

# Basic linear algebra recap. Convergence rates. Line Search

## Seminar

Optimization for ML. Faculty of Computer Science. HSE University

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- $\langle A, B \rangle = \text{tr}(A^T B)$

# Convergence rate



Figure 1: Illustration of different convergence rates

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- Any convergent sequence, that is slower (faster) than any linearly convergent sequence has sublinear (superlinear) convergence

# Root test

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- The case  $q > 1$  is impossible.

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- If  $q$  does not exist, but  $q = \lim_{k \rightarrow \infty} \sup_k \frac{r_{k+1}}{r_k} < 1$ , then  $\{r_k\}_{k=m}^{\infty}$  has linear convergence with a constant not exceeding  $q$ .

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- If  $\lim_{k \rightarrow \infty} \inf_k \frac{r_{k+1}}{r_k} = 1$ , then  $\{r_k\}_{k=m}^{\infty}$  has sublinear convergence.
- The case  $\lim_{k \rightarrow \infty} \inf_k \frac{r_{k+1}}{r_k} > 1$  is impossible.
- In all other cases (i.e., when  $\lim_{k \rightarrow \infty} \inf_k \frac{r_{k+1}}{r_k} < 1 \leq \lim_{k \rightarrow \infty} \sup_k \frac{r_{k+1}}{r_k}$ ) we cannot claim anything concrete about the convergence rate  $\{r_k\}_{k=m}^{\infty}$ .

## Line search

Typical line search problem is finding the good value  $\alpha$  of the stepsize:

$$x_{k+1} = x_k - \alpha \nabla f(x_k)$$



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  - Curvature conditions
  - The idea behind backtracking line search

## Problem 1. Stupid important idea on matrix computations.

Suppose, you have the following expression

$$b = A_1 A_2 A_3 x,$$

where the  $A_1, A_2, A_3 \in \mathbb{R}^{3 \times 3}$  - random square dense matrices and  $x \in \mathbb{R}^n$  - vector. You need to compute  $b$ .

Which one way is the best to do it?

1.  $A_1 A_2 A_3 x$  (from left to right)

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3. It does not matter
4. The results of the first two options will not be the same.

Check the simple 📄 code snippet after all.

## Problem 2. Connection between Frobenius norm and singular values.

Let  $A \in \mathbb{R}^{m \times n}$ , and let  $q := \min\{m, n\}$ . Show that

$$\|A\|_F^2 = \sum_{i=1}^q \sigma_i^2(A),$$

where  $\sigma_1(A) \geq \dots \geq \sigma_q(A) \geq 0$  are the singular values of matrix  $A$ . Hint: use the connection between Frobenius norm and scalar product and SVD.

### Problem 3. Known your inner product.

Simplify the following expression:

$$\sum_{i=1}^n \langle S^{-1} a_i, a_i \rangle,$$

where  $S = \sum_{i=1}^n a_i a_i^T$ ,  $a_i \in \mathbb{R}^n$ ,  $\det(S) \neq 0$



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- $r_k = 0.707^{2^k}$

## Problem 5. One test is simpler, than another.

Determine the convergence or divergence of the following sequence:

$$r_k = \frac{1}{k^k}$$

## Problem 6. Quadratic convergence.

Show, that the following sequence does not have a quadratic convergence.

$$r_k = \frac{1}{3^{k^2}}$$