Real sequences (Part 1)

2023-2024

Outline

- General definitions
- Properties
- Limits of sequences
- Convergent and divergent sequences
- The monotone convergence theorem
- Limits Properties
- Operations on convergent sequences

Outline

- Limits and order
- Ajacent sequences

General definitions

Definition

A real sequence or a sequence of real numbers is defined as a function from $\mathbb{N}$*, the set of natural numbers, or from a subset A of* $\mathbb{N}$*, to* $\mathbb{R}$*. In other words* $f : n \in A \mapsto U_n = f(n) \in \mathbb{R}$.

It is customary to denote a sequence by $(U_n)_{n\in\mathbb{N}}$ *or more clearly* (U_n).

Examples

1. $$U_n = \sqrt{n-3}, \forall n \in \mathbb{N}; n \geq 3.$$

2. $$f(n) = U_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}$$

General definitions

3. $$f(n) = U_n = 1 + \frac{1}{n+1}$$

4. $$\begin{cases} U_0 = 1 \\ U_{n+1} = \sqrt{U_n + 1} \end{cases}$$

The real numbers U_0, U_1, … are called elements or terms of the sequence (U_n). The number U_n is called the n^{th} term of rank n of the sequence or general term.

General definitions

- *increasing sequence*, $\forall n \in \mathbb{N}, n \geq n_0\ U_{n+1} \geq U_n$
- *decreasing sequence*, $\forall n \in \mathbb{N}, n \geq n_0\ U_{n+1} \leq U_n$
- *Strictly increasing sequence*, $\forall n \in \mathbb{N}, n \geq n_0\ U_{n+1} > U_n$
- *Strictly decreasing sequence*, $\forall n \in \mathbb{N}, n \geq n_0\ U_{n+1} < U_n$
- *Stationary sequence*, $\exists\ p \in \forall n \in \mathbb{N}, n \geq p,\ U_{n+1} = U_n = U_p$
- *Periodic sequence*, $\exists\ k \in \mathbb{N}\ \ \forall n \in \mathbb{N},\ U_{n+k} = U_n$

General definitions

- *Upper bounded sequence,* $\exists M \in \mathbb{R} \forall n \in \mathbb{N}, U_n \leq M$
- *Lower bounded sequence,* $\exists m \in \mathbb{R} \forall n \in \mathbb{N}, U_n \geq m$
- *Bounded sequence,* $\exists \alpha \in \mathbb{R}_+^* \forall n \in \mathbb{N}, |U_n| \leq \alpha$
- *Monotone sequence, « increasing or decreasing sequence»*

General definitions

Remarks

If (U_n) is such that $U_n > 0$ and $\frac{U_{n+1}}{U_n} \leq 1$ then (U_n) is decreasing.

If (U_n) is such that $U_n > 0$ and $\frac{U_{n+1}}{U_n} \geq 1$ then (U_n) is increasing.

The sequence $U_n = (-1)^n$ is not monotone.

Properties

1) If $(U_n)_{n\in\mathbb{N}}, (V_n)_{n\in\mathbb{N}}$ are increasing then the sum

$(U_n + V_n)_{n\in\mathbb{N}}$ is increasing.

2) If $(U_n)_{n\in\mathbb{N}}, (V_n)_{n\in\mathbb{N}}$ are decreasing then $(U_n + V_n)_{n\in\mathbb{N}}$

is decreasing .

Properties

3) If $(U_n)_{n\in\mathbb{N}}, (V_n)_{n\in\mathbb{N}}$ are positive increasing sequences then $(U_n \cdot V_n)_{n\in\mathbb{N}}$ is increasing.

4) If (U_n) is increasing and $\forall n \in \mathbb{N}, n \geq n_0\ U_n > 0$

then $\left(\frac{1}{U_n}\right)_{n\in\mathbb{N}}$ is decreasing.

Examples

Study the monotony of the following sequences :

Exemple 1: $U_n = 1 + E\left[\frac{1}{n}\right], n \in \mathbb{N}^*.$

Exemple 2: $U_n = \dfrac{3n-1}{2n+3}$

Exemple 3: $U_n = n^2 + 2,$

Example 4: $U_n = \dfrac{1}{0!} + \dfrac{1}{1!} + \cdots + \dfrac{1}{n!}$

Limits of sequences

Definition

- $\lim U_n = l \iff$

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow |U_n - l| < \varepsilon$

- $\lim\limits_{n \to +\infty} U_n = +\infty \iff$

$\forall A > 0, \exists n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow U_n > A$

- $\lim\limits_{n \to +\infty} U_n = -\infty \iff$

$\forall B > 0, \exists n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow U_n < -B$

Limits of sequences

Examples

$$\lim_{n\to+\infty}\left(\frac{3n+4}{2n+1}\right)=\frac{3}{2}$$

$$\lim_{n\to+\infty} n^2+3n+1=+\infty$$

$$\lim_{n\to+\infty} -n^2+3n+1=-\infty$$

Limits of sequences

$$\lim_{n \to +\infty} sin\frac{1}{n} = 0$$

$$\lim_{n \to +\infty} \frac{n^2 + 1}{n - 10} = +\infty$$

Convergent and divergent sequences

Theorem

When it exists, the limit of a sequence is unique.

Definition

A sequence which admits a finite limit $l \in \mathbb{R}$ is said to be convergent. Otherwise, we say that the sequence is divergent.

Proposition

Any convergent sequence is bounded.

Observe

The converse is not always true $U_n = (-1)^n$

The monotone convergence theorem

Theorem (Monotone Convergence Theorem)

- If $(u_n)_{n\in\mathbb{N}}$ is increasing sequence, then $\lim\limits_{n\to+\infty} u_n$ exists and $(u_n)_{n\in\mathbb{N}}$ is convergent if and only if $(u_n)_{n\in\mathbb{N}}$ is upper bounded.

- if $(u_n)_{n\in\mathbb{N}}$ is a decreasing sequence, then $\lim\limits_{n\to+\infty} u_n$ exists and $(u_n)_{n\in\mathbb{N}}$ is convergent If and only if $(u_n)_{n\in\mathbb{N}}$ is lower bounded.

The monotone convergence theorem

Collary1

Any upper bounded and increasing sequence is convergent.
Any lower bounded and decreasing sequence is convergent.

Collary2

If $(u_n)_{n\in\mathbb{N}}$ is increasing and not upper bounded then

$$\lim_{n\to+\infty} u_n = +\infty$$

If $(u_n)_{n\in\mathbb{N}}$ is decreasing and not lower bounded then

$$\lim_{n\to+\infty} u_n = -\infty$$

Operations on convergent sequences

Proposition

1) $$n \to +\infty, \quad \left.\begin{array}{l} U_n \to \ell \\ V_n \to \ell' \\ a \in \mathbb{R} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} U_n + V_n \to \ell + \ell' \\ U_n.V_n \to \ell.\ell' \\ a.U_n \to a.\ell \end{array}\right.$$

2) $$n \to +\infty, \quad \left.\begin{array}{l} U_n \to \ell \\ V_n \to \ell' \ et \ \ell' \neq 0 \end{array}\right\} \Rightarrow \left\{ \frac{U_n}{V_n} \to \frac{\ell}{\ell'} \right.$$

Limits properties

Theorem

If $$\lim_{n\to+\infty} U_n = \ell \Rightarrow \lim_{n\to+\infty} |U_n| = |\ell|$$

Proof

$$\left||U_n| - |\ell|\right| \leq |U_n - \ell| < \varepsilon, \forall n \in \mathbb{N}, n > n_0$$

The converse is not always true. $$U_n = (-1)^n$$

Remark The converse is true only when the limit equals to zero

Limits and Order

Theorem (Order Limit Theorem)

1- If $\forall n \in \mathbb{N}, U_n \leq V_n, et \lim\limits_{n \to +\infty} U_n = \ell, \lim\limits_{n \to +\infty} V_n = \ell' \Rightarrow \ell \leq \ell'$

2- If $\forall n \in \mathbb{N}, U_n \leq 0 \; et \lim\limits_{n \to +\infty} U_n = \ell \Rightarrow \ell \leq 0$

3- If $\forall n \in \mathbb{N}, U_n \geq 0 \; et \lim\limits_{n \to +\infty} U_n = \ell \Rightarrow \ell \geq 0$

Limits and order

4- If $\forall n \in \mathbb{N}, U_n \leq V_n$. Then

$\lim\limits_{n \to +\infty} u_n = +\infty \Rightarrow \lim\limits_{n \to +\infty} v_n = +\infty$ and

$\lim\limits_{n \to +\infty} v_n = -\infty \Rightarrow \lim\limits_{n \to +\infty} u_n = -\infty$

Limits and order

Theorem (*Squeeze Theorem*)

$$(X_n)_{n\in\mathbb{N}}, \quad (U_n)_{n\in\mathbb{N}}, \quad (Y_n)_{n\in\mathbb{N}}$$

(1) $\quad n \geq n_0 \qquad X_n \leq U_n \leq Y_n$

(2) $\quad \lim\limits_{n\to+\infty} X_n = \lim\limits_{n\to+\infty} Y_n = \ell$

then

$$\lim_{n\to+\infty} U_n = \ell.$$

Limits and order

<u>Corrolary of *Squeeze Theorem*</u>

If $|u_n| \rightarrow 0$ and $(v_n)_{n\in\mathbb{N}}$ is bounded, then $u_n v_n \rightarrow 0$

Example

Example

$$U_n = \frac{\sin n}{n^2+5}.$$

On sait que:

$$-1 \le \sin n \le 1 \Rightarrow \frac{-1}{n^2+5} \le \frac{\sin n}{n^2+5} \le \frac{1}{n^2+5}$$

$$0 \le \ell \le 0 \Rightarrow \ell = \lim_{n\to+\infty} U_n = \lim_{n\to+\infty}\left(\frac{\sin n}{n^2+5}\right) = 0$$

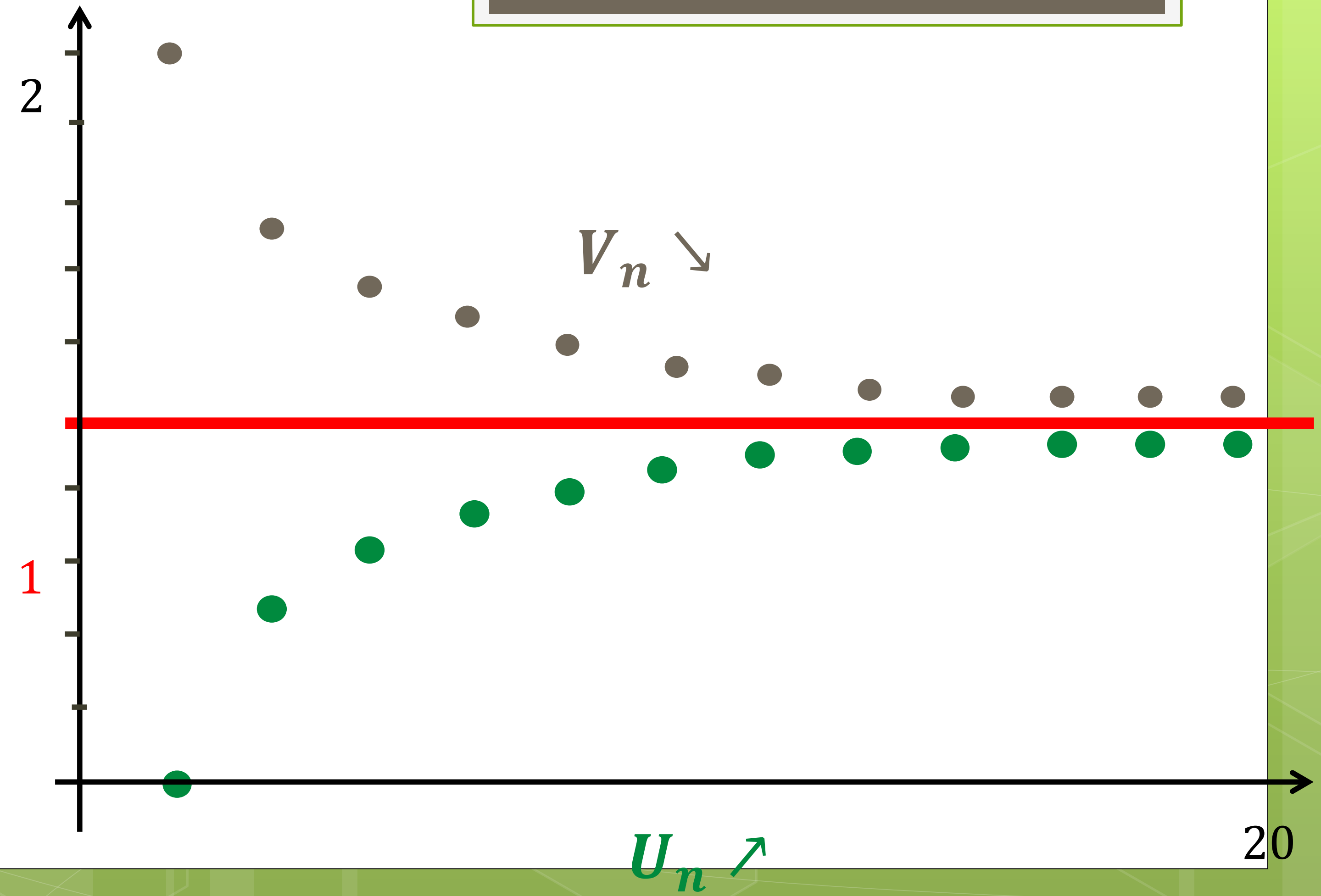
Adjacent sequences

Définition

Let $(U_n)_{n\in\mathbb{N}}$ *et* $(V_n)_{n\in\mathbb{N}}$ two real sequences.

We say that the two sequences are adjacent if the first is increasing and the second is decreasing and

$$\lim_{n\to+\infty}(U_n - V_n) = \lim_{n\to+\infty}(V_n - U_n) = 0.$$


2
1
$V_n \searrow$
$U_n \nearrow$
20

Adjacent sequences

Example 1

Set

$$U_n = 1 - \frac{1}{n}, V_n = 1 + \frac{1}{n}$$

then $(U_n)_{n\in\mathbb{N}}$ *et* $(V_n)_{n\in\mathbb{N}}$ are ajacent.

Adjacent sequences

Example 2

$$U_n = 1 + \sum_{k=1}^{n-1} \frac{1}{k^2 (k+1)^2}, V_n = U_n + \frac{1}{3 \cdot n^2}$$

$(U_n)_{n\in\mathbb{N}}$ et $(V_n)_{n\in\mathbb{N}}$ are ajacent.

Adjacent sequences

Example 3 Show that $(U_n)_{n\in\mathbb{N}}$ *et* $(V_n)_{n\in\mathbb{N}}$ are ajacent.

$$\begin{cases} U_0 = a > 0, V_0 = b > a > 0 \\ V_{n+1} = \dfrac{U_n + V_n}{2}; U_{n+1} = \dfrac{2}{\dfrac{1}{U_n} + \dfrac{1}{V_n}} \end{cases}$$

Adjacent sequences

3. Theorem

If $(U_n)_{n\in\mathbb{N}}$ et $(V_n)_{n\in\mathbb{N}}$ are two ajacent sequences such that $(U_n)_{n\in\mathbb{N}}$ is increasing and $(V_n)_{n\in\mathbb{N}}$ is decreasing then $\forall n\in\mathbb{N}, U_n \le V_n.$

Adjacent sequences

Theorem

If $(U_n)_{n\in\mathbb{N}}$ et $(V_n)_{n\in\mathbb{N}}$ are two ajacent sequences then they are convergent and converge to the same limit.