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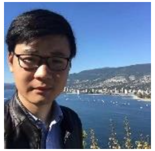
Lecture 4

KINODYNAMIC PATH FINDING



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Outline

Introduction

What is kinodynamic

Kinodynamic : Kinematic + Dynamic

The *kinodynamic planning* problem is to synthesize a robot motion subject to simultaneous *kinematic* constraints, such as avoiding obstacles, and *dynamics* constraints, such as modulus bounds on velocity, acceleration, and force. A kinodynamic solution is a mapping from time to generalized forces or accelerations.

—— *Kinodynamic Motion Planning*, Bruce Donald, Patrick Xavier, John Canny, John Reif

- Differentially constrained
- Up to force (acceleration)

Why kinodynamic planning ?

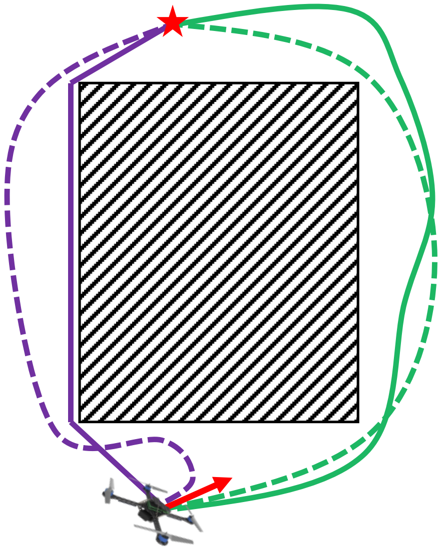
Straight-line connections between pairs of states are typically not valid trajectories due to the system’s differential constraints.

Ask: We have the back-end optimization, why kinodynamic planning?

Recall the old-school pipeline: Mission → Path Finding → Trajectory Optimization → Plan

Why kinodynamic planning ?

- Coarse-to-fine process
- Trajectory only optimizes locally
- Infeasible path means nothing to nonholonomic system

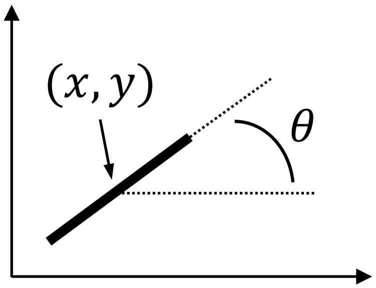


Examples

Unicycle and differential drive models:

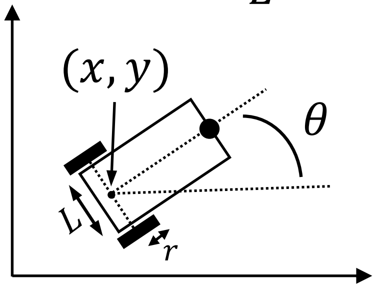
$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}cos\theta\\ sin\theta\\ 0\end{pmatrix}\cdot v + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\cdot \omega$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\frac{r}{2}(u_l+u_r)cos\theta\\ \frac{r}{2}(u_l+u_r)sin\theta\\ \frac{r}{L}(u_r-u_l)\end{pmatrix}$$



Unicycle

$$|v| \leq v_{max}$$
$$|\omega| \leq \omega_{max}$$



Differential drive

$$|u_l| \leq u_{l,max}$$
$$|u_r| \leq u_{r,max}$$

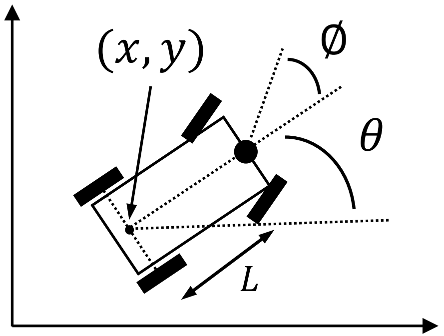
$$v = \frac{r}{2}(u_l+u_r) \qquad \omega = \frac{r}{L}(u_r-u_l)$$

http://planning.cs.uiuc.edu/node659.html

Examples

Simplified car model

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \frac{r}{L}\tan\emptyset \end{pmatrix}$$



$|v| \leq v_{max}, \quad |\emptyset| \leq \emptyset_{max} < \frac{\pi}{2}$ → Simple car model

$v \in \{-v_{max}, v_{max}\}, \quad |\emptyset| \leq \emptyset_{max} < \frac{\pi}{2}$ → Reeds & Shepp's car

$v = v_{max}, \quad |\emptyset| \leq \emptyset_{max} < \frac{\pi}{2}$ → Dubin's car

http://planning.cs.uiuc.edu/node658.html

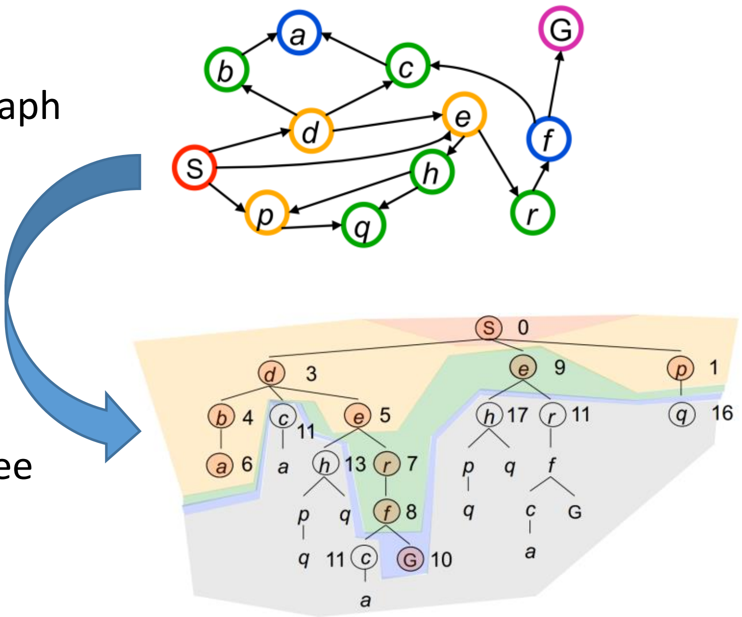
State Lattice Planning

Workflow

Basic idea

- Recall the search-based path finding method in L2
- For planning, how to build a graph?
- Is this graph doable for our real robot?

Search graph



Search tree

- Maintain a priority queue to store all the nodes to be expanded
- The heuristic function h(n) for all nodes are pre-defined
- The priority queue is initialized with the start state X_S
- Assign $g(X_S)=0$, and g(n)=infinite for all other nodes in the graph
- Loop
  - If the queue is empty, return FALSE; break;
  - Remove the node "n" with the lowest f(n)=g(n)+h(n) from the priority queue
  - Mark node "n" as expanded
  - If the node "n" is the goal state, return TRUE; break;
  - For all unexpanded neighbors "m" of node "n"
    - If g(m) = infinite
      - g(m)= g(n) + Cnm
      - Push node "m" into the queue
    - If $g(m) > g(n) + C_{nm}$
      - g(m)= g(n) + Cnm
  - end
- End Loop

Basic idea

- We have many weapons to attack graph search.
- Assume the robot a mass point is not satisfactory any more.
- We now require a graph with feasible motion connections.

- We manually create (build) a graph with all edges executable by the robot.

Forward direction: discrete (sample) in control space

Reverse direction: discrete (sample) in state space

- This is the basic motivation for all kinodyanmic planning.
- State lattice planning is the most straight-forward one.

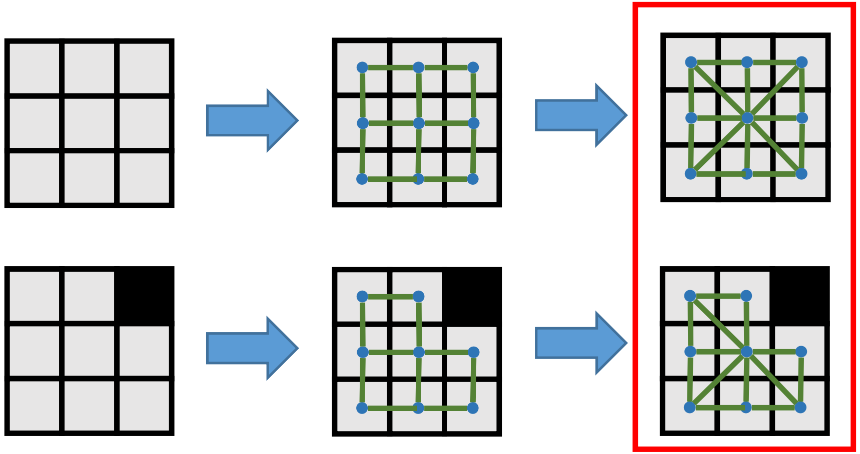
Connection with previous lectures

- We manually create (build) a graph with all edges executable by the robot.

Forward direction: discrete (sample) in control space

Reverse direction: discrete (sample) in state space

Things in L2



4 connection

8 connection

- Actually, this is a discretization of control space!
- We assume the robot can move in 4/8 directions

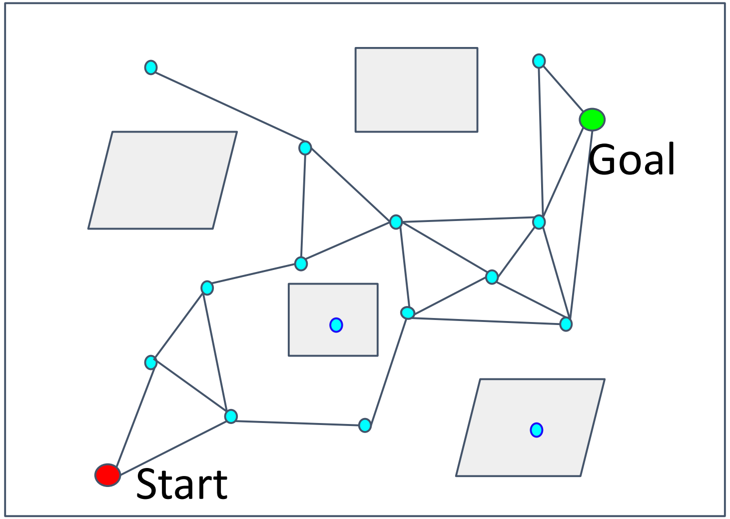
Connection with previous lectures

- We manually create (build) a graph with all edges executable by the robot.

Forward direction: discrete (sample) in control space

Reverse direction: discrete (sample) in state space

Things in L3



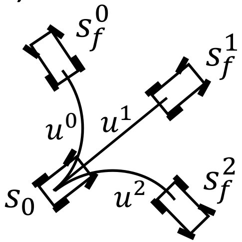
- Actually, this is a discretization of state space!
- Here the state is R^2, only position (x, y) is considered here.

Build the graph, sample in control vs. state space

For a robot model: $\dot{s} = f(s, u)$

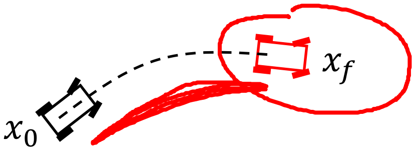
- The robot is differentially driven.
- We have an initial state s_0 of the robot.
- Generate feasible local motions by:

☐ Select a u, fix a time duration T, forward simulate the system (numerical integration).



- Forward simulation
- Fixed u, T
- No mission guidance,
- East to implement
- low planning efficiency

☐ Select a s_f, find the connection (a trajectory) between s_0 and s_f.



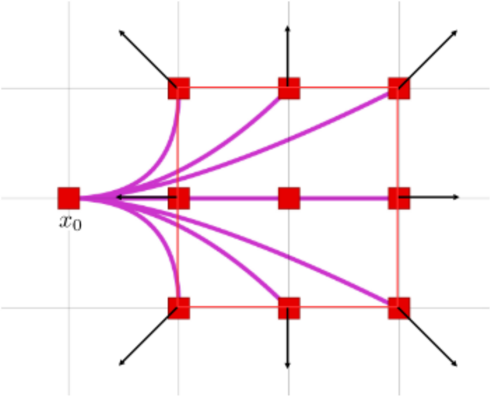
- Backward calculation
- Need calculate u, T
- Good mission guidance
- Hard to implement
- High planning efficiency

Sample in control space

State: $s = \begin{pmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$ **Input**: $u = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}$

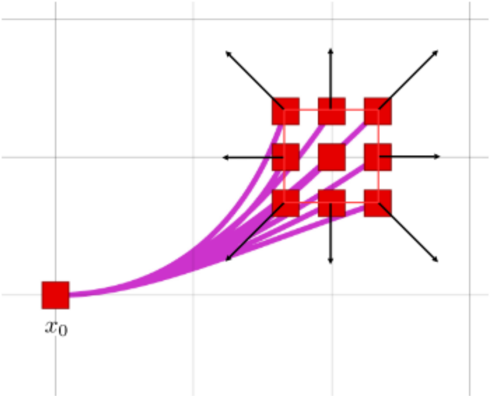
System equation: $\dot{s} = A \cdot s + B \cdot u$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Discretize acceleration

$$v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$



Discretize jerk

$$v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ a_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

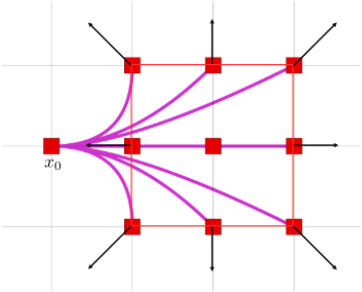
$$\dot{s} = A \cdot s + B \cdot u$$

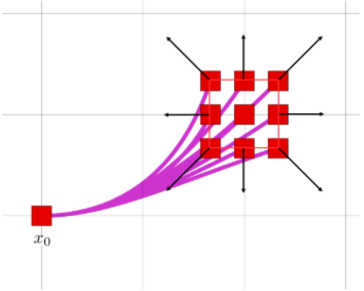
$$A = \begin{bmatrix} 0 & I_3 & 0 & \cdots & 0 \\ 0 & 0 & I_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & I_3 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_3 \end{bmatrix}$$

- Several-order integrator
- A nilpotent
- Very special, remember it

Search-based Motion Planning for Quadrotors using Linear Quadratic Minimum Time Control, Sikang Liu, Nikolay Atanasov, Kartik Mohta, and Vijay Kumar

Sample in control space





State: $s = \begin{pmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$ **Input**: $u = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}$

System equation: $\dot{s} = A \cdot s + B \cdot u$

$$\dot{s} = A \cdot s + B \cdot u$$

$$A = \begin{bmatrix} 0 & I_3 & 0 & \cdots & 0 \\ 0 & 0 & I_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & I_3 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_3 \end{bmatrix}$$

$$s(t) = \underbrace{e^{At}}_{F(t)} s_0 + \underbrace{\left[\int_0^t e^{A(t-\sigma)} B d\sigma \right]}_{G(t)} u_m$$

e^{At} : state transition matrix, critical to the integration.

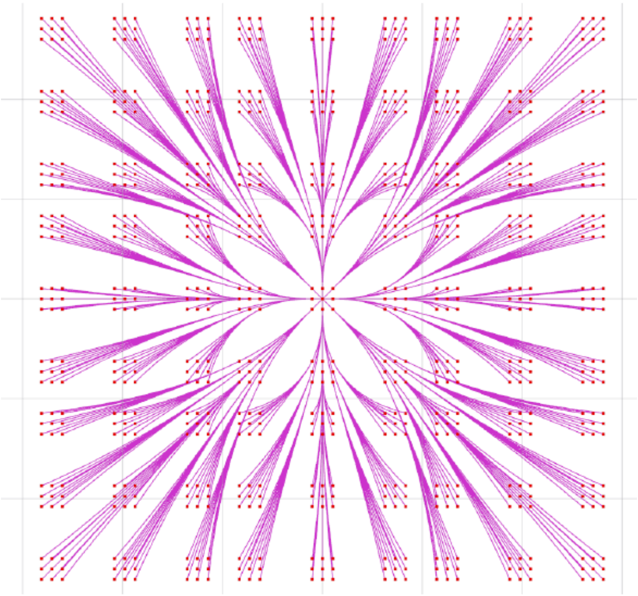
$$\boxed{e^{At} = I + \frac{At}{1!} + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots + \frac{(At)^k}{k!} + \cdots}$$

If matrix A $\in R^{n \times n}$ is nilpotent, i.e. $A^n = 0$, e^{At} has a closed-form expression in the form of an $(n-1)$ degree matrix polynomial in t.

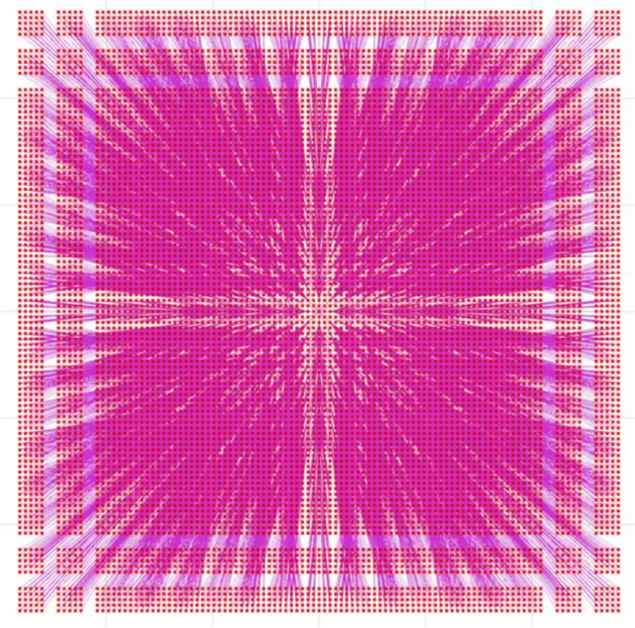
Search-based Motion Planning for Quadrotors using Linear Quadratic Minimum Time Control, Sikang Liu, Nikolay Atanasov, Kartik Mohta, and Vijay Kumar

Sample in control space

The lattice graph obtained by searching



9 discretization



25 discretization

Note

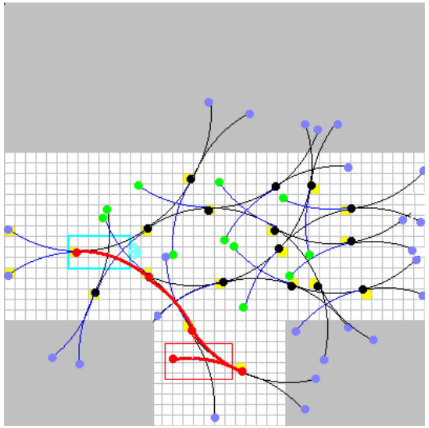
- During searching, the graph can be built when necessary.
- Create nodes (state) and edges (motion primitive) when nodes are newly discovered.
- Save computational time/space.

Sample in control space

- For every $s \in T$ from the search tree
- Pick a control vector u
- Integrate the equation over short duration
- Add collision-free motions to the search tree

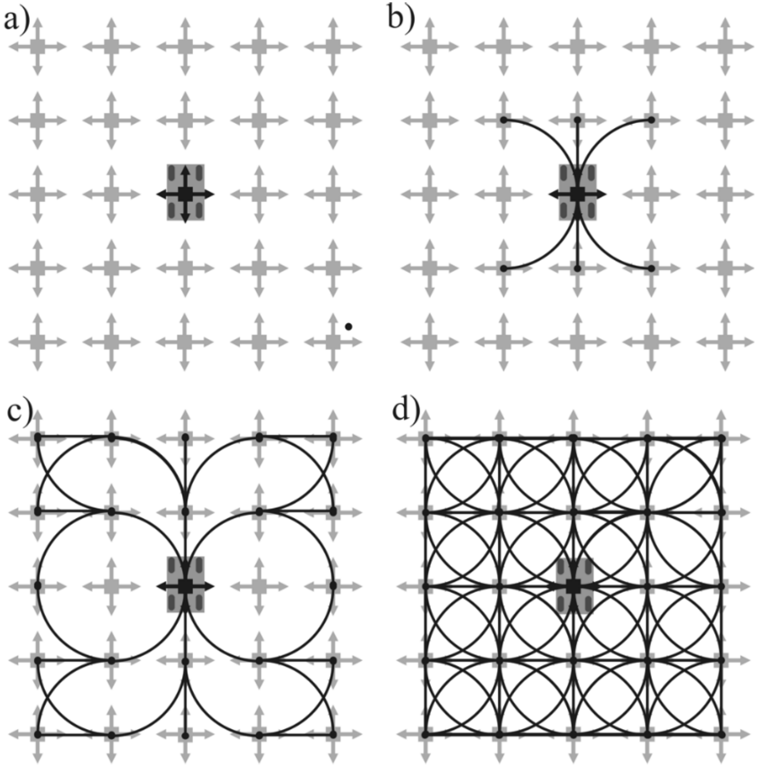
State: $s = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$ **Input**: $u = \begin{pmatrix} v \\ \emptyset \end{pmatrix}$

System equation: $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v \cdot cos\theta \\ v \cdot sin\theta \\ \frac{r}{L} \cdot tan\emptyset \end{pmatrix}$



1) Select a $s \in T$
2) Pick $v, \emptyset$ and τ
3) Integrate motion from s
4) Add result if collision-free

Sample in state space



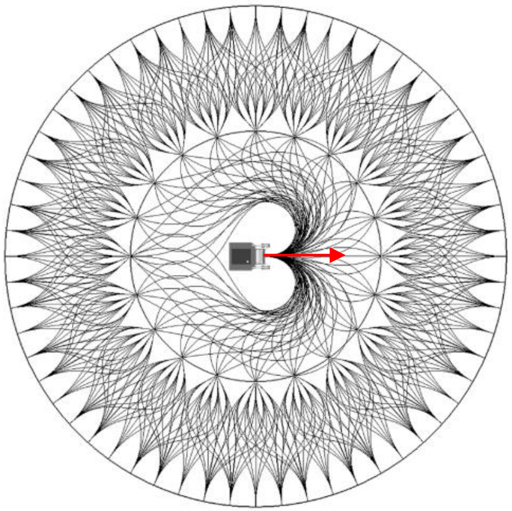
Build a lattice graph:

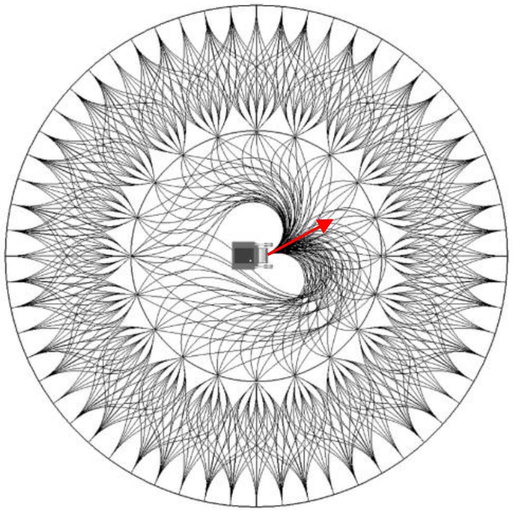
- Given an origin.
- for 8 neighbor nodes around the origin, feasible paths are found.
- extend outward to 24 neighbors.
- complete lattice.

Reeds-Shepp Car Model

Generating Near Minimal Spanning Control Sets for Constrained Motion Planning in Discrete State Spaces, Mihail Pivtoraiko and Alonzo Kelly

Sample in state space

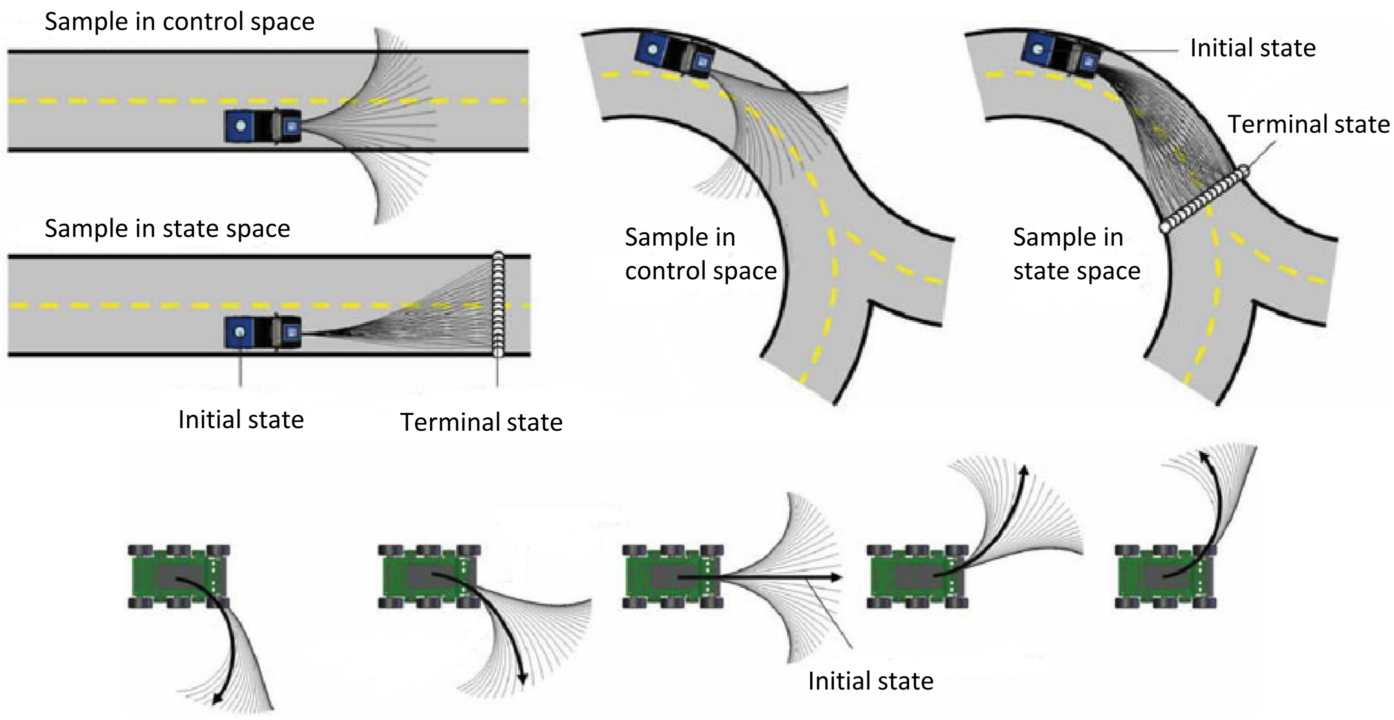




- Two layer lattice graph
- Only first layer is different
- Different initial states

Optimal Rough Terrain Trajectory Generation for Wheeled Mobile Robots, Thomas M. Howard Alonzo Kelly

Comparison



- Trajectories are denser in the direction of the initial angular velocity.
- Very similar outputs for several distinct inputs.

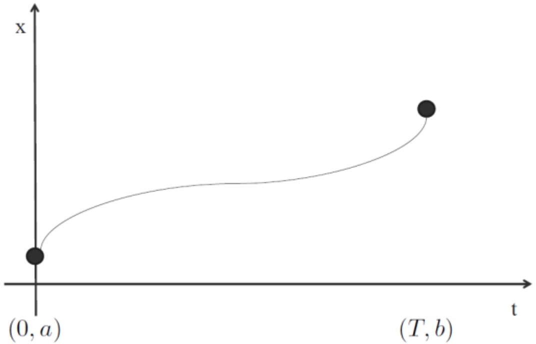
State Space Sampling of Feasible Motions for High-Performance Mobile Robot Navigation in Complex Environments,
Thomas M. Howard, Colin J. Green, and Alonzo Kelly

Boundary Value Problem

Boundary Value Problem (BVP)

- BVP is the basis of state sampled lattice planning.
- No general solution. Case by case design.
- Often evolve complicated numerical optimization.

- Design a trajectory $x(t)$ such that:
  - $x(0) = a$
  - $x(T) = b$



Boundary Value Problem (BVP)

- 5^{th} order polynomial trajectory:
  - $x(t) = c_5 t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0$

- Boundary conditions

	Position	Velocity	Acceleration
t = 0	a	0	0
t = T	b	0	0

- Solve:

$$\begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ T^5 & T^4 & T^3 & T^2 & T & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}$$

Optimal Boundary Value Problem (OBVP)

Modelling

Objective, minimize the integral of squared jerk:

$$J_{\Sigma} = \sum_{k=1}^{3} J_k, \quad J_k = \frac{1}{T}\int_0^T j_k(t)^2\, dt.$$

State: $s_k = (p_k, v_k, a_k)$ Input: $u_k = j_k$

System model: $\dot{s}_k = f_s(s_k, u_k) = (v_k, a_k, j_k)$

Solving

By $\boxed{\text{Pontryain's minimum principle}}$, we first introduce the costate: $\lambda = (\lambda_1, \lambda_2, \lambda_3)$

Define the Hamiltonian function:

$$\begin{aligned} H(s,u,\lambda) &= \frac{1}{T} j^2 + \lambda^T f_s(s,u) \\ &= \frac{1}{T} j^2 + \lambda_1 v + \lambda_2 a + \lambda_3 j \end{aligned}$$

minimum principle

$\dot{s}^*(t) = f(s^*(t), u^*(t)), \quad given: s^*(0) = s(0)$

$\lambda(t)$ is the solution of:

$$\dot{\lambda}(t) = -\nabla_s H(s^*(t), u^*(t), \lambda(t))$$

with the boundary condition of:

$$\lambda(T) = -\nabla h(s^*(T))$$

and the optimal control input is:

$$u^*(t) = \arg\min_{u(t)} H(s^*(t), u(t), \lambda(t))$$

A Computationally Efficient Motion Primitive for Quadrocopter Trajectory Generation, Mark W. Mueller, Markus Hehn, and Raffaello D'Andrea

Dynamic Programming and Optimal Control, D. P. Bertsekas

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Objective, minimize the integral of squared jerk:

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State: $s_k = (p_k, v_k, a_k)$ Input: $u_k = j_k$

System model: $\dot{s} = f_s(s,u) = (v, a, j)$

Solving

By Pontryain's minimum principle, we first introduce the costate: $\lambda = (\lambda_1, \lambda_2, \lambda_3)$

Define the Hamiltonian function:

$$H(s,u,\lambda) = \frac{1}{T} j^2 + \lambda^T f_s(s,u)$$
$$= \frac{1}{T} j^2 + \lambda_1 v + \lambda_2 a + \lambda_3 j$$

minimum principle

$\dot{s}^*(t) = f(s^*(t), u^*(t)), \quad given: s^*(0) = s(0)$

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$$u^*(t) = \arg\min_{u(t)} H(s^*(t), u(t), \lambda(t))$$

A Computationally Efficient Motion Primitive for Quadrocopter Trajectory Generation, Mark W. Mueller, Markus Hehn, and Raffaello D'Andrea

Dynamic Programming and Optimal Control, D. P. Bertsekas

Pontryagin's minimum principle

Generally:

$$J = \boxed{h(s(T))} + \boxed{\int_0^T g(s(t),u(t))\cdot dt}$$

final state

transition cost

Write the Hamiltonian and costate:

$$H(s,u,\lambda) = g(s,u) + \lambda^T f(s,u)$$

$$\lambda = (\lambda_1, \lambda_2, \lambda_3)$$

Suppose:

s^*: Optimal state

u^*: Optimal input

We have

minimum principle

$$\dot{s}^*(t) = f(s^*(t), u^*(t)), \quad given: s^*(0) = s(0)$$

$\lambda(t)$ is the solution of:

$$\dot{\lambda}(t) = -\nabla_s H(s^*(t), u^*(t), \lambda(t))$$

with the boundary condition of:

$$\lambda(T) = -\nabla h(s^*(T))$$

and the optimal control input is:

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Optimal Boundary Value Problem (OBVP)

Modelling

Objective, minimize the integral of squared jerk:

$$J_{\Sigma} = \sum_{k=1}^{3} J_k, \quad J_k = \frac{1}{T}\int_0^T j_k(t)^2\, dt.$$

State: $s_k = (p_k, v_k, a_k)$ Input: j_k

System equation: $\dot{s} = f_s(s,u) = (v, a, j)$

Solving

By Pontryain's minimum principle, we first introduce the costate: $\lambda = (\lambda_1, \lambda_2, \lambda_3)$

Define the Hamiltonian function:

$$H(s,u,\lambda) = \frac{1}{T} j^2 + \lambda^T f_s(s,u)$$

$$= \frac{1}{T} j^2 + \lambda_1 v + \lambda_2 a + \lambda_3 j$$

$$\dot{\lambda} = -\nabla_s H(\boxed{s^*}, \boxed{u^*}, \lambda) = (0, -\lambda_1, -\lambda_2)$$

Optimal state Optimal input

The costate is solved as:

$$\lambda(t) = \frac{1}{T}\begin{bmatrix} -2\alpha \\ 2\alpha t + 2\beta \\ -\alpha t^2 - 2\beta t - 2\gamma \end{bmatrix}$$

The optimal input is solved as:

$$u^*(t) = j^*(t) = \arg\min_{j(t)} H(s^*(t), j(t), \lambda(t))$$

$$= \frac{1}{2}\alpha t^2 + \beta t + \gamma$$

The optimal state trajectory is solved as:

$$s^*(t) = \begin{bmatrix} \frac{\alpha}{120} t^5 + \frac{\beta}{24} t^4 + \frac{\gamma}{6} t^3 + \frac{a_0}{2} t^2 + v_0 t + p_0 \\ \frac{\alpha}{24} t^4 + \frac{\beta}{6} t^3 + \frac{\gamma}{2} t^2 + a_0 t + v_0 \\ \frac{\alpha}{6} t^3 + \frac{\beta}{2} t^2 + \gamma t + a_0 \end{bmatrix}$$

Initial state: $s(0) = (p_0, v_0, a_0)$

A Computationally Efficient Motion Primitive for Quadrocopter Trajectory Generation, Mark W. Mueller, Markus Hehn, and Raffaello D'Andrea

Dynamic Programming and Optimal Control, D. P. Bertsekas

Optimal Boundary Value Problem (OBVP)

The cost:

$$J = \gamma^2 + \beta\gamma T + \frac{1}{3}\beta^2 T^2 + \frac{1}{3}\alpha\gamma T^2 + \frac{1}{4}\alpha\beta T^3 + \frac{1}{20}\alpha^2 T^4$$

J only depends on T, and the boundary states (known), so we can even get an optimal T!

How?

Polynomial function root finding problem.

α, β, γ Is solved as:

$$\begin{bmatrix} \frac{1}{120}T^5 & \frac{1}{24}T^4 & \frac{1}{6}T^3 \\ \frac{1}{24}T^4 & \frac{1}{6}T^3 & \frac{1}{2}T^2 \\ \frac{1}{6}T^3 & \frac{1}{2}T^2 & T \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \Delta p \\ \Delta v \\ \Delta a \end{bmatrix}$$

$$\begin{bmatrix} \Delta p \\ \Delta v \\ \Delta a \end{bmatrix} = \begin{bmatrix} p_f - p_0 - v_0 T - \frac{1}{2}a_0 T^2 \\ v_f - v_0 - a_0 T \\ a_f - a_0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \frac{1}{T^5} \begin{bmatrix} 720 & -360T & 60T^2 \\ -360T & 168T^2 & -24T^3 \\ 60T^2 & -24T^3 & 3T^4 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta v \\ \Delta a \end{bmatrix}$$

This derivation holds for fixed final state: $s(T) = (p_f, v_f, a_f)$

Similar solution can also be found when $s(T)$ is partially defined

Same solving process holds for $J_k = \int_0^T j_k(t)^2 dt + T$.

Optimal Boundary Value Problem (OBVP)

- Previous slides are about fixed final state problem.
- How about the final state is (partially)-free?
  - Did you notice where is the boundary condition?

$$\lambda(t) = -\nabla h(s^*(t))$$

For fixed final state problem:

$$h(s(T)) = \begin{cases} 0, & if \quad s = s(T) \\ \infty, & otherwise \end{cases}$$

Not differentiable

So we discard this condition, and use given $x(T)$ to directly solve for unknown variables

$$s^*(t) = \begin{bmatrix} \frac{\alpha}{120}t^5 + \frac{\beta}{24}t^4 + \frac{\gamma}{6}t^3 + \frac{a_0}{2}t^2 + v_0 t + p_0 \\ \frac{\alpha}{24}t^4 + \frac{\beta}{6}t^3 + \frac{\gamma}{2}t^2 + a_0 t + v_0 \\ \frac{\alpha}{6}t^3 + \frac{\beta}{2}t^2 + \gamma t + a_0 \end{bmatrix}$$

For (partially)-free final state problem:

$$given \quad s_i(T), \; i \in I$$

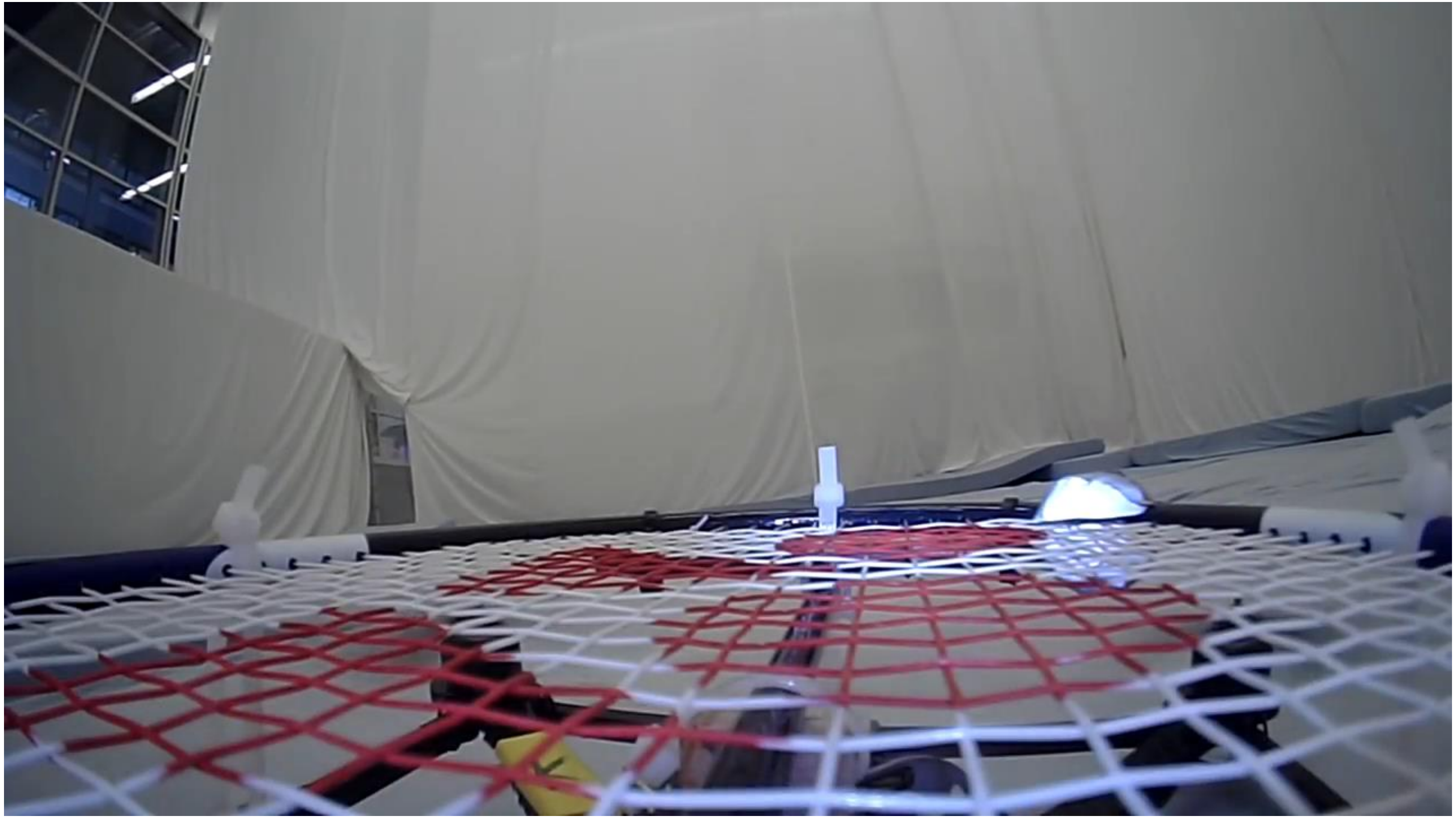
We have boundary condition for other costate:

$$\lambda_j(T) = \frac{\partial h(s^*(T))}{\partial s_j}, \quad for \; j \neq i$$

Then we solve this problem again.

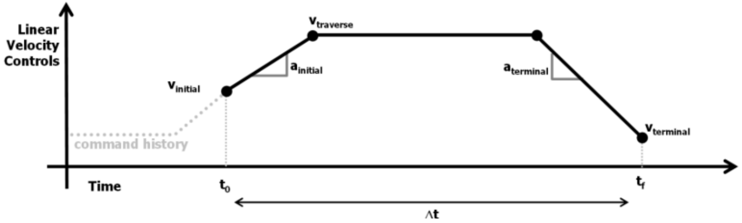
Application

Application



Another example

- Parametrize the control input
  - $\omega(t) = a + bt + ct^2 + dt^3 + \ldots.$
  - $v(t) =$

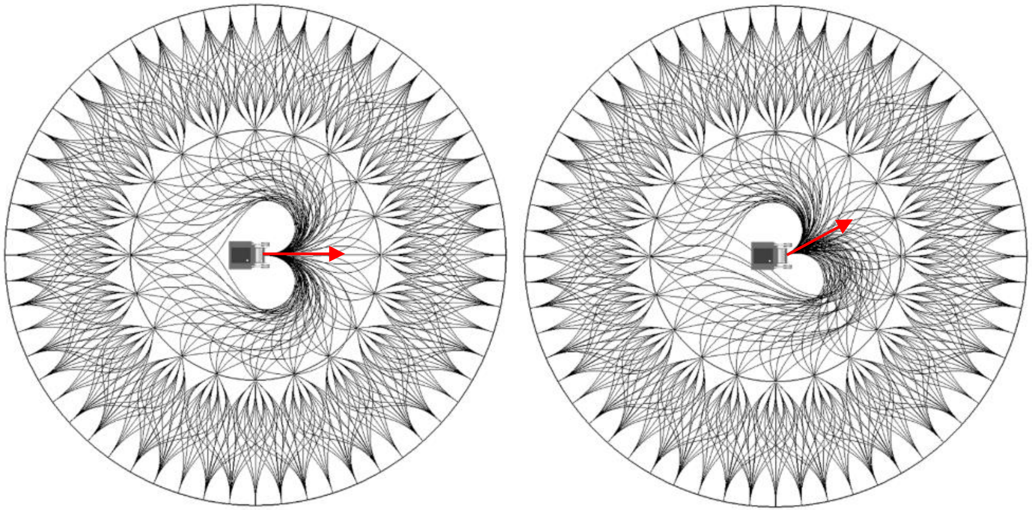


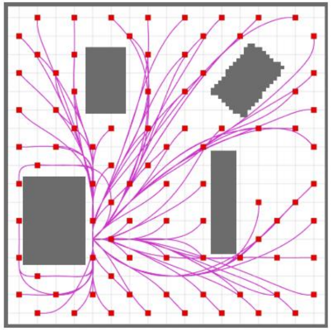
- Constrained Trajectory Generation
- Numerical difference evaluated Jacobian

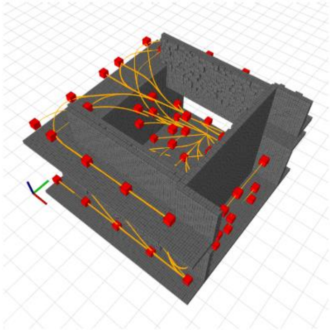
- Offline BVP, trajectory generation.
- Online search.

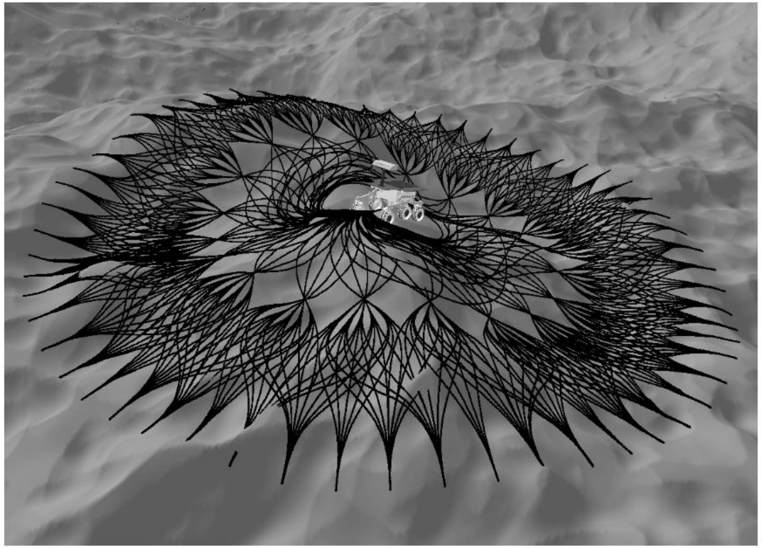
Optimal Rough Terrain Trajectory Generation for Wheeled Mobile Robots, Thomas M. Howard Alonzo Kelly

Graph search problem







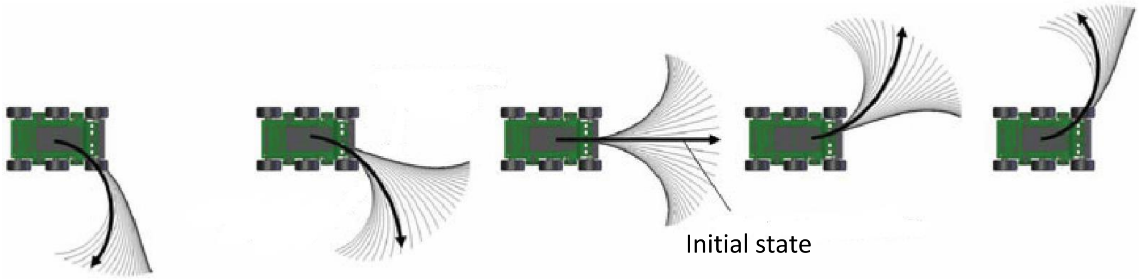


- Up to now, it ahs become a graph search problem.
- Every techniques learned in L2 can be applied here.

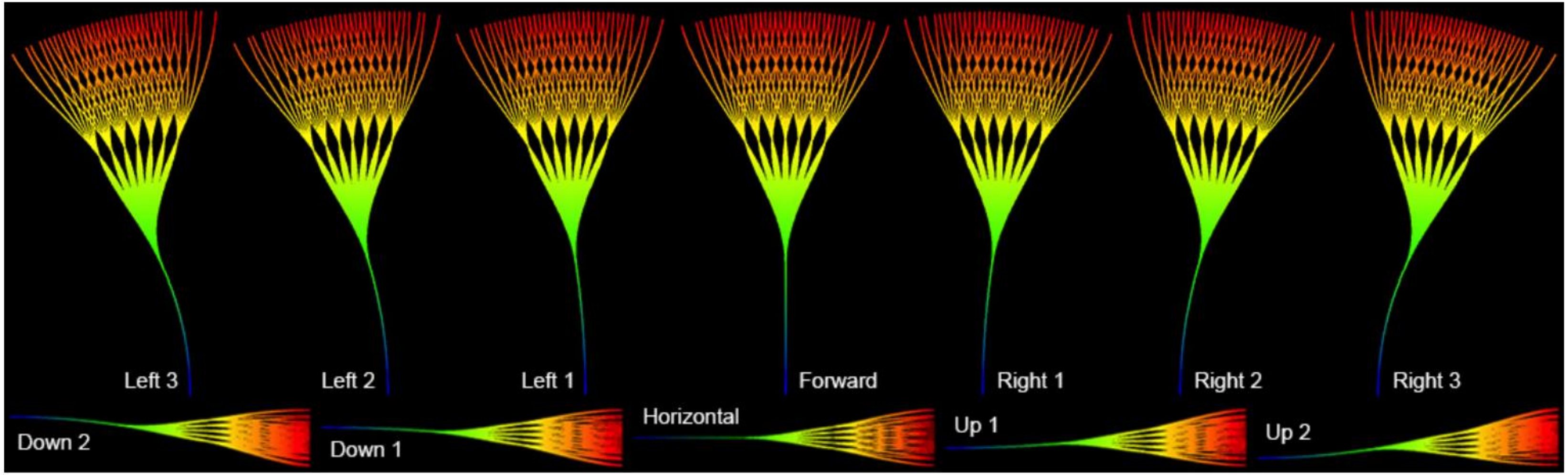
Trajectory library

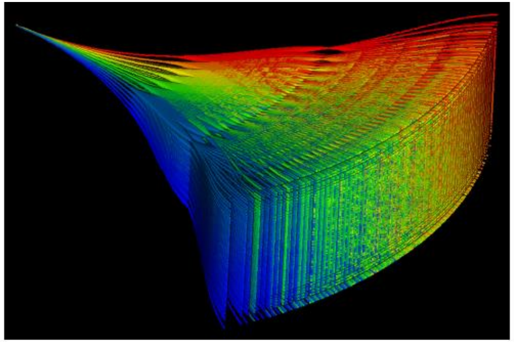
- Single layer lattice planning is a common option for local collision avoidance.
- No graph search, only trajectory selection.
- Rating each trajectory based on a multi-term cost function.

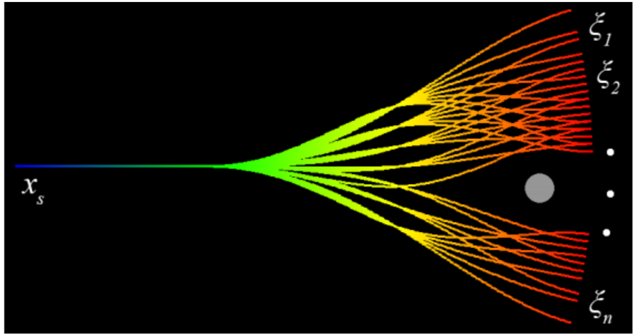
collision risk, information acquisition, comfort, energy, …



Trajectory library







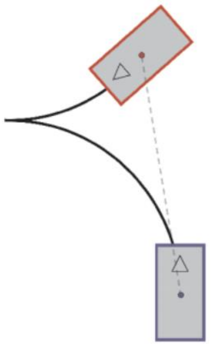
Maximum Likelihood Path Planning for Fast Aerial Maneuvers and Collision Avoidance Ji Zhang, Chen Hu, Rushat Gupta Chadha, and Sanjiv Singh

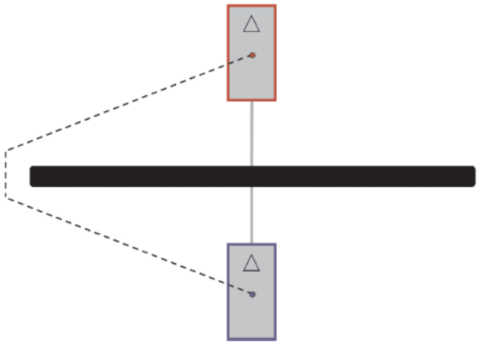
Heuristic

Heuristic design

Principle: solve an easier problem

- Assume no obstacle existence
- Assume no dynamic existence

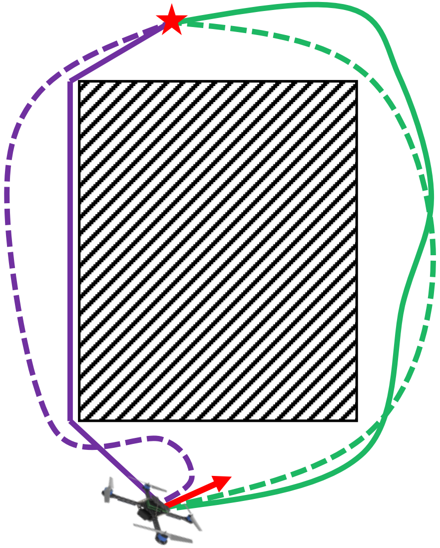




Planning Long Dynamically Feasible Maneuvers for Autonomous Vehicles, Maxim Likhachev, Dave Ferguson

Heuristic design

For every node (state), Ignoring the dynamic model and search the shortest path for it



Assume no obstacle existence

For every node (state), solve the OBVP to the planning target state as heuristic function h

- Maintain a priority queue to store all the nodes to be expanded
- The heuristic function h(n) for all nodes are pre-defined
- The priority queue is initialized with the start state X_S
- Assign $g(X_S)=0$, and g(n)=infinite for all other nodes in the graph
- Loop
  - If the queue is empty, return FALSE; break;
  - Remove the node "n" with the lowest $f(n)=\boxed{g(n)}+\boxed{h(n)}$ from the priority queue
  - Mark node "n" as expanded
  - If the node "n" is the goal state, return TRUE; break;
  - For all unexpanded neighbors "m" of node "n"
    - If g(m) = infinite
      - g(m)= g(n) + Cnm
      - Push node "m" into the queue
    - If $g(m) > g(n) + C_{nm}$
      - g(m)= g(n) + Cnm
  - end
- End Loop

Accumulate cost

Comparison

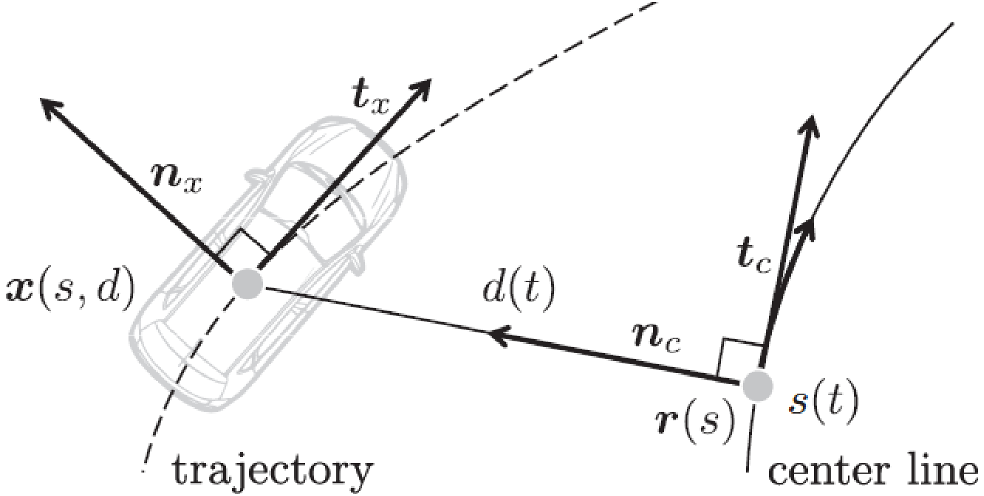
Euclidean 2D distance

non-holonomic-without-obstacles

Path Planning for Autonomous Vehicles in Unknown Semi-structured Environments, Dmitri Dolgov, Sebastian Thrun, Michael Montemerlo, James Diebel

Planning in Frenet-serret Frame

Frenet-serret frame



- dynamic reference frame.
- lateral and longitudinal independently.
- For lane following problem, the problem is decoupled.

✓ **Motion/control parametrization: quintic polynomial.**

$$d(t) = a_{d0} + a_{d1}t + a_{d2}t^2 + a_{d3}t^3 + a_{d4}t^4 + a_{d5}t^5$$

$$s(t) = a_{s0} + a_{s1}t + a_{s2}t^2 + a_{s3}t^3 + a_{s4}t^4 + a_{s5}t^5$$

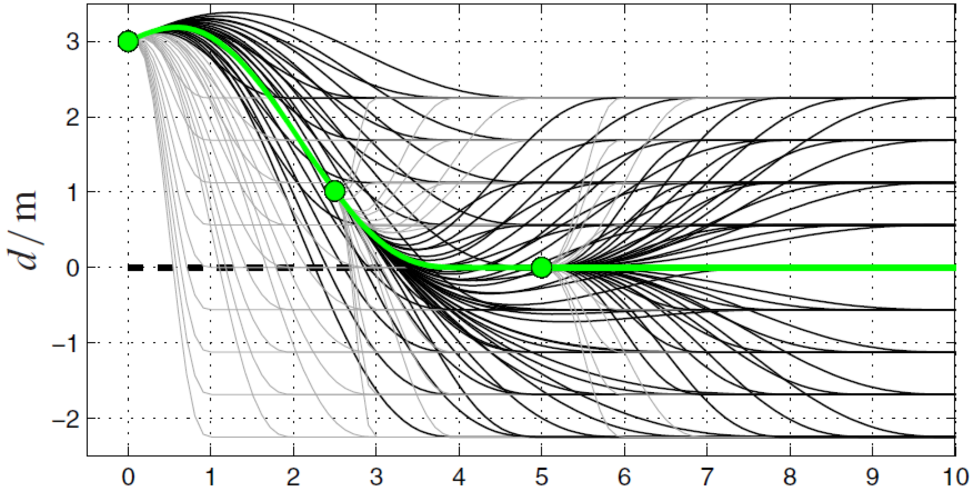
✓ **Solve the optimal control problem.**

We only discuss the lateral planning here,
for longitudinal planning, please refer to:

Optimal Trajectory Generation for Dynamic Street Scenarios in a Frenet Frame, Moritz Werling, Julius Ziegler, Sören Kammel, and Sebastian Thrun

Optimal trajectories for time-critical street scenarios using discretized terminal manifolds, Moritz Werling, Sören Kammel, Julius Ziegler and Lutz Gröll

Planning in Frenet-serret frame



Lateral trajectory

$$d(t) = a_{d0} + a_{d1}t + a_{d2}t^2 + a_{d3}t^3 + a_{d4}t^4 + a_{d5}t^5$$

Initial condition:

$$D(0) = \begin{pmatrix} d_0 & \dot{d}_0 & \ddot{d}_0 \end{pmatrix}$$

Terminate condition:

$$D(T) = \begin{pmatrix} d_f & \dot{d}_f & \ddot{d}_f \end{pmatrix}$$

Lane following:

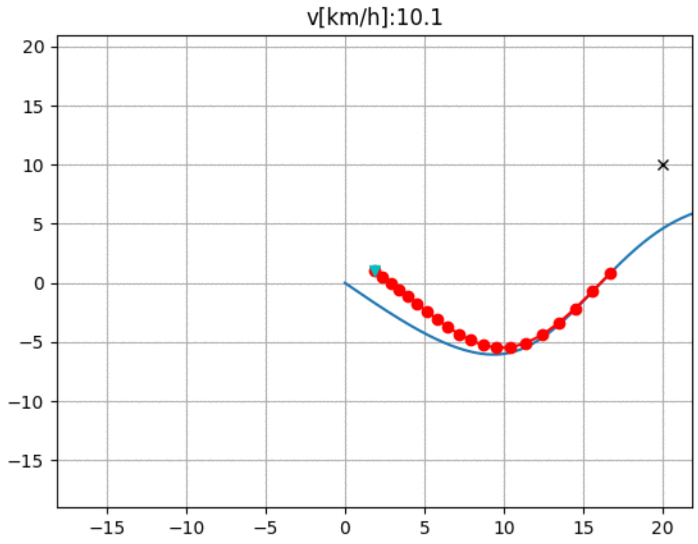
$$D(T) = \begin{pmatrix} d_f & 0 & 0 \end{pmatrix}$$

Recall what we learn previously:

$$\begin{bmatrix} T^3 & T^4 & T^5 \\ 3T^2 & 4T^3 & 5T^4 \\ 6T & 12T^2 & 20T^3 \end{bmatrix} \begin{bmatrix} a_{d3} \\ a_{d4} \\ a_{d5} \end{bmatrix} = \begin{bmatrix} \Delta p \\ \Delta v \\ \Delta a \end{bmatrix} \qquad \begin{bmatrix} \Delta p \\ \Delta v \\ \Delta a \end{bmatrix} = \begin{bmatrix} d_f - (d_0 + \dot{d}_0 T + \frac{1}{2}\ddot{d}_0 T^2) \\ \dot{d}_f - (\dot{d}_0 + \ddot{d}_0 T) \\ \ddot{d}_f - \ddot{d}_0 \end{bmatrix}$$

Planning in Frenet-serret frame

Example



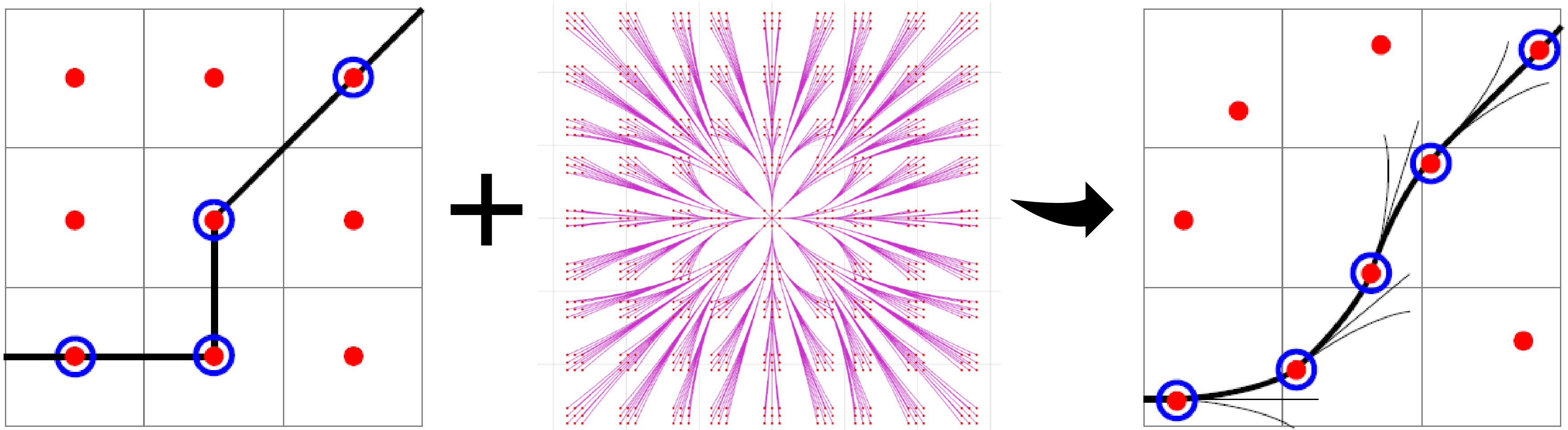
https://pythonrobotics.readthedocs.io/en/latest/modules/path_planning.html#optimal-trajectory-in-a-frenet-frame

Hybrid A*

Workflow

Basic idea

- Online generate a dense lattice costs too much time.
- How about prune some nodes?
- Define a rule to prune: use the grid map.



Practical Search Techniques in Path Planning for Autonomous Driving, Dmitri Dolgov, Sebastian Thrun
Path Planning for Autonomous Vehicles in Unknown Semi-structured Environments, Dmitri Dolgov, Sebastian Thrun

Detail

- Maintain a priority queue to store all the nodes to be expanded
- The heuristic function h(n) for all nodes are pre-defined
- The priority queue is initialized with the start state X_S
- Assign $g(X_S)=0$, and g(n)=infinite for all other nodes in the graph
- Loop
  - If the queue is empty, return FALSE; break;
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  - Mark node "n" as expanded
  - If the node "n" is the goal state, return TRUE; break;
  - For all unexpanded neighbors "m" of node "n"
    - If g(m) = infinite
      - $g(m)= g(n) + C_{nm}$
      - Push node "m" into the queue
      - +
    - If $g(m) > g(n) + C_{nm}$
      - $g(m)= g(n) + C_{nm}$
      - +
  - end
- End Loop

Choose a proper heuristic according to previous slides

Find neighbors by forward integrating the state in the node.

Record the state inside node "m"

Update the state inside node "m"

Heuristic design

(a) (b) (c) (d)

(a) 2D-Euclidean distance

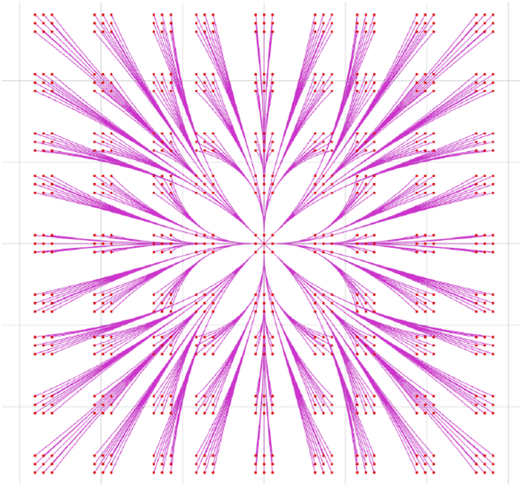
(b) non-holonomic-without-obstacles

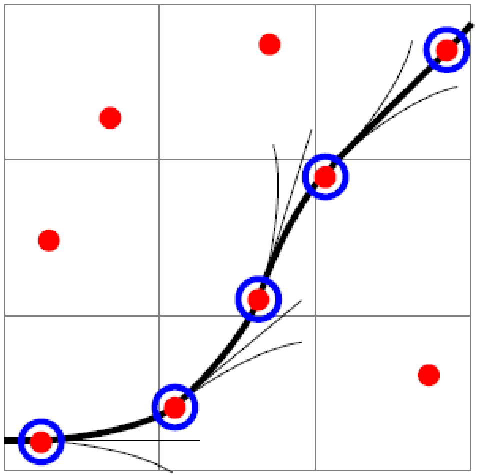
(c) non-holonomic-without-obstacles, bad performance in dead ends

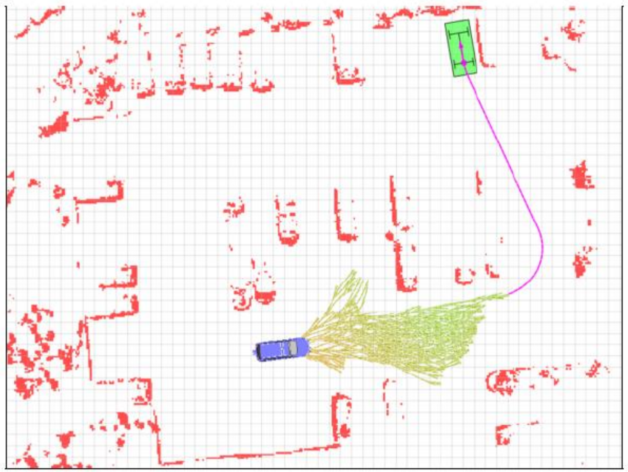
(d) non-holonomic-without-obstacles + holonomic-with-obstacles (2D shortest path)

Other tricks

Analytic Expansions: One shot heuristic
Add a state-driven bias towards the searching process





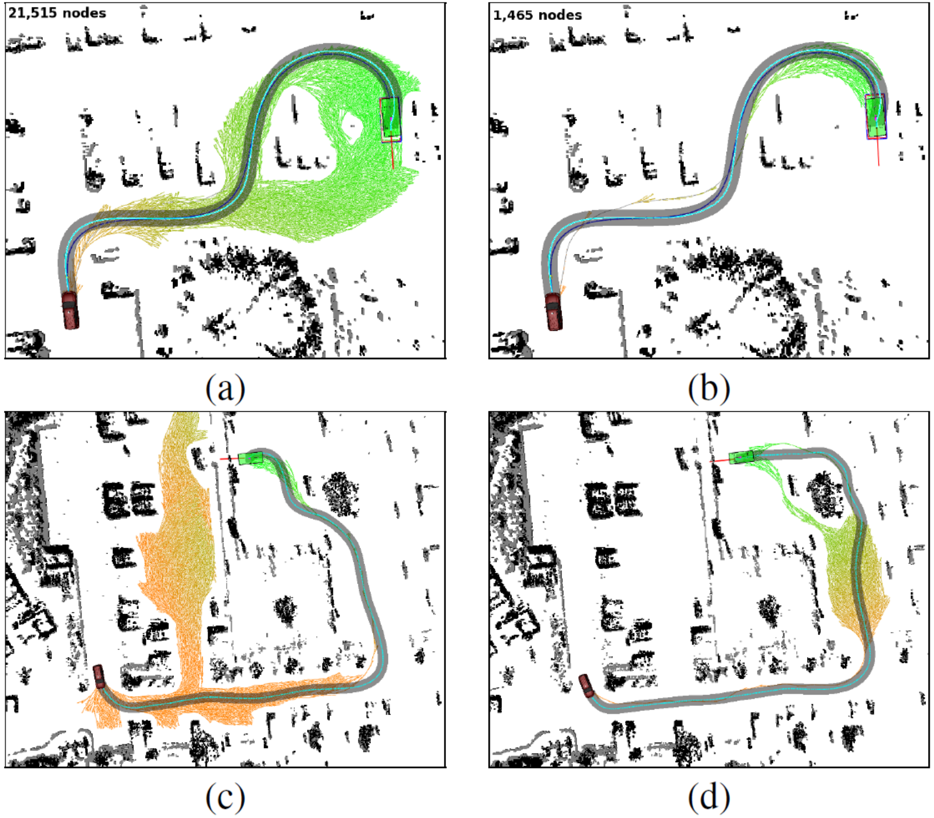


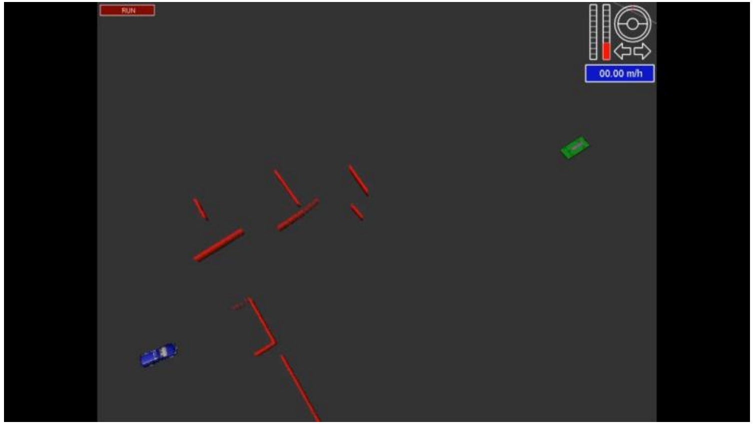
Control space sample (discretization) is kind of low-efficient, since no target biasing is encoded

How about we manually add (try) state space sample?

Application

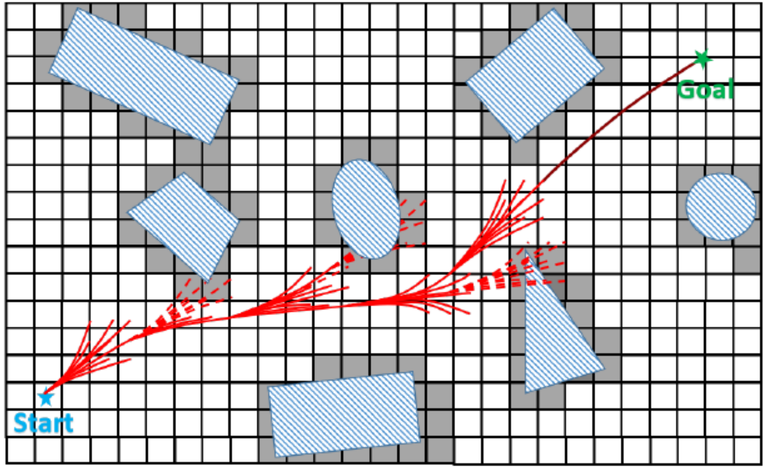
Autonomous car

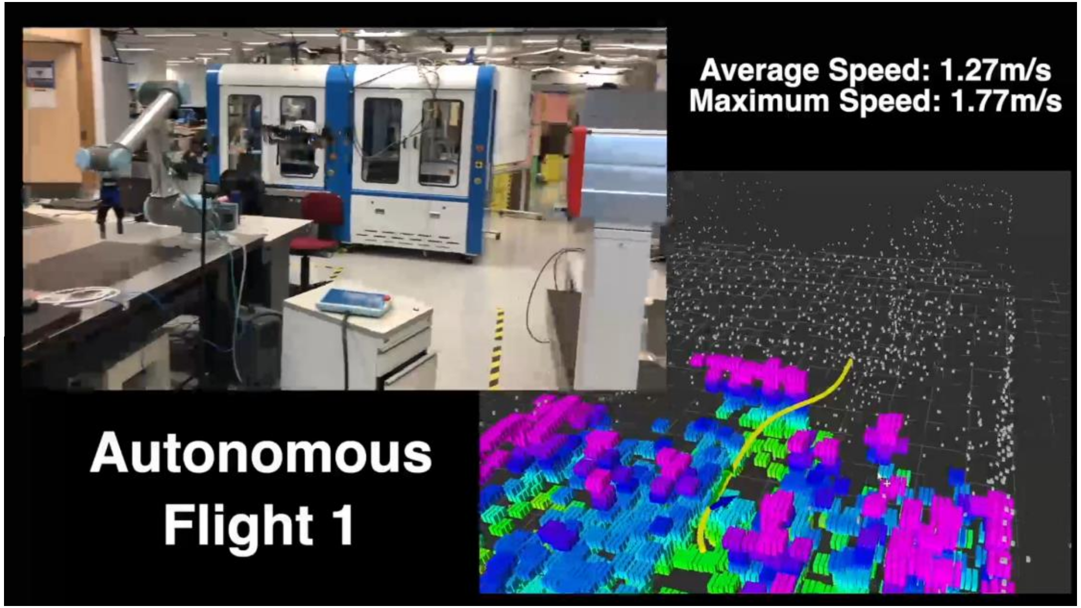




Practical Search Techniques in Path Planning for Autonomous Driving, Dmitri Dolgov, Sebastian Thrun

Autonomous UAV





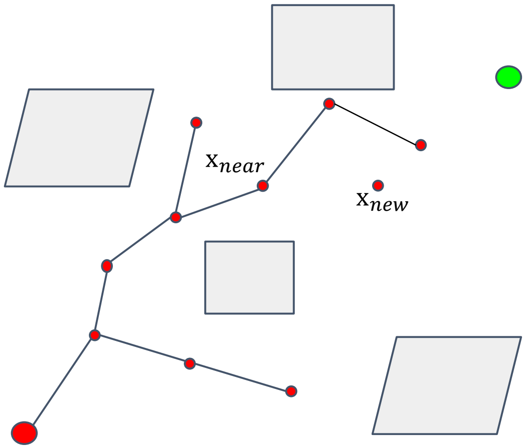
- As a promising front-end
- Careful engineering considerations
- Linear UAV model: nilpotent!
- Sophisticated C++ implementation

Robust and Efficient Quadrotor Trajectory Generation for Fast Autonomous Flight, Boyu Zhou, Fei Gao

https://github.com/HKUST-Aerial-Robotics/Fast-Planner

Kinodynamic RRT*

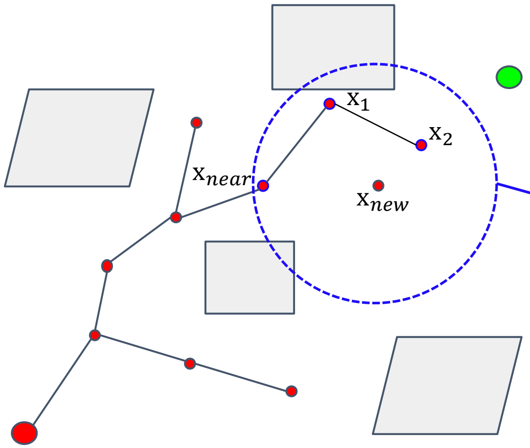
Review of RRT*



Algorithm 2: RRT Algorithm

Input: $\mathcal{M}$,x_{init},x_{goal}
Result: A path Γ from x_{init} to x_{goal}
$\mathcal{T}$.init();
for $i = 1$ *to* n **do**
  $x_{rand} \leftarrow Sample(\mathcal{M})$;
  $x_{near} \leftarrow Near(x_{rand}, \mathcal{T})$;
  $x_{new} \leftarrow Steer(x_{rand}, x_{near}, StepSize)$;
  if *CollisionFree*(x_{new}) **then**
    $X_{near} \leftarrow NearC(\mathcal{T}, x_{new})$;
    $x_{min} \leftarrow ChooseParent(X_{near}, x_{near}, x_{new})$;
    $\mathcal{T}.addNodEdge(x_{min}, x_{new})$;
    $\mathcal{T}.rewire()$;

Review of RRT*



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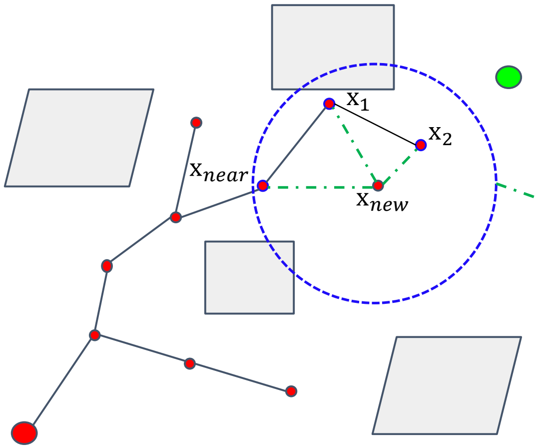
Review of RRT*



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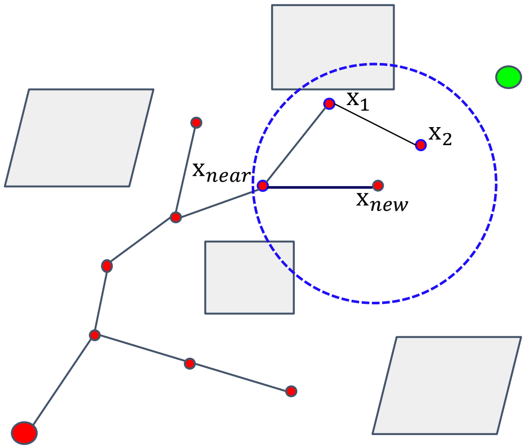
Review of RRT*



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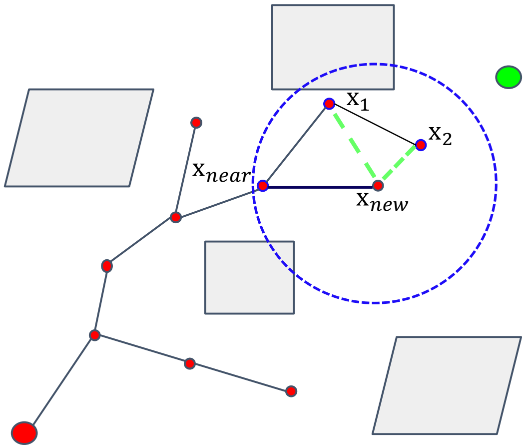
Review of RRT*



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Result: A path Γ from x_{init} to x_{goal}

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- $x_{new} \leftarrow Steer(x_{rand}, x_{near}, StepSize)$;
- **if** *CollisionFree*(x_{new}) **then**
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  - $x_{min} \leftarrow ChooseParent(X_{near}, x_{near}, x_{new})$;
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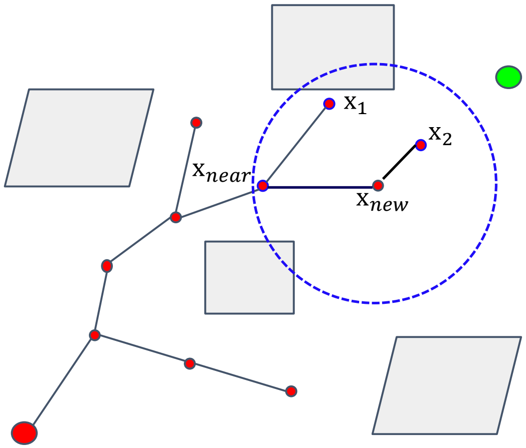
Review of RRT*



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    $x_{min} \leftarrow ChooseParent(X_{near}, x_{near}, x_{new})$;
    $\mathcal{T}.addNodEdge(x_{min}, x_{new})$;
    $\mathcal{T}.rewire()$;

Workflow

Similar to RRT* but different in details

Algorithm 2: RRT Algorithm

Input: $\mathcal{M}$,x_{init},x_{goal}
Result: A path Γ from x_{init} to x_{goal}
$\mathcal{T}$.init();
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    $x_{new} \leftarrow Steer(x_{rand}, x_{near}, StepSize)$;
    if *CollisionFree*(x_{new}) **then**
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        $x_{min} \leftarrow ChooseParent(X_{near}, x_{near}, x_{new})$;
        $\mathcal{T}.addNodEdge(x_{min}, x_{new})$;
        $\mathcal{T}.rewire()$;

Kinodynamic RRT*

Input: E, x_init, x_goal

Output: A trajectory T from x_init to x_goal

T.init();
for i = 1 to n **do**
    x_rand ← Sample(E);
    X_near ← Near(T, x_rand);
    x_min ← ChooseParent(X_near, x_rand);
    T.addNode(x_rand);
    T.rewire();

Problems when it comes to motion constraints

1. How to "Sample"

Kinodynamic RRT*
Input: E, x_init, x_goal **Output:** A trajectory T from x_init to x_goal T.init(); **for** i = 1 to n **do** x_rand ← Sample(E); X_near ← Near(T, x_rand); x_min ← ChooseParent(X_near, x_rand); T.addNode(x_rand); T.rewire();

LTI system state-space equation:

$$\dot{x}(t) = Ax(t) + Bu(t) + c$$

For example for double integrator systems,

$$x = \begin{bmatrix} p \\ v \end{bmatrix}, A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

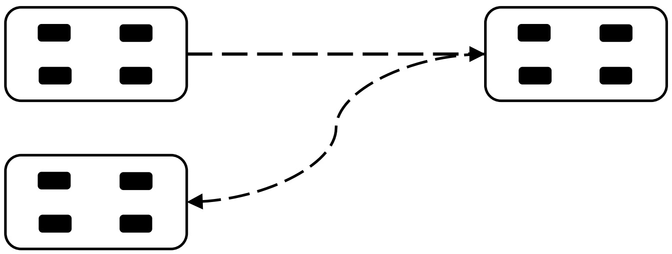
Instead of sampling in Euclidean space like RRT, it requires to sample in full state space.

Problems when it comes to motion constraints

2. How to define "Near"

Kinodynamic RRT*
Input: E, x_init, x_goal
Output: A trajectory T from x_init to x_goal

```
T.init();
for i = 1 to n do
    x_rand ← Sample(E);
    X_near ← Near(T, x_rand);
    x_min ← ChooseParent(X_near, x_rand);
    T.addNode(x_rand);
    T.rewire();
```



A car can not move sideways

If without motion constraints, Euclidean distance or Manhattan distance can be used.

In state space with motion constraints, bringing in optimal control.

Problems when it comes to motion constraints

2. How to define “Near”

If bring optimal control, we can define **cost functions** of transferring from states to states.

$$c[\pi] = \int_0^{\tau} \left(1 + u(t)^T R u(t)\right) dt$$

Typically, a quadratic form of time-energy optimal is adopted.

Two states are near if the cost of transferring from one state to the other is small.
(Note that the cost may be different if transfer reversely)

Problems when it comes to motion constraints

2. How to define "Near"

$$c[\pi] = \int_0^{\tau} \left(1 + u(t)^T R u(t)\right) dt$$

If we know the arriving time **τ** and the control policy **u(t)** of transferring, we can calculate the cost.

And thankfully, it's all in classic optimal control solutions.(OBVP)

Unbounded Optimal control solutions

2.1 Fixed final state x1, fixed final time τ

optimal control policy $u^*(t)$

$$u^*(t) = R^{-1}B^T exp[A^T(\tau - t)]G(\tau)^{-1}[x_1 - \bar{x}(\tau)].$$

Where $G(t)$ is the weighted controllability Gramian:

$$G(t) = \int_0^t exp[A(t - t')]BR^{-1}B^T exp[A^T(t - t')]dt'.$$

Which is the solution to the Lyapunov equation:

$$\dot{G}(t) = AG(t) + G(t)A^T + BR^{-1}B^T, G(0) = 0.$$

F. Lewis, V. Syrmos. Optimal Control. John Wiley & Sons

Unbounded Optimal control solutions

2.1 Fixed final state x1, fixed final time τ

$$u^*(t) = R^{-1}B^T exp[A^T(\tau - t)]G(\tau)^{-1}[x_1 - \bar{x}(\tau)].$$

And $\bar{x}(t)$ describe what the state x would be at time t if no control input were applied:

$$\bar{x}(t) = exp(At)x_0 + \int_0^t exp[A(t-t')]c dt'.$$

Which is the solution to the differential equation:

$$\dot{\bar{x}}(t) = A\bar{x}(t) + c, \bar{x}(0) = x_0$$

F. Lewis, V. Syrmos. Optimal Control. John Wiley & Sons

Unbounded Optimal control solutions

2.2 Fixed final state x1, free final time τ

If we want to find the optimal arrival time τ, we do this by filling in the control policy $u^*(t)$into the cost function $c[\pi]$ and evaluating the integral:

$$c[\tau] = \tau + [x_1 - \bar{x}(\tau)]^T G(t)^{-1}[x_1 - \bar{x}(\tau)].$$

The optimal τ is found by taking the derivative of $c[\tau]$ with respect to τ:

$$\dot{c}[\tau] = 1 - 2(Ax_1 + c)^T d(\tau) - d(\tau)^T BR^{-1}B^T d(\tau).$$

Where

$$d(\tau) = G(t)^{-1}[x_1 - \bar{x}(\tau)].$$

Unbounded Optimal control solutions

2.2 Fixed final state x1, free final time τ

Solve $\dot{c}[\tau] = 0$ for τ^*.
Noted that the function $\mathrm{c}[\tau]$ may have multiple local minima.
And for a double integrator system, it's a 4^{th} order polynomial.

Given the optimal arrival time τ^* as defined above, it again turns into a fixed final state, fixed final time problem.

Problems when it comes to motion constraints

3. How to "ChooseParent"

```
Kinodynamic RRT*
Input: E, x_init, x_goal
Output: A trajectory T from x_init to x_goal

T.init();
for i = 1 to n do
    x_rand ← Sample(E);
    X_near ← Near(T, x_rand);
    x_min ← ChooseParent(X_near, x_rand);
    T.addNode(x);
    T.rewire();
```

Now if we sample a random state, we can calculate control policy and cost from those state-nodes in the tree to the sampled state.

Choose one with the minimal cost and **check $x(t)$ and $u(t)$ are in bounds**.

If no qualified parent found, sample another state.

Problems when it comes to motion constraints

4. How to find near nodes efficiently

```
Kinodynamic RRT*
Input: E, x_init, x_goal
Output: A trajectory T from x_init to x_goal

T.init();
for i = 1 to n do
    x_rand ← Sample(E);
    X_near ← Near(T, x_rand);
    x_min ← ChooseParent(X_near, x_rand);
    T.addNode(x);
    T.rewire();
```

Every time we sample a random state x_rand, it requires to check every node in the tree to find its parent, that is solving a OBVP for each node, which is not efficient.

Problems when it comes to motion constraints

4. How to find near nodes efficiently

```
Kinodynamic RRT*
Input: E, x_init, x_goal
Output: A trajectory T from x_init to x_goal
T.init();
for i = 1 to n do
    x_rand ← Sample(E);
    X_near ← Near(T, x_rand);
    x_min ← ChooseParent(X_near, x_rand);
    T.addNode(x);
    T.rewire();
```

If we set a **cost tolerance** r, we can actually calculate bounds of the states (forward-reachable set) that can be reached by x_rand and bounds of the states (backward-reachable set) that can reach x_rand with cost less than r.

And if we store nodes in form of a kd-tree, we can then do range query in the tree.

Problems when it comes to motion constraints

4. How to find near nodes efficiently

$$c[\tau] = \tau + [x_1 - \bar{x}(\tau)]^T G(t)^{-1}[x_1 - \bar{x}(\tau)].$$

This formula describes how cost of transferring from state $\boldsymbol{x_0}$ to state $\boldsymbol{x_1}$ changes with arrival time $\boldsymbol{\tau}$.

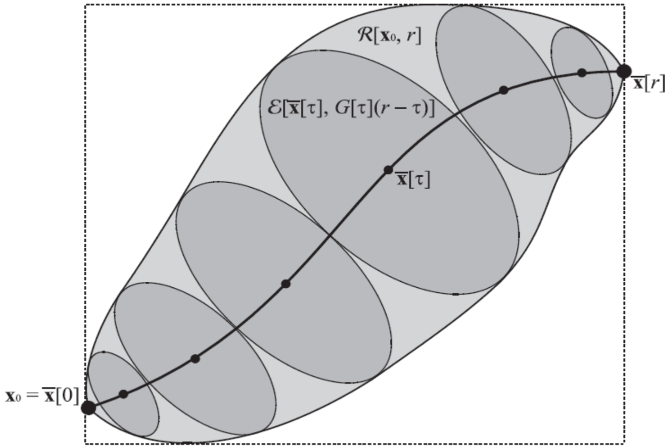
We can see that given initial state $\boldsymbol{x_0}$, cost tolerance $\boldsymbol{r}$ and arrival time $\boldsymbol{\tau}$, the forward-reachable set of $\boldsymbol{x_0}$ is:

$$\begin{aligned} &\{x_1 \mid \tau + [x_1 - \bar{x}(\tau)]^T G(t)^{-1}[x_1 - \bar{x}(\tau)] < r\} \\ &= \left\{x_1 \mid [x_1 - \bar{x}(\tau)]^T \frac{G(t)^{-1}}{r-\tau} [x_1 - \bar{x}(\tau)] < 1\right\}. \\ &= \mathcal{E}[\bar{x}(\tau), G(t)(r-\tau)]. \end{aligned}$$

Problems when it comes to motion constraints

4. How to find near nodes efficiently

$$\{x_1 \,|\tau + [x_1 - \bar{x}(\tau)]^T G(t)^{-1}[x_1 - \bar{x}(\tau)] < r\}$$
$$= \left\{x_1 \,|\, [x_1 - \bar{x}(\tau)]^T \frac{G(t)^{-1}}{r-\tau}[x_1 - \bar{x}(\tau)] < 1\right\}.$$
$$= \mathcal{E}[\bar{x}(\tau), G(t)(r-\tau)].$$



where $\mathcal{E}[x, M]$ is an **ellipsoid** with center x and positive definite weight matrix M, formally defined as:

$$\mathcal{E}[x, M] = \{x' \mid (x' - x)^T M^{-1}(x' - x) < 1\}.$$

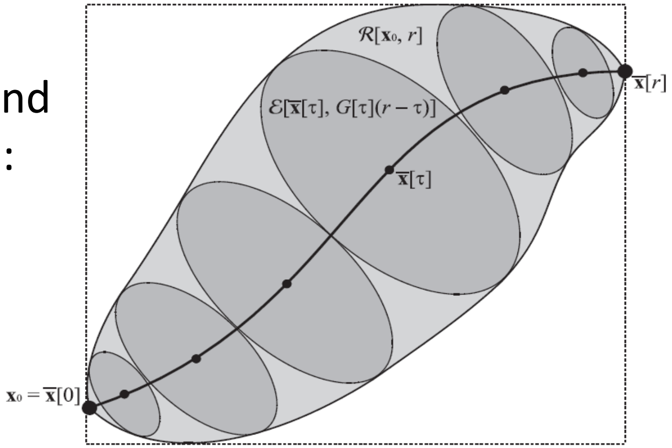
Hence, the forward-reachable set is the union of high dimensional ellipsoids for all possible arrival times $\boldsymbol{\tau}$.

Problems when it comes to motion constraints

4. How to find near nodes efficiently

For simplification, we sample several τs and calculate axis-aligned bounding box of the ellipsoids for each τ and update the maximum and minimum in each dimension:

$$\prod_{k=1}^{n} \begin{bmatrix} min\{0<\tau<r\}\left(\bar{x}(\tau)_k - \sqrt{G[\tau]_{(k,k)}(r-\tau)}\right), \\ max\{0<\tau<r\}\left(\bar{x}(\tau)_k + \sqrt{G[\tau]_{(k,k)}(r-\tau)}\right) \end{bmatrix}.$$



Similar for the calculation of the backward-reachable set.

Problems when it comes to motion constraints

4. How to find near nodes efficiently

```
Kinodynamic RRT*
Input: E, x_init, x_goal
Output: A trajectory T from x_init to x_goal
T.init();
for i = 1 to n do
    x_rand ← Sample(E);
    X_near ← Near(T, x_rand);
    x_min ← ChooseParent(X_near, x_rand);
    T.addNode(x);
    T.rewire();
```

When do "Near" query and "ChooseParent", X_near can be found from the **backward-reachable set** of x_rand.

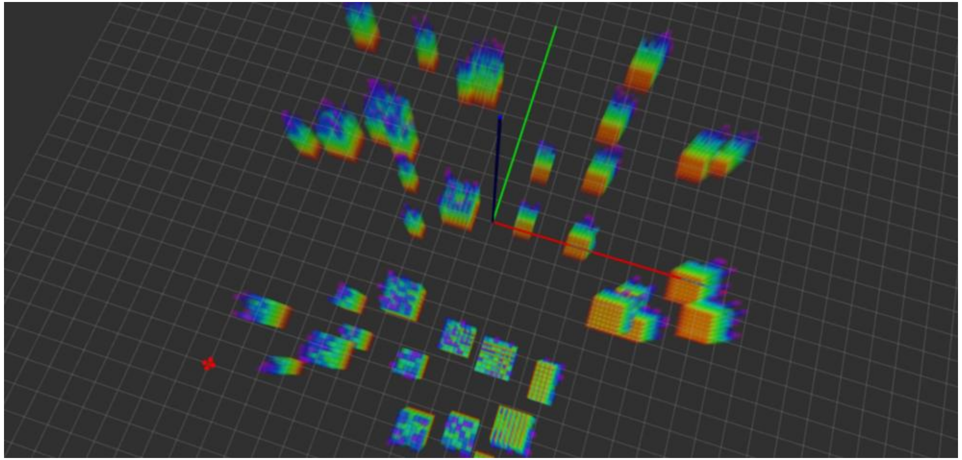
Problems when it comes to motion constraints

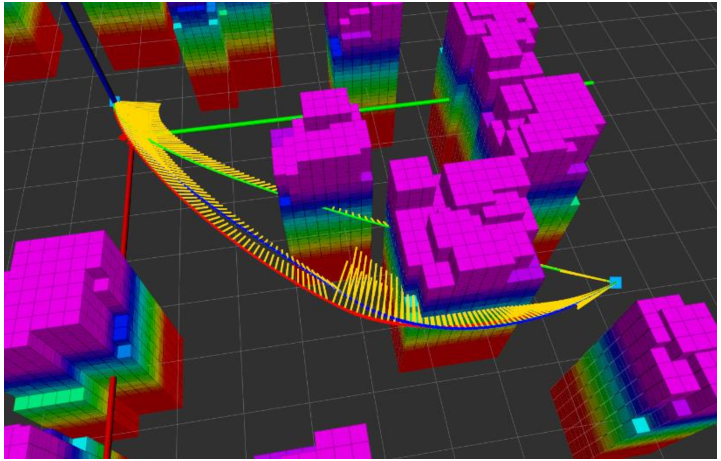
5. How to "Rewire"

```
Kinodynamic RRT*
Input: E, x_init, x_goal
Output: A trajectory T from x_init to x_goal
T.init();
for i = 1 to n do
    x_rand ← Sample(E);
    X_near ← Near(T, x_rand);
    x_min ← ChooseParent(X_near, x_rand);
    T.addNode(x);
    T.rewire();
```

When "Rewire", we calculate the **forward-reachable set** of x_rand, and solve OBVPs.

Demos





The green curve takes no account of the obstacles;
The red curve is the result of the kinodynamic trajectory planner;
The blue curve is the first feasible trajectory found by the kinodynamic trajectory planner;
The yellow lines are the control inputs in every control points.

Homework

Local lattice planner

Homework 1

- For the OBVP problem stated in slides p.25-p.29, please get the optimal solution (control, state, and time) for partially free final state case.

- Suppose the position is fixed, velocity and acceleration are free here.

Local lattice planner

Homework 2

- Build an ego-graph of the linear modeled robot.
- Select the best trajectory closest to the planning target.

Thanks for Listening!