CHAPTER 3

IMPROPER OR GENERALIZED INTEGRALS

Mathematical Analysis 1 , ENSIA 2024

IMPROPER OR GENERALIZED INTEGRALS

Introduction

In the study of Riemann integrals, functions bounded on closed and bounded intervals [a, b] have been considered. This chapter addresses cases where one of the two mentioned conditions is not satisfied.

GENERALIZED INTEGRALS

Definition

Let $]a, b[$ be an interval in $\mathbb{R}$, with $-\infty \leq a < b \leq +\infty$, and f a function defined on $]a, b[$. If one of the two conditions:

1. the interval $]a, b[$ is unbounded,
2. the function f is not bounded on $]a, b[$,

is satisfied, we say that the integral

$$\int_a^b f(x)\, dx$$

is an improper integral.

Example

The following integrals are improper integrals. (a)

$$\int_0^1 \frac{1}{x}\, dx;$$

(b)

$$\int_0^\infty \frac{1}{x^2+1}\, dx;$$

(c)

$$\int_0^\infty \frac{1}{\sqrt{x(x^2+1)}}\, dx.$$

For (a), the function $x \mapsto \frac{1}{x}$ is unbounded as x approaches 0, so we say this integral has a singularity at 0.

GENERALIZED INTEGRALS

Example

For (b), the interval $[0, \infty[$ is unbounded, so we say this integral has a singularity at $+\infty$.

For (c), the interval $]0, \infty[$ is unbounded, and the function $x \mapsto \sqrt{\frac{1}{x(x^2+1)}}$ is unbounded as x approaches 0. We say this integral has two singularities, at 0 and $+\infty$.

Integral with a Single Singularity

Definition

Let f be a function defined on the interval $[a, b[$, where $-\infty < a < b \leq +\infty$. If b is infinite or if f is unbounded as x approaches b, we say that the integral has a singularity at b, denoted by

$$\int_a^{*b} f(x)\, dx.$$

Definition

Let f be a function defined on $[a, b[$. We say that f is locally integrable on $[a, b[$ if it is integrable over every closed and bounded interval $[\alpha, \beta] \subset [a, b[$.

This is denoted by

$$f \in R_{\text{loc}}([a, b[),$$

indicating that f is locally Riemann-integrable on $[a, b[$.
For the rest of this section, we consider only locally integrable functions.

Convergent Integrals

Let $f \in R_{\text{loc}}([a, b[)$, and let

$$F(t) = \int_a^t f(x)\, dx.$$

The function F is well-defined on $[a, b[$ because f is locally integrable on $[a, b[$.

Definition

If the limit $\lim_{t \to b^-} F(t)$ exists and is finite, we say that the improper integral

$$\int_a^{*b} f(x)\, dx$$

is convergent. Moreover,

$$\int_a^{*b} f(x)\, dx = \lim_{t \to b^-} F(t) = \lim_{t \to b^-} \int_a^t f(x)\, dx.$$

If the integral is not convergent, we say it is divergent.

Example

Study of the integral $\int_1^{+\infty} \frac{1}{1+x^2}\, dx$:
1. Let $F(t) = \int_1^t \frac{1}{1+x^2}\, dx = \arctan(t) - \frac{\pi}{4}$. The limit $\lim_{t\to+\infty} F(t) = \frac{\pi}{4}$ exists and is finite. Therefore, the integral $\int_1^{+\infty} \frac{1}{1+x^2}\, dx$ is convergent, and $\int_1^{+\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{4}$.
2. Study of the integral $\int_0^1 \frac{1}{x}\, dx$:
Let $F(t) = \int_t^1 \frac{1}{x}\, dx = -\ln(t)$. The limit $\lim_{t\to 0^+} F(t) = +\infty$, so the integral $\int_0^1 \frac{1}{x}\, dx$ is divergent.

Convergent Integrals

Example

3. Study of the following Riemann integral:

$$I_\alpha = \int_1^{+\infty} \frac{1}{x^\alpha}\, dx$$

We have $F(t) = \int_1^t \frac{1}{x^\alpha}\, dx = \begin{cases} \ln(t) & \text{if } \alpha = 1, \\ \frac{1}{1-\alpha}\left(t^{1-\alpha} - 1\right) & \text{if } \alpha \neq 1. \end{cases}$

The limit $\lim_{t \to +\infty} F(t)$ is finite if and only if $\alpha > 1$. Thus, the Riemann integral I_α is convergent if, and only if, $\alpha > 1$, and in this case, $I_\alpha = \frac{1}{\alpha - 1}$.

4. Study of the Riemann integral:

$$J_\alpha = \int_0^1 \frac{1}{x^\alpha}\, dx$$

Example

For the Riemann integral:

$$J_\alpha = \int_0^1 \frac{1}{x^\alpha}\, dx$$

We have:

$$F(t) = \int_t^1 \frac{1}{x^\alpha}\, dx = \begin{cases} -\ln(t) & \text{if } \alpha = 1, \\ \frac{1}{1-\alpha}\left(1 - t^{1-\alpha}\right) & \text{if } \alpha \neq 1. \end{cases}$$

The limit $\lim_{t\to 0} F(t)$ is finite if and only if $\alpha < 1$. Therefore, the Riemann integral J_α is convergent if, and only if, $\alpha < 1$. In the case of convergence, we have $J_\alpha = \frac{1}{1-\alpha}$.

Properties:

Theorem

1. If the integral $\int_a^{*b} f(x)\,dx$ is convergent, then for any $c \in]a, b[$, the integral $\int_c^{*b} f(x)\,dx$ is convergent, and we have:

$$\int_a^{*b} f(x)\,dx = \int_a^{c} f(x)\,dx + \int_c^{*b} f(x)\,dx$$

2. If there exists $c \in]a, b[$ such that the integral $\int_c^{*b} f(x)\,dx$ is convergent, then the integral $\int_a^{*b} f(x)\,dx$ is convergent. Moreover,

$$\int_a^{*b} f(x)\,dx = \int_a^{c} f(x)\,dx + \int_c^{*b} f(x)\,dx$$

Properties:

Theorem

Let $f, g \in R_{loc}([a, b[)$. If the integrals $\int_a^{*b} f(x)\,dx$ and $\int_a^{*b} g(x)\,dx$ are convergent, then for any $\alpha, \beta \in \mathbb{R}$, the integral

$$\int_a^{*b} [\alpha f(x) + \beta g(x)]\,dx$$

is convergent, and we have:

$$\int_a^{*b} [\alpha f(x) + \beta g(x)]\,dx = \alpha \int_a^{*b} f(x)\,dx + \beta \int_a^{*b} g(x)\,dx$$

Positive Functions

It is said that the set of convergent integrals forms a vector space.If we can calculate the "partial" integral $\int_a^t f(x)\,dx$ for all $t \in]a,b[$, then we can study the convergence of the integral $\int_a^{\star b} f(x)\,dx$ by taking the limit. However, in many cases, calculating $\int_a^t f(x)\,dx$ directly might not be feasible. Therefore, we need convergence criteria that do not rely on the calculation of the "partial" integral $\int_a^t f(x)\,dx$. Moreover, there are criteria that are particularly useful for functions with a constant sign in the vicinity of a singularity. For simplicity, let's assume these functions are positive.

Positive Functions

According to the theorem above, it is sufficient for the function f to be positive in the vicinity of the singularity. However, for ease of exposition, let's assume that f is positive on $[a, b[$.

Theorem

Let F be the function that is the primitive of f vanishing at a, defined by:

$$F(t) = \int_a^t f(x)\,dx$$

We then have:
Let $f \in R_{loc}([a, b[)$ be a positive function on $]a, b[$. The improper integral $\int_a^{\star b} f(x)\,dx$ is convergent if, and only if, the function F, is upper bounded on $]a, b[$.

Positive Functions

The following theorem is known as the comparison theorem and is fundamental in the study of integrals of positive functions.

Theorem

Let f and $g \in R_{loc}([a,b[)$. Suppose $0 \leq f(x) \leq g(x)$ for all $x \in [a,b[$. Then we have the implication:

$$\int_a^{\star b} g(x)\,dx \text{ is convergent} \Rightarrow \int_a^{\star b} f(x)\,dx \text{ is convergent}$$

Example

Study of the Integral

Consider the integral

$$\int_1^\infty \frac{1}{x^3 + x^2}\, dx.$$

We have

$$\frac{1}{x^3 + x^2} \leq \frac{1}{1 + x^2}, \quad \forall x \in]1, +\infty[.$$

The function

$$F(t) = \int_1^t \frac{1}{1 + x^2}\, dx = \arctan t - \frac{\pi}{4} \leq \frac{\pi}{4},$$

being bounded on $]1, +\infty[$, implies that the integral $\int_1^\infty \frac{1}{1+x^2}\, dx$ is convergent. Consequently, by the comparison theorem, we deduce that the integral $\int_1^\infty \frac{1}{x^3+x^2}\, dx$ is also convergent.]

Positive Functions

The following theorem uses the comparison of f in the vicinity of the singularity with a function whose nature is known.

Theorem

Let f and $g \in R_{loc}([a, b[)$ be two positive functions, and let

$$l = \lim_{x \to b^-} \frac{f(x)}{g(x)}.$$

Then, the following results hold:
1. If $0 < l < +\infty$, then

$$\int_a^{\star b} f(x)\,dx \text{ is convergent} \Leftrightarrow \int_a^{\star b} g(x)\,dx \text{ is convergent.}$$

We say that the two integrals are of the same nature.

Positive Functions

Theorem

2. If $l = 0$, then

$$\int_a^{\star b} g(x)\,dx \text{ is convergent} \Rightarrow \int_a^{\star b} f(x)\,dx \text{ is convergent.}$$

3. If $l = +\infty$, then

$$\int_a^{\star b} g(x)\,dx \text{ is divergent} \Rightarrow \int_a^{\star b} f(x)\,dx \text{ is divergent.}$$

Example

$$\int_1^\infty \ln\left(1+\frac{1}{x}\right)\, dx.$$

We observe that

$$\ln\left(1+\frac{1}{x}\right) > 0, \quad \forall x \in (1, +\infty),$$

so we can use comparison theorems. As $x \to +\infty$, and using a Taylor expansion at 0, we have

$$\ln\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + o\left(\frac{1}{2x^2}\right).$$

For $g(x) = \frac{1}{x}$, we have

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 1,$$

Example

so, according to the previous theorem, the two integrals

$$\int_1^\infty \ln\left(1+\frac{1}{x}\right)\, dx$$

and

$$\int_1^\infty \frac{1}{x}\, dx$$

are of the same nature. Since the second integral is a divergent Riemann integral ($\alpha = 1$), then the first integral is also divergent.

Example

$$\int_1^\infty x^3 e^{-x}\, dx.$$

We know that $\int_1^\infty e^{-x}\, dx$ is convergent because

Positive Functions

Example

The function $x \mapsto e^{-x}$ dominates, as x goes to infinity, any polynomial. For example,

$$\lim_{x \to +\infty} \frac{x^5}{e^x} = 0.$$

So, for $g(x) = \frac{1}{x^2}$, we have

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{x^5}{e^x} = 0.$$

We are in the second case of the theorem. Knowing that the integral

$$\int_1^\infty \frac{1}{x^2}\, dx$$

is a convergent Riemann integral, then our integral is also convergent.

Functions of Any Sign

In this section, we cannot use the theorems mentioned previously, but we still have some results that we can apply. The first one is:

Theorem

Let $f \in \mathcal{R}_{loc}([a, b[)$. The integral $\int_a^{\star b} f(x)\, dx$ is convergent if, and only if, the following Cauchy condition is satisfied:

$$\forall \varepsilon > 0, \exists c \in]a, b[; \forall x', x'', x', x'' \in]c, b[\Rightarrow \left| \int_{x'}^{x''} f(x)\, dx \right| < \varepsilon.$$

In order to use the comparison theorems, we study the integral of the absolute value of f to reduce it to the case of an integral of a positive function. Therefore, we study:

$$\int_a^{\star b} |f(x)|\, dx$$

and we give the following definition:

Functions of Any Sign

Definition

If the integral

$$\int_a^{\star b} |f(x)|\, dx$$

is convergent, we say that the integral $\int_a^{\star b} f(x)\, dx$ is absolutely convergent.

Functions of Any Sign

Example

The integral

$$\int_1^\infty \frac{\sin x}{x^2}\, dx$$

is absolutely convergent. Indeed,

$$\left|\frac{\sin x}{x^2}\right| \leq \frac{1}{x^2},$$

and the Riemann integral $\int_1^\infty \frac{1}{x^2}\, dx$ is convergent.

Functions of Any Sign

Theorem

If the integral $\int_a^{\star b} f(x)\,dx$ is absolutely convergent, then it is convergent. We write

$$\int_a^{\star b} |f(x)|\,dx \text{ is convergent } \Rightarrow \int_a^{\star b} f(x)\,dx \text{ is convergent.}$$

Functions of Any Sign

The following theorem provides sufficient conditions for the convergence of the integral of a function of any sign.

Theorem

Let $f, g \in \mathcal{R}_{loc}([a, b[)$ satisfying:
- The function f is monotonic on $]a, b[$, and $\lim_{x \to b} f(x) = 0$.
- There exists $M > 0$ such that $\int_a^t g(x)\, dx \leq M$ for all $t \in]a, b[$. Then, the integral

$$\int_a^b f(x)g(x)\, dx$$

is convergent.

Example

Study the integral

$$I = \int_1^{\infty} \frac{\cos(3x)}{\sqrt{x}}\, dx.$$

1) Let's show that this integral is convergent. We have

$$\int_1^{t} \cos(3x)\, dx \leq 2.$$

Example

The function $x \mapsto \sqrt{\frac{1}{x}}$ is decreasing on $[1, +\infty[$ and $\lim_{x \to +\infty} \frac{1}{\sqrt{x}} = 0$.
Therefore, according to the Abel criterion, this integral is convergent.
2) Let's show that it is not absolutely convergent. We have,

$$|\cos(3x)| \geq \cos^2(3x) = \frac{1+\cos(6x)}{2},$$

so,

$$\frac{|\cos(3x)|}{\sqrt{x}} \geq \frac{1}{2\sqrt{x}} + \frac{\cos(6x)}{2\sqrt{x}}.$$

On one hand, the Riemann integral

$$\int_1^\infty \frac{1}{2\sqrt{x}}\, dx$$

is divergent, and on the other hand, the integral

Example

$$\int_1^\infty \frac{\cos(6x)}{2\sqrt{x}}\,dx$$

is convergent by Abel's criterion. Therefore, the integral

$$\int_1^\infty \left|\frac{\cos(3x)}{\sqrt{x}}\right| dx$$

is divergent.
This integral is an example of semi-convergent integrals.

Integrals with Two Singularities at the Boundaries

The integral with two singularities at the boundaries of the interval $]a, b[$ is denoted as

$$\int_{\star a}^{\star b} f(x)\, dx.$$

Example

a)

$$\int_{-\infty}^{2} \frac{1}{x-2}\, dx;$$

b)

$$\int_{-1}^{3} \frac{1}{\sqrt{(x+1)(3-x)}}\, dx;$$

c)

$$\int_{-\infty}^{\infty} x^2\, dx.$$

Integrals with Two Singularities at the Boundaries

Each of these three integrals has two singularities at the boundaries. For example, in case (a), the interval $]-\infty, 2[$ is unbounded, so the integral has a singularity at $v(-\infty)$, and the function is unbounded at $v(2)$, so the integral has a singularity at $v(2)$.

Integrals with Two Singularities at the Boundaries

Definition

Let $f \in R_{\text{loc}}(]a, b[)$. If there exists $c_0 \in]a, b[$ such that both integrals

$$\int_{\star a}^{c_0} f(x)\, dx \quad \text{and} \quad \int_{c_0}^{\star b} f(x)\, dx$$

are convergent, then the integral

$$\int_{\star a}^{\star b} f(x)\, dx$$

is convergent.

**Remark **

In fact, the specific choice of c_0 is irrelevant and can be replaced by any other number $c \in]a, b[$, as confirmed by the following proposition.

Integrals with Two Singularities at the Boundaries

Theorem

Let $f \in R_{loc}(]a,b[)$. The integral $\int_{\star a}^{\star b} f(x)\,dx$ is convergent if, and only if, for every $c \in]a,b[$, each of the integrals

$$\int_{\star a}^{c} f(x)\,dx \quad \text{and} \quad \int_{c}^{\star b} f(x)\,dx$$

is convergent.

Now, let's examine the integrals given as examples.

Integrals with Two Singularities at the Boundaries

Example

a)

$$\int_{-\infty}^{2} \frac{1}{x-2}\, dx.$$

We know from the previous discussion that the integral

$$\int_{-\infty}^{0} \frac{1}{x-2}\, dx$$

is divergent. Therefore, it is sufficient to conclude that the integral

$$\int_{-\infty}^{2} \frac{1}{x-2}\, dx$$

is also divergent.

Integrals with Two Singularities at the Boundaries

Example

b)

$$\int_{-1}^{3} \frac{1}{\sqrt{(x+1)(3-x)}}\, dx.$$

Both integrals (verify)

$$\int_{-1}^{0} \frac{1}{\sqrt{(x+1)(3-x)}}\, dx$$

and

$$\int_{0}^{3} \frac{1}{\sqrt{(x+1)(3-x)}}\, dx$$

are convergent.

Singularities within the Interval

Definition

Let [a, b] be a closed and bounded interval, and $c_0 \in]a, b[$. We say that a function f is locally integrable on [a, b] $\backslash(c_0$ if it is integrable on every closed and bounded interval of [a, c_0[and on every closed and bounded interval of $]c_0, b]$. We denote this as: $f \in R_{\text{loc}}([a, b] \setminus c_0)$.

Definition

If the function f is not bounded at a point c_0, we say that the integral

$$\int_a^b f(x)\,dx$$

has a singularity at c_0.

Definition

Let $f \in R_{\text{loc}}([a, b] \setminus c_0)$. If each of the integrals

$$\int_a^{\star c_0} f(x)\, dx \quad \text{and} \quad \int_{\star c_0}^{b} f(x)\, dx$$

is convergent, we say that the integral

$$\int_a^b f(x)\, dx$$

is convergent.

Singularities within the Interval

Example

Consider the integral

$$\int_{-1}^{4} \frac{1}{x-1}\, dx.$$

This integral has a singularity at $c_0 = 1$. Additionally, we have the decomposition

$$\int_{-1}^{4} \frac{1}{x-1}\, dx = \int_{-1}^{1} \frac{1}{x-1}\, dx + \int_{1}^{4} \frac{1}{x-1}\, dx.$$

Now, analyzing the integral

$$\int_{-1}^{1} \frac{1}{x-1}\, dx,$$

we find that it is a divergent improper Riemann integral. As a result, the entire original integral is divergent.

Singularities within the Interval

Example

Study of the integral $\int_{-2}^{3} f(x)\,dx$ where the function f is given by:

$$f(x) = \begin{cases} \frac{1}{\sqrt[3]{x}} & \text{if } x < 0 \\ \frac{1}{x} & \text{if } x > 0 \end{cases}$$

The integral $\int_{-2}^{\star 0} \frac{dx}{\sqrt[3]{x}}$ is convergent, and the integral $\int_{\star 0}^{3} \frac{dx}{x}$ is divergent. Therefore, the integral $\int_{-2}^{3} f(x)\,dx$ is divergent.

Definition

Let $f \in R_{\mathrm{loc}}([a,b] \setminus c_0)$. The following equivalence holds:

$$\int_b^a f(x)\,dx \text{ is convergent } \Leftrightarrow \lim_{\varepsilon \to 0^+} \int_a^{c_0-\varepsilon} f(x)\,dx + \lim_{\delta \to 0^+} \int_{c_0+\delta}^{b} f(x)\,dx \text{ exists}$$

Singularities within the Interval

Example

Study of the integral $\int_{-2}^{3} f(x)\,dx$ where the function f is given by:

$$f(x) = \begin{cases} \frac{1}{\sqrt[3]{x}} & \text{if } x < 0 \\ \frac{1}{\sqrt{x}} & \text{if } x > 0 \end{cases}$$

We have:

$$\lim_{\varepsilon \to 0^+} \int_{-2}^{-\varepsilon} \frac{dx}{\sqrt[3]{x}} + \lim_{\delta \to 0^+} \int_{\delta}^{3} \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \to 0} \frac{3}{2}\left(\varepsilon^{\frac{2}{3}} - 2^{\frac{2}{3}}\right) + \lim_{\delta \to 0} 2\left(\sqrt{3} - \sqrt{\delta}\right)$$

$= 2\sqrt{3} - \frac{3}{2}2^{\frac{2}{3}}$.Therefore, this integral is convergent.

**Remark ** In the limit of the previous definition, the numbers ε and δ are independent. In the case where $\delta = \varepsilon$, we have the following definition.

Cauchy Principal Value

Definition

(Cauchy Principal Value):** Let $f \in R_{loc}([a, b] \backslash c_0)$. If

$$\lim_{\epsilon \to 0^+} \left(\int_a^{c_0 - \epsilon} f(x)\, dx + \int_{c_0 + \epsilon}^{b} f(x)\, dx \right) = I$$

exists and is finite, then the integral

$$\int_a^b f(x)\, dx$$

has a Cauchy Principal Value, denoted as

$$I = \text{v.p.} \int_a^b f(x)\, dx.$$

In this case, it is said that the integral converges in the sense of the Cauchy Principal Value.

Cauchy Principal Value

Example

Calculating the Cauchy Principal Value

$$\int_{-3}^{4} \frac{1}{x}\, dx.$$

This integral has a singularity at zero. Evaluating the limit

$$\lim_{\epsilon \to 0} \left(\int_{-3}^{-\epsilon} \frac{1}{x}\, dx + \int_{\epsilon}^{4} \frac{1}{x}\, dx \right) = \lim_{\epsilon \to 0} \left(\ln \epsilon - \ln 3 + \ln 4 - \ln \epsilon \right) = \ln \frac{4}{3},$$

we find that the integral has a Cauchy Principal Value given by

$$\text{v.p.} \int_{-3}^{4} \frac{1}{x}\, dx = \ln \frac{4}{3}.$$

Note that although this integral is divergent, it possesses a Cauchy Principal Value.