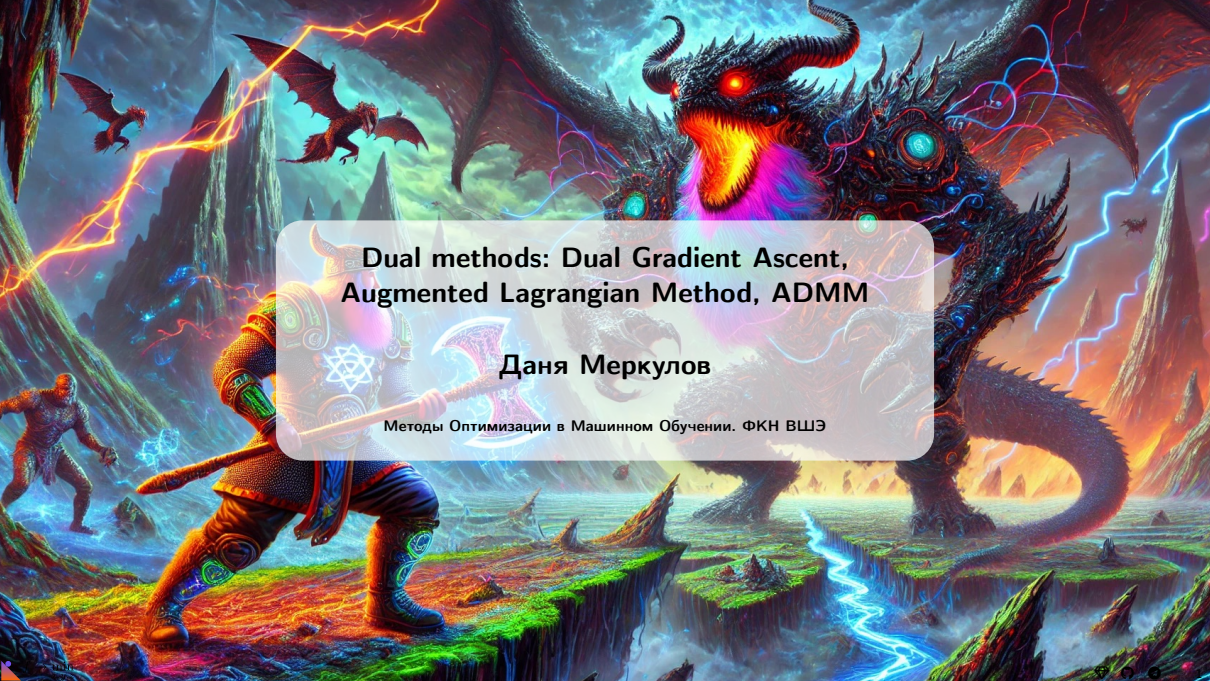

# Dual methods: Dual Gradient Ascent, Augmented Lagrangian Method, ADMM

Даня Меркулов

Методы Оптимизации в Машинном Обучении. ФКН ВШЭ

# Introduction to dual methods

# Why do we want to solve dual problems?

## Primal problem

$$\begin{aligned} & f_0(x) \to \min_{x \in \mathbb{R}^n} \\ \text{s.t.} \quad & f_i(x) \leq 0, \ i = 1, \dots, m \\ & h_i(x) = 0, \ i = 1, \dots, p \end{aligned}$$

## Dual problem

$$g(\lambda, \nu) = \min_{x \in \mathcal{D}} L(x, \lambda, \nu) =$$

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$$\text{s.t. } \lambda \succeq 0$$

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- **Duality Gap.** The difference between the primal and dual solutions (duality gap) provides valuable information about the solution's optimality.

# Conjugate functions

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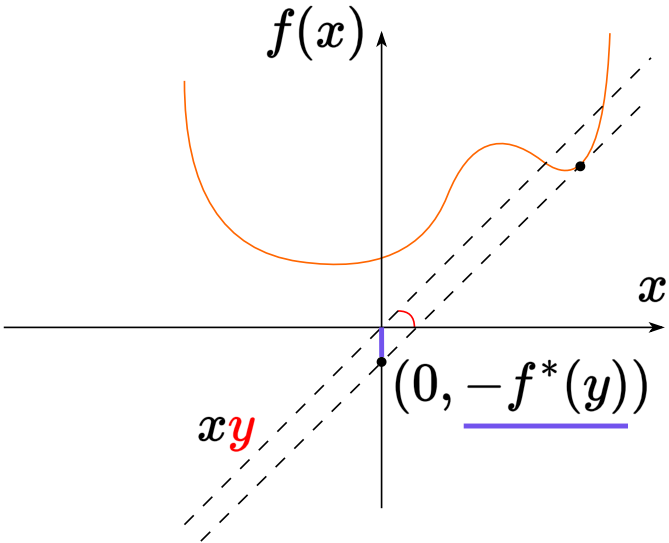


Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function defined by

$$f^*(y) = \max_x \left[ y^T x - f(x) \right]$$

is called its conjugate.

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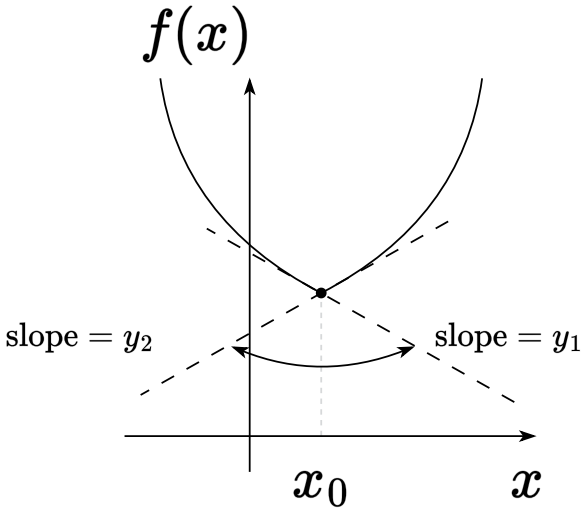

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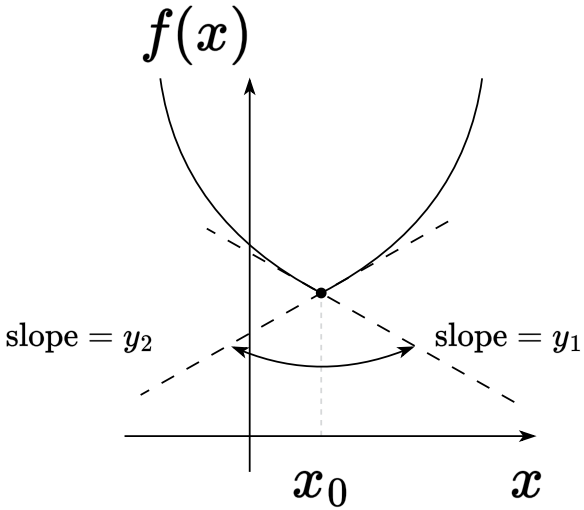


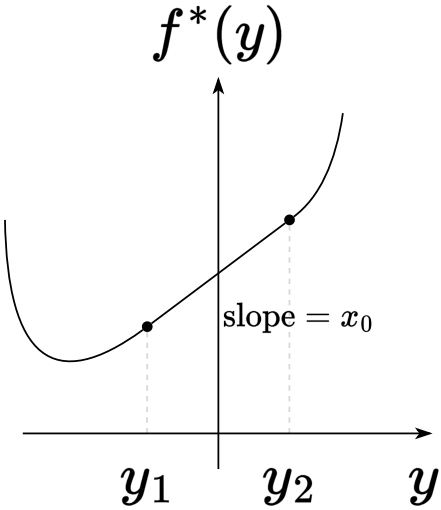

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# Conjugate function properties (proofs)

We will show that $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$, assuming that $f$ is convex and closed.

- **Proof of** $\Leftarrow$: Suppose $y \in \partial f(x)$. Then $x \in M_y$, the set of maximizers of $y^T z - f(z)$ over $z$. But

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Clearly $y \in \partial f(x) \Leftrightarrow x \in \arg\min_z \{f(z) - y^T z\}$

Lastly, if $f$ is strictly convex, then we know that $f(z) - y^T z$ has a unique minimizer over $z$, and this must be $\nabla f^*(y)$.

# Dual ascent

# Dual (sub)gradient method

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- Proximal gradients and acceleration can be applied as they would usually.

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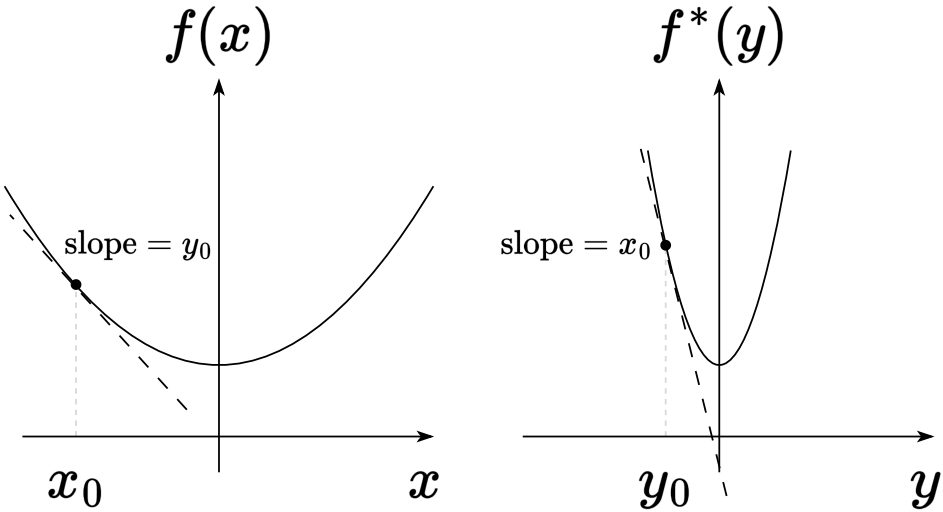

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Adding these together, using the Cauchy-Schwarz inequality, and rearranging shows that

$$\|x_u - x_v\|^2 \leq \frac{1}{\mu}\|u - v\|^2$$

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**Proof of "$\Leftarrow$"**: for simplicity, call $g = f^*$ and $L = \frac{1}{\mu}$. As $\nabla g$ is Lipschitz with constant $L$, so is $g_x(z) = g(z) - \nabla g(x)^T z$, hence

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Let $u = \nabla f(x)$, $v = \nabla g(y)$; then $x \in \partial g^*(u)$, $y \in \partial g^*(v)$, and the above reads $(x - y)^T (u - v) \geq \frac{\|u - v\|^2}{L}$, implying the result.

# Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent: (This is ignoring the role of $A$, and thus reflects the case when the singular values of $A$ are all close to 1. To be more precise, the step sizes here should be: $\frac{\mu}{\sigma_{\max}(A)^2}$ (first case) and $\frac{2}{\frac{\sigma_{\max}(A)^2}{\mu} + \frac{\sigma_{\min}(A)^2}{L}}$ (second case).)

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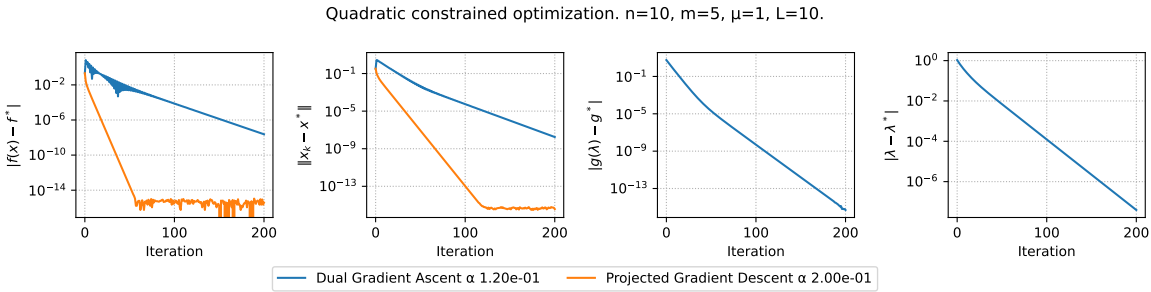
- Note that this describes convergence in the dual. Convergence in the primal requires more assumptions

# Example: equality constrained quadratic minimization.

$$f(x) = \frac{1}{2}x^T Ax - b^T x \to \min_{x \in \mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m \times n}, m < n.$$




We need to find a minimum of a quadratic function in some linear subspace, defined by the solution of linear equation $Cx = d$. This is a conditional optimization problem, we start from strongly convex setting.

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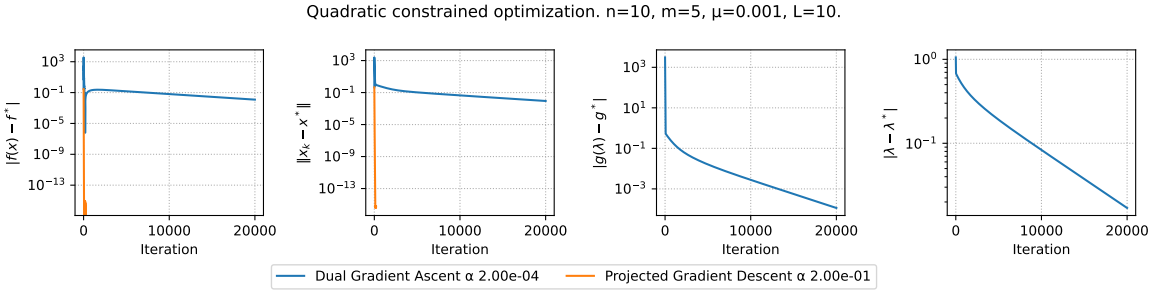


Situation is getting worse as soon as we loose strong convexity, the dual convergence will still be linear, but the rate is very low.

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Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into $B$ separate problems:

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Can think of these steps as:

- **Broadcast:** Send $u$ to each of the $B$ processors, each optimizes in parallel to find $x_i$.
- **Gather:** Collect $A_i x_i$ from each processor, update the global dual variable $u$.

# Inequality constraints

Consider the optimization problem:

$$\min_x \sum_{i=1}^{B} f_i(x_i) \quad \text{subject to} \quad \sum_{i=1}^{B} A_i x_i \leq b$$

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$$u^k = \left( u^{k-1} + \alpha_k \left( \sum_{i=1}^{B} A_i x_i^k - b \right) \right)_+$$

where $(u)_+$ denotes the positive part of $u$, i.e., $(u_+)_i = \max\{0, u_i\}$, for $i = 1, \ldots, m$.

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  - **Never let prices get negative**; hence the use of the positive part notation $(\cdot)_+$.

# Augmented Lagrangian method

# Augmented Lagrangian method aka method of multipliers

**Dual ascent disadvantage:** convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

$$\begin{aligned} &\min_x f(x) + \frac{\rho}{2}\|Ax - b\|^2 \\ &\text{s.t. } Ax = b \end{aligned}$$

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**Dual gradient ascent:** The iterative updates are given by:

$$\begin{aligned} x_k &= \arg\min_x \left[ f(x) + (u_{k-1})^T Ax + \frac{\rho}{2}\|Ax - b\|^2 \right] \\ u_k &= u_{k-1} + \rho(Ax_k - b) \end{aligned}$$

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**Notice step size choice $\alpha_k = \rho$ in dual algorithm. Why?**

Since $x_k$ minimizes the function:

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over $x$, we have the stationarity condition:

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This represents the stationarity condition for the original primal problem; under mild conditions, $Ax_k - b \to 0$ as $k \to \infty$, so the KKT conditions are satisfied in the limit and $x_k$, $u_k$ converge to the solutions.

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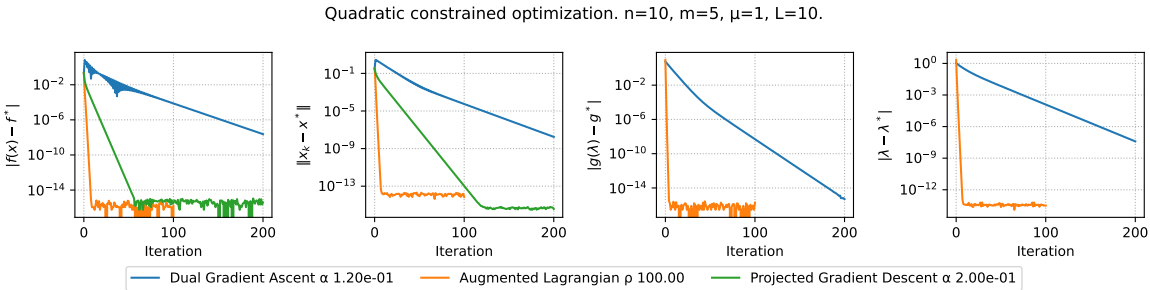
- **Advantage:** The augmented Lagrangian gives better convergence.
- **Disadvantage:** We lose decomposability! (Separability is ruined)

# Example: equality constrained quadratic minimization.

$$f(x) = \frac{1}{2}x^T Ax - b^T x \to \min_{x \in \mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m \times n}, m < n.$$




One can see, clear numerical superiority of the Augmented Lagrangian method both in convex and strongly convex case.

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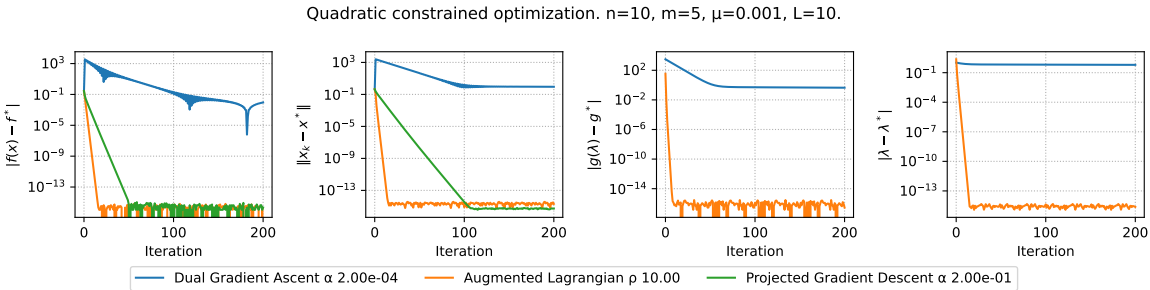


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# Introduction to ADMM

# Alternating Direction Method of Multipliers (ADMM)

**Alternating direction method of multipliers** or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

$$\min_{x,z} f(x) + g(z)$$

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where $\rho > 0$ is a parameter. The augmented Lagrangian for this problem is defined as:

$$L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|^2$$

# Alternating Direction Method of Multipliers (ADMM)

**ADMM repeats the following steps, for $k = 1, 2, 3, ...$:**

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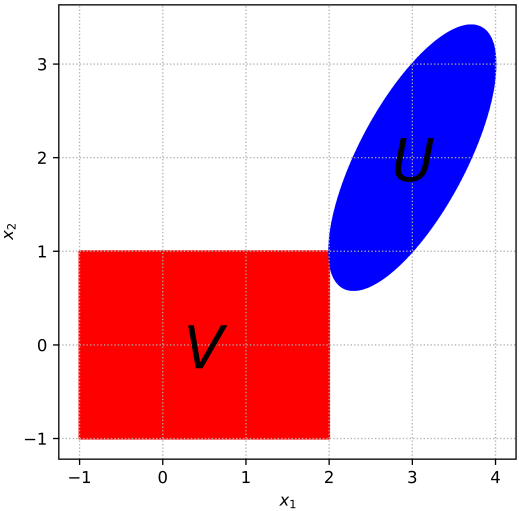
**Note:** The usual method of multipliers would replace the first two steps by a joint minimization:

$$(x^{(k)}, z^{(k)}) = \arg\min_{x,z} L_\rho(x, z, u^{(k-1)})$$

# Example: Alternating Projections




Consider finding a point in the intersection of convex sets $U, V \subseteq \mathbb{R}^n$:

$$\min_x I_U(x) + I_V(x)$$

To transform this problem into ADMM form, we express it as:

$$\min_{x,z} I_U(x) + I_V(z) \quad \text{subject to} \quad x - z = 0$$

Each ADMM cycle involves two projections:

$$x_k = \arg\min_x P_U\left(z_{k-1} - w_{k-1}\right)$$
$$z_k = \arg\min_z P_V\left(x_k + w_{k-1}\right)$$
$$w_k = w_{k-1} + x_k - z_k$$

# Sources


- Ryan Tibshirani. Convex Optimization 10-725