Linear Algebra

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Introduction

- Linear Algebra can be viewed as the study of vector spaces and linear maps between them.
- Linear Algebra has applications in almost all of the scientific fields.
- In particular, Linear Algebra plays a fundamental role in many areas of computer science, including artificial intelligence and machine learning.
- This course is devoted to the study of four principal objects : Vector spaces, Linear maps, Matrices and Systems of linear equations. We will see how these objects are all related to each other.

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Lecture 1 : Vector Spaces

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Contents

Introduction

The beauty and power of Linear Algebra will be seen more clearly when you view $\mathbb{R}^n$ **as only one of a variety of vector spaces** that arise naturally in applied problems. Actually, a study of vector spaces is not much different from a study of $\mathbb{R}^n$ itself, because you can use your geometric experience with $\mathbb{R}^2$ and $\mathbb{R}^3$ to visualize many general concepts. [LLM]

Vector Space Structure

Throughout this lecture, K will denote an arbitrary field.

Definition 1

A **vector space over $\boldsymbol{K}$**, or a **$\boldsymbol{K}$- vector space**, is a commutative group $(V,\oplus)$ equipped with a function $K \times V \to V; (\alpha, x) \mapsto \alpha \otimes x$ such that

(i) $\forall\ \alpha, \beta \in K, \forall x \in V, (\alpha + \beta) \otimes x = (\alpha \otimes x) \oplus (\beta \otimes x)$;

(ii) $\forall\ \alpha, \beta \in K, \forall x \in V, (\alpha \cdot \beta) \otimes x = \alpha \otimes (\beta \otimes x)$;

(iii) $\forall x, y \in V, \forall \alpha \in K, \alpha \otimes (x \oplus y) = (\alpha \otimes x) \oplus (\alpha \otimes \mathrm{y})$;

$(iv) 1 \otimes x = x, \forall x \in V$.

Vector Space Structure

Remark 1 : The elements of V are called **vectors** and the elements of K are called **scalars**. The multiplication $\otimes$ is called **multiplication by a scalar**.

Remark 2 : In the definition of a K- vector space V, if we replace the field K by a commutative ring A, we obtain the definition of a module over the ring A.

From now on, we will denote the operation $\oplus$ by $+$ and the operation $\otimes$ by $\cdot$.

Examples

Example 1.

Take $K = \mathbb{R}$ and $V = \mathbb{R}^2$, and set

$$\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2, (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

and

$$\forall\, \alpha \in \mathbb{R}, \forall (x_1, x_2) \in \mathbb{R}^2, \alpha(x_1, x_2) = (\alpha x_1, \alpha x_2).$$

Then $\mathbb{R}^2$ is an $\mathbb{R}$-vector space.

Examples

Example 2.

More generally, let K be any field and let n be a positive integer.

The set K^n of n-tuples of elements of K is a vector space over K with

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$

and

$$\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n),$$

for all elements $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ of K^n and for all elements α of K.

In particular, K is a vector space over itself.

Examples

Example 3.

Let $K[X]$ be the set of **polynomials** over K with an indeterminate X.

Let $P(X) = a_0 + a_1X + \cdots + a_n X^n, Q(X) = b_0 + b_1X + \cdots + b_m X^m \in K[X]$ and $\alpha \in K$.

We define on $K[X]$ the following operations :

$$P(X) + Q(X) = \sum_{i=0}^{r}(a_i + b_i)X^i, \text{ where } r = \max(m, n).$$

$$\alpha P(X) = (\alpha a_0) + (\alpha a_1)X + \cdots + (\alpha a_n)X^n.$$

Endowed with these two operations, $K[X]$ is a K- vector space.

Examples

Example 4.

Let X be a set and V be a K- vector space. Denote by $\mathcal{F}(X, V)$ the set of **functions** from X to V. Let $f, g \in \mathcal{F}(X, V)$ and $\alpha \in K$. We define the sum of f and g by

$$(f+g)(x) = f(x) + g(x), \forall x \in X,$$

and the multiplication by a scalar by

$$(\alpha f)(x) = \alpha f(x), \forall x \in X.$$

Endowed with these two operations, $\mathcal{F}(X, V)$ is a K- vector space.

In particular, with these operations, the set $\mathcal{F}(\mathbb{N}, \mathbb{R})$ of **real sequences** has an $\mathbb{R}$-vector space structure.

Properties

Proposition 1

Let V be a K-vector space. Then we have : $\forall x \in V, \forall \alpha \in K$,

1- $0_K \cdot x = 0_V$;

2- $\alpha \cdot 0_V = 0_V$;

3- $\alpha \cdot x = 0_V \Leftrightarrow \alpha = 0_K$ or $x = 0_V$;

4- $(-\alpha) \cdot x = \alpha \cdot (-x) = -(\alpha \cdot x)$.

Subspaces

Definition 2

Let V be a K- vector space and U be a subset of V. We say that U is a **subspace** of V if

1- $\forall u, v \in U, u + v \in U$;

2- $\forall u \in U, \forall \alpha \in K, \alpha \cdot u \in U$;

3- $0_V \in U$.

Remark 3

This definition means that U is a K-vector space under the inherited operations. Check it !

Examples

Example 5

V and $\{0\}$ are subspaces of the vector space V.

Example 6

The set

$$U = \{(x, y) \in \mathbb{R}^2, 2x + y = 0\}$$

is a subspace of $\mathbb{R}^2$.

Examples

Example 7

Let n be a positive integer. Then the set $K_n[X]$ of polynomials over K of degree $\leq n$ is a subspace of $K[X]$.

Example 8

The set

$$U = \{(x, y) \in \mathbb{R}^2, x + y = 1\}$$

is **not** a subspace of $\mathbb{R}^2$.

Intersection of subspaces

Proposition 2

The intersection of two subspaces of a K- vector space V is a subspace of V.

Remark 4 . The union of two subspaces of a K- vector space V is not in general a subspace of V. (See exercise 5, Worksheet 1)

Sum of subspaces

Definition 3

Let U and W be two subspaces of a K-vector space V. The set
$$U + W = \{u + w : u \in U, w \in W\}$$
is called **sum** of U and W.

Proposition 3

If U and W are two subspaces of a K-vector space V, then $U + W$ is a subspace of V.

Linear combination of vectors

Definition 4.

Let V be a K- vector space and $v \in V$. We say that v is a **linear combination** of $v_1, v_2, \cdots, v_n \in V$ if there exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in K$ such that

$$v = \sum_{i=1}^{n} \alpha_i v_i.$$

Example 9

Take $V = \mathbb{R}^2$, $v = (2,3)$, $e_1 = (1,0)$ and $e_2 = (0,1)$. We have

$$v = 2(1,0) + 3(0,1).$$

Then v is a linear combination of e_1 and e_2.

Spanning set

Definition 5

Let V be a K-vector space and $A \subseteq V$ a subset of V . Then the **span of $\boldsymbol{A}$ in $\boldsymbol{V}$** is the set of all finite linear combinations of elements of A.

Notation :

The span of A in V is denoted by $\langle A \rangle$, or *Span* (A).

When A is a finite set of vectors, that is $A = \{v_1, v_2, \cdots, v_n\}$, we write simply $\langle v_1, v_2, \cdots, v_n \rangle$, or *Span* $\{v_1, v_2, \cdots, v_n\}$.

Convention :

By convention, we set $\langle \emptyset \rangle = \{0\}$.

Spanning set

From the previous definition, we have

$$\langle A \rangle = \left\{ \sum_{i=1}^{n} \alpha_i u_i : \alpha_i \in K, u_i \in A, n \in \mathbb{N}^* \right\}.$$

Proposition 4

Let V be a K-vector space and $A \subseteq V$ a subset of V . Then the span of A in V is the smallest subspace of V containing A.

Spanning set

Example 10

$\mathbb{R}^n$ is spanned by the vectors

$$e_1 = (1,0,\cdots,0), e_2 = (0,1,\cdots,0), \cdots, e_n = (0,0,\cdots,1) \in \mathbb{R}^n.$$

Example 11

$K_n[X]$ is spanned by the vectors

$$1, X, \cdots, X^n.$$

Linear independence

Definition 6.

Let V be a K- vector space and $v_1, v_2, \cdots, v_n \in V$. We say that the vectors $v_1, v_2, \cdots, v_n$ are **linearly independent** $(L.I.)$ if we have : for all $\alpha_1, \alpha_2, \cdots, \alpha_n \in K$,

$$\sum_{i=1}^{n} \alpha_i v_i = 0_V \implies \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0_K.$$

Otherwise, we say that $v_1, v_2, \cdots, v_n$ are **linearly dependent** $(L.D.)$.

Linear dependence

From the previous definition, by taking the negation, we can see that the vectors $v_1, v_2, \cdots, v_n$ are linearly dependent if there exist n scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in K$, **not all zero**, for which

$$\sum_{i=1}^{n} \alpha_i v_i = 0_V.$$

Examples

Example 12

Set $V = \mathbb{R}^2$, $v_1 = (1,2), v_2 = (-1,1)$.

The vectors v_1 and v_2 are linearly independent.

Example 13

Set $V = K[X]$, $P_1 = X, P_2 = X + 1$.

The polynomyals P_1 and P_2 are linearly independent.

Examples

Example 14

Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of functions from $\mathbb{R}$ to $\mathbb{R}$, and consider the two functions defined by

$$f(x) = e^x,\ \ g(x) = e^{2x} \text{ for all } x \in \mathbb{R}.$$

Then f and g are linearly independent.

Example 15

The n vectors

$$e_1 = (1,0,\cdots,0), e_2 = (0,1,\cdots,0), \cdots, e_n = (0,0,\cdots,1) \in \mathbb{R}^n$$

are linearly independent.

Examples

Example 16

Set $V = \mathbb{R}^2, v_1 = (1,2), v_2 = (-1,1)$ and $v_3 = (3,3)$.

The vectors v_1, v_2, v_3 are linearly dependent. Indeed,

$$2v_1 - v_2 - v_3 = (0,0).$$

Finite-dimensional vector spaces

Definition 7

A vector space V is said to be **finite-dimensional** if it is spanned by a finite number of vectors.

Example 17

$\mathbb{R}^n$ is spanned by the n vectors

$$e_1 = (1,0,\cdots,0), e_2 = (0,1,\cdots,0), \cdots, e_n = (0,0,\cdots,1),$$

so it is a finite-dimensional vector space.

Example 18

$K[X]$ is not a finite-dimensional vector space.

Basis of a vector space

Definition 8

Let V be a K- vector space and $v_1, v_2, \cdots, v_n \in V$. We say that the set $\{v_1, v_2, \cdots, v_n\}$ is a **basis** of V if we have

1-The vectors $v_1, v_2, \cdots, v_n$ are linearly independent ;

2-The vectors $v_1, v_2, \cdots, v_n$ span V.

Examples

Example 19

The vectors

$$e_1 = (1,0,\cdots,0), e_2 = (0,1,\cdots,0),\cdots, e_n = (0,0,\cdots,1) \in \mathbb{R}^n$$

form a basis of $\mathbb{R}^n$, called **standard basis**.

Example 20

The set $\{1, X, \cdots, X^n\}$ is a basis of $K_n[X]$.

Basis of a vector space

Theorem 1

Let V be a K- vector space. Then, the vectors $v_1,\ v_2,\ \cdots, v_n$ of V constitute a basis of V if, and only if , given any vector u of V, there **exist uniquely** determined scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in K$ such that

$$u = \sum_{i=1}^{n} \alpha_i v_i.$$

Remark 4

If $\{v_1,\ v_2,\ \cdots, v_n\}$ is a basis of V and $u = \sum_{i=1}^{n} \alpha_i v_i$, then the scalars $\alpha_1, \alpha_2,\ \cdots, \alpha_n$ are called **coordinates** or **components** of u in the basis $\{v_1,\ v_2,\ \cdots, v_n\}$.

Existence of basis

Theorem 2

Let V be a finite-dimensional K-vector space, and A be a finite subset of V which spans V. Let L be a subset of linearly independent vectors of A. Then, there exists a basis B of V so that

$$L \subseteq B \subseteq A.$$

Theorem 3

Any finite-dimensional K-vector space has a basis.

Incomplete basis Theorem

Theorem 4

Let V be a finite-dimensional K- vector space, and $v_1, v_2, \cdots, v_p \in V$ linearly independent vectors. Then, one can find vectors of V, v_{p+1}, $v_{p+2}, \cdots, v_n$, such that $\{v_1, v_2, \cdots, v_n\}$ is a basis of V.

Dimension

Theorem 5

Let V be a finite-dimensional K-vector space. Then All basis of V have the same number of vectors.

Definition 9

The **dimension** of a finite-dimensional vector space is defined to be number of elements in any basis of that vector space. The dimension is defined to be zero in the case where the vector space consists of just the zero element.

Notation

The dimension of V is denoted by $\dim V$.

Examples

Example 21.

$\mathbb{R}^n$ is of dimension n.

Example 22.

$K_n[X]$ is of dimension $n+1$.

Example 23.

The subspace $U = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$ of $\mathbb{R}^2$ is of dimension 1.

Basis with known dimension

Theorem 6

Let V be a K-vector space of dimension n, then we have

1- Any n linearly independent vectors of V form a basis of V.

2- Any n vectors of V which span V form a basis of V.

Minimal spanning set and Maximal linearly independent set

Definition 10

A spanning set of a vector space V is said to be **minimal** if it has no proper subset which forms a spanning set of V.

A set $\{v_1, v_2, \cdots, v_n\}$ of linearly independent vectors of V is said to be **maximal** if for all $v \in V$, the vectors $v, v_1, v_2, \cdots, v_n$ are linearly dependent.

Theorem 7

Let V be a finite-dimensional vector space. Then we have :

1- Any minimal spanning set of V forms a basis of V.

2- Any maximal set of linearly independent vectors of V forms a basis of V.

Dimension and subspaces

Theorem 8

Let V be a finite-dimensional K-vector space and let U be a subspace of V. Then we have

1- $\dim(U) \leq \dim(V)$;

2- $\dim(U) = \dim(V) \Longrightarrow U = E$.

Rank of a family of vectors

Definition 11

Let V be a finite dimentional K-vector space and $v_1, v_2, \cdots, v_m \in V$. The **rank** of $\{v_1, v_2, \cdots, v_m\}$ is the dimension of the subspace of V spanned by $v_1, v_2, \cdots, v_m$.

Computing the rank of a family of vectors

To compute the rank of $\{v_1, v_2, \cdots, v_m\}$, we will use **Gaussian Elimination** to obtain **Row Echelon Form**.

Set

$$
\begin{aligned}
&v_1 = a_{11}\ a_{12}\ \cdots\ a_{1j}\ \cdots\ a_{1n}\\
&v_2 = a_{21}\ a_{22}\ \cdots\ a_{2j}\ \cdots\ a_{2n}\\
&\vdots\\
&v_i = a_{i1}\ a_{i2}\ \cdots\ a_{ij}\ \cdots\ a_{in}\\
&\vdots\\
&v_m = a_{m1}\ a_{m2}\ \cdots\ a_{mj}\ \cdots\ a_{mn},
\end{aligned}
$$

where the a_{ij}, $1 \leq j \leq n$, are the components of v_i in a fixed basis.

Row Echelon Form

Definition 12

A family of vectors is said to be in the row-echelon form (REF) if the two following conditions are satisfied :

1- If a row is zero, then all the following rows are zeros.

2- If the row i has its first nonzero coefficient in column j, then the first nonzero coefficient of row $i+1$ is in column $k>j$.

Definition 13

When the vectors are in the REF, the first nonzero element of each row is called a **pivot**.

Elementary row operations

To obtain the REF, we use a sequence of **elementary row operations**.

Definition 14

There are three types of elementary row operations :

1- Exchanging two rows ;

2- Multiplying a row by a nonzero scalar ;

3- Adding a multiple of one row to another row.

Computing the rank of a family of vectors

Proposition 5

The rank of $\{v_1, v_2, \cdots, v_m\}$ is the number of nonzero rows of the Row Echelon Form.

Gaussian Elimination Example

Example 24

Determine the rank of the vectors :

$v_1 = (1, 2, 1, 2, 1), v_2 = (2, 4, 2, 1, 5), v_3 = (1, 0, 1, 1, 0), v_4 = (0, 1, 0, 0, 1).$

$$v'_1 = v_1,\ v'_2 = \ v_2 - 2v_1, v'_3 = v_3 - \ v_1, v'_4 = v_4$$

$$\begin{pmatrix} {\color{red}1} & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 1 & 5 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & -2 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Gaussian Elimination Example

$$v''_1 = v'_1\,,\ v''_2 = v'_4\,,\ v''_3 = v'_3,\ v''_4 = v'_2$$

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & -2 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 & -1 \\ 0 & 0 & 0 & -3 & 3 \end{pmatrix}$$

Gaussian Elimination Example

$$v'''_1 = v''_1\,, v'''_2 = v''_2\,, v'''_3 = v''_3 + 2v''_2, v'''_4 = v''_4$$

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & \color{red}{1} & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 & -1 \\ 0 & 0 & 0 & -3 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -3 & 3 \end{pmatrix}.$$

Gaussian Elimination Example

$$v''''_1 = v'''_1\,,\, v''''_2 = v'''_2\,,\, v''''_3 = v'''_3,\, v''''_4 = v'''_4 - 3v'''_3$$

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -3 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, since we have 3 nonzero rows, rank $\{v_1, v_2, v_3, v_4\} = 3$.

Direct sum of subspaces

Definition 15

Let W_1, W_2 be two subspaces of a K-vector space V. The sum W_1+W_2 is called **direct** if $W_1 \cap W_2 = \{0_V\}$.

In particular, a vector space V is said to be the **direct sum** of two subspaces W_1 and W_2 if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0_V\}$.

When V is a direct sum of W_1 and W_2, we write $V = W_1 \oplus W_2$.

Theorem 9

Suppose W_1 and W_2 are subspaces of a vector space V. Then $V = W_1 \oplus W_2$ if, and only if, every vector in V can be written in a unique way as $w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$.

Dimension of sum of subspaces

Theorem 10

If W_1, W_2 are subspaces of a finite-dimensional vector space V, then we have

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$