Lecture 2: List algorithms using recursion and list comprehensions

Søren Haagerup

Department of Mathematics and Computer Science
University of Southern Denmark, Odense

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Expressions, patterns and types

Primitive types: $Int, Integer, Double, Float, Char$

Usage of primitive types:

$$\begin{aligned} c &:: Int \\ c &= 42 \end{aligned}$$

$$\begin{aligned} f &:: Int \to Int \\ f\ x &= 1337 - x \end{aligned}$$

Expressions, patterns and types

Primitive types: $Int, Integer, Double, Float, Char$

Pattern matching on primitive types:

$$
\begin{aligned}
&sumTo \quad :: Int \rightarrow Int \\
&sumTo\ 0 = 0 \\
&sumTo\ n = n + sumTo\ (n - 1)
\end{aligned}
$$

Expressions, patterns and types

Non-primitive types: Monomorphic: (without type variables)

$$\textbf{data}\ \mathit{Bool} = \mathit{False} \mid \mathit{True}$$

Pattern matching on non-primitive types:

$$\begin{aligned}
&\mathit{invert} && :: \mathit{Bool} \rightarrow \mathit{Bool} \\
&\mathit{invert\ False} && = \mathit{True} \\
&\mathit{invert\ True} && = \mathit{False}
\end{aligned}$$

$$\begin{aligned}
&\mathit{invert} && :: \mathit{Bool} \rightarrow \mathit{Bool} \\
&\mathit{invert\ False} && = \mathit{True} \\
&\mathit{invert}\ _ && = \mathit{False}
\end{aligned}$$

Wildcards are written as _

Expressions, patterns and types

Non-primitive types: Polymorphic: (with type variables)

$$\begin{array}{lll}
\textbf{data}\ \mathit{Maybe}\ a & = \mathit{Nothing} & \mid \mathit{Just}\ a \\
\textbf{data}\ \mathit{List}\ a & = \mathit{Nil} & \mid \mathit{Cons}\ a\ (\mathit{List}\ a) \\
\textbf{data}\ [a] & = [\,] & \mid (a : [a]) \\
\textbf{data}\ (a, b) & = (a, b) &
\end{array}$$

Example 1/2

$$\begin{array}{l}
\mathit{maybeAdd} :: \mathit{Maybe\ Int} \rightarrow \mathit{Maybe\ Int} \rightarrow \mathit{Maybe\ Int} \\
\mathit{maybeAdd}\ (\mathit{Just}\ x)\ (\mathit{Just}\ y) = \mathit{Just}\ (x + y) \\
\mathit{maybeAdd}\ _\ _ \qquad\qquad\qquad\quad = \mathit{Nothing}
\end{array}$$

Example 2/2

$$\begin{array}{l}
f :: (\mathit{Int}, \mathit{Int}) \rightarrow \mathit{Int} \\
f\ (0, y) = y \\
f\ (x, y) = (x - 1, x * y)
\end{array}$$

Expressions, patterns and types

	Type	Value constructors	Pattern / expression
Tuple	(a, b)	$(,)::a \to b \to (a, b)$	(x, y)
List	$[a]$	$[\,] :: [a]$	$[\,]$
		$(:) :: a \to [a] \to [a]$	$(x : xs)$
Bool	*Bool*	$True :: Bool$	*True*
		$False :: Bool$	*False*
Maybe	*Maybe a*	$Nothing :: Maybe\ a$	*Nothing*
		$Just :: a \to Maybe\ a$	$(Just\ x)$

- Capitalized words: Specific type
- Lowercase words: Type variable

When “specializing” a type, all occurrences of a type variable in the type expression must be replaced with the same type.

Brush up: Types

- **Monomorphic types:**
  - Int, $Integer$, $Bool$, $Char$, $Float$, $Double$, $String$
- **Polymorphic types:**
  - $[a]$, $Maybe\ a$, (a, b)
  - lowercase letters are *type variables* which can be replaced by any other type to construct a new type
  - $[[a]]$, $[[[a]]]$, $[Maybe\ a]$, $Maybe\ (a, b)$, $Maybe\ Int$ etc. are valid types

Brush up: Function types

An n-argument function is a one-argument function which returns a $(n-1)$-argument function.

$$add :: Int \to (Int \to Int)$$
$$add\ x\ y = x + y$$

Evaluate by calling add 40 2.

The same function in JavaScript would look like this:

```
function add(x) {
    return function(y) {
        return x+y;
    }
}
```

Evaluate by calling `add(40)(2)`.

Brush up: Function types

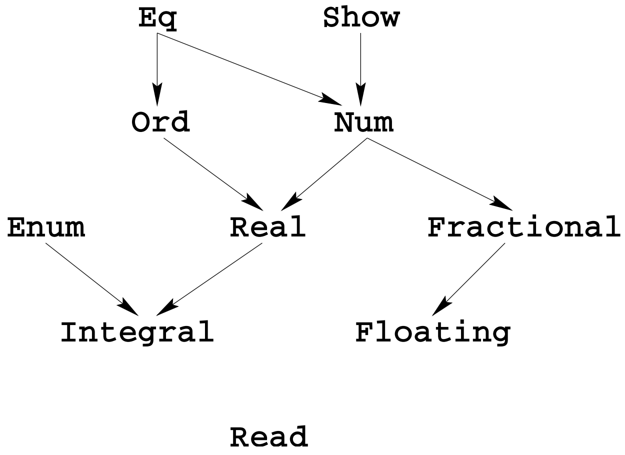
- **Monomorphic functions:** $words :: String \to [String]$
- **Polymorphic functions:**
  - $length :: [a] \to Int$
  - $(:) :: a \to [a] \to [a]$
    - This type signature ensures that lists can only be constructed with elements of the same type.
  - Can we make the following functions?
    - $sum :: [a] \to a$
    - $sort :: [a] \to [a]$

Type classes - Constraining the type of a function

- *Eq a* - all types *a* for which $(\equiv)$ is defined
- *Ord a* - all types *a* for which $(\leqslant)$ is defined
- *Num a* - all types *a* for which
  $(+), (*), abs, signum, fromInteger, negate$ are defined

$$
\begin{aligned}
&sum, product &&:: Num\ a \Rightarrow [a] \rightarrow a\\
&sum\ [\,] &&= 0\\
&sum\ (x : xs) &&= x + sum\ xs\\
&product\ [\,] &&= 1\\
&product\ (x : xs) &&= x * product\ xs
\end{aligned}
$$

Type classes - Constraining the type of a function



RECURSIVE LIST FUNCTIONS

Prelude: take and *drop*

$$
\begin{aligned}
&take :: Int \to [a] \to [a] \\
&take\ n\ \_ \mid n \leqslant 0 = [\,] \\
&take\ \_\ [\,] \qquad\quad\ = [\,] \\
&take\ n\ (x : xs) \quad = x : take\ (n-1)\ xs
\end{aligned}
$$

$$
\begin{aligned}
take\ 3\ [1,2,3,4,5] &\equiv 1 : take\ 2\ [2,3,4,5] \\
&\equiv 1 : (2 : take\ 1\ [3,4,5]) \\
&\equiv 1 : (2 : (3 : take\ 0\ [4,5])) \\
&\equiv 1 : (2 : (3 : [\,])) \\
&\equiv [1,2,3]
\end{aligned}
$$

If the input list has length m, how many reductions are made?

Prelude: *take* and *drop*

$$
\begin{aligned}
&drop :: Int \to [a] \to [a] \\
&drop\ n\ xs \mid n \leqslant 0 = xs \\
&drop\ \_\ [\,] \qquad\qquad = [\,] \\
&drop\ n\ (\_ : xs) \quad\ = drop\ (n-1)\ xs
\end{aligned}
$$

$$
\begin{aligned}
drop\ 3\ [1,2,3,4,5] &\equiv drop\ 2\ [2,3,4,5] \\
&\equiv drop\ 1\ [3,4,5] \\
&\equiv drop\ 0\ [4,5] \\
&\equiv [4,5]
\end{aligned}
$$

If the input list has length m, how many reductions are made?

Prelude: *takeWhile* and *dropWhile*

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathit{takeWhile} :: (a \rightarrow \mathit{Bool}) \rightarrow [a] \rightarrow [a]} \\
\multicolumn{2}{l}{\mathit{takeWhile}\ \_\ [\,] = [\,]} \\
\multicolumn{2}{l}{\mathit{takeWhile}\ p\ (x : \mathit{xs})} \\
\mid p\ x & = x : \mathit{takeWhile}\ p\ \mathit{xs} \\
\mid \mathit{otherwise} & = [\,]
\end{array}
$$

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathit{dropWhile} :: (a \rightarrow \mathit{Bool}) \rightarrow [a] \rightarrow [a]} \\
\multicolumn{2}{l}{\mathit{dropWhile}\ \_\ [\,] = [\,]} \\
\multicolumn{2}{l}{\mathit{dropWhile}\ p\ (x : \mathit{xs})} \\
\mid p\ x & = \mathit{dropWhile}\ p\ \mathit{xs} \\
\mid \mathit{otherwise} & = \mathit{xs}
\end{array}
$$

How many reductions are made?

Prelude: $(+\!\!+)$, *concat* and *reverse*

- $(+\!\!+) :: [a] \to [a] \to [a]$

  $[\,] +\!\!+ ys \quad\quad = ys$

  $(x : xs) +\!\!+ ys = x : (xs +\!\!+ ys)$

- $concat :: [[a]] \to [a]$

  $concat\ [\,] \quad\quad\quad = [\,]$

  $concat\ (xs : xss) = xs +\!\!+ concat\ xss$

- $reverse :: [a] \to [a]$

  $reverse\ [\,] \quad\quad = [\,]$

  $reverse\ (x : xs) = reverse\ xs +\!\!+ [x]$

$(+\!\!+)$ - running the algorithm

$$
\begin{aligned}
&(+\!\!+) :: [a] \to [a] \\
&[\,] +\!\!+ ys \hphantom{xxxx} = ys \\
&(x : xs) +\!\!+ ys = x : (xs +\!\!+ ys)
\end{aligned}
$$

$$
\begin{aligned}
[1,2,3] +\!\!+ ys &\equiv 1 : ([2,3] +\!\!+ ys) \\
&\equiv 1 : (2 : ([3] +\!\!+ ys)) \\
&\equiv 1 : (2 : (3 : ([\,] +\!\!+ ys))) \\
&\equiv 1 : (2 : (3 : ys))
\end{aligned}
$$

How many reductions?

reverse - running the algorithm

$$
\begin{aligned}
&\mathit{reverse} :: [a] \to [a]\\
&\mathit{reverse}\ [\,] \qquad\quad = [\,]\\
&\mathit{reverse}\ (x : xs) = \mathit{reverse}\ xs +\!\!+ [x]
\end{aligned}
$$

$$
\begin{aligned}
\mathit{reverse}\ [1,2,3] &\equiv \mathit{reverse}\ [2,3] +\!\!+ [1]\\
&\equiv (\mathit{reverse}\ [3] +\!\!+ [2]) +\!\!+ [1]\\
&\equiv ((\mathit{reverse}\ [\,] +\!\!+ [3]) +\!\!+ [2]) +\!\!+ [1]\\
&\equiv (([\,] +\!\!+ [3]) +\!\!+ [2]) +\!\!+ [1]\\
&\equiv \ \ldots\\
&\equiv [3,2,1]
\end{aligned}
$$

How many reductions?

Example: *trim*

$$
\begin{aligned}
\mathit{ltrim}\ \mathit{xs} &= \mathit{dropWhile}\ (\equiv \texttt{' '})\ \mathit{xs} \\
\mathit{rtrim}\ \mathit{xs} &= \mathit{reverse}\ (\mathit{ltrim}\ (\mathit{reverse}\ \mathit{xs})) \\
\mathit{trim}\ \mathit{xs} &= \mathit{rtrim}\ (\mathit{ltrim}\ \mathit{xs})
\end{aligned}
$$

Example: *trim*

$$
\begin{aligned}
\mathit{ltrim}\ \mathit{xs} &= \mathit{dropWhile}\ (\equiv \text{'\ \ '})\ \mathit{xs} \\
\mathit{rtrim}\ \mathit{xs} &= \mathit{reverse}\ \$\ \mathit{ltrim}\ \$\ \mathit{reverse}\ \mathit{xs} \\
\mathit{trim}\ \mathit{xs} &= \mathit{rtrim}\ \$\ \mathit{ltrim}\ \mathit{xs}
\end{aligned}
$$

Application operator. This operator is redundant, since ordinary application $(f\ x)$ means the same as $(f\ \$\ x)$. However, $ has low, right-associative binding precedence, so it sometimes allows parentheses to be omitted

Example: *trim*

$$
\begin{aligned}
ltrim &= dropWhile\ (\equiv \text{' '}) \\
rtrim &= reverse \circ ltrim \circ reverse \\
trim &= rtrim \circ ltrim
\end{aligned}
$$

Point-free style. Sometimes it makes the code mode readable. Sometimes it doesn't (this is the reason, that some people call it *pointless style*).

Example: *left*, *right*, *mid* (inspired by VBScript)

$$
\begin{aligned}
\mathit{left}\ n &= \mathit{take}\ n \\
\mathit{right}\ n &= \mathit{reverse} \circ \mathit{take}\ n \circ \mathit{reverse} \\
\mathit{mid}\ s\ n &= \mathit{take}\ n \circ \mathit{drop}\ s
\end{aligned}
$$

Examples:

$$
\begin{aligned}
\mathit{left}\ 3\ \texttt{"abcde"} &= \texttt{"abc"} \\
\mathit{right}\ 3\ \texttt{"abcde"} &= \texttt{"cde"} \\
\mathit{mid}\ 2\ 2\ \texttt{"abcde"} &= \texttt{"cd"}
\end{aligned}
$$

Example: *substr* (inspired by PHP)

Description

```
string substr ( string $string , int $start [, int $length ] )
```

Returns the portion of **`string`** specified by the **`start`** and **`length`** parameters.

$$
\begin{aligned}
&\mathit{substr} :: [a] \rightarrow \mathit{Int} \rightarrow \mathit{Maybe\ Int} \rightarrow [a] \\
&\mathit{substr}\ \mathit{xs}\ s\ \mathit{Nothing} = \mathit{drop}\ s\ \mathit{xs} \\
&\mathit{substr}\ \mathit{xs}\ s\ (\mathit{Just}\ l) \quad = \mathit{take}\ l\ (\mathit{substr}\ \mathit{xs}\ s\ \mathit{Nothing})
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{substr}\ \texttt{"abracadabra"}\ 5\ \mathit{Nothing} = \texttt{"adabra"} \\
&\mathit{substr}\ \texttt{"abracadabra"}\ 5\ (\mathit{Just}\ 4) = \texttt{"adab"}
\end{aligned}
$$

Example: *substr* (inspired by PHP)

But *substr* should work with negative offsets/lengths as well.

$$
\begin{aligned}
&\mathit{substr}\ \texttt{"abcdef"}\ (-1)\ \mathit{Nothing} &&= \texttt{"f"}\\
&\mathit{substr}\ \texttt{"abcdef"}\ (-2)\ \mathit{Nothing} &&= \texttt{"ef"}\\
&\mathit{substr}\ \texttt{"abcdef"}\ (-3)\ (\mathit{Just}\ 1) &&= \texttt{"d"}\\
&\mathit{substr}\ \texttt{"abcdef"}\ 0\ (\mathit{Just}\ (-1)) &&= \texttt{"abcde"}\\
&\mathit{substr}\ \texttt{"abcdef"}\ 2\ (\mathit{Just}\ (-1)) &&= \texttt{"cde"}\\
&\mathit{substr}\ \texttt{"abcdef"}\ 4\ (\mathit{Just}\ (-4)) &&= \texttt{""}\\
&\mathit{substr}\ \texttt{"abcdef"}\ (-3)\ (\mathit{Just}\ (-1)) &&= \texttt{"de"}
\end{aligned}
$$

Example: *substr* (inspired by PHP)

But *substr* should work with negative offsets/lengths as well.

$$
\begin{aligned}
&\mathit{substr} :: [a] \to \mathit{Int} \to \mathit{Maybe}\ \mathit{Int} \to [a] \\
&\mathit{substr}\ \mathit{xs}\ s\ \mathit{Nothing} = \mathit{drop}\ (\mathit{nonneg}\ \mathit{xs}\ s)\ \mathit{xs} \\
&\mathit{substr}\ \mathit{xs}\ s\ (\mathit{Just}\ l) \quad = \mathit{take}\ (\mathit{nonneg}\ \mathit{xs}'\ l)\ \mathit{xs}' \\
&\quad \textbf{where}\ \mathit{xs}' = \mathit{substr}\ \mathit{xs}\ s\ \mathit{Nothing}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{nonneg} :: [a] \to \mathit{Int} \to \mathit{Int} \\
&\mathit{nonneg}\ \mathit{xs}\ n \\
&\quad \mid n < 0 \qquad\quad = \mathit{max}\ 0\ (\mathit{length}\ \mathit{xs} + n) \\
&\quad \mid \mathit{otherwise} = n
\end{aligned}
$$

Prelude: zip

$$
\begin{aligned}
&zip :: [a] \to [b] \to [(a,b)] \\
&zip\ [\,]\ \_ && = [\,] \\
&zip\ \_\ [\,] && = [\,] \\
&zip\ (x:xs)\ (y:ys) && = (x,y) : zip\ xs\ ys
\end{aligned}
$$

$> zip\ [1\,..\,5]$ `"abcd"`

$[(1,{}'\texttt{a}'),(2,{}'\texttt{b}'),(3,{}'\texttt{c}'),(4,{}'\texttt{d}')]$

Prelude: zipWith

$$
\begin{aligned}
&\mathit{zipWith} :: (a \to b \to c) \to [a] \to [b] \to [c] \\
&\mathit{zipWith}\ \_\ [\,]\ \_ && = [\,] \\
&\mathit{zipWith}\ \_\ \_\ [\,] && = [\,] \\
&\mathit{zipWith}\ f\ (x : xs)\ (y : ys) && = f\ x\ y : \mathit{zipWith}\ f\ xs\ ys
\end{aligned}
$$

$$\mathit{zip} = \mathit{zipWith}\ (,)$$

$$> \mathit{zipWith}\ (+)\ [1\,..\,5]\ [5,4\,..\,1]$$
$$[6,6,6,6,6]$$

Insertion sort

$$
\begin{aligned}
&\mathit{insert} :: \mathit{Ord}\ a \Rightarrow a \to [a] \to [a]\\
&\mathit{insert}\ x\ [\,] && = [x]\\
&\mathit{insert}\ x\ (y : ys) \mid x \leqslant y && = x : y : ys\\
&\phantom{\mathit{insert}\ x\ (y : ys)} \mid \mathit{otherwise} && = y : \mathit{insert}\ x\ ys
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{isort} :: \mathit{Ord}\ a \Rightarrow [a] \to [a]\\
&\mathit{isort}\ [\,] && = [\,]\\
&\mathit{isort}\ (x : xs) && = \mathit{insert}\ x\ (\mathit{isort}\ xs)
\end{aligned}
$$

Merge sort

$$
\begin{array}{l}
merge :: Ord\ a \Rightarrow [a] \rightarrow [a] \rightarrow [a] \\
merge\ xs\ [\,] \qquad\qquad\qquad\qquad = xs \\
merge\ [\,]\ ys \qquad\qquad\qquad\qquad = ys \\
merge\ (x : xs)\ (y : ys) \mid x \leqslant y \quad\ \ = x : merge\ xs\ (y : ys) \\
\qquad\qquad\qquad\qquad\quad \mid otherwise = y : merge\ (x : xs)\ ys
\end{array}
$$

$$
\begin{array}{l}
msort :: Ord\ a \Rightarrow [a] \rightarrow [a] \\
msort\ [\,] = [\,] \\
msort\ [x] = [x] \\
msort\ xs = merge\ (msort\ ys)\ (msort\ zs) \\
\quad \mathbf{where} \\
\qquad (ys, zs) = splitAt\ (length\ xs\ \text{`}div\text{`}\ 2)\ xs
\end{array}
$$

Lab this Friday: Polynomials

A polynomial $p : \mathbb{R} \rightarrow \mathbb{R}$ with degree n is a function

$$p(x) = a_0 x^0 + a_1 x^1 + \ldots + a_n x^n$$

where $a_0 \ldots a_n$ are constants in $\mathbb{R}$, $a_n \neq 0$.
In Haskell we define a type synonym

$$\textbf{type}\ \mathit{Poly}\ a = [a]$$

and let a polynomial be defined by the list of its coefficients

$$p :: \mathit{Num}\ a \Rightarrow \mathit{Poly}\ a$$
$$p = [a0, a1 \ldots an]$$

Lab this Friday: Polynomials

Examples:

- $5 + 2x + 3x^2$ is represented by $[5, 2, 3]$
- $-2 + x^2$ is represented by $[-2, 0, 1]$
- 0 is represented by $[]$

Lab this Friday: Polynomials

Think about this in the break:

1. We discover that $-2 + x^2$ can be represented by infinitely many lists:
   $[-2, 0, 1], [-2, 0, 1, 0], [-2, 0, 1, 0, 0], [-2, 0, 1, 0, 0, 0] \ldots.$

   Inspired by *trim*, write a function *canonical* that converts a polynomial to its smallest representation.
2. We want to define addition of polynomials, such that

   $$(5 + 2x + 3x^2) + (-2 + x) = 3 + 3x + 3x^2$$

   i.e.

   $$\mathit{add}\ [5, 2, 3]\ [-2, 1] = [5 + (-2), 2 + 1, 3] = [3, 3, 3]$$

   Modify *zip* to implement *add*.

LIST COMPREHENSIONS

Introduction

In mathematics, the set of square numbers up to 5^2 is

$$\{x^2 \mid x \in \{1, \ldots, 5\}\}$$

In Haskell, the list of square numbers up to 5^2 can be written

$$[x * x \mid x \leftarrow [1 \mathinner{..} 5]]$$

We say

- $\mid$ "such that"
- $\leftarrow$ "is drawn from"
- $x \leftarrow xs$ is a "generator"

Cartesian product

$$cartesian\ xs\ ys = [(x,y) \mid x \leftarrow xs, y \leftarrow ys]$$

$> cartesian\ [1..3]$ `"abc"`

$[(1,\texttt{'a'}),(1,\texttt{'b'}),(1,\texttt{'c'}),$
$(2,\texttt{'a'}),(2,\texttt{'b'}),(2,\texttt{'c'}),$
$(3,\texttt{'a'}),(3,\texttt{'b'}),(3,\texttt{'c'})]$

Ordering matters!

$$cartesian'\ xs\ ys = [(x,y) \mid y \leftarrow ys, x \leftarrow xs]$$

$> cartesian'\ [1..3]$ `"abc"`

$[(1,\texttt{'a'}),(2,\texttt{'a'}),(3,\texttt{'a'}),$
$(1,\texttt{'b'}),(2,\texttt{'b'}),(3,\texttt{'b'}),$
$(1,\texttt{'c'}),(2,\texttt{'c'}),(3,\texttt{'c'})]$

Finding index of elements in a list

$$elemIndices :: Eq\ a \Rightarrow a \rightarrow [a] \rightarrow [Int]$$
$$elemIndices\ xs\ y = [i \mid (i, x) \leftarrow zip\ [0\, ..\, length\ xs]\ xs, x \equiv y]$$

The boolean expression $x \equiv y$ is called a **guard**.

$$> elemIndices\ [3, 4, 2, 1, 4, 5]\ 4$$
$$[2, 5]$$

$$
\begin{aligned}
\mathit{pythags}\ n = \ & [\,(x, y, z) \\
& \mid z \leftarrow [1\,..\,n], \\
& \ x \leftarrow [1\,..\,z], \\
& \ y \leftarrow [x\,..\,z], \\
& \ x * x + y * y \equiv z * z]
\end{aligned}
$$

$> \mathit{pythags}\ 15$

$[(3,4,5),(6,8,10),(5,12,13),(9,12,15)]$

```
          start
          |______________________...
          |  |       |
     z    1  2       3
          |  |___    |_________
          |  |   |   |      |   |
     x    1  1   2   1      2   3
          |  |__ |   |___   |_  |
          |  | | |   | | |  | | |
     y    1  1 2 2   1 2 3  2 3 3
```

Prelude functions
- here implemented using list comprehensions

- $zipWith\ f\ xs\ ys = [f\ a\ b \mid (a, b) \leftarrow zip\ xs\ ys]$
  Example: $zipWith\ (+)\ [2, 1, 3]\ [3, 1, 2] = [5, 2, 5]$
- $concat\ xss = [x \mid xs \leftarrow xss, x \leftarrow xs]$
  Example: $concat\ [[1], [1, 2], [1, 2, 3]] = [1, 1, 2, 1, 2, 3]$
- $map\ f\ xs = [f\ x \mid x \leftarrow xs]$
  Example: $map\ (*3)\ [1, 2, 3, 4] = [3, 6, 9, 12]$
- $filter\ p\ xs = [x \mid x \leftarrow xs, p\ x]$
  Example: $filter\ even\ [6, 2, 7, 5, 2] = [6, 2, 2]$

Checking if a list is sorted

$$sorted\ xs = and\ [x \leqslant y \mid (x, y) \leftarrow zip\ xs\ (tail\ xs)]$$

$$\begin{aligned} sorted\ [2, 3, 1] &\equiv and\ [True, False] \\ &\equiv False \end{aligned}$$

Pascal's triangle

$$\begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$\begin{aligned}(a+b)^3 &= \binom{3}{0} b^3 + \binom{3}{1} ab^2 + \binom{3}{2} + a^2 b + \binom{3}{3} a^3 \\ &= b^3 + 3ab^2 + 3ba^2 + a^3\end{aligned}$$

Pascal's triangle

$$\begin{array}{c}\binom{0}{0}\\ \binom{1}{0}\ \binom{1}{1}\\ \binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}\\ \binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}\end{array} \qquad \begin{array}{c}1\\ 1\ 1\\ 1\ 2\ 1\\ 1\ 3\ 3\ 1\end{array}$$

$$\mathit{pascal}\ \mathit{xs} = [1] \mathbin{+\!\!+} [x + y \mid (x, y) \leftarrow \mathit{zip}\ \mathit{xs}\ (\mathit{tail}\ \mathit{xs})] \mathbin{+\!\!+} [1]$$

$$\begin{aligned}
\mathit{pascal}\ [1] &= [1, 1]\\
\mathit{pascal}\ [1, 1] &= [1, 2, 1]\\
\mathit{pascal}\ [1, 2, 1] &= [1, 3, 3, 1]\\
\mathit{pascal}\ [1, 3, 3, 1] &= [1, 4, 6, 4, 1]\\
\mathit{pascal}\ [1, 4, 6, 4, 1] &= [1, 5, 10, 10, 5, 1]
\end{aligned}$$

Prime numbers

A *prime number* p is a number where its only divisors are 1 and p.

$$\begin{aligned}
\mathit{divisors}\ n &= [x \mid x \leftarrow [1\,..\,n], n\ \text{`}\mathit{mod}\text{`}\ x \equiv 0] \\
\mathit{prime}\ n &= \mathit{divisors}\ n \equiv [1, n] \\
\mathit{primes}\ n &= [x \mid x \leftarrow [2\,..\,n], \mathit{prime}\ x]
\end{aligned}$$

Caesar cipher

$\textbf{import}\ \mathit{Data.Char}\ (\mathit{ord}, \mathit{chr}, \mathit{isLower})$

$\mathit{char2int} :: \mathit{Char} \rightarrow \mathit{Int}$
$\mathit{char2int}\ c = \mathit{ord}\ c - \mathit{ord}\ \texttt{'a'}$ -- a=0, b=1 ...

$\mathit{int2char} :: \mathit{Int} \rightarrow \mathit{Char}$
$\mathit{int2char}\ n = \mathit{chr}\ (\mathit{ord}\ \texttt{'a'} + n)$ -- 0=a, 1=b ...

$$\begin{aligned} \mathit{shift}\ n\ c \mid & \mathit{isLower}\ c = \mathit{int2char}\ ((\mathit{char2int}\ c + n)\ \text{`}\mathit{mod}\text{`}\ 26) \\ \mid & \mathit{otherwise} = c \end{aligned}$$

$\mathit{encode}\ n\ \mathit{xs} = [\mathit{shift}\ n\ x \mid x \leftarrow \mathit{xs}]$
$\mathit{decode}\ n\ \mathit{xs} = [\mathit{shift}\ (-n)\ x \mid x \leftarrow \mathit{xs}]$

$\mathit{encode}\ 3$ `"haskell er fantastisk"`
$\equiv$ `"kdvnhoo hu idqwdvwlvn"`

Generating bitstrings

$$
\begin{aligned}
\mathit{bitstrings}\ 0 &= [[\,]] \\
\mathit{bitstrings}\ n &= [b : bs \mid b \leftarrow [0,1], bs \leftarrow \mathit{bitstrings}\ (n-1)]
\end{aligned}
$$

$$
\begin{aligned}
\mathit{bitstrings}\ 0 &\equiv [\,] \\
\mathit{bitstrings}\ 1 &\equiv [[0,1]] \\
\mathit{bitstrings}\ 2 &\equiv [[0,0],[0,1],[1,0],[1,1]] \\
\mathit{bitstrings}\ 3 &\equiv [[0,0,0],[0,0,1],[0,1,0],[0,1,1], \\
&\qquad [1,0,0],[1,0,1],[1,1,0],[1,1,1]]
\end{aligned}
$$

Finding the transpose of a matrix

$$
\begin{aligned}
&\mathit{transpose} :: [[a]] \to [[a]] \\
&\mathit{transpose}\ [\,] && = [\,] \\
&\mathit{transpose}\ ([\,] : \mathit{xss}) && = \mathit{transpose}\ \mathit{xss} \\
&\mathit{transpose}\ \mathit{xss} && = \quad [x \mid (x \!: \_) \leftarrow \mathit{xss}] \\
& && \quad\ : \mathit{transpose}\ [\mathit{xs} \mid (\_ : \mathit{xs}) \leftarrow \mathit{xss}]
\end{aligned}
$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Finding the transpose of a matrix

$$
\begin{aligned}
&\mathit{transpose} :: [[a]] \to [[a]] \\
&\mathit{transpose}\ [\,] &&= [\,] \\
&\mathit{transpose}\ ([\,] : \mathit{xss}) &&= \mathit{transpose}\ \mathit{xss} \\
&\mathit{transpose}\ \mathit{xss} &&= \ [x \mid (x{:}\ \_) \leftarrow \mathit{xss}] \\
& && \quad : \mathit{transpose}\ [\mathit{xs} \mid (\_ : \mathit{xs}) \leftarrow \mathit{xss}]
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{transpose}\ [[1,2,3],[4,5,6]] \\
&\quad \equiv [1,4] : \mathit{transpose}\ [[2,3],[5,6]] \\
&\quad \equiv [1,4] : [2,5] : \mathit{transpose}\ [[3],[6]] \\
&\quad \equiv [1,4] : [2,5] : [3,6] : \mathit{transpose}\ [[\,],[\,]] \\
&\quad \equiv [1,4] : [2,5] : [3,6] : \mathit{transpose}\ [[\,]] \\
&\quad \equiv [1,4] : [2,5] : [3,6] : \mathit{transpose}\ [\,] \\
&\quad \equiv [1,4] : [2,5] : [3,6] : [\,] \\
&\quad \equiv [[1,4],[2,5],[3,6]]
\end{aligned}
$$

Generating permutations

$$
\begin{aligned}
&\mathit{permutations}\ [\,] = [[\,]]\\
&\mathit{permutations}\ (x : xs) = [ys' +\!\!+\ x : ys'' \mid\\
&\quad ys \leftarrow \mathit{permutations}\ xs,\\
&\quad i \leftarrow [0\mathinner{..}\mathit{length}\ ys],\\
&\quad \mathbf{let}\ (ys', ys'') = \mathit{splitAt}\ i\ ys]
\end{aligned}
$$

$> \mathit{permutations}\ [\,]$

$[[\,]]$

$> \mathit{permutations}\ [1]$

$[[1]]$

$> \mathit{permutations}\ [1, 2]$

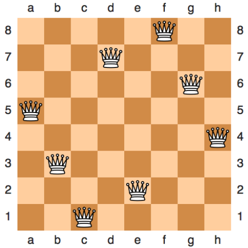
$[[1, 2], [2, 1]]$

$> \mathit{permutations}\ [1, 2, 3]$

$[[1, 2, 3], [2, 1, 3], [2, 3, 1], [1, 3, 2], [3, 1, 2], [3, 2, 1]]$

Solving the n-queens problem

The **eight queens puzzle** is the problem of placing eight chess queens on an 8×8 chessboard so that no two queens threaten each other. Thus, a solution requires that **no two queens share the same row, column, or diagonal**. The eight queens puzzle is an example of the more general **n-queens problem** of placing n queens on an $n \times n$ chessboard.



Backtracking - n-queens problem

$$
\begin{aligned}
&\mathit{validExtensions}\ n\ qs = [q : qs \mid q \leftarrow [1\,..\,n] \setminus\setminus qs, q\ \text{`}\mathit{notDiag}\text{`}\ qs] \\
&\quad \textbf{where} \\
&\qquad q\ \text{`}\mathit{notDiag}\text{`}\ qs = \mathit{and}\ \ [\ \mathit{abs}\ (q - qi) \not\equiv i \\
&\qquad\qquad\qquad\qquad\qquad\quad \mid (qi, i) \leftarrow qs\ \text{`}\mathit{zip}\text{`}\ [1\,..\,n]] \\
&\mathit{queens}'\ n\ 0 = [[\,]] \\
&\mathit{queens}'\ n\ i = [\ qs' \\
&\qquad\qquad\quad \mid qs \leftarrow \mathit{queens}'\ n\ (i - 1), \\
&\qquad\qquad\qquad qs' \leftarrow \mathit{validExtensions}\ n\ qs] \\
&\mathit{queens}\ n = \mathit{queens}'\ n\ n
\end{aligned}
$$

$> \mathit{queens}\ 8$

$[[4, 2, 7, 3, 6, 8, 5, 1], [5, 2, 4, 7, 3, 8, 6, 1], [3, 5, 2, 8, 6, 4, 7, 1]...]$