PageRank Project, v.1.1

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Consider a directed graph G, PageRank problem is to find main eigenvector x^* of a column-stochastic matrix $A \in \mathbb{R}^{n\times n}: A_{i,j} \geq 0, \ \sum_{i=1,n} A_{i,j} = 1$

$$Ax^* = x^*. \tag{1}$$

The solution is exists due to Perron-Frobenius theorem, but not necessarily being unique.

Solution set of (1) is the same as solution set of optimization problem in unit simplex $S_n = \{x : \sum_{i=1}^n x_i = 1, \ \ x_i \geq 0\}$

$$\operatorname{Arg}\min_{x\in S_n} \|Ax - x\|_2 = \operatorname{Arg}\min_{x\in S_n} \|Ax - x\|_2^2. \tag{2}$$

Modification of the problem is *robust* PageRank [2].

$$\min_{x\in S_n} \|Ax - x\|_2 + \lambda\|x\| \tag{3}$$

with small λ.

The matrix A is usually sparsed.

Note that the PageRank problem (2) may be reformulated in various ways:

- as solution of linear equation system problem: suppose that $x_n \neq 0$, then solve
$$(A - I)\begin{bmatrix} \widehat{x} \\ 1 \end{bmatrix} = 0,$$
then take as solution $x^* = \begin{bmatrix} \widehat{x} \\ 1 \end{bmatrix} / \| \begin{bmatrix} \widehat{x} \\ 1 \end{bmatrix} \|_1$.
- as non-smooth problem
$$\min_{x\in S_n} \|Ax - x\|_\infty,$$
- as saddle-point problem (similar approach is valid for robust PageRank)
$$\min_{x\in S_n}\max_{y\in S_{2n}} y^T \begin{bmatrix} A - I \\ -A + I \end{bmatrix} x,$$
or
$$\min_{x\in S_n}\max_{y\in B_n} y^T Ax,$$

- or as projection on linear space

$$\min_{\substack{Ax - x = 0 \\ (e,x) = 1}} \|x\|_2^2$$

with vector $e = (1, 1, \ldots, 1)$.

2 Robust PageRank

The main eigenvector may be very sensitive to matrix A perturbations [2]. This can be taken into account in form of Min-max problem

$$x^* = \arg\min_x \max_{\Delta \in \boldsymbol{\Delta}} \|(A + \Delta)x - x\|_2, \tag{4}$$

for Frobenius matrix norm the target function

$$f_1(x) = \max_{\|\Delta\|_F \le \varepsilon, A \text{ is stochastic}} \|(A + \Delta)x - x\|_2$$

can be approximated as

$$f_2(x) = \|Ax - x\|_2 + \varepsilon \|x\|_2 \to \min_x .$$

As soon $f_1(x) \le f_2(x)$, minimization of latter function lead to approximate solution of problem (4).

Other assumptions on perturbation matrix Δ lead to similar optimization problems, cf. [2].

3 Methods

The simplest method is Power-like Iterations:

$$x^{k+1} = Ax^k, \; x^0 \in S_n. \tag{5}$$

however, its convergence is proved only for class of matrices, e.g. irreducible matrices. In such case convergence speed is linear and depends on magnitude of second largest eigenvalue $|\lambda_2| < 1$ of matrix A.

However, the Power-like Iterations (5) are useful for any matrix. If we take average of generated vectors, resulting sequence

$$\overline{x}^k = \frac{1}{k} \sum_{i=0}^{k-1} x_k,$$

is convergent to a solution x^* of problem (2) for all stochastic A.

3.1 Mirror Descent

Consider problem in form (2). For unit simplex is taken common distance-generating function

$$d(x) = \ln(n) + \sum_{i=1}^{n} x_i \ln x_i,$$

with parametrized by parameter β prox-mapping

$$Mirr_x(\beta, v) = \arg\min_{x \in S_n} (v^T x + \beta d(x)).$$

The mapping can be calculated in explicit form. Denote $z = Mirr_x(\beta, v)$, then (check, also cf. Lecture 3-4 slides)

$$z_i = \frac{e^{-v_i/\beta}}{\sum_{i=1}^{n} e^{-v_i/\beta}}.$$

The mirror descent algorithm (cf. Lecture 2, slide 18) may be written as following [3]. Note that step-size parameter β_k is non-constant.

3.1.1 Mirror Descent Algorithm

1. Fix $k = 0, x^k = \frac{1}{n}e, \overline{x}^k = v^k = (0, 0, \ldots, 0), \beta_0 = 2/\sqrt{\ln(n)}$.
2. Calculate subgradient of target function $\partial f(x^k)$, put
$$v^{k+1} = v^k + \partial f(x^k),$$
3. Update
$$x^{k+1} = Mirr_{x_k}(\beta_k, v^k),$$
$$\beta_{k+1} = \beta_0 \sqrt{k+1},$$
$$\overline{x}^{k+1} = \frac{k}{k+1}\overline{x}^k + \frac{1}{k+1}.x^k.$$

Vectors $\overline{x}^k$ converge to PageRank vector x^*.

3.2 Stochastic Mirror Descent Algorithm

The variation of Mirror Descent algorithm instead of true subgradients $\partial f(x^k)$ uses stochastic subgradients $\xi_k(x^k)$, provided that $\mathbb{E}\xi^k(x^k) \in \partial f(x^k)$. Bound on error $\mathbb{E}f(x^k) - f^*$ is dependant on constant M, such that either

$$\|\xi^k(x^k)\|_* \leq M,$$

either

$$\|\xi^k(x^k)\|_*^2 \leq M^2,$$

4 Project challenges

Given matrix A, find its main eigenvector and main robust eigenvector for small $\lambda \approx 1/2n$.

- Justify constants used in Mirror Descent Algorithm/Stochastic Mirror Descent Algorithm,
- Apply three or more different algorithms to obtain solution of PageRank problem, including deterministic and stochastic Mirror Descent, evaluate theoretic upper bounds,
- Apply two or more different algorithms to obtain solution of Robust PageRank problem and evaluate its theoretic upper bounds,
- (optionally) compare robust PageRank solution with non-robust one (explore dependence on λ).

References

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