Chapter 5 : Eigenvalues and Eigenvectors

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Eigenvalues and Eigenvectors

Throughout this lecture, K will denote $\mathbb{R}$ or $\mathbb{C}$, and $V = K^n$.

Definition 1

Let $A \in \mathcal{M}_n(K)$. If there exist $\lambda \in K$ and a nonzero column vector $x \in K^n$ satisfying

$$Ax = \lambda x,$$

we say that λ is an eigenvalue of A and x is an eigenvector of A corresponding to λ.

Example

Example 1

Consider the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}.$$

We can easily verify that

$$\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, 4 is an eigenvalue of A, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to 4.

Characteristic Polynomial

Definition 2

The characteristic polynomial of the matrix A is defined by

$$P_A(X) = \det(A - XI)\,.$$

Exercise 1

Show that the characteristic polynomial is of degree n and its leading coefficient is $(-1)^n$.

The characteristic polynomial allows us to find all the eigenvalues of a given matrix as the following result shows.

Proposition 1

A scalar $\lambda \in K$ is an eigenvalue of A if and only if $P_A(\lambda) = 0$.

Example

Example 2

Consider again the matrix of example 1, we have

$$P_A(X) = \det(A - XI) = \begin{vmatrix} 2-X & 2 \\ 5 & -1-X \end{vmatrix} = X^2 - X - 12.$$

Then, the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 4$.

Example

Example 3

Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have

$$P_A(X) = \det(A - XI_2) = \begin{vmatrix} -X & 1 \\ -1 & -X \end{vmatrix} = X^2 + 1.$$

Then, the eigenvalues of A are $\lambda_1 = i$ and $\lambda_2 = -i$.

Similar Matrices

Definition 3

Let $A, B \in \mathcal{M}_n(K)$. We say that A is similar to B if there exists an invertible matrix $P \in \mathcal{M}_n(K)$ such that $B = P^{-1}AP$.

Obviously, the relation « similar to » is an equivalence relation, so we can use the term « A and B are similar ».

Proposition 2

If $A, B \in \mathcal{M}_n(K)$ are similar, then they have the same characteristic polynomial and then the same eigenvalues with the same multiplicities.

Properties of Eigenvalues

Proposition 3

Assume that the matrix $A \in \mathcal{M}_n(K)$ has n (non necessarily distinct) eigenvalues $\lambda_1, \cdots, \lambda_n$, then we have

1) the sum of the eigenvalues is the trace of A : $\lambda_1 + \cdots + \lambda_n = Tr(A)$;

2) the product of the eigenvalues is the determinant of A : $\lambda_1 \cdots \lambda_n = \det(A)$;

3) if A is triangular, then its eigenvalues are the entries of the diagonal ;

4) if λ is an eigenvalue of an invertible matrix A, then λ^{-1} is an eigenvalue of A^{-1};

5) if λ is an eigenvalue of a matrix A, and m a positive integer, then λ^m is an eigenvalue of A^m.

Linear Independence of Eigenvectors

Proposition 4

If $v_1, \cdots, v_k$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_k$ of $A \in \mathcal{M}_n(K)$, then the set $\{v_1, \cdots, v_k\}$ is linearly independent.

Eigenspace

Proposition 5

Let λ be an eigenvalue of A. Then the set

$$V_\lambda=\{v \in V : Av = \lambda v\}$$

is a subspace of V.

Definition 4

The subspace V_λ is called eigenspace corresponding to λ.

Example

Example 4

Consider again the matrix of example 1, then

$$V_4 = \{v \in K^2 : Av = 4v\}.$$

Set $v = \begin{pmatrix} x \\ y \end{pmatrix}$, then we have

$$Av = 4v \Leftrightarrow \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 2x + 2y = 4x \\ 5x - y = 4y \end{cases} \Leftrightarrow \begin{cases} -2x + 2y = 0 \\ 5x - 5y = 0 \end{cases} \Leftrightarrow x = y.$$

Therefore V_4 is the subspace spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\dim(V_4) = 1$.

Example

Example 5

Consider again the matrix of Example 3, then

$$V_{\lambda_1} = V_i = \{v \in \mathbb{C}^2 : Av = iv\}.$$

Set $v = \binom{x}{y}$, then we have

$$Av = iv \Leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = i \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} y = ix \\ -x = iy \end{cases} \Leftrightarrow y = ix.$$

Therefore V_i is the subspace of $\mathbb{C}^2$ spanned by $\begin{pmatrix} 1 \\ i \end{pmatrix}$, and $\dim(V_i) = 1$.

Diagonalization

Definition 5

A square matrix $A \in \mathcal{M}_n(K)$ is said to be diagonalizable if it is similar to a diagonal matrix, that is, there exists an invertible matrix $P \in \mathcal{M}_n(K)$ such that $P^{-1}AP$ is a diagonal matrix.

Remark 1

Let f be the endomorphism of V represented by the matrix A. Then « A is diagonalizable » means that there exists a basis B of V consisting of eigenvectors such that $\mathcal{M}(f, B)$ is diagonal. Moreover, we can take P as matrix whose columns are eigenvectors $v_1, \cdots, v_n$ corresponding respectively to eigenvalues $\lambda_1, \cdots, \lambda_n$ and we have

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \cdots, \lambda_n).$$

Diagonalizable Endomorphism

Definition 5

An endomorphism f of V is said to be diagonalizable if there exists a basis $\mathcal{B}$ of V such that the corresponding matrix $\mathcal{M}(f, \mathcal{B})$ is diagonal.

Diagonalization : A Sufficient Condition

Theorem 1

If a matrix $A \in \mathcal{M}_n(K)$ has n distinct eigenvalues, then it is diagonalizable.

Diagonalization : A Necessary and Sufficient Condition

Theorem 2

Let $\lambda_1, \cdots, \lambda_k$ the distinct eigenvalues of a matrix $A \in \mathcal{M}_n(K)$, and denote by $m_1, \cdots, m_k$ their respectives multiplicities as roots of the characteristic polynomial of A. Then A is diagonalizable if and only if

$$\dim\left(V_{\lambda_i}\right) = m_i$$

for all $i \in \{1, \cdots, k\}$.

Notice that the multiplicity of an eigenvalue is called algebraic multiplicity while the dimension of its corresponding eigenspace is called geometric multiplicity.

A Characterization of Diagonalizability

We say that V is a direct sum of its subspaces $V_1, \cdots, V_k$, denoted

$$V = V_1 \oplus \cdots \oplus V_k,$$

if every vector $x \in V$ can be written uniquely as

$$x = x_1 + \cdots + x_k, \text{ where } x_i \in V_i \text{ for } i \in \{1, \cdots, k\}.$$

Theorem 3

Let f be an endomorphism of V and let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues of f. Then the following are equivalent :

1) f is diagonalizable ;

2) V has a basis consisting of eigenvectors of f ;

3) $V = \oplus_{j=1}^{k} V_{\lambda_j}$;

4) $\sum_{j=1}^{k} \dim V_{\lambda_j} = \dim V$.

Examples

Example 6

We have seen that the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$$

has two distinct eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 4$. Then, by Theorem 1, A is diagonalizable.

We have also seen that $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to λ_2, and we can easily find that $v_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ is an eigenvector corresponding to λ_1. Thus, we can take

$$P = \begin{pmatrix} -2 & 1 \\ 5 & 1 \end{pmatrix},$$

and we obtain

$$P^{-1}AP = \frac{-1}{7}\begin{pmatrix} 1 & -1 \\ -5 & -2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}.$$

Complex Eigenvalues of Real Matrices

If $v = (x_1, \cdots, x_n) \in \mathbb{C}^n$, the conjugate of v is $\bar{v} = (\overline{x_1}, \cdots, \overline{x_n})$.

Proposition 6

Let $A \in \mathcal{M}_n(\mathbb{R})$ and suppose that λ is a non-real complex eigenvalue of A with multiplicity m. Then we have

- The conjugate $\bar{\lambda}$ is an eigenvalue of A with multiplicity m.
- A vector v is an eigenvector corresponding to λ if and only if its conjugate $\bar{v}$ is an eigenvector corresponding to $\bar{\lambda}$.
- The corresponding eigenspaces V_λ and $V_{\bar{\lambda}}$ have the same dimension. More precisely, $\{v_1, \ldots, v_k\}$ is a basis of V_λ if and only if $\{\bar{v}_1, \ldots, \bar{v}_k\}$ is a basis of $V_{\bar{\lambda}}$.

Example

Example 7

Consider once again the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of Example 3, then, from Proposition 3, we have

- $\lambda_2 = -i$ is an eigenvalue of A with multiplicity 1.
- $\bar{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector of A.
- V_{-i} is the subspace of $\mathbb{C}^2$ spanned by $\begin{pmatrix} 1 \\ -i \end{pmatrix}$, and $\dim(V_{-i}) = 1$.

Example

Example 8

We have seen that the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

has two distinct eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$. Then, by Theorem 1, A is diagonalizable.

We have also seen that $v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector corresponding to λ_1, and $v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector corresponding to λ_2. Thus, we can take

$$P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

and we obtain

$$P^{-1}AP = \frac{i}{2}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Triangularization

Definition 6

An endomorphism f of V is said to be triangularizable if there exists a basis $\mathcal{B}$ of V such that the corresponding matrix $\mathcal{M}(f,\mathcal{B})$ is (upper) triangular.

Theorem 4

An endomorphism f of V is triangularizable if and only if its characteristic polynomial $P_f(X)$ can be written as a product of linear factors, that is

$$P_f(X) = (-1)^n(X-\lambda_1)\cdots(X-\lambda_n),$$

where $\lambda_1,\cdots,\lambda_n$ are (not necessarily distinct) elements of K.

Proof

$\Longrightarrow)$

If A is triangularizable, then there is a basis of V such that $P^{-1}AP = T$ is upper triangular. We have

$$T = \begin{pmatrix} \lambda_1 & * & * \\ \vdots & \ddots & * \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

$P_A(X) = P_T(X) = (\lambda_1 - \mathrm{X}) \dots (\lambda_n - \mathrm{X})$. Then P_A is a product of linear factors.

Proof

$\Longleftarrow$)

By induction, for $n = 1$, this is true.

We suppose the implication is true for $n - 1$, and prove that it is true for n.

Let $\mathrm{A} \in \mathcal{M}_n(K)$ with $P_A(X) = (\lambda_1 - \mathrm{X}) \ldots (\lambda_n - \mathrm{X})$ a product of linear factors.

$P_A(\lambda_1) = 0$, then λ_1 is an eigenvalue of A. Let v_1 be an eigenvector corresponding to λ_1. Since $v_1 \neq 0$, we can find $w_2, \cdots, w_n \in V$ such that $\mathcal{B} = \{v_1, w_2, \cdots, w_n\}$ is a basis of V. Then we have

$$B = Mat(f, \mathcal{B}) = \begin{bmatrix} \lambda_1 & L \\ (0) & M \end{bmatrix} \in \mathcal{M}_n(K).$$

Proof

Since $P_A(X) = P_B(X)$, then

$$P_A(X) = \det(\mathrm{B} - \mathrm{X}I_n) = \begin{vmatrix} \lambda_1 - X & L \\ (0) & M - XI_{n-1} \end{vmatrix}$$
$$= (\lambda_1 - \mathrm{X})\det(\mathrm{M} - \mathrm{X}I_{n-1}) = (\lambda_1 - \mathrm{X})P_M(X)$$

As $P_A(X)$ is a product of linear factors, then $P_M(X)$ is.

By assumption induction , M is triangularizable. Therefore, there exists $\mathrm{Q} \in \mathcal{M}_{n-1}(K)$, invertible, such that $Q^{-1}MQ$ is triangular.

Set $P = \begin{bmatrix} 1 & (0) \\ (0) & Q \end{bmatrix} \in \mathcal{M}_n(K)$.

Proof

P is invertible with inverse

$P^{-1} = \begin{bmatrix} 1 & (0) \\ (0) & Q^{-1} \end{bmatrix}$. We have then

$$C := P^{-1}BP = \begin{bmatrix} 1 & (0) \\ (0) & Q^{-1} \end{bmatrix} \begin{bmatrix} \lambda_1 & L \\ (0) & M \end{bmatrix} \begin{bmatrix} 1 & (0) \\ (0) & Q \end{bmatrix} = \begin{vmatrix} \lambda_1 & LQ \\ (0) & Q^{-1}MQ \end{vmatrix}$$

is triangular, and we have C similar to B and B similar to A, then C similar to A, as required.

Example

Example 9

The matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is triangularizable over $\mathbb{C}$ but not over $\mathbb{R}$.

Polynomials of Matrices and Polynomials of Endomorphisms

Given $P(X) = \sum_{j=0}^{m} a_j X^j \in K[X], A \in \mathcal{M}_n(K)$ and an endomorphism f of V, we write

$$P(A) := \sum_{j=0}^{m} a_j A^j$$

and

$$P(f) := \sum_{j=0}^{m} a_j f^j,$$

where $A^0 = I_n, f^0 = Id_V$, and $f^j = f \circ \cdots \circ f$ (j times).

Properties of Polynomials of Endomorphisms

Let $P(X), Q(X) \in K[X]$ and let f be an endomorphism of V, then we have

$$\big(P(X).Q(X)\big)(f) = P(f) \circ Q(f).$$

Although the composition is not commutative in general, in our case, thanks to the commutativity of the product of polynomials, we have

$$P(f) \circ Q(f) = Q(f) \circ P(f).$$

More generally, for m polynomials we have

$$(P_1(X) \cdots P_m(X))(f) = P_1(f) \circ \cdots \circ P_m(f).$$

Eigenvalues and Roots of Polynomials

Theorem 5

Let V be a finite dimensional vector space over K and let f be an endomorphism of V. Let $P(X) \in K[X]$ a nonzero polynomial verifying

$$P(f) = 0.$$

Then every eigenvalue of f is a root of $P(X)$.

Characteristic Polynomial of Endomorphism

Definition 7

Let V be a finite dimensional vector space over K and let f be an endomorphism of V. Let $\mathcal{B}$ be a basis of V. Then the characteristic polynomial of f is given by

$$P_f(X) = \det(\mathcal{M}(f, \mathcal{B}) - XI_n).$$

The Cayley-Hamilton Theorem

Theorem 6 (Cayley-Hamilton Theorem)

Let V be a finite dimensional vector space over K and let f be an endomorphism of V. Then we have

$$P_f(f) = 0.$$

In matrix form :

Let $A \in \mathcal{M}_n(K)$, then we have

$$P_A(A) = 0.$$

Proof

From the fondamental Theorem of Algebra, P_f is a product of linear factors, then there exists a basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ of V such that f is triangularizable.

$$P_f(X) = (\lambda_1 - \mathrm{X}) \cdots (\lambda_n - \mathrm{X})$$

$$P_f(f) = (\lambda_1 Id_V - f) \circ \cdots \circ (\lambda_n Id_V - f)$$

Set $g_i = (\lambda_1 Id_V - f) \circ \cdots \circ (\lambda_i Id_V - f), \quad (i = 1, \cdots, n).$

In particular $g_n = P_f(f)$ and we want to prove that $g_n = 0$.

We will prove by induction on $i \in \{1, \cdots, n\}$ that

$\wp \quad (i): \ g_i(v_1) = g_i(v_2) = \cdots = g_i(v_i) = 0$ for all $i \in \{1, \cdots, n\}$.

Proof

For i=1, it is true.

Let $i \in \{1, \cdots, n-1\}$. We suppose that $\wp\ (i)$ is true and we prove that $\wp\ (i+1)$ is true.

We have

$$g_{i+1} = g_i \circ (\lambda_{i+1} Id_V - f) = (\lambda_{i+1} Id_V - f) \circ g_i,$$

then $g_{i+1}(v_1) = g_{i+1}(v_2) = \cdots = g_{i+1}(v_i) = 0$

since $\wp\ (i)$ is true.

We have also $g_{i+1}(v_{i+1}) = g_i(\lambda_{i+1} v_{i+1} - f(v_{i+1})) = 0.$

Therefore $\wp\ (i)$ is true for all $i \in \{1, \cdots, n\}$, thus $g_n = 0$.

Example

Example 6

The matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

has characteristic polynomial $P_A(X) = X^2 + 1$. We have

$$P_A(A) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0.$$

Minimal Polynomial

The Cayley-Hamilton Theorem allows us to define the following.

Definition 6

The minimal polynomial of an endomorphism f is the monic polynomial $m_f(X)$ of least degree such that $m_f(f) = 0$.

Minimal Polynomial Properties

Proposition 7

Let f be an endomorphism of V.

1) If λ is an eigenvalue of f, then $m_f(\lambda) = 0$.

2) Let $P(X) \in K[X]$ such that $P(f) = 0$. Then

$$m_f(X) | P(X).$$

In particular, $m_f(X) | P_f(X)$, so we have

$$\deg m_f(X) \leq \dim V.$$

Determining the Minimal Polynomial

Suppose that the characteristic polynomial of f has the form

$$P_f(X) = (-1)^n (X-\lambda_1)^{\alpha_1} \dots (X-\lambda_k)^{\alpha_k},$$

where $\lambda_1, \cdots, \lambda_k$ are distinct elements of K, then the minimal polynomial of f has the form

$$m_f(X) = (X-\lambda_1)^{\beta_1} \dots (X-\lambda_k)^{\beta_k},$$

where $1 \leq \beta_i \leq \alpha_i$ for all $i \in \{1, \cdots, k\}$.

To determine the minimal polynomial, we check if

$$(X-\lambda_1)^{c_1} \dots (X-\lambda_k)^{c_k} (f) = 0 \qquad (1)$$

by giving increasing values to the c_i until we get the equation (1).

Example

Example 7

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is given by $P_A(X) = (1 - X)^3$. We have

$$(X - 1)(A) = A - I_3 \neq 0,$$

$$(X - 1)^2(A) = (A - I_3)^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = 0.$$

So the minimal polynomial of f is $m_A(X) = (X - 1)^2$.

Decomposition Theorem

Theorem 7

Let f be an endomorphism of V and let $P(X) \in K[X]$. Suppose that

$$P(X) = P_1(X)P_2(X)\cdots P_k(X),$$

where the polynomials $P_1(X), P_2(X), \cdots, P_k(X)$ are pairwise coprime. Then we have

$$\ker P(f) = \ker P_1(f) \oplus \ker P_2(f) \oplus \cdots \oplus \ker P_k(f).$$

A New Characterization of the Diagonalizability

Theorem 8

An endomorphism f is diagonalizable if and only if its minimal polynomial is the product of distinct linear factors.

Proof

Assume that all the roots of $m_f(X)$ are distinct. Write

$$m_f(X) = (X - \lambda_1) \quad \ldots (X - \lambda_p) \quad .$$

$(X - \lambda_i)$ and $(X - \lambda_j)$ are coprime for all $i \neq j$.

By the decomposition theorem, we have

$$\ker m_f(f) = \ker(f - \lambda_1 Id_V) \oplus \cdots \oplus \ker\big(f - \lambda_p Id_V\big).$$

$$= V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_p}$$

Since $m_f(f) = 0$, then $\ker m_f(f) = V$, so $V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_p}$.

By theorem 3, f is diagonalisable.

Proof

Suppose f diagonalisable. Thus there is a basis B of V such that

$$M(f,B) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \lambda_p & 0 \\ 0 & 0 & \cdots & 0 & \lambda_p \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \lambda_p & 0 \\ 0 & 0 & \cdots & 0 & \lambda_p \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 I_{k_1} & & (0) \\ & \ddots & \\ (0) & & \lambda_p I_{k_p} \end{pmatrix}$$

Where $\lambda_1, \lambda_2, \cdots, \lambda_k \in K$ are all distinct eigenvalues of f.

Proof

For all $i \in \{1, \cdots, p\}$

$$M(f,B) - \lambda_i I_n = \begin{pmatrix} (\lambda_1 - \lambda_i) I_{k_1} & & (0) \\ & (0) & \\ (0) & & (\lambda_p - \lambda_i) I_{k_p} \end{pmatrix}.$$

Then $(M(f,B) - \lambda_1 I_n) \cdots \left(M(f,B) - \lambda_p I_n\right) = 0.$

So $M(f,B)$ is a root of the polynomial

$$Q(X) = (X - \lambda_1) \cdots (X - \lambda_p).$$

By proposition 7, $m_f(X) / Q(X)$ so $\deg m_f(X) \leq \deg Q(X)$.

By proposition 7, $m_f(\lambda_i) = 0$ for all $i \in \{1, \cdots, p\}$.

Since the λ_i are all distinct then $Q(X) \,/\, m_f(X)$.

Proof

Since $Q(X)$ and $m_f(X)$ are monic then $m_f(X) = Q(X)$.

This proves that $m_f(X)$ has only simple roots.

Example

Example 8

The matrix of Example 7 is not diagonalizable.

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

is given by $P_A(X) = (1 - X)^2(2 - X)$. We have

$$(X-1)(X-2)(A) = (A - I_3)(A - 2I_3) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Then the minimal polynomial of f is $m_A(X) = (X-1)(X-2)$, and so A is diagonalizable.

Jordan Block

Definition 7

A Jordan block is a square matrix of order p of the form

$$J(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix},$$

where $\lambda \in K$ and $p \in \mathbb{N}^*$.

Jordan Block Properties

Proposition 8

We have

1)

$$\left(J(\lambda) - \lambda I_p\right)^{p-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and $\left(J(\lambda) - \lambda I_p\right)^p = 0.$

2) $P_{J(\lambda)}(X) = (\lambda - X)^p = (-1)^p (X - \lambda)^p.$

3) $m_{J(\lambda)}(X) = (X - \lambda)^p.$

Jordan Matrix

Definition 8

A Jordan matrix is a block diagonal matrix of the form

$$\begin{pmatrix} J_1(\lambda_1) & \cdots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \cdots & J_p(\lambda_p) \end{pmatrix}$$

where $J_i(\lambda_i)$ is a Jordan block for $i \in \{1, \cdots, p\}$.

Remark 2

The Jordan blocks can have distinct orders, and some λ_i can be equal.

Example of Jordan Matrix

Example 9

Here is a Jordan matrix

$$A = \begin{pmatrix} -3 & 1 & & & & & & & \\ 0 & -3 & & & & & & & \\ & & -3 & 1 & 0 & & & (0) & \\ & & 0 & -3 & 1 & & & & \\ & & 0 & 0 & -3 & & & & \\ & & & & & 2 & & & \\ & & & & & & 5 & 1 & 0 \\ & (0) & & & & & 0 & 5 & 1 \\ & & & & & & 0 & 0 & 5 \end{pmatrix}$$

Example of Jordan Matrix

Example 9

The matrix A has

- A Jordan block 2×2 corresponding to the eigenvalue -3.
- A Jordan block 3×3 corresponding to the same eigenvalue -3.
- A Jordan block 1×1 corresponding to the eigenvalue 2.
- A Jordan block 3×3 corresponding to the eigenvalue 5.

Existence of Jordan Form

Theorem 9

Let $A \in \mathcal{M}_n(K)$, and suppose that the characteristic polynomial of A is a product of linear factors. Then A is similar to a Jordan matrix, that is, there exists an invertible matrix $P \in \mathcal{M}_n(K)$ such that

$$P^{-1}AP = \begin{pmatrix} J_1 & \cdots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \cdots & J_p \end{pmatrix},$$

where the $J_i, 1 \leq i \leq p$, are Jordan blocks.

Existence of Jordan Form

We can rephrase Theorem 9 as follows.

Theorem 9 bis

Let f be an endomorphism of V and suppose that the characteristic polynomial of f is a product of linear factors. Then there exists a basis $\mathcal{B}$ of V such that the matrix of f with respect to that basis is a Jordan matrix, that is,

$$M(f, \mathcal{B}) = \begin{pmatrix} J_1 & \cdots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \cdots & J_p \end{pmatrix},$$

where the $J_i, 1 \leq i \leq p$, are Jordan blocks.

Some Properties

Proposition 9

- The λ_i are the eigenvalues of A.
- The Jordan form of A is unique up to reordering the Jordan blocks.
- The number of blocks corresponding to λ is equal to $\dim(V_\lambda)$.
- The sum of the orders of the Jordan blocks corresponding to λ is the multiplicity of λ as a root of the characteristic polynomial of A.
- The order of the greatest Jordan block corresponding to λ is the multiplicity of λ as a root of the minimal polynomial of A.

Algorithm Determining the Jordan Form

The following algorithm allows us to determine a Jordan form for each square matrix with entries in $\mathbb{C}$.

In what follows, A is an $n \times n$ matrix with entries in $\mathbb{C}$, λ is an eigenvalue of A, and $N = A - \lambda I_n$.

For convenience, we denote a matrix and the endomorphism associated to it by the same notation.

Step 1

Step 1

We determine the subspaces of $\mathbb{C}^n$

$$\ker N \subsetneq \ker N^2 \subsetneq \cdots \subsetneq \ker N^m$$

where m is the least positive integer so that $\ker N^m$ has maximal dimension among the subspaces $\ker N^i, i \geq 1$.

Remark

If k is the multiplicity of the eigenvalue λ then $m \leq k$.

Step 2 (1)

Step 2

(1) We complete a basis of $\ker N^{m-1}$ to obtain a basis of $\ker N^m$.
Denote by $(v_1, v_2, \cdots)$ that completion.

$$ker\ N^m = ker\ N^{m-1} \oplus < v_1, v_2, \cdots >$$

Then $(Nv_1, Nv_2, \cdots)$ is an ordered set of L.I. vectors of $\ker N^{m-1}$, and $\langle Nv_1, Nv_2, \cdots \rangle \cap \ker N^{m-2} = \{0\}$.

Step 2 (2)

Step 2

(2) We complete a basis of

$$\ker N^{m-2} \oplus \langle Nv_1, Nv_2, \cdots \rangle$$

to obtain a basis of $\ker N^{m-1}$.

Denote by $(w_1, w_2, \cdots)$ that completion.

$$\ker N^{m-1} = \ker N^{m-2} \oplus \langle Nv_1, Nv_2, \cdots \rangle \oplus \langle w_1, w_2, \cdots \rangle$$

Then $(N^2v_1, N^2v_2, \cdots, Nw_1, Nw_2, \cdots)$ is an ordered set of L.I. vectors of $\ker N^{m-2}$, and

$$\langle N^2v_1, N^2v_2, \cdots, Nw_1, Nw_2, \cdots \rangle \cap \ker N^{m-3} = \{0\}.$$

Step 2 (3)

Step 2

(3) We complete a basis of

$$\ker N^{m-3} \oplus \langle N^2 v_1, N^2 v_2, \cdots, N w_1, N w_2, \cdots \rangle$$

to obtain a basis of $\ker N^{m-2}$.

Denote by $(x_1, x_2, \cdots)$ that completion.

$$\ker N^{m-2} = \ker N^{m-3} \oplus \langle N^2 v_1, N^2 v_2, \cdots, N w_1, N w_2, \cdots \rangle \oplus \langle x_1, x_2, \cdots \rangle$$

Then $(N^3 v_1, N^3 v_2, \cdots, N^2 w_1, N^2 w_2, \cdots, N x_1, N x_2, \cdots)$ is an ordered set of L.I. vectors of $\ker N^{m-3}$, and

$$\langle N^3 v_1, N^3 v_2, \cdots, N^2 w_1, N^2 w_2, \cdots, N x_1, N x_2, \cdots \rangle \cap \ker N^{m-4} = \{0\}.$$

Step 2 (⋮)

Step 2

$\vdots$

$\vdots$

$\vdots$

Step 2 (s-1)

Step 2

(s-1) We complete a basis of

$$\ker N \oplus \langle N^{m-2}v_1, N^{m-2}v_2, \cdots, N^{m-3}w_1, N^{m-3}w_2, \cdots \rangle$$

to obtain a basis of $\ker N^2$.

Denote by $(y_1, y_2, \cdots)$ that completion.

$$\ker N^2 = \ker N \oplus \langle N^{m-2}v_1, N^{m-2}v_2, \cdots, N^{m-3}w_1, N^{m-3}w_2, \cdots \rangle \oplus \langle y_1, y_2, \cdots \rangle$$

Then $(N^{m-1}v_1, N^{m-1}v_2, \cdots, N^{m-2}w_1, N^{m-2}w_2, \cdots, Ny_1, Ny_2, \cdots)$ is an ordered set of L.I. vectors of $\ker N$.

Step 2 (s)

Step 2

(s) At this last intermediate step, we complete the ordered set of L.I. vectors of $\ker N$ obtained in the previous step :

$$(N^{m-1}v_1, N^{m-1}v_2, \cdots, N^{m-2}w_1, N^{m-2}w_2, \cdots, Ny_1, Ny_2, \cdots)$$

to obtain a basis of $\ker N$.

Denote by $(z_1, z_2, \cdots)$ that completion.

$$\ker N = \langle N^{m-1}v_1, N^{m-1}v_2, \cdots, N^{m-2}w_1, N^{m-2}w_2, \cdots, Ny_1, Ny_2, \cdots \rangle \oplus \langle z_1, z_2, \cdots \rangle$$

So

$$(N^{m-1}v_1, N^{m-1}v_2, \cdots, N^{m-2}w_1, N^{m-2}w_2, \cdots, Ny_1, Ny_2, \cdots, z_1, z_2, \cdots)$$

is a basis of $\ker N$.

Step 3

Step 3

By adjoining to that basis of $\ker N$ all the bases of the complementary subspaces of $\ker N^i$ in $\ker N^{i+1}$ $(1 \leq i \leq m-1)$ obtained in the previous steps, we get a basis of $\ker N^m$, but we must arrange its vectors as follows :

$$\left(\begin{array}{c} N^{m-1}v_1, N^{m-2}v_1, \cdots, v_1, N^{m-1}v_2, N^{m-2}v_2, \cdots, v_2, \cdots, \cdots \\ N^{m-2}w_1, N^{m-3}w_1, \cdots, w_1, N^{m-2}w_2, N^{m-3}w_2, \cdots, w_2, \cdots, \cdots \\ \vdots \\ Ny_1,\ y_1, Ny_2, y_2, \cdots \\ z_1, z_2, \cdots \end{array}\right).$$

This is a basis giving all Jordan blocks corresponding to the eigenvalue λ.

Step 4

Step 4

Finally, a Jordan basis for A is simply the union of all the bases corresponding to all the distinct eigenvalues of A.

Examples

1) $A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}$;

2) $A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$;

3) $A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}$;

Examples

4) $A = \begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}$;

5) $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

6) $A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 2 & -1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$.

Example

Example 10

Find the Jordan matrix corresponding to the matrix

$$A = \begin{pmatrix} 4 & 3 & -2 \\ -3 & -1 & 3 \\ 2 & 3 & 0 \end{pmatrix}.$$

Example

Example 11

Find the Jordan matrix corresponding to the matrix

$$A = \begin{pmatrix} 5 & 0 & 4 & -2 & -3 \\ -2 & 3 & -3 & 2 & 4 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 1 & 0 & 2 & -1 & 1 \end{pmatrix}.$$

Matrix Exponential

Definition 9

Let $A \in \mathcal{M}_n(\mathbb{C})$. The matrix exponential of A is given by

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

Remark 3

This definition is formal, but we can prove that this series converges.

Matrix Exponential of a Diagonal Matrix

Proposition 10

If $D \in \mathcal{M}_n(\mathbb{C})$ is a diagonal matrix of the form

$$D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix},$$

then

$$e^D = \begin{pmatrix} e^{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n} \end{pmatrix}.$$

Matrix Exponential of a Diagonalizable Matrix

Proposition 11

1) If $A, P \in \mathcal{M}_n(\mathbb{C})$ with P invertible,then

$$e^{P^{-1}AP} = P^{-1}e^{A}P.$$

2) If $A \in \mathcal{M}_n(\mathbb{C})$ is diagonalizable, then there exist $D, P \in \mathcal{M}_n(\mathbb{C})$ with D diagonal and P invertible such that

$$e^{A} = Pe^{D}P^{-1}.$$

Example

Example 12

The matrix

$$A = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}$$

has characteristic polynomial $P_A(X) = X^2 + 2X - 8$. It has 2 distinct eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -4$, then it is diagonalizable. The vectors $v_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors corresponding to λ_1 and λ_2, respectively, so we have

$$D = P^{-1}AP,$$

where

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}, P = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } P^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix}.$$

Example

Example 12 (continued)

Therefore

$$e^{A} = e^{PDP^{-1}} = Pe^{D}P^{-1}.$$

Since

$$e^{D} = \begin{pmatrix} e^{2} & 0 \\ 0 & e^{-4} \end{pmatrix},$$

then

$$e^{A} = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{2} & 0 \\ 0 & e^{-4} \end{pmatrix} \frac{1}{6} \begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 5e^{2} + e^{-4} & 5e^{2} - 5e^{-4} \\ e^{2} - e^{-4} & e^{2} + 5e^{-4} \end{pmatrix}.$$

Some Properties

Proposition 12

We have the following properties :

1) $e^{0} = I$.

2) $A^{m} e^{A} = e^{A} A^{m}$, for all $m \in \mathbb{N}$.

3) ${}^{t}(e^{A}) = e^{({}^{t}A)}$.

4) If $AB = BA$, then $Ae^{B} = e^{B} A$ and $e^{A} e^{B} = e^{B} e^{A}$.

Inverse of Matrix Exponential

Proposition 13

1) Let $s, t \in \mathbb{C}$ and $A \in \mathcal{M}_n(\mathbb{C})$, then we have

$$e^{A(s+t)} = e^{As}e^{At}.$$

2) For all $A \in \mathcal{M}_n(\mathbb{C})$, e^A is invertible and we have

$$(e^A)^{-1} = e^{-A}.$$

Proof

Proof.

$$e^{sA}\, e^{tA} = \sum_{j\geq 0} \frac{(sA)^j}{j!} \sum_{k\geq 0} \frac{(tA)^k}{k!} = \sum_{j\geq 0} \sum_{k\geq 0} \frac{(s)^j (t)^k A^{j+k}}{j!k!}$$

Set $j + k = n \Leftrightarrow k = n - j\ (k \geq 0 \Leftrightarrow n \geq j)$.

$$e^{sA}\, e^{sB} = \sum_{j\geq 0} \sum_{n\geq j} \frac{(s)^j (t)^{n-j} A^n}{j!(n-j)!} \frac{n!}{n!}$$

$$= \sum_{n\geq 0} \frac{A^n}{n!} \sum_{j=0}^{n} \frac{n!\,(s)^j (t)^{n-j}}{j!\,(n-j)!}$$

$$\sum_{n\geq 0} \frac{A^n}{n!} (s+t)^n = \sum_{n\geq 0} \frac{(A(s+t))^n}{n!} = e^{A(s+t)}.$$

Matrix Exponential of sum of Matrices

Theorem 10

Let $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $AB = BA$, then we have

$$e^{A+B} = e^A e^B.$$

Proof

Set $g(t) = e^{(A+B)t} e^{-Bt} e^{-At}$

$$g'(t) = (A+B)e^{(A+B)t}e^{-Bt}e^{-At} + (-\mathrm{B})e^{-Bt}e^{(A+B)t}e^{-At} + (-\mathrm{A})e^{-At}e^{(A+B)t}e^{-Bt} = 0$$

Then $g(t)$ is constant. We have $g(0) = I$

So $g(1) = I = e^{(A+B)}e^{-B}e^{-A}$.

Systems of Differential Equations

Definition 10

The system of differential equations

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\ \qquad\qquad\qquad \cdots \\ x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) \end{cases}$$

where $x_1, \cdots, x_n$ are differentiable functions of the variable t with derivatives $x_1{}', \cdots, x_n{}'$ and the a_{ij} are constants, is called linear homogeneous differential system.

Matrix Differential Equation

We can write a linear homogeneous differential system as a matrix differential equation

$$X'(t) = AX(t),$$

where

$$X(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, X'(t) = \begin{pmatrix} x_1{}'(t) \\ \vdots \\ x_n{}'(t) \end{pmatrix}, \text{ and } A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Solution of a System of Differential Equations

Proposition 14

The vector function

$$X(t) = e^{tA}X(0)$$

is a solution of the linear homogeneous differential system

$$X'(t) = AX(t).$$

Systems of Differential Equations

Example 13

Let (S) be the system of differential equations

$$\begin{cases} x_1'(t) = x_1(t) + 5x_2(t) \\ x_2'(t) = x_1(t) - 3x_2(t) \end{cases}$$

with initial conditions $x_1(0) = 1$ and $x_2(0) = 2$.

This system can be written as

$$X'(t) = AX(t) \text{ with } X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } A = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}.$$

By Proposition 11, the vector function

$$X(t) = e^{tA}X(0)$$

is a solution of (S).

Systems of Differential Equations

Example 13

We have seen in Example 12 that

$$D = P^{-1}AP,$$

with

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}, P = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } P^{-1} = \tfrac{1}{6}\begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix}.$$

Then

$$e^{tA} = e^{PtDP^{-1}} = Pe^{tD}P^{-1} = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-4t} \end{pmatrix}\frac{1}{6}\begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix}.$$

Therefore the solution of the system is given by

$$X(t) = \frac{1}{6}\begin{pmatrix} 5e^{2t} + e^{-4t} & 5e^{2t} - 5e^{-4t} \\ e^{2t} - e^{-4t} & e^{2t} + 5e^{-4t} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2}e^{2t} - \frac{3}{2}e^{-4t} \\ \frac{1}{2}e^{2t} + \frac{3}{2}e^{-4t} \end{pmatrix}.$$