# Real sequences (Part 3)

# 2023-2024

# Outline

- Subsequences
- Bolzano-Weierstrass property
- Cauchy Sequences
- Limit superior and Limit inferior

# Subsequences

**Definition** (Subsequence).
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence, regarded as a function

$u : \mathbb{N} \to \mathbb{R}$. A subsequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ is a composite

function $u \circ \varphi : \mathbb{N} \to \mathbb{R}$ where $\varphi : \mathbb{N} \to \mathbb{N}$ is any strictly increasing function.

Observe:

- A useful inequality: The definition implies a simple inequality that is use-ful in proofs: $\varphi(n) \geq n$ for all $n$. A formal proof might induction.

# Subsequences

**Example**

If $U_n = (-1)^n$ then $U_{2n} = (-1)^{2n} = 1$

is a subsequence of $(U_n)_{n \in \mathbb{N}}$

# Subsequences

## Theorem

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence and L a number.

(a) If $(u_n)_{n\in\mathbb{N}}$ converges to L, then every subsequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ converges to L, too.

(b) If $(u_n)_{n\in\mathbb{N}}$ diverges to $\pm\infty$, then every subsequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ diverges to $\pm\infty$, too.

(c) If $(u_n)_{n\in\mathbb{N}}$ has subsequences converging to different limits, then $(u_n)_{n\in\mathbb{N}}$ diverges.

# Subsequences

**Proposition**

The two subsequences $(u_{2n})_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ converge to the same limit $\ell$ if and only if the sequence $(u_n)_{n\in\mathbb{N}}$ converges to the limit $\ell$.

# Bolzano-Weierstrass property

**Theorem.**

Every bounded sequence of real numbers $(u_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

# Bolzano-Weierstrass property

Proof.

Suppose $U_n$ is a bounded sequence in $\mathbb{R}$. $\exists$M such that $-M \leq U_n \leq M$, n = 1, 2, · · · . Select $U_{n0} = U_1$.

- Bisect I := [−M, M] into [−M, 0] and [0, M].
- At least one of these (either [−M, 0] or [0, M]) must contain $U_n$ for infinitely many indices n.
- Call it $I_1$ and select n1 > n0 with $U_{n1} \in I_0$.
- Continue in this way to get a subsequence $U_{nk}$ such that • $I_0 \supset I_1 \supset I_2 \supset I_3 \cdots$
- $I_k = [a_k , b_k ]$ with $|I_k| = 2^{-k}$M.
- Choose $n_0 < n_1 < n_2 < \cdots$ with $U_{nk} \in I_k$ .
- Since $a_k \leq a_{k+1} \leq$ M (monotone and bounded), $a_k \rightarrow$ x.
- Since $U_{nk} \in I_k$ and $|I_k| = 2^{-k}$M,

we have $|U_{nk} - x| < |U_{nk} - a_k| + |a_k - x| \leq 2^{-k-1}M + |a_k - x| \rightarrow 0$ as $k \rightarrow +\infty$

# Cauchy Sequences

Definition

A real sequence $(U_n)$ called a Cauchy sequence if

$$\forall \varepsilon, \exists \mathrm{N} \in \mathbb{N}, \forall n \in \mathbb{N},$$

$$\mathrm{n} > \mathrm{N} \text{ and } \mathrm{m} > \mathrm{N} \Rightarrow |U_n - U_m| < \varepsilon.$$

# Cauchy Sequences

**Proposition.**

Any convergent sequence is a Cauchy sequence.

Proof.

Assume $U_n \to \alpha$. Let $\varepsilon > 0$ be given.

- $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |U_n - \alpha| < \frac{\varepsilon}{2}$.
- $n, m \geq N \Rightarrow |U_n - U_m| \leq |\alpha - U_n| + |\alpha - U_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

**Lemma.** ( Boundedness of Cauchy sequence)

If $U_n$ is a Cauchy sequence, $U_n$ is bounded.

# Cauchy Sequences

**Theorem.** ( Completeness)

Any Cauchy sequence in $\mathbb{R}$ converges to an element in [a, b].

Proof

# Cauchy Sequences

Corollary

Let $(u_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$ two reals sequences with $\lim_{n\to+\infty} \alpha_n=0$. If there exists N $\in \mathbb{N}$ such that for all

$n, p \in \mathbb{N}$ with n $\geq$ N : $|U_{n+p} - U_n| \leq \alpha_n$

Then $(u_n)_{n\in\mathbb{N}}$ is a cauchy sequence.

Proof

# Cauchy Sequences

<u>Example</u>

Show using the cauchy criterion that the sequence :

$\boldsymbol{u_n}$=1+$\frac{1}{4}$+$\frac{1}{9}+\cdots+\frac{1}{n^2}$ is convergent.

# Limit superior and Limit inferior

Proposition

Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence, and let

$$a_n = \sup\{u_k \mid k \geq n\}$$
$$b_n = \inf\{u_k \mid k \geq n\}.$$

Then,

1. $(a_n)$ is monotone decreasing and bounded, and $(b_n)$ is monotone increasing and bounded.(ie$(a_n)$and$(b_n)$are convergent.)
2. $\lim\limits_{n\to+\infty} b_n \leq \lim\limits_{n\to+\infty} a_n$

Proof

# Limit superior and Limit inferior

## Definition (Limsup/Liminf)

Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence. We define, if the limits exist,

$$\lim_{n\to+\infty} \sup u_n = \lim_{n\to+\infty} (\sup\{ u_k \mid k \geq n\}) = \lim_{n\to+\infty} a_n$$

$$\lim_{n\to+\infty} \inf u_n = \lim_{n\to+\infty} (\inf\{u_k \mid k \geq n\}) = \lim_{n\to+\infty} b_n$$

These are called the limit superior and limit inferior respectively.

# Limit superior and Limit inferior

## Theorem

If $(u_n)_{n\in\mathbb{N}}$ is not bounded above, then

$$\lim_{n\to+\infty} a_n = +\infty$$

Similarly, if $(u_n)_{n\in\mathbb{N}}$ is not bounded below, then

$$\lim_{n\to+\infty} b_n = -\infty$$

Where $a_n$ and $b_n$ are defined in the previous proposition.

# Limit superior and Limit inferior

- <u>Remark</u>

By the previous theorem, we see that if $(u_n)_{n\in\mathbb{N}}$ is not bounded above then

$$\lim_{n\to+\infty} \sup u_n = +\infty$$

Similarly, if $(u_n)_{n\in\mathbb{N}}$ is not bounded below, then

$$\lim_{n\to+\infty} \inf u_n = -\infty$$

# Limit superior and Limit inferior

Let's consider a few examples.

Example 1

Let $\boldsymbol{u_n} = (-1)^n$. Calculate the lim inf and lim sup of this sequence.

Proof: Notice that $\{(-1)^k \mid k \geq n\} = \{-1, 1\}$. Thus, the supremum of these sets is always 1 and the infimum is always $-1$. Therefore,

$$\lim_{n\to+\infty} \mathbf{sup}\, \boldsymbol{u_n} = 1$$

and

$$\lim_{n\to+\infty} \boldsymbol{inf}\, \boldsymbol{u_n} = -1.$$

# Limit superior and Limit inferior

- Example 2

Let $u_n = \frac{1}{n}$. Calculate the lim inf and lim sup of this sequence.

Proof: We may do this directly:

$\sup\{\frac{1}{k} \mid k \geq n\} = \frac{1}{n} \rightarrow 0 \Rightarrow \lim_{n \to +\infty} \sup u_n = 0.$

$\inf\{\frac{1}{k} \mid k \geq n\} = 0 \rightarrow 0 \Rightarrow \lim_{n \to +\infty} \inf u_n = 0.$

# Limit superior and Limit inferior

Theorem

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence and $\ell \in \mathbb{R}$ . The following are equivalent:

1. $\lim_{n\to+\infty} \sup u_n = \ell$.

2. For any $\varepsilon>0$, there exists $N \in \mathbb{R}$ such that

$$u_n<\ell+\varepsilon \text{ for all } n\geq N,$$

and there exists a subsequence $(u_{nk})$ of $(u_n)$ such that

$$\lim_{k\to+\infty} u_{nk}=\ell.$$

Proof

# Limit superior and Limit inferior

Theorem

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence and $\ell \in \mathbb{R}$. The following are equivalent:

1. $\lim\limits_{n\to+\infty} \inf u_n = \ell$.

2. For any $\varepsilon>0$, there exists $N \in \mathbb{R}$ such that

$$u_n > \ell - \varepsilon \text{ for all } n \geq N,$$

and there exists a subsequence $(u_{nk})$ of $(u_n)$ such that

$$\lim_{k\to+\infty} u_{nk} = \ell.$$

Proof

Add proof here and it will automatically be hidden

# Limit superior and Limit inferior

## Corollary

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence. Then

$$\lim_{n\to+\infty} u_n=\ell \quad \text{if and only if} \quad \lim_{n\to+\infty} \sup u_n = \lim_{n\to+\infty} \inf u_n = \ell$$

**Proof**

Add proof here and it will automatically be hidden

# Limit superior and Limit inferior

Corollary

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence.

1. *Suppose* $\lim_{n\to+\infty} \sup u_n = \ell$ and $(u_{nk})$ a subsequence of $(u_n)$

with $\lim_{k\to+\infty} u_{nk} = \ell'$. Then $\ell' \leq \ell$

2. *Suppose* $\lim_{n\to+\infty} \inf u_n = \ell$ and $(u_{nk})$ a subsequence of $(u_n)$

with $\lim_{k\to+\infty} u_{nk} = \ell'$. Then $\ell' \geq \ell$

# Limit superior and Limit inferior

**Theorem**

Suppose $(u_n)_{n\in\mathbb{N}}$ is a sequence and such that $u_n > 0$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} \sup \frac{u_{n+1}}{u_n} = \ell < 1. \text{ Then}$$

$$\lim_{n\to+\infty} u_n = 0.$$

**Proof**

# Limit superior and Limit inferior

## Theorem

Suppose $(u_n)_{n\in\mathbb{N}}$ is a sequence and such that $u_n>0$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} \inf \frac{u_{n+1}}{u_n} = \ell > 1. \text{ Then}$$

$$\lim_{n\to+\infty} u_n = +\infty.$$

## Proof

Add proof here and it will automatically be hidden

# Limit superior and Limit inferior

- <u>Example</u>

Given a real positive number α, define

$$\boldsymbol{u_n} = \frac{\alpha^n}{n!}, \boldsymbol{n} \in \mathbb{N}$$

$$\lim_{n\to+\infty} \mathbf{sup} \frac{\boldsymbol{u_{n+1}}}{\boldsymbol{u_n}} = \lim_{n\to+\infty} \frac{\alpha}{n+1} = 0$$

$$\Rightarrow \lim_{n\to+\infty} \boldsymbol{u_n} = 0.$$