# Lecture 3 : Matrices

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# Definition

Throughout this lecture, $K$ will denote an arbitrary field (usually $\mathbb{R}$ or $\mathbb{C}$).

Definition 1

A matrix $A$ is an $\boldsymbol{m \times n}$ **array** of scalars from a given field $K$ of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The individual values $a_{ij} \in K$ in the matrix are called **entries**.

The **size** of the array is $\boldsymbol{m \times n}$, where $m$ is the number of rows and $n$ is the number of columns.

# Row matrix and column matrix

Definition 2

A **row matrix** of size $1 \times n$ is a matrix of the form $\begin{pmatrix} a_{11} & \cdots & a_{1n} \end{pmatrix}$

A **column matrix** of size $m \times 1$ is a matrix of the form $\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$

Example 1

$\begin{pmatrix} 1 & 0 & 3 \end{pmatrix}$ is a row matrix of size $1 \times 3$ ;

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a column matrix of size $2 \times 1$.

# Notation

Notation

A matrix $A$ of size $m \times n$ is denoted by $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ or $(a_{ij})$ when we know the number of rows and columns.

Example 2

$K = \mathbb{R}$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$$

is a matrix of size $2 \times 3$.

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

is a matrix of size $2 \times 2$.

# Square matrices

If $m = n$, the matrix is called **square**. In this case we have

a) A matrix $A$ is said to be **diagonal** if $a_{ij} = 0$ for $i \neq j$.

Example 3

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

# Special matrices

b) A **diagonal matrix** $A$ may be denoted by $diag(d_1, d_2, \dots, d_n)$, where $a_{ii} = d_i$ and $a_{ij} = 0$ for $i \neq j$.

The diagonal matrix $diag(1, 1, \dots, 1)$ is called the **identity matrix** and is denoted by $I_n$. We have then

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

$0 = diag(0, \dots, 0)$ is called the **zero matrix**.

# Triangular matrices

c) A square matrix $A$ is said to be **lower triangular** if $a_{ij} = 0$ for $i < j$.

Example 4

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 0 \end{pmatrix}$$

d) A square matrix A is said to be **upper triangular** if $a_{ij} = 0$ for $i > j$.

Example 5

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{pmatrix}$$

# Symmetric matrices

e) A square matrix A is **called symmetric** if $a_{ij} = a_{ji}$.

Example 6

$$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & -1 \\ 3 & -1 & 0 \end{pmatrix}.$$

f) $E_{ij}$ is the matrix with 1 in the $(i, j)$ position and zeros in all other positions.

Example 7

$$E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

# Notation

**Notation**

The set of all $m \times n$ matrices is denoted by $\mathcal{M}_{m,n}(K)$, where $K$ is the underlying field. In the case where $m = n$, we write $\mathcal{M}_n(K)$ to denote the matrices of size $n \times n$.

# Matrix operations

## Equality

Two matrices $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$, $B = (b_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ are **equal** if, and only if,

$a_{ij} = b_{ij}$ for all $i \in \{1,2, \cdots, m\}$ and $j \in \{1,2, \cdots, n\}$.

# Matrix operations

## Sum of matrices

Let $A = \left(a_{ij}\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}, B = \left(b_{ij}\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} \in \mathcal{M}_{m,n}(K)$. Then

$$A + B = \left(a_{ij} + b_{ij}\right)_{ij}$$

## Example 8

$$\begin{pmatrix} 1 & -1 & 0 \\ 3 & 1 & 2 \\ 0 & 4 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 3 \\ -1 & -2 & 5 \\ 3 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 \\ 2 & -1 & 7 \\ 3 & 4 & -3 \end{pmatrix}$$

# Matrix operations

## Multiplication by a scalar

Let $A = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}} \in \mathcal{M}_{m,n}(K)$ and $\alpha \in K$. Then

$$\alpha A = (\alpha a_{ij})_{ij}$$

## Example 9

$$(-2)\begin{pmatrix} 1 & -1 & 0 \\ 3 & 1 & 2 \\ 0 & 4 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 0 \\ -6 & -2 & -4 \\ 0 & -8 & 4 \end{pmatrix}$$

# Matrix operations

## Product of two matrices

Let $A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}} \in \mathcal{M}_{m,n}(K)$ and $B = (b_{ij})_{\substack{1\le i\le n \\ 1\le j\le r}} \in \mathcal{M}_{n,r}(K)$.

The product $AB$ is defined as follows :

$$AB = (c_{ij})_{\substack{1\le i\le m \\ 1\le j\le r}} \in \mathcal{M}_{m,r}(K),$$

where

$$c_{ij} = \sum_{k=1}^{k=n} a_{ik} b_{kj}.$$

# Product of matrices

Example 10

Set $A = \begin{pmatrix} 2 & 3 & 1 \\ -3 & 0 & -1 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R})$ and $B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix} \in \mathcal{M}_{3,2}(\mathbb{R})$

$$AB = \begin{pmatrix} 2 \times 1 + 3 \times 2 + 1 \times 1 & 2 \times -1 + 3 \times 1 + 1 \times 3 \\ -3 \times 1 + 0 \times 2 + (-1) \times 1 & -3 \times (-1) + 0 \times 1 + (-1) \times 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 9 & 4 \\ -4 & 0 \end{pmatrix}$$

# Product of matrices

Proposition 1

The product of matrices is associative.

# Product of matrices

The product of two matrices is **not commutative**.

Example 11

Set $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$; $B = \begin{pmatrix} -1 & 2 \\ 1 & 4 \end{pmatrix}$

$$AB = \begin{pmatrix} -2 & -2 \\ 1 & 16 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 3 & 7 \\ 9 & 11 \end{pmatrix}.$$

We have $AB \neq BA$.

# Vector space of matrices

Theorem 1

$\mathcal{M}_{m,n}(K)$ endowed with the addition and the multiplication by a scalar is a $K$- vector space.

Theorem 2

The vector space $\mathcal{M}_{m,n}(K)$ has dimension $mn$.

The matrices $E_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$, form a basis of $\mathcal{M}_{m,n}(K)$.

# Ring of square matrices

**Theorem 3**

$(\mathcal{M}_n(K), +, \cdot)$ is a non commutative ring.

**Remark 1**

$(\mathcal{M}_n(K), +, \cdot)$ has zero divisors.

**Example 12**

Set $A = \begin{pmatrix} 0 & -1 \\ 0 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix}$.

We have $A \times B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ with $A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $B \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

# Invertible Matrices

Definition 4

Let $A \in \mathcal{M}_n(K)$. The square matrix $A$ is said to be invertible if there exists a matrix $B \in \mathcal{M}_n(K)$ such that $AB = BA = I_n$.

In this case $B$ is called the inverse matrix of $A$.

If $A$ is invertible, $B$ is denoted by $A^{-1}$.

The set of invertibles matrices of size $n \times n$ is denoted by $GL(n, K)$.

# Invertible Matrices

### Example 13

The inverse of $A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ is

$$B = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}.$$

We have $AB = BA = I_2$.

# Invertible Matrices

**Proposition 2**

Let $A, B \in \mathcal{M}_n(K)$ two invertible matrices. Then $AB$ is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

**Proposition 3**

Let $A, B \in \mathcal{M}_n(K)$. If $AB = I_n$, then $BA = I_n$.

# Matrix power

Let $A$ be a matrix of size $n \times n$ and $m \in \mathbb{N}$.

We define

$A^m = A \cdot A \cdot \dots \cdot A$ ($m$ times),

and

$A^0 = I_n$.

If $A$ is invertible, we define

$A^{-m} = (A^{-1})^m = A^{-1} \cdot A^{-1} \cdot \dots \cdot A^{-1}$($m$ times).

# Matrix power

Theorem 4

Let $A$ be an invertible matrix. Then

1. $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$
2. $A^m$ is invertible and $(A^m)^{-1} = A^{-m}$.
3. $kA$ is invertible if $k \neq 0$ and $(kA)^{-1} = \frac{1}{k}A^{-1}$.

# Row Reduced Echelon Form (RREF)

Definition 9

We say that a matrix in Row Echelon Form (REF) is in Row Reduced Echelon Form (RREF) if it satisfies the following additional conditions :

1. All the pivots are 1.
2. Each pivot is the only nonzero entry in its column.

Theorem 5

The Row Reduced Echelon Form of a matrix is unique.

# Examples

## Example 14

The matrix

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is in RREF, while the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Is not in RREF.

# Computing the inverse of a matrix using RREF

Gauss method for inverting a matrix

Let $A$ be an invertible square matrix of size $n$. To determine the inverse of $A$, we use the Gauss method putting the matrix $A$ in the Row Reduced Echelon Form, which is $I_n$. We then do the same operations in the same order on $I_n$, and we end up with the inverse of A.

In practice, we do both operations at the same time, as we will see in the next example.

# Computing the inverse of a matrix using RREF

Example 15

$$R_2 := R_2 - 4R_1; R_3 := R_3 + R_1$$

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 4 & 0 & -1 & \vdots & 0 & 1 & 0 \\ -1 & 2 & 2 & \vdots & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & -8 & -5 & \vdots & -4 & 1 & 0 \\ 0 & 4 & 3 & \vdots & 1 & 0 & 1 \end{pmatrix}$$

# Computing the inverse of a matrix using RREF

$$R_2 := \frac{-1}{8} R_2$$

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & -8 & -5 & \vdots & -4 & 1 & 0 \\ 0 & 4 & 3 & \vdots & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & \vdots & \frac{1}{2} & \frac{-1}{8} & 0 \\ 0 & 4 & 3 & \vdots & 1 & 0 & 1 \end{pmatrix}$$

# Computing the inverse of a matrix using RREF

$$R_3 := R_3 - 4R_2$$

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & \vdots & \frac{1}{2} & \frac{-1}{8} & 0 \\ 0 & 4 & 3 & \vdots & 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & \vdots & \frac{1}{2} & \frac{-1}{8} & 0 \\ 0 & 0 & \frac{1}{2} & \vdots & -1 & \frac{1}{2} & 1 \end{pmatrix}$$

# Computing the inverse of a matrix using RREF

$$R_3 := 2R_3$$

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & \vdots & \frac{1}{2} & \frac{-1}{8} & 0 \\ 0 & 0 & \frac{1}{2} & \vdots & -1 & \frac{1}{2} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & \vdots & \frac{1}{2} & \frac{-1}{8} & 0 \\ 0 & 0 & 1 & \vdots & -2 & 1 & 2 \end{pmatrix}$$

# Computing the inverse of a matrix using RREF

$$R_2 := R_2 - \frac{5}{8} R_3 ; R_1 \coloneqq R_1 - R_3$$

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & \vdots & \frac{1}{2} & \frac{-1}{8} & 0 \\ 0 & 0 & 1 & \vdots & -2 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & \vdots & 3 & -1 & -2 \\ 0 & 1 & 0 & \vdots & \frac{7}{4} & \frac{-3}{4} & \frac{-5}{4} \\ 0 & 0 & 1 & \vdots & -2 & 1 & 2 \end{pmatrix}$$

# Computing the inverse of a matrix using RREF

$$R_1 \coloneqq R_1 - 2R_2$$

$$\begin{pmatrix} 1 & 2 & 0 & \vdots & 3 & -1 & -2 \\ 0 & 1 & 0 & \vdots & \frac{7}{4} & \frac{-3}{4} & \frac{-5}{4} \\ 0 & 0 & 1 & \vdots & -2 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \vdots & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \vdots & \frac{7}{4} & \frac{-3}{4} & \frac{-5}{4} \\ 0 & 0 & 1 & \vdots & -2 & 1 & 2 \end{pmatrix}.$$

Then the inverse of $A$ is given by

$$A^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 2 & 2 \\ 7 & -3 & -5 \\ -8 & 4 & 8 \end{pmatrix}.$$

# Matrix transpose

Let $A = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}} \in \mathcal{M}_{m,n}(K)$.

**Definition 5**

The **transpose** of $A$, denoted by $A^T$, is a matrix of size $n \times m$ defined by

$$A^T = (a'_{ij})_{\substack{1\le i\le n\\ 1\le j\le m}},\ a'_{ij} = a_{ji}$$

**Example 16**

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 0 & -2 \end{pmatrix}; \qquad A^T = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \end{pmatrix}$$

# Matrix transpose

Proposition 4

Let $A, B \in \mathcal{M}_{m,n}(K)$.

1) $(A+B)^T = A^T + B^T$.
2) $(kA)^T = k(A)^T$.
3) $(A^T)^T = A$.
4) If $A \in \mathcal{M}_{m,n}(K)$ and $B \in \mathcal{M}_{n,r}(K)$, then $(AB)^T = (B)^T(A)^T$
5) If $m = n$ and $A$ invertible, then $(A^T)^{-1} = (A^{-1})^T$.

# Matrix Trace

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

be a square matrix of size $n \times n$.

Definition 6

The **Trace** of $A$ denoted by $Tr(A)$ is defined as follows :

$$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

# Matrix Trace

## Example 17

Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 1 & 2 \\ 0 & 4 & -2 \end{pmatrix},$$

then

$$Tr(A) = 1 + 1 + (-2) = 0$$

# Matrix Trace Properties

Proposition 5

Let $A, B \in \mathcal{M}_n(K)$.

*1)* $Tr(A+B) = Tr(A) + Tr(B)$.

*2)* $Tr(\alpha A) = \alpha Tr(A), \forall \alpha \in K$.

*3)* $Tr(A^T) = Tr(A)$.

*4)* $Tr(AB) = Tr(BA)$.

# Matrix of a linear map

Let $V$ and $W$ be finite dimensional $K$- vector spaces and let $f : V \longrightarrow W$ be a linear map.

Let $B = (v_1, v_2, \cdots, v_n)$ a basis of $V$ and $B' = (w_1, w_2, \cdots, w_m)$ a basis of $W$.

Each vector $f(v_i)$, $i \in \{1,2,\cdots,n\}$, can be written with respect to the basis $(w_1, w_2, \cdots, w_m)$ of $W$ as follows :

$$f(v_1)=a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m$$
$$f(v_2)=a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m$$
$$\vdots$$
$$f(v_n) = a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m$$

# Matrix of a linear map

**Definition 7**

The matrix

$$\mathcal{M}(f, B, B') = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

is called the matrix representing the linear map $f$ with respect to the bases $B$ and $B$’ of $V$ and $W$ respectively.

# Matrix of a linear map

## Example 18

Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear map defined by

$$f(x,y) = (x-y, 2x+y, x+3y).$$

We have

$$f(1,0) = (1,2,1) = e'_1 + 2e'_2 + e'_3;$$

$$f(0,1) = (-1,1,3) = -e'_1 + e'_2 + 3e'_3.$$

The matrix representing $f$ with respect to the standard bases $B = (e_1, e_2)$ and $B' = (e'_1, e'_2, e'_3)$ of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively is :

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix} = \mathcal{M}(f, B, B')$$

# Matrix of the identity map

Remark 2

Let $V$ be a $K$-vector space of dimension $n$ and $Id_V : V \longrightarrow V$ the identity map. Then the matrix representing $Id_V$ with respect to a basis $B$ of $V$ is the identity matrix $I_n$.

# Composed linear maps matrix

Proposition 6

Let $U$, $V$ and $W$ be finite-dimensional vector spaces over $K$ with bases $B_1 = (u_1, \cdots, u_n)$ , $B_2 = (v_1, \cdots, v_m)$ and $B_3 = (w_1, \cdots, w_r)$ respectively. Let $f: U \to V$ and $g : V \to W$ be two linear maps, $A$ the matrix representing $f$ with respect to $B_1$ and $B_2$ , and $B$ the matrix representing $g$ with respect to $B_2$ and $B_3$. Then $BA$ is the matrix representing the linear map $g \circ f: U \to W$ with respect to $B_1$ and $B_3$.

# Composed linear maps matrix

## Example 19

Let $f\colon\ \mathbb{R}^2 \longrightarrow \mathbb{R}^3, g\colon\ \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear maps defined by

$f(x, y) = (x + 2y, x - y, x)$ and $g(x, y, z) = (x + y + z, x - y - z)$.

Let $A, B$, the matrices representing $f$ and $g$ with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$. Then

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

The matrix representing $g \circ f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is $BA = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$

# The isomorphism between vector space of matrices and vector space of linear maps

**Theorem 6**

Let $V$ and $W$ be finite-dimensional vector spaces over $K$ with (ordered) bases $B = (v_1, v_2, \cdots, v_n)$ and $B' = \{w_1, w_2, \cdots, w_m\}$ respectively. Then, there exists an isomorphism $\varphi$ from the $K$-vector space $\mathcal{L}(V, W)$ of linear maps to $\mathcal{M}_{m,n}(K)$ that sends a linear map $f: V \longrightarrow W$ to the matrix representing $f$ with respect to $B$ and $B'$.

# Example

## Example 20

Let $A = \begin{pmatrix} 2 & 3 & 1 \\ -3 & 0 & -1 \end{pmatrix}$ the matrix representing the linear map $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ with respect to the standard bases $B = \{e_1, e_2, e_3\}$ and $B' = \{e'_1, e'_2\}$ of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Then

$$f(x, y, z) = A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (2x + 3y + z, -3x - z)$$

# Some properties

Theorem 7

A matrix $A \in \mathcal{M}_n(K)$ is invertible if, and only if, the linear map representing $A$ is bijective.

Proposition 7

Let $A \in \mathcal{M}_n(K)$. The matrix $A$ is invertible if, and only if, the column vectors of $A$ are linearly independent.

# Matrix rank

Definition 8

The rank of a matrix $A \in \mathcal{M}_{p,q}(K)$ is the rank of the linear map representing $A$. It is denoted by $r(A)$.

Theorem 8

For any matrix $A$, we have $r(A) = r(A^T)$.

Proposition 8

Let $A \in \mathcal{M}_n(K)$. The matrix $A$ is invertible if, and only if, the rank of $A$ equals to $n$.

# Change of coordinates matrix

Let $V$ be a $K$-vector space of dimension $n$. Let $B = (v_1, v_2, \cdots, v_n)$ and $B' = (v'_1, v'_2, \cdots, v'_n)$ be two ordered bases of $V$ such that

$$\begin{gathered} v'_1 = a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n \\ v'_2 = a_{12}v_1 + a_{22}v_2 + \cdots + a_{n2}v_n \\ \vdots \\ v'_n = a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{nn}v_n. \end{gathered}$$

# Change of coordinates matrix

**Definition 10**

The matrix

$$P_{B'\longrightarrow B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

changing $B'$ coordinates into $B$ coordinates is called the change of coordinates matrix.

**Proposition 9**

The matrix $P$ is invertible.

# Change of basis

Theorem 9

Let $V$ and $W$ be $K$-vector spaces of dimension $n$. Let $f: V \longrightarrow W$ be a linear map. Let $B = (v_1, v_2, \cdots, v_n), B' = (v'_1, v'_2, \cdots, v'_n)$ be two ordered bases of $V$ and $R = (w_1, w_2, \cdots, w_m)$, $R' = (w'_1, w'_2, \cdots, w'_m)$ be two ordered bases of $W$. Denote by $P$ the matrix changing $B$ coordinates into $B'$ coordinates and by $Q$ the matrix changing $R$ coordinates into $R'$ coordinates. Then

$$\mathcal{M}(f, B', R') = Q^{-1}\mathcal{M}(f, B, R)P.$$

# Determinant

Definition 11

Let $A = \left(a_{ij}\right)_{1 \leq i,j \leq n}$ be a square matrix. Denote by $A_{i,j}$ the **submatrix** of $A$ of size $n-1$ obtained by deleting row $i$ and column $j$. Then the **determinant** of $A$, denoted by $\det(A)$ or $|A|$, is the scalar defined recursively as :

- If $n = 1, \det(A) = a_{11}$.
- For all $n \geq 2, \det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det\left(A_{1,j}\right)$.

Example 21

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

# Minor and Cofactor

Definition 12

The scalar

$$\det\left(A_{i,j}\right)$$

is called a **minor**.

The scalar

$$(-1)^{i+j}\det\left(A_{i,j}\right)$$

is called a **cofactor**.

# Laplace expansion

Theorem 10

Expansion along row $i$ :

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det\left(A_{i,j}\right).$$

Expansion along column $j$ :

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det\left(A_{i,j}\right).$$

# Example

Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 1 & 2 \\ 0 & 4 & -2 \end{pmatrix}.$$

Computing along the first row :

$$\det(A) = 1.\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} + 0.\begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} = -10 - 6 = -16.$$

Computing along the second column :

$$\det(A) = (-1)(-1)\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} + (-1).4\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}$$
$$= -6 - 2 - 8 = -16.$$

# Numerical note

By today's standards, a square matrix of size $25$ is small. Yet it would be impossible to calculate its determinant by cofactor expansion. In general, a cofactor expansion requires more than $n!$ multiplications, and $25!$ is approximately $1{,}5 \times 10^{25}$.

If a computer performs one trillion multiplications per second, it would have to run for more than $500\,000$ years to compute a determinant of size $25$ by this method. Fortunately, there are faster methods, as we'll soon discover.

# Properties of determinants

Theorem 11 (Row operations)

Let $A$ be a square matrix

a. If a multiple of one row of $A$ is added to another row to produce a matrix $B$, then $\det(B) = \det(A)$.

b. If two rows of $A$ are interchanged to produce $B$, then $\det(B) = -\det(A)$.

c. If one row of $A$ is multiplied by $k$ to produce $B$, then $\det(B) = k\det(A)$.

# Determinant of REF Matrix

**Proposition 10**

If $A$ is a triangular matrix, the $\det(A)$ is the product of the entries of the diagonal of $A$.

**Corollary 1**

Let $U$ be the REF of a square matrix $A$, and suppose that $r$ interchanges have been performed. Then we have

$$\det(A) = \begin{cases} (-1)^r \times \text{product of pivots in } U, & \text{when } A \text{ is invertible} \\ 0 & \text{, when } A \text{ is not invertible.} \end{cases}$$

# Numerical note

**1.** Most computer programs that compute $\det(A)$ for a general matrix $A$ use the method of corollary above.

**2.** It can be shown that evaluation of a determinant of size $n$ using row operations requires about $2n^3/3$ arithmetic operations. Any modern microcomputer can calculate a determinant of size $25$ in a fraction of a second, since only about $10\,000$ operations are required.

# Example

## Example 23

Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 3 & -3 & 2 \\ 2 & 4 & -2 \end{pmatrix}.$$

Then

$$\det(A) = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 6 & -4 \end{vmatrix} = - \begin{vmatrix} 1 & -1 & 1 \\ 0 & 6 & -4 \\ 0 & 0 & -1 \end{vmatrix} = -\big(1 \times 6 \times (-1)\big) = 6.$$

# Determinant and Invertibility

Theorem 12

A square matrix $A$ is invertible if and only if $\det(A) \neq 0$.

# Determinant and transpose

**Theorem 13**

If $A$ is a square matrix, then

$$\det(A^T) = \det(A).$$

# Determinants and matrix products

Theorem 14

If $A$ and $B$ are square matrices of size $n$, then

$$\det(AB) = \det(A)\det(B).$$

Corollary 2

If $A$ is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

# An inverse formula

**Definition 13**

Let $A$ be an $n \times n$ matrix. The **adjugate** matrix of $A$, denoted by $\mathrm{adj}(A)$, is the transpose of the matrix of cofactors of $A$ ; that is,

$$\mathrm{adj}(A) = \left( \left( (-1)^{i+j} \det\left(A_{i,j}\right) \right)_{1 \le i,j \le n} \right)^T$$

**Theorem 15**

Let $A$ be an invertible matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(\mathrm{A}).$$

# Example

Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 1 & 2 \\ 0 & 4 & -2 \end{pmatrix}.$$

We have seen that $\det(A) = -16$. The cofactors

$$C_{ij} = (-1)^{i+j} \det\big(A_{i,j}\big)$$

of $A$ are

$$C_{11} = \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} = -10, C_{12} = -\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} = 6, C_{13} = \begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} = 12,$$

$$C_{21} = -\begin{vmatrix} -1 & 0 \\ 4 & -2 \end{vmatrix} = -2, C_{22} = \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2, C_{23} = -\begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} = -4,$$

# Example

$$C_{31} = \begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} = -2, C_{32} = -\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = -2, C_{33} = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 4.$$

Then the inverse of $A$ is

$$A^{-1} = \frac{-1}{16}\begin{pmatrix} -10 & -2 & -2 \\ 6 & -2 & -2 \\ 12 & -4 & 4 \end{pmatrix}.$$