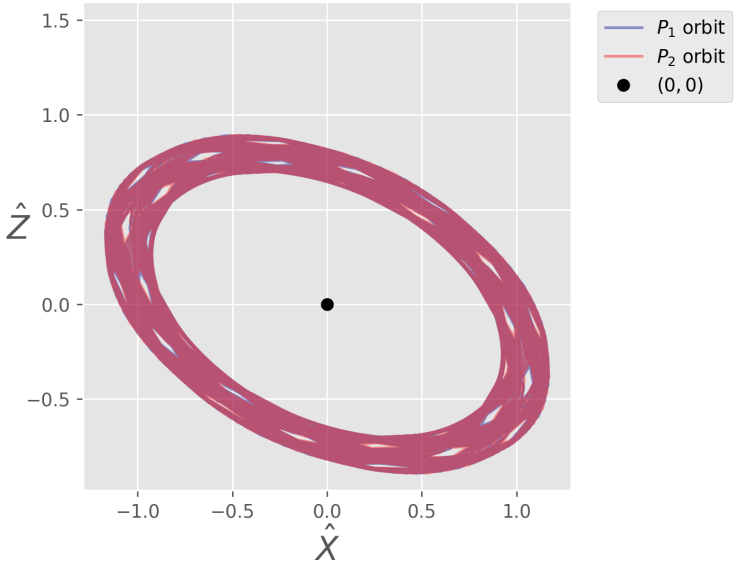
We choose initial total energy and find the initial velocities :

1. Model A1 : $E_0 = -E^c_{total}/2$, $V_i^2(0) = \frac{1}{4}E^c_{total} = 12.5625$,
2. Model A2 : $E_0 = -E^c_{total}/18$, $V_i^2(0) = \frac{1}{36}E^c_{total} = 1.3958$.

Equal mass case (Model A1)



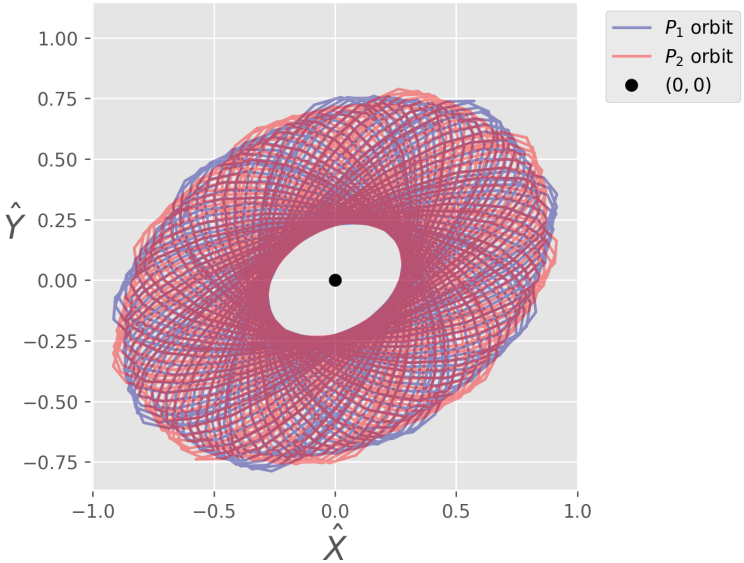
(c) The orbit on $\hat{X}\hat{Y}$-plane.



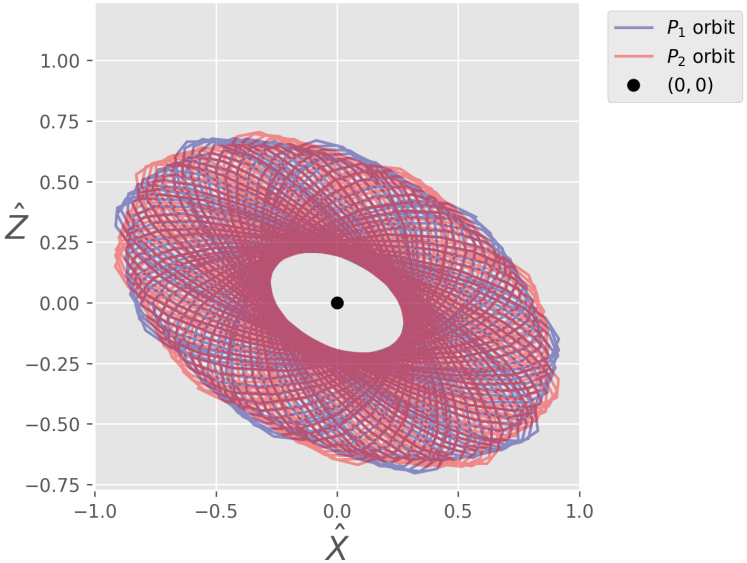
(d) The orbit on $\hat{X}\hat{Z}$-plane.

Figure: Plane orbits of Model A1 from $t = 0$ to $t = 100$.

Equal mass case (Model A2)



(a) The orbit on $\hat{X}\hat{Y}$-plane.



(b) The orbit on $\hat{X}\hat{Z}$-plane.

Figure: Plane orbits of Model A2 from $t = 0$ to $t = 100$.

Equal mass case

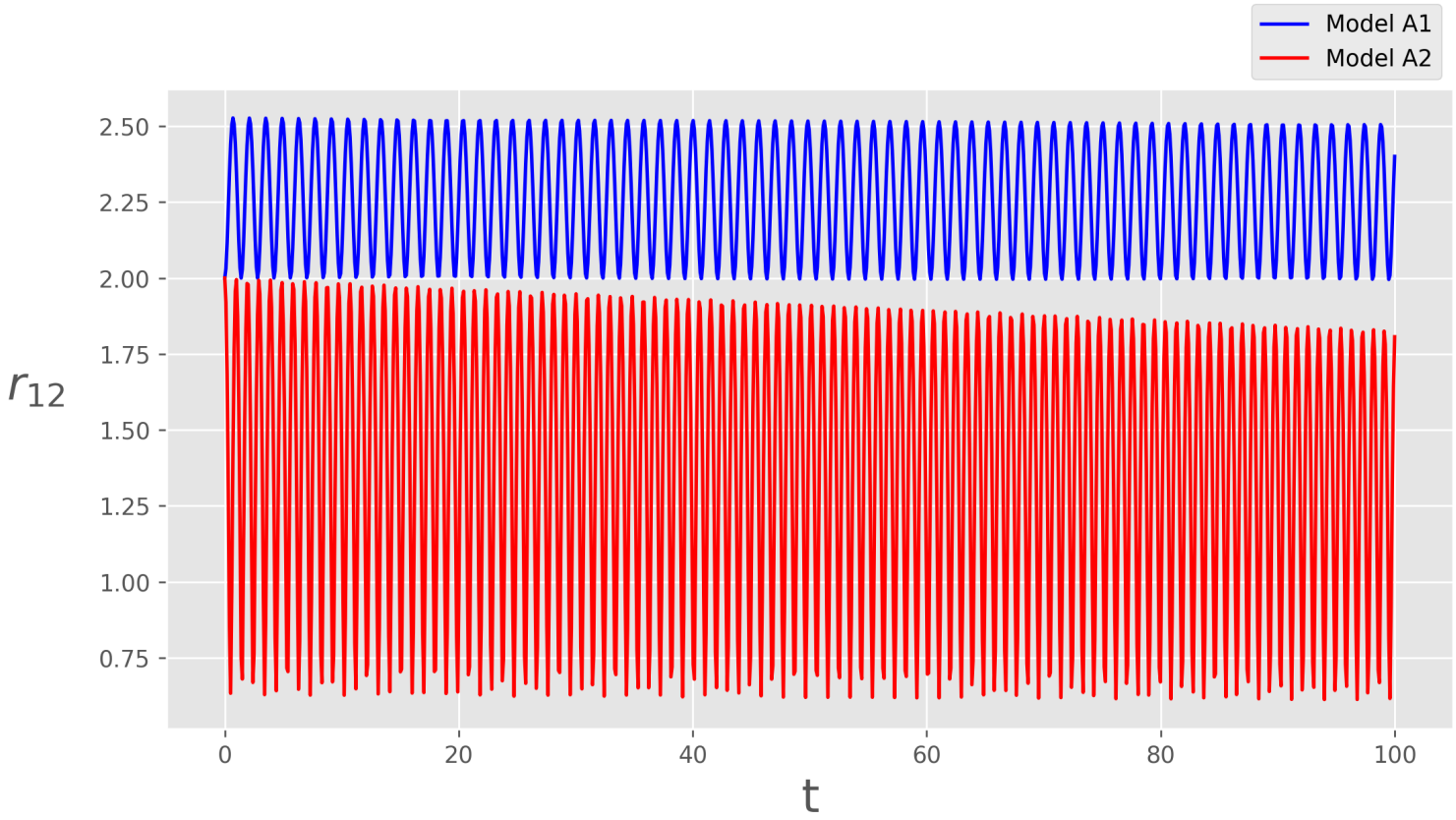


Figure: The length of r_{12} from $t = 0$ to $t = 100$.

Equal mass case

We obtain the orbital radial velocity ratio

$$|\vec{v}_{1,\hat{z}}|/|\vec{v}_{2,\hat{z}}| = 1 = m_2/m_1$$

for those two models.

Model	$\mathbf{E_0}$	$\mathbf{V_i^2(0)}$	**maximum** $|v_{1,\hat{z}}|/|v_{2,\hat{z}}|$	**minimum** $|v_{1,\hat{z}}|/|v_{2,\hat{z}}|$
Model A1	-25.1250	12.5625	1.0000	1.0000
Model A2	-2.7917	1.3958	1.0000	1.0000

Table: Model with $m_1 = m_2 = 0.5$.

Unequal masses cases

We suppose that $\mathbf{m_1} = \alpha$ and $\mathbf{m_2} = \mathbf{1} - \alpha$.

Let the positions of two particles P_1, P_2 are

- $\vec{r_1}(0) = (1 - \alpha, 0, 0)$,
- $\vec{r_2}(0) = (-\alpha, 0, 0)$.

Let rotation angles be

- $\omega = \frac{\pi}{6}$,
- $I = \frac{\pi}{4}$,
- $\Omega = \frac{\pi}{6}$.

Unequal masses cases

Let α = 0.2 for Model D1 and Model D2.

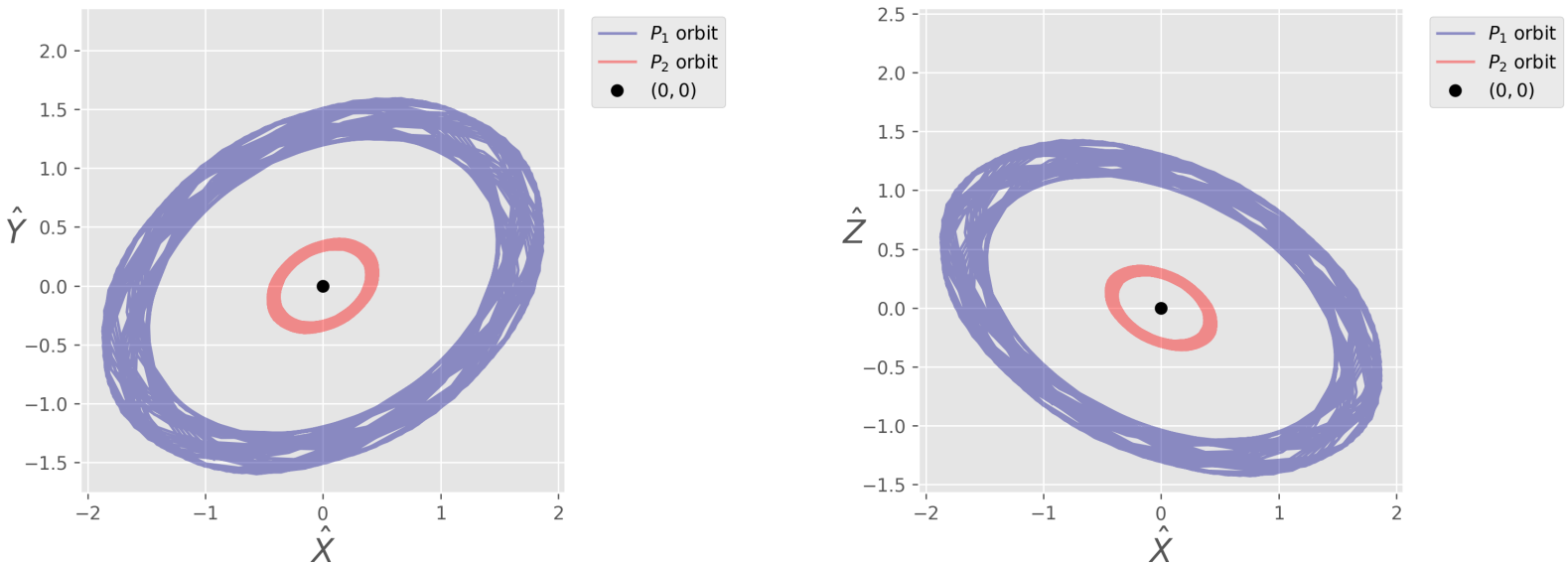
After rotation, the initial positions of two particles P_1, P_2 are

- $\vec{r_1}(0) = (0.9172, 1.1827, 0.5657)$,
- $\vec{r_2}(0) = (-0.2293, -0.2957, -0.1414)$.

We choose initial total energy and find the initial velocities :

1. Model D1 : $E_0 = -E^c_{total}/2$, $V_i^2(0) = \frac{1}{4}E^c_{total} = 12.5625$,
2. Model D2 : $E_0 = -E^c_{total}/18$, $V_i^2(0) = \frac{1}{36}E^c_{total} = 1.3958$.

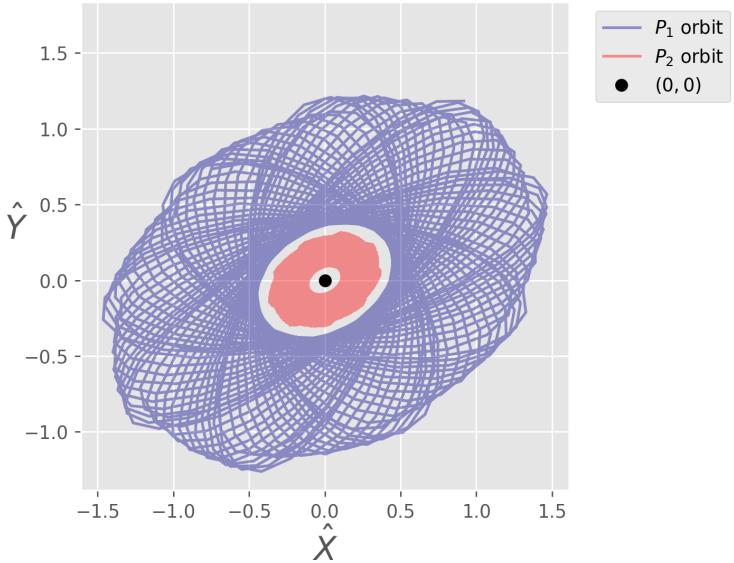
Unequal masses cases (Model D1)



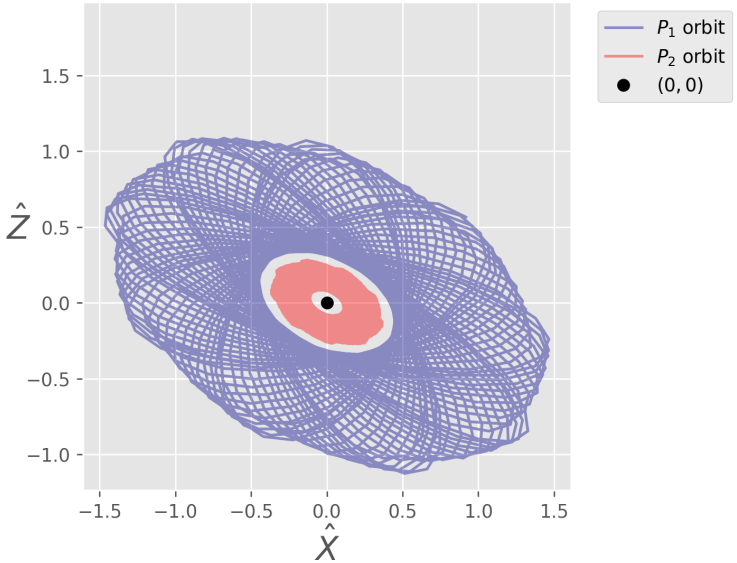
(a) The orbit on $\hat{X}\hat{Y}$-plane. (b) The orbit on $\hat{X}\hat{Z}$-plane.

Figure: Plane orbits of Model D1 from $t = 0$ to $t = 100$.

Unequal masses cases (Model D2)



(a) The orbit on $\hat{X}\hat{Y}$-plane.



(b) The orbit on $\hat{X}\hat{Z}$-plane.

Figure: Plane orbits of Model D2 from $t = 0$ to $t = 100$.

Unequal masses cases

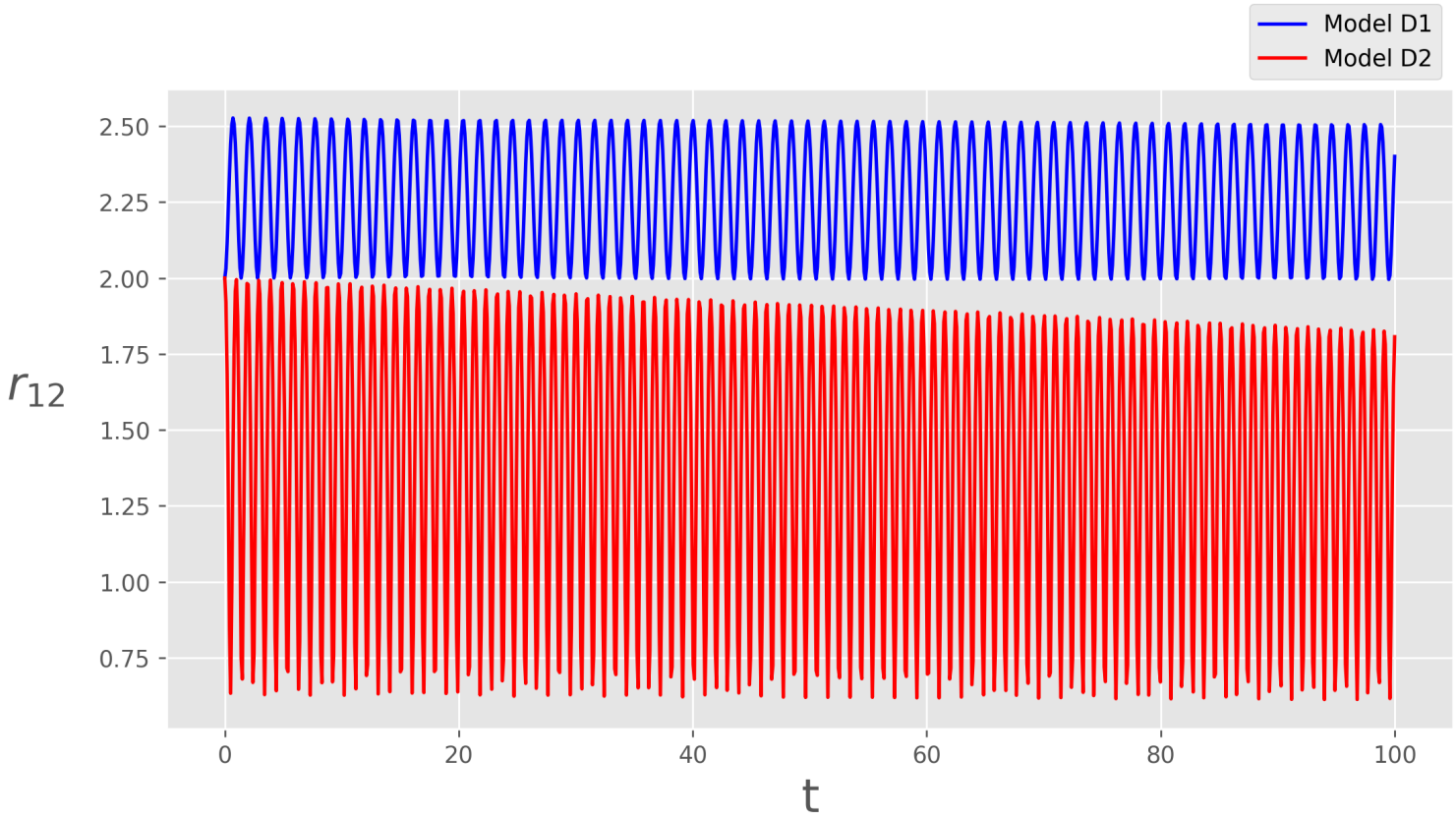
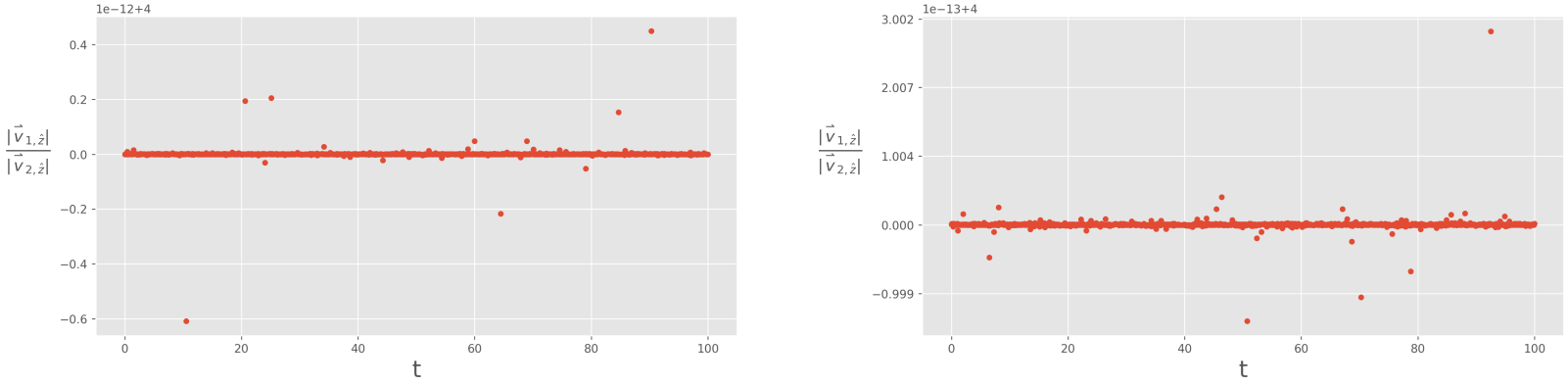


Figure: The length of r_{12} from $t = 0$ to $t = 100$.

Unequal masses cases



(a) Model D1.

(b) Model D2.

Figure: The orbital radial velocity ratio.

Unequal masses cases

Model	α	$\mathbf{E_0}$	$\mathbf{V_i^2(0)}$	**maximum** $\lvert v_{1,\hat{z}}\rvert/\lvert v_{2,\hat{z}}\rvert$	**minimum** $\lvert v_{1,\hat{z}}\rvert/\lvert v_{2,\hat{z}}\rvert$
B1	0.4	-25.1250	12.5625	$1.5 + 6.3283 \cdot 10^{-14}$	$1.5 - 7.6161 \cdot 10^{-14}$
B2		-2.7917	1.3958	$1.5 + 7.8826 \cdot 10^{-14}$	$1.5 - 1.7675 \cdot 10^{-13}$
C1	0.3	-25.1250	12.5625	$2.3333 + 2.0295 \cdot 10^{-13}$	$2.3333 - 8.8374 \cdot 10^{-14}$
C2		-2.7917	1.3958	$2.3333 + 1.5676 \cdot 10^{-13}$	$2.3333 - 7.0166 \cdot 10^{-14}$
D1	0.2	-25.1250	12.5625	$4.0 + 4.4942 \cdot 10^{-13}$	$4.0 - 6.0885 \cdot 10^{-13}$
D2		-2.7917	1.3958	$4.0 + 2.8244 \cdot 10^{-13}$	$4.0 - 1.4033 \cdot 10^{-13}$

Table: Models with $m_1 = \alpha$, $m_2 = 1 - \alpha$.

Results

Discuss the relation between velocity ratio and mass ratio in

- Circular orbit.
- Elliptical orbit.
- Circular orbit with the galactic potential.
- Elliptical orbit with the galactic potential.

We present all results that the mass ratio of the two particles exhibits an inverse proportionality to the velocity ratio.