Taylor Formula

Taylor Formula

- Taylor formula with integral remainder
- Taylor formula with Lagrange remainder
- Taylor formula with Young remainder

Taylor Formula

THEOREM (Taylor formula with integral remainder**)**

Let f a function defined on an open interval I of $\mathbb{R}$

and verify : f is class C^{n+1} on I, and let x_0 and $x \in I$

Then we have:

$$f(x) = \sum_{k=0}^{n} (x - x_0)^k \frac{f^{(k)}(x_0)}{k!} + R_n(x, x_0)$$

Taylor Formula

where

$$R_n(x, x_0) = \int_{x_0}^{x} \frac{f^{(n+1)}(t)(x-t)^n}{n!} dt$$

Taylor Formula

The above formula is called the Taylor formula with integral remainder. The polynomial:

$$p_n(x) = \sum_{k=0}^{n} (x - x_0)^k \frac{f^{(k)}(x_0)}{k!}$$

is called a regular or a smooth part of f

Taylor Formula

THEOREM (Taylor formula with Lagrange remainder)

Let f a function defined on an open interval I of $\mathbb{R}$ and

verify : f is classe C^n on I, and $f^{(n+1)}$ exists on I and let x_0 and $x \in I$

then there exists $c = x_0 + \theta(x - x_0),\ \theta \in [0,1]$ such that :

$$f(x) = \sum_{k=0}^{n} (x - x_0)^k \frac{f^{(k)}(x_0)}{k!} + (x - x_0)^{n+1} \frac{f^{(n+1)}(c)}{(n+1)!}$$

Taylor Formula

the terme

$$R_n(x, x_0) = (x - x_0)^{n+1} \frac{f^{(n+1)}(c)}{(n+1)!}$$

is called Lagrange remainder

Remark

For n=0 this is exactly the statement of mean value theorem.

Taylor Formula

- Applications

For $f(x)=e^x$ is of C^∞ classe on $\mathbb{R}$ then we can we can apply the previous formula.

Alors, for all x∈ $\mathbb{R}$ $\exists c=tx,\ t\in[0,1]$ such that

$$e^x=\sum_{k=0}^{n}\frac{x^k}{k!}+\frac{x^{n+1}e^c}{(n+1)!}$$

We have $\forall n\in\mathbb{N}\ \ (e^x)^{(n)}=e^x$

Taylor Formula

let $f(x) = cosx$. Same we can apply the previous formula.
for all x$\in \mathbb{R}$ *and* $\forall n \in \mathbb{N}$ because f is of class C^∞ on $\mathbb{R}$

Furthermore we know that

$\forall n \in \mathbb{N}$

$$f^{(n)}(0) = \cos\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & n = 2k+1 \\ (-1)^k & n = 2k \end{cases}$$

then we have,

$$\cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + \frac{x^{2n+1}}{(2n+1)!} \cos\left(c + (2n+1)\frac{\pi}{2}\right)$$

Taylor Formula

- For $f(x)=\sin x$, we have :

$$f^{(n)}(0)=\sin\left(n\frac{\pi}{2}\right)=\begin{cases} 0 & n=2k \\ (-1)^k & n=2k+1 \end{cases}$$

So for all x∈ $\mathbb{R}$ $\exists c=tx,\ t\in\left]0,1\right[$ such that

$$\sin x=\sum_{k=0}^{n}(-1)^k\frac{x^{k+1}}{(2k+1)!}+\frac{x^{2n+3}}{(2n+3)!}\sin\left(c+(2n+3)\frac{\pi}{2}\right)$$

Let $f(x)=(1+x)^{\alpha}$, $\alpha\epsilon\mathbb{R}$. we show by induction that

$$f^{(k)}(0)=\alpha(\alpha-1)(\alpha-2)........(\alpha-k+1)$$

Taylor Formula

$$(1+x)^{\alpha} = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^{2}+\ldots\ldots\ldots+\frac{\alpha(\alpha-1)(\alpha-2)\ldots\ldots(\alpha-n+1)}{n!}x^{n}$$

$$+\frac{\alpha(\alpha-1)(\alpha-2)\ldots\ldots(\alpha-n)}{(n+1)!}(1+c)^{\alpha-n-1}$$

If we take ,

$$\alpha=-1$$

$$f^{(k)}(0)=(-1)^{k}k!$$

$$\frac{1}{(1+x)}=1-x+x^{2}-x^{3}+\ldots\ldots+(-1)^{n}x^{n}+(-1)^{n+1}x^{n+1}\frac{1}{(1+c)^{n+2}}$$

Taylor Formula

ii. For $\alpha = \mathbf{\frac{1}{2}}$, we show by induction that

$$\frac{f^{(k)}(0)}{k!} = (-1)^{k-1}\frac{1.3.5 \dots (2k-3)}{2.4.6 \dots 2k}$$

We then obtain

$$\sqrt{(1+x)}$$
$$= 1 + \frac{x}{2} + \frac{x^2}{2.4} + \frac{x^3}{2.4.6} + \cdots + (-1)^{n-1}\frac{1.3.5 \dots (2n-3)}{2.4.6 \dots 2n}x^n$$

$$+(-1)^n\frac{1.3.5 \dots (2n-1)}{2.4.6 \dots 2n(2n+2)}(1+c)^{\frac{1}{2}-n-1}$$

Taylor Formula

iii. For $\boldsymbol{\alpha = -\frac{1}{2}}$, we show by induction that

$$\frac{f^{(k)}(0)}{k!} = (-1)^k \frac{1.3.5 \ldots (2k-1)}{2.4.6 \ldots 2k}$$

We then obtain

$$\frac{1}{\sqrt{(1+x)}} =$$

$$1 - \frac{x}{2} + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \cdots + (-1)^n \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} x^n$$

$$+(-1)^n \frac{1.3.5 \ldots (2n+1)}{2.4.6 \ldots 2n(2n+2)} (1+c)^{\frac{1}{2}-n-1}$$

Taylor Formula

- Corollary

If in addition we have $\left|f^{(n+1)}\right|$ is upper bounded by a real M, then

$$|f(x) - P_n(x)| \leq M \frac{|x - x_0|^{n+1}}{(n+1)!}.$$

- Example :

Approximate the value of sin(0.01).

$f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x,$
$f'''(x) = -\cos x, f''''(x) = -\sin x$

$$f(0) = 0, f'^{(0)} = 1, f''(0) = 0, f'''(0) = -1$$

Taylor formula at x_0=0 at order 3 is

$$f(x) = 0 + 1.x + 0.\frac{x^2}{2!} - 1.\frac{x^3}{3!} + \frac{f^{(4)}(c)}{4!}x^4$$

Taylor Formula

$$f(x) = x - \frac{x^3}{6} + \frac{f^{(4)}(c)}{4!}x^4$$

Let's apply this to the value $x = 0{,}01$

$$\sin(0{,}01) \approx 0{,}01 - \frac{(0{,}01)^3}{6} = 0{,}00999983333$$

As $\left|f^{(4)}(c)\right| \leq 1$ then the previous corollary allows us to estimate the error committed for $x = 0{,}01$:

$$\left|\sin(0{,}01) - \left(0{,}01 - \frac{(0{,}01)^3}{6}\right)\right| \leq \frac{(0{,}01)^4}{4!} \approx 4.10^{-4}$$

Taylor Formula

THEOREM (Taylor formula with Young remainder)

Let f a function defined on an open interval

$I\ of$ $\mathbb{R}$ and verify : f is class C^n on I, and let

$x_0 \in I$ then for all $x \in I$:

Taylor Formula

there exist a function

$$\psi : v(0) \to \mathbb{R}$$

$$h \to \psi(h)$$

with

$$\lim_{h \to 0} \psi(h) = 0$$

such that we have :

$$f(x) = \sum_{k=0}^{n} (x - x_0)^k \frac{f^{(k)}(x_0)}{k!} + (x - x_0)^n \psi(x - x_0)$$

Taylor Formula

The expression

$$R_n\left(x, x_0\right) = \left(x - x_0\right)^n \psi\left(x - x_0\right)$$

is called Young remainder.

Taylor Formula

Landau Notation

- Let f and g two fonctions defined on a neighborhood of x_0. We say that f is negligible compared to g at v(x_0) if there exists a function

$$\psi : v(0) \to \mathbb{R}$$

$$h \to \psi(h)$$

with

$$\lim_{x \to 0} \psi(h) = 0$$

Taylor Formula

such as

$$f(x) = g(x)\psi(x - x_0)$$

in this case we note

$$f = o(g) \text{ sur } \mathrm{v}(x_0)$$

- If g does not vanish in a neighborhood of x_0 except perhaps at x_0, we have

$$f = o(g) \text{ sur } \mathrm{v}(x_0) \Leftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

Taylor Formula

Examples :

$$x^2 = o(x) \quad au \;\; v(0) \; because \;\; \lim_{x\to 0} \frac{x^2}{x} = 0$$

2) In general

If $k < n, \;\; (x - x_0)^n = o\left((x - x_0)^k\right)$

$$at \;\; v(x_0) \;\; because \; \lim_{x\to x_0} \frac{(x - x_0)^n}{(x - x_0)^k} = 0$$

if $k < n, \quad x^k = o(x^n) \;\; at \;\; v(+\infty)$

Taylor Formula

- Using Landau notations, the Taylor–Young formula is written :

$$f(x) = \sum_{k=0}^{n} (x - x_0)^k \frac{f^{(k)}(x_0)}{k!} + o(x - x_0)^n$$

Indeed

$$\lim_{x \to x_0} \frac{R_n(x, x_0)}{(x - x_0)^n} = \lim_{x \to x_0} \psi(x - x_0) = 0$$

In practice, we often use this notation.

Taylor Formula

Examples :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$\sin x$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

Taylor Formula

local study

- Extremum

A function f has an extremum at x_0

if $f(x) - f(x_0)$ keeps a constant sign on a neighborhood of x_0 of the type $\tau_{x_0} =]x_0 - \alpha, x_0 + \alpha[,\ \alpha > 0,$

Let's write the Taylor-Yong formula

$$f(x) - f(x_0) = (x - x_0)\frac{f'(x_0)}{1!} + (x - x_0)^2\frac{f''(x_0)}{2!} + \cdots$$

$$+(x - x_0)^n\frac{f^{(n)}(x_0)}{n!} + o(x - x_0)^n \qquad (1)$$

Taylor Formula

1) If $f'(x_0) \neq 0$, as $x - x_0$ changes the

sign on τ_{x_0}, then $f(x) - f(x_0)$changes

the sign on τ_{x_0}, and therefore f has no extrema at x_0.

Taylor Formula

2) If $f'(x_0) = 0$ and $f''(x_0) \neq 0$, then, formula (1)

is written :

$$f(x) - f(x_0) = (x - x_0)^2 \frac{f''(x_0)}{2!} + \cdots + (x - x_0)^n \frac{f^{(n)}(x_0)}{n!}$$

$$+o\big((x - x_0)^n\big)$$

Taylor Formula

As $(x - x_0)^2$ does not change sign on τ_{x_0}, then $f(x) - f(x_0)$ keeps a constant sign on τ_{x_0}, so f admits an extrema at x_0, moreover

i. if $f''(x_0) > 0$, admits a local minimum at x_0,

ii. Si $f''(x_0) < 0$, admits a local maximum at x_0.

Taylor Formula

If $f'(x_0) = f''(x_0) = 0$ et $f^{(3)}(x_0) \neq 0$, we have :

$$f(x) - f(x_0)$$

$$= (x - x_0)^3 \frac{f^{(3)}(x_0)}{3!} + \cdots + (x - x_0)^n \frac{f^{(n)}(x_0)}{n!}$$

$$+ o\big((x - x_0)^n\big)$$

In this case $f(x) - f(x_0)$ changes sign on τ_{x_0},

so f has no extrema at x_0.

Taylor Formula

Note that in this case, $f'(x_0) = 0$ but f

does not admit an extremum at x_0 .We then deduce that, for the existence of an extremum, the condition $f'(x_0) = 0$ **is necessary but not sufficient.**

Taylor Formula

- in the general case
- Let $k \geq 1$ the smallest integer such that
- $f^{(k)}(x_0) \neq 0$. Then

$$f(x) - f(x_0)$$

$$= (x - x_0)^k \frac{f^{(k)}(x_0)}{k!} + \cdots + (x - x_0)^n \frac{f^{(n)}(x_0)}{n!}$$

$$+o\left((x - x_0)^n\right)$$

Taylor Formula

So,

i. If k is odd, f has no extremum at x_0.

ii. If k is even, f has an extremum at x_0.

Moreover,

a. If $f^{(k)}(x_0) > 0$, f has a minimum at x_0.

b. If $f^{(k)}(x_0) < 0$, f has a maximum enx_0.

Taylor Formula

- Examples :

$$f(x) = \sin x, \quad x_0 = \frac{3\pi}{2},$$

We have

$$f(x_0) = -1, \qquad f'(x_0) = 0 \;\; and \quad f''(x_0) = 1,$$

Then

$$\sin x - \sin\frac{3\pi}{2} = \frac{1}{2!}\left(x - \frac{3\pi}{2}\right)^2 + o\left(\left(x - \frac{3\pi}{2}\right)^2\right)$$

We can clearly see that $\sin x - \sin\frac{3\pi}{2} \geq 0$ on τ_{x_0}, then the function $\sin x$ admits a minimum at $x_0 = \frac{3\pi}{2}$.

Taylor Formula

$$f(x) = \sin x\,, \qquad\qquad x_0 = \pi$$

We have

$$f(x_0) = 0, f'(x_0) = -1 \;\; and \; f''(x_0) = 0,$$

So,

$$\sin x - \sin \pi = -(x-\pi) + o\big((x-\pi)\big)$$

then $\sin x - \sin \pi$ changes sign on τ_{x_0} then f

has no extrema at $x_0 = \pi$.

Taylor Formula

- <u>Position of the curve relative to its tangent line :</u>

We know that the tangent line equation (Δ), to the curve C_f at a point $M_0(x_0, f(x_0))$ is given by

$$(\Delta) : y = f(x_0) + (x - x_0)f'(x_0)$$

In fact it is the polynomial of degree ≤ 1 of the formula Taylor at point x_0.

Taylor Formula

Study sign of $f(x_0) - y$:

We have,

$$f(x) - y = (x - x_0)^2 \frac{f''(x_0)}{2!} + (x - x_0)^3 \frac{f^{(3)}(x_0)}{3!} + \cdots$$

$$+(x - x_0)^n \frac{f^{(n)}(x_0)}{n!} + o\big((x - x_0)^n\big)$$

Taylor Formula

1) If $f''(x_0) \neq 0$, then $f(x) - y$ does not change sign on τ_{x_0}, and therefore the curve C_f is on the same side with respect to its tangent line at the point M_0.

Moreover,

i. If $f''(x_0) > 0$, then $f(x) - y \geq 0$ on τ_{x_0}, so the curve C_f is **above its tangent line at the point** M_0.

ii. If $f''(x_0) < 0$, then $f(x) - y \leq 0$ on τ_{x_0}, so the curve C_f is **below its tangent line at the point** M_0.

Taylor Formula

If $f''(x_0) = 0$ and $f^{(3)}(x_0) \neq 0$ then

$$f(x) - y$$

$$= (x - x_0)^3 \frac{f^{(3)}(x_0)}{3!} + \cdots + (x - x_0)^n \frac{f^{(n)}(x_0)}{n!}$$

$$+ o\left((x - x_0)^n\right)$$

We can clearly see that $f(x) - y$ changes sign

On τ_{x_0}, so the curve C_f intersects its tangent line at a point M_0.

In this case we say the curve C_f admits a point of inflection at point M_0.

Taylor Formula

3) If $f''(x_0) = f^{(3)}(x_0) = 0$ and $f^{(4)}(x_0) \neq 0$, then $f(x) - y$ does not change sign on τ_{x_0}, and so the curve C_f is on the same side with respect to its tangent line at the point M_0.It is identical to the first case.

Note also that, in this case, $f''(x_0) = 0$ but the curve has no inflection point at M_0. So, for the existence of the inflection point, **the condition $f''(x_0) = 0$ is only necessary but not sufficient.**

Taylor Formula

- <u>In the general case</u>

Let $k \geq 2$ the smallest integer such that $f^{(k)}(x_0) \neq 0$,
then

$$f(x) - y$$

$$= (x - x_0)^k \frac{f^{(k)}(x_0)}{k!} + \cdots + (x - x_0)^n \frac{f^{(n)}(x_0)}{n!} + o\big((x - x_0)^n\big)$$

Taylor Formula

So,

i. If k is odd, the curve C_f admits an inflection point at M_0.

ii. If k is even, the curve C_f does not admit an inflection point at M_0, moreover

a)If $f^{(k)}(x_0) > 0$, the curve is **above its tangent line at the point** M_0.

b) If $f^{(k)}(x_0) < 0$, is **below its tangent line at the point** M_0.

Taylor Formula

- Example:

$$f(x) = \cos x\,,\;\; x_0 = \frac{\pi}{4},$$

We have :

$$\cos x = \frac{\sqrt{2}}{2} - \left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} - \left(x - \frac{\pi}{4}\right)^2 \frac{\sqrt{2}}{2} + o\left(\left(x - \frac{\pi}{4}\right)^2\right)$$

Taylor Formula

The tangente line equation to the cure C_f at a point M_0 is

$$y = \frac{\sqrt{2}}{2} - \left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} = -x + \frac{\sqrt{2}}{2}\left(1 + \frac{\pi}{4}\right)$$

and

$$\cos x - y = -\left(x - \frac{\pi}{4}\right)^2 \frac{\sqrt{2}}{2} + o\left(\left(x - \frac{\pi}{4}\right)^2\right)$$

We can clearly see that $\cos x - y \leq 0$ on τ_{x_0}, the the curve

C_f **below its tangent line on $\boldsymbol{\tau_{x_0}}$**

Limited developments

Definitions and properties

- INTRODUCTION
- Usual limited development
- Properties and examples
- Taylor–Young theorem
- Necessary condition for the existence of a LD

Limited developments

- INTRODUCTION

We have seen that, via Taylor's formula, we can approximate a function f by a polynomial. But in Taylor's formula, we require the function f to be regular or smooth, that is to say differentiable up to order n on a neighborhood of the point x_0. On the other hand, this regularity is not required to obtain a limited development of a function. This is the advantage of limited developments which make it possible to approach regular or non-regular functions using polynomials. The other advantage is to use simple D.L. calculation techniques to find the Taylor expansions, for regular functions, without knowing the derivatives of f at x_0, and consequently, we can simply obtain derivatives of f at x_0. We start with zero-limited developments.

Limited developments

- Developments limited at zero
- ***Definition***

Let f be a function defined on neighborhood of *0 ie defined on* $v(0)$.

We say that f admits a limited expansion of order *n* at 0 if, there exists

$(n+1)$ real numbers $(a_0, a_1, \dots, a_n)$ and a function

Limited developments

$$\varphi : v(0) \to \mathbb{R}$$

$$x \to \varphi(x)$$

with

$$\lim_{x \to 0} \varphi(x) = 0$$

Such that

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + x^n \varphi(x) \quad (1)$$

Limited developments

- The polynomial

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is called the **regular part** of the limited expansion of f.

Limited developments

- The expression $x^n\varphi(x)$ is called **the remainder of order $\boldsymbol{n}$** of the L.D of f.

- The lowest degree monomial of the regular part is called the **Principal part** of the L.D of f.

Limited developments

Remark

due to the fact that

$$\lim_{x\to 0}\frac{x^n\varphi(x)}{x^n}=\lim_{x\to 0}\varphi(x)=0$$

Limited developments

Then (1) is written in the form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + o(x^n) \quad (2)$$

Limited developments

- Example 1

$$(1 - x^{n+1})$$

$$= (1 - x)(1 + x + x^2 + x^3 + x^4 + \cdots + x^n)$$

We have

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n$$

Limited developments

$then$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \frac{x^{n+1}}{1-x}$$

or

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + x^n\,\frac{x}{1-x}$$

which is written :

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + x^n\varphi(x)$$

with $\varphi(x) = \frac{x}{1-x}$ $\quad and \quad \lim_{x\to 0} \varphi(x) = 0$

So the function $f(x) = \frac{1}{1-x}$ admits a L.D of order n

Limited developments

Example 2

Let the function defined by

$$f(x) = \begin{cases} 1 - 2x + 5x^4 + 3x^7 cos\frac{1}{x} & si \quad x \neq 0 \\ 0 & si \quad x = 0 \end{cases} \quad (3)$$

$$f(x) = 1 - 2x + 0x^2 + 0x^3 + x^3\left(5x + 3x^4 cos\frac{1}{x}\right)$$

$$= 1 - 2x + o(x^3)$$

Limited developments

In fact this function admits aD.L. d'ordre k pour tout

$k = 0,1, \dots, 6$. En effet, $f(x)$ s'écrit :

$$f(x) = 1 - 2x + 5x^4 + x^6\left(3x\cos\frac{1}{x}\right)$$

$$= 1 - 2x + 5x^4 + o(x^6)$$

Limited developments

But f does not admit a L.D of order 7 at 0, because

$$f(x) = 1 - 2x + 5x^4 + 0x^7 + 3x^7 cos\frac{1}{x}$$

but

$$\lim_{x\to 0} \cos\left(\frac{1}{x}\right) \neq 0$$

Limited developments

- **Noticed :**

The function f admits a D.L. at zero up to order 6 without it being continuous at zero. **This is the advantage of L.D** compared to the Taylor formula which requires a lot of regularity to the function f.

Limited developments

- **Proposition :**

If f admits a limited expansion of order n at zero,

then this **expansion is unique**.

Limited developments

- Limited expansions of even and odd functions

Let f be a function defined on neighborhood of 0 and which admits the L.D to order *n at 0*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + o(x^n)$$

I. *if f is even then* $a_1 = a_3 = \cdots = a_{2k+1} = 0$

II. *If f is odd then*

$$a_0 = a_2 = \cdots = a_{2k} = 0$$

Limited developments

Proof:

Suppose f is even. For all $x \neq 0$ we can write

$f(x) = f(-x) = a_0 - a_1 x + a_2 x^2 - a_3 x^3 \ldots +$

$(-1)^n a_n x^n + (-1)^n x^n \varphi(-x)$. Noting that

$\lim_{x \to 0}((-1)^n \varphi(-x)) = 0$, we obtain thanks to the

uniqueness of the L.D, that the coefficients

$$a_1 = a_3 = \cdots = a_{2k+1} = 0.$$

Limited developments

- Limited developments obtained by Taylor's formula

Proposition

If the function f satisfies the assumptions of the Taylor-Yong theorem, at $x_0 = 0$, then the function f has a limited expansion in a neighborhood of 0, of order n. This limited development is given by the formula of Taylor-Young.

In addition we have the following relation

$$a_k = \frac{f^{(k)}(0)}{k!}, \qquad k = 0, \dots, n,$$

Limited developments

Examples :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + o(x^n)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdot\cdot + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

Limited developments

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$(1+x)^\alpha$$

$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 \ldots$$

$$+ \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + o(x^n) \qquad (**)$$

Les développements limités

- ***Exercice***

En déduire les D.L. des fonctions suivantes

$$f_1(x) = \frac{1}{1+x}$$

$$f_2(x) = \frac{1}{1-x}$$

$$f_3(x) = \sqrt{1+x}$$

$$f_4(x) = \frac{1}{\sqrt{1+x}}$$

$$f_5(x) = \frac{1}{\sqrt{1-x^2}}$$

Limited developments

$$f_6(x) = \frac{1}{1 + x^2}$$

$$f_7(x) = \frac{1}{1 - x^2}$$

$$f_8(x) = \sqrt{x - 1}$$

$$f_9(x) = \sqrt{1 + x^2}$$

Limited developments

$$f_{10}(x) = \sqrt{1 - x^2}$$

$$f_{11}(x) = \frac{1}{\sqrt{1 - x}}$$

$$f_{12}(x) = \frac{1}{\sqrt{1 + x^2}}$$

$$f_{13}(x) = \frac{1}{\sqrt{1 - x^2}}$$

Limited developments

- Solution :

1) On prend α=-1

$$f_1(x) = \frac{1}{1+x}$$

$$f_1(x)$$
$$= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \cdots$$
$$+ (-1)^n x^n + o(x^n)$$

Limited developments

Solution :

2) $f_2(x) = \frac{1}{1-x}$

$f_2(x)$

$= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots + x^n$

$+ o(x^n)$

Limited developments

Solution :

3) $f_3(x) = \sqrt{1+x}$

$$f_3(x)$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$$

$$- (-1)^{n+1}\frac{1}{2.4.6\ldots(2n)}x^n + o(x^{2n})$$

Limited developments

Solution :

4) $f_4(x) = \frac{1}{\sqrt{1+x}}$

$$f_4(x)$$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$$

$$+ (-1)^n \frac{1.3.5.7\ldots(2n-1)}{2.4.6\ldots(2n)}x^n + o(x^n)$$

Limited developments

Solution :

5) $f_5(x) = \frac{1}{\sqrt{1-x^2}}$

$$f_5(x)$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots$$

$$+ \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n.(2n+1)}x^{2n+1} + o(x^{2n+2})$$

Limited developments

- <u>Solution :</u>

1. $f_6(x) = \frac{1}{1+x^2}$

$$f_6(x)$$

$$= 1 - x^2 + x^4 + x^6 + \cdots + (-1)^n x^{2n}$$

$$+ o(x^{2n+1})$$

Limited developments

2) $f_7(x) = \frac{1}{1-x^2}$

$f_7(x)$

$= 1 + x^2 + x^4 + x^6 + \cdots + x^{2n} + o(x^{2n+1})$

Limited developments

3) $f_8(x) = \sqrt{1-x}$

$f_8(x)$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \cdots - \frac{1}{2.4.6\ldots(2n)}x^n$$

$$+ o(x^{2n})$$

Limited developments

4) $f_9(x) = \sqrt{1 + x^2}$

$$f_9(x)$$

$$= 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 + \cdots$$

$$+ (-1)^{n+1}\frac{1.3.5.7 \ldots (2n-3)}{2.4.6 \ldots (2n)}x^{2n} + o(x^{2n})$$

Limited developments

5) $f_{10}(x) = \sqrt{1 - x^2}$

$$f_{10}(x)$$

$$= 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 + \cdots$$

$$- \frac{1.3.5.7\ldots(2n-3)}{2.4.6\ldots(2n)}x^{2n} + o(x^{2n})$$

Limited developments

6) $f_{11}(x) = \frac{1}{\sqrt{1-x}}$

$$f_{11}(x)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots$$

$$+ \frac{1.3.5.7 \ldots (2n-1)}{2.4.6 \ldots (2n)} x^n + o(x^{2n})$$

Limited developments

7) $f_{12}(x) = \frac{1}{\sqrt{1+x^2}}$

$f_{12}(x)$

$$= 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \cdots$$

$$+ (-1)^n \frac{1.3.5.7 \ldots (2n-1)}{2.4.6 \ldots (2n)} x^{2n} + o(x^{2n})$$

Limited developments

8) $f_{13}(x) = \frac{1}{\sqrt{1-x^2}}$

$$f_{13}(x)$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots$$

$$+ \frac{1.3.5.7\ldots(2n-1)}{2.4.6\ldots(2n)}x^{2n} + o(x^{2n})$$

Limited developments

- ***Noticed***

We saw above that a function admits a L.D, at zero, without it being continuous at zero. But there is also a condition that the function f must satisfy to admit a L.D, at zero.

Limited developments

- Necessary condition for the existence of a L.D

Proposition

If f has a limited expansion at 0

then $\lim_{x \to 0} f(x)$ **exists and is finite**.

Limited developments

- **Noticed**

This resul of this proposition allows us to recognize functions that do not have limited devlopment. For example, the following functions do not have limited development in 0:

Limited developments

$$\ln x \, , \frac{1}{x} \, , \operatorname{cot}g\, x \, , \sin\frac{1}{x} \, , \ldots$$

Because the first 3 functions have infinite limits at zero while the last one does not admit a limit at zero.

Limited developments

- Limited Developpement at $x_0 \neq 0$
- Limited developments to the neighborhood of infinity
- Generalized Limited developments
- Application of Limited Developments

Limited developments

➢ Limited Developpement at $x_0 \neq 0$

- ***Définition***

Let f be a function defined on neighborhood of x_0 ie

$defined\ on$ $v(x_0)$. We say that f admits a limited expansion

of order *n* at x_0 if, there exists $(n+1)$ real numbers

$(a_0, a_1, \dots, a_n)$ and a function

Limited developments

$$\varphi : v(0) \to \mathbb{R}$$

$$x \to \varphi(x)$$

with

$$\lim_{x \to x_0} \varphi(x - x_0) = 0$$

Such that

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x$$

Limited developments

- The polynomial

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots +$$

$$a_n(x - x_0)^n$$

is called the **regular part** of the limited expansion of f.

Limited developments

- The expression $(x - x_0)^n \varphi(x - x_0)$ is called **the remainder of order $\boldsymbol{n}$** of the L.D of f.

- The lowest degree monomial of the regular part is called the **Principal part** of the L.D of f.

Limited developments

Remark

due to the fact that

$$\lim_{x \to x_0} \frac{(x - x_0)^n \varphi(x - x_0)}{(x - x_0)^n} = \lim_{x \to x_0} \varphi(x - x_0) = 0$$

Limited developments

Then (1) is written in the form:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + o((x - x_0)^n) \qquad (2)$$

Limited developments

- **Proposition :**

A function f admits a limited expansion, of orde n

at x_0, if and only if, the function

$g(t) = f(t + x_0)$ admits a limited expansion, of

order n at zero.

Limited developments

Example 1

Calculate the DL of order 4 of the function

$f(x) = \cos x$ en $x_0 = \frac{\pi}{6}$. On pose $t = x - \frac{\pi}{6}$ et

We have : $\cos x = \cos\left(t + \frac{\pi}{6}\right)$

$$= \cos\left(\frac{\pi}{6}\right)\cos(t) - \sin\left(\frac{\pi}{6}\right)\sin(t)$$

$$= \frac{\sqrt{3}}{2}\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} + o(t^4)\right] - \frac{1}{2}\left[t - \frac{t^3}{3!} + o(t^4)\right]$$

Limited developments

$$\cos x = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2$$

$$+\frac{1}{12}\left(x - \frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{48}\left(x - \frac{\pi}{6}\right)^4 + o\left(\left(x - \frac{\pi}{6}\right)^4\right)$$

Example 2

Determine the L.D at order 3 of e^x at $x_0 = -2.$

we set

$t = x + 2$

Limited developments

$$e^x = e^{-2}e^t$$

$$=e^{-2}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + o(t^3)\right]$$

$$=e^{-2}\left[1 + (x+2) + \frac{(x+2)^2}{2!} + \frac{(x+2)^3}{3!} + o\left((x+2)^3\right)\right]$$

Limited developments

➢ Limited developments to the neighborhood of infinity

- ***Definition***

Let f be a function defined on $]\alpha, +\infty[$. On dit que f We say that f admits a limited expansion of order n in the neighborhood of $+\infty$ si, if, there exists n+1 real numbers $(a_0, a_1, \dots, a_n)$ and a function

Limited developments

$$\varphi: v(0) \longrightarrow \mathbb{R}$$

$$x \longmapsto \varphi(x)$$

with

$$\lim_{x \to +\infty} \varphi\left(\frac{1}{x}\right) = 0$$

Such that

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} + \frac{1}{x_n}\varphi\left(\frac{1}{x}\right)$$

Limited developments

Noticed

The expression for f is written in the form:

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} + o\left(\frac{1}{x^n}\right)$$

Limited developments

- **Proposition :**

A function f admits a limited expansion, of orde n

at $+\infty$, if and only if, the function

$g(t) = f\left(\frac{1}{t}\right)$ admits a limited expansion, of order

n at zero.

Limited developments

Noticed

We have the same definition for the development limited to the neighborhood of $-\infty$, for a function defined on an interval of the form $]-\infty, \alpha[$.

Limited developments

- Generalized Limited developments
- Let's start with an example

Exemple :

The function $cotg\, x$ does not admit a limited expansion in zero, because $\lim_{x \to 0} cotg\, x$ is not finite. But using limited expansions, for example of order 5, in zero of the functions cos and sin we have:

Limited developments

$$cotg\, x = \frac{1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)}{x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)}$$

$$= \frac{1}{x} \frac{1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} + o(x^4)}$$

We can clearly see that the function $x\, cotg\, x$ admits a limited expansion of order 4 in Zero.

Limited developments

- ***Definition***

Suppose that the function f has no zero-limited expansion, but there exists α>0 such that

$$x^{\alpha} f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + o(x^n)$$

then,

$$f(x) = \frac{a_0}{x^{\alpha}} + \frac{a_1}{x^{\alpha-1}} + \frac{a_2}{x^{\alpha-2}} + \cdots + \frac{a_n}{x^{\alpha-n}} + o(x^{\alpha-n})$$

is called generalized limited expansion of f at zero.

Limited developments

For the previous example, using division according to the asending powers to order 4, we have:

$$x\,cotg\,x = 1 - \frac{x^2}{3} - \frac{17}{3.5!}x^4 + o(x^4)$$

We obtain :

$$cotg\,x = \frac{1}{x} - \frac{x}{3} - \frac{17}{3.5!}x^3 + o(x^3)$$

Limited developments

- Application of Limited Developments
- Find the limits.

Generally are the limits of indeterminate form.

It is always possible, with a change of variable, to return to a limit when x tends towards 0.

Limited developments

- <u>Examples.</u>

1. Calculete $\lim_{x \to \frac{\pi}{2}} (x - \frac{\pi}{2}) \tan x$.

We see that this limit is of the indeterminate form $0 \cdot \infty$. we set $\mathrm{X} = x - \frac{\pi}{2}$,

$$(x - \frac{\pi}{2}) \tan x = X \tan(X + \frac{\pi}{2})$$

Limited developments

$$X\frac{\sin(X+\frac{\pi}{2})}{\cos(X+\frac{\pi}{2})} = -\frac{X}{\tan X}$$

On connait le DL de $\tan X$ en 0

$$\tan X \ = X + \frac{X^3}{3} + o(X^3)$$

En remplaçant on a

$$\lim_{x\to\frac{\pi}{2}}(x-\frac{\pi}{2})\tan x = \lim_{X\to 0}\frac{X}{\tan X}$$

Limited developments

$$\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \tan x$$

$$= -\lim_{X \to 0} \frac{X}{\tan X} =$$

$$\lim_{X \to 0} \frac{1}{1 + \frac{X^2}{3} + o(X^2)} = -1$$

Limited developments

2. Find $\lim_{x \to +\infty} x^2 \left(e^{1/x} - e^{\frac{1}{1+x}}\right)$. We set

$X = \frac{1}{x}$. Then we have $x \to +\infty$ if only if $X \to 0$

$$x^2 \left(e^{1/x} - e^{\frac{1}{1+x}}\right) = \frac{1}{X^2}\left(e^X - e^{\frac{X}{1+X}}\right)$$

It is sufficient to calculate the DL at a certain order of :

$$\frac{1}{X^2}\left(e^X - e^{\frac{X}{1+X}}\right)$$

Limited developments

We have :

$$\frac{X}{1+X} = X - X^2 + o(X^2)$$

$$e^Y = 1 + Y + \frac{Y^2}{2!} + o(Y^2)$$

$$e^{\frac{X}{1+X}} = 1 + X - \frac{X^2}{2!} + o(X^2)$$

so $$\frac{1}{X^2}\left(e^X - e^{\frac{X}{1+X}}\right) = 1 + o(X^2)$$

Therefore

$$\lim_{x\to+\infty} x^2\left(e^{1/x} - e^{\frac{1}{1+x}}\right) = 1.$$

Limited developments

Position of the curve relative to a tangent line

Suppose that f admits a DL of order n at x_0

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

$+a_n(x - x_0)^n + o((x - x_0)^n)$ avec $n \geq 2$. This implies that f (or its extension if f is not defined at x_0) is continuous and differentiable at x_0,

with $a_0 = f(x_0)$ and $a_1 = f'(x_0)$ so the line tangent equation is $y = a_0 + a_1(x - x_0)$

hence the sign of

$$f(x) - \left(a_0 + a_1(x - x_0)\right)$$

Limited developments

Can be deduced, at neighborhood of x_0 of the sign of

$a_2(x-x_0)^2+\cdots+a_n(x-x_0)^n+o((x-x_0)^n)$

Let m be the smallest integer such that $a_m \neq 0$. So we have

If m is even then the sign of

$$\Delta = f(x)-\left(a_0+a_1(x-x_0)\right)$$

is locally of the same sign as a_m and we have

1. If $a_m > 0$ then $\Delta \geq 0$ locally and therefore the curve is locally above its tangent line.
2. Si $a_m < 0$ alors $\Delta \leq 0$ locally and therefore the curve is locally below its tangent line.

- If m is odd the the curve untersect its tangent line at $\left(x_0, f(x_0)\right)$ it is an inflection tangent line.

Limited developments

Position de la courbe par rapport à une asymptote

Supposons que f *admet une asymptote d'équation*

$$y = a_0 x + a_1$$

Pour trouver a_0 et a_1 en utilisant la méthode des DL à l'ordre 1 de la fonction $Xf\left(\frac{1}{X}\right)$, $\left(X = \frac{1}{x}\right)$

Si $Xf\left(\frac{1}{X}\right) = a_0 + a_1 X + o(X)$ en 0,

on a $\frac{1}{x} f(x) = a_0 + a_1 \frac{1}{x} + o\left(\frac{1}{x}\right)$ au $V(+\infty)$

On voit que $\lim_{x \to +\infty} \frac{f(x)}{x} = a_0$

et

$$\lim_{x \to +\infty} f(x) - a_0 x = a_1$$

Limited developments

Position de la courbe par rapport à l'asymptote

on doit calculer un DL d'ordre supérieure de $Xf\left(\frac{1}{X}\right)$ en 0.

Si $Xf\left(\frac{1}{X}\right)=a_0 + a_1X + a_2X^2 + \cdots + a_nX^n + o(X^n)$.

$f(x)$-$(a_0x + a_1) = a_2\frac{1}{x} \ldots + a_n\frac{1}{x^{n-1}} + o\left(\frac{1}{x^{n-1}}\right)$

Soit m le plus petit entier tel que $a_m \neq 0$. Alors

- Si $a_m > 0$ alors la courbe est au dessus de l'asymptote au voisinage de $+\infty$.
- Si $a_m < 0$ alors la courbe est au dessous de l'asymptote au voisinage de $+\infty$.

Limited developments

Example :

Déterminons, lorsque $x \longrightarrow -\infty$, l'équation de

l' asymptote et sa position par rapport à la courbe

de la fonction f définie par

$$f(x) = \sqrt[3]{8x^3 + 4x^2 - 1} - x$$

Limited developments

On pose $t = \frac{1}{x} \to 0$. On a alors

$$\sqrt[3]{8x^3 + 4x^2 - 1} = \sqrt[3]{\frac{8}{t^3} + \frac{4}{t^2} - 1},$$

$$= \frac{2}{t}\sqrt[3]{1 + \frac{t}{2} - \frac{t^3}{8}},$$

$$= \frac{2}{t}\left[1 + \frac{1}{3}\left(\frac{t}{2} - \frac{t^3}{8}\right) - \frac{1}{9}\left(\frac{t}{2} - \frac{t^3}{8}\right)^2 + o(t^2)\right],$$

$$= \frac{2}{t}\left[1 + \frac{t}{6} - \frac{t^2}{36} + o(t^2)\right]$$

$$= \frac{2}{t} + \frac{1}{3} - \frac{t}{18} + o(t)$$

Limited developments

On obtient alors,

$$f(x) = x + \frac{1}{3} - \frac{1}{18x} + o\left(\frac{1}{x}\right)$$

L'équation de l'asymptote à la courbe, lorsque $x \to -\infty$ est :

$$y = x + \frac{1}{3}$$

De plus

$$f(x) - y = -\frac{1}{18x} + o\left(\frac{1}{x}\right) \to 0^+$$

La courbe est donc au dessus de l'asymptote, lorsque $x \to -\infty$.

Limited developments

Tests d'évaluations:

Donner la réponse, en la justifiant, aux questions suivantes :

Test 1:

La fonction $x \longmapsto \ln x$ a-t-elle un développement limité en zéro?

Rép :

La fonction $x \longmapsto \ln x$, n'admet pas de développement limité en zéro, car la condition nécessaire d'existence du DL n'est pas vérifiée c-a-d $\lim\limits_{x \to 0} \ln x$ n'est pas finie.

Limited developments

Test 2 :

La fonction $x \longmapsto \sqrt{x}$ a-t-elle un développement limité d'ordre 1 en zéro?

Rép :

La fonction $x \longmapsto \sqrt{x}$, n'admet pas de développement limité d'ordre 1 en zéro.

Car on ne peut pas la mettre sous la forme :

$\sqrt{x}=a_0 + a_1 x + x\varphi(x)$ avec $a_0 = a_1 \ = 0$ et $\lim_{x \to 0} \varphi(x) = 0$

Les développements limités

On peut la mettre sous cette forme avec

$\varphi(x) = \frac{1}{\sqrt{x}}$, mais $\lim\limits_{x \to 0} \varphi(x) = +\infty$,

Donc la fonction $x \longmapsto \sqrt{x}$ n'admet pas de DL d'ordre 1 en zéro.

Test 3 :

La fonction $x \longmapsto \sqrt{x^7}$ a-t-elle un développement limité en zéro d'ordre 3 d'ordre 4?

Limited developments

Rép :

Oui elle admet un développement limité d'ordre 3 mais pas d'ordre 4 en zéro. Car

$\sqrt{x^7}=a_0 + a_1x + a_2x^2 + a_3x^3 + x^3\ \varphi(x)$, avec

$$a_0 = a_1 = a_2 = a_3 = 0,$$

$$\varphi(x) = x^{\frac{1}{2}} \text{ et } \lim_{x \to 0} \varphi(x) = 0,$$

Donc elle admet un DL d'ordre 3 en zéro.

Les développements limités

Encore en peut écrire :

$\sqrt{x^7}$=$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + x^4\ \varphi(x)$,

avec $a_0 = a_1 = a_2 = a_3 = a_4 = 0$ et $\varphi(x) = x^{-\frac{1}{2}}$

mais

$$\lim_{x \to 0} \varphi(x) = +\infty\,.$$

Donc elle n'admet de DL d'ordre 4.

Limited developments

Test 4 :

La fonction $x \longmapsto e^{-1/x^2}$ posséde-t-elle un développement limité en zéro d'ordre n?

Rép :

oui car on a :

$$e^{-1/x^2} = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + x^n \, \varphi(x),$$

avec $a_0 = a_1 = a_2 = a_3 = a_4 = \cdots = a_n = 0$ et

$$\varphi(x) = x^{-n} \, e^{-1/x^2} \text{ et } \lim_{x \to 0} \varphi(x) = 0$$

Limited developments

Donc

$$e^{-1/x^2} = o(x^n)$$

La fonction e^{-1/x^2} possède un DL d'ordre n pour tout entier n.

Limited Developments

Operation on Limited Development

- Sum on L.D
- Product on L.D
- Quotient on L.D
- Composition on L.D
- Primitivation on L.D
- Differentiability on L.D

Limited Developments

Operation on Limited Development

- ***Theoreme***

Let f and g be two functions having limited expansions from order n at zero. Then

1. The functions $f+g$ et $f.g$ **admit** limited expansions from order n at zero.

2.If $\lim_{x \to 0} g(x) \neq 0$ then the functions $\frac{1}{g}$ and $\frac{f}{g}$ **admit limited expansions at zero**.

Limited Developments

Operation on Limited Development

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moreover,

- The regular part of the limited expansion of $f+g$ is the sum of the regular parts of the limited expansions of f and g.

Limited Developments

Operation on Limited Development

- The regular part of the limited expansion of fg is obtained from the product of the regular parts of the limited expansions of f and g, keeping only the monomials of degrees less than or equal to n.

Limited Developments

Operation on Limited Development

- The regular part of the limited expansion of $\frac{f}{g}$ is the quotient of order n of the division according to the asending powers of the regular parts of the limited expansions of f and g.

Limited Developments

Operation on Limited Development

- ***Example 1***

To obtain the limited expansion of order 4 of the function

$$f(x) = e^x \sin x$$

We multiply the regular parts of the L.D. of e^x and $\sin x$. So we have

Limited Developments

Operation on Limited Development

$$e^x \sin x$$
$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)\left(x - \frac{x^3}{3!}\right)$$
$$+ o(x^4)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)x - \frac{x^3}{3!}(1 + x)$$
$$+ o(x^4)$$

$$= x + x^2 + \frac{x^3}{3} + o(x^4)$$

Limited Developments

Operation on Limited Development

- ***Example 2***

Using division according to asending powers of

$$\sin x = \left(x - \frac{x^3}{3!}\right) + o(x^4)$$

by

$$\cos x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) + o(x^4)$$

Limited Developments

Operation on Limited Development

we obtain

$$tg\, x = x + \frac{x^3}{3} + o(x^4)$$

- <u>Attention</u>

The previous criterion simply says that

$\lim_{x \to 0} g(x) \neq 0$, then $\frac{f}{g}$ admit a L.D at ordre *n* at 0 it does not tell us if $\lim_{x \to 0} g(x) = 0$, then $\frac{f}{g}$ does not admit a D.L at order n in 0!! It is possible that $\lim_{x \to 0} g(x) = 0$, with $\frac{f}{g}$ admits a L.D to ordre *n* at 0

Limited Developments

Operation on Limited Development

- Example :

The function $\frac{\sin x}{x}$ admit a L.D to order 3 at 0.

Treatment of the case $\lim_{x\to 0} g(x) = 0.$

1. $\lim_{x\to 0} f(x) \neq 0$. In this case $\frac{f}{g}$ does not admit L.D to order n at 0, because $\lim_{x\to 0} \frac{f(x)}{g(x)} = \pm\infty$.

2. $\lim_{x\to 0} f(x) = 0$. In this case the D.L of f is of the form

$$f(x) = a_p x^p + \cdots + a_n x^n + x^n \varphi(x)$$

Limited Developments

Operation on Limited Development.

- Treatment of the case $\lim_{x \to 0} g(x) = 0.$

And that of g is of the form

$$g(x) = b_q x^q + \cdots + b_n x^n + x^n \psi(x)$$

with $a_p \neq 0$ and $b_q \neq 0$. We treat the quotient $\frac{f}{g}$ according to the values of p and q.

- If p<q but $a_p \neq 0$, *we show that*

$\lim_{x \to 0} \frac{f(x)}{g(x)} = \pm\infty$ and therefore $\frac{f}{g}$ does not admit a L.D to order n at 0.

Limited Developments

Operation on Limited Development

- <u>Treatment of the case</u> $\lim_{x \to 0} g(x) = 0.$
- If $p \geq$ q we reduce the quotient $\frac{f}{g}$ to the form $\frac{f_1}{g_1}$ with $\lim_{x \to 0} g_1(x) \neq 0$. So to calculate the D.L of $\frac{f}{g}$ to order n at 0, we calculate the D.L of f and g to order n + q, and then we use the method ofdivision according to increasing powers.

Limited Developments

Operation on Limited Development

- Example

Calculons le D.L de $\frac{\ln(1+x)}{\sin x}$ à l'ordre 3 en 0. Il faut déterminer q tel que $b_q \neq 0$ dans le DL de $\sin x$. On a

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$$

Par conséquent le premier coefficient non-nul est b_1. Donc $q = 1$. Donc on doit calculer le D.L de $\ln(1+x)$ et $\sin x$ à l'ordre 3+ $q = 4$

Limited Developments

Operation on Limited Development

- Example

We have : $\sin x = x - \frac{x^3}{3!} + o(x^4)$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathrm{o}(x^4),$$

So

$$\frac{\ln(1+x)}{\sin x} = \frac{1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \mathrm{o}(x^3)}{1 - \frac{x^2}{3!} + o(x^3)}$$

Therefore we have a LD of order 3 at the top and bottom and with $\lim_{x \to 0} g_1(x) \neq 0$, where $g_1(x) = 1 - \frac{x^2}{3!} + o\left(x^3\right)$ we can apply the previous criterion and make the division according to increasing powers.

Limited Developments

Operation on Limited Development

- ***Proposition*** *(*L.D of composition function*)*

If the functions f and u have limited expansions of order n and if $\lim_{x \to 0} u(x) = 0$ then the function g= $f \circ u$ **admits a limited expansion of order $\boldsymbol{n}$.** Furthermore, the regular part of the LD of g is obtained from the regular part of the LD of f by substituting the regular part of the LD of u and keeping only the monomials of degrees less than or equal to n.

Limited Developments

Operation on Limited Development

- ***Example 1***

The L.D of order 4 of the function $e^{\sin x}$ is obtained as follows:

$$e^x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) + o(x^4)$$

$$\sin x = \left(x - \frac{x^3}{3!}\right) + o(x^4)$$

Limited Developments

Operation on Limited Development

$$e^{\sin x}$$

$$= 1 + \left(x - \frac{x^3}{3!}\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!}\right)^2$$

$$+ \frac{1}{3!}\left(x - \frac{x^3}{3!}\right)^3 + \frac{1}{4!}\left(x - \frac{x^3}{3!}\right)^4 + o(x^4)$$

Limited Developments

Operation on Limited Development

$$e^{\sin x}$$

$$= 1 + x + \frac{x^2}{2} + \left(\frac{1}{3!} - \frac{1}{3!}\right)x^3$$

$$+ \left(\frac{1}{4!} - \frac{1}{3!}\right)x^4 + o(x^4)$$

$$= 1 + x + \frac{x^2}{2} - \frac{3}{4!}x^4 + o(x^4)$$

Limited Developments
Operation on Limited Development

➢ Integration of D.L.

Let f be an integrable function on neighborhood of zero, and let

$$F(x) = \int_0^x f(t)dt$$

the primitive of f, vanishing at zero, we have the following result which gives the L.D of F.

Limited Developments

Operation on Limited Development.

➢ Integration of L.D

- ***Proposition***

Let f be an integrable function on a neighborhood of zero, if f admits a limited expansion of order n at zero then its antiderivative $\boldsymbol{F}$ admits a limited expansion of order $\boldsymbol{n}+\mathbf{1}$ at zero. Furthermore, the regular part of the D.L. of F is the antiderivative, vanishing at zero, of the regular part of the limited expansion of f.

Limited Developments

Operation on Limited Development

- Integration d'un D.L.

- ***Example***

$$\frac{1}{1+x^2}$$

$$= 1 - x^2 + x^4 + \cdots + (-1)^n x^{2n}$$

$$+ o(x^{2n+1})$$

Limited Developments
Operation on Limited Development

- Integration d'un D.L.

We obtain

$$Arctg\ x$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \mathrm{o}(x^{2n+2})$$

Limited Developments
Operation on Limited Development

- ***Noticed***

$$(Arctg)^{(n)}(0)$$

$$= \begin{cases} 0 & si \quad n = 2k, \\ (-1)^k (2k)! & si \quad n = 2k+1 \end{cases}$$

Limited Developments

Operation on Limited Development

➢ Integration of L.D

- Example

1) Determine the L.D of order 5 at zero of the following functions:

$$Arcsin\,x, \ln(1+x)\ et\ Argsh\,x,$$

2) Determine the derivative of order 5 at zero of the function

$$f(x) = (Arcsin\,x)\ln(1+x)$$

Limited Developments

Operation on Limited Development

- Integration of L.D.

Solution :

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + o(x^5)$$

Limited Developments

Operation on Limited Development

- Integration of D.L.

Solution :

We obain

$$Arcsin\, x = x + \frac{x^3}{6} + \frac{3x^5}{40} + o(x^6)$$

Limited Developments

Operation on Limited Development

- Intégration d'un D.L.

- Solution :

Sachant que

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + o(x^4)$$

On obtient

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \mathrm{o}(x^5)$$

OPERATIONS SUR LES D.L.

➢ Intégration d'un D.L.

- Solution :

$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + o(x^5)$$

On obtient

$$Argsh\, x = x - \frac{x^3}{6} + \frac{3x^5}{40} + \mathrm{o}(x^6)$$

Limited Developments

Operation on Limited Development

- Intégration d'un D.L.

- Solution :

We write the limited expansion of order 5 of the function

$$f(x) = Arcsin\,x\,\ln(1+x)$$

Which is obtained by multiplying the regular parts of the L.D of $Arcsin\,x$ and $\ln(1+x)$

Limited Developments

Operation on Limited Development

➢ Integration of D.L.

- Solution :

We obtain:

$$Arcsin\,x\ln(1+x)=\left(x+\frac{x^3}{6}+\frac{3x^5}{40}\right)\left(x-\right.$$

Limited Developments

Operation on Limited Development

- Integration of L.D
- Solution :

$$Arcsin\, x \ln(1+x)$$

$$= \left(x^2 - \frac{x^3}{2} + \frac{x^4}{2} - \frac{x^5}{3}\right) + o(x^5)$$

but $$a_k = \frac{f^{(k)}(0)}{k!}$$

Limited Developments

Operation on Limited Development

- Intégration d'un D.L.

- Solution :

Which give

$$\left(Arcsin\,x\ln(1+x)\right)^{(5)}(0) = -\frac{5!}{3}$$

Limited Developments
Operation on Limited Development

➢ Derivation d'un D.L.

- ***Proposition***

suppose that

f admits a limited expansion of order n at zero,

1. f $is\ differentiable\ on\ a\ neighborhood\ of\ zero.$

2. f' admits a limited expansion of order $n-1$, at zero.

Limited Developments

Operation on Limited Development.

Differentiability of L.D

Then the regular part of the limited expansion of f' is the derivative of the regular part of the limited expansion of f. Indeed, because the regular part of the limited expansion of f is an antiderivative of the regular part of the limited expansion of f'.

Limited Developments
Operation on Limited Development

<u>Differentiability of L.D</u>

- ***<u>Noticed</u>***

1. The fact that f has a limited expansion of order n does not imply that f is differentiable.
2. Even if f is differentiable in the neighborhood of zero, the function f' may not admit a limited expansion at zero

Limited Developments

Operation on Limited Development

Differentiability of L.D

- Example1

Calculete the D.L to order 3 of the function

$$f(x) = \frac{1}{(1-x)^2}$$

We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + o(x^4)$$

Limited Developments

Operation on Limited Development

Differentiability of L.D

- Example 1

as $\frac{1}{1-x}$ is classe C^4 , so we

applies the previous criterion and by derivation

we have :

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + o(x^3)$$

Limited Developments
Operation on Limited Development

Differentiability of L.D

- Example 2

Find the D.L of the function $\tan x$ *at order 5 by derivation. In fact we have*

$$\tan x = -(\ln(\cos))'(x)$$

Limited Developments

Operation on Limited Development

<u>Differentiability of L.D</u>

- <u>Example 2</u>

Le D.L à l'ordre 6 de :

$$\ln(\cos(x)) = \ln\left(1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + o(x^6)\right)\right)$$

$$= \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right) - \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right)^2$$

$$+ \frac{1}{3}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right)^3 + o(x^6)$$

Limited Developments

Operation on Limited Development

Differentiability of L.D

- Example 2

$$\ln(\cos(x)) = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{90} + o(x^6)$$

The opposite of the derivative gives

$$tg\, x = x + \frac{x^3}{3} + \frac{x^5}{15} + o(x^5)$$