

# Proximal gradient method

Daniil Merkulov

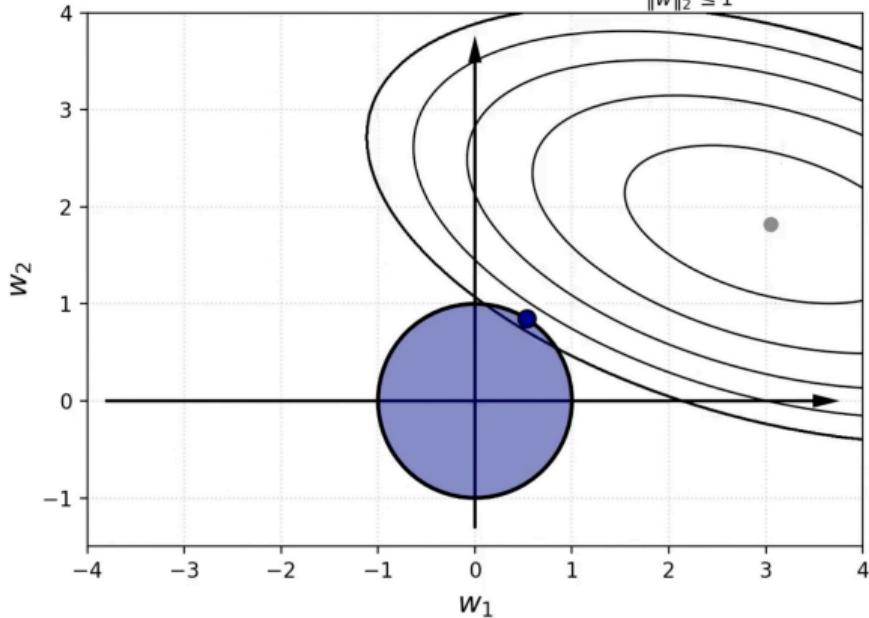
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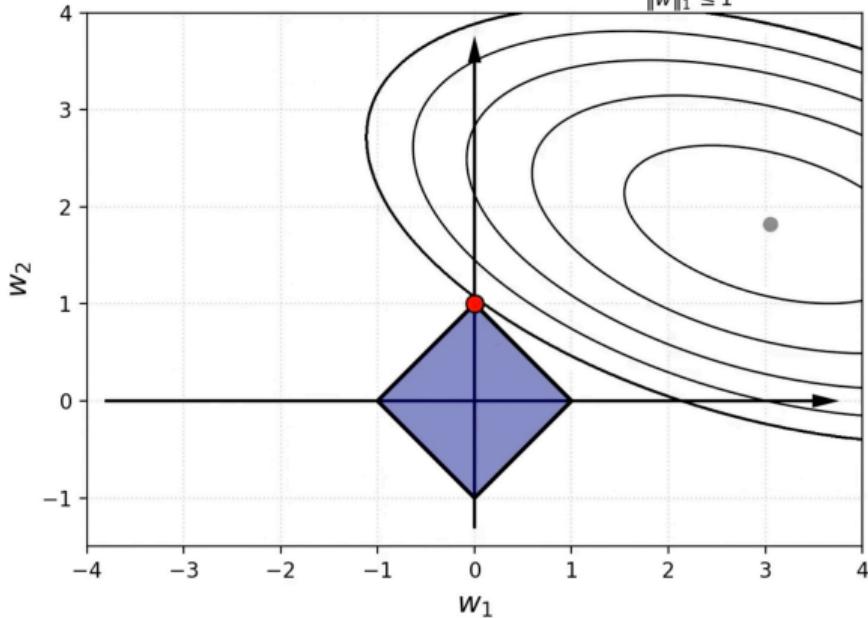
## Non-smooth problems

$\ell_1$  induces sparsity

$\ell_2$  regularization.  $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_2 \leq 1}$



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@fminxyz

## Subgradient method

Subgradient Method:

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$x_{k+1} = x_k - \alpha_k g_k, \quad g_k \in \partial f(x_k)$$

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convex (non-smooth)

$$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$

$$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

strongly convex (non-smooth)

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$$\frac{1}{K} \quad \frac{1}{K^2}$$

GD      AGD

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### Theorem

Assume that  $f$  is  $G$ -Lipschitz and convex, then  
Subgradient method converges as:

where

$$\bullet \quad \alpha = \frac{R}{G\sqrt{k}}$$

$$f(\bar{x}) - f^* \leq \frac{GR}{\sqrt{k}},$$

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- $\alpha = \frac{R}{G\sqrt{k}}$
- $R = \|x_0 - x^*\|$
- $\bar{x} = \frac{1}{k} \sum_{i=0}^{k-1} x_i$

## Non-smooth convex optimization lower bounds

convex (non-smooth)	strongly convex (non-smooth)
$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$	$f(x_k) - f^* \sim \mathcal{O}\left(\frac{1}{k}\right)$
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- Subgradient method is optimal for the problems above.
- One can use Mirror Descent (a generalization of the subgradient method to a possibly non-Euclidian distance) with the same convergence rate to better fit the geometry of the problem.
- However, we can achieve standard gradient descent rate  $\mathcal{O}\left(\frac{1}{k}\right)$  (and even accelerated version  $\mathcal{O}\left(\frac{1}{k^2}\right)$ ) if we will exploit the structure of the problem.

## Proximal mapping intuition

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

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$$\frac{x_{k+1} - x_k}{\alpha} = -\nabla f(x_k)$$

Leads to ordinary Gradient Descent method

$$x_{k+1} - x_k = -\alpha \nabla f(x_k)$$

$$x_{k+1} = x_k - \alpha \nabla f(x_k)$$

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$$f(x)$$

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$$\nabla \left[ \frac{1}{2\alpha} \|x - x_k\|_2^2 + f(x) \right] \Big|_{x=x_{k+1}} = 0$$

$g(x)$

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$$x_{k+1} = \arg \min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

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$$g(x) \quad \frac{x_{k+1} - x_k}{\alpha} + \nabla f(x_{k+1}) = 0$$

$$\nabla g(x_{k+1}) = 0 \quad \left. \frac{x - x_k}{\alpha} + \nabla f(x) \right|_{x=x_{k+1}} = 0$$

$$\nabla \left[ \frac{1}{2\alpha} \|x - x_k\|_2^2 + f(x) \right] \Big|_{x=x_{k+1}} = 0$$

$$x_{k+1} = \arg \min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

! Proximal operator

$$x_{k+1} = \text{prox}_{f,\alpha}(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

## Proximal operator visualization

$$\text{Prox}_f(x) = \operatorname{argmin}_{x'} \frac{1}{2} \|x - x'\|^2 + f(x')$$

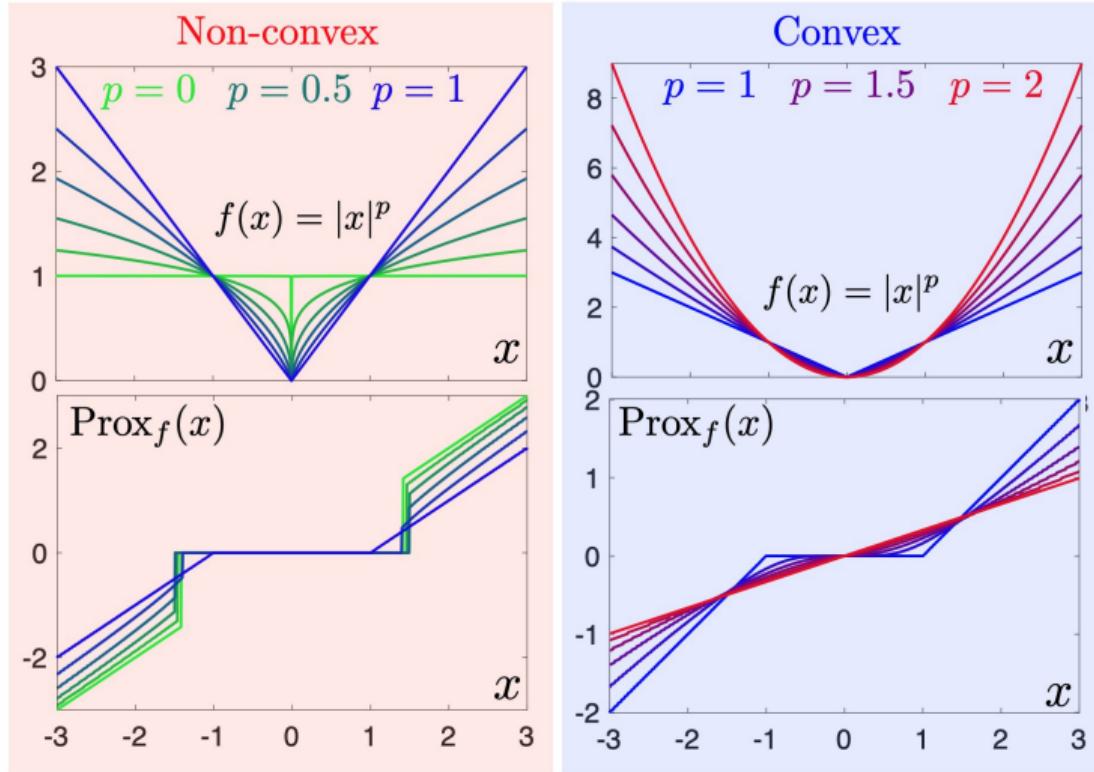


Figure 1: Source

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нeбнeд gueKР. Энepг



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$$x_{k+1} = (I + \alpha \nabla f)^{-1} x_k \xrightarrow{\alpha \rightarrow 0} (I - \alpha \nabla f)x_k$$

$$(1+x)^{-1} \approx 1-x \quad x \rightarrow 0$$

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- **GD from proximal method.** Back to the discretization:

$$x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k)$$

↑  
using Hessian

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Thus, we have a usual gradient descent with  $\alpha \rightarrow 0$ :  $x_{k+1} = x_k - \alpha \nabla f(x_k)$

- **Newton from proximal method.** Now let's consider proximal mapping of a second order Taylor approximation of the function  $f_{x_k}^{II}(x)$ :

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$$x_{k+1} = \text{prox}_{f_{x_k}^{II}, \alpha}(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2} \langle \nabla^2 f(x_k)(x - x_k), x - x_k \rangle + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

$f_{x_k}^{II}(x)$

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$$\nabla f(x_k) + \nabla^2 f(x_k)(x - x_k) + \frac{1}{\alpha}(x - x_k) \Big|_{x=x_{k+1}} = 0$$

$$x_{k+1} = x_k - \left[ \nabla^2 f(x_k) + \frac{1}{\alpha} I \right]^{-1} \nabla f(x_k) \quad \alpha \rightarrow \infty$$

$$\Rightarrow x_{k+1} = x_k - \left[ \nabla^2 f(x_k) \right]^{-1} \nabla f(x_k)$$

## From projections to proximity

Let  $\mathbb{I}_S$  be the indicator function for closed, convex  $S$ . Recall orthogonal projection  $\pi_S(y)$

## From projections to proximity

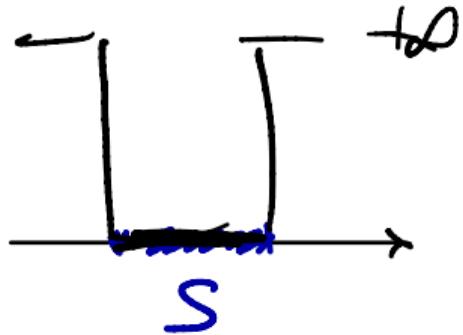
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With the following notation of indicator function

$$\mathbb{I}_S(x) = \begin{cases} 0, & x \in S, \\ \infty, & x \notin S, \end{cases}$$

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Rewrite orthogonal projection  $\pi_S(y)$  as

$$\boxed{\pi_S(y) := \arg \min_{x \in \mathbb{R}^n} \left( \frac{1}{2} \|x - y\|^2 + \mathbb{I}_S(x) \right)}$$

$x \notin S$   $\underset{x \notin S}{\text{argmin}} \infty$

$x \in S$   $\underset{x \in S}{\text{argmin}} \|x - y\|_2^2$

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Rewrite orthogonal projection  $\pi_S(y)$  as

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Proximity: Replace  $\mathbb{I}_S$  by some convex function!

$$\text{prox}_r(y) = \text{prox}_{r,1}(y) := \arg \min_x \frac{1}{2} \|x - y\|^2 + r(x)$$

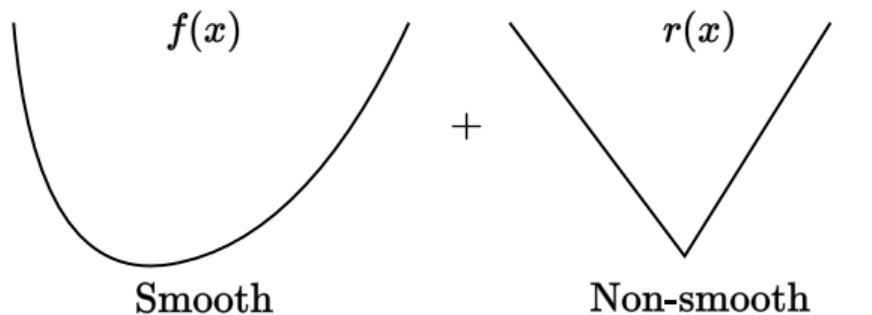
## Regularized / Composite Objectives

Many nonsmooth problems take the form

$$\min_{x \in \mathbb{R}^n} \varphi(x) = f(x) + r(x)$$

- Lasso, L1-LS, compressed sensing

$$f(x) = \frac{1}{2} \|Ax - b\|_2^2, r(x) = \lambda \|x\|_1$$



PROX-  
friendly

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Many nonsmooth problems take the form

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- **Lasso, L1-LS, compressed sensing**

$$f(x) = \frac{1}{2} \|Ax - b\|_2^2, r(x) = \lambda \|x\|_1$$

- **L1-Logistic regression, sparse LR**

$$f(x) = -y \log h(x) - (1-y) \log(1-h(x)), r(x) = \lambda \|x\|_1$$



## Proximal mapping intuition

Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

$$\begin{aligned} \min \varphi(x) &= f(x) + r(x) \\ 0 &\in \partial \varphi(x^*) \end{aligned}$$

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+

$x^*$

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$$x^* - \alpha \nabla f(x^*) \in (I + \alpha \partial r)(x^*)$$

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Optimality conditions:

$$(I + \alpha \partial r)^{-1} = \text{prox}_{\alpha r}$$

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$$\xrightarrow{x^* = (I + \alpha \partial r)^{-1}(x^* - \alpha \nabla f(x^*))}$$

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$$x^* \in \alpha \nabla f(x^*) + (I + \alpha \partial r)(x^*)$$

$$x^* - \alpha \nabla f(x^*) \in (I + \alpha \partial r)(x^*)$$

$$x^* = (I + \alpha \partial r)^{-1}(x^* - \alpha \nabla f(x^*))$$

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

Which leads to the proximal gradient method:

$$x_{k+1} = \text{prox}_{r,\alpha}(x_k - \alpha \nabla f(x_k))$$

And this method converges at a rate of  $\mathcal{O}(\frac{1}{k})!$



## Proximal mapping intuition

Optimality conditions:

$$0 \in \nabla f(x^*) + \partial r(x^*)$$

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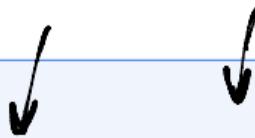
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 Another form of proximal operator



$$\text{prox}_{f,\alpha}(x_k) = \text{prox}_{\alpha f}(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ \alpha f(x) + \frac{1}{2} \|x - x_k\|_2^2 \right]$$

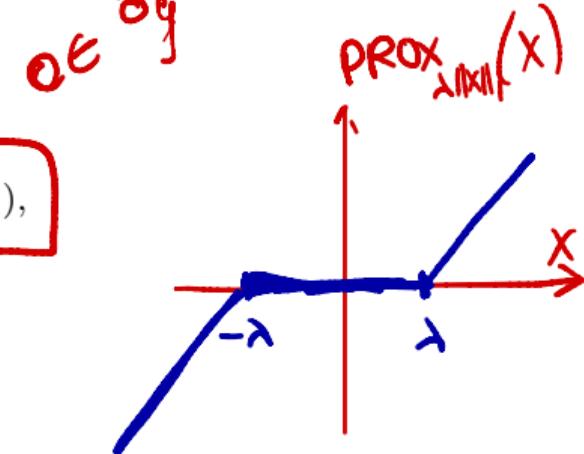
$$\text{prox}_f(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2} \|x - x_k\|_2^2 \right]$$

Proximal operators examples  $\text{PROX}_{\lambda\|x\|_1}(x) = \underset{y \in \mathbb{R}^n}{\operatorname{argmin}} \left[ \lambda \|y\|_1 + \frac{1}{2} \|x - y\|^2 \right]$

- $r(x) = \lambda \|x\|_1, \underline{\lambda > 0}$

$$[\text{prox}_r(x)]_i = [|x_i| - \lambda]_+ \cdot \text{sign}(x_i),$$

which is also known as soft-thresholding operator.



soft-thresholding

## Proximal operators examples

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## Proximal operators examples

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- $r(x) = \mathbb{I}_S(x).$

$$\text{prox}_r(x_k - \alpha \nabla f(x_k)) = \text{proj}_r(x_k - \alpha \nabla f(x_k))$$

$$\text{PROX}_r(x) =$$

$$= \underset{y \in \mathbb{R}^n}{\text{argmin}} \left[ r(y) + \frac{1}{2} \|x - y\|^2 \right] =$$

$$\underset{y \in S}{\text{argmin}} \|x - y\|^2$$

$$r = I \quad = \underset{y \in \mathbb{R}^n}{\text{argmin}} \left[ I_S(y) + \frac{1}{2} \|x - y\|^2 \right]$$

## Proximal operator properties

### Theorem

Let  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\text{prox}_r$  is defined. If there exists such an  $\hat{x} \in \mathbb{R}^n$  that  $r(\hat{x}) < +\infty$ . Then, the proximal operator is uniquely defined (i.e., it always returns a single unique value).

### Proof:

# Proximal operator properties

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The proximal operator returns the minimum of some optimization problem.

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It is strongly convex, meaning it has exactly one unique minimum (the existence of  $\hat{x}$  is necessary for  $r(\tilde{x}) + \frac{1}{2}\|x - \tilde{x}\|_2^2$  to take a finite value somewhere).

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- $\text{prox}_r(x) = y$ ,

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1. Let's establish the equivalence between the first and second conditions. The first condition can be rewritten as

$$y = \arg \min_{\tilde{x} \in \mathbb{R}^d} \left( r(\tilde{x}) + \frac{1}{2} \|x - \tilde{x}\|^2 \right).$$

From the optimality condition for the convex function  $r$ , this is equivalent to:

$$0 \in \partial \left( r(\tilde{x}) + \frac{1}{2} \|x - \tilde{x}\|^2 \right) \Big|_{\tilde{x}=y} = \partial r(y) + y - x.$$

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2. From the definition of the subdifferential, for any subgradient  $g \in \partial f(y)$  and for any  $z \in \mathbb{R}^d$ :

$$\langle g, z - y \rangle \leq r(z) - r(y).$$

In particular, this holds true for  $g = x - y$ . Conversely, it is also clear: for  $g = x - y$ , the above relationship holds, which means  $g \in \partial r(y)$ .

# Proximal operator properties

## Theorem

The operator  $\text{prox}_r(x)$  is firmly nonexpansive (FNE)

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2^2 \leq \langle \text{prox}_r(x) - \text{prox}_r(y), x - y \rangle$$

and nonexpansive:

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2 \leq \|x - y\|_2$$

## Proof

1. Let  $u = \text{prox}_r(x)$ , and  $v = \text{prox}_r(y)$ . Then, from the previous property:

$$\langle x - u, z_1 - u \rangle \leq r(z_1) - r(u)$$

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- Substitute  $z_1 = v$  and  $z_2 = u$ . Summing up, we get:

$$\langle x - u, v - u \rangle + \langle y - v, u - v \rangle \leq 0,$$

$$\langle x - y, v - u \rangle + \|v - u\|_2^2 \leq 0.$$

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- Which is exactly what we need to prove after substitution of  $u, v$ .

$$\langle x - u, z_1 - u \rangle \leq r(z_1) - r(u) \quad \|u - v\|_2^2 \leq \langle x - y, u - v \rangle$$

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- The last point comes from simple Cauchy-Bunyakovsky-Schwarz for the last inequality.

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## Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be convex functions. Additionally, assume that  $f$  is continuously differentiable and  $L$ -smooth, and for  $r$ ,  $\text{prox}_r$  is defined. Then,  $x^*$  is a solution to the composite optimization problem if and only if, for any  $\alpha > 0$ , it satisfies:

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## Proof

1. Optimality conditions:

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### 2. Recall from the previous lemma:

$$\text{prox}_r(x) = y \Leftrightarrow x - y \in \partial r(y)$$

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### 3. Finally,

$$x^* = \text{prox}_{\alpha r}(x^* - \alpha \nabla f(x^*)) = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

## Convergence tools

### Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an  $L$ -smooth convex function. Then, for any  $x, y \in \mathbb{R}^n$ , the following inequality holds:

$$f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y) \text{ or, equivalently,}$$

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### Proof

1. To prove this, we'll consider another function  $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$ . It is obviously a convex function (as a sum of convex functions). And it is easy to verify, that it is an  $L$ -smooth function by definition, since  $\nabla \varphi(y) = \nabla f(y) - \nabla f(x)$  and  $\|\nabla \varphi(y_1) - \nabla \varphi(y_2)\| = \|\nabla f(y_1) - \nabla f(y_2)\| \leq L\|y_1 - y_2\|$ .

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$$x := y, y := y - \frac{1}{L} \nabla \varphi(y) \quad \varphi\left(y - \frac{1}{L} \nabla \varphi(y)\right) \leq \varphi(y) + \left\langle \nabla \varphi(y), -\frac{1}{L} \nabla \varphi(y) \right\rangle + \frac{1}{2L} \|\nabla \varphi(y)\|_2^2$$

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2. Now let's consider the smoothness parabolic property for the  $\varphi(y)$  function:

$$\varphi(y) \leq \varphi(x) + \langle \nabla \varphi(x), y - x \rangle + \frac{L}{2} \|y - x\|_2^2$$

$$x := y, y := y - \frac{1}{L} \nabla \varphi(y) \quad \varphi\left(y - \frac{1}{L} \nabla \varphi(y)\right) \leq \varphi(y) + \left\langle \nabla \varphi(y), -\frac{1}{L} \nabla \varphi(y) \right\rangle + \frac{1}{2L} \|\nabla \varphi(y)\|_2^2$$

$$\varphi\left(y - \frac{1}{L} \nabla \varphi(y)\right) \leq \varphi(y) - \frac{1}{2L} \|\nabla \varphi(y)\|_2^2$$

3. From the first order optimality conditions for the convex function  $\nabla \varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any  $x$ , the minimum of the function  $\varphi(y)$  is at the point  $y = x$ . Therefore:

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## Convergence tools

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The lemma has been proved. From the first view it does not make a lot of geometrical sense, but we will use it as a convenient tool to bound the difference between gradients.

# Convergence tools

## Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\mathbb{R}^n$ . Then, the function  $f$  is  $\mu$ -strongly convex if and only if for any  $x, y \in \mathbb{R}^d$  the following holds:

$$\text{Strongly convex case } \mu > 0 \quad \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$$

$$\text{Convex case } \mu = 0 \quad \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$$

## Proof

1. We will only give the proof for the strongly convex case, the convex one follows from it with setting  $\mu = 0$ . We start from necessity. For the strongly convex function

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|x - y\|_2^2$$

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|_2^2$$

$$\text{sum} \quad \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$$

## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

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$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle = \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle$$

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Thus, we have a strong convexity criterion satisfied

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$$\text{switch } x \text{ and } y \quad - \langle \nabla f(x), x - y \rangle \leq - \left( f(x) - f(y) + \frac{\mu}{2} \|x - y\|_2^2 \right)$$

# Convergence

## Theorem

Consider the proximal gradient method

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

For the criterion  $\varphi(x) = f(x) + r(x)$ , we assume:

- $f$  is convex, differentiable,  $\text{dom}(f) = \mathbb{R}^n$ , and  $\nabla f$  is Lipschitz continuous with constant  $L > 0$ .
- $r$  is convex, and  $\text{prox}_{\alpha r}(x_k) = \arg \min_{x \in \mathbb{R}^n} [\alpha r(x) + \frac{1}{2} \|x - x_k\|_2^2]$  can be evaluated.

Proximal gradient descent with fixed step size  $\alpha = 1/L$  satisfies

$$\varphi(x_k) - \varphi^* \leq \frac{L\|x_0 - x^*\|^2}{2k},$$

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Proximal gradient descent has a convergence rate of  $O(1/k)$  or  $O(1/\varepsilon)$ . This matches the gradient descent rate!  
(But remember the proximal operation cost)

# Convergence

## Proof

1. Let's introduce the **gradient mapping**, denoted as  $G_\alpha(x)$ , acts as a “gradient-like object”:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

$$x_{k+1} = x_k - \alpha G_\alpha(x_k).$$

where  $G_\alpha(x)$  is:

$$G_\alpha(x) = \frac{1}{\alpha} (x - \text{prox}_{\alpha r}(x - \alpha \nabla f(x)))$$

Observe that  $G_\alpha(x) = 0$  if and only if  $x$  is optimal. Therefore,  $G_\alpha$  is analogous to  $\nabla f$ . If  $x$  is locally optimal, then  $G_\alpha(x) = 0$  even for nonconvex  $f$ . This demonstrates that the proximal gradient method effectively combines gradient descent on  $f$  with the proximal operator of  $r$ , allowing it to handle non-differentiable components effectively.

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# Convergence

## Proof

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7. Now it is easy to verify, that when  $x = x_k$  we have monotonic decrease for the proximal gradient algorithm:

$$\varphi(x_{k+1}) \leq \varphi(x_k) - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2$$

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## Convergence

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$$\begin{aligned}\varphi(x_{k+1}) &\leq \varphi(x^*) + \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2 \\ \varphi(x_{k+1}) - \varphi(x^*) &\leq \langle G_\alpha(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2 \\ &\leq \frac{1}{2\alpha} [2\langle \alpha G_\alpha(x_k), x_k - x^* \rangle - \|\alpha G_\alpha(x_k)\|_2^2] \\ &\leq \frac{1}{2\alpha} [2\langle \alpha G_\alpha(x_k), x_k - x^* \rangle - \|\alpha G_\alpha(x_k)\|_2^2 - \|x_k - x^*\|_2^2 + \|x_k - x^*\|_2^2] \\ &\leq \frac{1}{2\alpha} [-\|x_k - x^* - \alpha G_\alpha(x_k)\|_2^2 + \|x_k - x^*\|_2^2] \\ &\leq \frac{1}{2\alpha} [\|x_k - x^*\|_2^2 - \|x_{k+1} - x^*\|_2^2]\end{aligned}$$

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9. Now we write the bound above for all iterations  $i \in 0, k - 1$  and sum them:

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# Convergence

## Theorem

Consider the proximal gradient method

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

For the criterion  $\varphi(x) = f(x) + r(x)$ , we assume:

- $f$  is  $\mu$ -strongly convex, differentiable,  $\text{dom}(f) = \mathbb{R}^n$ , and  $\nabla f$  is Lipschitz continuous with constant  $L > 0$ .
- $r$  is convex, and  $\text{prox}_{\alpha r}(x_k) = \arg \min_{x \in \mathbb{R}^n} [\alpha r(x) + \frac{1}{2} \|x - x_k\|_2^2]$  can be evaluated.

Proximal gradient descent with fixed step size  $\alpha \leq 1/L$  satisfies

$$\|x_{k+1} - x^*\|_2^2 \leq (1 - \alpha\mu)^k \|x_0 - x^*\|_2^2$$

This is exactly gradient descent convergence rate. Note, that the original problem is even non-smooth!

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$$\text{smoothness } \|\nabla f(x_k) - \nabla f(x^*)\|_2^2 \leq 2L(f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle)$$

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4. Due to convexity of  $f$ :  $f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle \geq 0$ . Therefore, if we use  $\alpha \leq \frac{1}{L}$ :

$$\|x_{k+1} - x^*\|_2^2 \leq (1 - \alpha\mu) \|x_k - x^*\|^2,$$

which is exactly linear convergence of the method with up to  $1 - \frac{\mu}{L}$  convergence rate.

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Let  $x_0 = y_0 \in \text{dom}(r)$ . For  $k \geq 1$ :

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- Uses extra “memory” for interpolation
  - Same computational cost as ordinary prox-grad
  - Convergence rate theoretically optimal

Example: ISTA = PROX grad method  
+  $\ell_1$

### Iterative Shrinkage-Thresholding Algorithm (ISTA)

ISTA is a popular method for solving optimization problems involving L1 regularization, such as Lasso. It combines gradient descent with a shrinkage operator to handle the non-smooth L1 penalty effectively.

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- **Application:**

- Efficient for sparse signal recovery, image processing, and compressed sensing.

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- **Application:**

- Especially useful for large-scale problems in machine learning and signal processing where the L1 penalty induces sparsity.

## Example: Matrix Completion

### Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

- **Problem Formulation:**

$$\min_X \frac{1}{2} \|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda \|X\|_*,$$

where  $P_\Omega$  projects onto the observed set  $\Omega$ , and  $\|\cdot\|_*$  denotes the nuclear norm.

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- Widely used in recommender systems, image recovery, and other domains where data is naturally matrix-formed but partially observed.

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If we allow the proximal operator to be inexact (numerically), then it is true that we can solve any nonsmooth optimization problem. But this is not better from the point of view of theory than solving the problem by subgradient descent, because some auxiliary method (for example, the same subgradient descent) is used to solve the proximal subproblem.

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- Further reading: Proximal operator splitting, Douglas-Rachford splitting, Best approximation problem, Three operator splitting.