Can Emmy (∃) beat the devil (∀) in tennis?

A case for teaching analysis concepts with a better story

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Abstract

An important concept of analysis is the use of existential and universal quantifiers. Mastery of both the concept as well as the gritty "arithmetic" techniques are both vital components for performing elementary analysis proofs, like showing boundedness and continuity of maps, or convergence of sequences. We propose a way to reduce this twofold cognitive burden by making the general concept more accessible. The main ingredients of this didactical approach are personification (both as a cognitive shortcut and as a mnemonic device), doodlification, and discovery fiction (helping to distinguish between the thought process spawning a proof, and the polished product that is usually presented in textbooks). The goal of this manuscript is to spark a discussion about the way we teach analysis, and whether we can improve on our current practice.

What follows is an exposition for a didactical device that can be applied in a classroom setting with the goal of teaching mathematical proofs using existential (∃) and universal (∀) quantifiers, by personifying them in a way that is consistent with their role in mathematical proofs – as ∃mmy and de∀il (abbreviations come with due apologies to typography nerds). This technique will not help with the actual work involved in designing a specific ε-δ proof, but it may reduce mental load during the first few encounters with this concept by making the concept more accessible by placing it in a setting which is more readily humanly comprehensible.

A long list of caveats is in order: The author is not trained in the science of learning psychology, he is ignorant about most of the relevant literature in this field, and makes no attempt for empirical validation.

The author has learned about mathematics by absorbing conventional calculus and analysis lectures – in some sense despite this fact. His struggle and his own teaching experience informs the content in this manuscript. He does not claim to propose a silver bullet or to having solved a meaningful fraction of the problem of teaching difficult mathematics.

1 Proving boundedness of a map: Beating the devil in tennis

The following two situations are isomorphic. (The proof of this statement is left to the interested reader.)

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Your name is ∃mmy. The de∀il is challenging you to beat him at Tennis, but with your serve alone.

The rules are:

∃mmy has to perform such a miraculously good serve that

any return of the de∀il

is futile and ∃mmy wins the match.

The match starts.

You perform a serve.

The de∀il tries to return,

but no use: He misses the ball, and you win.

Your professor is asking you to prove that the map $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto \frac{1}{1+x^2}$ is bounded from above.

This means that you need to show the following:

$\exists M \in \mathbb{R}$ (There exists a constant M),

$\forall x \in \mathbb{R}$ (no matter which point is chosen),

we have $f(x) \leq M$. $\square$

We start with the proof now.

Let $M = 1$ (our choice)

and given *any* $x \in \mathbb{R}$ (not our choice),

we have $x^2 \geq 0$. This means that $1 + x^2 \geq 1$, and thus $\frac{1}{1+x^2} \leq 1 = M$. QED

There are a few things to note here:

- First we have to make sure that we {understand the rules of the game, know what we need to do in order prove a statement}. Then we need to actually {play, perform the proof}.
- It is possible to {lose the match, botch the proof}: If ∃mmy makes a bad serve, she will lose! Similarly, if we start out with a bad choice of M in the proof (e.g. $M = 0$), then we won't be able to finish the proof. This does not mean that the statement is wrong (just like a match is not hopeless only because there is technically a possibility of losing).
- Even if we {watch 10,000 tennis matches, read 10,000 mathematical proofs}, we can stay very bad {tennis players, mathematicians}.

2 Proving continuity of a function: Defeating the devil at chess

Our protagonist has overcome their challenge. Alas, there is more to come! (Isomorphy of the following two scenarios is an easy corollary from the corresponding statement in Section 1.)

∃mmy has been challenged to a game of chess by the De∀il. He is confronting her with a very specific chess situation:

The De∀il will start with a first move of his choosing.

∃mmy will then counter with a reply which has to be such a powerful move that

no matter how good the De∀il's next move is,

∃mmy will have won the game. Checkmate

You are trying to prove continuity of some function f involving existential (∃) and universal (∀) quantifiers.

$\forall x_0 \in \mathbb{R}, \varepsilon > 0$ (No matter which point $x_0 \in \mathbb{R}$ and which tolerance $\varepsilon > 0$),

$\exists \delta > 0$ (we can always find a value $\delta > 0$),

$\forall x \in \mathbb{R}, |x - x_0| < \delta$: (such that for any choice of x_0 under the condition that $|x - x_0| < \delta$)

we have $|f(x) - f(x_0)| < \varepsilon$. $\square$

Note how the statement of continuity, as written down in the right column with existential and universal quantifiers, indeed resembles a chess game (or a tennis match, or a boxing match, or actually any turn-based contest between two opponents). The universal quantifier $\forall$ acts as an adversary: We cannot choose the object that is supplied by this quantifier, we have to live with it. The existential quantifier allows us to react to that (and to everything preceding it). We can choose an object which may or may not help us with the statements succeeding it. After that, the second universal quantifier is another adversarial choice. It may depend on our previous choice, and it can be "as bad as it can be". The difficult task in performing this proof is choosing (in the line marked with the existential quantifier) a $\delta > 0$

- depending on x_0 and ε (supplied by "the devil" beforehand) and
- good enough to be able to cope with the devil's choice of x coming afterwards.

3 Good and bad moves

We can see how this is indeed very similar to a chess match: Each one of our turns must be a reaction to previous turns and we must and can work with the reality of the current state of the game (we cannot move a rook on the board if we don't have any left). At the same time, we have to look ahead and make a (good enough) turn that can cope with the worst (for us) possible future adversarial moves. It is completely possible to botch up a proof by making a wrong choice, just like making a bad move will make us lose a chess match, as the following example shows. Here, we are (without success) trying to prove that $f : x \mapsto x^2$ is a continuous function.

The de∀il makes a powerful first move.	We consider, e.g., $x_0 = 3$ and $\varepsilon = 10^{-10}$.
∃mmy responds with a bad move.	We set $\delta = 10$. (This is a mistake)
The de∀il sees his chance and counters.	If we consider $x = 5$ (which is a possible choice, since $\lvert x - x_0 \rvert = 2 < 10$), then
∃mmy loses.	$\lvert f(x) - f(x_0) \rvert = \lvert 25 - 9 \rvert = 16 > \varepsilon$, which unfortunately fails to show that f is continuous at $x_0 = 3$, although this is in fact the case.

Just as there are chess matches that are lost only due to an avoidable mistake, it is possible to not be able to prove a (true) statement just because we have not found the right idea. Growing as a mathematician means learning how to recognize opportunities just like an experienced chess player will not miss an opening created by their opponent. In some sense, being a mathematician is harder: A human chess opponent will sometimes make a mistake (which means that you can be a chess grand master even if you sometimes make a mistake, you just need to make fewer mistakes than your opponents on average), but mathematical rigour dictates that we need to prove a statement without any gaps (if we want to prove continuity everywhere, it is not sufficient to prove continuity on all odd integers).

4 Lost matches

Furthermore, just like there are games that cannot be won anymore (when all our moves can be countered), there are proofs that are impossible to do. For example, it is impossible to prove the following (that the set of natural numbers is bounded):

$$\exists M \in \mathbb{R}\ \forall k \in \mathbb{N}:\ k \leq M.$$

We cannot choose a real number which dominates every single integer. In our metaphor: For any $M \in \mathbb{R}$ that we can start with, the devil can find a $k \in \mathbb{N}$ such that $k \leq M$ is `False`, i.e. $k > M$ (e.g. by setting $k = \mathtt{round}(M) + 1$). But the statement is true if we swap the order:

$$\forall k \in \mathbb{N}\ \exists M \in \mathbb{R}:\ k \leq M.$$

Indeed, no matter which integer k the devil chooses, we can find a real number M such that indeed $k \leq M$ (for example, by setting $M = k + 1$). This is the difference between a true statement and a wrong statement: We may hope to prove true statements, but wrong statements cannot be proven.

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