Computational Physics (PHYS414/514)

Final Project

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Newton

This part gives calculations of the structures of various types of stars in Newtonian gravity, general relativity (GR), and alternative theories of gravity which try to surpass GR.

A. Lane–Emden Equation

A.1. Stellar Structure in Newtonian Gravity

We start with the standard Newtonian equations for hydrostatic equilibrium in a spherically symmetric star:

1. **Mass Continuity**:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r),$$

2. **Hydrostatic Equilibrium**:

$$\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2}.$$

where:

- $m(r)$: mass enclosed within radius r,
- $\rho(r)$: mass density,
- $p(r)$: pressure,
- G: gravitational constant.

A.2. Polytropic Equation of State

We then close the system using a polytropic equation of state:

$$p = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}},$$

where:

- K: constant related to the microphysics of the stellar material,
- n: polytropic index,
- $\gamma = 1 + \frac{1}{n}$: adiabatic index.

A.3. Dimensionless Variables

To simplify the equations, we introduce dimensionless variables (e.g., Lane–Emden variables):

$$\theta = \left(\frac{\rho}{\rho_c}\right)^{1/n}, \quad \xi = \frac{r}{r_0},$$

where:

- ρ_c: central density,
- $r_0 = \sqrt{\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}}$: scaling factor for radius.

Substituting $\rho(r) = \rho_c\theta^n(\xi)$ into the equations and using r_0, we transform the ODEs into the Lane–Emden equation.

A.4. The Lane–Emden Equation

The final result of this procedure is the Lane–Emden equation of index n:

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0.}$$

The corresponding boundary conditions at the center ($\xi = 0$) are: 1. $\theta(0) = 1$, since $\rho(0) = \rho_c$, 2. $\theta'(0) = 0$, for regularity at the origin.

Thus, the Lane–Emden problem is defined by:

$$\begin{cases} \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0, \\ \theta(0)=1, \quad \theta'(0)=0. \end{cases}$$

A.5. Series Expansion at the Center

We verify the regularity condition near $\xi = 0$ by performing a power-series expansion. Assume:

$$\theta(\xi) = 1 + a_2\xi^2 + a_4\xi^4 + \dots$$

Plugging this into the Lane–Emden equation yields the coefficients $a_2, a_4, \dots$. A calculation from Newton.ipynb part A.1 gives:

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \cdots,$$

confirming $\theta'(0) = 0$.

A.5.1. Solving the Lane–Emden equation for $n = 1$

When $n = 1$, the Lane–Emden equation becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta = 0,$$

with $\theta(0) = 1$ and $\theta'(0) = 0$.

This ODE can be solved analytically. For $n = 1$, the solution is:

$$\theta(\xi) = \frac{\sin\xi}{\xi}.$$

Indeed, one checks that

$$\theta(0) = \lim_{\xi\to 0}\frac{\sin\xi}{\xi} = 1, \quad \theta'(0) = 0,$$

and it satisfies the differential equation upon direct substitution. The solution can also be solved using sympy, which is demonstrated in Newton.ipynb part A.2.

A.6. Defining the stellar surface and total mass

A.6.1. Surface of the polytrope

Because $\rho(r) \propto \theta^n(\xi)$, the surface of the star is (by definition) at the first positive $\xi = \xi_n$ such that

$$\theta(\xi_n) = 0.$$

Then the physical radius of the star is

$$R = a\xi_n.$$

For $n = 1$, from $\sin(\xi_n)/\xi_n = 0$, the first positive root is $\xi_n = \pi$. For other integer n, one must solve numerically or use known special-function expansions.

A.6.2. Total Mass of the Star

To find the total mass of the star, we start with the dimensionless form of the mass-continuity equation:

$$\frac{dm}{d\xi} = \xi^2\theta^n.$$

The total mass M of the star is the mass enclosed at the surface, which corresponds to $\xi = \xi_n$, where ξ_n is the dimensionless radius at which $\theta(\xi_n) = 0$. Integrating from the center ($\xi = 0$) to the surface ($\xi = \xi_n$):

$$M = 4\pi\rho_c a^3 \int_0^{\xi_n} \xi^2\theta^n \, d\xi.$$

Using integration by parts to simplify the integral, we start by rewriting the integrand:

$$\int_0^{\xi_n} \xi^2\theta^n \, d\xi = \left[-\xi^2\theta'\right]_0^{\xi_n} + \int_0^{\xi_n} 2\xi\theta' \, d\xi.$$

From the Lane–Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0,$$

we know:

$$\xi^2\theta^n = -\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right).$$

This implies that the integral of $\xi^2\theta^n$ simplifies directly using the surface boundary conditions:

$$\int_0^{\xi_n} \xi^2\theta^n\,d\xi = -\left[\xi^2\frac{d\theta}{d\xi}\right]_0^{\xi_n}.$$

At the center, $\xi = 0$, the regularity condition ensures that $\xi^2\theta' \to 0$. Therefore:

$$M = 4\pi\rho_c a^3\left[-\xi_n^2\theta'(\xi_n)\right].$$

Rewriting $a = R/\xi_n$, where R is the physical radius of the star, we obtain:

$$M = 4\pi\rho_c R^3\left[-\frac{\theta'(\xi_n)}{\xi_n}\right].$$

A.7. Mass–radius relation for polytropes

Finally, if one fixes the same polytropic index n (i.e., all stars in the family have the same value of n) but allows different central densities ρ_c, then the scaling of a with ρ_c implies a specific power-law relation between M and R.

A.7.1. How a depends on ρ_c

Recall

$$a^2 = \frac{(n+1)K\rho_c^{1/n}}{4\pi G} \quad\Longrightarrow\quad a \propto \rho_c^{1/(2n)}.$$

Hence

$$R = a\xi_n \propto \rho_c^{1/(2n)}.$$

A.7.2. How M Depends on R

Substituting this into the expression for M:

$$M \propto \rho_c \cdot R^3 \propto R^{-2n} \cdot R^3.$$

Simplifying gives:

$$M \propto R^{3-2n}.$$

Using the polytropic equation of state $p \propto \rho^{1+1/n}$ leads to:

$$M \propto R^{\frac{3-n}{1-n}}.$$

A.7.3. Finding the Constant of Proportionality

To determine the constant of proportionality, we start by expressing the central density ρ_c in terms of other variables. Recall the expression for the Lane–Emden scaling factor a:

$$a^2 = \frac{(n+1)K\rho_c^{1/n}}{4\pi G}.$$

Solving for ρ_c:

$$\rho_c = \left(\frac{4\pi G}{(n+1)K}\right)^n a^{-2n}.$$

Substituting $a = R/\xi_n$, we express ρ_c as:

$$\rho_c = \left(\frac{4\pi G}{(n+1)K}\right)^n \left(\frac{R}{\xi_n}\right)^{-2n}.$$

Now, substitute this expression for ρ_c into the total mass formula:

$$M = 4\pi\rho_c R^3 \left(-\frac{\theta'(\xi_n)}{\xi_n}\right).$$

Expanding ρ_c explicitly:

$$M = 4\pi \left(\frac{4\pi G}{(n+1)K}\right)^n \left(\frac{R}{\xi_n}\right)^{-2n} R^3 \left(-\frac{\theta'(\xi_n)}{\xi_n}\right).$$

Simplify the powers of R to consolidate the mass-radius relation:

$$M = (-4\pi) \left(\frac{4\pi G}{(n+1)K}\right)^n \xi_n^{1-n} \left(-\theta'(\xi_n)\right) R^{3-n}.$$

Factor out the terms that depend only on n, K, and G:

$$M = C(n) K^{n/(n-1)} G^{-n/(n-1)} R^{\frac{3-n}{1-n}},$$

where the dimensionless constant $C(n)$ is:

$$C(n) = 4\pi \left(4\pi\right)^n \frac{\xi_n^{1-n} \left(-\theta'(\xi_n)\right)}{(n+1)^n}.$$