# Lecture 7: Methods for combining codes.

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# Outline

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# Asymptotic regime, $n \to \infty$

$\frac{d}{n} \to \delta$ (relative minimum distance), $\frac{k}{n} \to R$ (code rate).

## Definition

A code family $\{\mathcal{C}_n\}$ is said to be *asymptotically good* if there exist constants $R, \delta > 0$:

- $\frac{k_n}{n} \geq R > 0$;
- $\frac{d_n}{n} \geq \delta > 0$;

# Are the codes we already know asymptotically good?

1. $(n = 2^m - 1, k = 2^m - m - 1, d = 3)_2$ Hamming codes
   - $R = \frac{2^m - m - 1}{2^m - 1} \to 1;$
   - $\delta = \frac{3}{2^m - 1} \to 0.$
2. $(n = 2^m, k, d)_2$ $RM(m, s)$ code
   - $k = \sum_{i=0}^{s} \binom{m}{i} = V_s;$
   - $d = 2^{m-s};$
   - $R = \frac{V_r}{2^m}$
   - $\delta = 2^{-s}.$

**Statement**

*Hamming and RM codes are asymptotically bad.*

# Are the codes we already know asymptotically good?

BCH codes:

- $t = \text{const.}$ Hamming bound

$$n - k \geq t \log n + O(1).$$

BCH code

$$n - k \leq t \log n + O(1).$$

BCH codes are good!

- $t$ grows with $n$

**Theorem**

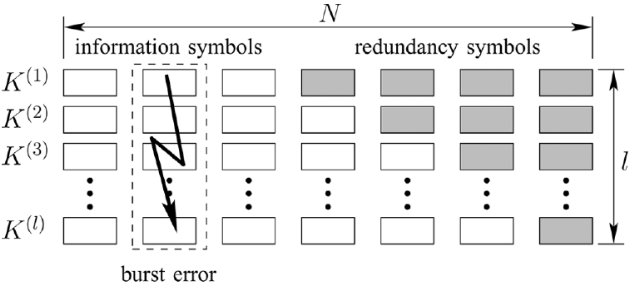
*Let $n \to \infty$ and $\delta > 0$, then the rate of BCH code $R \to 0$.*

BCH codes are asymptotically bad.

Solution: combine existing codes and construct new asymptotically good codes!

# Outline

# Interleaved codes

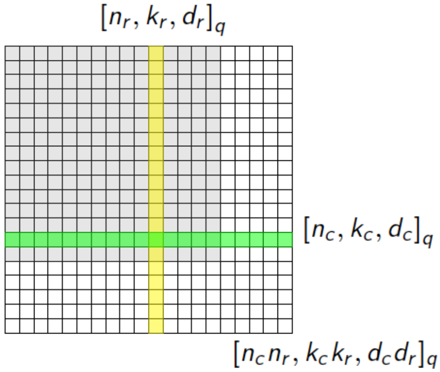



$$R = \frac{\sum R_i}{\ell}.$$

# Outline

# Construction

A codeword of a product code is a matrix whose rows are codewords of the first component code and whose columns are codewords of the second component code.

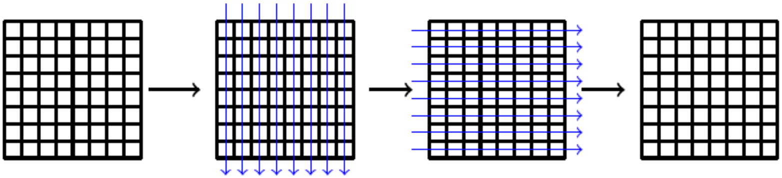


# Parameters

Consider a product code $\mathcal{C}$ constructed from row code $\mathcal{C}_r$ and column code $\mathcal{C}_c$, then

$$\begin{aligned} n(\mathcal{C}) &= n_r n_c \\ R(\mathcal{C}) &= R(\mathcal{C}_r)R(\mathcal{C}_c) \\ d(\mathcal{C}) &\geq d(\mathcal{C}_r)d(\mathcal{C}_c) \end{aligned}$$

# Iterative decoder



# Generator matrix

**Statement**

*Let $G_r$ and $G_c$ be generator matrices of a row code $\mathcal{C}_r$ and a column code $\mathcal{C}_c$, then*

$$G = G_r \otimes G_c.$$

Recall the Kronecker product definition. Let $\mathbf{X} = [x_{i,j}]$ be of size $m_x \times n_x$, $\mathbf{Y} = [y_{i,j}]$ be of size $m_y \times n_y$, then

$$X \otimes Y = \begin{bmatrix} x_{1,1}Y & x_{1,2}Y & \dots & x_{1,n_x}Y \\ x_{2,1}Y & x_{2,2}Y & \dots & x_{2,n_x}Y \\ \vdots & \vdots & \ddots & \vdots \\ x_{m_x,1}Y & x_{m_x,2}Y & \dots & x_{m_x,n_x}Y \end{bmatrix}$$

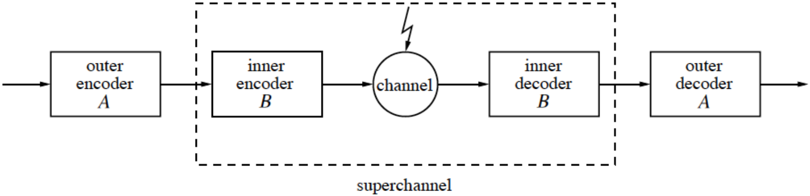
# Product of cyclic codes

**Statement**

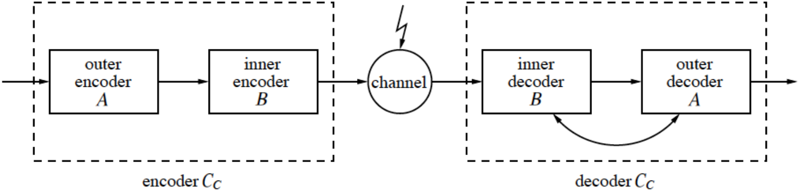
*Let $\mathcal{C}_r$ and $\mathcal{C}_c$ be cyclic codes with $(n_r, n_c) = 1$, then $\mathcal{C} = \mathcal{C}_r \otimes \mathcal{C}_c$ is also cyclic.*

# Outline

# Forney's view



# Concatenation according to Blokh and Zyablov



# Construction

Inner and outer codes:

- Outer $(n_a, k_a)$ code $A$ over $\mathbb{F}_{q^{k_b}}$.
- Inner $(n_b, k_b)$ code $B$ over $\mathbb{F}_q$.

Parameters of $A \diamond B$:

$$\begin{aligned} N &= n_a n_b \text{ (symbols from } \mathbb{F}_q) \\ R &= R_a R_b \\ D &\geq d_a d_b \end{aligned}$$

# Zyablov bound

**Theorem**

*Let $R \in (0,1)$, then it is possible to construct a code of rate $R$ and distance*

$$\delta(R) = \max_{R \leq r \leq 1} \left(1 - \frac{R}{r}\right) h^{-1}(1-r),$$

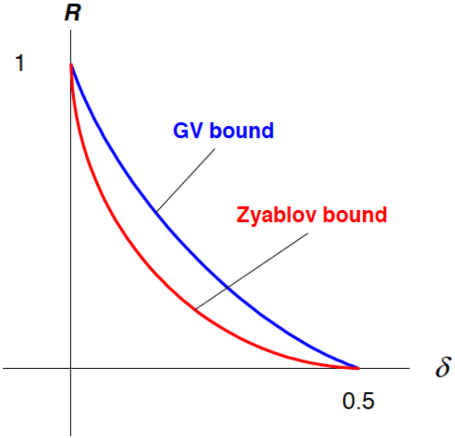
*where $h^{-1}$ is the inverse of entropy function.*

**Proof.**

Outer code: $[n_a, k_a]$ RS code over $\mathbb{F}_Q$, $Q = q^{k_b}$, $n_a = Q - 1$.

Inner code: $[n_b, k_b]$ code over $\mathbb{F}_q$, which meets the VG bound. □

# Zyablov bound



# Decoder

Decoder all the inner codes, then decode the outer code.

With this decoder we can decode up to $\frac{d_a d_b}{4}$ errors.

How to improve the number of correctable errors? We want to decode up to $D/2$.

# Generalized minimum distance decoder

Forget for a while about concatenated codes. Assume we are give a code $\mathcal{C}$ of length $n$. We send a codeword $\mathbf{x}$ and received a sequence $\mathbf{y}$ with errors. Assume are also provided with the reliabilities $\alpha_i$, $i = 1, \ldots, n$.

$$f(\mathbf{x}, \mathbf{y}, \alpha) = \sum_{i=1}^{n} \alpha_i \phi(x_i, y_i),$$

where $\phi(x_i, y_i) = +1$ if $x_i = y_i$ and $\phi(x_i, y_i) = -1$ otherwise.

# Generalized minimum distance decoder

$$d(\mathbf{x}, \mathbf{y}, \alpha) = n - f(\mathbf{x}, \mathbf{y}, \alpha).$$

**Theorem**

*Forney*
*Assume we are given an $(n, k, d)$ code $\mathcal{C}$. If there exists a codeword $c$, such that $d(c, \mathbf{y}, \alpha) < d$, then $c$ will be recovered from one of the decoding trials.*

*In a trial $i$, $i = 0, \ldots, d-1$, we erase $i$ least reliable symbols.*

# GMD for concatenated codes

Let us decode all the inner codes. How to introduce the reliabilities for outer code?

Let us consider the first inner code. $\mathbf{y}$ is the sequence to be decoded, $\hat{\mathbf{x}}$ is the decoding result.

$$\begin{aligned}\alpha_1 &= 0, \ \text{(decoding failure)}\\ \alpha_1 &= n_b - d(\mathbf{y}, \hat{\mathbf{x}}), \ \text{(otherwise)}\end{aligned}$$

This method allows to decode up to $D/2$.

# Outline

# Generalized concatenated codes

The main difference:

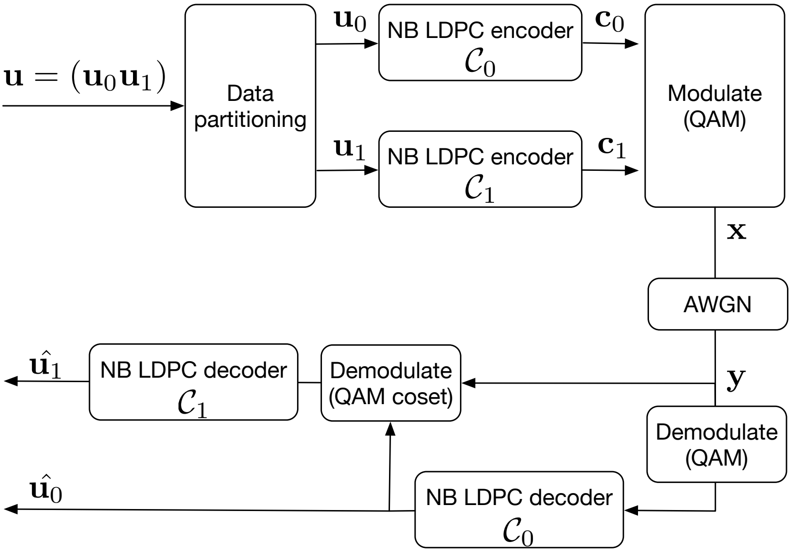
- $\ell$ outer codes $A_i$, $i = 1, \ldots, \ell$;
- $\ell$ nested inner codes

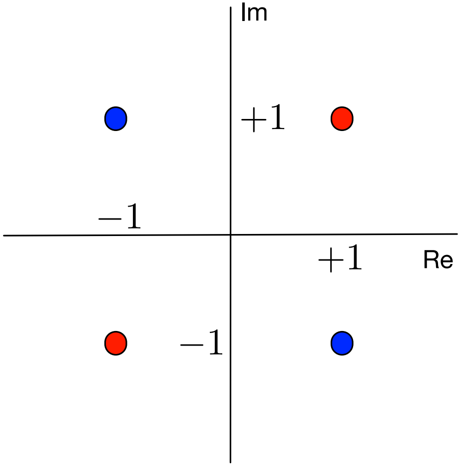
$$B_1 \subset B_2 \subset \ldots \subset B_\ell.$$

Distance estimate

$$D \geq \min_{1 \leq i \leq \ell} d_b^{(i)} d_a^{(i)}.$$

# Multilevel coding




Im
+1
−1
+1
Re
−1

# Outline

# Problem 1

Consider a product code which consists of single parity check codes with lenghts $n_1$ and $n_2$. What are the parameters of this code? How to decode it?

# Problem 2

Consider a repetition code $R_9$ of length 9. The code can be presented as a concatenation of $R_3$ and $R_3$. We first encode the sourse steam with $R_3$ and then encode resulting output with $R_3$. Evaluate the probability of error for this decoder and compare it with probability of error for optimal decoder $R_9$

# Outline

# Soft decoding

In case of *hard decision decoding* demodulator makes a hard decision and provides the decoder with the symbols of the field. In case of *soft decision decoding* demodulator provides the decoder with probability distributions.

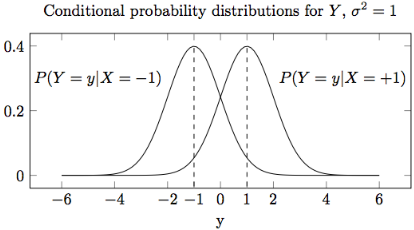
# Example: Binary input AWGN channel

$X \in \{-1, +1\}$

$$X \longrightarrow \bigoplus \longrightarrow Y$$

$$\uparrow$$

$$Z \sim N(0, \sigma^2)$$

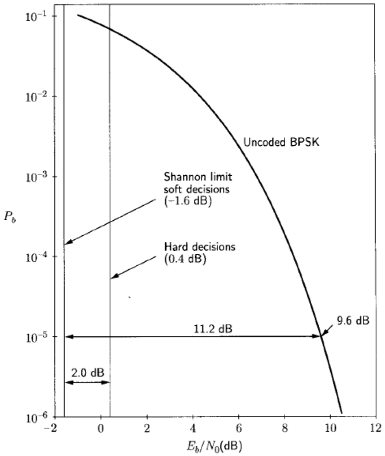


# LLR calculation

Assume again, that 0 is sent with the level $+1$ and 0 is sent with the level $-1$. Let us calculate the log likelihood ratio given $y$

$$
\begin{aligned}
LLR &= \log \frac{\Pr(X=+1|y)}{\Pr(X=-1|y)} \\
&= \log \frac{\Pr(y|X=+1)\Pr(X=+1)\Pr(y)}{\Pr(y|X=-1)\Pr(X=-1)\Pr(y)} \\
&= \log \frac{f(y|+1)}{f(y|-1)} \\
&= \frac{(y+1)^2-(y-1)^2}{2\sigma^2} \\
&= \frac{2}{\sigma^2} y
\end{aligned}
$$

# Soft vs hard decoding




Consider again BI-AWGN channel. We can use LLR values or make a hard decision ($+1$ if $LLR > 0$, $-1$ otherwise). The difference is approx. 2 dB!

Thank you for your attention!