Real Number system

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Axiomatic definition of real numbers

Definition

The set of real numbers , denoted $\mathbb{R}$, is a commutative field endowed with two operations : $(x, y) \mapsto x + y$ (addition), $(x, y) \mapsto xy$ (multiplication)

and an order relation denoted $\leq$ satisfying the following axioms :

1) $\mathbb{Q} \subset \mathbb{R}$
2) $(\mathbb{R}, \leq)$ is totally ordered
3) If $x \leq \mathrm{y}$ then $x + z \leq y + z, \forall z \in \mathbb{R}$
4) If $0 \leq \mathrm{x}$ and $0 \leq \mathrm{y}$ then $0 \leq \mathrm{xy}$
5) Supremum's axiom : Any nonempty upper bounded subset of $\mathbb{R}$ admits a supremum.

Axiomatic definition of real numbers

Remark: the conditions 3) and 4) mean that there is a compatibility between the order relation and the operations defined on $\mathbb{R}$.

Notations :

- The relation $x \leq \mathrm{y}$ and $x \neq y$ is denoted $x < y$.
- We write $x < y$ or $y > x$.
- A real number is called positive if $\mathrm{x} > 0$ and negative if $x < 0$.
- A real number is called nonnegative if $\mathrm{x} \geq 0$ and nonpositive if $x \leq 0$.
- $x - y = x + (-y)$
- $\frac{a}{b} = ab^{-1}$

Absolute value

Definition :

The absolute value of a real number x, denoted $|x|$, is defined as follows :

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x \leq 0 \end{cases}$$

We have

1. $-|x| \leq x \leq |x|, \ \forall x \in \mathbb{R}$
2. $|x| = |-x|, \ \forall x \in \mathbb{R}$

Some fundamental properties

1) Inequalities :

- $x + z \leq y + z \Longrightarrow x \ \leq \mathrm{y}$

In particular $x + z = y + z \Longrightarrow x \ = \mathrm{y}$

- $x \leq \mathrm{y} \Leftrightarrow -x \geq -\mathrm{y} \Leftrightarrow y - x \geq 0$
- $|x| \leq a \Leftrightarrow -a \leq x \leq a$
- $\forall x, y :\ |x + y| \leq |x|+|y|$
- $\forall x, y : \big||x|-|y|\big| \leq |x - y|$
- Let $x < y$. Then

$$xz < yz \ \text{ if } \mathrm{z} > 0$$

$$xz > \mathrm{yz} \ \text{ if } \mathrm{z} < 0$$

- $\forall x, y : |xy| = |x||y|$
- For all $a \geq 0$, then $a^2 \geq 0$.

Upper bound and lower bound

Let E be a noempty subset of $\mathbb{R}$.

Definition

We say that E is upper bounded if there exists M such that

$$\forall x \in E : x \leq \mathrm{M}$$

M is called an upper bound of E.

Definition

We say that E is lower bounded if there exists m such that

$$\forall x \in E : m \leq \mathrm{x}$$

m is called a lower bound of E.

Definition

E is said to be bounded if E is upper bounded and lower bounded.

Supremum and Infimum

Let E be a noempty subset of $\mathbb{R}$.

Definition (Supremum and infimum)

The least upper bound of E is called **supremum** of E. When it exists, it is unique and we denote it by $\sup E$.

The greatest lower bound of E is called **infimum** of E. When it exists, it is unique and we denote it by $\inf E$.

Supremum and infimum

Proposition (Caracterization of the supremum).

Let E be an upper bounded subset of $\mathbb{R}$.

$SupE = M \Leftrightarrow$

1) M is an upper bound of E
2) $\forall\varepsilon > 0, \exists x \in E$ such that $x + \varepsilon > M$

Supremum and infimum

Theorem The least upper bound theorem

Every set of real number $E \subset \mathbb{R}$ that is bounded above has a least upper bound, sup E.

Proof

Theorem The greatest lower bound theorem

Every set of real number $E \subset \mathbb{R}$ that is bounded below has a greatest lower bound, inf E.

Proof

Supremum and infimum

Proposition (Caracterisation of the infimum).

Let E be a lower bounded subset of $\mathbb{R}$.

$inf E = m \Leftrightarrow$

1) m is a lower bound of E
2) $\forall \varepsilon > 0, \exists x \in E$ such that $x - \varepsilon < m$

Properties of supremum and infimum

Proposition

Let A and B be two bounded subsets of $\mathbb{R}$. Suppose $A \cap B \neq \emptyset$. Denote by

$-A = \{-x, x \in A\}$; $A + B = \{x + y, x \in A \; et \; x \in B\}$.

Then $A \cup B$, $A \cap B$, $-A$ and $A + B$ are bounded and we have

1) $A \subset B \quad \Longrightarrow \quad supA \leq supB$ and $infA \geq infB$
2) $sup(A \cup B) = max(supA, supB)$ and $inf(A \cup B) = min(infA, infB)$
3) $sup(A \cap B) \leq min(supA, supB)$ and $inf(A \cap B) \geq max(infA, infB)$
4) $sup(-A) = -infA$ and $inf(-A) = -supA$
5) $sup(A + B) = supA + supB$ and $inf(A + B) = infA + infB$

Archimedean property

Archimedean property :

$\forall x \in \mathbb{R}, \exists\, n \in \mathbb{N} - \{0\}$, such that $n > x$.

$\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}_+ - \{0\}, \exists\, n \in \mathbb{N} - \{0\}$, such that $ny > x$.

In other words, the set $\mathbb{N}$ is not upper bounded. In this case, we say that $\mathbb{R}$ is Archimedean.

Corollary $\forall \varepsilon > 0, \exists\, n \in \mathbb{N} - \{0\}$, such that $0 < \frac{1}{n} < \varepsilon$

Remark *This property allows us to define the integer part of a real number.*

Proposition

Let $x \in \mathbb{R}$. Then there exist a unique integer denoted by $\lfloor x \rfloor$ suth that

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

The integer $\lfloor x \rfloor$ is called integer part of a real number x.

Integer part of a real number

Properties

1) $\forall m \in \mathbb{Z},\ \lfloor m \rfloor = m$
2) $\forall x \in \mathbb{R}, \forall m \in \mathbb{Z},\ \lfloor x + m \rfloor = \lfloor x \rfloor + m$
3) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$
4) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x \leq y \Longrightarrow \lfloor x \rfloor \leq \lfloor y \rfloor$

Definition

The function denoted $\{x\}$ defined on $\mathbb{R}$ by $\{x\} = x - \lfloor x \rfloor$ is called the fractional part of x.

Examples

Examples :

Let

$E = \left\{\frac{1}{n}, n \in \mathbb{N}^*\right\}$,

$\mathrm{F} = \left\{\frac{n+2}{n-1}, n \in \mathbb{N} \quad , n \geq 2\right\}$,

$\mathrm{G} = \left\{\frac{pq}{p^2+q^2}, (p,q) \in \mathbb{N}^* \times \mathbb{N}^*\right\}$.

Determine $\inf E$, $\sup E$, $\max E$, $\min E$.

Intervals

In mathematics, a (real) interval is a set that contains all real numbers lying between any two numbers, more precisely

Definition *Let a and b be two real numbers such that* $b > a$.

The set $\{x : a < x < b\}$ *is called open interval and it is denoted by* $]a, b[$.

The set $[a, b] = \{x : a \leq x \leq b\}$ *is called closed interval (compact interval).*

The sets $[a, b[= \{x : a \leq x < b\}$, $]a, b] = \{x : a < x \leq b\}$, *are called (respectively right and left) half-open intervals.*

For all intervals, the points a and b are called endpoints. If $a = b$, we set by definition $[a, a] = \{a\}$ (*degenerate closed interval*) and $]a, a[= \emptyset$.

Intervals

Definition

The length of the interval (a, b) (closed, open, or half-open) is given by the real number $b\ -\ a$.

Definition

The midpoint of an inteval $[a, b]$ (or $]a, b[$) is a point $c \in [a, b]$ (or $]a, b[$) such that $c = \frac{a+b}{2}$.

Intervals

Definition

The set $\{x : x \leq a\}$ is a left unbounded closed interval, noted $]-\infty, a]$.

The set $\{x : x < a\}$ is a left unbounded open interval, noted $]-\infty, a[$.

The set $\{x : x \geq a\}$ is a right unbounded closed interval, noted $[a,+\infty[$.

The set $\{x : x > a\}$ is a right unbounded open interval, noted $]a,+\infty[$.

The set $\mathbb{R}$ is also denoted $]-\infty,+\infty[$. $-\infty$ and $+\infty$ represent infinity numbers.

Intervals

Infimum and supremum of intervals

From the definition of the supremum and infimum, we deduce that an interval of the form (a, b) is bounded and admits a supremum and an infimum. We have

- $inf\,]a, b[= a$ and $sup\,]a, b[= b$
- $inf\,[a, b] = min[a, b] = a$ and $sup[a, b] = max[a, b] = b$
- $inf\,]a, b] = a$ and $sup\,]a, b] = max\,]a, b] = b$
- $inf\,[a, b[= min[a, b[= a$ and $sup[a, b[= b$

Intervals

- An interval of the form $(a, +\infty[$ is lower bounded but not upper bounded, it admits an infimum but not a supremum

$$\inf(]a, +\infty[\,) = a$$

$$\inf([a, +\infty[) \; = \min([a, +\infty, [) = a$$

- An interval of the form $]-\infty, b)$ is upper bounded but not lower bounded, it admits a supremum but not an infimum

$$\text{sup}(]-\infty, b[) = b$$

$$\text{sup}(]-\infty, b]) = \max(]-\infty, b]) = b$$

Density of $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ in $\mathbb{R}$

Theorem

For every $x, y \in \mathbb{R}$ such that $x < y$, there exists a rational r such that

$x < r < y$.

We say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Theorem

For every $x, y \in \mathbb{R}$ such that $x < y$, there exists a rational number ir such that

$x < ir < y$.

We say that $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$.

Extended real number line

Definition

The extended real number line is obtained from the real number line $\mathbb{R}$ by

adding two infinity elements $+\infty$ and $-\infty$, endowed by the totally order relation extended from that of $\mathbb{R}$ to $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,+\infty\}$, where $\overline{\mathbb{R}}$ denotes the extended real number line.

Operations on $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,+\infty\}$ are defined by

$$x + (+\infty) = +\infty + x = +\infty, \forall x \in \mathbb{R}$$

$$x + (-\infty) = -\infty + x = -\infty, \forall x \in \mathbb{R}$$

Extended real number line

- $x\,(\pm\infty) \; = (\pm\infty)\,x \; = \begin{cases} \pm\infty \; if \; x > 0 \\ \mp\infty \; if \; x < 0 \end{cases}$
- $(+\,\infty) + (+\,\infty) = \; +\,\infty,$
- $(-\,\infty) + (-\,\infty) = \; -\,\infty,$
- $(\pm\,\infty)\;(\pm\,\infty) = \; +\,\infty,$
- $(\pm\,\infty)\;(\mp\infty) = \; -\,\infty.$
- As the sum $(+\,\infty) + (-\,\infty)$ and the product $0\;(\pm\,\infty)$ are not well defined, so $\overline{\mathbb{R}}$ does not have any algebraic structure.

Topology of the line $\mathbb{R}$

Definition

A subset A of $\mathbb{R}$ is said to be open if it is empty or if for every $x \in A$ there exists an open interval containing x and contained in A.

Definition

An interval centered at a is of the form $]a-h, a+h[,\ h > 0.$

Definition (*neighbourhood)*

Let $a \in \mathbb{R}$. Any subset of $\mathbb{R}$ containing an interval centered at a is called a *neighbourhood of a.*

Example *The interval $]-\varepsilon, +\varepsilon[$ $(\varepsilon > 0)$ is a neighbourhood of 0. The interval $]-\frac{1}{n}, +\frac{1}{n}[$, $(n > 0)$ is a neighbourhood of* 0.

Topology of the line $\mathbb{R}$

Definition (Adherent point)

Let A be a subset of $\mathbb{R}$. A point a of $\mathbb{R}$ is said to be an adherent point to A if

$\forall h > 0, A \cap]a-h, a+h[\neq \emptyset$. We write $a \in \underline{A}$.

All elements of A are adherent to A.

Example

$A =]0,1[$

$a = 0 \in \underline{A}$ since $\forall h > 0, A \cap]-h, h[\neq \emptyset$

$a = 1 \in \underline{A}$ since $\forall h > 0, A \cap]1-h, 1+h[\neq \emptyset$

Topology of the line $\mathbb{R}$

Definition (Accumulation point)

Let A be a subset of $\mathbb{R}$. A point a of $\mathbb{R}$ is said to be an accumulation point of A if

$\forall h > 0, A \cap (]a-h, a+h[\setminus \{a\}) \neq \emptyset.$

Example

$A =]0,1[$

$a = 0$ is an accumulation point of A since $\forall h > 0, A \cap]-h, h[\neq \emptyset$

$a = 1$ is an accumulation point of A since $\forall h > 0, A \cap]1-h, 1+h[\neq \emptyset$