

# 1 Fréchet spaces

Let  $X$  be a topological vector space with topology  $\mathcal{T}$ . A few preliminary definitions:

**Definition 1.** A set  $C$  is called

- balanced if for all  $x \in C$  and all  $|\lambda| \leq 1$  also  $\lambda x \in C$ .
- convex if for all  $x, y \in C$  and all  $0 \leq t \leq 1$ , also  $tx + (1 - t)y \in C$ .
- absorbent if  $\bigcup_{t>0} tC = X$ .
- absolutely convex, if  $C$  is both convex and balanced. This is the same as saying that for every  $x, y \in C$  and real numbers  $|t| + |s| = 1$ ,  $tx + sy \in C$ . The best image for this is that  $C$  contains every parallelogram with vertices  $x$  and  $y$  symmetric to the origin.

**Definition 2.** • A collection of sets  $\mathcal{B} \subset \mathcal{T}$  is called a base of the topology  $\mathcal{T}$  if for every open set  $U \in \mathcal{T}$  there is a  $B \in \mathcal{B}$  with  $B \subset U$ .

- A collection of sets  $\mathcal{B}_x \subset \mathcal{T}$  is called a local base at  $x$  of the topology  $\mathcal{T}$  if for every neighborhood  $U \in \mathcal{T}$  of  $x$  (i.e.  $x \in U$ ) there is a  $B \in \mathcal{B}_x$  with  $B \subset U$ .

We are going to compare two definitions of locally convex spaces.

**Definition 3.** A topological vector space  $X$  is called *locally convex* if the origin has a local base of absolutely convex absorbent sets.

**Definition 4.** A topological vector space  $X$  is called *locally convex* if there is a family of seminorms generating the topology.

**Lemma 1.** Definitions 3 and 4 are equivalent.

*Proof.* Proof of 3  $\Rightarrow$  4. Define the Minkowski gauge of a set  $C \in X$  as

$$\mu_C(x) = \inf\{\lambda > 0 : x \in \lambda C\}.$$

Then we can show that the origin's local base of absolutely convex absorbent sets  $\mathcal{B}_0 = \{B_a, a \in I\}$  constitute a family of seminorms via their Minkowski gauges  $\{\mu_{B_a}, a \in I\}$  and the latter generates the topology of  $X$ . XXX

Proof of 4  $\Rightarrow$  3 Using the family of seminorms we can easily build a local base of absolutely convex absorbent sets for the origin. XXX □

**Remark 1.** 1. In Definition 4, note that the family of seminorms is not necessarily countable.

2. If the family of seminorms  $\|\cdot\|_k$  generating the topology is *countable*, then this constitutes a pseudometric generating the topology. To see this, define

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|x - y\|_k}{1 + \|x - y\|_k}. \quad (1)$$

Note that this pseudo metric is neither unique (so the metrization is not unique) nor homogenous (so we can't define a suitable norm).

3. As translation is continuous in any t.v.s, we can use a base for the origin as a base for any point by translation.

**Lemma 2.** Let  $X$  be a locally convex TVS with its family of seminorms  $(\|\cdot\|_{\alpha})_{\alpha \in I}$ . Then  $X$  is Hausdorff if and only if

$$\forall \alpha \in I : \|x\|_{\alpha} = 0 \Leftrightarrow x = 0 \quad (2)$$

*Proof.* Let  $X$  be Hausdorff. We show (2). If for every  $\alpha \in I$ ,  $\|x\|_{\alpha} = 0$  and we assume that  $x \neq 0$ , we know by the Hausdorff property that there exists an open neighborhood  $U_x$  of  $x$  and another open neighborhood of 0 such that they do not intersect. As the topology is generated by the seminorms, we can set  $U_x = \{u : \|u - x\|_{\alpha} < \varepsilon \forall \alpha \leq K\}$  for some  $K \in \mathbb{N}, \varepsilon > 0$ . Now obviously  $0 \in U_x$  as  $\|x\|_{\alpha} = 0$  for all  $\alpha$ . Hence there cannot exist any neighborhood of 0 such that we can separate  $x$  and 0. The other direction of (2) is obvious by the definition of seminorms.

Now suppose (2) holds. If we set  $x \neq y$ , by (2) we know that there is at least one  $\alpha \in I$  such that  $\|x - y\|_{\alpha} = M \neq 0$ .

By setting  $U_x = \{u \in X : \|u - x\|_{\alpha} < M/2\}$  and  $V_y = \{v \in X : \|v - y\|_{\alpha} < M/2\}$ , we have  $U_x \cap V_y = \emptyset$  by the triangle inequality (assume there is a  $w$  both in  $U_x$  and  $V_y$ , then one can show that  $\|x - y\|_{\alpha} \leq \|x - w\|_{\alpha} + \|w - y\|_{\alpha} < M$ ). □

We are going to compare two definitions of Fréchet spaces.

**Definition 5.** A Fréchet space  $X$  is a topological vector space  $X$  such that

1.  $X$  is locally convex, i.e.  $0 \in X$  has a local base of absorbent and absolutely convex sets.
2. The topology  $\mathcal{T}$  can be induced by a translation invariant metric  $d$ , i.e.  $d(x+a, y+a) = d(x, y)$ .
3.  $(X, d)$  is a complete metric space.

**Definition 6.** A Fréchet space  $X$  is a topological vector space  $X$  such that

1. The topology can be induced by a countable family of seminorms  $\|\cdot\|_k$ , i.e.  $U$  open if and only if for all  $u \in U$  there is an integer  $K \geq 0$  and an  $\varepsilon > 0$  such that

$$\{v : \|v - u\|_k < \varepsilon : k \leq K\} \subset U$$

2.  $X$  is Hausdorff.
3.  $X$  is complete w.r.t. the family of seminorms  $\|\cdot\|_k$ , i.e. if  $(x_m)_m$  is a Cauchy sequence w.r.t. all seminorms, then there exists an  $x \in X$  with  $x_n \rightarrow x$  w.r.t.  $\|\cdot\|_k$  (and by property 2 even  $x_n \rightarrow x$ ).

*Remark 2.* 1. Note that the items 1-3 are not all “parallelly equivalent”, but as a set, they are equivalent:

- 5.1 and 5.2  $\Rightarrow$  6.1 (the countability property of the family of seminorms is stronger than plain local convexity)
- 5.2  $\Rightarrow$  6.2 (metrizable implies Hausdorff)
- 6.1 and 6.2  $\Rightarrow$  5.2
- 6.1  $\Rightarrow$  5.1
- 5.3  $\Leftrightarrow$  6.3

**Lemma 3.** *Definitions 5 and 6 are equivalent.*

*Proof.* **5  $\Rightarrow$  6.**

ad 1.): From the equivalent definitions of locally convex vector spaces we know that 6.1 is *almost* local convexity. The trouble is that we claim the existence of a *countable* family of seminorms whereas local convexity just gives some family of seminorms. By combining 5.1 (local convexity in its set-theoretic version) and 5.2 (metrizable), we obtain 6.1.

Indeed, the metric  $d$  allows us to define small balls  $B_n = \{x : d(x, 0) < \frac{1}{n}\}$  for each  $n \in \mathbb{N}$ . As the topology can be induced by  $d$ , we know that there is an open set inside each  $B_n$  and by local convexity we know of the existence of an absorbent and absolutely convex set  $C_n \subset B_n$ . Those sets in turn define via their Minkowski gauges  $\mu_n$  a countable family of seminorms. Equivalence of the topology of  $X$  and the topology generated by the family  $\mu_n$  follows from the fact that the  $C_n$  are a local base of the topology of  $X$ .

ad 2.): Every metric (or metrizable) space is Hausdorff.

ad 3.): This is a direct consequence of completeness of  $d$ .

**6  $\Rightarrow$  5** ad 1.): As noted above, 6.1 is slightly stronger than 5.1.

ad 2.): By 6.2 we know by lemma 2 that the countable family of seminorms in 6.1 fulfills

$$\forall n \in \mathbb{N} : \|x\|_k = 0 \Leftrightarrow x = 0.$$

This property makes the pseudometric in (1) (which is one possible pseudometrization) a proper metric. The fact that the topology of  $X$  is induced by  $d$  is equivalent to 6.1.  $\square$

## 2 The space $\mathbb{R}^\infty$

**Lemma 4.** *The topological vector space  $W = \mathbb{R}^\infty = \{(x_n)_{n \in \mathbb{N}} : x_n \in \mathbb{R} : \forall n \in \mathbb{N}\}$  with the product topology (i.e. the topology generated by all sets of the form  $U_1 \times \cdots \times U_m \times \mathbb{R}^\infty$  with all  $U_i \subset \mathbb{R}$  open) is a Fréchet space. One possible metrization is*

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|} \quad (3)$$

*Proof.* We will prove this by showing that  $\mathbb{R}^\infty$  fulfills every item in definition 6.

ad 1.): We define  $\|x\|_k := |x_k|$ , i.e. the absolute value of the  $k$ -th item. This is obviously a seminorm on  $\mathbb{R}^\infty$ . We show that the product topology can be generated by this family of seminorms.

Let  $U'$  be an open set in the product topology. Then  $U'$  is a union of sets of the form  $U_1 \times \cdots \times U_m \times \mathbb{R}^\infty$ . It suffices to consider only open sets out of this basis. Let  $U = U_1 \times \cdots \times U_m \times \mathbb{R}^\infty$ . Choose a point  $u \in U$ , i.e.  $u = (u_1, \dots, u_m, u_{m+1}, \dots)$  where  $u_i \in U_i$  for  $i = 1, \dots, m$ . Define  $\delta = \min_{i=1}^m \text{dist}(u_i, \partial U_i)$ , where  $\text{dist}$  is the distance function on  $\mathbb{R}$ . Then the set  $\{v : \|v - u\|_k < \delta : k \leq m\}$  is a subset of  $U$ , which constitutes the first part of Definition 6.1. The other direction is obvious by definition of the product topology.

ad 2.): By lemma 2 we know that we only need to show that if and only if for given  $u \in \mathbb{R}^\infty$  and all  $n \in \mathbb{N}$  we have  $\|u\|_k = |u_k| = 0$ , then  $u = 0 \in \mathbb{R}^\infty$ . But this is obvious.

ad 3.): This is easily shown by using completeness of  $\mathbb{R}$  in each dimension of  $\mathbb{R}^\infty$ .  $\square$

**Lemma 5.** *The space  $\mathbb{R}^\infty$  with the product topology is a polish space.*

*Proof.*  $\mathbb{R}^\infty$  is separable as  $\mathbb{Q}^\infty$  is countable and dense and it is completely metrizable by definition of Fréchet spaces.  $\square$

**Lemma 6.** *The product topology of  $\mathbb{R}^\infty$  cannot be generated by a norm.*

*Proof.* Open sets of  $\mathbb{R}^\infty$  are necessarily unbounded but balls defined by any norm must be (by definition) bounded.  $\square$

**Lemma 7.** *The Borel  $\sigma$ -algebra is the same as the product  $\sigma$ -algebra.*

*Proof.* The Borel- $\sigma$ -algebra is the  $\sigma$ -algebra generated by open sets (i.e. sets of the form  $U_1 \times \cdots \times U_m \times \mathbb{R}^\infty$ ), which is the same as the product *sigma*-algebra.  $\square$

We write  $e_i$  for the element of  $W$  with  $e_i(j) = \delta_{ij}$ . Any element  $x \in W$  can be written as  $x = \sum_{i=1}^\infty x_i e_i$ .

**Lemma 8.** *Every continuous linear functional  $f \in W^*$  is of the form*

$$f(x) = \sum_{i=1}^n a_i x_i \quad (4)$$

for some  $a_1, \dots, a_n \in \mathbb{R}$  and with the notation  $W \ni x = (x_1, x_2, \dots)$ . Thus  $W^*$  can be identified with  $c_{00}$ , the set of all real sequences which are eventually zero.

*Proof.* Let  $f \in W^*$ . Define  $a_i = f(e_i)$ . We show that only finitely many  $a_i$  are nonzero. Indeed, assume that for any  $N \in \mathbb{N}$  there is an  $i_N > N$  such that  $a_{i_N} \neq 0$ .

As we assumed  $f$  to be continuous, for any  $\varepsilon > 0$  there is an open neighborhood  $U$  of 0 such that

$$\sup_{u \in U} |f(u)| < \varepsilon. \quad (5)$$

By the form of the topology, this  $U$  is of the form

$$U = \bigotimes_{i=1}^n U_i \times \mathbb{R}^\infty$$

for some  $n \in \mathbb{N}$ . Now we can define a sequence  $(u_N)_N$  of elements  $u_N$  (which are sequences) in  $W$  as  $u_N = \frac{1}{a_{i_N}} \cdot N \cdot e_{i_N}$ , i.e. the  $N$ -th element is the sequence which has the value  $\frac{N}{a_{i_N}}$  at the  $i_N$ -th position and 0 elsewhere. For  $M$  high enough,  $(u_N)_{N \geq M}$  is a sequence in  $U$ . But

$$f(u_N) = \frac{N}{a_{i_N}} \cdot f(e_{i_N}) = N \xrightarrow{N \rightarrow \infty} \infty,$$

hence (5) is violated and  $f$  cannot be continuous.

This means that  $f$  acts on only a finite number of basis elements of  $W$  and as  $f$  is linear we can write it in the form (4).  $\square$

We choose the measure  $\mu$  to be an infinite product of Gaussian measures with variance 1 and mean 0. So the projections of elements  $x \in W$  on the coordinates are i.i.d. Gaussians.

**Lemma 9.**  $\mu$  is a Gaussian measure. The covariance form of  $\mu$  is given by

$$q(f, g) = \sum_{i=1}^{\infty} f(e_i)g(e_i)$$

where the sum is actually finite.

*Proof.* We argue by using characteristic functions.

$$\begin{aligned} \int_W e^{if(x)} \mu(dx) &= \int_{\mathbb{R}^\infty} e^{i \sum_{j=1}^n f(e_j)x_j} \bigotimes_{j=1}^{\infty} N(0, 1)(dx_j) = \prod_{j=1}^n \int_{\mathbb{R}} e^{if(e_j)x_j} N(0, 1)(dx_j) \\ &= \prod_{j=1}^n e^{-\frac{f(e_j)^2}{2}} = e^{-\frac{1}{2} \sum_{j=1}^n f(e_j)^2} \end{aligned}$$

and thus the covariance form is  $q(f, f) = \sum_{j=1}^n f(e_j)^2$  and hence  $q(f, g) = \sum_{j=1}^{\infty} f(e_j)g(e_j)$  where the sum runs until the largest non-zero entry of both  $f$  and  $g$ .  $\square$

**Lemma 10.**  $\mu$  has full support, i.e. there is no open set other than the empty set having zero measure.

*Proof.* Let  $U$  be open in  $W$ , i.e.  $U = \bigotimes_{i=1}^n U_i \times \mathbb{R}^\infty$  with  $U_j$  open in  $\mathbb{R}$ . Then  $\mu(U) = \prod_{j=1}^n N(0, 1)(U_j) > 0$  as  $N(0, 1)$  has full support on  $\mathbb{R}$ .  $\square$

*Remark 3.*  $q$  is actually positive definite: The only  $f \in W^*$  with  $q(f, f) = 0$  is  $f = 0$ . This means that  $i : W^* \hookrightarrow L^2(W, \mu)$  with inner product  $q$  is an injection in the sense that  $i(W^*) \subset L^2$  is isomorphic to  $W^*$ . This is actually cool because normally there may be  $0 \neq f \in W^*$  with  $0 = f \in L^2$  (for example in spaces with degenerate Gaussians like  $W = \mathbb{R}^2$  with  $N(0, 1) \otimes \delta_0$ ). Here, choosing  $f$  as the projection on the second coordinate (written as  $f = (0, 1)^T$ ) is not 0 as a linear functional in  $W^* = \mathbb{R}^2$  but  $f(x) = 0$   $\mu$ -almost surely and thus  $0 = f \in L^2$  as a square-integrable random variable. So, to reiterate,  $W^* \hookrightarrow L^2(W, \mu)$  is an injection and we can view  $W^*$  as a **subspace** of  $L^2$ . This is because we don't lose "information" by viewing  $f$  as an element in  $L^2$ . In the example above with  $\mathbb{R}^2$  we do lose information because after identifying  $W^*$  with a subset of  $L^2$ , we can't distinguish  $(0, 1)^T$  and  $(0, 0)^T$  as random variables in  $L^2$ , although they are quite distinct in  $W^* = \mathbb{R}^2$ .

*Remark 4.* As was pointed out, we can think of  $W^*$  as a subset of  $L^2(W, \mu)$  with inner product  $q$ . But:  $W^*$  is not complete in the  $q$  inner product. Let's define  $K = \overline{W^*}^{L^2(W, \mu)}$ , the  $L^2$ -closure of  $W^*$ .

**Lemma 11.**  $K$  consists of all functions  $f : W \rightarrow \mathbb{R}$  of the form

$$f(x) = \sum_{j=1}^{\infty} a_j x_j \tag{6}$$

with  $\sum_{j=1}^{\infty} |a_j|^2 < \infty$ .

*Proof.* As  $W^*$  can be identified with  $c_{00}$  and  $L^2(W, \mu)$  with inner product  $q$  can be identified with  $l^2$ , the space of square summable sequences, this is equivalent to  $\overline{c_{00}}^{l^2} = l^2$ , which is standard.  $\square$

*Remark 5.* Note one subtle point about (6): For an arbitrary  $x \in W$ , the sum may not converge. It does, however, converge for  $\mu$ -a.e.  $x \in W$ . Indeed: Sums of independent random variables (and the  $a_j \cdot x_j$  are i.i.d Gaussians) converge a.s. as soon as the sum of their variances converge. Thus, we only need to show

$$\sum_{j=1}^{\infty} a_j^2 \cdot \mathbb{E} x_j^2 < \infty$$

which follows immediately by  $\mathbb{E} x_j^2 = 1$  and the condition on  $(a_j)_j$ .

By the same calculation as in lemma 9, we can show that  $f$  is a Gaussian random variable with covariance form  $q(f, g) = \sum_{j=1}^{\infty} f(e_j)g(e_j)$  where now the sum may contain infinitely many terms (but still is of finite value by the Cauchy-Schwarz inequality in  $l^2$ ).