4. Derivatives

4.1. Defining the Derivative at a point.

4.1.1. Derivative at a point

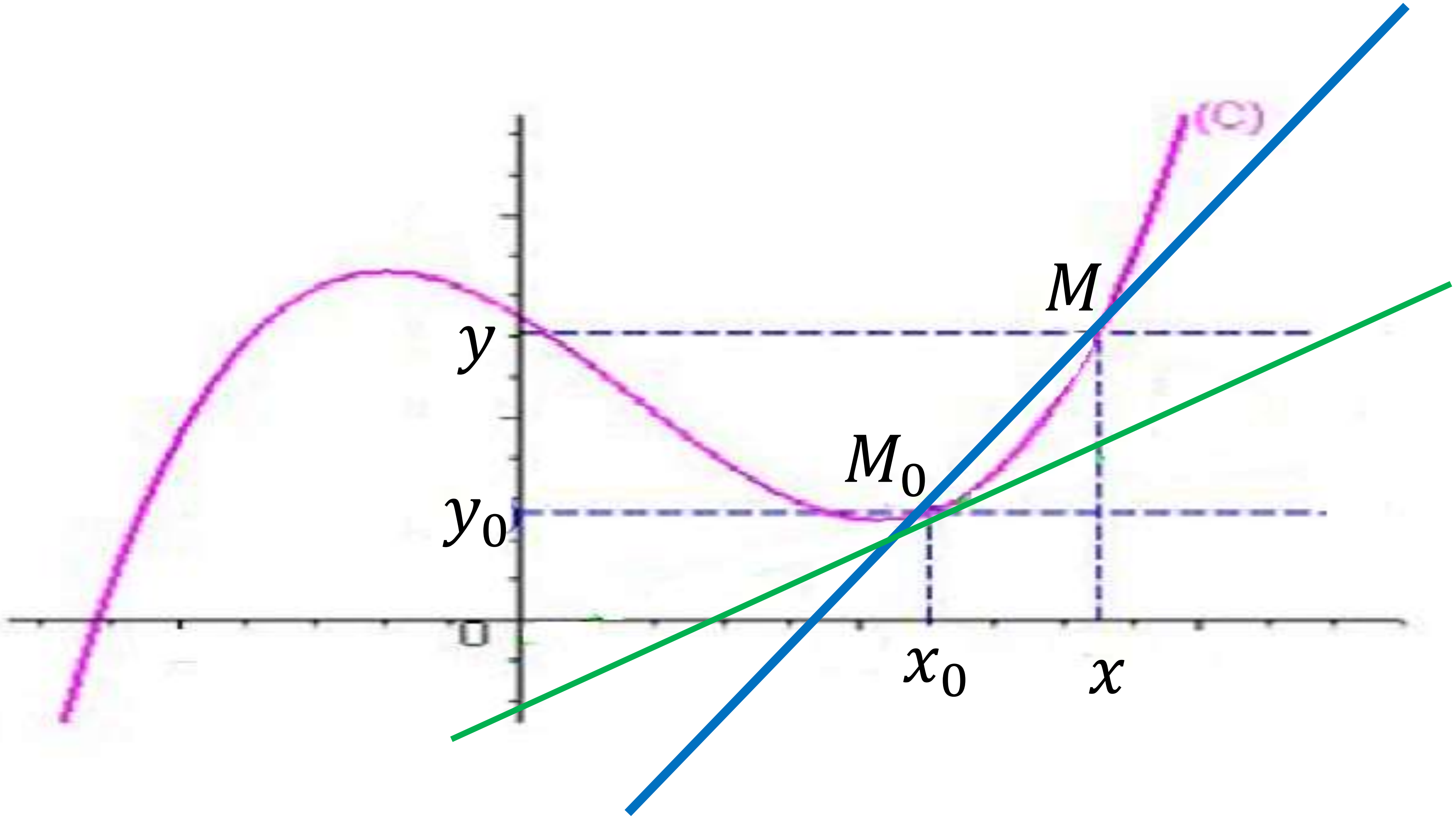
Definition : Let f be a function defined on an open interval $]a, b[$, and let $x_0 \in]a, b[$. We say $\boldsymbol{f}$ is differentiable at $\boldsymbol{x_0}$, with derivative $\boldsymbol{f'}(\boldsymbol{x_0})$, if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Exists.

4. Derivatives

Geometric Interpretation for the derivative



4. Derivatives

Definition bis : Let f be a function defined on an open interval $]a, b[$, and let $x_0 \in]a, b[$. We say $\boldsymbol{f}$ is differentiable at $\boldsymbol{x_0}$, with derivative $\boldsymbol{f'}(\boldsymbol{x_0})$, if the limit

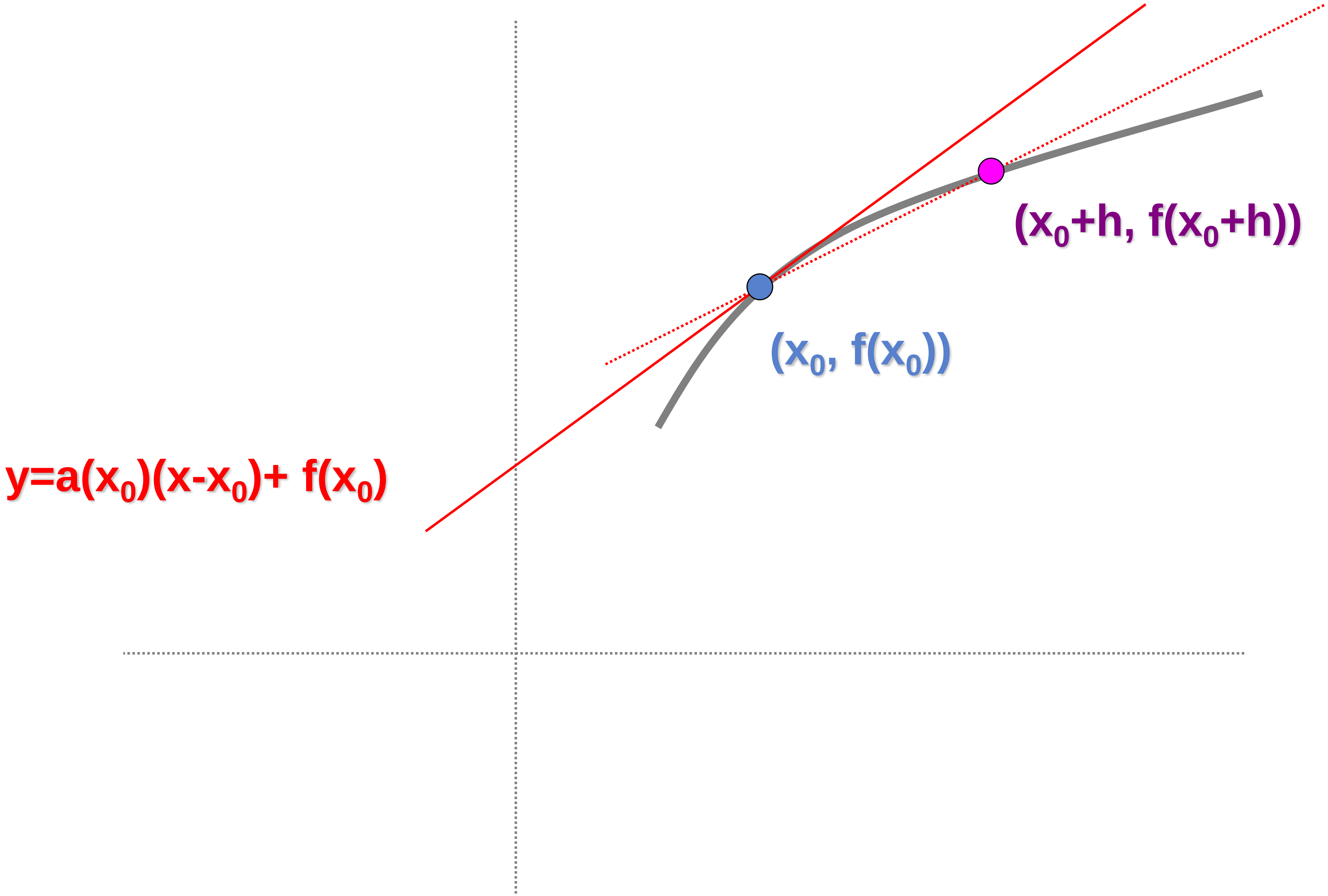
$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Exists.

This limit is denoted $f'(x_0)$ (or also $\frac{df}{dx}(x_0)$) and is called ***the derivative*** *of $f(x)$ at $x = x_0$.*

4. Derivatives

Geometric Interpretation



4. Derivatives

Example: Find, if possible, derivatives at $x_0 = 1$ for the function $f(x) = x^3 - 1$

$$f'(1) = \lim_{x\to 1} \frac{f(x) - \overset{0}{\overset{\|}{f(1)}}}{x-1}$$

$$= \lim_{x\to 1} \frac{x^3 - 1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = 3$$

$$and \;\; f'(x) = 3x^2 \qquad \Rightarrow \qquad f'(1) = 3$$

Derivatives

Left Hand And Right Hand Derivatives

Definition (Right derivative)

Let $f : [a,b[\to \mathbb{R}$. *A function* ***f* is right differentiable at *a*** *if the right limit :* $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$ *exists.* When it does exist, then that limit is called the *right derivative* of f. *And denoted by* $f'_d(a)$. *We will say* f *is defferentiable on* $[a,b[$ *when* f *is differentiable on* $]a,b[$ *and is right differentiable at* a.

4. Dérivabilité

Left Hand And Right Hand Derivatives

Definition (Left derivative)

Let $f :]a, b] \rightarrow \mathbb{R}$. *A function* ***$f$* is left defferentiable at *b*** *if the left limit* $\lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$ *exists.* When it does exist, then that limit is called the *left derivative* of $\boldsymbol{f}$, $and\ denoted\ by$ $\boldsymbol{f'_g(b)}$. *We will say* f *is defferentiable on* $]a, b]$ *when* f *is defferentiable on* $]a, b[$*and is right defferentiable at* b.

4. Derivatives

Example :

if $f(x) = |x| = \begin{cases} x & si\ x > 0 \\ -x & si\ x \leq 0 \end{cases}$,

$then\ f_d{}'(0) = 1$ *et* $f'_g(0) = -1$.

The following result is immediate

Proposition *: Let* $f : I \to \mathbb{R}$, $and\ a \in I$. *For the map f to be differentiable at a, it is necessary and sufficient that*

f be both left and right differentiable at a and the left and right derivative are equal $ie\ f'_g(a) = f'_d(a)$. *Under these conditions, we have* $f'(a) = f'_g(a) = f'_d(a)$.

4. Derivatives

Derivative on an interval

Definition: *Let f be a function defined on an interval I. We say that* ***a function f is differentiable on an interval I*** *if only if f is differentiable for all point a in that inerval I.*

Proposition : *Let* $f :]a,b[\to \mathbb{R}$ *and let* $x_0 \in]a,b[$.

- *If f is differentiable at x_0 then f is continuous at x_0.*
- if f *is differentiable on I then f is continuous on I.*

4. Derivatives

Combinations of Differentiable Functions

Theorem:(Algebraic Differentiability Theorem)
Let f and g be functions defined on an interval A, and assume both are differentiable at some point $x \in$ A. Then

- $(f+g)'(x) = f'(x) + g'(x)$,
- $(\alpha f)'(x) = \alpha f'(x), \alpha \in \mathbb{R}$,
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$,
- $\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{f(x)^2}, \quad si\ \ f(x) \neq 0.$
- $\left(\frac{f}{g}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad si\ g(x) \neq 0.$

4. Derivatives

Chain Rule

Proposition : Let $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ satisfy $f(I) \subseteq J$ so that the composition $g \circ f$ is defined. If f is differentiable at $x_0 \in I$ and if g is differentiable at $f(x_0) \in J$, then $g \circ f$ is differentiable at x_0 *with*

$$(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0).$$

4. Derivatives

Examples : *Soit* $h(x) = e^{-\frac{x^2}{2}}$

We put $f(x) = \frac{-x^2}{2}$ $\quad$ *and* $g(x) = e^x$.

$$h'(x) = (g \circ f)'(x) = g'(f(x))f'(x) = -x\, e^{-\frac{x^2}{2}}$$

4. Derivatives

Derivatives of inverse functions

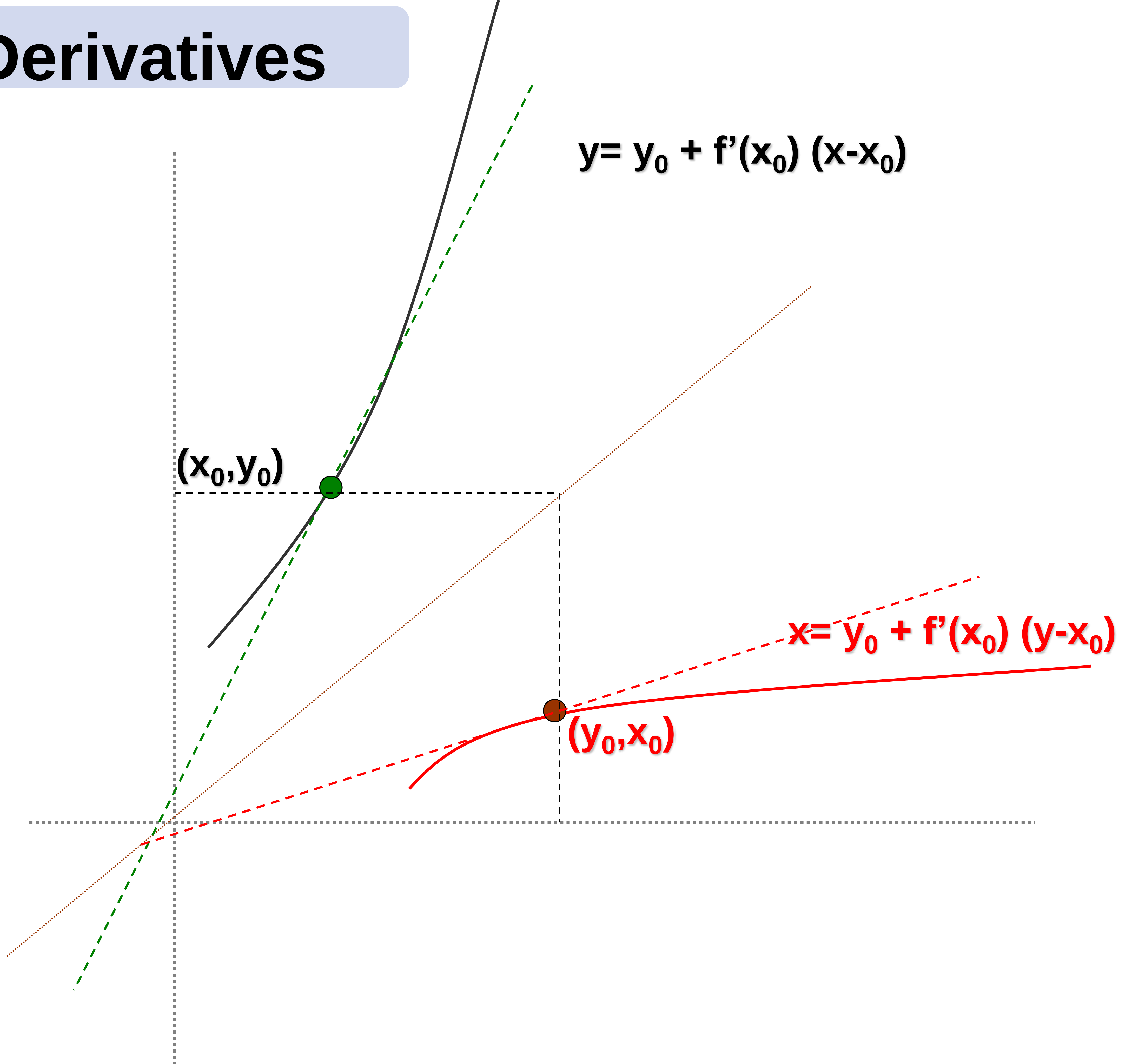
Let $f : I \to J$ and $\boldsymbol{f^{-1}}: J \to I$ be inverse functions, where I and J are open intervals.

Suppose f is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$.

Then $\boldsymbol{f^{-1}}$ is differentiable at $y_0 = f(x_0)$

$$\Rightarrow \quad (f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

4. Derivatives



4. Derivatives

Derivatives of usuels functions :

Fonction	Fonction dérivée
$x^n, n \in \mathbb{Z}$	nx^{n-1}
$x^\alpha, \alpha \in \mathbb{R}$	$\alpha x^{\alpha-1}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{cos^2(x)} = 1 + tan^2(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$

4. Derivatives

Local and Global Maxima and Minima

Definition

Let I be an interval, like (a,b) or [a,a] for example, and let the function f(x) be defined for all x∈ I. Now let x_0 ∈ I. Then

- we say that f(x) has a *global (or absolute) minimum on the interval* I at the point x= x_0

if f(x)≥ f(x_0) for all x∈ I.

- Similarly, we say that f(x) *has a global (or absolute) maximum **on*** I at x=x_0

if f(x)≤ f(x_0) for all x∈ I.

4. Derivatives

Local and Global Maxima and Minima

- we say that f(x) has a *local minimum on the interval* I at the point x= x_0 if f(x)≥ $f(x_0)$ for all x∈ I that are near x_0 .
- Similarly, we say that f(x) *has a local maximum on* I at x=x_0 if f(x)≤ $f(x_0)$ for all x∈ I. that are near x_0 .

The global maxima and minima of a function are called the global extrema of the function, while the local maxima and minima are called the local extrema.

4. Derivatives

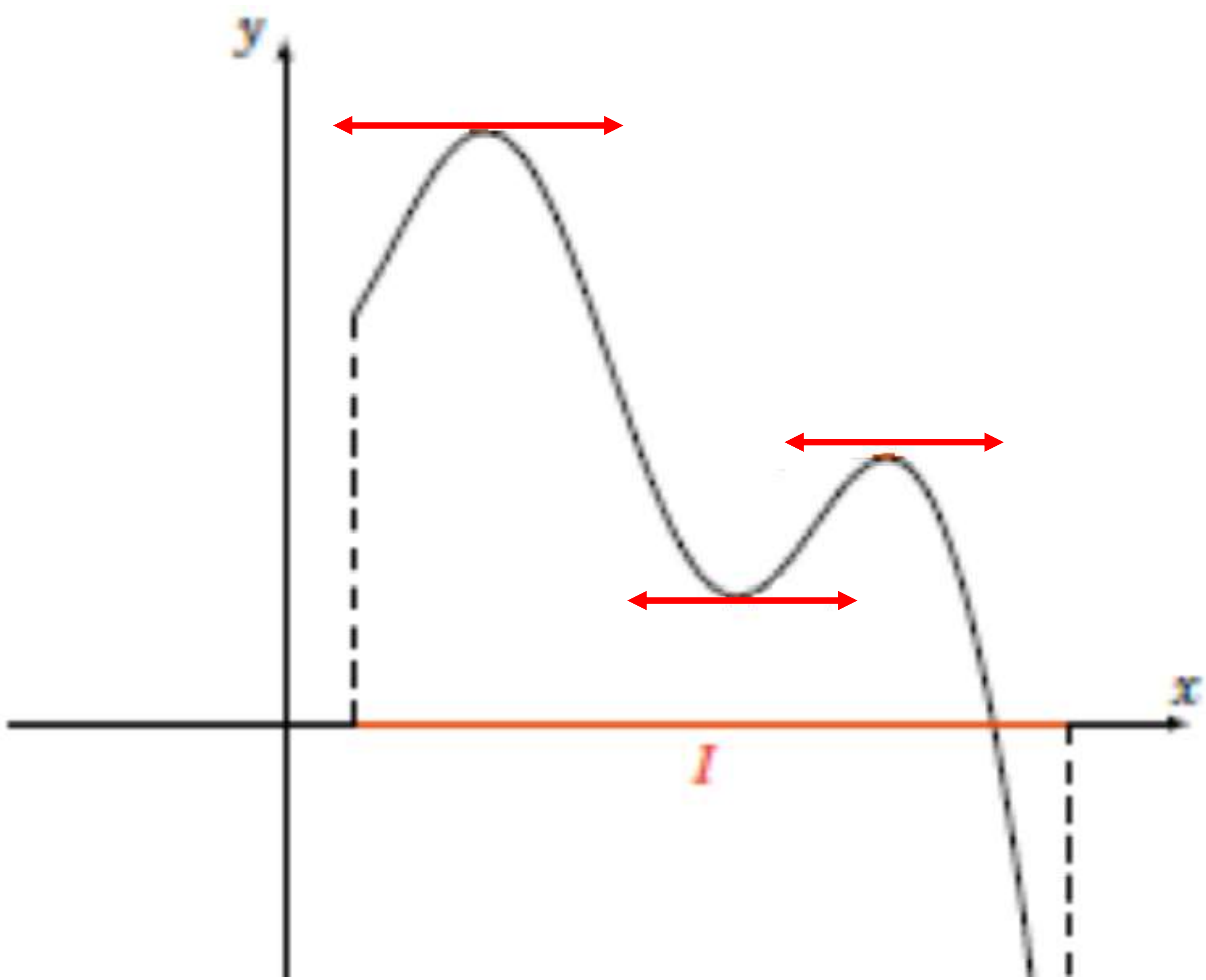
Theorem :

Let f be differentiable on an open interval $]a, b[$. If f If attains a local maximum (or a local minimum) at some point $x_0 \in]a, b[$,

then $\boldsymbol{f'(x_0) = 0}$.

But the converse is generally false!!

4. Derivatives



4. Derivatives

Rolle’s Theorem
(Michel Rolle, 1652–1719)

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ *be continuous on* $[a, b]$ and differentiable on $]a, b[$, if $f(a) = f(b)$ then there exists a point $c \in]a, b[$ *where* $f'(c) = 0$.

Mean Value Theorem

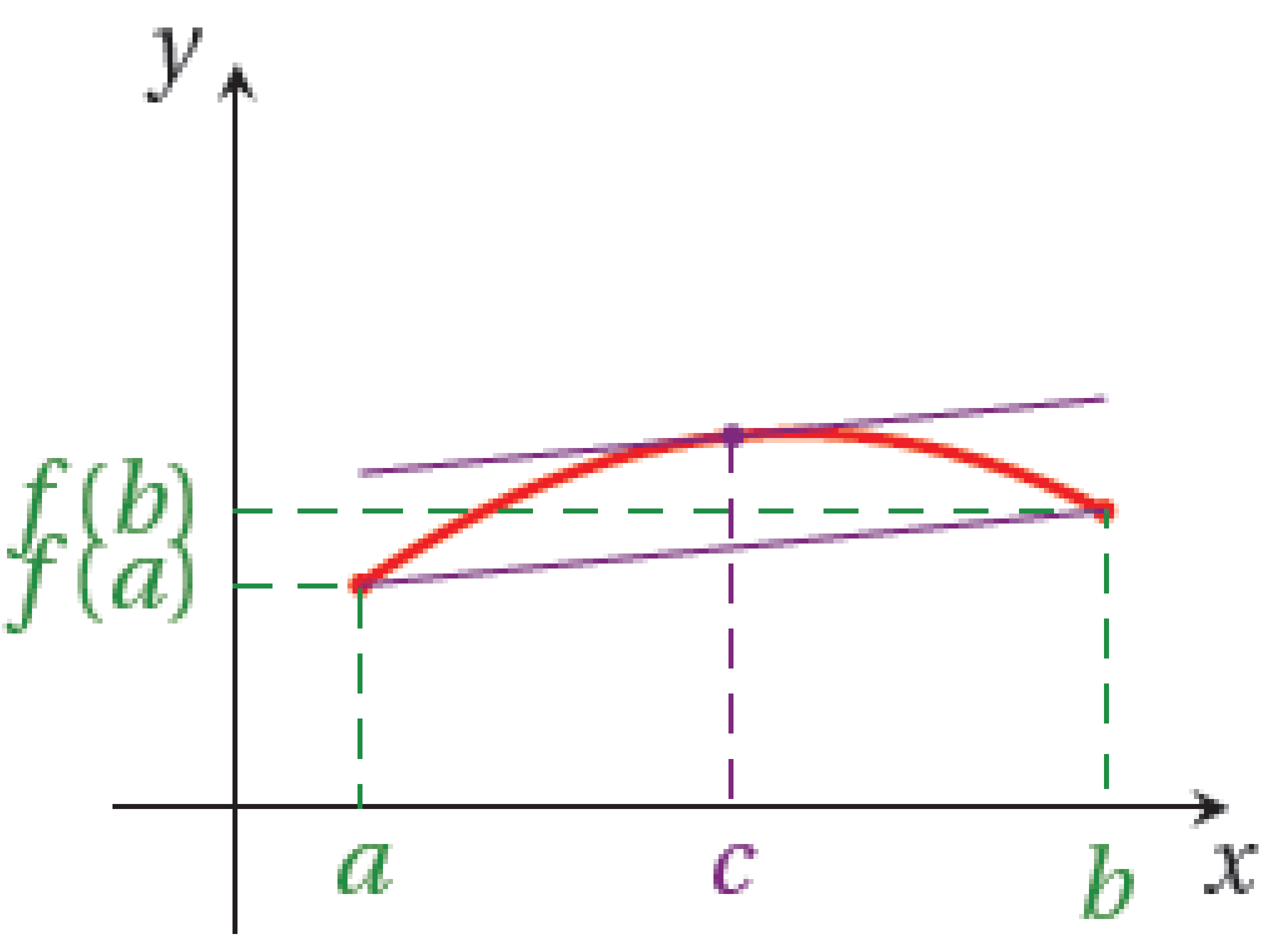
Theorem

If f is continuous on [a, b] and differentiable on $]a,b[$ then there exists a point $c \in]a,b[$ where

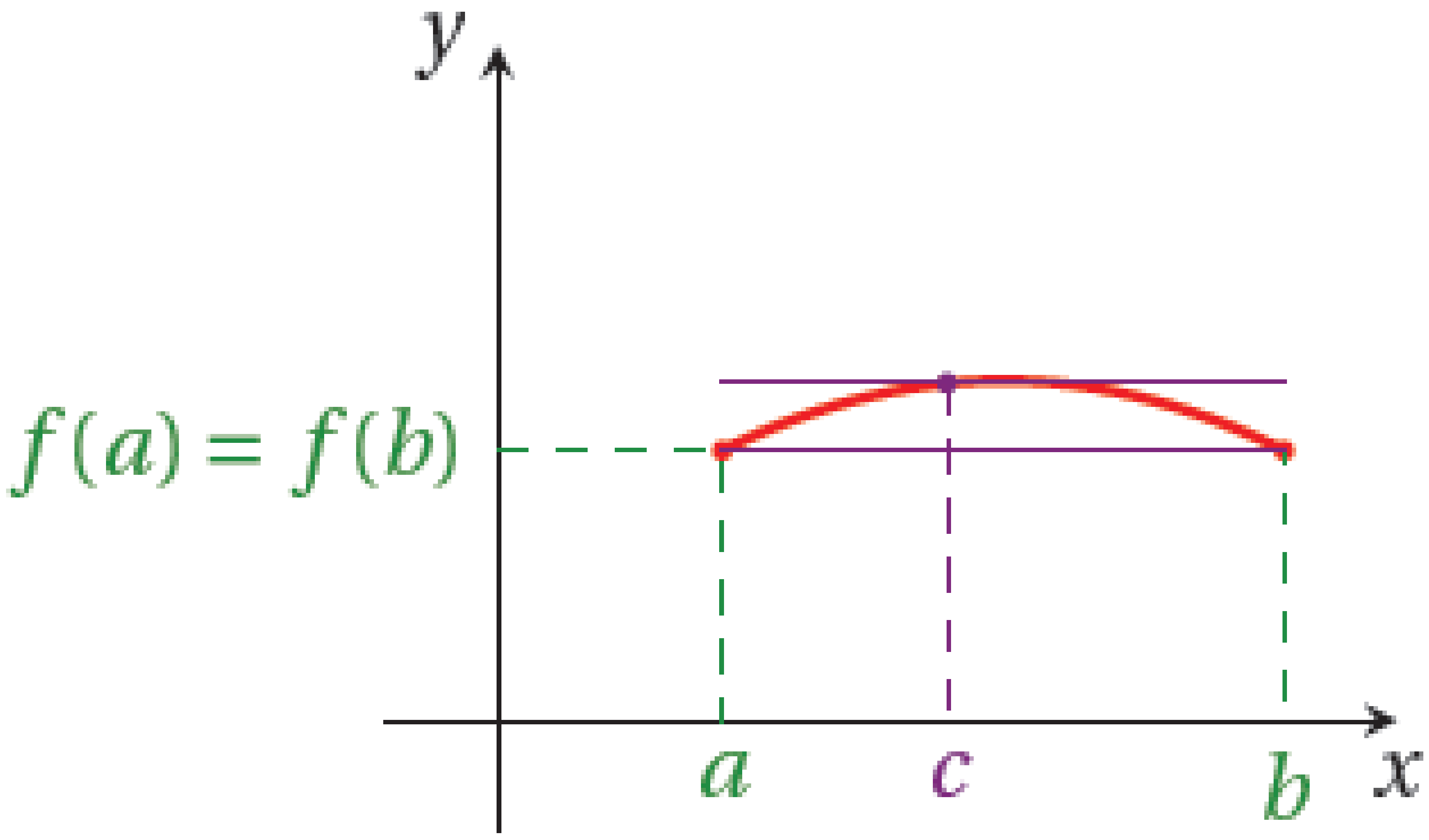
$$f(b) - f(a) = f'(c)(b - a).$$

4. Derivatives

L’égalité des accroissements finis, écrite sur un pont de Pékin.



4. Derivatives



4. Derivatives

Proposition

Suppose that $f : \mathbf{I} \rightarrow \mathbb{R}$ is differentiable at all points of an open interval $\mathbf{I}$.

(i) $f'(x) \geq 0$ for all $x \in \mathbf{I}$ if and only if f is increasing on $\mathbf{I}$.

(ii) If $f'(x) > 0$ for all $x \in \mathbf{I}$, then f is strictly increasing on $\mathbf{I}$.

(iii) $f'(x) \leq 0$ for all $x \in \mathbf{I}$ if and only if f is decreasing on $\mathbf{I}$.

(iv) If $f'(x) < 0$ for all $x \in \mathbf{I}$, then f is strictly decreasing on $\mathbf{I}$.

(v) If $f'(x) \neq 0$ for all $x \in \mathbf{I}$, then f is monotone on $\mathbf{I}$.

4. Derivatives

L'Hospital's Rules

Proposition : *Let* $f\ ,g\ :\ I\ \rightarrow \mathbb{R}$ *be twoo functions differenciables and let* $x_0 \in I$. *we suppose that*

– $f\ (x_0) = \ g(x_0) = 0,$

– $\forall x\ \in\ I\ \backslash\{x_0\}, g'(x)\ \neq 0.$

$$if \quad \lim_{x\rightarrow x_0} \frac{f'(x)}{g'(x)} = l \quad (l \in \mathbb{R}) \Longrightarrow \lim_{x\rightarrow x_0} \frac{f(x)}{g(x)} = l$$

4. Derivatives

Example :

$$\lim_{x\to 0}\frac{\overset{f(x)}{\overset{\parallel}{\ln(1+x)}}}{\underset{\underset{g(x)}{}}{\underset{\parallel}{\sqrt{x}}}} \qquad = \lim_{x\to 0}\frac{\overset{f'(x)}{\overset{\parallel}{(\ln(1+x))'}}}{\underset{\underset{g'(x)}{}}{\underset{\parallel}{(\sqrt{x})'}}}$$

$$= \lim_{x\to 0}\frac{\frac{1}{1+x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0}\frac{2\sqrt{x}}{1+x} = 0$$

4. Derivatives

Higher Derivatives

if f is a function defined on an interval open I of $\mathbb{R}$
is differentiable at every point x_0 of I.

and furthermore f' is differentiable at x_0

We say that f is twice differentiable at x_0
and we put $f''(x_0) = (f')'(x_0)$

4. Derivatives

Higher Derivatives

if f is a function defined on an interval open I of $\mathbb{R}$
is twice differentiable at every point x_0 of I.

and furthermore **f”** is differentiable at $\mathbf{x_0}$

We say that f is three time differentiable at x_0
and we put $f'''(x_0) = (f'')'(x_0)$

4. Derivatives

Higher Derivatives

if f is a function defined on an interval open I of $\mathbb{R}$ is n-time differentiable at every point x_0 of I.

and that moreover the n-th derivative function $f^{(n)}$ is differentiable at x_0

We say that f is (n+1) times differentiable at x_0 and we put $f^{(n+1)}(x_0) = (f^{(n)})'(x_0)$

4. Derivatives

Smoothness

Differentiability class
Functions of the classe C^{∞} on an open interval of $\mathbb{R}$

- Consider an open set I on the real line $\mathbb{R}$ and a function f defined on I with real values. Let k be q non negative integer. The function f is said to be differentiability class C^k if the derivatives f', f,''….. $f^{(k)}$ exists and are continuous on I.
- If f is k-differenciable on I then it is at least in the class C^{k-1} since f', f,''….. $f^{(k-1)}$ are continuous on I.
- The function is said to be infinitely differentiable, smooth or of class C^{∞} if it has derivatives of all orders on I.

4. Derivatives

Example : *The functions* exp, sin, cos, *polynomes function admit derivatives of any order. We will say that they are indefinitely differentiable.*

4. Derivatives

Theorem : Leibniz formula

$$(fg)^{(n)} = f^{(n)}g + \binom{n}{1} f^{(n-1)}g' + \cdots + \binom{n}{k} f^{(n-k)}g^{(k)} + \cdots + fg^{(n)}.$$

in other words

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}g^{(k)}.$$

Where

$$\binom{n}{k} = C_n^k = \frac{n!}{k!(n-k)!}$$

are called binomial coefficients.

4. Derivatives

Example :

Calculate the n-th derivative of $f(x) = ln(1+x)$

$$f'(x) = 1/(1+x) = (1+x)^{-1}$$

$$f''(x) = (-1)(1+x)^{-2}.$$

$$f^{(3)} = (-1)(-2)(1+x)^{-3}$$

.

.

.

$$f^{(n)} = (-1)^{n}\frac{(n-1)!}{(1+x)^{n}}$$