Computational Physics (PHYS514) Final Project

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Newton

This part gives calculations of the structures of various types of stars in Newtonian gravity, general relativity (GR), and alternative theories of gravity which try to surpass GR.

1 From the Stellar Structure Equations to the Lane–Emden Equation

1.1 Stellar Structure in Newtonian Gravity

We start with the standard Newtonian equations for hydrostatic equilibrium in a spherically symmetric star:

1. **Mass Continuity**:
$$\frac{dm}{dr} = 4\pi r^2 \rho(r),$$
2. **Hydrostatic Equilibrium**:
$$\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2}.$$

where:

- $m(r)$: mass enclosed within radius r,
- $\rho(r)$: mass density,
- $p(r)$: pressure,
- G: gravitational constant.

1.2 Polytropic Equation of State

We then close the system using a polytropic equation of state:

$$p = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}},$$

where:

- K: constant related to the microphysics of the stellar material,
- n: polytropic index,
- $\gamma = 1 + \frac{1}{n}$: adiabatic index.

1.3 Dimensionless Variables

To simplify the equations, we introduce dimensionless variables (e.g., Lane–Emden variables):

$$\theta = \left(\frac{\rho}{\rho_c}\right)^{1/n}, \quad \xi = \frac{r}{r_0},$$

where:

- ρ_c: central density,
- $r_0 = \sqrt{\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}}$: scaling factor for radius.

Substituting $\rho(r) = \rho_c\theta^n(\xi)$ into the equations and using r_0, we transform the ODEs into the Lane–Emden equation.

1.4 The Lane–Emden Equation

The final result of this procedure is the Lane–Emden equation of index n:

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0.}$$

The corresponding boundary conditions at the center ($\xi = 0$) are: 1. $\theta(0) = 1$, since $\rho(0) = \rho_c$, 2. $\theta'(0) = 0$, for regularity at the origin.

Thus, the Lane–Emden problem is defined by:

$$\begin{cases} \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0, \\ \theta(0) = 1, \quad \theta'(0) = 0. \end{cases}$$

1.5 Series Expansion at the Center

We verify the regularity condition near $\xi = 0$ by performing a power-series expansion. Assume:

$$\theta(\xi) = 1 + a_2\xi^2 + a_4\xi^4 + \dots$$

Plugging this into the Lane–Emden equation yields the coefficients $a_2, a_4, \dots$. A calculation from Newton.ipynb part A gives:

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \cdots,$$

confirming $\theta'(0) = 0$.