

Dual methods: Dual Gradient Ascent, Augmented Lagrangian Method, ADMM.

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Why do we want to solve dual problems?

Primal problem



$$\begin{aligned} f_0(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } f_i(x) &\leq 0, i = 1, \dots, m \\ h_i(x) &= 0, i = 1, \dots, p \end{aligned}$$

Dual problem

Beerga barugas

$$\begin{aligned} g(\lambda, \nu) &= \min_{x \in \mathcal{D}} L(x, \lambda, \nu) = \\ \min_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right) &\rightarrow \max_{\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p} \\ \text{s.t. } \lambda &\succeq 0 \end{aligned}$$

- **Shadow Prices.** In economics and resource allocation problems, dual variables can be interpreted as shadow prices, providing economic insights into resource utilization and constraints.

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Poznáme ho v mnoha aplikacích
m = 5
p = 5

Primal problem

$$n = 10^6$$

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- **Market Equilibrium.** Dual problems often represent market equilibrium conditions, making them essential for economic modeling and analysis.
- **Dual Problems Provide Bounds.** Dual problems often offer bounds on the optimal value of the primal problem. This can be useful for assessing the quality of approximate solutions.
- **Duality Gap.** The difference between the primal and dual solutions (duality gap) provides valuable information about the solution's optimality.

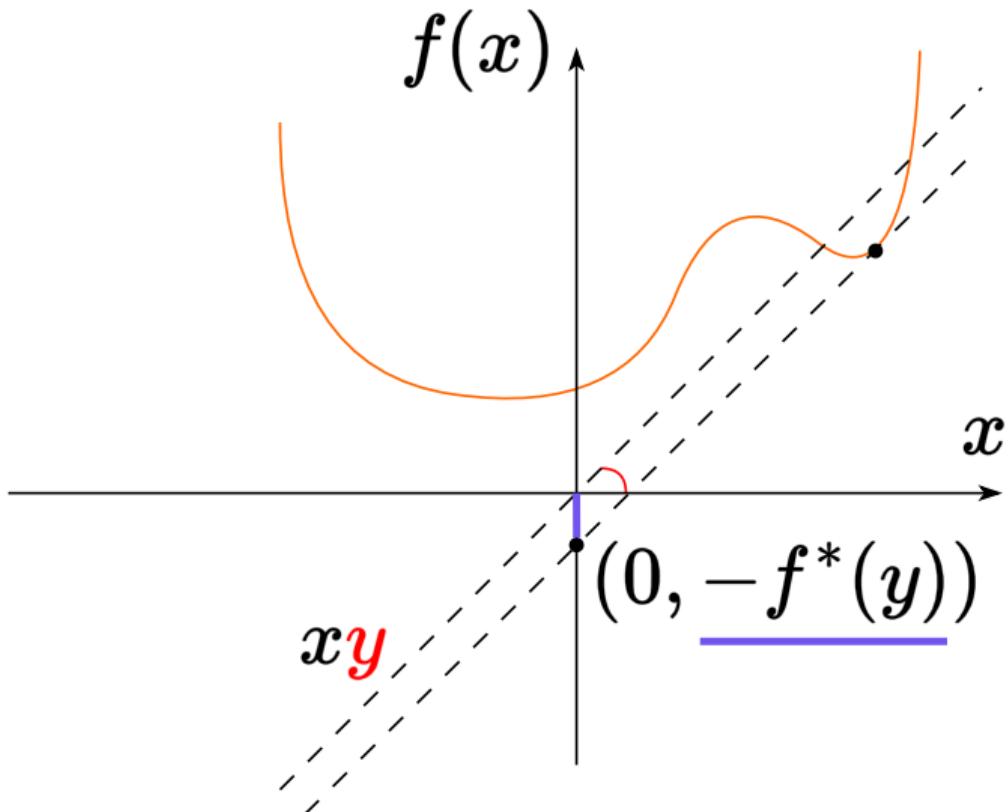
Conjugate functions

компактная функция

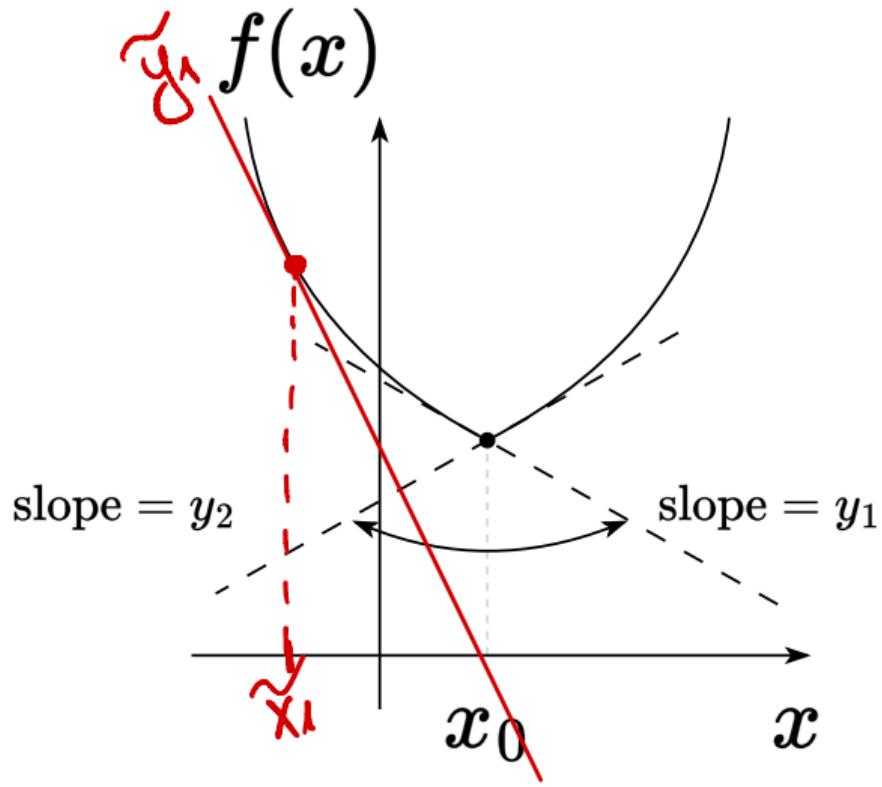
Recall that given $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the function defined by

$$f^*(y) = \max_x [y^T x - f(x)]$$

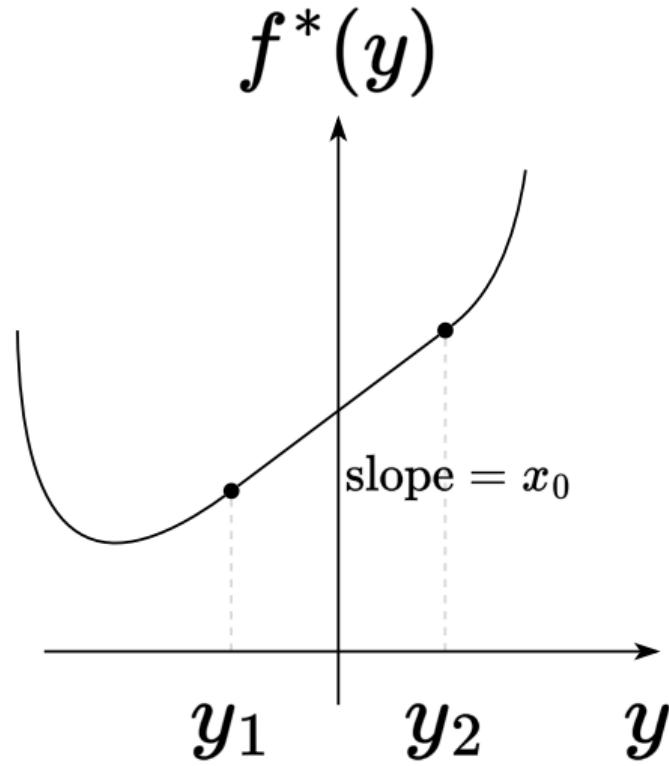
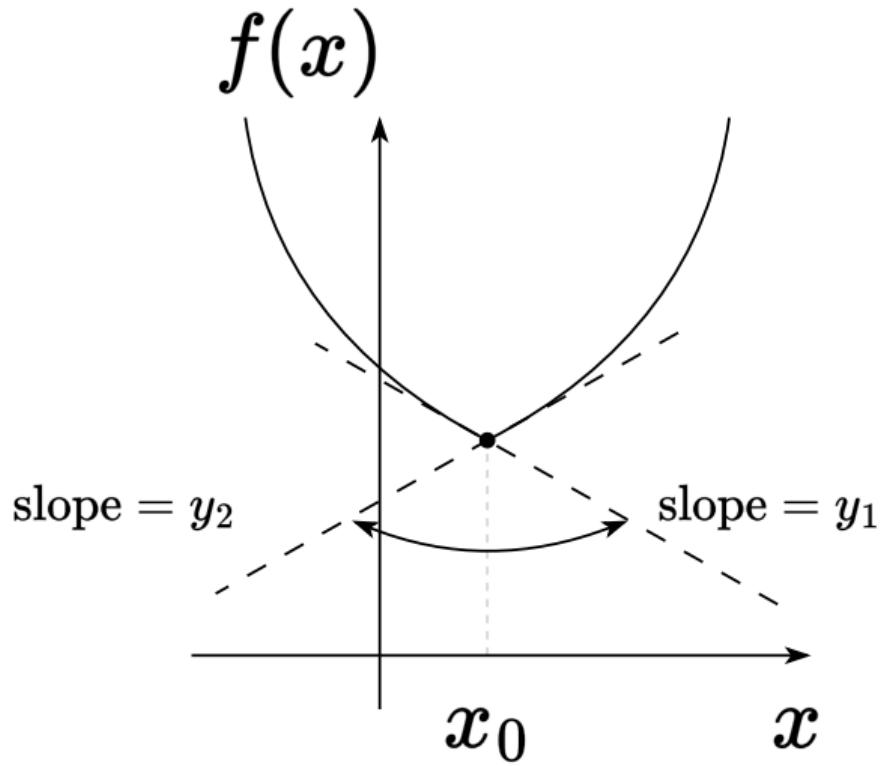
is called its conjugate.



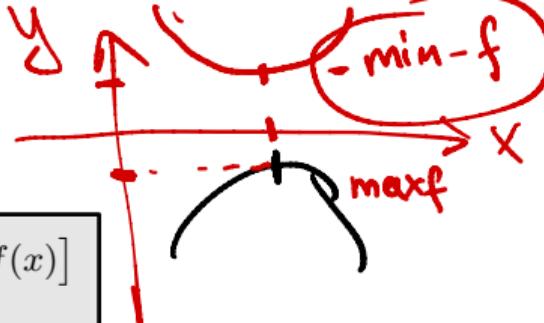
Geometrical intuition



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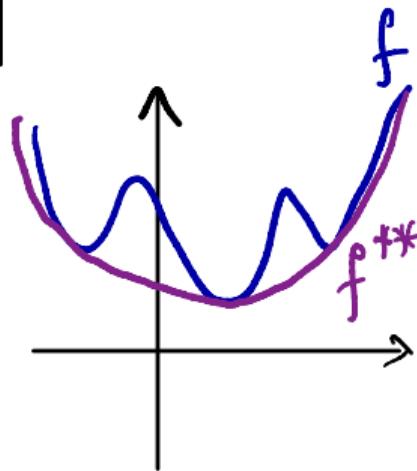
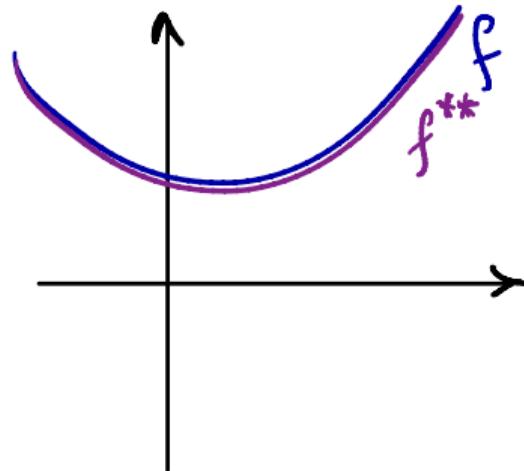
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$$-f^*(y) = \min_x [f(x) - y^T x]$$

- If f is closed and convex, then $f^{**} = f$. Also,

$$x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x) \Leftrightarrow x \in \arg \min_z [f(z) - y^T z]$$



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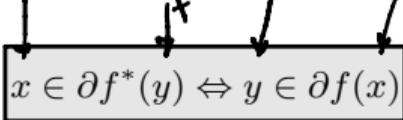
- If f is strictly convex, then

$$\nabla f^*(y) = \arg \min_z [f(z) - y^T z]$$

Conjugate function properties (proofs)

$$f^*(y) = \max_z y^T z - f(z)$$

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Yon harakoma rec touga



We will show that $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$ assuming that f is convex and closed.

- **Proof of \Leftarrow :** Suppose $y \in \partial f(x)$. Then $x \in M_y$, the set of maximizers of $y^T z - f(z)$ over z . But

$$f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{and} \quad \partial f^*(y) = \text{cl}(\text{conv}(\bigcup_{z \in M_y} \{z\})).$$

Thus $x \in \partial f^*(y)$.

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- **Proof of \Rightarrow :** From what we showed above, if $x \in \partial f^*(y)$, then $y \in \partial f^*(x)$, but $f^{**} = f$.

$$\text{ecm } y \in \partial f(x) \implies x \in \partial f^*(y)$$

$$\text{ngt6 } f := f^* \quad y \in \partial f^*(x) \Rightarrow x \in \partial f^{**}(y)$$

$\partial f(y)$

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- **Proof of \Rightarrow :** From what we showed above, if $x \in \partial f^*(y)$, then $y \in \partial f^*(x)$, but $f^{**} = f$.

Clearly $y \in \partial f(x) \Leftrightarrow x \in \arg \min_z \{f(z) - y^T z\}$

Lastly, if f is strictly convex, then we know that $f(z) - y^T z$ has a unique minimizer over z , and this must be $\nabla f^*(y)$.

Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

Consider the problem:

$$\min_x f(x) \quad \text{subject to} \quad Ax = b$$

$$A \in \mathbb{R}^{m \times n}$$

насторону градиент. здагою 1) $L(x, \lambda) = f(x) + \lambda^T(Ax - b)$

2) $g(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda) = \inf_x \left[f(x) + (A^T \lambda)^T x - \lambda^T b \right] =$
 $= \inf_x \left(f(x) - (-A^T \lambda)^T x - \lambda^T b \right) =$
 $= -f^*(-A^T \lambda) - \lambda^T b$

3) Другий. здага: $-f^*(-A^T \lambda) - \lambda^T b \xrightarrow[\lambda \in \mathbb{R}^n]{\max}$

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where f^* is the conjugate of f . Defining $g(u) = -f^*(-A^T u) - b^T u$, note that:

$$\partial g(u) = A\partial f^*(-A^T u) - b$$

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Dual ascent method for maximizing dual objective:

- Step sizes α_k , $k = 1, 2, 3, \dots$, are chosen in standard ways.

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$$x_k \in \arg \min_x [f(x) + (u_{k-1})^T Ax]$$

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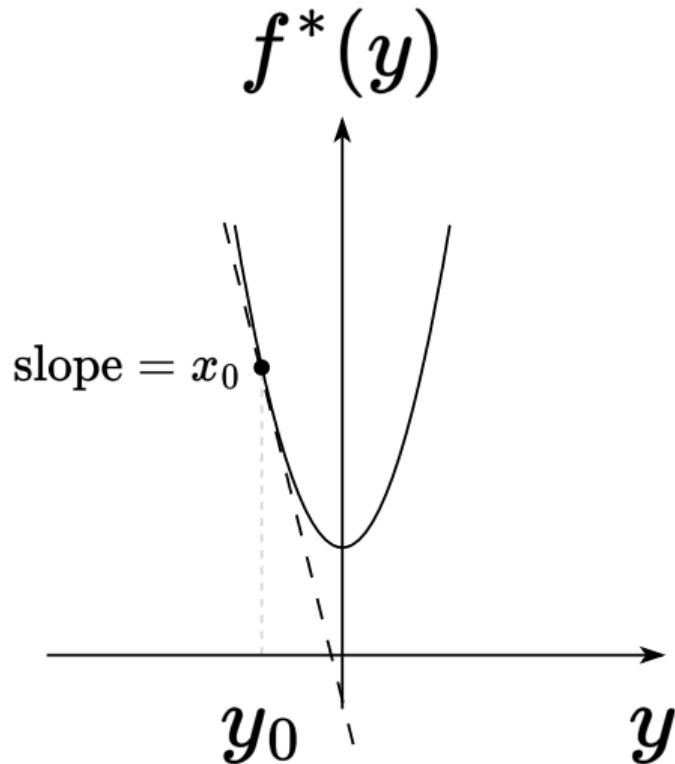
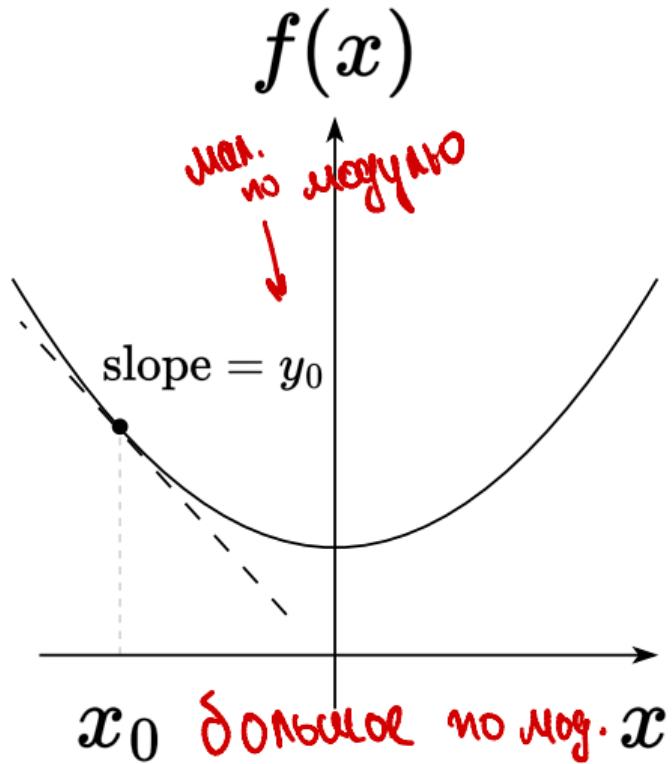
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- Step sizes α_k , $k = 1, 2, 3, \dots$, are chosen in standard ways.
- Proximal gradients and acceleration can be applied as they would usually.

Slopes of f and f^*



Slopes of f and f^*

Assume that f is a closed and convex function. Then f is strongly convex with parameter $\mu \Leftrightarrow \nabla f^*$ is Lipschitz with parameter $1/\mu$.

НР ГМОУ
МУР

$f - \mu$ сильногол.

$f - L$ сильногол.

ДВОІСВ.
МУД

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Proof of “ \Rightarrow ”: Recall, if g is strongly convex with minimizer x , then

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Hence, defining $x_u = \nabla f^*(u)$ and $x_v = \nabla f^*(v)$,

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Adding these together, using the Cauchy-Schwarz inequality, and rearranging shows that

$$\|x_u - x_v\|^2 \leq \frac{1}{\mu} \|u - v\|^2$$

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Proof of " \Leftarrow : for simplicity, call $g = f^*$ and $L = \frac{1}{\mu}$. As ∇g is Lipschitz with constant L , so is $g_x(z) = g(z) - \nabla g(x)^T z$, hence

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Minimizing each side over z , and rearranging, gives

$$\frac{1}{2L} \|\nabla g(x) - \nabla g(y)\|^2 \leq g(y) - g(x) + \nabla g(x)^T (x - y)$$

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Exchanging roles of x , y , and adding together, gives

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Let $u = \nabla f(x)$, $v = \nabla g(y)$; then $x \in \partial g^*(u)$, $y \in \partial g^*(v)$, and the above reads $(x - y)^T (u - v) \geq \frac{\|u - v\|^2}{L}$, implying the result.

Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent: (This is ignoring the role of A , and thus reflects the case when the singular values of A are all close to 1. To be more precise, the step sizes here should be: $\frac{\mu}{\sigma_{\max}(A)^2}$ (first case) and $\frac{2}{\mu + \frac{\sigma_{\min}(A)^2}{L}}$ (second case).)

- If f is strongly convex with parameter μ , then dual gradient ascent with constant step sizes $\alpha_k = \mu$ converges at sublinear rate $O(\frac{1}{\epsilon})$.

$$x^2 + |x|$$

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- If f is strongly convex with parameter μ and ∇f is Lipschitz with parameter L , then dual gradient ascent with step sizes $\alpha_k = \frac{2}{\mu + \frac{1}{L}}$ converges at linear rate $O(\log(\frac{1}{\epsilon}))$.

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- If f is strongly convex with parameter μ and ∇f is Lipschitz with parameter L , then dual gradient ascent with step sizes $\alpha_k = \frac{2}{\frac{1}{\mu} + \frac{1}{L}}$ converges at linear rate $O(\log(\frac{1}{\epsilon}))$.
- Note that this describes convergence in the dual. Convergence in the primal requires more assumptions

Example: equality constrained quadratic minimization.

$$\lambda(A) \in [\mu; L]$$

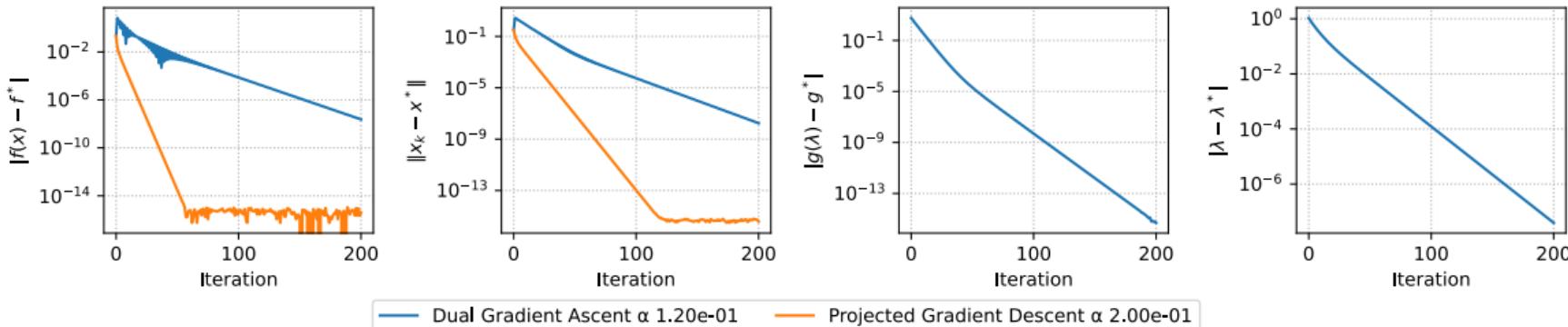
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$$f(x) = \frac{1}{2}x^T Ax - b^T x \rightarrow \min_{x \in \mathbb{R}^n} \quad \text{subject to} \quad Cx = d, \quad A \in \mathbb{S}_+^n, C \in \mathbb{R}^{m \times n}, m < n.$$

$$\lambda = 4$$

Quadratic constrained optimization. $n=10, m=5, \mu=1, L=10$.



We need to find a minimum of a quadratic function in some linear subspace, defined by the solution of linear equation $Cx = d$. This is a conditional optimization problem, we start from strongly convex setting.

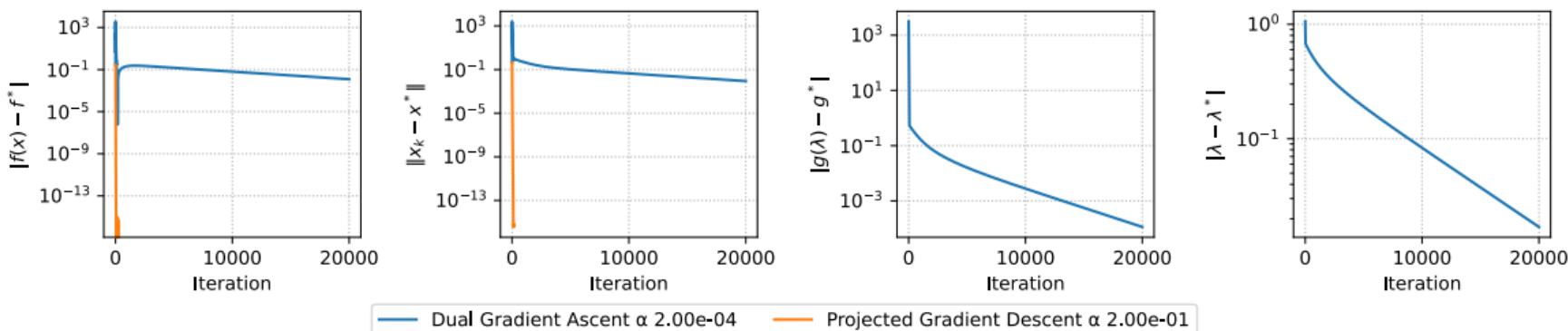
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$$\frac{1}{L} + \frac{1}{\mu}$$

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Quadratic constrained optimization. $n=10, m=5, \mu=0.001, L=10$.

$$\mu = 1e-3$$



Situation is getting worse as soon as we loose strong convexity, the dual convergence will still be linear, but the rate is very low.

Dual decomposition

$B=n$

Consider

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad Ax = b$$

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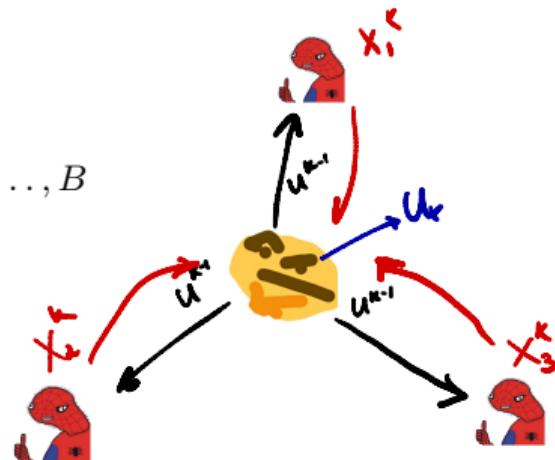
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- **Broadcast:** Send u to each of the B processors, each optimizes in parallel to find x_i .
- **Gather:** Collect $A_i x_i$ from each processor, update the global dual variable u .

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Consider the optimization problem:

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$$u^k = \left(u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b \right) \right)_+$$

where $(u)_+$ denotes the positive part of u , i.e., $(u)_+ = \max\{0, u_i\}$, for $i = 1, \dots, m$.

↑ нозеркүнде үшін $u \geq 0$

Price Coordination Interpretation (Vandenberghe)

- **System Overview:** Consider a system with B units, where each unit independently chooses its decision variable x_i , which determines how to allocate its goods.

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 - Never let prices get negative; hence the use of the positive part notation $(\cdot)_+$.

Augmented Lagrangian method aka method of multipliers

Dual ascent disadvantage: convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

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$$\min_x f(x) + \frac{\rho}{2} \|Ax - b\|^2$$

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Dual gradient ascent: The iterative updates are given by:

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Augmented Lagrangian method aka method of multipliers

Notice step size choice $\alpha_k = \rho$ in dual algorithm. Why?

Since x_k minimizes the function:

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over x , we have the stationarity condition:

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This represents the stationarity condition for the original primal problem; under mild conditions, $Ax_k - b \rightarrow 0$ as $k \rightarrow \infty$, so the KKT conditions are satisfied in the limit and x_k, u_k converge to the solutions.

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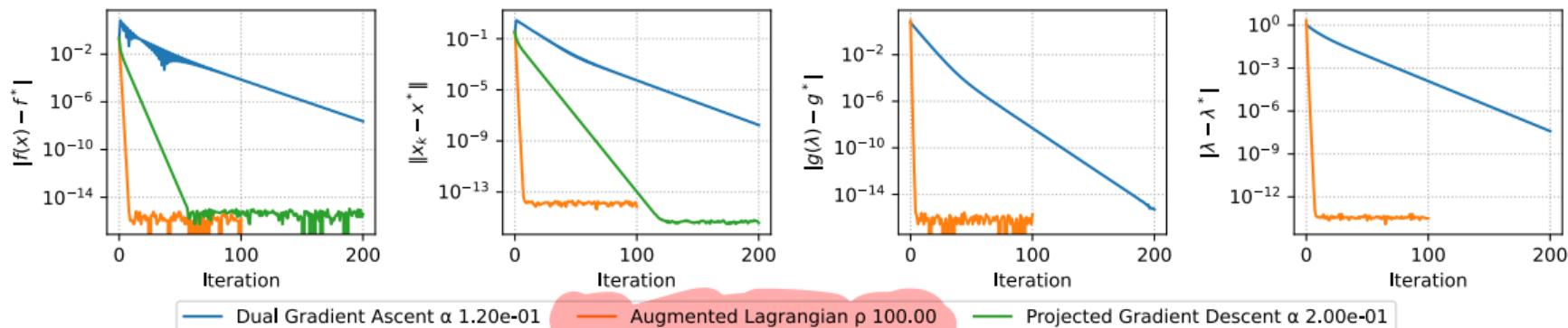
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- **Disadvantage:** We lose decomposability! (Separability is ruined)

Example: equality constrained quadratic minimization.

$$f(x) = \frac{1}{2}x^T Ax - b^T x \rightarrow \min_{x \in \mathbb{R}^n} \quad \text{subject to} \quad Cx = d, \quad A \in \mathbb{S}_+^n, C \in \mathbb{R}^{m \times n}, m < n.$$

Quadratic constrained optimization. $n=10, m=5, \mu=1, L=10$.

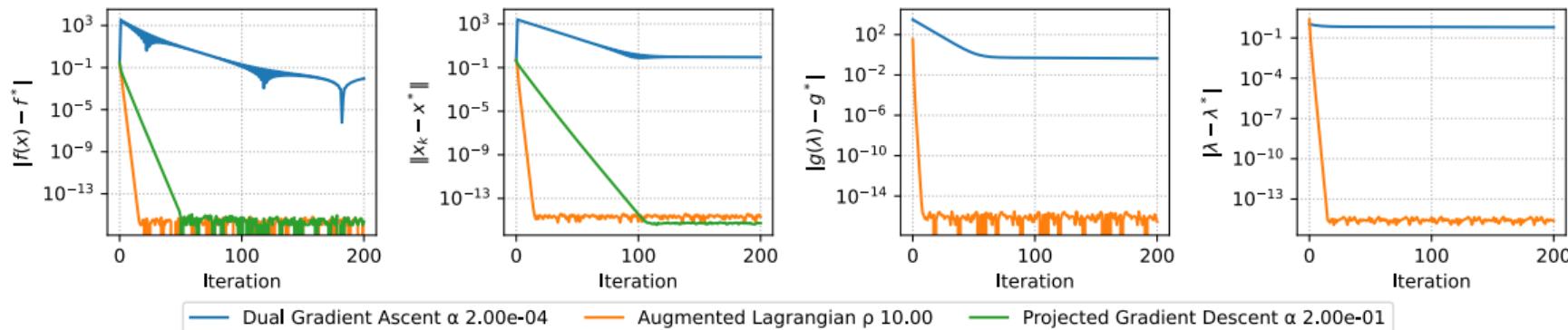


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Alternating Direction Method of Multipliers (ADMM)

Alternating direction method of multipliers or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

$$\begin{aligned} & \min_{x,z} f(x) + g(z) \\ \text{s.t. } & Ax + Bz = c \end{aligned}$$

$$\min_{x \in \mathbb{R}^n} f(x) + V(x)$$

↑ PROX.

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where $\rho > 0$ is a parameter. The augmented Lagrangian for this problem is defined as:

$$L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2} \|Ax + Bz - c\|^2$$

Alternating Direction Method of Multipliers (ADMM)

ADMM repeats the following steps, for $k = 1, 2, 3, \dots$:

1. Update x :

$$x_k = \arg \min_x L_\rho(x, z_{k-1}, u_{k-1})$$

$$\cancel{x} \cdot y$$

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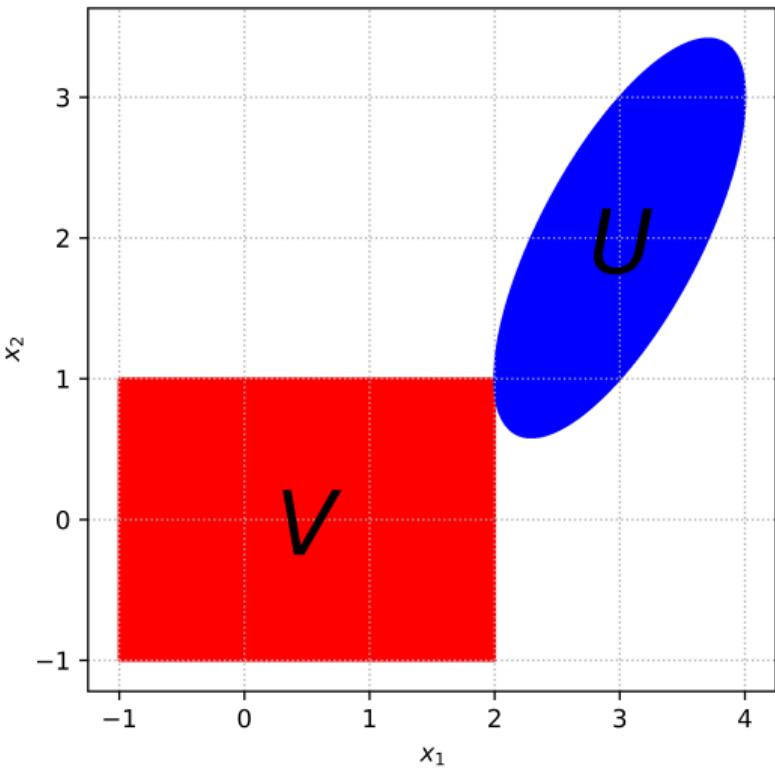
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Note: The usual method of multipliers would replace the first two steps by a joint minimization:

$$(x^{(k)}, z^{(k)}) = \arg \min_{x, z} L_\rho(x, z, u^{(k-1)})$$

Example: Alternating Projections



Consider finding a point in the intersection of convex sets $U, V \subseteq \mathbb{R}^n$:

$$\min_x I_U(x) + I_V(x)$$

To transform this problem into ADMM form, we express it as:

$$Ax + Bz = c$$

$$\min_{x,z} I_U(x) + I_V(z) \quad \text{subject to} \quad x - z = 0$$

Each ADMM cycle involves two projections:

$$\left\{ \begin{array}{l} x_k = \arg \min_x P_U(z_{k-1} - w_{k-1}) \\ z_k = \arg \min_z P_V(x_k + w_{k-1}) \\ w_k = w_{k-1} + x_k - z_k \end{array} \right.$$

Sources

- Ryan Tibshirani. Convex Optimization 10-725