Lecture 4 : Combinatorics and counting

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INTRODUCTION, BASIC COUNTING PRINCIPLES

Sum Rule Principle: If an event E can occur in m ways, another event F can occur in n ways, and suppose both events can not occur simultaneously. Then E or F can occur in $m + n$ ways.

Example :

Suppose there are 8 male professors and 5 female professors teaching a calculus class. A student can choose a calculus professor in $8 + 5 = 13$ ways.

INTRODUCTION, BASIC COUNTING PRINCIPLES

Product Rule Principle:

Suppose an event E can occur in m different ways and associated with each way of occurring of E, another event F can occur in n different ways, then the total number of occurrence of two events in the given order is $m \times n$.

Example :

In how many ways can an organization containing 26 members elect a president, treasurer, and secretary (assuming no person is elected to more than one position)?

The president can be elected in 26 different ways; following this, the treasurer can be elected in 25 different ways (since the person chosen president is not eligible to be treasurer); and, following this, the secretary can be

elected in 24 different ways. Thus, by the above principle of counting, there are

$$26 \times 25 \times 24 = 15600$$

different ways in which the organization can elect the officers.

INTRODUCTION, BASIC COUNTING PRINCIPLES

There is a set theoretical interpretation of the above two counting principles. Specifically, suppose

$n(A)$ denotes the number of elements in a set A. Then:

1) Sum Rule Principle: If A and B are disjoint finite sets, then

$$n(A \cup B) = n(A) + n(B)$$

2) Product Rule Principle: Let $A \times B$ be the Cartesian product of sets A and B. Then

$$n(A \times B) = n(A) \times n(B)$$

FACTORIAL NOTATION

The product of the positive integers from 1 to n inclusive is denoted by $n!$ (read "n factorial"):

$$n! = 1 \times 2 \times 3 \times (n-2) \times (n-1) \times n$$

In other words, $n!$ is defined by

$$1! = 1 \text{ and } n! = n \times (n-1)!$$

It is also convenient to define $0! = 1.$

Example :

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

BINOMIAL COEFFICIENTS

The symbol $\binom{n}{r}$ read "nCr", where r and n are positive integers with $r \leq n$, is defined as follows

$$\binom{n}{r} = \frac{\mathrm{n}(n-1)(n-2)\ldots(\mathrm{n}-\mathrm{r}+1)}{1\,.2.3\ldots(\mathrm{r}-1)\mathrm{r}}\ .$$

We have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

But $n - (n - r) = r$, hence we have the following important relation:

$$\binom{n}{r} = \binom{n}{n-r}$$

or, in other words, if $a + b = n$ then $\binom{n}{a} = \binom{n}{b}$.

BINOMIAL COEFFICIENTS

Binomial Coefficients and Pascal's Triangle

The numbers $\binom{n}{r}$ are called the binomial coefficients since they appear as the coefficients in the expansion of $(a\ +\ b)^n$. Specifically, one can prove that

$$(a\ +\ b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \ , \forall a, b \in \mathbb{R}$$

The coefficients of the successive powers of $a + b$ can be arranged in a triangular array of numbers, called Pascal's triangle. The numbers in Pascal's triangle have the following intersecting properties:

1) The first number and the last number in each row is 1.

BINOMIAL COEFFICIENTS

Triangle de Pascal

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
...

a b

Binôme de Newton

$(a+b)^0 = 1$
$(a+b)^1 = a + b$
$(a+b)^2 = a^2 + 2ab + b^2$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$
$(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$
...

2) Every other number in the array can be obtained by adding the two numbers appearing directly above it. For example

$$10 = 4 + 6, 15 = 5 + 10, 20 = 10 + 10.$$

BINOMIAL COEFFICIENTS

Since the numbers appearing in Pascal's triangle are binomial coefficients, property $2)$ of Pascal's triangle comes from the following theorem

Theorem

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Permutations

Definitions

Any arrangement of a set of n distinct objects in a given order is called a permutation of the objects.

Any arrangement of any $r < n$ of these objects in a given order is called an r −permutation.

The number of r −permutations of n objects is denoted by $P(n, r)$.

Permutations

The first element in an r −permutation of n objects can be chosen in n different ways; following this, the second element in the permutation can be chosen in $(n-1)$ ways; and, following this, the third element in the permutation can be chosen in $(n — 2)$ ways. Continuing in this manner, we have that the r^{th} (last) element in the r-permutation can be chosen in $n—(r—1) = n — r + 1$ ways. Thus, by the fundamental principle of counting, we have

$$P(n,r) = n(n-1)(n-2) \ldots (n-r+1).$$

Thus we have proven the following theorem.

Theorem $P(n,r) = n(n-1)(n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$

If $r = n$ then $P(n,n) = n!$

Corollary There are $n!$ permutations of n distinct objects.

Permutations

Consider, for example, the set of letters a, b, c, and d. Then:

- ❖ $bdca, dcba$ and $acdb$ are $4-$ permutations.

$P(4,4) = 4!$, there are 24 $4-$permutation of the 4 letters.

- ❖ bad, adb, cbd and bca are $3-$permutations.

$P(4,3) = \frac{4!}{(4-3)!} = 4!$, there are 24 $3-$permutation of the 4 letters.

- ❖ ad, cb, da and bd are $2-$permutations.

$P(4,2) = \frac{4!}{(4-2)!} = 4 \times 3$, there are 12 $2-$permutation of the 4 letters.

Permutations

Permutations with repetitions

Let denote $P(n; n_1, n_2, \dots, n_r)$ the number of permutations of n objects of which n_1 are alike, n_2 are alike, …, n_r are alike. The general formula follows:

Theorem :

$$P(n; n_1, n_2, \dots, n_r) = \frac{n!}{n_1! n_2! \dots . n_r!}$$

Example :

How many seven-letter words can be formed using the letters of the word "BENZENE"?

We seek the number of permutations of seven objects of which three are alike (the three Es), and two are alike (the two Ns). The number of such words is

$$P(7; 3,2) = \frac{7!}{3!\,2!} = 420.$$

COMBINATIONS

Definition

Let E be a set of n objects. An $r-$ combination of these n objects, denoted $C(n,r)$, is the number of subsets of cardinal r included in E.

For example, the $3-$ combinations of the letters a,b,c,d are

$$\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}$$

Formula for $C(n,r)$

$$\mathrm{C}(n,r)=\frac{P(n,r)}{r!}=\binom{n}{r}$$

COMBINATIONS

Example

A farmer buys 3 cows, 2 sheeps and 4 hens from a man who has 6 cows, 5 sheeps and 8 hens. How many choices does the farmer have?

The farmer can choose the cows in $\binom{6}{3}$ ways, the sheeps in $\binom{5}{2}$ ways, and the hens in $\binom{8}{4}$ ways.

Hence altogether he can choose the animals in

$$\binom{6}{3}.\binom{5}{2}.\binom{8}{4} = 20 - 10 - 70 = \ 14000 \ \text{ways}$$

THE PIGEONHOLE PRINCIPLE

Pigeonhole Principle:

If n pigeonholes are occupied by $(n+1)$ or more pigeons, then at least one pigeonhole is occupied by more than one pigeon.

Generalized Pigeonhole Principle:

If n pigeonholes are occupied by $kn+1$ or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by $k+1$ or more pigeons.

Example

Find the minimum number of students in a class to be sure that three of them are born in the same month.

Here the $n = 12$ months are the pigeonholes and $k+1 = 3$, or $k = 2$. Hence among any $kn+1 = 25$ students (pigeons), three of them are born in the same month.

THE INCLUSION-EXCLUSION PRINCIPLE

Suppose $n(A)$ denotes the number of elements in a set A.

Let A and B be any finite sets. Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

In other words, to find the number $n(A \cup B)$ of elements in the union $A \cup B$, we add $n(A)$ and $n(B)$ and then we subtract $n(A \cap B)$ that is, we "include" $n(A)$ and $n(B)$, and we "exclude" $n(A \cap B)$. This follows from the fact that, when we add $n(A)$ and $n(B)$, we have counted the elements of $A \cap B$ twice. This

principle holds for any number of sets. We first state it for three sets.

Theorem: For any finite sets $A, B, C,$ we have

$$n(A \cup B \cup C) =$$

$$n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C).$$

THE INCLUSION-EXCLUSION PRINCIPLE

Example

Find the number of mathematics students at a college taking at least one of the languages French, German, and Russian given the following data:

65 study French, 45 study German, 42 study Russian,

20 study French and German, 25 study French and Russian, 15 study German and Russian,

8 study all three languages,

We want to find $n(F \cup G \cup R)$ where $F, G,$ and R denote the sets of students studying French, German, and Russian, respectively.

By the inclusion-exclusion principle,

$$n(F \cup G \cup R)$$
$$= n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$$
$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$$

THE INCLUSION-EXCLUSION PRINCIPLE (generalization)

Now, suppose we have any finite number of finite sets, say, $A_1, A_2, \ldots, A_m$. Let s_k be the sum of the cardinalities

$$n(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$$

of all possible k −tuple intersections of the given m sets. Then we have the following general inclusion-exclusion principle.

Theorem

$$n(A_1 \cup A_2 \cup \cdots \cup A_m) = s_1 - s_2 + s_3 - \cdots + (-1)^{m-1} s_m$$