Convexity: convex sets, convex functions. Polyak - Lojasiewicz Condition.

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Affine set

Suppose x_1, x_2 are two points in $\mathbb{R}^{\ltimes}$. Then the line passing through them is defined as follows:

$$x = \theta x_1 + (1-\theta)x_2, \theta \in \mathbb{R}$$

The set A is called **affine** if for any x_1, x_2 from A the line passing through them also lies in A, i.e.

$$\forall \theta \in \mathbb{R}, \forall x_1, x_2 \in A : \theta x_1 + (1-\theta)x_2 \in A$$

Example

- $\mathbb{R}^n$ is an affine set.

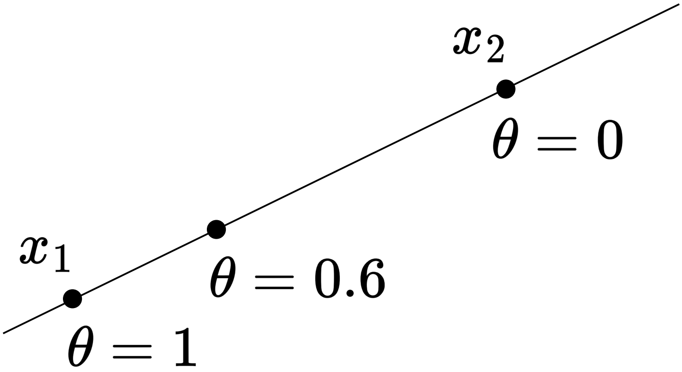

Figure 1: Illustration of a line between two vectors x_1 and x_2

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Example

- $\mathbb{R}^n$ is an affine set.
- The set of solutions $\{x \mid \mathbf{A}x = \mathbf{b}\}$ is also an affine set.

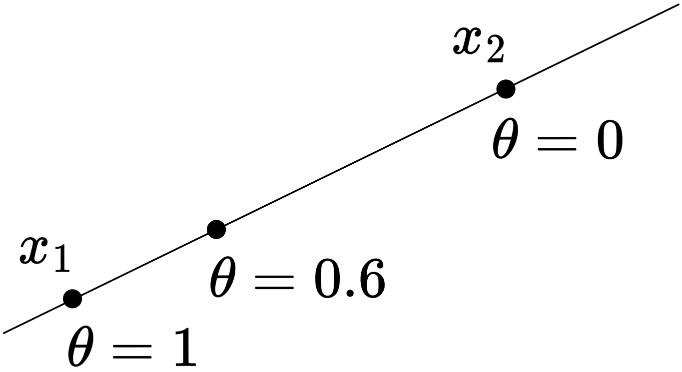

Figure 1: Illustration of a line between two vectors x_1 and x_2

Cone

A non-empty set S is called a cone, if:

$$\forall x \in S,\ \theta \geq 0 \quad \to \quad \theta x \in S$$

For any point in cone it also contains beam through this point.

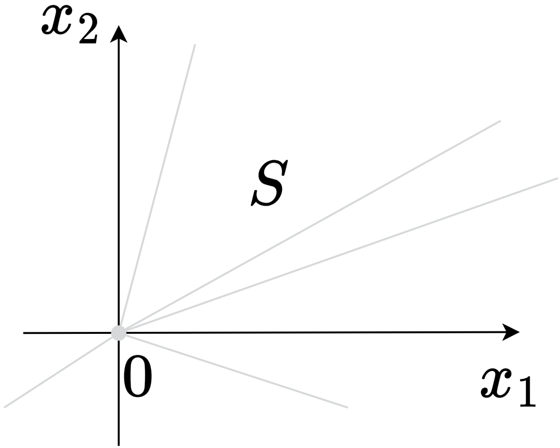

Figure 2: Illustration of a cone

Convex cone

The set S is called a convex cone, if:

$$\forall x_1, x_2 \in S,\ \theta_1, \theta_2 \geq 0 \quad \to \quad \theta_1 x_1 + \theta_2 x_2 \in S$$

Convex cone is just like cone, but it is also convex.

Example

- $\mathbb{R}^n$

Convex cone: set that contains all conic combinations of points in the set

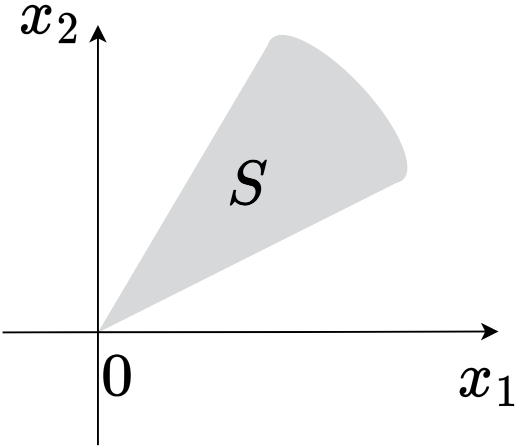

Figure 3: Illustration of a convex cone

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Example

- $\mathbb{R}^n$
- Affine sets, containing 0

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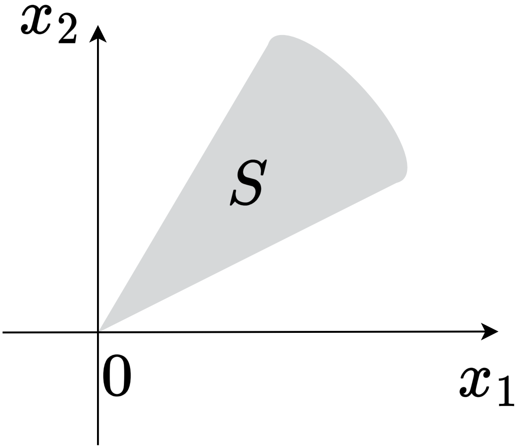

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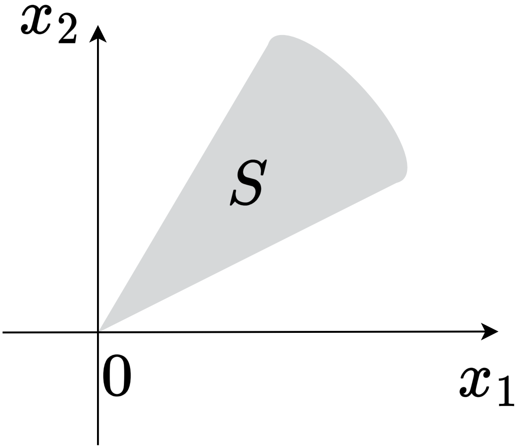

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Example

- $\mathbb{R}^n$
- Affine sets, containing 0
- Ray
- $\mathbf{S}^n_+$ - the set of symmetric positive semi-definite matrices

Convex cone: set that contains all conic combinations of points in the set

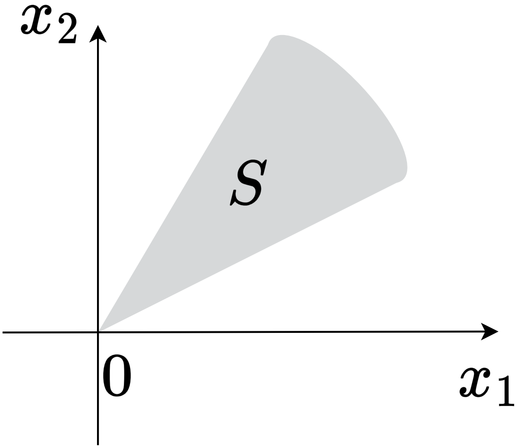


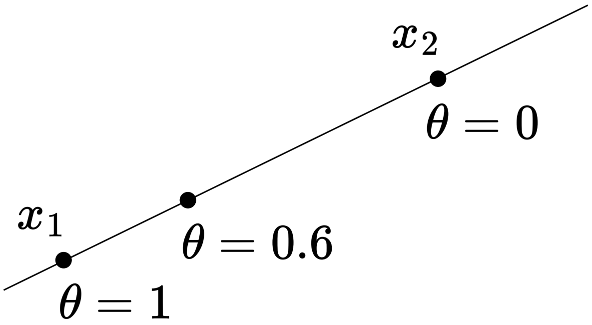
Figure 3: Illustration of a convex cone

Line segment

Suppose x_1, x_2 are two points in $\mathbb{R}^n$.
Then the line segment between them is defined as follows:

$$x = \theta x_1 + (1 - \theta) x_2, \; \theta \in [0, 1]$$

Convex set contains line segment between any two points in the set.



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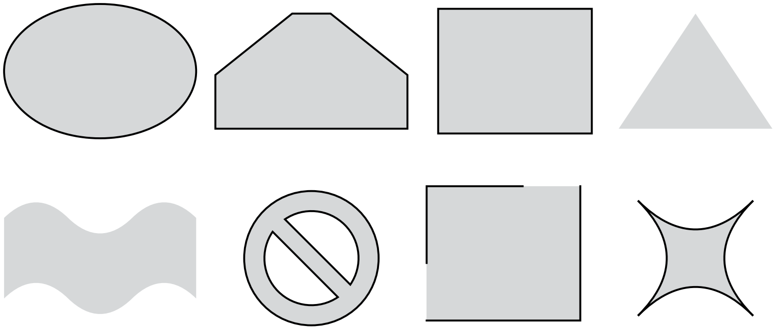

Figure 5: Top: examples of convex sets. Bottom: examples of non-convex sets.

Example

An empty set and a set from a single vector are convex by definition.

Example

Any affine set, a ray, a line segment - they all are convex sets.

Convex combination

Let $x_1, x_2, \ldots, x_k \in S$, then the point $\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k$ is called the convex combination of points $x_1, x_2, \ldots, x_k$ if $\sum\limits_{i=1}^{k} \theta_i = 1, \ \theta_i \geq 0$.

Convex hull

The set of all convex combinations of points from S is called the convex hull of the set S.

$$\mathbf{conv}(S) = \left\{ \sum_{i=1}^{k}\theta_i x_i \mid x_i \in S, \sum_{i=1}^{k}\theta_i = 1,\ \theta_i \ge 0 \right\}$$

- The set $\mathbf{conv}(S)$ is the smallest convex set containing S.

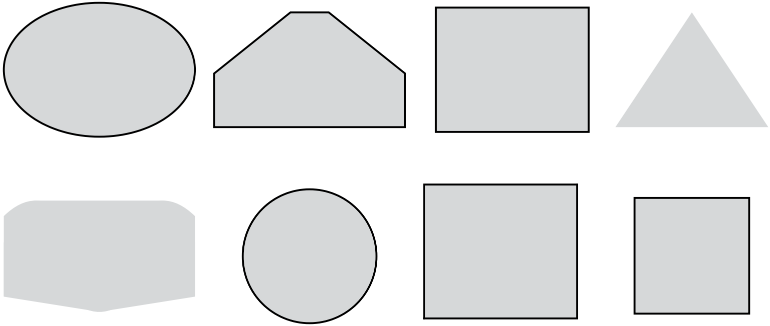

Figure 6: Top: convex hulls of the convex sets. Bottom: convex hull of the non-convex sets.

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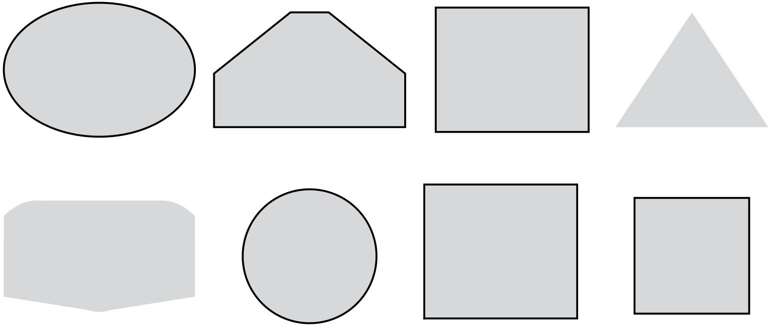
- The set $\mathbf{conv}(S)$ is the smallest convex set containing S.
- The set S is convex if and only if $S = \mathbf{conv}(S)$.



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Minkowski addition

The Minkowski sum of two sets of vectors S_1 and S_2 in Euclidean space is formed by adding each vector in S_1 to each vector in S_2.

$$S_1 + S_2 = \{\mathbf{s_1} + \mathbf{s_2} \mid \mathbf{s_1} \in S_1,\ \mathbf{s_2} \in S_2\}$$

Similarly, one can define a linear combination of the sets.

Example

We will work in the $\mathbb{R}^2$ space. Let's define:

$$S_1 := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$$

This is a unit circle centered at the origin. And:

$$S_2 := \{x \in \mathbb{R}^2 : -4 \leq x_1 \leq -1, -3 \leq x_2 \leq -1\}$$

This represents a rectangle. The sum of the sets S_1 and S_2 will form an enlarged rectangle S_2 with rounded corners. The resulting set will be convex.

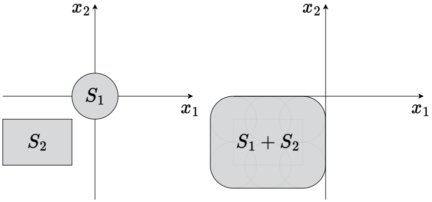

Figure 7: $S = S_1 + S_2$

Finding convexity

In practice, it is very important to understand whether a specific set is convex or not. Two approaches are used for this depending on the context.

- By definition.

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- By definition.
- Show that S is derived from simple convex sets using operations that preserve convexity.

Finding convexity by definition

$$x_1, x_2 \in S,\ 0 \leq \theta \leq 1 \quad \rightarrow \quad \theta x_1 + (1-\theta)x_2 \in S$$

Example

Prove, that ball in $\mathbb{R}^n$ (i.e. the following set $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\| \leq r\}$) - is convex.

Exercises

Which of the sets are convex:

- Stripe, $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$

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- A set of points whose distance to a given point does not exceed a certain part of the distance to another given point is $\{x \in \mathbb{R}^n \mid \|x - a\|_2 \leq \theta\|x - b\|_2, a, b \in \mathbb{R}^n, 0 \leq 1\}$

Operations, that preserve convexity

The linear combination of convex sets is convex Let there be 2 convex sets S_x, S_y, let the set

$$S = \{s \mid s = c_1 x + c_2 y,\ x \in S_x,\ y \in S_y,\ c_1, c_2 \in \mathbb{R}\}$$

Take two points from S: $s_1 = c_1 x_1 + c_2 y_1, s_2 = c_1 x_2 + c_2 y_2$ and prove that the segment between them $\theta s_1 + (1 - \theta) s_2, \theta \in [0, 1]$ also belongs to S

$$\theta s_1 + (1 - \theta) s_2$$

$$\theta(c_1 x_1 + c_2 y_1) + (1 - \theta)(c_1 x_2 + c_2 y_2)$$

$$c_1(\theta x_1 + (1 - \theta) x_2) + c_2(\theta y_1 + (1 - \theta) y_2)$$

$$c_1 x + c_2 y \in S$$

The intersection of any (!) number of convex sets is convex

If the desired intersection is empty or contains one point, the property is proved by definition. Otherwise, take 2 points and a segment between them. These points must lie in all intersecting sets, and since they are all convex, the segment between them lies in all sets and, therefore, in their intersection.

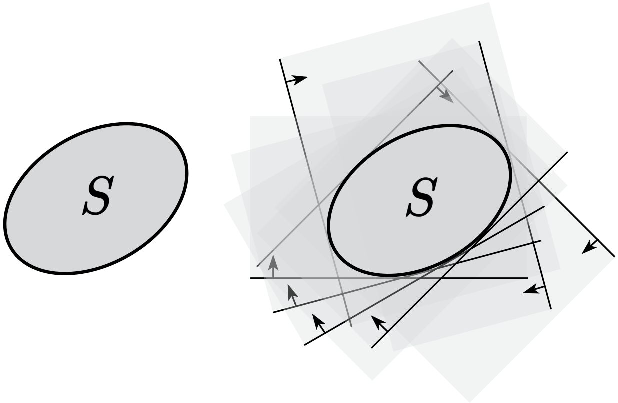

Figure 8: Intersection of halfplanes

The image of the convex set under affine mapping is convex

$$S \subseteq \mathbb{R}^n \text{ convex } \quad \to \quad f(S) = \{f(x) \mid x \in S\} \text{ convex } \quad (f(x) = \mathbf{A}x + \mathbf{b})$$

Examples of affine functions: extension, projection, transposition, set of solutions of linear matrix inequality $\{x \mid x_1 A_1 + \ldots + x_m A_m \preceq B\}$. Here $A_i, B \in \mathbf{S}^p$ are symmetric matrices $p \times p$.

Note also that the prototype of the convex set under affine mapping is also convex.

$$S \subseteq \mathbb{R}^m \text{ convex } \to f^{-1}(S) = \{x \in \mathbb{R}^n \mid f(x) \in S\} \text{ convex } \; (f(x) = \mathbf{A}x + \mathbf{b})$$

Example

Let $x \in \mathbb{R}$ is a random variable with a given probability distribution of $\mathbb{P}(x = a_i) = p_i$, where $i = 1, \ldots, n$, and $a_1 < \ldots < a_n$. It is said that the probability vector of outcomes of $p \in \mathbb{R}^n$ belongs to the probabilistic simplex, i.e.

$$P = \{p \mid \mathbf{1}^T p = 1, p \succeq 0\} = \{p \mid p_1 + \ldots + p_n = 1, p_i \geq 0\}.$$

Determine if the following sets of p are convex:

- $\mathbb{P}(x > \alpha) \leq \beta$

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Determine if the following sets of p are convex:

- $\mathbb{P}(x > \alpha) \le \beta$
- $\mathbb{E}|x^{201}| \le \alpha \mathbb{E}|x|$
- $\mathbb{E}|x^2| \ge \alpha \mathbb{V}x \ge \alpha$

Jensen's inequality

The function $f(x)$, **which is defined on the convex set** $S \subseteq \mathbb{R}^n$, is called **convex** on S, if:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$.
If the above inequality holds as strict inequality $x_1 \neq x_2$ and $0 < \lambda < 1$, then the function is called strictly convex on S.

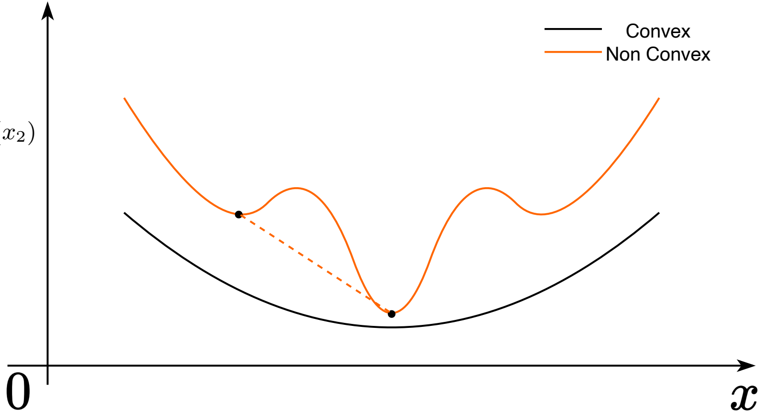

Figure 9: Difference between convex and non-convex function

Jensen's inequality

Theorem

Let $f(x)$ be a convex function on a convex set $X \subseteq \mathbb{R}^n$ and let $x_i \in X, 1 \leq i \leq m$, be arbitrary points from X. Then

$$f\left(\sum_{i=1}^{m} \lambda_i x_i\right) \leq \sum_{i=1}^{m} \lambda_i f(x_i)$$

for any $\lambda = [\lambda_1, \ldots, \lambda_m] \in \Delta_m$ - probability simplex.

Proof

1. First, note that the point $\sum_{i=1}^{m} \lambda_i x_i$ as a convex combination of points from the convex set X belongs to X.

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Proof

1. First, note that the point $\sum_{i=1}^{m} \lambda_i x_i$ as a convex combination of points from the convex set X belongs to X.
2. We will prove this by induction. For $m = 1$, the statement is obviously true, and for $m = 2$, it follows from the definition of a convex function.

Jensen's inequality

3. Assume it is true for all m up to $m = k$, and we will prove it for $m = k + 1$. Let $\lambda \in \Delta k + 1$ and

$$x = \sum_{i=1}^{k+1} \lambda_i x_i = \sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_{k+1}.$$

Assuming $0 < \lambda_{k+1} < 1$, as otherwise, it reduces to previously considered cases, we have

$$x = \lambda_{k+1} x_{k+1} + (1 - \lambda_{k+1}) \bar{x},$$

where $\bar{x} = \sum_{i=1}^{k} \gamma_i x_i$ and $\gamma_i = \frac{\lambda_i}{1 - \lambda_{k+1}} \geq 0, 1 \leq i \leq k$.

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4. Since $\lambda \in \Delta_{k+1}$, then $\gamma = [\gamma_1, \ldots, \gamma_k] \in \Delta_k$. Therefore $\bar{x} \in X$ and by the convexity of $f(x)$ and the induction hypothesis:

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = f\left(\lambda_{k+1} x_{k+1} + (1 - \lambda_{k+1})\bar{x}\right) \leq \lambda_{k+1} f(x_{k+1}) + (1 - \lambda_{k+1}) f(\bar{x}) \leq \sum_{i=1}^{k+1} \lambda_i f(x_i)$$

Thus, initial inequality is satisfied for $m = k+1$ as well.

Examples of convex functions

- $f(x) = x^p, \; p > 1, \; x \in \mathbb{R}_+$

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- $f(x) = -\ln x, \; x \in \mathbb{R}_{++}$
- $f(x) = x \ln x, \; x \in \mathbb{R}_{++}$
- The sum of the largest k coordinates $f(x) = x_{(1)} + \ldots + x_{(k)}, \; x \in \mathbb{R}^n$

Examples of convex functions

- $f(x) = x^p,\ p > 1,\ x \in \mathbb{R}_+$
- $f(x) = \|x\|^p,\ p > 1, x \in \mathbb{R}^n$
- $f(x) = e^{cx},\ c \in \mathbb{R}, x \in \mathbb{R}$
- $f(x) = -\ln x,\ x \in \mathbb{R}_{++}$
- $f(x) = x \ln x,\ x \in \mathbb{R}_{++}$
- The sum of the largest k coordinates $f(x) = x_{(1)} + \ldots + x_{(k)},\ x \in \mathbb{R}^n$
- $f(X) = \lambda_{max}(X),\ X = X^T$

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- The sum of the largest k coordinates $f(x) = x_{(1)} + \ldots + x_{(k)},\ x \in \mathbb{R}^n$
- $f(X) = \lambda_{max}(X),\ X = X^T$
- $f(X) = -\log\det X,\ X \in S^n_{++}$

Epigraph

For the function $f(x)$, defined on $S \subseteq \mathbb{R}^n$, the following set:

$$\text{epi } f = \{[x, \mu] \in S \times \mathbb{R} : f(x) \leq \mu\}$$

is called **epigraph** of the function $f(x)$.

Convexity of the epigraph is the convexity of the function

For a function $f(x)$, defined on a convex set X, to be convex on X, it is necessary and sufficient that the epigraph of f is a convex set.

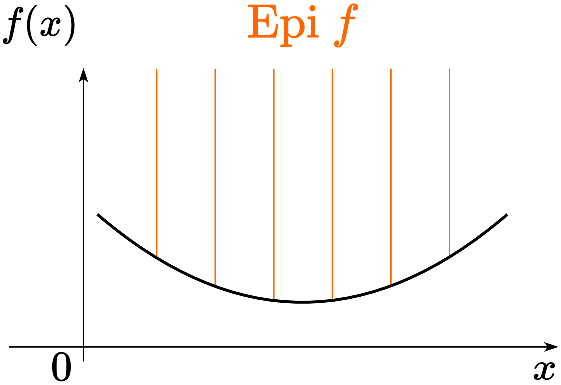

Figure 10: Epigraph of a function

Convexity of the epigraph is the convexity of the function

1. **Necessity**: Assume $f(x)$ is convex on X. Take any two arbitrary points $[x_1, \mu_1] \in \text{epi} f$ and $[x_2, \mu_2] \in \text{epi} f$. Also take $0 \leq \lambda \leq 1$ and denote $x_\lambda = \lambda x_1 + (1-\lambda)x_2, \mu_\lambda = \lambda\mu_1 + (1-\lambda)\mu_2$. Then,

$$
\lambda \begin{bmatrix} x_1 \\ \mu_1 \end{bmatrix} + (1-\lambda) \begin{bmatrix} x_2 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} x_\lambda \\ \mu_\lambda \end{bmatrix}.
$$

From the convexity of the set X, it follows that $x_\lambda \in X$. Moreover, since $f(x)$ is a convex function,

$$
f(x_\lambda) \leq \lambda f(x_1) + (1-\lambda)f(x_2) \leq \lambda\mu_1 + (1-\lambda)\mu_2 = \mu_\lambda
$$

Inequality above indicates that $\begin{bmatrix} x_\lambda \\ \mu_\lambda \end{bmatrix} \in \text{epi} f$. Thus, the epigraph of f is a convex set.

Convexity of the epigraph is the convexity of the function

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Inequality above indicates that $\begin{bmatrix} x_\lambda \\ \mu_\lambda \end{bmatrix} \in \text{epi} f$. Thus, the epigraph of f is a convex set.

2. **Sufficiency**: Assume the epigraph of f, epi$f$, is a convex set. Then, from the membership of the points $[x_1, \mu_1]$ and $[x_2, \mu_2]$ in the epigraph of f, it follows that

$$\begin{bmatrix} x_\lambda \\ \mu_\lambda \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \mu_1 \end{bmatrix} + (1-\lambda) \begin{bmatrix} x_2 \\ \mu_2 \end{bmatrix} \in \text{epi} f$$

for any $0 \leq \lambda \leq 1$, i.e., $f(x_\lambda) \leq \mu_\lambda = \lambda\mu_1 + (1-\lambda)\mu_2$. But this is true for all $\mu_1 \geq f(x_1)$ and $\mu_2 \geq f(x_2)$, particularly when $\mu_1 = f(x_1)$ and $\mu_2 = f(x_2)$. Hence we arrive at the inequality

Sublevel set

For the function $f(x)$, defined on $S \subseteq \mathbb{R}^n$, the following set:

$$\mathcal{L}_\beta = \{x \in S : f(x) \leq \beta\}$$

is called **sublevel set** or Lebesgue set of the function $f(x)$.

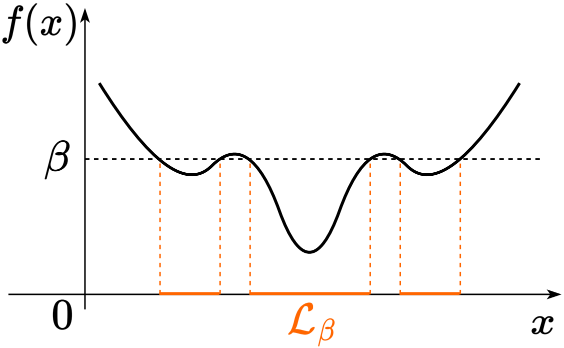

Figure 11: Sublevel set of a function with respect to level β

First-order differential criterion of convexity

The differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is convex if and only if $\forall x, y \in S$:

$$f(y) \geq f(x) + \nabla f^T(x)(y - x)$$

Let $y = x + \Delta x$, then the criterion will become more tractable:

$$f(x + \Delta x) \geq f(x) + \nabla f^T(x)\Delta x$$

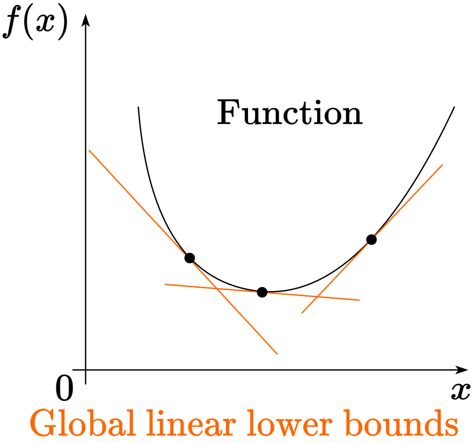

Figure 12: Convex function is greater or equal than Taylor linear approximation at any point

Second-order differential criterion of convexity

Twice differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is convex if and only if $\forall x \in \mathbf{int}(S) \neq \emptyset$:

$$\nabla^2 f(x) \succeq 0$$

In other words, $\forall y \in \mathbb{R}^n$:

$$\langle y, \nabla^2 f(x) y \rangle \geq 0$$

Connection with epigraph

The function is convex if and only if its epigraph is a convex set.

Example

Let a norm $\|\cdot\|$ be defined in the space U. Consider the set:

$$K := \{(x,t) \in U \times \mathbb{R}^+ : \|x\| \leq t\}$$

which represents the epigraph of the function $x \mapsto \|x\|$. This set is called the cone norm. According to the statement above, the set K is convex.
In the case where $U = \mathbb{R}^n$ and $\|x\| = \|x\|_2$ (Euclidean norm), the abstract set K transitions into the set:

$$\{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+ : \|x\|_2 \leq t\}$$

Connection with sublevel set

If $f(x)$ - is a convex function defined on the convex set $S \subseteq \mathbb{R}^n$, then for any β sublevel set $\mathcal{L}_\beta$ is convex.

The function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is closed if and only if for any β sublevel set $\mathcal{L}_\beta$ is closed.

Reduction to a line

$f : S \to \mathbb{R}$ is convex if and only if S is a convex set and the function $g(t) = f(x + tv)$ defined on $\{t \mid x + tv \in S\}$ is convex for any $x \in S, v \in \mathbb{R}^n$, which allows checking convexity of the scalar function to establish convexity of the vector function.

Strong convexity

$f(x)$, **defined on the convex set** $S \subseteq \mathbb{R}^n$, is called μ-strongly convex (strongly convex) on S, if:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) - \frac{\mu}{2}\lambda(1-\lambda)\|x_1 - x_2\|^2$$

for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$ for some $\mu > 0$.

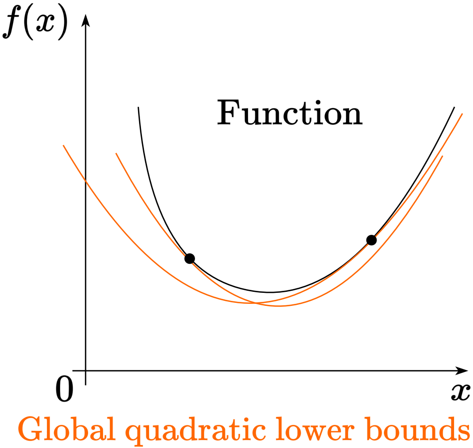

Figure 13: Strongly convex function is greater or equal than Taylor quadratic approximation at any point

First-order differential criterion of strong convexity

Differentiable $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is μ-strongly convex if and only if $\forall x, y \in S$:

$$f(y) \geq f(x) + \nabla f^T(x)(y - x) + \frac{\mu}{2}\|y - x\|^2$$

Let $y = x + \Delta x$, then the criterion will become more tractable:

$$f(x + \Delta x) \geq f(x) + \nabla f^T(x)\Delta x + \frac{\mu}{2}\|\Delta x\|^2$$

Theorem

Let $f(x)$ be a differentiable function on a convex set $X \subseteq \mathbb{R}^n$. Then $f(x)$ is strongly convex on X with a constant $\mu > 0$ if and only if

$$f(x) - f(x_0) \geq \langle \nabla f(x_0), x - x_0 \rangle + \frac{\mu}{2}\|x - x_0\|^2$$

for all $x, x_0 \in X$.

Proof of first-order differential criterion of strong convexity

Necessity: Let $0 < \lambda \leq 1$. According to the definition of a strongly convex function,

$$
f(\lambda x + (1-\lambda)x_0) \leq \lambda f(x) + (1-\lambda) f(x_0) - \frac{\mu}{2}\lambda(1-\lambda)\|x - x_0\|^2
$$

or equivalently,

$$
f(x) - f(x_0) - \frac{\mu}{2}(1-\lambda)\|x - x_0\|^2 \geq \frac{1}{\lambda}[f(\lambda x + (1-\lambda)x_0) - f(x_0)] =
$$

$$
= \frac{1}{\lambda}[f(x_0 + \lambda(x - x_0)) - f(x_0)] = \frac{1}{\lambda}[\lambda\langle \nabla f(x_0), x - x_0\rangle + o(\lambda)] =
$$

$$
= \langle \nabla f(x_0), x - x_0\rangle + \frac{o(\lambda)}{\lambda}.
$$

Thus, taking the limit as $\lambda \downarrow 0$, we arrive at the initial statement.

Proof of first-order differential criterion of strong convexity

Sufficiency: Assume the inequality in the theorem is satisfied for all $x, x_0 \in X$. Take $x_0 = \lambda x_1 + (1-\lambda)x_2$, where $x_1, x_2 \in X$, $0 \leq \lambda \leq 1$. According to the inequality, the following inequalities hold:

$$
f(x_1) - f(x_0) \geq \langle \nabla f(x_0), x_1 - x_0 \rangle + \frac{\mu}{2}\|x_1 - x_0\|^2,
$$

$$
f(x_2) - f(x_0) \geq \langle \nabla f(x_0), x_2 - x_0 \rangle + \frac{\mu}{2}\|x_2 - x_0\|^2.
$$

Multiplying the first inequality by λ and the second by $1-\lambda$ and adding them, considering that

$$
x_1 - x_0 = (1-\lambda)(x_1 - x_2), \quad x_2 - x_0 = \lambda(x_2 - x_1),
$$

and $\lambda(1-\lambda)^2 + \lambda^2(1-\lambda) = \lambda(1-\lambda)$, we get

$$
\begin{aligned}
\lambda f(x_1) + (1-\lambda) f(x_2) - f(x_0) - \frac{\mu}{2}\lambda(1-\lambda)\|x_1 - x_2\|^2 \geq \\
\langle \nabla f(x_0), \lambda x_1 + (1-\lambda)x_2 - x_0 \rangle = 0.
\end{aligned}
$$

Thus, inequality from the definition of a strongly convex function is satisfied. It is important to mention, that $\mu = 0$ stands for the convex case and corresponding differential criterion.

Second-order differential criterion of strong convexity

Twice differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is called μ-strongly convex if and only if $\forall x \in \mathbf{int}(S) \neq \emptyset$:

$$\nabla^2 f(x) \succeq \mu I$$

In other words:

$$\langle y, \nabla^2 f(x) y \rangle \geq \mu \|y\|^2$$

Theorem

Let $X \subseteq \mathbb{R}^n$ be a convex set, with $\text{int}X \neq \emptyset$. Furthermore, let $f(x)$ be a twice continuously differentiable function on X. Then $f(x)$ is strongly convex on X with a constant $\mu > 0$ if and only if

$$\langle y, \nabla^2 f(x) y \rangle \geq \mu \|y\|^2$$

for all $x \in X$ and $y \in \mathbb{R}^n$.

Proof of second-order differential criterion of strong convexity

The target inequality is trivial when $y = \mathbf{0}_n$, hence we assume $y \neq \mathbf{0}_n$.

Necessity: Assume initially that x is an interior point of X. Then $x + \alpha y \in X$ for all $y \in \mathbb{R}^n$ and sufficiently small α. Since $f(x)$ is twice differentiable,

$$f(x + \alpha y) = f(x) + \alpha \langle \nabla f(x), y \rangle + \frac{\alpha^2}{2} \langle y, \nabla^2 f(x) y \rangle + o(\alpha^2).$$

Based on the first order criterion of strong convexity, we have

$$\frac{\alpha^2}{2} \langle y, \nabla^2 f(x) y \rangle + o(\alpha^2) = f(x + \alpha y) - f(x) - \alpha \langle \nabla f(x), y \rangle \geq \frac{\mu}{2} \alpha^2 \|y\|^2.$$

This inequality reduces to the target inequality after dividing both sides by α^2 and taking the limit as $\alpha \downarrow 0$.

If $x \in X$ but $x \notin \text{int}X$, consider a sequence $\{x_k\}$ such that $x_k \in \text{int}X$ and $x_k \to x$ as $k \to \infty$. Then, we arrive at the target inequality after taking the limit.

Proof of second-order differential criterion of strong convexity

Sufficiency: Using Taylor's formula with the Lagrange remainder and the target inequality, we obtain for $x + y \in X$:

$$f(x+y) - f(x) - \langle \nabla f(x), y \rangle = \frac{1}{2} \langle y, \nabla^2 f(x + \alpha y) y \rangle \geq \frac{\mu}{2} \|y\|^2,$$

where $0 \leq \alpha \leq 1$. Therefore,

$$f(x+y) - f(x) \geq \langle \nabla f(x), y \rangle + \frac{\mu}{2} \|y\|^2.$$

Consequently, by the first order criterion of strong convexity, the function $f(x)$ is strongly convex with a constant μ. It is important to mention, that $\mu = 0$ stands for the convex case and corresponding differential criterion.

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$$
f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i)
$$

for $\alpha_i \geq 0; \quad \sum\limits_{i=1}^{n} \alpha_i = 1$ (probability simplex)

For the infinite dimension case:

$$
f\left(\int\limits_{S} x p(x) dx\right) \leq \int\limits_{S} f(x) p(x) dx
$$

If the integrals exist and $p(x) \geq 0, \quad \int\limits_{S} p(x) dx = 1.$

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- If the function $f(x)$ and the set S are convex, then any local minimum $x^* = \text{arg}\min\limits_{x \in S} f(x)$ will be the global one. Strong convexity guarantees the uniqueness of the solution.
- Let $f(x)$ - be a convex function on a convex set $S \subseteq \mathbb{R}^n$. Then $f(x)$ is continuous $\forall x \in \textbf{ri}(S)$.

Operations that preserve convexity

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- If $f(x)$ is convex on S, then $g(x, t) = t f(x/t)$ - is convex with $x/t \in S, t > 0$.

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- If $f(x)$ is convex on S, then $g(x, t) = t f(x/t)$ - is convex with $x/t \in S, t > 0$.
- Let $f_1 : S_1 \to \mathbb{R}$ and $f_2 : S_2 \to \mathbb{R}$, where range$(f_1) \subseteq S_2$. If f_1 and f_2 are convex, and f_2 is increasing, then $f_2 \circ f_1$ is convex on S_1.

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- Pseudoconvexity: $\langle \nabla f(y), x - y \rangle \geq 0 \longrightarrow f(x) \geq f(y)$
- Discrete convexity: $f : \mathbb{Z}^n \to \mathbb{Z}$; "convexity + matroid theory."

Examples

Example

Show, that $f(x) = c^\top x + b$ is convex and concave.

Examples

Example

Show, that $f(x) = x^\top A x$, where $A \succeq 0$ - is convex on $\mathbb{R}^n$.

Examples

Example

Show, that $f(A) = \lambda_{max}(A)$ - is convex, if $A \in S^n$.

PL inequality holds if the following condition is satisfied for some $\mu > 0$,

$$\|\nabla f(x)\|^2 \geq \mu(f(x) - f^*) \forall x$$

The example of a function, that satisfies the PL-condition, but is not convex.

$$f(x, y) = \frac{(y - \sin x)^2}{2}$$

Convex optimization problem

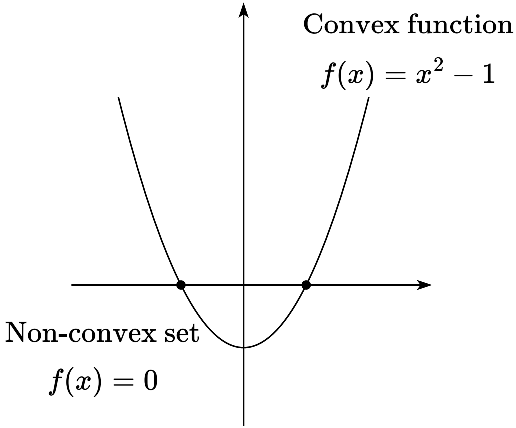

Figure 14: The idea behind the definition of a convex optimization problem

Note, that there is an agreement in notation of mathematical programming. The problems of the following type are called **Convex optimization problem**:

$$
\begin{aligned}
&f_0(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } &f_i(x) \leq 0, \; i = 1, \ldots, m \\
&Ax = b,
\end{aligned}
\tag{COP}
$$

where all the functions $f_0(x), f_1(x), \ldots, f_m(x)$ are convex and all the equality constraints are affine. It sounds a bit strange, but not all convex problems are convex optimization problems.

$$
f_0(x) \to \min_{x \in S}, \tag{CP}
$$

where $f_0(x)$ is a convex function, defined on the convex set S. The necessity of affine equality constraint is essential.

Linear Least Squares

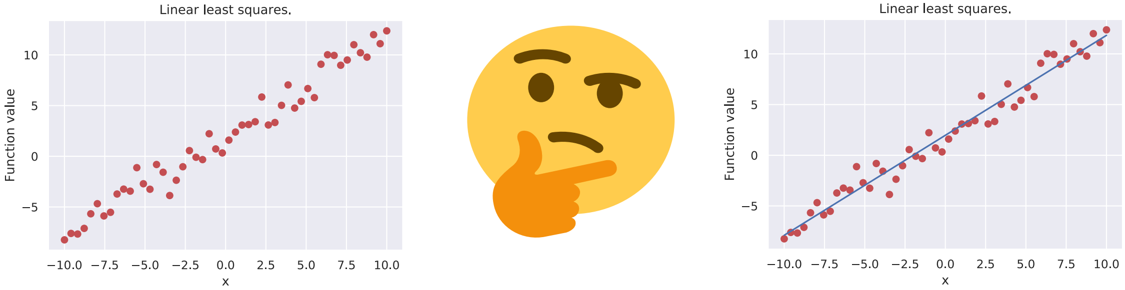

Figure 15: Illustration

Neural networks?