Proximal Gradient Method.

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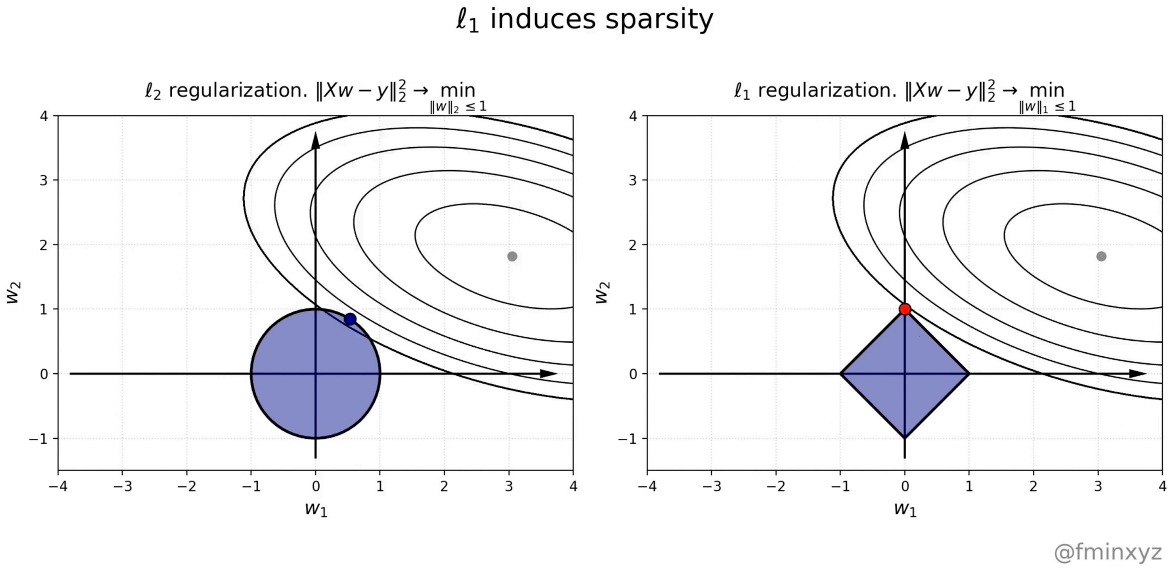
Методы Оптимизации в Машинном Обучении. ФКН ВШЭ

Subgradient method

Non-smooth problems



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convex (non-smooth)	strongly convex (non-smooth)
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Theorem

Assume that f is G-Lipschitz and convex, then Subgradient method converges as:

$$f(\overline{x}) - f^* \leq \frac{GR}{\sqrt{k}},$$

where

- $\alpha = \frac{R}{G\sqrt{k}}$

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- Subgradient method is optimal for the problems above.
- One can use Mirror Descent (a generalization of the subgradient method to a possiby non-Euclidian distance) with the same convergence rate to better fit the geometry of the problem.
- However, we can achieve standard gradient descent rate $\mathcal{O}\left(\frac{1}{k}\right)$ (and even accelerated version $\mathcal{O}\left(\frac{1}{k^2}\right)$) if we will exploit the structure of the problem.

Proximal operator

Proximal mapping intuition

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

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! Proximal operator

$$\text{prox}_{f,\alpha}(x_k) = \text{arg}\min_{x \in \mathbb{R}^n} \left[f(x) + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

Proximal operator visualization

$$\text{Prox}_f(x) = \underset{x'}{\text{argmin}} \frac{1}{2}\|x - x'\|^2 + f(x')$$

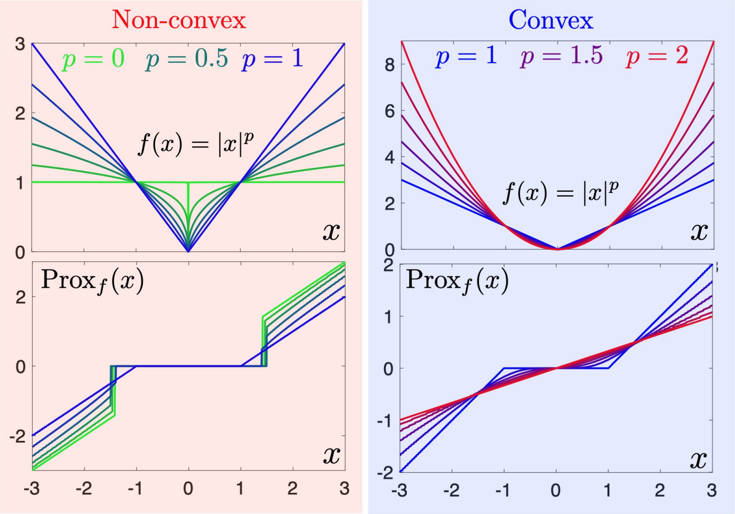

Рисунок 1: Source

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- **GD from proximal method.** Back to the discretization:

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Proximity: Replace $\mathbb{I}_S$ by some convex function!

$$\text{prox}_r(y) = \text{prox}_{r,1}(y) := \arg\min \frac{1}{2}\|x - y\|^2 + r(x)$$

Composite optimization

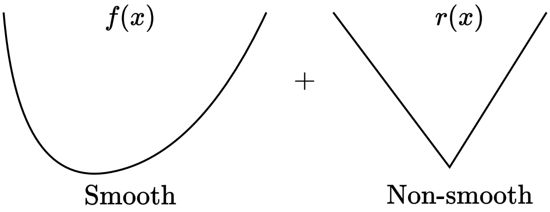
Regularized / Composite Objectives

Many nonsmooth problems take the form

$$\min_{x \in \mathbb{R}^n} \varphi(x) = f(x) + r(x)$$

- **Lasso, L1-LS, compressed sensing**

$$f(x) = \frac{1}{2}\|Ax - b\|_2^2, r(x) = \lambda\|x\|_1$$



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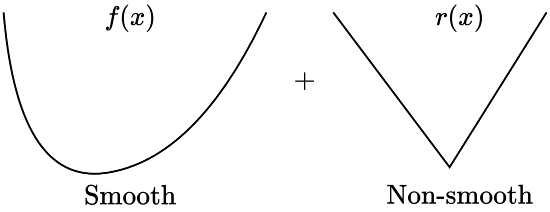
- **L1-Logistic regression, sparse LR**

$$f(x) = -y \log h(x) - (1-y)\log(1-h(x)), r(x) = \lambda\|x\|_1$$



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Which leads to the proximal gradient method:

$$
x_{k+1} = \text{prox}_{r,\alpha}(x_k - \alpha \nabla f(x_k))
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And this method converges at a rate of $\mathcal{O}(\frac{1}{k})$!

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x^* &= (I + \alpha \partial r)^{-1}(x^* - \alpha \nabla f(x^*)) \\
x^* &= \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))
\end{aligned}
$$

Which leads to the proximal gradient method:

$$
x_{k+1} = \text{prox}_{r,\alpha}(x_k - \alpha \nabla f(x_k))
$$

And this method converges at a rate of $\mathcal{O}(\frac{1}{k})$!

Another form of proximal operator

$$
\text{prox}_{f,\alpha}(x_k) = \text{prox}_{\alpha f}(x_k) = \text{arg}\min_{x \in \mathbb{R}^n} \left[\alpha f(x) + \frac{1}{2} \|x - x_k\|_2^2 \right] \qquad \text{prox}_{f}(x_k) = \text{arg}\min_{x \in \mathbb{R}^n} \left[f(x) + \frac{1}{2} \|x - x_k\|_2^2 \right]
$$

Proximal operators examples

- $r(x) = \lambda \|x\|_1$, $\lambda > 0$

$$[\text{prox}_r(x)]_i = [|x_i| - \lambda]_+ \cdot \text{sign}(x_i),$$

which is also known as soft-thresholding operator.

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- $r(x) = \frac{\lambda}{2}\|x\|_2^2$, $\lambda > 0$

$$\text{prox}_r(x) = \frac{x}{1+\lambda}.$$

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- $r(x) = \frac{\lambda}{2}\|x\|_2^2$, $\lambda > 0$

$$\text{prox}_r(x) = \frac{x}{1+\lambda}.$$

- $r(x) = \mathbb{I}_S(x)$.

$$\text{prox}_r(x_k - \alpha \nabla f(x_k)) = \text{proj}_r(x_k - \alpha \nabla f(x_k))$$

Proximal operator properties

Theorem

Let $r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function for which prox_r is defined. If there exists such an $\hat{x} \in \mathbb{R}^n$ that $r(x) < +\infty$. Then, the proximal operator is uniquely defined (i.e., it always returns a single unique value).

Proof:

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The proximal operator returns the minimum of some optimization problem.

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Question: What can be said about this problem?

Proximal operator properties

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The proximal operator returns the minimum of some optimization problem.

Question: What can be said about this problem?

It is strongly convex, meaning it has exactly one unique minimum (the existence of $\hat{x}$ is necessary for $r(\tilde{x}) + \frac{1}{2}\|x - \tilde{x}\|_2^2$ to take a finite value somewhere).

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Theorem

Let $r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function for which prox_r is defined. Then, for any $x, y \in \mathbb{R}^n$, the following three conditions are equivalent:

- $\text{prox}_r(x) = y$,

Proof

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- $\langle x - y, z - y \rangle \leq r(z) - r(y)$ for any $z \in \mathbb{R}^n$.

Proof

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Proof

1. Let's establish the equivalence between the first and second conditions.The first condition can be rewritten as

$$y = \arg\min_{\tilde{x} \in \mathbb{R}^d} \left(r(\tilde{x}) + \frac{1}{2}\|x - \tilde{x}\|^2 \right).$$

From the optimality condition for the convex function r, this is equivalent to:

$$0 \in \partial \left(r(\tilde{x}) + \frac{1}{2}\|x - \tilde{x}\|^2 \right)\Big|_{\tilde{x}=y} = \partial r(y) + y - x.$$

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2. From the definition of the subdifferential, for any subgradient $g \in \partial f(y)$ and for any $z \in \mathbb{R}^d$:

$$\langle g, z - y \rangle \leq r(z) - r(y).$$

In particular, this holds true for $g = x - y$. Conversely, it is also clear: for $g = x - y$, the above relationship holds, which means $g \in \partial r(y)$.

Proximal operator properties

Theorem

The operator $\text{prox}_r(x)$ is firmly nonexpansive (FNE)

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2^2 \leq \langle \text{prox}_r(x) - \text{prox}_r(y), x - y \rangle$$

and nonexpansive:

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2 \leq \|x - y\|_2$$

Proof

1. Let $u = \text{prox}_r(x)$, and $v = \text{prox}_r(y)$. Then, from the previous property:

$$\langle x - u, z_1 - u \rangle \leq r(z_1) - r(u)$$
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2. Substitute $z_1 = v$ and $z_2 = u$. Summing up, we get:

$$\begin{aligned} \langle x - u, v - u \rangle + \langle y - v, u - v \rangle &\leq 0, \\ \langle x - y, v - u \rangle + \|v - u\|_2^2 &\leq 0. \end{aligned}$$

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3. Which is exactly what we need to prove after substitution of u, v.

$$\|u - v\|_2^2 \leq \langle x - y, u - v \rangle$$

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4. The last point comes from simple Cauchy-Bunyakovsky-Schwarz for the last inequality.

Proximal operator properties

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex functions. Additionally, assume that f is continuously differentiable and L-smooth, and for r, prox_r is defined. Then, x^* is a solution to the composite optimization problem if and only if, for any $\alpha > 0$, it satisfies:

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

Proof

1. Optimality conditions:

Proximal operator properties

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$$0 \in \nabla f(x^*) + \partial r(x^*)$$

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Proof

1. Optimality conditions:

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Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex functions. Additionally, assume that f is continuously differentiable and L-smooth, and for r, prox_r is defined. Then, x^* is a solution to the composite optimization problem if and only if, for any $\alpha > 0$, it satisfies:

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\end{aligned}$$

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2. Recall from the previous lemma:

$$\text{prox}_r(x) = y \Leftrightarrow x - y \in \partial r(y)$$

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2. Recall from the previous lemma:

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3. Finally,

$$x^* = \text{prox}_{\alpha r}(x^* - \alpha \nabla f(x^*)) = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

Theoretical tools for convergence analysis

Convergence tools

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an L-smooth convex function. Then, for any $x, y \in \mathbb{R}^n$, the following inequality holds:

$$
f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y) \text{ or, equivalently,}
$$
$$
\|\nabla f(y) - \nabla f(x)\|_2^2 = \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L \left(f(x) - f(y) - \langle \nabla f(y), x - y \rangle \right)
$$

Proof

1. To prove this, we'll consider another function $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$. It is obviously a convex function (as a sum of convex functions). And it is easy to verify, that it is an L-smooth function by definition, since $\nabla \varphi(y) = \nabla f(y) - \nabla f(x)$ and $\|\nabla \varphi(y_1) - \nabla \varphi(y_2)\| = \|\nabla f(y_1) - \nabla f(y_2)\| \leq L\|y_1 - y_2\|$.

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Convergence tools 💎💎💎💎

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$$
\stackrel{x := y, y := y - \frac{1}{L}\nabla\varphi(y)}{} \quad \varphi\left(y - \frac{1}{L}\nabla \varphi(y)\right) \leq \varphi(y) + \left\langle \nabla \varphi(y), -\frac{1}{L}\nabla \varphi(y)\right\rangle + \frac{1}{2L}\|\nabla \varphi(y)\|_2^2
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Theorem

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\begin{aligned}
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{\scriptstyle x:=y,\, y:=y - \frac{1}{L}\nabla \varphi(y)} \quad & \varphi\left(y - \frac{1}{L}\nabla \varphi(y)\right) \leq \varphi(y) + \left\langle \nabla \varphi(y), -\frac{1}{L}\nabla \varphi(y)\right\rangle + \frac{1}{2L}\|\nabla \varphi(y)\|_2^2 \\
& \varphi\left(y - \frac{1}{L}\nabla \varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla \varphi(y)\|_2^2
\end{aligned}
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Convergence tools

3. From the first order optimality conditions for the convex function $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$. We can conclude, that for any x, the minimum of the function $\varphi(y)$ is at the point $y = x$. Therefore:

$$\varphi(x) \leq \varphi\left(y - \frac{1}{L}\nabla\varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla\varphi(y)\|_2^2$$

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$$f(x) + \langle \nabla f(x), y - x\rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y)$$

Convergence tools 💎💎💎💎

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$$\varphi(x) \leq \varphi\left(y - \frac{1}{L}\nabla\varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla\varphi(y)\|_2^2$$

4. Now, substitute $\varphi(y) = f(y) - \langle \nabla f(x), y\rangle$:

$$f(x) - \langle \nabla f(x), x\rangle \leq f(y) - \langle \nabla f(x), y\rangle - \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|_2^2$$

$$f(x) + \langle \nabla f(x), y - x\rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y)$$

$$\|\nabla f(y) - \nabla f(x)\|_2^2 \leq 2L\left(f(y) - f(x) - \langle \nabla f(x), y - x\rangle\right)$$

Convergence tools

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& \|\nabla f(y) - \nabla f(x)\|_2^2 \leq 2L\left(f(y) - f(x) - \langle \nabla f(x), y - x\rangle\right) \\
\text{switch x and y} \quad & \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L\left(f(x) - f(y) - \langle \nabla f(y), x - y\rangle\right)
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\end{aligned}$$

The lemma has been proved. From the first view it does not make a lot of geometrical sense, but we will use it as a convenient tool to bound the difference between gradients.

Convergence tools 💎💎💎💎

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable on $\mathbb{R}^n$. Then, the function f is μ-strongly convex if and only if for any $x, y \in \mathbb{R}^d$ the following holds:

$$\begin{aligned} \text{Strongly convex case } \mu > 0 \quad & \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2 \\ \text{Convex case } \mu = 0 \quad & \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0 \end{aligned}$$

Proof

1. We will only give the proof for the strongly convex case, the convex one follows from it with setting $\mu = 0$. We start from necessity. For the strongly convex function

$$\begin{aligned} & f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2}\|x - y\|_2^2 \\ & f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2}\|x - y\|_2^2 \\ \text{sum} \quad & \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2 \end{aligned}$$

Convergence tools

2. For the sufficiency we assume, that $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$. Using Newton-Leibniz theorem $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x-y)), x - y \rangle dt$:

Convergence tools

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$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle = \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle$$

Convergence tools 💎💎💎💎

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&\geq \int_0^1 t^{-1} \mu \|t(x-y)\|^2 dt
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Convergence tools 💎💎💎💎

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Convergence tools 💎💎💎💎

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Thus, we have a strong convexity criterion satisfied

$$
f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|_2^2
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Convergence tools 💎💎💎💎

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Convergence tools

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$$
\text{switch x and y} \qquad -\langle \nabla f(x), x-y \rangle \leq -\left(f(x) - f(y) + \frac{\mu}{2} \|x-y\|_2^2 \right)
$$

Proximal Gradient Method. Convex case

Convergence

Theorem

Consider the proximal gradient method

$$x_{k+1} = \text{prox}_{\alpha r}\left(x_k - \alpha \nabla f(x_k)\right)$$

For the criterion $\varphi(x) = f(x) + r(x)$, we assume:

- f is convex, differentiable, $\text{dom}(f) = \mathbb{R}^n$, and ∇f is Lipschitz continuous with constant $L > 0$.
- r is convex, and $\text{prox}_{\alpha r}(x_k) = \text{arg}\min\limits_{x \in \mathbb{R}^n} \left[\alpha r(x) + \frac{1}{2}\|x - x_k\|_2^2\right]$ can be evaluated.

Proximal gradient descent with fixed step size $\alpha = 1/L$ satisfies

$$\varphi(x_k) - \varphi^* \leq \frac{L\|x_0 - x^*\|^2}{2k},$$

Proximal gradient descent has a convergence rate of $O(1/k)$ or $O(1/\varepsilon)$. This matches the gradient descent rate! (But remember the proximal operation cost)

Convergence

Proof

1. Let's introduce the **gradient mapping**, denoted as $G_\alpha(x)$, acts as a "gradient-like object":

$$
\begin{aligned}
x_{k+1} &= \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) \\
x_{k+1} &= x_k - \alpha G_\alpha(x_k).
\end{aligned}
$$

where $G_\alpha(x)$ is:

$$
G_\alpha(x) = \frac{1}{\alpha}\left(x - \text{prox}_{\alpha r}\left(x - \alpha \nabla f\left(x\right)\right)\right)
$$

Observe that $G_\alpha(x) = 0$ if and only if x is optimal. Therefore, G_α is analogous to ∇f. If x is locally optimal, then $G_\alpha(x) = 0$ even for nonconvex f. This demonstrates that the proximal gradient method effectively combines gradient descent on f with the proximal operator of r, allowing it to handle non-differentiable components effectively.

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2. We will use smoothness and convexity of f for some arbitrary point x:

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\text{smoothness} \quad f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2}\|x_{k+1} - x_k\|_2^2
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Convergence

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&\text{convexity } f(x) \geq f(x_k) + \langle \nabla f(x_k), x - x_k \rangle \quad \leq f(x) - \langle \nabla f(x_k), x - x_k \rangle + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{\alpha^2 L}{2}\|G_\alpha(x_k)\|_2^2
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{\scriptstyle \text{smoothness}} \quad f(x_{k+1}) &\leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2}\|x_{k+1} - x_k\|_2^2 \\
{\scriptstyle \text{convexity } f(x) \geq f(x_k) + \langle \nabla f(x_k), x - x_k \rangle} \quad &\leq f(x) - \langle \nabla f(x_k), x - x_k \rangle + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{\alpha^2 L}{2}\|G_\alpha(x_k)\|_2^2 \\
&\leq f(x) + \langle \nabla f(x_k), x_{k+1} - x \rangle + \frac{\alpha^2 L}{2}\|G_\alpha(x_k)\|_2^2
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$$

$$
f(x_{k+1}) + r(x_{k+1}) \leq f(x) + r(x) - \langle G_\alpha(x_k), x - x_k + \alpha G_\alpha(x_k) \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2
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Convergence

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$\alpha \leq \frac{1}{L} \Rightarrow \frac{\alpha}{2} \left(\alpha L - 2 \right) \leq -\frac{\alpha}{2}$

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\end{aligned}
$$

7. Now it is easy to verify, that when $x = x_k$ we have monotonic decrease for the proximal gradient algorithm:

$$
\varphi(x_{k+1}) \leq \varphi(x_k) - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2
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Convergence

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\varphi(x_{k+1}) - \varphi(x^*) &\leq \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2}\|G_\alpha(x_k)\|_2^2 \\
&\leq \frac{1}{2\alpha}\left[2\langle \alpha G_\alpha(x_k), x_k - x^* \rangle - \|\alpha G_\alpha(x_k)\|_2^2\right]
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&\leq \frac{1}{2\alpha}\left[-\|x_k - x^* - \alpha G_\alpha(x_k)\|_2^2 + \|x_k - x^*\|_2^2\right]
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&\le \frac{1}{2\alpha}\left[\|x_k - x^*\|_2^2 - \|x_{k+1} - x^*\|_2^2\right]
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&\leq \frac{1}{2\alpha} \|x_0 - x^*\|_2^2
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$$
\begin{aligned}
\sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] &\leq \frac{1}{2\alpha} \left[\|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2 \right] \\
&\leq \frac{1}{2\alpha} \|x_0 - x^*\|_2^2
\end{aligned}
$$

10. Since $\varphi(x_k)$ is a decreasing sequence, it follows that:

Convergence

9. Now we write the bound above for all iterations $i \in 0, k-1$ and sum them:

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\sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] &\leq \frac{1}{2\alpha} \left[\|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2 \right] \\
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$$

$$
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Convergence 💎💎💎💎💎

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\varphi(x_k) &\leq \frac{1}{k} \sum_{i=0}^{k-1} \varphi(x_{i+1}) \\
\varphi(x_k) - \varphi(x^*) &\leq \frac{1}{k} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] \leq \frac{\|x_0 - x^*\|_2^2}{2\alpha k}
\end{aligned}
$$

Convergence

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\end{aligned}
$$

Which is a standard $\frac{L\|x_0 - x^*\|_2^2}{2k}$ with $\alpha = \frac{1}{L}$, or, $\mathcal{O}\left(\frac{1}{k}\right)$ rate for smooth convex problems with Gradient Descent!

Proximal Gradient Method. Strongly convex case

Convergence

Theorem

Consider the proximal gradient method

$$
x_{k+1} = \text{prox}_{\alpha r}\left(x_k - \alpha \nabla f(x_k)\right)
$$

For the criterion $\varphi(x) = f(x) + r(x)$, we assume:

- f is μ-strongly convex, differentiable, $\text{dom}(f) = \mathbb{R}^n$, and ∇f is Lipschitz continuous with constant $L > 0$.
- r is convex, and $\text{prox}_{\alpha r}(x_k) = \arg\min\limits_{x \in \mathbb{R}^n} \left[\alpha r(x) + \frac{1}{2}\|x - x_k\|_2^2\right]$ can be evaluated.

Proximal gradient descent with fixed step size $\alpha \leq 1/L$ satisfies

$$
\|x_k - x^*\|_2^2 \leq (1 - \alpha\mu)^k \|x_0 - x^*\|_2^2
$$

This is exactly gradient descent convergence rate. Note, that the original problem is even non-smooth!

Convergence

Proof

1. Considering the distance to the solution and using the stationary point lemm:

Convergence

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1. Considering the distance to the solution and using the stationary point lemm:

$$
\|x_{k+1} - x^*\|_2^2 = \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2
$$

Proof

1. Considering the distance to the solution and using the stationary point lemm:

$$
\begin{aligned}
\|x_{k+1} - x^*\|_2^2 &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2 \\
\text{stationary point lemm} &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - \text{prox}_{\alpha f}(x^* - \alpha \nabla f(x^*))\|_2^2
\end{aligned}
$$

Convergence

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\text{stationary point lemm} &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - \text{prox}_{\alpha f}(x^* - \alpha \nabla f(x^*))\|_2^2 \\
\text{nonexpansiveness} &\leq \|x_k - \alpha \nabla f(x_k) - x^* + \alpha \nabla f(x^*)\|_2^2
\end{aligned}
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&= \|x_k - x^*\|^2 - 2\alpha \langle \nabla f(x_k) - \nabla f(x^*), x_k - x^* \rangle + \alpha^2 \|\nabla f(x_k) - \nabla f(x^*)\|_2^2
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2. Now we use smoothness from the convergence tools and strong convexity:

Convergence

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&= \|x_k - x^*\|^2 - 2\alpha \langle \nabla f(x_k) - \nabla f(x^*), x_k - x^* \rangle + \alpha^2 \|\nabla f(x_k) - \nabla f(x^*)\|_2^2
\end{aligned}
$$

2. Now we use smoothness from the convergence tools and strong convexity:

$$
\text{smoothness} \quad \|\nabla f(x_k) - \nabla f(x^*)\|_2^2 \leq 2L \left(f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle \right)
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Convergence

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\text{strong convexity} \quad & -\langle \nabla f(x_k) - \nabla f(x^*), x_k - x^* \rangle \leq -\left(f(x_k) - f(x^*) + \frac{\mu}{2}\|x_k - x^*\|_2^2\right) - \langle \nabla f(x^*), x_k - x^* \rangle
\end{aligned}
$$

Convergence

3. Substitute it:

Convergence

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$$
\begin{aligned}
\|x_{k+1} - x^*\|_2^2 \leq \|x_k - x^*\|^2 - 2\alpha \left(f(x_k) - f(x^*) + \frac{\mu}{2}\|x_k - x^*\|_2^2 \right) - 2\alpha\langle \nabla f(x^*), x_k - x^* \rangle + \\
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&+ \alpha^2 2L \left(f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^*\rangle \right) \\
&\leq (1 - \alpha\mu)\|x_k - x^*\|^2 + 2\alpha(\alpha L - 1) \left(f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^*\rangle \right)
\end{aligned}
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+ \alpha^2 2L \left(f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^*\rangle \right) \\
\leq (1 - \alpha\mu)\|x_k - x^*\|^2 + 2\alpha(\alpha L - 1) \left(f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^*\rangle \right)
\end{aligned}
$$

4. Due to convexity of f: $f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^*\rangle \geq 0$. Therefore, if we use $\alpha \leq \frac{1}{L}$:

$$
\|x_{k+1} - x^*\|_2^2 \leq (1 - \alpha\mu)\|x_k - x^*\|^2,
$$

which is exactly linear convergence of the method with up to $1 - \frac{\mu}{L}$ convergence rate.

Accelerated Proximal Gradient – *convex* objective

Accelerated Proximal Gradient Method

Let $f : \mathbb{R}^n \to \mathbb{R}$ be **convex** and L-**smooth**, $r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, closed and convex, $\varphi(x) = f(x) + r(x)$ admit a minimiser $x^\star$, and suppose $\text{prox}_{\alpha r}$ is easy to evaluate for $\alpha > 0$. With any $x_0 \in \text{dom}\, r$ define the sequence

$$
\begin{aligned}
t_0 &= 1, \qquad y_0 = x_0, \\
x_k &= \text{prox}_{\frac{1}{L} r}\left(y_{k-1} - \tfrac{1}{L}\nabla f(y_{k-1})\right), \\
t_k &= \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2}, \\
y_k &= x_k + \frac{t_{k-1} - 1}{t_k}(x_k - x_{k-1}), \qquad k \geq 1.
\end{aligned}
$$

Then for every $k \geq 1$

$$
\boxed{\varphi(x_k) - \varphi(x^\star) \leq \frac{2L \|x_0 - x^\star\|_2^2}{(k+1)^2}}
$$

Accelerated Proximal Gradient – μ-*strongly convex* objective

Accelerated Proximal Gradient Method

Assume in addition that f is μ-**strongly convex** ($\mu > 0$).
Set the step $\alpha = \frac{1}{L}$ and the fixed momentum parameter

$$\beta = \frac{\sqrt{L/\mu} - 1}{\sqrt{L/\mu} + 1}.$$

Generate the iterates for $k \geq 0$ (take $x_{-1} = x_0$):

$$y_k = x_k + \beta\,(x_k - x_{k-1}),$$
$$x_{k+1} = \text{prox}_{\alpha r}(y_k - \alpha \nabla f(y_k)).$$

Then for every $k \geq 0$

$$\boxed{\varphi(x_k) - \varphi(x^\star) \;\leq\; \left(1 - \sqrt{\frac{\mu}{L}}\right)^k \left(\varphi(x_0) - \varphi(x^\star) + \frac{\mu}{2}\|x_0 - x^\star\|_2^2\right)}$$

Numerical experiments

Quadratic case

$$f(x) = \frac{1}{2m}\|Ax - b\|_2^2 + \lambda\|x\|_1 \to \min_{x \in \mathbb{R}^n}, \qquad A \in \mathbb{R}^{m \times n}, \quad \lambda\left(\tfrac{1}{m}A^TA\right) \in [\mu; L].$$

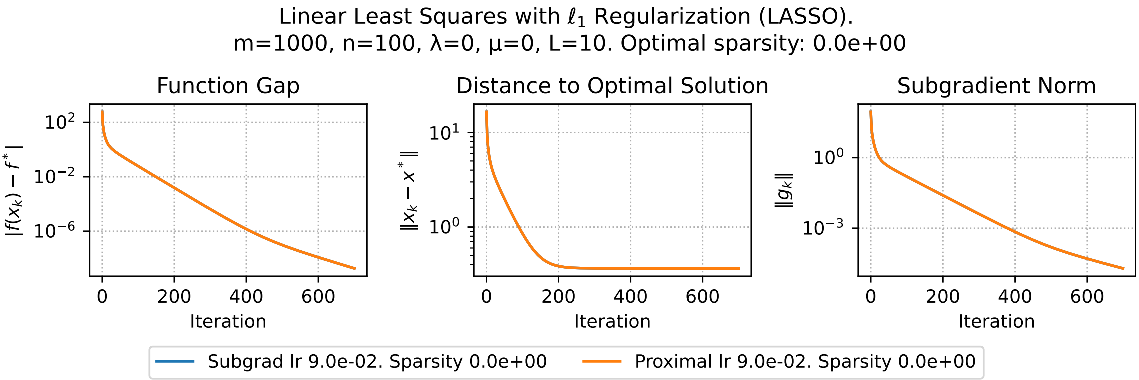


Рисунок 2: Smooth convex case. Sublinear convergence, no convergence in domain, no difference between subgradient and proximal methods

Quadratic case

$$f(x) = \frac{1}{2m}\|Ax - b\|_2^2 + \lambda\|x\|_1 \to \min_{x \in \mathbb{R}^n}, \qquad A \in \mathbb{R}^{m \times n}, \quad \lambda\left(\tfrac{1}{m}A^T A\right) \in [\mu; L].$$

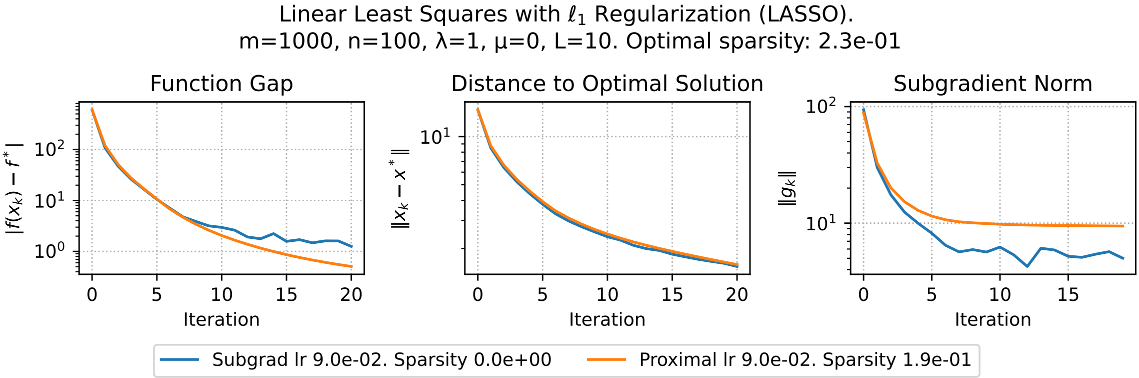


Рисунок 3: Non-smooth convex case. Sublinear convergence. At the beginning, the subgradient method and proximal method are close.

Quadratic case

$$f(x) = \frac{1}{2m}\|Ax - b\|_2^2 + \lambda\|x\|_1 \to \min_{x \in \mathbb{R}^n}, \qquad A \in \mathbb{R}^{m \times n}, \quad \lambda\left(\tfrac{1}{m}A^TA\right) \in [\mu; L].$$

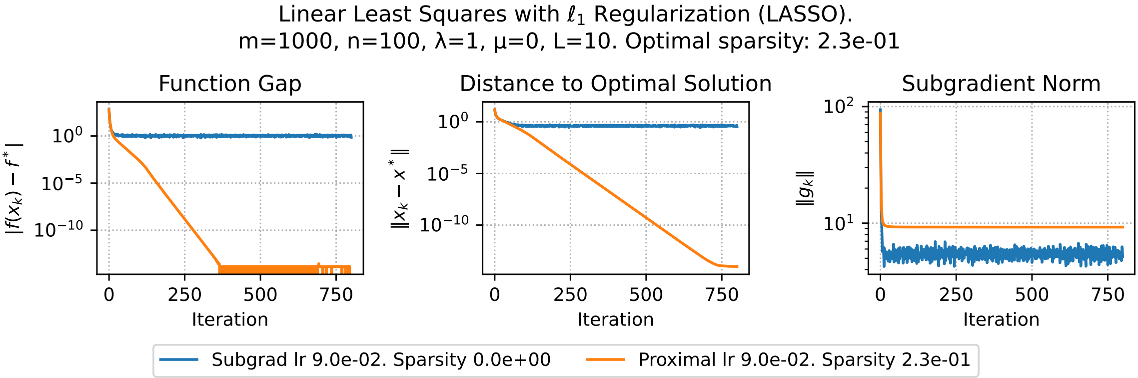


Рисунок 4: Non-smooth convex case. If we take more iterations, the proximal method converges with the constant learning rate, which is not the case for the subgradient method. The difference is tremendous, while the iteration complexity is the same.

Binary logistic regression

$$f(x) = \frac{1}{m}\sum_{i=1}^{m} \log(1 + \exp(-b_i(A_i x))) + \lambda\|x\|_1 \to \min_{x\in\mathbb{R}^n}, \qquad A_i \in \mathbb{R}^n, \quad b_i \in \{-1, 1\}$$

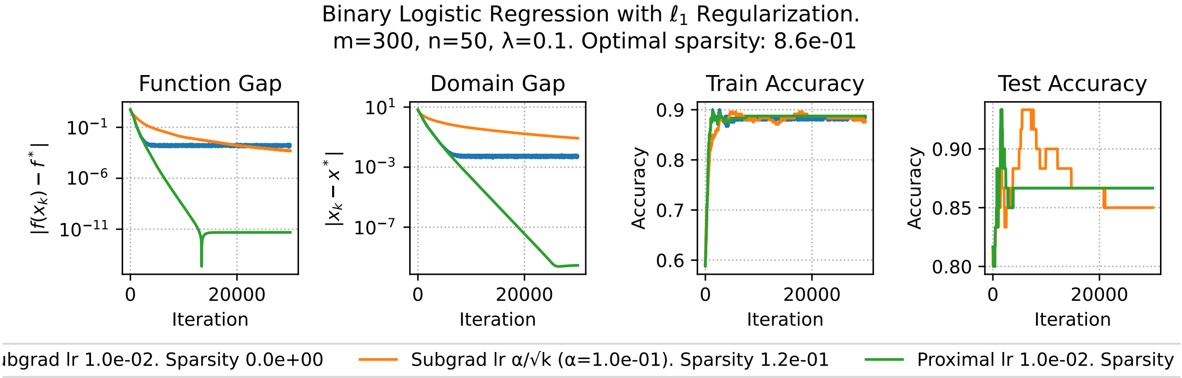
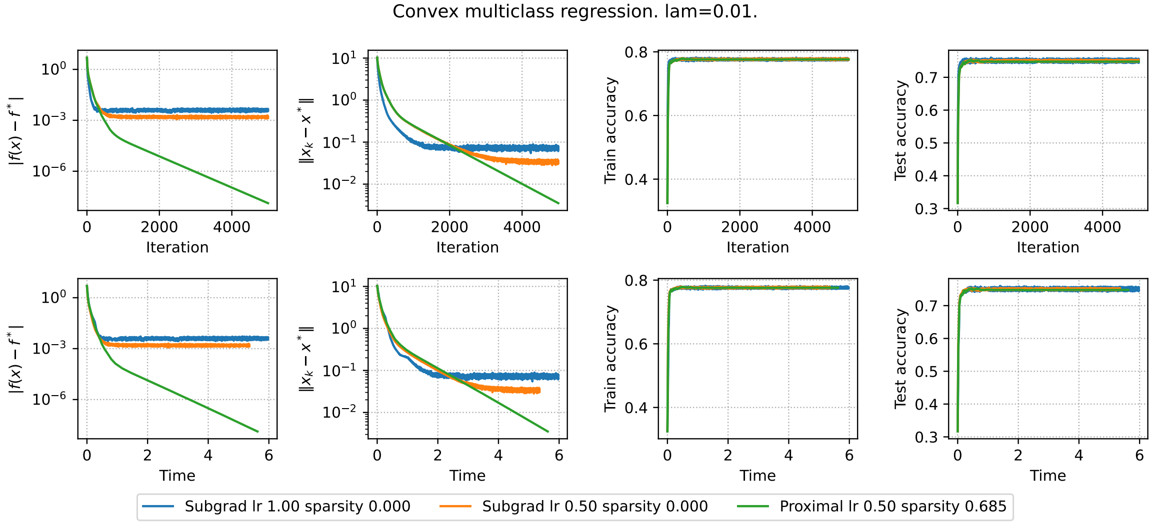


Рисунок 5: Logistic regression with ℓ_1 regularization

Softmax multiclass regression



Example: ISTA

Iterative Shrinkage-Thresholding Algorithm (ISTA)

ISTA is a popular method for solving optimization problems involving L1 regularization, such as Lasso. It combines gradient descent with a shrinkage operator to handle the non-smooth L1 penalty effectively.

- **Algorithm**:

Example: ISTA

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- **Algorithm**:
  - Given x_0, for $k \geq 0$, repeat:
$$x_{k+1} = \text{prox}_{\lambda\alpha\|\cdot\|_1}\left(x_k - \alpha \nabla f(x_k)\right),$$
    where $\text{prox}_{\lambda\alpha\|\cdot\|_1}(v)$ applies soft thresholding to each component of v.

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- **Convergence**:

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- **Convergence**:
    - Converges at a rate of $O(1/k)$ for suitable step size α.

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- **Application**:

Example: ISTA

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- **Convergence**:
  - Converges at a rate of $O(1/k)$ for suitable step size α.
- **Application**:
  - Efficient for sparse signal recovery, image processing, and compressed sensing.

Example: FISTA

Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)

FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

- **Algorithm**:

Example: FISTA

Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)

FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

- **Algorithm**:
    - Initialize $x_0 = y_0$, $t_0 = 1$.

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  - Initialize $x_0 = y_0$, $t_0 = 1$.
  - For $k \geq 1$, update:

$$
\begin{aligned}
x_k &= \text{prox}_{\lambda\alpha\|\cdot\|_1}\left(y_{k-1} - \alpha\nabla f(y_{k-1})\right), \\
t_k &= \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2}, \\
y_k &= x_k + \frac{t_{k-1} - 1}{t_k}(x_k - x_{k-1}).
\end{aligned}
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\end{aligned}
$$

- **Convergence**:
  - Improves the convergence rate to $O(1/k^2)$.

Example: FISTA

Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)

FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

- **Algorithm**:
  - Initialize $x_0 = y_0$, $t_0 = 1$.
  - For $k \geq 1$, update:

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x_k &= \text{prox}_{\lambda\alpha\|\cdot\|_1}\left(y_{k-1} - \alpha\nabla f(y_{k-1})\right), \\
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y_k &= x_k + \frac{t_{k-1} - 1}{t_k}(x_k - x_{k-1}).
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- **Convergence**:
  - Improves the convergence rate to $O(1/k^2)$.
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- **Convergence**:
  - Improves the convergence rate to $O(1/k^2)$.
- **Application**:
  - Especially useful for large-scale problems in machine learning and signal processing where the L1 penalty induces sparsity.

Example: Matrix Completion

Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

- **Problem Formulation**:

$$\min_X \frac{1}{2}\|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda\|X\|_*,$$

  where P_Ω projects onto the observed set Ω, and $\|\cdot\|_*$ denotes the nuclear norm.

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- **Application**:
  - Widely used in recommender systems, image recovery, and other domains where data is naturally matrix-formed but partially observed.

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- Proximal method is a general modern framework for many numerical methods. Further development includes accelerated, stochastic, primal-dual modifications and etc.
- Further reading: Proximal operator splitting, Douglas-Rachford splitting, Best approximation problem, Three operator splitting.