Probability and Statistics

ARRAR - REMITA
National Higher School of Artificial Inteligence

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Part 2 : Introduction to probability

Any experiment that can be repeated, under the same conditions, and whose outcome cannot be predicted is called a random experiment.
The set of all possible outcomes (possible results) of a random experiment, denoted Ω, is called the space of trial (fundamental set or also space of eventualities).
Any element ω of Ω is called an elementary outcome or result.
The trial space Ω can be finite, countable infinite, or have the power of continuity.

Example

When you throw a coin once $\Omega = \{P, F\}$.

Example

In the case where the throws stop as soon as you get heads
$\Omega = \{P, \{F, P\}, \{F, F, P\}, \{F, F, F, P\}, \{F, F, F, F, P\}, \cdots\}$.

Example

When observing the life of a lamp $\Omega =]0, \infty[= \mathbb{R}_+^*$.

We call an event all subset A of the trial space Ω associated with a random experiment .
The set $\mathcal{P}(\Omega)$ of events in Ω is stable for the usual operations on sets. It is therefore natural to think of defining a probability measure on this family.

- Let there be an event A associated with a random experiment, whose trial space is Ω, we say that A is realized if the elementary outcome ω that occurred as a result of the event belongs to the set A.
- The non-realization of the event A is also an event denoted $\overline{A}$.
- Ω is called the certain event and $\emptyset$ is called the impossible event.

- If the realization of event A leads to the realization of another event B, we say that event A implies event B, and we note $A \subset B$.
- Two events A and B are said to be identical if they either happen or do not happen at the same time, and are denoted $A = B$.
- The intersection of two events A and B is an event noted $A \cap B$, which occurs if and only if both events are realized simultaneously ; i.e.

$$A \cap B = \{\omega \in \Omega : \omega \in A \text{ and } \omega \in B\}.$$

- Two events A and B are said to be incompatible if the realization of one leads to the non-realization of the other. That is to say, their intersection is the impossible event $\emptyset$.

Definition

The events $A_1, A_2, \cdots, A_n, \cdots$, are said to be 2 by 2 incompatible if they are such that

$$A_i \cap A_j = \emptyset, \forall i, j \in \mathbb{N}^* \text{ with } i \neq j.$$

- The union of two events A and B is an event noted $A \cup B$, which occurs if at least one of these two events are realized simultaneously ; that is to say

$$A \cup B = \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B\}.$$

- The difference of two events A and B is an event noted $A - B$, which happens when A occurs and B does not occur.
- The symmetric difference of two events A and B is an event denoted $A \triangle B$, which occurs if one and only one of the two events takes place,

$$\begin{aligned} A \triangle B &= (A - B) \cup (B - A) \\ &= \left(A \cap \overline{B}\right) \cup \left(B \cap \overline{A}\right). \end{aligned}$$

Example

A coin is thrown 3 times in a row, independently. We consider the events

A : The first roll gives the Heads (Pile) side

B : Tails (Face) appear at least twice

C : The first roll gives the Tails side

D : Tails appears at most once

E : Tails appears at most three times

F : Face appears exactly three times

G : Face appears at least four times.

1. Determine Ω and the events listed.
2. Do some operations.

Let Ω be a space of trial, associated to a certain random experiment, and let us consider a countable finite or infinite sequence of events $A_1, \cdots, A_n, \cdots$.

Definition

A (finite or infinite countable) collection of events defines a complete system of events, for the trial space Ω, if it satisfies the following conditions :

1. $A_i \neq \emptyset, \forall i \in \{1, 2, \cdots\}$, (no event is impossible) ;
2. $A_i \cap A_j = \emptyset, \forall i \neq j \in \{1, 2, \cdots\}$, (the events are two by two incompatible) ;
3. $\bigcup\limits_{i \geq 1} A_i = \Omega$, (their union is the certain event Ω).

Example

$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$
$C_1 = \{A, \overline{A}\}$ where A any non-empty part of Ω.
$C_2 = \{\{\omega_1\}, \{\omega_2, \omega_3, \omega_4\}\}$
$C_3 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_1, \omega_2, \omega_3\}\}$
$C_4 = \{A \cap B, \overline{A} \cap \overline{B}, A \Delta B\}$.

Definition

A non-empty family $\mathcal{A}$, of parts of the event space Ω, is an algebra of events if the following conditions are satisfied

1. $\forall A \in \mathcal{A}, \overline{A} \in \mathcal{A}$ ($\mathcal{A}$ is stable by passing to the complementary) ;
2. $\forall A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cup A_2 \in \mathcal{A}$ ($\mathcal{A}$ is stable with respect to the union of two events).

Remarque : particular algebras

1. Trivial (rough) algebra $\Omega; \mathcal{A} = \{\Omega, \emptyset\}$;
2. Finest algebra $\mathcal{B} = \mathcal{P}(\Omega)$;
3. Bernoulli algebra : A random experiment with two outcomes is called a Bernoulli experiment, $\Omega = \{\omega_1, \omega_2\}$, the associated algebra is given by $\mathcal{P}(\Omega) = \{\Omega, \emptyset, \{\omega_1\}, \{\omega_2\}\}$.

If $\mathcal{A}$ is an algebra of events, on an trial space Ω. Then we have

Theorem

i $\Omega \in \mathcal{A}$ *and* $\emptyset \in \mathcal{A}$;

ii $\forall A_1, A_2 \in \mathcal{A} \Rightarrow A_1 - A_2 \in \mathcal{A}$ *($\mathcal{A}$ is stable with respect to the difference of two events)* ;

iii $\forall A_1, A_2, \cdots A_n \in \mathcal{A} \Rightarrow \cup_{i=1}^{n} A_i \in \mathcal{A}$ *($\mathcal{A}$ is stable with respect to the finite union of events)* ;

iv $\forall A_1, A_2, \cdots A_n \in \mathcal{A} \Rightarrow \cap_{i=1}^{n} A_i \in \mathcal{A}$ *($\mathcal{A}$ is stable with respect to the finite intersection of events).*

Example

Let Ω be a trial space associated with a random experiment.

a. Consider the complete system A_1 and A_2 for Ω. Construct the algebra generated by this system. How many elements does it have ?
b. Construct the smallest algebra containing the complete system A_1, A_2 and A_3. How many elements does it have ?

Example

Consider two events A_1 and A_2 associated with a randomized experiment $\mathcal{E}$, whose trial space is Ω and let be the event family

$$\begin{aligned} C_1 &= \{\Omega, \emptyset, A_1, A_2, \overline{A}_1, \overline{A}_2, A_1 \cup A_2, A_1 \cup \overline{A}_2, \overline{A}_1 \cup A_2, \overline{A}_1 \cup \overline{A}_2, \\ &\quad A_1 \cap A_2, A_1 \cap \overline{A}_2, \overline{A}_1 \cap \overline{A}_2, (A_1 \cap A_2) \cup \left(\overline{A}_1 \cap \overline{A}_2\right)\} \end{aligned}$$

a. Is the event class C_1 an algebra ?

b. In case C_1 is not an algebra, complete it to get one.

Definition

A non-empty family $\mathcal{A}$ of parts of the trial space Ω is a $\sigma-$ algebra of events if the following conditions are satisfied

1. $\forall A \in \mathcal{A} \Longrightarrow \overline{A} \in \mathcal{A}$, ($\mathcal{A}$ is stable by passing to the complementary) ;
2. $\forall A_1, A_2, \cdots, A_n, \cdots \in \mathcal{A} \Longrightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{A}$, ($\mathcal{A}$ is stable with respect to the infinite or countable union).

Theorem

Let $\mathcal{A}$ a $\sigma-$algebra of events on Ω then

a. *$\mathcal{A}$ is an algebra of events on Ω.*

b. *$\mathcal{A}$ is stable with respect to the intersection of an infinite number of its elements.*

Démonstration.

a. Obvious.

b. $\forall A_1, \cdots, A_n, \cdots \in \mathcal{A} \Longrightarrow \overline{A}_1, \cdots, \overline{A}_n, \cdots \in \mathcal{A}$

$$\Longrightarrow \bigcup_{i=1}^{\infty} \overline{A}_i \in \mathcal{A} \Longrightarrow \overline{\bigcup_{i=1}^{\infty} \overline{A}_i} \in \mathcal{A} \Longrightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}.$$

$\square$

Theorem

If $\mathcal{A}_i, i \in I$ (where I is an arbitrary part of $\mathbb{N}$), is a $\sigma-$ algebra of events associated at the event space Ω. So, the intesection $\cap_{i \in I} \mathcal{A}_i$ is also a $\sigma-$ algebra on Ω.

Démonstration.

Let $\mathcal{A}_i\,; i \in I$ be the $\sigma-$algebras $\mathcal{A}_i\,; i \in I$ and define $\mathcal{B} = \cap_{i \in I} \mathcal{A}_i$.

a. $$\forall B \in \mathcal{B} = \cap_{i \in I} \mathcal{A}_i \Longrightarrow B \in \mathcal{A}_i, \forall i \in I$$
$$\Longrightarrow \overline{B} \in \mathcal{A}_i, \forall i \in I \Longrightarrow \overline{B} \in \cap_{i \in I} \mathcal{A}_i, \forall i \in I.$$

□

Démonstration.

b. $\forall B_1, \dots, B_n, \dots \in \mathcal{B} \Longrightarrow B_1 \in \cap_{i\in I}\mathcal{A}_i, \cdots, B_n \in \cap_{i\in I}\mathcal{A}_i, \cdots$
$\Longrightarrow B_1, \dots, B_n, \dots \in \mathcal{A}_i, \forall i \in I$
$\Longrightarrow \bigcup_{i=1}^{\infty} B_i \in \mathcal{A}_i, \forall i \in I \Longrightarrow \bigcup_{i=1}^{\infty} B_i \in \cap_{i\in I}\mathcal{A}_i = \mathcal{B}.$

□

- The number of possibilities to order p elements taken among n with repetition, is called arrangements with repetition noted $\mathbb{A}_n^p = n^p$.
- The number of possibilities to order p elements taken among n without repetition, is called arrangements without repetition noted

$$A_n^p = \frac{n!}{(n-p)!} = n\,(n-1)\,(n-2)\cdots(n-p+1)\,.$$

  - If $p = n$; $\mathbb{A}_n^n = n^n$ and is called permutation with repetition of n objects.
  - If $p = n$; $A_n^n = n!$ and is called permutation without repetition of n objects, also denoted $\mathcal{P}_n$.

- The number of possibilities to choose p elements among n without taking twice the same elements and without ordering them, is called combinations of p objects among n discernible objects is :

  - Cases without repetition :
    $C_n^p = \frac{A_n^p}{p!} = \frac{n!}{p!(n-p)!}; 0 \leq p \leq n.$
  - Cases with repetition :
    $C_{n+p-1}^p = \frac{A_{n+p-1}^p}{p!} = \frac{(n+p-1)!}{p!(n-1)!}; p \in \mathbb{N}^*.$

Let Ω be a trial space, finite or infinite (countable or not), associated to a random experiment $\mathcal{E}$, and consider an algebra ($\sigma-$algebra) $\mathcal{A}$ associated with Ω.

Let Ω be a trial space, finite or infinite (countable or not), associated to a random experiment $\mathcal{E}$, and consider an algebra ($\sigma-$algebra) $\mathcal{A}$ associated with Ω. The couple $(\Omega, \mathcal{A})$ is called a probabilizable space.

Let the set function $\mathbb{P}$ defined on $\mathcal{A}$ and with values in $\mathbb{R}$ by

$$\begin{aligned} \mathbb{P} \quad : \quad \mathcal{A} &\longrightarrow \mathbb{R} \\ A &\longmapsto \mathbb{P}\left(A\right) \end{aligned}$$

Definition

The application $\mathbb{P}$ satisfying the following conditions

1. $\forall A \in \mathcal{A}; 0 \leq \mathbb{P}(A) \leq 1$
2. $\mathbb{P}(\Omega) = 1$
3. $\forall A_1, A_2, \cdots, A_n, \cdots \in \mathcal{A}$ with $A_i \cap A_j = \emptyset, i \neq j$ we have $\mathbb{P}\left(\bigcup_{i \geq 1} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ ($\sigma-$additivity property).

is called a probability law (function, measure) and the triplet $(\Omega, \mathcal{A}, \mathbb{P})$ is called a probability space.

Property. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For all the events A and B, the following relations are always verified :

1. $\mathbb{P}(\emptyset) = 0$.
2. $\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A)$.
3. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
4. $A \subseteq B \Longrightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$.

Let a finite trial space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$. $(\Omega, \mathcal{P}(\Omega))$ a space on which we can define a probability $\mathbb{P}$ by giving the numbers $\mathbb{P}(\omega_i) = p_i$ such that $\forall i = 1, \cdots, n; p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$.

Definition

We call uniform probability the fuction

$$\begin{aligned} \mathbb{P} : \mathcal{P}(\Omega) &\longrightarrow [0,1] \\ \omega_i &\longrightarrow \frac{1}{n} \end{aligned}$$

More generally, we define the probability $\mathbb{P}$ in the following form $\forall A \in \mathcal{P}(\Omega), \mathbb{P}(A) = \frac{Card(A)}{Card(\Omega)}$.

Let A and B two events from a trial space Ω such that $\mathbb{P}(A) = 0,6$ and $\mathbb{P}(B) = 0,3$.

1. Choose for $\mathbb{P}(A \cup B)$ one of the following values $0,2$ or $0,8$ then calculate $\mathbb{P}\left(\overline{A \cup B}\right)$.
2. Choose for $\mathbb{P}(A \cap B)$ one of the following values $0,2$ or $0,8$ then calculate $\mathbb{P}(A \cup B)$.

One card is drawn from a deck of 52 cards. Let the events be
$C = \{$the card drawn is a heart$\}$, $F = \{$the card drawn is a figure$\}$ whatever the color.
Calculate $\mathbb{P}(C), \mathbb{P}(F), \mathbb{P}(C \cap F)$ and $\mathbb{P}(C \cup F)$.

Let H be an event such that $\mathbb{P}(H) \neq 0$. For all event A, we define

$$\mathbb{P}(A/H) = \frac{\mathbb{P}(A \cap H)}{\mathbb{P}(H)},$$

called the conditional probability of the event A knowing H.

Example

We throw two dice and observe that the obtained sum is equal to 6. What is the probability that one of the two dice has given the result 2 ?

Let

$B = \{\text{sum of the dice} = 6\} = \{(2;4), (4;2), (1;5), (5;1), (3;3)\}$

$A = \{\text{one of the two dice gives } 2\}$

$A \cap B = \{(2;4), (4;2)\}$hence $\mathbb{P}(A/B) = \frac{2}{5}$.

Definition

Let $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ be a probability space and H a fixed event such that $\mathbb{P}(H) \neq 0$. Then the function $\mathbb{P}(\cdot/H)$ defined by

$$\begin{array}{rcl} \mathbb{P}(\cdot/H) : \mathcal{P}(\Omega) & \longrightarrow & [0,1] \\ A & \longrightarrow & \mathbb{P}(A/H) \end{array}$$

is a new probability on $(\Omega, \mathcal{P}(\Omega))$.

Proposition. The function $\mathbb{P}\left(\cdot/H\right)$ verify

1. $\mathbb{P}\left(\emptyset/H\right) = 0, \mathbb{P}\left(\Omega/H\right) = 1$, and if $A \supseteq H, \mathbb{P}\left(A/H\right) = 1$.
2. Let $A_1, \cdots, A_n \cdots$ a sequence of events two by two incompatible, then $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i/H\right) = \sum_{i=1}^{n} \mathbb{P}\left(A_i/H\right)$.
3. For all $A \in \mathcal{P}\left(\Omega\right), \mathbb{P}\left(\overline{A}/H\right) = 1 - \mathbb{P}\left(A/H\right)$.
4. For all $A \in \mathcal{P}\left(\Omega\right)$ and $B \in \mathcal{P}\left(\Omega\right)$, if $A \subset B, \mathbb{P}\left(A/H\right) \leq \mathbb{P}\left(B/H\right)$.
5. For all $A \in \mathcal{P}\left(\Omega\right)$ and $B \in \mathcal{P}\left(\Omega\right)$, $\mathbb{P}\left(A \cup B/H\right) = \mathbb{P}\left(A/H\right) + \mathbb{P}\left(B/H\right) - \mathbb{P}\left(A \cap B/H\right)$.
6. Let A and B be non-zero probability events, then $\mathbb{P}\left(A \cap B\right) = \mathbb{P}\left(A/B\right)\mathbb{P}\left(B\right) = \mathbb{P}\left(B/A\right)\mathbb{P}\left(A\right)$.

Let be the events $A_1, \cdots, A_n$ such that they form a partition of the fundamental set Ω, i.e.
$\forall i \neq j, A_i \cap A_j = \emptyset$ and $\bigcup\limits_{i=1}^{n} A_i = \Omega$.
Let B be any event, from $\Omega = A_1 \cup A_2 \cup \cdots \cup A_n$ and from $B \cap \Omega = B$,we deduce

$$\begin{aligned} B &= B \cap (A_1 \cup A_2 \cup \cdots \cup A_n) \\ &= (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n). \end{aligned}$$

The events $B \cap A_1, B \cap A_2, \cdots, B \cap A_n$ are two by two incompatible so we obtain the **formula of total probability**

$$\mathbb{P}(B) = \mathbb{P}(B \cap A_1) + \mathbb{P}(B \cap A_2) + \cdots + \mathbb{P}(B \cap A_n).$$

From $\mathbb{P}(B/A_i)\mathbb{P}(A_i) = \mathbb{P}(B \cap A_i), \forall i = 1, \cdots, n$ and the formula of total probability, we have

$$\mathbb{P}(B) = \mathbb{P}(B/A_1)\mathbb{P}(A_1) + \cdots + \mathbb{P}(B/A_n)\mathbb{P}(A_n).$$

and since

$$\mathbb{P}(B \cap A_i) = \mathbb{P}(A_i/B)\mathbb{P}(B) = \mathbb{P}(B/A_i)\mathbb{P}(A_i)$$

we get the **Bayes formula**

$$\mathbb{P}(A_i/B) = \frac{\mathbb{P}(B/A_i)\mathbb{P}(A_i)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(B/A_i)\mathbb{P}(A_i)}{\mathbb{P}(B/A_1)\mathbb{P}(A_1) + \mathbb{P}(B/A_2)\mathbb{P}(A_2) + \cdots + \mathbb{P}(B/A_n)\mathbb{P}(A_n)}$$

Definition

Let $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ be a probability space. Two events A and B from this space are said to be independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

Proposition.

1. If A and B are independent, it is the same for the pairs of events A and $\overline{B}$, $\overline{A}$ and B, $\overline{A}$ and $\overline{B}$.
2. in the case where A and B have non-zero probabilities, A and B are independent if and only if

$$\mathbb{P}(A/B) = \mathbb{P}(A) \text{ or } \mathbb{P}(B/A) = \mathbb{P}(B).$$

A student must take an oral exam with one of the three teachers T_1, T_2 and T_3. The choice of the teacher is random. We estimate that the student's chances of success are 60% with T_1, 40% with T_2 and 55% with T_3.

1. What is the probability that the student succeeds in his oral exam ?
2. If he succeeds, what is the probability that he passed the exam with T_2 ?
3. If he failed, what is the probability that he did not take his oral with T_1 ?

Consider the random drawing of a card from a deck of 32 cards. Let A be the event "draw an ace", B the event "draw a hearts" and C the event "draw a red ace".
Study the independence of the events A, B and C.