“Young man, in Mathematics you don’t understand things. You just get used to them.”
John von Neumann

Lecture 1 : Logic

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Number sets

Natural Numbers

$$\mathbb{N} = \{0,1,2, \dots\}$$

Integers

$$\mathbb{Z} = \{\dots, -2, -1,0,1,2, \dots\}$$

Rational Numbers

$$\mathbb{Q} = \left\{\frac{p}{q}, p \in \mathbb{Z}, q \in N^*\right\}$$

Decimal Numbers

$$D = \left\{\frac{a}{10^n}, a \in \mathbb{Z}, n \in \mathbb{N}\right\}$$

Real Numbers : the set of Rational Numbers with the set of Irrational Numbers adjoined.

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$$

Number sets

Complex Numbers - A complex number is a number which can be written in the form $a + bi$ where a and b are real numbers and i is the square root of -1. The set of complex numbers is denoted $\mathbb{C}$.

Notations

$\mathbb{N}^* = \mathbb{N} \setminus \{0\}; \mathbb{Z}^* = \mathbb{Z} \setminus \{0\};\ \mathbb{Q}^* = \mathbb{Q} \setminus \{0\};\ \mathbb{R}^* = \mathbb{R} \setminus \{0\};\ \mathbb{C}^* = \mathbb{C} \setminus \{0\}$

$A_+ : A = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are positive numbers

$A_- : A = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are negative numbers

Definitions

Proposition

A proposition is a mathematically precise statement that is either true or false, but not both.

Examples

- The statement "$1 + 1 > 3$" is false, while the statement "$5 > 3$" is true. Both statements are propositions.
- The statement "What a great book!" is not a proposition. Someone is simply expressing an opinion.
- The statement "$x + 1 < 7$" is not a proposition. The truth value of this statement relies on what the variable x is assigned.

Examples

- The statement "It is cold outside" is not a proposition, because it lacks precision. Without a definite definition of cold, one cannot assign to the statement a truth value.
- The statement "$5 + 7$" is not a proposition, because it does not have a truth value.
- The statement "$5 + 7 = 12$" is a proposition, because it has a truth value, namely true.

Definitions

Paradox

A paradox is a statement that cannot be assigned a truth value.

Example 1

“This statement is false”

If true, than false and if false, then true.

Example 2

A classic example of a paradox is the barber paradox. It states:

“The barber shaves every man in town who does not shave himself.”

If the barber shaves himself, then this means the barber does not shave himself and if he does not shave himself then he must shave himself. Neither is possible and so it is a paradox.

Definitions

We are particularly interested in combining propositions by operators.

Definition

A compound proposition is a statement obtained by combining propositions with logical operators.

operators

1. ∧ conjunction (and)

Truth Table :

p	q	p ∧ q	q ∧ p
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

This time, for p ∧ q to be true, we need both p and q to be true.

Note that ∧ is also commutative.

Operators

2. ∨ disjunction (or)

Truth Table:

p	q	p ∨ q	q ∨ p
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

The disjunction operator returns T when at least one of the two propositions p or q is true.

The operator ∨ is commutative.

Operators

Remark

More than two propositions can be joined using logical operators.

In this instances, it is important to be careful about how they are grouped.

Example

$\big((2+1=4)\ \land (2+2=5)\big) \lor (5-2=3)$ is true

$(2+1=4)\ \land ((2+2=5) \lor (5-2=3))$ is false

Operators

3. ¬ negation (not)

Truth Table:

p	¬ p
T	F
F	T

Operators

4. $\oplus$ exclusive (xor)

Truth Table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

The statement is true if and only if exactly one of the statements is true.

Operators

5. Implication ($\Longrightarrow$)

The statement $\neg p \vee q$ is denoted by $p \Longrightarrow q$.

We say

1) p implies q
2) If p, then q
3) The truth of p is a necessary condition for the truth of q

Operators

Truth table

p	q	$\neg p$	$p \Longrightarrow q$ $(\neg p \vee q)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

The converse of $p \Longrightarrow q$ is $q \Longrightarrow p$.

The contrapositive of $p \Longrightarrow q$ is $\neg q \Longrightarrow \neg p$

Operators

6. Equivalence ($\Leftrightarrow$)

$p \Leftrightarrow q$ means ($p \Rightarrow q$ and $q \Rightarrow p$).

Truth table

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Equivalent Propositions

Two propositions are equivalent if they have identical truth tables.

Properties

1) $(p \wedge q) \Leftrightarrow (q \wedge p)$
2) $(p \vee q) \Leftrightarrow (q \vee p)$
3) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
4) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
5) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
6) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
7) $\neg(\neg p) \Leftrightarrow p$
8) $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$
9) $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$
10) $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$
11) $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
12) $((p \Leftrightarrow q) \wedge (q \Leftrightarrow r)) \Rightarrow (p \Leftrightarrow r)$

An amizing exercise

The Island of Knights and Knaves.

On the island, Knights always tell the truth and Knaves always lie.

You meet two islanders A and B. A says :

I am a Knave but he is not

You need to decide what are A and B.

Predicates and Quantifiers

Definiton

A predicate is a statement that contains variables. A predicate may be true or false depending on the values of these variables.

Example

$$\forall x \in \mathbb{R}, x^2 \geq 0$$

Predicates and Quantifiers

- Universal quantifier

$$\forall\, x \text{ (for all x), } P(x)$$

means that $P(x)$ is true for all possible values of x.

Predicates and Quantifiers

- Existential quantifier

$$\exists\, x \;\; \text{(there exists } x\text{)},\; P(x)$$

Means that there exists an x where $P(x)$ is true.

Sometimes, we will use also

$$\exists!\, x, P(x)$$

It means that there exists a unique x where $P(x)$ is true.

Predicates and Quantifiers

Consider the universal statement

$$\forall x , P(x).$$

This asserts that $P(x)$ is true for all values of x . Hence, if it is false, then this means that there exists an x such that $P(x)$ is false.

Similarly, the existential statement

$$\exists x , P(x)$$

asserts that there is an x where $P(x)$ is true. Hense, if it is false, this means that for all values of x , $P(x)$ is false, that is $\neg P(x)$ is true . Therefore, we have the following :

Predicates and Quantifiers

$$\neg(\forall\, x, P(x)) \Leftrightarrow (\exists\, x, \neg P(x))$$

$$\neg(\exists\, x, P(x)) \Leftrightarrow (\forall\, x, \neg P(x))$$

Predicates and Quantifiers

Examples

Statement : $\forall\ x\ \in\ \mathbb{R},\ 2x < 3$

Negation : $\exists\ x\ \in\ \mathbb{R},\ 2x\ \geq 3$

Statement :$\exists\ x\ \in\ \mathbb{R},\ 2^{x}\ = 256$

Negation :$\forall x\ \in\ \mathbb{R},\ 2^{x} \neq 256$

Predicates and Quantifiers

Some statement involve several quantifiers.

The statement

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y > x$$

means that for all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $y > x$.

This statement is true.

The order of the quantifiers is very important.

The statement

$$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y > x$$

is false since the negation

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, y \leq x$$

is true.

Predicates and Quantifiers

Example 1

The predicate

$$\forall\, x \in \mathbb{Z}, \qquad (x^2 = 5 \implies x = 2)$$

is true.

Example 2

The predicate

$$\exists\, x \in \mathbb{R}, x^2 = 2$$

is true, but the predicate

$$\exists\, x \in \mathbb{Z}, x^2 = 2$$

is false.

Methods of proof

- Direct Proof

$$A \Longrightarrow B$$

Example

Let $n \in \mathbb{N}$. Show that

$$n \text{ is odd} \Longrightarrow n^2 \text{ is odd}$$

- Proof by contrapositive

We show $\neg B \Longrightarrow \neg A$ instead of $A \Longrightarrow B$

Example

Let $n \in \mathbb{N}^*$.

Show that

$$n^2 - 1 \text{ is not divisible by } 8 \Longrightarrow n \text{ is even}$$

Methods of proof

- Proof by contradiction

To show that p is true, we suppose that p is false and than $\neg p$ is true.

We show that $\neg p \Longrightarrow \mathrm{q}$, where q is false.

So $\neg \mathrm{q} \Longrightarrow p$ is true.

As $\neg \mathrm{q}$ is true then p is true.

Example

$$\sqrt{2} \notin \mathbb{Q}.$$

Methods of proof

- By induction

Let $P(n)$ be a logical statement for each $n \in \mathbb{N}$. The principle of mathematical induction states that $P(n)$ is true for all $n \in \mathbb{N}$ if :

1. $P(0)$ is true, and
2. $P(n) \implies P(n+1)\ for\ all\ n \in \mathbb{N}$

Example

Show that for all $n \in \mathbb{N}^*$, we have $2^{n-1} \leq n! \leq n^n$.

Methods of proof

5. By giving a counter example

To show that a proposition p is false, we must show that $\neg p$ is true.

Example 1

Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ a function defined by $f(x) = x^2$.

Show that the proposition :

$$\forall\, x, y \in \mathbb{R}, f(x) = f(y) \Longrightarrow x = y$$

is false.

Example 2

Show that the proposition

$$\forall\, y \in \mathbb{R}, \exists x \in \mathbb{R},\ f(x) = y$$

is false.