

Team Contest Reference

Universität zu Lübeck



Team: No Output

Inhaltsverzeichnis

1 Mathematische Algorithmen	2	6 2-SAT-Solver	8
1.1 Primzahlen	2	6.1 2-Sat mit SCC	8
1.1.1 Sieb des Eratosthenes $\mathcal{O}(n^2)$	2	6.2 Hilfsalgorithmen	8
1.1.2 Primzahlentest	2	6.2.1 Erzeugen eines Graphens	8
1.2 Binomial Koeffizient	2	6.2.2 Indexumrechnung	9
1.3 Eulersche φ -Funktion	2	6.3 Suchen eines Pfades	9
2 Mathematisch Formeln und Gesetze	2	6.4 Algorithmus zum Prüfen der Erfüllbarkeit	9
2.1 Catalan	2	6.5 Algorithmus zur Belegung einer 2-CNF	9
2.2 kgV und ggT	2	7 Verschiedenes	9
2.3 modulare Exponentiation	2	7.1 Potenzmenge	9
2.4 Modulare Arithmetik	2	7.2 LongestCommonSubsequence	9
2.4.1 Erweiterter Euklidischer Algorithmus	3	7.3 LongestCommonSubstring	10
2.5 Kombinatorik	3	7.4 LongestIncreasingSubsequence	10
3 Datenstrukturen	3	7.5 Permutation & Sequenzen	11
3.1 Fenwick Tree (Binary Indexed Tree)	3	7.6 Knuth-Morris Pratt	11
4 Graphen	3	8 Formatierung & Sonstiges	12
4.1 planare Graphen	3	8.1 Ausgabeformatierung mit JAVA - DecimalFormat	12
4.2 Topologische Sortierung	3	8.2 Ausgabeformatierung mit printf	12
4.3 Prim (Minimum Spanning Tree)	3	8.3 C++ Eingabe ohne bekannt Länge	13
4.4 Kruskal	4		
4.5 Floyd-Warshall ($\mathcal{O}(n^3)$)	4		
4.6 Dijkstra	4		
4.7 Belman-Ford	5		
4.8 MaxFlow	5		
4.9 Bipartite Matching	5		
4.10 Bitonic TSP	6		
5 Geometrie	6		
5.1 Kreuzprodukt, Skalarprodukt	6		
5.2 Orthogonale Projektion	6		
5.3 Rotation	6		
5.4 Geradenschnittpunkt	6		
5.5 Zusammenhang Kreuzprodukt & Sinus	7		
5.6 Dreiecksfläche	7		
5.7 Graham Scan (Convex Hull)	7		
5.8 Line Intersection	7		
5.9 Punkt in Polygon	7		
5.10 Fläche eines Polygons	8		
		MD5: cat <string> tr -d [:space:] md5sum	
		Testscript <ProblemClass>:	
		<pre>#!/bin/bash homefolder="/home/team" folder="\$homefolder/test-session" cd \$homefolder/workspace/\$1/src/ javac -encoding UTF-8 -d . "\$1.java" cat \$folder/\$1/sample.in java -Xrs -Xss8m - Xmx1272864k \$1 > \$folder/\$1/myout.out diff -u \$folder/\$1/sample.out \$folder/\$1/myout.out</pre>	

1 Mathematische Algorithmen

1.1 Primzahlen

Für Primzahlen gilt immer (aber nicht nur für Primzahlen)

$$a^p \equiv a \pmod{p} \quad \text{bzw.} \quad a^{p-1} \equiv 1 \pmod{p}.$$

1.1.1 Sieb des Eratosthenes $\mathcal{O}(n^2)$

```
1 static boolean[] sieve(int until) {
2     boolean[] a = new boolean[until + 1];
3     Arrays.fill(a, true); a[1]=false; a[0]=false;
4     for (int i = 2; i < Math.sqrt(a.length); i++) {
5         if (a[i]) {
6             for (int j = i * i; j < a.length; j += i) a[j] =
                false;
7         }
8     }
9     return a; // a[i] == true, iff. i is prime. a[0] is
                ignored
10 }
```

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1.1.2 Primzahlentest

```
1 static boolean isPrim(int p) {
2     if (p < 2 || p > 2 && p % 2 == 0) return false;
3     for (int i = 3; i <= Math.sqrt(p); i += 2)
4         if (p % i == 0) return false;
5     return true;
6 }
```

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1.2 Binomial Koeffizient

```
1 static int[][] mem = new int[MAX_N][(MAX_N + 1) / 2];
2 static int binoCo(int n, int k) {
3     if (k < 0 || k > n) return 0;
4     if (2 * k > n) binoCo(n, n - k);
5     if (mem[n][k] > 0) return mem[n][k];
6     int ret = 1;
7     for (int i = 1; i <= k; i++) {
8         ret *= n - k + i;
9         ret /= i;
10        mem[n][i] = ret;
11    }
12    return ret;
13 }
```

MD5: 3a459246143bbdc49336d77c9b2720e4

1.3 Eulersche φ -Funktion

$$\varphi(n \in \mathbb{N}) := |\{a \in \mathbb{N} | 1 \leq a \leq n \wedge \text{ggT}(a, n) = 1\}|$$

$$\varphi(n \cdot m) = \varphi(n) \cdot \varphi(m)$$

```
1 #include <iostream>
2 #include <cmath>
3 using namespace std;
4 int phi(int);
5 int main(){
6     int n;
```

```
7     while((cin>>n)!=0) cout << phi(n) << endl;
8     return 0;
9 }
10
11 int phi(int n){
12     int coprime = 1;
13     int primes[] = {2,3,5,7,11,13}; //...
14     int primessizes = 6; //anpassen !
15     //zusätzlich Primfaktorzerlegung v. n
16     for(int i =0; i<primessizes; i++){
17         int anz = 0;
18         while(n % primes[i] == 0){
19             n = n / primes[i];
20             anz ++;
21             cout<<"p: " <<primes[i]<<endl;
22         }
23         if(anz>0)
24             coprime *= ((int) pow((double) primes[i],
25                 (double)(anz-1))*(primes[i] -
26 1));
27         if(n==1) break;
28     }
29     if(n != 1){
30         coprime *= (n - 1);
31     }
32     return coprime;
33 }
```

2 Mathematisch Formeln und Gesetze

2.1 Catalan

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \prod_{k=2}^n (n+k)/k$$

$$C_{n+1} = \frac{4n+2}{n+2} C_n = \sum_{k=0}^n C_k C_{n-k}$$

2.2 kgV und ggT

$$\text{ggT}(n, m) \cdot \text{kgV}(m, n) = |m \cdot n|$$

2.3 modulare Exponentiation

$$b^e \equiv c \pmod{m}$$

$$b^e = b^{(\sum_{i=0}^{n-1} a_i 2^i)} = \prod_{i=0}^{n-1} (b^{2^i})^{a_i}$$

```
1 function modular_pow(base, exponent, modulus)
2     result := 1
3     while exponent > 0
4         if (exponent mod 2 == 1):
5             result := (result * base) mod modulus
6             exponent := exponent >> 1
7             base = (base * base) mod modulus
8     return result
```

2.4 Modulare Arithmetik

Bedeutung der größten gemeinsamen Teiler ($d = \text{ggT}(a, b)$, $s, t := \text{EEA}(a, b)$):

$$d = \text{ggT}(a, b) = as + bt.$$

Verwendung zur Berechnung des inversen Elements b^{-1} zu b bezüglich der Basis einer Restklassengruppe $a \in \mathbb{P}$ ($1 \equiv b^{-1}b$)

mod 1).:

$$d = 1 \Rightarrow 1 \equiv t \cdot b \pmod{a} \Rightarrow b^{-1} := t$$

$d \neq 1 \Rightarrow b^{-1}$ existiert nicht bzgl a, b .

2.4.1 Erweiterter Euklidischer Algorithmus

```

1 static int[] eea(int a, int b) {
2     int[] dst = new int[3];
3     if (b == 0) {
4         dst[0] = a;
5         dst[1] = 1;
6         return dst; // a, 1, 0
7     }
8     dst = eea(b, a % b);
9     int tmp = dst[2];
10    dst[2] = dst[1] - ((a / b) * dst[2]);
11    dst[1] = tmp;
12    return dst;
13 }
```

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2.5 Kombinatorik

	mit ZL	ohne ZL
Variat.	n^k	$\frac{n!}{(n-k)!}$
Kombinat.	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$

3 Datenstrukturen

3.1 Fenwick Tree (Binary Indexed Tree)

```

1 class FenwickTree {
2     private int[] values;
3     private int n;
4     public FenwickTree(int n) {
5         this.n = n;
6         values = new int[n];
7     }
8     public int get(int i) { //get value of i
9         int x = values[i];
10        while (i > 0) {
11            x += values[i];
12            i -= i & -i;
13        }
14        return x;
15    }
16    public void add(int i, int x) { // add x to interval
17        // [i, n]
18        if (i == 0) values[0] += x;
19        else {
20            while (i < n) {
21                values[i] += x;
22                i += i & -i;
23            }
24        }
25    }
26 }
```

MD5: da8d56a0188958c7d35409b7a6fb7a9c

4 Graphen

Graph $G = (V, E)$ mit Kanten E und Knoten V . i.A.: $n = |V(G)|, m = |E|$

Es gilt: $m = n - 1$ gdw. G Baum; $2 \mid \deg(v \in V)$ gdw. ex. Eulerkreis und G (stark, falls gerichtet) zusammenhängend.

4.1 planare Graphen

$|E| \leq 3|V| - 6$ (notwendige Bedingung) oder Eulersche Polyederformel $|V| + |F| - |E| = 2$

4.2 Topologische Sortierung

```

1 static List<Integer> topoSort(Map<Integer, List<
2     Integer>> edges,
3     Map<Integer, List<Integer>> revedges) {
4     Queue<Integer> q = new LinkedList<Integer>();
5     List<Integer> ret = new LinkedList<Integer>();
6     Map<Integer, Integer> indeg = new HashMap<Integer,
7         Integer>();
8     for (int v : revedges.keySet()) {
9         indeg.put(v, revedges.get(v).size());
10        if (revedges.get(v).size() == 0)
11            q.add(v);
12    }
13    while (!q.isEmpty()) {
14        int tmp = q.poll();
15        ret.add(tmp);
16        for (int dest : edges.get(tmp)) {
17            indeg.put(dest, indeg.get(dest) - 1);
18            if (indeg.get(dest) == 0)
19                q.add(dest);
20        }
21    }
22    return ret;
23 }
```

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4.3 Prim (Minimum Spanning Tree)

```

1 #define WHITE 0
2 #define BLACK 1
3 #define INF INT_MAX
4
5 int baum( int **matrix, int N){
6     int i, sum = 0;
7
8     int color[N];
9     int dist[N];
10
11    // markiere alle Knoten ausser 0 als unbesucht
12    color[0] = BLACK;
13    for( i=1; i<N; i++){
14        color[i] = WHITE;
15        dist[i] = INF;
16    }
17
18    // berechne den Rand
19    for( i=1; i<N; i++){
20        if( dist[i] > matrix[i][nextIndex]){
21            dist[i] = matrix[i][nextIndex];
22        }
23    }
24
25    while( 1){
26        int nextDist = INF, nextIndex = -1;
27
```

```

28  /* Den naechsten Knoten waehlen */
29  for(i=0; i<N; i++){
30      if( color[i] != WHITE) continue;
31
32      if( dist[i] < nextDist){
33          nextDist = dist[i];
34          nextIndex = i;
35      }
36  }
37
38  /* Abbruchbedingung */
39  if( nextIndex == -1) break;
40
41  /* Knoten in MST aufnehmen */
42  color[nextIndex] = RED;
43  sum += nextDist;
44
45  /* naechste kuerzeste Distanzen berechnen */
46  for( i=0; i<N; i++){
47      if( i == nextIndex || color[i] == BLACK )
48          continue;
49
50      if( dist[i] > matrix[i][nextIndex]){
51          dist[i] = matrix[i][nextIndex];
52      }
53  }
54
55  return sum;
56 }

```

4.4 Kruskal

```

1  public static LinkedList<Edge> kruskal(LinkedList<Edge>
    > adjList, int root, int nodeCount) {
2      LinkedList<SortedSet<Integer>> branches = new
        LinkedList<SortedSet<Integer>>();
3      for (int i = 0; i < nodeCount; i++) {
4          branches.add(new TreeSet<Integer>());
5          branches.get(branches.size() - 1).add(i);
6      }
7
8      PriorityQueue<Edge> edges = new PriorityQueue<Edge>
        >(1, new Comparator<Edge>() {
9          @Override
10         public int compare(Edge e1, Edge e2) {
11             if (e1.weight <= e2.weight) {
12                 return -1;
13             } else {
14                 return 1;
15             }
16         }
17     });
18     edges.addAll(adjList);
19     LinkedList<Edge> result = new LinkedList<Edge>();
20
21     while (branches.size() > 1) {
22         Edge min = edges.remove();
23
24         SortedSet<Integer> from = null;
25         for (SortedSet<Integer> branchFrom : branches) {
26             if (branchFrom.contains(min.from)) {
27                 if (!branchFrom.contains(min.to)) {
28                     from = branchFrom;
29                     break;
30                 }
31             }

```

```

32     }
33
34     if (from != null) {
35         for (SortedSet<Integer> branchTo : branches) {
36             if (!(from.equals(branchTo))) {
37                 if (branchTo.contains(min.to)) {
38                     from.addAll(branchTo);
39                     branches.remove(branchTo);
40                     result.add(min);
41                     break;
42                 }
43             }
44         }
45     }
46 }
47
48 return result;
49 }

```

4.5 Floyd-Warshal ($\mathcal{O}(n^3)$)

```

1  for(int i = 0; i<n; i++)
2      for(int j = 0; j<n; j++)
3          if((i,j) ∈ E(G)){
4              d[i,j] = w[i,j];
5          }
6          else
7              d[i,j] = ∞
8  for(int k = 0; k<n; k++)
9      for(int i = 0; i<n; i++)
10         for(int j = 0; j<n; j++)
11             d[i,j] = min (d[i,j], d[i,k] + d[k,j]);

```

4.6 Dijkstra

- alle kürzesten Wege von einem Knoten aus in $\mathcal{O}(\#Kanten + \#Knoten)$
- negative Kanten:
 - auf alle Kantengewichte $|min| + 1$ (damit 0 nicht entsteht)
 - Kantenzahl zum Ziel mitspeichern

$$\frac{\text{Weglänge}}{\text{Kantenzahl} \cdot (|min| + 1)}$$

```

1  // look for shortest distance from a to b in adjacency
    matrix
2  // visited nodes for breadth first search
3  bool nodeVisited[26];
4  for (int k=0; k<26; k++) {
5      nodeVisited[k]=false;
6  }
7  queue<int> searchQueue;
8  queue<string> outputQueue;
9  searchQueue.push(aNumber); // start search from a
10 string start="";
11 start += a[0];
12 outputQueue.push(start);
13 string outputString;
14 while (searchQueue.empty()==false && nodeVisited[
    bNumber]==false) {
15     int node=searchQueue.front();
16     searchQueue.pop();
17     string nodeString=outputQueue.front();
18     outputQueue.pop();

```

```

19     for (int k=0; k<26; k++) {
20         if (cities[node][k]==true &&
            nodeVisited[k]==false) {
21             searchQueue.push(k);
22             nodeVisited[k]=true;
23             char addToOutput=k+'A';
24             string s=nodeString;
25             s += addToOutput;
26             outputQueue.push(s);
27             if (k==bNumber) {
28                 outputString=s;
29             }
30         }
31     }
32 }
33 cout << outputString << "\n";

```

4.7 Belman-Ford

```

1 procedure BellmanFord(list vertices, list edges,
    vertex source)
2 // This implementation takes in a graph,
    represented as lists of vertices
3 // and edges, and modifies the vertices so that
    their distance and
4 // predecessor attributes store the shortest paths.
5
6 // Step 1: initialize graph
7 for each vertex v in vertices:
8     if v is source then v.distance := 0
9     else v.distance := infinity
10    v.predecessor := null
11
12 // Step 2: relax edges repeatedly
13 for i from 1 to size(vertices)-1:
14     for each edge uv in edges: // uv is the edge
        from u to v
15         u := uv.source
16         v := uv.destination
17         if u.distance + uv.weight < v.distance:
18             v.distance := u.distance + uv.weight
19             v.predecessor := u
20
21 // Step 3: check for negative-weight cycles
22 for each edge uv in edges:
23     u := uv.source
24     v := uv.destination
25     if u.distance + uv.weight < v.distance:
26         error "Graph contains a negative-weight
            cycle"

```

4.8 MaxFlow

```

1 public class Flow {
2     static class Edge {
3         int c;
4         int f = 0;
5         Vertex s;
6         Vertex d;
7         Edge(int cap, Vertex source, Vertex dest) {
8             c = cap;
9             s = source;
10            d = dest;
11        }
12        int res(Vertex v) {

```

```

13        if (v == d) return f;
14        else return c - f;
15    }
16 }
17 static class Vertex {
18     List<Edge> lks = new ArrayList<Edge>();
19 }
20 static int maxFlow(Vertex so, Vertex si) {
21     ff: while (true) {
22         HashMap<Vertex, Edge> etp = new HashMap<Vertex,
            Edge>();
23         List<Vertex> fringe = new ArrayList<Vertex>();
24         fringe.add(so);
25         etp.put(so, null);
26         int minRes = Integer.MAX_VALUE;
27         boolean foundrp = false;
28         bfs: while (!fringe.isEmpty()) {
29             List<Vertex> newFringe = new ArrayList<Vertex>
                >();
30             for (Vertex v : fringe) {
31                 for (Edge e : v.lks) {
32                     Vertex child = (e.d == v) ? e.s : e.d;
33                     if (!etp.containsKey(child) && e.res(v) >
                        0) {
34                         etp.put(child, e);
35                         newFringe.add(child);
36                         minRes = Math.min(minRes, e.res(v));
37                         if (child == si) {
38                             foundrp = true;
39                             break bfs;
40                         }
41                     }
42                 }
43             }
44             fringe = newFringe;
45         }
46         if (!foundrp) break ff;
47         Vertex nxt = si;
48         while (nxt != so) {
49             Vertex prv = nxt;
50             Edge edge = etp.get(prv);
51             if (edge.s == prv) {
52                 edge.f = edge.f - minRes;
53                 nxt = edge.d;
54             } else {
55                 edge.f = edge.f + minRes;
56                 nxt = edge.s;
57             }
58         }
59         int flow = 0;
60         for (Edge e : so.lks) {
61             flow += e.f;
62         }
63     }
64 }

```

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4.9 Bipartite Matching

```

1 import java.util.*;
2
3
4 public class BPM {
5     int m, n;
6     boolean[][] graph;
7     boolean seen[];
8     int matchL[]; //What left vertex i is matched
        to (or -1 if unmatched)

```

```

9      int matchR[];    //What right vertex j is matched
                        to (or -1 if unmatched)
10
11     int maximumMatching() {
12         //Read input and populate graph[][]
13         //Set m to be the size of L, n to be the
            size of R
14         Arrays.fill(matchL, -1);
15         Arrays.fill(matchR, -1);
16
17         int count = 0;
18         for (int i = 0; i < m; i++) {
19             Arrays.fill(seen, false);
20             if (bpm(i)) count++;
21         }
22         return count;
23     }
24
25     boolean bpm(int u) {
26         //try to match with all vertices on right
            side
27         for (int v = 0; v < n; v++) {
28             if (!graph[u][v] || seen[v]) continue;
29             seen[v] = true;
30             //match u and v, if v is unassigned, or
                if v's match on the left side can be
                reassigned to another right vertex
31             if (matchR[v] == -1 || bpm(matchR[v])) {
32                 matchL[u] = v;
33                 matchR[v] = u;
34                 return true;
35             }
36         }
37         return false;
38     }
39
40     public void run(){
41
42         Scanner sc = new Scanner(System.in).useLocale(
            Locale.US);
43         int T = sc.nextInt();
44         while(T-->0){
45             n = sc.nextInt();
46             m = sc.nextInt();
47             int K = sc.nextInt();
48             graph = new boolean [m][n];
49             matchL = new int[m];
50             matchR = new int[n];
51             seen = new boolean[n];
52             while(K-->0){
53                 int y = (int)sc.nextDouble();
54                 int x = (int)sc.nextDouble();
55                 graph[x][y] = true;
56             }
57             System.out.println(maximumMatching());
58         }
59         sc.close();
60     }
61
62     public static void main(String[] args){
63         (new BPM()).run();
64     }
65 }
66 }

```

MD5: -----

4.10 Bitonic TSP

All nodes n_i are sorted in x -direction; $d(i, j)$ is the distance:

```

1 public static double bitonic(double[][] d) {
2     int N = d.length;
3     double[][] B = new double[N][N];
4     for (int j = 0; j < N; j++) {
5         for (int i = 0; i <= j; i++) {
6             if (i < j - 1)
7                 B[i][j] = B[i][j - 1] + d[j - 1][j];
8             else {
9                 double min = 0;
10                for (int k = 0; k < j; k++) {
11                    double r = B[k][i] + d[k][j];
12                    if (min > r || k == 0)
13                        min = r;
14                }
15                B[i][j] = min;
16            }
17        }
18    }
19    return B[N-1][N-1];

```

MD5: 49fca508fb184da171e4c8e18b6ca4c7

5 Geometrie

5.1 Kreuzprodukt, Skalarprodukt

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}, \quad \langle a, b \rangle = \sum a_i b_i = |a||b| \cos(\angle(a, b))$$

5.2 Orthogonale Projektion

r_0 : Ortsvektor; u : Richtungsvektor; n : Normalenvektor

$$P_g(\vec{x}) = \vec{r}_0 + \frac{(\vec{x} - \vec{r}_0) \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$P_g(\vec{x}) = \vec{x} - \frac{(\vec{x} - \vec{r}_0) \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} \text{ (nur 2D bzw. 3D auf Ebene)}$$

5.3 Rotation

```

1 static Point rotate(Point v, double a) {
2     double cos = Math.cos(a);
3     double sin = Math.sin(a);
4     double x = cos * v.x - sin * v.y;
5     double y = sin * v.x + cos * v.y;
6     return new Point(x, y);
7 }

```

5.4 Geradenschnittpunkt

$$g_1 : ax + by = c; g_2 : px + qx = r; \Rightarrow \vec{p} = \frac{1}{aq-bp} \begin{pmatrix} x = cq - br \\ y = ar - cp \end{pmatrix}$$

$$g_1 : \vec{p} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} + s \begin{pmatrix} s_x \\ s_y \end{pmatrix} \quad g_2 : \vec{p} = \begin{pmatrix} q_x \\ q_y \end{pmatrix} + t \begin{pmatrix} t_x \\ t_y \end{pmatrix} \quad w_x = (r_x - q_x), w_y = (r_y - q_y)$$

$$\Rightarrow D = (s_x t_y - t_x s_y), D_s = (t_x w_y - t_y w_x), D_t = (s_y w_x - s_x w_y); s = D_s / D, t = D_t / D$$

5.5 Zusammenhang Kreuzprodukt & Sinus

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \angle(\vec{a}, \vec{b})$$

5.6 Dreiecksfläche

$$F = \sqrt{s(s-a)(s-b)(s-c)}; s = \frac{a+b+c}{2}$$

5.7 Graham Scan (Convex Hull)

```

1  public static class Point implements Comparable<
    Point> {
2      double x, y, r;
3      Point p0;
4      public Point(double x, double y) {
5          this.x = x;
6          this.y = y;
7      }
8      public int compareTo(Point p) {
9          double s = ccw(p0, p, this);
10         if (s != 0) return (int) Math.signum(s);
11         else return (int) Math.signum(p.r - r);
12     }
13     public static double dist(Point a, Point b) {
14         double x = a.x - b.x;
15         double y = a.y - b.y;
16         return Math.sqrt(x * x + y * y);
17     }
18     public static double ccw(Point a, Point b, Point c)
19     {
20         return (b.x - a.x) * (c.y - a.y) - (b.y - a.y) * (
21             c.x - a.x);
22     }
23     static List<Point> graham(List<Point> P) {
24         Point p0 = P.get(0);
25         for (int i = 1; i < P.size(); i++) {
26             Point p = P.get(i);
27             if (p.y < p0.y || (p.y == p0.y && p.x < p0.x)) {
28                 p0 = p;
29             }
30         }
31         P.remove(p0);
32         for (Point p : P) {
33             p.r = dist(p0, p);
34             p.p0 = p0;
35         }
36         Collections.sort(P);
37         Iterator<Point> I = P.iterator();
38         Point f = I.next();
39         while (I.hasNext()) {
40             Point p = I.next();
41             if (ccw(p0, p, f) == 0) {
42                 I.remove();
43             } else {
44                 f = p;
45             }
46         }
47         LinkedList<Point> S = new LinkedList<Point>();
48         if (P.isEmpty()) {
49             S.add(p0);
50         } else {
51             S.push(p0);
52             S.push(P.get(0));
53             for (int i = 1; i < P.size(); i++) {
54                 Point b = S.pop();
55                 Point a = S.peek();
56                 S.push(b);
57                 while (ccw(a, b, P.get(i)) <= 0) {
58                     S.pop();
59                 }
60             }
61         }
62     }

```

```

55         b = S.pop();
56         a = S.peek();
57         S.push(b);
58     }
59     S.push(P.get(i));
60 } }
61 return S;
62 }

```

MD5: fa3b15e54ec7447485870a1978f8aac4

5.8 Line Intersection

• Mehr als 2 Linien:

- findet nicht alle Intersection Points, aber immer wenn einer existiert, dann angegeben

- $O(n \log n + l \log n)$

• 2 Linien:

- line intersection (test if possible!)

- Achtung: beide Reihenfolgen testen: if ((checkLines(readLines[j],newLine) == true) && (checkLines(newLine,readLines[j]) == true))

```

1 struct line {
2     int x0;
3     int y0;
4     int x1;
5     int y1;
6 };
7
8 // prueft, ob sich die Linien schneiden koennen
9 bool checkLines(line a, line b) {
10     // Vektor Linie a
11     int x0 = a.x1 - a.x0;
12     int y0 = a.y1 - a.y0;
13     // Vektor zu Startpunkt b
14     int x1 = b.x0 - a.x0;
15     int y1 = b.y0 - a.y0;
16     // Vektor zu Endpunkt b
17     int x2 = b.x1 - a.x0;
18     int y2 = b.y1 - a.y0;
19     // Kreuzprodukte berechnen
20     int crossProduct1 = x0 * y1 + x1 * y0;
21     int crossProduct2 = x0 * y2 + x2 * y0;
22     // Wenn ein Produkt negativ, das andere positiv ist
23     // , koennen sich die Linien schneiden
24     if (crossProduct1 * crossProduct2 < 0) {
25         return true;
26     }
27     return false;
28 }

```

5.9 Punkt in Polygon

KreuzProdTest: -1: $A \rightarrow R$ schneidet BC (ausser unterer Endpunkt); 0: A auf BC ; +1: sonst

PiP: Input: $P[i]$ ($x[i], y[i]$); $P[0]=P[n]$; Output: -1: Q außerhalb Polygon, 0: Q auf Polygon, +1: Q innerhalb des Polygons


```

1  public static int KreuzProdTest(double ax, double ay
    , double bx, double by,
2    double cx, double cy) {
3    if (ay == by && by == cy) {
4      if ((bx <= ax && ax <= cx) || (cx <= ax && ax <=
        bx))
5        return 0;
6      else
7        return +1;
8    }
9    if (by > cy) { double tmpx = bx; double tmpy = by; bx = cx; by =
        cy; cx = tmpx; cy = tmpy; }
10   if (ay == by && ax == bx) return 0;
11   if (ay <= by || ay > cy) return +1;
12   double delta = (bx - ax) * (cy - ay) - (by - ay) * (cx - ax);
13   if (delta > 0) return -1; else if (delta < 0) return +1;
        else return 0;
14 }
15 public static int PunktInPoly(double[] x, double[] y,
    double qx, double qy) {
16   int n = x.length - 1;
17   int t = -1;
18   for (int i = 0; i <= n - 1; i++) {
19     t = t * KreuzProdTest(qx, qy, x[i], y[i], x[i +
        1], y[i + 1]); }
20   return t;
21 }

```

MD5: 38a79d6979334bc6a01381e15eef6e04

5.10 Fläche eines Polygons

Input: Polygon-Koordinaten sortiert im Uhrzeigersinn

```

1  static double area(List<Point> p) {
2    double a = 0;
3    Point q = p.get(p.size() - 1);
4    Point r;
5    for (Point r : p) {
6      a += (q.x + r.x) * (q.y - r.y);
7      q = r;
8    }
9    return a / -2;
10 }

```

MD5: 1f1dbdaaf78726c57e3e0ece63fe1cb3

6 2-SAT-Solver

6.1 2-Sat mit SCC

```

1  public class D_Manha {
2    static class Node {
3      ArrayList<Node> out = new ArrayList<Node>();
4      ArrayList<Node> in = new ArrayList<Node>();
5      int var;
6      boolean explored = false;
7      boolean discovered = false;
8      int CCC;
9      public Node(int v, String n) {
10        var = v;
11        name = n;
12      }
13    }
14    static void impl(Node x, Node y) {
15      x.out.add(y);

```

```

        y.in.add(x);
    }
    public static void main(String[] args) {
        Scanner in = new Scanner(System.in);
        int n = in.nextInt();
        while (n-- > 0) {
            ArrayList<Node> graph; //TODO :
                implikationsgraph
            // Kosaraju
            S = new ArrayList<Node>();
            for (Node v : graph) {
                if (!v.explored) {
                    DFS(v);
                }
            }
            for (Node v : graph) {
                v.explored = false;
                v.discovered = false;
            }
            int CCCidx = 0;
            do {
                ArrayList<Node> CCC = new ArrayList<Node>();
                DFSTrans(S.get(S.size()-1), CCC, CCCidx++);
                S.removeAll(CCC);
            } while (!S.isEmpty());

            boolean possible = true;
            for (int i = 1; i <= s; i++) {
                if (st.get(i).CCC == sf.get(i).CCC) {
                    possible = false;
                }
            }
            for (int i = 1; i <= a; i++) {
                if (at.get(i).CCC == af.get(i).CCC) {
                    possible = false;
                }
            }
            if (possible) {
                System.out.println("Yes");
            } else {
                System.out.println("No");
            }
        }
        static ArrayList<Node> S;
        public static void DFS(Node v) {
            v.discovered = true;
            for (Node u : v.out) {
                if (!u.discovered) {
                    DFS(u);
                }
            }
            v.explored = true;
            S.add(v);
        }
        public static void DFSTrans(Node v, ArrayList<Node>
            CCC, int CCCidx) {
            v.discovered = true;
            for (Node u : v.in) {
                if (!u.discovered) {
                    DFSTrans(u, CCC, CCCidx);
                }
            }
            v.explored = true;
            CCC.add(v);
            v.CCC = CCCidx;
        }
    }

```

6.2 Hilfsalgorithmen

6.2.1 Erzeugen eines Graphens

```

1 SAT2Graph( $\varphi = (\alpha_1 \vee \beta_1) \wedge \dots \wedge (\alpha_m, \beta_m)$ ) {
2   G: Graph als Adjazenzliste
3   for(int i = 0 < m; i++){
4     jede Klausel liefert zwei Implikationen
5     Fuege Kanten  $(-\alpha_i, \beta_i), (-\beta_i, \alpha_i)$  zu G hinzu.
6 }

```

6.2.2 Indexumrechnung

```

1 /** rechnet den Index fuer den Array Zugriff um */
2 idx(int i) := n + i + ((i > 0) ? (-1) : 0)

```

6.3 Suchen eines Pfades

```

1 /**
2  Prueft mithilfe einer Breitensuche ob ein Weg
3  von Knoten x nach -x existiert
4  */
5  boolean BFSSATCheck(SATGraph G, int x) {
6    boolean[] seen = new boolean[2 * n];
7    Queue<Integer> queue;
8    queue.add(x); seen[idx(x)] = true;
9    while (!queue.isEmpty()) {
10     Integer q = queue.poll();
11     for (Integer p : G.get(idx(q))) {
12       if (!seen[idx(p)]) {
13         queue.add(p);
14         seen[idx(p)] = true;
15       }
16       if (p == -x) return true;
17     }
18   }
19   return seen[idx(-x)];
20 }

```

6.4 Algorithmus zum Prüfen der Erfüllbarkeit

```

1 /**
2  Prueft ob fuer eine 2-CNF eine Belegung existiert
3  */
4  boolean SAT2Check( $\varphi = (\alpha_1 \vee \beta_1) \wedge \dots \wedge (\alpha_n, \beta_n)$ ) {
5    SAT2Graph G( $\varphi$ ) erzeugen
6    for(int i = 0 < n; i++){
7      if(BFSSATCheck(G,i) && BFSSATCheck(G,-i))
8        return false;
9      //Es gibt einen  $i \rightarrow -i$  und  $-i \rightarrow i$  Weg
10     return true;
11 }

```

6.5 Algorithmus zur Belegung einer 2-CNF

```

1 /**
2  * Ermittelt falls moeglich eine gueltige Belegung fuer
3  * eine 2-CNF
4  */
5  Solve2SAT( $\varphi = (\alpha_1 \vee \beta_1) \wedge \dots \wedge (\alpha_n, \beta_n)$ ) {
6    SAT2Graph G( $\varphi$ ) erzeugen
7    vars = [0,...,0] //(2*n) Variablenbelegung
8    assigned = [false,...,false] //(n+1) Belegung
9    zugewiesen?
10
11   for (int x = 1; x < n + 1; x++) {

```

```

oldVars = vars.clone();oldAssigned = assigned.
clone();
if (assign(vars, assigned, x))
  continue; //x:=1

vars = oldVars;assigned = oldAssigned;
if (!assign(vars, assigned, -x))
  return null; //x:=0 liefert auch keine Loesung
}
return vars; //gueltige Belegung
}
/**
* Belegt die Variable x mit 1 und liefert false,
* falls dies nicht moeglich ist
* WICHTIG: Parameter werden veraendert (Referenzen
  uebergeben!).
*/
boolean assign(ArrayList<Integer> vars,
  ArrayList<Boolean> assigned, int x) {
  int xi = (x < 0) ? -x : x;
  if (assigned[xi]) return (vars[idx(x)] == 1);
  //Belege x, -x mit 0,1:
  vars[idx(x)] = 1;vars[idx(-x)] = 0;
  assigned[xi] = true;
  for (Integer k : G.get(idx(x))) {
    if (!assign(vars, assigned, k)) {
      //Belegung nicht weiter moeglich
      assigned[xi] = false; return false;
    }
  }
  return true;
}

```

7 Verschiedenes

7.1 Potenzmenge

```

1 static <T> Iterator<List<T>> powerSet(final List<T> l)
2 {
3   return new Iterator<List<T>>() {
4     int i; // careful: i becomes 2^l.size()
5     public boolean hasNext() {
6       return i < (1 << l.size());
7     }
8     public List<T> next() {
9       Vector<T> temp = new Vector<T>();
10      for (int j = 0; j < l.size(); j++)
11        if (((i >> j) & 1) == 1)
12          temp.add(l.get(j));
13      i++;
14      return temp;
15    }
16    public void remove() {}
17  };
18 }

```

7.2 LongestCommonSubsequence

```

1 #include <iostream>
2 #include <vector>
3 #include <string>
4 #include <sstream>
5 #include <algorithm>
6 #include <iterator>

```

```

7 using namespace std;
8 #define MAX(a,b) (a > b) ? a : b
9
10 string X,Y;
11 vector< vector<int> > c(101, vector<int>(101,0));
12 int m,n,ctr;
13
14 int LCS(){
15     m = X.length(),n=Y.length();
16     c.resize(m+1);
17     for(int i = 0; i<n+1; i++) {
18         c[i].resize(n+1);
19         c[i][0] = 0;
20     }
21     int i,j;
22     for (i=0;i<=m;i++)
23         for (j=0;j<=n;j++)
24             c[i][j]=0;
25
26     for (i=1;i<=m;i++)
27         for (j=1;j<=n;j++)
28         {
29             if (X[i-1]==Y[j-1])
30                 c[i][j]=c[i-1][j-1]+1;
31             else
32                 c[i][j]=max(c[i][j-1],c[i-1][j]);
33         }
34     return c[m][n];
35 }
36 /** Print a single LCS */
37 void printLCS(int i,int j){
38     if (i==0 || j==0)
39         return;
40     if (X[i-1]==Y[j-1])
41     {
42         printLCS(i-1,j-1);
43         cout<<X[i-1];
44     }
45     else if (c[i][j]==c[i-1][j])
46         printLCS(i-1,j);
47     else
48         printLCS(i,j-1);
49 }
50
51 int main(){
52     while(cin>>X>>Y) {
53         cout << "Length:_" << LCS() << endl;
54         printLCS(m,n);
55         cout<<endl ;
56     }}

```

7.3 LongestCommonSubstring

```

1 private static List<String> longestCommonSubstring(
2     String S1, String S2)
3 {
4     List<String> ret = new ArrayList<String>();
5     List<Integer> idx = new ArrayList<Integer>();
6     int Start = 0;
7     int Max = 0;
8     for (int i = 0; i < S1.length(); i++)
9     {
10         for (int j = 0; j < S2.length(); j++)
11         {
12             int x = 0;
13             while (S1.charAt(i + x) == S2.charAt(j +
14                 x))

```

```

13         {
14             x++;
15             if (((i + x) >= S1.length()) || ((j
16                 + x) >= S2.length())) break;
17         }
18         if (x > Max)
19         {
20             Max = x;
21             Start = i;
22             idx.clear();
23             idx.add(Start);
24         } else if (x==Max){
25             Start = i;
26             idx.add(Start);
27         }
28     }
29     HashSet<String> set = new HashSet<String>(idx.
30         size(),1f);
31     for(Integer start : idx){
32         String substr = S1.substring(start,start+Max);
33         if(!set.contains(substr)){
34             ret.add(substr);
35             set.add(substr);
36         }
37     }
38     Collections.sort(ret);
39     //return S1.substring(Start, (Start + Max));
40     return ret;

```

7.4 LongestIncreasingSubsequence

```

1 #include <vector>
2 using namespace std;
3
4 /** finde LIS in O(n log k)
5 *a: Sequenz (in)
6 *b: LIS (out)
7 */
8 void find_lis(vector<int> &a, vector<int> &b)
9 {
10     vector<int> p(a.size());
11     int u, v;
12     if (a.empty()) return;
13     b.push_back(0);
14
15     for (size_t i = 1; i < a.size(); i++)
16     {
17         // ist naechstes Element a[i] groesser als
18         letztes der aktuelle LIS
19         // a[b.back()], fuege es (Index) an "b" an.
20         if (a[b.back()] < a[i]) {
21             p[i] = b.back();
22             b.push_back(i);
23             continue;
24         }
25
26         // finde kleinstes El. in LIS (index in b)
27         welches gerade groesser als a[i] ist
28         // binaere suche |b|<=k => O(log k)
29         for (u = 0, v = b.size()-1; u < v;)
30         {
31             int c = (u + v) / 2;
32             if (a[b[c]] < a[i]) u=c+1; else v=c;
33         }

```

```

33         // aktualisiere b falls neuer Wert kleiner als
           vorheriger kleinerer Wert
34     if (a[i] < a[b[u]])
35     {
36         if (u > 0) p[i] = b[u-1];
37         b[u] = i;
38     }
39 }
40
41 for (u = b.size(), v = b.back(); u--; v = p[v]) b[u]
    = v;
42 }
43
44 #include <cstdio>
45 int main()
46 {
47     int a[] = { 1, 9, 3, 8, 11, 4, 5, 6, 4, 19, 7, 1, 7
48         };
49     vector<int> seq(a, a+sizeof(a)/sizeof(a[0])); // seq
50         : Eingabesequent
51     vector<int> lis; // lis
52         : Index Vektor fuer LIS
53     find_lis(seq, lis);
54     //Sequenz ausgeben:
55     for (size_t i = 0; i < lis.size(); i++)
56         printf("%d_", seq[lis[i]]);
57         printf("\n");
58     return 0;
59 }

```

7.5 Permutation & Sequenzen

```

1 import java.util.Scanner;
2 public class PermsAndSequ {
3     public static void main(String[] args) {
4         Scanner sc = new Scanner(System.in);
5         int n;
6         while ((n = sc.nextInt()) != 0) {
7             int k = sc.nextInt();
8             Sequences(n, k);
9             Permutations(n);
10        }
11    }
12
13    public static void Sequences(int n, int k) {
14        int[] x = new int[k];
15        for (int i = 0; i < k; i++)
16            x[i] = 1;
17        Print(x);
18        while (true) {
19            boolean lastX = true;
20            for (int i = 0; i < k; i++)
21                if (x[i] != n) {
22                    lastX = false;
23                    break;
24                }
25            if (lastX)
26                break;
27            int p = k - 1;
28            while (!(x[p] < n))
29                p--;
30            x[p] = x[p] + 1;
31            for (int i = p + 1; i < k; i++)
32                x[i] = 1;
33            Print(x);
34        }
35    }

```

```

36    }
37    public static void Permutations(int n) {
38        int[] x = new int[n];
39        for (int i = 0; i < n; i++)
40            x[i] = i + 1;
41        Print(x);
42        while (true) {
43            boolean lastX = true;
44            for (int i = 0; i < n - 1; i++)
45                if (x[i] < x[i + 1]) {
46                    lastX = false;
47                    break;
48                }
49            if (lastX) break;
50            int k = n - 1 - 1;
51            while (x[k] > x[k + 1]) k--;
52            int t = k + 1;
53            while (t < (n - 1) && x[t + 1] > x[k])
54                t++;
55            int tmp = x[k];
56            x[k] = x[t];
57            x[t] = tmp;
58            // reverse x[k+1] ... x[n-1]
59            for (int i = 0; i <= ((n - 1) - (k + 1)) / 2; i++) {
60                tmp = x[k + 1 + i];
61                x[k + 1 + i] = x[n - 1 - i];
62                x[n - 1 - i] = tmp;
63            }
64            Print(x);
65        }
66    }
67    public static void Print(int[] x) {
68        for (int i = 0; i < x.length; i++)
69            System.out.print(x[i] + "_");
70        System.out.println("");
71    }
72 }

```

7.6 Knuth-Morris Pratt

Finds the first occurrence of the pattern in the text.

```

1 int match(String text, String pattern, int[] jump) {
2     int j = 0;
3     if (text.length() == 0)
4         return -1;
5     for (int i = 0; i < text.length(); i++) {
6         while (j > 0 && pattern.charAt(j) != text.charAt(i))
7             j = jump[j - 1];
8         if (pattern.charAt(j) == text.charAt(i))
9             j++;
10        if (j == pattern.length())
11            return i - pattern.length() + 1;
12    }
13    return -1;
14
15    // Computes the jump function
16    int[] computeJump(String pattern) {
17        int[] jump = new int[pattern.length()];
18        int j = 0;
19        for (int i = 1; i < pattern.length(); i++) {
20            while (j > 0 && pattern.charAt(j) != pattern.
21                charAt(i))
22                j = jump[j - 1];

```

```

22  if (pattern.charAt(j) == pattern.charAt(i))      25  return jump;}
23      j++;
24  jump[i] = j;}

```

MD5: b5b9ca67a1df2c7c2913615bf1ed8a5b

8 Formatierung & Sonstiges

8.1 Ausgabeformatierung mit JAVA - DecimalFormat

Symbol	Bedeutung
0	(Ziffer) – unbelegt wird eine Null angezeigt. (0.234=(00.00)=>00.23)
#	(Ziffer) – unbelegt bleibt leer, (keine unnötigen nullen).
.	Dezimaltrenner.
,	Gruppert die Ziffern (eine Gruppe ist so groß wie der Abstand von ",ßu ".).
;	Trennzeichen. Links Muster für pos., rechts für neg. Zahlen
-	Das Standardzeichen für Negativpräfix
%	Prozentwert.
‰	Promille.
X	Alle anderen Zeichen X können ganz normal benutzt werden.
'	Ausmarkieren von speziellen Symbolen im Präfix oder Suffix

8.2 Ausgabeformatierung mit printf

%d %i	Decimal signed integer.	
%o	Octal int.	+ Vorzeichen immer ausgeben.
%x %X	Hex int.	blank pos. Zahlen mit Leerzeichen beg.
%u	Unsigned int.	# verschiedene Bedeutung:
%c	Character.	%#o (Oktal) 0 Präfix wird eingefügt.
%s	String. siehe unten.	%#x (Hex) 0x Präfix bei !=0
%f	double	%#X (Hex) 0X Präfix bei !=0
%e %E	double.	%#e Dezimalpunkt immer anzeigen.
%g %G	double.	%#E Dezimalpunkt immer anzeigen.
		%#f Dezimalpunkt immer anzeigen.
-	linksbündig.	%#g
0	Felder mit 0 ausfüllen (an Stelle von Leerzeichen).	%#G Dezimalpunkt immer anzeigen. Nullen nach Dzmpt. bleiben

```

1  int i = 123;
2  printf( "%d|_|_|%d\n" , i, -i); // |123| | -123|
3  printf( "%5d|_|%5d\n" , i, -i); // | 123| | -123|
4  printf( "|%-5d|_|%5d\n" , i, -i); // |123| | -123|
5  printf( "|%+-5d|_|%+-5d\n" , i, -i); // |+123| | -123|
6  printf( "|%05d|_|%05d\n\n" , i, -i); // |00123| | -0123|
7  printf( "|%X|_|%x\n" , 0xabc, 0xabc ); // |ABC| |abc|
8  printf( "|%08x|_|%#x\n\n" , 0xabc, 0xabc ); // |00000abc| |0xabc|
9  double d = 1234.5678;
10 printf( "%f|_|%f\n" , d, -d); // |1234,567800| | -1234,567800|
11 printf( "|%.2f|_|%.2f\n" , d, -d); // |1234,57| | -1234,57|
12 printf( "|%10f|_|%10f\n" , d, -d); // |1234,567800| | -1234,567800|
13 printf( "|%10.2f|_|%10.2f\n" , d, -d); // | 1234,57| | -1234,57|
14 printf( "|%010.2f|_|%010.2f\n",d, -d); // |0001234,57| | -001234,57|
15 String s = "Monsterbacke";
16 printf( "\n|s|\n" , s ); // |Monsterbacke|
17 printf( "|%20s|\n" , s ); // | Monsterbacke|
18 printf( "|%-20s|\n" , s ); // |Monsterbacke|
19 printf( "|%7s|\n" , s ); // |Monsterbacke|
20 printf( "|%.7s|\n" , s ); // |Monster|
21 printf( "|%20.7s|\n" , s ); // | Monster|

```

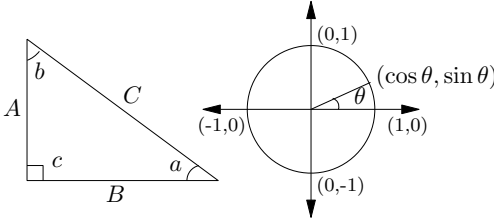
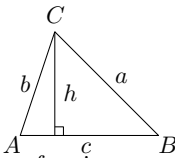
8.3 C++ Eingabe ohne bekannt Länge

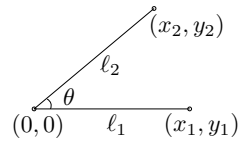
```
1 #include <iostream>
2 #include <sstream>
3 #include <istream>
4 #include <string>
5 #include <vector>
6 #include <cstdlib>
7
8 using namespace std;
9 int main(){
10     string s;
11     do{
12         getline(cin,s);
13         istream* ss;
14         ss = new istream( s );
15         while (!ss->eof())
16         {
17             string xs;
18             getline( *ss, xs, '_' ); // try to read the next field into it
19
20             int x = atoi(xs.c_str());
21             cout<<"_"<<xs;
22         }
23         cout<<endl;
24     } while(!cin.eof());
25 }
```

Theoretical Computer Science Cheat Sheet		
Definitions		Series
$f(n) = O(g(n))$	iff \exists positive c, n_0 such that $0 \leq f(n) \leq cg(n) \forall n \geq n_0$.	$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$
$f(n) = \Omega(g(n))$	iff \exists positive c, n_0 such that $f(n) \geq cg(n) \geq 0 \forall n \geq n_0$.	In general:
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.	$\sum_{i=1}^n i^m = \frac{1}{m+1} \left[(n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$
$f(n) = o(g(n))$	iff $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$.	$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}.$
$\lim_{n \rightarrow \infty} a_n = a$	iff $\forall \epsilon > 0, \exists n_0$ such that $ a_n - a < \epsilon, \forall n \geq n_0$.	Geometric series:
$\sup S$	least $b \in \mathbb{R}$ such that $b \geq s, \forall s \in S$.	$\sum_{i=0}^n c^i = \frac{c^{n+1} - 1}{c - 1}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1 - c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1 - c}, \quad c < 1,$
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \leq s, \forall s \in S$.	$\sum_{i=0}^n ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad c < 1.$
$\liminf_{n \rightarrow \infty} a_n$	$\lim_{n \rightarrow \infty} \inf\{a_i \mid i \geq n, i \in \mathbb{N}\}.$	Harmonic series:
$\limsup_{n \rightarrow \infty} a_n$	$\lim_{n \rightarrow \infty} \sup\{a_i \mid i \geq n, i \in \mathbb{N}\}.$	$H_n = \sum_{i=1}^n \frac{1}{i}, \quad \sum_{i=1}^n iH_i = \frac{n(n+1)}{2} H_n - \frac{n(n-1)}{4}.$
$\binom{n}{k}$	Combinations: Size k sub-sets of a size n set.	$\sum_{i=1}^n H_i = (n+1)H_n - n, \quad \sum_{i=1}^n \binom{i}{m} H_i = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right).$
$[n_k]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.	1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad 2. \sum_{k=0}^n \binom{n}{k} = 2^n, \quad 3. \binom{n}{k} = \binom{n}{n-k},$
$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.	4. $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}, \quad 5. \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$
$\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle$	1st order Eulerian numbers: Permutations $\pi_1 \pi_2 \dots \pi_n$ on $\{1, 2, \dots, n\}$ with k ascents.	6. $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}, \quad 7. \sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n},$
$\langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle$	2nd order Eulerian numbers.	8. $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}, \quad 9. \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n},$
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.	10. $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}, \quad 11. \left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1,$
14. $\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix} \right] = (n-1)!,$	15. $\left[\begin{smallmatrix} n \\ 2 \end{smallmatrix} \right] = (n-1)!H_{n-1},$	16. $\left[\begin{smallmatrix} n \\ n \end{smallmatrix} \right] = 1, \quad 17. \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \geq \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\},$
18. $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = (n-1) \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right] + \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right],$	19. $\left\{ \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right\} = \left[\begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right] = \binom{n}{2},$	20. $\sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = n!, \quad 21. C_n = \frac{1}{n+1} \binom{2n}{n},$
22. $\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \rangle = \langle \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \rangle = 1,$	23. $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle = \langle \begin{smallmatrix} n \\ n-1-k \end{smallmatrix} \rangle,$	24. $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle = (k+1) \langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \rangle + (n-k) \langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \rangle,$
25. $\langle \begin{smallmatrix} 0 \\ k \end{smallmatrix} \rangle = \begin{cases} 1 & \text{if } k=0, \\ 0 & \text{otherwise} \end{cases}$	26. $\langle \begin{smallmatrix} n \\ 1 \end{smallmatrix} \rangle = 2^n - n - 1,$	27. $\langle \begin{smallmatrix} n \\ 2 \end{smallmatrix} \rangle = 3^n - (n+1)2^n + \binom{n+1}{2},$
28. $x^n = \sum_{k=0}^n \langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle \binom{x+k}{n},$	29. $\langle \begin{smallmatrix} n \\ m \end{smallmatrix} \rangle = \sum_{k=0}^m \binom{n+1}{k} (m+1-k)^n (-1)^k,$	30. $m! \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} = \sum_{k=0}^n \langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle \binom{k}{n-m},$
31. $\langle \begin{smallmatrix} n \\ m \end{smallmatrix} \rangle = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \binom{n-k}{m} (-1)^{n-k-m} k!,$	32. $\langle\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \rangle\rangle = 1,$	33. $\langle\langle \begin{smallmatrix} n \\ n \end{smallmatrix} \rangle\rangle = 0 \text{ for } n \neq 0,$
34. $\langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle = (k+1) \langle\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \rangle\rangle + (2n-1-k) \langle\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \rangle\rangle,$	35. $\sum_{k=0}^n \langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle = \frac{(2n)^n}{2^n},$	
36. $\left\{ \begin{smallmatrix} x \\ x-n \end{smallmatrix} \right\} = \sum_{k=0}^n \langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle \binom{x+n-1-k}{2n},$	37. $\left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{smallmatrix} k \\ m \end{smallmatrix} \right\} = \sum_{k=0}^n \left\{ \begin{smallmatrix} k \\ m \end{smallmatrix} \right\} (m+1)^{n-k},$	

Theoretical Computer Science Cheat Sheet		
Identities Cont.		Trees
<p>38. $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{pmatrix} k \\ m \end{pmatrix} = \sum_{k=0}^n \begin{bmatrix} k \\ m \end{bmatrix} n^{n-k} = n! \sum_{k=0}^n \frac{1}{k!} \begin{bmatrix} k \\ m \end{bmatrix},$</p> <p>40. $\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\} (-1)^{n-k},$</p> <p>42. $\left\{ \begin{matrix} m+n+1 \\ m \end{matrix} \right\} = \sum_{k=0}^m k \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\},$</p> <p>44. $\binom{n}{m} = \sum_k \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{m-k},$</p> <p>46. $\left\{ \begin{matrix} n \\ n-m \end{matrix} \right\} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \begin{bmatrix} m+k \\ k \end{bmatrix},$</p> <p>48. $\left\{ \begin{matrix} n \\ \ell+m \end{matrix} \right\} \binom{\ell+m}{\ell} = \sum_k \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} \left\{ \begin{matrix} n-k \\ m \end{matrix} \right\} \binom{n}{k},$</p>	<p>39. $\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \begin{pmatrix} x+k \\ 2n \end{pmatrix},$</p> <p>41. $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \binom{k}{m} (-1)^{m-k},$</p> <p>43. $\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^m k(n+k) \begin{bmatrix} n+k \\ k \end{bmatrix},$</p> <p>45. $(n-m)! \binom{n}{m} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} (-1)^{m-k}, \text{ for } n \geq m,$</p> <p>47. $\begin{bmatrix} n \\ n-m \end{bmatrix} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \left\{ \begin{matrix} m+k \\ k \end{matrix} \right\},$</p> <p>49. $\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} \binom{n}{k}.$</p>	<p>Every tree with n vertices has $n-1$ edges.</p> <p>Kraft inequality: If the depths of the leaves of a binary tree are d_1, \dots, d_n:</p> $\sum_{i=1}^n 2^{-d_i} \leq 1,$ <p>and equality holds only if every internal node has 2 sons.</p>
Recurrences		
<p>Master method:</p> $T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1$ <p>If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then</p> $T(n) = \Theta(n^{\log_b a}).$ <p>If $f(n) = \Theta(n^{\log_b a})$ then</p> $T(n) = \Theta(n^{\log_b a} \log_2 n).$ <p>If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \leq cf(n)$ for large n, then</p> $T(n) = \Theta(f(n)).$ <p>Substitution (example): Consider the following recurrence</p> $T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$ <p>Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have</p> $t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$ <p>Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get</p> $\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$ <p>Substituting we find</p> $u_{i+1} = \frac{1}{2} + u_i, \quad u_1 = \frac{1}{2},$ <p>which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$.</p> <p>Summing factors (example): Consider the following recurrence</p> $T(n) = 3T(n/2) + n, \quad T(1) = 1.$ <p>Rewrite so that all terms involving T are on the left side</p> $T(n) - 3T(n/2) = n.$ <p>Now expand the recurrence, and choose a factor which makes the left side “telescope”</p>	<p>1($T(n) - 3T(n/2) = n$)</p> $3(T(n/2) - 3T(n/4) = n/2)$ $\vdots \quad \vdots \quad \vdots$ $3^{\log_2 n-1} (T(2) - 3T(1) = 2)$ <p>Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$.</p> <p>Summing the right side we get</p> $\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\frac{3}{2}\right)^i.$ <p>Let $c = \frac{3}{2}$. Then we have</p> $n \sum_{i=0}^{m-1} c^i = n \left(\frac{c^m - 1}{c - 1} \right)$ $= 2n(c^{\log_2 n} - 1)$ $= 2n(c^{(k-1)\log_2 n} - 1)$ $= 2n^k - 2n,$ <p>and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider</p> $T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$ <p>Note that</p> $T_{i+1} = 1 + \sum_{j=0}^i T_j.$ <p>Subtracting we find</p> $T_{i+1} - T_i = 1 + \sum_{j=0}^i T_j - 1 - \sum_{j=0}^{i-1} T_j$ $= T_i.$ <p>And so $T_{i+1} = 2T_i = 2^{i+1}$.</p>	<p>Generating functions:</p> <ol style="list-style-type: none"> 1. Multiply both sides of the equation by x^i. 2. Sum both sides over all i for which the equation is valid. 3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$. 3. Rewrite the equation in terms of the generating function $G(x)$. 4. Solve for $G(x)$. 5. The coefficient of x^i in $G(x)$ is g_i. <p>Example:</p> $g_{i+1} = 2g_i + 1, \quad g_0 = 0.$ <p>Multiply and sum:</p> $\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i.$ <p>We choose $G(x) = \sum_{i \geq 0} x^i g_i$. Rewrite in terms of $G(x)$:</p> $\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i.$ <p>Simplify:</p> $\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$ <p>Solve for $G(x)$:</p> $G(x) = \frac{x}{(1-x)(1-2x)}.$ <p>Expand this using partial fractions:</p> $G(x) = x \left(\frac{2}{1-2x} - \frac{1}{1-x} \right)$ $= x \left(2 \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i \right)$ $= \sum_{i \geq 0} (2^{i+1} - 1) x^{i+1}.$ <p>So $g_i = 2^i - 1$.</p>

Theoretical Computer Science Cheat Sheet				
$\pi \approx 3.14159,$		$e \approx 2.71828,$	$\gamma \approx 0.57721,$	$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803,$
				$\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$
i	2^i	p_i	General	Probability
1	2	2	Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$): $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30},$ $B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$	Continuous distributions: If
2	4	3		$\Pr[a < X < b] = \int_a^b p(x) dx,$
3	8	5		then p is the probability density function of X . If
4	16	7	Change of base, quadratic formula: $\log_b x = \frac{\log_a x}{\log_a b}, \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$	$\Pr[X < a] = P(a),$
5	32	11	Euler's number e :	then P is the distribution function of X . If P and p both exist then
6	64	13	$e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$	$P(a) = \int_{-\infty}^a p(x) dx.$
7	128	17	$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$	Expectation: If X is discrete
8	256	19	$(1 + \frac{1}{n})^n < e < (1 + \frac{1}{n})^{n+1}.$	$E[g(X)] = \sum_x g(x) \Pr[X = x].$
9	512	23	$(1 + \frac{1}{n})^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$	If X continuous then
10	1,024	29	Harmonic numbers:	$E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x) dx = \int_{-\infty}^{\infty} g(x) dP(x).$
11	2,048	31	$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \dots$	Variance, standard deviation:
12	4,096	37		$\text{VAR}[X] = E[X^2] - E[X]^2,$
13	8,192	41		$\sigma = \sqrt{\text{VAR}[X]}.$
14	16,384	43		For events A and B :
15	32,768	47		$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$
16	65,536	53		$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$
17	131,072	59		iff A and B are independent.
18	262,144	61		$\Pr[A B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$
19	524,288	67	Factorial, Stirling's approximation:	For random variables X and Y :
20	1,048,576	71	$1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots$	$E[X \cdot Y] = E[X] \cdot E[Y],$
21	2,097,152	73		if X and Y are independent.
22	4,194,304	79	$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$	$E[X + Y] = E[X] + E[Y],$
23	8,388,608	83	Ackermann's function and inverse:	$E[cX] = c E[X].$
24	16,777,216	89	$a(i, j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$	Bayes' theorem:
25	33,554,432	97	$\alpha(i) = \min\{j \mid a(j, j) \geq i\}.$	$\Pr[A_i B] = \frac{\Pr[B A_i] \Pr[A_i]}{\sum_{j=1}^n \Pr[A_j] \Pr[B A_j]}.$
26	67,108,864	101		Inclusion-exclusion:
27	134,217,728	103		$\Pr\left[\bigvee_{i=1}^n X_i\right] = \sum_{i=1}^n \Pr[X_i] +$
28	268,435,456	107	Binomial distribution:	$\sum_{k=2}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} \Pr\left[\bigwedge_{j=1}^k X_{i_j}\right].$
29	536,870,912	109	$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \quad q = 1 - p,$	Moment inequalities:
30	1,073,741,824	113	$E[X] = \sum_{k=1}^n k \binom{n}{k} p^k q^{n-k} = np.$	$\Pr[X \geq \lambda E[X]] \leq \frac{1}{\lambda},$
31	2,147,483,648	127	Poisson distribution:	$\Pr[X - E[X] \geq \lambda \cdot \sigma] \leq \frac{1}{\lambda^2}.$
32	4,294,967,296	131	$\Pr[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad E[X] = \lambda.$	Geometric distribution:
Pascal's Triangle			Normal (Gaussian) distribution:	$\Pr[X = k] = pq^{k-1}, \quad q = 1 - p,$
1			$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad E[X] = \mu.$	$E[X] = \sum_{k=1}^{\infty} k p q^{k-1} = \frac{1}{p}.$
1 1			The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is	
1 2 1			$nH_n.$	
1 3 3 1				
1 4 6 4 1				
1 5 10 10 5 1				
1 6 15 20 15 6 1				
1 7 21 35 35 21 7 1				
1 8 28 56 70 56 28 8 1				
1 9 36 84 126 126 84 36 9 1				
1 10 45 120 210 252 210 120 45 10 1				

Theoretical Computer Science Cheat Sheet																										
Trigonometry	Matrices	More Trig.																								
<div></div> <p>Pythagorean theorem: $C^2 = A^2 + B^2$.</p> <p>Definitions:</p> $\sin a = A/C, \quad \cos a = B/C,$ $\csc a = C/A, \quad \sec a = C/B,$ $\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$ <p>Area, radius of inscribed circle:</p> $\frac{1}{2}AB, \quad \frac{AB}{A+B+C}.$ <p>Identities:</p> $\sin x = \frac{1}{\csc x}, \quad \cos x = \frac{1}{\sec x},$ $\tan x = \frac{1}{\cot x}, \quad \sin^2 x + \cos^2 x = 1,$ $1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x,$ $\sin x = \cos\left(\frac{\pi}{2} - x\right), \quad \sin x = \sin(\pi - x),$ $\cos x = -\cos(\pi - x), \quad \tan x = \cot\left(\frac{\pi}{2} - x\right),$ $\cot x = -\cot(\pi - x), \quad \csc x = \cot \frac{x}{2} - \cot x,$ $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$ $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$ $\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$ $\sin 2x = 2 \sin x \cos x, \quad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$ $\cos 2x = \cos^2 x - \sin^2 x, \quad \cos 2x = 2 \cos^2 x - 1,$ $\cos 2x = 1 - 2 \sin^2 x, \quad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$ $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \quad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$ $\sin(x+y) \sin(x-y) = \sin^2 x - \sin^2 y,$ $\cos(x+y) \cos(x-y) = \cos^2 x - \sin^2 y.$ <p>Euler's equation:</p> $e^{ix} = \cos x + i \sin x, \quad e^{i\pi} = -1.$	<p>Multiplication:</p> $C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}.$ <p>Determinants: $\det A \neq 0$ iff A is non-singular.</p> $\det A \cdot B = \det A \cdot \det B,$ $\det A = \sum_{\pi} \prod_{i=1}^n \text{sign}(\pi) a_{i,\pi(i)}.$ <p>2×2 and 3×3 determinant:</p> $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$ $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} a & b \\ d & e \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$ $= aei + bfg + cdh - ceg - fha - ibd.$ <p>Permanents:</p> $\text{perm } A = \sum_{\pi} \prod_{i=1}^n a_{i,\pi(i)}.$	<div></div> <p>Law of cosines: $c^2 = a^2 + b^2 - 2ab \cos C.$</p> <p>Area:</p> $A = \frac{1}{2}hc,$ $= \frac{1}{2}ab \sin C,$ $= \frac{c^2 \sin A \sin B}{2 \sin C}.$ <p>Heron's formula:</p> $A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$ $s = \frac{1}{2}(a + b + c),$ $s_a = s - a,$ $s_b = s - b,$ $s_c = s - c.$ <p>More identities:</p> $\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$ $\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$ $\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$ $= \frac{1 - \cos x}{\sin x},$ $= \frac{\sin x}{1 + \cos x},$ $\cot \frac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$ $= \frac{1 + \cos x}{\sin x},$ $= \frac{\sin x}{1 - \cos x},$ $\sin x = \frac{e^{ix} - e^{-ix}}{2i},$ $\cos x = \frac{e^{ix} + e^{-ix}}{2},$ $\tan x = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$ $= -i \frac{e^{2ix} - 1}{e^{2ix} + 1},$ $\sin x = \frac{\sinh ix}{i},$ $\cos x = \cosh ix,$ $\tan x = \frac{\tanh ix}{i}.$																								
Hyperbolic Functions																										
<p>Definitions:</p> $\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$ $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \text{csch } x = \frac{1}{\sinh x},$ $\text{sech } x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}.$ <p>Identities:</p> $\cosh^2 x - \sinh^2 x = 1, \quad \tanh^2 x + \text{sech}^2 x = 1,$ $\coth^2 x - \text{csch}^2 x = 1, \quad \sinh(-x) = -\sinh x,$ $\cosh(-x) = \cosh x, \quad \tanh(-x) = -\tanh x,$ $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$ $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$ $\sinh 2x = 2 \sinh x \cosh x,$ $\cosh 2x = \cosh^2 x + \sinh^2 x,$ $\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x},$ $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$ $2 \sinh^2 \frac{x}{2} = \cosh x - 1, \quad 2 \cosh^2 \frac{x}{2} = \cosh x + 1.$																										
<table><tr><th>θ</th><th>$\sin \theta$</th><th>$\cos \theta$</th><th>$\tan \theta$</th></tr><tr><td>0</td><td>0</td><td>1</td><td>0</td></tr><tr><td>$\frac{\pi}{6}$</td><td>$\frac{1}{2}$</td><td>$\frac{\sqrt{3}}{2}$</td><td>$\frac{\sqrt{3}}{3}$</td></tr><tr><td>$\frac{\pi}{4}$</td><td>$\frac{\sqrt{2}}{2}$</td><td>$\frac{\sqrt{2}}{2}$</td><td>1</td></tr><tr><td>$\frac{\pi}{3}$</td><td>$\frac{\sqrt{3}}{2}$</td><td>$\frac{1}{2}$</td><td>$\sqrt{3}$</td></tr><tr><td>$\frac{\pi}{2}$</td><td>1</td><td>0</td><td>∞</td></tr></table>	θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	0	0	1	0	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\pi}{2}$	1	0	∞	<p>... in mathematics you don't understand things, you just get used to them.</p> <p>– J. von Neumann</p>	
θ	$\sin \theta$	$\cos \theta$	$\tan \theta$																							
0	0	1	0																							
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$																							
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Theoretical Computer Science Cheat Sheet		
Number Theory	Graph Theory	
<p>The Chinese remainder theorem: There exists a number C such that:</p> $C \equiv r_1 \pmod{m_1}$ \vdots $C \equiv r_n \pmod{m_n}$ <p>if m_i and m_j are relatively prime for $i \neq j$.</p> <p>Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then</p> $\phi(x) = \prod_{i=1}^n p_i^{e_i-1} (p_i - 1).$ <p>Euler's theorem: If a and b are relatively prime then</p> $1 \equiv a^{\phi(b)} \pmod{b}.$ <p>Fermat's theorem:</p> $1 \equiv a^{p-1} \pmod{p}.$ <p>The Euclidean algorithm: if $a > b$ are integers then</p> $\gcd(a, b) = \gcd(a \bmod b, b).$ <p>If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then</p> $S(x) = \sum_{d x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$ <p>Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.</p> <p>Wilson's theorem: n is a prime iff</p> $(n - 1)! \equiv -1 \pmod{n}.$ <p>Möbius inversion:</p> $\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of } r \text{ distinct primes.} \end{cases}$ <p>If</p> $G(a) = \sum_{d a} F(d),$ <p>then</p> $F(a) = \sum_{d a} \mu(d) G\left(\frac{a}{d}\right).$ <p>Prime numbers:</p> $p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$ $\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$	<p>Definitions:</p> <p><i>Loop</i> An edge connecting a vertex to itself.</p> <p><i>Directed</i> Each edge has a direction.</p> <p><i>Simple</i> Graph with no loops or multi-edges.</p> <p><i>Walk</i> A sequence $v_0 e_1 v_1 \dots e_\ell v_\ell$.</p> <p><i>Trail</i> A walk with distinct edges.</p> <p><i>Path</i> A trail with distinct vertices.</p> <p><i>Connected</i> A graph where there exists a path between any two vertices.</p> <p><i>Component</i> A maximal connected subgraph.</p> <p><i>Tree</i> A connected acyclic graph.</p> <p><i>Free tree</i> A tree with no root.</p> <p><i>DAG</i> Directed acyclic graph.</p> <p><i>Eulerian</i> Graph with a trail visiting each edge exactly once.</p> <p><i>Hamiltonian</i> Graph with a cycle visiting each vertex exactly once.</p> <p><i>Cut</i> A set of edges whose removal increases the number of components.</p> <p><i>Cut-set</i> A minimal cut.</p> <p><i>Cut edge</i> A size 1 cut.</p> <p><i>k-Connected</i> A graph connected with the removal of any $k - 1$ vertices.</p> <p><i>k-Tough</i> $\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq S$.</p> <p><i>k-Regular</i> A graph where all vertices have degree k.</p> <p><i>k-Factor</i> A k-regular spanning subgraph.</p> <p><i>Matching</i> A set of edges, no two of which are adjacent.</p> <p><i>Clique</i> A set of vertices, all of which are adjacent.</p> <p><i>Ind. set</i> A set of vertices, none of which are adjacent.</p> <p><i>Vertex cover</i> A set of vertices which cover all edges.</p> <p><i>Planar graph</i> A graph which can be embedded in the plane.</p> <p><i>Plane graph</i> An embedding of a planar graph.</p> <hr/> $\sum_{v \in V} \deg(v) = 2m.$ <p>If G is planar then $n - m + f = 2$, so</p> $f \leq 2n - 4, \quad m \leq 3n - 6.$ <p>Any planar graph has a vertex with degree ≤ 5.</p>	<p>Notation:</p> <p>$E(G)$ Edge set</p> <p>$V(G)$ Vertex set</p> <p>$c(G)$ Number of components</p> <p>$G[S]$ Induced subgraph</p> <p>$\deg(v)$ Degree of v</p> <p>$\Delta(G)$ Maximum degree</p> <p>$\delta(G)$ Minimum degree</p> <p>$\chi(G)$ Chromatic number</p> <p>$\chi_E(G)$ Edge chromatic number</p> <p>G^c Complement graph</p> <p>K_n Complete graph</p> <p>K_{n_1, n_2} Complete bipartite graph</p> <p>$r(k, \ell)$ Ramsey number</p> <hr/> <p>Geometry</p> <p>Projective coordinates: triples (x, y, z), not all x, y and z zero.</p> $(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$ <p>Cartesian Projective</p> $(x, y) \quad (x, y, 1)$ $y = mx + b \quad (m, -1, b)$ $x = c \quad (1, 0, -c)$ <p>Distance formula, L_p and L_∞ metric:</p> $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$ $[x_1 - x_0 ^p + y_1 - y_0 ^p]^{1/p},$ $\lim_{p \rightarrow \infty} [x_1 - x_0 ^p + y_1 - y_0 ^p]^{1/p}.$ <p>Area of triangle $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2):</p> $\frac{1}{2} \text{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$ <p>Angle formed by three points:</p>  $\cos \theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$ <p>Line through two points (x_0, y_0) and (x_1, y_1):</p> $\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$ <p>Area of circle, volume of sphere:</p> $A = \pi r^2, \quad V = \frac{4}{3} \pi r^3.$ <hr/> <p>If I have seen farther than others, it is because I have stood on the shoulders of giants.</p> <p>– Issac Newton</p>

Theoretical Computer Science Cheat Sheet	
π	Calculus
<p>Wallis' identity:</p> $\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$ <p>Brouncker's continued fraction expansion:</p> $\frac{\pi}{4} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \cdots}}}}$ <p>Gregory's series:</p> $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$ <p>Newton's series:</p> $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$ <p>Sharp's series:</p> $\frac{\pi}{6} = \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots \right)$ <p>Euler's series:</p> $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$ $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$ $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$	<p>Derivatives:</p> <ol style="list-style-type: none"> $\frac{d(cu)}{dx} = c \frac{du}{dx},$ $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx},$ $\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx},$ $\frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx},$ $\frac{d(u/v)}{dx} = \frac{v(\frac{du}{dx}) - u(\frac{dv}{dx})}{v^2},$ $\frac{d(e^{cu})}{dx} = ce^{cu} \frac{du}{dx},$ $\frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx},$ $\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx},$ $\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx},$ $\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx},$ $\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx},$ $\frac{d(\sec u)}{dx} = \tan u \sec u \frac{du}{dx},$ $\frac{d(\csc u)}{dx} = -\cot u \csc u \frac{du}{dx},$ $\frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx},$ $\frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx},$ $\frac{d(\arctan u)}{dx} = \frac{1}{1+u^2} \frac{du}{dx},$ $\frac{d(\operatorname{arccot} u)}{dx} = \frac{-1}{1+u^2} \frac{du}{dx},$ $\frac{d(\operatorname{arcsec} u)}{dx} = \frac{1}{u\sqrt{1-u^2}} \frac{du}{dx},$ $\frac{d(\operatorname{arccsc} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx},$ $\frac{d(\sinh u)}{dx} = \cosh u \frac{du}{dx},$ $\frac{d(\cosh u)}{dx} = \sinh u \frac{du}{dx},$ $\frac{d(\tanh u)}{dx} = \operatorname{sech}^2 u \frac{du}{dx},$ $\frac{d(\coth u)}{dx} = -\operatorname{csch}^2 u \frac{du}{dx},$ $\frac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u \frac{du}{dx},$ $\frac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u \frac{du}{dx},$ $\frac{d(\operatorname{arcsinh} u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx},$ $\frac{d(\operatorname{arccosh} u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx},$ $\frac{d(\operatorname{arctanh} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx},$ $\frac{d(\operatorname{arcoth} u)}{dx} = \frac{1}{u^2-1} \frac{du}{dx},$ $\frac{d(\operatorname{arcsech} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx},$ $\frac{d(\operatorname{arccsch} u)}{dx} = \frac{-1}{ u \sqrt{1+u^2}} \frac{du}{dx}.$ <p>Integrals:</p> <ol style="list-style-type: none"> $\int cu \, dx = c \int u \, dx,$ $\int (u+v) \, dx = \int u \, dx + \int v \, dx,$ $\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1,$ $\int \frac{1}{x} \, dx = \ln x,$ $\int e^x \, dx = e^x,$ $\int \frac{dx}{1+x^2} = \arctan x,$ $\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx,$ $\int \sin x \, dx = -\cos x,$ $\int \cos x \, dx = \sin x,$ $\int \tan x \, dx = -\ln \cos x ,$ $\int \cot x \, dx = \ln \cos x ,$ $\int \sec x \, dx = \ln \sec x + \tan x ,$ $\int \csc x \, dx = \ln \csc x + \cot x ,$ $\int \arcsin \frac{x}{a} \, dx = \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0,$
<p>Partial Fractions</p> <p>Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining</p> $\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$ <p>where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:</p> $\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$ <p>where</p> $A = \left[\frac{N(x)}{D(x)} \right]_{x=a}.$ <p>For a repeated factor:</p> $\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$ <p>where</p> $A_k = \frac{1}{k!} \left[\frac{d^k}{dx^k} \left(\frac{N(x)}{D(x)} \right) \right]_{x=a}.$	
<p>The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.</p> <p>– George Bernard Shaw</p>	

Theoretical Computer Science Cheat Sheet

Calculus Cont.

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$
16. $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$
17. $\int \sin^2(ax) dx = \frac{1}{2a}(ax - \sin(ax) \cos(ax)),$
18. $\int \cos^2(ax) dx = \frac{1}{2a}(ax + \sin(ax) \cos(ax)),$
19. $\int \sec^2 x dx = \tan x,$
20. $\int \csc^2 x dx = -\cot x,$
21. $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx,$
22. $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx,$
23. $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx, \quad n \neq 1,$
24. $\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx, \quad n \neq 1,$
25. $\int \sec^n x dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, \quad n \neq 1,$
26. $\int \csc^n x dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx, \quad n \neq 1,$
27. $\int \sinh x dx = \cosh x,$
28. $\int \cosh x dx = \sinh x,$
29. $\int \tanh x dx = \ln |\cosh x|,$
30. $\int \coth x dx = \ln |\sinh x|,$
31. $\int \operatorname{sech} x dx = \arctan \sinh x,$
32. $\int \operatorname{csch} x dx = \ln \left| \tanh \frac{x}{2} \right|,$
33. $\int \sinh^2 x dx = \frac{1}{4} \sinh(2x) - \frac{1}{2} x,$
34. $\int \cosh^2 x dx = \frac{1}{4} \sinh(2x) + \frac{1}{2} x,$
35. $\int \operatorname{sech}^2 x dx = \tanh x,$
36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$
37. $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2|,$
38. $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$
39. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left(x + \sqrt{a^2 + x^2} \right), \quad a > 0,$
40. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$
41. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
42. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
43. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$
44. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|,$
45. $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}},$
46. $\int \sqrt{a^2 \pm x^2} dx = \frac{x}{2} \sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left| x + \sqrt{a^2 \pm x^2} \right|,$
47. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right|, \quad a > 0,$
48. $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left| \frac{x}{a+bx} \right|,$
49. $\int x \sqrt{a+bx} dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2},$
50. $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{1}{x\sqrt{a+bx}} dx,$
51. $\int \frac{x}{\sqrt{a+bx}} dx = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right|, \quad a > 0,$
52. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
53. $\int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} (a^2 - x^2)^{3/2},$
54. $\int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
55. $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
56. $\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$
57. $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
58. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|,$
59. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$
60. $\int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} (x^2 \pm a^2)^{3/2},$
61. $\int \frac{dx}{x \sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right|,$

Theoretical Computer Science Cheat Sheet		
Calculus Cont.		Finite Calculus
62. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \arccos \frac{a}{ x }, \quad a > 0,$	63. $\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x},$	Difference, shift operators: $\Delta f(x) = f(x+1) - f(x),$ $E f(x) = f(x+1).$
64. $\int \frac{x dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2},$	65. $\int \frac{\sqrt{x^2 \pm a^2}}{x^4} dx = \mp \frac{(x^2 + a^2)^{3/2}}{3a^2 x^3},$	Fundamental Theorem: $f(x) = \Delta F(x) \Leftrightarrow \sum f(x) \delta x = F(x) + C.$
66. $\int \frac{dx}{ax^2 + bx + c} = \begin{cases} \frac{1}{\sqrt{b^2 - 4ac}} \ln \left \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right , & \text{if } b^2 > 4ac, \\ \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}}, & \text{if } b^2 < 4ac, \end{cases}$		$\sum_a^b f(x) \delta x = \sum_{i=a}^{b-1} f(i).$
67. $\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \left 2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right , & \text{if } a > 0, \\ \frac{1}{\sqrt{-a}} \arcsin \frac{-2ax - b}{\sqrt{b^2 - 4ac}}, & \text{if } a < 0, \end{cases}$		Differences: $\Delta(cu) = c\Delta u, \quad \Delta(u+v) = \Delta u + \Delta v,$ $\Delta(uv) = u\Delta v + E v \Delta u,$ $\Delta(x^n) = nx^{n-1},$ $\Delta(H_x) = x^{-1}, \quad \Delta(2^x) = 2^x,$ $\Delta(c^x) = (c-1)c^x, \quad \Delta\binom{x}{m} = \binom{x}{m-1}.$
68. $\int \sqrt{ax^2 + bx + c} dx = \frac{2ax + b}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a} \int \frac{dx}{\sqrt{ax^2 + bx + c}},$		Sums: $\sum cu \delta x = c \sum u \delta x,$ $\sum (u+v) \delta x = \sum u \delta x + \sum v \delta x,$ $\sum u \Delta v \delta x = uv - \sum E v \Delta u \delta x,$ $\sum x^n \delta x = \frac{x^{n+1}}{n+1}, \quad \sum x^{-1} \delta x = H_x,$ $\sum c^x \delta x = \frac{c^x}{c-1}, \quad \sum \binom{x}{m} \delta x = \binom{x}{m+1}.$
69. $\int \frac{x dx}{\sqrt{ax^2 + bx + c}} = \frac{\sqrt{ax^2 + bx + c}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}},$		Falling Factorial Powers: $x^{\overline{n}} = x(x-1) \cdots (x-n+1), \quad n > 0,$ $x^{\overline{0}} = 1,$ $x^{\overline{n}} = \frac{1}{(x+1) \cdots (x+ n)}, \quad n < 0,$ $x^{\overline{n+m}} = x^{\overline{n}}(x-m)^{\overline{n}}.$
70. $\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{-1}{\sqrt{c}} \ln \left \frac{2\sqrt{c}\sqrt{ax^2 + bx + c} + bx + 2c}{x} \right , & \text{if } c > 0, \\ \frac{1}{\sqrt{-c}} \arcsin \frac{bx + 2c}{ x \sqrt{b^2 - 4ac}}, & \text{if } c < 0, \end{cases}$		Rising Factorial Powers: $x^{\overline{n}} = x(x+1) \cdots (x+n-1), \quad n > 0,$ $x^{\overline{0}} = 1,$ $x^{\overline{n}} = \frac{1}{(x-1) \cdots (x- n)}, \quad n < 0,$ $x^{\overline{n+m}} = x^{\overline{n}}(x+m)^{\overline{n}}.$
71. $\int x^3 \sqrt{x^2 + a^2} dx = (\frac{1}{3}x^2 - \frac{2}{15}a^2)(x^2 + a^2)^{3/2},$		Conversion: $x^{\overline{n}} = (-1)^n (-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$ $= 1/(x+1)^{\overline{-n}},$ $x^{\overline{n}} = (-1)^n (-x)^{\overline{n}} = (x+n-1)^{\overline{n}}$ $= 1/(x-1)^{\overline{-n}},$ $x^n = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\overline{k}} = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^{n-k} x^{\overline{k}},$ $x^{\overline{n}} = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] (-1)^{n-k} x^k,$ $x^{\overline{n}} = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] x^k.$
72. $\int x^n \sin(ax) dx = -\frac{1}{a} x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) dx,$		
73. $\int x^n \cos(ax) dx = \frac{1}{a} x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) dx,$		
74. $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx,$		
75. $\int x^n \ln(ax) dx = x^{n+1} \left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2} \right),$		
76. $\int x^n (\ln ax)^m dx = \frac{x^{n+1}}{n+1} (\ln ax)^m - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} dx.$		
$x^1 = x^1$	$x^{\overline{1}} = x^1$	
$x^2 = x^2 + x^1$	$x^{\overline{2}} = x^2 - x^{\overline{1}}$	
$x^3 = x^3 + 3x^2 + x^1$	$x^{\overline{3}} = x^3 - 3x^{\overline{2}} + x^{\overline{1}}$	
$x^4 = x^4 + 6x^3 + 7x^2 + x^1$	$x^{\overline{4}} = x^4 - 6x^{\overline{3}} + 7x^{\overline{2}} - x^{\overline{1}}$	
$x^5 = x^5 + 15x^4 + 25x^3 + 10x^2 + x^1$	$x^{\overline{5}} = x^5 - 15x^{\overline{4}} + 25x^{\overline{3}} - 10x^{\overline{2}} + x^{\overline{1}}$	
$x^{\overline{1}} = x^1$	$x^1 = x^1$	
$x^{\overline{2}} = x^2 + x^1$	$x^2 = x^2 - x^1$	
$x^{\overline{3}} = x^3 + 3x^2 + 2x^1$	$x^3 = x^3 - 3x^2 + 2x^1$	
$x^{\overline{4}} = x^4 + 6x^3 + 11x^2 + 6x^1$	$x^4 = x^4 - 6x^3 + 11x^2 - 6x^1$	
$x^{\overline{5}} = x^5 + 10x^4 + 35x^3 + 50x^2 + 24x^1$	$x^5 = x^5 - 10x^4 + 35x^3 - 50x^2 + 24x^1$	

Theoretical Computer Science Cheat Sheet		
Series		
Taylor's series:		Ordinary power series:
$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \dots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$		$A(x) = \sum_{i=0}^{\infty} a_i x^i.$
Expansions:		Exponential power series:
$\frac{1}{1-x}$	$= 1 + x + x^2 + x^3 + x^4 + \dots$	$= \sum_{i=0}^{\infty} x^i,$
$\frac{1}{1-cx}$	$= 1 + cx + c^2x^2 + c^3x^3 + \dots$	$= \sum_{i=0}^{\infty} c^i x^i,$
$\frac{1}{1-x^n}$	$= 1 + x^n + x^{2n} + x^{3n} + \dots$	$= \sum_{i=0}^{\infty} x^{ni},$
$\frac{x}{(1-x)^2}$	$= x + 2x^2 + 3x^3 + 4x^4 + \dots$	$= \sum_{i=0}^{\infty} i x^i,$
$\sum_{k=0}^n \binom{n}{k} \frac{k! z^k}{(1-z)^{k+1}}$	$= x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \dots$	$= \sum_{i=0}^{\infty} i^n x^i,$
e^x	$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$	$= \sum_{i=0}^{\infty} \frac{x^i}{i!},$
$\ln(1+x)$	$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$	$= \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i},$
$\ln \frac{1}{1-x}$	$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots$	$= \sum_{i=1}^{\infty} \frac{x^i}{i},$
$\sin x$	$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$	$= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$
$\cos x$	$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$	$= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$
$\tan^{-1} x$	$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$	$= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$
$(1+x)^n$	$= 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$	$= \sum_{i=0}^{\infty} \binom{n}{i} x^i,$
$\frac{1}{(1-x)^{n+1}}$	$= 1 + (n+1)x + \binom{n+2}{2}x^2 + \dots$	$= \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$
$\frac{x}{e^x - 1}$	$= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots$	$= \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$
$\frac{1}{2x}(1 - \sqrt{1-4x})$	$= 1 + x + 2x^2 + 5x^3 + \dots$	$= \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{2i}{i} x^i,$
$\frac{1}{\sqrt{1-4x}}$	$= 1 + 2x + 6x^2 + 20x^3 + \dots$	$= \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$
$\frac{1}{\sqrt{1-4x}} \left(\frac{1 - \sqrt{1-4x}}{2x} \right)^n$	$= 1 + (2+n)x + \binom{4+n}{2}x^2 + \dots$	$= \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$
$\frac{1}{1-x} \ln \frac{1}{1-x}$	$= x + \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{25}{12}x^4 + \dots$	$= \sum_{i=1}^{\infty} H_i x^i,$
$\frac{1}{2} \left(\ln \frac{1}{1-x} \right)^2$	$= \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{11}{24}x^4 + \dots$	$= \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i},$
$\frac{x}{1-x-x^2}$	$= x + x^2 + 2x^3 + 3x^4 + \dots$	$= \sum_{i=0}^{\infty} F_i x^i,$
$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2}$	$= F_n x + F_{2n} x^2 + F_{3n} x^3 + \dots$	$= \sum_{i=0}^{\infty} F_{ni} x^i.$
		Binomial theorem:
		$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$
		Difference of like powers:
		$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$
		For ordinary power series:
		$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$
		$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$
		$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$
		$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$
		$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$
		$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$
		$\int A(x) dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$
		$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$
		$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$
		Summation: If $b_i = \sum_{j=0}^i a_j$ then
		$B(x) = \frac{1}{1-x} A(x).$
		Convolution:
		$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^i a_j b_{i-j} \right) x^i.$
		God made the natural numbers; all the rest is the work of man. – Leopold Kronecker

Theoretical Computer Science Cheat Sheet		
Series		Escher's Knot
Expansions:		
$\frac{1}{(1-x)^{n+1}} \ln \frac{1}{1-x}$	$= \sum_{i=0}^{\infty} (H_{n+i} - H_n) \binom{n+i}{i} x^i,$	$\left(\frac{1}{x}\right)^{-n} = \sum_{i=0}^{\infty} \left\{ \begin{matrix} i \\ n \end{matrix} \right\} x^i,$
$x^{\overline{n}}$	$= \sum_{i=0}^{\infty} \left[\begin{matrix} n \\ i \end{matrix} \right] x^i,$	$(e^x - 1)^n = \sum_{i=0}^{\infty} \left\{ \begin{matrix} i \\ n \end{matrix} \right\} \frac{n! x^i}{i!},$
$\left(\ln \frac{1}{1-x}\right)^n$	$= \sum_{i=0}^{\infty} \left[\begin{matrix} i \\ n \end{matrix} \right] \frac{n! x^i}{i!},$	$x \cot x = \sum_{i=0}^{\infty} \frac{(-4)^i B_{2i} x^{2i}}{(2i)!},$
$\tan x$	$= \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i} (2^{2i} - 1) B_{2i} x^{2i-1}}{(2i)!},$	$\zeta(x) = \sum_{i=1}^{\infty} \frac{1}{i^x},$
$\frac{1}{\zeta(x)}$	$= \sum_{i=1}^{\infty} \frac{\mu(i)}{i^x},$	$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty} \frac{\phi(i)}{i^x},$
$\zeta(x)$	$= \prod_p \frac{1}{1-p^{-x}},$	
$\zeta^2(x)$	$= \sum_{i=1}^{\infty} \frac{d(i)}{x^i} \quad \text{where } d(n) = \sum_{d n} 1,$	
$\zeta(x)\zeta(x-1)$	$= \sum_{i=1}^{\infty} \frac{S(i)}{x^i} \quad \text{where } S(n) = \sum_{d n} d,$	
$\zeta(2n)$	$= \frac{2^{2n-1} B_{2n} }{(2n)!} \pi^{2n}, \quad n \in \mathbb{N},$	
$\frac{x}{\sin x}$	$= \sum_{i=0}^{\infty} (-1)^{i-1} \frac{(4^i - 2) B_{2i} x^{2i}}{(2i)!},$	
$\left(\frac{1 - \sqrt{1-4x}}{2x}\right)^n$	$= \sum_{i=0}^{\infty} \frac{n(2i+n-1)!}{i!(n+i)!} x^i,$	
$e^x \sin x$	$= \sum_{i=1}^{\infty} \frac{2^{i/2} \sin \frac{i\pi}{4}}{i!} x^i,$	
$\sqrt{\frac{1 - \sqrt{1-x}}{x}}$	$= \sum_{i=0}^{\infty} \frac{(4i)!}{16^i \sqrt{2} (2i)!(2i+1)!} x^i,$	
$\left(\frac{\arcsin x}{x}\right)^2$	$= \sum_{i=0}^{\infty} \frac{4^i i!^2}{(i+1)(2i+1)!} x^{2i}.$	
Cramer's Rule		Stieltjes Integration
If we have equations:		If G is continuous in the interval $[a, b]$ and F is nondecreasing then
$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$		$\int_a^b G(x) dF(x)$
$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$		exists. If $a \leq b \leq c$ then
\vdots		$\int_a^c G(x) dF(x) = \int_a^b G(x) dF(x) + \int_b^c G(x) dF(x).$
$a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n$		If the integrals involved exist
Let $A = (a_{i,j})$ and B be the column matrix (b_i) . Then there is a unique solution iff $\det A \neq 0$. Let A_i be A with column i replaced by B . Then		$\int_a^b (G(x) + H(x)) dF(x) = \int_a^b G(x) dF(x) + \int_a^b H(x) dF(x),$
$x_i = \frac{\det A_i}{\det A}.$		$\int_a^b G(x) d(F(x) + H(x)) = \int_a^b G(x) dF(x) + \int_a^b G(x) dH(x),$
		$\int_a^b c \cdot G(x) dF(x) = \int_a^b G(x) d(c \cdot F(x)) = c \int_a^b G(x) dF(x),$
		$\int_a^b G(x) dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x) dG(x).$
		If the integrals involved exist, and F possesses a derivative F' at every point in $[a, b]$ then
		$\int_a^b G(x) dF(x) = \int_a^b G(x) F'(x) dx.$
Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius. – William Blake (The Marriage of Heaven and Hell)		Fibonacci Numbers
		1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...
		Definitions:
		$F_i = F_{i-1} + F_{i-2}, \quad F_0 = F_1 = 1,$
		$F_{-i} = (-1)^{i-1} F_i,$
		$F_i = \frac{1}{\sqrt{5}} \left(\phi^i - \hat{\phi}^i \right),$
		Cassini's identity: for $i > 0$:
		$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$
		Additive rule:
		$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n,$
		$F_{2n} = F_n F_{n+1} + F_{n-1} F_n.$
		Calculation by matrices:
		$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$