Computational Physics (PHYS514) Final Project

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Newton

This part gives calculations of the structures of various types of stars in Newtonian gravity, general relativity (GR), and alternative theories of gravity which try to surpass GR.

1 From the stellar structure equations to the Lane–Emden equation

1.1 Stellar structure in Newtonian gravity

We start with the standard Newtonian equations for hydrostatic equilibrium in a spherically symmetric star:
1. **Mass continuity:**

$$\frac{dm}{dr} = 4\pi r^2 \rho(r),$$

2. **Hydrostatic equilibrium:**

$$\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2}.$$

Here: - $m(r)$ is the mass enclosed within radius r, - $\rho(r)$ is the mass density, - $p(r)$ is the pressure, - G is the gravitational constant.

1.2 Polytropic equation of state

We then close the system using a polytropic equation of state:

$$p = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}},$$

where - K is a constant (related to the microphysics of the stellar material), - n is called the polytropic index, - $\gamma = 1 + \frac{1}{n}$.

1.3 Dimensionless variables

To transform these ODEs into the Lane–Emden equation, one introduces dimensionless variables that factor out the central values. Let

$$\rho(r) = \rho_c \theta^n(\xi),$$

where $\rho_c = \rho(0)$ is the central density, and define

$$r = a\xi,$$

for some length scale a that will be determined shortly. The quantity $\theta(\xi)$ is a dimensionless function satisfying $\theta(0) = 1$.

From the polytropic EOS, the central pressure is

$$p_c = K\rho_c^\gamma.$$

A convenient choice for a is

$$a^2 = \frac{(n+1)K\rho_c^{\frac{1}{n}}}{4\pi G}.$$

Substituting $\rho(r) = \rho_c \theta^n(\xi)$ into the equations and using the chosen a, one arrives at a single, dimensionless ODE for $\theta(\xi)$.

1.4 The Lane–Emden equation

The final result of this procedure is the Lane–Emden equation of index n:

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0.}$$

The corresponding boundary conditions at the center ($\xi = 0$) are: 1. $\theta(0) = 1$, since $\rho(0) = \rho_c$, 2. $\theta'(0) = 0$, for regularity at the origin.

Hence, we have everything that defines the Lane–Emden problem:

$$\begin{cases} \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0, \\ \theta(0) = 1, \quad \theta'(0) = 0. \end{cases}$$