# Subgradient and Subdifferencial

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

# Main notions recap

**Main notions**

For a domain set $E \in \mathbb{R}^n$ and a function $f : E \to \mathbb{R}$:

- A vector $g \in \mathbb{R}^n$ is called **subgradient** of the function $f$ at $x \in E$ if $\forall y \in E$

$$
f(y) \geq f(x) + g^T(y - x)
$$

- A set $\partial f(x)$ is called **subdifferential** of the function $f$ at $x \in E$ if:

$$
\partial f(x) = \{g \in \mathbb{R}^n \mid f(y) \geq f(x) + g^T(y - x)\} \forall y \in E
$$

- $f(\cdot)$ is called **subdifferentiable** at point $x \in E$ if $\partial f(x) \neq$

# Connection between subdifferentiation and convexity

**Connection between subdifferentiation and convexity**

If $f : E \to \mathbb{R}$ is subdifferentiable on the **convex** subset $S \in E$ then $f$ is convex on $S$.

- The inverse is generally incorrect
- There is no sense to derive the subgradient of nonconvex function.

# Connection between subdifferentiation and differentiation

**CConnection between subdifferentiation and differentiation**

1) If $f : E \to \mathbb{R}$ is convex and differentiable at $x \in \text{int } E$ then $\partial f(x) = \{\Delta f(x)\}$
2) If $f : E \to \mathbb{R}$ is convex and for $x \in \text{int } E$ $\partial f(x) = \{s\}$ then $f$ is differentiable at $x$ and $\Delta f(x) = s$

- Derive the subdifferencial of a differentiable function is overkill.

# Problem 1

**Question**

Find the subgradient of the function

$$f(x) = -\sqrt{x}$$

# Subdifferentiation rules

1) $f : E \to \mathbb{R}$, $x \in E$, $c > 0$

$$\Rightarrow \partial(cf)(x) = c\partial f(x)$$

2) $f : F \to \mathbb{R}$, $g : G \to \mathbb{R}$, $x \in F \bigcap G$

$$\Rightarrow \partial(f + g)(x) \supseteq \partial f(x) + \partial g(x)$$

3) $T : V \to W = Ax + b$, $g : W \to \mathbb{R}$, $x_0 \in V$

$$\Rightarrow \partial(g \circ T)(x_0) \supseteq A^* \partial(g)(T(x_0))$$

4) $f(x) = \max(f_1(x), \ldots, f_m(x))$, $I(x) = \{i \in 1 \ldots m | f_i(x) = f(x)\}$

$$\Rightarrow \partial f(x) \supseteq \text{Conv}(\bigcup_{i \in I(x)} \partial f_i(x))$$

**When is equality reached?**

If abovementioned functions are convex and $x$ is inner point then all inequalities turn into equalities.

# Problem 2

**Question**

Find the subgradient of the function $f(x) + g(x)$ if

$$f(x) = -\sqrt{x} \text{ when } x \geq 0$$

$$g(x) = -\sqrt{-x} \text{ when } x \leq 0$$

# Problem 3

**Question**

1) Find the subgradient of the function $f(x) = ||Ax - b||_1$;
2) For task $f(x) = \frac{1}{2}||Ax - b||_2^2 + \lambda||x||_1 \to \min_x$ say which lambdas lead to $x_{opt} = 0$

# Problem 4

**Question**

Check the differentiability of the function

$$f(A) = \sup_{||x||_2 = 1} x^T A x, \text{ where } A \in \mathbb{S}^n, x \in \mathbb{R}^n$$