# Computational Physics (PHYS414/514)

# Final Project

Mehmet Eren Erken

Koç University

# Newton

This part gives calculations of the structures of various types of stars in Newtonian gravity, general relativity (GR), and alternative theories of gravity which try to surpass GR.

## A. Lane–Emden Equation

### A.1. Stellar Structure in Newtonian Gravity

We start with the standard Newtonian equations for hydrostatic equilibrium in a spherically symmetric star:

1. **Mass Continuity**:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r),$$

2. **Hydrostatic Equilibrium**:

$$\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2}.$$

where:

- $m(r)$: mass enclosed within radius $r$,
- $\rho(r)$: mass density,
- $p(r)$: pressure,
- $G$: gravitational constant.

## A.2. Polytropic Equation of State

We then close the system using a polytropic equation of state:

$$p = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}},$$

where:

- $K$: constant related to the microphysics of the stellar material,
- $n$: polytropic index,
- $\gamma = 1 + \frac{1}{n}$: adiabatic index.

## A.3. Dimensionless Variables

To simplify the equations, we introduce dimensionless variables (e.g., Lane–Emden variables):

$$\theta = \left(\frac{\rho}{\rho_c}\right)^{1/n}, \quad \xi = \frac{r}{r_0},$$

where:

- $\rho_c$: central density,
- $r_0 = \sqrt{\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}}$: scaling factor for radius.

Substituting $\rho(r) = \rho_c\theta^n(\xi)$ into the equations and using $r_0$, we transform the ODEs into the Lane–Emden equation.

## A.4. The Lane–Emden Equation

The final result of this procedure is the Lane–Emden equation of index $n$:

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0.}$$

The corresponding boundary conditions at the center ($\xi = 0$) are: 1. $\theta(0) = 1$, since $\rho(0) = \rho_c$, 2. $\theta'(0) = 0$, for regularity at the origin.

Thus, the Lane–Emden problem is defined by:

$$\begin{cases} \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0, \\ \theta(0)=1, \quad \theta'(0)=0. \end{cases}$$

## A.5. Series Expansion at the Center

We verify the regularity condition near $\xi = 0$ by performing a power-series expansion. Assume:

$$\theta(\xi) = 1 + a_2\xi^2 + a_4\xi^4 + \dots$$

Plugging this into the Lane–Emden equation yields the coefficients $a_2, a_4, \dots$. A calculation from Newton.ipynb part A.1 gives:

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \cdots,$$

confirming $\theta'(0) = 0$.

### A.5.1. Solving the Lane–Emden equation for $n = 1$

When $n = 1$, the Lane–Emden equation becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta = 0,$$

with $\theta(0) = 1$ and $\theta'(0) = 0$.

This ODE can be solved analytically. For $n = 1$, the solution is:

$$\theta(\xi) = \frac{\sin\xi}{\xi}.$$

Indeed, one checks that

$$\theta(0) = \lim_{\xi\to 0}\frac{\sin\xi}{\xi} = 1, \quad \theta'(0) = 0,$$

and it satisfies the differential equation upon direct substitution. The solution can also be solved using sympy, which is demonstrated in Newton.ipynb part A.2.

## A.6. Defining the stellar surface and total mass

### A.6.1. Surface of the polytrope

Because $\rho(r) \propto \theta^n(\xi)$, the surface of the star is (by definition) at the first positive $\xi = \xi_n$ such that

$$\theta(\xi_n) = 0.$$

Then the physical radius of the star is

$$R = a\xi_n.$$

For $n = 1$, from $\sin(\xi_n)/\xi_n = 0$, the first positive root is $\xi_n = \pi$. For other integer $n$, one must solve numerically or use known special-function expansions.

### A.6.2. Total Mass of the Star

To find the total mass of the star, we start with the dimensionless form of the mass-continuity equation:

$$\frac{dm}{d\xi} = \xi^2\theta^n.$$

The total mass $M$ of the star is the mass enclosed at the surface, which corresponds to $\xi = \xi_n$, where $\xi_n$ is the dimensionless radius at which $\theta(\xi_n) = 0$. Integrating from the center ($\xi = 0$) to the surface ($\xi = \xi_n$):

$$M = 4\pi\rho_c a^3 \int_0^{\xi_n} \xi^2\theta^n \, d\xi.$$

Using integration by parts to simplify the integral, we start by rewriting the integrand:

$$\int_0^{\xi_n} \xi^2\theta^n \, d\xi = \left[-\xi^2\theta'\right]_0^{\xi_n} + \int_0^{\xi_n} 2\xi\theta' \, d\xi.$$

From the Lane–Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0,$$

we know:

$$\xi^2\theta^n = -\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right).$$

This implies that the integral of $\xi^2\theta^n$ simplifies directly using the surface boundary conditions:

$$\int_0^{\xi_n} \xi^2\theta^n \, d\xi = -\left[\xi^2\frac{d\theta}{d\xi}\right]_0^{\xi_n}.$$

At the center, $\xi = 0$, the regularity condition ensures that $\xi^2\theta' \to 0$. Therefore:

$$M = 4\pi\rho_c a^3 \left[-\xi_n^2\theta'(\xi_n)\right].$$

Rewriting $a = R/\xi_n$, where $R$ is the physical radius of the star, we obtain:

$$M = 4\pi\rho_c R^3 \left[-\frac{\theta'(\xi_n)}{\xi_n}\right].$$

## A.7. Mass–radius relation for polytropes

Finally, if one fixes the same polytropic index $n$ (i.e., all stars in the family have the same value of $n$) but allows different central densities $\rho_c$, then the scaling of $a$ with $\rho_c$ implies a specific power-law relation between $M$ and $R$.

### A.7.1. How $a$ depends on $\rho_c$

Recall

$$a^2 = \frac{(n+1)K\rho_c^{1/n}}{4\pi G} \quad \Longrightarrow \quad a \propto \rho_c^{1/(2n)}.$$

Hence

$$R = a\xi_n \propto \rho_c^{1/(2n)}.$$

### A.7.2. How $M$ Depends on $R$

Substituting this into the expression for $M$:

$$M \propto \rho_c \cdot R^3 \propto R^{-2n} \cdot R^3.$$

Simplifying gives:

$$M \propto R^{3-2n}.$$

Using the polytropic equation of state $p \propto \rho^{1+1/n}$ leads to:

$$M \propto R^{\frac{3-n}{1-n}}.$$

### A.7.3. Finding the Constant of Proportionality

To determine the constant of proportionality, we start by expressing the central density $\rho_c$ in terms of other variables. Recall the expression for the Lane–Emden scaling factor $a$:

$$a^2 = \frac{(n+1)K\rho_c^{1/n}}{4\pi G}.$$

Solving for $\rho_c$:

$$\rho_c = \left(\frac{4\pi G}{(n+1)K}\right)^n a^{-2n}.$$

Substituting $a = R/\xi_n$, we express $\rho_c$ as:

$$\rho_c = \left(\frac{4\pi G}{(n+1)K}\right)^n \left(\frac{R}{\xi_n}\right)^{-2n}.$$

Now, substitute this expression for $\rho_c$ into the total mass formula:

$$M = 4\pi \rho_c R^3 \left( -\frac{\theta'(\xi_n)}{\xi_n} \right).$$

Expanding $\rho_c$ explicitly:

$$M = 4\pi \left( \frac{4\pi G}{(n+1)K} \right)^n \left( \frac{R}{\xi_n} \right)^{-2n} R^3 \left( -\frac{\theta'(\xi_n)}{\xi_n} \right).$$

Simplify the powers of $R$ to consolidate the mass-radius relation:

$$M = (-4\pi) \left( \frac{4\pi G}{(n+1)K} \right)^n \xi_n^{1-n} \left( -\theta'(\xi_n) \right) R^{3-n}.$$

Factor out the terms that depend only on $n$, $K$, and $G$:

$$M = C(n) K^{n/(n-1)} G^{-n/(n-1)} R^{\frac{3-n}{1-n}},$$

where the dimensionless constant $C(n)$ is:

$$C(n) = 4\pi \, (4\pi)^n \, \frac{\xi_n^{1-n} \left( -\theta'(\xi_n) \right)}{(n+1)^n}.$$

# B. White Dwarf Data Fitting

## B.1. Converting $\log g$ to Radius

The surface gravity of a star $g$ (in CGS units) is related to its mass $M$ and radius $R$ through Newtonian gravity:

$$g = \frac{GM}{R^2},$$

where $G$ is the gravitational constant. For white dwarfs, $g$ is typically given as $\log g$ (base-10 logarithm in CGS units), and the mass $M$ is expressed in solar masses ($M_\odot$). To compute the radius $R$ in Earth radii ($R_\oplus$), we follow these steps:

1. Compute $g$ in cm/s$^2$ using:

$$g = 10^{\log g}.$$

2. Convert $M$ from solar masses to grams:

$$M_{\text{grams}} = \left(\frac{M}{M_\odot}\right) \times 1.989 \times 10^{33}.$$

3. Solve for $R$ in cm:

$$R = \sqrt{\frac{GM_{\text{grams}}}{g}}.$$

4. Convert $R$ to Earth radii:

$$R_{R_\oplus} = \frac{R}{6.371 \times 10^8}.$$

## B.2. Figure: Plot of White Dwarf Data

Below, the plotted results of the White Dwarf data can be seen. The plotting is done on Newton.ipynb part B.

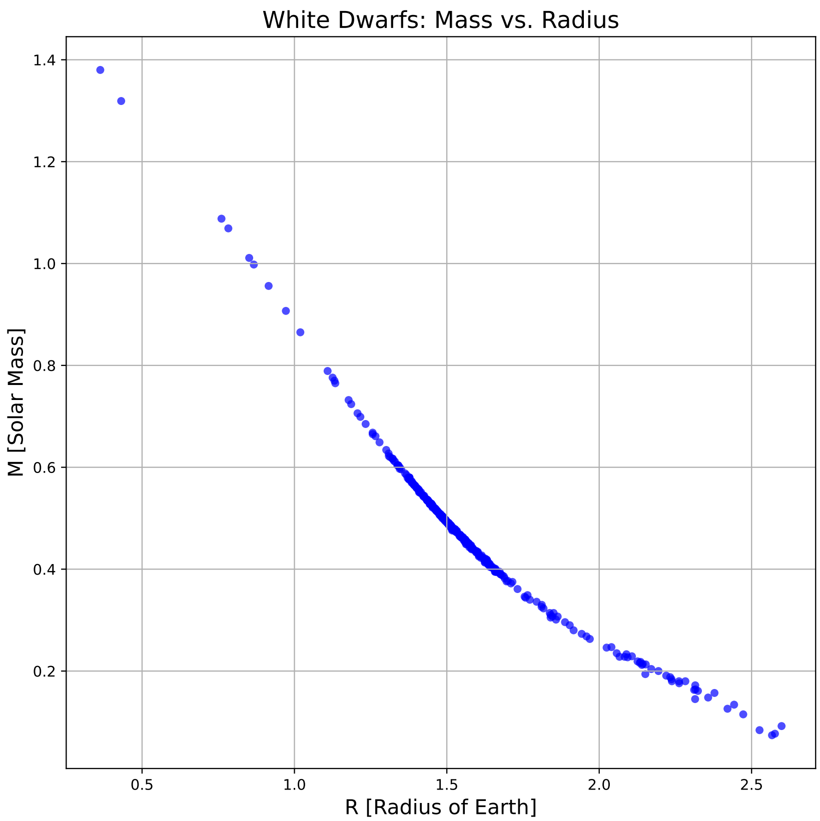


Figure 1: Visualization of the white dwarf data.

# C. Polytropic Fit for White Dwarfs

We have the EOS for cold WDs (in CGS units for pressure):

$$\boxed{P(\rho) \;=\; C\left[x\,(2x^2-3)\,\sqrt{1+x^2}\;+\;3\,\sinh^{-1}(x)\right], \quad x \;=\; \left(\frac{\rho}{D}\right)^{1/q}.}$$

We want the leading term for **small** $x$. This corresponds to **low-mass** WDs where the density $\rho$ is still well below the scale set by $D$.

## C.1. Series Expansion for Small $x$

Let's expand, term by term, for $x \to 0$.

### C.1.1. Inside the Bracket $\left[\ldots\right]$:

$$\begin{aligned} x\,(2\,x^2-3)\,\sqrt{1+x^2} &= \; x\left(-3+2\,x^2\right)\left[1+\tfrac{x^2}{2}-\tfrac{x^4}{8}+\ldots\right] \\ &= x\left[-3+2x^2+\ldots\right]\left[1+\tfrac{x^2}{2}+\ldots\right] \\ &= -3\,x+0.5\,x^3+1.375\,x^5+\ldots \quad \text{(we keep up to } x^5 \text{ terms).} \end{aligned}$$

### C.1.2. The Term $3\,\sinh^{-1}(x)$:

Recall for small $x$,

$$\sinh^{-1}(x) = x - x^3/6 + (3/40)\,x^5 + \ldots.$$

Multiplying by 3 yields:

$$3\,\sinh^{-1}(x) \;\approx\; 3\,x \;-\; \tfrac{1}{2}\,x^3 \;+\; 0.225\,x^5 + \ldots$$

### C.1.3. Summation:

Combining the terms:

$$\begin{aligned}\left[x\,(2x^2-3)\,\sqrt{1+x^2}\right] \;+\; 3\,\sinh^{-1}(x) \;&=\; \underbrace{(-3x+3x)}_{0}\\ &\quad+\; \underbrace{(0.5\,x^3-0.5\,x^3)}_{0}\\ &\quad+\; \underbrace{(1.375+0.225)\,x^5}_{1.6\,x^5} \;+\; \ldots\end{aligned}$$

Hence, the first **nonzero** term is:

$$= \frac{8}{5}\,x^5 \;+\; \mathcal{O}(x^7).^{1}$$

Therefore:

$$P(\rho) \;\approx\; C\,\frac{8}{5}\,x^5 \quad \text{for } x \to 0.$$

### C.1.4. Rewrite $x = \left(\frac{\rho}{D}\right)^{1/q}$:

$$P(\rho) \;\approx\; \frac{8C}{5}\left(\frac{\rho}{D}\right)^{\frac{5}{q}} \;=\; \underbrace{\left[\frac{8C}{5}\,\frac{1}{D^{5/q}}\right]}_{K_\star}\,\rho^{\,\frac{5}{q}}$$

[1] This derivation is also done by symbolic calculations through `sympy` in Newton.ipynb part C.1 through series expansion.

Hence the polytropic index in the exponent:

$$P \ \propto \ \rho^{5/q}$$

Compare to a standard polytrope:

$$P \ \propto \ \rho^{1+\frac{1}{n_\star}} \quad \Longrightarrow \quad 1 + \frac{1}{n_\star} \ = \ \frac{5}{q}$$

Solve for $n_\star$:

$$\frac{1}{n_\star} = \frac{5}{q} - 1 = \frac{5-q}{q} \implies n_\star = \frac{q}{5-q}$$

And the overall coefficient:

$$K_\star \ = \ \frac{8\,C}{5}\,\frac{1}{D^{5/q}}.$$

Thus:

$$\boxed{n_\star \ = \ \frac{q}{5-q}, \quad K_\star \ = \ \frac{8\,C}{5}\,D^{-\frac{5}{q}}}$$

These final relations are exactly what we wanted to show.

## C.2. Figure: Line Fitting to White Dwarf Data

Below, the fitted line to the White Dwarf data can be seen. The plotting is done in Newton.ipynb, part C.2.

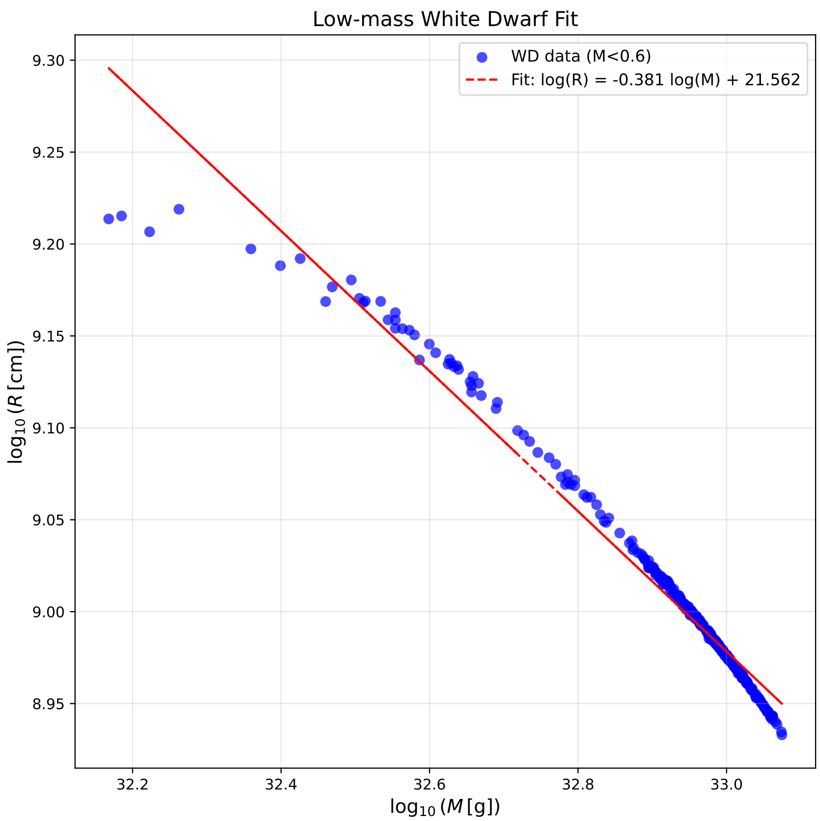


Figure 2: Fitting a line through the White Dwarf data.

The results of the fitting process are as follows. The inferred polytropic index is $n_\star = 1.5521$. The effective scaling constant is given by

$$c = 10^{\text{intercept}} = 3.6434 \times 10^{21}\,\mathrm{cm\,g^{-1/2}}.$$

The effective constant $K_\star$ was calculated using the scaling relation:

$$K_\star = \frac{4\pi G c^2}{(n_\star + 1)^2 \xi_1^2},$$

where $\xi_1$ is the first zero of the Lane-Emden equation for $n_\star = 1.55$ ($\xi_1 \approx 3.65$).

Finally, the calculated value of $K_\star$ is:

$$K_\star = 1.285 \times 10^{35}\,\mathrm{erg\,cm}^{3(1-\mathrm{n}_\star)}\,\mathrm{g}^{-\mathrm{n}_\star}.$$

## C.3. Figure: Central Density $\rho_c$ vs Mass $M$

The relationship between Central Density $\rho_c$ and Mass $M$ for White Dwarfs is shown below. The plots are generated in Newton.ipynb, parts C.2.2 and C.2.3.

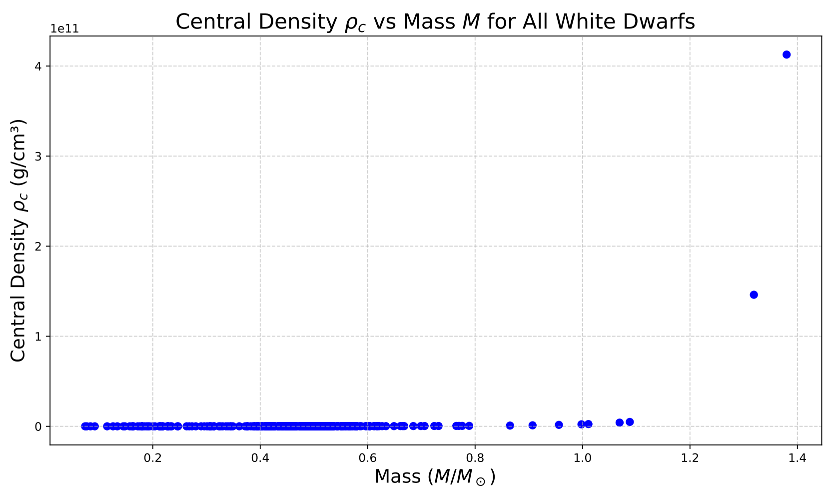


Figure 3: Plot of Central Density $\rho_c$ vs Mass $M$ for White Dwarfs. Full dataset shown to highlight the low-mass turning point.

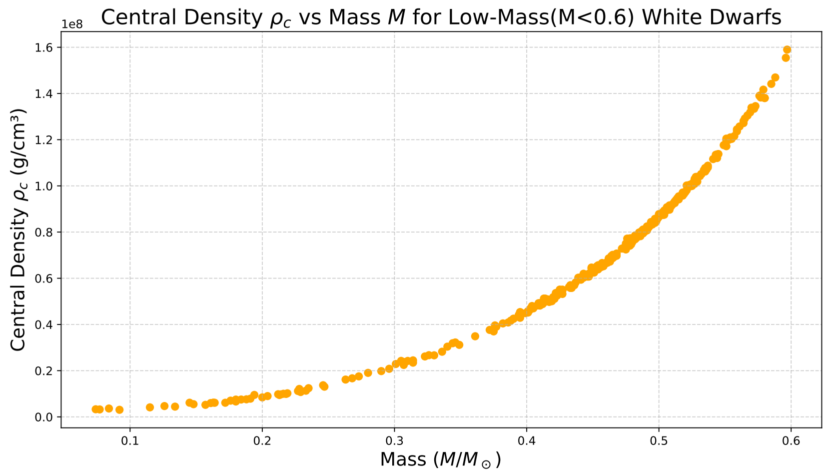


Figure 4: Focus on low-mass White Dwarf data.

# D. Finding best fits of $C$ and $D$

## D.1. We Already Know $q$

Previously, we assumed $q \approx 3$. Substituting into the EOS for Cold WDs, we have:

$$\boxed{P(\rho) \;=\; C\left[x\left(2x^2-3\right)\sqrt{x^2+1} \;+\; 3\,\sinh^{-1}(x)\right], \quad x = \left(\tfrac{\rho}{D}\right)^{1/q}.}$$

Once $D$ is chosen, we compute:

$$C \;=\; \frac{5}{8}\,K_\star\left[D^{5/q}\right] \quad\Longrightarrow\quad \text{(thus, } P(\rho) \text{ is fully specified).}$$

## D.2. Hydrostatic Equilibrium IVP

We solve the usual ODEs (in CGS units):

$$\begin{cases} \dfrac{dP}{dr} = -\,\rho(r)\,\dfrac{G\,m(r)}{r^2}, \\ \dfrac{dm}{dr} = 4\pi\,r^2\,\rho(r), \end{cases}$$

starting from $r = 0$ with

$$\rho(0) = \rho_c, \quad m(0) = 0.$$

We integrate outward until $\rho \approx 0$ (the star's surface). At each step:

$$P(r) = P(\rho(r)) \;=\; \text{(EOS for Cold WDs)},$$

and invert $P \leftrightarrow \rho$ numerically. Since we have an explicit expression for $P(\rho)$, we can invert $\rho(P)$ either using a small root-finding method inside the ODE solver or through a direct formula if the equation is rearranged appropriately. Alternatively, $\rho$ can be kept as the integration variable, and $d\rho/dr$ can be derived from $dP/d\rho$.

## D.3. Getting $(R, M)$

Once $\rho$ becomes negligible, we record:

$$R_\star(\rho_c), \quad M_\star(\rho_c).$$

By scanning over different $\rho_c$ values, we produce discrete points:

$$\big\{(R_i, M_i)\big\}_{i=1}^{10}.$$

This process is performed **once** for a single choice of $D$.

## D.4. Final Result

The actual values of $C$ and $D$, calculated theoretically, are as follows:

$$C = \frac{m^4 e^5}{24\pi^2\hbar^3} = 6.002332185660436 \times 10^{21},$$

$$D = \frac{\mu m^3 e^3 \mu_e}{3\pi^2\hbar^3} = 1.947864333345182 \times 10^{9}.$$

Through computations from Newton.ipynb, part D, best-fitted values are as follows:

$$\hat{D} = 3.3333 \times 10^{-6},$$

$$\hat{C} = 1.6938 \times 10^{12}.$$

# E. Plotting the Full $R$ vs $M$ Curve

## E.1. Full Electron-Degenerate EOS in the Relativistic Limit

We start with the general ($T = 0$) electron degeneracy pressure:

$$P(\rho) \ = \ C\Big[x\,(2x^2{-}3)\,\sqrt{1+x^2} + 3\,\sinh^{-1}(x)\Big], \quad x \ = \ \left(\tfrac{\rho}{D}\right)^{1/q}, \quad \text{with } q = 3.$$

**Nonrelativistic limit ($x \ll 1$).** By following part C.1.4 and substituting $q = 3$, one finds $P \propto \rho^{5/3}$, i.e. a polytrope with $n = \frac{3}{2}$ .

**Ultrarelativistic limit ($x \gg 1$).** Expanding the expression in the $P(\rho)$:

$$x\,(2x^2 - 3)\,\sqrt{1+x^2} \ + \ 3\,\sinh^{-1}(x)$$

for $x \to \infty$ using symoblic computations from Newton.ipynb, part E.1: gives a leading term *proportional to* $x^4$:

$$x\,(2x^2 - 3)\,\sqrt{1+x^2} \ \approx \ 2\,x^4 \quad \text{as } x \to \infty,$$

After substituting $x^3 = \rho/D$, we get:

$$P(\rho) \ \propto \ \rho^{4/3},$$

which corresponds to a polytrope of index $n = 3$.

Thus, the **relativistic limit** yields $P(\rho) \propto \rho^{4/3}$, and $n = 3$.

## E.2. Chandrasekhar Mass via $n = 3$ Polytrope

From polytrope theory (Lane–Emden approach), the mass-radius relation for a polytrope of index $n$ is summarized as:

$$M \;\propto\; R^{\frac{3-n}{1-n}}.$$

For $n < 3$, this results in a continuous family of solutions. However, when $n = 3$, the exponent becomes:

$$\frac{3-3}{1-3} \;=\; 0,$$

which means $M$ becomes a **constant**, independent of $R$. Physically, this implies the star's mass saturates at a maximum value in the ultrarelativistic regime, known as the **Chandrasekhar mass**, $M_{\mathrm{Ch}}$.

### E.2.1. Analytical Formula

From detailed polytrope analysis, the Chandrasekhar mass is given (in CGS units, for $\mu_e = 2$) by:

$$M_{\mathrm{Ch}} \;=\; \underbrace{\kappa}_{\text{a constant of nature}} \left(\mu_e\right)^{-2},$$

where $\kappa$ depends on $\hbar$, $c$, $m_e$, etc. Substituting:

$$C \;=\; \frac{m_e^4\, c^5}{24\pi^2\, \hbar^3}, \quad D \;=\; \frac{m_u\, m_e^3\, c^3\, \mu_e}{3\pi^2\, \hbar^3}, \quad \mu_e = 2,$$

and performing the polytrope calculations for $n = 3$, we recover the well-known result:

$$M_{\mathrm{Ch}} \;\approx\; 1.44\, M_{\odot}$$

## E.3. Numerical Confirmation

When solving the full EOS over a range of central densities $\rho_c$, the White Dwarf mass $M(\rho_c)$ initially increases with $\rho_c$ but eventually peaks and begins to **decrease** for very high $\rho_c$. This peak corresponds to the Chandrasekhar limit, beyond which no stable solutions exist.