Duality

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> *The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedure.*

Preface to Mécanique analytique

Figure 1: Joseph-Louis Lagrange

Motivation

Duality lets us associate to any constrained optimization problem a concave maximization problem, whose solutions lower bound the optimal value of the original problem. What is interesting is that there are cases, when one can solve the primal problem by first solving the dual one. Now, consider a general constrained optimization problem:

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As a consequence:

$$\max_{y \in \Omega} g(y) \leq \min_{x \in S} f(x)$$

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And the Lagrangian, associated with this problem:

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L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) = f_0(x) + \lambda^\top f(x) + \nu^\top h(x)
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Dual function

We assume $\mathcal{D} = \bigcap\limits_{i=0}^{m} \textbf{dom } f_i \cap \bigcap\limits_{i=1}^{p} \textbf{dom } h_i$ is nonempty. We define the Lagrange dual function (or just dual function) $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ as the minimum value of the Lagrangian over x: for $\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p$

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$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

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When the Lagrangian is unbounded below in x, the dual function takes on the value $-\infty$. Since the dual function is the pointwise infimum of a family of affine functions of (λ, ν), it is concave, even when the original problem is not convex.

Dual function as a lower bound

Let us show, that the dual function yields lower bounds on the optimal value p^* of the original problem for any $\lambda \succeq 0, \nu$. Suppose some $\hat{x}$ is a feasible point for the original problem, i.e., $f_i(\hat{x}) \leq 0$ and $h_i(\hat{x}) = 0$, $\lambda \succeq 0$. Then we have:

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The term "dual feasible", to describe a pair (λ, ν) with $\lambda \succeq 0$ and $g(\lambda, \nu) > -\infty$, now makes sense. It means, as the name implies, that (λ, ν) is feasible for the dual problem. We refer to (λ^*, ν^*) as dual optimal or optimal Lagrange multipliers if they are optimal for the above problem.

Summary

	Primal	Dual
Function	$f_0(x)$	$g(\lambda, \nu) = \min\limits_{x \in \mathcal{D}} L(x, \lambda, \nu)$
Variables	$x \in S \subseteq \mathbb{R}^{\ltimes}$	$\lambda \in \mathbb{R}^m_+, \nu \in \mathbb{R}^p$
Constraints	$f_i(x) \leq 0, \; i = 1, \ldots, m$ $h_i(x) = 0, \; i = 1, \ldots, p$	$\lambda_i \geq 0, \forall i \in \overline{1, m}$
Problem	$f_0(x) \to \min\limits_{x \in \mathbb{R}^n}$ s.t. $f_i(x) \leq 0, \; i = 1, \ldots, m$ $h_i(x) = 0, \; i = 1, \ldots, p$	$g(\lambda, \nu) \to \max\limits_{\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p}$ s.t. $\lambda \succeq 0$
Optimal	x^* if feasible, $p^* = f_0(x^*)$	λ^*, ν^* if $\max$ is achieved, $d^* = g(\lambda^*, \nu^*)$

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This problem is devoid of inequality constraints, presenting m linear equality constraints instead. The Lagrangian is expressed as $L(x, \nu) = x^T x + \nu^T (Ax - b)$, spanning the domain $\mathbb{R}^n \times \mathbb{R}^m$. The dual function is denoted by $g(\nu) = \inf_x L(x, \nu)$. Given that $L(x, \nu)$ manifests as a convex quadratic function in terms of x, the minimizing x can be derived from the optimality condition

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-(1/4)\nu^T AA^T \nu - b^T \nu \leq \inf\{x^T x \mid Ax = b\}.
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Which is a simple non-trivial lower bound without any problem solving.

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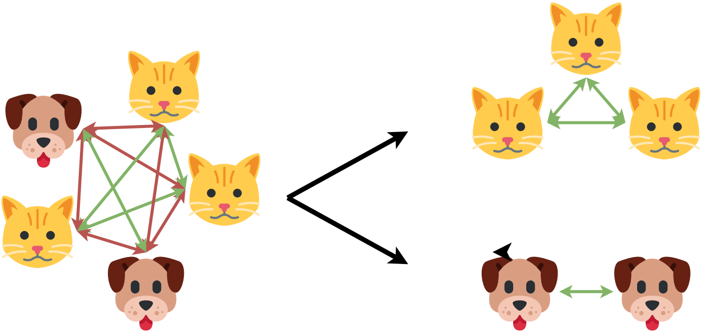

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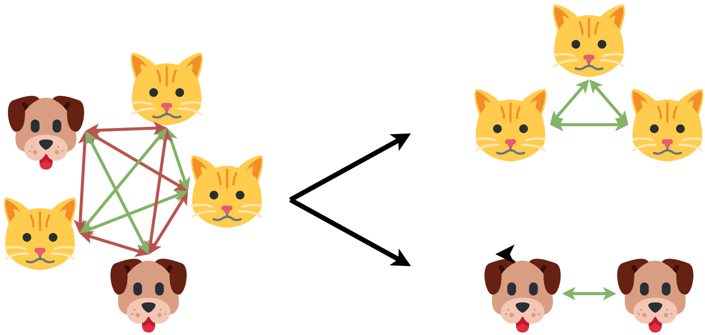

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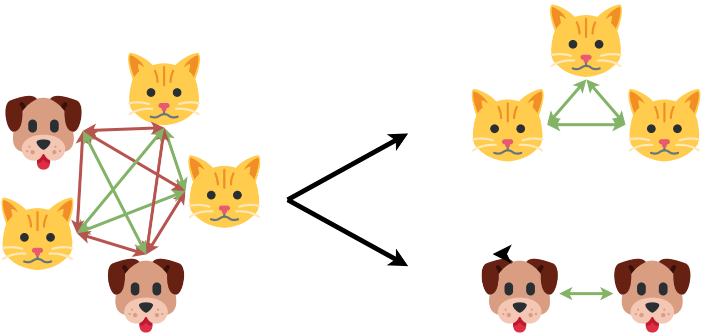

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The coefficient W_{ij} in the matrix represents the expense associated with placing elements i and j in the same partition, while $-W_{ij}$ signifies the cost of segregating them. The objective encapsulates the aggregate cost across all pairs of elements, and the challenge posed by problem is to find the partition that minimizes the total cost.

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We now derive the dual function for this problem. The Lagrangian is expressed as

$$L(x,\nu) = x^T W x + \sum_{i=1}^{n} \nu_i (x_i^2 - 1) = x^T (W + \text{diag}(\nu)) x - \mathbf{1}^T \nu.$$

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By minimizing over x, we procure the Lagrange dual function:

$$g(\nu) = \inf_x x^T (W + \text{diag}(\nu)) x - \mathbf{1}^T \nu = \begin{cases} -\mathbf{1}^T \nu & \text{if } W + \text{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise,} \end{cases}$$

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This renders a simple bound on the optimal value p^*: $p^* \geq -\mathbf{1}^T \nu = n\lambda_{\text{min}}(W)$.

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$$g(\nu) = \inf_x x^T(W + \mathrm{diag}(\nu))x - \mathbf{1}^T\nu = \begin{cases} -\mathbf{1}^T\nu & \text{if } W + \mathrm{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise,} \end{cases}$$

exploiting the realization that the infimum of a quadratic form is either zero (when the form is positive semidefinite) or $-\infty$ (when it's not).

This dual function furnishes lower bounds on the optimal value of the problem. For instance, we can adopt the particular value of the dual variable

$$\nu = -\lambda_{\min}(W)\mathbf{1}$$

which is dual feasible, since $W + \mathrm{diag}(\nu) = W - \lambda_{\min}(W)I \succeq 0$.

This renders a simple bound on the optimal value p^*: $p^* \geq -\mathbf{1}^T\nu = n\lambda_{\min}(W)$.

The code for the problem is available here Open in Colab

Strong duality

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- Several sufficient conditions known!
- "Easy" necessary and sufficient conditions: unknown.

Strong duality in linear least squares

Exercise

In the Least-squares solution of linear equations example above calculate the primal optimum p^* and the dual optimum d^* and check whether this problem has strong duality or not.

Useful features of duality

- **Construction of lower bound on solution of the primal problem.**

  It could be very complicated to solve the initial problem. But if we have the dual problem, we can take an arbitrary $y \in \Omega$ and substitute it in $g(y)$ - we'll immediately obtain some lower bound.

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- **Dual function is always concave**

  As a pointwise minimum of affine functions.

Slater's condition

Theorem

If for a convex optimization problem (i.e., assuming minimization, f_0, f_i are convex and h_i are affine), there exists a point x such that $h(x) = 0$ and $f_i(x) < 0$ (existance of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.

An example of convex problem, when Slater's condition does not hold

Example

$$\min\{f_0(x) = x \mid f_1(x) = \frac{x^2}{2} \leq 0\},$$

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Example

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\min\{f_0(x) = x \mid f_1(x) = \frac{x^2}{2} \leq 0\},
$$

The only point in the budget set is: $x^* = 0$. However, it is impossible to find a non-negative $\lambda^* \geq 0$, such that

$$
\nabla f_0(0) + \lambda^* \nabla f_1(0) = 1 + \lambda^* x = 0.
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A nonconvex quadratic problem with strong duality

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Lagrangian and dual function

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\begin{split}
& -\sum_{i=1}^n \frac{(q_i^\top b)^2}{\lambda_i + \lambda} - \lambda \to \max_{\lambda \in \mathbb{R}} \\
\text{s.t. } & \lambda \geq -\lambda_{min}(A)
\end{split}
$$

Sensitivity analysis

Let us switch from the original optimization problem

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\begin{split}
& f_0(x) \to \min_{x \in \mathbb{R}^n} \\
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One can even show, that when P is convex optimization problem, $p^*(u, v)$ is a convex function.

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$$\begin{split} p^*(0,0) &= p^* = d^* = g(\lambda^*, \nu^*) \leq \\ & \leq L(x, \lambda^*, \nu^*) = \\ & = f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x) \leq \\ & \leq f_0(x) + \sum_{i=1}^m \lambda_i^* u_i + \sum_{i=1}^p \nu_i^* v_i \end{split}$$

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And taking the optimal x for the perturbed problem, we have:

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p^*(u,v) \geq p^*(0,0) - {\lambda^*}^T u - {\nu^*}^T v \tag{1}
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In scenarios where strong duality holds, we can draw several insights about the sensitivity of optimal solutions in relation to the Lagrange multipliers. These insights are derived from the inequality expressed in equation above:

- **Impact of Tightening a Constraint (Large $\lambda_i^\star$)**:
  When the ith constraint's Lagrange multiplier, $\lambda_i^\star$, holds a substantial value, and if this constraint is tightened (choosing $u_i < 0$), there is a guarantee that the optimal value, denoted by $p^\star(u, v)$, will significantly increase.

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These interpretations provide a framework for understanding how changes in constraints, reflected through their corresponding Lagrange multipliers, impact the optimal solution in problems where strong duality holds.

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However, if $f_i(x^*) = 0$, meaning the constraint is precisely met at the optimum, then the situation is different. The value of the i-th optimal Lagrange multiplier, λ_i^*, gives us insight into how 'sensitive' or 'active' this constraint is. A small λ_i^* indicates that slight adjustments to the constraint won't significantly affect the optimal value. Conversely, a large λ_i^* implies that even minor changes to the constraint can have a significant impact on the optimal solution.

Mixed strategies for matrix games

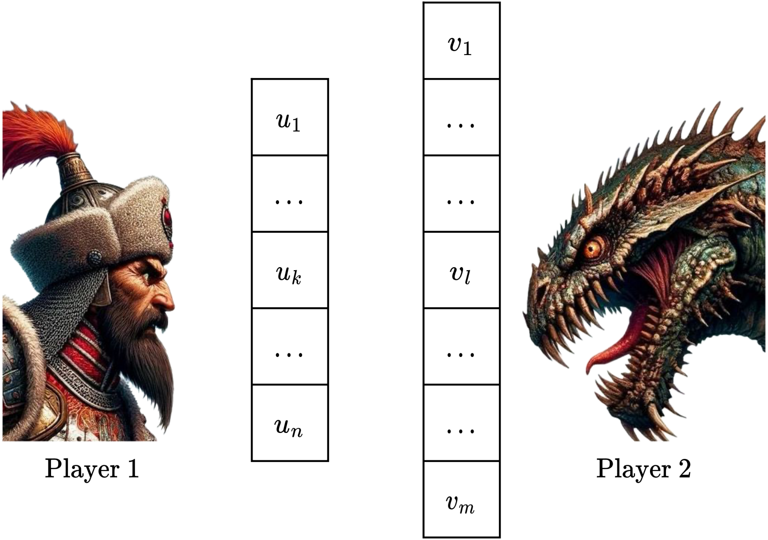

Figure 3: The scheme of a mixed strategy matrix game

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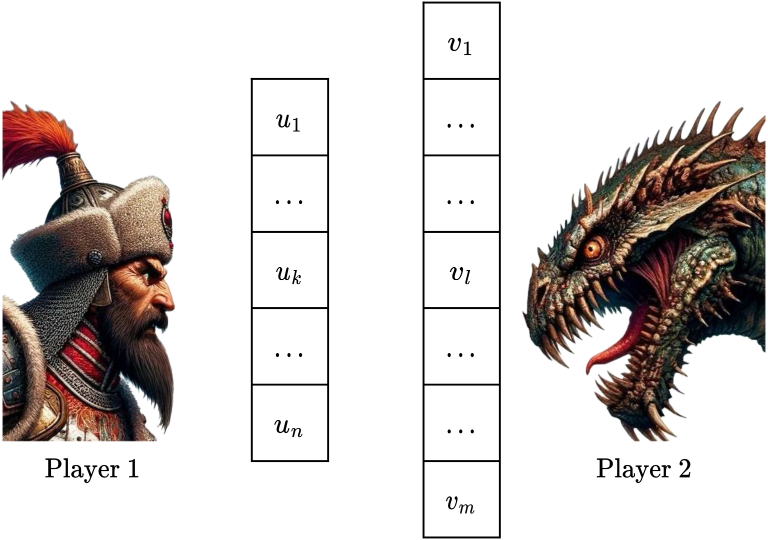

Figure 3: The scheme of a mixed strategy matrix game

In zero-sum matrix games, players 1 and 2 choose actions from sets $\{1, \dots, n\}$ and $\{1, \dots, m\}$, respectively. The outcome is a payment from player 1 to player 2, determined by a payoff matrix $P \in \mathbb{R}^{n \times m}$. Each player aims to use mixed strategies, choosing actions according to a probability distribution: player 1 uses probabilities u_k for each action i, and player 2 uses v_l.

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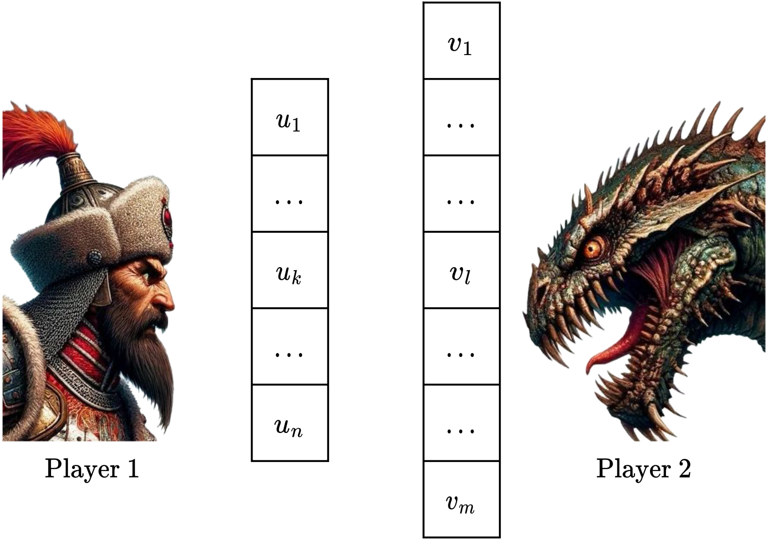

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The expected payoff from player 1 to player 2 is given by $\sum_{k=1}^{n} \sum_{l=1}^{m} u_k v_l P_{kl} = u^T P v$. Player 1 seeks to minimize this expected payoff, while player 2 aims to maximize it.

Mixed strategies for matrix games. Player 1's Perspective

Player 1

$$\begin{array}{|c|}\hline u_1 \\ \hline \ldots \\ \hline u_k \\ \hline \ldots \\ \hline u_n \\ \hline \end{array}$$

Assuming player 2 knows player 1's strategy u, player 2 will choose v to maximize $u^T P v$. The worst-case expected payoff is thus:

$$\max_{v \geq 0, 1^T v = 1} u^T P v = \max_{i=1,\ldots,m} (P^T u)_i$$

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Player 1's optimal strategy minimizes this worst-case payoff, leading to the optimization problem:

$$\begin{aligned} &\min \max_{i=1,\dots,m} (P^T u)_i \\ &\text{s.t. } u \geq 0 \\ &1^T u = 1 \end{aligned} \tag{3}$$

This forms a convex optimization problem with the optimal value denoted as p_1^*.

Mixed strategies for matrix games. Player 2's Perspective

v_1
$\dots$
$\dots$
v_l
$\dots$
$\dots$
v_m

Player 2

Conversely, if player 1 knows player 2's strategy v, the goal is to minimize $u^T P v$. This leads to:

$$\min_{u \geq 0, 1^T u = 1} u^T P v = \min_{i=1,\dots,n} (Pv)_i$$

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Player 2 then maximizes this to get the largest guaranteed payoff, solving the optimization problem:

$$
\begin{aligned}
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&\text{s.t. } v \geq 0 \\
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\end{aligned}
\tag{4}
$$

The optimal value here is p_2^*.

Mixed strategies for matrix games

Duality and Equivalence

It's generally advantageous to know the opponent's strategy, but surprisingly, in mixed strategy matrix games, this advantage disappears. The key lies in duality: the problems above are Lagrange duals. By formulating player 1's problem as a linear program and introducing Lagrange multipliers, we find that the dual problem matches player 2's problem. Due to strong duality in feasible linear programs, $p_1^* = p_2^*$, showing no advantage in knowing the opponent's strategy.

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Conclusion

This formulation shows that the Lagrange dual problem is equivalent to problem Equation 4. Given the feasibility of these linear programs, strong duality holds, meaning the optimal values of Equation 3 and Equation 4 are equal.