Lecture 5 : Eigenvalues and Eigenvectors

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Eigenvalues and Eigenvectors

Throughout this lecture, K will denote $\mathbb{R}$ or $\mathbb{C}$, and $V = K^n$.

Definition 1

Let $A \in \mathcal{M}_n(K)$. If there exist $\lambda \in K$ and a nonzero column vector $x \in K^n$ satisfying

$$Ax = \lambda x,$$

we say that λ is an eigenvalue of A and x is an eigenvector of A corresponding to λ.

Example

Example 1

Consider the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}.$$

We can easily verify that

$$\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, 4 is an eigenvalue of A, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to 4.

Characteristic Polynomial

Definition 2

The characteristic polynomial of the matrix A is defined by

$$P_A(t) = \det(A - tI)\,.$$

Exercise 1

Show that the characteristic polynomial is of degree n and its leading coefficient is $(-1)^n$.

The characteristic polynomial allows us to find all the eigenvalues of a given matrix as the following result shows.

Proposition 1

A scalar $\lambda \in K$ is an eigenvalue of A if and only if $P_A(\lambda) = 0$.

Example

Example 2

Consider again the matrix of example 1, we have

$$P_A(t) = \det(A - tI) = \begin{vmatrix} 2-t & 2 \\ 5 & -1-t \end{vmatrix} = t^2 - t - 12.$$

Then, the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 4$.

Similar Matrices

Definition 3

Let $A, B \in \mathcal{M}_n(K)$. We say that A is similar to B if there exists an invertible matrix $P \in \mathcal{M}_n(K)$ such that $B = P^{-1}AP$.

Obviously, the relation « similar to » is an equivalence relation, so we can use the term « A and B are similar ».

Proposition 2

If $A, B \in \mathcal{M}_n(K)$ are similar, the they have the same characteristic polynomial and then the same eigenvalues with the same multiplicities.

Properties of Eigenvalues

Proposition 3

Assume that the matrix $A \in \mathcal{M}_n(K)$ has n (non necessarily distinct) eigenvalues $\lambda_1, \cdots, \lambda_n$, then we have

1) the sum of the eigenvalues is the trace of A : $\lambda_1 + \cdots + \lambda_n = Tr(A)$;

2) the product of the eigenvalues is the determinant of A : $\lambda_1 \cdots \lambda_n = \det(A)$;

3) if A is triangular, then its eigenvalues are the entries of the diagonal ;

4) if λ is an eigenvalue of an invertible matrix A, then λ^{-1} is an eigenvalue of A^{-1};

5) if λ is an eigenvalue of a matrix A, and m a positive integer, then λ^m is an eigenvalue of A^m.

Linear Independence of Eigenvectors

Proposition 4

If $v_1, \cdots, v_k$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_k$ of $A \in \mathcal{M}_n(K)$, then the set $\{v_1, \cdots, v_k\}$ is linearly independent.

Eigenspace

Proposition 5

Let λ be an eigenvalue of A. Then the set

$$V_\lambda = \{v \in V : Av = \lambda v\}$$

is a subspace of V.

Definition 4

The subspace V_λ is called eigenspace corresponding to λ.

Example

Example 3

Consider again the matrix of example 1, then

$$V_4 = \{v \in K^2 : Av = 4v\}.$$

Set $v = \binom{x}{y}$, then we have

$$Av = 4v \Leftrightarrow \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 2x + 2y = 4x \\ 5x - y = 4y \end{cases} \Leftrightarrow \begin{cases} -2x + 2y = 0 \\ 5x - 5y = 0 \end{cases} \Leftrightarrow x = y.$$

Therefore V_4 is the subspace spanned by $\binom{1}{1}$, and $\dim(V_4) = 1$.

Diagonalization

Definition 5

A square matrix $A \in \mathcal{M}_n(K)$ is said to be diagonalizable if it is similar to a diagonal matrix, that is, there exists an invertible matrix $P \in \mathcal{M}_n(K)$ such that $P^{-1}AP$ is a diagonal matrix.

Remark 1

Let f be the endomorphism of V represented by the matrix A. Then « A is diagonalizable » means that there exists a basis B of V consisting of eigenvectors such that $\mathcal{M}(f, B)$ is diagonal. Moreover, we can take P as matrix whose columns are eigenvectors $v_1, \cdots, v_n$ corresponding respectively to eigenvalues $\lambda_1, \cdots, \lambda_n$ and we have

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \cdots, \lambda_n).$$

Diagonalization : A Sufficient Condition

Theorem 1

If a matrix $A \in \mathcal{M}_n(K)$ has n distinct eigenvalues, then it is diagonalizable.

Diagonalization : A Necessary and Sufficient Condition

Theorem 2

Let $\lambda_1, \cdots, \lambda_k$ the distinct eigenvalues of a matrix $A \in \mathcal{M}_n(K)$, and denote by $m_1, \cdots, m_k$ their respectives multiplicities as roots of the characteristic polynomial of A. Then A is diagonalizable if and only if

$$\dim\left(V_{\lambda_i}\right) = m_i$$

for all $i \in \{1, \cdots, k\}$.

Examples

Example 4

We have seen that the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$$

has two distinct eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 4$. Then, by Theorem 1, A is diagonalizable. We have also seen that $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to λ_2, and we can easily find that $v_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ is an eigenvector corresponding to λ_1. Thus, we can take

$$P = \begin{pmatrix} -2 & 1 \\ 5 & 1 \end{pmatrix},$$

and we obtain

$$P^{-1}AP = \frac{-1}{7}\begin{pmatrix} 1 & -1 \\ -5 & -2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}.$$