1 Fréchet spaces

Let X be a topological vector space with topology $\mathcal{T}$. A few preliminary definitions:

Definition 1. A set C is called

- balanced if for all $x \in C$ and all $|\lambda| \leq 1$ also $\lambda x \in C$.
- convex if for all $x, y \in C$ and all $0 \leq t \leq 1$, also $tx + (1-t)y \in C$.
- absorbent if $\bigcup_{t>0} tC = X$.
- absolutely convex, if C is both convex and balanced. This is the same as saying that for every $x, y \in C$ and real numbers $|t| + |s| = 1$, $tx + sy \in C$. The best image for this is that C contains every parallelogramm with vertices x and y symmetric to the origin.

Definition 2.
- A collection of sets $\mathcal{B} \subset \mathcal{T}$ is called a base of the topology $\mathcal{T}$ if for every open set $U \in \mathcal{T}$ there is a $B \in \mathcal{B}$ with $B \in U$.
- A collection of sets $\mathcal{B}_x \subset \mathcal{T}$ is called a local base at x of the topology $\mathcal{T}$ if for every neighborhood $U \in \mathcal{T}$ of x (i.e. $x \in U$) there is a $B \in \mathcal{B}$ with $B \in U$.

We are going to compare two definitions of locally convex spaces.

Definition 3. A topogical vector space X is called *locally convex* if the origin has a local base of absolutely convex absorbent sets.

Definition 4. A topological vector space X is called *locally convex* if there is a family of seminorms generating the topology.

Lemma 1. *Definitions* 3 *and* 4 *are equivalent.*

Proof. Proof of 3 $\Rightarrow$ 4. Define the Minkowski gauge of a set $C \in X$ as

$$\mu_C(x) = \inf\{\lambda > 0 : x \in \lambda C\}.$$

Then we can show that the origin's local base of absolutely convex absorbent sets $\mathcal{B}_0 = \{B_a, a \in I\}$ constitute a family of seminorms via their Minkowski gauges $\{\mu_{B_a}, a \in I\}$ and the latter generates the topology of X. XXX

Proof of 4 $\Rightarrow$ 3 Using the family of seminorms we can easily build a local base of absolutely convex absorbent sets for the origin.XXX □

Remark 1.
1. In Definition 4, note that the family of seminorms is not necessarily countable.
2. If the family of seminorms $\|\cdot\|_k$ generating the topology is *countable*, then this constitutes a pseudometric generating the topology. To see this, define

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|x-y\|_k}{1 + \|x-y\|_k}. \tag{1}$$

   Note that this pseudo metric is neither unique (so the metrization is not unique) nor homogenous (so we can't define a suitable norm).
3. As translation is continuous in any t.v.s, we can use a base for the origin as a base for any point by translation.

Lemma 2. *Let X be a locally convex TVS with its family of seminorms $(\|\cdot\|_\alpha)_{\alpha \in I}$. Then X is Hausdorff if and only if*

$$\forall \alpha \in I: \ \|x\|_\alpha = 0 \quad \Leftrightarrow \quad x = 0 \tag{2}$$

Proof. Let X be Hausdorff. We show (2). If for every $\alpha \in I$, $\|x\|_\alpha = 0$ and we assume that $x \neq 0$, we know by the Hausdorff property that there exists an open neighborhood U_x of x and another open neighborhood of 0 such that they do not intersect. As the topology is generated by the seminorms, we can set $U_x = \{u : \ \|u - x\|_\alpha < \varepsilon \ \forall k \leq K\}$ for some $K \in \mathbb{N}, \varepsilon > 0$. Now obviously $0 \in U_x$ as $\|x\|_\alpha = 0$ for all α. Hence there cannot exist any neighborhood of 0 such that we can separate x and 0. The other direction of (2) is obvious by the definition of seminorms.

Now suppose (2) holds. If we set $x \neq y$, by (2) we know that there is at least one $\alpha \in I$ such that $\|x - y\|_\alpha = M \neq 0$.

By setting $U_x = \{u \in X : \ \|u - x\|_\alpha < M/2\}$ and $V_y = \{v \in X : \ \|v - y\|_\alpha < M/2\}$, we have $U_x \cap V_y = \emptyset$ by the triangle inequality (assume there is a w both in U_x and V_y, then one can show that $\|x-y\|_\alpha \leq \|x-w\|_\alpha + \|w-y\|_\alpha < M$). □

We are going to compare two definitions of Fréchet spaces.

Definition 5. A Fréchet space X is a topological vector space X such that

1. X is locally convex, i.e. $0 \in X$ has a local base of absorbent and absolutely convex sets.
2. The topology $\mathcal{T}$ can be induced by a translation invariant metric d, i.e. $d(x+a, y+a) = d(x,y)$.
3. (X, d) is a complete metric space.

Definition 6. A Fréchet space X is a topological vector space X such that

1. The topology can be induced by a countable family of seminorms $\|\cdot\|_k$, i.e. U open if and only if for all $u \in U$ there is an integer $K \geq 0$ and an $\varepsilon > 0$ such that
$$\{v : \|v-u\|_k < \varepsilon : \ k \leq K\} \subset U$$
2. X is Hausdorff.
3. X is complete w.r.t. the family of seminorms $\|\cdot\|_k$, i.e. if $(x_m)_m$ is a Cauchy sequence w.r.t. all seminorms, then there exists an $x \in X$ with $x_n \to x$ w.r.t. $\|\cdot\|_k$ (and by property 2 even $x_n \to x$).

Remark 2. 1. Note that the items 1-3 are not all "parallely equivalent", but as a set, they are equivalent:

- 5.1 and 5.2 $\Rightarrow$ 6.1 (the countability property of the family of seminorms is stronger than plain local convexity)
- 5.2 $\Rightarrow$ 6.2 (metrizability implies Hausdorff)
- 6.1 and 6.2 $\Rightarrow$ 5.2
- 6.1$\Rightarrow$ 5.1
- 5.3 $\Leftrightarrow$ 6.3

Lemma 3. *Definitions* 5 *and* 6 *are equivalent.*

Proof. **5 $\Rightarrow$ 6.**
ad 1.): From the equivalent definitions of locally convex vector spaces we know that 6.1 is *almost* local convexity. The trouble is that we claim the existence of a *countable* family of seminorms whereas local convexity just gives some family of seminorms. By combining 5.1 (local convexity in its set-theoretic version) and 5.2 (metrizability), we obtain 6.1.

Indeed, the metric d allows us to define small balls $B_n = \{x : \ d(x,0) < \frac{1}{n}\}$ for each $n \in \mathbb{N}$. As the topology can be induced by d, we know that there is an open set inside each B_n and by local convexity we know of the existence of an absorbent and absolutely convex set $C_n \subset B_n$. Those sets in turn define via their Minkowski gauges μ_n a countable family of seminorms. Equivalence of the topology of X and the topology generated by the family μ_n follows from the fact that the C_n are a local base of the topology of X.

ad 2.): Every metric (or metrizable) space is Hausdorff.

ad 3.): This is a direct consequence of completeness of d.

6 $\Rightarrow$ 5 ad 1.): As noted above, 6.1 is slightly stronger than 5.1.

ad 2.): By 6.2 we know by lemma 2 that the countable family of seminorms in 6.1 fulfills

$$\forall n \in \mathbb{N} : \ \|x\|_k = 0 \quad \Leftrightarrow \quad x = 0.$$

This property makes the pseudometric in (1) (which is one possible pseudometrization) a proper metric. The fact that the topology of X is induced by d is equivalent to 6.1. □

2 The space $\mathbb{R}^\infty$

Lemma 4. *The topological vector space* $\mathbb{R}^\infty = \{(x_n)_{n\in\mathbb{N}} : \ x_n \in \mathbb{R} : \ \forall n \in \mathbb{N}\}$ *with the product topology (i.e. the topology generated by all sets of the form* $U_1 \times \cdots \times U_m \times \mathbb{R}^\infty$ *with all* $U_i \subset \mathbb{R}$ *open) is a Fréchet space. One possible metrization is*

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|} \tag{3}$$

Proof. We will prove this by showing that $\mathbb{R}^\infty$ fulfills every item in definition 6.

ad 1.): We define $\|x\|_k := |x_k|$, i.e. the absolute value of the k-th item. This is obviously a seminorm on $\mathbb{R}^\infty$. We show that the product topology an be generated by this family of seminorms.

Let U' be an open set in the product topology. Then U' is a union of sets of the form $U_1 \times \cdots \times U_m \times \mathbb{R}^\infty$. It suffices to consider only open sets out of this basis. Let $U = U_1 \times \cdots \times U_m \times \mathbb{R}^\infty$. Choose a point $u \in U$, i.e. $u = (u_1, \ldots, u_m, u_{m+1}, \ldots)$ where $u_i \in U_i$ for $i = 1, \ldots, m$. Define $\delta = \min_{i=1}^M \operatorname{dist}(u_i, \partial U_i)$, where dist is the distance function on $\mathbb{R}$. Then the set $\{v : \|v - u\|_k < \delta : k \leq M\}$ is a subset of U, which constitutes the first part of Definition 6.1. The other direction is obvious by definition of the product topology.

ad 2.): By lemma 2 we know that we only need to show that if and only if for given $u \in \mathbb{R}^\infty$ and all $n \in \mathbb{N}$ we have $\|u\|_k = |u_k| = 0$, then $u = 0 \in \mathbb{R}^\infty$. But this is obvious.

ad 3.): This is easily shown by using completeness of $\mathbb{R}$ in each dimension of $\mathbb{R}^\infty$. □

Lemma 5. *The space $\mathbb{R}^\infty$ with the product topology is a polish space.*

Proof. $\mathbb{R}^\infty$ is separable as $\mathbb{Q}^\infty$ is countable and dense and it is completely metrizable by definition of Fréchet spaces. □

Lemma 6. *The product topology of $\mathbb{R}^\infty$ cannot be generated by a norm.*

Proof. Open sets of $\mathbb{R}^\infty$ are necessarily unbounded but balls defined by any norm must be (by definition) bounded. □

Lemma 7. *The Borel σ-algebra is the same as the product σ-algebra.*

Proof. The Borel-σ-algebra is the σ-algebra generated by open sets (i.e. sets of the form $U_1 \times \cdots \times U_m \times \mathbb{R}^\infty$), which is the same as the product *sigma*-algebra. □