Lecture 4 : Systems of Linear Equations

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Contents

Linear Equation

Throughout this lecture, K will denote an arbitrary infinite field (usually $\mathbb{R}$ or $\mathbb{C}$).

Definition 1

A linear equation in variables $x_1, x_2, \cdots, x_m$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_mx_m = b,$$

where $a_1, a_2, \cdots, a_m$ and b are elements of K. The constant a_i is called the coefficient of x_i and b is called the constant term of the equation.

Example 1

$x + 2y - 5z = 3$ is a linear equation.

$x - yz = 0$ and $2x + y^2 = 1$ are not linear equations.

Systems of Linear Equations

Definition 2

A system of linear equations (or linear system) is a collection of linear equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = b_2 \\ \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = b_n \end{cases} \qquad (S),$$

where the $a_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$, are elements of K. More precisely, we say that (S) is a linear system of *n* equations in *m* variables (or unknowns).

A solution of a linear system (S) is a tuple $(\alpha_1, \alpha_2, \cdots, \alpha_m) \in K^m$ that makes each equation a true statement when the values $\alpha_1, \alpha_2, \cdots, \alpha_m$ are substituted for $x_1, x_2, \cdots, x_m$. The set of all solutions of a linear system is called the solution set of the system.

A linear system is said to be consistent if it has at least one solution, and is said to be inconsistent if it has no solution.

Equivalent Systems

Definition 3

Two linear systems (S) and (S') in same variables are said to be equivalent if their solution sets are the same.

So, to solve a system (S), we will transform it into a simpler equivalent system (S').

Matrix Form of Linear System

The linear system (S) can be written in the matrix form

$$AX = B,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Definition 4

The matrix A is called coefficient matrix of (S).

Linear Map and Linear System

Let $V = K^m$ and $W = K^n$, and let B and B' two bases of V and W respectively. Suppose that $x_1, x_2, \cdots, x_m$ are the coordinates of the unknown vector $v \in V$ in the basis B and $b_1, b_2, \cdots, b_n$ the coordinates of the given vector $w \in W$ in the basis B'. Let $f: V \to W$ be the linear map represented by the matrix A with respect to the bases B and B'. Then the linear system (S) can be written as

$$f(v) = w.$$

So, solving the system (S) is equivalent to finding the preimage of w under f. In particular, when $w = 0$, the solution set of (S) is the kernel of f.

Rank of a Linear System

Definition 5

The rank of the system (S) is the rank of the matrix A, which is the rank of the linear map f. It is denoted by $r(S) = r$.

We know that for any matrix M, we have $r(M) = r(M^T)$. This gives the following result.

Proposition 1

For the linear system (S), we have

$$r \leq m \text{ and } r \leq n.$$

Homogeneous System

Definition 6

We say that a system (S) is homogeneous when $B = 0$. Otherwise, the system is said to be nonhomogeneous.

Theorem 1

The solution set of a homogeneous system is a subspace of K^m of dimension $m - r$.

Proof

Indeed, the solution set is the kernel of the linear map representing the matrix A. Its dimension is deduced from the rank theorem.

Structure of Solution Set

Definition 7

A linear system $AX = B$ is called nonhomogeneous if $B \neq 0$. The homogeneous linear system $AX = 0$ is called its corresponding homogeneous linear system.

Theorem 2

Suppose that one solution v_0 of the linear system (S): $AX = B$ is known. Then, the solution set of (S) is given by :

$$\mathcal{S} = \{v_0 + v : v \in \ker(f)\}.$$

Augmented Matrix of Linear System

Definition 8

The augmented matrix of a linear system $AX = B$ is given by

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & b_n \end{array}\right).$$

Solving a Linear System

Theorem 3

Let A' be the RREF of the augmented matrix of the linear system (S), and let (S') be the linear system having A' as augmented matrix. Then the linear systems (S) and (S') are equivalent.

Now, since (S) and (S') have the same solution set, we will solve the linear system (S'), which is in general much simpler than the linear system (S).

Solving a NonHomogeneous System

The matrix A' has the form

$$\left(\begin{array}{ccccccccccccccccccc|c}
0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & \cdots & * & * & 0 & \cdots & b'_1 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & \cdots & * & * & 0 & \cdots & b'_2 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & * & \cdots & \cdots & * & * & 0 & \cdots & b'_3 \\
 & & & & & & & & & & & & & & \cdots & & & & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & b'_r \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & & & & & & & & & & & & & & \cdots & & & & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0
\end{array}\right),$$

where the $*$ represent arbitrary scalars.

Denote by $j_1, j_2, \cdots, j_r$ the indices of columns that contain the pivots, where $r = rank(A')$, and

$$1 \leq j_1 < j_2 < \cdots < j_r \leq m+1.$$

Solving a NonHomogeneous System

We have then :

- The zero rows correspond to equations of the form
$$0x_1 + 0x_2 + \cdots + 0x_m = 0,$$
so we can neglect them, and consider only the r nonzero rows.

- If the last pivot belongs to the last column, then the last non trivial equation is
$$0x_1 + 0x_2 + \cdots + 0x_m = 1,$$
so the linear system is inconsistent.

Solving a NonHomogeneous System

- If the last pivot doesn't belong to the last column, the solution set of the linear system is given by

$$\begin{cases} x_{j_1} = -\sum_{j\neq j_1,\cdots,j_r} a'_{1j}x_j + b'_1 \\ x_{j_2} = -\sum_{j\neq j_1,\cdots,j_r} a'_{2j}x_j + b'_2 \\ \vdots \\ x_{j_r} = -\sum_{j\neq j_1,\cdots,j_r} a'_{rj}x_j + b'_r, \end{cases}$$

where the variables $x_j, j \neq j_1, \cdots, j_r$, called free variables, can take arbitrary values.

In particular, if $r = m$, there is a unique solution given by

$$(x_1, x_2 \cdots, x_m) = (b'_1, b'_2, \cdots, b'_m).$$

Solving a Homogeneous System

When the linear system is homogeneous, the last column is zero, and we have only two cases

1. $r = m$. In this case, the linear system has $(0,0,\cdots,0)$ as unique solution.
2. $r < m$. In this case, the solution set is a subspace of K^m of dimension $m - r$.

Examples

In the following examples, we take $K = \mathbb{R}$.

Notice that in practice, when we solve a linear system, we do not always need to perform completely the RREF.

Example 2

The linear system

$$(S_1) \quad \begin{cases} 2x + y - 2z + 3w = 1 \\ 3x + 2y - z + 2w = 4 \\ 3x + 3y + 3z - 3w = 5 \end{cases}$$

is inconsistent.

Examples

Example 3

The linear system

$$(S_2) \quad \begin{cases} x + 2y - 3z = 4 \\ x + 3y + z = 11 \\ 2x + 5y - 4z = 13 \\ 2x + 6y + 2z = 22 \end{cases}$$

has the unique solution $(1,3,1)$.

Examples

Example 4

Consider the linear system

$$(S_3) \quad \begin{cases} x + 2y - 2z + 3w = 2 \\ 2x + 4y - 3z + 4w = 5 \\ 5x + 10y - 8z + 11w = 12. \end{cases}$$

The solution set of (S_3) is

$$\mathcal{S} = \{\alpha\,(-2,1,0,0) + \beta\,(1,0,2,1) + (4,0,1,0), \qquad \alpha\,,\beta \in \mathbb{R}\}.$$

Cramer's Rule

For any $n \times n$ matrix A and any B in K^n, let $A_j(B)$ be the matrix obtained from A by replacing column j by the vector B. We have then, if we denote by $A^{(1)}, A^{(2)}, \cdots, A^{(n)}$ the columns of A :

$$A_j(B) = \left(A^{(1)}, \cdots, A^{(j-1)}, B, A^{(j+1)}, \cdots, A^{(n)}\right).$$

Theorem 4

Let A be an invertible $n \times n$ matrix. For any B in K^n, the unique solution $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in K^n$ of the linear system $AX = B$ is given by

$$\alpha_j = \frac{\det(A_j(B))}{\det(A)}, \qquad j = 1, 2, \cdots, n.$$

Example

Example 5

The linear system

$$(S) \quad \begin{cases} x - y + z = 7 \\ 4x - 2y + z = 3 \\ 2x - 3y + 5z = 2 \end{cases}$$

has the unique solution $\left(-\frac{41}{3}, -37, \frac{49}{3}\right)$.