



# Proximal Gradient Method.

Даня Меркулов

Методы Оптимизации в Машинном Обучении. ФКН ВШЭ

## Subgradient method

# Non-smooth problems

$\ell_1$  induces sparsity

$\ell_2$  regularization.  $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_2 \leq 1}$



$\ell_1$  regularization.  $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_1 \leq 1}$



@fminxyz

# Subgradient method

Subgradient Method:

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$x_{k+1} = x_k - \alpha_k g_k, \quad g_k \in \partial f(x_k)$$

# Subgradient method

Subgradient Method:

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$x_{k+1} = x_k - \alpha_k g_k, \quad g_k \in \partial f(x_k)$$

---

convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

---

strongly convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{k}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

---

# Subgradient method

Subgradient Method:

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$x_{k+1} = x_k - \alpha_k g_k, \quad g_k \in \partial f(x_k)$$

---

convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

---

strongly convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{k}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

---

## Theorem

Assume that  $f$  is  $G$ -Lipschitz and convex, then  
Subgradient method converges as:

$$f(\bar{x}) - f^* \leq \frac{GR}{\sqrt{k}},$$

where

- $\alpha = \frac{R}{G\sqrt{k}}$

# Subgradient method

Subgradient Method:

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$x_{k+1} = x_k - \alpha_k g_k, \quad g_k \in \partial f(x_k)$$

---

convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

---

strongly convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{k}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

---

## Theorem

Assume that  $f$  is  $G$ -Lipschitz and convex, then  
Subgradient method converges as:

$$f(\bar{x}) - f^* \leq \frac{GR}{\sqrt{k}},$$

where

- $\alpha = \frac{R}{G\sqrt{k}}$
- $R = \|x_0 - x^*\|$

# Subgradient method

Subgradient Method:

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$x_{k+1} = x_k - \alpha_k g_k, \quad g_k \in \partial f(x_k)$$

convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

strongly convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{k}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

## i Theorem

Assume that  $f$  is  $G$ -Lipschitz and convex, then  
Subgradient method converges as:

$$f(\bar{x}) - f^* \leq \frac{GR}{\sqrt{k}},$$

where

- $\alpha = \frac{R}{G\sqrt{k}}$
- $R = \|x_0 - x^*\|$
- $\bar{x} = \frac{1}{k} \sum_{i=0}^{k-1} x_i$



# Non-smooth convex optimization lower bounds

---

convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

---

---

strongly convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{k}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

---

# Non-smooth convex optimization lower bounds

---

convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

---

strongly convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{k}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

---

- Subgradient method is optimal for the problems above.

# Non-smooth convex optimization lower bounds

convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

strongly convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{k}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

- Subgradient method is optimal for the problems above.
- One can use Mirror Descent (a generalization of the subgradient method to a possibly non-Euclidian distance) with the same convergence rate to better fit the geometry of the problem.

# Non-smooth convex optimization lower bounds

convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

strongly convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{k}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

- Subgradient method is optimal for the problems above.
- One can use Mirror Descent (a generalization of the subgradient method to a possibly non-Euclidian distance) with the same convergence rate to better fit the geometry of the problem.
- However, we can achieve standard gradient descent rate  $\mathcal{O}\left(\frac{1}{k}\right)$  (and even accelerated version  $\mathcal{O}\left(\frac{1}{k^2}\right)$ ) if we will exploit the structure of the problem.

## Proximal operator

# Proximal mapping intuition

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

## Proximal mapping intuition

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_k)$$

Leads to ordinary Gradient Descent method

# Proximal mapping intuition

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_k)$$

Leads to ordinary Gradient Descent method

Implicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_{k+1})$$



## Proximal mapping intuition

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_k)$$

Leads to ordinary Gradient Descent method

Implicit Euler discretization:

$$\begin{aligned}\frac{x_{k+1} - x_k}{\alpha} &= -\nabla f(x_{k+1}) \\ \frac{x_{k+1} - x_k}{\alpha} + \nabla f(x_{k+1}) &= 0\end{aligned}$$

# Proximal mapping intuition

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_k)$$

Leads to ordinary Gradient Descent method

Implicit Euler discretization:

$$\begin{aligned}\frac{x_{k+1} - x_k}{\alpha} &= -\nabla f(x_{k+1}) \\ \frac{x_{k+1} - x_k}{\alpha} + \nabla f(x_{k+1}) &= 0 \\ \frac{x - x_k}{\alpha} + \nabla f(x) \Big|_{x=x_{k+1}} &= 0\end{aligned}$$

# Proximal mapping intuition

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_k)$$

Leads to ordinary Gradient Descent method

Implicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_{k+1})$$

$$\frac{x_{k+1} - x_k}{\alpha} + \nabla f(x_{k+1}) = 0$$

$$\frac{x - x_k}{\alpha} + \nabla f(x) \Big|_{x=x_{k+1}} = 0$$

$$\nabla \left[ \frac{1}{2\alpha} \|x - x_k\|_2^2 + f(x) \right] \Big|_{x=x_{k+1}} = 0$$

# Proximal mapping intuition

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_k)$$

Leads to ordinary Gradient Descent method

Implicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_{k+1})$$

$$\frac{x_{k+1} - x_k}{\alpha} + \nabla f(x_{k+1}) = 0$$

$$\frac{x - x_k}{\alpha} + \nabla f(x) \Big|_{x=x_{k+1}} = 0$$

$$\nabla \left[ \frac{1}{2\alpha} \|x - x_k\|_2^2 + f(x) \right] \Big|_{x=x_{k+1}} = 0$$

$$x_{k+1} = \arg \min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

# Proximal mapping intuition

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_k)$$

Leads to ordinary Gradient Descent method

Implicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_{k+1})$$

$$\frac{x_{k+1} - x_k}{\alpha} + \nabla f(x_{k+1}) = 0$$

$$\frac{x - x_k}{\alpha} + \nabla f(x) \Big|_{x=x_{k+1}} = 0$$

$$\nabla \left[ \frac{1}{2\alpha} \|x - x_k\|_2^2 + f(x) \right] \Big|_{x=x_{k+1}} = 0$$

$$x_{k+1} = \arg \min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

## Proximal mapping intuition

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_k)$$

Leads to ordinary Gradient Descent method

Implicit Euler discretization:

$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_{k+1})$$

$$\frac{x_{k+1} - x_k}{\alpha} + \nabla f(x_{k+1}) = 0$$

$$\frac{x - x_k}{\alpha} + \nabla f(x) \Big|_{x=x_{k+1}} = 0$$

$$\nabla \left[ \frac{1}{2\alpha} \|x - x_k\|_2^2 + f(x) \right] \Big|_{x=x_{k+1}} = 0$$

$$x_{k+1} = \arg \min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

! Proximal operator

$$\text{prox}_{f,\alpha}(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

## Proximal operator visualization

$$\text{Prox}_f(x) = \underset{x'}{\operatorname{argmin}} \frac{1}{2} \|x - x'\|^2 + f(x')$$



Рисунок 1: Source

## Proximal mapping intuition

- **GD from proximal method.** Back to the discretization:



## Proximal mapping intuition

- **GD from proximal method.** Back to the discretization:

$$x_{k+1} + \alpha \nabla f(x_{k+1}) = x_k$$

## Proximal mapping intuition

- **GD from proximal method.** Back to the discretization:

$$x_{k+1} + \alpha \nabla f(x_{k+1}) = x_k$$

$$(I + \alpha \nabla f)(x_{k+1}) = x_k$$

## Proximal mapping intuition

- **GD from proximal method.** Back to the discretization:

$$x_{k+1} + \alpha \nabla f(x_{k+1}) = x_k$$

$$(I + \alpha \nabla f)(x_{k+1}) = x_k$$

$$x_{k+1} = (I + \alpha \nabla f)^{-1} x_k \stackrel{\alpha \rightarrow 0}{\approx} (I - \alpha \nabla f) x_k$$

## Proximal mapping intuition

- **GD from proximal method.** Back to the discretization:

$$x_{k+1} + \alpha \nabla f(x_{k+1}) = x_k$$

$$(I + \alpha \nabla f)(x_{k+1}) = x_k$$

$$x_{k+1} = (I + \alpha \nabla f)^{-1} x_k \stackrel{\alpha \rightarrow 0}{\approx} (I - \alpha \nabla f) x_k$$

# Proximal mapping intuition

- **GD from proximal method.** Back to the discretization:

$$x_{k+1} + \alpha \nabla f(x_{k+1}) = x_k$$

$$(I + \alpha \nabla f)(x_{k+1}) = x_k$$

$$x_{k+1} = (I + \alpha \nabla f)^{-1} x_k \stackrel{\alpha \rightarrow 0}{\approx} (I - \alpha \nabla f) x_k$$

Thus, we have a usual gradient descent with  $\alpha \rightarrow 0$ :  $x_{k+1} = x_k - \alpha \nabla f(x_k)$

- **Newton from proximal method.** Now let's consider proximal mapping of a second order Taylor approximation of the function  $f_{x_k}^{II}(x)$ :

## Proximal mapping intuition

- **GD from proximal method.** Back to the discretization:

$$\begin{aligned}x_{k+1} + \alpha \nabla f(x_{k+1}) &= x_k \\(I + \alpha \nabla f)(x_{k+1}) &= x_k \\x_{k+1} &= (I + \alpha \nabla f)^{-1} x_k \stackrel{\alpha \rightarrow 0}{\approx} (I - \alpha \nabla f) x_k\end{aligned}$$

Thus, we have a usual gradient descent with  $\alpha \rightarrow 0$ :  $x_{k+1} = x_k - \alpha \nabla f(x_k)$

- **Newton from proximal method.** Now let's consider proximal mapping of a second order Taylor approximation of the function  $f_{x_k}^{II}(x)$ :

$$x_{k+1} = \text{prox}_{f_{x_k}^{II}, \alpha}(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2} \langle \nabla^2 f(x_k)(x - x_k), x - x_k \rangle + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

## Proximal mapping intuition

- **GD from proximal method.** Back to the discretization:

$$\begin{aligned}x_{k+1} + \alpha \nabla f(x_{k+1}) &= x_k \\(I + \alpha \nabla f)(x_{k+1}) &= x_k \\x_{k+1} &= (I + \alpha \nabla f)^{-1} x_k \stackrel{\alpha \rightarrow 0}{\approx} (I - \alpha \nabla f) x_k\end{aligned}$$

Thus, we have a usual gradient descent with  $\alpha \rightarrow 0$ :  $x_{k+1} = x_k - \alpha \nabla f(x_k)$

- **Newton from proximal method.** Now let's consider proximal mapping of a second order Taylor approximation of the function  $f_{x_k}^{II}(x)$ :

$$\begin{aligned}x_{k+1} = \text{prox}_{f_{x_k}^{II}, \alpha}(x_k) &= \arg \min_{x \in \mathbb{R}^n} \left[ f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2} \langle \nabla^2 f(x_k)(x - x_k), x - x_k \rangle + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right] \\ \nabla f(x_k) + \nabla^2 f(x_k)(x - x_k) + \frac{1}{\alpha}(x - x_k) \Big|_{x=x_{k+1}} &= 0\end{aligned}$$

## Proximal mapping intuition

- **GD from proximal method.** Back to the discretization:

$$\begin{aligned}x_{k+1} + \alpha \nabla f(x_{k+1}) &= x_k \\(I + \alpha \nabla f)(x_{k+1}) &= x_k \\x_{k+1} &= (I + \alpha \nabla f)^{-1} x_k \stackrel{\alpha \rightarrow 0}{\approx} (I - \alpha \nabla f) x_k\end{aligned}$$

Thus, we have a usual gradient descent with  $\alpha \rightarrow 0$ :  $x_{k+1} = x_k - \alpha \nabla f(x_k)$

- **Newton from proximal method.** Now let's consider proximal mapping of a second order Taylor approximation of the function  $f_{x_k}^{II}(x)$ :

$$\begin{aligned}x_{k+1} = \text{prox}_{f_{x_k}^{II}, \alpha}(x_k) &= \arg \min_{x \in \mathbb{R}^n} \left[ f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2} \langle \nabla^2 f(x_k)(x - x_k), x - x_k \rangle + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right] \\ \nabla f(x_k) + \nabla^2 f(x_k)(x - x_k) + \frac{1}{\alpha}(x - x_k) &\Big|_{x=x_{k+1}} = 0 \\ x_{k+1} &= x_k - \left[ \nabla^2 f(x_k) + \frac{1}{\alpha} I \right]^{-1} \nabla f(x_k)\end{aligned}$$



## From projections to proximity

Let  $\mathbb{I}_S$  be the indicator function for closed, convex  $S$ . Recall orthogonal projection  $\pi_S(y)$

## From projections to proximity

Let  $\mathbb{I}_S$  be the indicator function for closed, convex  $S$ . Recall orthogonal projection  $\pi_S(y)$

$$\pi_S(y) := \arg \min_{x \in S} \frac{1}{2} \|x - y\|_2^2.$$

## From projections to proximity

Let  $\mathbb{I}_S$  be the indicator function for closed, convex  $S$ . Recall orthogonal projection  $\pi_S(y)$

$$\pi_S(y) := \arg \min_{x \in S} \frac{1}{2} \|x - y\|_2^2.$$

With the following notation of indicator function

$$\mathbb{I}_S(x) = \begin{cases} 0, & x \in S, \\ \infty, & x \notin S, \end{cases}$$

## From projections to proximity

Let  $\mathbb{I}_S$  be the indicator function for closed, convex  $S$ . Recall orthogonal projection  $\pi_S(y)$

$$\pi_S(y) := \arg \min_{x \in S} \frac{1}{2} \|x - y\|_2^2.$$

With the following notation of indicator function

$$\mathbb{I}_S(x) = \begin{cases} 0, & x \in S, \\ \infty, & x \notin S, \end{cases}$$

Rewrite orthogonal projection  $\pi_S(y)$  as

$$\pi_S(y) := \arg \min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - y\|^2 + \mathbb{I}_S(x).$$

## From projections to proximity

Let  $\mathbb{I}_S$  be the indicator function for closed, convex  $S$ . Recall orthogonal projection  $\pi_S(y)$

$$\pi_S(y) := \arg \min_{x \in S} \frac{1}{2} \|x - y\|_2^2.$$

With the following notation of indicator function

$$\mathbb{I}_S(x) = \begin{cases} 0, & x \in S, \\ \infty, & x \notin S, \end{cases}$$

Rewrite orthogonal projection  $\pi_S(y)$  as

$$\pi_S(y) := \arg \min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - y\|^2 + \mathbb{I}_S(x).$$

Proximity: Replace  $\mathbb{I}_S$  by some convex function!

$$\text{prox}_r(y) = \text{prox}_{r,1}(y) := \arg \min \frac{1}{2} \|x - y\|^2 + r(x)$$

## Composite optimization

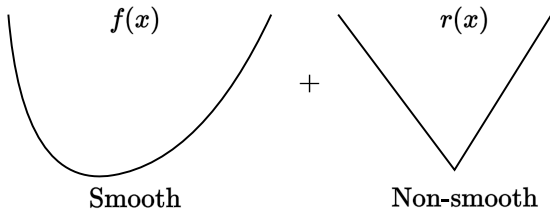
# Regularized / Composite Objectives

Many nonsmooth problems take the form

$$\min_{x \in \mathbb{R}^n} \varphi(x) = f(x) + r(x)$$

- **Lasso, L1-LS, compressed sensing**

$$f(x) = \frac{1}{2} \|Ax - b\|_2^2, r(x) = \lambda \|x\|_1$$



# Regularized / Composite Objectives

Many nonsmooth problems take the form

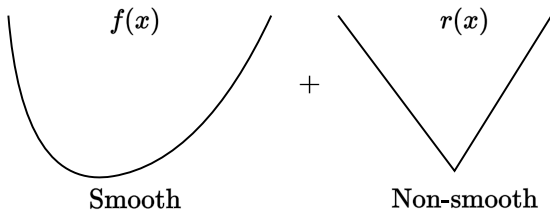
$$\min_{x \in \mathbb{R}^n} \varphi(x) = f(x) + r(x)$$

- **Lasso, L1-LS, compressed sensing**

$$f(x) = \frac{1}{2} \|Ax - b\|_2^2, r(x) = \lambda \|x\|_1$$

- **L1-Logistic regression, sparse LR**

$$f(x) = -y \log h(x) - (1-y) \log(1-h(x)), r(x) = \lambda \|x\|_1$$





# Proximal mapping intuition

Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

## Proximal mapping intuition

Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

$$0 \in \alpha \nabla f(x^*) + \alpha \partial r(x^*)$$

# Proximal mapping intuition

Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

$$0 \in \alpha \nabla f(x^*) + \alpha \partial r(x^*)$$

$$x^* \in \alpha \nabla f(x^*) + (I + \alpha \partial r)(x^*)$$

## Proximal mapping intuition

Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

$$0 \in \alpha \nabla f(x^*) + \alpha \partial r(x^*)$$

$$x^* \in \alpha \nabla f(x^*) + (I + \alpha \partial r)(x^*)$$

$$x^* - \alpha \nabla f(x^*) \in (I + \alpha \partial r)(x^*)$$

## Proximal mapping intuition

Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

$$0 \in \alpha \nabla f(x^*) + \alpha \partial r(x^*)$$

$$x^* \in \alpha \nabla f(x^*) + (I + \alpha \partial r)(x^*)$$

$$x^* - \alpha \nabla f(x^*) \in (I + \alpha \partial r)(x^*)$$

$$x^* = (I + \alpha \partial r)^{-1}(x^* - \alpha \nabla f(x^*))$$

## Proximal mapping intuition

Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

$$0 \in \alpha \nabla f(x^*) + \alpha \partial r(x^*)$$

$$x^* \in \alpha \nabla f(x^*) + (I + \alpha \partial r)(x^*)$$

$$x^* - \alpha \nabla f(x^*) \in (I + \alpha \partial r)(x^*)$$

$$x^* = (I + \alpha \partial r)^{-1}(x^* - \alpha \nabla f(x^*))$$

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

## Proximal mapping intuition

Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

$$0 \in \alpha \nabla f(x^*) + \alpha \partial r(x^*)$$

$$x^* \in \alpha \nabla f(x^*) + (I + \alpha \partial r)(x^*)$$

$$x^* - \alpha \nabla f(x^*) \in (I + \alpha \partial r)(x^*)$$

$$x^* = (I + \alpha \partial r)^{-1}(x^* - \alpha \nabla f(x^*))$$

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

## Proximal mapping intuition

Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

$$0 \in \alpha \nabla f(x^*) + \alpha \partial r(x^*)$$

$$x^* \in \alpha \nabla f(x^*) + (I + \alpha \partial r)(x^*)$$

$$x^* - \alpha \nabla f(x^*) \in (I + \alpha \partial r)(x^*)$$

$$x^* = (I + \alpha \partial r)^{-1}(x^* - \alpha \nabla f(x^*))$$

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

Which leads to the proximal gradient method:

$$x_{k+1} = \text{prox}_{r,\alpha}(x_k - \alpha \nabla f(x_k))$$

And this method converges at a rate of  $\mathcal{O}(\frac{1}{k})$ !



## Proximal mapping intuition

Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

$$0 \in \alpha \nabla f(x^*) + \alpha \partial r(x^*)$$

$$x^* \in \alpha \nabla f(x^*) + (I + \alpha \partial r)(x^*)$$

$$x^* - \alpha \nabla f(x^*) \in (I + \alpha \partial r)(x^*)$$

$$x^* = (I + \alpha \partial r)^{-1}(x^* - \alpha \nabla f(x^*))$$

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

Which leads to the proximal gradient method:

$$x_{k+1} = \text{prox}_{r,\alpha}(x_k - \alpha \nabla f(x_k))$$

And this method converges at a rate of  $\mathcal{O}(\frac{1}{k})$ !

**i** Another form of proximal operator

$$\text{prox}_{f,\alpha}(x_k) = \text{prox}_{\alpha f}(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ \alpha f(x) + \frac{1}{2} \|x - x_k\|_2^2 \right] \quad \text{prox}_f(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2} \|x - x_k\|_2^2 \right]$$

## Proximal operators examples

- $r(x) = \lambda \|x\|_1, \lambda > 0$

$$[\text{prox}_r(x)]_i = [|x_i| - \lambda]_+ \cdot \text{sign}(x_i),$$

which is also known as soft-thresholding operator.

## Proximal operators examples

- $r(x) = \lambda \|x\|_1, \lambda > 0$

$$[\text{prox}_r(x)]_i = [|x_i| - \lambda]_+ \cdot \text{sign}(x_i),$$

which is also known as soft-thresholding operator.

- $r(x) = \frac{\lambda}{2} \|x\|_2^2, \lambda > 0$

$$\text{prox}_r(x) = \frac{x}{1 + \lambda}.$$

## Proximal operators examples

- $r(x) = \lambda \|x\|_1, \lambda > 0$

$$[\text{prox}_r(x)]_i = [|x_i| - \lambda]_+ \cdot \text{sign}(x_i),$$

which is also known as soft-thresholding operator.

- $r(x) = \frac{\lambda}{2} \|x\|_2^2, \lambda > 0$

$$\text{prox}_r(x) = \frac{x}{1 + \lambda}.$$

- $r(x) = \mathbb{I}_S(x).$

$$\text{prox}_r(x_k - \alpha \nabla f(x_k)) = \text{proj}_r(x_k - \alpha \nabla f(x_k))$$

# Proximal operator properties

## **i** Theorem

Let  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\text{prox}_r$  is defined. If there exists such an  $\hat{x} \in \mathbb{R}^n$  that  $r(\hat{x}) < +\infty$ . Then, the proximal operator is uniquely defined (i.e., it always returns a single unique value).

**Proof:**

# Proximal operator properties

## Theorem

Let  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\text{prox}_r$  is defined. If there exists such an  $\hat{x} \in \mathbb{R}^n$  that  $r(\hat{x}) < +\infty$ . Then, the proximal operator is uniquely defined (i.e., it always returns a single unique value).

## Proof:

The proximal operator returns the minimum of some optimization problem.

# Proximal operator properties

## Theorem

Let  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\text{prox}_r$  is defined. If there exists such an  $\hat{x} \in \mathbb{R}^n$  that  $r(\hat{x}) < +\infty$ . Then, the proximal operator is uniquely defined (i.e., it always returns a single unique value).

## Proof:

The proximal operator returns the minimum of some optimization problem.

Question: What can be said about this problem?

# Proximal operator properties

## Theorem

Let  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\text{prox}_r$  is defined. If there exists such an  $\hat{x} \in \mathbb{R}^n$  that  $r(\hat{x}) < +\infty$ . Then, the proximal operator is uniquely defined (i.e., it always returns a single unique value).

## Proof:

The proximal operator returns the minimum of some optimization problem.

Question: What can be said about this problem?

It is strongly convex, meaning it has exactly one unique minimum (the existence of  $\hat{x}$  is necessary for  $r(\tilde{x}) + \frac{1}{2}\|x - \tilde{x}\|_2^2$  to take a finite value somewhere).



# Proximal operator properties

## i Theorem

Let  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\text{prox}_r$  is defined. Then, for any  $x, y \in \mathbb{R}^n$ , the following three conditions are equivalent:

- $\text{prox}_r(x) = y,$

## Proof

# Proximal operator properties

## Theorem

Let  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\text{prox}_r$  is defined. Then, for any  $x, y \in \mathbb{R}^n$ , the following three conditions are equivalent:

- $\text{prox}_r(x) = y$ ,
- $x - y \in \partial r(y)$ ,

## Proof

# Proximal operator properties

## i Theorem

Let  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\text{prox}_r$  is defined. Then, for any  $x, y \in \mathbb{R}^n$ , the following three conditions are equivalent:

- $\text{prox}_r(x) = y$ ,
- $x - y \in \partial r(y)$ ,
- $\langle x - y, z - y \rangle \leq r(z) - r(y)$  for any  $z \in \mathbb{R}^n$ .

## Proof

# Proximal operator properties

## i Theorem

Let  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\text{prox}_r$  is defined. Then, for any  $x, y \in \mathbb{R}^n$ , the following three conditions are equivalent:

- $\text{prox}_r(x) = y$ ,
- $x - y \in \partial r(y)$ ,
- $\langle x - y, z - y \rangle \leq r(z) - r(y)$  for any  $z \in \mathbb{R}^n$ .

## Proof

1. Let's establish the equivalence between the first and second conditions. The first condition can be rewritten as

$$y = \arg \min_{\tilde{x} \in \mathbb{R}^d} \left( r(\tilde{x}) + \frac{1}{2} \|x - \tilde{x}\|^2 \right).$$

From the optimality condition for the convex function  $r$ , this is equivalent to:

$$0 \in \partial \left( r(\tilde{x}) + \frac{1}{2} \|x - \tilde{x}\|^2 \right) \Big|_{\tilde{x}=y} = \partial r(y) + y - x.$$

# Proximal operator properties

## i Theorem

Let  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\text{prox}_r$  is defined. Then, for any  $x, y \in \mathbb{R}^n$ , the following three conditions are equivalent:

- $\text{prox}_r(x) = y$ ,
- $x - y \in \partial r(y)$ ,
- $\langle x - y, z - y \rangle \leq r(z) - r(y)$  for any  $z \in \mathbb{R}^n$ .

## Proof

1. Let's establish the equivalence between the first and second conditions. The first condition can be rewritten as

$$y = \arg \min_{\tilde{x} \in \mathbb{R}^d} \left( r(\tilde{x}) + \frac{1}{2} \|x - \tilde{x}\|^2 \right).$$

From the optimality condition for the convex function  $r$ , this is equivalent to:

$$0 \in \partial \left( r(\tilde{x}) + \frac{1}{2} \|x - \tilde{x}\|^2 \right) \Big|_{\tilde{x}=y} = \partial r(y) + y - x.$$

2. From the definition of the subdifferential, for any subgradient  $g \in \partial f(y)$  and for any  $z \in \mathbb{R}^d$ :

$$\langle g, z - y \rangle \leq r(z) - r(y).$$

In particular, this holds true for  $g = x - y$ . Conversely, it is also clear: for  $g = x - y$ , the above relationship holds, which means  $g \in \partial r(y)$ .

# Proximal operator properties

## i Theorem

The operator  $\text{prox}_r(x)$  is firmly nonexpansive (FNE)

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2^2 \leq \langle \text{prox}_r(x) - \text{prox}_r(y), x - y \rangle$$

and nonexpansive:

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2 \leq \|x - y\|_2$$

## Proof

1. Let  $u = \text{prox}_r(x)$ , and  $v = \text{prox}_r(y)$ . Then, from the previous property:

$$\langle x - u, z_1 - u \rangle \leq r(z_1) - r(u)$$

$$\langle y - v, z_2 - v \rangle \leq r(z_2) - r(v).$$

# Proximal operator properties

## i Theorem

The operator  $\text{prox}_r(x)$  is firmly nonexpansive (FNE)

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2^2 \leq \langle \text{prox}_r(x) - \text{prox}_r(y), x - y \rangle$$

and nonexpansive:

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2 \leq \|x - y\|_2$$

## Proof

1. Let  $u = \text{prox}_r(x)$ , and  $v = \text{prox}_r(y)$ . Then, from the previous property:

$$\langle x - u, z_1 - u \rangle \leq r(z_1) - r(u)$$

$$\langle y - v, z_2 - v \rangle \leq r(z_2) - r(v).$$

2. Substitute  $z_1 = v$  and  $z_2 = u$ . Summing up, we get:

$$\langle x - u, v - u \rangle + \langle y - v, u - v \rangle \leq 0,$$

$$\langle x - y, v - u \rangle + \|v - u\|_2^2 \leq 0.$$

# Proximal operator properties

## i Theorem

The operator  $\text{prox}_r(x)$  is firmly nonexpansive (FNE)

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2^2 \leq \langle \text{prox}_r(x) - \text{prox}_r(y), x - y \rangle$$

and nonexpansive:

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2 \leq \|x - y\|_2$$

## Proof

1. Let  $u = \text{prox}_r(x)$ , and  $v = \text{prox}_r(y)$ . Then, from the previous property:
3. Which is exactly what we need to prove after substitution of  $u, v$ .

$$\langle x - u, z_1 - u \rangle \leq r(z_1) - r(u)$$

$$\langle y - v, z_2 - v \rangle \leq r(z_2) - r(v).$$

$$\|u - v\|_2^2 \leq \langle x - y, u - v \rangle$$

2. Substitute  $z_1 = v$  and  $z_2 = u$ . Summing up, we get:

$$\langle x - u, v - u \rangle + \langle y - v, u - v \rangle \leq 0,$$

$$\langle x - y, v - u \rangle + \|v - u\|_2^2 \leq 0.$$



# Proximal operator properties

## i Theorem

The operator  $\text{prox}_r(x)$  is firmly nonexpansive (FNE)

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2^2 \leq \langle \text{prox}_r(x) - \text{prox}_r(y), x - y \rangle$$

and nonexpansive:

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2 \leq \|x - y\|_2$$

## Proof

1. Let  $u = \text{prox}_r(x)$ , and  $v = \text{prox}_r(y)$ . Then, from the previous property:

$$\langle x - u, z_1 - u \rangle \leq r(z_1) - r(u)$$

$$\langle y - v, z_2 - v \rangle \leq r(z_2) - r(v).$$

2. Substitute  $z_1 = v$  and  $z_2 = u$ . Summing up, we get:

$$\langle x - u, v - u \rangle + \langle y - v, u - v \rangle \leq 0,$$

$$\langle x - y, v - u \rangle + \|v - u\|_2^2 \leq 0.$$

3. Which is exactly what we need to prove after substitution of  $u, v$ .

$$\|u - v\|_2^2 \leq \langle x - y, u - v \rangle$$

4. The last point comes from simple Cauchy-Bunyakovsky-Schwarz for the last inequality.

# Proximal operator properties

## i Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be convex functions. Additionally, assume that  $f$  is continuously differentiable and  $L$ -smooth, and for  $r$ ,  $\text{prox}_r$  is defined. Then,  $x^*$  is a solution to the composite optimization problem if and only if, for any  $\alpha > 0$ , it satisfies:

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

## Proof

1. Optimality conditions:

# Proximal operator properties

## i Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be convex functions. Additionally, assume that  $f$  is continuously differentiable and  $L$ -smooth, and for  $r$ ,  $\text{prox}_r$  is defined. Then,  $x^*$  is a solution to the composite optimization problem if and only if, for any  $\alpha > 0$ , it satisfies:

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

## Proof

1. Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

# Proximal operator properties

## i Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be convex functions. Additionally, assume that  $f$  is continuously differentiable and  $L$ -smooth, and for  $r$ ,  $\text{prox}_r$  is defined. Then,  $x^*$  is a solution to the composite optimization problem if and only if, for any  $\alpha > 0$ , it satisfies:

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

## Proof

### 1. Optimality conditions:

$$\begin{aligned} 0 &\in \nabla f(x^*) + \partial r(x^*) \\ -\alpha \nabla f(x^*) &\in \alpha \partial r(x^*) \end{aligned}$$

# Proximal operator properties

## i Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be convex functions. Additionally, assume that  $f$  is continuously differentiable and  $L$ -smooth, and for  $r$ ,  $\text{prox}_r$  is defined. Then,  $x^*$  is a solution to the composite optimization problem if and only if, for any  $\alpha > 0$ , it satisfies:

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

## Proof

### 1. Optimality conditions:

$$\begin{aligned} 0 &\in \nabla f(x^*) + \partial r(x^*) \\ -\alpha \nabla f(x^*) &\in \alpha \partial r(x^*) \\ x^* - \alpha \nabla f(x^*) - x^* &\in \alpha \partial r(x^*) \end{aligned}$$

# Proximal operator properties

## i Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be convex functions. Additionally, assume that  $f$  is continuously differentiable and  $L$ -smooth, and for  $r$ ,  $\text{prox}_r$  is defined. Then,  $x^*$  is a solution to the composite optimization problem if and only if, for any  $\alpha > 0$ , it satisfies:

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

## Proof

1. Optimality conditions:

$$\begin{aligned} 0 &\in \nabla f(x^*) + \partial r(x^*) \\ -\alpha \nabla f(x^*) &\in \alpha \partial r(x^*) \\ x^* - \alpha \nabla f(x^*) - x^* &\in \alpha \partial r(x^*) \end{aligned}$$

2. Recall from the previous lemma:

$$\text{prox}_r(x) = y \Leftrightarrow x - y \in \partial r(y)$$

# Proximal operator properties

## i Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be convex functions. Additionally, assume that  $f$  is continuously differentiable and  $L$ -smooth, and for  $r$ ,  $\text{prox}_r$  is defined. Then,  $x^*$  is a solution to the composite optimization problem if and only if, for any  $\alpha > 0$ , it satisfies:

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

## Proof

1. Optimality conditions:

$$\begin{aligned} 0 &\in \nabla f(x^*) + \partial r(x^*) \\ -\alpha \nabla f(x^*) &\in \alpha \partial r(x^*) \\ x^* - \alpha \nabla f(x^*) - x^* &\in \alpha \partial r(x^*) \end{aligned}$$

2. Recall from the previous lemma:

$$\text{prox}_r(x) = y \Leftrightarrow x - y \in \partial r(y)$$

3. Finally,

$$x^* = \text{prox}_{\alpha r}(x^* - \alpha \nabla f(x^*)) = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

## Theoretical tools for convergence analysis





## i Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an  $L$ -smooth convex function. Then, for any  $x, y \in \mathbb{R}^n$ , the following inequality holds:

$$f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y) \text{ or, equivalently,}$$
$$\|\nabla f(y) - \nabla f(x)\|_2^2 = \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L (f(x) - f(y) - \langle \nabla f(y), x - y \rangle)$$

## Proof

1. To prove this, we'll consider another function  $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$ . It is obviously a convex function (as a sum of convex functions). And it is easy to verify, that it is an  $L$ -smooth function by definition, since  $\nabla \varphi(y) = \nabla f(y) - \nabla f(x)$  and  $\|\nabla \varphi(y_1) - \nabla \varphi(y_2)\| = \|\nabla f(y_1) - \nabla f(y_2)\| \leq L\|y_1 - y_2\|$ .



### i Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an  $L$ -smooth convex function. Then, for any  $x, y \in \mathbb{R}^n$ , the following inequality holds:

$$f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y) \text{ or, equivalently,}$$
$$\|\nabla f(y) - \nabla f(x)\|_2^2 = \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L (f(x) - f(y) - \langle \nabla f(y), x - y \rangle)$$

### Proof

1. To prove this, we'll consider another function  $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$ . It is obviously a convex function (as a sum of convex functions). And it is easy to verify, that it is an  $L$ -smooth function by definition, since  $\nabla \varphi(y) = \nabla f(y) - \nabla f(x)$  and  $\|\nabla \varphi(y_1) - \nabla \varphi(y_2)\| = \|\nabla f(y_1) - \nabla f(y_2)\| \leq L\|y_1 - y_2\|$ .
2. Now let's consider the smoothness parabolic property for the  $\varphi(y)$  function:



## i Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an  $L$ -smooth convex function. Then, for any  $x, y \in \mathbb{R}^n$ , the following inequality holds:

$$f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y) \text{ or, equivalently,}$$
$$\|\nabla f(y) - \nabla f(x)\|_2^2 = \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L (f(x) - f(y) - \langle \nabla f(y), x - y \rangle)$$

## Proof

1. To prove this, we'll consider another function  $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$ . It is obviously a convex function (as a sum of convex functions). And it is easy to verify, that it is an  $L$ -smooth function by definition, since  $\nabla \varphi(y) = \nabla f(y) - \nabla f(x)$  and  $\|\nabla \varphi(y_1) - \nabla \varphi(y_2)\| = \|\nabla f(y_1) - \nabla f(y_2)\| \leq L\|y_1 - y_2\|$ .
2. Now let's consider the smoothness parabolic property for the  $\varphi(y)$  function:

$$\varphi(y) \leq \varphi(x) + \langle \nabla \varphi(x), y - x \rangle + \frac{L}{2} \|y - x\|_2^2$$

## Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an  $L$ -smooth convex function. Then, for any  $x, y \in \mathbb{R}^n$ , the following inequality holds:

$$f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y) \text{ or, equivalently,}$$

$$\|\nabla f(y) - \nabla f(x)\|_2^2 = \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L (f(x) - f(y) - \langle \nabla f(y), x - y \rangle)$$

## Proof

1. To prove this, we'll consider another function  $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$ . It is obviously a convex function (as a sum of convex functions). And it is easy to verify, that it is an  $L$ -smooth function by definition, since  $\nabla \varphi(y) = \nabla f(y) - \nabla f(x)$  and  $\|\nabla \varphi(y_1) - \nabla \varphi(y_2)\| = \|\nabla f(y_1) - \nabla f(y_2)\| \leq L\|y_1 - y_2\|$ .
2. Now let's consider the smoothness parabolic property for the  $\varphi(y)$  function:

$$\varphi(y) \leq \varphi(x) + \langle \nabla \varphi(x), y - x \rangle + \frac{L}{2} \|y - x\|_2^2$$

$$\varphi\left(y - \frac{1}{L} \nabla \varphi(y)\right) \leq \varphi(y) + \left\langle \nabla \varphi(y), -\frac{1}{L} \nabla \varphi(y) \right\rangle + \frac{1}{2L} \|\nabla \varphi(y)\|_2^2$$

## Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an  $L$ -smooth convex function. Then, for any  $x, y \in \mathbb{R}^n$ , the following inequality holds:

$$f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y) \text{ or, equivalently,}$$

$$\|\nabla f(y) - \nabla f(x)\|_2^2 = \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L (f(x) - f(y) - \langle \nabla f(y), x - y \rangle)$$

## Proof

1. To prove this, we'll consider another function  $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$ . It is obviously a convex function (as a sum of convex functions). And it is easy to verify, that it is an  $L$ -smooth function by definition, since  $\nabla \varphi(y) = \nabla f(y) - \nabla f(x)$  and  $\|\nabla \varphi(y_1) - \nabla \varphi(y_2)\| = \|\nabla f(y_1) - \nabla f(y_2)\| \leq L\|y_1 - y_2\|$ .
2. Now let's consider the smoothness parabolic property for the  $\varphi(y)$  function:

$$\varphi(y) \leq \varphi(x) + \langle \nabla \varphi(x), y - x \rangle + \frac{L}{2} \|y - x\|_2^2$$

$$x:=y, y:=y - \frac{1}{L} \nabla \varphi(y) \quad \varphi\left(y - \frac{1}{L} \nabla \varphi(y)\right) \leq \varphi(y) + \left\langle \nabla \varphi(y), -\frac{1}{L} \nabla \varphi(y) \right\rangle + \frac{1}{2L} \|\nabla \varphi(y)\|_2^2$$

$$\varphi\left(y - \frac{1}{L} \nabla \varphi(y)\right) \leq \varphi(y) - \frac{1}{2L} \|\nabla \varphi(y)\|_2^2$$

## Convergence tools

3. From the first order optimality conditions for the convex function  $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any  $x$ , the minimum of the function  $\varphi(y)$  is at the point  $y = x$ . Therefore:

$$\varphi(x) \leq \varphi\left(y - \frac{1}{L}\nabla\varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla\varphi(y)\|_2^2$$

## Convergence tools

3. From the first order optimality conditions for the convex function  $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any  $x$ , the minimum of the function  $\varphi(y)$  is at the point  $y = x$ . Therefore:

$$\varphi(x) \leq \varphi\left(y - \frac{1}{L}\nabla\varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla\varphi(y)\|_2^2$$

4. Now, substitute  $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$ :

## Convergence tools

3. From the first order optimality conditions for the convex function  $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any  $x$ , the minimum of the function  $\varphi(y)$  is at the point  $y = x$ . Therefore:

$$\varphi(x) \leq \varphi\left(y - \frac{1}{L}\nabla\varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla\varphi(y)\|_2^2$$

4. Now, substitute  $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$ :

$$f(x) - \langle \nabla f(x), x \rangle \leq f(y) - \langle \nabla f(x), y \rangle - \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|_2^2$$



3. From the first order optimality conditions for the convex function  $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any  $x$ , the minimum of the function  $\varphi(y)$  is at the point  $y = x$ . Therefore:

$$\varphi(x) \leq \varphi\left(y - \frac{1}{L}\nabla\varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla\varphi(y)\|_2^2$$

4. Now, substitute  $\varphi(y) = f(y) - \langle\nabla f(x), y\rangle$ :

$$f(x) - \langle\nabla f(x), x\rangle \leq f(y) - \langle\nabla f(x), y\rangle - \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|_2^2$$

$$f(x) + \langle\nabla f(x), y - x\rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y)$$

3. From the first order optimality conditions for the convex function  $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any  $x$ , the minimum of the function  $\varphi(y)$  is at the point  $y = x$ . Therefore:

$$\varphi(x) \leq \varphi\left(y - \frac{1}{L}\nabla\varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla\varphi(y)\|_2^2$$

4. Now, substitute  $\varphi(y) = f(y) - \langle\nabla f(x), y\rangle$ :

$$f(x) - \langle\nabla f(x), x\rangle \leq f(y) - \langle\nabla f(x), y\rangle - \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|_2^2$$

$$f(x) + \langle\nabla f(x), y - x\rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y)$$

$$\|\nabla f(y) - \nabla f(x)\|_2^2 \leq 2L(f(y) - f(x) - \langle\nabla f(x), y - x\rangle)$$

3. From the first order optimality conditions for the convex function  $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any  $x$ , the minimum of the function  $\varphi(y)$  is at the point  $y = x$ . Therefore:

$$\varphi(x) \leq \varphi\left(y - \frac{1}{L}\nabla\varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla\varphi(y)\|_2^2$$

4. Now, substitute  $\varphi(y) = f(y) - \langle\nabla f(x), y\rangle$ :

$$f(x) - \langle\nabla f(x), x\rangle \leq f(y) - \langle\nabla f(x), y\rangle - \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|_2^2$$

$$f(x) + \langle\nabla f(x), y - x\rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y)$$

$$\|\nabla f(y) - \nabla f(x)\|_2^2 \leq 2L(f(y) - f(x) - \langle\nabla f(x), y - x\rangle)$$

switch  $x$  and  $y$      $\|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L(f(x) - f(y) - \langle\nabla f(y), x - y\rangle)$

3. From the first order optimality conditions for the convex function  $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any  $x$ , the minimum of the function  $\varphi(y)$  is at the point  $y = x$ . Therefore:

$$\varphi(x) \leq \varphi\left(y - \frac{1}{L}\nabla\varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla\varphi(y)\|_2^2$$

4. Now, substitute  $\varphi(y) = f(y) - \langle\nabla f(x), y\rangle$ :

$$f(x) - \langle\nabla f(x), x\rangle \leq f(y) - \langle\nabla f(x), y\rangle - \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|_2^2$$

$$f(x) + \langle\nabla f(x), y - x\rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y)$$

$$\|\nabla f(y) - \nabla f(x)\|_2^2 \leq 2L(f(y) - f(x) - \langle\nabla f(x), y - x\rangle)$$

switch  $x$  and  $y$

$$\|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L(f(x) - f(y) - \langle\nabla f(y), x - y\rangle)$$

3. From the first order optimality conditions for the convex function  $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any  $x$ , the minimum of the function  $\varphi(y)$  is at the point  $y = x$ . Therefore:

$$\varphi(x) \leq \varphi\left(y - \frac{1}{L}\nabla\varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla\varphi(y)\|_2^2$$

4. Now, substitute  $\varphi(y) = f(y) - \langle\nabla f(x), y\rangle$ :

$$f(x) - \langle\nabla f(x), x\rangle \leq f(y) - \langle\nabla f(x), y\rangle - \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|_2^2$$

$$f(x) + \langle\nabla f(x), y - x\rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y)$$

$$\|\nabla f(y) - \nabla f(x)\|_2^2 \leq 2L(f(y) - f(x) - \langle\nabla f(x), y - x\rangle)$$

switch  $x$  and  $y$

$$\|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L(f(x) - f(y) - \langle\nabla f(y), x - y\rangle)$$

The lemma has been proved. From the first view it does not make a lot of geometrical sense, but we will use it as a convenient tool to bound the difference between gradients.



## i Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\mathbb{R}^n$ . Then, the function  $f$  is  $\mu$ -strongly convex if and only if for any  $x, y \in \mathbb{R}^d$  the following holds:

$$\text{Strongly convex case } \mu > 0 \quad \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$$

$$\text{Convex case } \mu = 0 \quad \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$$

## Proof

1. We will only give the proof for the strongly convex case, the convex one follows from it with setting  $\mu = 0$ . We start from necessity. For the strongly convex function

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|x - y\|_2^2$$

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|_2^2$$

$$\text{sum} \quad \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$$

## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle = \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle$$



## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle = \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle$$
$$\langle \nabla f(y), x - y \rangle = \int_0^1 \langle \nabla f(y), x - y \rangle dt \quad \quad = \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(y), (x - y) \rangle dt$$

## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

$$\begin{aligned} f(x) - f(y) - \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle \\ \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y), x - y \rangle dt \\ y + t(x - y) - y &= t(x - y) \end{aligned}$$
$$\begin{aligned} &= \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(y), (x - y) \rangle dt \\ &= \int_0^1 t^{-1} \langle \nabla f(y + t(x - y)) - \nabla f(y), t(x - y) \rangle dt \end{aligned}$$

## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

$$\begin{aligned} f(x) - f(y) - \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle \\ \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y), x - y \rangle dt \\ y + t(x - y) - y &= t(x - y) \\ &= \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(y), t(x - y) \rangle dt \\ &\geq \int_0^1 t^{-1} \mu \|t(x - y)\|^2 dt \end{aligned}$$

## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

$$\begin{aligned} f(x) - f(y) - \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle \\ \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y), x - y \rangle dt \\ y + t(x - y) - y &= t(x - y) \\ &= \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(y), t(x - y) \rangle dt \\ &\geq \int_0^1 t^{-1} \mu \|t(x - y)\|^2 dt = \mu \|x - y\|^2 \int_0^1 t dt \end{aligned}$$

## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

$$\begin{aligned} f(x) - f(y) - \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle \\ \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y), x - y \rangle dt \\ y + t(x - y) - y &= t(x - y) \\ &= \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(y), t(x - y) \rangle dt \\ &\geq \int_0^1 t^{-1} \mu \|t(x - y)\|^2 dt = \mu \|x - y\|^2 \int_0^1 t dt = \frac{\mu}{2} \|x - y\|_2^2 \end{aligned}$$

## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

$$\begin{aligned} f(x) - f(y) - \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle \\ \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y), x - y \rangle dt \\ y + t(x - y) - y &= t(x - y) \\ &= \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(y), t(x - y) \rangle dt \\ &\geq \int_0^1 t^{-1} \mu \|t(x - y)\|^2 dt = \mu \|x - y\|^2 \int_0^1 t dt = \frac{\mu}{2} \|x - y\|_2^2 \end{aligned}$$

## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

$$\begin{aligned} f(x) - f(y) - \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle \\ \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y), x - y \rangle dt \\ y + t(x - y) - y &= t(x - y) \\ &= \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(y), t(x - y) \rangle dt \\ &\geq \int_0^1 t^{-1} \mu \|t(x - y)\|^2 dt = \mu \|x - y\|^2 \int_0^1 t dt = \frac{\mu}{2} \|x - y\|_2^2 \end{aligned}$$

Thus, we have a strong convexity criterion satisfied

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|_2^2$$

## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

$$\begin{aligned} f(x) - f(y) - \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle \\ \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y), x - y \rangle dt \\ y + t(x - y) - y &= t(x - y) \\ &= \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(y), t(x - y) \rangle dt \\ &\geq \int_0^1 t^{-1} \mu \|t(x - y)\|^2 dt = \mu \|x - y\|^2 \int_0^1 t dt = \frac{\mu}{2} \|x - y\|_2^2 \end{aligned}$$

Thus, we have a strong convexity criterion satisfied

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|_2^2 \text{ or, equivalently:}$$



## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

$$\begin{aligned} f(x) - f(y) - \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle \\ \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y), x - y \rangle dt \\ y + t(x - y) - y &= t(x - y) \\ &= \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(y), t(x - y) \rangle dt \\ &\geq \int_0^1 t^{-1} \mu \|t(x - y)\|^2 dt = \mu \|x - y\|^2 \int_0^1 t dt = \frac{\mu}{2} \|x - y\|_2^2 \end{aligned}$$

Thus, we have a strong convexity criterion satisfied

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|_2^2 \text{ or, equivalently:}$$

$$\text{switch } x \text{ and } y \quad - \langle \nabla f(x), x - y \rangle \leq - \left( f(x) - f(y) + \frac{\mu}{2} \|x - y\|_2^2 \right)$$

## Proximal Gradient Method. Convex case

# Convergence

## i Theorem

Consider the proximal gradient method

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

For the criterion  $\varphi(x) = f(x) + r(x)$ , we assume:

- $f$  is convex, differentiable,  $\text{dom}(f) = \mathbb{R}^n$ , and  $\nabla f$  is Lipschitz continuous with constant  $L > 0$ .
- $r$  is convex, and  $\text{prox}_{\alpha r}(x_k) = \arg \min_{x \in \mathbb{R}^n} [\alpha r(x) + \frac{1}{2} \|x - x_k\|_2^2]$  can be evaluated.

Proximal gradient descent with fixed step size  $\alpha = 1/L$  satisfies

$$\varphi(x_k) - \varphi^* \leq \frac{L \|x_0 - x^*\|^2}{2k},$$

Proximal gradient descent has a convergence rate of  $O(1/k)$  or  $O(1/\varepsilon)$ . This matches the gradient descent rate! (But remember the proximal operation cost)

# Convergence

## Proof

1. Let's introduce the **gradient mapping**, denoted as  $G_\alpha(x)$ , acts as a “gradient-like object”:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

$$x_{k+1} = x_k - \alpha G_\alpha(x_k).$$

where  $G_\alpha(x)$  is:

$$G_\alpha(x) = \frac{1}{\alpha} (x - \text{prox}_{\alpha r}(x - \alpha \nabla f(x)))$$

Observe that  $G_\alpha(x) = 0$  if and only if  $x$  is optimal. Therefore,  $G_\alpha$  is analogous to  $\nabla f$ . If  $x$  is locally optimal, then  $G_\alpha(x) = 0$  even for nonconvex  $f$ . This demonstrates that the proximal gradient method effectively combines gradient descent on  $f$  with the proximal operator of  $r$ , allowing it to handle non-differentiable components effectively.

# Convergence

## Proof

1. Let's introduce the **gradient mapping**, denoted as  $G_\alpha(x)$ , acts as a “gradient-like object”:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

$$x_{k+1} = x_k - \alpha G_\alpha(x_k).$$

where  $G_\alpha(x)$  is:

$$G_\alpha(x) = \frac{1}{\alpha} (x - \text{prox}_{\alpha r}(x - \alpha \nabla f(x)))$$

Observe that  $G_\alpha(x) = 0$  if and only if  $x$  is optimal. Therefore,  $G_\alpha$  is analogous to  $\nabla f$ . If  $x$  is locally optimal, then  $G_\alpha(x) = 0$  even for nonconvex  $f$ . This demonstrates that the proximal gradient method effectively combines gradient descent on  $f$  with the proximal operator of  $r$ , allowing it to handle non-differentiable components effectively.

2. We will use smoothness and convexity of  $f$  for some arbitrary point  $x$ :

# Convergence

## Proof

1. Let's introduce the **gradient mapping**, denoted as  $G_\alpha(x)$ , acts as a “gradient-like object”:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

$$x_{k+1} = x_k - \alpha G_\alpha(x_k).$$

where  $G_\alpha(x)$  is:

$$G_\alpha(x) = \frac{1}{\alpha} (x - \text{prox}_{\alpha r}(x - \alpha \nabla f(x)))$$

Observe that  $G_\alpha(x) = 0$  if and only if  $x$  is optimal. Therefore,  $G_\alpha$  is analogous to  $\nabla f$ . If  $x$  is locally optimal, then  $G_\alpha(x) = 0$  even for nonconvex  $f$ . This demonstrates that the proximal gradient method effectively combines gradient descent on  $f$  with the proximal operator of  $r$ , allowing it to handle non-differentiable components effectively.

2. We will use smoothness and convexity of  $f$  for some arbitrary point  $x$ :

$$\text{smoothness} \quad f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|_2^2$$

# Convergence

## Proof

1. Let's introduce the **gradient mapping**, denoted as  $G_\alpha(x)$ , acts as a “gradient-like object”:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

$$x_{k+1} = x_k - \alpha G_\alpha(x_k).$$

where  $G_\alpha(x)$  is:

$$G_\alpha(x) = \frac{1}{\alpha} (x - \text{prox}_{\alpha r}(x - \alpha \nabla f(x)))$$

Observe that  $G_\alpha(x) = 0$  if and only if  $x$  is optimal. Therefore,  $G_\alpha$  is analogous to  $\nabla f$ . If  $x$  is locally optimal, then  $G_\alpha(x) = 0$  even for nonconvex  $f$ . This demonstrates that the proximal gradient method effectively combines gradient descent on  $f$  with the proximal operator of  $r$ , allowing it to handle non-differentiable components effectively.

2. We will use smoothness and convexity of  $f$  for some arbitrary point  $x$ :

$$\text{smoothness} \quad f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|_2^2$$

$$\text{convexity} \quad f(x) \geq f(x_k) + \langle \nabla f(x_k), x - x_k \rangle$$

# Convergence

## Proof

1. Let's introduce the **gradient mapping**, denoted as  $G_\alpha(x)$ , acts as a “gradient-like object”:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

$$x_{k+1} = x_k - \alpha G_\alpha(x_k).$$

where  $G_\alpha(x)$  is:

$$G_\alpha(x) = \frac{1}{\alpha} (x - \text{prox}_{\alpha r}(x - \alpha \nabla f(x)))$$

Observe that  $G_\alpha(x) = 0$  if and only if  $x$  is optimal. Therefore,  $G_\alpha$  is analogous to  $\nabla f$ . If  $x$  is locally optimal, then  $G_\alpha(x) = 0$  even for nonconvex  $f$ . This demonstrates that the proximal gradient method effectively combines gradient descent on  $f$  with the proximal operator of  $r$ , allowing it to handle non-differentiable components effectively.

2. We will use smoothness and convexity of  $f$  for some arbitrary point  $x$ :

$$\text{smoothness } f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|_2^2$$

$$\text{convexity } f(x) \geq f(x_k) + \langle \nabla f(x_k), x - x_k \rangle \quad \leq f(x) - \langle \nabla f(x_k), x - x_k \rangle + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2$$



# Convergence

## Proof

1. Let's introduce the **gradient mapping**, denoted as  $G_\alpha(x)$ , acts as a "gradient-like object":

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

$$x_{k+1} = x_k - \alpha G_\alpha(x_k).$$

where  $G_\alpha(x)$  is:

$$G_\alpha(x) = \frac{1}{\alpha} (x - \text{prox}_{\alpha r}(x - \alpha \nabla f(x)))$$

Observe that  $G_\alpha(x) = 0$  if and only if  $x$  is optimal. Therefore,  $G_\alpha$  is analogous to  $\nabla f$ . If  $x$  is locally optimal, then  $G_\alpha(x) = 0$  even for nonconvex  $f$ . This demonstrates that the proximal gradient method effectively combines gradient descent on  $f$  with the proximal operator of  $r$ , allowing it to handle non-differentiable components effectively.

2. We will use smoothness and convexity of  $f$  for some arbitrary point  $x$ :

$$\text{smoothness } f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|_2^2$$

$$\text{convexity } f(x) \geq f(x_k) + \langle \nabla f(x_k), x - x_k \rangle \leq f(x) - \langle \nabla f(x_k), x - x_k \rangle + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2$$

$$\leq f(x) + \langle \nabla f(x_k), x_{k+1} - x \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2$$

## Convergence

3. Now we will use a proximal map property, which was proven before:

## Convergence

3. Now we will use a proximal map property, which was proven before:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) \quad \Leftrightarrow \quad x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1})$$

## Convergence

3. Now we will use a proximal map property, which was proven before:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) \quad \Leftrightarrow \quad x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1})$$

$$\text{Since } x_{k+1} - x_k = -\alpha G_\alpha(x_k) \quad \Rightarrow \quad \alpha G_\alpha(x_k) - \alpha \nabla f(x_k) \in \partial \alpha r(x_{k+1})$$

## Convergence

3. Now we will use a proximal map property, which was proven before:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) \quad \Leftrightarrow \quad x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1})$$

$$\text{Since } x_{k+1} - x_k = -\alpha G_\alpha(x_k) \quad \Rightarrow \quad \alpha G_\alpha(x_k) - \alpha \nabla f(x_k) \in \partial \alpha r(x_{k+1})$$

$$G_\alpha(x_k) - \nabla f(x_k) \in \partial r(x_{k+1})$$

## Convergence

3. Now we will use a proximal map property, which was proven before:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) \quad \Leftrightarrow \quad x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1})$$

$$\text{Since } x_{k+1} - x_k = -\alpha G_\alpha(x_k) \quad \Rightarrow \quad \alpha G_\alpha(x_k) - \alpha \nabla f(x_k) \in \partial \alpha r(x_{k+1})$$

$$G_\alpha(x_k) - \nabla f(x_k) \in \partial r(x_{k+1})$$

4. By the definition of the subgradient of convex function  $r$  for any point  $x$ :

## Convergence

3. Now we will use a proximal map property, which was proven before:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) \quad \Leftrightarrow \quad x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1})$$

$$\text{Since } x_{k+1} - x_k = -\alpha G_\alpha(x_k) \quad \Rightarrow \quad \alpha G_\alpha(x_k) - \alpha \nabla f(x_k) \in \partial \alpha r(x_{k+1})$$

$$G_\alpha(x_k) - \nabla f(x_k) \in \partial r(x_{k+1})$$

4. By the definition of the subgradient of convex function  $r$  for any point  $x$ :

$$r(x) \geq r(x_{k+1}) + \langle g, x - x_{k+1} \rangle, \quad g \in \partial r(x_{k+1})$$

## Convergence

3. Now we will use a proximal map property, which was proven before:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) \quad \Leftrightarrow \quad x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1})$$

$$\text{Since } x_{k+1} - x_k = -\alpha G_\alpha(x_k) \quad \Rightarrow \quad \alpha G_\alpha(x_k) - \alpha \nabla f(x_k) \in \partial \alpha r(x_{k+1})$$

$$G_\alpha(x_k) - \nabla f(x_k) \in \partial r(x_{k+1})$$

4. By the definition of the subgradient of convex function  $r$  for any point  $x$ :

$$r(x) \geq r(x_{k+1}) + \langle g, x - x_{k+1} \rangle, \quad g \in \partial r(x_{k+1})$$

substitute specific subgradient

$$r(x) \geq r(x_{k+1}) + \langle G_\alpha(x_k) - \nabla f(x), x - x_{k+1} \rangle$$



## Convergence

3. Now we will use a proximal map property, which was proven before:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) \quad \Leftrightarrow \quad x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1})$$

$$\text{Since } x_{k+1} - x_k = -\alpha G_\alpha(x_k) \quad \Rightarrow \quad \alpha G_\alpha(x_k) - \alpha \nabla f(x_k) \in \partial \alpha r(x_{k+1})$$

$$G_\alpha(x_k) - \nabla f(x_k) \in \partial r(x_{k+1})$$

4. By the definition of the subgradient of convex function  $r$  for any point  $x$ :

$$r(x) \geq r(x_{k+1}) + \langle g, x - x_{k+1} \rangle, \quad g \in \partial r(x_{k+1})$$

substitute specific subgradient

$$r(x) \geq r(x_{k+1}) + \langle G_\alpha(x_k) - \nabla f(x_k), x - x_{k+1} \rangle$$

$$r(x) \geq r(x_{k+1}) + \langle G_\alpha(x_k), x - x_{k+1} \rangle - \langle \nabla f(x_k), x - x_{k+1} \rangle$$

## Convergence

3. Now we will use a proximal map property, which was proven before:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) \quad \Leftrightarrow \quad x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1})$$

$$\text{Since } x_{k+1} - x_k = -\alpha G_\alpha(x_k) \quad \Rightarrow \quad \alpha G_\alpha(x_k) - \alpha \nabla f(x_k) \in \partial \alpha r(x_{k+1})$$

$$G_\alpha(x_k) - \nabla f(x_k) \in \partial r(x_{k+1})$$

4. By the definition of the subgradient of convex function  $r$  for any point  $x$ :

$$r(x) \geq r(x_{k+1}) + \langle g, x - x_{k+1} \rangle, \quad g \in \partial r(x_{k+1})$$

substitute specific subgradient

$$r(x) \geq r(x_{k+1}) + \langle G_\alpha(x_k) - \nabla f(x), x - x_{k+1} \rangle$$

$$r(x) \geq r(x_{k+1}) + \langle G_\alpha(x_k), x - x_{k+1} \rangle - \langle \nabla f(x), x - x_{k+1} \rangle$$

$$\langle \nabla f(x), x_{k+1} - x \rangle \leq r(x) - r(x_{k+1}) - \langle G_\alpha(x_k), x - x_{k+1} \rangle$$

## Convergence

3. Now we will use a proximal map property, which was proven before:

$$\begin{aligned}x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) &\Leftrightarrow x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1}) \\ \text{Since } x_{k+1} - x_k = -\alpha G_\alpha(x_k) &\Rightarrow \alpha G_\alpha(x_k) - \alpha \nabla f(x_k) \in \partial \alpha r(x_{k+1}) \\ &G_\alpha(x_k) - \nabla f(x_k) \in \partial r(x_{k+1})\end{aligned}$$

4. By the definition of the subgradient of convex function  $r$  for any point  $x$ :

$$\begin{aligned}r(x) &\geq r(x_{k+1}) + \langle g, x - x_{k+1} \rangle, \quad g \in \partial r(x_{k+1}) \\ \text{substitute specific subgradient} \quad r(x) &\geq r(x_{k+1}) + \langle G_\alpha(x_k) - \nabla f(x), x - x_{k+1} \rangle \\ r(x) &\geq r(x_{k+1}) + \langle G_\alpha(x_k), x - x_{k+1} \rangle - \langle \nabla f(x), x - x_{k+1} \rangle \\ &\langle \nabla f(x), x_{k+1} - x \rangle \leq r(x) - r(x_{k+1}) - \langle G_\alpha(x_k), x - x_{k+1} \rangle\end{aligned}$$

5. Taking into account the above bound we return back to the smoothness and convexity:

## Convergence

3. Now we will use a proximal map property, which was proven before:

$$\begin{aligned}x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) &\Leftrightarrow x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1}) \\ \text{Since } x_{k+1} - x_k = -\alpha G_\alpha(x_k) &\Rightarrow \alpha G_\alpha(x_k) - \alpha \nabla f(x_k) \in \partial \alpha r(x_{k+1}) \\ &G_\alpha(x_k) - \nabla f(x_k) \in \partial r(x_{k+1})\end{aligned}$$

4. By the definition of the subgradient of convex function  $r$  for any point  $x$ :

$$\begin{aligned}r(x) &\geq r(x_{k+1}) + \langle g, x - x_{k+1} \rangle, \quad g \in \partial r(x_{k+1}) \\ \text{substitute specific subgradient} \quad r(x) &\geq r(x_{k+1}) + \langle G_\alpha(x_k) - \nabla f(x), x - x_{k+1} \rangle \\ r(x) &\geq r(x_{k+1}) + \langle G_\alpha(x_k), x - x_{k+1} \rangle - \langle \nabla f(x), x - x_{k+1} \rangle \\ &\langle \nabla f(x), x_{k+1} - x \rangle \leq r(x) - r(x_{k+1}) - \langle G_\alpha(x_k), x - x_{k+1} \rangle\end{aligned}$$

5. Taking into account the above bound we return back to the smoothness and convexity:

$$f(x_{k+1}) \leq f(x) + \langle \nabla f(x_k), x_{k+1} - x \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2$$

## Convergence

3. Now we will use a proximal map property, which was proven before:

$$\begin{aligned}x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) &\Leftrightarrow x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1}) \\ \text{Since } x_{k+1} - x_k = -\alpha G_\alpha(x_k) &\Rightarrow \alpha G_\alpha(x_k) - \alpha \nabla f(x_k) \in \partial \alpha r(x_{k+1}) \\ &G_\alpha(x_k) - \nabla f(x_k) \in \partial r(x_{k+1})\end{aligned}$$

4. By the definition of the subgradient of convex function  $r$  for any point  $x$ :

$$\begin{aligned}r(x) &\geq r(x_{k+1}) + \langle g, x - x_{k+1} \rangle, \quad g \in \partial r(x_{k+1}) \\ \text{substitute specific subgradient} \quad r(x) &\geq r(x_{k+1}) + \langle G_\alpha(x_k) - \nabla f(x), x - x_{k+1} \rangle \\ r(x) &\geq r(x_{k+1}) + \langle G_\alpha(x_k), x - x_{k+1} \rangle - \langle \nabla f(x), x - x_{k+1} \rangle \\ \langle \nabla f(x), x_{k+1} - x \rangle &\leq r(x) - r(x_{k+1}) - \langle G_\alpha(x_k), x - x_{k+1} \rangle\end{aligned}$$

5. Taking into account the above bound we return back to the smoothness and convexity:

$$\begin{aligned}f(x_{k+1}) &\leq f(x) + \langle \nabla f(x_k), x_{k+1} - x \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2 \\ f(x_{k+1}) &\leq f(x) + r(x) - r(x_{k+1}) - \langle G_\alpha(x_k), x - x_{k+1} \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2\end{aligned}$$

## Convergence

3. Now we will use a proximal map property, which was proven before:

$$\begin{aligned}x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) &\Leftrightarrow x_k - \alpha \nabla f(x_k) - x_{k+1} \in \partial \alpha r(x_{k+1}) \\ \text{Since } x_{k+1} - x_k = -\alpha G_\alpha(x_k) &\Rightarrow \alpha G_\alpha(x_k) - \alpha \nabla f(x_k) \in \partial \alpha r(x_{k+1}) \\ &G_\alpha(x_k) - \nabla f(x_k) \in \partial r(x_{k+1})\end{aligned}$$

4. By the definition of the subgradient of convex function  $r$  for any point  $x$ :

$$\begin{aligned}r(x) &\geq r(x_{k+1}) + \langle g, x - x_{k+1} \rangle, \quad g \in \partial r(x_{k+1}) \\ \text{substitute specific subgradient} \quad r(x) &\geq r(x_{k+1}) + \langle G_\alpha(x_k) - \nabla f(x), x - x_{k+1} \rangle \\ r(x) &\geq r(x_{k+1}) + \langle G_\alpha(x_k), x - x_{k+1} \rangle - \langle \nabla f(x), x - x_{k+1} \rangle \\ \langle \nabla f(x), x_{k+1} - x \rangle &\leq r(x) - r(x_{k+1}) - \langle G_\alpha(x_k), x - x_{k+1} \rangle\end{aligned}$$

5. Taking into account the above bound we return back to the smoothness and convexity:

$$\begin{aligned}f(x_{k+1}) &\leq f(x) + \langle \nabla f(x_k), x_{k+1} - x \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2 \\ f(x_{k+1}) &\leq f(x) + r(x) - r(x_{k+1}) - \langle G_\alpha(x_k), x - x_{k+1} \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2 \\ f(x_{k+1}) + r(x_{k+1}) &\leq f(x) + r(x) - \langle G_\alpha(x_k), x - x_k + \alpha G_\alpha(x_k) \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2\end{aligned}$$

- Using  $\varphi(x) = f(x) + r(x)$  we can now prove extremely useful inequality, which will allow us to demonstrate monotonic decrease of the iteration:



6. Using  $\varphi(x) = f(x) + r(x)$  we can now prove extremely useful inequality, which will allow us to demonstrate monotonic decrease of the iteration:

$$\varphi(x_{k+1}) \leq \varphi(x) - \langle G_\alpha(x_k), x - x_k \rangle - \langle G_\alpha(x_k), \alpha G_\alpha(x_k) \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2$$





6. Using  $\varphi(x) = f(x) + r(x)$  we can now prove extremely useful inequality, which will allow us to demonstrate monotonic decrease of the iteration:

$$\begin{aligned}\varphi(x_{k+1}) &\leq \varphi(x) - \langle G_\alpha(x_k), x - x_k \rangle - \langle G_\alpha(x_k), \alpha G_\alpha(x_k) \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2 \\ \varphi(x_{k+1}) &\leq \varphi(x) + \langle G_\alpha(x_k), x_k - x \rangle + \frac{\alpha}{2} (\alpha L - 2) \|G_\alpha(x_k)\|_2^2\end{aligned}$$



6. Using  $\varphi(x) = f(x) + r(x)$  we can now prove extremely useful inequality, which will allow us to demonstrate monotonic decrease of the iteration:

$$\varphi(x_{k+1}) \leq \varphi(x) - \langle G_\alpha(x_k), x - x_k \rangle - \langle G_\alpha(x_k), \alpha G_\alpha(x_k) \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2$$

$$\varphi(x_{k+1}) \leq \varphi(x) + \langle G_\alpha(x_k), x_k - x \rangle + \frac{\alpha}{2} (\alpha L - 2) \|G_\alpha(x_k)\|_2^2$$

$$\alpha \leq \frac{1}{L} \Rightarrow \frac{\alpha}{2} (\alpha L - 2) \leq -\frac{\alpha}{2}$$

6. Using  $\varphi(x) = f(x) + r(x)$  we can now prove extremely useful inequality, which will allow us to demonstrate monotonic decrease of the iteration:

$$\varphi(x_{k+1}) \leq \varphi(x) - \langle G_\alpha(x_k), x - x_k \rangle - \langle G_\alpha(x_k), \alpha G_\alpha(x_k) \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2$$

$$\varphi(x_{k+1}) \leq \varphi(x) + \langle G_\alpha(x_k), x_k - x \rangle + \frac{\alpha}{2} (\alpha L - 2) \|G_\alpha(x_k)\|_2^2$$

$$\alpha \leq \frac{1}{L} \Rightarrow \frac{\alpha}{2} (\alpha L - 2) \leq -\frac{\alpha}{2} \quad \varphi(x_{k+1}) \leq \varphi(x) + \langle G_\alpha(x_k), x_k - x \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2$$

6. Using  $\varphi(x) = f(x) + r(x)$  we can now prove extremely useful inequality, which will allow us to demonstrate monotonic decrease of the iteration:

$$\varphi(x_{k+1}) \leq \varphi(x) - \langle G_\alpha(x_k), x - x_k \rangle - \langle G_\alpha(x_k), \alpha G_\alpha(x_k) \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2$$

$$\varphi(x_{k+1}) \leq \varphi(x) + \langle G_\alpha(x_k), x_k - x \rangle + \frac{\alpha}{2} (\alpha L - 2) \|G_\alpha(x_k)\|_2^2$$

$$\alpha \leq \frac{1}{L} \Rightarrow \frac{\alpha}{2} (\alpha L - 2) \leq -\frac{\alpha}{2} \quad \varphi(x_{k+1}) \leq \varphi(x) + \langle G_\alpha(x_k), x_k - x \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2$$

7. Now it is easy to verify, that when  $x = x_k$  we have monotonic decrease for the proximal gradient algorithm:

$$\varphi(x_{k+1}) \leq \varphi(x_k) - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2$$

8. When  $x = x^*$ :



8. When  $x = x^*$ :

$$\varphi(x_{k+1}) \leq \varphi(x^*) + \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2$$



8. When  $x = x^*$ :

$$\varphi(x_{k+1}) \leq \varphi(x^*) + \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2$$

$$\varphi(x_{k+1}) - \varphi(x^*) \leq \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2$$



8. When  $x = x^*$ :

$$\begin{aligned}\varphi(x_{k+1}) &\leq \varphi(x^*) + \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2 \\ \varphi(x_{k+1}) - \varphi(x^*) &\leq \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2 \\ &\leq \frac{1}{2\alpha} [2\langle \alpha G_\alpha(x_k), x_k - x^* \rangle - \|\alpha G_\alpha(x_k)\|_2^2]\end{aligned}$$



8. When  $x = x^*$ :

$$\begin{aligned}\varphi(x_{k+1}) &\leq \varphi(x^*) + \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2 \\ \varphi(x_{k+1}) - \varphi(x^*) &\leq \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2 \\ &\leq \frac{1}{2\alpha} [2\langle \alpha G_\alpha(x_k), x_k - x^* \rangle - \|\alpha G_\alpha(x_k)\|_2^2] \\ &\leq \frac{1}{2\alpha} [2\langle \alpha G_\alpha(x_k), x_k - x^* \rangle - \|\alpha G_\alpha(x_k)\|_2^2 - \|x_k - x^*\|_2^2 + \|x_k - x^*\|_2^2]\end{aligned}$$

8. When  $x = x^*$ :

$$\begin{aligned}\varphi(x_{k+1}) &\leq \varphi(x^*) + \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2 \\ \varphi(x_{k+1}) - \varphi(x^*) &\leq \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2 \\ &\leq \frac{1}{2\alpha} [2\langle \alpha G_\alpha(x_k), x_k - x^* \rangle - \|\alpha G_\alpha(x_k)\|_2^2] \\ &\leq \frac{1}{2\alpha} [2\langle \alpha G_\alpha(x_k), x_k - x^* \rangle - \|\alpha G_\alpha(x_k)\|_2^2 - \|x_k - x^*\|_2^2 + \|x_k - x^*\|_2^2] \\ &\leq \frac{1}{2\alpha} [-\|x_k - x^* - \alpha G_\alpha(x_k)\|_2^2 + \|x_k - x^*\|_2^2]\end{aligned}$$

8. When  $x = x^*$ :

$$\begin{aligned}
 \varphi(x_{k+1}) &\leq \varphi(x^*) + \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2 \\
 \varphi(x_{k+1}) - \varphi(x^*) &\leq \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2 \\
 &\leq \frac{1}{2\alpha} [2\langle \alpha G_\alpha(x_k), x_k - x^* \rangle - \|\alpha G_\alpha(x_k)\|_2^2] \\
 &\leq \frac{1}{2\alpha} [2\langle \alpha G_\alpha(x_k), x_k - x^* \rangle - \|\alpha G_\alpha(x_k)\|_2^2 - \|x_k - x^*\|_2^2 + \|x_k - x^*\|_2^2] \\
 &\leq \frac{1}{2\alpha} [-\|x_k - x^* - \alpha G_\alpha(x_k)\|_2^2 + \|x_k - x^*\|_2^2] \\
 &\leq \frac{1}{2\alpha} [\|x_k - x^*\|_2^2 - \|x_{k+1} - x^*\|_2^2]
 \end{aligned}$$

## Convergence

9. Now we write the bound above for all iterations  $i \in 0, k - 1$  and sum them:

## Convergence

9. Now we write the bound above for all iterations  $i \in 0, k-1$  and sum them:

$$\sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] \leq \frac{1}{2\alpha} [\|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2]$$

## Convergence

9. Now we write the bound above for all iterations  $i \in 0, k-1$  and sum them:

$$\begin{aligned} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] &\leq \frac{1}{2\alpha} [\|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2] \\ &\leq \frac{1}{2\alpha} \|x_0 - x^*\|_2^2 \end{aligned}$$

## Convergence

9. Now we write the bound above for all iterations  $i \in 0, k-1$  and sum them:

$$\begin{aligned} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] &\leq \frac{1}{2\alpha} [\|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2] \\ &\leq \frac{1}{2\alpha} \|x_0 - x^*\|_2^2 \end{aligned}$$

10. Since  $\varphi(x_k)$  is a decreasing sequence, it follows that:

## Convergence

9. Now we write the bound above for all iterations  $i \in 0, k-1$  and sum them:

$$\begin{aligned} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] &\leq \frac{1}{2\alpha} [\|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2] \\ &\leq \frac{1}{2\alpha} \|x_0 - x^*\|_2^2 \end{aligned}$$

10. Since  $\varphi(x_k)$  is a decreasing sequence, it follows that:

$$\sum_{i=0}^{k-1} \varphi(x_k) = k\varphi(x_k) \leq \sum_{i=0}^{k-1} \varphi(x_{i+1})$$



## Convergence

9. Now we write the bound above for all iterations  $i \in 0, k-1$  and sum them:

$$\begin{aligned} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] &\leq \frac{1}{2\alpha} [\|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2] \\ &\leq \frac{1}{2\alpha} \|x_0 - x^*\|_2^2 \end{aligned}$$

10. Since  $\varphi(x_k)$  is a decreasing sequence, it follows that:

$$\begin{aligned} \sum_{i=0}^{k-1} \varphi(x_k) &= k\varphi(x_k) \leq \sum_{i=0}^{k-1} \varphi(x_{i+1}) \\ \varphi(x_k) &\leq \frac{1}{k} \sum_{i=0}^{k-1} \varphi(x_{i+1}) \end{aligned}$$

## Convergence

9. Now we write the bound above for all iterations  $i \in 0, k-1$  and sum them:

$$\begin{aligned} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] &\leq \frac{1}{2\alpha} [\|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2] \\ &\leq \frac{1}{2\alpha} \|x_0 - x^*\|_2^2 \end{aligned}$$

10. Since  $\varphi(x_k)$  is a decreasing sequence, it follows that:

$$\begin{aligned} \sum_{i=0}^{k-1} \varphi(x_k) &= k\varphi(x_k) \leq \sum_{i=0}^{k-1} \varphi(x_{i+1}) \\ \varphi(x_k) &\leq \frac{1}{k} \sum_{i=0}^{k-1} \varphi(x_{i+1}) \\ \varphi(x_k) - \varphi(x^*) &\leq \frac{1}{k} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] \leq \frac{\|x_0 - x^*\|_2^2}{2\alpha k} \end{aligned}$$

## Convergence

9. Now we write the bound above for all iterations  $i \in 0, k-1$  and sum them:

$$\begin{aligned} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] &\leq \frac{1}{2\alpha} [\|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2] \\ &\leq \frac{1}{2\alpha} \|x_0 - x^*\|_2^2 \end{aligned}$$

10. Since  $\varphi(x_k)$  is a decreasing sequence, it follows that:

$$\begin{aligned} \sum_{i=0}^{k-1} \varphi(x_k) &= k\varphi(x_k) \leq \sum_{i=0}^{k-1} \varphi(x_{i+1}) \\ \varphi(x_k) &\leq \frac{1}{k} \sum_{i=0}^{k-1} \varphi(x_{i+1}) \\ \varphi(x_k) - \varphi(x^*) &\leq \frac{1}{k} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] \leq \frac{\|x_0 - x^*\|_2^2}{2\alpha k} \end{aligned}$$

## Convergence

9. Now we write the bound above for all iterations  $i \in 0, k-1$  and sum them:

$$\begin{aligned} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] &\leq \frac{1}{2\alpha} [\|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2] \\ &\leq \frac{1}{2\alpha} \|x_0 - x^*\|_2^2 \end{aligned}$$

10. Since  $\varphi(x_k)$  is a decreasing sequence, it follows that:

$$\begin{aligned} \sum_{i=0}^{k-1} \varphi(x_k) &= k\varphi(x_k) \leq \sum_{i=0}^{k-1} \varphi(x_{i+1}) \\ \varphi(x_k) &\leq \frac{1}{k} \sum_{i=0}^{k-1} \varphi(x_{i+1}) \\ \varphi(x_k) - \varphi(x^*) &\leq \frac{1}{k} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] \leq \frac{\|x_0 - x^*\|_2^2}{2\alpha k} \end{aligned}$$

Which is a standard  $\frac{L\|x_0 - x^*\|_2^2}{2k}$  with  $\alpha = \frac{1}{L}$ , or,  $\mathcal{O}\left(\frac{1}{k}\right)$  rate for smooth convex problems with Gradient Descent!

## Proximal Gradient Method. Strongly convex case

# Convergence

## i Theorem

Consider the proximal gradient method

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

For the criterion  $\varphi(x) = f(x) + r(x)$ , we assume:

- $f$  is  $\mu$ -strongly convex, differentiable,  $\text{dom}(f) = \mathbb{R}^n$ , and  $\nabla f$  is Lipschitz continuous with constant  $L > 0$ .
- $r$  is convex, and  $\text{prox}_{\alpha r}(x_k) = \arg \min_{x \in \mathbb{R}^n} [\alpha r(x) + \frac{1}{2} \|x - x_k\|_2^2]$  can be evaluated.

Proximal gradient descent with fixed step size  $\alpha \leq 1/L$  satisfies

$$\|x_k - x^*\|_2^2 \leq (1 - \alpha\mu)^k \|x_0 - x^*\|_2^2$$

This is exactly gradient descent convergence rate. Note, that the original problem is even non-smooth!

## Proof

1. Considering the distance to the solution and using the stationary point lemm:

## Proof

1. Considering the distance to the solution and using the stationary point lemm:

$$\|x_{k+1} - x^*\|_2^2 = \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2$$



## Proof

1. Considering the distance to the solution and using the stationary point lemm:

$$\begin{aligned}\|x_{k+1} - x^*\|_2^2 &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2 \\ \text{stationary point lemm} &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - \text{prox}_{\alpha f}(x^* - \alpha \nabla f(x^*))\|_2^2\end{aligned}$$

## Proof

1. Considering the distance to the solution and using the stationary point lemm:

$$\begin{aligned} \|x_{k+1} - x^*\|_2^2 &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2 \\ \text{stationary point lemm} &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - \text{prox}_{\alpha f}(x^* - \alpha \nabla f(x^*))\|_2^2 \\ \text{nonexpansiveness} &\leq \|x_k - \alpha \nabla f(x_k) - x^* + \alpha \nabla f(x^*)\|_2^2 \end{aligned}$$

## Proof

1. Considering the distance to the solution and using the stationary point lemm:

$$\begin{aligned}
 \|x_{k+1} - x^*\|_2^2 &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2 \\
 \text{stationary point lemm} &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - \text{prox}_{\alpha f}(x^* - \alpha \nabla f(x^*))\|_2^2 \\
 \text{nonexpansiveness} &\leq \|x_k - \alpha \nabla f(x_k) - x^* + \alpha \nabla f(x^*)\|_2^2 \\
 &= \|x_k - x^*\|^2 - 2\alpha \langle \nabla f(x_k) - \nabla f(x^*), x_k - x^* \rangle + \alpha^2 \|\nabla f(x_k) - \nabla f(x^*)\|_2^2
 \end{aligned}$$

## Proof

1. Considering the distance to the solution and using the stationary point lemm:

$$\begin{aligned}
 \|x_{k+1} - x^*\|_2^2 &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2 \\
 \text{stationary point lemm} &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - \text{prox}_{\alpha f}(x^* - \alpha \nabla f(x^*))\|_2^2 \\
 \text{nonexpansiveness} &\leq \|x_k - \alpha \nabla f(x_k) - x^* + \alpha \nabla f(x^*)\|_2^2 \\
 &= \|x_k - x^*\|^2 - 2\alpha \langle \nabla f(x_k) - \nabla f(x^*), x_k - x^* \rangle + \alpha^2 \|\nabla f(x_k) - \nabla f(x^*)\|_2^2
 \end{aligned}$$

2. Now we use smoothness from the convergence tools and strong convexity:

## Proof

1. Considering the distance to the solution and using the stationary point lemm:

$$\begin{aligned}
 \|x_{k+1} - x^*\|_2^2 &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2 \\
 \text{stationary point lemm} &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - \text{prox}_{\alpha f}(x^* - \alpha \nabla f(x^*))\|_2^2 \\
 \text{nonexpansiveness} &\leq \|x_k - \alpha \nabla f(x_k) - x^* + \alpha \nabla f(x^*)\|_2^2 \\
 &= \|x_k - x^*\|^2 - 2\alpha \langle \nabla f(x_k) - \nabla f(x^*), x_k - x^* \rangle + \alpha^2 \|\nabla f(x_k) - \nabla f(x^*)\|_2^2
 \end{aligned}$$

2. Now we use smoothness from the convergence tools and strong convexity:

$$\text{smoothness} \quad \|\nabla f(x_k) - \nabla f(x^*)\|_2^2 \leq 2L (f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle)$$

## Proof

1. Considering the distance to the solution and using the stationary point lemm:

$$\begin{aligned}
 \|x_{k+1} - x^*\|_2^2 &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2 \\
 \text{stationary point lemm} &= \|\text{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - \text{prox}_{\alpha f}(x^* - \alpha \nabla f(x^*))\|_2^2 \\
 \text{nonexpansiveness} &\leq \|x_k - \alpha \nabla f(x_k) - x^* + \alpha \nabla f(x^*)\|_2^2 \\
 &= \|x_k - x^*\|^2 - 2\alpha \langle \nabla f(x_k) - \nabla f(x^*), x_k - x^* \rangle + \alpha^2 \|\nabla f(x_k) - \nabla f(x^*)\|_2^2
 \end{aligned}$$

2. Now we use smoothness from the convergence tools and strong convexity:

$$\begin{aligned}
 \text{smoothness} \quad \|\nabla f(x_k) - \nabla f(x^*)\|_2^2 &\leq 2L (f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle) \\
 \text{strong convexity} \quad -\langle \nabla f(x_k) - \nabla f(x^*), x_k - x^* \rangle &\leq -\left(f(x_k) - f(x^*) + \frac{\mu}{2} \|x_k - x^*\|_2^2\right) - \langle \nabla f(x^*), x_k - x^* \rangle
 \end{aligned}$$

3. Substitute it:

3. Substitute it:

$$\begin{aligned}\|x_{k+1} - x^*\|_2^2 &\leq \|x_k - x^*\|^2 - 2\alpha \left( f(x_k) - f(x^*) + \frac{\mu}{2} \|x_k - x^*\|_2^2 \right) - 2\alpha \langle \nabla f(x^*), x_k - x^* \rangle + \\ &\quad + \alpha^2 2L (f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle)\end{aligned}$$



## 3. Substitute it:

$$\begin{aligned}\|x_{k+1} - x^*\|_2^2 &\leq \|x_k - x^*\|^2 - 2\alpha \left( f(x_k) - f(x^*) + \frac{\mu}{2} \|x_k - x^*\|_2^2 \right) - 2\alpha \langle \nabla f(x^*), x_k - x^* \rangle + \\ &\quad + \alpha^2 2L (f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle) \\ &\leq (1 - \alpha\mu) \|x_k - x^*\|^2 + 2\alpha(\alpha L - 1) (f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle)\end{aligned}$$

3. Substitute it:

$$\begin{aligned}\|x_{k+1} - x^*\|_2^2 &\leq \|x_k - x^*\|^2 - 2\alpha \left( f(x_k) - f(x^*) + \frac{\mu}{2} \|x_k - x^*\|_2^2 \right) - 2\alpha \langle \nabla f(x^*), x_k - x^* \rangle + \\ &\quad + \alpha^2 2L (f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle) \\ &\leq (1 - \alpha\mu) \|x_k - x^*\|^2 + 2\alpha(\alpha L - 1) (f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle)\end{aligned}$$

4. Due to convexity of  $f$ :  $f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle \geq 0$ . Therefore, if we use  $\alpha \leq \frac{1}{L}$ :

$$\|x_{k+1} - x^*\|_2^2 \leq (1 - \alpha\mu) \|x_k - x^*\|^2,$$

which is exactly linear convergence of the method with up to  $1 - \frac{\mu}{L}$  convergence rate.

# Accelerated Proximal Gradient – *convex* objective

## i Accelerated Proximal Gradient Method

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be **convex** and  $L$ -**smooth**,  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be proper, closed and convex,  $\varphi(x) = f(x) + r(x)$  admit a minimiser  $x^*$ , and suppose  $\text{prox}_{\alpha r}$  is easy to evaluate for  $\alpha > 0$ . With any  $x_0 \in \text{dom } r$  define the sequence

$$\begin{aligned}t_0 &= 1, & y_0 &= x_0, \\x_k &= \text{prox}_{\frac{1}{L}r}(y_{k-1} - \frac{1}{L}\nabla f(y_{k-1})), \\t_k &= \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2}, \\y_k &= x_k + \frac{t_{k-1} - 1}{t_k}(x_k - x_{k-1}), & k &\geq 1.\end{aligned}$$

Then for every  $k \geq 1$

$$\varphi(x_k) - \varphi(x^*) \leq \frac{2L \|x_0 - x^*\|_2^2}{(k+1)^2}$$

# Accelerated Proximal Gradient – $\mu$ -strongly convex objective

## i Accelerated Proximal Gradient Method

Assume in addition that  $f$  is  $\mu$ -strongly convex ( $\mu > 0$ ).

Set the step  $\alpha = \frac{1}{L}$  and the fixed momentum parameter

$$\beta = \frac{\sqrt{L/\mu} - 1}{\sqrt{L/\mu} + 1}.$$

Generate the iterates for  $k \geq 0$  (take  $x_{-1} = x_0$ ):

$$\begin{aligned} y_k &= x_k + \beta(x_k - x_{k-1}), \\ x_{k+1} &= \text{prox}_{\alpha r}(y_k - \alpha \nabla f(y_k)). \end{aligned}$$

Then for every  $k \geq 0$

$$\varphi(x_k) - \varphi(x^*) \leq \left(1 - \sqrt{\frac{\mu}{L}}\right)^k \left(\varphi(x_0) - \varphi(x^*) + \frac{\mu}{2} \|x_0 - x^*\|_2^2\right)$$

## Numerical experiments

## Quadratic case

$$f(x) = \frac{1}{2m} \|Ax - b\|_2^2 + \lambda \|x\|_1 \rightarrow \min_{x \in \mathbb{R}^n}, \quad A \in \mathbb{R}^{m \times n}, \quad \lambda \left( \frac{1}{m} A^T A \right) \in [\mu; L].$$

Linear Least Squares with  $\ell_1$  Regularization (LASSO).  
m=1000, n=100,  $\lambda=0$ ,  $\mu=0$ , L=10. Optimal sparsity: 0.0e+00

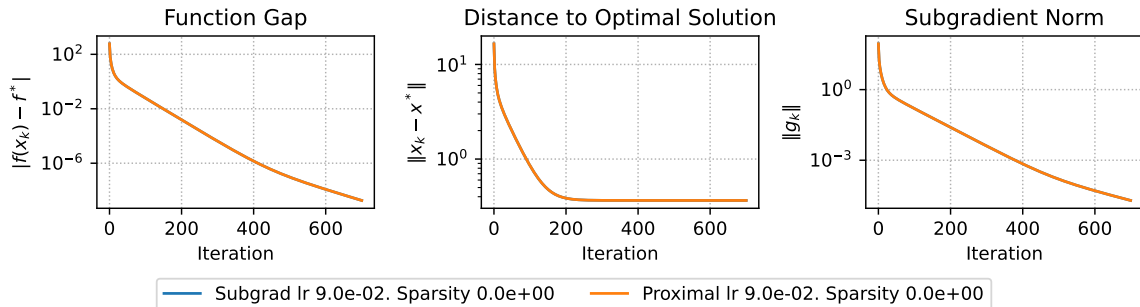


Рисунок 2: Smooth convex case. Sublinear convergence, no convergence in domain, no difference between subgradient and proximal methods

## Quadratic case

$$f(x) = \frac{1}{2m} \|Ax - b\|_2^2 + \lambda \|x\|_1 \rightarrow \min_{x \in \mathbb{R}^n}, \quad A \in \mathbb{R}^{m \times n}, \quad \lambda \left( \frac{1}{m} A^T A \right) \in [\mu; L].$$

Linear Least Squares with  $\ell_1$  Regularization (LASSO).  
 $m=1000$ ,  $n=100$ ,  $\lambda=1$ ,  $\mu=0$ ,  $L=10$ . Optimal sparsity:  $2.3e-01$

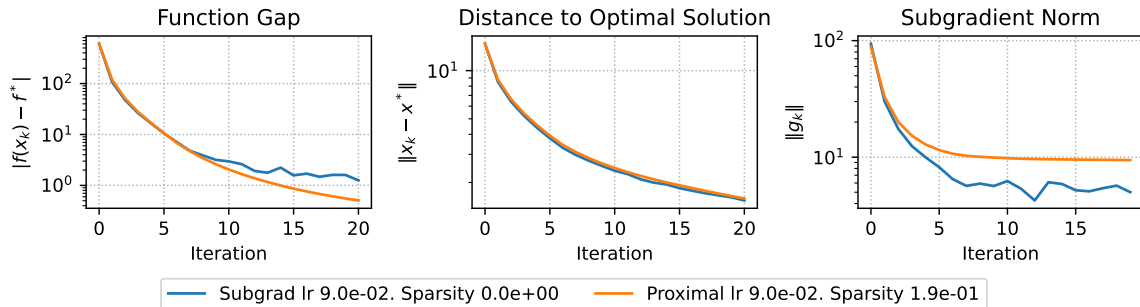


Рисунок 3: Non-smooth convex case. Sublinear convergence. At the beginning, the subgradient method and proximal method are close.

## Quadratic case

$$f(x) = \frac{1}{2m} \|Ax - b\|_2^2 + \lambda \|x\|_1 \rightarrow \min_{x \in \mathbb{R}^n}, \quad A \in \mathbb{R}^{m \times n}, \quad \lambda \left( \frac{1}{m} A^T A \right) \in [\mu; L].$$

Linear Least Squares with  $\ell_1$  Regularization (LASSO).  
m=1000, n=100,  $\lambda=1$ ,  $\mu=0$ , L=10. Optimal sparsity: 2.3e-01

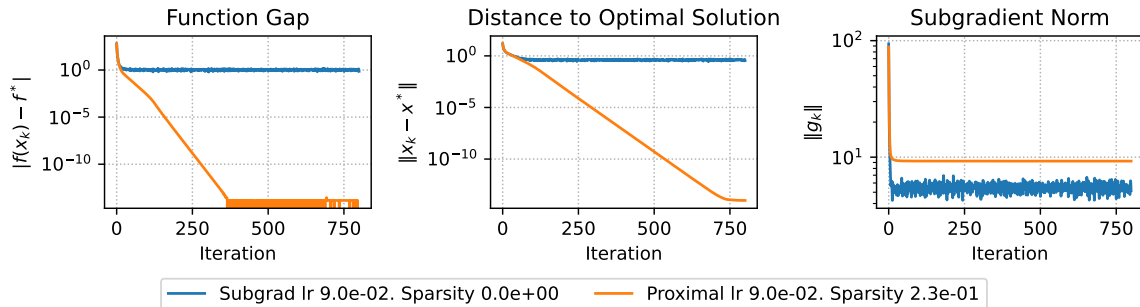


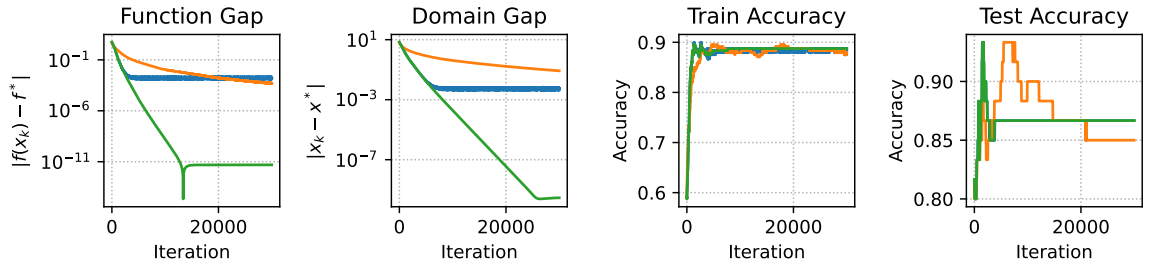
Рисунок 4: Non-smooth convex case. If we take more iterations, the proximal method converges with the constant learning rate, which is not the case for the subgradient method. The difference is tremendous, while the iteration complexity is the same.



## Binary logistic regression

$$f(x) = \frac{1}{m} \sum_{i=1}^m \log(1 + \exp(-b_i(A_i x))) + \lambda \|x\|_1 \rightarrow \min_{x \in \mathbb{R}^n}, \quad A_i \in \mathbb{R}^n, \quad b_i \in \{-1, 1\}$$

Binary Logistic Regression with  $\ell_1$  Regularization.  
 $m=300$ ,  $n=50$ ,  $\lambda=0.1$ . Optimal sparsity:  $8.6e-01$



lbgrad Ir  $1.0e-02$ . Sparsity  $0.0e+00$

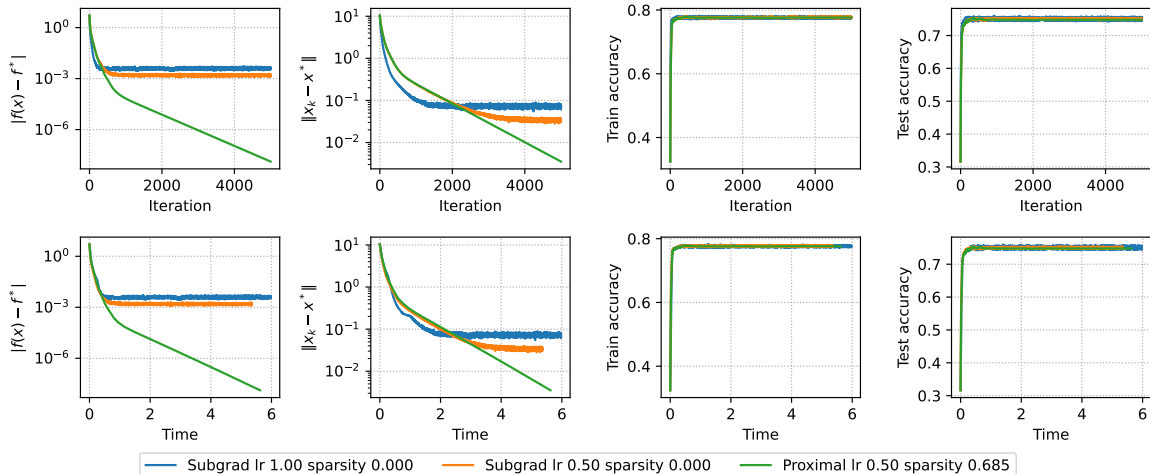
— Subgrad Ir  $\alpha/\sqrt{k}$  ( $\alpha=1.0e-01$ ). Sparsity  $1.2e-01$

— Proximal Ir  $1.0e-02$ . Sparsity

Рисунок 5: Logistic regression with  $\ell_1$  regularization

# Softmax multiclass regression

Convex multiclass regression. lam=0.01.



## Example: ISTA

### Iterative Shrinkage-Thresholding Algorithm (ISTA)

ISTA is a popular method for solving optimization problems involving L1 regularization, such as Lasso. It combines gradient descent with a shrinkage operator to handle the non-smooth L1 penalty effectively.

- **Algorithm:**

# Example: ISTA

## Iterative Shrinkage-Thresholding Algorithm (ISTA)

ISTA is a popular method for solving optimization problems involving L1 regularization, such as Lasso. It combines gradient descent with a shrinkage operator to handle the non-smooth L1 penalty effectively.

- **Algorithm:**

- Given  $x_0$ , for  $k \geq 0$ , repeat:

$$x_{k+1} = \text{prox}_{\lambda\alpha\|\cdot\|_1}(x_k - \alpha\nabla f(x_k)),$$

where  $\text{prox}_{\lambda\alpha\|\cdot\|_1}(v)$  applies soft thresholding to each component of  $v$ .

# Example: ISTA

## Iterative Shrinkage-Thresholding Algorithm (ISTA)

ISTA is a popular method for solving optimization problems involving L1 regularization, such as Lasso. It combines gradient descent with a shrinkage operator to handle the non-smooth L1 penalty effectively.

- **Algorithm:**

- Given  $x_0$ , for  $k \geq 0$ , repeat:

$$x_{k+1} = \text{prox}_{\lambda\alpha\|\cdot\|_1}(x_k - \alpha\nabla f(x_k)),$$

where  $\text{prox}_{\lambda\alpha\|\cdot\|_1}(v)$  applies soft thresholding to each component of  $v$ .

- **Convergence:**

# Example: ISTA

## Iterative Shrinkage-Thresholding Algorithm (ISTA)

ISTA is a popular method for solving optimization problems involving L1 regularization, such as Lasso. It combines gradient descent with a shrinkage operator to handle the non-smooth L1 penalty effectively.

- **Algorithm:**

- Given  $x_0$ , for  $k \geq 0$ , repeat:

$$x_{k+1} = \text{prox}_{\lambda\alpha\|\cdot\|_1}(x_k - \alpha\nabla f(x_k)),$$

where  $\text{prox}_{\lambda\alpha\|\cdot\|_1}(v)$  applies soft thresholding to each component of  $v$ .

- **Convergence:**

- Converges at a rate of  $O(1/k)$  for suitable step size  $\alpha$ .

# Example: ISTA

## Iterative Shrinkage-Thresholding Algorithm (ISTA)

ISTA is a popular method for solving optimization problems involving L1 regularization, such as Lasso. It combines gradient descent with a shrinkage operator to handle the non-smooth L1 penalty effectively.

- **Algorithm:**

- Given  $x_0$ , for  $k \geq 0$ , repeat:

$$x_{k+1} = \text{prox}_{\lambda\alpha\|\cdot\|_1}(x_k - \alpha\nabla f(x_k)),$$

where  $\text{prox}_{\lambda\alpha\|\cdot\|_1}(v)$  applies soft thresholding to each component of  $v$ .

- **Convergence:**

- Converges at a rate of  $O(1/k)$  for suitable step size  $\alpha$ .

- **Application:**

# Example: ISTA

## Iterative Shrinkage-Thresholding Algorithm (ISTA)

ISTA is a popular method for solving optimization problems involving L1 regularization, such as Lasso. It combines gradient descent with a shrinkage operator to handle the non-smooth L1 penalty effectively.

- **Algorithm:**

- Given  $x_0$ , for  $k \geq 0$ , repeat:

$$x_{k+1} = \text{prox}_{\lambda\alpha\|\cdot\|_1}(x_k - \alpha\nabla f(x_k)),$$

where  $\text{prox}_{\lambda\alpha\|\cdot\|_1}(v)$  applies soft thresholding to each component of  $v$ .

- **Convergence:**

- Converges at a rate of  $O(1/k)$  for suitable step size  $\alpha$ .

- **Application:**

- Efficient for sparse signal recovery, image processing, and compressed sensing.



## Example: FISTA

### Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)

FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

- **Algorithm:**

## Example: FISTA

### Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)

FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

- **Algorithm:**
  - Initialize  $x_0 = y_0$ ,  $t_0 = 1$ .

## Example: FISTA

### Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)

FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

- **Algorithm:**

- Initialize  $x_0 = y_0$ ,  $t_0 = 1$ .
- For  $k \geq 1$ , update:

$$x_k = \text{prox}_{\lambda\alpha\|\cdot\|_1}(y_{k-1} - \alpha\nabla f(y_{k-1})),$$

$$t_k = \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2},$$

$$y_k = x_k + \frac{t_{k-1} - 1}{t_k}(x_k - x_{k-1}).$$

## Example: FISTA

### Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)

FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

- **Algorithm:**

- Initialize  $x_0 = y_0, t_0 = 1$ .
- For  $k \geq 1$ , update:

$$x_k = \text{prox}_{\lambda\alpha\|\cdot\|_1}(y_{k-1} - \alpha\nabla f(y_{k-1})),$$

$$t_k = \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2},$$

$$y_k = x_k + \frac{t_{k-1} - 1}{t_k}(x_k - x_{k-1}).$$

- **Convergence:**

## Example: FISTA

### Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)

FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

- **Algorithm:**

- Initialize  $x_0 = y_0$ ,  $t_0 = 1$ .
- For  $k \geq 1$ , update:

$$x_k = \text{prox}_{\lambda\alpha\|\cdot\|_1}(y_{k-1} - \alpha\nabla f(y_{k-1})),$$

$$t_k = \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2},$$

$$y_k = x_k + \frac{t_{k-1} - 1}{t_k}(x_k - x_{k-1}).$$

- **Convergence:**

- Improves the convergence rate to  $O(1/k^2)$ .

## Example: FISTA

### Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)

FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

- **Algorithm:**

- Initialize  $x_0 = y_0$ ,  $t_0 = 1$ .
- For  $k \geq 1$ , update:

$$x_k = \text{prox}_{\lambda\alpha\|\cdot\|_1}(y_{k-1} - \alpha\nabla f(y_{k-1})),$$

$$t_k = \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2},$$

$$y_k = x_k + \frac{t_{k-1} - 1}{t_k}(x_k - x_{k-1}).$$

- **Convergence:**

- Improves the convergence rate to  $O(1/k^2)$ .

- **Application:**

## Example: FISTA

### Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)

FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

- **Algorithm:**

- Initialize  $x_0 = y_0, t_0 = 1$ .
- For  $k \geq 1$ , update:

$$x_k = \text{prox}_{\lambda\alpha\|\cdot\|_1}(y_{k-1} - \alpha\nabla f(y_{k-1})),$$

$$t_k = \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2},$$

$$y_k = x_k + \frac{t_{k-1} - 1}{t_k}(x_k - x_{k-1}).$$

- **Convergence:**

- Improves the convergence rate to  $O(1/k^2)$ .

- **Application:**

- Especially useful for large-scale problems in machine learning and signal processing where the L1 penalty induces sparsity.

# Example: Matrix Completion

## Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

- **Problem Formulation:**

$$\min_X \frac{1}{2} \|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda \|X\|_*,$$

where  $P_\Omega$  projects onto the observed set  $\Omega$ , and  $\|\cdot\|_*$  denotes the nuclear norm.



# Example: Matrix Completion

## Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

- **Problem Formulation:**

$$\min_X \frac{1}{2} \|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda \|X\|_*,$$

where  $P_\Omega$  projects onto the observed set  $\Omega$ , and  $\|\cdot\|_*$  denotes the nuclear norm.

- **Proximal Operator:**

# Example: Matrix Completion

## Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

- **Problem Formulation:**

$$\min_X \frac{1}{2} \|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda \|X\|_*,$$

where  $P_\Omega$  projects onto the observed set  $\Omega$ , and  $\|\cdot\|_*$  denotes the nuclear norm.

- **Proximal Operator:**

- The proximal operator for the nuclear norm involves singular value decomposition (SVD) and soft-thresholding of the singular values.

# Example: Matrix Completion

## Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

- **Problem Formulation:**

$$\min_X \frac{1}{2} \|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda \|X\|_*,$$

where  $P_\Omega$  projects onto the observed set  $\Omega$ , and  $\|\cdot\|_*$  denotes the nuclear norm.

- **Proximal Operator:**

- The proximal operator for the nuclear norm involves singular value decomposition (SVD) and soft-thresholding of the singular values.

- **Algorithm:**

# Example: Matrix Completion

## Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

- **Problem Formulation:**

$$\min_X \frac{1}{2} \|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda \|X\|_*,$$

where  $P_\Omega$  projects onto the observed set  $\Omega$ , and  $\|\cdot\|_*$  denotes the nuclear norm.

- **Proximal Operator:**

- The proximal operator for the nuclear norm involves singular value decomposition (SVD) and soft-thresholding of the singular values.

- **Algorithm:**

- Similar proximal gradient or accelerated proximal gradient methods can be applied, where the main computational effort lies in performing partial SVDs.

# Example: Matrix Completion

## Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

- **Problem Formulation:**

$$\min_X \frac{1}{2} \|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda \|X\|_*,$$

where  $P_\Omega$  projects onto the observed set  $\Omega$ , and  $\|\cdot\|_*$  denotes the nuclear norm.

- **Proximal Operator:**

- The proximal operator for the nuclear norm involves singular value decomposition (SVD) and soft-thresholding of the singular values.

- **Algorithm:**

- Similar proximal gradient or accelerated proximal gradient methods can be applied, where the main computational effort lies in performing partial SVDs.

- **Application:**

# Example: Matrix Completion

## Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

- **Problem Formulation:**

$$\min_X \frac{1}{2} \|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda \|X\|_*,$$

where  $P_\Omega$  projects onto the observed set  $\Omega$ , and  $\|\cdot\|_*$  denotes the nuclear norm.

- **Proximal Operator:**

- The proximal operator for the nuclear norm involves singular value decomposition (SVD) and soft-thresholding of the singular values.

- **Algorithm:**

- Similar proximal gradient or accelerated proximal gradient methods can be applied, where the main computational effort lies in performing partial SVDs.

- **Application:**

- Widely used in recommender systems, image recovery, and other domains where data is naturally matrix-formed but partially observed.

## Summary

- If we exploit the structure of the problem, we may beat the lower bounds for the unstructured problem.

## Summary

- If we exploit the structure of the problem, we may beat the lower bounds for the unstructured problem.
- Proximal gradient method for a composite problem with an  $L$ -smooth convex function  $f$  and a convex proximal friendly function  $r$  has the same convergence as the gradient descent method for the function  $f$ . The smoothness/non-smoothness properties of  $r$  do not affect convergence.



## Summary

- If we exploit the structure of the problem, we may beat the lower bounds for the unstructured problem.
- Proximal gradient method for a composite problem with an  $L$ -smooth convex function  $f$  and a convex proximal friendly function  $r$  has the same convergence as the gradient descent method for the function  $f$ . The smoothness/non-smoothness properties of  $r$  do not affect convergence.
- It seems that by putting  $f = 0$ , any nonsmooth problem can be solved using such a method. Question: is this true?

## Summary

- If we exploit the structure of the problem, we may beat the lower bounds for the unstructured problem.
- Proximal gradient method for a composite problem with an  $L$ -smooth convex function  $f$  and a convex proximal friendly function  $r$  has the same convergence as the gradient descent method for the function  $f$ . The smoothness/non-smoothness properties of  $r$  do not affect convergence.
- It seems that by putting  $f = 0$ , any nonsmooth problem can be solved using such a method. Question: is this true?

## Summary

- If we exploit the structure of the problem, we may beat the lower bounds for the unstructured problem.
- Proximal gradient method for a composite problem with an  $L$ -smooth convex function  $f$  and a convex proximal friendly function  $r$  has the same convergence as the gradient descent method for the function  $f$ . The smoothness/non-smoothness properties of  $r$  do not affect convergence.
- It seems that by putting  $f = 0$ , any nonsmooth problem can be solved using such a method. Question: is this true?

If we allow the proximal operator to be inexact (numerically), then it is true that we can solve any nonsmooth optimization problem. But this is not better from the point of view of theory than solving the problem by subgradient descent, because some auxiliary method (for example, the same subgradient descent) is used to solve the proximal subproblem.

- Proximal method is a general modern framework for many numerical methods. Further development includes accelerated, stochastic, primal-dual modifications and etc.

# Summary

- If we exploit the structure of the problem, we may beat the lower bounds for the unstructured problem.
- Proximal gradient method for a composite problem with an  $L$ -smooth convex function  $f$  and a convex proximal friendly function  $r$  has the same convergence as the gradient descent method for the function  $f$ . The smoothness/non-smoothness properties of  $r$  do not affect convergence.
- It seems that by putting  $f = 0$ , any nonsmooth problem can be solved using such a method. Question: is this true?

If we allow the proximal operator to be inexact (numerically), then it is true that we can solve any nonsmooth optimization problem. But this is not better from the point of view of theory than solving the problem by subgradient descent, because some auxiliary method (for example, the same subgradient descent) is used to solve the proximal subproblem.

- Proximal method is a general modern framework for many numerical methods. Further development includes accelerated, stochastic, primal-dual modifications and etc.
- Further reading: Proximal operator splitting, Douglas-Rachford splitting, Best approximation problem, Three operator splitting.