Matrix Derivatives. Automatic Differentiation

Seminar

Optimization for ML. Faculty of Computer Science. HSE University



Theory recap. Differential

• Differential $df(x)[\cdot]: U \to V$ in point $x \in U$ for $f(\cdot): U \to V$:

$$f(x+h) - f(x) = \underbrace{df(x)[h]}_{\text{differential}} + \overline{o}(||h||)$$

Canonical form of the differential:

$U \to V$	\mathbb{R}	\mathbb{R}^n	$\mathbb{R}^{n imes m}$
R	f'(x)dx	$\nabla f(x)dx$	$\nabla f(x)dx$
\mathbb{R}^n $\mathbb{R}^{n imes m}$	$ \nabla f(x)^T dx tr(\nabla f(X)^T dX) $	J(x)dx	_

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Theory recap. Differentiation Rules

• Useful differentiation rules and standard derivatives:

Differentiation Rules	Standard Derivatives	
$dA = 0$ $d(\alpha X) = \alpha(dX)$	$d(\langle A, X \rangle) = \langle A, dX \rangle$ $d(\langle Ax, x \rangle) = \langle (A + A^{T})x, dx \rangle$	
d(AXB) = A(dX)B	$d(Det(X)) = Det(X)\langle X^{-T}, dX \rangle$	
$d(X+Y) = dX + dY$ $d(X^T) = (dX)^T$	$d(X^{-1}) = -X^{-1}(dX)X^{-1}$	
d(XY) = (dX)Y + X(dY)		
$d(\langle X, Y \rangle) = \langle dX, Y \rangle + \langle X, dY \rangle$ $d\left(\frac{X}{\phi}\right) = \frac{\phi dX - (d\phi)X}{\phi^2}$		



Matrix Calculus. Problem 1

Example

Find $\nabla f(x)$, if $f(x) = \frac{1}{2}x^TAx + b^Tx + c$.

Matrix Calculus. Problem 2

Example

Find $\nabla f(X)$, if $f(X) = tr(AX^{-1}B)$

• $h(x) = f(g(x)) \Rightarrow dh(x_0)[dx] = df(g(x_0))[dg(x_0)[dx]]$

Matrix Calculus. Problem 3

Example

Find the gradient $\nabla f(x)$ and hessian $\nabla^2 f(x)$, if $f(x) = \frac{1}{2} ||x||_2^3$

•
$$d^2f(x)[h_1, h_2] = d\left(df(x)[\underbrace{h_1}_{\text{fixed when take outer }d(\cdot)}]\right)[h_2]$$

• Canonic form for $f: \mathbb{R}^n \to \mathbb{R}$: $d^2f(x)[h_1, h_2] = h_1^T \underbrace{\nabla^2 f(x)}_{} h_2$



Automatic Differentiation. Forward mode

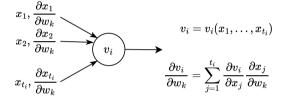


Figure 1: Illustration of forward chain rule to calculate the derivative of the function v_i with respect to w_k .

- Uses the forward chain rule
- Has complexity $d \times \mathcal{O}(T)$ operations



Automatic Differentiation. Reverse mode

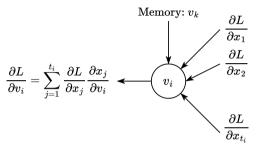


Figure 2: Illustration of reverse chain rule to calculate the derivative of the function L with respect to the node v_i .

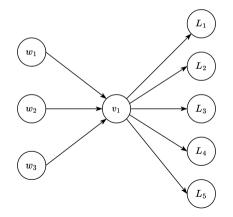
- Uses the backward chain rule
- Stores the information from the forward pass
- Has complexity $\mathcal{O}(T)$ operations



Automatic Differentiation. Problem 1

Example

Which of the AD modes would you choose (forward/reverse) for the following computational graph of primitive arithmetic operations?





Automatic Differentiation. Problem 2

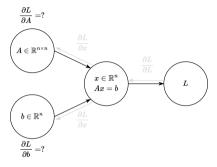


Figure 4: x could be found as a solution of linear system

Suppose, we have an invertible matrix A and a vector b, the vector x is the solution of the linear system Ax = b, namely one can write down an analytical solution $x = A^{-1}b$.

Find the derivatives $\frac{\partial L}{\partial A}, \frac{\partial L}{\partial b}.$



Automatic Differentiation. Problem 3

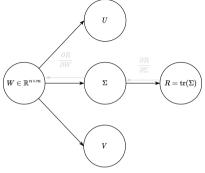


Figure 5: Computation graph for singular regularizer

Suppose, we have the rectangular matrix $W \in \mathbb{R}^{m \times n}$, which has a singular value decomposition:

$$W = U\Sigma V^T$$
, $U^TU = I$, $V^TV = I$, $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_{\min(m,n)})$

The regularizer $R(W)=\operatorname{tr}(\Sigma)$ in any loss function encourages low rank solutions. Find the derivative $\frac{\partial R}{\partial W}$.

Computation experiment with JAX

 $\bullet \ \mathsf{JAX} \ \mathsf{docs:} \ \mathsf{https:}//\mathsf{jax.readthedocs.io/en/latest/notebooks/quickstart.\mathsf{html}}$

Automatic Differentiation Problems

