Dual methods: Dual Gradient Ascent, Augmented Lagrangian Method, ADMM.

Daniil Merkulov

Optimization for ML. Faculty of Computer Science. HSE University

Why do we want to solve dual problems?

Primal problem

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& f_0(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } \quad & f_i(x) \leq 0, \; i = 1, \ldots, m \\
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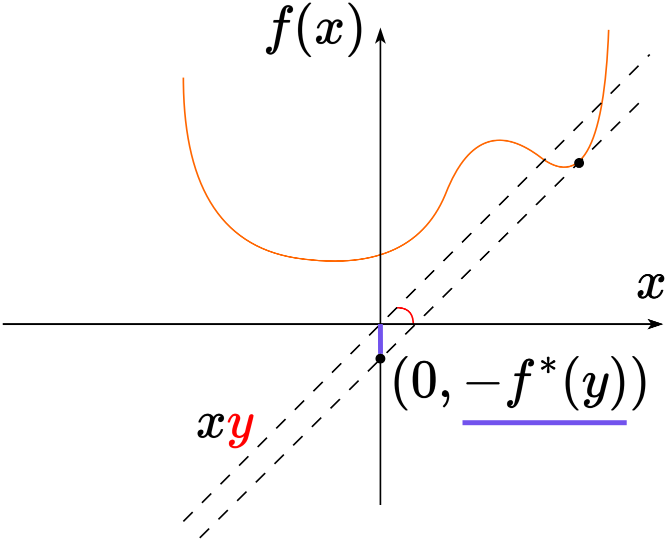
- **Duality Gap.** The difference between the primal and dual solutions (duality gap) provides valuable information about the solution's optimality.

Conjugate functions

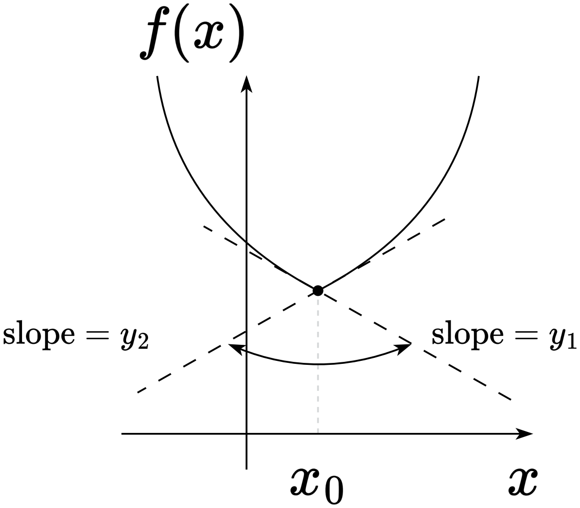


Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function defined by

$$f^*(y) = \max_x \left[y^T x - f(x) \right]$$

is called its conjugate.

Geometrical intution



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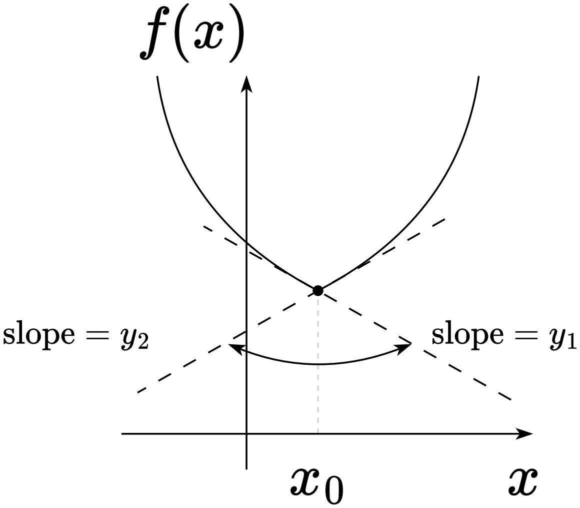

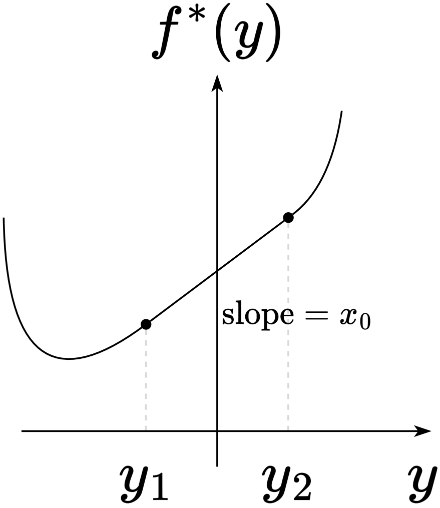

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- If f is strictly convex, then

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Conjugate function properties (proofs)

We will show that $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$, assuming that f is convex and closed.

- **Proof of** $\Leftarrow$: Suppose $y \in \partial f(x)$. Then $x \in M_y$, the set of maximizers of $y^T z - f(z)$ over z. But

$$f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{and} \quad \partial f^*(y) = \text{cl}(\text{conv}(\bigcup_{z \in M_y} \{z\})).$$

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Clearly $y \in \partial f(x) \Leftrightarrow x \in \arg\min_z \{f(z) - y^T z\}$

Lastly, if f is strictly convex, then we know that $f(z) - y^T z$ has a unique minimizer over z, and this must be $\nabla f^*(y)$.

Dual (sub)gradient method

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- Step sizes α_k, $k = 1, 2, 3, \ldots$, are chosen in standard ways.

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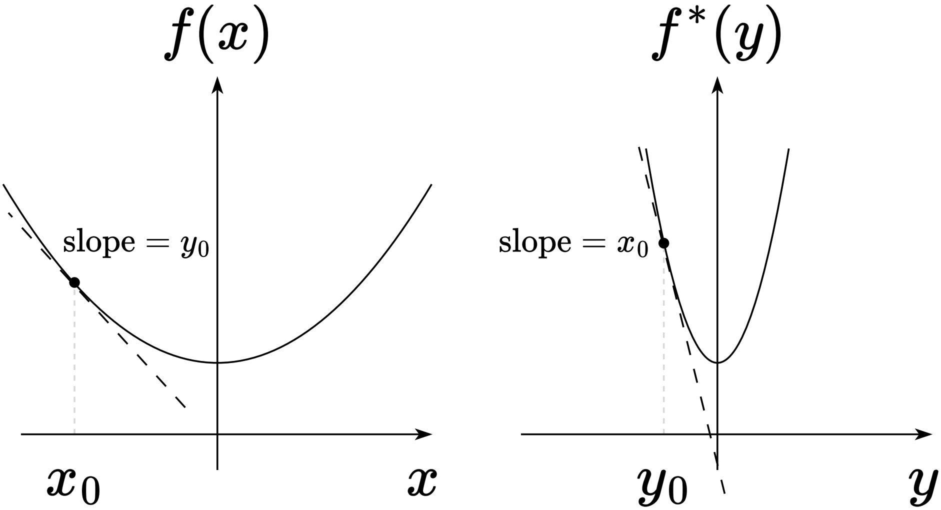
- Proximal gradients and acceleration can be applied as they would usually.

Slopes of f and f^*



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Adding these together, using the Cauchy-Schwarz inequality, and rearranging shows that

$$\|x_u - x_v\|^2 \leq \frac{1}{\mu}\|u - v\|^2$$

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Proof of "$\Leftarrow$": for simplicity, call $g = f^*$ and $L = \frac{1}{\mu}$. As ∇g is Lipschitz with constant L, so is $g_x(z) = g(z) - \nabla g(x)^T z$, hence

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Let $u = \nabla f(x)$, $v = \nabla g(y)$; then $x \in \partial g^*(u)$, $y \in \partial g^*(v)$, and the above reads $(x - y)^T (u - v) \geq \frac{\|u - v\|^2}{L}$, implying the result.

Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent: (This is ignoring the role of A, and thus reflects the case when the singular values of A are all close to 1. To be more precise, the step sizes here should be: $\frac{\mu}{\sigma_{\max}(A)^2}$ (first case) and $\frac{2}{\frac{\sigma_{\max}(A)^2}{\mu} + \frac{\sigma_{\min}(A)^2}{L}}$ (second case).)

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- Note that this describes convergence in the dual. Convergence in the primal requires more assumptions

Example: equality constrained quadratic minimization.

$$
f(x) = \frac{1}{2}x^T A x - b^T x \to \min_{x \in \mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m \times n}, m < n.
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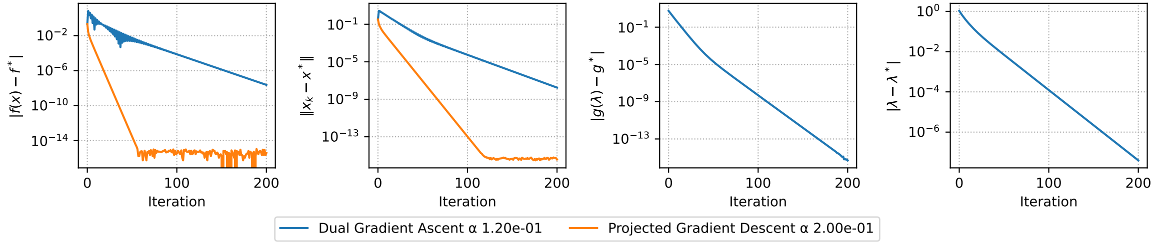
Quadratic constrained optimization. n=10, m=5, μ=1, L=10.



We need to find a minimum of a quadratic function in some linear subspace, defined by the solution of linear equation $Cx = d$. This is a conditional optimization problem, we start from strongly convex setting.

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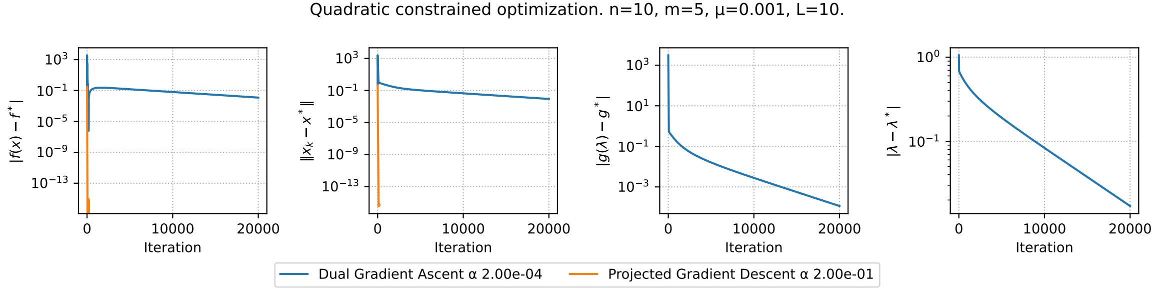

Situation is getting worse as soon as we loose strong convexity, the dual convergence will still be linear, but the rate is very low.

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$$\Rightarrow x_i^{\text{new}} \in \arg\min_{x_i} \left(f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \ldots, B$$

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Can think of these steps as:

- **Broadcast:** Send u to each of the B processors, each optimizes in parallel to find x_i.

Dual decomposition

Consider

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Here $x = (x_1, \ldots, x_B) \in \mathbb{R}^n$ divides into B blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition A accordingly:

$$A = [A_1 \ldots A_B], \text{ where } A_i \in \mathbb{R}^{m \times n_i}$$

Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into B separate problems:

$$x^{\text{new}} \in \arg\min_x \left(\sum_{i=1}^B f_i(x_i) + u^T Ax \right)$$

$$\Rightarrow x_i^{\text{new}} \in \arg\min_{x_i} \left(f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \ldots, B$$

$$x_i^k \in \arg\min_{x_i} \left(f_i(x_i) + (u^{k-1})^T A_i x_i \right), \quad i = 1, \ldots, B$$

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Can think of these steps as:

- **Broadcast:** Send u to each of the B processors, each optimizes in parallel to find x_i.
- **Gather:** Collect $A_i x_i$ from each processor, update the global dual variable u.

Inequality constraints

Consider the optimization problem:

$$\min_x \sum_{i=1}^{B} f_i(x_i) \quad \text{subject to} \quad \sum_{i=1}^{B} A_i x_i \leq b$$

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Using **dual decomposition**, specifically the **projected subgradient method**, the iterative steps can be expressed as:

- The primal update step:

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- The dual update step:

$$u^k = \left(u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b \right) \right)_+$$

where $(u)_+$ denotes the positive part of u, i.e., $(u_+)_i = \max\{0, u_i\}$, for $i = 1, \dots, m$.

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- **System Overview**: Consider a system with B units, where each unit independently chooses its decision variable x_i, which determines how to allocate its goods.

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  - **Never let prices get negative**; hence the use of the positive part notation $(\cdot)_+$.

Augmented Lagrangian method aka method of multipliers

Dual ascent disadvantage: convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

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\begin{aligned}
&\min_x f(x) + \frac{\rho}{2}\|Ax - b\|^2 \\
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Dual gradient ascent: The iterative updates are given by:

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Notice step size choice $\alpha_k = \rho$ in dual algorithm. Why?

Since x_k minimizes the function:

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over x, we have the stationarity condition:

$$0 \in \partial f(x_k) + A^T \left(u_{k-1} + \rho(Ax_k - b)\right)$$

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This represents the stationarity condition for the original primal problem; under mild conditions, $Ax_k - b \to 0$ as $k \to \infty$, so the KKT conditions are satisfied in the limit and x_k, u_k converge to the solutions.

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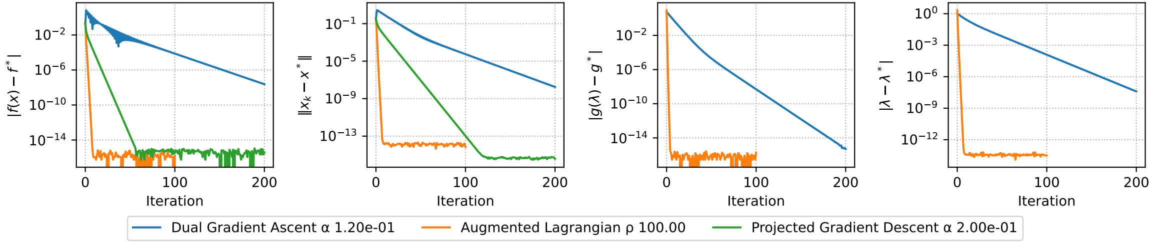
- **Disadvantage:** We lose decomposability! (Separability is ruined)

Example: equality constrained quadratic minimization.

$$f(x) = \frac{1}{2}x^T Ax - b^T x \to \min_{x \in \mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m \times n}, m < n.$$

Quadratic constrained optimization. n=10, m=5, μ=1, L=10.



One can see, clear numerical superiority of the Augmented Lagrangian method both in convex and strongly convex case.

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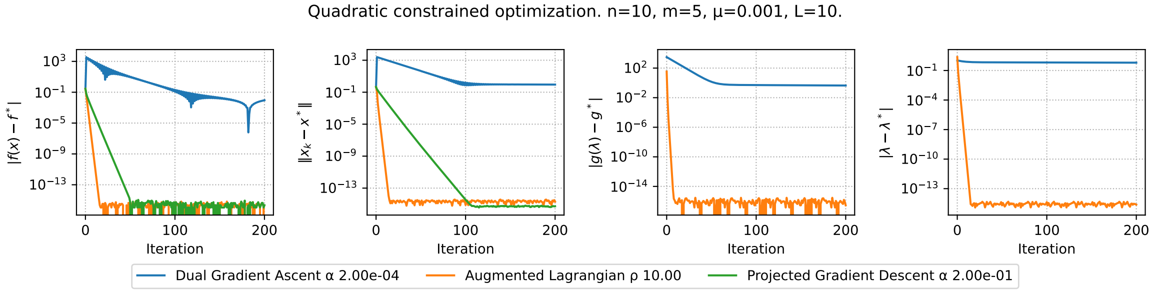

One can see, clear numerical superiority of the Augmented Lagrangian method both in convex and strongly convex case.

Alternating Direction Method of Multipliers (ADMM)

Alternating direction method of multipliers or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

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where $\rho > 0$ is a parameter. The augmented Lagrangian for this problem is defined as:

$$
L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|^2
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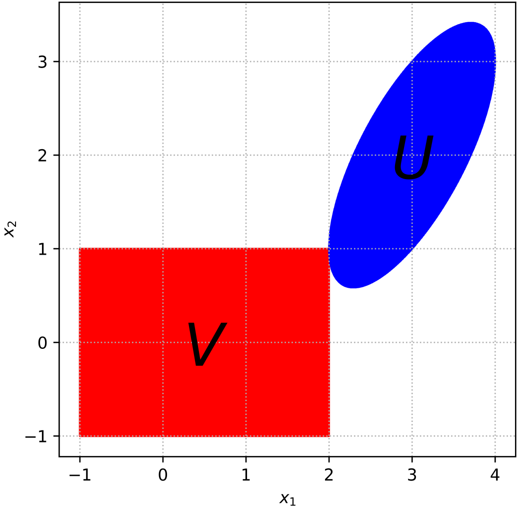
Note: The usual method of multipliers would replace the first two steps by a joint minimization:

$$(x^{(k)}, z^{(k)}) = \arg\min_{x,z} L_\rho(x, z, u^{(k-1)})$$

Example: Alternating Projections



Consider finding a point in the intersection of convex sets $U, V \subseteq \mathbb{R}^n$:

$$\min_x I_U(x) + I_V(x)$$

To transform this problem into ADMM form, we express it as:

$$\min_{x,z} I_U(x) + I_V(z) \quad \text{subject to} \quad x - z = 0$$

Each ADMM cycle involves two projections:

$$x_k = \arg\min_x P_U\left(z_{k-1} - w_{k-1}\right)$$
$$z_k = \arg\min_z P_V\left(x_k + w_{k-1}\right)$$
$$w_k = w_{k-1} + x_k - z_k$$

Sources

- Ryan Tibshirani. Convex Optimization 10-725