CHAPTER 2

PRIMITIVES AND INTEGRALS

Mathematical Analysis 1 , ENSIA 2024

PRIMITIVES AND INTEGRALS

Part I

THE PRIMITIVE OF FUNCTION

The subject of this course and subsequent ones is integral calculus, given its significance in physics, chemistry, electronics, signal processing, etc. The outline for this session is as follows:

1. **Primitive of a Function**
2. **Primitives of the Same Function**
3. **Primitive with a Given Value**
4. **Existence of Primitives**
5. **Indefinite Integral and Properties**
6. **Primitives and Areas**
7. **Integral of a Continuous Function**

THE PRIMITIVE OF FUNCTION

Let I denote an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function.

Definition

A primitive of f on I is any function $F : I \to \mathbb{R}$ that is differentiable on I and such that:

$$\forall x \in I, \quad F'(x) = f(x).$$

Example

$$F : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3 + \frac{1}{2}x^2 - x$$

is a primitive on $\mathbb{R}$ of $f : x \mapsto 3x^2 + x - 1$.

Example

$$F :]0, +\infty[\to \mathbb{R}, \quad x \mapsto x \ln(x) - x$$

is a primitive on $]0, +\infty[$ of $f : x \mapsto \ln(x)$.

Example

$$F : \mathbb{R} \to \mathbb{R}, \quad x \mapsto F(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a primitive on $\mathbb{R}$ of $f : x \mapsto \begin{cases} 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

Primitives of the Same Function

Definition

Let $f : I \to \mathbb{R}$ be a function having a primitive F on I. Then, it has infinitely many primitives on I.
The set of all these primitives is exactly the set of functions from $I \to \mathbb{R}$ in the form $F + C$, where C is a real constant.

$$(I \to \mathbb{R}), \quad (x \mapsto F(x) + C), \quad C \in \mathbb{R}$$

Proof.

Let F be a primitive of f on I, and let C be a real number.
It is clear that the function $x \in I \mapsto G(x) = F(x) + C$ is a primitive of f on I.
Conversely, let G be a primitive of f on I.
We have:

$$(G(x) - F(x))' = G'(x) - F'(x) = f(x) - f(x) = 0$$

Primitives of the Same Function

Proof.

This implies:

$$\exists C \in \mathbb{R} \quad \text{such that} \quad \forall x \in I, \quad G(x) - F(x) = C$$

Therefore, for every x in I, $G(x) = F(x) + C$. □

Example

The primitives of the function f defined on $\mathbb{R}$ by:

$$f(x) = \frac{1}{1+x^2} - x$$

are the functions:

$$F(x) = \arctan(x) - \frac{1}{2}x^2 + C, \quad C \in \mathbb{R}$$

Example

The primitives of the function f defined on $\mathbb{R}$ by:

$$f(x) = xe^{x^2}$$

are the functions:

$$F(x) = \frac{1}{2}e^{x^2} + C, \quad C \in \mathbb{R}$$

Note: The assumption I being an interval is essential.
Consider the functions F and G defined on $\mathbb{R}^*$ by:

$$F(x) = \frac{1}{x}$$

and

$$G(x) = \begin{cases} \frac{1}{x} + 1 & \text{if } x > 0 \\ \frac{1}{x} & \text{if } x < 0 \end{cases}$$

The functions F and G are differentiable on $\mathbb{R}^*$ and share the same derivative function, namely $-\frac{1}{x^2}$.
However, there exists no real C such that $F(x) = G(x) + C$.

Primitive with a Given Value

There exists one and only one primitive that takes a given value $y_0 \in \mathbb{R}$ at a given point $x_0 \in I$.

Proof.

Let F be a primitive of f on I, and let $x_0 \in I$. We seek a primitive G of f on I such that $G(x_0) = y_0$, where $y_0 \in \mathbb{R}$.
According to the previous statement:

$$\exists C \in \mathbb{R}, \quad \forall x \in I, \quad G(x) = F(x) + C$$

$$G(x_0) = F(x_0) + C = y_0 \implies C = y_0 - F(x_0)$$

Therefore, for every $x \in I$, the function $x \mapsto F(x) + (y_0 - F(x_0))$ is the unique primitive of f on I such that $G(x_0) = y_0$. □

Example

The antiderivatives of the function f defined on $\mathbb{R}$ by:

$$f(x) = \frac{1}{1+x^2} - x$$

are the functions: $F(x) = Arctanx - \frac{1}{2}x^2 + C,\ C \in \mathbb{R}$

$$F(1) = 0 \quad \Longleftrightarrow \quad \frac{\pi}{4} - \frac{1}{2} + C = 0 \quad \Longleftrightarrow \quad C = \frac{1}{2} - \frac{\pi}{4}$$

Therefore: $x \in \mathbb{R} \longrightarrow Arctanx - \frac{1}{2}x^2 + \frac{1}{2} - \frac{\pi}{4}$ is the unique antiderivative of f that equals 0 at the point $x_0 = 1$.

Existence of Primitives

Theorem

Every continuous function on an interval I in $\mathbb{R}$ has antiderivatives.

Example

Examples of functions with antiderivatives on $\mathbb{R}$:

$$f_1(x) = \mathrm{e}^{-x^2}$$
$$f_2(x) = \cos\left(x^2\right)$$
$$f_3(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

Indefinite Integral and Properties

Let I be an interval in $\mathbb{R}$ and $f : \ \to \mathbb{R}$ be a function.

Definition

The "indefinite integral" of f on I, denoted by $\int \ \mathrm{d}x$, is the set of primitives of f on I. For a primitive F of f on I, we have:

$$\int f(x)\,\mathrm{d}x = \{x \in \ \mid F(x) + C,\ C \in \mathbb{R}\}.$$

In accordance with the convention:

$$\int f(x)\,\mathrm{d}x = F(x) + C,\ C \in \mathbb{R}.$$

Indefinite Integral and Properties

Example

$$\int \left(\frac{1}{1+x^2} - x \right) \mathrm{d}x = Arctan x - \frac{1}{2}x^2 + C, \quad C \in \mathbb{R}, \quad I = \mathbb{R}.$$

Example

$$\int \sqrt{x}\, \mathrm{d}x = \frac{2}{3}x^{3/2} + C, \quad C \in \mathbb{R}, \quad I = [0, +\infty[.$$

Example

$$\int x\mathrm{e}^{-x^2}\, \mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-x^2} + C, \quad C \in \mathbb{R}, \quad I = \mathbb{R}.$$

Theorem

1. *If F and G are primitives on I of f and g respectively, then $F+G$ is a primitive on of $f+g$. We have:*

$$\int (f(x)+g(x))\,\mathrm{d}x = F(x)+G(x)+C = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

2. *For any $\lambda \in \mathbb{R}$, λF is a primitive on I of λf.*
"We have:"

$$\int \lambda f(x)\,\mathrm{d}x = \lambda F(x) + \mathcal{C}.$$

"For $\lambda \neq 0$, we have:"

$$\int \lambda f(x)\,\mathrm{d}x = \lambda \int f(x)\,\mathrm{d}x.$$

Indefinite Integral and Properties

Proof.

It follows from the relations:

$$(F+F)' = F' + G' = f + g \quad \text{and} \quad (\lambda F)' = \lambda F' = \lambda F$$

□

Example

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n$$

Result:

$$\int P(x)\,\mathrm{d}x = a_0 x + \frac{a_1}{2}x^2 + \ldots + \frac{a_n}{n+1}x^{n+1} + \mathcal{C}.$$

Indefinite Integral and Properties

Example

$$\int (\cos^2 x + x + \sin x)\,\mathrm{d}x = \int \left(\frac{1 + \cos 2x}{2} + x + \sin x \right)\,\mathrm{d}x$$

Result:

$$= \frac{x}{2} + \frac{\sin 2x}{4} + \frac{x^2}{2} - \cos x + \mathcal{C}.$$

Indefinite Integral and Properties

1. $\int x^m \, dx, = \frac{1}{m+1} x^{m+1} + c; \; m \in \mathbb{Z}, \; m \neq -1$
2. $\int \frac{1}{x} \, dx = \log |x| + c$
3. $\int \sin x dx = -\cos x + c$
4. $\int \cos x dx = \sin x + c$
5. $\int e^x \, dx = e^x + c$
6. $\int \sinh(x) \, dx = \cosh(x) + c$
7. $\int \cosh(x) \, dx = \sinh(x) + c$
8. $\int \frac{-1}{\sin^2 x} = \cot x \, dx + c$
9. $\int \frac{1}{(1+x^2)} \, dx = \arctan x + c$
10. $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + c$
11. $\int \frac{1}{\cos^2 x} \, dx = \tan x + c$
12. $\int \frac{-1}{\sqrt{1-x^2}} \, dx = \arccos x + c$

Indefinite Integral and Properties

13. $\int \frac{1}{\cosh^2 x} dx = \tanh x + c$
14. $\int \frac{1}{\sqrt{x^2+1}}\, dx = \operatorname{argsh} x + c$
15. $\int \frac{1}{\sqrt{x^2-1}}\, dx = \operatorname{argch} x + c$
16. $\int \frac{1}{1-x^2}\, dx = \operatorname{Argth} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + c$
17. $\int \tan(x)\, dx = -\ln |\cos(x)| + c$
18. $\int \cot(x)\, dx = \ln |\sin(x)| + c$
19. $\int \sec(x)\, dx = \ln |\sec(x) + \tan(x)| + c$
20. $\int \csc(x)\, dx = -\ln |\csc(x) + \cot(x)| + c$

Primitives and Areas

We assume the existence of regions D in the plane called "squareable," meaning that we can assign a positive real number called the "area of D" to them. In particular:
If $f : [a, b] \to \mathbb{R}$ $(a < b)$ is continuous and positive, and if F is a primitive of f on $[a, b]$, then:

$$D = \{(x, y) \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\} \text{ is squareable.}$$

The area of D is given by:

$$\mathcal{A}(D) = F(b) - F(a)$$

Let $h > 0$ be such that $[X, X + h] \subset [a, b]$. According to the Intermediate Value Theorem, there exist c_X and d_X in $[X, X + h]$ such that:

$$f(c_X) = \inf_{x \in [X, X+h]} f(x) \quad \text{and} \quad f(d_X) = \sup_{x \in [X, X+h]} f(x).$$

Primitives and Areas

We have:

$$f(c_X)h \leq g(X+h) - g(X) \leq f(d_X)h$$

This implies:

$$f(c_X) \leq \frac{g(X+h) - g(X)}{h} \leq f(d_X)$$

Which further implies:

$$\lim_{h \to 0} \frac{g(X+h) - g(X)}{h} = f(X).$$

This implies that g is a primitive of f on $[a, b]$.
The same reasoning holds for $h < 0$.

Primitives and Areas

Let F be a primitive of f on $[a, b]$. There exists a constant C such that $F(x) = g(x) + C$ for $x \in [a, b]$. In particular,

$$F(a) = g(a) + C = C \quad \left(\text{since } \int_a^a f(x)\,dx = 0\right).$$

Therefore, $F(b) - F(a) = g(b) =$ the definite integral of f over $[a, b]$.

Example

$$\int_0^2 (1 + x^2)\,dx = \left[\frac{x}{2} + \frac{x^3}{3}\right]_0^2 = \frac{14}{3}.$$

Integral of a Continuous Function

Definition

Let $f : I \to \mathbb{R}$ be a continuous function, and F be a primitive of f. Let a and b be two real numbers in I.
The real number $F(b) - F(a)$ does not depend on the chosen primitive. It is denoted by:

$$\int_a^b f(x)\,dx$$

and is called the integral of f over $[a, b]$.

Indeed, if G is another primitive of f on I, we know that there exists a constant $C \in \mathbb{R}$ such that $G = F + C$, and therefore,

$$G(b) - G(a) = F(b) - F(a).$$

Integral of a Continuous Function

Example

$$\int_0^{\pi/2} \sin(x)\,dx = -[\cos(\pi/2) - \cos(0)] = 1$$

Example

$$\int_0^1 \frac{1}{1+x^2}\,dx = [\arctan(1) - \arctan(0)] = \frac{\pi}{4}$$

Properties:
Let f and g be two functions continuous on $[a, b]$, and $\lambda \in \mathbb{R}$. Then, we have the following properties:

1. Linearity
2. Positivity
3. Order
4. Chasles' Relation
5. Mean Value Formulas

Integral of a Continuous Function-**Linearity**

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Proof.

Since f and g are continuous on $[a, b]$, they admit primitives F and G.

$$\begin{aligned}\int_a^b (f(x) + g(x))\,dx &= (F + G)(b) - (F + G)(a) \\ &= (F(b) - F(a)) + (G(b) - G(a)) \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

$\square$

Integral of a Continuous Function-**Linearity**

$$\int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx$$

Proof.

Since λF is a primitive of λf, we have:

$$\int_a^b \lambda f(x)\, dx = (\lambda F)(b) - (\lambda F)(a)$$

$$= \lambda(F(b) - F(a))$$

$$= \lambda \int_a^b f(x)\, dx$$

$\square$

Integral of a Continuous Function-**Positivity**

1. $$f \geq 0 \implies \int_a^b f(x)\,dx \geq 0, \quad (a < b).$$

2. $$f \geq 0 \text{ and } f \neq 0 \implies \int_a^b f(x)\,dx > 0.$$

Proof.

$$f \geq 0 \text{ on } [a, b] \implies F \text{ is increasing on } [a, b]$$

$$\implies \int_a^b f(x)\,dx = F(b) - F(a) \geq 0.$$

Suppose $\int_a^b f(x)\,dx = 0$.

$$\implies F(a) = F(b)$$

$$\implies F \text{ is constant on } [a, b] \implies F' = f \equiv 0$$

Integral of a Continuous Function -**Positivity**

Proof.

This contradicts the assumption $f \neq 0 \implies \int_a^b f(x)\,dx > 0.$

$\square$

$$f \leq g \implies \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx, \quad (a < b).$$

Consequence of the previous properties applied to $g - f$. In particular:
$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

Example

Consider the limit $\lim_{n\to\infty} I_n$ where $I_n = \int_0^\pi \frac{\sin x}{n+x}\,dx$.
The function $x \in [0, \pi] \mapsto \frac{\sin x}{n+x}$ is continuous, and we have $0 \leq \frac{\sin x}{n+x} \leq \frac{1}{n}$.

$$\implies 0 \leq \int_0^\pi \frac{\sin x}{n+x}\,dx \leq \int_0^\pi \frac{1}{n}\,dx = \frac{\pi}{n} \implies \lim_{n\to\infty} I_n = 0.$$

Example

Consider the limit $\lim_{n\to\infty} I_n$ where $I_n = \int_1^{1+\frac{1}{n}} \sqrt{1+x^n}\,dx$, for $n \geq 1$. We know that the sequence $(1+\frac{1}{n})^n$ is increasing and converges to the number e.

$$1 \leq x \leq 1+\frac{1}{n} \implies 1 \leq x^n \leq \left(1+\frac{1}{n}\right)^n \leq e < 3$$

$$\implies \sqrt{2} \leq \sqrt{1+x^n} < 2$$

$$\int_1^{1+\frac{1}{n}} \sqrt{2}\,dx \leq \int_1^{1+\frac{1}{n}} \sqrt{1+x^n}\,dx < \int_1^{1+\frac{1}{n}} 2\,dx$$

$$\implies \frac{\sqrt{2}}{n} \leq I_n < \frac{2}{n} \implies \lim_{n\to\infty} I_n = 0.$$

Integral of a Continuous Function-**Order**

Example

Calculate $\lim_{a\to+\infty} I(a)$, where $I(a) = \int_a^{2a} \frac{1}{\sqrt{1+x^2+x^4}}\, dx$, $a > 0$.

The function $x \mapsto \frac{1}{\sqrt{1+x^2+x^4}}$ is continuous on $\mathbb{R}$. Moreover:

$$\frac{1}{x^2+1} < \frac{1}{\sqrt{1+x^2+x^4}} < \frac{1}{x^2}.$$

$$\int_a^{2a} \frac{1}{x^2+1}\, dx < \int_a^{2a} \frac{1}{\sqrt{1+x^2+x^4}}\, dx < \int_a^{2a} \frac{1}{x^2}\, dx.$$

$$\arctan(2a) - \arctan(a) < I(a) < \frac{1}{a} - \frac{1}{2a} = \frac{1}{2a}.$$

As $\lim_{x\to+\infty} \arctan(x) = \frac{\pi}{2}$, we conclude that $\lim_{a\to+\infty} I(a) = 0$.

Let f be continuous on I, and let a, b, c be three real numbers in I. We have:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$= (F(b) - F(c)) + (F(c) - F(a))$$

$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

$\square$

Integral of a Continuous Function - **Mean Value Formulas**

For $c \in]a, b[$, we have $\frac{1}{b-a} \int_a^b f(x)\, dx = f(c)$.

Proof.

By definition:

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

According to the Mean Value Theorem applied to F on $[a, b]$, there exists $c \in]a, b[$ such that:

$$F(b) - F(a) = (b-a)F'(c) = (b-a)f(c).$$

Therefore:

$$\frac{1}{b-a} \int_a^b f(x)\, dx = f(c).$$

□

Integral of a Continuous Function- **Mean Value Formulas**

Example

Let $c \in [0,1]$. We claim that

$$\int_0^1 (x-1/2)^2\,dx = (c-1/2)^2$$

The existence of c follows from the mean value theorem applied to $f(x) = (x-1/2)^2$.

$$\int_0^1 (x-1/2)^2\,dx = (c-1/2)^2 \iff \int_0^1 (x^2 - x + 1/4)\,dx = (c-1/2)^2$$

$$\iff (c-1/2)^2 = 1/12 \iff \begin{cases} c_1 = 1/2 + 1/\sqrt{12} \\ c_2 = 1/2 - 1/\sqrt{12} \end{cases}$$

It is clear that c_1 and $c_2 \in [0,1]$.

Integral of a Continuous Function- **Mean Value Formulas**

Example

Calculate:

$$\lim_{a \to 0} \frac{\sin a}{1 - \cos a} \int_0^a \cos(x^2)\, dx$$

According to the mean value theorem:

$$\exists c \in (0, a) \quad \text{s.t.} \quad \int_0^a \cos(x^2)\, dx = a \cos(c^2)$$

$$\implies \frac{\sin a}{1 - \cos a} \int_0^a \cos(x^2)\, dx = \frac{a \sin a}{1 - \cos a} \cos(c^2)$$

As $a \to 0$, $c \to 0$, and $\lim_{a \to 0} \cos(c^2) = 1$.

$$\therefore \lim_{a \to 0} \frac{\sin a}{1 - \cos a} \int_0^a \cos(x^2)\, dx = 2$$

Integral of a Continuous Function- **Mean Value Formulas**

Given $g \geq 0$ and $g \not\equiv 0$, there exists $c \in [a, b]$ such that:

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

Proof.

Since f is continuous on $[a, b]$, it is bounded on this interval and attains its lower bound m and upper bound M.

$$g \geq 0 \implies \forall x \in [a, b],\ m \cdot g(x) \leq f(x)g(x) \leq M \cdot g(x)$$

$$\implies m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx$$

Given $g \not\equiv 0$, we have:

Integral of a Continuous Function- **Mean Value Formulas**

By the Intermediate Value Theorem:

$$\exists c \in [a, b] \quad \text{s.t.} \quad \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} = f(c).$$

Hence:

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

Example

Define $I(t)$ for $t > 0$ by $I(t) = \int_{2t}^{2t^2} \frac{\cos x}{x}\,dx$.
Using the mean value theorem applied to $f(x) = \cos x$ and $g(x) = \frac{1}{x}$, there exists $c \in [t, 2t]$ such that:

$$I(t) = \cos c \int_{2t}^{2t^2} \frac{1}{x}\,dx.$$

Integral of a Continuous Function- **Mean Value Formulas**

Let f and g be two continuous functions on $[a, b]$ with f increasing and $g \geq 0$. Then, there exists $c \in [a, b]$ such that:

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^c g(x)\,dx + f(b)\int_c^b g(x)\,dx.$$

Indeed, consider the auxiliary function h defined on $[a, b]$ by:

$$h(t) = f(a)\int_a^t g(x)\,dx + f(b)\int_t^b g(x)\,dx - \int_a^b f(x)g(x)\,dx.$$

For all $x \in [a, b]$, we have $f(a)g(x) \leq f(x)g(x) \leq f(b)g(x)$, which implies:

$$f(a)\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq f(b)\int_a^b g(x)\,dx.$$

Thus, h is continuous on $[a, b]$ and $h(a)h(b) \leq 0$. By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that $h(c) = 0$.

Classroom questions

Respond true or false

1. If f has a primitive on an interval I, then f is continuous on I.
2. If F denotes a primitive of f on I, then F is continuous on I.
3. If f^2 has a primitive on I, then the same is true for f.
4. If f is continuous on $[a, b]$ and $\int_a^b f(x)\,dx > 0$, then $f \geq 0$.
5. If f is continuous on $[a, b]$ and $f \geq 0$, then $\int_a^b f(x)\,dx \geq 0$.

Classroom questions

1. If f is continuous on $[a, b]$ and $f \geq 0$, then:

$$\int_a^b f(x)\,dx = 0 \implies f \equiv 0.$$

2. If f and g are continuous and differentiable on $[a, b]$ such that $f \leq g$, then:

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

   Also, for all x in $[a, b]$, $f'(x) \leq g'(x)$.
3. $\int_a^b \sqrt{(x-a)(b-x)}\,dx = \frac{\pi}{2}\left(\frac{b-a}{2}\right)^2.$
4. $\int_0^2 \frac{1}{\sqrt{1-x^2}}\,dx = \int_0^2 (\arcsin x)'\,dx = \arcsin 2.$
5. $\int_{-1}^1 \frac{1}{x^3}\,dx = 0.$

Classroom questions

1. If f and g are continuous on $[a, b]$, then:
$$\int_a^b f(x)g(x)\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right).$$
2. $\int_0^{\pi} \frac{\sin x}{x^4 - x^2 + 1}\,dx > 0.$
3. $\int_0^{\pi} (x - |\cos x|)\,dx = \frac{\pi^2}{2} - 2.$
4. In the mean value theorems, the real number c is unique.

CHAPTER 2

PRIMITIVES AND INTEGRALS

Mathematical Analysis 1 , ENSIA 2024

PRIMITIVES AND INTEGRALS

Part II

Integration techniques

1. **Integration by parts**
2. **Reduction formulas**
3. **Change of variable**
4. **Cases of even, odd, periodic functions**

Integration techniques

"I" denotes an interval in $\mathbb{R}$, and $f : I \to \mathbb{R}$ is a function.

Definition

We will say that f is of class C^1 on I if f is differentiable on I and its derivative f' is continuous on I.

This definition is useful in the two theorems that will follow:
1. Integration by Parts 2. Change of Variable

Integration by Parts

Let f and g be two functions of class C^1 on I. We have the following formula, known as the integration by parts formula:

$$\int_1 f(x)g'(x)\,dx = f(x)g(x) - \int_1 f'(x)g(x)\,dx$$

For any $a \in I$ and $b \in I$,

$$\int_a^b f(x)g'(x)\,dx = \Big|f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

where $\Big|f(x)g(x)\Big|_a^b = f(b)g(b) - f(a)g(a)$.

Integration by Parts

Example

$$\int_1 x\cos^2(x)\,dx = \int_1 x\left(\frac{1+\cos(2x)}{2}\right)\,dx = \frac{1}{2}\int_1 x\,dx + \frac{1}{2}\int_1 x\cos(2x)\,dx$$

To calculate a primitive of $x\cos^2(2x)$, we use integration by parts with:

$$\begin{cases} f(x) = x & \Rightarrow \quad f'(x) = 1 \\ g(x) = \frac{\sin(2x)}{2} & \Rightarrow \quad g'(x) = \cos(2x) \end{cases}$$

Applying the integration by parts formula:

$$\int x\cos(2x)\,dx = \frac{x\sin(2x)}{2} - \int \frac{\sin(2x)}{2}\,dx$$

Example

$$\int x\cos^2(x)\,dx = x\frac{\sin(2x)}{2} - \int_1 \frac{\sin(2x)}{2}\,dx$$
$$= x\frac{\sin(2x)}{2} + \frac{\cos(2x)}{4} + K$$

Where K is the constant of integration.
Therefore,

$$\int_1 x\cos^2(x)\,dx = x\frac{\sin(2x)}{2} + \frac{\cos(2x)}{4} + K$$

Hence,

$$\int_1 x\cos^2(x)\,dx = \frac{x^2}{4} + \frac{1}{2}\left(x\frac{\sin(2x)}{2} + \frac{\cos(2x)}{4}\right) + K$$

Simplifying further:

$$\int_1 x\cos^2(x)\,dx = \frac{x^2}{4} + \frac{x}{4}\sin(2x) + \frac{1}{8}\cos(2x) + K$$

Example

$$\int_0^1 \arctan(x)\,dx = x\arctan(x)\big|_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx$$
$$= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$$

Hence:

$$\int_0^1 \arctan(x)\,dx = \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$$

Where $\frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx = \frac{1}{2}\log(1+x^2)\Big|_0^1 = \frac{1}{2}\log(2) - \frac{1}{2}\log(1) = \frac{1}{2}\log(2)$.
Therefore:

$$\int_0^1 \arctan(x)\,dx = \frac{\pi}{4} - \frac{1}{2}\cdot\frac{1}{2}\log(2) = \frac{\pi}{4} - \frac{1}{4}\log(2)$$

Example

$$\int_0^{\frac{\pi}{3}} \frac{x}{\cos^2(x)}\,dx = x\tan(x)\Big|_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan(x)\,dx$$
$$= \frac{\pi}{3}\tan\left(\frac{\pi}{3}\right) + \log(\cos(x))\Big|_0^{\frac{\pi}{3}}$$

Now, $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, and $\log(\cos(\pi/3)) = \log\left(\frac{1}{2}\right) = -\log(2)$.
Hence:

$$\int_0^{\frac{\pi}{3}} \frac{x}{\cos^2(x)}\,dx = \frac{\pi}{3}\sqrt{3} - \log(2)$$

Integration by Parts

Example

Antiderivatives of $F(x) = (x^2 + x + 1)\sin(x)$, $x \in \mathbb{R}$.
We perform integration by parts initially by setting:

$$\begin{cases} f(x) = x^2 + x + 1 & \Rightarrow \quad f'(x) = 2x + 1 \\ g(x) = -\cos(x) & \Rightarrow \quad g'(x) = -\sin(x) \end{cases}$$

This yields:

$$\int (x^2 + x + 1)\sin(x)\,dx = -(x^2 + x + 1)\cos(x) + \int (2x + 1)\cos(x)\,dx$$

Continuing the integration by parts, one would set $u = 2x + 1$ and $dv = \cos(x)\,dx$.

Example

To find an antiderivative of $(2x+1)\cos(x)$, we integrate by parts again, setting:

$$\begin{cases} f(x) = 2x+1 & \Rightarrow \quad f'(x) = 2 \\ g(x) = \sin(x) & \Rightarrow \quad g'(x) = \cos(x) \end{cases}$$

This gives:

$$\int (2x+1)\cos(x)\,dx = (2x+1)\sin(x) - \int 2\sin(x)\,dx = (2x+1)\sin(x) + 2$$

Finally:

$$\int (x^2+x+1)\sin(x)\,dx = -(x^2+x+1)\cos(x) + (2x+1)\sin(x) + 2\cos(x)$$

Integration by Parts

Consequence of the integration by parts theorem:
Let I be an interval of $\mathbb{R}$, and $f : I \to \mathbb{R}$ be a continuous function on I.
Let F_1 be an antiderivative of f, F_2 be an antiderivative of F_1, F_3 be an antiderivative of F_2, and so on.
Consider a polynomial $P(x) = a_0 + a_1 x + \ldots + a_n x^n$.
Then, the formula is given by:

$$\int f(x)P(x)\,dx = P(x)F_1(x) - P'(x)F_2(x) + P''(x)F_3(x) + \ldots + (-1)^n P^{(n)}$$

Integration by Parts

Proof.

"Let's set:"

$$\begin{cases} u = P & \Rightarrow \quad u' = P' \\ v = F_1 & \Rightarrow \quad v' = f \end{cases}$$

Then, we have the first integration by parts:

$$\int_1 f(x)P(x)\,dx = P(x)F_1(x) - \int_1 F_1(x)P'(x)\,dx$$

We integrate by parts a second time by setting:

$$\begin{cases} u = P' & \Rightarrow \quad u' = P'' \\ v = F_2 & \Rightarrow \quad v' = F_1 \end{cases}$$

This leads to:

$$\int f(x)P(x)\,dx = P(x)F_1(x) - P(x)F_1(x) + \int F_2(x)P''(x)\,dx$$

Particular case:

$$f(x) = e^{\lambda x}, \text{ where } \lambda \neq 0$$

According to the previous formula:

$$\int e^{\lambda x} P(x)\, dx = \frac{1}{\lambda}\left(P(x) - \frac{P'(x)}{\lambda} + \frac{P''(x)}{\lambda^2} - \frac{P'''(x)}{\lambda^3} + \ldots + (-1)^n \frac{P^{(n)}(x)}{\lambda^n}\right.$$

$$= \frac{e^{\lambda x}}{\lambda}\left(P(x) - \frac{P'(x)}{\lambda} + \frac{P''(x)}{\lambda^2} - \frac{P'''(x)}{\lambda^3} + \ldots + (-1)^n \frac{P^{(n)}(x)}{\lambda^n}\right) + K$$

The expression in parentheses is a polynomial of the same degree as P.

Integration by Parts

Rather than applying this formula, we seek an antiderivative of $e^{\lambda x}P(x)$ in the form of a product $e^{\lambda x}Q(x)$ where Q is a polynomial of the same degree as P.
The coefficients of Q are obtained by differentiation and identification of the coefficients of monomials of the same degree.

Example

$$\int e^{-x}(x^2+x)\,dx = (ax^2+bx+c)e^{-x}+K$$

When we differentiate the right-hand side, we obtain $e^{-x}(x^2+x)$. The coefficients a, b, and c can be determined by identifying the coefficients of monomials of the same degree in $e^{-x}(ax^2+bx+c)$.

Integration by Parts

Example

Example (continued):

$$(2ax+b)e^{-x}-(ax^2+bx+c)e^{-x}=(-ax^2+(2a-b)x+(b-c))e^{-x}$$

$= \mathrm{e}^{-x}(x^2+x)$

This gives:

$$-ax^2+(2a-b)x+(b-c)=x^2+x$$

By identification, we obtain:

$$\begin{cases} -a=1 \\ 2a-b=1 \\ b-c=0 \end{cases}$$

Example

Example (continued): Which simplifies to:

$$\begin{cases} a = -1 \\ b = -3 \\ c = -3 \end{cases}$$

"In conclusion:"

$$\int e^{-x}(x^2 + x)\,dx = -(x^2 + 3x + 3)e^{-x} + K$$

$$f(x) = \cos(\lambda x) \text{ where } \lambda \neq 0$$

According to the previous formula, we have:

$$\int P(x)\cos(\lambda x)\,dx = \frac{\sin(\lambda x)}{\lambda}\left(P(x) - \frac{1}{\lambda^2}P''(x) + \frac{1}{\lambda^4}P^{(4)}(x) + \right.$$

$$\left. \ldots + \frac{\cos(\lambda x)}{\lambda}\left(\frac{P'(x)}{\lambda} - \frac{1}{\lambda^3}P'''(x) + \frac{1}{\lambda^5}P^{(5)}(x) + \ldots\right) + K\right.$$

To compute an antiderivative of $P(x)\cos(\lambda x)$, it is preferable to use the following, but first, we provide the following definition.

Integration by Parts

$$f, F : I \longrightarrow \mathbb{C}$$

Definition

F is an antiderivative of f if and only if $\mathrm{Re}(F)$ is an antiderivative of $\mathrm{Re}(f)$ and $\mathrm{Im}(F)$ is an antiderivative of $\mathrm{Im}(f)$.

$$\int P(x)\cos(\lambda x)\,dx = \mathrm{Re}\left(\int P(x)e^{i\lambda x}\,dx\right)$$

$$\int P(x)e^{i\lambda x}\,dx = Q(x)e^{i\lambda x} + K$$

Where Q is a complex-coefficient polynomial of the same degree as P, and K is a complex constant.

Example

$$I = \int_1 (x^2 + x - 1)\cos(2x)\,dx = \mathrm{Re}\left(\int_1 (x^2 + x - 1)e^{2ix}\,dx\right)$$

$$\int_1 (x^2 + x - 1)e^{2ix}\,dx = (ax^2 + bx + c)e^{2ix} + K$$

where a, b, c, K are complex numbers.
After differentiation, we get:

$$2iax^2 + (2a + 2ib)x + (b + 2ic) = x^2 + x - 1$$

By identification, we find: $a = -i/2, b = -i/2 + 1/2, c = 3i/4 + 1/4$
Therefore, the result is:

$$I = \left(\frac{1}{2}x + \frac{1}{4}\right)\cos(2x) + \left(\frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}\right)\sin(2x) + C_1$$

where C_1 is an arbitrary constant.

Integration by Parts

Formula of Taylor with Integral remainder:

Theorem

Let $n \in \mathbb{N}$ and $f : [a, b] \to \mathbb{R}$ be a $C^{(n+1)}$ function, meaning $f', f'', \ldots, f^{(n+1)}$ exist and are continuous on $[a, b]$.
We have:

$$f(b) = f(a) + \frac{(b-a)}{1!}f'(a) + \frac{(b-a)^2}{2!}f''(a) + \ldots + \frac{(b-a)^n}{n!}f^{(n)}(a) + \frac{1}{n!}$$

$$\int_a^b f^{(n+1)}(x) \cdot (b-x)^n \, dx$$

Proof.

Proof: Induction and integration by parts reasoning. $\square$

Integration by Parts

Example

$$f(x) = \log(1+x), \quad x > -1.$$

Then:

$$\log(2) = \lim_{n\to\infty}\left(1 - \frac{1}{2} + \frac{1}{3} + \ldots + \frac{(-1)^{n-1}}{n}\right)$$

We apply the Taylor formula with Lagrange remainder on $[0,1]$. For all $n \geq 1$ and $x \in [0,1]$, we have:

$$f^{(n)}(x) = \frac{(-1)^{n-1}}{(1+x)^n}\underset{\Longrightarrow}{(n-1)!} f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

$$\log(2) = \left(1 - \frac{1}{2} + \frac{1}{3} + \ldots + \frac{(-1)^{n-1}}{n}\right) + \frac{1}{n!}\int_0^1 \frac{(-1)^n}{(1+x)^{n+1}} n!(1-x)^n \, dx$$

It can be shown that the last expression tends to 0 after simplification and bounding.

Reduction formulas

Let $I_n = \int_1 (f_n(x))^n \, dx$, where n is a natural number, and f_n is a continuous function on I. If, during integration by parts, we manage to express I_n in terms of I_{n-k} and certain functions, we say that we have obtained a reduction formula.

Example

"For" $x > 0$, we define $I_n = \int_1 (\log x)^n \, dx$. Establish a reduction formula and then calculate I_n.
Let's consider $I_{n+1} = \int_1 (\log x)^{n+1} \, dx$.

Example

Example (continued): We integrate by parts, setting:

$$\begin{cases} f(x) = (\log x)^{n+1} & \Rightarrow f'(x) = (n+1)\frac{(\log x)^n}{x} \\ g(x) = x & \Rightarrow g'(x) = 1 \end{cases}$$

This gives us:

$$I_{n+1} = x(\log x)^{n+1} - (n+1)I_n$$

Another integration by parts applied to I_n yields:

$$I_n = x(\log x)^n - nI_{n-1}$$

Combining these results, we get:

$$I_{n+1} = x(\log x)^{n+1} - (n+1)x(\log x)^n + (n+1)nI_{n-1} + \ldots + C$$

Example

Let $I_n = \int_0^1 x^n \sqrt{1-x}\,dx$. For $n \geq 1$, we seek a reduction formula using integration by parts. Let's set:

$$\begin{cases} f(x) = x^n & \Rightarrow f'(x) = nx^{n-1} \\ g(x) = -\frac{2}{3}(1-x)^{3/2} & \Rightarrow g'(x) = \sqrt{1-x} \end{cases}$$

$$I_n = -\frac{2}{3}x^n(1-x)^{3/2}\Big|_0^1 + \frac{2n}{3}\int_0^1 x^{n-1}(1-x)^{3/2}\,dx$$

Reduction formulas

Example

Example (continued):

$$I_n = \frac{2n}{3}\int_0^1 x^{n-1}(1-x)^{3/2}\,dx = \frac{2n}{3}\int_0^1 x^{n-1}(1-x)\sqrt{1-x}\,dx$$

$$I_n = \frac{2n}{3}(I_{n-1} - I_n)$$

$$I_n = \frac{2n}{3+2n} I_{n-1}$$

Reduction formulas

Example

Example (continued): Therefore:

$$I_1 = \frac{2}{5} I_0 = \frac{2}{5} \cdot \frac{2}{3}$$

$$I_2 = \frac{4}{7} I_1 = \frac{4}{7} \cdot \frac{2}{5} \cdot \frac{2}{3}$$

$$I_3 = \frac{6}{9} I_2 = \frac{6}{9} \cdot \frac{4}{7} \cdot \frac{2}{5} \cdot \frac{2}{3}$$

In general:

$$I_n = \frac{2^{n+1} \cdot n!}{3 \cdot 5 \cdot 7 \cdots (2n+3)}$$

Example

(Wallis integrals):

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n(x)\, dx$$

For $n \geq 2$, we use integration by parts with:

$$f(x) = \cos^{n-1}(x) \quad \Rightarrow \quad f'(x) = -(n-1)\cos^{n-2}(x)\sin(x)$$

$$g(x) = \sin(x) \quad \Rightarrow \quad g'(x) = \cos(x)$$

$$I_n = \sin(x)\cos^{n-1}(x)\Big|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}(x)\sin^2(x)\, dx$$

The term $\sin(x)\cos^{n-1}(x)\Big|_0^{\frac{\pi}{2}}$ evaluates to zero since $\sin(0) = \sin\left(\frac{\pi}{2}\right) = 0$. Therefore:

$$I_n = (n-1)\int^{\frac{\pi}{2}} \cos^{n-2}(x)\sin^2(x)\, dx$$

Reduction formulas

Example

(continued):

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_2 = \frac{1}{2} I_0 = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I_4 = \frac{3}{4} I_2 = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I_6 = \frac{5}{6} I_4 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Reduction formulas

Example

(continued):

$$I_3 = \frac{2}{3} I_1 = \frac{2}{3}$$

$$I_5 = \frac{3}{4} I_3 = \frac{3}{4} \cdot \frac{2}{3}$$

$$I_7 = \frac{5}{6} I_5 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{2}{3}$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_{2p} = \frac{(2p)!}{2^{2p}(p!)^2} \cdot \frac{\pi}{2}$$

$$I_{2p+1} = \frac{2^{2p}(p!)^2}{(2p+1)!}$$

Reduction formulas

Example

(continued): We also have:

$$I_{2p+2} \leq I_{2p+1} \leq I_{2p}$$

$$\frac{2p+1}{2p+2} = \frac{I_{2p+2}}{I_{2p}} \leq \frac{I_{2p+1}}{I_{2p}} \leq 1$$

$$\lim_{p\to\infty} \frac{I_{2p+1}}{I_{2p}} = 1$$

$$\lim_{p\to\infty} \frac{(1\cdot 2\cdot 4\cdot \ldots \cdot 2p)^2}{(1\cdot 3\cdot 5\cdot \ldots \cdot (2p-1))^2\cdot(2p+1)} = \frac{\pi}{2}$$

Change of variable

Let I and J be two intervals of $\mathbb{R}$. Let $\varphi : J \to I$ be a C^1-class function, meaning that φ is differentiable, and its derivative is continuous on .
Let $f : I \to \mathbb{R}$ be a continuous function on .
1. If F is an antiderivative of f on I, then $F \circ \varphi$ is an antiderivative on J of $(f \circ \varphi) \cdot \varphi'$. This is expressed mathematically as:

$$\int (f \circ \varphi) \cdot \varphi' \, dt = (F \circ \varphi)(t) + C,$$

where C is the constant of integration.
2. For any $c, d \in J$, the equality holds:

$$\int_c^d f(\varphi(t)) \cdot \varphi'(t) \, dt = \int_{\varphi(c)}^{\varphi(d)} f(x) \, dx.$$

Change of variable

Proof.

1. Result from the derivative of a composite function:

$$(F \circ \varphi)'(t) = F'(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t)) \cdot \varphi'(t)$$

2. Result from the definition of the integral of a continuous function:

$$\int_c^d f(\varphi(t)) \cdot \varphi'(t)\, dt = F(\varphi(d)) - F(\varphi(c)) = \int_{\varphi(c)}^{\varphi(d)} f(x)\, dx$$

$\square$

Change of variable

First Aspect of the Change of Variable Theorem:
To find an antiderivative of a function g that is continuous on an interval J, one examines whether it can be expressed in the form:
$g(x) = f(\varphi(x))\varphi'(x)$, where:

$$\varphi : J \to I$$

is a C^1-class function, and f is continuous on I.
Knowing an antiderivative F of f implies the knowledge of an antiderivative of g, namely $F \circ \varphi$.

$$\int g(x)\,dx = \int f(\varphi(x))\varphi'(x)\,dx = F(\varphi(x)) + C$$

In practice, one often substitutes $\varphi(x)$ with t and $\varphi'(x)\,dx$ with dt.

$$\int g(x)\,dx = \int f(\varphi(x))\varphi'(x)\,dx = \int f(t)\,dt = F(t) + C$$

Change of variable

Example

$$\text{Evaluate } \int_0^1 x e^{x^2}\, dx$$

1. Perform a change of variable by letting $t = x^2$, then $dt = 2x\, dx$.

$$\int_0^1 x e^{x^2}\, dx = \frac{1}{2}\int_0^1 2x e^{x^2} \sqrt{24}\, dx = \frac{1}{2}\int_0^1 e^t\, dt$$

2. Now, calculate the integral with respect to t.

$$\frac{1}{2}\int_0^1 e^t\, dt = \frac{1}{2}\left[e^t\right]_0^1 = \frac{1}{2}\left(e^1 - 1\right)$$

Change of variable

Example

Evaluate

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^2 x}\, dx$$

$$= -\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos^2 x}\, dx$$

$$= -\int_0^{\frac{\pi}{4}} \frac{(\cos x)'}{\cos^2 x}\, dx$$

$$= -\int_1^{\frac{\sqrt{2}}{2}} \frac{1}{t^2}\, dt$$

Change of variable

$$= -\left[-\frac{1}{t}\right]_{\frac{\sqrt{2}}{2}}^{1}$$

$$= \sqrt{2} - 1$$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^2 x}\, dx = \sqrt{2} - 1$$

Example

Change of variable

Example

$$\text{Evaluate } \int_{1/e}^{e} \frac{1}{x\sqrt{1-\log^2(x)}}\, dx \text{ where } x \in \left]\frac{1}{e}, e\right[$$

$$\int_1 \frac{1}{x\sqrt{1-\log^2 x}}\, dx = \int_1 \frac{(\log x)'}{\sqrt{1-\log^2 x}}\, dx = \int_1 \frac{1}{\sqrt{1-t^2}}\, dt$$

$$\int_1 \frac{1}{x\sqrt{1-\log^2 x}}\, dx = \operatorname{Arcsin}(t) + C$$

$$\int_1 \frac{1}{x\sqrt{1-\log^2 x}}\, dx = \operatorname{Arcsin}(\log x) + C$$

Change of variable

Second aspect of the change of variable theorem: When we don't have a simple primitive for f, we express the original variable x in terms of a new variable t, i.e., $x = \varphi(t)$ with $\varphi : J \to I$ being a C^1-class and bijective function. We assume that the function $(f \circ \varphi)\varphi'$ has a simple-to-compute primitive G. In this case:

$$G \circ \varphi^{-1}$$

is a primitive of f. In practice, we substitute x with $\varphi(t)$ and dx with $\varphi'(t)dt$ in the expression of f.

Change of variable

Example

Given the function $f(x) = \sqrt{1-x^2}$, find the definite integral $\int_0^1 f(x)\,dx$. Let's make the substitution $x = \sin(t)$, where t belongs to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2(t)} \cdot \cos(t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\cos^2(t)} \cdot \cos(t)\,dt = \int_0^{\frac{\pi}{2}} |\cos(t)| \cdot \cos(t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \cos^2(t)\,dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos(2t)}{2}\,dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2t))\,dt$$

Example

$$= \frac{1}{2}\left(\left[t + \frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}}\right)$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{2}\right) - (0+0)\right)$$

$$= \frac{\pi}{4}$$

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\arcsin(x) + \frac{1}{4}\sin(2\arcsin(x)) + C$$

$$= \frac{1}{2}\arcsin(x) + \frac{1}{4}x\sqrt{1-x^2} + C$$

Change of variable

Example

(Continued): We can express the result differently for $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, as follows:

$$\sin^2(t) = 2\sin(t)\cos(t), \quad \cos^2(t) = 2\sin(t)\cos(t), \quad \sqrt{1-\sin^2(t)} = 2x\sqrt{}$$

Now, let's rewrite the result:

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2} + C$$

$$\int_0^1 \sqrt{1-x^2}\,dx = \left[\frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2}\right]_0^1$$

$$= \frac{\pi}{4}$$

Note: It's not necessary to find the expression of the primitive in terms of

Change of variable

Example

$$\int \frac{1}{(1+x^2)\sqrt{1+x^2}}\,dx, \quad x \in \mathbb{R}$$

Let's make the substitution $x = \tan(t)$, where t belongs to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\int \frac{1}{(1+x^2)\sqrt{1+x^2}}\,dx =$$

$$\int \frac{1+\tan^2(t)}{(1+\tan^2(t))\sqrt{1+\tan^2(t)}}\,dt = \int \frac{1}{\sqrt{1+\tan^2(t)}}\,dt$$

Change of variable

Example

This integral simplifies to $\int \cos(t)\, dt = \sin(t) + C$.
Now, substituting back $t = \arctan(x)$:

$$\sin(t) + C = \sin(\arctan(x)) + C$$

Therefore,

$$\int \frac{1}{(1+x^2)\sqrt{1+x^2}}\, dx = \sin(\arctan(x)) + C$$

Change of variable

Example

$$\int \frac{\sqrt{x}}{\sqrt{1-x^3}}\,dx, \quad x \in \,]0,1[$$

Perform a substitution by letting $x = t^{2/3}$ where $t \in \,]0,1[$, which implies $dx = \frac{2}{3}t^{-1/3}dt$.

$$\int \frac{\sqrt{x}}{\sqrt{1-x^3}}\,dx = \int \frac{\frac{2}{3}}{\sqrt{1-t^2}}\,dt$$

$$= \int \frac{2}{3\sqrt{1-t^2}}\,dt$$

$$= \int \frac{2}{3\sqrt{1-t^2}} \cdot \,dt$$

Change of variable

Example

$$= \int \frac{2}{\sqrt{1-t^2}}\, dt$$

$$= \frac{2}{3} \arcsin(t) + C$$

Substitute back $x = t^{2/3}$:

$$= \frac{2}{3} \arcsin\left(\sqrt{x^3}\right) + C$$

Therefore,

$$\int \frac{\sqrt{x}}{\sqrt{1-x^3}}\, dx = \frac{2}{3} \arcsin\left(\sqrt{x^3}\right) + C$$

Cases of even, odd, periodic functions

For a continuous and even function $f : \mathbb{R} \longrightarrow \mathbb{R}$, the following holds for any $a \in \mathbb{R}$:

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Proof.

Indeed, the proof is as follows:

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

In the first integral, we perform the variable change $x = -t$, which implies $dx = -dt$:

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(-t)\,dt$$

Then, renaming the variable t to x in the last expression, we get: □

Cases of even, odd, periodic functions

Proof.

$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-t)\,dt = \int_{0}^{a} f(x)\,dx$$

By combining this with the second integral, we have:

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Thus, the property is established for a continuous and even function. $\square$

Cases of even, odd, periodic functions

For an odd continuous function:
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and odd function. Then, for any $a \in \mathbb{R}$:

$$\int_{-a}^{a} f(x)\,dx = 0$$

The proof is identical to the previous case.
For a periodic continuous function:
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and periodic function, meaning there exists $T > 0$ such that:

$$f(x+T) = f(x) \quad \text{for all} \quad x \in \mathbb{R}$$

Then, for any $a \in \mathbb{R}$:

$$\int_{a}^{a+T} f(x)\,dx = \int_{0}^{T} f(x)\,dx$$

Proof.

$$\int_a^{a+T} f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^T f(x)\,dx + \int_T^{a+T} f(x)\,dx$$

Since f is periodic, $f(x+T) = f(x)$. Therefore,
$\int_a^0 f(x)\,dx = -\int_0^a f(x)\,dx$ and $\int_T^{a+T} f(x)\,dx = \int_0^a f(x)\,dx$.
Substituting these results back into the original expression:

$$\int_a^{a+T} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^T f(x)\,dx + \int_0^a f(x)\,dx$$

Simplifying:

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$$

Thus, the property is established for a continuous and periodic function. □

Cases of even, odd, periodic functions

Proof.

For the last integral, we make the substitution $x = u + T$ where $dx = du$:

$$\int_T^{a+T} f(x)\,dx = \int_0^a f(u+T)\,du = \int_0^a f(u)\,du = \int_0^a f(x)\,dx = -\int_a^0 f(x)$$

□

Example

$$I_n = \int_0^{2\pi} f(\sin(nx))\,dx = \int_0^{2\pi} f(\cos(nx))\,dx$$

Assuming f is a continuous function on $\mathbb{R}$, consider f being a function of cosine. Let's make the substitution $x = u + \frac{\pi}{2n}$ where $dx = du$:

Cases of even, odd, periodic functions

Example

continued

$$I_n = \int_{-\frac{\pi}{2n}}^{2\pi-\frac{\pi}{2n}} f(\sin(nu + \frac{\pi}{2}))\, du = \int_{-\frac{\pi}{2n}}^{2\pi-\frac{\pi}{2n}} f(\cos(nu))\, du$$

Now, I_n is expressed in terms of the function f evaluated at $\cos(nu)$ over the interval $\left[-\frac{\pi}{2n}, 2\pi - \frac{\pi}{2n}\right]$.

$$I_n = \int_0^{2\pi} f(\cos(nx))\, dx$$

This demonstrates the periodicity property when dealing with trigonometric functions in the integral.

Cases of even, odd, periodic functions

Example

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function such that $f(a+b-x) = f(x)$ for all $x \in [a, b]$.
then $\int_a^b x \cdot f(x)\,dx = \frac{a+b}{2} \int_a^b f(x)\,dx$

Proof.

If we make the substitution $x = a + b - u$, then $dx = -du$. Therefore:

$$\int_a^b x \cdot f(x)\,dx = -\int_b^a (a+b-u) \cdot f(a+b-u)\,du$$

Now, using the property $f(a+b-x) = f(x)$:

$$= \int_a^b (a+b-u) \cdot f(u)\,du$$

□

Cases of even, odd, periodic functions

Proof.

Expanding:

$$\int_a^b x \cdot f(x)\, dx = \int_a^b (a+b) \cdot f(u)\, du - \int_a^b u \cdot f(u)\, du$$

$$= (a+b) \int_a^b f(x)\, du - \int_a^b x \cdot f(x)\, du$$

$$= \frac{a+b}{2} \int_a^b f(x)\, dx$$

Thus, the claim is proven. $\square$

Cases of even, odd, periodic functions

$$\int_{-\pi}^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx$$

Apply the symmetry property $f(x) = f(-x)$:

$$= 2 \int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx$$

Now, using the claimed property:

$$= \pi \int_{0}^{\pi} \frac{\sin x}{1 + \cos^2 x} \, dx$$

Cases of even, odd, periodic functions

Example

(Continued):

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

$$= -\pi \int_0^{\pi} \frac{-\sin x}{1+\cos^2 x}\, dx$$

$$= -\pi \int_1^{-1} \frac{1}{1+t^2}\, dt$$

$$= \pi \left[\arctan(t)\right]_{-1}^{1}$$

$$= \frac{\pi^2}{2}$$

CHAPTER 2

PRIMITIVES AND INTEGRALS

Mathematical Analysis 1 , ENSIA 2024

PRIMITIVES AND INTEGRALS

Part III

1. **Polynomial Reminders**
2. **Decomposition of a Rational Function:**
3. **Antiderivatives of a Rational Function**

Polynomial Reminders

Definition

A polynomial in one complex variable z is any expression of the form:

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

Where: $a_0, a_1, a_2, \ldots, a_n \in \mathbb{C}$ are called the coefficients of the polynomial.
If $a_n \neq 0$, we say that the polynomial is of degree n.
The derivative of the polynomial P, denoted as P', is defined as:

$$P'(z) = a_1 + 2a_2 z + \ldots + na_n z^{(n-1)}$$

This represents the derivative of P with respect to z.

Roots of a Polynomial

We say that $\lambda \in \mathbb{C}$ is a root of the polynomial P if $P(\lambda) = 0$. This definition is equivalent to P being divisible by $(z - \lambda)$, meaning there exists a polynomial Q such that $P(z) = (z - \lambda)Q(z)$.

The roots of a polynomial can be either simple or multiple.

We say that $\lambda \in \mathbb{C}$ is a simple root if:

$$P(z) = (z - \lambda)Q(z), \quad Q(\lambda) \neq 0 \implies \begin{cases} P(\lambda) = 0 \\ P'(\lambda) \neq 0 \end{cases}$$

We say that $\lambda \in \mathbb{C}$ is a double root if:

$$P(z) = (z - \lambda)^2 Q(z), \quad Q(\lambda) \neq 0 \implies \begin{cases} P(\lambda) = P'(\lambda) = 0 \\ P''(\lambda) \neq 0 \end{cases}$$

Polynomial Reminders

More generally:
We will say that $\lambda \in \mathbb{C}$ is a root of multiplicity or order $k \in \mathbb{N}^*$ if:

$$P(z) = (z-\lambda)^k Q(z), \quad Q(\lambda) \neq 0$$

$$\implies \begin{cases} P(\lambda) = P'(\lambda) = \ldots = P^{(k-1)}(\lambda) = 0 \\ P^{(k)}(\lambda) \neq 0 \end{cases}$$

Example

$$P(z) = z^3 - 1$$

There are 3 simple roots: $z_1 = 1$, $z_2 = \frac{-1-i\sqrt{3}}{2}$, and $z_3 = \frac{-1+i\sqrt{3}}{2}$.

Example

$$P(z) = z^5 - 3z^4 + 4z^3 - 4z^2 + 3z - 1$$

$z_1 = 1$ is a triple root because $P(1) = P'(1) = P''(1) = 0$, and

Example

$$P(z) = z^4 + 4z^3 + mz^2 + nz + 2 \text{ where } m, n \in \mathbb{N}$$

Choose m and n such that -1 is a double root.
We need to have $P(-1) = P'(-1) = 0$ and $P''(-1) \neq 0$.

$$P'(z) = 4z^3 + 12z^2 + 2mz + n$$

$$P''(z) = 12z^2 + 24z + 2m$$

Example

Setting $z = -1$:

$$P'(-1) = 4(-1)^3 + 12(-1)^2 + 2m(-1) + n = 0$$
$$P''(-1) = 12(-1)^2 + 24(-1) + 2m \neq 0$$

Solving the equations:

$$m - n - 1 = 0$$
$$8 - 2m + n = 0$$

This simplifies to:

$$m = 7$$
$$n = 6$$

So, $m = 7$ and $n = 6$ satisfy the conditions.
Also, $-1 + i$ and $-1 - i$ are also roots of P.

Polynomial Reminders

Case of Polynomials with Real Coefficients

Let $n \in \mathbb{N}^*$ and $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$.

If $\lambda \in \mathbb{C}$ is a root of the polynomial, then the conjugate $\overline{\lambda}$ is also a root.
Proof:

$$P(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n = 0$$

Taking the conjugate of both sides:

$$\overline{P(\lambda)} = \overline{a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n} = 0$$

$$\overline{P(\lambda)} = \overline{a_0} + \overline{a_1}\overline{\lambda} + \overline{a_2}\overline{\lambda}^2 + \ldots + \overline{a_n}\overline{\lambda}^n = P(\overline{\lambda}) = 0$$

As a consequence, if $\lambda \in \mathbb{C}$ is a root of order $k \geq 1$, then the conjugate $\overline{\lambda}$ is also a root of order k of the polynomial.

Polynomial Factorization

Consider $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$, where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{C}$, a polynomial of degree $n \geq 1$. It is assumed that it has at least one complex root, i.e., $\exists \lambda \in \mathbb{C}$ such that $P(\lambda) = 0$.
Note: This result is not valid in $\mathbb{R}$.
For example, consider $P_1(x) = x^2 + x + 1$ and $P_2(x) = x^4 + 1$.
The polynomials $P_1(x)$ and $P_2(x)$ do not have real roots. However, in $\mathbb{C}$, $P_1(x)$ has complex roots, and $P_2(x)$ has complex roots, even though they are not real.
This emphasizes that, unlike in $\mathbb{R}$, in $\mathbb{C}$, every non-constant polynomial has at least one complex root.

Polynomial Reminders

Consequences:
If $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{C}$ is a polynomial of degree $n \geq 1$, then it has n distinct or non-distinct roots.
If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the distinct roots with respective orders $k_1, k_2, \ldots, k_p$ (where $k_1 + k_2 + \ldots + k_p = n$), then:

$$P(z) = a_n (z - \lambda_1)^{k_1} (z - \lambda_2)^{k_2} \ldots (z - \lambda_p)^{k_p}$$

Now, if $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ is a polynomial of degree $n \geq 1$, and $\lambda \in \mathbb{C}$ is a root of multiplicity k, then the conjugate $\overline{\lambda}$ is also a root.
In the expression of P, you find the factor $(z - \lambda)^k (z - \overline{\lambda})^k$.

Polynomial Reminders

Particularly, in the case where P is real, i.e., of the form:

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$, $x \in \mathbb{R}$,
Then, the factorization of such a polynomial in $\mathbb{R}$ gives us factors of the form $(x-a)^k$ or $(x^2+px+q)^m$ with $\Delta < 0$.

Example

$$P(x) = x^4 + 1$$

$$P(x) = (x^2+1)^2 - 2x^2 = (x^2 - x\sqrt{2} + 1)(x^2 + x\sqrt{2} + 1)$$

In this example, the factorization involves quadratic factors with $\Delta < 0$.

Polynomial Reminders

Example

$$P(x) = x^5 + 1$$

Let's find the roots in $\mathbb{C}$. Suppose z is such a root.
$z = \rho e^{i\theta} \implies z^5 = \rho^5 e^{5i\theta}$.

$$z^5 = -1 \implies \rho^5 e^{5i\theta} = e^{i\pi} \implies \begin{cases} \rho = 1 \\ \theta = \frac{\pi}{5} + \frac{2\pi k}{5}, \quad k = 0, 1, 2, 3, 4 \end{cases}$$

So, we have 5 roots:

$$Z_1 = e^{i\pi/5}, \quad Z_2 = (Z_1)^* = e^{-i\pi/5}, \quad Z_3 = e^{i(3\pi/5)}, \quad Z_4 = (Z_3)^* = e^{-i3\pi/}$$

Therefore, the factorization of $P(x)$ is:

$$P(x) = (x+1)(x^2 - 2x\cos(\pi/5) + 1)(x^2 - 2x\cos(3\pi/5) + 1)$$

Fraction Rationale

Consider P and Q, two real polynomials that are coprime (have no common factors).

The function $F : x \mapsto F(x) = \frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$, is called a rational function. F is said to be proper if the degree of P is strictly less than the degree of Q ($\deg(P) < \deg(Q)$).

The roots of Q are called the "poles" of F. A root of order k of the denominator is called a "pole" of order k.

Decomposition of a Rational Function

Example

$$F(x) = \frac{x^2+1}{x(x-1)^2(x+1)^3}$$

$x = 0$ is a simple pole
$x = 1$ is a double pole
$x = -1$ is a triple pole

Decomposition of a Rational Function

Definition

A partial (or elementary) rational fraction is a fraction of the form:

$$\frac{A}{(x-a)^k} \quad \text{or} \quad \frac{Cx+D}{(x^2+px+q)^j}$$

where $k, j \in \mathbb{N}^*$ and $\Delta = p^2 - 4q < 0$.
Let $F(x) = \frac{P(x)}{Q(x)}$ be a proper rational fraction. We know that the denominator can be factored into factors of the form $(x-a)^k$ or $(x^2+px+q)^j$ where $\Delta < 0$. The former are called simple elements of the first kind, and the latter are simple elements of the second kind.

Decomposition of a Rational Function

Theorem

Every proper rational fraction can be uniquely expressed as the sum of partial rational fractions. For simplicity, the procedure is outlined as follows:
For each simple element of the first kind $(x-a)^k$ *in the partial fraction decomposition, there should be a sum of the form:*

$$\frac{A_k}{(x-a)^k} + \frac{A_{k-1}}{(x-a)^{k-1}} + \ldots + \frac{A_1}{(x-a)}$$

For each simple element of the second kind $(x^2+px+q)^j$ *with* $\Delta < 0$ *in the partial fraction decomposition, there should be a sum of the form:*

$$\frac{C_1x+D_1}{x^2+px+q} + \frac{C_2x+D_2}{(x^2+px+q)^2} + \ldots + \frac{C_jx+D_j}{(x^2+px+q)^j}$$

Decomposition of a Rational Function

Example

$$F(x) = \frac{x^2+1}{x^2(x-1)(x+1)^3}$$

$$F(x) = \frac{A_2}{x^2} + \frac{A_1}{x} + \frac{B_1}{(x-1)} + \frac{C_3}{(x+1)^3} + \frac{C_2}{(x+1)^2} + \frac{C_1}{(x+1)}$$

Example

$$F(x) = \frac{x^2+x+1}{(x-1)^2(x-2)^3(x^2+1)^2}$$

$$F(x) = \frac{A_2}{(x-1)^2} + \frac{A_1}{(x-1)} + \frac{B_3}{(x-2)^3} +$$

$\cdot \frac{B_2}{(x-2)^2} + \frac{B_1}{(x-2)} + \frac{C_1x+D_1}{(x^2+1)} + \frac{C_2x+D_2}{(x^2+1)^2}$

Calculating Coefficients

Method of Undetermined Coefficients: In the decomposition of F into partial fractions, you multiply both sides by the denominator, expand, and then identify the coefficients of the monomials of the same power.

Example

Decompose the rational function into partial fractions:

$$F(x) = \frac{x^2+1}{x(x-1)(x+2)}$$

We have:

$$F(x) = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$$

Now, you would proceed to find the values of A, B, and C.

Decomposition of a Rational Function

Example

(continued)

$$x^2 + 1 = (A + B + C)x^2 + (A + 2B - C)x - 2A$$

By identification, we get the system of equations:

$$-2A = 1$$

$$A + 2B - C = 0$$

$$A + B + C = 1$$

Decomposition of a Rational Function

Example

(continued) Solving this system:

$$A = -\frac{1}{2}$$

$$B = \frac{2}{3}$$

$$C = \frac{5}{6}$$

Therefore, the partial fraction decomposition is:

$$\frac{x^2+1}{x(x-1)(x+2)} = -\frac{1}{2x} + \frac{2}{3(x-1)} + \frac{5}{6(x+2)}$$

Method of Division According to Increasing Powers:
The previous method can be quite lengthy. Here's an alternative method for calculating coefficients for the pole of order k, $k \geq 1$.
Suppose the denominator is $Q(x) = (x-a)^k Q_1(x)$ where $Q_1(a) \neq 0$, meaning that a is a pole of order k.
The partial fraction for this pole is of the form:

$$\frac{A_k}{(x-a)^k} + \frac{A_{k-1}}{(x-a)^{k-1}} + \ldots + \frac{A_1}{(x-a)}$$

Definition

Decomposition of a Rational Function

Two Cases: 1. **$k = 1$:** There is only one coefficient to calculate, A_1:

$$A_1 = \lim_{x \to a} (x - a) F(x) = \frac{P(a)}{Q(a)}$$

2. **$k > 1$:** In the rational fraction $P(x)/Q_1(x) = (x - a)^k F(x)$, set $x = a + y$ and perform division according to increasing powers of y up to order $k - 1$.

Example

$$F(x) = \frac{x^2 + 1}{x^2(x-1)(x+1)^3}$$

$$F(x) = \frac{A_2}{x^2} + \frac{A_1}{x} + \frac{B_1}{(x-1)} + \frac{C_3}{(x+1)^3} + \frac{C_2}{(x+1)^2} + \frac{C_1}{(x+1)}$$

Decomposition of a Rational Function

Example

(continued):**

$$F(x) = \frac{x^2+1}{x^2(x-1)(x+1)^3}$$

$$F(x) = \frac{A_2}{x^2} + \frac{A_1}{x} + \frac{B_1}{(x-1)} + \frac{C_3}{(x+1)^3} + \frac{C_2}{(x+1)^2} + \frac{C_1}{(x+1)}$$

For the pole at $x = 1$:

$$B_1 = \lim_{x\to 1}(x-1)F(x) = \lim_{x\to 1} \frac{x^2+1}{x^2(x+1)^3} = \frac{1}{4}$$

Example

(continued):** Since $x = 0$ is a double pole, we perform division according to increasing powers up to order 1:

$$F(x) = \frac{1+x^2}{(x-1)(x+1)^3} = \frac{1+x^2}{(x-1)(1+3x+\dots)} = \frac{1+x^2}{(-1-2x+\dots)}$$

$= -1 + 2x + \dots$

This implies $A_2 = -1$ and $A_1 = 2$.

Decomposition of a Rational Function

Example

(continued)

$$F(x) = \frac{x^2+1}{x^2(x-1)(x+1)^3}$$

For the triple pole at $x=-1$, we set $x+1=y$, meaning $x=-1+y$, and then perform division according to increasing powers up to order 2:

$$\frac{x^2+1}{x^2(x-1)} = \frac{(-1+y)^2+1}{(-1+y)^2(-2+y)} = \frac{2-2y+y^2}{-2+5y-4y^2+\dots}$$

This implies $C_3=-1$, $C_2=-\frac{3}{2}$, and $C_1=-\frac{9}{4}$.

Decomposition of a Rational Function

Example

$$F(x) = \frac{x^3 - 1}{x^3(x+1)^2(x^2+1)}$$

For the triple pole at $x = 0$:

$$F(x) = \frac{A_3}{x^3} + \frac{A_2}{x^2} + \frac{A_1}{x} + \frac{B_1}{(x+1)} + \frac{B_2}{(x+1)^2} + \frac{C_1 x + D_1}{x^2+1}$$

We perform division according to increasing powers up to order 2 of $x^3 F(x)$:

$$x^3 F(x) = \frac{-1 + x^3}{1 + 2x + x^2}(1 + x^2) = \frac{-1 + x^3}{1 + 2x + 2x^2 + \dots}$$

This implies $A_3 = -1$, $A_2 = 2$, and $A_1 = -2$.

Decomposition of a Rational Function

Example

(continued):

$$(x+1)^2 F(x) = \frac{x^3-1}{x^3(x^2+1)} = \frac{(-1+y)^3-1}{(-1+y)^3((-1+y)^2+1)} = \frac{-2+3y+\ldots}{-2+8y+\ldots}$$

This implies $B_2 = 1$ and $B_1 = \frac{5}{2}$.

Decomposition of a Rational Function

Example

(continued): We have two coefficients left to determine. We can either give x two values other than 0 and -1, or we can use:

$$\lim_{x \to +\infty} xF(x) = 0 = A_1 + B_1 + C_1$$

From this, we find that $C_1 = -A_1 - B_1 = 2 - \frac{5}{2} = -\frac{1}{2}$.
To find D_1, we give x the value 1.

$$F(1) = 0 = A_3 + A_2 + A_1 + \frac{B_1}{2} + \frac{B_2}{4} + \frac{C_1}{2} + D_1$$

Solving for D_1, we get $D_1 = -\frac{3}{4}$.

Decomposition of a Rational Function

Example

$$F(x) = \frac{1}{(x^2-1)(x^2+1)}$$

We have:

$$F(x) = \frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

Due to the parity of the rational function, we know that $F(-x) = F(x)$, which implies $B = -A$ and $C = 0$.
Therefore, $F(x) = \frac{A}{x-1} - \frac{A}{x+1} + \frac{D}{x^2+1}$.
By finding the limit as x approaches 1, we get $A = \frac{1}{4}$.
To find D, we give x the value 0.

$$F(0) = -1 = -2A + D = -\frac{1}{2} + D$$

Solving for D, we find $D = -\frac{1}{2}$

Decomposition of a Rational Function

** general case**
Let $F(x) = \frac{P(x)}{Q(x)}$ be a rational function, whether proper or not.
If $\deg(P) < \deg(Q)$, we directly proceed to partial fraction decomposition.
If $\deg(P) > \deg(Q)$, we perform either polynomial long division or division following the decreasing powers of $P(x)$ by $Q(x)$. This results in two unique polynomials $E(x)$ (quotient) and $R(x)$ (remainder) such that $P(x) = E(x)Q(x) + R(x)$, where $\deg(R) < \deg(Q)$.
This implies $F(x) = \frac{P(x)}{Q(x)} = E(x) + \frac{R(x)}{Q(x)} = E(x) + F_1(x)$, where $F_1(x)$ is proper and can be expressed as a sum of partial fractions.

Decomposition of a Rational Function

Example

$$f(x) = \frac{x^4 + x - 1}{x^3 + 2x^2 - x - 2} = x - 2 + \frac{5x^2 + x - 5}{(x+2)(x^2-1)}$$

We have $F_1(x) = \frac{5x^2+x-5}{(x^3+2x^2-x-2)} = \frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x+1)}$. Solving for A, B, and C using the limits:

$$A = \lim_{x \to -2} (x+2) F_1(x) = \frac{13}{3}$$

$$B = \lim_{x \to 1} (x-1) F_1(x) = \frac{1}{6}$$

$$C = \lim_{x \to -1} (x+1) F_1(x) = \frac{1}{2}$$

Decomposition of a Rational Function

Example

Thus, the decomposition is:

$$f(x) = x - 2 + \frac{13}{3(x+2)} + \frac{1}{6(x-1)} + \frac{1}{2(x+1)}$$

Antiderivatives of a Rational Function

The decomposition of $F(x) = \frac{P(x)}{Q(x)}$ generally results in a polynomial part $E(x)$ and partial fractions of the form:

$$\frac{A}{(x-a)^k} \text{ or } \frac{Cx+D}{(x^2+px+q)^j}$$

where $k, j \in \mathbb{N}^*$ and $\Delta = p^2 - 4q < 0$.
If $E(x) = a_0 + a_1 x + \ldots + a_i x^i$,

$$\int E(x)\, dx = a_0 x + \frac{a_1}{2} x^2 + \ldots + \frac{a_i}{i+1} x^{i+1} + C$$

Similarly,

$$\int \frac{1}{(x-a)^k}\, dx = \begin{cases} \ln|x-a| + C & \text{if } k = 1 \\ \frac{1}{1-k}(x-a)^{1-k} + C & \text{if } k \neq 1 \end{cases}$$

Therefore, we need to find the antiderivatives of $\frac{Ax+B}{(x^2+px+q)^m}$. We have:

$$\int \frac{Ax+B}{(x^2+px+q)^m}\,dx = \int \frac{\frac{A}{2}(2x+p)+B-\frac{A}{2}p}{(x^2+px+q)^m}\,dx$$

$$= \frac{A}{2}\int \frac{2x+p}{(x^2+p+q)^m}\,dx + (B-\frac{A}{2}p)\int \frac{1}{(x^2+px+q)^m}\,dx$$

$$\int \frac{2x+p}{(x^2+px+q)^m}\,dx = \int \frac{\varphi'(x)}{\varphi(x)^m}\,dx$$

$$= \int \frac{dt}{t^m} = \begin{cases} \ln|t| + C & \text{if } m = 1 \\ \frac{1}{1-m}t^{1-m} + C & \text{if } m \neq 1 \end{cases}$$

Antiderivatives of a Rational Function

Therefore:

$$\int \frac{2x+p}{(x^2+px+q)^m}\,dx$$

$$= \begin{cases} \ln\left|x^2+px+q\right| + C & \text{if } m = 1 \\ \frac{1}{1-m}(x^2+px+q)^{1-m} + C & \text{if } m \neq 1 \end{cases}$$

Now, for the antiderivatives of $\frac{1}{(x^2+px+q)^m}$:

$$x^2+px+q = (x+\frac{p}{2})^2 + (q-\frac{p^2}{4})$$

$$x^2+px+q = (x+\frac{p}{2})^2 + \alpha^2$$

where $\alpha = \sqrt{q-\frac{p^2}{4}}$.

Antiderivatives of a Rational Function

This expression can also be written as:

$$x^2 + px + q = \alpha^2(1 + (\frac{x + \frac{p}{2}}{\alpha})^2)$$

$$\int \frac{1}{(x^2 + px + q)^m}\, dx = \int \frac{1}{\alpha^{2m}(1 + (\frac{x+\frac{p}{2}}{\alpha})^2)^m}\, dx$$

Antiderivatives of a Rational Function

Let's perform the substitution $\frac{x+\frac{p}{2}}{\alpha} = t$, which implies $x = \alpha t - \frac{p}{2}$. Then, calculate the differential dx in terms of dt:

$$dx = \alpha dt$$

Now, substitute x and dx in the integral:

$$\int \frac{1}{(x^2 + px + q)^m} \, dx$$

$$= \alpha^{1-2m} \int \frac{1}{(1 + t^2)^m} \, dt$$

Now, define $I_m = \int_1 \frac{1}{(1+t^2)^m} \, dt$. Rewrite the integral in terms of I_m:

$$\alpha^{1-2m} \int \frac{1}{(1 + t^2)^m} \, dt = \alpha^{2m-1} I_m$$

for m = 1

$$I_1 = \int \frac{1}{1+t^2} dt$$

$$I_1 = \arctan(t) + C$$

Let's evaluate I_{m+1} for $m > 1$ step by step.
Given:

$$I_{m+1} = \int \frac{1+t^2-t^2}{(1+t^2)^{m+1}} dt$$

$$I_{m+1} = \int \frac{1}{(1+t^2)^m} dt - \frac{1}{2} \int t \frac{2t}{(1+t^2)^{m+1}} dt$$

Antiderivatives of a Rational Function

Let's perform integration by parts with the chosen u and dv:

$$u = t \implies du = dt$$
$$dv = \frac{2t}{(1+t^2)^{m+1}}\,dt \implies v = \frac{-1}{m}\frac{1}{(1+t^2)^m}$$

Now, apply the integration by parts formula:

$$I_{m+1} = I_m - \frac{1}{2}(\frac{-1}{m}\frac{t}{(1+t^2)^m} + \frac{1}{m}\int \frac{1}{(1+t^2)^m}dt)$$

$$I_{m+1} = I_m(1 - \frac{1}{2m}) + \frac{1}{2m}\frac{t}{(1+t^2)^m}$$

$$I_{m+1} = \frac{2m-1}{2m}I_m + \frac{1}{2m}\frac{t}{(1+t^2)^m}$$

Antiderivatives of a Rational Function

Example

Primitives of $f(x) = \frac{x^3-1}{x^3(x+1)^2(x^2+1)}$ are obtained by decomposing it into partial fractions, as seen in Example 2. In this regard, the expression for $f(x)$ is as follows:

$$f(x) = -\frac{1}{x^3} + \frac{2}{x^2} - \frac{2}{x} + \frac{5/2}{(x+1)} + \frac{1}{(x+1)^2} - \frac{1/2x + 3/4}{(x^2+1)}$$

The indefinite integral of $f(x)$ is given by:

$$\int f(x)\,dx = \frac{1}{2x^2} - \frac{2}{x} - 2\ln(x) + \frac{5}{2}\ln(1+x) - \frac{1}{x+1} - \frac{1}{4}\int \frac{2x}{x^2+1}\,dx - \frac{3}{4}$$

$\int \frac{1}{x^2+1}\,dx$

Antiderivatives of a Rational Function

Example

(continued)

$$\int f(x)\,dx = \frac{1}{2x^2} - \frac{2}{x} - 2\ln(x) + \frac{5}{2}\ln(1+x) - \frac{1}{x+1} - \frac{1}{4}\ln(x^2+1) - \frac{3}{4}$$

$\arctan(x) + C$
where C is the constant of integration.

Antiderivatives of a Rational Function

Example

Decomposition and indefinite integral of $f(x) = \frac{x^2+1}{(x-1)^2(x-2)(x^2+x+1)}$
In this case, $f(x)$ can be expressed as the sum of partial fractions:

$$f(x) = \frac{A_2}{(x-1)^2} + \frac{A_1}{(x-1)} + \frac{B}{(x-2)} + \frac{Cx+D}{x^2+x+1}$$

Calculation of coefficients:

$$B = \lim_{x \to 2} (x-2) f(x) = \frac{5}{7}$$

$$(x-1)^2 f(x) = \frac{1+x^2}{(x-2)(x^2+x+1)} = -\frac{2}{3} - \frac{2}{3} y$$

where x =y+1.

Antiderivatives of a Rational Function

Example

$$A_1 = A_2 = -\frac{2}{3}$$

$$\lim_{x\to\infty} xf(x) = 0 = A_1 + B + C \implies C = -A_1 - B = -\frac{1}{21}$$

$$f(0) = -\frac{1}{2} = A_2 - A_1 - \frac{1}{2}B + D \implies D = -\frac{1}{2}(-A_2 + A_1 + \frac{1}{2}B) = -\frac{1}{7}$$

Therefore, the coefficients are $A_1 = A_2 = -\frac{2}{3}$, $B = \frac{5}{7}$, $C = -\frac{1}{21}$, and $D = -\frac{1}{7}$.

Antiderivatives of a Rational Function

Example

Example 2 (continued): Indefinite Integrals of $f(x) = \frac{x^2+1}{(x-1)^2(x-2)(x^2+x+1)}$
In this case, the partial fraction decomposition of $f(x)$ was found to be:

$$f(x) = -\frac{2}{3}\cdot\frac{1}{(x-1)^2} - \frac{2}{3}\cdot\frac{1}{(x-1)} + \frac{5}{7}\cdot\frac{1}{(x-2)} - \frac{2x+1}{42(x^2+x+1)} - \frac{5}{42}$$

$\cdot\frac{1}{(x^2+x+1)}$

This expression can be simplified further:

$$f(x) = -\frac{2}{3}\cdot\frac{1}{(x-1)^2} - \frac{2}{3}\cdot\frac{1}{(x-1)} + \frac{5}{7}\cdot\frac{1}{(x-2)} - \frac{1}{42}\cdot\frac{2x+1}{(x^2+x+1)} - \frac{5}{42}$$

$\cdot\frac{1}{(x^2+x+1)}$

Antiderivatives of a Rational Function

Example

(continued):

$$\int f(x)\,dx = -\int \frac{2}{3(x-1)^2}\,dx - \int \frac{2}{3(x-1)}\,dx + \int \frac{5}{7(x-2)}\,dx - \frac{1}{42}$$

$\int \frac{(2x+1)}{(x^2+x+1)}\,dx - \int \frac{5}{42(x^2+x+1)}\,dx$

$$\int f(x)\,dx = \frac{2}{3(x-1)} - \frac{2}{3}\ln|x-1| + \frac{5}{7}\ln|x-2| - \frac{1}{42}\ln(x^2+x+1) -$$

$\int \frac{5}{42(x^2+x+1)}\,dx$

Example

(continued):

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}=\frac{3}{4}\left(1+\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}}\right)=\frac{3}{4}\left(1+\left(\frac{2x+1}{\sqrt{3}}\right)^2\right)$$

$$\int \frac{1}{x^2+x+1}\,dx=\frac{4}{3}\int \frac{1}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}\,dx$$

We make the substitution $t=\frac{2x+1}{\sqrt{3}}$, which implies $x=\frac{t\sqrt{3}-1}{2}$ and $dx=\frac{\sqrt{3}}{2}dt$.

Antiderivatives of a Rational Function

Example

(continued):

$$\int \frac{1}{x^2+x+1}\,dx = \frac{2}{3}\frac{1}{(x-1)} - \frac{2}{3}\ln|x-1| + \frac{5}{7}\ln|x-2|$$

$$-\frac{1}{42}\ln(x^2+x+1) - \frac{5}{21\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

So, the final result is:

$$\int \frac{x^2+1}{(x-1)^2(x-2)(x^2+x+1)}\,dx = \frac{2}{3}\frac{1}{(x-1)} - \frac{2}{3}\ln|x-1| + \frac{5}{7}\ln|x-2|$$

$$-\frac{1}{42}\ln(x^2+x+1) - \frac{5}{21\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Antiderivatives of a Rational Function

Example

$$f(x) = \frac{x^2+2}{(x-1)(x^2+1)^2}$$

$$f(x) = \frac{A}{(x-1)} + \frac{B_1x+C_1}{(x^2+1)} + \frac{B_2x+C_2}{(x^2+1)^2}$$

$$A = \lim_{x\to 1}(x-1)f(x) = \frac{3}{4}$$

$$\lim_{x\to i}(x^2+1)^2 f(x) = -\frac{1}{2}(1+i) = B_2 i + C_2 \implies B_2 = C_2 = -\frac{1}{2}$$

$$\lim_{x\to\infty} xf(x) = 0 = A + B_1 \implies B_1 = -\frac{3}{4}$$

Antiderivatives of a Rational Function

Example

(continued):

$$f(0) = -2 = -A + C_1 + C_2$$

$$\implies C_1 = -2 + A - C_2 = -\frac{3}{4}$$

Hence:

$$f(x) = \frac{3}{4} \cdot \frac{1}{(x-1)} - \frac{3}{8} \cdot \frac{2x}{(x^2+1)} - \frac{3}{4} \cdot \frac{1}{(x^2+1)} - \frac{1}{4} \cdot \frac{2x}{(x^2+1)^2} - \frac{1}{2}$$

$\cdot \frac{1}{(x^2+1)^2}$

Antiderivatives of a Rational Function

Example

(continued):

$$\int f(x)\,dx = \frac{3}{4}\ln|x-1| - \frac{3}{8}\ln(x^2+1) - \frac{3}{4}\arctan(x) + \frac{1}{4}\frac{1}{1+x^2}$$

$$-\frac{1}{2}\int_1 \frac{1}{(x^2+1)^2}\,dx$$

To compute $\int \frac{1}{(x^2+1)^2}\,dx$, we apply the formula:

$$I_{m+1} = \frac{2m-1}{2m} I_m + \frac{1}{2m}\frac{t}{(1+t^2)^m}$$

Where $I_m = \int \frac{1}{(x^2+1)^m}\,dx$.

Antiderivatives of a Rational Function

Example

(continued):

$$I_2 = \frac{1}{2} I_1 + \frac{1}{2} \frac{x}{1 + x^2}$$

$$\frac{1}{2} I_1 = \frac{1}{2} \arctan(x) + C$$

In conclusion, we have:

$$\int f(x)\, dx = \frac{3}{4} \ln|x-1| - \frac{3}{8} \ln(x^2+1) - \frac{3}{4} \arctan(x) + \frac{1}{4} \frac{1}{1+x^2}$$

$$-\frac{1}{4} \arctan(x) - \frac{1}{4} \frac{x}{1+x^2} + K$$

where K is the constant of integration.

CHAPTER 2

PRIMITIVES AND INTEGRALS

Mathematical Analysis 1 , ENSIA 2024

PRIMITIVES AND INTEGRALS

Part IV

1. **Rational fractions in sine and cosine**
2. **BIOCHE Rules**
3. **Trigonometric polynomials**
4. **Rational functions in hyperbolic sine and cosine**
5. **Functions containing radicals**

Rational Functions in Sine and Cosine

Definition

A polynomial function of two real variables is defined as an application $P : \mathbb{R}^2 \longrightarrow \mathbb{R}$ where $(x, y) \longrightarrow P(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} \cdot x^i \cdot y^j$, where $a_{ij} \in \mathbb{R}$.
A rational function of two real variables is defined as the quotient of two polynomial functions of two real variables.

Rational Functions in Sine and Cosine

"General Method: Let $(x, y) \longrightarrow R(x, y) = \frac{P(x,y)}{Q(x,y)}$ be a rational function of two real variables. We aim to find the antiderivatives of:

$$f(x) = R(\cos(x), \sin(x))$$

where $x \in I \subset]-\pi, \pi[$ and f is continuous on I.
By making the change of variable:

$$t = \tan\left(\frac{x}{2}\right) \quad \Rightarrow \quad x = 2\arctan(t)$$

we reduce the problem to the calculation of antiderivatives of a rational function in t."

Rational Functions in Sine and Cosine

"We have:

$$\sin(x) = \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)} = \frac{2t}{1+t^2}$$

$$\cos(x) = \frac{\sin^2\left(\frac{x}{2}\right)-\cos^2\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)} = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{2}{1+t^2}dt$$

Where $t = \tan\left(\frac{x}{2}\right)$.
Therefore,

$$\int R(\cos(x), \sin(x))dx = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{2}{1+t^2}dt$$

Note: Instead of taking $I \subset]\pi, \pi[$, we could have taken $I_k \subset]\pi + 2k\pi, \pi + 2k\pi[$, and set in this case..."

Rational Functions in Sine and Cosine

Example

$$\int \frac{1}{2+\cos(x)}\,dx \quad , \quad x \in]-\pi, \pi[$$

$$\int \frac{1}{2+\cos(x)}\,dx = \int \frac{\frac{2}{1+t^2}}{\frac{2+(1-t^2)}{1+t^2}}\,dt = 2\int \frac{1}{3+t^2}\,dt$$

Let $t = \sqrt{3}u$, which implies $dt = \sqrt{3}du$.

$$\Rightarrow \int \frac{1}{1+\left(\frac{t}{\sqrt{3}}\right)^2}\,dt = \int \frac{\sqrt{3}}{1+u^2}\,du = \sqrt{3}\arctan(u) + C$$

$$\Rightarrow \int \frac{1}{2+\cos(x)}\,dx = \frac{2}{3}\int \frac{1}{1+\left(\frac{t}{\sqrt{3}}\right)^2}\,dt = \frac{2}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right) + C$$

BIOCHE Rules

"In the search for primitives of rational functions involving sine and cosine, before performing the variable change $t = \tan\left(\frac{x}{2}\right)$, which can be lengthy, we check if other variable changes are possible.
It suffices that the expression $\omega(x) = R(\cos(x), \sin(x))\, dx$ is invariant under one of the following cases:

$$\omega(-x) = \omega(x)$$

In such a case, we can use the following property:

$$\int R(\cos(x), \sin(x))\, dx = -\int g(\cos(x)) \sin(x)\, dx$$

We set $t = \cos(x)$ to simplify the integration."

BIOCHE Rules

$$\omega(\pi - x) = \omega(x)$$

In this case:

$$\int R(\cos(x), \sin(x))\, dx = \int g(\sin(x)) \cos(x)\, dx$$

$$= \int g(\sin(x))\, (\sin(x))'\, dx$$

We set $t = \sin(x)$.

$$\omega(\pi + x) = \omega(x)$$

In this case:

$$\int R(\cos(x), \sin(x))\, dx = \int g(\tan(x)) \tan(x)'\, dx$$

We set $t = \tan(x)$.
It is essential to remember that:

$$\frac{d}{dx}(-x) = -\frac{d}{dx}x$$

$$\frac{d}{dx}(\pi - x) = -\frac{d}{dx}x$$

$$\frac{d}{dx}(\pi + x) = \frac{d}{dx}x$$

Example

Antiderivatives of $f(x) = \frac{1}{\sin(x)}$, where $x \in]0, \pi[$.

$$\omega(x) = \frac{1}{\sin(x)}\, dx$$

satisfies:

$$\omega(-x) = \omega(x)$$

.

$$\int \frac{1}{\sin(x)}\, dx = \int \frac{\sin(x)}{\sin^2(x)}\, dx = -\int \frac{(\cos(x))'}{1-\cos^2(x)}\, dx = -\int \frac{1}{1-t^2}\, dt$$

$$\int \frac{1}{\sin(x)}\, dx = \frac{1}{2}\int \frac{1}{t-1}\, dt - \frac{1}{2}\int \frac{1}{t+1}\, dt = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

BIOCHE Rules

Example

$$\int \frac{1}{\sin(x)}\,dx = \frac{1}{2}\ln\left|\frac{\cos(x)-1}{\cos(x)+1}\right| + C = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$

The last equality comes from the relation:

$$\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$$

.

Example

Antiderivatives of $f(x) = \frac{1}{\sin(x)(1+\cos^2(x))}$, where $x \in]0, \pi[$.

$$\omega(x) = \frac{1}{\sin(x)(1+\cos^2(x))}\, dx$$

satisfies:

$$\omega(-x) = \omega(x)$$

.

$$\int \frac{1}{\sin(x)(1+\cos^2(x))}\, dx = \int \frac{\sin(x)}{\sin^2(x)(1+\cos^2(x))}\, dx$$

$$= -\int \frac{(\cos(x))'}{(1-\cos^2(x))(1+\cos^2(x))}\, dx$$

Example

$$\int \frac{1}{\sin(x)(1+\cos^2(x))}\,dx = -\int \frac{1}{(1-t^2)(1+t^2)}\,dt$$

$$= -\int \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{Ct+D}{(1+t^2)}\,dt$$

$$= -\int \left(\frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{Ct+D}{(1+t^2)}\right)\,dt$$

BIOCHE Rules

Example

(continued):

$$A = \lim_{t \to 1} \frac{(1-t)}{(1-t^2)(1+t^2)} = \frac{1}{4}$$

$$B = \lim_{t \to -1} \frac{(1+t)}{(1-t^2)(1+t^2)} = \frac{1}{4}$$

$$\lim_{t \to \infty} \frac{t}{(1-t^2)(1+t^2)} = 0 = -A + B + C \implies C = 0$$

For $t = 0$, we find $1 = A + B + D \implies D = \frac{1}{2}$.
Therefore:

$$\int \frac{1}{\sin(x)(1+\cos^2(x))}\, dx = \frac{1}{4} \ln\left(\frac{1-\cos(x)}{1+\cos(x)}\right) - \frac{1}{2} \arctan(\cos(x)) + C$$

BIOCHE Rules

Example

Antiderivatives of $f(x) = \frac{1}{\cos(x)}$, where $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

$$\omega(x) = \frac{1}{\cos(x)}\, dx$$

satisfies:

$$\omega(\pi - x) = \omega(x)$$

.

$$\int \frac{1}{\cos(x)}\, dx = \int \frac{\cos(x)}{\cos^2(x)}\, dx = \int \frac{(\sin(x))'}{1 - \sin^2(x)}\, dx = \int \frac{1}{1 - t^2}\, dt$$

Example

$$\int \frac{1}{\cos(x)}\,dx = \frac{1}{2}\int \frac{1}{1+t}\,dt - \frac{1}{2}\int \frac{1}{t-1}\,dt = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + C$$

$$\int \frac{1}{\cos(x)}\,dx = \frac{1}{2}\ln\left(\frac{1+\sin(x)}{1-\sin(x)}\right) + C$$

BIOCHE Rules

Example

Antiderivatives of $f(x) = \frac{1}{\sin^2(x)+3\cos^2(x)}$, where $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

$$\omega(x) = \frac{1}{\sin^2(x) + 3\cos^2(x)}\, dx$$

satisfies:

$$\omega(\pi + x) = \omega(x)$$

.

$$\int \frac{1}{\sin^2(x) + 3\cos^2(x)}\, dx = \int \frac{1}{\cos^2(x)(3 + \tan^2(x))}\, dx$$

$$= \int \frac{(\tan(x))'}{3 + \tan^2(x)}\, dx = \int \frac{1}{3 + t^2}\, dt$$

$$\int \frac{1}{\sin^2(x) + 3\cos^2(x)}\, dx = \frac{1}{3} \int \frac{1}{1 + \left(t \right)^2}\, dt$$

Example

(continued): Let's continue with the integration:
Let $t = \sqrt{3}u$, then $dt = \sqrt{3}du$.

$$\int \frac{1}{1+\left(\frac{t}{\sqrt{3}}\right)^2}\,dt = \int \frac{\sqrt{3}}{1+u^2}\,du = \sqrt{3}\arctan(u) + C$$

Now, substitute back $t = \sqrt{3}u$:

$$\int \frac{1}{1+\left(\frac{t}{\sqrt{3}}\right)^2}\,dt = \sqrt{3}\arctan\left(\frac{t}{\sqrt{3}}\right) + C$$

Finally, multiply by $\frac{1}{3}$:

$$\int \frac{1}{\sin^2(x)+3\cos^2(x)}\,dx = \frac{1}{\sqrt{3}}\arctan\left(\frac{\tan(x)}{\sqrt{3}}\right) + C$$

Trigonometric polynomials

It is about calculating:

$$I_{(n,m)} = \int \sin^n(x)\cos^m(x)\,dx$$

1st case: $n = 2p+1$

$$I_{(n,m)} = \int \sin^{2p}(x)\cos^m(x)\sin(x)\,dx$$

$$I_{(n,m)} = \int (1-\cos^2(x))^p\cos^m(x)\sin(x)\,dx$$

$$I_{(n,m)} = -\int (1-\cos^2(x))^p\cos^m(x)(\cos(x))'\,dx$$

$$I_{(n,m)} = -\int (1-t^2)^p t^m\,dt$$

Trigonometric polynomials

2nd case: $m = 2q + 1$

$$I_{(n,m)} = \int \sin^n(x) \cos^{2q}(x) \cos(x)\, dx$$

$$I_{(n,m)} = \int \sin^n(x)(1 - \sin^2(x))^q (\sin(x))'\, dx$$

$$I_{(n,m)} = \int t^n (1 - t^2)^q\, dt$$

3rd case: $n = 2p$ and $m = 2q$ To do this, we switch to complex numbers:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Then, we linearize the expression $\sin^n(x) \cos^m(x)$.

Example

$$\int \sin^3(x)\cos^4(x)\,dx = -\int (1-\cos^2(x))\cos^4(x)(-\sin(x))\,dx$$

$$= -\int (1-t^2)t^4\,dt = -\frac{t^5}{5} + \frac{t^7}{7} + C$$

$$= -\frac{\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C$$

Trigonometric polynomials

Example

$$\int \sin^4 x \cos^5 x \, dx = \int_1 \sin^4 x \; \cos^4 x \cos x) \, dx$$

$$= \int \sin^4 x \; \cos^4 x \sin' x) \, dx$$

$$= \int \sin^4 x \, (1 - \sin^2 x)^2 \sin' x) \, dx$$

$$= \int \sin^4 x \, (1 - \sin^2 x)^2 (\sin' x)) \, dx$$

$$= \int t^4 (1 - t^2)^2 \, dt)$$

$$= \frac{1}{9} t^9 - \frac{2}{7} t^7 + \frac{1}{5} t^5 + C$$

$$= \frac{1}{9} \sin^9 x - \frac{2}{7} \sin^7 x + \frac{1}{5} \sin^5 x + C$$

Trigonometric polynomials

Example

$$\int \sin^2 x \cos^4 x \, dx$$

$$\sin^2 x \cos^4 x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 \left(\frac{e^{ix} + e^{-ix}}{2}\right)^4$$

$$= \frac{1}{32}(1 - \cos^2 x)(e^{4ix} + 4e^{2ix} + 6e^{ix}e^{-ix} + 4e^{-2ix} + e^{-4ix})$$

$$= \frac{1}{32}(1 - \cos 2x)(2\cos 4x + 8\cos 2x + 6)$$

$$= \frac{1}{16}(1 - \cos 2x)(\cos 4x + 4\cos 2x + 3)$$

Trigonometric polynomials

Example

(continuation):

$$\sin^2 x \cos^4 x = -\frac{1}{16}\cos 2x \cos 4x - \frac{1}{4}\cos^2 2x$$

$$-\frac{3}{16}\cos 2x + \frac{1}{16}(\cos 4x + 4\cos 2x + 3)$$

Using trigonometric identities:

$$\cos 2x \cos 4x = \frac{1}{2}(\cos 6x + \cos 2x)$$

$$\cos^2 2x = \frac{1 + \cos 4x}{2}$$

$$\sin^2 x \cos^4 x = \frac{1}{32}(-\cos 6x - 2\cos 4x + \cos 2x + 2)$$

Trigonometric polynomials

Example

(continuation):

$$\int \sin^2 x \cos^4 x \, dx = \int \frac{1}{32}(-\cos 6x - 2\cos 4x + \cos 2x + 2)\sqrt{24}\, dx$$

$$= -\frac{1}{192}\sin 6x - \frac{1}{64}\sin 4x + \frac{1}{64}\sin 2x + \frac{1}{16}x + C$$

Rational functions in hyperbolic sine and cosine

The goal is to find the primitives of $x \mapsto R(\cosh(x), \sinh(x))$, where R is a rational function of two real variables.
The change of variable $t = e^x$ or even $t = \tanh\left(\frac{x}{2}\right)$ leads to:

$$\cosh(x) = \frac{1+t^2}{1-t^2}, \quad \sinh(x) = \frac{2t}{1-t^2}, \quad \text{and} \quad dx = \frac{2dt}{1-t^2}$$

This leads us to primitives of rational functions in t.

Rational functions in hyperbolic sine and cosine

However, before undertaking such variable changes, it is more advantageous to see if other changes are possible, such as $t = \cosh(x)$, $t = \sinh(x)$, or $t = \tanh(x)$.
To do this, we replace $\cosh(x)$ with $\cos(x)$, $\sinh(x)$ with $\sin(x)$, and then apply the rules of BIOCHE seen previously.

Example

Calculate:

$$\int \frac{1}{1+\cosh(x)}\, dx$$

$$\int \frac{1}{1+\cosh(x)}\, dx = 2\int \frac{e^x}{e^{2x}+2e^x+1}\, dx$$

Example

(continuation):

$$2\int \frac{e^x}{(e^x+1)^2}\,dx = 2\int \frac{e^x}{(e^x+1)^2}\,dx = -\frac{2}{1+e^x} + C$$

Hence:

$$\int \frac{1}{1+\cosh(x)}\,dx = -\frac{2}{1+e^x} + C.$$

Rational functions in hyperbolic sine and cosine

Example

Calculate:

$$\int \frac{\sinh^3(x)}{\cosh(x)}\, dx$$

$$\int \frac{\sinh^3(x)}{\cosh(x)}\, dx = \int \frac{\sinh^2(x)}{\cosh(x)} \sinh(x)\, dx = \int (\cosh^2(x) - 1)\, dx = \int \frac{t^2-1}{t}\, d$$

$$= \int (t - \frac{1}{t})\, dt = \frac{1}{2}t^2 - \ln|t| + C = \frac{1}{2}\cosh^2(x) - \ln|\cosh(x)| + C$$

Example

$$\int \frac{1}{\cosh 3x - \cosh x}\, dx$$

Let $\omega(x) = \frac{1}{\cosh 3x - \cosh x}$ such that $\omega(\pi - x) = \omega(x)$.
Applying the BIOCHE rule, let $t = \sin x$ for the trigonometric function $\frac{1}{\cos 3x - \cos x}$.
By analogy, let $t = \sinh x$.

$$\cosh 3x = \cosh(2x + x) = \cosh 2x \cosh x + \sinh x \sinh 2x$$

$$\cosh 3x = (\cosh^2 x + \sinh^2 x) \cosh x + 2 \cosh x \sinh^2 x$$

$$\cosh 3x = \cosh x + 4 \sinh^2 x \cosh x$$

Rational functions in hyperbolic sine and cosine

Example

(continued):

$$\cosh(3x) - \cosh(x) = 4\sinh^2(x)\cosh(x)$$

$$\int \frac{1}{\cosh^3(x) - \cosh(x)}\,dx = \frac{1}{4}\int \frac{1}{\sinh^2(x)\cosh(x)}\,dx$$

$$= \frac{1}{4}\int \frac{\cosh(x)}{\sinh^2(x)\cosh^2(x)}\,dx$$

Example

(continued): So, the integral becomes:

$$= \frac{1}{4}\int \frac{1}{t^2(1+t^2)}\,dt$$

$$= \frac{1}{4}\int \left(\frac{1}{t^2} - \frac{1}{1+t^2}\right)\,dt$$

$$= -\frac{1}{4t} - \frac{1}{4}\arctan(t) + C$$

$$= -\frac{1}{4\sinh(x)} - \frac{1}{4}\arctan(\sinh(x)) + C$$

Functions containing radicals

To find the primitives of $\int R(x, \sqrt{ax^2+bx+c})\, dx$, where R is a rational function of two real variables, we refer to the canonical expression of the trinomial ax^2+bx+c, which can be transformed into one of the three forms depending on the cases:
1. **First case:** $\sqrt{k^2-t^2}$ - We can set $t = k\cos(\theta)$ with $\theta \in [0, \pi]$ or $t = k\sin(\theta)$ with $\theta \in [-\pi/2, \pi/2]$.
2. **Second case:** $\sqrt{k^2+t^2}$ - We set $t = k\sinh(\theta)$ with $\theta \in \mathbb{R}$.
3. **Third case:** $\sqrt{t^2-k^2}$ - If $t \geq k$, we set $t = k\cosh(\theta)$ with $\theta \geq 0$.
- If $t \leq -k$, we set $t = -k\cosh(\theta)$ with $\theta \geq 0$.
Depending on the case, we perform the appropriate variable change to simplify the expression and make the calculation of the primitive more accessible.

Functions containing radicals

Example

Calculete the integral $\int \frac{1}{\sqrt{x^2+2x+3}}\,dx$,

$$\int \frac{1}{\sqrt{x^2+2x+3}}\,dx = \int \frac{1}{\sqrt{2+(x+1)^2}}\,dx = \int \frac{1}{\sqrt{2}\sqrt{1+\left(\frac{x+1}{\sqrt{2}}\right)^2}}\,dx$$

Let's make the substitution: $\frac{x+1}{\sqrt{2}} = t$, which implies $dx = \sqrt{2}\,dt$.

$$\int \frac{1}{\sqrt{x^2+2x+3}}\,dx = \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}}{\sqrt{1+t^2}}\,dt = \operatorname{Argsh} t + C = \operatorname{Argsh}\left(\frac{x+1}{\sqrt{2}}\right) -$$

Functions containing radicals

Example

Now, we want to find $\int \frac{3x+2}{\sqrt{x^2+2x+3}}\,dx$.

$$\int \frac{3x+2}{\sqrt{x^2+2x+3}}\,dx = \int \frac{\frac{3}{2}(2x+2)-1}{\sqrt{x^2+2x+3}}\,dx$$

$$= \frac{3}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx - \int \frac{1}{\sqrt{x^2+2x+3}}\,dx$$

$$= \frac{3}{2}\left(2\sqrt{x^2+2x+3}\right) + C_1 - \operatorname{Argsh}\left(\frac{x+1}{\sqrt{2}}\right) + C_2$$

$$= 3\sqrt{x^2+2x+3} - \operatorname{Argsh}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

Functions containing radicals

Example

$$\int \frac{1}{\sqrt{x^2+2x+3}}\,dx$$

$$\int \frac{1}{\sqrt{1+2x-x^2}}\,dx = \int_1 \frac{1}{\sqrt{2-(x-1)^2}}\,dx$$

We set:

$$x-1=\sqrt{2}\sin\theta \quad \left(\text{with } |\theta|<\frac{\pi}{2}\right) \implies dx=\sqrt{2}\cos\theta\,d\theta$$

Functions containing radicals

Example

$$\implies \int \frac{1}{\sqrt{1+2x-x^2}}\,dx = \int \frac{\sqrt{2}\cos\theta}{\sqrt{2-2\sin^2\theta}}\,d\theta = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta$$

$$= \int d\theta = \theta + C = \text{Arcsin}\left(\frac{x-1}{\sqrt{2}}\right) + C$$

Example

Calculete

$$\int \sqrt{x^2+3x+2}\,dx, \quad x \geq -1$$

$$\int \sqrt{x^2+3x+2}\,dx = \int \sqrt{(x+3/2)^2-(1/2)^2}\,dx$$

We set:

$$x+3/2 = 1/2\cosh\theta \quad (\text{for } \theta \geq 0) \implies dx = 1/2\sinh\theta\,d\theta$$

$$\int \sqrt{x^2+3x+2}\,dx = \int \frac{1}{2}\sqrt{(1/2\cosh\theta)^2-(1/2)^2}\sinh\theta\,d\theta$$

Functions containing radicals

Example

Continued

$$\int \sqrt{x^2 + 3x + 2}\, dx = \frac{1}{4} \int \sqrt{\cosh^2\theta - 1}\, \sinh\theta \, d\theta$$

$$= \frac{1}{4} \int \sqrt{\sinh^2\theta}\, \sinh\theta \, d\theta$$

$$= \frac{1}{4} \int \sinh^2\theta \, d\theta$$

$$= \frac{1}{8} \int (\cosh 2\theta - 1) \, d\theta$$

$$= \frac{1}{16} \sinh 2\theta - \frac{1}{8}\theta + C$$

Functions containing radicals

Example

Continued

$$= \frac{1}{8}\cosh\theta\sqrt{\cosh^2\theta - 1} - \frac{1}{8}\theta + C$$

$$= \frac{1}{8}(2x+3)\sqrt{(2x+3)^2 - 1} - \frac{1}{8}\text{Argcosh}(2x+3) + C$$

$$= \frac{1}{4}(2x+3)\sqrt{x^2+3x+2} - \frac{1}{8}\text{Argcosh}(2x+3) + C$$

Functions containing radicals

Example

$$\int \frac{1}{(4x-x^2)^{3/2}}\,dx = \int_1 \frac{1}{(4-(x-2)^2)^{3/2}}\,dx$$

We set:

$$x-2 = 2\sin\theta, |\theta| < \frac{\pi}{2} \implies dx = 2\cos\theta d\theta$$

$$\int \frac{1}{(4x-x^2)^{3/2}}\,dx = \int \frac{2\cos\theta}{(4-4\sin^2\theta)^{3/2}}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{(1-\sin^2\theta)^{3/2}}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{(\cos^3\theta)}\,d\theta$$

Example

Continued

$$\int \frac{1}{(4x-x^2)^{3/2}}\,dx = \frac{1}{4}\int \frac{1}{\cos^2\theta}\,d\theta = \frac{1}{4}\tan\theta + C = \frac{1}{4}\frac{\sin\theta}{\cos\theta} + C$$

$$= \frac{1}{4}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} + C = \frac{1}{4}\frac{\frac{x-2}{2}}{\sqrt{4x-x^2}} + C = \frac{1}{4}\frac{x-2}{\sqrt{4x-x^2}} + C$$

Example

$$\int \frac{1}{x\sqrt{x^2+x+1}}\,dx = \int \frac{1}{x\sqrt{3/4+\left(x+\frac{1}{2}\right)^2}}\,dx = \frac{2}{\sqrt{3}}\int \frac{1}{x\sqrt{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}}$$

We let $\frac{2x+1}{\sqrt{3}} = \sinh\theta$ which implies $dx = \frac{\sqrt{3}}{2}\cosh\theta\,d\theta$. Substituting this, we get:

$$\int \frac{1}{x\sqrt{x^2+x+1}}\,dx = \frac{2}{\sqrt{3}}\int \frac{1}{\sinh\theta - \frac{1}{\sqrt{3}}}\,d\theta$$

Functions containing radicals

Example

Continued We make the substitution $t = e^{\theta}$ which implies $d\theta = \frac{1}{t}dt$. Substituting this, we get:

$$\frac{2}{\sqrt{3}} \int \frac{1}{\sinh\theta - \frac{1}{\sqrt{3}}}\, d\theta = \frac{4}{\sqrt{3}} \int_1 \frac{1}{(t - \frac{1}{\sqrt{3}})^2 - \frac{4}{3}}\, dt$$

$$= \frac{4}{\sqrt{3}} \int \frac{1}{(t - \sqrt{3})(t + \frac{1}{\sqrt{3}})}\, dt$$

$$= \int \left(\frac{1}{t - \sqrt{3}} - \frac{1}{t + \frac{1}{\sqrt{3}}} \right) dt$$

$$= \ln \left| \frac{t - \sqrt{3}}{t + \frac{1}{\sqrt{3}}} \right| + C$$

Functions containing radicals

Example

Continued

$$\theta = \operatorname{ArgSh}\left(\frac{2x+1}{\sqrt{3}}\right) = \ln\left(\frac{2x+1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sqrt{x^2+x+1}\right)$$

$$\Rightarrow t = e^{\theta} = \frac{2x+1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sqrt{x^2+x+1}$$

$$\int \frac{1}{x\sqrt{x^2+x+1}}\,dx = \ln\left|\frac{x-1+\sqrt{x^2+x+1}}{x+1+\sqrt{x^2+x+1}}\right| + C$$

To find the primitives of functions of the form $\frac{P_n(x)}{\sqrt{ax^2+bx+c}}$, we can use the reduction formula:

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

To determine the coefficients, we differentiate, multiply by $\sqrt{ax^2+bx+c}$, and then identify terms.

Functions containing radicals

Example

$$\int \frac{x^2+1}{\sqrt{x^2+2x+3}}\,dx = (ax+b)\sqrt{x^2+2x+3} + \lambda \int_1 \frac{1}{\sqrt{x^2+2x+3}}\,dx$$

We differentiate:

$$\frac{x^2+1}{\sqrt{x^2+2x+3}} = a\sqrt{x^2+2x+3} + (ax+b)\frac{x+1}{\sqrt{x^2+2x+3}} + \frac{\lambda}{\sqrt{x^2+2x+3}}$$

Functions containing radicals

Example

$$x^2 + 1 = a(x^2 + 2x + 3) + (ax + b)(x + 1) + \lambda$$

By identification, we find:

$$\begin{cases} 2a = 1 \\ 3a + b = 0 \\ 3a + b + \lambda = 1 \end{cases}$$

$$\begin{cases} a = \frac{1}{2} \\ b = -\frac{3}{2} \\ \lambda = 1 \end{cases}$$

Functions containing radicals

Example

Continued

$$\int \frac{x^2+1}{\sqrt{x^2+2x+3}}dx = \left(\frac{1}{2}x - \frac{3}{2}\right)\sqrt{x^2+2x+3} + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

The integral $\int_1 \frac{1}{\sqrt{x^2+2x+3}}dx$ has been calculated. (see Example 1, primitives of: $x \mapsto Arsh(x, \sqrt{ax^2+bx+c}))$

$$\int \frac{1}{\sqrt{x^2+2x+3}}dx = Arsh\left(\frac{x+1}{\sqrt{2}}\right) + C$$

Hence:

$$\int \frac{x^2+1}{\sqrt{x^2+2x+3}}dx = \left(\frac{1}{2}x - \frac{3}{2}\right)\sqrt{x^2+2x+3} + Arsh\left(\frac{x+1}{\sqrt{2}}\right) + C$$

Functions containing radicals

Primitives of functions in the form $R\left(x, \left(\frac{ax+b}{cx+d}\right)^{r_1}, \left(\frac{ax+b}{cx+d}\right)^{r_2}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{r_n}\right)$ with $ad - bc \neq 0$ and $r_1, r_2, \ldots, r_n$ as positive rational numbers.
Let $\frac{ax+b}{cx+d} = t^m$, where m is the least common denominator.

Example

$$\int \frac{1}{x\sqrt{\frac{1-x}{1+x}}}\,dx$$

We set:

$$\frac{1-x}{1+x} = t^2 \implies x = \frac{1-t^2}{1+t^2} \implies dx = -\frac{4t}{(1+t^2)^2}\,dt$$

Thus,

$$\int \frac{1}{x\sqrt{\frac{1-x}{1+x}}}\,dx$$

$=$

$$\int \frac{4t^2}{(1-t^2)(1+t^2)}\,dt$$

Example

(continued):

$$\int \frac{1}{x\sqrt{\frac{1-x}{1+x}}}\,dx = -\int \left(\frac{A}{1-t} + \frac{B}{1+t} + \frac{Ct+D}{1+t^2}\right)\,dt$$

Where: $A = B = \frac{1}{4}$, $C = 0$, and $D = -\frac{1}{2}$

$$\int \frac{1}{x\sqrt{\frac{1-x}{1+x}}}\,dx = -\int \left(\frac{\frac{1}{4}}{1-t} + \frac{\frac{1}{4}}{1+t} - \frac{\frac{1}{2}}{1+t^2}\right)\,dt$$

Functions containing radicals

Example

(continued):

$$\int \frac{1}{x\sqrt{\frac{1-x}{1+x}}}\,dx = -\left(\frac{1}{4}\ln|1+t| - \frac{1}{4}\ln|1-t| - \frac{1}{2}\arctan(t)\right) + C$$

$$\int \frac{1}{x\sqrt{\frac{1-x}{1+x}}}\,dx = \frac{1}{4}\ln\left(\frac{1-t}{1+t}\right) - \frac{1}{2}\arctan(t) + C$$

$$\int \frac{1}{x\sqrt{\frac{1-x}{1+x}}}\,dx = \frac{1}{4}\ln\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) - \frac{1}{2}\arctan\left(\sqrt{\frac{1-x}{1+x}}\right) + C$$

Example

$$\int \frac{1}{(\sqrt[3]{x^2}(2+3\sqrt[3]{x}))}\, dx, \quad x > 0$$

Let $x = t^3$, then $dx = 3t^2\, dt$:

$$\int \frac{1}{(\sqrt[3]{x^2}(2+3\sqrt[3]{x}))}\, dx = \int \frac{3t^2}{t^2(2+3t)}\, dt = \int \frac{3}{(2+3t)}\, dt$$

$$\int \frac{1}{(\sqrt[3]{x^2}(2+3\sqrt[3]{x}))}\, dx = \ln|2+3t| + C = \ln\left|2+3\sqrt[3]{x}\right| + C$$