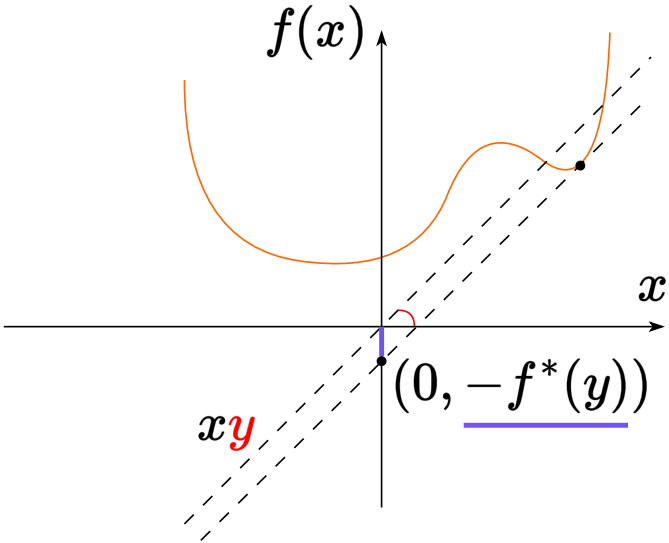
# Conjugate functions. Dual (sub)gradient method. Augmented Lagrangian method. ADMM.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

# Definition




Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function defined by

$$f^*(y) = \max_x \left[ y^T x - f(x) \right]$$

is called its conjugate.

# Conjugate function properties

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- Conjugates appear frequently in dual programs, since

$$-f^*(y) = \min_x \left[ f(x) - y^T x \right]$$

- If $f$ is closed and convex, then $f^{**} = f$. Also,

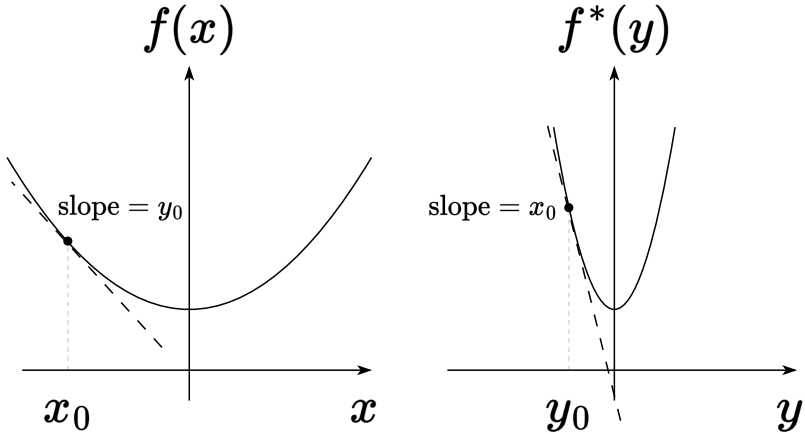
$$x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x) \Leftrightarrow x \in \arg\min_z \left[ f(z) - y^T z \right]$$

- If $f$ is strictly convex, then

$$\nabla f^*(y) = \arg\min_z \left[ f(z) - y^T z \right]$$

## Slopes of $f$ and $f^*$

Assume that $f$ is a closed and convex function. Then $f$ is strongly convex with parameter $\mu \Leftrightarrow \nabla f^*$ is Lipschitz with parameter $1/\mu$.



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$$\left(\delta\left([s = a]\right) - b\right)^* = \sup_{s \in \mathbf{dom} f_2(s)} \left(y^T s - \delta\left([s = a]\right) + b\right) = a^T y + b$$

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$$f^*(s) = \sup_{x \in \mathbb{R}^n} \left(sx - \log(1 + \exp(x))\right)$$

$$f^*(s) = \begin{cases} \infty, & \text{if } s < 0 \\ 0, & \text{if } s = 0 \\ 0, & \text{if } s = 1 \\ \infty, & \text{if } s > 1 \\ \textbf{?}, & \text{if } 0 < s < 1 \end{cases}$$

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$s \in (0, 1)$:

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Thus,

$$f^*(s) = \begin{cases} 0, & \text{if } s \in \{0, 1\} \\ s \log s + (1 - s) \log(1 - s), & \text{if } 0 < s < 1 \\ \infty, & \text{else} \end{cases}$$

$$\mathrm{dom} f^*(s) = [0, 1]$$

# Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

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$$\min_x \quad f(x) \quad \text{subject to} \quad Ax = b$$

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Its dual problem is:

$$\max_u \quad -f^*(-A^T u) - b^T u$$

where $f^*$ is the conjugate of $f$. Defining $g(u) = -f^*(-A^T u) - b^T u$, note that:

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Dual ascent method for maximizing dual objective:

$$x_k \in \arg\min_x \left[ f(x) + (u_{k-1})^T A x \right]$$

$$u_k = u_{k-1} + \alpha_k (A x_k - b)$$

- Step sizes $\alpha_k$, $k = 1, 2, 3, \ldots$, are chosen in standard ways.
- Proximal gradients and acceleration can be applied as they would usually.

# Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent:[1]

- If $f$ is strongly convex with parameter $\mu$, then dual gradient ascent with constant step sizes $\alpha_k = \mu$ converges at sublinear rate $O(\frac{1}{\epsilon})$.
- If $f$ is strongly convex with parameter $\mu$ and $\nabla f$ is Lipschitz with parameter $L$, then dual gradient ascent with step sizes $\alpha_k = \frac{2}{\frac{1}{\mu}+\frac{1}{L}}$ converges at linear rate $O(\log(\frac{1}{\epsilon}))$.

Note that this describes convergence in the dual. (Convergence in the primal requires more assumptions)

[1] This is ignoring the role of $A$, and thus reflects the case when the singular values of $A$ are all close to 1. To be more precise, the step sizes here should be: $\frac{\mu}{\sigma_{\max}(A)^2}$ (first case) and $\frac{2}{\frac{\sigma_{\max}(A)^2}{\mu}+\frac{\sigma_{\min}(A)^2}{L}}$ (second case).

# Dual decomposition

Consider

$$\min_x \sum_{i=1}^{B} f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Here $x = (x_1, \ldots, x_B) \in \mathbb{R}^n$ divides into $B$ blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition $A$ accordingly:

$$A = [A_1 \ldots A_B], \text{ where } A_i \in \mathbb{R}^{m \times n_i}$$

Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into $B$ separate problems:

$$x^{\text{new}} \in \arg\min_x \left( \sum_{i=1}^{B} f_i(x_i) + u^T A x \right)$$

$$\Rightarrow x_i^{\text{new}} \in \arg\min_{x_i} \left( f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \ldots, B$$

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$$x_i^k \in \arg\min_{x_i} \left( f_i(x_i) + (u^{k-1})^T A_i x_i \right), \quad i = 1, \ldots, B$$

$$u_i^k = u_i^{k-1} + \alpha_k \left( A_i x_i^k - b_i \right), \quad i = 1, \ldots, B$$

Can think of these steps as:

- **Broadcast:** Send $u$ to each of the $B$ processors, each optimizes in parallel to find $x_i$.
- **Gather:** Collect $A_i x_i$ from each processor, update the global dual variable $u$.

# Inequality constraints

Consider the optimization problem:

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Using **dual decomposition**, specifically the **projected subgradient method**, the iterative steps can be expressed as:

- The primal update step:

$$x_i^k \in \arg\min_{x_i} \left[ f_i(x_i) + \left(u^{k-1}\right)^T A_i x_i \right], \quad i = 1, \ldots, B$$

- The dual update step:

$$u^k = \left( u^{k-1} + \alpha_k \left( \sum_{i=1}^{B} A_i x_i^k - b \right) \right)_+$$

where $(u)_+$ denotes the positive part of $u$, i.e., $(u_+)_i = \max\{0, u_i\}$, for $i = 1, \ldots, m$.

# Augmented Lagrangian method aka method of multipliers

**Dual ascent disadvantage:** convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

$$\min_x f(x) + \frac{\rho}{2}\|Ax - b\|^2$$
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**Dual gradient ascent:** The iterative updates are given by:

$$\begin{aligned} x_k &= \arg\min_x \left[ f(x) + (u_{k-1})^T Ax + \frac{\rho}{2}\|Ax - b\|^2 \right] \\ u_k &= u_{k-1} + \rho(Ax_k - b) \end{aligned}$$

- **Advantage:** The augmented Lagrangian gives better convergence.
- **Disadvantage:** We lose decomposability! (Separability is ruined)
- **Notice** step size choice $\alpha_k = \rho$ in dual algorithm.

# Colab Example

- Dual subgradient and Augmented Lagrangian methods Comparison Open in Colab.

# Alternating Direction Method of Multipliers (ADMM)

**Alternating direction method of multipliers** or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

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where $\rho > 0$ is a parameter. The augmented Lagrangian for this problem is defined as:

$$L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|^2$$

# Alternating Direction Method of Multipliers (ADMM)

**ADMM repeats the following steps, for $k = 1, 2, 3, \ldots$:**

1. Update $x$:

$$x_k = \arg\min_x L_\rho(x, z_{k-1}, u_{k-1})$$

2. Update $z$:

$$z_k = \arg\min_z L_\rho(x_k, z, u_{k-1})$$

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**Note:** The usual method of multipliers would replace the first two steps by a joint minimization:

$$(x^{(k)}, z^{(k)}) = \arg\min_{x,z} L_\rho(x, z, u^{(k-1)})$$

# ADMM Summary

- ADMM is one of the key and popular recent optimization methods.
- It is implemented in many solvers and is often used as a default method.
- The non-standard formulation of the problem itself, for which ADMM is invented, turns out to include many important special cases. "Unusual" variable $y$ often plays the role of an auxiliary variable.
- Here the penalty is an additional modification to stabilize and accelerate convergence. It is not necessary to make $\rho$ very large.