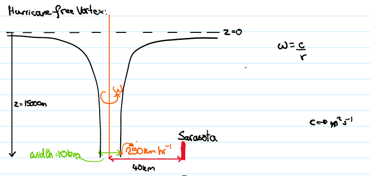
# MATE60005 Problem Sheet Raw Solutions

17 February 2025 19:29

Hurricane-free Vortex:




$\omega = \frac{c}{r}$

$c \to m^2 s^{-1}$

i) Wind speed $v = 250\ km\ hr^{-1}$, $r = \frac{10}{2} = 5km$

$\therefore$ using $v = \frac{c}{r} \to c = vr = 250 \cdot 5 = 1250\ km^2 hr^{-1}$

Strength of vortex, $\pi c = 1250\pi\ km^2 hr^{-1} = \frac{1250\pi \cdot (1000)^2}{3600}\ m^2 s^{-1}$

$= 1.09 \times 10^6\ m^2 s^{-1}$ (3sf)

ii) Circulation at $r = 300km$:

Circulation $= \int_0^{2\pi r} v\,dx = 2\pi r \frac{c}{r} = 2\pi c = 2 \cdot 1.09 \times 10^6$

$= \underline{2.18 \times 10^6\ m^2 s^{-1}}$ (3sf) (Radius independant of circulation in integration path)

iii) Wind speed $v_\theta = \frac{c}{r} = \frac{1250}{40} = \underline{\underline{31.25 km\ hr^{-1}}}$ $= \frac{31.25 \times 1000}{3600} = 8.68 ms^{-1}$ (3sf)

iv) Vorticity: $\nabla \times \underline{v}$

Using polar coordinates: $= \frac{1}{r}\begin{vmatrix} i & rj & k \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ v_r & rv_\theta & v_z \end{vmatrix}$ $(v_r, v_\theta, v_z)$

Wind speed from (iii) is $v_\theta = 31.25 km\ hr^{-1} (= 8.68 ms^{-1}) = \frac{c}{r}$

$\nabla \times \underline{v} = \left(\cancel{\frac{\partial v_z}{\partial \theta}}^{0} - \cancel{\frac{\partial (rv_\theta)}{\partial z}}^{0}\right)\vec{i} - r\left(\cancel{\frac{\partial v_z}{\partial r}}^{0} - \cancel{\frac{\partial v_r}{\partial z}}^{0}\right)\vec{j} + \left(\frac{\partial (rv_\theta)}{\partial r} - \cancel{\frac{\partial v_r}{\partial \theta}}^{0}\right)\vec{k}$

$\therefore \nabla \times \underline{v} = \frac{\partial (rv_\theta)}{\partial r} = \frac{\partial}{\partial r} \cdot c = 0$ $\therefore$ Vorticity $\nabla \times \underline{v} = 0$

b) i) $\frac{dp}{dz} = -\rho g = -\frac{pmg}{RT}$ $\rho = \frac{pm}{RT} \to$ specific density for given T, molecular mass m and pressure p.

$pV = nRT \to p = \frac{n}{VRT}$ since $\rho = \frac{m}{V} \to V = \frac{m}{\rho}$ $p = \frac{\rho n}{mRT}$

$dp = -\frac{pmg}{RT}\,dz \to \int_{p_0}^{p_1} \frac{1}{p}\,dp = -\frac{mg}{RT_0}\int_0^z \frac{1}{e^{-cz}}\,dz = -\frac{mg}{RT_0}\int_0^z e^{cz}\,dz$ since $T \approx T_0 \exp(-cz)$

$\ln\left(\frac{p}{p_0}\right) = -\frac{mg}{RT_0}\left[\frac{1}{c}\cdot \exp(cz)\right]_0^z = -\frac{mg}{RT_0 c}\left[\exp(cz) - 1\right]$

$\therefore \frac{p}{p_0} = \exp\left\{-\frac{mg}{RT_0 c}\left[\exp(cz) - 1\right]\right\} \to p = p_0 \exp\left\{-\frac{mg}{RT_0 c}\left[\exp(cz) - 1\right]\right\}$ , $T = T_0 \exp(-cz)$, $T_0 = 250K$

Sub in:

• $g = 3.71\ ms^{-2}$; $p_0$ (surface pressure) $= 700\ Pa$; $m = 0.04401$; $R = 8.314\ J mol^{-1} K^{-1}$.

Analytic Formula: $p = 700 \exp\left\{-\frac{0.04401 \cdot 3.71}{8.314 \cdot 250 \cdot 1.3 \times 10^{-5}}\left[\exp(1.3 \times 10^{-5} z) - 1\right]\right\}$

$p = 700 \exp\{-6.04\left[\exp(1.3 \times 10^{-5} z) - 1\right]\}$ (3sf)

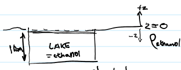
ii) $1 = 700 \exp\{-6.04\left[\exp(1.3 \times 10^{-5} z) - 1\right]\}$

$\frac{1}{700} = \exp\{-6.04\left[\exp(1.3 \times 10^{-5} z) - 1\right]\} \to \ln\left(\frac{1}{700}\right) = -6.04\left[\exp(1.3 \times 10^{-5} z) - 1\right] = -6.55$

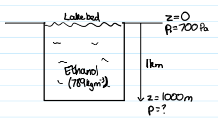
$\to \frac{-6.55}{-6.04} + 1 = \exp(1.3 \times 10^{-5} z) = 2.084$

$\therefore z = \frac{\ln(2.084)}{1.3 \times 10^{-5}} = 56488.$ $\to \underline{56500m}$ in altitude (3sf) (56.5km)

iii)




$\rho_{ethanol} = 789\ kgm^{-3}$




Gauge pressure = absolute pressure - atmospheric pressure (= 700Pa)

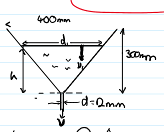
$\frac{dp}{dz} = -\rho g \to dp = -\rho g\,dz$

$\to p - p_0 = \rho g (z_0 - z)$

$p - 700 = 789 \cdot 3.71\left[-(-1000)\right] \to p = 2927190 + 700$

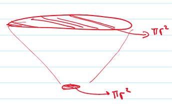
$\therefore = 2927890 Pa \approx 2.93\ MPa$ (3sf)

2)




For $m^3 s^{-1}$:

$A = \pi r^2 = \pi\left(\frac{2}{2} \cdot \frac{1}{1000}\right)^2 = 10^{-6}\pi$ ($mm^2 \to m^2$)




i) Discharge rate: $Q = Av = \pi v\ mm^3 s^{-1} = 10^{-6}\pi\ m^3 s^{-1}$

A = surface area (at exit)

ii) Conservation of mass $\to$ continuity equation: $\frac{d\rho}{dt} + \text{div}(\rho v) = 0$

$\to$ since fluid is incompressible ($\rho$ = constant)

$\text{div}\, v = 0 \therefore v_1 A_1 = v A_2$ — $A_2$ = cross sectional area at the exit nozzle.

$\to v_1 = \frac{vA_2}{A_1}$

$\pi r^2$ $d = 2mm$ $r = \frac{d}{2}$ $\therefore r^2 = \left(\frac{d}{2}\right)^2 = \frac{d^2}{4}$

$\pi r^2 = \pi\left(\frac{d}{2}\right)^2$ $A_2 = \frac{d^2\pi}{4}$ $A_1 = \frac{d_1^2\pi}{4}$ (for $v_1$) $= \frac{\left(\frac{4h}{3}\right)^2\pi}{4} = \frac{\frac{16h^2}{9}\pi}{4} = \frac{16h^2\pi}{36}$

$\therefore v_1 = \frac{vA_2}{A_1} = v \cdot \frac{\left(\frac{d^2\cancel{\pi}}{4}\right)}{\left(\frac{16h^2\cancel{\pi}}{36}\right)}$ $\to$ $v_1 = \frac{9d^2}{16h^2}v$

iii) Bernoulli's Equation: $\frac{1}{2}\rho v^2 + p + \cancel{\phi} = \frac{1}{2}\rho v^2 + p + \rho gh$ = constant.

$\frac{1}{2}\rho_1 v_1^2 + p_1 + \rho_1 gh = \frac{1}{2}\rho_1 v^2 + p_1 + \cancel{\rho g(0)}$

- Density is the same; water only in system. No pressure changes due to incompressibility. $\to \rho$ = const.

$\therefore \frac{1}{2}\rho v_1^2 + \rho gh = \frac{1}{2}\rho v^2$ sub (ii) $v_1 = \frac{9d^2 v}{16h^2}$; divide by $\rho$:

$\therefore \frac{1}{2}\cancel{\rho}\left(\frac{9d^2 v}{16h^2}\right)^2 + \cancel{\rho}gh = \frac{1}{2}\cancel{\rho}v^2 \to \frac{1}{2}\left(\frac{9d^2 v}{16h^2}\right)^2 + gh = \frac{1}{2}v^2 \to \frac{1}{2}\left(\frac{81d^4}{256h^4}\right)v^2 + gh = \frac{1}{2}v^2$

$\frac{1}{2}v_1^2 + gh = \frac{1}{2}v^2 \to v_1^2 + 2gh = v^2$

$\to \left(\frac{9d^2}{16h^2}\right)^2 v^2 + 2gh = v^2 \to 2gh = v^2 - \left(\frac{9d^2}{16h^2}\right)^2 v^2 = v^2\left(1 - \frac{81d^4}{256h^4}\right)$

$\to gh = \frac{1}{2}v^2\left[1 - \frac{81d^4}{256h^4}\right] \to 2gh = v^2 a \to v = \sqrt{\frac{2gh}{1 - \frac{81d^4}{256h^4}}}$

$\frac{4}{3} \div 2 = \frac{2}{3}$

$\therefore Q = Av = \pi \cdot \sqrt{\frac{2gh}{1 - \frac{81d^4}{256h^4}}}\ mm^3 s^{-1} = 10^{-6}\pi\sqrt{\frac{2gh}{1 - \frac{81d^4}{256h^4}}}\ m^3 s^{-1}$

$2\pi\int_{0.001}^{\frac{2}{3}h} v r \cdot dr = v\left[\left(\frac{2h}{3}\right)^2 - (1 \times 10^{-3})^2\right]$

$= v\left[\frac{4h^2}{9} - 10^{-6}\right]\pi\ m^3 s^{-1}$

in the case $d_1 >> d$

$4\left(\frac{4h}{3}\right)^4 \geq d^4$ $\equiv \frac{256h^4}{81} \geq d^4$

$1 \geq \frac{81d^4}{256h^4}$

iv) This $v$ result in (iii) is consistent with Torricelli's ($v = \sqrt{2gh}$).

$\to$ when $d_1 >> d$: $v = \sqrt{\frac{2gh}{1 - \frac{81 \cdot d^4}{256h^4}}}$ $ms^{-1}$

Extra

$\to$ when $d_1 >> d$: velocity $v$ is negligible at $d_1$ ($v \to 0$)

$\therefore v = \frac{\sqrt{2gh}}{\sqrt{1 - \frac{81d^4}{256h^4}}} \approx \sqrt{2gh}$

$\therefore \sqrt{1 - \frac{81d^4}{256h^4}} \approx 1 \to 1 - \frac{81d^4}{256h^4} \approx 1$

$$\sqrt{1-\frac{81d^4}{256h^4}} \qquad \sqrt{1-\frac{81d^4}{256h^4}}$$

$$= v\left[\frac{4h^2}{9} - 10^{-6}\right]\pi \quad m^3s^{-1}$$

$$1-\frac{81d^4}{256h^4}$$

iv) This result in (iii) is consistent with Torricelli's ($v=\sqrt{2gh}$).
→ when $d_1 >> d$: $v = \sqrt{\frac{2gh}{1-\frac{81\cdot 0.02}{256h^4}}}$ $ms^{-1}$

Extra → when $d_1 >> d$: velocity $v$ is negligible at $d_1$ ($v \to 0$)

$\therefore v = \frac{\sqrt{2gh}}{\sqrt{1-\frac{81d^4}{256h^4}}} \approx \sqrt{2gh}$

$\therefore \sqrt{1-\frac{81d^4}{256h^4}} \approx 1 \to 1-\frac{81d^4}{256h^4} \approx 1$

$\frac{81d^4}{256h^4} << 1$ (closes to 0)

$\therefore d^4 << \left(\frac{256h^4}{81}\right) \to d << \frac{16}{9}h$

$\therefore$ if $h = 0.01m$: $d = \frac{16\cdot 0.01}{9} = 0.01777$

given $d << h$ generally, set $d = 0.001m$ for graphs.

3) $v_x = A + By + \frac{1}{2}\frac{C}{\mu}y^2 \quad v_y = 0 \quad v_z = 0 \qquad (\underline{v} = v_x + v_y + v_z)$

i) Navier-Stokes Equation: $\rho\frac{d\underline{v}}{dt} = -\text{grad}\,p + \mu\nabla^2\underline{v} + f^v$

$= -\text{grad}\,p + \mu\nabla^2\underline{v} + \cancel{\rho g}^{\,0}$ ~ no gravity

At steady state: $\frac{d\underline{v}}{dt} = 0 \therefore \mu\nabla^2\underline{v} = \text{grad}\,p$

$\to \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}}^{\,0} + \frac{\partial^2 v}{\partial y^2} + \cancel{\frac{\partial^2 v}{\partial z^2}}^{\,0}\right) = \left(\frac{dp}{dx} + \cancel{\frac{dp}{dy}} + \cancel{\frac{dp}{dz}}^{\,0}\right) \therefore \mu\frac{d^2v}{dy^2} = \frac{dp}{dx}$

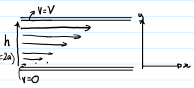
since $v_x$ only has >0 values.

$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_y}{\partial y^2}}^{\,0} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}^{\,0} = \frac{C}{\mu} \quad \to \cancel{\mu}\cdot\frac{C}{\cancel{\mu}} = \frac{dp}{dx}$

$\therefore \frac{dp}{dx} \equiv p_x \quad ; \frac{dp}{dx} = C$

↳ (or $p = \int C\,dx = Cx + D$)

(ii)




Boundaries:
① $v=0$ at $y=0$
② $v=V$ at $y=h$

$V = v_x = A + By + \frac{1}{2}\frac{C}{\mu}y^2$

Using ①: $0 = A + B(0) + \frac{1}{2}\frac{C}{\mu}(0)^2 \to A = 0$

using ②: $V = Bh + \frac{1}{2}\frac{C}{\mu}h^2$

$Bh = V - \frac{Ch^2}{2\mu} \qquad \therefore B = \frac{V}{h} - \frac{Ch}{2\mu}$

$\therefore v_x = \cancel{A}^{\,0} + By + \frac{C}{2\mu}y^2 \quad = \left(\frac{V}{h} - \frac{Ch}{2\mu}\right)y + \frac{C}{2\mu}y^2$

given $\frac{dp}{dx} = p_x = C$:

$v_x = \left(\frac{V}{h} - \frac{p_x h}{2\mu}\right)y + \frac{p_x}{2\mu}y^2$

iii) ⓐ $v_x' = \frac{v_x}{V}$ defines the relative velocity of the fluid flowing between plates (i.e. Couette flow) against the maximum velocity $V$ when $y=h$, starting from $y=0$ ($v=0$). It's a normalised value.

ⓑ $y' = \frac{y}{h}$ is the vertical height(s) of the fluid flow across the plates, relative to the maximum height (upper boundary) $y=h$, starting from $y=0$. It's also a normalised value.

ⓒ $p_x' = \frac{p_x h^2}{\mu V}$ is the relative pressure gradient of the fluid flow based on the height $h$. It has a square dependance, such that $p_x' \propto h^2$.

Substituting ⓐ, ⓑ and ⓒ to (ii);

$v_x = \left(\frac{V}{h} - \frac{p_x h}{2\mu}\right)y + \frac{p_x}{2\mu}y^2 \quad \to v_x' = \frac{1}{\cancel{V}}\left[\left(\frac{\cancel{V}}{h} - \frac{p_x'\cancel{\mu V}}{h^2}\cdot\frac{h}{2\cancel{\mu}}\right)y'\cdot h + \frac{p_x'\cancel{\mu V}}{h^2}\cdot\frac{1}{2\cancel{\mu}}\cdot(y'\cdot h)^2\right]$

$y'\left[1-\frac{1}{2}+\frac{y'}{2}\right] =$

$y'\left[1+\frac{1}{2}-\frac{y'}{2}\right]$
$\frac{3}{2}-\frac{y'}{2}$

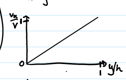
$v_x' = \left(\frac{1}{\cancel{h}} - \frac{p_x'\cancel{h}}{2h^2}\right)y'\cancel{h} + \frac{p_x'}{2\cancel{h^2}}y'^2\cancel{h^2}$

$v_x' = \left(1 - \frac{p_x'}{2}\right)y' + \frac{p_x'}{2}\cdot y'^2 \quad \to v_x' = y'\left(1 - \frac{p_x'}{2} + \frac{p_x' y'}{2}\right)$

$v_x' = y'\left[1 - \frac{p_x'}{2}(1-y')\right]$

Make a plot of $\frac{v_x}{V}$, $\frac{y}{h}$ when $p_x = 0$; $p_x' = 1$; $p_x' = -1$:

→ for $p_x = 0$ ($p_x' = 0$)
→ $v_x' = y'[1 - \cancel{\frac{0}{2}(1-y')}]$
$v_x' = y'$

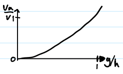



(Newtonian fluid-like)
→ velocity will increase linearly the higher the fluids position as $y \to h$

For $p_x' = 1$:
$v_x' = y'\left[1 - \frac{1}{2}(1-y')\right] = \frac{y'}{2} + \frac{y'^2}{2}$

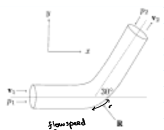

(Shear-thinning-like)
→ pressure gradient causes rather Logarithmic(ln) increase of fluid through plates, and flow rate decreases as $y \to h$.

For $p_x' = -1$:
$v_x' = y'\left[1 - \frac{(-1)}{2}(1-y')\right] = \frac{3y'}{2} - \frac{y'^2}{2}$




(Shear-thickening-like)
→ pressure gradient causes rather exponential increase of fluid through plates, and flow rate increases as $y \to h$.

4)




$d = 20cm$
$r_{curve} = 0.5m$
$L = 0.3m$
$\mu = 0.6 Pa\cdot s$
input gauge pressure: $2\times10^7 Pa$
$Q_{oil} = 0.01 m^3s^{-1}$
$\rho_{oil} = 900\, kgm^{-3}$

i) $Re = \frac{uL}{\nu} = \frac{\rho\cdot v\cdot l}{\mu}$ — flowspeed; characteristic length → diameter? ; kinematic visc.

$Q = Av_{ave} \Rightarrow v_{(ave)} = \frac{Q}{A} = \frac{0.01}{\pi(0.1)^2} = 0.318\,ms^{-1}$

$d = 20cm$
$= 0.2m$
$\therefore a = 0.2/2 = 0.1m$

$\therefore Re = \frac{900\cdot 0.318\cdot 0.2}{0.6} = 95.49$
$\approx \underline{95.5}$ (3sf)

ii) $De = Re\sqrt{\frac{d}{2r}} = 95.5\sqrt{\frac{0.2}{2\cdot 0.5}}$ $\qquad r = 0.5 \to$ curvature

$= \underline{42.7}$ (3sf)

$42.7 < 50$ $\therefore$ Follows Hagen-Poiseuille flow & is laminar.

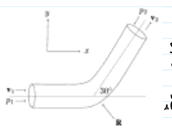
Pressure drop, $\Delta p$:

$\to \Delta p = \frac{8\mu L\cdot v_{ave}}{a^2} \qquad v_{ave} = \frac{Q}{\pi a^2} = 0.318\,ms^{-1}$ [given in (i)]

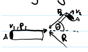
$= 8\cdot 0.6\cdot 3\cdot\frac{0.318}{0.1^2}$

$\Delta p = 457.92 \approx \underline{458\,Pa}$ (3sf)

(iii)




Key equation:
Sum of all forces acting on system = Rate of out flow of momentum through control surface.




$P = \frac{F}{A} \to F = Ap$

$R_x = R\cos 30$

$= \rho Q v$
momentum $= mv$
$\rho = \frac{m}{V} \to m = V\rho$
$\therefore = \rho V v$
↳ $kg\,m^{-3}\cdot ms^{-1}\cdot m^3s^{-1}$
$kg\,ms^{-2} = F$

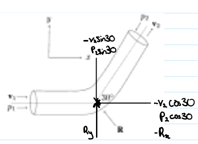
$R_x/R_y$ using Force balance:

x-direction: $p_1A - p_2A\cos 30 + R_x = \rho Qv_1 + \rho Qv_2\cos 30 \qquad (\cos 30 = \frac{\sqrt{3}}{2})$

$\to R_x = \rho_{oil}Q\left(v_1 + \frac{\sqrt{3}}{2}v_2\right) - A\left(p_1 - \frac{\sqrt{3}}{2}p_2\right)$

y-direction: $R_y - p_2A\sin 30 = \rho Qv_2\sin 30 \qquad (\sin 30 = \frac{1}{2})$

$\to R_y = \frac{1}{2}(p_2A + \rho Qv_2)$




Finding $v_2$ → Bernoulli's: $\frac{1}{2}\rho_{oil}v_1^2 + p_1 + \cancel{\rho gh} = \frac{1}{2}\rho_{oil}v_2^2 + p_2 + \cancel{\rho gh}$

$p_1 - p_2 = \frac{1}{2}\rho_{oil}(v_2^2 - v_1^2) \quad \to 458 = \frac{1}{2}900(v_2^2 - 0.318^2)$
($\Delta p$)

$\frac{458}{450} = v_2^2 - 0.318^2 \to v_2 = \sqrt{\frac{458}{450} + 0.318^2}$

$v_2 = 1.0578\,ms^{-1}$

↓

x-direction: $R_x = 900\cdot 0.01\left[0.318 + \frac{\sqrt{3}}{2}\cdot 1.0578\right] - 0.1^2\pi\left[2\times10^7 - \frac{\sqrt{3}}{2}(2\times10^7 - 458)\right]$
$= \underline{-84080.11\,N}$

$$\frac{458}{450} = v_2^2 - 0.318^2 \rightarrow v_2 = \sqrt{\frac{458}{450} + 0.318^2}$$

$$v_2 = 1.0578\,ms^{-1}$$

$\downarrow$

$x$-direction: $R_x = 900 \cdot 0.01\left[0.318 + \frac{\sqrt{3}}{2} \cdot 1.0578\right] - 0.1^2\pi\left[2\times10^7 - \frac{\sqrt{3}}{2}(2\times10^7 - 458)\right]$

$= \underline{-84080.11\,N}$

$y$-direction: $R_y = \frac{1}{2}\left[(2\times10^7 - 457.92) \cdot 0.1^2\pi + 900 \cdot 0.01 \cdot 1.0578\right]$

$= \underline{314161.59\,N}$

Using Trig $R = \sqrt{(-84080)^2 + 314162^2} \rightarrow R = 325218.3445\,N$

$= \underline{325\,kN}$ (3sf)