2.Notions of limits

2.1. Limit of functions

$$f(x) = x^2 \implies f(0{,}1) = 0{,}01;$$

and $f(0{,}01) = 0{,}0001 \; and \;\; f(10^{-3}) = 10^{-6}$

This suggests that $\lim\limits_{x \to 0} x^2 = 0.$

2. Notions of limits

Definition :

Let f be a function whose domain includes an open interval I containing x_0, except perhaps for x = x_0. Let l be a number.

The limit of f, as x approaches x_0, equals l

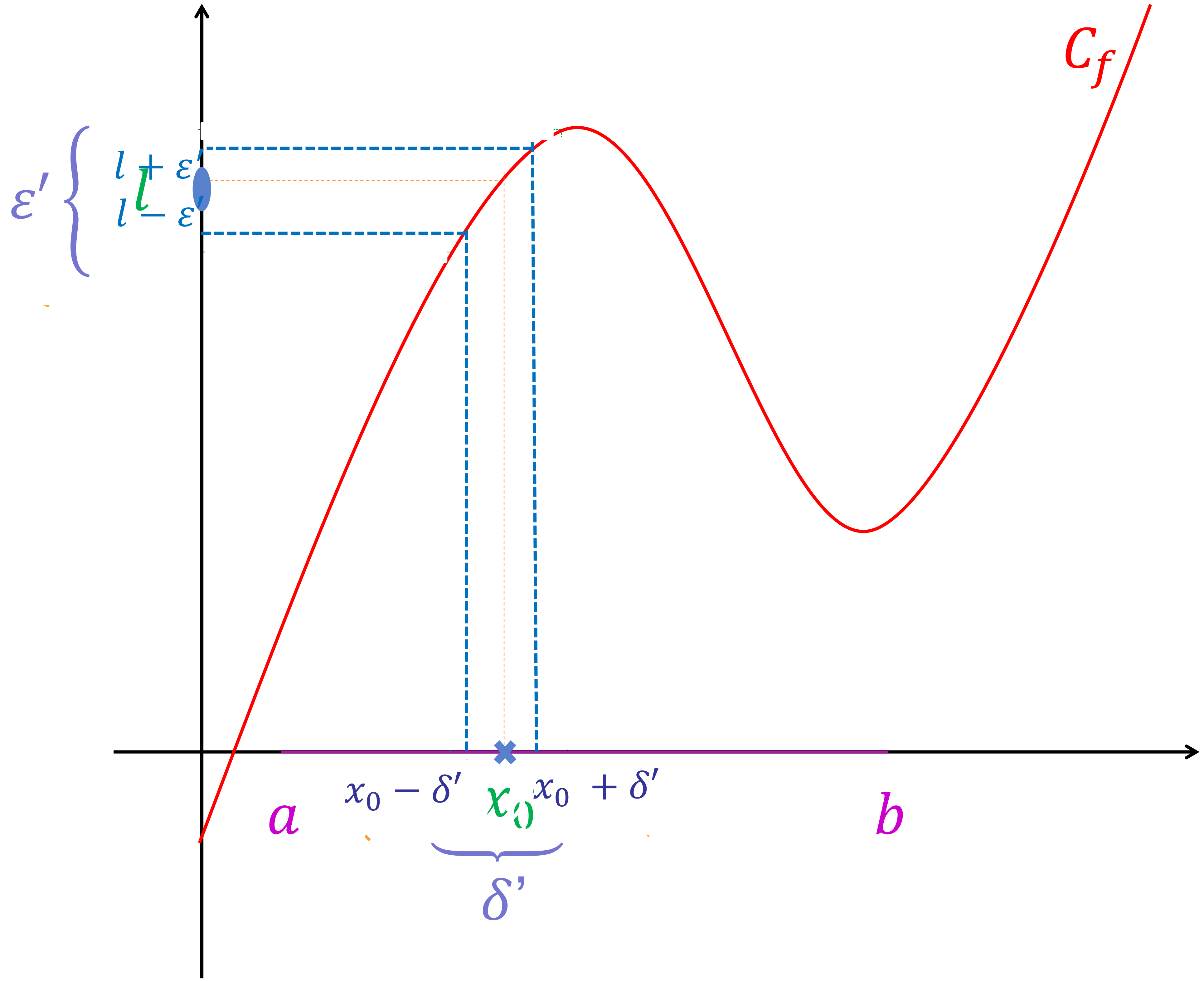
$$\Updownarrow$$

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| \leq \delta \Rightarrow |f(x) - l| \leq \varepsilon.$$

$$\Updownarrow$$

$$\lim_{x \to x_0} f(x) = l.$$

2. Notions of limits



2. Notions of limits

Example :

show that $\lim_{x \to 1} x + 2 = 3$

We have $a = 1$, $\ell = 3$ et $f(x) = x + 2.$

Let $\varepsilon > 0$, is there a $\delta > 0$ such that

$|x - 1| \leq \delta \Rightarrow |f(x) - 3| \leq \varepsilon$??

$|f(x) - 3| \leq \varepsilon \Leftrightarrow$

$|(x + 2) - 3| = |x - 1| \leq \varepsilon$

so just take $\delta = \varepsilon$.

2. Notions of limits

Example : $$\lim_{x\to 1}\frac{1}{2x-3}=-1$$

$x-x_0$

We have, $\left|\frac{1}{2x-3}-(-1)\right|=\frac{2}{|2x-3|}|x-1|$

Bound above the term $\frac{2}{|2x-3|}$. Let , $x\in]1-\alpha,1+\alpha[$

We have $1-\alpha<x<1+\alpha$

$$\Rightarrow -1-2\alpha<2x-3<-1+2\alpha$$

just choose $\alpha>0$ such that

$-1+2\alpha<0,$ $\alpha=1/4$

$$-\frac{3}{2}<2x-3<-\frac{1}{2}\quad\Rightarrow |2x-3|\geq 1/2$$

Then , $|f(x)+1|\leq 4|x-1|$

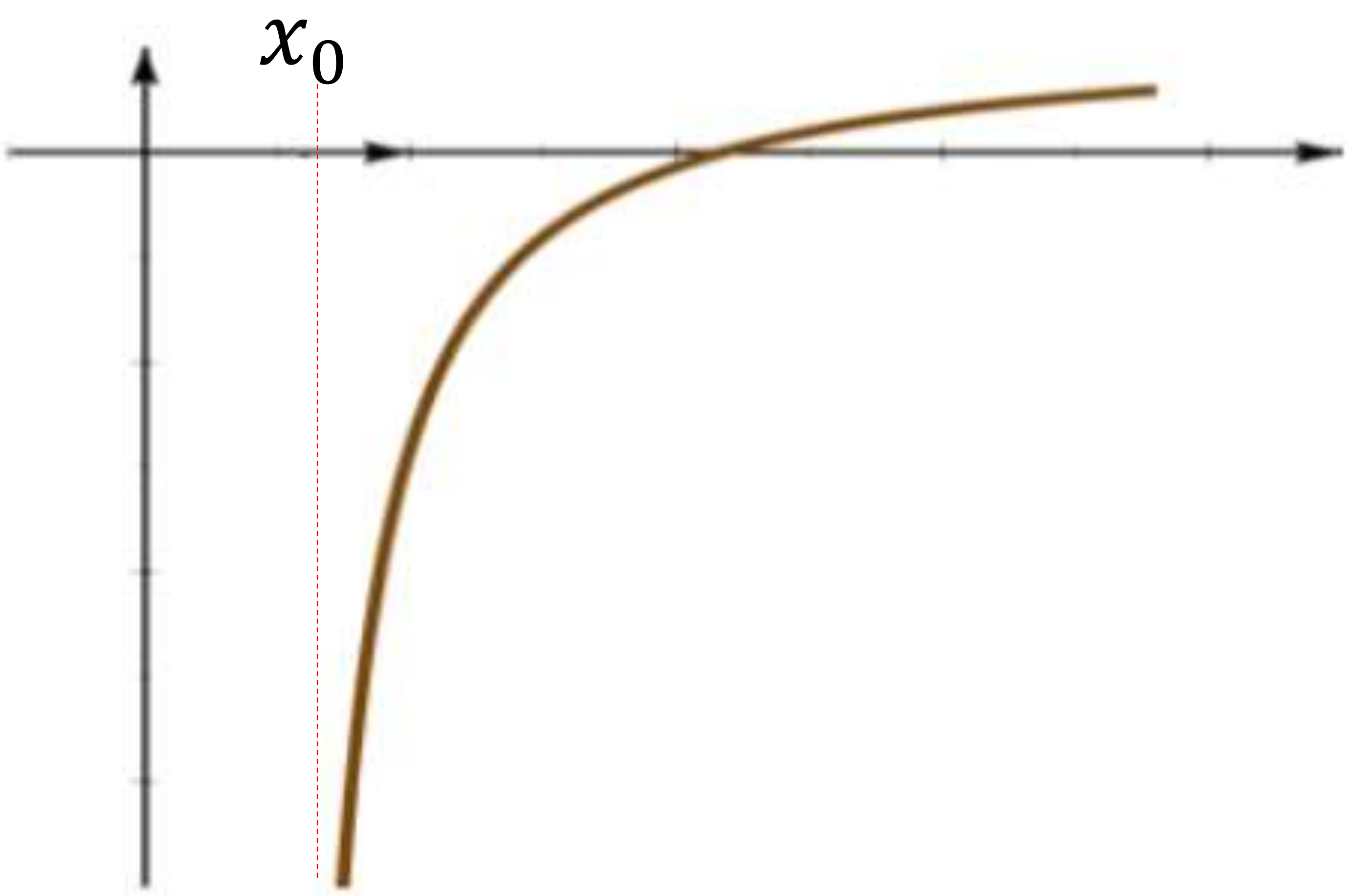
$$\delta=\varepsilon/4$$

2. Notions of limits

2.1.2. Infinite (two-sided) limit

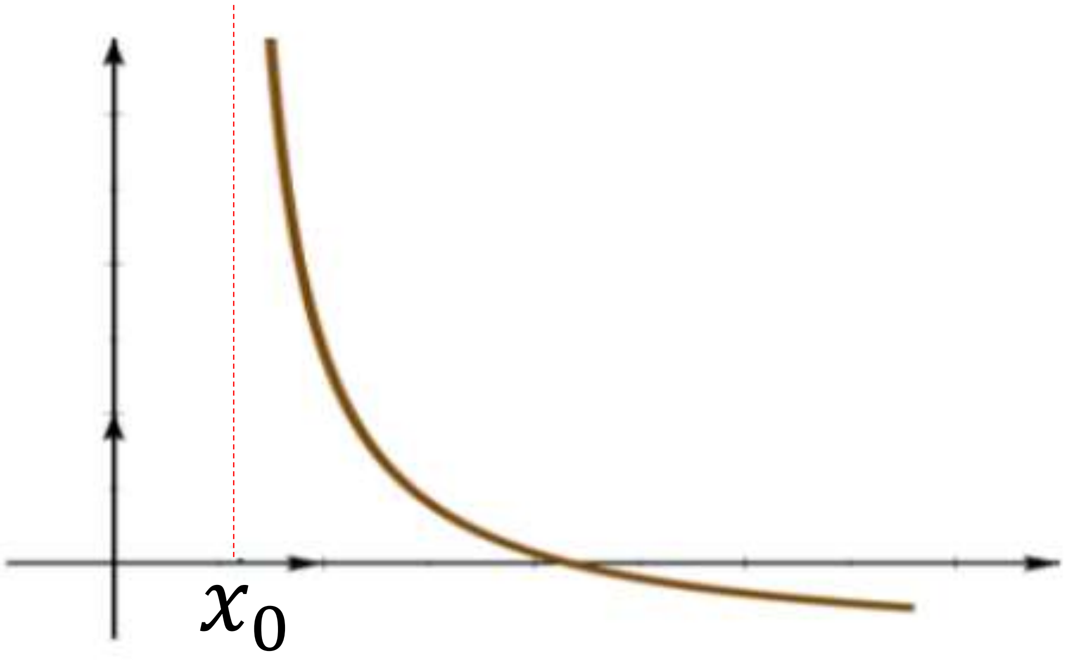
Let f(x) be defined for all inputs x in an open interval I containing x_0, except perhaps at x = x_0

✓ ***f has limit $-\infty$ at x_0***



2. Notions of limits

✓ ***f has limit $+\infty$ at x_0***



f has a **vertical asymptote** at $x = a$ ssi:

$$\lim_{x \to a} f(x) = \pm\infty.$$

2. Notions of limits

Example :

$$\lim_{x \to 0} \frac{1}{\sqrt{x}} = +\infty.$$

Let $A > 0$, *we search* $\delta > 0$ *such that* :

$$|x - 0| \leq \delta \Rightarrow |f(x)| \geq A\,.$$

$$\left|\frac{1}{\sqrt{x}}\right| \geq A \;\Rightarrow x \leq \frac{1}{A^2}.$$

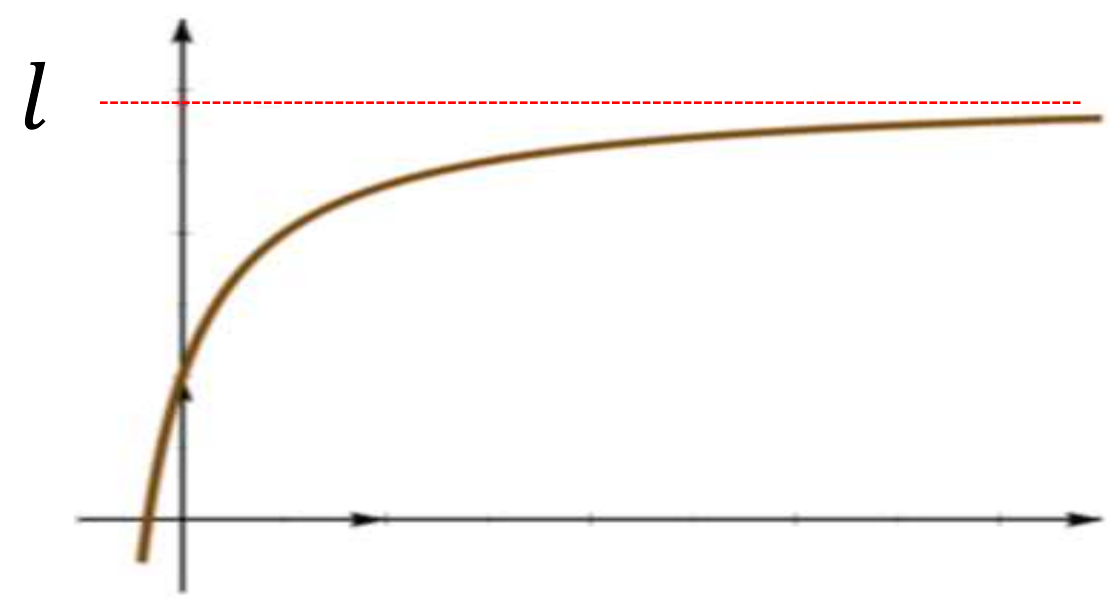
we can choose

$$\boldsymbol{\delta = \frac{1}{A^2}}$$

2. Notions of limits

2.2. Limit at infinity:

Let f(x) be defined for all inputs x in some open interval I = (b, +∞).



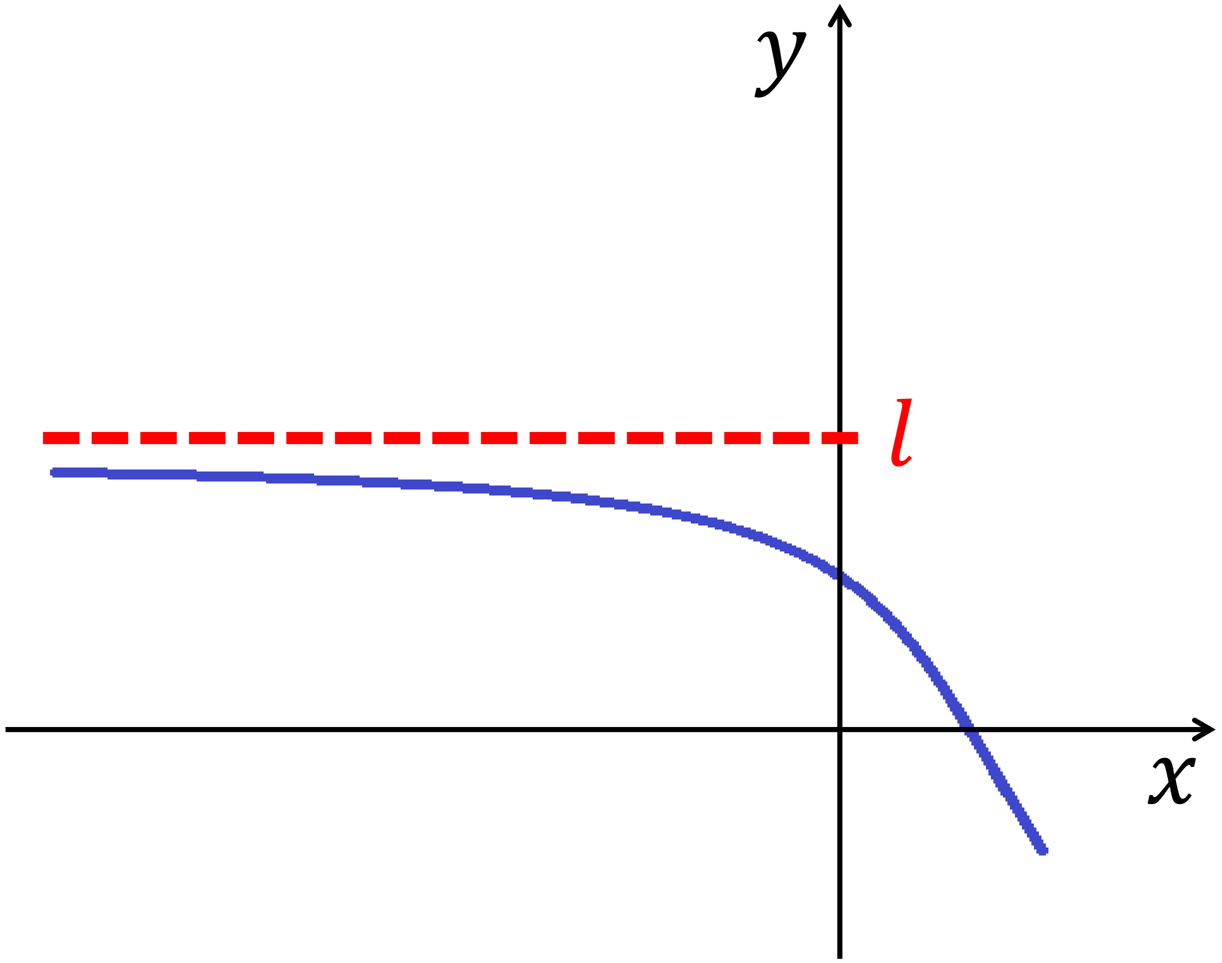
$$\lim_{x \to +\infty} f(x) = l.$$

2. Notions of limits

2.2. Limit at infinity:

Let f(x) be defined for all inputs x in some open interval I = (−∞, b).

f ***has limit*** l ***at*** $-\infty$



The function f *has a horizontal asymptote in* $y = l$ *if:*

$$\lim_{x \to \pm\infty} f(x) = l$$

2. Notions of limits

Example :

$$\textit{Show that } \lim_{x \to +\infty} \frac{x+1}{x-2} = 1.$$

$$\forall x \in \,]2, +\infty[, \textit{ we have } \left|\frac{x+1}{x-2} - 1\right| = \frac{3}{x-2}.$$

$$\varepsilon > 0, \exists ?\, A \in \mathbb{R}_+^* \textit{ such that :}$$

$$\forall x \in]2, +\infty[, \qquad x \geq A \Longrightarrow \left|\frac{x+1}{x-2} - 1\right| \leq \varepsilon.$$

$$\frac{3}{x-2} \leq \varepsilon, \textit{ c-à-d } x \geq 2 + \frac{3}{\varepsilon}.$$

$$A = 2 + \frac{3}{\varepsilon}.$$

2. Notions of limits

2.2.2. Infinite limit at infinity

Let f(x) be defined for all inputs x in some open interval I = (- ∞, ∞).

Definition

✓ f *hase a limit* $-\infty$ *at* $+\infty$

$$\Updownarrow$$

$$\forall A \in \mathbb{R}, \exists B \in \mathbb{R}, \forall x \in I, (x \geq B \Rightarrow f(x) \leq -A)$$

$$\Longleftrightarrow \quad \lim_{x \to +\infty} f(x) = -\infty$$

2. Notions of limits

✓ ***f has a limit*** $-\infty$ ***at*** $-\infty$

$$\Updownarrow$$

$$\forall A \in \mathbb{R}^{*}{}_{+}, \exists B \in \mathbb{R}^{*}{}_{+}, \forall x \in I,$$
$$(x \leq -B \Rightarrow f(x) \leq -A),$$

$$\Longleftrightarrow \quad \lim_{x \to -\infty} f(x) = -\infty \,.$$

2.3. Right-hand limit, Left-hand limit

Definition

f admits l for right-hand limit (resp. Left-hand limit) in a, if

$$\lim_{\substack{x \to a, \\ x > a}} f(x) = l \text{ or } \lim_{x \to a^+} f(x) = l$$

$$\text{(resp. } \lim_{\substack{x \to a \\ x < a}} f(x) = l \quad \text{or} \quad \lim_{x \to a^-} f(x) = l\text{).}$$

and we have

$$\lim_{x \to a^+} f(x) = l$$

$$\Updownarrow$$

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I,$$
$$(0 < x - a \leq \delta \Rightarrow |f(x) - l| \leq \varepsilon).$$

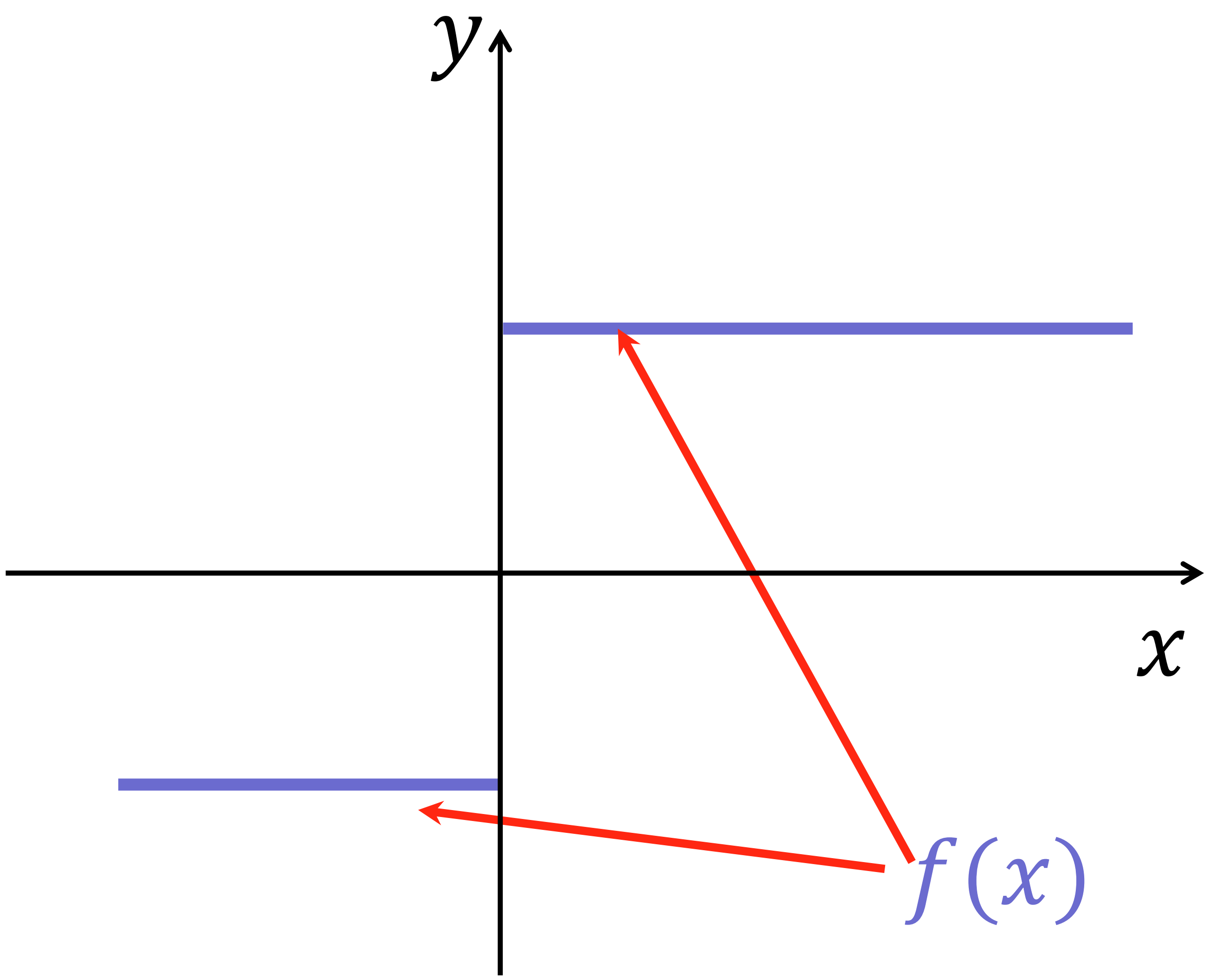
(resp. $\lim_{x \to a^-} f(x) = l$

$$\Updownarrow$$

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I,$$
$$(0 < a - x \leq \delta \Rightarrow |f(x) - l| \leq \varepsilon)$$ *).*

2. Notions of limits

Example :



$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{-x}{2x} = \boxed{-\frac{1}{2}}.$$

$$And \ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{2x} \boxed{= \frac{1}{2}}.$$

2.4. Properties :

2.4.1. The uniqueness of the limits

Proposition :

The limit of a function is unique if it exists.

Demonstration :

$$\lim_{x\to a} f(x) = l \text{ et } \lim_{x\to a} f(x) = l'$$

$$l < l',$$

We put $\varepsilon = (l - l')/2,$

$$\exists \alpha > 0, |x - a| < \alpha \; et \; x \in I \Rightarrow |f(x) - l| < \varepsilon$$

and

$$\exists \alpha' > 0, |x - a| < \alpha' \; et \; x \in I \Rightarrow |f(x) - l'| < \varepsilon$$

The interval

$$]a - \alpha, a + \alpha[\cap]a - \alpha', a + \alpha'[\neq \emptyset$$

Let $x_0 \in]a - \alpha, a + \alpha[\cap]a - \alpha', a + \alpha'[$

Et on obtient: $\begin{cases} |f(x_0) - l| < \varepsilon \\ |f(x_0) - l'| < \varepsilon \end{cases}$

$$|l - l'| = |l - f(x_0) + f(x_0) - l'|$$

$$< |l - f(x_0)| + |f(x_0) - l'|$$

$$< l - l'$$

2.4.2. Sequential limits in functions

Theorem (sequential criterion for limits)

Let $f: I \to \mathbb{R}$, *where* $I \subset \mathbb{R}$ *and suppose that* $a \in \mathbb{R}$, *is any accumulation point of* I. *Then*

$$\lim_{x \to a} f(x) = l.$$

$$\Updownarrow$$

$$\boldsymbol{\lim_{n \to +\infty} f(u_n) = l.}$$

For every sequence $(\boldsymbol{u_n})_{\boldsymbol{n} \in \mathbb{N}}$ *in* I with $\boldsymbol{u_n} \neq a$ *for all* $\boldsymbol{n} \in \mathbb{N}$

such that $\boldsymbol{\lim_{n \to +\infty} u_n = a}$.

Proof.

2. Notions of limits

- Corollary . Suppose that $f: I \to \mathbb{R}$ and $a \in \mathbb{R}$ is an accumulation point of I. Then $\lim_{x \to a} f(x)$ does not exist if either of the following conditions holds:

1- There are sequences $(\boldsymbol{u_n})_{\boldsymbol{n}\in\mathbb{N}}$, $(\boldsymbol{v_n})_{\boldsymbol{n}\in\mathbb{N}}$ in I with

$\boldsymbol{u_n} \neq a$, $\boldsymbol{v_n} \neq a$ such that $\boldsymbol{\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = a}$,

but $\boldsymbol{\lim_{n\to+\infty} f(u_n) \neq \lim_{n\to+\infty} f(u_n)}$

2-There is a sequence $(\boldsymbol{u_n})_{\boldsymbol{n}\in\mathbb{N}}$ in I with such $\boldsymbol{u_n} \neq a$

$\boldsymbol{\lim_{n\to+\infty} u_n = a}$, but the sequence $(\boldsymbol{f(u_n)})$ diverges.

2. Notions of limits

Example :

$$f(x) = sin\left(\frac{1}{x}\right).$$

$$\forall n \in \mathbb{N}^* \quad u_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \quad et\ v_n = \frac{1}{-\frac{\pi}{2} + 2n\pi}.$$

$$(u_n)\ et\ (v_n) \longrightarrow 0$$

But we have

$$\forall n \in \mathbb{N}^* \quad f(u_n) = \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1.$$

$$et\ f(v_n) = \sin(-\frac{\pi}{2} + 2n\pi) = -1.$$

2.4.3.Order properties.

As for limits of sequences, limits of functions preserve (non-strict) inequalities.

Proposition :

$f\, et\ g \in \mathcal{F}(I,\mathbb{R}),\ and\ a \in \mathbb{R} \cup \{-\infty, +\infty\}.$

$$if\ f(x) \le g(x)\ for\ all\ x \in I \text{ and } \lim_{x\to a} f(x), \lim_{x\to a} g(x)\ \ exist$$

$$Then$$

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

Proof.

2. Notions of limits

2.4.4. The squeeze principle

Theorem (gendarmes theorem):

$$f, g \; et \; h \in \mathcal{F}(I, \mathbb{R}) \quad and \; a \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

$$if \; f(x) \leq g(x) \leq h(x) \quad for \; all \; x \in I$$

If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = l$.

$$\Longrightarrow$$

$$\lim_{x\to a} g(x) = l$$

Proof.

2. Notions of limits

Example : *Find* $\lim_{x \to 0} xsin(\frac{1}{x})$.

the fonction sinus is bounded .

$$\forall t \in \mathbb{R}, -1 \leq \sin(t) \leq 1$$

c-à-d

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \Longrightarrow -x \leq xsin(\frac{1}{x}) \leq x$$

But $\lim_{x \to 0} x = \lim_{x \to 0}(-x) = 0.$

so

$$\lim_{x \to 0} xsin(\frac{1}{x}) = 0.$$

2. Notions of limits

- Limits of monotone functions

If a function $f:(a, \text{b}) \to \mathbb{R}$ is monotone on interval (a, b). Then the two limits $\lim_{x \to a^+} f(x)$

and $\lim_{x \to b^-} f(x)$ exists

Proof.

2. Notions of limits

2.5. Algebraic properties

$\lim\limits_{x\to a} f(x)$	$\lim\limits_{x\to a} g(x)$	$\lim\limits_{x\to a} f(x)+g(x)$	$\lim\limits_{x\to a}(f(x)-g(x))$	$\lim\limits_{x\to a} f(x)g(x)$
ℓ	ℓ'	$\ell+\ell'$	$\ell-\ell'$	$\ell\ell'$
$\mathbf{0}$	$-\infty$	$-\infty$	$+\infty$	F.I.
$\mathbf{0}$	$+\infty$	$+\infty$	$-\infty$	F.I.
$-\infty$	$+\infty$	F.I.	$-\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	F.I.	$-\infty$
$-\infty$	$-\infty$	$-\infty$	F.I.	$+\infty$

IF: indeterminate form (F.I)

2. Notions of limits

$\lim\limits_{x\to a} f(x)$	ℓ	ℓ	$+\infty$	$+\infty$	$-\infty$	$-\infty$	$\pm\infty$	$\ell > 0$ ou $+\infty$	$\ell < 0$ ou $-\infty$	$\ell > 0$ ou $+\infty$	$\ell < 0$ ou $-\infty$	0
$\lim\limits_{x\to a} g(x)$	ℓ'	$\pm\infty$	$\ell' > 0$	$\ell' < 0$	$\ell' > 0$	$\ell' < 0$	$\pm\infty$	0^+	0^+	0^-	0^-	0
$\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$	$\dfrac{\ell}{\ell'}$	0	$+\infty$	$-\infty$	$-\infty$	$+\infty$	F.I.	$+\infty$	$-\infty$	$-\infty$	$+\infty$	I.F

2.6. The indeterminate form (I.F)

$$(+\infty) + (-\infty),\ \frac{0}{0},\ \ \frac{\infty}{\infty},\ \ 0 \times (\pm\infty),\ \ (0^+)^{\,0},\ \ 1^{\infty}.$$

2. Notions of limits

2.7. Algebric methods pour for study the indeterminate form :

- ✓ **Simplification of algebric expressions :**

Exemple : $\lim\limits_{x\to 1} \frac{x^2-1}{x^2-3x+2} = \lim\limits_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{(x-2)\cancel{(x-1)}}$

$$= \lim_{x\to 1} \frac{(x+1)}{(x-2)} = -2$$

2. Notions of limits

- ✓ **Changement de variable :**

Exemple : $\lim\limits_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$. $\quad y=\sqrt[6]{x}\Rightarrow\begin{cases}\sqrt{x}=y^3\\ \sqrt[3]{x}=y^2\end{cases}$

$$\lim_{y\to 1}\frac{y^3-1}{y^2-1}=\lim_{y\to 1}\frac{\cancel{(y-1)}(y^2+y+1)}{\cancel{(y-1)}(y+1)}=\frac{3}{2}.$$

- ✓ **Using the conjugate :**

Exemple : $\lim\limits_{x\to+\infty}\sqrt{x+1}-\sqrt{x}$

$$=\lim_{x\to+\infty}\frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{(\sqrt{x+1}+\sqrt{x})}=0.$$

2. Notions of limits

- **Factoring :**

Example :

$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to+\infty}\frac{\cancel{x}}{-\cancel{x}(\sqrt{1+\frac{1}{x}})}=\lim_{x\to+\infty}\frac{-1}{\sqrt{1+\frac{1}{x}}}=-1.$$

- **Using the logarithms and exponentiels :**

Example :

$$\lim_{x\to+\infty}(1+x)^{\frac{1}{x}}=\lim_{x\to+\infty}e^{\frac{\ln(1+x)}{x}}=e.$$

1

2. Notions of limits

2.8. Some usual limits to know

$$\lim_{x\to 0}\frac{sinx}{x} = 1, \qquad \lim_{x\to 0}\frac{1-cosx}{x^2} = \frac{1}{2},$$

$$\lim_{x\to 0}\frac{cosx-1}{x} = 0, \qquad \lim_{x\to 0}\frac{tgx}{x} = 1\,.$$

2. Notions of limits

$$\lim_{x\to 0}\frac{ln(1+x)}{x} = 1\,,$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1.$$

$$\lim_{x\to 0^+} xlnx = 0.$$

$$\lim_{x\to +\infty}\frac{lnx}{x} = 0.$$

$$\lim_{x\to +\infty}\frac{e^x}{x} = +\infty.$$

$$\lim_{x\to -\infty} xe^x = 0.$$

2. Notions of limits

Proposition

- $$\lim_{x \to +\infty} \frac{ln^{\beta}}{x^{\alpha}} = 0, \quad \forall \alpha > 0 \; et \; \forall \beta \in \mathbb{R}$$

- $$\lim_{x \to 0^{+}} x^{\alpha} |lnx|^{\beta} = 0, \quad \forall \alpha > 0 \; et \; \forall \beta \in \mathbb{R}$$

- $$\lim_{x \to +\infty} \frac{e^{\beta x}}{x^{\alpha}} = +\infty, \quad \forall \alpha > 0 \; et \; \forall \beta \in \mathbb{R}$$

- $$\lim_{x \to +\infty} x^{\beta} e^{-\alpha x} = 0, \quad \forall \alpha > 0 \; et \; \forall \beta \in \mathbb{R}$$

2. Notions of limits

2.8. Theorem limit of composite functions:

$$f: I \to \mathbb{R}, g: J \to \mathbb{R} \ \ such\ that\ f(I) \subset J$$

$$if \lim_{x \to a} f(x) = b \ and \lim_{x \to b} g(x) = c,$$

$$a, b\ and\ c \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

$$Then \ \ \boldsymbol{\lim_{x \to a} g\big(f(x)\big) = c}\,.$$

2. Notions of limits

2.9. Equivalents in the neighborhood of a point

Definition :

$$f \ et \ g \in \mathcal{F}(I, \mathbb{R}),$$

are equivalents **on some reduced neighborhood of** $\boldsymbol{a}$

$$\Updownarrow$$

$$\exists h \ telle \ que \ \lim_{x \to a} h(x) = 1$$

and

$$\exists \delta > 0, \forall x \in I, (0 < |x - a| < \delta \Rightarrow f(x) = g(x)h(x))$$

we use the notation $\boldsymbol{f \sim g \ at \ V(a)}$

2. Notions of limits

Definition :

f and g ***are equivalents on*** **neighborhood of** $-\infty)$

$$\Updownarrow$$

$\exists B \in \mathbb{R}^*{}_+$ and a function h, defined on I, and verify

$x \leq -B$ such as $\lim\limits_{x\to-\infty} h(x) = 1$ and that $f(x) = g(x)h(x)$.

We use the notation $f \sim_{-\infty} g$

2. Notions of limits

Examples

- $x \mapsto x$ et $x \mapsto sinx$ are equivalents at $V(0)$.

$$sinx \sim x$$

- $x \mapsto x\ et\ x \mapsto ln(1 + x)\ are$ *equivalents at V(0).*

$$ln(1 + x) \sim x$$

- $x \mapsto x^4\ et\ \ x \mapsto x^4 + 2x - 5$

  are equivalents at $V(+\infty)$ *(or* $V(-\infty)$*).*

2. Notions of limits

Proposition

Si f et g ***are equivalents on*** **neighborhood** $\boldsymbol{a} \in \mathbb{R}$ *and that f has a limit at* $\boldsymbol{a}$*, then g also has this limit this limit at* $\boldsymbol{a}$

Properties:

- $f \sim f_1$ et $g \sim g_1 \Rightarrow fg \sim f_1 g_1$
- $f \sim f_1 \Longrightarrow \frac{1}{f} \sim \frac{1}{f_1}$ if f *and* f_1 do not zero near x_0
- $f \sim f_1$ et $g \sim g_1 \Longrightarrow \frac{f}{g} \sim \frac{f_1}{g_1}$

if g *et* g_1 do not zero near x_0.

- $f \sim f_1 \Longrightarrow f^n \sim {f_1}^n,\ n \in \mathbb{N}$
- $f \sim f_1 \Longrightarrow f^\alpha \sim {f_1}^\alpha,\ \alpha \in \mathbb{R}$

if f *et* f_1 are positives strictly au near x_0.

2. Notions of limits

Attention:

- $f \sim f_1$ et $g \sim g_1 \not\Rightarrow f + g \sim f_1 + g_1$
- $f \sim f_1 \not\Rightarrow e^f \sim e^{f_1}$
- $f \sim f_1 \not\Rightarrow ln(f) \sim \ln(f_1)$

Example: $\lim\limits_{x \to 0} \frac{\sin(x)}{\ln(1+x)} = ?$

We have $sinx \sim_0 x$ *and* $\ln(1+x) \sim_0 x$

$$\Longrightarrow \sin(x)/\ln(1+x) \sim 1 \text{ au } V(0)$$

$$\Longrightarrow \lim_{x \to 0} sin(x)/ln(1+x) = 1.$$