# 3. CONTINUITY

## 3.1. Continuity of a function at a point :

**Definition 1** :

*Let the function $f$ be defined for all input $x$ in an open interval $I$ and* ***a*** $\in I$ *. We say $f$ is* ***continuous at x=a*** *if*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I, (|x - a| \leq \delta \Rightarrow |f(x) - f(a)| \leq \varepsilon.$$

*ie* $f(x)$ *has a limit* $f(a)$ *at* $x = a$.

# 3. CONTINUITY

**Example :**

*Show, by using the definition of continuity, that the function* $f(x) = x^3 + x + 1$ *is continuous at* $0$,

*We look for* $\delta > 0$ *suth that:*

1

*inequality* $|x - 0| = |x| \leq \delta$ *implice* $|f(x) - f(0)| \leq \varepsilon$.

*Let* $\varepsilon > 0$. $\forall x \in \mathbb{R}$, *we have* $f(x) - f(0) = x^3 + x$ *;*

*moreover, if* $|x| \leq 1$, *we have* $|x|^3 \leq |x|$ *so, using the triangular inequality,*

$$|f(x) - f(0)| \leq |x^3 + x| \leq |x| + |x| = 2|x|$$

$$\Rightarrow |f(x) - f(0)| \leq 2|x|.$$

*thus,we choose* $\delta = \min(1, \frac{\varepsilon}{2})$

# 3. CONTINUITY

**Definition 1bis** :

Let $f$ be a function defined on an interval $I$ and $\boldsymbol{a} \in I$ .
**The function $\boldsymbol{f}$ is continuous at point $\boldsymbol{a}$** if and only if $f\ (a)$exists and both limits of $f(x)$ to the left and right of $a$ exist and are equal to $f\ (a)$.

# 3. CONTINUITY

**Examples:**

- ✓ *the constant function on an interval,*
- ✓ *sine and cosine functions on* $\mathbb{R}$,
- ✓ *the absolute value function* $|x|$ *on* $\mathbb{R}$,

## Discontinuity

**Definition**

*We say* $\boldsymbol{f}$ ***is discontinuous at*** $\boldsymbol{a}$ *if and only if* $\boldsymbol{f}$ *is not continuous at* $\boldsymbol{a}$ *, which is then called a point of discontinuity of* $\boldsymbol{f}$ *.*

# 3. CONTINUITY

## Discontinuity

The function f will be discontinuous at x = a in any of the following cases :

- (i) $\lim_{x \to a^+} f(x)$and $\lim_{x \to a^-} f(x)$ exist but are not equal.
- (ii) $\lim_{x \to a^+} f(x)$and $\lim_{x \to a^-} f(x)$exist  and are equal but not equal to f (a).
- iii) f (a) is not defined

# 3. CONTINUITY

**Example:**

*The function **E** called « integer part » function defined on $\mathbb{R}$ as follows:*

*For all* $x \in \mathbb{R}$ *:*

$\boldsymbol{E(x) = n}$*,if only if* $\boldsymbol{n\ is\ the\ largest\ integer\ less\ than\ or\ equal\ to\ x}$.

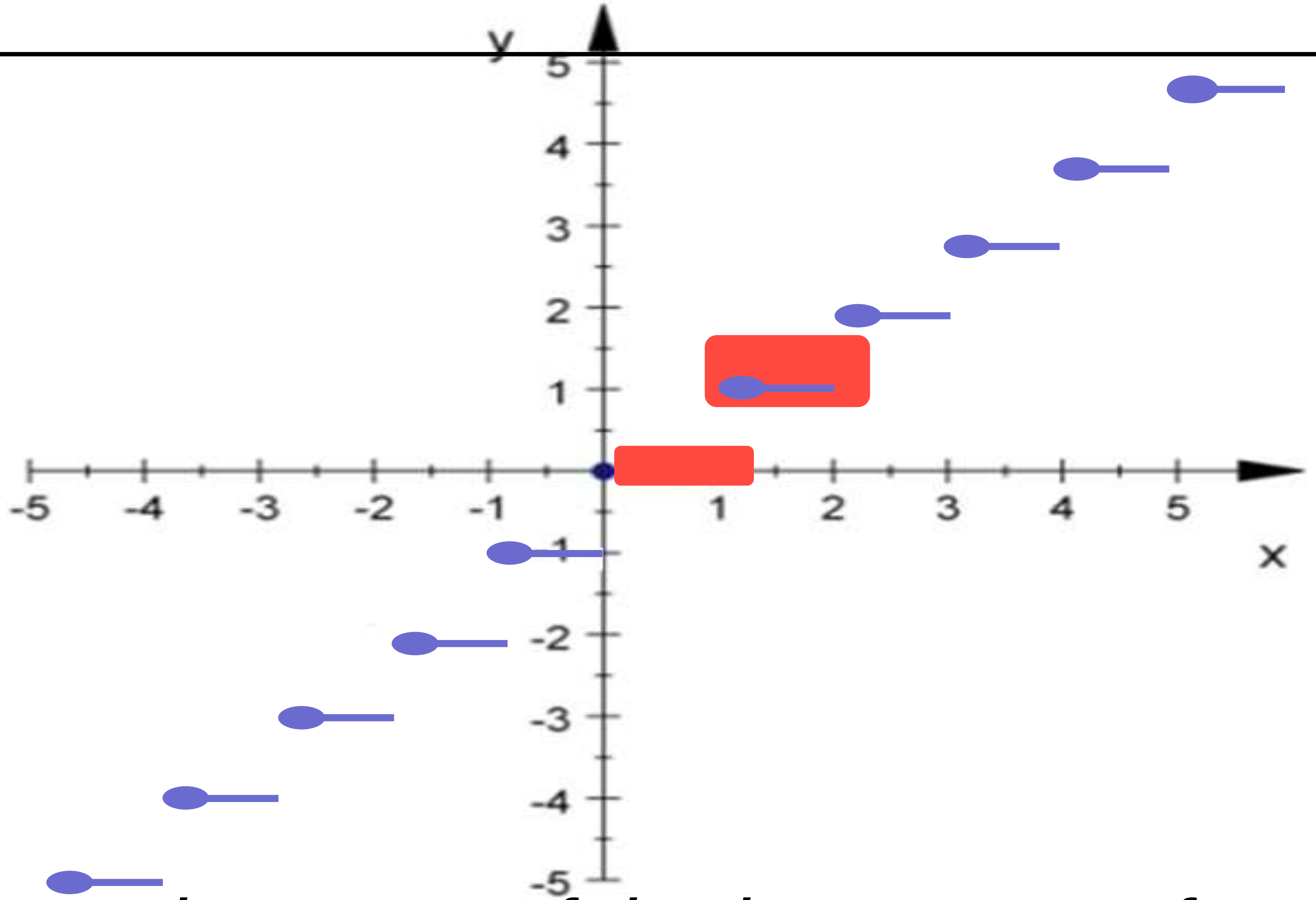
*In other words :*

$$\forall x \in \mathbb{R}: n \leq E(x) < n+1$$

*Thus :* $E(3.5) = 3$ *;* $E(\pi) = 3$ *;* $E(1/2) = 0$ *;*

$E(-\sqrt{3}) = -1$ *et* $E(-0,5) = ?$

# 3. CONTINUITY




*Construct the curve of the integer part function on the interval* $[-5\,,5[$.

*What do we see? On the interval* $[0\,,1[$ *and* $[1\,,2[$ *, then on the intervals of the form* $[n\,,n+1[, n \in \mathbb{Z}$.

*What is happening at point 1?*

*And at every point on the entire abscissa?*

# 3. CONTINUITY

## 3.2. Right Continuity and Left Continuity

- A function $f$ is right continuous at a point **a** if it is defined on an interval [$\boldsymbol{a}$ , b] lying to the right of $\boldsymbol{a}$ and if $\lim_{x \to a^+} f(x)$= f(a).

- Similarly it is left continuous at a if it is defined on an interval [d, a ] lying to the left of a and if $\lim_{x \to a^-} f(x)$= f(a).

# 3. CONTINUITY

- A function $f$ is continuous at a point x = **a** if **a** is in the domain of $f$ and:

- 1. If x = **a** is an interior point of the domain of $f$, then

$$\lim_{x \to \boldsymbol{a}} f(x) = f(a)$$

- 2. If x = **a** is not an interior point of the domain but is an endpoint of the domain, then $f$ must be right or left continuous at x = **a** , as appropriate.

**Remark** :

*The $\boldsymbol{f}$ is continuous at $\boldsymbol{a}$ if and only if it is continuous to the right and left at $\boldsymbol{a}$ .*

Example:

a function $f(x) = \begin{cases} x^2 & x < 1 \\ 1 & x = 1 \\ x & 1 < x \end{cases}$

Is continuous on $]-\infty, 1[ \cup ]1, +\infty[$ $\quad x^2$ , $x$ are continuous.

It remains to be seen if it is continuous at 1.

1) $f(1) \in \mathrm{dom}(f)$ car $f(1) = 1$ ✓

2) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 = 1$ $\qquad \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} x = 1$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$ ✓

3) $\lim\limits_{x \to 1} f(x) = 1 = f(1)$ ✓

So f(x) is continuous at 1.

# 3. CONTINUITY

Example:

$$f(x)=\begin{cases}3x-1 & x<2\\ x^2-6 & 2\leq x\end{cases}$$

1) $f(2)=(2)^2-6 \;\; =-2$ ✓

2) $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+} x^2-6 \;\; =-2$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 3x-1 \quad =3(2)-1 \;=5$$

$$\lim_{x\to 2^-} f(x)\neq\lim_{x\to 2^+} f(x) \quad \chi$$

So a function $f$ is not continuous at x = 2.

# 3. CONTINUITY

## 3.3. Characterizations of Continuity

**Proposition :**

Let $f$ : D $\longrightarrow \mathbb{R}$ and let a ∊ D, with D ⊆ $\mathbb{R}$

The following conditions are equivalent:

*1.* $f$ is continuous at a.

2. If $(u_n)_{n\in\mathbb{N}}$ is a sequence in D such that $u_n \longrightarrow$ a, then $f(u_n) \longrightarrow f(a)$.

Proof.

**Corollary** (Criterion for Discontinuity). Let $f$ : D $\longrightarrow \mathbb{R}$ and let a ∊ D a be a limit point of D. If there exists a sequence $(u_n)_{n\in\mathbb{N}} \subseteq$ D where $u_n \longrightarrow$ a but such that $f(u_n)$ does not converge to $f(a)$,we may conclude that $f$ is not continuous at a.

# 3. CONTINUITY

## 3.4. continuous extension to a point

**Definition :**

*Let $f$ be a function defined on I except at a point $a$ of $I$ and admitting a finite limit $l$ at $a$.*

We define a new function:

$$g(x) = \begin{cases} f(x) & for \quad x \neq a. \\ l & for \quad x \neq a \end{cases}$$

which is continuous at $a$. It is called the continuous extension of $f(x)$ to $a$.

**Example :**

*the function* $f(x) = \frac{ln(1+x)}{x}$ *is defined on*

$$D_f = ]-1,0[\cup]0,+\infty[.$$

*The functions* $x \mapsto ln(1+x)$ *et* $x \mapsto x$ *are continuous on* $D_f$

*and* $\forall x \in D_f, (x \neq 0)$ *so* $f$ *is continuous on* $D_f$ *.*

*since* $\lim_{x\to 0} f(x) = 1$ *so the function* $\tilde{f}$ *defined by*

$$\tilde{f}(x) = \begin{cases} \frac{ln(1+x)}{x} & if\ x \in ]-1,0[\cup]0,+\infty[ \\ 1 & if\ x = 0 \end{cases}$$

*is defined and continuous on* $]-1,+\infty[$.

# 3. CONTINUITY

## Continuity over an interval

**Definition :** Let $f : I \to \mathbb{R}$. *We say that* ***$f$ is continuous on $I$ if and only if $f$ is continuous at every point of $I$.***

**Notation :** *We denote by* $C(I, \mathbb{R})$ ***the set of functions*** *from* $I$ *to* $\mathbb{R}$ *continuous on* $I$.

# 3. CONTINUITY

## Algebraic combinations are continuous.

**Proposition :** *Let* $\lambda \in \mathbb{R}, f, g : I \to \mathbb{R}$

- ✓ *if* $f$ *is continuous on* $I$*, then* $|f|$ *is continuous on* $I$*.*
- ✓ *if* $f$ *and* $g$ *are continuous on* $I$*, then* $f + g$ $is$ *continuous sur* $I$*.*
- ✓ *if* $f$ *is continuous on* $I$*, then* $\lambda f$ *is continuous on* $I$*.*

# 3. CONTINUITY

- *If* $g$ *is continuous on* $I$ *and if* $(\forall x \in I, g(x) \neq 0)$, *then* $\frac{1}{g}$ *is continuous on* $I$.

- *if* $f$ *et* $g$ *are continuous on* $I$, *then* $fg$ *is continuous on* $I$.

- *if* $f$ *and* $g$ *are continuous on* $I$), *then* $\frac{f}{g}$ *is continuous on I, provided* $\forall x \in I, g(x) \neq 0$.

# 3. CONTINUITY

**Proposition:** ( Composition )

$Let\ f : I \to \mathbb{R}\ \ and\ g : J \to \mathbb{R}$ be functions such that $f(I) \subset J$

$if\ f\ is\ continuous\ on\ I\ and\ g\ is\ continuous\ on\ f(I),$

$then\ the\ composition\ of\ g\ with\ f,\ g \circ f\ is\ continuous\ on\ I.$

Proof.

# 3. CONTINUITY

**Uniform continuity**

Uniform continuity is a subtle but powerful strengthening of continuity.

**Definition**

$Let\ f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$. Then $f$ is uniformly

continuous on $I$ if for every $\varepsilon > 0$ there exists a $\delta > 0$

such that

$|x - y| < \delta$ and $x, y \in I$ implies that $|f(x) - f(y)| < \varepsilon$

# 3. CONTINUITY

Proposition:

A function $f : I \to \mathbb{R}$ is not uniformly continuous on $I$ if and only if there exists $\varepsilon > 0$ and sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in $I$ such that $\lim_{n\to+\infty} |x_n - y_n| = 0$ and $| f(x_n) - f(y_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$.

Example 1: Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = x^2$. Then $f$ is uniformly continuous on [0, 1]. To prove this, note that for all $x$, $y \in [0, 1]$ we have

$| x^2 - y^2 | = | x + y | | x - y | \leq 2| x - y |$, so we can take $\delta = \varepsilon /2$

in the definition of uniform continuity. Similarly, f(x) = x 2 is

uniformly continuous on any bounded set.

# 3. CONTINUITY

***Example 2:*** The function $f(x) = x^2$ is continuous but not uniformly continuous on $\mathbb{R}$. We have already proved that f is continuous on $\mathbb{R}$ (it's a polynomial).

To prove that $f$ is not uniformly continuous, let $x_n$ = n, $y_n$ = n + $1/n$ .

Then $\lim\limits_{n\to+\infty} |x_n - y_n| = 0$ , but $|f(x_n) - f(y_n)| \geq 2$ for every $n \in \mathbb{N}$. It follows from Proposition above that f is not uniformly continuous on $\mathbb{R}$.

***Example 3:*** The function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 1/x$ is continuous but not uniformly continuous on (0, 1]. It is continuous on (0, 1] since it's a rational function whose denominator x is nonzero in (0, 1]. To prove that f is not uniformly continuous, we define $x_n, y_n \in (0, 1]$ for $n \in \mathbb{N}$ by $x_n = 1/n$ , $y_n = 1/n+1$ . $\lim\limits_{n\to+\infty} |x_n - y_n| = 0$ , but $|f(x_n) - f(y_n)| = (n + 1) - n = 1$ for every $n \in \mathbb{N}$. It follows from Proposition above that $f$ is not uniformly continuous on (0, 1].

# 3. CONTINUITY

## Properties of Continuous Functions

**Proposition** Let $f$ be continuous on the closed and bounded interval $[a, b]$. Then $f$ is bounded on [a, b]; that is, there exist numbers m and M such that

$$m \leq f(x) \leq M \text{ for all } x \in [a, b]$$

Proof.

**Theorem (Heine)**

If $f : I \to \mathbb{R}$ is continuous and $I \subset \mathbb{R}$ is a closed bounded interval ($I$ is a compact of $\mathbb{R}$), then $f$ is uniformly continuous on $I$.

Proof.

# 3. CONTINUITY

## Lipshitz Functions

**Definition** let $f : I \to \mathbb{R}$ be a function and $I \subset \mathbb{R}$. If there exists a constante M > 0 such that

$$|f(x) - f(y)| < M|x - y| \qquad \forall\, x, y \in I$$

then $f$ is said to be a Lipshitz function on $I$.

**Theorem**

Every Lipshitz function is uniformly continuous,

Proof.

# 3. CONTINUITY

**Lipshitz Functions**
**Examples**

- $f(x) = x^2$ is Lipshitz continuous on $[0,1]$. But is not Lipshitz on $\mathbb{R}$.

- $f(x) = \sqrt{x}$ is not Lipshitz on $\mathbb{R}^+$. But is uniformaly continuous on $\mathbb{R}^+$

# 3. CONTINUITY

## Intermediate value theorem (IVT)

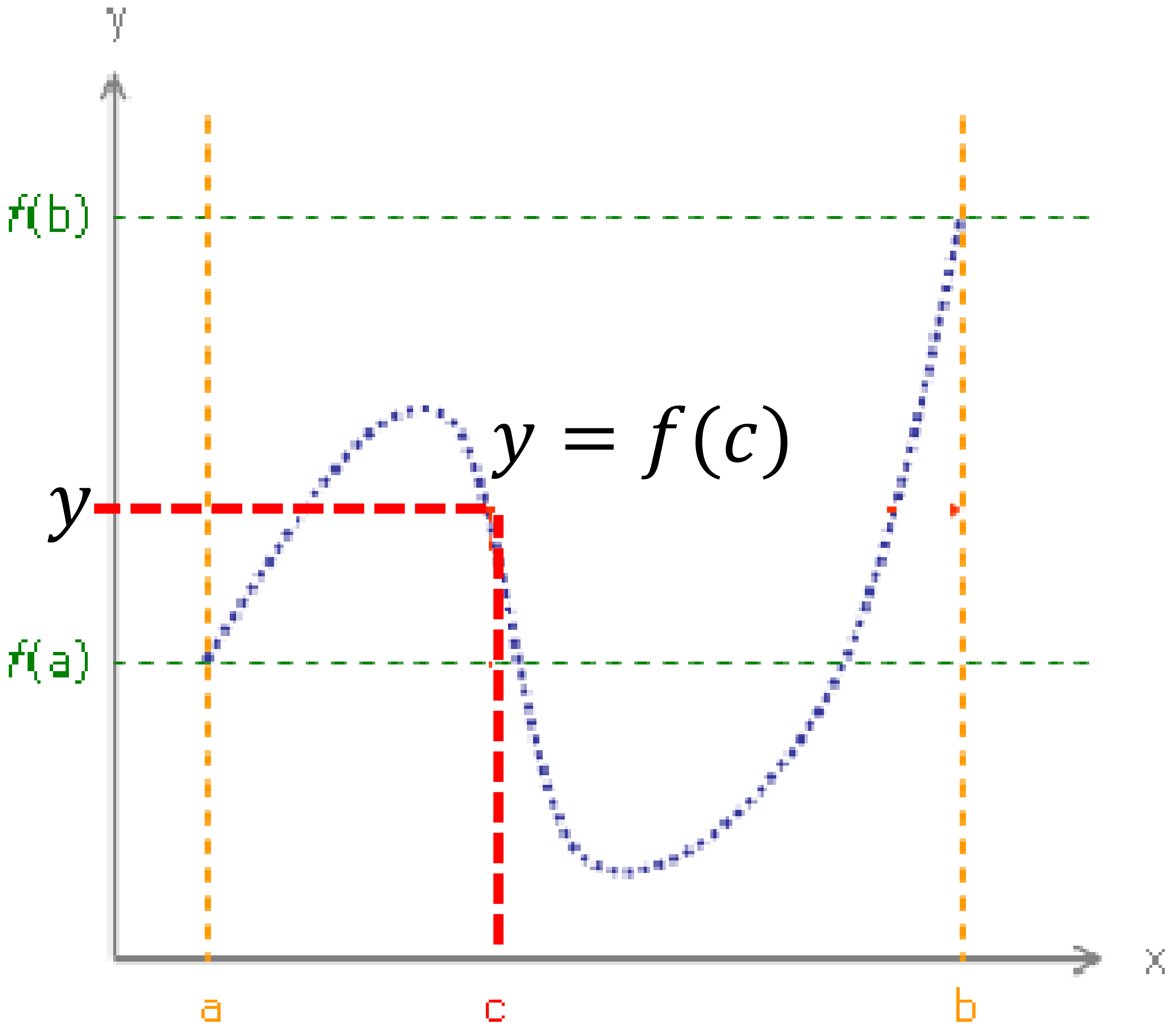
**Theoreme :**

Let $f$ be continuous on $[a, b]$, with f(a) = $f(a) \neq f(b)$; let y be any number between $f(a)$ and $f(b)$ (i.e., $f(a)$ < y < $f(b)$ or $f(b)$ < y < $f(a)$ ). Then there exists c in (a, b) with $f(c)$ = y.

Proof.

**Remark :** *The real* c *is not necessarily unique.*

# 3. CONTINUITY



# 3. CONTINUITY

**Corollary:**

$Let\ f : I \to \mathbb{R}$ be continuous.If $f(a)$ and $f(b)$ have opposite sign, then there is at least one point $c \in$ (a, b) such that $f(c) = 0$

Theorem (Extreme value theorem (EVT)).

If $f$ is continuous on [a, b], then $f$ assumes both a maximum

and a minimum value on [a, b]. That is, there exist

$x_{min}$ and $x_{max}$ in [a, b] such that

$$f(x_{min}) \leq f(x) \leq f(x_{max}) \text{ for all } x \in [a, b].$$

Proof.

# 3. CONTINUITY

**Example:**

Let the fonction $f(x) = x^3 + x + 1$.

The equation $f(x) = 0$ does it admit a solution on [- 1,0]?

1. the fonction $f(x)$ is continuous on $\mathbb{R}$, particularly on [-1,0]

2. $f(-1) = -1$ et $f(0) = 1 \Rightarrow f(-1)f(0) < 0$

Then *there exists* $c \in [a, b]$ *suth that* $f(c) = 0$

therefore the equation $f(x) = 0$ admit a solution on [-1,0]

# 3. CONTINUITY

***Example :***

*Let $f$ be a continuous function on $[0,1]$ and with values in $[0,1]$. Show that the equation $f(x) = x$ has at least one solution.*

*Let's introduce the function* $g(x) = f(x) - x$;

$g$ *is continuous on* $[0,1]$ *as the difference of two continuous functions*

*moreover,* $g(0) = f(0) \in [0,1]$ *so* $g(0) \geq 0$
$and\ g(1) = f(1) - 1 \leq 0\ because\ f(1) \in [0,1]$.

*so* $g(0)g(1) \leq 0$ *and by the intermediate value theorem, there exists at least one point* $x_0 \in [0,1]$ *such that* $g(x_0) = 0$

# 3. CONTINUITY

**Proposition:**

If $I$ is an interval and $f : I \rightarrow \mathbb{R}$ is continuous, then the range of $f$ is either a single point (if $f$ is constant) or an interval.

Proof.

**Lemma**

If $I$ is an interval and $f : I \rightarrow \mathbb{R}$ is monotone, then $f(I)$ is an interval if and only if $f$ is continuous.

Proof.

# Inverse Function Theorem

**Theorem (Inverse Function Theorem)** :

*Assume that* $f$ *is continuous and strictly monotonic on the interval* $I$ *then there exists a inverse function to* $f$

$f^{-1}: f(I) \to I$, continuous and *similarly monotonous.*

Proof.

# 3. CONTINUITY

**Remarks on image of interval :**

- ✓ *We consider the function* $f : x \mapsto x^2$ *continuous on* $\mathbb{R}$*. Then* $f(]-1,2]) = [0,4]$ *:*

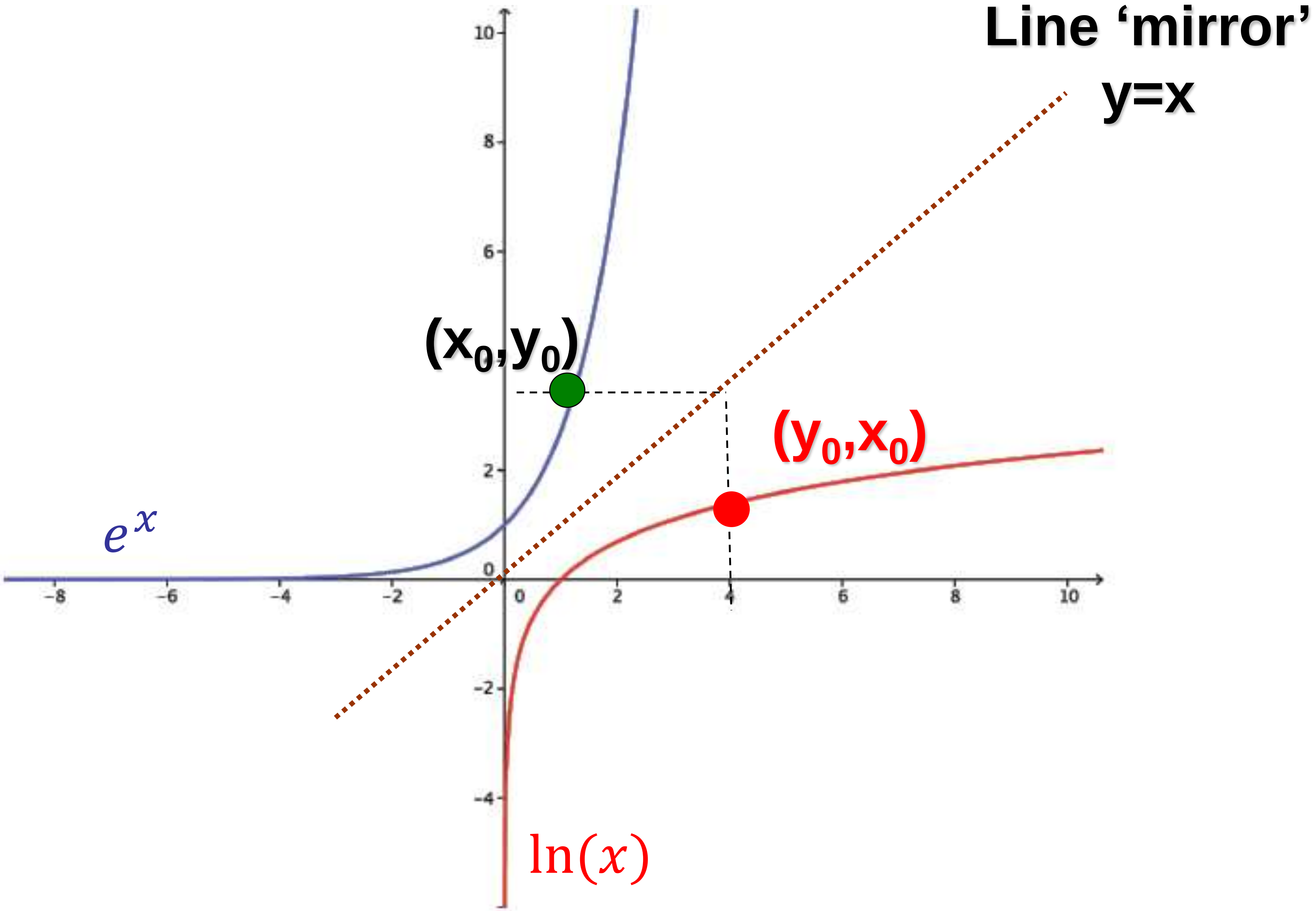
*the image of interval* $]-1,2]$ *under a continuous function* $f$ *is indeed an interval, but it is not of the same nature. (one is closed, the other is half-open or half closed).*

# 3. CONTINUITY

- ✓ *We consider the function* $g : x \mapsto 1/x$ *continuous on the bounded interval* $]0, 1]$.

*we find* $g(]0,1]) = [1, +\infty[$ *unbounded interval, which is also not open on the same side ! So the image of interval under a continuous function does not necessarily have the same properties as the starting interval.*

# 3. CONTINUITY



# 3. CONTINUITY

***Example :***

*The exponentiel function performs a bijection from* $\mathbb{R}$ *to* $\mathbb{R}_+^*$ *because :*

- *It is continues on* $\mathbb{R}$*.*
- *It is strictly monotonic on* $\mathbb{R}$ *.*

*Its inverse bijection is the function* $Ln$ *.*