Lab 9:

1- show that for any simple grap G (not necessarily connected) having n vertices and m edges, if $m \geq n$, then G contains a cycle.

solution:

→ show that G contains no cycle, then $m < n$. we have already proven this result when G is connected. suppose G is not connected with components $G_1, G_2, \ldots G_K$, with $k > 1$. since G contains no cycle, none of the components contains a cycle either; therefore, for each i, we have that G_i is a tree. Let m_i denote the number of edges in G_i and n_i the number of vertices in G_i. Then for each i, $m_i = n_i - 1$ therfore

$$m = m_1 + m_2 + \ldots + m_k = (n_1 - 1) + (n_2 - 1) + \_\_ + (n_k - 1)$$
$$= (n_1 + n_2 + \cdots + n_k) - K = n - k < n - 1 < n.$$

2- Suppose $G = (V, E)$ is a connected simple graph. suppose $S = (V_S, E_S)$ and $T = (V_T, E_T)$ are subtrees of G with no vertices in common (in other words, V_S and V_T are disjoint). show that for any edge (x, y) in E for which x is in V_S & y is in V_T, the subgraph obtained by forming the union of S, T and the edge (x, y) (namely, $U = (V_S \cup V_T, E_S \cup E_T \cup \{(x,y)\})$) is also a tree.

solution:

→ U is connected: If u, v are vertices in U, the only difficult case is the one in which u is in V_S and v is in V_T (or conversely). To find a path in that case, get a path p from u to x, another path q from y to v and then combined path $p \cup \{(x,y)\} \cup q$ is the needed path from u to v in U.

U contains no cycle; it suffices to show that U has the right number of edges. Let $n_S = |V_S|$, $m_S = |E_S|$, and $n_T = |V_T|$, $m_T = |E_T|$. Then U has $n_S + n_T$ vertices. We show U has $n_S + n_T - 1$ edges.

Let m be the number of edges in U. Then

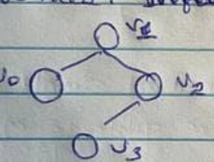
$$m = m_S + m_T + 1 = (n_S - 1) + (n_T - 1) + 1 = n_S + n_T - 1 = n - 1,$$

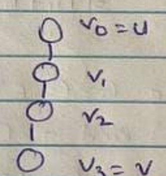
as required.

Problem 5: prove that if T is a tree with at least 2 vertices, T has at least 2 vertices having degree 1.

proof by contradiction.

Let T be a nontrivial tree, that is, one with at least 2 vertices. ~~[illegible]~~,





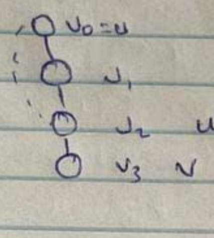
Let P be a path in T of the longest possible length. It's alright if your tree has more than one path of equal length, so long as we choose on with the longest possible length. As mentined above, u and v represent the beginning and end vertices of P. So our P is u, v_1, v_2, v

* suppose, contrary to the proposition, that there are NOT two vertices of degree one. This brings us to why we care about P. The longest path, in any tree, contains a start and end vertex of degree one.

Both ends of P must be vertices with degree one, satisfying the "at least 2 vertices of degree one" part of the original proposition. But we're supposing that there are NOT two vertices of degree one.

that means either u or v need to be adjacent to one more vertex in T. Let's pick u, the start vertex and we will call the adjacent vertex u'.

which vertex is u'? u is already adjacent to a vertex on the path $P(v_1)$. If u' was another vertex on the path P. then we'd get a cycle



we can't have a cycle in the tree, since a tree is an undirected acyclic graph. contradiction

Or we can show it like this

<u>Case one</u>: there are no vertex of degree one. since $n-1$ edges in tree total degree of any tree $2(n-1)$. But for this case since no vertex has degree 1 then every vertex have at least a degree of 2 and since there are n vertices, the total degree is $\geq 2n$ which is contradiction.

Case Two: there is only one vertex of degree one. similarly the total degree of any tree has to be $2(n-1)$. then there are $(n-1)$ vertices with which have degree of ≥ 2 while only one vertex with degree of one. Thus summing up to find the total of the vertices, we have total degree of vertices is $\geq 2(n-1)+1 = 2n-1$

contradiction.