Real Analysis

Brief summary:
Real analysis is the branch of mathematical analysis which studies real numbers, sequences, series of real numbers and real functions. It is distinguished from complex analysis which focuses on complex numbers and their functions. This report will include properties of real-valued sequences and functions, ranging from limit, convergence and continuity to differentiability and integrability

1.Sequences and Limits

The notion of sequence is central in real analysis. Among sequences, convergent sequences are a particularly important branch. Based on convergent sequences, the notion of a convergent series will also be introduced in this chapter.

1.1 Sequences

In General, a sequence is function f:N→R, with a domain being usually taken to be natural numbers. The general term of the function is denoted as $(a_n)=(a_n)_{n\in N}=(a_1,a_2,a_3,\cdots)$

1.2 Sequences converging to a limit

A sequence tending to a limit is said to be convergent. Otherwise it is divergent.

By definition, a sequence $(a_n)_{n\in N}$ converges to a limit $a \in R$ if for all $\epsilon \geq 0$, there always exists a natural number N, such that for any natural number n, if n>N, then $|a_n - a| < \epsilon$. We can write this as $\lim(a_n) = a$ or a_n approaches a as n approaches positive infinity. As we can imply from this definition, the value of a_n fluctuates around the limit a without deviating from a farther than ϵ. This range is thus called the ϵ-neighbourhood of the point a_n. Specially, sequences whose terms tend to 0 are called null sequences. In other words, $(a_n)_{n\in N}$ is a null sequence if and only if for any $\epsilon > 0$, all the elements of the sequence belong to the ϵ-neighbourhood of zero, apart from only finitely many.

Convergent sequences have several properties.

- A sequence can have at most one limit. In other words, the limit of a sequence is unique

- Any convergent sequence is bounded. A sequence is said to be bounded only if there exists a real number M such that for all natural number n, $|a_n| < M$
- If the limit of a sequence is not zero, then there exists a natural number N such that for any natural number n, if n > N, then $|a_n|>1\backslash 2\ |a|$

Proof: Fix ∈ to be 1\2 |a| > 0. Then there exists a natural number N such that for all n > N,

$$\frac{|a|}{2} > |a - a_n| \geq |a| - |a_n|$$

This implies that $|a_n|$ > |a| - 1\2 |a| = 1\2 |a|.

Proceeding from this, we can get further implications of this theorem

$$\left(\frac{|a|}{2} > |a - a_n|\right) \Leftrightarrow \left(a - \frac{|a|}{2} < a_n < a + \frac{|a|}{2}\right)$$

Consider two cases:

Case 1: a > 0, then for n>N $\quad a_n > a - \frac{|a|}{2} = \frac{a}{2}.$

Case 2: a < 0, then for n>N $\quad a_n < a + \frac{|a|}{2} = \frac{a}{2}.$

- For any two real sequences a_n and b_n which tend to a and b respectively, if for all natural numbers n, $a_n \geq b_n$, then a ≥ b.
- The Sandwich rule: if all any natural number n, $a_n \leq b_n \leq c_n$ and both a_n and c_n tend to a, then b_n also tends to a

Proof: Let ∈ > 0.

Then we can find two natural numbers such that

$$[(n > N_1) \Rightarrow (a - \varepsilon < a_n)] \wedge [(n > N_2) \Rightarrow (c_n < a + \varepsilon)].$$

Then choosing N to be
max{N_1,N_2}, it's easy to deduce that

$$(a - \varepsilon < a_n \leq b_n \leq c_n < a + \varepsilon) \Rightarrow (|b_n - a| < \varepsilon)$$

- For any two real sequences a_n and b_n which tend to a and b respectively,

  A. The limit of (a_n + b_n) tends to (a + b)

$$\left|\frac{a_n}{b_n}-\frac{a}{b}\right|<\varepsilon.$$

B. The limit of $(a_n - b_n)$ tends to $(a - b)$

C. The limit of (a_nb_n) tends to ab

D. The limit of (a_n / b_n) tends to a / b, if $b \neq 0$

- Proof for A : Choosing N_1 such that for all $n>N_1$, $|a_n - a| < \epsilon/2$. Choosing N_2 such that for all $n>N_2$, $|b_n - b| < \epsilon/2$.

  Then for all $n>\max\{N_1,N_2\}$,

  $$|a_n+b_n-(a+b)|\leq|a_n-a|+|b_n-b|<\varepsilon.$$

- Proof for B: Similar to proof for A, choosing N_1 such that for all $n>N_1$, $|a_n - a| < \epsilon/2$. Choosing N_2 such that for all $n>N_2$, $|b_n - b| < \epsilon/2$.

  Then $|a_n - a| + |b_n - b| < \epsilon$ and $|a_n - a| + |b_n - b| >= |(a_n - b_n) - (a - b)|$.

  Hence, $|(a_n - b_n) - (a - b)| < \epsilon$, for all $n>\max\{N_1,N_2\}$

- Proof for C: Since both sequences a_n and b_n are convergence, they are also bounded. Let K be such that for n, $|a_n|<=K$ and $|b_n|<=K$.

  Then we have,

  $$|a_nb_n-ab|=|a_nb_n-ab_n+ab_n-a_nb_n|$$
  $$\leq|a_n-a||b_n|+|a||b_n-b|\leq K|a_n-a|+K|b_n-b|.$$

  Choosing N_1 such that for all $n>N_1$, $K|a_n - a| < \epsilon/2$ and N_2 such that for all $n>N_2$, $K|b_n - b| < \epsilon/2$. Then for all

  $$|a_nb_n-ab|<\varepsilon.$$

- Proof for D:

$$\left|\frac{a_n}{b_n}-\frac{a}{b}\right|=\left|\frac{a_nb-ab_n}{b_nb}\right|\leq\frac{|a_n-a||b|+|a||b_n-b|}{|b_nb|}.$$

Choose N_1 such that for $n>N_1$,

$$|b_n|>\frac{1}{2}|b|\ \ \text{(possible by Theorem 3.1.4).}$$

Choose N_2 such that for $n>N_2$,

$$|a_n-a|<\frac{1}{4}|b|\varepsilon.$$

Choose N_3 such that for $n>N_3$,

$$|a||b_n-b|<\frac{1}{4}b^2\varepsilon.$$

Then for $n>N:=\max\{N_1,N_2,N_3\}$,

1.3 Monotone sequence

A sequence of real numbers $(a_n)_{n \in N}$ is said to be increasing if for all natural numbers n, $a_n \leq a_{n+1}$. A sequence of real numbers $(a_n)_{n \in N}$ is said to be decreasing if for all natural numbers n, $a_n \geq a_{n+1}$. By definition, a sequence of real numbers $(a_n)_{n \in N}$ is said to be monotone if it is either increasing or decreasing.

If an increasing sequence is bounded above or a decreasing sequence is bounded below, then it is a convergent sequence. Although such theorems appeal to natural instincts, proof for them is still very important:

Put $A = \{a_n : n \in \mathbb{N}^+\}$.

As A is bounded above, we define $\alpha = \sup\{a_n : n \in \mathbb{N}^+\} \in \mathbb{R}$.

Let $\epsilon > 0$. Since $\alpha - \varepsilon$ is is not an upper bound for A,

there exists a natural number $a_N > \alpha - \varepsilon$. N such that

By monotonicity, for all n >= N, $\alpha - \varepsilon < a_N \leqslant a_n \leqslant \alpha$

1.4 Series

The series $\sum a_n$ is convergent if the sequence of the partial sums $(s_n)_{n \in N}$ is convergent, where $s_n = a_1+a_2+a_3 \cdots +a_n$. Otherwise, it is divergent.

There are many ways to test the convergence of a series. The proof for different theorems may be based on other theorems proven to be true. The purpose of introducing new theorems using ideas we have already grasped is for simplicity so that for different series, we can often find the easiest way to test the convergence:

- the series $\sum a_n$ is convergent if $\lim(a_n) = 0$

Proof for this theorem is reflected by the idea (and can be used to prove the following theorem as well):

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

- the series $\sum a_n$ is convergent if $\lim(s_{2n}-s_n) = 0$.
- Comparison test 1: Let $\sum a_n$ and $\sum b_n$ be two series of positive terms. Assume that for all natural numbers n, $a_n \leq b_n$. If $\sum b_n$ is convergent, then $\sum a_n$ is convergent. If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Proof: Let $s_n = \sum_{k=1}^{n} a_k$, $s'_n = \sum_{k=1}^{n} b_k$. Then

$$(\forall\, n \in \mathbb{N})\ (s_n \leq s'_n).$$

- Comparison test 2: Let $\sum a_n$ and $\sum b_n$ be two series of positive terms. Assume that $\lim(a_n/b_n) = L > 0$. $\sum a_n$ is convergent if and only if $\sum b_n$ is convergent. This is largely because by definition of the limit of a sequence:

$$(\exists\, N \in \mathbb{N})(\forall\, n > N)\ \left(\frac{1}{2}L < \frac{a_n}{b_n} < \frac{3}{2}L\right)$$

- Root test: Suppose $\lim(a_n\hat{}(1/n))=L$. If $L < 1$, $\sum a_n$ converges; If $L > 1$, $\sum a_n$ diverges; If $L = 1$, no conclusion can be drawn.

Proof: Suppose $L < 1$. Choose r such that $L<r<1$. Then

$$(\exists\, N \in \mathbb{N})(\forall\, n \in \mathbb{N})[(n > N) \Rightarrow (\sqrt[n]{a_n} < r)].$$

So for all $n > N$, $a_n < r\hat{}n$. Hence, the convergence follows from the comparison test with the convergent series $\sum_{n=1}^{\infty} r^n$

Next suppose $L < 1$. $(\exists\, N \in \mathbb{N})(\forall\, n \in \mathbb{N})[(n > N) \Rightarrow (\sqrt[n]{a_n} > 1)].$

So $a_n >1$ for all $n>N$.

Hence, the series diverges.

- Ratio test: Suppose $\lim(a_{n+1}/a_n)=L$. If $L < 1$, $\sum a_n$ converges; If $L > 1$, $\sum a_n$ diverges; If $L = 1$, no conclusion can be drawn.

1. Limits of functions and continuity

2.1 Limits of functions

- Let f be a real-valued function defined on$E \subset R$. We say that f(x) tends to L as x approaches x0, or that the limit of f(x) as x approaches x0 is L. if, for any $\epsilon>0$, there exists $\delta>0$ such that for all $x \in E, 0<|x-x0|<\delta$implies that $|f(x)-L| < \epsilon$.
- Some particular properties are very helpful in calculating the limits of functions: The sum, difference, product and quotient of two functions are equal to the sum, difference, product and quotient of the limits of two functions.

2.2 Continuous functions

- lim(x→p)f(x)=f(p). In other words, for any $\epsilon>0$, there exists $\delta>0$ such that for all $x \in Df$, $0<|x-p|<\delta$ implies that $|f(x)-f(p)| < \epsilon$.
- Let f and g be continuous at $a \in R$. Then $(f \pm g)$, $(f*g)$,(f/g) are also continuous at a. In addition, if g is continuous at a and f is continuous at b=g(a), fg is also continuous at a.
- Furthermore, for continuous functions on closed intervals [a,b], if f is continuous at every point of (a,b) and lim(x→a+)f(x)=f(a) and lim(x→b-)f(x)=f(b), then f is a continuous function on [a,b]. Meanwhile, two theorems apply to this function, namely, the intermediate value theorem and the boundedness theorem. In cases of the intermediate value theorem, if f(a) < f(b), for any real number $y \in [f(a),f(b)]$, there exists a real number $x \in [a,b]$ such that f(x)=y. As for the boundedness theorem, it is stated that the range of f possesses a supremum and infimum.

Differential Calculus

A function f:R→R is differentiable at a if lim(h→0)[(f(a+h)−f(a))/h] exists in R. The limit is the derivative of f at a, written as f'(a)

When discussing the differentiability, there are many theorems that will prove to be useful.

- If f is differentiable at a, then f is continuous at a.
- (product rule). If f and g are differentiable at a, then $f \cdot g$ is also differentiable at a, and $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.
- If g is differentiable at a and g(a)/= 0, then $\varphi = 1/g$ is also differentiable at a, and $\varphi'(a) = (1/g)'(a) = -g'(a)/[g(a)]$^2.

- (quotient rule). If f and g are differentiable at a and g(a)/= 0, then φ = f/g is also differentiable at a, and φ′(a)=(f/g)′ (a)=[f′(a) · g(a) − f(a) · g′(a)]/[g(a)]^2. The proof of this theorem follows from the previous two.

- (chain rule) If g is differentiable at a ∈ R and f is differentiable at g(a), then f ∘ g is differentiable at a and (f ∘ g)′(a) = f′(g(a)) · g′(a).

Moreover, differentiable functions also exhibit some common properties.

- (Rolle's theorem) If f is continuous on [a, b] and differentiable on (a, b), and f(a) = f(b), then there exists n ∈(a,b) such that f′(n) = 0

- (Mean value theorem) If f is continuous on [a, b] and differentiable on (a, b), then there exists n ∈(a,b) such that

  f′(n)=[f(b)−f(a)]/(b-a)

- (Cauchy mean value theorem). If f and g are continuous on [a, b] and differentiable on (a, b), then there exists x_0 ∈(a,b) such that

  [f(b) − f(a)]g′(x_0) = [g(b) − g(a)]f′(x_0) .

- (L'Hˆopital's rule) Let f and g be differentiable functions in the neigh- bourhood of a ∈ R. Assume that (i) g′(x)/= 0 in some neighbourhood of a; (ii) f(a) = g(a) = 0;

  (iii) lim(x→a)(f′(x)/g′(x)) exists.

  lim(x→a)(f(x)/g(x))= lim(x→a)(f′(x)/g′(x))

  L'Hˆopital's rule is a useful tool for computing limits

(Taylor's theorem) Let h > 0 and p ≥ 1. Suppose that f and its deriva- tives up to order n − 1 are continuous on [a,a + h] and that $f^{(n)}$ exists on (a,a + h). Then there exists a number t ∈ (0, 1) such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n$$

where

$$R_n = \frac{h^n(1-t)^{n-p}}{p(n-1)!}f^{(n)}(a+th).$$

Let $\sum_{n=1}^{\infty} a_n$ *be an absolutely convergent series*

The polynomial

$$p(x) := f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)$$

is called the **Taylor polynomial** for f of degree $n-1$ based at the point a.

2. Series

3.1 Series of positive and negative terms

- Alternating series

(Alternating series test) Let $(a_n)_n$ be a sequence of positive numbers such that

(i) $(\forall n \in \mathbb{N})\ \left(a_n \geq a_{n+1}\right)$;

(ii) $\lim_{n\to\infty} a_n = 0$.

Then the series

$$a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1}a_n + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

converges.

- Absolute convergence:

The series $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

A convergent series which is not absolutely convergent is said to be **conditionally convergent**.

- Rearrangement theorem:

with sum s. *Then any rearrangement of* $\sum_{n=1}^{\infty} a_n$ *is also convergent with sum* s.

- Multiplication of series:

(Product series) *Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be absolutely convergent series* *with sums a and b respectively. Then the* **product series**

$$a_1\, b_1 + a_2\, b_1 + a_2\, b_2 + a_1\, b_2 + a_1\, b_3 + a_2\, b_3 + \dots$$

is absolutely convergent with sum $a\, b$.

3.2 Power series

By definition, a power series is a series of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n,$$

where the coefficients a_n and variable x are real numbers.

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series.

Then exactly one of the following possibilities occurs

(i) The series converges only when $x = 0$.

(ii) The series converges absolutely for all $x \in \mathbb{R}$.

(iii) There exists $r > 0$ such that the series is absolutely convergent for all $x \in \mathbb{R}$ such that $|x| < r$ and is divergent for all $x \in \mathbb{R}$ such that $|x| > r$.

3. Elementary functions

- Exponential function

We define the **exponential function** $\exp : \mathbb{R} \to \mathbb{R}$ via the power series

$$\exp(x) := 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- **(multiplicative property)** *For all $x, y \in \mathbb{R}$,*

$$\exp(x) \cdot \exp(y) = \exp(x + y).$$

(i) $\exp(0) = 1$.

(ii) $\exp(-x) = 1/\exp(x)$ *for all* $x \in \mathbb{R}$.

(iii) $\exp(x) > 0$ *for all* $x \in \mathbb{R}$.

- *The exponential function $f(x) = \exp(x)$ is everywhere differentiable and*

$$f'(a) = \exp(a)$$

for each $a \in \mathbb{R}$.

- The logarithms

Definition 7.2.1. The function $f : \mathbb{R} \to (0, +\infty)$; $x \to e^x$ is a bijection. Its inverse $g : (0, +\infty) \to \mathbb{R}$ is called the **logarithm function**. We write $g(x) = \log(x)$. Thus, for $x > 0$,

$$y = \log(x) \text{ if and only if } e^y = x.$$

The logarithm function is a differentiable and increasing function with the following properties:

(i) $\log(x\,y) = \log(x) + \log(y)$ *for all* $x, y \in (0, +\infty)$.

(ii) $\log(x\,/\,y) = \log(x) - \log(y)$ *for all* $x, y \in (0, +\infty)$.

(iii) $\log(1) = 0$.

(iv) $(\log x)' = 1/x$.

(v) $\lim_{x\to+\infty} \log x = +\infty$.

(vi) $\lim_{x\downarrow 0} \log x = -\infty$.

(vii) $\frac{\log x}{x^k} \to 0$ *as* $x \to +\infty$ *for each* $k \in \mathbb{N}$.

4. The Riemann Integral

5.1 Definition of integral:

Let a < b. A partition of the interval [a, b] is a finite collection of points in [a,b] one of which is a and one of which is b. The point of a partition can be numbered $x_0, x_1, \ldots, x_n$ so that $a=x_0 <x_1 <x_2 <\cdots<x_{n-1} <x_n =b$.

Let $P = \{x_0, \ldots, x_n\}$ be a partition on $[a, b]$. Let

$$m_i = \inf\{f(x)\,|, x_{i-1} \le x \le x_i\},$$
$$M_i = \sup\{f(x)\,|, x_{i-1} \le x \le x_i\}.$$

The lower sum of f for P is defined as

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

The upper sum of f for P is defined as

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

The integral sum of f for the partition P and $\{\xi_i \in [x_{i-1}, x_i] \,|\, (x_i)_{i=0}^n \in P\}$ is

$$\sigma(f,P,\xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

Denote

$$m := \inf_{x\in[a,b]} f(x),\ M := \sup_{x\in[a,b]} f(x).$$

The following inequality is obviously true

$$(\forall i \in \{1,2,\dots,n\}) \big[m \le m_i \le f(\xi_i) \le M_i \le M\big].$$

Therefore we have

$$m(b-a) = \sum_{i=1}^{n} m(x_i - x_{i-1}) \le \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$
$$\le \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} M(x_i - x_{i-1}) = M(b-a),$$

which implies that for every partition and for every choice $\{\xi_i \in [x_{i-1}, x_i] \,|\, (x_i)_{i=0}^n \in P\}$ the following inequality holds

$$m(b-a) \le L(f,P) \le \sigma(f,P,\xi) \le U(f,P) \le M(b-a).$$

In other words, the sets of real numbers

$$\{L(f,P) \,|\, P\},\ \{U(f,P) \,|\, P\}$$

are bounded.

The upper integral of f over $[a,b]$ is

$$J := \inf\{U(f,P) \,|\, P\},$$

the lower integral of f over $[a,b]$ is

$$j := \sup\{L(f,P) \,|\, P\}.$$

A function $f : [a,b] \to \mathbb{R}$ is called Riemann integrable if

$$J = j.$$

The common value is called integral of f over $[a,b]$ and is denoted by

$$\int_a^b f(x)dx.$$

5.2 Criterion of integrability

A function f : [a, b] → R is Riemann integrable if and only if for any ε > 0 there exists a partition P of [a, b] such that U (f, P) – L(f, P) < ε.

5.3 Integrable functions

Let P be a partition of [a,b]. The length of the greatest subinterval of [a, b] under the partition P is called the norm of the partition P and is denoted by ‖P ‖

Let f : [a, b] → R be continuous. Then f is Riemann integrable.

5.4 Elementary properties of the integral

- Let a < c < b. Let f be integrable on [a, b]. Then f is integrable on [a, c] and on [c, b] and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

  Conversely, if f is integrable on $[a, c]$ and on $[c, b]$ then it is integrable on $[a, b]$.

- *Let f and g be integrable on $[a, b]$. Then $f + g$ is also integrable on $[a, b]$*

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

- *Let f, g be integrable on $[a, b]$ and*

$$(\forall x \in [a, b])\,(f(x) \le g(x)).$$

  Then

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

- *Let f be integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$ and*

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

- Let $f : [a,b] \to R$ be integrable and $(\forall x \in [a,b])(m \leq f(x) \leq M)$. Let $g : [m,M] \to R$ be continuous. Then $h : [a,b] \to R$ defined by $h(x) = g(f(x))$ is integrable.

5.5 Integration as the inverse to differentiation

Let f be integrable on $[a,b]$ and let F be defined on $[a,b]$ by

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on $[a,b]$.

Let f be integrable on $[a,b]$ and let F be defined on $[a,b]$ by

$$F(x) = \int_a^x f(t)dt.$$

Let f be continuous at $c \in [a,b]$. Then F is differentiable at c and

$$F'(c) = f(c).$$

Let f be continuous on [a,b] and f = g′ for some function g defined on [a,b]. Then for $x \in [a,b]$

$$\int_a^x f(t)dt = g(x) - g(a).$$

Let f be integrable on $[a,b]$ and $f = g'$ for some function g defined on $[a,b]$.

$$\int_a^b f(x)dx = g(b) - g(a).$$