ShanghaiTech University

CS101 Algorithms and Data Structures

Fall 2019

Homework 8

Due date: 23:59, November 17, 2019

1. Please write your solutions in English.
2. Submit your solutions to gradescope.com.
3. Set your FULL Name to your Chinese name and your STUDENT ID correctly in Account Settings.
4. If you want to submit a handwritten version, scan it clearly. Camscanner is recommended.
5. When submitting, match your solutions to the according problem numbers correctly.
6. No late submission will be accepted.
7. Violations to any of above may result in zero score.

1: (3*2'+4') Dijkstra's Algorithm

Question 1. *Judge whether the following statement is true or false and explain why. Give a counter-example if it is false.*

(a) *Suppose G is strongly connected with integer edge weights, and has shortest paths from some vertex v (i.e. a finite weight shortest path exists from v to all nodes). Then shortest paths can be found from every vertex to every other vertex.*

Solution:

True. Since G has shortest paths from some vertices v, there should be no negative cycle in G, otherwise the shortest path is undefined. Since G is a strongly connected graph, there must be a path between any pair of vertices and the above property implies that there must be a shortest path between any pair of vertices.

(b) If G is a connected and undirected graph without negative cycles, we can apply Dijkstra's algorithm to find the shortest path.

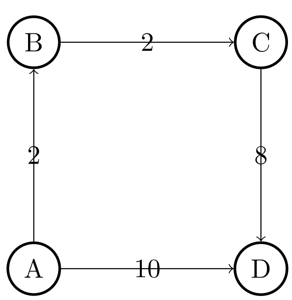
Solution:

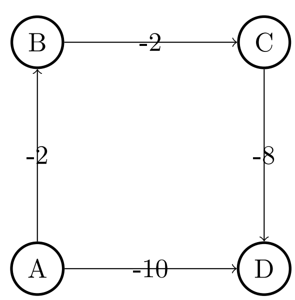
True. Suppose there's no shortest path between any pair of vertices, then those two vertices are either unconnected or there's a negative edge between and both conditions are contradictory to the question. Thus there must be a shortest path. The Dijkstra's algorithm always finds a shortest edge between the visited and unvisited parts so it can find that shortest path.

(c) Suppose G is a DAG. We can find the longest path by negating all edge lengths and then run Dijkstra's algorithm from every source node.

Solution:

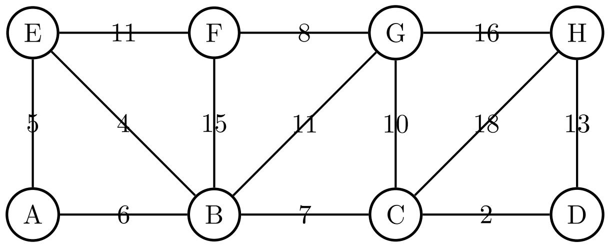
False. Consider the following two graphs:





By negating all edge lengths of the left graph, we obtain the right graph.
By running the Dijkstra's algorithm on the right graph, the longest path from vertex A to vertex D is $A - D$, however, the actual longest path from A to D is $A - B - C - D$, which can be obtained form the left graph.

Question 2. *Given a weighted graph below, please run Dijkstra's algorithm using vertex A as the source. Write down the vertices in the order which they are marked and the updated distances at each step.*



Solution:

step	vertex
1	A
2	E
3	B
4	C
5	D
6	F
7	G
8	H

step	dist[A]	dist[B]	dist[C]	dist[D]	dist[E]	dist[F]	dist[G]	dist[H]
1	0	6	∞	∞	5	∞	∞	∞
2	0	6	∞	∞	5	16	∞	∞
3	0	6	13	∞	5	16	17	∞
4	0	6	13	15	5	16	17	31
5	0	6	13	15	5	16	17	28
6	0	6	13	15	5	16	17	28
7	0	6	13	15	5	16	17	28
8	0	6	13	15	5	16	17	28

2: (2'+3') Floyd-Warshall Algorithm

Question 3. *Let $G = (V, E)$ be a connected, undirected graph with edge weights $w : E \to \mathbb{Z}$. Which of the following statements are True about the Floyd-Warshall algorithm applied to G?*

(A) Since G is undirected, we cannot apply Floyd-Warshall algorithm.

(B) Since G is undirected, Floyd-Warshall will be asymptotically faster than on directed graphs.

(C) Since G is undirected, Floyd-Warshall will be unable to detect negative-weight cycles.

(D) None of the above.

Solution: C

Question 4. *Consider the following implementation of the Floyd-Warshall algorithm. Assume $w_{ij} = \infty$ where there is no edge between vertex i and vertex j, and assume $w_{ii} = 0$ for every vertex i.*

Algorithm 1 Floyd-Warshall

```
for i = 1 to n do
  for j = 1 to n do
    A[i, j, 0] = w_ij
    P[i, j] = −1
  end for
end for
for k = 1 to n do
  for i = 1 to n do
    for j = 1 to n do
      A[i, j, k] = A[i, j, k − 1]
      if A[i, j, k] > A[i, k, k − 1] + A[k, j, k − 1] then
        A[i, j, k] = A[i, k, k − 1] + A[k, j, k − 1]
        P[i, j] = k
      end if
    end for
  end for
end for
```

Assume matrix P, the output of the above algorithm is given. Consider the following matrix for graph G with 7 vertices. What is the shortest path from vertex 5 to vertex 7 in graph G?

P	1	2	3	4	5	6	7
1	-1	5	4	-1	4	4	-1
2	5	-1	5	5	-1	5	-1
3	4	5	-1	-1	-1	-1	6
4	-1	5	-1	-1	3	3	1
5	4	-1	-1	3	-1	3	6
6	4	5	-1	3	3	-1	-1
7	-1	-1	6	1	6	-1	-1

Solution:

The shortest path is $\mathbf{5-3-6-7}$. Since $P[5,7] == 6$, the shortest path must pass vertex 6; Since $P[6,7] == -1$, which means edge $6-7$ is the shortest path from vertex 6 to vertex 7; Since the $P[5,6] == 3$, the path from vertex 5 to vertex 6 must pass vertex 3; Since $P[5,3] == -1$, the edge $5-3$ is part of the path; Since $P[3,6] == -1$, the edge $3-6$ is part of the path; Thus the next vertex of vertex 5 is vertex 3, the next vertex of vertex 3 is vertex 6, the next vertex of vertex 6 is vertex 7; Therefore, the shortest path from vertex 5 to vertex 7 is $5-3-6-7$.

3: (3'+3'+4') Shortest Path

Question 5. *Consider a weighted undirected graph with positive edge weights and let (u, v) be an edge in the graph. It is known that the shortest path from the source vertex s to u has weight 53 and the shortest path from s to v has weight 65. Which is the range of the weight the edge (u, v)?*

Solution:
$\boldsymbol{weight(u, v) >= 12.}$ Motivated by triangular, we can obtain $dist(s, v) + weight(v, u) >= dist(s, u)$, thus $weight(u, v) >= dist(s, u) - dist(s, v) = 12$.
Therefore, $weight(u, v) >= 12$.

Question 6. *Consider the weighted undirected graph with 4 vertices, where the weight of edge $\{i, j\}$ is given by the entry $W_{i,j}$ in the matrix W*

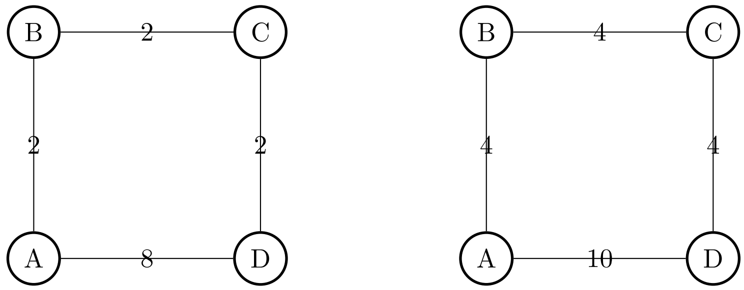
$$W = \begin{bmatrix} 0 & 2 & 8 & 6 \\ 2 & 0 & 5 & 8 \\ 8 & 5 & 0 & x \\ 6 & 8 & x & 0 \end{bmatrix}$$

We want to find the largest possible integer value of x, for which at least one shortest path between some pairs of vertices will definitely contain the edge with weight x. What is this largest possible integer value of such x? Explain your reason briefly. ***When breaking tie, the path may be random.***

Solution:
$\boldsymbol{max\{x\} = 12.}$ Since the edge of weight x must be contained in at least one shortest path between some pair of vertices, i.e. the edge between the third and fourth vertices is part of the shortest path, then consider the marginal situation, when the weight of that edge is equal to the weight of the "shortest path" from the third vertex to the fourth vertex through all the other vertices but doesn't contain x, then the shortest path from third vertex to the fourth vertex will not contain x because x is too large and will be ignored, so the maximum of x is that weight minus 1. The "shortest path" mentioned is: 3rd - 2nd - 1st - 4th.
Therefore, $max\{x\} == 5 + 2 + 6 - 1 = 12$.

Question 7. *Suppose $G = (V, E)$ is a weighted graph and T is its shortest-path tree from source s. If we increase all weights in G by the same amount, i.e., $\forall e \in E$, $w'_e = w_e + c$. Is T still the shortest-path tree (from source s) of the new graph? If yes, prove the statement. Otherwise, give a counter example.*

Solution:
No. Consider the following two graphs:



The left graph is the original G and the path from vertex A to vertex D in its shortest-path tree T is $A - B - C - D$; The right graph is obtained from increase all weights in the left graph by 2, the the path from vertex A to vertex D in its shortest-path tree T' is $A - D$. Therefore T is not the shortest-path tree of the new graph.