Lab 12

problem 1: Ore's theorem implies that graphs with "many edges" tend to be hamiltonian. is it true that every dense graph is hamiltonian? prove your answer.

No, Not every dense graph has a hamiltonian cycle. if we have a disconnected graph ~~of size~~ G that each components have a complete subgraph of G then it is dense but we can't form a cycle to reach every edge.

Assuming the graph is connected.

$G = (V, E)$ it is dense if $m = C(n,2) = \frac{n(n-1)}{2}$

If we take the degree of vertices to be average

$$\deg(v) = m/n \Rightarrow \frac{n(n-1)}{2(n)} = \frac{n-1}{2}$$

by ore's theorem

$$\deg(u) + \deg(v) \geq n$$

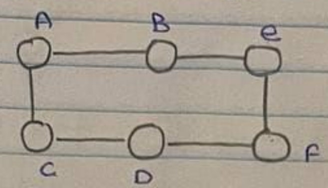
$$\frac{n-1}{2} + \frac{n-1}{2} \geq n$$

$$n-1 \geq n$$

contradicts

So, being dense doesn't show existance of hamiltonian cycle.

problem 2: graph G having $n=6$ vertices



A) is G hamiltonian ?

Yes, because there exists a simple cycle that connects every vertices in G.

B) can we find 2 non adjacent vertices that sum of whose degree is less than G?

Ore's theorem: $deg(u) + deg(v) \geq n$, where u & v are non adjacent ~~and~~ then the graph has a hamiltonian cycle.

$n=6$ $\quad deg(u) + deg(v) = deg(A) + deg(F) = 4$

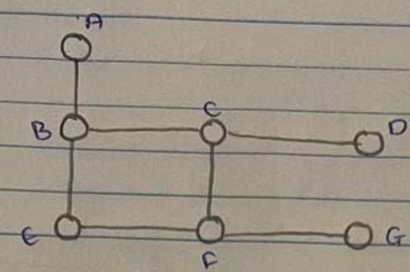
$4 < 6$

C) do these fact contradict Ore's theorem? explain.

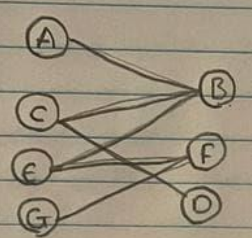
No, Because ore's theorem suggests that if deg(u) and deg(v), assuming u,v are non adjacent vertices if their sum is greater or equal to n, (n -> no of vertex) then it has hamiltonian cycle.

it doesn't suggest that there is no other criteria that ~~[illegible]~~ a graph can be showen to have hamiltonian cycle other than that.

problem 3: consider the undirected graph G below



a) Is G bipartite? If so, exhibit the partition (X,Y) of the vertices, and re-draw the graph using this partition so that, bipartiteness of the graph is obvious.



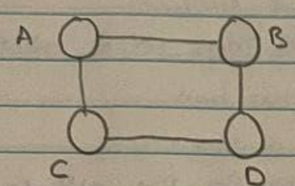
B) Exhibit a maximum matching in G

maximum matching is AB, CD, FG
size = 3

c) Exhibit a minimum vertex cover for G.

MVC = u = {B, G, D} size = 3
So, König's theorem that if there is maximum matching of size n then the minimum vertex cover will also be of that same size. holds (given a graph is bipartite).

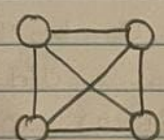
problem 4: show that hamiltonian cycle problem is polynomial reducable to TSP by considering the following Hamiltonian graph - an instance of hamiltonian cycle - and transforming it to TSP instance in polynomial time so that a solution to the HC problem yields a solution to the TSP problem and conversely.



Given :

$G = (V, E)$ with n vertices, we obtain a TSP problem.

→ we need a complete graph $H = K_n$, lets think of getting H by adding the missing vertices to G.



⇒ G is a subgraph of H

$H = K_4$

$H = (V_H, E_H)$

→ Let's define k and cost function c.

Let $k = 0$

$c : E_H \rightarrow N$ $\quad c(e) := \begin{cases} 0 & \text{if } e \in E \\ 1 & \text{if } e \notin E \end{cases}$

it takes $O(n^2)$ time to do this.

Now, let's verify that a solution to HC problem yields a solution to TSP problem.

- Suppose c is hamiltonian cycle in G, we need to. (G is a solution)

- check c is HC in H(H → New graph for TSP)
- since $V = V_H$, is C spanning? yes
- if c is still simple. yes

Check $\sum_{e \in C} c(e) \leq k$

since each e in C belongs to E, $c(e) = 0$

$\sum c(e) = 0 = k$

∴ we verified that c is the soln to the TSP problem we created.

Then, let's verify that, the solution to the TSP problem yields a solution to the HC problem.

Suppose c is a solution to the TSP problem.

So, $\sum_{e \in C} c(e) \leq k = 0$ by the mapping we did earlier.

summation of integers (positive) can only be zero iff each edge weight is valued to be zero.

→ e in c belongs to E, ⇒ $c \subseteq E$

So, c is a hamiltonian cycle that creats a graph G from H.

∴ we have a soln to the HC problem.

problem 5: Show that TSP is NP-complete. (Hint: use the relationship between TSP and Hamiltonian cycle discussed in the slides. You may assume that the Hamiltonian cycle problem is NP-complete)

Solution:

assume hamiltonian cycle is NP complete.

and TSP: given a graph G with cost function $c: E \to N$ and positive integer k, is there a hamiltonian cycle c in G so that the sum of costs of the edge in c is at most k?

is the decision problem.

on problem 4 we proved that hamiltonian cycle is polynomial reducable to TSP.

remember, ~~transitivity eq.~~

To show that a given problem is NP-complete you can show that it is reducable to another NP complete problem.

Since HC is NP-complete, because VC is NP complete

$$\text{Vertex Cover} \xrightarrow{\text{poly}} HC \xrightarrow{\text{poly}} TSP$$

By the transitivity rule

TSP is also NP-complete.

problem 6: Show that the worst case for vertex cover Approx can happen by giving an example of a graph G which has these properties:

a. G has a smallest vertex cover of size s

b. vertex Cover Approx outputs size 2^*s as its approximation to optimal size.

Solution:

Consider the following disconnected graph with two edges and four vertices. The smallest vertex cover has size 2 but the vertex cover approximation algorithm outputs a vertex cover of size = 4