Discrete 2D FNO

1. Function Spaces in 2D
   $\overline{C}[0,1]=C[0,1]_{(0,1)\times(0,1)}$
   $C_p(\mathbb{R})=\mathbb{R}$ + functions from $\overline{C}[0,1]$
   These f functions in $C_p(\mathbb{R})$ are continuous except the possible discontinuity points $\{k:\in\mathbb{Z}^2\}$, in these points we define $f\in C_p(\mathbb{R})$ as
   $f(k_1,k_2)=\dfrac{f(0,0)+f(0,1)+f(1,0)+f(1,1)}{2},\ \ k=(k_1,k_2)^T\in Z^2$
   Let $C_{p,K}$ be the component-wise periodic extension to $\mathbb{R}$ of $f=(f_1,\dots,f_K)$, where $f_j\in\overline{C}[0,1],\ \ j=1,\dots,K$
   Let the numbers $I,O\in N$ and the functions $r\in C_{p,O\times I},\ u\in C_{p,I}$ be given
2. Discrete FNO
   1. The Convolution ($r\in C_{p,O\times I},\ \ u\in C_{p,I},\ \ v=r*u\in C_{p,O},\forall x_1,x_2\in\mathbb{R},$)
      $v(x_1,x_2)=(r*u)(x_1,x_2)=\int\int_R r(x_1-y_1,x_2-y_2)*u(y_1,y_2)\,dy_1dy_2\ \ x_1,x_2\in\mathbb{R}$
   2. The Convolution Component-wise
      $v_l(x_1,x_2)=(r*u)_l(x_1,x_2)=\int\int_R (r(x_1-y_1,x_2-y_2)u(y_1,y_2))_l\,dy_1dy_2=$
      $=\int\int_R \sum_{j=1}^{I} r_{l,j}(x_1-y_1,x_2-y_2)\cdot u_j(y_1,y_2)\,dy_1dy_2=$
      $=\sum_{j=1}^{I}(r_{l,j}*u_j)(x_1,x_2)\ \ x_1,x_2\in\mathbb{R}$
   3. 2 with the Fourier Transform
      $v_l(x_1,x_2)=\mathcal{F}^{-1}(\mathcal{F}(r)(y_1,y_2)*\mathcal{F}(u)(y_1,y_2)_l)(x_1,x_2)=$
      $=\mathcal{F}^{-1}(\sum_{j=1}^{I}\mathcal{F}(r_{l,j})(y_1,y_2)*\mathcal{F}(u_j)(y_1,y_2))(x_1,x_2)=$
      $=(F_r(u))_l(x_1,x_2)\ \ x_1,x_2\in\mathbb{R}$
   4. 3 in Vector Form
      $v(x_1,x_2)=\mathcal{F}^{-1}(\mathcal{F}(r)*\mathcal{F}(u))=F_r(u)(x_1,x_2)\ \ x_1,x_2\in\mathbb{R}$
3. FNO for continuous functions
   Let $u(x_1,x_2)=(u_1(x_1,x_2),\dots,u_I(x_1,x_2))^T=(u_j(x_1,x_2))^T\in\mathbb{R}^I\ \ x_1,x_2\in R,j=1\dots I$
   given by its Fourier series component-wisely, for example:
   $u_j(x_1,x_2)=\sum_{k_1,k_2\in Z}(U_{k_1,k_2,j}\cdot e^{i\cdot2\cdot k_1\cdot\pi\cdot x_1}\cdot e^{i\cdot2\cdot k_2\cdot\pi\cdot x_2})$
   Similarly let $r(x_1,x_2)=(r_{l,j}(x_1,x_2))\in\mathbb{R}^{O\times I}$, where
   $r_{l,j}(x_1,x_2)=\sum_{k_1,k_2\in Z}(R_{k_1,k_2,l,j}\cdot e^{i\cdot2\cdot k_1\cdot\pi\cdot x_1}\cdot e^{i\cdot2\cdot k_2\cdot\pi\cdot x_2}),\ \ x_1,x_2\in\mathbb{R}$
   Now the convolution r and u, denoted as $v=(v_1,\dots,v_O)^T$ is defined by the following formula
   $v_l(x_1,x_2)=\sum_{j=1}^{I}(r_{l,j}*u_j)(x_1,x_2)=\sum_{j=1}^{I}(\sum_{k_1,k_2\in Z}(R_{k_1,k_2,l,j}\cdot e^{i\cdot2\cdot k_1\cdot\pi\cdot x_1}\cdot e^{i\cdot2\cdot k_2\cdot\pi\cdot x_2}))*$
   $*(\sum_{k_1',k_2'\in Z}(U_{k_1',k_2',j}\cdot e^{i\cdot2\cdot k_1'\cdot\pi\cdot x_1}\cdot e^{i\cdot2\cdot k_2'\cdot\pi\cdot x_2}))=\sum_{j=1}^{I}\sum_{k_1,k_2\in Z}R_{k_1,k_2,l,j}\cdot U_{k_1,k_2,j}\cdot e^{i\cdot2\cdot k_1\cdot\pi\cdot x_1}\cdot e^{i\cdot2\cdot k_2\cdot\pi\cdot x_2}=$
   $=\sum_{k_1,k_2\in Z}\sum_{j=1}^{I}R_{k_1,k_2,l,j}\cdot U_{k_1,k_2,j}\cdot e^{i\cdot2\cdot k_1\cdot\pi\cdot x_1}\cdot e^{i\cdot2\cdot k_2\cdot\pi\cdot x_2}=\bar{F}_r(u)$
   since $(e^{i\cdot2\cdot k_1\cdot\pi\cdot x_1}\cdot e^{i\cdot2\cdot k_2\cdot\pi\cdot x_2})*(e^{i\cdot2\cdot k_1'\cdot\pi\cdot x_1}\cdot e^{i\cdot2\cdot k_2'\cdot\pi\cdot x_2})=\begin{cases} e^{i\cdot2\cdot k_1\cdot\pi\cdot x_1}\cdot e^{i\cdot2\cdot k_2\cdot\pi\cdot x_2}, if\ k_1=k_1'\wedge k_2=k_2' \\ 0, otherwise\end{cases}$