

Introduction to dual methods





#### Primal problem

$$\begin{aligned} f_0(x) &\to \min_{x \in \mathbb{R}^n} \\ \text{s.t.} & f_i(x) \leq 0, \ i=1,\ldots,m \\ & h_i(x) = 0, \ i=1,\ldots,p \end{aligned}$$

#### Dual problem

$$\begin{split} g(\lambda,\nu) &= \min_{x \in \mathcal{D}} L(x,\lambda,\nu) = \\ \min_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right) &\to \max_{\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p} \\ &\text{s.t. } \lambda \succ 0 \end{split}$$

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- **Dual Problems Provide Bounds**. Dual problems often offer bounds on the optimal value of the primal problem. This can be useful for assessing the quality of approximate solutions.



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- Market Equilibrium. Dual problems often represent market equilibrium conditions, making them essential for economic modeling and analysis.
- **Dual Problems Provide Bounds.** Dual problems often offer bounds on the optimal value of the primal problem. This can be useful for assessing the quality of approximate solutions.
- **Duality Gap.** The difference between the primal and dual solutions (duality gap) provides valuable information about the solution's optimality.

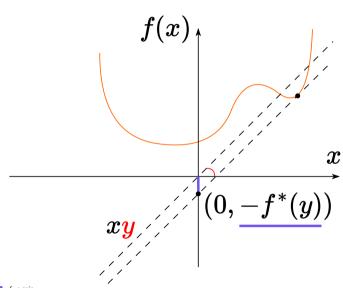


**Conjugate functions** 





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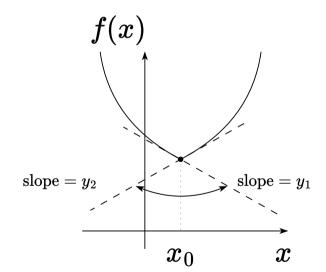


Recall that given  $f:\mathbb{R}^n\to\mathbb{R},$  the function defined by

$$f^*(y) = \max_x \left[ y^T x - f(x) \right]$$

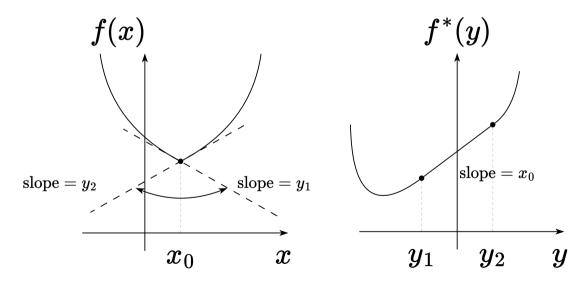
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• If f is closed and convex, then  $f^{**} = f$ . Also,

$$x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x) \Leftrightarrow x \in \arg\min_{z} \left[ f(z) - y^T z \right]$$



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• If f is strictly convex, then

$$\nabla f^*(y) = \arg\min_{z} \left[ f(z) - y^T z \right]$$

We will show that  $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$ , assuming that f is convex and closed.

• **Proof of**  $\Leftarrow$ : Suppose  $y \in \partial f(x)$ . Then  $x \in M_y$ , the set of maximizers of  $y^Tz - f(z)$  over z. But

$$f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{ and } \quad \partial f^*(y) = \operatorname{cl}(\operatorname{conv}(\bigcup_{z \in M_+} \{z\})).$$

Thus  $x \in \partial f^*(y)$ .



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• Proof of  $\Leftarrow$ : Suppose  $y \in \partial f(x)$ . Then  $x \in M_v$ , the set of maximizers of  $y^Tz - f(z)$  over z. But

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Clearly  $y \in \partial f(x) \Leftrightarrow x \in \arg\min_z \{f(z) - y^T z\}$ 

Lastly, if f is strictly convex, then we know that  $f(z) - y^T z$  has a unique minimizer over z, and this must be  $\nabla f^*(y)$ .

 $f \to \min_{x,y,z}$  Conjugate functions



Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

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where  $f^{*}$  is the conjugate of f. Defining  $g(u)=-f^{*}(-A^{T}u)-b^{T}u$ , note that:

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Dual ascent method for maximizing dual objective:

 $x_k \in \arg\min_x \left[ f(x) + (u_{k-1})^T Ax \right]$   $u_k = u_{k-1} + \alpha_k (Ax_k - b)$ 

• Step sizes  $\alpha_k$ , k=1,2,3,..., are chosen in standard ways.

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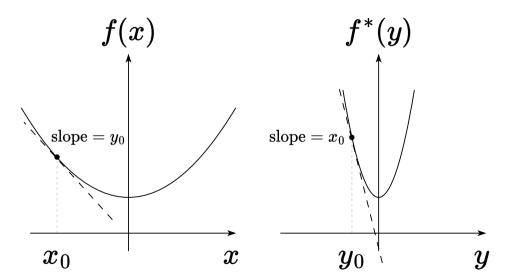
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- Step sizes  $\alpha_k, \ k=1,2,3,...,$  are chosen in standard ways.
  - Proximal gradients and acceleration can be applied as they would usually.

# ${\bf Slopes} \ {\bf of} \ f \ {\bf and} \ f^*$



Assume that f is a closed and convex function. Then f is strongly convex with parameter  $\mu \Leftrightarrow \nabla f^*$  is Lipschitz with parameter  $1/\mu$ .

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Hence, defining  $x_u = \nabla f^*(u)$  and  $x_v = \nabla f^*(v)$ ,

$$f(x_v) - u^T x_v \geq f(x_u) - u^T x_u + \frac{\mu}{2} \|x_u - x_v\|^2$$

$$f(x_u) - v^T x_u \ge f(x_v) - v^T x_v + \frac{\mu}{2} \|x_u - x_v\|^2$$

 $f \to \min_{x,y,z}$  Dual ascent

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$$f(x_u) - v^T x_u \geq f(x_v) - v^T x_v + \frac{\mu}{2} \|x_u - x_v\|^2$$

Adding these together, using the Cauchy-Schwarz inequality, and rearranging shows that

$$||x_u - x_v||^2 \le \frac{1}{u} ||u - v||^2$$

 $f \to \min_{x,y,z}$  Dual ascent

**Proof of "\Leftarrow"**: for simplicity, call  $g=f^*$  and  $L=\frac{1}{\mu}$ . As  $\nabla g$  is Lipschitz with constant L, so is  $g_x(z) = g(z) - \nabla g(x)^T z$ , hence

$$g_x(z) \leq g_x(y) + \nabla g_x(y)^T(z-y) + \frac{L}{2}\|z-y\|_2^2$$

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Minimizing each side over z, and rearranging, gives

$$\frac{1}{2L}\|\nabla g(x) - \nabla g(y)\|^2 \leq g(y) - g(x) + \nabla g(x)^T(x-y)$$

 $f \to \min_{x,y,z}$ 

**Proof of "** $\Leftarrow$ ": for simplicity, call  $g=f^*$  and  $L=\frac{1}{n}$ . As  $\nabla g$  is Lipschitz with constant L, so is  $q_x(z) = q(z) - \nabla q(x)^T z$ , hence

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Exchanging roles of x, y, and adding together, gives

$$\frac{1}{I}\|\nabla g(x) - \nabla g(y)\|^2 \leq (\nabla g(x) - \nabla g(y))^T(x-y)$$



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Let  $u = \nabla f(x)$ ,  $v = \nabla g(y)$ ; then  $x \in \partial g^*(u)$ ,  $y \in \partial g^*(v)$ , and the above reads  $(x-y)^T(u-v) \geq \frac{\|u-v\|^2}{L}$ , implying the result.

### **Convergence guarantees**

The following results hold from combining the last fact with what we already know about gradient descent: (This is ignoring the role of A, and thus reflects the case when the singular values of A are all close to 1. To be more precise, the step sizes here should be:  $\frac{\mu}{\sigma_{\max}(A)^2}$  (first case) and  $\frac{2}{\sigma_{\max}(A)^2 + \sigma_{\min}(A)^2}$  (second case).)

• If f is strongly convex with parameter  $\mu$ , then dual gradient ascent with constant step sizes  $\alpha_k = \mu$  converges at sublinear rate  $O(\frac{1}{\epsilon})$ .

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- If f is strongly convex with parameter  $\mu$  and  $\nabla f$  is Lipschitz with parameter L, then dual gradient ascent with step sizes  $\alpha_k = \frac{2}{\frac{1}{2} + \frac{1}{r}}$  converges at linear rate  $O(\log(\frac{1}{\epsilon}))$ .

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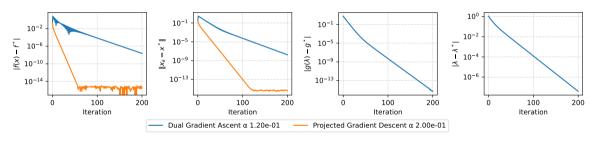
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- Note that this describes convergence in the dual. Convergence in the primal requires more assumptions

## Example: equality constrained quadratic minimization.

$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m \times n}, m < n.$$

Quadratic constrained optimization. n=10, m=5,  $\mu$ =1, L=10.

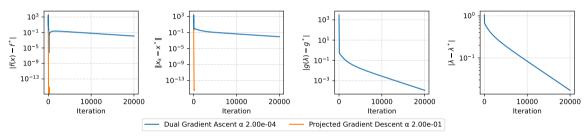


We need to find a minimum of a quadratic function in some linear subspace, defined by the solution of linear equation Cx = d. This is a conditional optimization problem, we start from strongly convex setting.

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Quadratic constrained optimization. n=10, m=5,  $\mu$ =0.001, L=10.



Situation is getting worse as soon as we loose strong convexity, the dual convergence will still be linear, but the rate is very low.



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$$\min_{x} \sum_{i=1}^{B} f_i(x_i) \quad \text{subject to} \quad Ax = b$$



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Here  $x=(x_1,\ldots,x_B)\in\mathbb{R}^n$  divides into B blocks of variables, with each  $x_i\in\mathbb{R}^{n_i}$ . We can also partition A accordingly:

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Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into B separate problems:

$$\begin{split} x^{\mathsf{new}} \in \arg\min_{x} \left( \sum_{i=1}^{B} f_i(x_i) + u^T A x \right) \\ \Rightarrow x_i^{\mathsf{new}} \in \arg\min_{x_i} \left( f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \dots, B \\ x_i^k \in \arg\min_{x} \left( f_i(x_i) + (u^{k-1})^T A_i x_i \right), \quad i = 1, \dots, B \end{split}$$

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$$I(J_i(x_i) + u \mid A_i x_i), \quad i = 1, \dots, I$$

$$x_i^k \in \arg\min_{x_i} \left( f_i(x_i) + (u^{k-1})^T A_i x_i \right), \quad i = 1, \dots, B$$

Can think of these steps as: • **Broadcast:** Send u to each of the B processors,

each optimizes in parallel to find  $x_i$ .

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$$A = [A_1 \dots A_B], \text{ where } A_i \in \mathbb{R}^{m \times n_i}$$

Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into B separate problems:

$$\begin{split} x^{\mathsf{new}} &\in \arg\min_{x} \left( \sum_{i=1}^{B} f_i(x_i) + u^T A x \right) \\ \Rightarrow x_i^{\mathsf{new}} &\in \arg\min_{x} \left( f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \dots, B \end{split}$$

Can think of these steps as:

update the global dual variable u.

 $u^k = u^{k-1} + \alpha_k \left( \sum_{i=1}^B A_i x_i^k - b \right)$ 

 $x_i^k \in \arg\min_{\boldsymbol{x}} \left( f_i(\boldsymbol{x}_i) + (\boldsymbol{u}^{k-1})^T A_i \boldsymbol{x}_i \right), \quad i = 1, \dots, B$ 

#### **Inequality constraints**

#### Consider the optimization problem:

$$\min_{x} \sum_{i=1}^{B} f_i(x_i) \quad \text{subject to} \quad \sum_{i=1}^{B} A_i x_i \leq b$$



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The primal update step:

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The dual update step:

$$u^k = \left(u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b\right)\right)_+$$

where  $(u)_+$  denotes the positive part of u, i.e.,  $(u_+)_i = \max\{0,u_i\}$ , for  $i=1,\dots,m$ .



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  - Never let prices get negative; hence the use of the positive part notation (.).



# Augmented Lagrangian method





Dual ascent disadvantage: convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

$$\min_{x} f(x) + \frac{\rho}{2} \|Ax - b\|^2$$
 s.t.  $Ax = b$ 





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$$\min_{x} f(x) + \frac{\rho}{2} ||Ax - b||^{2}$$
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Dual gradient ascent: The iterative updates are given by:

$$\begin{split} x_k &= \arg\min_{x} \left[ f(x) + (u_{k-1})^T A x + \frac{\rho}{2} \|Ax - b\|^2 \right] \\ u_k &= u_{k-1} + \rho (Ax_k - b) \end{split}$$



#### Notice step size choice $\alpha_k = \rho$ in dual algorithm. Why?

Since  $x_k$  minimizes the function:

$$f(x) + (u_{k-1})^T A x + \frac{\rho}{2} \|Ax - b\|^2$$

over x, we have the stationarity condition:

$$0 \in \partial f(x_k) + A^T \left( u_{k-1} + \rho (Ax_k - b) \right)$$

which simplifies to:

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This represents the stationarity condition for the original primal problem; under mild conditions,  $Ax_k-b\to 0$  as  $k\to\infty$ , so the KKT conditions are satisfied in the limit and  $x_k$ ,  $u_k$  converge to the solutions.

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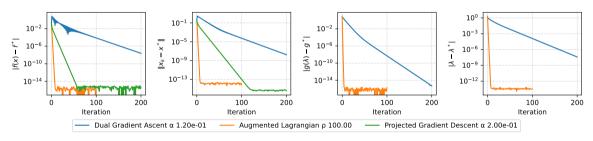
- Advantage: The augmented Lagrangian gives better convergence.
- **Disadvantage:** We lose decomposability! (Separability is ruined)



#### Example: equality constrained quadratic minimization.

$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m \times n}, m < n.$$

Quadratic constrained optimization. n=10, m=5,  $\mu$ =1, L=10.



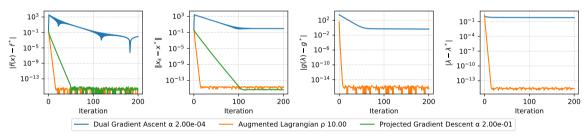
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Quadratic constrained optimization. n=10, m=5,  $\mu$ =0.001, L=10.



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#### Introduction to ADMM







**Alternating direction method of multipliers** or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

$$\min_{x,z} f(x) + g(z)$$

$$\mathrm{s.t.}\ Ax+Bz=c$$



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We augment the objective to include a penalty term for constraint violation:

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where  $\rho > 0$  is a parameter. The augmented Lagrangian for this problem is defined as:

$$L_{\rho}(x,z,u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|^2$$



#### ADMM repeats the following steps, for k = 1, 2, 3, ...:

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$$x_k = \arg\min_x L_\rho(x, z_{k-1}, u_{k-1})$$



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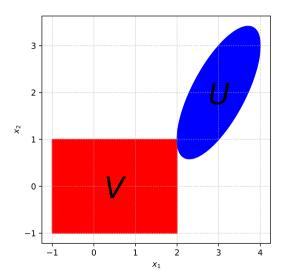
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Note: The usual method of multipliers would replace the first two steps by a joint minimization:

$$(x^{(k)}, z^{(k)}) = \arg\min_{x, z} L_{\rho}(x, z, u^{(k-1)})$$



#### **Example: Alternating Projections**



Consider finding a point in the intersection of convex sets  $U, V \subseteq \mathbb{R}^n$ :

$$\min_{x} I_{U}(x) + I_{V}(x)$$

To transform this problem into ADMM form, we express it as:

$$\min_{x,z} I_U(x) + I_V(z) \quad \text{subject to} \quad x-z = 0$$

Each ADMM cycle involves two projections:

$$\begin{split} x_k &= \arg\min_x P_U \left( z_{k-1} - w_{k-1} \right) \\ z_k &= \arg\min_z P_V \left( x_k + w_{k-1} \right) \\ w_k &= w_{k-1} + x_k - z_k \end{split}$$



#### **Sources**

• Ryan Tibshirani. Convex Optimization 10-725



