

Computational Physics (PHYS414/514)

Final Project

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# Newton

This part gives calculations of the structures of various types of stars in Newtonian gravity, general relativity (GR), and alternative theories of gravity which try to surpass GR.

## A. Lane–Emden Equation

### A.1. Stellar Structure in Newtonian Gravity

We start with the standard Newtonian equations for hydrostatic equilibrium in a spherically symmetric star:

1. **Mass Continuity:**

$$\frac{dm}{dr} = 4\pi r^2 \rho(r),$$

2. **Hydrostatic Equilibrium:**

$$\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2}.$$

where:

- $m(r)$ : mass enclosed within radius  $r$ ,
- $\rho(r)$ : mass density,
- $p(r)$ : pressure,
- $G$ : gravitational constant.

## A.2. Polytropic Equation of State

We then close the system using a polytropic equation of state:

$$p = K\rho^\gamma = K\rho^{1+\frac{1}{n}},$$

where:

- $K$ : constant related to the microphysics of the stellar material,
- $n$ : polytropic index,
- $\gamma = 1 + \frac{1}{n}$ : adiabatic index.

## A.3. Dimensionless Variables

To simplify the equations, we introduce dimensionless variables (e.g., Lane–Emden variables):

$$\theta = \left(\frac{\rho}{\rho_c}\right)^{1/n}, \quad \xi = \frac{r}{r_0},$$

where:

- $\rho_c$ : central density,
- $r_0 = \sqrt{\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}}$ : scaling factor for radius.

Substituting  $\rho(r) = \rho_c \theta^n(\xi)$  into the equations and using  $r_0$ , we transform the ODEs into the Lane–Emden equation.

#### A.4. The Lane–Emden Equation

The final result of this procedure is the Lane–Emden equation of index  $n$ :

$$\boxed{\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0.}$$

The corresponding boundary conditions at the center ( $\xi = 0$ ) are: 1.  $\theta(0) = 1$ , since  $\rho(0) = \rho_c$ , 2.  $\theta'(0) = 0$ , for regularity at the origin.

Thus, the Lane–Emden problem is defined by:

$$\begin{cases} \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0, \\ \theta(0) = 1, \quad \theta'(0) = 0. \end{cases}$$

#### A.5. Series Expansion at the Center

We verify the regularity condition near  $\xi = 0$  by performing a power-series expansion. Assume:

$$\theta(\xi) = 1 + a_2 \xi^2 + a_4 \xi^4 + \dots$$

Plugging this into the Lane–Emden equation yields the coefficients  $a_2, a_4, \dots$

A calculation from Newton.ipynb part A.1 gives:

$$\theta(\xi) = 1 - \frac{1}{6} \xi^2 + \frac{n}{120} \xi^4 - \dots,$$

confirming  $\theta'(0) = 0$ .

### A.5.1. Solving the Lane–Emden equation for $n = 1$

When  $n = 1$ , the Lane–Emden equation becomes

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) + \theta = 0,$$

with  $\theta(0) = 1$  and  $\theta'(0) = 0$ .

This ODE can be solved analytically. For  $n = 1$ , the solution is:

$$\theta(\xi) = \frac{\sin \xi}{\xi}.$$

Indeed, one checks that

$$\theta(0) = \lim_{\xi \rightarrow 0} \frac{\sin \xi}{\xi} = 1, \quad \theta'(0) = 0,$$

and it satisfies the differential equation upon direct substitution. The solution can also be solved using sympy, which is demonstrated in Newton.ipynb part A.2.

## A.6. Defining the stellar surface and total mass

### A.6.1. Surface of the polytrope

Because  $\rho(r) \propto \theta^n(\xi)$ , the surface of the star is (by definition) at the first positive  $\xi = \xi_n$  such that

$$\theta(\xi_n) = 0.$$

Then the physical radius of the star is

$$R = a\xi_n.$$

For  $n = 1$ , from  $\sin(\xi_n)/\xi_n = 0$ , the first positive root is  $\xi_n = \pi$ . For other integer  $n$ , one must solve numerically or use known special-function expansions.

### A.6.2. Total Mass of the Star

To find the total mass of the star, we start with the dimensionless form of the mass-continuity equation:

$$\frac{dm}{d\xi} = \xi^2 \theta^n.$$

The total mass  $M$  of the star is the mass enclosed at the surface, which corresponds to  $\xi = \xi_n$ , where  $\xi_n$  is the dimensionless radius at which  $\theta(\xi_n) = 0$ . Integrating from the center ( $\xi = 0$ ) to the surface ( $\xi = \xi_n$ ):

$$M = 4\pi\rho_c a^3 \int_0^{\xi_n} \xi^2 \theta^n d\xi.$$

Using integration by parts to simplify the integral, we start by rewriting the integrand:

$$\int_0^{\xi_n} \xi^2 \theta^n d\xi = [-\xi^2 \theta']_0^{\xi_n} + \int_0^{\xi_n} 2\xi \theta' d\xi.$$

From the Lane–Emden equation:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0,$$

we know:

$$\xi^2 \theta^n = -\frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right).$$

This implies that the integral of  $\xi^2 \theta^n$  simplifies directly using the surface boundary conditions:

$$\int_0^{\xi_n} \xi^2 \theta^n d\xi = - \left[ \xi^2 \frac{d\theta}{d\xi} \right]_0^{\xi_n}.$$

At the center,  $\xi = 0$ , the regularity condition ensures that  $\xi^2 \theta' \rightarrow 0$ . Therefore:

$$M = 4\pi \rho_c a^3 \left[ -\xi_n^2 \theta'(\xi_n) \right].$$

Rewriting  $a = R/\xi_n$ , where  $R$  is the physical radius of the star, we obtain:

$$M = 4\pi \rho_c R^3 \left[ -\frac{\theta'(\xi_n)}{\xi_n} \right].$$

## A.7. Mass–radius relation for polytropes

Finally, if one fixes the same polytropic index  $n$  (i.e., all stars in the family have the same value of  $n$ ) but allows different central densities  $\rho_c$ , then the scaling of  $a$  with  $\rho_c$  implies a specific power-law relation between  $M$  and  $R$ .

### A.7.1. How $a$ depends on $\rho_c$

Recall

$$a^2 = \frac{(n+1)K\rho_c^{1/n}}{4\pi G} \implies a \propto \rho_c^{1/(2n)}.$$

Hence

$$R = a\xi_n \propto \rho_c^{1/(2n)}.$$

### A.7.2. How $M$ Depends on $R$

Substituting this into the expression for  $M$ :

$$M \propto \rho_c \cdot R^3 \propto R^{-2n} \cdot R^3.$$

Simplifying gives:

$$M \propto R^{3-2n}.$$

Using the polytropic equation of state  $p \propto \rho^{1+1/n}$  leads to:

$$M \propto R^{\frac{3-n}{1-n}}.$$

### A.7.3. Finding the Constant of Proportionality

To determine the constant of proportionality, we start by expressing the central density  $\rho_c$  in terms of other variables. Recall the expression for the Lane–Emden scaling factor  $a$ :

$$a^2 = \frac{(n+1)K\rho_c^{1/n}}{4\pi G}.$$

Solving for  $\rho_c$ :

$$\rho_c = \left( \frac{4\pi G}{(n+1)K} \right)^n a^{-2n}.$$

Substituting  $a = R/\xi_n$ , we express  $\rho_c$  as:

$$\rho_c = \left( \frac{4\pi G}{(n+1)K} \right)^n \left( \frac{R}{\xi_n} \right)^{-2n}.$$



Now, substitute this expression for  $\rho_c$  into the total mass formula:

$$M = 4\pi\rho_c R^3 \left( -\frac{\theta'(\xi_n)}{\xi_n} \right).$$

Expanding  $\rho_c$  explicitly:

$$M = 4\pi \left( \frac{4\pi G}{(n+1)K} \right)^n \left( \frac{R}{\xi_n} \right)^{-2n} R^3 \left( -\frac{\theta'(\xi_n)}{\xi_n} \right).$$

Simplify the powers of  $R$  to consolidate the mass-radius relation:

$$M = (-4\pi) \left( \frac{4\pi G}{(n+1)K} \right)^n \xi_n^{1-n} (-\theta'(\xi_n)) R^{3-n}.$$

Factor out the terms that depend only on  $n$ ,  $K$ , and  $G$ :

$$M = C(n) K^{n/(n-1)} G^{-n/(n-1)} R^{\frac{3-n}{1-n}},$$

where the dimensionless constant  $C(n)$  is:

$$C(n) = 4\pi (4\pi)^n \frac{\xi_n^{1-n} (-\theta'(\xi_n))}{(n+1)^n}.$$

## B. White Dwarf Data Fitting

### B.1. Converting $\log g$ to Radius

The surface gravity of a star  $g$  (in CGS units) is related to its mass  $M$  and radius  $R$  through Newtonian gravity:

$$g = \frac{GM}{R^2},$$

where  $G$  is the gravitational constant. For white dwarfs,  $g$  is typically given as  $\log g$  (base-10 logarithm in CGS units), and the mass  $M$  is expressed in solar masses ( $M_\odot$ ). To compute the radius  $R$  in Earth radii ( $R_\oplus$ ), we follow these steps:

1. Compute  $g$  in  $\text{cm/s}^2$  using:

$$g = 10^{\log g}.$$

2. Convert  $M$  from solar masses to grams:

$$M_{\text{grams}} = \left( \frac{M}{M_\odot} \right) \times 1.989 \times 10^{33}.$$

3. Solve for  $R$  in cm:

$$R = \sqrt{\frac{GM_{\text{grams}}}{g}}.$$

4. Convert  $R$  to Earth radii:

$$R_{R_\oplus} = \frac{R}{6.371 \times 10^8}.$$

## B.2. Figure: Plot of White Dwarf Data

Below, the plotted results of the White Dwarf data can be seen. The plotting is done on Newton.ipynb part B.

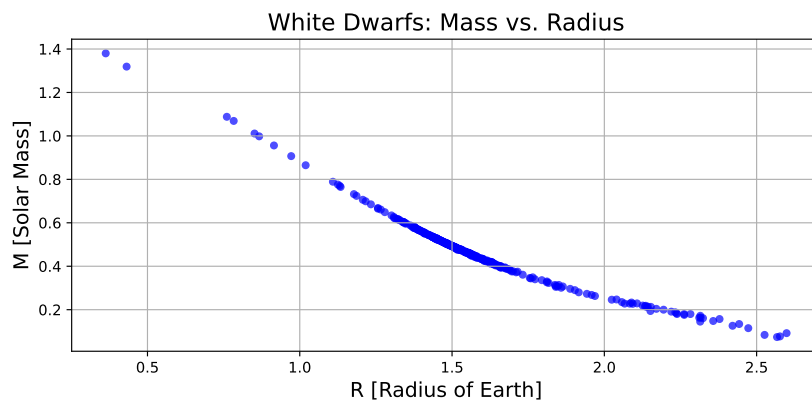


Figure 1: Visualization of the white dwarf data.