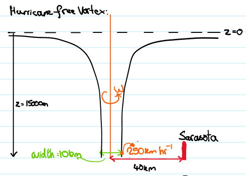
MATE60005 Problem Sheet Raw Solutions

17 February 2025 19:29

Hurricane-free Vortex:



$\omega = \frac{c}{r}$

$c \to m^2 s^{-1}$

i) Wind speed $v = 250\ km\ hr^{-1}$, $r = \frac{10}{2} = 5km$

$\therefore$ using $v = \frac{c}{r} \to c = vr = 250 \cdot 5 = 1250\ km^2 hr^{-1}$

Strength of vortex, $\pi c = 1250\pi\ km^2 hr^{-1} = \frac{1250\pi \cdot (1000)^2}{3600}\ m^2 s^{-1}$

$= 1.09 \times 10^6\ m^2 s^{-1}$ (3sf)

ii) Circulation at $r = 300km$:

Circulation $= \int_0^{2\pi r} v\, dx = 2\pi r \frac{c}{r} = 2\pi c = 2 \cdot 1.09 \times 10^6$

$= 2.18 \times 10^6\ m^2 s^{-1}$ (3sf)

(Radius independant of circulation in integration path)

iii) Wind speed $v_\theta = \frac{c}{r} = \frac{1250}{40} = 31.25 km\ hr^{-1} = \frac{31.25 \times 1000}{3600} = 8.68 ms^{-1}$ (3sf)

iv) Vorticity: $\nabla \times V$

Using polar coordinates: (v_r, v_θ, v_z) $= \frac{1}{r}\begin{vmatrix} i & rj & k \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ v_r & rv_\theta & v_z \end{vmatrix}$

Wind speed from (iii) is $v_\theta = 31.25 km\ hr^{-1} (= 8.68 ms^{-1}) = \frac{c}{r}$

$\nabla \times \underline{V} = \left(\cancel{\frac{\partial v_z}{\partial \theta}}^{0} - \cancel{\frac{\partial (v_\theta)}{\partial z}}^{0}\right)\vec{i} - r\left(\cancel{\frac{\partial v_z}{\partial r}}^{0} - \cancel{\frac{\partial v_r}{\partial z}}^{0}\right)\vec{j} + \left(\frac{\partial (rv_\theta)}{\partial r} - \cancel{\frac{\partial v_r}{\partial \theta}}^{0}\right)\vec{k}$

$\therefore \nabla \times \underline{v} = \frac{\partial (rv_\theta)}{\partial r} = \frac{\partial}{\partial r} \cdot c = 0$ $\therefore$ Vorticity $\nabla \times \underline{v} = 0$

b) i) $\frac{dp}{dz} = -\rho g = -\frac{pmg}{RT}$ $\quad \rho = \frac{pm}{RT} \to$ specific density for given T, molecular mass m and pressure p.

$pV = nRT \to p = \frac{n}{VRT}$ $\quad$ since $\rho = \frac{m}{V} \to V = \frac{m}{\rho}$ $\quad \rho = \frac{\rho n}{mRT}$

$dp = -\frac{pmg}{RT}\, dz \to \int_{p_0}^{p_1} \frac{1}{p}\, dp = -\frac{mg}{RT_0}\int_0^z \frac{1}{e^{-cz}}\, dz = -\frac{mg}{RT_0}\int_0^z e^{cz}\, dz$ since $T \approx T_0 \exp(-cz)$

$\ln\left(\frac{p}{p_0}\right) = -\frac{mg}{RT_0}\left[\frac{1}{c} \cdot \exp(cz)\right]_0^z = -\frac{mg}{RT_0 c}[\exp(cz) - 1]$

$\therefore \frac{p}{p_0} = \exp\left\{-\frac{mg}{RT_0 c}[\exp(cz) - 1]\right\} \to p = p_0 \exp\left\{-\frac{mg}{RT_0 c}[\exp(cz) - 1]\right\}$

$, T = T_0 \exp(-cz)$, $T_0 = 250K$

Sub in:

- $g = 3.71 ms^{-2}$; p_0 (surface pressure) $= 700\ Pa$; $m = 0.04401$; $R = 8.314 J mol^{-1} K^{-1}$.

Analytic Formula: $p = 700 \exp\left\{-\frac{0.04401 \cdot 3.71}{8.314 \cdot 250 \cdot 1.3 \times 10^{-5}}[\exp(1.3 \times 10^{-5} z) - 1]\right\}$

$p = 700 \exp\{-6.04[\exp(1.3 \times 10^{-5} z) - 1]\}$ (3sf)

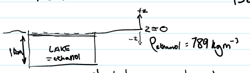
ii) $1 = 700 \exp\{-6.04[\exp(1.3 \times 10^{-5} z) - 1]\}$

$\frac{1}{700} = \exp\{6.04[\exp(1.3 \times 10^{-5} z) - 1]\} \to \ln\left(\frac{1}{700}\right) = -6.04[\exp(1.3 \times 10^{-5} z) - 1] = -6.55$

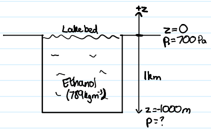
$\to \frac{-6.55}{-6.04} + 1 = \exp(1.3 \times 10^{-5} z) = 2.084$

$\therefore z = \frac{\ln(2.084)}{1.3 \times 10^{-5}} = 56488.$ $\to$ 56500m in altitude (3sf) (56.5km)

iii)



$\rho_{ethanol} = 789\ kgm^{-3}$



Gauge pressure = absolute pressure - atmospheric pressure $= 700Pa$

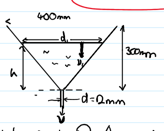
$\frac{dp}{dz} = -\rho g \to dp = -\rho g\, dz$

$\to p - p_0 = \rho g(z_0 - z)$

$p - 700 = 789 \cdot 3.71[-(-1000)] \to p = 2927190 + 700$

$\therefore = 2927890 Pa \approx 2.93 MPa$ (3sf)

2)



For $m^3 s^{-1}$:

$A = \pi r^2 = \pi\left(\frac{2}{2} \cdot \frac{1}{1000}\right)^2 = 10^{-6}\pi$ $\quad$ $mm^2 \to m^2$

i) Discharge rate: $Q = Av = \pi v\ mm^3 s^{-1} = 10^{-6}\pi\ m^3 s^{-1}$

A = surface area (at exit)

ii) Conservation of mass $\to$ continuity equation: $\frac{d\rho}{dt} + \text{div}(\rho v) = 0$

$\to$ since fluid is incompressible (ρ = constant)

$\text{div}\, v = 0 \therefore v_1 A_1 = v A_2$ $\quad$ $-A_2$ = cross sectional area at the exit nozzle.

$\to v_1 = \frac{vA_2}{A_1}$

$\pi r^2 = \pi\left(\frac{d}{2}\right)^2$ $\quad A_2 = \frac{d^2\pi}{4}$ $\quad$ πr^2 $d = 2mm$ $\quad r = \frac{d}{2} \therefore r^2 = \left(\frac{d}{2}\right)^2 = \frac{d^2}{4}$

A_1 (for v_1) $= \frac{d_1^2\pi}{4} = \frac{\left(\frac{4h}{3}\right)^2\pi}{4} = \frac{\frac{16h^2}{9}\pi}{4} = \frac{16h^2\pi}{36}$

$\therefore v_1 = \frac{vA_2}{A_1} = v \cdot \frac{\left(\frac{d^2\cancel{\pi}}{4}\right)}{\left(\frac{16h^2\cancel{\pi}}{36}\right)} \to \boxed{v_1 = \frac{9d^2}{16h^2} v}$

iii) Bernoulli's Equation: $\frac{1}{2}\rho v^2 + p + \phi^v = \frac{1}{2}\rho v^2 + p + \rho gh$ = constant.

$\frac{1}{2}\rho_1 v_1^2 + p_1 + \rho_1 gh = \frac{1}{2}\rho_1 v^2 + p_1 + \cancel{\rho g(0)}$

- Density is the same; water only in system. No pressure changes due to incompressibility. (p = constant)

$\therefore \frac{1}{2}\rho v_1^2 + \rho gh = \frac{1}{2}\rho v^2$ $\quad$ sub (ii) $v_1 = \frac{9d^2 v}{16h^2}$; divide by ρ:

$\therefore \frac{1}{2}\cancel{\rho}\left(\frac{9d^2 v}{16h^2}\right)^2 + \cancel{\rho}gh = \frac{1}{2}\cancel{\rho}v^2 \to \frac{1}{2}\left(\frac{9d^2 v}{16h^2}\right)^2 + gh = \frac{1}{2}v^2 \to \frac{1}{2}\left(\frac{81d^4}{256h^4}\right)v^2 + gh = \frac{1}{2}v^2$

$\to gh = \frac{1}{2}v^2\left[1 - \frac{81d^4}{256h^4}\right] \to 2gh = v^2 a \to v = \sqrt{\frac{2gh}{1 - \frac{81d^4}{256h^4}}}$

$\therefore Q = Av = \pi \cdot \sqrt{\frac{2gh}{1 - \frac{81d^4}{256h^4}}}\ mm^3 s^{-1} = 10^{-6}\pi\sqrt{\frac{2gh}{1 - \frac{81d^4}{256h^4}}}\ m^3 s^{-1}$

iv) If $d \to 0$: $v = \lim_{d \to 0}\sqrt{\frac{2gh}{1 - \left(\frac{81d^4}{256h^4}\right)}} = \sqrt{\frac{2gh}{1 - \frac{0}{256}}} = \sqrt{\frac{2gh}{1}}$

$\sqrt{1-\frac{81d^4}{256h^4}}$ $\qquad$ $\sqrt{1-\frac{81d^4}{256h^4}}$

iv) If $d \to 0$: $v = \lim_{d\to 0} \sqrt{\dfrac{2gh}{1-\left(\frac{81d^4}{256h^4}\right)}} = \sqrt{\dfrac{2gh}{1-\frac{0}{256}}} = \sqrt{\dfrac{2gh}{1}}$

$\therefore v = \lim_{d\to 0} \to v = \sqrt{2gh}$, agrees with Torricelli's result.

3) $v_x = A + By + \frac{1}{2}\frac{C}{\mu}y^2 \quad v_y = 0 \quad v_z = 0 \qquad (\underline{v} = v_x + v_y + v_z)$

i) Navier-Stokes Equation: $\rho \dfrac{d\underline{v}}{dt} = -\text{grad}\, p + \mu \nabla^2 \underline{v} + f^v$

$= -\text{grad}\, p + \mu \nabla^2 \underline{v} + \cancel{\rho g}^{\,0}$ ~ no gravity

At steady state: $\dfrac{d\underline{v}}{dt} = 0 \therefore \mu \nabla^2 \underline{v} = \text{grad}\, p$

$\to \mu\left(\cancel{\dfrac{\partial^2 v}{\partial x^2}} + \dfrac{\partial^2 v}{\partial y^2} + \cancel{\dfrac{\partial^2 v}{\partial z^2}}\right) = \left(\dfrac{\partial p}{\partial x} + \cancel{\dfrac{\partial p}{\partial y}} + \cancel{\dfrac{\partial p}{\partial z}}\right) \therefore \mu \dfrac{\partial^2 v}{\partial y^2} = \dfrac{\partial p}{\partial x}$

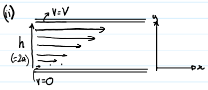
since v_x only has >0 values.

$\dfrac{\partial^2 v}{\partial y^2} = \dfrac{\partial^2 v_x}{\partial y^2} + \cancel{\dfrac{\partial^2 v_y}{\partial y^2}} + \cancel{\dfrac{\partial^2 v_z}{\partial z^2}} = \dfrac{C}{\mu} \quad \to \cancel{\mu} \cdot \dfrac{C}{\cancel{\mu}} = \dfrac{\partial p}{\partial x}$

$\therefore \dfrac{dp}{dx} \equiv p_x \quad ; \dfrac{dp}{dx} = C$

or $p = \int C\, dx = Cx + D$

(i)



Boundaries:
① $v = 0$ at $y = 0$ $\qquad v = v_x = A + By + \frac{1}{2}\frac{C}{\mu}y^2$
② $v = V$ at $y = h$

Using ①: $0 = A + B(0) + \frac{1}{2}\frac{C}{\mu}(0)^2 \to A = 0$

using ②: $V = Bh + \frac{1}{2}\frac{C}{\mu}h^2$

$Bh = V - \dfrac{Ch^2}{2\mu} \qquad \therefore B = \dfrac{V}{h} - \dfrac{Ch}{2\mu}$

$\therefore v_x = \cancel{A}^{\,0} + By + \dfrac{C}{2\mu}y^2 \quad = \left(\dfrac{V}{h} - \dfrac{Ch}{2\mu}\right)y + \dfrac{C}{2\mu}y^2$

given $\dfrac{dp}{dx} = p_x = C$:

$v_x = \left(\dfrac{V}{h} - \dfrac{p_x h}{2\mu}\right)y + \dfrac{p_x}{2\mu}y^2$

iii) ⓐ $v_x' = \dfrac{v_x}{V}$ defines the relative velocity of the fluid flowing between plates (i.e. Couette flow) against the maximum velocity V when $y = h$, starting from $y = 0$ ($v = 0$). It's a normalised value.

ⓑ $y' = \dfrac{y}{h}$ is the vertical height of the fluid flow across the plates, relative to the maximum height (upper boundary) $y = h$, starting from $y = 0$. It's also a normalised value.

ⓒ $p_x' = \dfrac{p_x h^2}{\mu V}$ is the relative pressure gradient of the fluid flow based on the height h. It has a square dependance, such that $p_x' \propto h^2$.

Substituting ⓐ, ⓑ and ⓒ to (ii);

$v_x = \left(\dfrac{V}{h} - \dfrac{p_x h}{2\mu}\right)y + \dfrac{p_x}{2\mu}y^2 \quad \to v_x' = \dfrac{1}{\cancel{V}}\left[\left(\dfrac{\cancel{V}}{h} - \dfrac{p_x' \cancel{\mu V} \cdot h}{h^2 \cdot 2\cancel{\mu}}\right)y' h + \dfrac{p_x' \cancel{\mu V}}{h^2} \cdot \dfrac{1}{2\cancel{\mu}} \cdot (y'h)^2\right]$

$v_x' = \left(\dfrac{1}{\cancel{h}} - \dfrac{p_x' h}{2h^2}\right)y'\cancel{h} + \dfrac{p_x'}{2\cancel{h}}y'^2\cancel{h}$

$v_x' = \left(1 - \dfrac{p_x'}{2}\right)y' + \dfrac{p_x'}{2} \cdot y'^2 \quad \to v_x' = y'\left(1 - \dfrac{p_x'}{2} + \dfrac{p_x' y'}{2}\right)$

$v_x' = y'\left[1 - \dfrac{p_x'}{2}(1 - y')\right]$

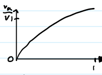
Make a plot of $\frac{v_x}{V}$, $\frac{y}{h}$ when $p_x = 0$; $p_x' = 1$, $p_x' = -1$:

For $p_x = 0$ ($p_x' = 0$)
$\to v_x' = y'[1 - \cancel{0(1-y')}]$
$v_x' = y'$

(Newtonian fluid-like)
→ velocity will increase linearly the higher the fluid's position as $y \to h$

$y'[1 - \frac{1}{2} + \frac{y'}{2}] =$

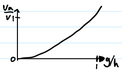
For $p_x' = 1$:
$v_x' = y'\left[1 - \dfrac{1}{2}(1 - y')\right] = \dfrac{y'}{2} + \dfrac{y'^2}{2}$



(Shear-thinning-like)
→ pressure gradient causes rather Logarithmic(ln) increase of fluid through plates, and flow rate decreases as $y \to h$.

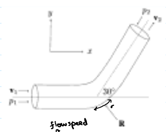
$y'[1 + \frac{1}{2} - \frac{y'}{2}]$
$\frac{3}{2} - \frac{y'}{2}$

For $p_x' = -1$:
$v_x' = y'\left[1 - \dfrac{(-1)}{2}(1 - y')\right] = \dfrac{3y'}{2} - \dfrac{y'^2}{2}$



(Shear-thickening-like)
→ pressure gradient causes rather exponential increase of fluid through plates, and flow rate increases as $y \to h$.

4)



$d = 20\text{cm}$
$r_{curve} = 0.5\text{m}$
$L = 0.3\text{m}$
$\mu = 0.6\,\text{Pa}\cdot\text{s}$
input gauge pressure: $2\times10^7\,\text{Pa}$
$Q_{oil} = 0.01\,\text{m}^3\text{s}^{-1}$
$\rho_{oil} = 900\,\text{kgm}^{-3}$

i) $Re = \dfrac{ul}{\nu} = \dfrac{\rho \cdot v \cdot l}{\mu}$ — flow speed; characteristic length → diameter! ; ν: kinematic visc.

$Q = A v_{ave} \Rightarrow \underset{(ave)}{v} = \dfrac{Q}{A} = \dfrac{0.01}{\pi(0.1)^2} = 0.318\,\text{ms}^{-1}$ $\qquad$ $d = 20\text{cm} = 0.2\text{m}$ $\therefore a = 0.2/2 = 0.1\text{m}$

$\therefore Re = \dfrac{900 \cdot 0.318 \cdot 0.2}{0.6} = 95.49 \approx \underline{95.5}$ (3sf)

ii) $De = Re\sqrt{\dfrac{d}{2r}} = 95.5\sqrt{\dfrac{0.2}{2\cdot 0.5}}$ $\qquad r = 0.5$ → curvature

$= \underline{42.7}$ (3sf)

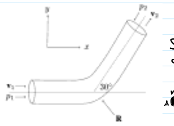
$42.7 < 50$ ∴ Follows Hagen-Poiseuille flow & is laminar.

Pressure drop, Δp:
$\to \Delta p = 8\mu L \cdot \dfrac{v_{ave}}{a^2} \qquad v_{ave} = \dfrac{Q}{\pi a^2} = 0.318\,\text{ms}^{-1}$ [given in (i)]

$= 8 \cdot 0.6 \cdot 3 \cdot \dfrac{0.318}{0.1^2}$

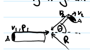
$\Delta p = 457.92 \approx \underline{458\,\text{Pa}}$ (3sf)

iii)



Key equation:
Sum of all forces acting on system = Rate of out flow of momentum through control surface.

~~$= \rho Q v$~~
momentum $= mv$ $\quad \therefore = \rho V v$
$\rho = \frac{m}{V} \to m = V\rho$ $\quad$ $\text{kgm}^{-3} \cdot \text{ms}^{-1} \cdot \text{m}^3\text{s}^{-1}$ $\quad \text{kgms}^{-2} = F$



$p = \dfrac{F}{A} \to F = Ap$

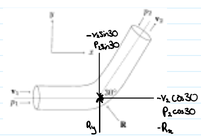
$R_x = R\cos 30$

R_x / R_y using Force balance:
x-direction: $p_1 A - p_2 A\cos 30 + R_x = \rho Q v_1 + \rho Q v_2 \cos 30$ $\qquad (\cos 30 = \frac{\sqrt{3}}{2})$

$\to R_x = \rho_{oil} Q\left(v_1 + \frac{\sqrt{3}}{2}v_2\right) - A\left(p_1 - \frac{\sqrt{3}}{2}p_2\right)$

y-direction: $R_y - p_2 A\sin 30 = \rho Q v_2 \sin 30$ $\qquad (\sin 30 = \frac{1}{2})$

$\to R_y = \frac{1}{2}(p_2 A + \rho Q v_2)$



Finding v_2 → Bernoulli's: $\frac{1}{2}\rho_{oil} v_1^2 + p_1 + \cancel{\rho g h} = \frac{1}{2}\rho_{oil} v_2^2 + p_2 + \cancel{\rho g h}$

$\underset{(\Delta p)}{p_1 - p_2} = \frac{1}{2}\rho_{oil}(v_2^2 - v_1^2) \quad \to 458 = \frac{1}{2}900(v_2^2 - 0.318^2)$

$\dfrac{458}{450} = v_2^2 - 0.318^2 \to v_2 = \sqrt{\dfrac{458}{450} + 0.318^2}$

$v_2 = 1.0578\,\text{ms}^{-1}$

x-direction: $R_x = 900 \cdot 0.01\left[0.318 + \frac{\sqrt{3}}{2} \cdot 1.0578\right] - 0.1^2\pi\left[2\times10^7 - \frac{\sqrt{3}}{2}(2\times10^7 - 458)\right]$
$= \underline{-84080.11\,\text{N}}$

$$\frac{458}{450} = v_2^2 - 0.318^2 \rightarrow v_2 = \sqrt{\frac{458}{450} + 0.318^2}$$

$$v_2 = 1.0578\ ms^{-1}$$

$\downarrow$

x-direction: $R_x = 900 \cdot 0.01 \left[0.318 + \frac{\sqrt{3}}{2} \cdot 1.0578 \right] - 0.1^2 \pi \left[2\times10^7 - \frac{\sqrt{3}}{2}(2\times10^7 - 458) \right]$

$= \underline{-84080.11\ N}$

y-direction: $R_y = \frac{1}{2}\left[(2\times10^7 - 457.92) \cdot 0.1^2 \pi + 900 \cdot 0.01 \cdot 1.0578 \right]$

$= \underline{314161.59\ N}$

Using Trig $R = \sqrt{(-84080)^2 + 314162^2} \rightarrow R = 325218.3445\ N$

$= \underline{325\ kN}$ (3sf)