MATH462 Homework 0

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Work on the following background problems. You don't have to turn in anything

1. Find all the solutions $y(x)$ of the following first order differential equations:
   (a) $y' + 5y = 0$
   (b) $(1 + x)y' = y, \quad x \geq 0$
   (c) $y' = \frac{1}{1+x^2}$
   (d) $y' = 2xy^2$
   (e) $yy' = x$

2. Find the solution of each of the following initial value problems
   (a) $y' + 4y = 0, \quad y(0) = 1$
   (b) $y'' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -1$
   (c) $y'' - 4y = 0, \quad y(0) = 1, \quad y'(0) = -1$

3. Compute the second partial derivatives $u_{xx}, u_{xt}, u_{tx}, u_{tt}$ for the following functions. Be sure to take note that $u_{xt} = u_{tx}$.
   (a) $u(x, t) = \arctan(xt)$
   (b) $u(x, t) = x \ln(xt)$
   (c) $u(x, t) = e^{x/t} + t \sin(x^2)$

4. Use the Multivariable Chain Rule to compute the following:
   (a) Suppose $f(x, y)$ is a function of two variables and $g(t)$ is defined by the formula

   $$g(t) = f(e^{-t}, t^2 + 1)$$

   Express the derivative $g'(t)$ in terms of the partial derivatives of f.

   (b) Suppose $f(x, y)$ is a function of two variables and $g(s, t)$ is defined by the formula

   $$g(s, t) = f(1 + st, 5t)$$

   Express the partial derivatives $g_s(s, t)$ and $g_t(s, t)$ in terms of the partial derivatives of f.

5. Suppose $u = f(x, y)$, but we would like to view u as a function of the polar coordinates (r, θ).

   (a) Apply the Multivariable Chain Rule to $u = f(r\cos, r\sin)$ to express $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. You should obtain

$$\frac{\partial u}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\frac{\partial u}{\partial y}$$

   (b) Solve the system of equations in the previous part for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ to show

$$\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} + -\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

   (c) Rewrite the four equations from the previous two parts by converting the coefficients in front of each partial derivative from (r, θ) to (x, y) using $x = r\cos\theta$ and $y = r\sin\theta$. For example,

$$\begin{aligned}\frac{\partial u}{\partial r} &= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \\ &= \frac{x}{r}\frac{\partial u}{\partial x} + \frac{y}{r}\frac{\partial u}{\partial y} \\ &= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial y}\end{aligned}$$

6. Use the Fundamental Theorem of Calculus to evaluate the following derivatives.

   (a) $\frac{d}{dx}\displaystyle\int_0^x e^{-s^2}\,ds$

   (b) $\frac{d}{dx}\displaystyle\int_0^{x^2} \sqrt{u}\cos(4u)du$

7. Let $c > 0$ be a constant and let $\phi(w)$ be a twice-differentiable function of a single variable. Let

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)]$$

Evaluate u_{tt} and u_{xx} and show that u satisfies the wave equation: $u_{tt} = c^2 u_{xx}$. Additionally, show that u satisfies the two "initial conditions" $u(x, 0) = \phi(x)$ and $u_t(x, 0) = 0$.

8. Let $c > 0$ be a constant and let $\psi(w)$ be a differentiable function of a single variable. Let

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(w)dw$$

Evaluate u_{tt} and u_{xx} and show that u satisfies the wave equation: $u_{tt} = c^2 u_{xx}$. Additionally, show that u satisfies the two "initial conditions" $u(x,0) = 0$ and $u_t(x,0) = \psi(x)$.

9. Let $k > 0$ be a positive constant.

   (a) Show that for any real number r, the function $u(x,t) = e^{rx+kr^2t}$ is a solution of the diffusion equation (heat equation) $u_t = ku_{xx}$.

   (b) Show that for any real number r, the functions $u(x,t) = e^{kr^2t}\cos(rx)$ and $u(x,t) = e^{-kr^2t}\sin(rx)$ are also solutions of $u_t = ku_{xx}$.

10. Let $c > 0$ be a positive constant.

    (a) Show that for any real number r, the function $u(x,t) = e^{rx+crt}$ is a solution of the wave equation $u_{tt} = c^2 u_{xx}$.

    (b) Show that for any real number r, the functions

    $$u(x,t) = \sin(crt)\sin(rx), \quad \sin(crt)\cos(rx), \quad \cos(crt)\sin(rx), \quad \text{and} \quad \cos(crt)\cos(rx)$$

    are also solutions of $u_{tt} = c^2 u_{xx}$.

11. Let k and V be constants. Suppose that $u(x,t)$ is a solution to the "heat equation with convection"

$$u_t - ku_{xx} + Vu_x = 0$$

Show that the function $w(x,t) = u(x+Vt,t)$ is a solution to the (regular) heat equation. That is, show that $w_t = kw_{xx}$.