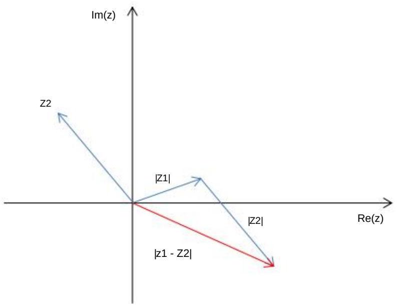
Mathematics 2 for Mechanical Engineers - Ubung 2 (L osungen)¨

Complex Numbers (Part 2)

Task 1 Graphically justify the statement that is valid for all ¨ z_1, z_2 2 **C :**

$$|z_1 z_2| ÿ |z_1| + |z_2|$$

Triangle inequality: In a triangle, one side cannot be longer than the sum of the other two.



Task 2 What is the distance between $z_1 = 8$ 5 i and $z_2 = 4$ 2 i in the Gaussian number plane?

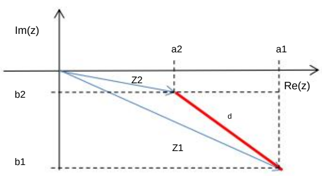
Lt. According to Pythagoras, the sketch applies:

$$d^2 = (Re(z_1)\ Re(z_2))^2 + (Im(z_1)\ Im(z_2))^2$$
$$= (8\ 4)^2 + (5\ (2))^2$$
$$= 16 + 9$$
$$= 25$$

) $d = 5$

If you generalize this procedure, you get $z_1 = a_1 + i\, b_1$ and $z_2 = a_2 + i\, b_2$ as a formula for the distance for any two complex numbers

$$|z_1 z_2| = q\ |a_1\ a_2|^2 + |b_1\ b_2|^2$$



Task 3 Calculate

a) $\frac{1}{5+4\,i}$

$$\frac{1}{5+4\,i} \underset{\text{expand}}{\overset{\text{mit conjugiert complex}}{=}} \frac{1}{5+4\,i\;5} \cdot \frac{5\;4\,i}{5\;4\,i}$$

$$= \frac{4\,i}{52+42}$$

$$= \frac{5\;4\,i}{41}$$

$$= \frac{5}{41} \quad \frac{4}{41}\,i$$

b) $\frac{(3+2i\,)(2i)}{5+i}$

$$\frac{(3+2\,i)(2\,i)}{5+i} = \frac{6\;3\,i+4\,i+2}{5+i}$$

$$= \frac{8+i}{5+i}$$

$$\underset{\text{expand}}{\overset{\text{mit conjugiert complex}}{=}} \frac{(8+i)(5\,i)}{(5+i)(5\,i)}$$

$$= \frac{40\;8\,i+5\,i+1}{25+1}$$

$$= \frac{41}{26} \quad \frac{3}{26}\,i$$

Task 4 Calculate the complex numbers

$$z1 = 1\ i;\ z2 = 4\ i;\ z3 = 4ei\ ;\ z4 = 3ei^{\frac{4}{3p}}$$

into exponential or Cartesian form. Determine the conjugate complex numbers. Sketch the points in the Gaussian number plane.

a) $z1 = 1\ i$

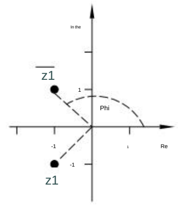
conjugate complex number: $\overline{z1} = 1 + i$

Amount or length: ¨ $|z1| = |\overline{z1}| = p\overline{(1)2 + 12} = p2$

Angle with positive x-axis (3rd or 2nd quadrant, see sketch):

$fz1 = 45 + 180 \hat{=} \frac{5p}{4}$, $\quad f\overline{z1} = 45 + 90 \hat{=} \frac{3p}{4}$

) $z1 = 1\ i = p2 \cdot no^{\frac{5}{4p}}$, $\quad \overline{z1} = 1 + i = p2 \cdot no^{\frac{3}{4p}}$



b) $z2 = 4\ i$

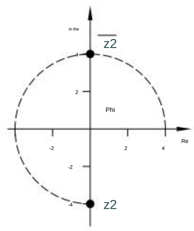
conjugate complex number: $\overline{z2} = 4\ i$

Amount or length: ¨ $|z2| = |\overline{z2}| = p\overline{(4)2} = 4$

Angle with positive x-axis:

$fz2 = 270 \hat{=} \frac{3p}{2}$, $\quad f\overline{z2} = 90 \hat{=} \frac{p}{2}$

) $z2 = 4\ i = 4ei^{\frac{3}{2p}}$, $\quad \overline{z2} = 4i = 4ei^{\frac{p}{2}}$



c) $z3 = 4ei$

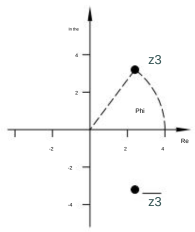
conjugate complex number: $\overline{z3} = 4e^{i}$

Amount or length: ¨ $|z3| = |\overline{z3}| = \overline{4}$

Angle with positive x-axis: $fz3 = 1$ in radians (!!!) ($1 \hat{=} 57.3$)

$f\overline{z3} = 1 \hat{=} 57.3 = 302.7$

) $z3 = 4ei = 4(\cos(1) + i \cdot \sin(1))$, $\quad \overline{z3} = 4e^{i} = 4(\cos(1) + i \cdot \sin(1))$



d) z4 = 3ei $\frac{4}{3p}$

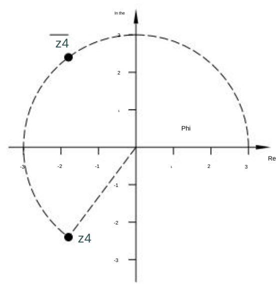
conjugate complex number: $\overline{z4}$ = 3e 4 i 3p

or length: ¨ |z4| = |$\overline{z4}$| = 3 Amount

Angle with positive x-axis: fz4 = $\frac{4}{3p}$ =ˆ 240, f$\overline{z4}$ = $\frac{4}{3p}$ = $\frac{2}{3p}$ =ˆ 120

) z4 = 3ei $\frac{4}{3p}$ = 3 ÿ cos ÿ 4 3p ÿ + i · sin ÿ 4 3p ÿÿ = 3 ÿ 1 3p 2 $\frac{p3\ i}{2}$ ÿ = 2 ÿ $\frac{3}{2}$ i $\frac{3\ p3}{2}$

$\overline{z4}$ = 3e i $\frac{4}{3p}$ = 3 cos $\frac{2}{3p}$ + i · sin $\frac{2}{}$ = 3 ÿ 1 $\frac{}{2}$ + i $\frac{p3}{}$ = $\frac{3}{2}$ + i 2 $\frac{3\ p3}{}$



Exercise 5 Use Euler's formula to show that the following applies to the real angle functions:

cos (f) = not f + e i f / 2

sin (f) = no f It is i f / 2 i

tan (f) = 1/i · no f It is i f / not f + e i f

Euler's formula is: ei f = cos (f) + i sin (f). This gives = cos (f) + i sin (f) + cos (f) i sin (f) = 2 cos (f)

not f + e i f

) cos (f) = 1/2 ÿ no f + e i f ÿ

no f It is i f = cos (f) + i sin (f) (cos (f) + i sin (f)) = 2 i sin (f)

) sin (f) = 1/2 i ÿ no f It is i f ÿ

) tan (f) = sin / cos 1

= 2 i (cos (f) i sin (f)) / 1 (cos (f) + i sin (f))

= 2 1/i (cos (f) i sin (f)) cos (f) + i sin / (f)

Task 6 In three-phase current there are neutral conductors and three conductors whose voltages have the same frequency and the same amplitude, but are each shifted by phase. This means that the voltages are on the different conductors 2p/3

u1(t) = U0 (cos (wt) + i sin (wt))

u2(t) = U0 ÿ cos ÿ wt + u3(t) 2/3 p • + i sin ÿ wt + p • + i 2/3 p •• p

= U0 ÿ cos ÿ wt + 4/3 sin ÿ wt + 4/3 ••

at. Show that at all times the sum of the voltages is neutralized, ie u1(t) + u2(t) + u3(t) = 0 for all ¨ t 2 **R.**

Representation in exponential form: ;

U0ei wt u2(t) = U0ei(wt+2 3p); u3(t) = U0ei(wt+4 3p) u1(t) =

This results in

u1(t) + u2(t) + u3(t) = U0ei wt · ÿ 1 + no 2/3d + no 4/3p ÿ

(With = 0 ?

and it applies

$1 + \text{no}\ \frac{2}{3}d + \text{no}\ \frac{4}{3}p$

$= 1 + \cos ÿ2\ 3\ p • + i \cdot \sin ÿ2\ 3\ p • + \cos ÿ4\ 3\ p • + i \cdot \sin ÿ4\ 3\ p •$

$= 1 \quad \frac{1}{2} + i\ \frac{p3}{2} \quad \frac{1}{2} \quad \frac{p3\ i}{2}$

$= 0$

Task 7 Calculate the trigonometric representations of the complex numbers

$z1 = 1 + i,\ z2 = p\ 27 + 3\ i$ and $z3 = 36$

and use them to determine z2 $\frac{z1^{10} \cdot z4\ 2}{3}$. Also give the result in Cartesian form.

$|z1| = q\ (1)2 + 12 = p\ 2$

3 fz1 = 135 =ˆ p (angle bisector 2nd/4th quadrant) 4

$) z1 = 1 + i = p\ 2\ ÿ \cos (\frac{3}{4}\ 3\ p) + i \sin (\frac{}{4}\ p) •$

$|z2| = qp\ 27^2 + 32 = p27 + 9 = 6$

$fz2 = \arctan (\frac{3}{p27}) = \arctan (\frac{1}{p3}) = \frac{p}{6}$, (1. Quadrant) 6

$) z2 = 6\ ÿ \cos (\frac{p}{6}) + i \sin (\frac{p}{6})\ ÿ$

$|z3| = 36$ fz3

= 0 (lies on the real axis)) $z3 = 36 (\cos (0) + i \sin$
(0))

And this can be used to calculate (lengths are raised to a power, angles are multiplied!):

$$\frac{z_1^{10} \cdot z_2^{4}}{z_2^{3}} = \frac{p2^{-10} \cos\left(\frac{30p}{4}\right) + i \sin\left(\frac{30p}{4}\right) 64\, ÿ \cos\left(\frac{4p}{6}\right) + i \sin\left(\frac{4p}{6}\right) ÿ}{362}$$

$$= \frac{p2^{-10} \cdot 64}{362} \cdot no^{\frac{30}{4}p} \cdot no^{\frac{4}{6}p.m}$$

$$= 25 \cdot no^{\frac{30}{4}p + i\, 6p\frac{4}{}}$$

$$= 25ei(15 \tfrac{}{2} + 2\ 3)p$$

$$= 32ei(45+4 \tfrac{}{6})p$$

$$= 32ei^{\frac{49}{6}p}$$ (Multiples of $2p$ can be subtracted from the argument)

$$= 32ei^{\frac{p}{6}}$$

$$= 32\, ÿ \cos\left(\frac{p}{6}\right) + i \sin\left(\frac{p}{6}\right) ÿ$$

$$= 32 \left(\frac{p\overline{3}}{2} + \frac{1}{2} i\right)!$$

$$= 16p\overline{3} + 16\, i$$

Task 8 Calculate

a) $ÿ\ 1 + p\overline{3}\ iÿ^3$

with := $\begin{pmatrix} 0 & \overline{} 1 \\ @1 + p\,3\,i & A \end{pmatrix}^3$ $\underbrace{}_{z =: v}$

$$) |w| = q\ \overline{(1)2 + p\,3}^{\,2} = 2 \text{ and } fw = \arctan(p\,3) + p = \frac{2}{3} p \text{ (2. Quadrant)}$$

$$) w = 2\, ÿ \cos\left(\frac{2}{3}p\right) + i \sin\left(\frac{2}{3}p\right) \bullet$$

$$) z = w^3 = 8 (\cos(2p) + i \sin(2p)) = 8$$

b) $2^{1000}\cdot\left(\frac{5+3\sqrt{3}\,i}{4}-\frac{1}{1+\sqrt{3}\,i}\right)^{999}$

$$z = 2^{1000}\Bigg(\underbrace{\frac{5+3\sqrt{3}\,i}{4}-\frac{1}{1+\sqrt{3}\,i}}_{=:w}\Bigg)^{999}$$

$$\Rightarrow w = \frac{5+3\sqrt{3}\,i}{4}-\frac{1}{1+\sqrt{3}\,i}\cdot\frac{1-\sqrt{3}\,i}{1-\sqrt{3}\,i}$$

$$= \frac{5+3\sqrt{3}\,i}{4}-\frac{1-\sqrt{3}\,i}{4}$$

$$= \frac{4+4\sqrt{3}\,i}{4}$$

$$= 1+\sqrt{3}\,i$$

$\Rightarrow |w| = 2$

$\Rightarrow \varphi_w = \arctan(\sqrt{3}) = \frac{\pi}{3}$ (1. Quadrant)

$$\Rightarrow z = 2^{1000}\cdot 2^{999}\cdot\left(\cos\left(999\frac{\pi}{3}\right)+i\sin\left(999\frac{\pi}{3}\right)\right)$$

$$= \frac{1}{2}\left(\cos(333\pi)+i\sin(333\pi)\right) \text{ (multiples of } 2\pi \text{ can be subtracted)}$$

$$= \frac{1}{2}\left(\cos(\pi)+i\sin(\pi)\right)$$

$$= \frac{1}{2}\cdot(-1)$$

$$= -\frac{1}{2}$$

Task 9

a) Calculate all third roots of the number 125 i in trigonometric and card index scher Form.

First you should put the number into trigonometric representation

$w := -125\,i$ neg. vertical axis $\Rightarrow \varphi_w = \frac{3}{2}\pi, r_w = 125$

$$\Rightarrow w = 125\cdot\left(\cos\left(\frac{3}{2}\pi\right)+i\sin\left(\frac{3}{2}\pi\right)\right).$$

In general, the following applies to the angle φ and the length r of the nth roots of unity:

$$\varphi_k = \frac{\varphi + 2k\pi}{n} \quad k = 0,\dots,n-1, \text{ and } r = \sqrt[n]{r}$$

This gives you 3 roots of unity z_0, z_1 and z_2 with $n = 3$, $k = 0, 1, 2$:

$$f0 = \frac{\frac{3}{2}p}{3} = \frac{p}{2}$$

r0 = p3 125 = 5 (applies to all 3 solutions) ¨

) z0 = 5 ÿ cos ($\frac{p}{2}$) + i sin ($\frac{p}{2}$) ÿ
= 5 i

$$f1 = \frac{\frac{3}{2}p + 2p}{3} = \frac{7}{6}p$$

r1 = r0 = 5

) z1 = 5 ÿ cos ($\frac{7}{6}$ p) + i sin ($\frac{7}{6}$ p) •
= 5 ÿ 1 $\frac{}{2}$ p 3 $\frac{1}{2}$ i •
= $\frac{5}{2}$ p 3 $\frac{5}{2}$ i

$$f2 = \frac{\frac{3}{2}p + 4p}{3} = \frac{11}{6}p$$

r2 = r0 = 5

) z2 = 5 ÿ cos ($\frac{11}{6}$ p) + i sin ($\frac{11}{6}$ p) •
= 5 ÿ1 2 p 3 $\frac{1}{2}$ i •
= $\frac{5}{2}$ p 3 $\frac{5}{2}$ i

b) Give the polar representation of the complex number z = 18(1 + p3 i) and calculate the square roots of this number.

z = 18 ÿ 1 + p 3 iÿ
= 18 18p 3 i

) r̂z = | 18| · q 12 + p 3 ^{2} = 18 · p 4 = 36

) f̂z = arctan (p3) + p = 1 $\frac{4}{3}$ p (3. Quadrant) 3

The “18” has no influence on the angle, but the negative sign does!

Using the general formula (see part a.) follows

$$z0 = 6 \text{ ÿ } \cos \left(\frac{2}{3} 2 p\right) + i \sin \left(\overline{3} p\right) \bullet$$

$$(= 6 \text{ ÿ } 1 \frac{}{2} + \frac{1}{2} p\, 3 i \bullet$$

$$= 3 + 3 p\, 3 \bar{i}$$

$$z1 = 6 \text{ ÿ } \cos (= 6 \left(\frac{5}{3} 5 p\right) + i \sin \left(\overline{3}\, 3 p\right) \bullet$$

$$\text{ÿ}1\ 2 \ \frac{1}{2} p\, 3 i \bullet$$

$$= 3\ 3 p\, 3 \bar{i}$$