Complex numbers

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Notations and terminology

Definitions

A complex number is a number of the form :

$$x + iy, \text{ where } x, y \in \mathbb{R} \text{ and } i^2 = -1,$$

x is called the real part of z and is denoted by $Re(z)$.

y is called the imaginary part of z and is denoted by $Im(z)$.

Numbers of the form $iy,\ y \in \mathbb{R}^*$, are called pure imaginary.

Examples

$$i, i+1, 2-i$$

are complex numbers.

-2 is a complex number, we can write it

$-2 = -2 + 0i.$

We have $\mathbb{R} \subset \mathbb{C}$.

Conjugate and modulus of a complex number

Definition

Let $z = x + iy \in \mathbb{C}$. $Set\ \bar{z} = x - iy$ and $|z| = \sqrt{x^2 + y^2}$.

$\bar{z}$ is called the conjugate of z and $|z|$, the modulus of z.

Example

If $z = 3 + 4i$, then
$Re(z) = 3$
$Im(z) = 4$

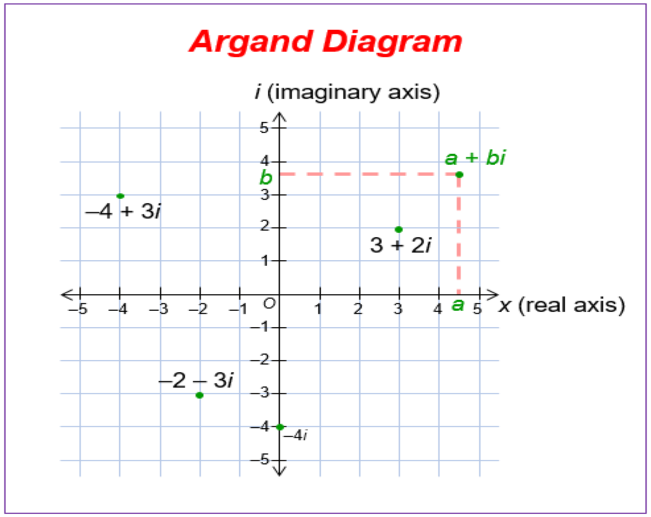
$\bar{z} = 3 - 4i$
$|z| = 5$

Properties of $\bar{z}, |z|$

- $z\bar{z} = |z|^2$
- $Re(z) = \frac{z+\bar{z}}{2}, Im(z) = \frac{z-\bar{z}}{2}$
- $\overline{z+w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z}\bar{w}$
- $|zw| = |z||w|$
- $|z+w| \leq |z| + |w|$

The Argand diagram

We obtain a geometric model for the complex numbers by representing a+ib by the point (a, b) in the real plane with coordinates a and b.



Geometric interpretation of the addition

Let $z = x + iy$, w$= u + iv,$ two complex numbers.

Definition

The sum of z and w is defined by

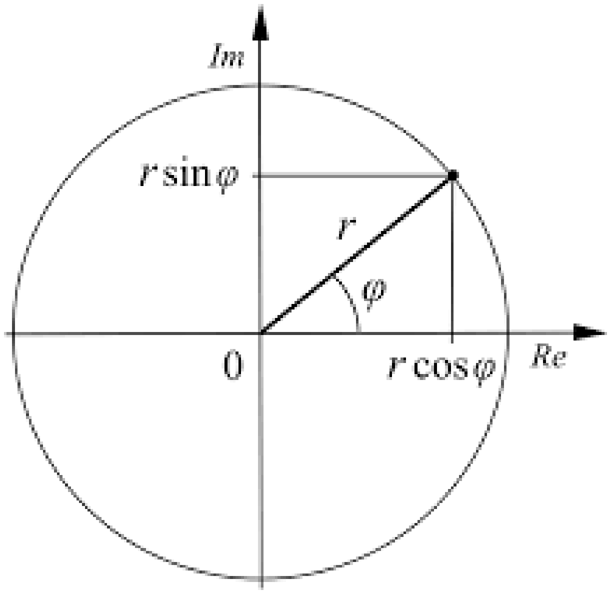
$$z + w = (x + u) + (y + v)i.$$

In the Argand diagram, it appears as the vector sum of z and w.

The complex number $z + w$ is represented geometrically as the fourth vertex of the parallelogram formed by $0, z$ and w.

Polar form

Suppose that the complex number $z = x + iy$ on the Argand diagram has polar coodinates r, φ.



Definition

We call r the modulus of z and φ the argument of z, denoted by $\arg(z)$.

Pythagoras'theorem gives $|z| = \sqrt{x^2 + y^2}$.

We have $cos\varphi = \frac{x}{r}$ and $sin\varphi = \frac{y}{r}$.

$|z|$ is a single valued.

$\arg(z)$ is many valued ($\varphi+2n\pi, n \in \mathbb{Z}$).

We define the principal value of $\arg(z)$ to be that value of φ which satisfies $-\pi < \varphi < \pi$.

Therefore,

$z = rcos\varphi + irsin\varphi = r(cos\varphi + \ isin\varphi) = re^{i\varphi}$.

The formula

$$e^{i\theta} = cos\theta + \ isin\theta$$

is called Euler's formula.

We call the representation

$$z = re^{i\theta},$$

the polar form for z, and

the representation $z = x + iy$, the cartesian or algebraic form for z.

Example

$z = 1 + i$

$|z| = \sqrt{2}$

$arg(z) = \frac{\pi}{4} + 2n\pi,\ n \in \mathbb{Z}.$

The principal value of $\arg(1 + i) = \frac{\pi}{4}$.

Remark

$$\theta = \theta' \ (\text{mod } 2\pi) \Leftrightarrow \exists k \in \mathbb{Z}, \theta = \theta' + 2k\pi \Leftrightarrow \begin{cases} cos\theta = cos\theta' \\ sin\theta = sin\theta' \end{cases}$$

Properties of the argument

- $\arg(zz') = \arg(z) + \arg(z') \ (mod\ 2\pi)$
- $\arg(z^n) = n \arg(z) \ (mod\ 2\pi)$
- $\arg(\bar{z}) = -\arg z \ (mod\ 2\pi)$

De Moivre's Theorem

An immediate consequence of Euler's formula is a theorem known as De Moivre's theorem :

$$(cos\theta + \; isin\theta)^n = (e^{i\theta})^n = e^{in\theta} = cosn\theta + isinn\theta, n \; \in \mathbb{Z}.$$

Exercise :

1) Linearise cos^2xsin^2x.

2) Give $sin3x$ in terms of powers of $sinx$ and $cosx$.

Square roots of a complex number

Proposition

Let $z = a + ib$ be a complex number, then z admits two square roots ω and $-\omega$.

Proof

Let $\omega = x + iy$. We search for x, y such that $\omega^2 = z$.

We have

$$\omega^2 = z \Leftrightarrow (x + iy)^2 = a + ib \Leftrightarrow$$

$$x^2 - y^2 = a \ \ and \ \ 2x\,y = b$$

We can add the equation $|\omega|^2 = |z|$,

$$x^2 + y^2 = \sqrt{a^2 + b^2}$$

Thus

$$\begin{cases} x^2 - y^2 = a & (1) \\ 2xy = b & (2) \\ x^2 + y^2 = \sqrt{a^2 + b^2} & (3) \end{cases}$$

$$\Longleftrightarrow$$

$$\begin{cases} 2x^2 = \sqrt{a^2 + b^2} + a \\ 2y^2 = \sqrt{a^2 + b^2} - a \\ 2xy = b \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} x = \pm\dfrac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}+a} \\ y = \pm\dfrac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}-a} \\ 2xy = b \end{cases}$$

In all cases, the square roots of z are ω and $-\omega$.

Example

The square roots of i are $\frac{\sqrt{2}}{2}(1+i)$ and $-\frac{\sqrt{2}}{2}(1+i)$.

Proposition

The equation $az^2 + bz + c = 0$, $a, b, c \in \mathbb{C}$ et $a \neq 0$, admits two solutions $z_1, z_2 \in \mathbb{C}$.

Let $\Delta = b^2 - 4ac$ and $\delta \in \mathbb{C}$ a square root of Δ. Then the solutions are

$$z_1 = \frac{-b+\delta}{2a} \text{ and } z_2 = \frac{-b-\delta}{2a}$$

If $\Delta = 0$ then the solution $z = z_1 = z_2 = -b/2a$ is unique.

Example

Give the solutions of the equation

$$z^2 + z + 1 = 0$$

n^{th} roots of the unity

Definition

Let $z = re^{i\theta} \in \mathbb{C}$. A n^{th} root of z is a number ω such that $\omega^n = z$.

Proposition

The equation $\omega^n = z$ admits n solutions $\omega_0, \omega_1, \cdots, \omega_{n-1}$ given by

$$\omega_k = r^{1/n} e^{i\frac{\theta + 2k\pi}{n}}, k = 0, \cdots, n-1.$$

Example

$The\ n^{\text{th}}$ roots of 1 are the solutions of the equation : $z^n = 1$.

They are given by

$$\omega_k = e^{i\frac{2k\pi}{n}},\ k = 0, \cdots, n-1.$$

$e^{i\frac{2\pi}{n}}$ is called the primitive n^{th} root of 1.

Lemma

Set $\omega = e^{i\frac{2\pi}{n}}$. We have

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$$