

# Proximal Gradient Method. Proximal operator

## Seminar

Optimization for ML. Faculty of Computer Science. HSE University

# Regularized / Composite Objectives

Many nonsmooth problems take the form

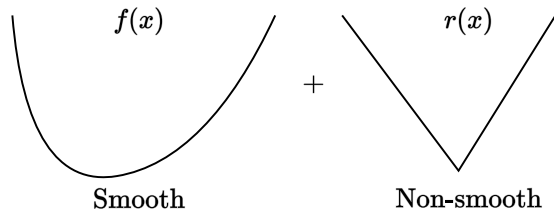
$$\min_{x \in \mathbb{R}^n} \varphi(x) = f(x) + r(x)$$

- **Lasso, L1-LS, compressed sensing**

$$f(x) = \frac{1}{2} \|Ax - b\|_2^2, r(x) = \lambda \|x\|_1$$

- **L1-Logistic regression, sparse LR**

$$f(x) = -y \log h(x) - (1-y) \log(1-h(x)), r(x) = \lambda \|x\|_1$$



# Non-smooth convex optimization lower bounds

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convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$
$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

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- Subgradient method is optimal for the problems above.
- One can use Mirror Descent (a generalization of the subgradient method to a possibly non-Euclidian distance) with the same convergence rate to better fit the geometry of the problem.
- However, we can achieve standard gradient descent rate  $\mathcal{O}\left(\frac{1}{k}\right)$  (and even accelerated version  $\mathcal{O}\left(\frac{1}{k^2}\right)$ ) if we will exploit the structure of the problem.

# Proximal operator

## i Proximal operator

For a convex set  $E \in \mathbb{R}^n$  and a convex function  $f : E \rightarrow \mathbb{R}$  operator  $\text{prox}_f(x)$  s.t.

$$\text{prox}_f(x) = \underset{y \in E}{\operatorname{argmin}} \left[ f(y) + \frac{1}{2} \|y - x\|_2^2 \right]$$

is called **proximal operator** for function  $f$  at point  $x$

## From projections to proximity

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With the following notation of indicator function

$$\mathbb{I}_S(x) = \begin{cases} 0, & x \in S, \\ \infty, & x \notin S, \end{cases}$$

Rewrite orthogonal projection  $\pi_S(y)$  as

$$\pi_S(y) := \arg \min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - y\|^2 + \mathbb{I}_S(x).$$



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Proximity: Replace  $\mathbb{I}_S$  by some convex function!

$$\text{prox}_r(y) = \text{prox}_{r,1}(y) := \arg \min \frac{1}{2} \|x - y\|^2 + r(x)$$

# Proximal Gradient Method

## 💡 Proximal Gradient Method Theorem

Consider the proximal gradient method

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

for the criterion  $\phi(x) = f(x) + r(x)$  s.t.: 1.  $f$  is convex, differentiable with Lipschitz gradients; 1.  $r$  is convex and prox-friendly. Then Proximal Gradient Method with fixed step size  $\alpha = \frac{1}{L}$  converges with rate  $O(\frac{1}{k})$

# ISTA and FISTA

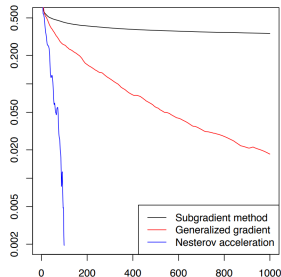
Methods for solving problems involving  $L1$  regularization (e.g. Lasso).

**ISTA** (Iterative Shrinkage-Thresholding Algorithm)

- Step:

$$x_{k+1} = \text{prox}_{\alpha\lambda||\cdot||_1}(x_k - \alpha\nabla f(x_k))$$

- Convergence:  $O(\frac{1}{k})$



Composite optimization

**FISTA** (Fast Iterative Shrinkage-Thresholding Algorithm)

- Step:

$$x_{k+1} = \text{prox}_{\alpha\lambda||\cdot||_1}(y_k - \alpha\nabla f(y_k)),$$

$$t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2},$$

$$y_{k+1} = x_{k+1} + \frac{t_k - 1}{t_{k+1}}(x_{k+1} - x_k)$$

- Convergence:  $O(\frac{1}{k^2})$

## Problem 1. ReLU in prox

### Question

Find the  $\text{prox}_f(x)$  for  $f(x) = \lambda \max(0, x)$ :

$$\text{prox}_{\lambda \max(0, \cdot)}(x) = \underset{y \in \mathbb{R}}{\operatorname{argmin}} \left[ \frac{1}{2} \|y - x\|^2 + \lambda \max(0, y) \right]$$


## Problem 2. Grouped $l_1$ -regularizer

### Question

Find the  $\text{prox}_f(x)$  for  $f(x) = \|x\|_{1/2} = \sum_{g=0}^G \|x_g\|_2$  where  $x \in \mathbb{R}^n = [\underbrace{x_1, x_2, \dots}_1, \underbrace{\dots}_g, \dots, \underbrace{x_{n-2}, x_{n-1}, x_n}_G]$ :

$$\text{prox}_{\|x\|_{1/2}}(x) = \underset{y \in \mathbb{R}}{\text{argmin}} \left[ \frac{1}{2} \|y - x\|_2^2 + \sum_{g=0}^G \|y_g\|_2 \right]$$

# Linear Least Squares with $L_1$ -regularizer

Proximal Methods Comparison for Linear Least Squares with  $L_1$ -regularizer  Open in Colab.