CHAPTER 2

SETS AND FUNCTIONS

ALGEBRA, ENSIA 2023

SETS AND ELEMENTS

SETS AND ELEMENTS

In mathematics, we often encounter "sets", for example, real numbers form a set. Defining a set formally is a delicate matter, so we will use "naive" set theory, based on the intuitive properties of sets.

Definition

A **set** is a collection of objects called **elements**.

SETS AND ELEMENTS

We use uppercase letters to label sets, and elements will usually be represented by lowercase letters. When a is an element of a set A, we write

$$a \in A.$$

Otherwise, we write

$$a \notin A.$$

If A contains no elements, it is the **empty set**, denoted by $\emptyset$.
Two sets are **equal** if they have exactly the same elements. In other words,

$$A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B).$$

All the elements that we will consider are assumed to belong to a **universe set** U.
We use the bracket notation $\{\}$ to refer to a set.

SETS AND ELEMENTS

Example

The sets $\{1, 2, 3\}$ and $\{3, 2, 1\}$ are the same, because the ordering does not matter. The set $\{1, 1, 2, 3, 3\}$ is also the same set as $\{1, 2, 3\}$, because we are not interested in repetitions.

One may specify a set **explicitly**, that is by listing all the elements the set contains, or **implicitly**, using a predicate :

$$\{x : P(x)\}.$$

This notation is also known as **set-builder** notation.

Example

$A = \{1, 2\}$, $\mathbb{N} = \{0, 1, 2, \cdots\}$ are explicit descriptions.
The set $\{x : x \text{ is a prime number }\}$ is implicit.

Cardinality

Definition

The **Cardinality** $|A|$ of a set A is the number of distinct elements of A. If $|A|$ is finite, then A is said to be **finite**. Otherwise, A is said to be **infinite**.

Example

1. $|\varnothing| = 0$ while $|\{\varnothing\}| = 1$.
2. $|\{1, 2, 5\}| = 3$.
3. The set of prime numbers is infinite.

SET OPERATIONS

SET OPERATIONS

We now use operators (connectives) to define the **set operations**. These allow us to build new set from given ones.

Definition

The **union** of A and B is

$$A \cup B = \{x : x \in A \vee x \in B\}.$$

Definition

The **intersection** of A and B is

$$A \cap B = \{x : x \in A \wedge x \in B\}.$$

SET OPERATIONS

Example

If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then

$$A \cup B = \{1, 2, 3, 4, 5, 6\}$$

and

$$A \cap B = \{3, 4\}.$$

These operations of union and intersection can be illustrated with **Venn diagrams**.

SET OPERATIONS

Definition

The sets A and B are **disjoint** when $A \cap B = \emptyset$.

Definition

The **set difference** of B from A is

$$A - B = \{x : x \in A \wedge x \notin B\}.$$

The **complement** of A is defined as

$$\overline{A} = U - A = \{x : x \in U \wedge x \notin A\}.$$

Read $A - B$ as "A minus B".

SET OPERATIONS

Example

Let $U = \{1, 2, \cdots, 10\}$, $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7, 8\}$. Then

$$A - B = \{1, 2\}$$

and

$$\overline{A} = \{6, 7, 8, 9, 10\}.$$

Use Venn diagram to illustrate.

Example

$\mathbb{R} - \mathbb{Q}$: irrational numbers.

Some properties

As with the logical operations, we need **an order** to make sense of expressions that involve many operations.

Example

If we take $U = \{1, 2, , 3, 4, 5\}$, $A = \{5\}$, $B = \{3, 4, 5\}$ and $C = \{2, 3\}$, then

$$A \cup (B \cap C) = \{5\} \cup \{3\} = \{3, 5\},$$

while

$$(A \cup B) \cap C = \{3, 4, 5\} \cap \{2, 3\} = \{3\}.$$

This shows that, in general,

$$A \cup (B \cap C) \neq (A \cup B) \cap C.$$

Therefore, we **cannot write** expressions as $A \cup B \cap C$.

Some properties

However, since, as we will see,

$$A \cup (B \cup C) = (A \cup B) \cup C,$$

and

$$A \cap (B \cap C) = (A \cap B) \cap C,$$

then we can write $A \cup B \cup C$ and $A \cap B \cap C$.

Some properties

From the properties of the logical operators we derive the following.

Theorem

Let U be the universe set, and let A, B, C be sets. Then we have :

1. $A \cap B = B \cap A$
2. $A \cup B = B \cup A$
3. $\overline{(\overline{A})} = A$
4. $\overline{A \cap B} = \overline{A} \cup \overline{B}$
5. $\overline{A \cup B} = \overline{A} \cap \overline{B}$
6. $(A \cap B) \cap C = A \cap (B \cap C)$
7. $(A \cup B) \cup C = A \cup (B \cup C)$
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Some properties

Theorem

Let A and B be finite sets. Then we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

CARTESIAN PRODUCTS

CARTESIAN PRODUCTS

Let A and B be sets. Given elements $a \in A$ and $b \in B$, we call (a, b) an **ordered** pair. In this context, a and b are called **coordinates**.

Definition (Kuratowski, 1921)

If $a \in A$ and $b \in B$,

$$(a, b) = \{\{a\}, \{a, b\}\}$$

We have then

$$(a, b) = (a', b') \Leftrightarrow a = a' \text{ and } b = b'$$

Definition

The **Cartesian product** of A and B is

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

The **Cartesian product** $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is called the **Cartesian plane**.

CARTESIAN PRODUCTS

Example

If $A = \{1, 2\}$ and $B = \{0, 1, 2\}$,

$$A \times B = \{1, 2\} \times \{0, 1, 2\} = \{(1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}.$$

If $A = \{1, 2, 7\}$ and $B = \{\emptyset, \{1, 5\}\}$,

$$A \times B = \{(1, \emptyset), (1, \{1, 5\}), (2, \emptyset), (2, \{1, 5\}), (7, \emptyset), (7, \{1, 5\})\}.$$

CARTESIAN PRODUCTS

We generalize definition of an ordered pair by defining

$$(a, b, c) = \{\{a\}, \{a, b\}, \{a, b, c\}\},$$

$$(a, b, c, d) = \{\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\},$$

and for $n \in \mathbb{N}$,

$$(a_1, a_2, \cdots, a_n) = \{\{a_1\}, \{a_1, a_2\}, \cdots, \{a_1, a_2, \cdots, a_n\}\},$$

which is called and **ordered** $n-$**tuple**. Then

$$A \times B \times C = \{(a, b, c), a \in A \wedge b \in B \wedge c \in C\},$$

$$A \times B \times C \times D = \{(a, b, c, d), a \in A \wedge b \in B \wedge c \in C \wedge d \in D\},$$

and

$$A^n = \{(a_1, a_2, \cdots, a_n) : a_i \in A \wedge i = 1, 2, \cdots, n\}.$$

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) is the **Cartesian** $n-$**space**.

SUBSETS

SUBSETS

Let A and B be sets. We say that A is a **subset** of B, and we write $A \subseteq B$, when every element of A is an element of B.

Definition

$$A \subseteq B \Leftrightarrow \forall x, (x \in A \Rightarrow x \in B).$$

If A is not a subset of B, we write $A \nsubseteq B$

SUBSETS

We have then

$$A \not\subseteq B \Leftrightarrow \exists x, x \in A \wedge x \notin B.$$

Example

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 5\}$. Then $A \not\subseteq B$ because $\exists\ 3 \in A$ and $3 \notin B$.

When $A \subseteq B$ but $A \neq B$, we say that A is a **proper subset** of B, and we write $A \subset B$).

Example

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

FAMILIES OF SETS

The elements of a set can be sets themselves. We call such a collection a **family of sets** and often use capital script letters to name. For example, let

$$\mathcal{E} = \{\{1,2,3\},\{2,3,4\},\{3,4,5\}\}.$$

The set $\mathcal{E}$ has three elements :$\{1,2,3\},\{2,3,4\},\{3,4,5\}$.
Families of sets can have infinitely many elements. For example, let

$$\mathcal{F} = \{[n, n+1[: n \in \mathbb{Z}\}.$$

The set $\mathbb{Z}$ plays the role of an **index set**, a set whose only purpose is to enumerate the elements of the family. Each element of an index set is called an **index**. If we let $I = \mathbb{Z}$ and $A_i = [i, i+1[$, the family can be written as

$$\mathcal{F} = \{A_i : i \in I\}.$$

SUBSETS

Theorem

Let A, B, C be sets. Then

1. $A \subseteq B \Leftrightarrow \overline{B} \subseteq \overline{A}$.
2. $\emptyset \subseteq A$.
3. $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$.
4. *If* $A \subseteq B$ *and* $B \subseteq C$, *then* $A \subseteq C$.

FAMILIES OF SETS

There is a natural way to construct a family of sets. Take a set A. The collection of all subsets of A is called the **power set** of A and denoted by $\mathcal{P}(A)$.

Definition

For any set A,

$$\mathcal{P}(A) = \{B, B \subseteq A\}.$$

Example

$$\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

FUNCTIONS

Functions

Let E, F be sets. A **function** $f : E \to F$ assigns to **each** $x \in E$ a **unique** element $f(x) \in F$. Functions are also called maps, mappings, or transformations.

Definition

Let $f : E \to F$ be a function. Then E is called the **domain** of f and F is called the **codomain** of f.

We write $f : x \mapsto f(x)$ to indicate that is the function that maps x to $f(x)$.

Functions

Definition

Let $f : E \to F$ be a function.

1. If $x \in E$, $f(x)$ is the **image** of x under f.
2. If $y \in F$ is such that $y = f(x)$ for some $x \in E$, then x is the **preimage** of y under f.

Example

Let $E = \{1, 2, 3\}$ and $F = \{a, b\}$. Then we can define a function $f : E \to F$ by setting $f(1) = f(2) = a$ and $f(3) = b$.
a is the image of 1 under f.
1 is the preimage of a under f.

This can be represented by the following pictures.

Definition

Let $f : E \to F$ be a function and $A \subset E$. The **restriction** of f to A is the function denoted $f_{|_A} : A \to F$ defined by $f_{|_A}(x) = f(x), \forall x \in A$.

Image and inverse image of a set

Definition

For a function $f : E \longrightarrow F, A \subseteq E$, and $B \subseteq F$, the **image of** A is

$$f(A) = \{y \in F : \exists x \in A, y = f(x)\}.$$

The **inverse image of** B is

$$f^{-1}(B) = \{x \in E : f(x) \in B\}.$$

Example

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$, Then

1. $f([-1, 1]) = [0, 1]$.
2. $f^{-1}(\{1\}) = \{1, -1\}$.
3. $f^{-1}(\{-1\}) = \emptyset$.

Some properties

Let $f : E \to F$ be a function.

1. $\forall A, B \in \mathcal{P}(E), A \subset B \Rightarrow f(A) \subset f(B)$.
2. $\forall A, B \in \mathcal{P}(E), f(A \cap B) \subset f(A) \cap f(B)$.
3. $\forall A, B \in \mathcal{P}(E), f(A \cup B) = f(A) \cup f(B)$.
4. $\forall A, B \in \mathcal{P}(F), f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
5. $\forall A, B \in \mathcal{P}(F), f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Remarkable examples

1. The **identity function** on a set E is the function $Id_E : E \to E$ defined by $Id_E(x) = x$, $\forall x \in E$.
2. If $E \subseteq F$, the **inclusion map** is the function $i : E \to F$ defined by $i(x) = x$, $\forall x \in E$.
3. Let $E = E_1 \times E_2 \times \cdots \times E_n$. Define, for each i, $\pi_i : E \to E_i$ as follows :

   $$\pi_i(x_1, x_2, \cdots, x_n) = x_i.$$

   The function π_i is the i^{th} **projection**.
4. A **constant** function is a map $f : E \to F$ such that $f(x) = c, \forall x \in E$, where $c \in F$ is fixed.

Remarkable examples

5 Suppose $A \subseteq E$. The **characteristic function** of A, $\chi_A : E \to \{0, 1\}$, is defined by

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

6 A **boolean function** is a function

$$f : \{0, 1\}^n \to \{0, 1\},$$

where n is a positive integer.
For $n = 2$, we can define the following functions :

i $f(0, 0) = 0, f(1, 0) = 1, f(0, 1) = 1, f(1, 1) = 1.$

ii $g(0, 0) = 0, g(1, 0) = 0, g(0, 1) = 0, g(1, 1) = 1.$

iii $h(0, 0) = 0, h(1, 0) = 1, h(0, 1) = 1, h(1, 1) = 0.$

Did you recognize these functions ?

Injective function

Definition

A function $f : E \to F$ is **injective** if we have

$$\forall x_1, x_2 \in E, f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

An injection is also known as a **one to one** function.

Example

The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$, $\forall x \in \mathbb{Z}$, is not injective since $f(1) = f(-1)$.

The function $g : \mathbb{N} \to \mathbb{N}$ defined by $g(x) = x^2$, $\forall x \in \mathbb{N}$, is injective.

Surjective function

Definition

A function $f : E \to F$ is **surjective** if we have

$$\forall y \in F, \exists x \in E, y = f(x).$$

A surjection is also known as an **onto** function. From the definition, f is surjective if, and only, if $f(E) = F$.

Example

The function $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = 2x$ is surjective. Indeed,

$$\forall y \in \mathbb{Q}, \exists x = \frac{y}{2} \in \mathbb{Q} : f(x) = 2 \cdot \frac{y}{2} = y.$$

The function $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = 2x$ is not surjective. Indeed,

$$\exists y = 1 \in \mathbb{Z}, \forall x \in \mathbb{Z}, g(x) = 2x \neq 1.$$

Bijective function

Definition

A function that is both injective and surjective is said to be **bijective**.

Example

$$\begin{array}{rccc} f & : & [0,+\infty[& \rightarrow & [0,+\infty[\\ & & x & \mapsto & f(x) = x^2 \end{array}$$

is bijective.

Inverse function

Theorem

Let $f : E \to F$ be a function. Then f is bijective if and only if

$$\forall y \in F, \exists! x \in E : f(x) = y.$$

From this theorem, we obtain a unique function $f^{-1} : F \to E$ defined by :

$$f(x) = y \Leftrightarrow x = f^{-1}(y).$$

Definition

f^{-1} is called the inverse of f.

Inverse function

Example

$f : [0, +\infty[\longrightarrow [0, +\infty[$ defined by $f(x) = x^2$ is bijective.
Its inverse is given by $f^{-1}(x) = \sqrt{x}$.

Composition

Definition

Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be functions. The **composed function** $g \circ f : E \longrightarrow G$ is defined by :

$$\forall x \in E, g \circ f(x) = g(f(x)).$$

Example

Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = x^2$ and $g(x) = x + 1$. Then

$$(g \circ f)(x) = g(x^2) = x^2 + 1,$$

while

$$(f \circ g)(x) = f(x+1) = (x+1)^2 = x^2 + 2x + 1.$$

Therefore, in general,

$$g \circ f \neq f \circ g.$$

Composition

Theorem

Let $f : E \longrightarrow F$, $g : F \longrightarrow G$ *and* $h : G \longrightarrow H$ *be functions. Then we have*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Theorem

Let $f : E \longrightarrow F$ *be a bijective function, then* $f^{-1} \circ f = Id_E$ *and* $f \circ f^{-1} = Id_F$.