CHAPTER 4

Ordinary Differential Equations

Mathematical Analysis 1 , ENSIA 2024

Ordinary Differential Equation

1. **Definition and generalities**
2. **First-order equations**
3. **Differential equations with separated variables**
4. **Homogeneous Differential Equations:**
5. **Equations reducing to homogeneous equations**
6. **Differential Equations: Linear First-Order Equations.**
7. **Differential Equations: Second-Order Linear Equations**

Definition

ordinary differential equation of nth order, defined on an interval I of $\mathbb{R}$, any relation between the variable $x \in I$, an unknown function $y(x)$, and its successive derivatives $y^{(k)}(x)$, $k \in \{1, \ldots, n\}$, of the form

$$F(x, y(x), \ldots, y^{(n-1)}(x), y^{(n)}(x)) = 0$$

where $x \in I$, and F is a function with $n + 2$ variables.
In the simplest case, it allows expressing, for every x in I, the value of $y^{(n)}(x)$ in terms of $y(x)$, …, $y^{(n-1)}(x)$, and the variable x, in the so-called normalized form:

$$y^{(n)} = f(x, y, \ldots, y^{(n-1)}) \quad \text{for every} \quad x \in I \quad (4.1)$$

where f is a function of $n + 1$ variables.

Ordinary differential equation

Example

The equation $y' = -2xy$ over is a first-order differential equation because it only involves the first derivative.
The equation $y = \frac{1}{2\sqrt{x-1}}y'' - 5x$ over $]1, +\infty[$ is a second-order differential equation.

Ordinary differential equation

Definition

A solution (or integral) of the differential equation (4.1) is any function $\phi \in C^n(I;\mathbb{R})$, (a function $\phi : I \to \mathbb{R}$, n times continuously differentiable) such that, for every $x \in I$, we have $\phi^{(n)}(x) = f(x, \phi(x), \dots, \phi^{(n-1)}(x))$.

Example

1) It is easy to verify that the function $\phi(x) = e^{2x}$ is a solution of the differential equation $y' = 2y$, $x \in \mathbb{R}$. Furthermore, we notice that any function of the form $\phi_c(x) = ce^{2x}$, where c is any real constant, is a solution to this equation.
2) Functions of the form $y(x) = a\cos x + b\sin x$, with $a, b \in \mathbb{R}$, are solutions to the second-order differential equation $y'' + y = 0$.

Remark Here, we are only interested in first-order equations and particular second-order equations called linear.

First-order equations

Definition

A first-order differential equation is an equation of the form

$$F(x, y(x), y'(x)) = 0. \quad (4.2)$$

Definition

The general solution of a first-order differential equation is a function denoted as

$$y(x) = \phi(x, c)$$

dependent on an arbitrary constant c, of class C^1, and satisfies equation (4.2). In the following, we study some types of first-order equations.

Differential equations with separated variables

Definition

A differential equation is said to have separated variables if it can be written in the form:

$$u(y)dy = v(x)dx, \quad (4.2.1)$$

where u and v are two continuous functions, and where $\frac{dy}{dx}$ denotes y'.

Example

The equation defined on $I =]1, +\infty[$ by

$$xy' \ln x = (3 \ln x + 1)y$$

is an equation with separated variables because we can "separate the variables" x and y by dividing by $yx \ln x$, which is possible if, and only if, $y \neq 0$. Thus, we obtain: For $(x, y) \in]1, +\infty[\times \mathbb{R}^*$

Example

$$xy' \ln x = (3 \ln x + 1)y \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3 \ln x + 1}{x \ln x} \Rightarrow \frac{1}{y}dy = \frac{3 \ln x + 1}{x \ln x}dx$$

Solution method: To solve a differential equation with separated variables, it suffices to integrate both sides of equation (4.2.1) separately.

$$u(y)dy = v(x)dx \Rightarrow \int u(y)dy = \int v(x)dx \Rightarrow U(y) = V(x) + c$$

where c is a real constant, and if possible, express y as a function of x.

Example

1) Solve on $I =]1, +\infty[$ the differential equation from the previous example:

$$xy' \ln x = (3\ln x + 1)y.$$

After separating the variables, we integrate both sides, yielding

$$\int \frac{1}{y} dy = \int \frac{3\ln x + 1}{x \ln x} dx \Rightarrow \ln|y| + c_1 = 3\ln|x| + \ln|\ln x| + c_2$$

$$\Rightarrow \ln|y| = 3\ln x + \ln(\ln x) + c_3 \Rightarrow \ln|y| = \ln(x^3 \ln x) + c_3$$

Example

By exponentiating the last equality, we obtain the final solution as:

$$y = Cx^3 \ln x \quad \text{with} \quad C \in \mathbb{R}^*,$$

where $C = \pm e^{c_3}$ takes into account both possibilities for $|y|$. Furthermore, since the identically zero function $y = 0$ is a solution of equation (9.2.3), a solution that can be obtained by considering C in $\mathbb{R}$. Thus, the general solution of equation on I is given by:

$$y = Cx^3 \ln x \quad \text{with} \quad C \in \mathbb{R}.$$

Example

2) Solve the equation

$$y - \frac{y'}{2x} = 1 \quad \text{over} \quad \mathbb{R}^*.$$

We have:

$$y - \frac{dy}{2xdx} = 1, \quad x \in \mathbb{R}^* \Rightarrow y - \frac{dy}{2xdx} = 1 \Rightarrow \frac{dy}{y-1} = 2xdx \quad \text{with} \quad y \neq 1.$$

Differential equations with separated variables

Example

By integration, we obtain:

$$\int \frac{1}{y-1} dy = \int 2x dx \Rightarrow \ln|y-1| = x^2 + K \quad \text{with} \quad K \in \mathbb{R}$$

$\Rightarrow |y-1| = e^{x^2} e^K \Rightarrow y - 1 = ce^{x^2} \quad \text{with} \quad c = \pm e^K \in \mathbb{R}^*$.As the constant function $y = 1$ is also a solution of equation , the general solution of this equation is given by:

$$y(x) = ce^{x^2} + 1 \quad \text{with} \quad c \in \mathbb{R}.$$

Definition

A function $f(x, y)$ is said to be homogeneous of degree n with respect to the variables x and y if for all $\lambda \in \mathbb{R}$,

$$f(\lambda x, \lambda y) = \lambda^n f(x, y).$$

1. The function $f(x, y) = \sqrt[3]{x^3 + y^3}$ is homogeneous of degree 1 because

$$f(\lambda x, \lambda y) = \sqrt[3]{\lambda^3 x^3 + \lambda^3 y^3} = \lambda f(x, y).$$

2. The function $f(x, y) = x^2 - y^2/xy$ is homogeneous of degree 0 because

$$\frac{(\lambda x)^2 - (\lambda y)^2}{\lambda x \lambda y} = \frac{x^2 - y^2}{xy}.$$

Homogeneous Differential Equations:

Definition

A first-order differential equation

$$y' = f(x, y) \quad (4.3)$$

is called homogeneous if the function $f(x, y)$ is homogeneous of degree zero.

Resolution Method: By hypothesis, $f(\lambda x, \lambda y) = f(x, y)$. Setting $\lambda = 1/x$ in this identity, we obtain:

$$f(x, y) = f\left(1, \frac{y}{x}\right)$$

This implies that a homogeneous function of degree zero depends only on the ratio y/x.

Homogeneous Differential Equations

To solve a homogeneous differential equation, we use the change of unknown function:

$$z(x) = \frac{y(x)}{x}$$

which gives

$$y' = xz' + z.$$

Replacing this into equation (4.3), we obtain the following equation:

$$xz' + z = f\left(1, z\right)$$

which is a separable variables equation. Indeed, this equation can be written in the form

$$\frac{dz}{f\left(1, z\right) - z} = \frac{dx}{x}.$$

We solve this equation using the method for solving separable variables equations. Then, we return to the function $y(x)$.

Example

Consider the following differential equation:

$$y' = \frac{x^2 - y^2}{xy}.$$

We have seen previously that the function

$$f(x, y) = \frac{x^2 - y^2}{xy}$$

is homogeneous of degree zero, thus this equation is homogeneous. By setting $\lambda = 1/x$, the equation becomes:

$$y' = \frac{1 - \frac{y^2}{x^2}}{\frac{y}{x}}$$

Example

We then set

$$z = \frac{y}{x} \Rightarrow y' = xz' + z,$$

and after calculation, the equation becomes

$$xz' + z = \frac{1 - z^2}{z}$$

This equation is separable. Indeed, it can be written as:

$$\frac{zdz}{1 - 2z^2} = \frac{dx}{x}.$$

Equations reducing to homogeneous equations

Equations of the form:

$$y' = \frac{ax + by + c}{a_1 x + b_1 y + c_1} \quad (4.2.3)$$

where (c, c_1) are two non-zero real constants.
Remark: Note that if $c = c_1 = 0$, the equation is homogeneous, as the function

$$f(x, y) = \frac{ax + by}{a_1 x + b_1 y}$$

is indeed homogeneous of degree zero.
Let us look for a change of variable and function of the type

$$x = t + h \quad \text{and} \quad y = z + k, \quad (h, k) \in \mathbb{R}^2$$

Equations reducing to homogeneous equations

so that the function $g(t,z) = f(x,y)$ becomes homogeneous of degree zero. The first term of equation (4.2.3) is written as:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dt} \cdot \frac{dt}{dx} = \frac{dz}{dt},$$

and the equation then becomes:

$$\frac{dz}{dt} = \frac{at + bz + ah + bk + c}{a_1 t + b_1 z + a_1 h + b_1 k + c_1}.$$

This equation is homogeneous if, and only if, h and k satisfy the system

$$\begin{cases} ah + bk + c = 0 \\ a_1 h + b_1 k + c_1 = 0. \end{cases} \qquad (4.2.4)$$

This system has a unique solution if, and only if, its determinant

$$\Delta = \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = ab_1 - ba_1$$

is non-zero.

Equations reducing to homogeneous equations

First case: If $\Delta \neq 0$. In this case, the system has a unique solution given by

$$h = \frac{1}{\Delta} \begin{vmatrix} -c & b \\ -c_1 & b_1 \end{vmatrix} = \frac{bc_1 - cb_1}{\Delta}$$

and

$$k = \frac{1}{\Delta} \begin{vmatrix} a & -c \\ a_1 & -c_1 \end{vmatrix} = \frac{ca_1 - ac_1}{\Delta}.$$

Then, using this change of variable, equation (4.2.3) becomes

$$z' = \frac{at + bz}{a_1 t + b_1 z}, \quad (4.2.5)$$

which is a homogeneous equation, as the function

$$g(t, z) = \frac{at + bz}{a_1 t + b_1 z}$$

is homogeneous of degree zero. Using the method for solving homogeneous equations, we determine the solution $z(t)$ of equation (4.2.5), and then we return to the original variables.

Equations reducing to homogeneous equations

Example

Solve the differential equation:

$$y' = \frac{x+y-3}{x-y-1}$$

We have,

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

so the system

$$\begin{cases} h+k-3=0 \\ h-k-1=0 \end{cases}$$

admits a unique solution given by

$$h = -\frac{1}{2}\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = 2 \quad \text{and} \quad k = -\frac{1}{2}\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1.$$

Example

Then, by setting

$$x = t + 2 \quad \text{and} \quad y = z + 1,$$

equation becomes

$$z' = \frac{t + z}{t - z}$$

which is a homogeneous equation. We solve this equation and determine the solution $y(x)$ of equation.

Equations reducing to homogeneous equations

Second case: If $\Delta = 0$. In this case, the system (4.2.4) has no solution and thus the method used previously does not work. We will approach it differently.

1. If $a \neq 0$ and $b \neq 0$ then

$$ab_1 - ba_1 = 0 \implies \exists \lambda \in \mathbb{R}; \quad \frac{a_1}{a} = \frac{b_1}{b} = \lambda.$$

Then the equation (4.2.3) becomes:

$$y' = \frac{ax + by + c}{\lambda(ax + by) + c_1}.$$

By setting

$$z = ax + by \quad \Rightarrow \quad y' = \frac{z' - a}{b}$$

Equations reducing to homogeneous equations

This equation can be written as:

$$\frac{dz}{dx} = b\frac{z + c_1}{\lambda z + c_1} + a$$

It's a separated variables equation because it can be written as:

$$\frac{\lambda z + c_1}{(b + \lambda a)z + bc + ac_1} dz = dx$$

Equations reducing to homogeneous equations

2. If $a = 0 \;\; a_1 b = 0$. (i) if $a_1 = 0$, then equation (4.2.3) is written in the form:

$$y' = \frac{by + c}{b_1 y + c_1}$$

which is a separable equation, as it can be written as:

$$\frac{b_1 y + c_1}{by + c}\, dy = dx.$$

(ii) if $b = 0$, then equation (4.2.3) is in the form:

$$y' = \frac{c}{a_1 x + b_1 y + c_1}.$$

We then set

$$z = a_1 x + b_1 y \quad \Rightarrow \quad y' = \frac{z' - a_1}{b_1},$$

obtaining the following separable equation:

$$z' = b\frac{c}{z + c_1} + a_1.$$

Equations reducing to homogeneous equations

Note that in the case where $\Delta = 0$, we directly move to separable equations.

Example

2. Integrate the following differential equation:

$$y' = \frac{x - 2y + 1}{-2x + 4y - 3}$$

We have

$$\Delta = \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 0$$

In this case, it is simple to notice that $\lambda = -2$. Equation can then be written as:

$$y' = \frac{x - 2y + 1}{-2(x - 2y) - 3}.$$

We set $z = x - 2y$, yielding $y' = \frac{1 - z'}{2}$.

Example

We then obtain the equation:

$$z' = \frac{-4z - 5}{-2z - 3},$$

which is a separable equation, as it can be written as:

$$\frac{2z + 3}{4z + 5}\, dz = dx.$$

Example

2. Solve the equation

$$y' = \frac{y+1}{-3y+2}$$

We are in the case where $a = a1 = 0$. Note that this equation is separable.

3. Solve the differential equation

$$y' = \frac{3x-2}{5}$$

This equation is separable. Indeed, we have:

$$dy = \frac{3x-2}{5}\,dx.$$

In this example, $a1 = b1 = 0$.

Linear First-Order Equations

Definition

A linear first-order differential equation is an equation that can be written in the form:

$$a(x)y' + b(x)y = f(x) \quad (4.4)$$

where the functions $x \mapsto a(x), b(x)$, and $f(x)$ are continuous over the same interval I, on which the function $x \mapsto a(x)$ does not become zero.

The function $f(x)$ is called the right-hand side of the equation, and this equation, Equation (4.4) is called an equation with a right-hand side or a complete equation.

Definition

If $f(x) = 0$, the equation

$$a(x)y' + b(x)y = 0 \quad (4.5)$$

is called an equation without a right-hand side or homogeneous.

Solving the homogeneous equation. In fact, the homogeneous equation (4.5) is a separable equation. Indeed, it can be written in the form

$$\frac{dy}{y} = -\frac{b(x)}{a(x)}dx.$$

By integrating, we get

$$\ln|y| = -\int \frac{b(x)}{a(x)}dx + c_1, \quad c_1 \in \mathbb{R}.$$

Taking the exponential of both sides, we have

$$y(x) = C\exp\left(-\int \frac{b(x)}{a(x)}dx\right),$$

where $C = \pm\exp(c_1) \in \mathbb{R}^*$.

Since $y = 0$ is a solution to this equation, the general solution of the homogeneous equation is

$$y_h(x) = C \exp\left(-\int \frac{b(x)}{a(x)} dx\right), \quad C \in \mathbb{R}. \quad (4.6)$$

Example

Consider the equation:

$$(x+1)y' + 3xy = 0; \quad x \in I,$$

with $I =]-\infty, -1[$ or $I =]-1, +\infty[$. For $y \neq 0$, the equation can be written as:

$$\frac{dy}{y} = -\frac{3x}{x+1} dx$$

The general solution to this equation is

$$y_h(x) = C \exp\left(\int \left(-3 + \frac{3}{x+1}\right) dx\right) = C \exp(-3x)|x+1|^3, \quad C \in \mathbb{R}.$$

Solving the complete equation. Let's suppose we know a particular solution y_p of the complete equation (4.4). We have

$$a(x)y_p' + b(x)y_p = f(x) \quad (4.6.)$$

From equations (4.4) and (4.6), we deduce that the function $z = y - y_p$ is a solution to the homogeneous equation (4.5). Since $y = z + y_p$, we have the following result:

Theorem

The general solution of the complete equation (4.4) is the sum of the general solution of the associated homogeneous equation and a particular solution of the complete equation.

Finding a particular solution of the complete equation.

Lagrange's method, or variation of constants.

This method consists of finding a particular solution of the complete equation, starting from the general solution of the associated homogeneous equation, by varying the constant "C". That's why it's called this name. So, let

$$y_h(x) = C \exp\left(-\int \frac{b(x)}{a(x)} dx\right)$$

be the general solution of the associated homogeneous equation to (4.4). We seek a particular solution of the complete equation in the form

$$y_p(x) = C(x) \exp\left(-\int \frac{b(x)}{a(x)} dx\right),$$

where the unknown is the function $C(x)$.

We have

$$y_p'(x) = C'(x)\exp\left(-\int \frac{b(x)}{a(x)}dx\right) - C(x)\frac{b(x)}{a(x)}\exp\left(-\int \frac{b(x)}{a(x)}dx\right).$$

Substituting into (4.4), and since

$$C(x)\frac{b(x)}{a(x)}\exp\left(-\int \frac{b(x)}{a(x)}dx\right) - C(x)\frac{b(x)}{a(x)}\exp\left(-\int \frac{b(x)}{a(x)}dx\right) = 0,$$

we obtain

$$C'(x)a(x)\exp\left(-\int \frac{b(x)}{a(x)}dx\right) = f(x),$$

which gives

$$C(x) = \int \frac{f(x)}{a(x)}\exp\left(\int \frac{b(x)}{a(x)}dx\right)dx.$$

Example

Solve the following differential equation

$$y' + \frac{x}{1+x^2}y = \frac{1}{1+x^2}.$$

The associated homogeneous equation to above equation is

$$y' + \frac{x}{1+x^2}y = 0.$$

The general solution to this equation is

$$y_h(x) = C\exp\left(-\int \frac{x}{1+x^2}dx\right) = \frac{C}{\sqrt{1+x^2}}$$

Example

Let's find a particular solution of the complete equation in the form

$$y_p(x) = \frac{C(x)}{\sqrt{1+x^2}}$$

We have

$$y_p'(x) = \frac{C'(x)}{\sqrt{1+x^2}} - C(x)x(1+x^2)^{-\frac{3}{2}}.$$

Substituting into the complete equation, we get

$$C'(x) = \frac{1}{\sqrt{1+x^2}} \Rightarrow C(x) = \text{Argsh}\ x.$$

Example

A particular solution of the complete equation is

$$y_p(x) = \frac{\operatorname{Argsh} x}{\sqrt{1+x^2}}.$$

The general solution to the complete equation is given by

$$y(x) = y_h(x) + y_p(x) = \frac{C}{\sqrt{1+x^2}} + \frac{\operatorname{Argsh} x}{\sqrt{1+x^2}}.$$

Equation of Bernoulli

Definition

A Bernoulli equation is an equation of the form

$$a(x)y' + b(x)y = y^n f(x), \quad (4.7)$$

where the functions $x \mapsto a(x)$, $x \mapsto b(x)$, and $x \mapsto f(x)$ satisfy the assumptions made for linear equations.

Remark: 1. For $n = 0$ or $n = 1$, this equation reduces to a complete linear equation or a homogeneous linear equation, respectively. 2. In general, $n \in \mathbb{N}$. But we can take $n = \alpha \in \mathbb{R}$, considering only positive solutions.

Solving the Bernoulli equation

By dividing both sides of this equation by y^n, we obtain:

$$a(x)y'y^{-n} + b(x)y^{1-n} = f(x).$$

Linear First-Order Equations

We then set

$$z = y^{1-n} \Rightarrow z' = (1-n)\frac{y'}{y^n},$$

the equation becomes

$$a(x)z' + (1-n)b(x)z = (1-n)f(x),$$

which is a first-order linear equation with a non-constant coefficient. We solve this equation and then revert to the unknown $y(x)$.

Example

Solve the equation

$$y' + xy = x^3y^3.$$

Dividing both sides by y^3, we obtain:

$$y^{-3}y' + xy^{-2} = x^3.$$

Example

Now let's set

$$z = y^{-2} \Rightarrow z' = -2y^{-3}.$$

Example

By replacing z with its value, we obtain

$$y^2(x) = \frac{1}{x^2 + 1 + Ce^{x^2}}, \quad C \in \mathbb{R}.$$

The set of solutions to the equation (9.2.25) is given by

$$S = \left\{ y^2(x) = \frac{1}{Ce^{x^2} + x^2 + 1}, \text{ where } C \in \mathbb{R}, y = 0. \right\}$$

Equation of Riccati

Definition

A Riccati equation is an equation of the form:

$$a(x)y' + b(x)y = g(x) + y^2 f(x), \quad (4.8)$$

where the functions $x \mapsto a(x)$, $x \mapsto b(x)$, $x \mapsto f(x)$, and $x \mapsto g(x)$ satisfy the assumptions made for linear equations.

Equation of Riccati

Remark

1. If the function g is zero, this equation becomes a particular case of the Bernoulli equation (with $n = 2$).
2. If we don't know a particular solution y_1, we cannot solve this equation. Moreover, unlike linear equations, we cannot search for a particular solution except in very particular cases.
3. However, if we know a particular solution, we can find all the solutions to this equation (the general solution).

Equation of Riccati

Solving the Riccati equation Suppose we know a particular solution y_1 of this equation. We seek the general solution in the form

$$y(x) = u(x) + y_1(x).$$

Since y_1 is a particular solution of equation (4.8), it satisfies

$$a(x)y_1' + b(x)y_1 = g(x) + y_1^2 f(x).$$

By replacing $y(x)$ with its value in equation (4.8), we deduce that the function u satisfies the following Bernoulli equation:

$$a(x)u' + [b(x) - 2y_1 f(x)]u = u^2 f(x).$$

By solving the preceding Bernoulli equation, we obtain the general solution of the Riccati equation.

Equation of Riccati

Example

Solve the equation

$$y' + 3y = -y^2 - 2,$$

where $y_1 = -1$ is a particular solution. Then let's search for the solution in the form

$$y = u - 1.$$

By replacing it into equation, we obtain the following Bernoulli equation satisfied by the function u:

$$u' + u = -u^2.$$

Equation of Riccati

Example

The general solution to this equation is

$$u(x) = \frac{1}{ce^x - 1}.$$

The general solution to equation is given by

$$y(x) = \frac{1}{ce^x - 1} - 1, \quad c \in \mathbb{R}, \quad ce^x - 1 \neq 0.$$

Definition

A second-order linear equation defined on an interval I of $\mathbb{R}$ is an equation of the form

$$a(x)y'' + b(x)y' + c(x)y = f(x), \quad (4.9)$$

where $x \in I$, and the functions $x \mapsto a(x)$, $x \mapsto b(x)$, $x \mapsto c(x)$, and $x \mapsto f(x)$ are continuous functions on I, and the function $x \mapsto a(x)$ does not vanish on I.

- The functions $x \mapsto a(x)$, $x \mapsto b(x)$, and $x \mapsto c(x)$ are called the coefficients of the equation. - The function $x \mapsto f(x)$ is called the right-hand side of the equation.

Solving these Equations

Remark Like all linear equations, if y_p is a particular solution of equation (4.9), then the function

$$z = y - y_p$$

is a solution of the associated homogeneous equation:

$$a(x)y'' + b(x)y' + c(x)y = 0, \quad \text{where} \quad x \in I$$

As with first-order linear equations, we have the following result:

Theorem

The general solution of the complete equation is the sum of the general solution of the associated homogeneous equation and a particular solution of the complete equation.

Solving the Homogeneous Equation

Definition

Two solutions y_1 and y_2 are said to be independent if they are non-zero and satisfy:

$$\frac{y_1(x)}{y_2(x)} \neq \text{Const.}$$

We have the theorem:

Theorem

If y_1 and y_2 are two independent solutions of the homogeneous equation, then the general solution of this equation is of the form

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

with c_1 and c_2 being two arbitrary constants.

For the resolution of the homogeneous equation, two distinct cases are observed:

- Case where the coefficients of the equation are constants.
- Case where the coefficients of the equation are functions of x not all constants.

Second-Order Linear Differential Equation with Constant Coefficients

They are in the form:

$$ay'' + by' + cy = 0. \quad (4.10)$$

Where a, b, and c are real constants.

The characteristic equation of equation (4.10) is:

$$ar^2 + br + c = 0 \quad (4.11)$$

Which has the discriminant:

$$\Delta = b^2 - 4ac.$$

The following theorem gives the general solution of equation (4.10).

Theorem

1. If r_1 and r_2 are two distinct real roots of the characteristic equation (4.11), the general solution of equation (4.10) is given by:

$$y_h(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad c_1, c_2 \in \mathbb{R}.$$

2. If r is a double root of equation (4.11), the general solution of equation (4.10) is in the form:

$$y_h(x) = c_1 e^{rx} + c_2 x e^{rx}, \quad c_1, c_2 \in \mathbb{R}.$$

Theorem

3. If r_1 and r_2 are two complex conjugate roots of equation (4.11) ($r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$), The general solution of equation (4.10) is given by:

$$y_h(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x), \quad c_1, c_2 \in \mathbb{R}.$$

Example

Solve the following differential equations:

1. $y'' - 3y' + 2y = 0$
2. $y'' + 2y' + y = 0$
3. $y'' + 2y' + 5y = 0$
4. $y'' + y = 0$

Second-Order Linear Equations

Example

1. $y'' - 3y' + 2y = 0$, Its characteristic equation is: $r^2 - 3r + 2 = 0$ which has two distinct roots $r_1 = 1$ and $r_2 = 2$. Therefore, the homogeneous equation has two particular and independent solutions $y_1(x) = e^x$ and $y_2(x) = e^{2x}$. The general solution is thus given by

$$y(x) = c_1 e^x + c_2 e^{2x}; \quad c_1, c_2 \in \mathbb{R}.$$

Example

2. $y'' + 2y' + y = 0$, Its characteristic equation has a double root $r = -1$, and the general solution is given by

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}; \quad c_1, c_2 \in \mathbb{R}.$$

Example

3. $y'' + 2y' + 5y = 0$, Its characteristic equation has two complex conjugate roots: $r_1 = -1 + 2i$ and $r_2 = -1 - 2i$, and the general solution is given by

$$y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x), \quad c_1, c_2 \in \mathbb{R}.$$

Example

4. $y'' + y = 0$, the characteristic equation has two complex and conjugate roots: $r_1 = i$ and $r_2 = -i$. So the general solution of the equation is

$$y(x) = c_1 \cos x + c_2 \sin x; \quad c_1, c_2 \in \mathbb{R}.$$

Finding a particular solution of the complete equation

The search for a particular solution of the equation with constant coefficients, with the following forcing term:

$$ay'' + by' + cy = f(x). \quad (4.12)$$

is generally done as in the case of a first-order equation, using the method of variation of parameters (Lagrange's method), a method that essentially relies on the following assumption: Let y_1 and y_2 be two independent solutions of the homogeneous equation associated with equation (4.12), and

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

be the general solution of the homogeneous equation. We seek a particular solution in the form

$$y_p(x) = c_1(x) y_1(x) + c_2(x) y_2(x) \quad (4.13)$$

where the unknown functions $x \mapsto c_1(x)$ and $x \mapsto c_2(x)$ are differentiable functions on the interval of resolution. Substituting into equation (4.12) leads to a two-variable equation, which is not convenient. We then make the following assumption called Lagrange's assumption:

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0.$$

Thus, we have

$$y_p'(x) = c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0 \quad \text{(Lagrange's assumption)}$$

$$+c_1(x)y_1'(x) + c_2(x)y_2'(x)$$

$$y_p''(x) = c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + c_1(x)y_1''(x) + c_2(x)y_2''(x)$$

Substituting y_p into (4.12) yields:

$$a(c_1'(x)y_1'(x) + c_2'(x)y_2'(x)) + c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) = f$$

Since y_1 and y_2 are solutions of the homogeneous equation, we obtain the following equation:

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{a}.$$

Thus, we deduce that the functions $c_1'(x)$ and $c_2'(x)$ are solutions of the system of equations:

$$\begin{cases} c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{a} \end{cases}$$

The determinant of this system is given by:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

called the Wronskian of the functions y_1 and y_2.
The system (I) has a unique solution if, and only if, the determinant $W(x)$ is non-zero. The answer to this question is given by the following theorem:

Theorem

The Wronskian $W(x)$ is zero if, and only if, the functions $y_1(x)$ and $y_2(x)$ are linearly dependent.
Since the functions y_1 and y_2 are independent, the system (I) has a unique solution which we can compute as follows:

$$c_1'(x) = \frac{0 \cdot y_2'(x) - \frac{f(x)}{a} \cdot y_2(x)}{W(x)} = \frac{-y_2(x) f(x)}{aW(x)}$$

$$c_2'(x) = -\frac{y_1'(x) \cdot 0 + y_1(x) \cdot \frac{f(x)}{a}}{W(x)} = \frac{y_1(x) f(x)}{aW(x)}$$

We integrate these functions and substitute them into (4.13), obtaining a particular solution of equation (4.12).

Example

Solve the following differential equation:

$$y'' + y = \sin x$$

The solution to the homogeneous equation is given by:

$$y_h(x) = c_1 \cos x + c_2 \sin x.$$

We search for a particular solution y_p using the method of variation of parameters. We set

$$y_p(x) = c_1(x) \cos x + c_2(x) \sin x.$$

Second-Order Linear Equations

Example

The functions $c_1'(x)$ and $c_2'(x)$ are solutions of the system

$$\begin{cases} c_1'(x)\cos x + c_2'(x)\sin x = 0 \\ c_1'(x)(-\sin x) + c_2'(x)\cos x = \sin x \end{cases}$$

The determinant of this system is

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

The solution of the system (I) is given by

$$c_1'(x) = \frac{-\sin^2 x}{1} = \frac{1}{2}(\cos 2x - 1),$$

$$c_2'(x) = \frac{\sin x \cos x}{1} = \frac{\sin 2x}{2}.$$

Second-Order Linear Equations

Example

Upon integration, we obtain

$$c_1(x) = \frac{1}{2}\int (\cos 2x - 1)\, dx = \frac{\sin 2x}{4} - \frac{x}{2},$$

$$c_2(x) = \frac{1}{2}\int \sin 2x\, dx = -\frac{\cos 2x}{4}.$$

A particular solution of equation is thus:

$$y_p(x) = \left(\frac{\sin 2x}{4} - \frac{x}{2}\right)\cos x - \frac{\cos 2x}{4}\sin x = \frac{1}{4}[\sin 2x \cos x - \cos 2x \sin x] -$$

The general solution to this equation is:

$$y(x) = c_1 \cos x + c_2 \sin x + \frac{\sin x}{4} - \frac{x}{2}, \quad c_1, c_2 \in \mathbb{R}.$$

Product of a polynomial and an exponential We have the following theorem:

Theorem

In the case where the second term of equation (4.12) takes the form $f(x) = P_n(x)e^{\alpha x}$ *where* P_n *is a polynomial of degree* n, $n \in \mathbb{N}$, *and* $\alpha \in \mathbb{R}$.

We observe three cases:

a. If α *is not a root of the characteristic equation (4.11), then a particular solution of (4.12) is given by*

$$y_p(x) = Q_n(x)e^{\alpha x}$$

where Q_n *is a polynomial of degree* n.

Second-Order Linear Equations

Theorem

b. If α is a simple real root of the characteristic equation (4.11), then a particular solution of (4.12) can be chosen in the form

$$y_p(x) = xQ_n(x)e^{\alpha x}$$

where Q_n is a polynomial of degree n.

c. If α is a double root of the characteristic equation, a particular solution of the complete equation is in the form

$$y_p(x) = x^2 Q_n(x) e^{\alpha x}$$

where Q_n is a polynomial of degree n.
In summary, we look for a particular solution in the form

$$y_p(x) = x^k Q_n(x) e^{\alpha x}$$

where k is the multiplicity order of α as a root of the characteristic equation, with the convention that $k = 0$ if α is not a root.
Remark: If $f(x) = P_n(x)e^{\alpha x}$, in many examples, we find a particular solution in the form

$$y_p(x) = Q(x)e^{\alpha x}$$

where $e^{\alpha x}$ remains the same, $Q(x)$ is a polynomial, but the degree of this polynomial differs from one example to another. For this reason, we search for a particular solution in the form $y_p(x) = Q(x)e^{\alpha x}$ and check under what condition the degree of the polynomial Q is constrained.

Product of a polynomial, an exponential, and a sine or cosine function.

Theorem

1. In the case where the second term of equation (4.12) is of the form

$$f(x) = P_n(x)e^{\alpha x}\cos\beta x = Re(P_n(x)e^{rx})$$

where $r = \alpha + i\beta \in \mathbb{C}$, then a particular solution can be sought in the following form:

$$y_p(x) = Re(Z_p(x)),$$

where $Z_p(x)$ is given as follows:

Product of a polynomial, an exponential, and a sine or cosine function.

Theorem

a. If r is not a root of the characteristic equation, $Z_p(x)$ is given in the form

$$Z_p(x) = Q_n(x)e^{rx}$$

b. If r is a root of the characteristic equation, $Z_p(x)$ is given in the form

$$Z_p(x) = xQ_n(x)e^{rx}$$

2. In the case where

$$f(x) = P_n(x)e^{\alpha x}\sin\beta x = \mathrm{Im}(P_n(x)e^{rx})$$

then a particular solution can be sought in the following form:

$$y_p(x) = \mathrm{Im}(Z_p(x)),$$

and $Z_p(x)$ is given as before. The proof of this theorem is left as an exercise.

Method of superposition. Let $ay'' + by' + cy = f(x) \quad (4.14)$ be a linear differential equation where the right-hand side is the sum of two or more functions, for example

$$f(x) = f_1(x) + f_2(x).$$

To find a particular solution of equation (4.14), we can use the following result called the method of superposition.

Theorem

If y_1 is a particular solution of equation (4.14) with respect to the right-hand side $f_1(x)$ and y_2 is a particular solution of equation (4.14) with respect to the right-hand side $f_2(x)$, then

$$y_p = y_1 + y_2$$

is a particular solution of equation (4.14) with respect to the right-hand side $f(x) = f_1(x) + f_2(x)$.

Example

1. Solve the equation

$$y'' - 2y' + y = (9x^2 - 6x + 5)e^{-2x} + (3x - 2)e^x$$

The characteristic equation $r^2 - 2r + 1 = 0$ has a double root: $r = 1$. Therefore, the general solution to the homogeneous equation is:

$$y_h(x) = (c_1 + c_2 x)e^x, \quad c_1, c_2 \in \mathbb{R}$$

Since the right-hand side is the sum of two functions, we use the method of superposition.

Example

i) We seek a particular solution of this equation with respect to the right-hand side $f_1(x) = (9x^2 - 6x + 5)e^{-2x}$. Since $f_1(x)$ is in the form $P_2(x)e^{\alpha x}$ and $\alpha = -2$ is not a root of the characteristic equation, we look for a solution y_1 in the form:

$$y_1(x) = Q(x)e^{-2x}$$

where $Q(x) = a_0 + a_1x + a_2x^2$. We have:

$$y_1' = [Q'(x) - 2Q(x)]e^{-2x}$$

$$y_1'' = [Q''(x) - 4Q'(x) + 4Q(x)]e^{-2x}$$

Substituting into the equation, we obtain:

$$Q''(x) - 6Q'(x) + 9Q(x) = (9x^2 - 6x + 5)$$

Example

Therefore, for all $x \in \mathbb{R}$:

$$9a_2x^2 + (9a_1 - 12a_2)x + (2a_2 - 6a_1 + 9a_0) = 9x^2 - 6x + 5$$

By identification, we get:

$$\begin{cases} 9a_2 = 9 \\ 9a_1 - 12a_2 = -6 \\ 2a_2 - 6a_1 + 9a_0 = 5 \end{cases} \Rightarrow \begin{cases} a_2 = 1 \\ a_1 = \frac{2}{3} \\ a_0 = \frac{7}{9} \end{cases}$$

So,

$$y_1(x) = (x^2 + \frac{2}{3}x + \frac{7}{9})e^{-2x}$$

Example

ii) We seek a particular solution with respect to the right-hand side $f_2(x) = (3x-2)e^x$. Since $f_2(x)$ is in the form $P_1(x)e^{\alpha x}$ and $\alpha = 1$ is a double root of the characteristic equation, we look for a solution y_2 in the form:

$$y_2(x) = Q(x)e^x$$

where $Q(x) = x^2(a_0 + a_1 x)$. We have:

$$y_2' = [Q'(x) + Q(x)]e^x$$

$$y_2'' = [Q''(x) + 2Q'(x) + Q(x)]e^x$$

Substituting into the equation, we obtain:

$$Q''(x) = (3x-2)$$

Therefore, for all $x \in \mathbb{R}$:

Example

$$6a_1x + 2a_0 = 3x - 2$$

We then get:

$$a_1 = \frac{1}{2}$$

and $a_0 = -1$. So,

$$y_2(x) = \frac{x^2}{2}(x-2)e^x$$

Example

A particular solution of the equation is:

$$y_p(x) = (x^2 + \frac{2}{3}x + \frac{7}{9})e^{-2x} + \frac{x^2}{2}(x-2)e^x$$

The general solution is:

$$y(x) = (c_1 + c_2 x)e^{2x} + (x^2 + \frac{2}{3}x + \frac{7}{9})e^{-2x} + \frac{x^2}{2}(x-2)e^x$$

, where c_1, c_2 are real constants.

Example

2. To solve $y'' + y = 4\cos^3(x) + (6x-3)\sin(2x)$.
The characteristic equation has two complex roots: $r_1 = i$ and $r_2 = -i$.
Thus, the general solution to the homogeneous equation is:

$$y_h(x) = A\cos(x) + B\sin(x)$$

where $A, B \in \mathbb{R}$.
3. Seeking a particular solution of the equation . By linearizing $\cos^3(x)$, the right-hand side of this equation is written as:

$$f(x) = 3\cos(x) + \cos(3x) + (6x-3)\sin(2x) = f_1(x) + f_2(x) + f_3(x)$$

Using the method of superposition, we determine three particular solutions. The first, y_1, is relative to the right-hand side $f_1(x)$, the second, y_2, is relative to $f_2(x)$, and the third, y_3, is relative to $f_3(x)$. According to proposition

Second-Order Linear Equations

Example

$$y_1(x) = \mathrm{Re}(Z_1(x)), \quad y_2(x) = \mathrm{Re}(Z_2(x)), \quad \text{and} \quad y_3(x) = \mathrm{Im}(Z_3(x))$$

where $Z_1(x)$, $Z_2(x)$, and $Z_3(x)$ are, respectively, particular solutions of the equations

$$y'' + y = 3e^{ix}, \quad y'' + y = e^{i3x}, \quad \text{and} \quad y'' + y = (6x - 3)e^{i2x}$$

Let's calculate $Z_1(x)$. Since $r = i$ is a root of the characteristic equation, we seek $Z_1(x)$ in the form

$$Z_1(x) = Axe^{ix}$$

We have:

$$Z_1'(x) = (A + iAx)e^{ix} \quad \text{and} \quad Z_1''(x) = (2iA - Ax)e^{ix}$$

Example

Upon substitution and simplification by e^{ix}, we obtain:

$$2iA = 3 \implies A = -\frac{3}{2}i$$

So,

$$Z_1(x) = -\frac{3}{2}ie^{ix} = \frac{3}{2}x\sin(x) - \frac{3}{2}ix\cos(x)$$

Since $y_1(x) = \mathrm{Re}(Z_1(x))$, we have

$$y_1(x) = \frac{3}{2}x\sin x$$

Example

Let's calculate $Z_2(x)$. Since $r = 3i$ is not a root of the characteristic equation, we look for $Z_1(x)$ in the form

$$Z_2(x) = Ae^{3ix}$$

Then,

$$Z_2'(x) = 3iAe^{3ix} \quad \text{and} \quad Z_2''(x) = -9Ae^{3ix}$$

Substituting, we have

$$-8A = 1 \implies A = -\frac{1}{8}$$

Example

So,

$$Z_2(x) = -\frac{1}{8}e^{3ix} = -\frac{1}{8}\cos 3x - \frac{1}{8}i\sin 3x$$

Since $y_2(x) = \mathrm{Re}(Z_2(x))$, then

$$y_2(x) = -\frac{1}{8}\cos 3x$$

Example

Let's calculate $Z_3(x)$. Since $r = 2i$ is not a root of the characteristic equation, we have

$$Z_3(x) = (ax + b)e^{i2x}$$

We get:

$$Z_3'(x) = [a + 2i(ax + b)]e^{i2x} \quad \text{and} \quad Z_3''(x) = [4ia - 4(ax + b)]e^{i2x}$$

Upon substitution, we have

$$-3ax + (-3b + 4ia) = 6x - 3 \implies a = -2 \text{ and } b = 1 - \frac{8}{3}i$$

Example

Thus,

$$Z_3(x) = \left(1 - 2x - \frac{8}{3}i\right)(\cos 2x + i \sin 2x)$$

$$= (1-2x)\cos 2x + \frac{8}{3}\sin 2x + i\left((1-2x)\sin 2x - \frac{8}{3}\cos 2x\right)$$

So,

$$y_3(x) = \mathrm{Im}(Z_3(x)) = (1-2x)\sin 2x - \frac{8}{3}\cos 2x$$

A particular solution of the equation is

$$y_p(x) = y_1(x) + y_2(x) + y_3(x) = \frac{3}{2}x\sin x - \frac{1}{8}\cos 3x + (1-2x)\sin 2x - \frac{8}{3}\text{co}$$

Second-order linear differential equations with variable coefficients:

Definition

A second-order linear differential equation with variable coefficients is of the form:

$$a(x)y'' + b(x)y' + c(x)y = f(x) \quad (4.15)$$

where $x \in I$, and at least one of the functions $x \mapsto a(x)$, $b(x)$, $c(x)$ is not a constant.

Solving these equations: As previously seen, the general solution of a linear equation is the sum of the general solution of the associated homogeneous equation and a particular solution of the complete equation.

The associated homogeneous equation:

Definition

The associated homogeneous equation to equation (4.15) is of the form:

$$a(x)y'' + b(x)y' + c(x)y = 0, \quad \text{where } x \in I \quad (4.16)$$

Remark Unlike constant-coefficient equations, for equations with variable coefficients, solving the homogeneous equation poses a problem. In fact, except for a few cases, we generally do not know how to solve these equations.

In the following, we provide some cases where we can solve them. Recall that if we know two independent solutions y_1 and y_2 of the homogeneous equation (4.16), then the general solution of this equation is of the form:

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad c_1, c_2 \in \mathbb{R} \quad (4.17)$$

1) If we know a particular solution: Suppose y_1 is a known solution of the homogeneous equation. We seek a second solution in the form:

$$y_2 = u(x) y_1(x)$$

where $u(x)$ is not a constant.

We have:

$$y_2' = u'(x)y_1(x) + u(x)y_1'(x)$$
$$y_2'' = u''(x)y_1(x) + 2u'(x)y_1'(x) + u(x)y_1''(x)$$

Substituting into equation (4.16), since y_1 is a solution to this equation, we deduce that the function $u'(x)$ satisfies the following linear first-order differential equation:

$$a(x)y_1(x)u'' + [2a(x)y_1'(x) + b(x)y_1(x)]u' = 0$$

Solving the first-order linear differential equations gives:

$$u'(x) = \exp\left(-\int \frac{2a(x)y_1'(x) + b(x)y_1(x)}{a(x)y_1(x)}dx\right) \neq 0$$

Integrating this expression, we obtain:

$$u(x) = \int \left[\exp\left(-\int \frac{2a(x)y_1'(x) + b(x)y_1(x)}{a(x)y_1(x)} dx \right) \right] dx \neq \text{Cte.} \quad (4.18)$$

So the function $y_2 = u(x)y_1(x)$ is indeed a solution of equation (4.16) and is independent of $y_1(x)$. The general solution of the homogeneous equation is thus given by:

$$y(x) = c_1 y_1(x) + c_2 u(x) y_1(x), \quad c_1, c_2 \in \mathbb{R}$$

We thus have the following result:

Theorem

If a particular solution y_1 of the homogeneous equation (4.16) is known, then

$$y_2(x) = u(x)y_1(x)$$

is a second solution of this equation, independent of y_1, where $u(x)$ is given by formula (4.18).

Example

Solve the equation

$$x^2y'' - 7xy' + 15y = 0$$

knowing a particular solution $y_1 = x^3$. We seek a second solution to this equation in the form:

$$y_2(x) = x^3u(x)$$

We differentiate:

$$y_2'(x) = x^3u'(x) + 3x^2u(x)$$

$$y_2''(x) = x^3u''(x) + 6x^2u'(x) + 6xu(x)$$

Substituting into equation , we obtain:

$$x^2(x^3u''(x) + 6x^2u'(x) + 6xu(x)) - 7x(x^3u'(x) + 3x^2u(x)) + 15(x^3u(x)) =$$

This equation is a first-order equation for the function $z = u'$. A solution to this equation is $u'(x) = 2x$.

Example

Integrating and taking the integration constant to be zero, we obtain $u(x) = x^2$. Therefore, a particular solution of the equation is $y_2(x) = x^2 \cdot x^3 = x^5$. The general solution of equation is then $y(x) = c_1x^3 + c_2x^5$, where $c_1, c_2 \in \mathbb{R}$.

Variable Transformation Method

The purpose of this method is to search for a change of variables, if it exists, that can transform this equation into one with constant coefficients.

Assuming that the function $c(x)$ does not vanish on the interval I, we can write equation (4.16) in the form:

$$A(x)y''(x) + B(x)y'(x) + y(x) = 0 \quad (4.19)$$

with

$$A(x) = \frac{a(x)}{c(x)}$$

and

$$B(x) = \frac{b(x)}{c(x)}$$

.

Let's consider a variable change given in the form:

$$\begin{cases} g : I \subset \mathbb{R} \to \mathbb{R} \\ x \mapsto t = g(x) \\ t = g(x) \Leftrightarrow x = g^{-1}(t) \end{cases}$$

where g is assumed to be bijective, of class C^2, and g^{-1} exists.
We have:

$$y(x) = y(g^{-1}(t)),$$

so the function y is considered as a function u of the new variable t, and we set

$$y(x) = u(g(x)) = u(t)$$

The formulas for the derivatives of composite functions give, noting $u' = \frac{du}{dt}$:

$$y'(x) = u'(t)g'(x)$$

$$y''(x) = u''(t)(g'(x))^2 + u'(t)g''(x)$$

Indeed, we have

$$y'(x) = u'(g(x))g'(x) = u'(t)g'(x)$$

Similarly, we obtain:

$$y''(x) = u''(t)(g'(x))^2 + u'(t)g''(x)$$

Substituting into equation (4.19), we deduce that the function $u(t)$ satisfies the second-order differential equation:

$$A(x)(g'(x))^2u''(t) + [A(x)g''(x) + B(x)g'(x)]u'(t) + u(t) = 0. \quad (4.20)$$

This equation has constant coefficients if and only if the function g satisfies the following two conditions:

$$A(x)(g'(x))^2 = A \quad (4.21)$$

$$A(x)g''(x) + B(x)g'(x) = B \quad (4.22)$$

where A and B are constants.
It is evident that if conditions (4.21) and (4.22) are satisfied, equation (4.20) becomes

$$Au''(t) + Bu'(t) + u(t) = 0, \quad (4.23)$$

which is an equation with constant coefficients. Thus, equation (4.19) reduces to an equation with constant coefficients. We solve equation (4.23) and then return to the original variable $x = g^{-1}(t)$.

Example

Solve the Euler equation for $x > 0$:

$$x^2y'' - 7xy' + 5y = 0$$

We set

$$t = g(x) \implies x = g^{-1}(t)$$

From the preceding, equation becomes:

$$x^2(g'(x))^2u'' + x^2g''(x) - 7xg'(x)u' + 5u = 0$$

This equation becomes one with constant coefficients if and only if the function g satisfies the following conditions:

$$x^2(g'(x))^2 = A$$

$$x^2g''(x) - 7xg'(x) = B$$

Example

Notice that we can choose $A = 1$, then we have

$$x^2(g'(x))^2 = 1 \implies g'(x) = \frac{1}{x}$$

The second condition then becomes:

$$-1 - 7 = B \implies B = -8$$

The change of variable is:

$$t = g(x) = \ln(x) \implies x = e^t$$

With this change of variables, equation then becomes:

$$u'' - 8u' + 5u = 0$$

Example

The general solution to this equation is:

$$u(t) = c_1 e^{(4+\sqrt{11})t} + c_2 e^{(4-\sqrt{11})t} \quad ; \quad c_1, c_2 \in \mathbb{R}$$

Returning to the variable x, we obtain:

$$\begin{aligned} y(x) &= c_1 e^{(4+\sqrt{11})\ln x} + c_2 e^{(4-\sqrt{11})\ln x} \\ &= c_1 e^{\ln(x^{(4+\sqrt{11}))}} + c_2 e^{\ln(x^{(4-\sqrt{11}))}} \\ &= c_1 x^{(4+\sqrt{11})} + c_2 x^{(4-\sqrt{11})}, \quad c_1, c_2 \in \mathbb{R} \end{aligned}$$