# Hamming codes: review

The $(7, 4)$ binary Hamming code consists of $2^4 = 16$ 7-bit codewords that satisfy three parity-check equations.

$$\begin{aligned} c_1 \oplus c_3 \oplus c_5 \oplus c_7 &= 0 \\ c_2 \oplus c_3 \oplus c_6 \oplus c_7 &= 0 \\ c_4 \oplus c_5 \oplus c_6 \oplus c_7 &= 0 \end{aligned}$$

We can characterize the code using the parity-check matrix $H$:

$$\mathbf{c}H^T = [\, c_1 \; c_2 \; c_3 \; c_4 \; c_5 \; c_6 \; c_7 \,] \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}^T = 0$$

*Check bits* $c_1, c_2, c_4$ can be computed from $c_3, c_5, c_6, c_7$.

$$\begin{aligned} c_1 &= c_3 \oplus c_5 \oplus c_7 \\ c_2 &= c_3 \oplus c_6 \oplus c_7 \\ c_4 &= c_5 \oplus c_6 \oplus c_7 \end{aligned} \iff [\, c_1 \; c_2 \; c_4 \,] = [\, c_3 \; c_5 \; c_6 \; c_7 \,] \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

# Hamming codes: error detection and correction

Each codeword bit affects at least one equation. Therefore every single-bit error can be detected.

Each bit is checked by a *unique* set of equations. Therefore the error location can be determined by *which* parity-check equations fail.

*Definition*: the *syndrome* $\mathbf{s} = [\, s_0\, s_1\, s_2\,]$ of received vector $\mathbf{r} = [\, r_1\, r_2 \ldots r_7\,]$ is the binary vector that tells which parity-check equations are *not* satisfied.

$$\begin{aligned} s_0 &= r_1 \oplus r_3 \oplus r_5 \oplus r_7 \\ s_1 &= r_2 \oplus r_3 \oplus r_6 \oplus r_7 \\ s_2 &= r_4 \oplus r_5 \oplus r_6 \oplus r_7 \end{aligned} \iff [\, s_0\, s_1\, s_2\,] = [\, r_1\, \ldots\, r_7\,]\, H^T$$

When $\mathbf{s} = 0$, the decoder *assumes* that no error has occurred. This is the most likely conclusion under reasonable assumptions.

Each nonzero value of $\mathbf{s}$ corresponds to an error in exactly one of $2^3 - 1 = 7$ bit positions. The syndrome identifies the location of a single error.

For this parity-check matrix $H$, the syndrome $\mathbf{s} = [\, s_0\, s_1\, s_2\,]$ is the binary representation of the assumed error location (most significant bit is $s_2$).

# Hamming codes: minimum distance

Hamming codes can correct single errors. Thus $d^* \geq 2t+1 = 2 \cdot 1 + 1 = 3$.

When used for error detection only, Hamming codes detect double errors.

Fact: minimum distance is exactly 3. Therefore Hamming codes can either correct single errors or detect double errors (but not both simultaneously).

A Hamming code with $m$ parity-check bits has $2^m - 1$ nonzero syndromes, hence blocklength $n = 2^m - 1$. The rate quickly approaches 1 for large $n$.

| $m$ | $n$ | $k$ | rate |
|---|---|---|---|
| 2 | 3 | 1 | 0.3333 |
| 3 | 7 | 4 | 0.5714 |
| 4 | 15 | 11 | 0.7333 |
| 5 | 31 | 26 | 0.8387 |
| 6 | 63 | 57 | 0.9047 |
| 8 | 255 | 247 | 0.9686 |
| 15 | 32767 | 32752 | 0.9995 |
| 32 | 4294967295 | 4294967263 | 1.0000 |

# Extended (expanded, expurgated) Hamming codes

Two easy ways to "extend" a Hamming code:

- Add overall parity-check *bit*: $c_0 = c_1 \oplus \cdots \oplus c_7 \Leftrightarrow c_0 \oplus \cdots \oplus c_7 = 0$.

$$H_1 = \left[\begin{array}{c|ccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

  This *expanded* code has blocklength $8$ but same number of codewords. Code parameters: $(8, 4, 4)$, rate $1/2$.

- Add overall parity-check *equation*: $c_1 \oplus c_2 \oplus \cdots \oplus c_6 \oplus c_7 = 0$.

$$H_2 = \left[\begin{array}{ccccccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

  This *expurgated* code consists of Hamming codewords with even parity. Code parameters: $(7, 3, 4)$, rate $3/7$.

# Extended Hamming codes: minimum distance

Both expanded and expurgated Hamming codes are constructed by adding redundancy to code with minimum distance 3.

- The minimum distance of extended codes is no smaller, hence $\geq 3$.
- All codewords have even parity, so distance between codewords is even.

Therefore the minimum distance is an even number and so is $\geq 4$.

- Hamming codes contain codewords of weight 3.
- The additional parity-check bit increases distance by at most 1.

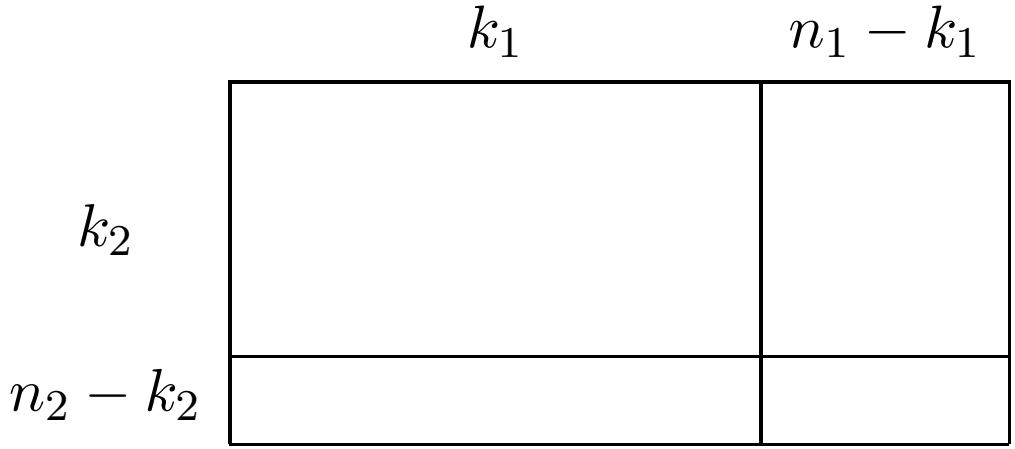
Therefore the minimum distance of extended Hamming codes is $d^* = 4$.

These codes can correct single errors and *simultaneously* detect double errors.

Double error is indicated by nonzero syndrome but even overall parity.

# General product codes

Let $\mathcal{C}_1$ be an $(n_1, k_1)$ block code and let $\mathcal{C}_2$ be an $(n_2, k_2)$ block code. The *product code* $\mathcal{C}_1 \otimes \mathcal{C}_2$ is an $(n_1 n_2, k_1 k_2)$ code.




Encoder (systematic) for product code:

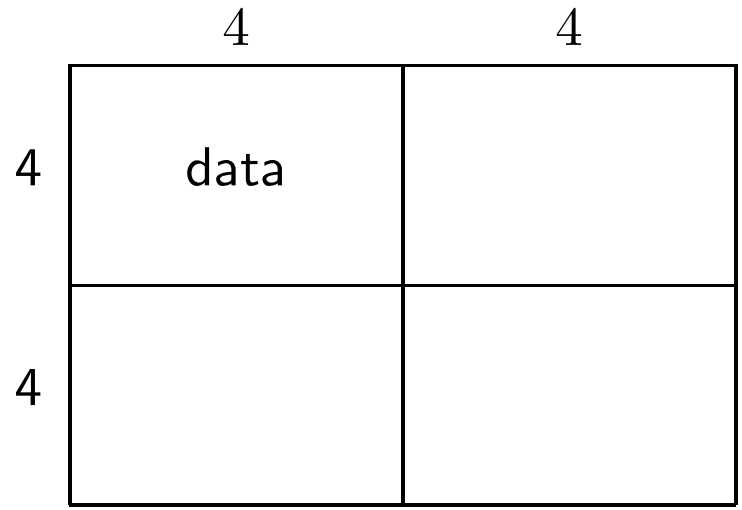
- First arrange $k_1 k_2$ information symbols in a $k_2 \times k_1$ array.
- Then encode first $k_2$ rows using code $\mathcal{C}_1$.
- Finally encode *all* $n_1$ columns using code $\mathcal{C}_2$.

Fact: the minimum distance of $\mathcal{C}_1 \otimes \mathcal{C}_2$ is $d^* = d_1^* \cdot d_2^*$.

By definition, every *column* is a codeword of $\mathcal{C}_2$. But if $\mathcal{C}_1$ and $\mathcal{C}_2$ are *linear* codes, then all *rows* are codewords of $\mathcal{C}_1$. This definition assumes systematic encoders for $\mathcal{C}_1$ and $\mathcal{C}_2$.

# General product code example

Consider the product of two $(8, 4, 4)$ expanded Hamming codes.




Product code parameters: $(n, k, d^*) = (64, 16, 16)$. Rate: $1/4$

Error correcting ability: $t = \lfloor (16 - 1)/2) \rfloor = 7$

Product codes can be decoded up to the guaranteed error correcting ability. The decoding procedure requires a column decoder that can correct both errors and erasures. (Blahut chapter 12.)

We will find more efficient codes; e.g., the (64,25,16) expanded BCH code needs only 39 check bits for same minimum distance.

# Nonbinary single error correcting code

The single check equation

$$c_1 + c_2 + \cdots + c_n = 0$$

allows detection of a single symbol error in a received $n$-tuple.

Furthermore, the syndrome $s$ defined by

$$s = r_1 + r_2 + \cdots + r_n$$

indicates the *magnitude* of the error. If the error is in location $i$ and the incorrect symbol is $r_i = c_i + e_i$, then

$$s \;=\; r_1 + r_2 + \cdots + r_n \;=\; c_1 + \cdots + (c_i + e_i) + \cdots + c_n \;=\; e_i \,.$$

The syndrome tells exactly *what* should be subtracted from the incorrect symbol in order to obtain a codeword.

What is not known is *where* the error is — which symbol is wrong.

# More equations needed

A second equation is needed to identify the error location. The effect of an error magnitude on the syndrome should be different for each location.

A reasonable choice for this second equation:

$$1 \cdot c_1 + 2 \cdot c_2 + \cdots + n \cdot c_n = 0\,.$$

Now every valid codeword satisfies two equations:

$$1 \cdot c_1 + 1 \cdot c_2 + \cdots + 1 \cdot c_n = 0$$
$$1 \cdot c_1 + 2 \cdot c_2 + \cdots + n \cdot c_n = 0$$

We can derive encoding equations to express $c_1, c_2$ in terms of $c_3, \ldots, c_n$.

*Example*: Let symbols be $4$-bit values with addition modulo $16$. For $n = 15$,

$$H = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & 15 \end{bmatrix}$$

is parity-check matrix for a code that can correct single symbols errors. Almost.

# Decoding procedure

Suppose there is a single error of magnitude $e_i \neq 0$ in location $i$.

The syndrome $\mathsf{s} = [\, s_0 \; s_1 \,]$ can be expressed in terms of unknowns $i$ and $e_i$:

$$s_0 = \sum_{j=1}^{n} r_j \;=\; e_i + \sum_{j=1}^{n} c_j \;=\; e_i$$

$$s_1 = \sum_{j=1}^{n} j r_j \;=\; i e_i + \sum_{j=1}^{n} j c_j \;=\; i e_i$$

We can determine $e_i$ and $i$ from the syndrome equations:

$$e_i = s_0$$

$$i \;=\; \frac{i e_i}{e_i} = \frac{s_1}{s_0}$$

Sadly, division is not always defined for modulo $16$ arithmetic. E.g., suppose $s_0 = 4$, $s_1 = 8$. Then $s_1 = i s_0 \bmod 16$ has four solutions:

$$2,\; 6,\; 10,\; 14\,.$$

We cannot be certain where the single error is located.

# Finite fields

This problem with division is solved by using a "better" multiplication.

We will define $\mathrm{GF}(16)$, the *field* of $16$ elements.

In $\mathrm{GF}(16)$, multiplication has an inverse operation of division, and most of the other familiar properties of arithmetic are valid.

Another approach: mod $17$ arithmetic with channel alphabet $\{0, 1, \ldots, 16\}$.

The "parity-check" matrix for a 1EC code over $\mathrm{GF}(17)$ is

$$H = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & 16 \end{bmatrix}.$$

The error pattern and location can be computed using the above equations:

$$\text{error pattern: } e_i = s_0$$

$$\text{error location: } i = \frac{s_1}{s_0}$$

Using either $\mathrm{GF}(16)$ or modulo $17$ arithmetic, these equations can be solved when $s_0 \neq 0$.

# Reed-Solomon codes

The codes over $\mathrm{GF}(16)$ and $\mathrm{GF}(17)$ are examples of *Reed-Solomon* codes.

Reed-Solomon codes use symbols from finite field $\mathrm{GF}(Q)$ and have $n = Q - 1$.

Each row of $H$ consists of consecutive powers of elements of $\mathrm{GF}(Q)$.

When the elements are chosen carefully, each additional check equation increases the minimum distance by $1$.

For example, the following parity-check matrix corresponds to $4$ equations:

$$H = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & 16 \\ 1 & 4 & 9 & \cdots & 256 \\ 1 & 8 & 27 & \cdots & 4096 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & 16 \\ 1 & 4 & 9 & \cdots & 1 \\ 1 & 8 & 10 & \cdots & 16 \end{bmatrix}$$

This PC matrix defines a code over $\mathrm{GF}(17)$ with minimum distance $5$.
It can correct two symbol errors in a codeword of length $16$.

Decoding procedures for Reed-Solomon codes are chief goal of this course.