Universal Nesterov's gradient method in general model conception

Alexander Tyurin [1], Pavel Dvurechensky [2], Alexander Gasnikov [3]

[1]HSE (Russian Federation)
[2]WIAS (Germany)
[3]MIPT (Russian Federation)

General optimization problem

General convex optimization problem:

$$F(x) \to \min_{x \in Q}$$

We denote x_* – the solution of this problem.

1. $F(x)$ is a convex function.
2. $Q \subseteq \mathbb{R}^n$ is a convex, closed set.

Examples:

1. Physics, Gas Network:
$$\frac{1}{3}\sum_{i=1}^{N} \alpha_i |x_i|^3 \to \min_{Ax=d}.$$
2. Machine Learning, Logistic regression:
$$\sum_{i=1}^{N} \log(1 + \exp(x_i^T w)) \to \min_{w \in \mathbb{R}^n}.$$

Bregman divergence

Prox-function $d(x)$: continuously differentiable on int$Q$ and 1-strongly convex function, i.e.

$$d(x) \geq d(y) + \langle \nabla d(y), x - y \rangle + \frac{1}{2} \|x - y\|^2, \;\; x, y \in \text{int } Q.$$

Bregman divergence $V(x, y) \stackrel{\text{def}}{=} d(x) - d(y) - \langle \nabla d(y), x - y \rangle$.
Examples:

- $V(x, y) = \frac{1}{2} \|x - y\|_2^2$
- $V(x, y) = \sum_{i=1}^{n} x_i \log \frac{x_i}{y_i}$ (KL-divergence).

Definition of (δ, L)-model

Definition

Pair $(F_\delta(y), \psi_\delta(x,y))$ is (δ, L)-model in a point y of a function F with respect to the norm $\|\|$ if for all $x \in Q$

$$0 \leq F(x) - F_\delta(y) - \psi_\delta(x,y) \leq \frac{L}{2}\|x-y\|^2 + \delta,$$

$$\psi_\delta(x,x) = 0, \;\; \forall x \in Q,$$

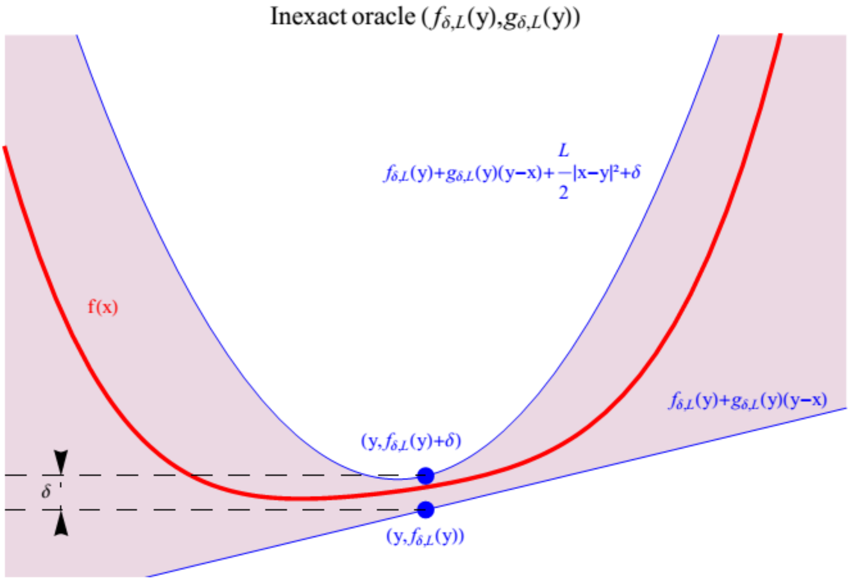
and $\psi_\delta(x,y)$ is a convex function of x for all $y \in Q$.

Definition (Devolder–Glineur–Nesterov, 2013)

Function F has (δ, L)-oracle in a point y if there exists a pair $(F_\delta(y), \nabla F_\delta(y))$ such that

$$0 \leq F(x) - F_\delta(y) - \langle \nabla F_\delta(y), x-y\rangle \leq \frac{L}{2}\|x-y\|^2 + \delta, \;\; \forall x \in Q.$$

Definition of (δ, L)-inexact oracle; Devolder, PhD Thesis'13



Definition of inexact solution

Definition

Let us assume an optimization task

$$\psi(x) \to \min_{x \in Q},$$

where ψ is a convex function. Then $\operatorname{Arg\,min}^{\widetilde{\delta}}_{x \in Q} \psi(x)$ is a set of $\widetilde{x}$ such that

$$\exists h \in \partial \psi(\widetilde{x}), \;\; \langle h, x_* - \widetilde{x} \rangle \geq -\widetilde{\delta}.$$

Arbitrary point from the $\operatorname{Arg\,min}^{\widetilde{\delta}}_{x \in Q} \psi(x)$ we define as $\arg\min^{\widetilde{\delta}}_{x \in Q} \psi(x)$.

From the definition follows that

$$\psi(\widetilde{x}) - \psi(x_*) \leq \widetilde{\delta}.$$

In the opposite direction, it is not always true.

Input: x_0, N – number of steps, $\{\delta_k\}_{k=0}^{N-1}$, $\{\widetilde{\delta}_k\}_{k=0}^{N-1}$ – sequences, and $L_0 > 0$.

$$\mathbf{0}\text{ - }\mathbf{step:}\ L_1 = \frac{L_0}{2}$$

$$\boldsymbol{k+1}\text{ - }\mathbf{step}\text{: } \alpha_{k+1} = \frac{1}{L_{k+1}}$$

$$\phi_{k+1}(x) = V(x, x_k) + \alpha_{k+1}\psi_{\delta_k}(x, x_k)$$

$$x_{k+1} = \arg\min_{x \in Q}{}^{\widetilde{\delta}_k} \phi_{k+1}(x)$$

$$\textbf{If } F_{\delta_k}(x_{k+1}) > F_{\delta_k}(x_k) + \psi_{\delta_k}(x_{k+1}, x_k) +$$

$$+ \frac{L_{k+1}}{2} \|x_{k+1} - x_k\|^2 + \delta_k, \textbf{ then } L_{k+1} = 2L_{k+1}$$

and repeat the current step.

Otherwise, $L_{k+2} = \dfrac{L_{k+1}}{2}$ **and go to the next step.**

Fast gradient descent

Input: x_0, N – number of steps, $\{\delta_k\}_{k=0}^{N-1}$, $\{\widetilde{\delta}_k\}_{k=0}^{N-1}$ – sequences, and $L_0 > 0$.

$$\textbf{0 - step: } y_0 = u_0 = x_0;\ L_1 = \frac{L_0}{2};\ \alpha_0 = 0;\ A_0 = \alpha_0$$

$$\boldsymbol{k+1}\textbf{ - step: Find } \alpha_{k+1} : A_k + \alpha_{k+1} = L_{k+1}\alpha_{k+1}^2$$

$$A_{k+1} = A_k + \alpha_{k+1}$$

$$y_{k+1} = \frac{\alpha_{k+1}u_k + A_k x_k}{A_{k+1}}$$

$$\phi_{k+1}(x) = V(x, u_k) + \alpha_{k+1}\psi_{\delta_k}(x, y_{k+1})$$

$$u_{k+1} = \arg\min_{x \in Q}{}^{\widetilde{\delta}_k} \phi_{k+1}(x),\ \ x_{k+1} = \frac{\alpha_{k+1}u_{k+1} + A_k x_k}{A_{k+1}}$$

$$\textbf{If } F_{\delta_k}(x_{k+1}) > F_{\delta_k}(y_{k+1}) + \psi_{\delta_k}(x_{k+1}, y_{k+1}) +$$

$$+\frac{L_{k+1}}{2}\left\|x_{k+1} - y_{k+1}\right\|^2 + \delta_k\textbf{, then } L_{k+1} = 2L_{k+1}$$

and repeat the current step.

$$\textbf{Otherwise, } L_{k+2} = \frac{L_{k+1}}{2} \textbf{ and go to the next step.}$$

Theorem (Gradient descent)

Let $V(x_*, x_0) \le R^2$ *where* x_0 *is a starting point,* x_* *is the closest point to* x_0 *in terms of a Bregman distance* V*, and*

$$\bar{x}_N = \frac{1}{N}\sum_{k=0}^{N-1} x_k.$$

For proposed algorithm we have

$$F(\bar{x}_N) - F(x_*) \le \frac{2LR^2}{N} + \frac{2L}{N}\sum_{k=0}^{N-1}\widetilde{\delta}_k + \frac{2}{N}\sum_{k=0}^{N-1}\delta_k.$$

Theorem (Fast gradient descent)

Let $V(x_*, x_0) \le R^2$ *where* x_0 *is a starting point and* x_* *is the closest point to* x_0 *in terms of a Bregman distance* V*. For proposed algorithm we have*

$$F(x_N) - F(x_*) \le \frac{8LR^2}{(N+1)^2} + \frac{8L\sum_{k=0}^{N-1}\widetilde{\delta}_k}{(N+1)^2} + \frac{2\sum_{k=0}^{N-1}\delta_k A_{k+1}}{A_N}.$$

Gradient descent vs Fast gradient descent

Let $\widetilde{\delta}_k = \widetilde{\delta}$ and $\delta_k = \delta$ then

$$F(\bar{x}_N) - F(x_*) \leq \frac{2LR^2}{N} + 2L\widetilde{\delta} + 2\delta,$$

$$F(x_N) - F(x_*) \leq \frac{8LR^2}{(N+1)^2} + \frac{8L\widetilde{\delta}}{N+1} + 2N\delta.$$

Fast gradient descent is more robust to $\widetilde{\delta}$, which appears when we solve an intermediate optimization problem, but fast gradient descent accumulates δ, which appears when we receive (δ, L)-model.

There are intermediate gradient methods with convergence rate equal to (in the case $\tilde{\delta} = 0$ – Devolder–Glineur–Nesterov, 2013)

$$\mathcal{O}(1)\frac{LR^2}{N^p} + \mathcal{O}(1)N^{1-p}\widetilde{\delta} + \mathcal{O}(1)N^{p-1}\delta,$$

where $p \in [1, 2]$ (D. Kamzolov, 2017).

Linear model (Nesterov, 1983)

Assume that F is a **smooth** convex function and the gradient of F is Lipschitz-continuous with parameter L, then

$$0 \le F(x) - F(y) - \langle \nabla F(y), x - y \rangle \le \frac{L}{2} \|x - y\|^2, \ \forall x, y \in Q.$$

We can take $\psi_{\delta_k}(x, y) = \langle \nabla F(y), x - y \rangle$, $F_{\delta_k}(y) = F(y)$, and $\delta_k = 0$ for all $k \ge 0$. Let us assume that we can find exact solutions in intermediate optimization problems thus $\widetilde{\delta}_k = 0$ for all $k \ge 0$ and

$$F(x_N) - F(x_*) \le \frac{8LR^2}{(N+1)^2}.$$

The bound is optimal.

Universal method (Nesterov, 2013)

Let

$$\|\nabla F(x) - \nabla F(y)\|_* \leq L_\nu \|x - y\|^\nu, \ \forall x, y \in Q.$$

Therefore,

$$0 \leq F(x) - F(y) - \langle \nabla F(y), x - y \rangle \leq \frac{L(\delta)}{2} \|x - y\|^2 + \delta, \ \forall x, y \in Q,$$

where

$$L(\delta) = L_\nu \left[\frac{L_\nu}{2\delta} \frac{1 - \nu}{1 + \nu} \right]^{\frac{1-\nu}{1+\nu}}$$

and $\delta > 0$ is a parameter.

Consequently, we can take $\psi_{\delta_k}(x, y) = \langle \nabla F(y), x - y \rangle$, $F_{\delta_k}(y) = F(y)$. Let us assume that $\widetilde{\delta}_k = 0$ for all $k \geq 0$. Let us take

$$\delta_k = \epsilon \frac{\alpha_{k+1}}{4A_{k+1}}, \ \forall k,$$

where ϵ is the accuracy of the solution by function.

Universal method (Nesterov, 2013)

In order to have

$$F(x_N) - F(x_*) \leq \epsilon,$$

it is enough to do

$$N \leq \inf_{\nu \in [0,1]} \left[64^{\frac{1+\nu}{1+3\nu}} \left(\frac{2 - 2\nu}{1+\nu} \right)^{\frac{1-\nu}{1+3\nu}} \left(\frac{L_\nu R^{1+\nu}}{\epsilon} \right)^{\frac{2}{1+3\nu}} \right]$$

steps.
The bound is optimal.

In fast gradient method we have

$$\phi_{k+1}(x) = V(x, u_k) + \alpha_{k+1}\psi_{\delta_k}(x, y_{k+1}).$$

Instead of it let us take (idea goes back to A. Nemirovski, 2013)

$$\widetilde{\phi}_{k+1}(x) = \alpha_{k+1}\psi_{\delta_k}(x, y_{k+1}).$$

Let us look at this substitution from the view of an error $\widetilde{\delta}_k$. We can show that it enough to take $\widetilde{\delta}_k = 2R_Q^2$ for all $k \geq 0$, where R_Q is a diameter of a set Q. Let us take

$$\delta_k = \epsilon\frac{\alpha_{k+1}}{4A_{k+1}}, \ \forall k,$$

where ϵ is the accuracy of the solution by function.
Finally,

$$N \leq \inf_{\nu\in(0,1]}\left[96^{\frac{1+\nu}{2\nu}}\left[\frac{2-2\nu}{1+\nu}\right]^{\frac{1-\nu}{2\nu}}\left(\frac{L_\nu R_Q^{1+\nu}}{\epsilon}\right)^{\frac{1}{\nu}}\right].$$

$$F(x) \overset{\text{def}}{=} f(x) + h(x) \to \min_{x \in Q},$$

where f is a smooth convex function and the gradient of f is Lipschitz-continuous with parameter L. Function h is a convex function. We can show

$$0 \leq F(x) - F(y) - \langle \nabla f(y), x - y \rangle - h(x) + h(y) \leq$$
$$\leq \frac{L}{2} \|x - y\|^2, \ \ \forall x, y \in Q.$$

We can take $\psi_{\delta_k}(x, y) = \langle \nabla f(y), x - y \rangle + h(x) - h(y)$, $F_{\delta_k}(y) = F(y)$, and $\delta_k = 0$ for all $k \geq 0$. On the one hand the methods do not change, but on the other hand auxiliary optimizations are more complicated.

Superposition of functions (Nemirovski–Nesterov, 1985)

$$F(x) \overset{\text{def}}{=} f(f_1(x), \ldots, f_m(x)) \to \min_{x \in Q},$$

where f_k is a smooth convex function and the gradient of f_k is Lipschitz-continuous with parameter L_k for all $k \geq 0$. Function f is a convex function, Lipschitz-continuous with parameter M with respect to L^1-norm, and monotonic function. Therefore,

$$0 \leq F(x) - F(y) - f(f_1(y) + \langle \nabla f_1(y), x - y \rangle, \ldots, f_m(y) +$$

$$+ \langle \nabla f_m(y), x - y \rangle) + F(y) \leq M \frac{\sum_{i=1}^m L_i}{2} \|x - y\|^2, \forall x, y \in Q.$$

We can take $\psi_{\delta_k}(x, y) =$
$f(f_1(y) + \langle \nabla f_1(y), x - y \rangle, \ldots, f_m(y) + \langle \nabla f_m(y), x - y \rangle) - F(y)$,
$F_{\delta_k}(y) = F(y)$, and $\delta_k = 0$ for all $k \geq 0$.

Proximal method with inexact solution in an intermediate optimization problem; Devolder–Glineur–Nesterov, 2013

Let us consider

$$f(x) \overset{\text{def}}{=} \min_{y \in Q} \underbrace{\left\{ \phi(y) + \frac{L}{2} \|y - x\|_2^2 \right\}}_{\Lambda(x,y)} \to \min_{x \in \mathbb{R}^n} . \tag{1}$$

Function ϕ is a convex function and

$$\max_{y \in Q} \left\{ \Lambda(x, y(x)) - \Lambda(x, y) + \frac{L}{2} \|y - y(x)\|_2^2 \right\} \leq \delta.$$

Then

$$\left(\phi(y(x)) + \frac{L}{2} \|y(x) - x\|_2^2 - \delta, \psi_\delta(z, x) = \langle L(x - y(x)), z - x \rangle \right)$$

is (δ, L)-model of $f(z)$ at the point x w.r.t 2-norm.

Let us consider

$$f(x) \stackrel{\text{def}}{=} \max_{y \in Q} \left[\langle x, b - Ay \rangle - \phi(y) \right] \to \min_{x \in \mathbb{R}^n}, \tag{2}$$

where $\phi(y)$ is a μ-strong convex function w.r.t. p-norm ($1 \le p \le 2$). Then f is a smooth convex function and the gradient of f is Lipschitz-continuous with parameter

$$L = \frac{1}{\mu} \max_{\|y\|_p \le 1} \|Ay\|_2^2 .$$

If $y_\delta(x)$ is a δ-solution of the max problem, then

$$(\langle x, b - Ay_\delta(x) \rangle - \phi(y_\delta(x)), \psi_\delta(z, x) = \langle b - Ay_\delta(x), z - x \rangle)$$

is $(\delta, 2L)$-model of $f(z)$ at the point x w.r.t 2-norm.

Let us consider

$$\phi(y) + \frac{\mu}{2}\|Ay - b\|_2^2 \to \min_{Ay=b,\, y\in Q}.$$

and it's dual problem

$$f(x) \overset{\text{def}}{=} \max_{y\in Q} \underbrace{\left(\langle x, b - Ay\rangle - \phi(y) - \frac{\mu}{2}\|Ay - b\|_2^2\right)}_{\Lambda(x,y)} \to \min_{x\in\mathbb{R}^n}.$$

If $y_\delta(x)$ is a solution in the sense of an inequality

$$\max_{y\in Q} \langle \nabla_y \Lambda(y_\delta(x), y), y - y_\delta(x)\rangle \le \delta,$$

then

$$\Big(\langle x, b - Ay_\delta(x)\rangle - \phi(y_\delta(x)) - \frac{\mu}{2}\|Ay_\delta(x) - b\|_2^2,$$
$$\psi_\delta(z, x) = \langle b - Ay_\delta(x), z - x\rangle\Big)$$

is (δ, μ^{-1})-model of $f(z)$ at the point x w.r.t 2-norm.

Let us consider

$$f(x) \overset{\text{def}}{=} \min_{y \in Q} F(y, x) \to \min_{x \in \mathbb{R}^n}.$$

- Set Q is convex and bounded.
- Function F is smooth and convex w.r.t. both variables.
- $\|\nabla F(y', x') - \nabla F(y, x)\|_2 \le L \|(y', x') - (y, x)\|_2, \ \forall y, y' \in Q, \ x, x' \in \mathbb{R}^n$.

If we can find a point $\widetilde{y}_\delta(x) \in Q$ such that

$$\langle \nabla_y F(\widetilde{y}_\delta(x), x), y - \widetilde{y}_\delta(x) \rangle \ge -\delta, \ \ \forall y \in Q,$$

then

$$F(\widetilde{y}_\delta(x), x) - f(x) \le \delta, \left\| \nabla f(x') - \nabla f(x) \right\|_2 \le L \left\| x' - x \right\|_2$$

and

$$(F(\widetilde{y}_\delta(x), x) - 2\delta, \psi_\delta(z, x) = \langle \nabla_y F(\widetilde{y}_\delta(x), x), z - x \rangle)$$

is $(6\delta, 2L)$-model of $f(x)$ at the point x w.r.t 2-norm.

Let us consider

$$f_\mu(x = (z,p)) = g(z,p) + \mu \sum_{k=1}^{n} z_k \ln z_k + \frac{\mu}{2}\|p\|_2^2 \to \min_{z \in S_n(1), p \geq 0}.$$

Let's introduce norm

$$\|x\|^2 = \|(z,p)\|^2 = \|z\|_1^2 + \|p\|_2^2.$$

Assume that

$$\|\nabla g(x_2) - \nabla g(x_1)\|_* \leq L\|x_2 - x_1\|, \quad L \leq \mu.$$

Then

$$\left(f_\mu(y), \langle \nabla g(y), x - y\rangle + (\mu - L)\sum_{k=1}^{n} z_k \ln z_k + \frac{\mu - L}{2}\|p\|_2^2\right).$$

is $(0, 2L)$-model of $f_\mu(x = (z,p))$ at the point y.

Third new example: Proximal Sinkhorn algorithm

Let introduce smoothed Wasserstein distance (Peyre–Cuturi, 2018)

$$W_\gamma(p,q) = \min_{\pi \in \Pi(p,q)} \left\{ \sum_{i,j=1}^{n} C_{ij}\pi_{ij} + \gamma \sum_{i,j} \pi_{ij} \log \frac{\pi_{ij}}{\pi_{ij}^0} \right\},$$

where

$$\Pi(p,q) = \{\pi \in \mathbb{R}_+^{n\times n},\ \sum_{j=1}^{n} \pi_{ij} = p,\ \sum_{i=1}^{n} \pi_{ij} = q\}.$$

It's well known (Franklin–Lorentz, 1989; Dvurechensky et al., 2018) that the complexity (a.o.) of the Sinkhorn algorithm for this problem is

$$O\left(n^2 \min\left\{\frac{\ln n}{\gamma\epsilon}, \exp\left(\frac{\tilde{C}}{\gamma}\right)\ln\left(\epsilon^{-1}\right)\right\}\right).$$

Third new example: Proximal Sinkhorn algorithm

If we want to calculate $W_0(p,q)$ with the precision ϵ one can calculate $W_\gamma(p,q)$ with $\gamma = \epsilon/4\ln n$ with the precision $\epsilon/2$. So the complexity in this case will be $\tilde{O}(n^2/\epsilon^2)$ a.o. Can we do better?

The answer is Yes!.

If we use **proximal model** of $f(x)$ at point y of the form

$$\psi_\delta(x,y) = f(x) - f(y)$$

and choose arbitrary $L > 0$. Then proximal gradient descent (Chen–Teboulle, 1993)

$$x^{k+1} = \arg\min_{x \in Q} \left\{ \psi_\delta(x, x^k) + LV(x, x^k) \right\}$$

will converges as $O(LR^2/k)$, $R^2 = V(x_*, x^0)$.

Third new example: Proximal Sinkhorn algorithm

So, let's apply proximal set up to $W_0(p,q)$ calculation. Put $x = \pi$, $Q = \Pi$, $L = \gamma$, choose $V = KL$. We obtain that proximal gradient method

$$\pi^{k+1} = \min_{\pi \in \Pi(p,q)} \left\{ \sum_{i,j=1}^{n} C_{ij}\pi_{ij} + \gamma \sum_{i,j} \pi_{ij} \log \frac{\pi_{ij}}{\pi_{ij}^{k}} \right\}$$

requires $O(\gamma \ln n/\epsilon)$ iterations to reach the precision ϵ. But, this assumes that we have to solve exactly auxiliary problem – i.e. we can calculate exactly $W_\gamma(p,q)$. But from above we know that if γ is not small (**Note:** we can choose γ as we want!) we can solve this problem with very high precision with the complexity $\tilde{O}(n^2)$. So the total number of a.o. will be $\tilde{O}(n^2/\epsilon)$ – that is better than for the simple Sinkhorn $\tilde{O}(n^2/\epsilon^2)$.

Auxiliary optimization problem

Let us take $\phi_{k+1}(x) = V(x, u_k) + \alpha_{k+1}\psi_{\delta_k}(x, y_{k+1})$. From

$$\langle \nabla \phi_{k+1}(x_\epsilon), x_\epsilon - x_* \rangle \leq \epsilon$$

we can get

$$\phi_{k+1}(x_\epsilon) - \phi_{k+1}(x_*) \leq \epsilon.$$

Let an intermediate optimization problem has Lipschitz-continuous with parameter M function. Let us assume x_ϵ is a ϵ-solution then

$$\langle \nabla \phi_{k+1}(x_\epsilon), x_\epsilon - x_* \rangle \leq \|\nabla \phi_{k+1}(x_\epsilon)\|_* \|x_\epsilon - x_*\| \leq M\sqrt{2\epsilon}.$$

If we have linear convergence speed for an auxiliary optimization problem, then we should do c times more steps where c is a small constant.

Thank you!