Lecture 11

Real functions of real variables-Limits-Continuity and Differentiability

Centents:

1. Real functions of real variables .
2. Concept of limits.
3. Continuity.
4. Differentiability.

1. Real functions of real variables.

1.1. Definitions-Examples:

A real function of a real variable is an

Maps $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}$.

1. Real functions of real variables

Definition:

$$U = \{x \in \mathbb{R}, f(x) is\ well\ defined\}$$

U ***is called the domaine of f .***

The plot of f

is $the\ subset\ denoted\ by\ C_f\ of$ $\mathbb{R}^2$

defined by

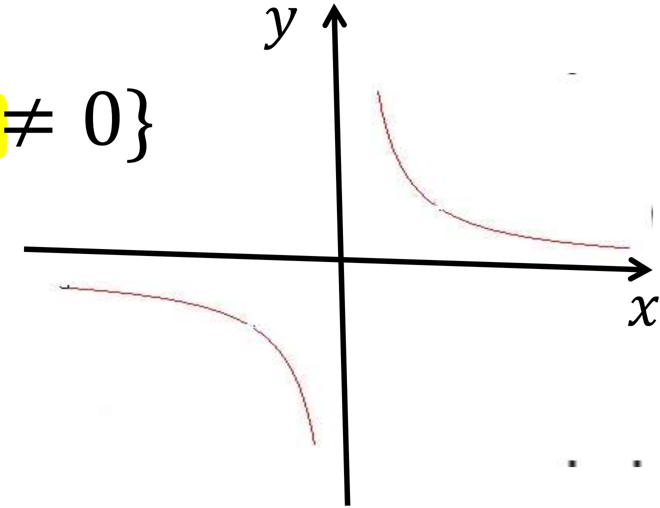
$$C_f = \{(x, f(x));\ x \in U\}$$

1 Real functions of real variables

Examples :

1) $f(x) = \dfrac{1}{x}$

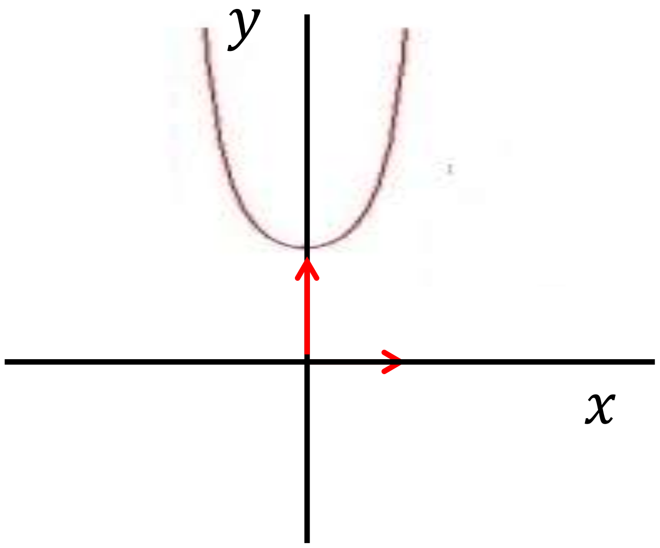
$$D_f = \{x \in \mathbb{R} \,;\, x \neq 0\}$$



$$D_f = \mathbb{R}^* =]-\infty, 0[\cup]0, +\infty[$$

1. Real functions of real variables

$2)\ f(x) = x^2 + 1$ is defined on $\mathbb{R}$.



1. Real functions of real variables.

$$3)\ f(x) = \sqrt{x^2 - 3x + 2}$$

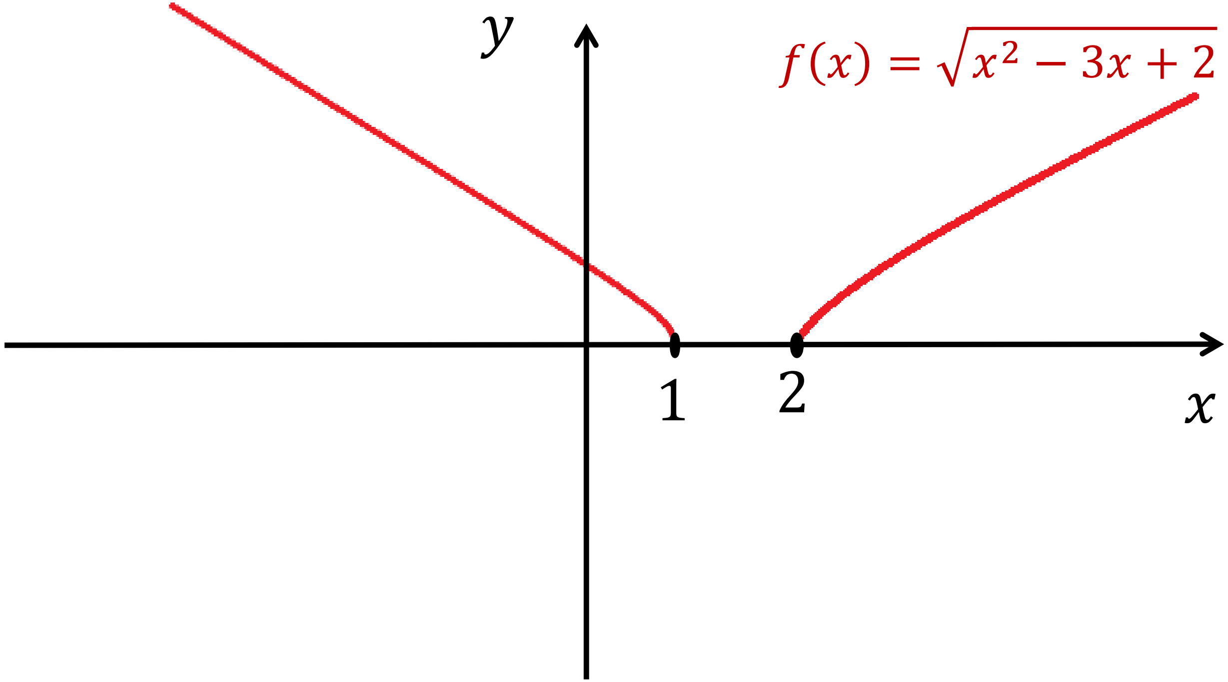
$$D_f = \{x \in \mathbb{R},\ x^2 - 3x + 2 \geq 0\}$$

$$\Delta = 1 \quad \Rightarrow x_1 = 1 \ et \ x_2 = 2.$$

x	$-\infty$ 1		2 $+\infty$
$x-1$	$-$	$+$	$+$
$x-2$	$-$	$-$	$+$
x^2-3x+2	$+$	$-$	$+$

$$D_f =]-\infty, 1] \cup [2, +\infty[\ .$$

1 Real functions of real variables.



1. Real functions of real variables

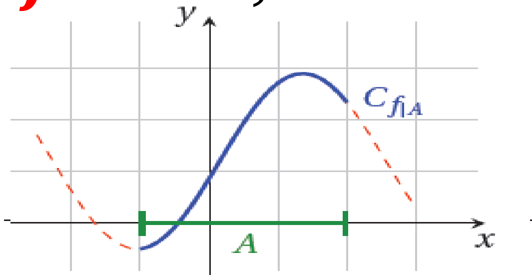
Restriction of a function

Definition:

Let f be a function and D_f the domaine of f . A a subset of D_f, $A \subset D_f$

We define the **restriction of f to A**,

$$f_{|A}: A \to \mathbb{R}$$
$$x \mapsto f_{|A}(x) = f(x)$$



1. Real functions of real variables

1.2. Combining functions

$$\mathcal{F}(\boldsymbol{I},\mathbb{R}) = \{\boldsymbol{f}: \boldsymbol{I} \longrightarrow \mathbb{R}\}$$

$$f\ et\ g \in \boldsymbol{\mathcal{F}}(\boldsymbol{I},\mathbb{R}),$$

$$\Downarrow$$

$$f = g \Leftrightarrow \forall x \in I, f(x) = g(x)$$

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- ✓ **Sums :**

$$(f+g)(x) = f(x) + g(x).$$

- ✓ **products :**

$$(fg)(x) = f(x)g(x).$$

- ✓ **Quotients:**

$$\forall x \in I; \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, avec\ g(x) \neq 0.$$

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✓ **inverse:**

$$f^{-1}(x) \neq [f(x)]^{-1} = \frac{1}{f(x)}$$

✓ **Multiplication by scalars :**

$$\forall \alpha \in \mathbb{R};\ \forall f \ \in \ \mathcal{F}(I,\mathbb{R}), \forall x \in I,$$
$$(\alpha f)(x) \ = \ \alpha f(x)$$

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1.2.2. Composition of function

Definition: $f : A \to B, g : B \to C$ two fonctions. Then the ***composite function (also called the compostion of g and f)*** is defined by:

$$(g \circ f)(x) = g(f(x)), x \in A.$$

$$\begin{tikzcd}
D_f \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] & f(D_f) \arrow[r, "g"] & \mathbb{R} \\
x \arrow[r] & f(x) \arrow[r] & g(f(x))
\end{tikzcd}$$

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Function Id_E

For all set E, we define the **identity function**

$$\mathrm{Id}_{\mathrm{E}}:\ \mathrm{E} \to \mathrm{E}$$

By

$$Id_E(x) = x, \qquad x \in E.$$

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Algebric operation for the identity function:

1. $f : A \to E, \quad Id_E \circ f = f$.

2. $f : A \to E, \quad f \circ Id_A = f$.

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The zero function

$$0(x) = 0, \forall x \in \mathbb{R}.$$

We have

$$0 + f = f + 0 = f, \forall f \in \mathcal{F}(I, \mathbb{R})$$

The unit function

$$1(x) = 1, \forall x \in \mathbb{R}.$$

We have

$$1 \cdot f = f \cdot 1 = f, \forall f \in \mathcal{F}(I, \mathbb{R})$$

1 Real functions of real variables

1.2.3. Functions and Bounds.

a. Bounded below, Bounded above

Definitions : $f : I \to \mathbb{R}$

✓ $f \text{ is bound below}$

$$\Updownarrow$$

$$\exists m \in \mathbb{R} \text{ such that: } \forall x \in I;\ f(x) \geq m.$$

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✓ f is ***bounded above***

$$\Updownarrow$$

$$\exists M \in \mathbb{R}\ tel\ que\quad \forall x \in I, f(x) \leq M$$

✓ f is ***bounded***

$$\Updownarrow$$

$$\exists (m, M) \in \mathbb{R}^2\ tel\ que$$
$$\forall x \in I, m \leq f(x) \leq M.$$

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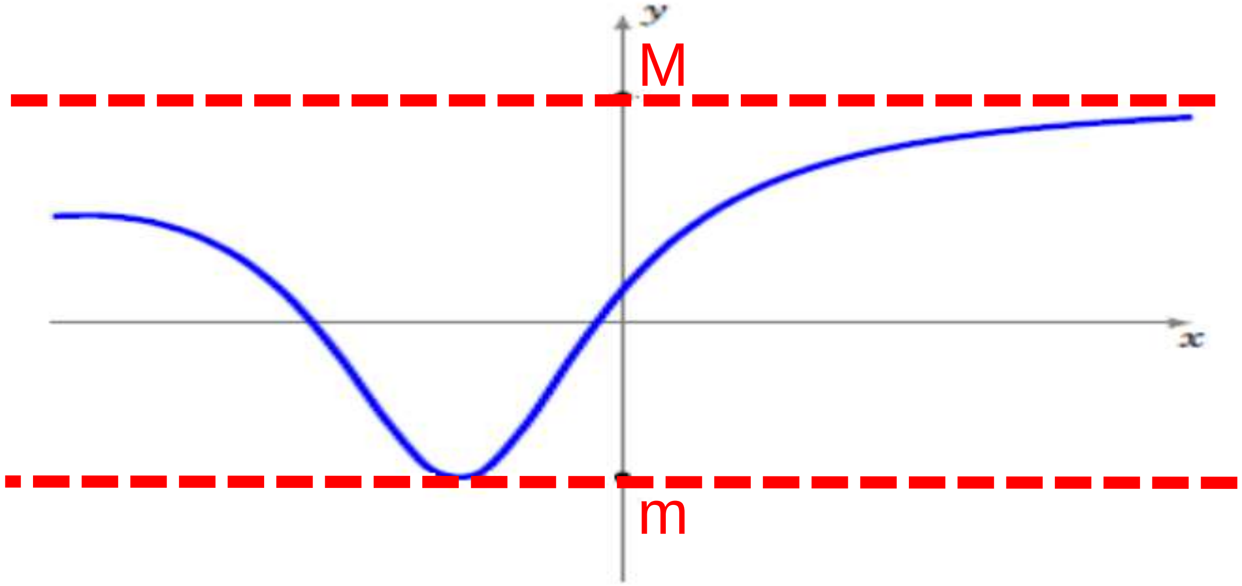
Geometric Interpretation :



1. Real functions of real variables

b. Increasing and decreasing functions

Definition:

Let I a subset of $\mathbb{R}$ and $f \in \mathcal{F}(I, \mathbb{R})$.

We say that f is

- ✓ ***Increasing*** $\Leftrightarrow \begin{cases} \forall (x, x') \in I^2, \\ (x < x' \Rightarrow f(x) \leq f(x')). \end{cases}$

- ✓ ***Decreasing*** $\Leftrightarrow$

$$\begin{cases} \forall (x, x') \in I^2, \\ (x < x' \Rightarrow f(x) \geq f(x')). \end{cases}$$

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✓ ***Strictly increasing*** :

$\Updownarrow$

$$\forall (x, x') \in I^2, (x < x' \Rightarrow f(x) < f(x')).$$

✓ ***Strictly decreasing***

$\Updownarrow$

$$\forall (x, x') \in I^2, (x < x' \Rightarrow f(x) > f(x')).$$

1 Real functions of real variables.

- ✓ ***monotonic*** *if only if is always increasing or always decreasing.*

- ✓ ***strictly monotonic*** *if only if is always strictly increasing or always strictly decreasing.*

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Example :

1. $f : \mathbb{R} \to \mathbb{R}; \quad x \longmapsto f(x) = x^3.$

is strictly increasing on $\mathbb{R}$.

2. $f : \mathbb{R} \to \mathbb{R}, \quad x \longmapsto f(x) = x^2.$

is neither increasing nor decreasing on $\mathbb{R}$.

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$$f(x) - f(x') = x^2 - x'^2 = (x - x')(x + x')$$

$$\frac{f(x) - f(x')}{x - x'} = (x + x')$$

$$\Rightarrow \quad f(x) - f(x') \leq 0$$

Then $f(x)$ is strictly decreasing sur $]-\infty, 0]$

1. Real functions of real variables

C. Odd and Even functions:

Definition: $f \in \mathcal{F}(I, \mathbb{R})$.

✓ f is ***even***

$\Updownarrow$

$$\forall x \in I \; et \; (-x) \in I \; ; \; f(-x) = f(x).$$

✓ f is ***odd***

$\Updownarrow$

$$\forall x \in I \; et \; (-x) \in I; \; f(-x) = -f(x).$$

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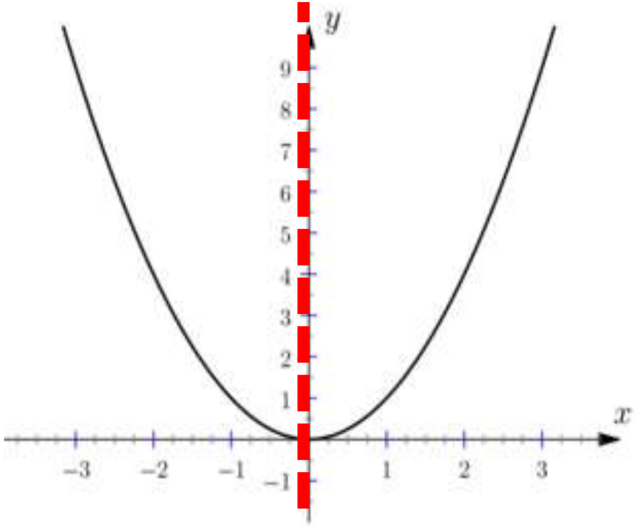
Example :

- $A\ function\ f(x) = x^2$

$is\ \ even,$

$$f(-x) = (-x)^2$$

$$= x^2 = f(x).$$



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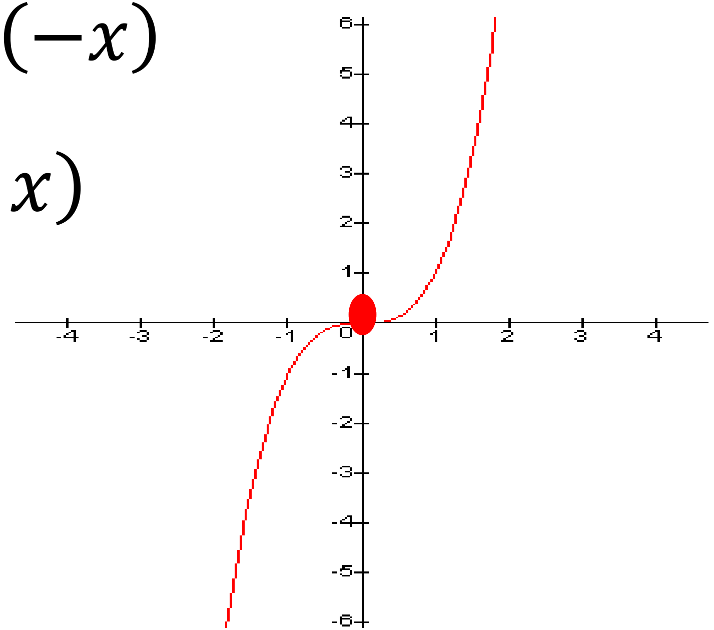
➤ $A\ \ function\ f(x) = x^3 + x\ is\ odd,$
$because$

$$f(-x) = (-x)^3 + (-x)$$

$$= -(x^3 + x)$$

$$= -f(x)$$



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d. Periodic Functions

A function y= f(x) is said to be a periodic function if there exists a positive real number P such that f(x + P) = f(x), for all x belongs to real numbers. The least value of the positive real number P is called the fundamental period of a function. This fundamental period of a function is also called the period of the function, at which the function repeats itself.

$$\forall x \in I; (x + P) \in I,$$
$$f(x + P) = f(x).$$

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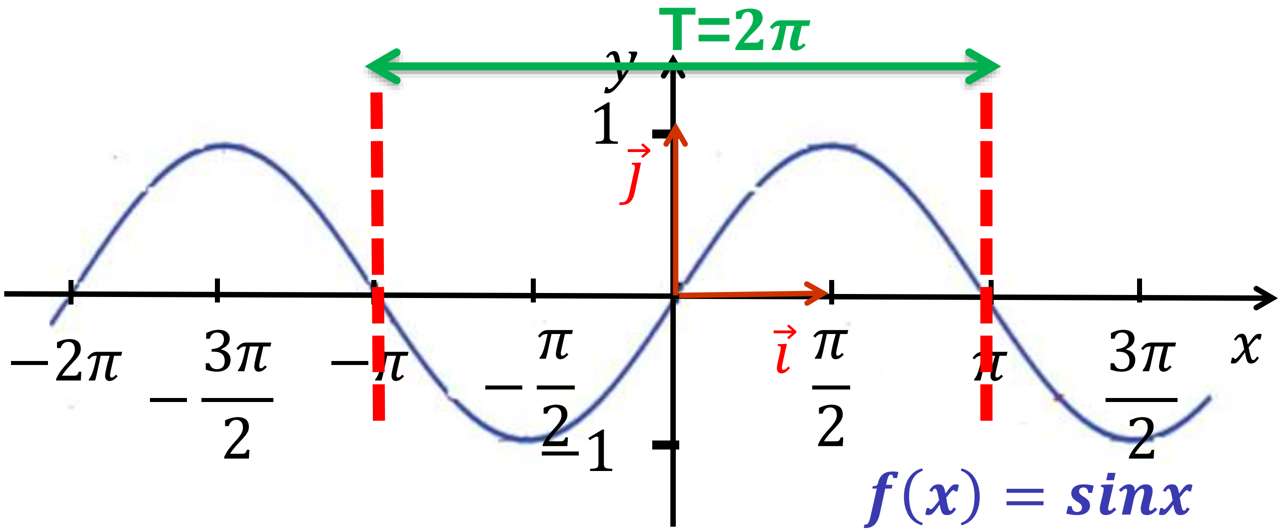
Example :

$The\ \ sine\ function\ and\ cosine\ function\ \ are$

$periodic\ \ with\ a\ period\ of\ \ 2\pi\ on\ \mathbb{R}.$



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