Lecture 2 : Linear Maps

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Contents

Definition of a linear map

Throughout this lecture, K will denote an arbitrary field.

Definition 1

Let V and W be K- vector spaces. A function $f: V \longrightarrow W$ is said to be a **linear map** if for any two vectors $u, v \in V$ and any scalar $\alpha \in K$, the following two conditions are satisfied :

1) $f(u+v) = f(u) + f(v)$;

2) $f(\alpha u) = \alpha f(u)$.

Definition of a linear map

Remark 1

A linear map is also called homomorphism of vector spaces.

Exercise 1

Show that the following statements are equivalent :

1) The map f is linear ;
2) $\forall u, v \in V$ and $\forall \alpha \in K, f(\alpha u + v) = \alpha f(u) + f(v)$;
3) $\forall u, v \in V$ and $\forall \alpha, \beta \in K, f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$.

Endomorphism, isomorphism and automorphism

Definition 2

A linear map from a vector space V to itself is called an **endomorphism of $\boldsymbol{V}$**.

A bijective linear map is called an **isomorphism**. Two K- vector spaces are **isomorphic** if there exists an isomorphism between them.

When the endomorphism is bijective, it is called an **automorphism**

Examples

Example 1

The identity map

$$Id_V : V \longrightarrow V$$
$$x \mapsto Id_V(x) = x$$

is a linear map and also an automorphism of the vector space V.

Example 2

The zero function

$$f: V \longrightarrow W$$
$$x \mapsto f(x) = 0_W$$

is linear.

Examples

Example 3

The map

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto f(x) = 2x$$

is an automorphism.

Example 4

The map

$$f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x, y) \mapsto f(x, y) = 2x + y$$

is linear

Examples

Example 5

The map

$$\begin{aligned} f: \mathbb{R} &\longrightarrow \mathbb{R} \\ x \mapsto\ & f(x) = \sin x \end{aligned}$$

is not linear.

Example 6

The map

$$\begin{aligned} f: \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) \mapsto\ & f(x, y) = xy \end{aligned}$$

is not linear.

Examples

Example 7

Consider the subspace

$$\mathcal{C}^{\infty}([a,b],\mathbb{R}) = \{f \in \mathcal{F}([a,b],\mathbb{R}) : f \text{ is infinitely differentiable}\},$$

and define

$$D : \mathcal{C}^{\infty}([a,b],\mathbb{R}) \to \mathcal{C}^{\infty}([a,b],\mathbb{R})$$
$$f \mapsto D(f),$$

where

$$D(f)(x) = f'(x), \forall x \in [a,b].$$

Then D is a linear map.

Examples

Example 8

Consider the subspace

$$\mathcal{C}([a,b],\mathbb{R}) = \{f \in \mathcal{F}([a,b],\mathbb{R}) : f \text{ is continuous}\},$$

and define

$$I : \mathcal{C}([a,b],\mathbb{R}) \to \mathcal{C}([a,b],\mathbb{R})$$
$$f \mapsto I(f),$$

where

$$I(f)(x) = \int_a^x f(t)dt, \forall x \in [a,b].$$

Then I is a linear map.

Composition of linear maps

Theorem 1

Let $f : U \longrightarrow V$ and $g : V \longrightarrow W$ be two linear maps. Then the composed function $g \circ f : U \longrightarrow W$ is a linear map.

Image and Kernel

Definition 3

Let $f\colon U \to V$ be a linear map. We define the **image of f** by

$$\operatorname{Im} f = \{y \in V : \exists x \in U \text{ such that } y = f(x)\},$$

and we define the **kernel of f** by

$$\operatorname{Ker} f = \{x \in U \colon f(x) = 0_V\}.$$

Some properties

Proposition 1

Let $f: V \to W$ be a linear map of K- vector spaces. Then we have :

1) $f(0_V) = 0_W$;

2) $\mathrm{Ker} f$ is a subspace of V ;

3) $\mathrm{Ker} f = \{0_V\}$ if, and only if f is injective ;

4) $\mathrm{Im} f$ is a subspace of W ;

5) If $\dim V < +\infty$ and $\{v_1, v_2, \cdots, v_n\}$ is a basis of V then

$$\mathrm{Im} f = \langle f(v_1), f(v_2), \cdots, f(v_n)\rangle.$$

Example

Example 9

Let

$$f: K_3[X] \to K_2[X]$$
$$f(P) = P'.$$

We have

$$\operatorname{Ker} f = \{P \in K_3[X]: f(P) = 0_V\} = \langle 1 \rangle,$$

then $\{1\}$ is a basis of $\operatorname{Ker} f$, so $\dim(\ker f) = 1$.

$$\operatorname{Im} f = \langle f(1), f(X), f(X^2), f(X^3) \rangle = \langle 1, 2X, 3X^2 \rangle,$$

then $\{1, 2X, 3X^2\}$ is a basis of $\operatorname{Im} f$, so $\dim(\operatorname{Im} f) = 3$.

Example

Example 10

Let

$$\varphi: \mathcal{C}^{\infty}([a,b],\mathbb{R}) \to \mathcal{C}^{\infty}([a,b],\mathbb{R})$$
$$f \mapsto \varphi(f) = f'' + f' - 2f.$$

Then φ is a linear map, and we have

$$\operatorname{Ker}\varphi = \{\alpha g + \beta h : \alpha, \beta \in \mathbb{R}\},$$

where $g(x) = e^{x}$ and $h(x) = e^{-2x}$ for all $x \in \mathbb{R}$.

Inverse of a linear map

Theorem 2

Let $f\colon U \rightarrow V$ be a linear map. If f is an isomorphism, then

$$f^{-1}\colon V \longrightarrow U$$

is also an isomorphism.

Values on a basis

Proposition 2

Let V be a finite-dimensional K- vector space and $\{v_1, v_2, \cdots, v_n\}$ be a basis of V. Let W be a K- vector space and $w_1, w_2, \cdots, w_n$ be vectors of W. Then, there exists a unique linear map f from V to W such that

$$f(v_i) = w_i, i = 1, \cdots, n.$$

Remark 2

The proposition means that a linear map is entirely defined by its values on a given basis.

Linear maps and bases

Theorem 3

Let V and W be K- vector spaces and let $\{v_1, v_2, \cdots, v_n\}$ be a basis of V. Let $w_1, w_2, \cdots, w_n$ be vectors of W and let f be the linear map from V to W given by $f(v_i) = w_i, i = 1, \cdots, n$. Then we have :

1) f is injective if, and only if, $w_1, w_2, \cdots, w_n$ are linearly independent ;
2) f is surjective if, and only if, $w_1, w_2, \cdots, w_n$ span W ;
3) f is bijective if, and only if, $\{w_1, w_2, \cdots, w_n\}$ is a basis of W.

Isomorphic spaces

Corollary 1

Let V, W be finite-dimensional vector spaces over K and let f be a linear map from V to W. The map f is an isomorphism from V to W if, and only if, the image of a basis of V is a basis of W.

Corollary 2

Two isomorphic finite-dimensional vector spaces over K have the same dimension.

Corollary 3

Any vector space of finite dimension n over K is isomorphic to K^n.

Rank Theorem

Definition 3

Let V and W be finite-dimensional vector spaces and $f: V \to W$ be a linear map. The **rank of $\boldsymbol{f}$**, denoted by $r(f)$, is defined as the dimension of $\mathrm{Im} f$.

Theorem 3

Let $f: V \to W$ be a linear map between finite-dimensional K-vector spaces. Then

$$r(f) = \dim(V) - \dim(\ker f).$$

Rank Properties

Theorem 4

Let $f: V \to W$ be a linear map between finite-dimensional K-vector spaces. Then we have :

1) $r(f) \leq \dim(V)$ and $r(f) \leq \dim(W)$;

2) $r(f) = \dim(V) \Leftrightarrow f$ is injective ;

3) $r(f) = \dim(W) \Leftrightarrow f$ is surjective.