Linear Programming. Simplex Algorithm. Applications.

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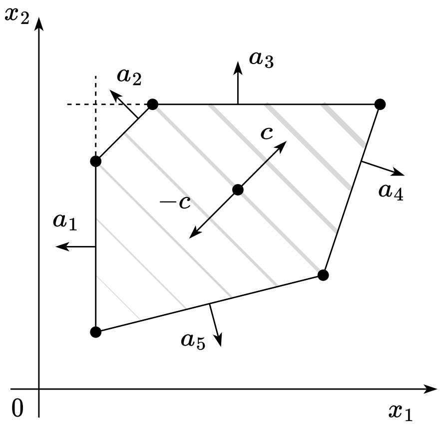
What is Linear Programming?



Generally speaking, all problems with linear objective and linear equalities/inequalities constraints could be considered as Linear Programming. However, there are some formulations.

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\begin{aligned}
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\text{s.t. } & Ax \leq b
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\tag{LP.Basic}
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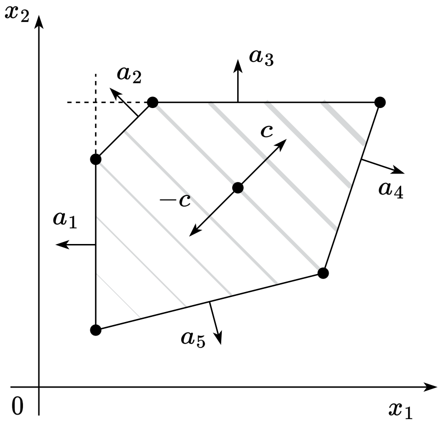

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Standard form. This form seems to be the most intuitive and geometric in terms of visualization. Let us have vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$.

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&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax = b \\
& x_i \geq 0, \; i = 1, \dots, n
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\tag{LP.Standard}
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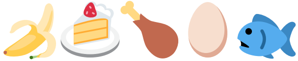
Example: Diet problem



$$\begin{array}{l} \text{Proteins} \\ \text{Carbs} \\ \text{Fats} \\ \text{Calories} \\ \text{Vitamin D} \end{array} \left[\begin{array}{c} \text{Amount per 100g} \\ W \in \mathbb{R}^{n \times p} \end{array} \right]$$

$c \in \mathbb{R}^p$, price per 100g

$r \in \mathbb{R}^n$, nutrient requirements

$x \in \mathbb{R}^p$, amount of products, 100g

$$\min_{x \in \mathbb{R}^p} c^T x$$
$$Wx \succeq r$$
$$x \succeq 0$$

Imagine, that you have to construct a diet plan from some set of products: bananas, cakes, chicken, eggs, fish. Each of the products has its vector of nutrients. Thus, all the food information could be processed through the matrix W. Let us also assume, that we have the vector of requirements for each of nutrients $r \in \mathbb{R}^n$. We need to find the cheapest configuration of the diet, which meets all the requirements:

$$\begin{aligned} &\min_{x \in \mathbb{R}^p} c^\top x \\ \text{s.t. } & Wx \succeq r \\ & x_i \geq 0, \; i = 1, \dots, n \end{aligned}$$

Open In Colab

Basic transformations

- Max-min

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\begin{aligned} &\min_{x \in \mathbb{R}^n} c^\top x \\ &\text{s.t. } Ax \leq b \end{aligned} \quad \leftrightarrow \quad \begin{aligned} &\max_{x \in \mathbb{R}^n} -c^\top x \\ &\text{s.t. } Ax \leq b \end{aligned}
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- Unsigned variables to nonnegative variables.

$$x \leftrightarrow \begin{cases} x = x_+ - x_- \\ x_+ \geq 0 \\ x_- \geq 0 \end{cases}$$

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\min_{x \in \mathbb{R}^n} \|Ax - b\|_\infty \leftrightarrow \min_{x \in \mathbb{R}^n} \max_i |a_i^\top x - b_i|
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ℓ_1 approximation problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_1 \leftrightarrow \min_{x \in \mathbb{R}^n} \sum_{i=1}^n |a_i^\top x - b_i|$$

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\begin{aligned}
&L(x, \nu, \lambda) = c^T x + \nu^T (Ax - b) - \lambda^T x \\
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Has the following dual:

$$
\begin{aligned}
&\max_{\nu \in \mathbb{R}^m} -b^\top \nu \\
\text{s.t. } & -A^T \nu \preceq c
\end{aligned}
\tag{2}
$$

Find the dual problem to the problem above (it should be the original LP). Also, write down KKT for the dual problem, to ensure, they are identical to the primal KKT.

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PROOF. For (i), suppose that Equation 1 has a finite optimal solution x^*. It follows from KKT that there are optimal vectors λ^* and ν^* such that (x^*, ν^*, λ^*) satisfies KKT. We noted above that KKT for Equation 1 and Equation 2 are equivalent. Moreover, $c^T x^* = (-A^T \nu^* + \lambda^*)^T x^* = -(\nu^*)^T A x^* = -b^T \nu^*$, as claimed.

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To prove (ii), suppose that the primal is unbounded, that is, there is a sequence of points x_k, $k = 1, 2, 3, \ldots$ such that

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$$c^T x_k \downarrow -\infty, \quad Ax_k = b, \quad x_k \geq 0.$$

Suppose too that the dual Equation 2 is feasible, that is, there exists a vector $\bar{\nu}$ such that $-A^T\bar{\nu} \leq c$. From the latter inequality together with $x_k \geq 0$, we have that $-\bar{\nu}^T Ax_k \leq c^T x_k$, and therefore

$$-\bar{\nu}^T b = -\bar{\nu}^T Ax_k \leq c^T x_k \downarrow -\infty,$$

yielding a contradiction. Hence, the dual must be infeasible. A similar argument can be used to show that the unboundedness of the dual implies the infeasibility of the primal.

Example: Transportation problem

The prototypical transportation problem deals with the distribution of a commodity from a set of sources to a set of destinations. The object is to minimize total transportation costs while satisfying constraints on the supplies available at each of the sources, and satisfying demand requirements at each of the destinations.

Figure 1: Western Europe Map. Open In Colab

Example: Transportation problem

Customer / Source	Arnhem [€/ton]	Gouda [€/ton]	Demand [tons]
London	n/a	2.5	125
Berlin	2.5	n/a	175
Maastricht	1.6	2.0	225
Amsterdam	1.4	1.0	250
Utrecht	0.8	1.0	225
The Hague	1.4	0.8	200
Supply [tons]	550 tons	700 tons	

$$\text{minimize:} \quad \text{Cost} = \sum_{c \in \text{Customers}} \sum_{s \in \text{Sources}} T[c,s]x[c,s]$$

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$$\sum_{s \in \text{Sources}} x[c,s] = \text{Demand}[c] \qquad \forall c \in \text{Customers}$$

This can be represented in the following graph:

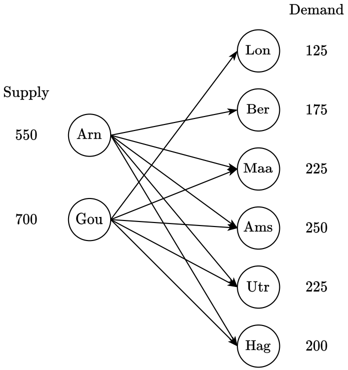
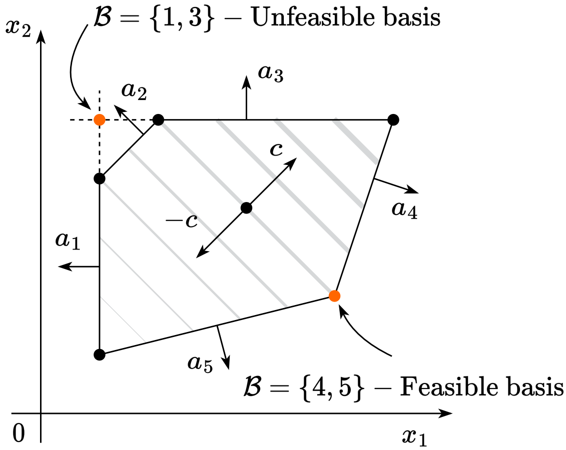


Figure 2: Graph associated with the problem

Geometry of simplex algorithm



We will consider the following simple formulation of LP, which is, in fact, dual to the Standard form:

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- Definition: a **basis** $\mathcal{B}$ is a subset of n (integer) numbers between 1 and m, so that $\text{rank} A_B = n$.

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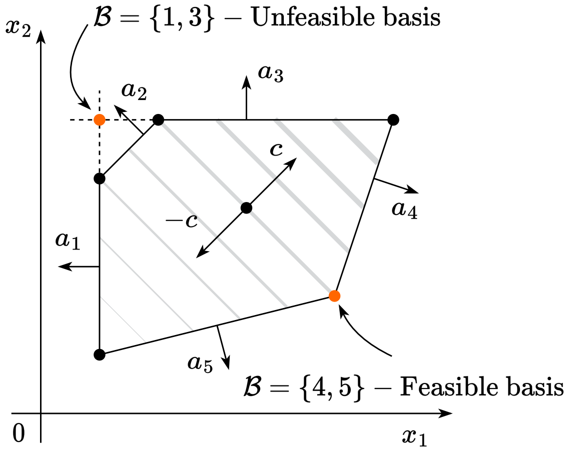

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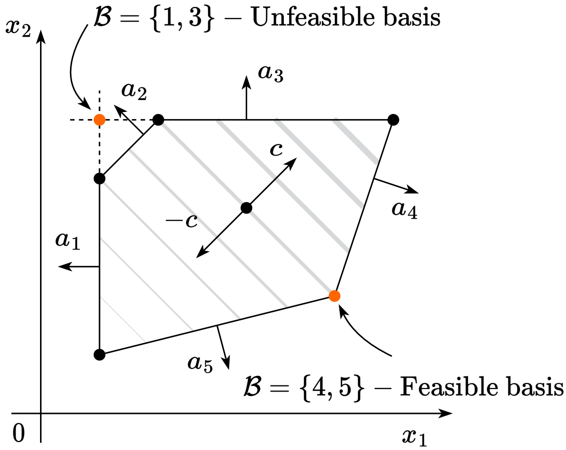

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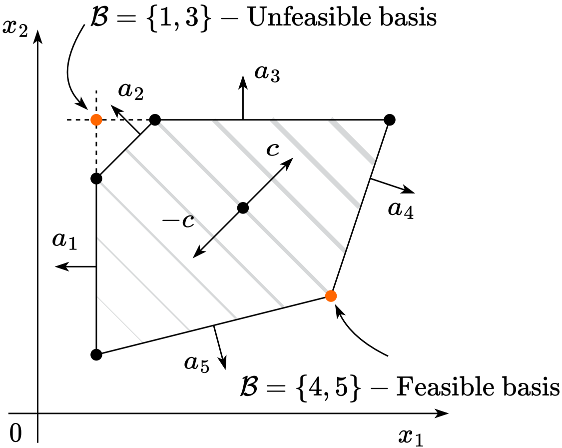

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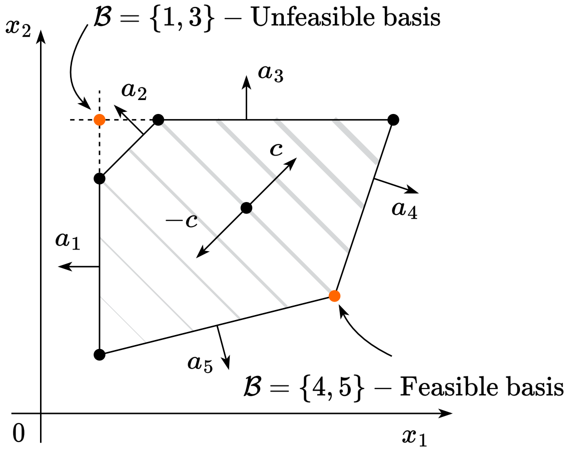

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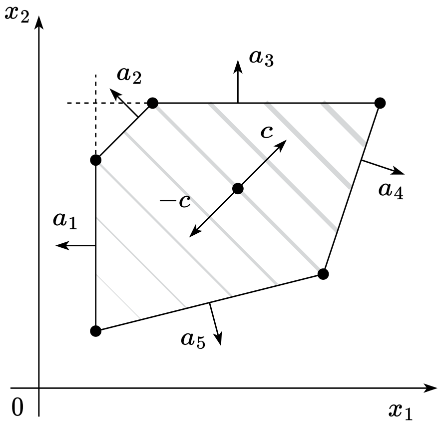
- A basis $\mathcal{B}$ is optimal if x_B is an optimum of the LP.Inequality.

The solution of LP if exists lies in the corner



Theorem

1. If Standard LP has a nonempty feasible region, then there is at least one basic feasible point

The high-level idea of the simplex method is following:

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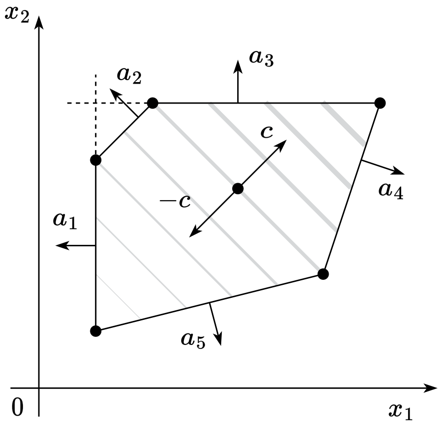

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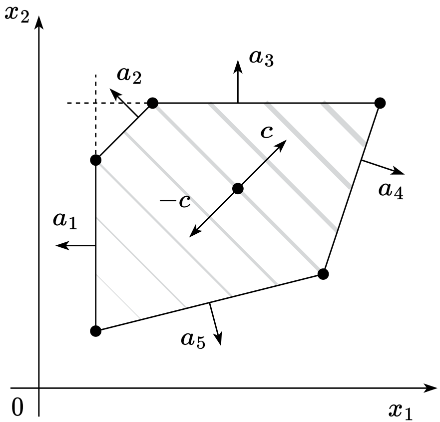

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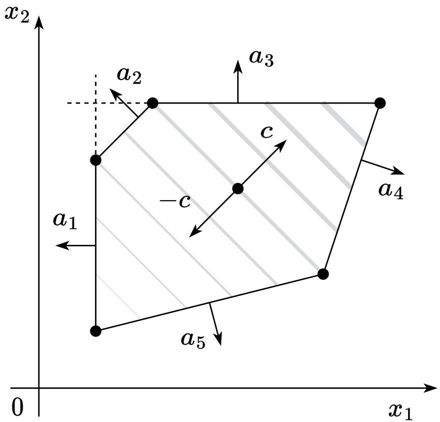

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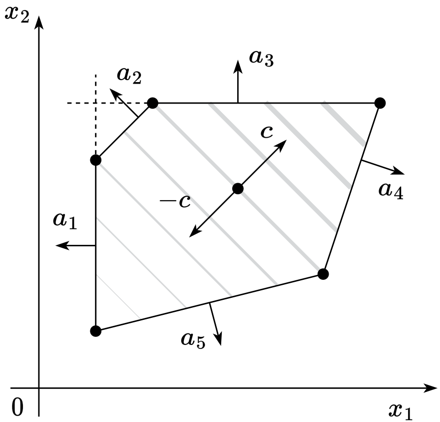

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For proof see Numerical Optimization by Jorge Nocedal and Stephen J. Wright theorem 13.2

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- Ensure, that you are in the corner.

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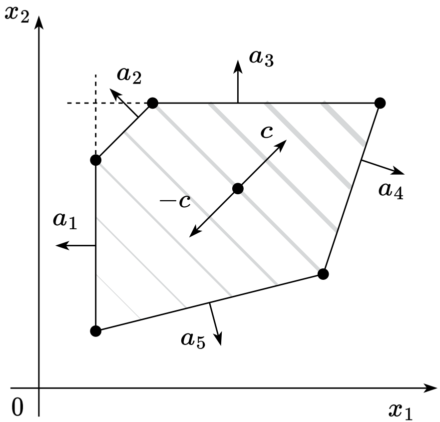

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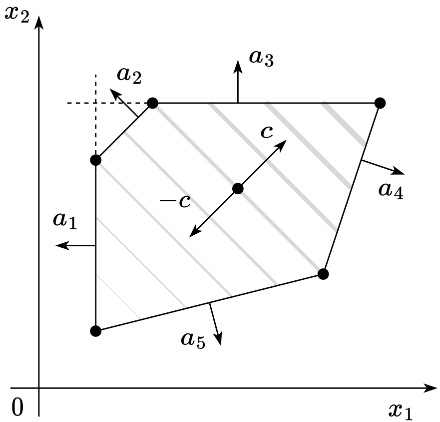

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- Check optimality.
- If necessary, switch the corner (change the basis).

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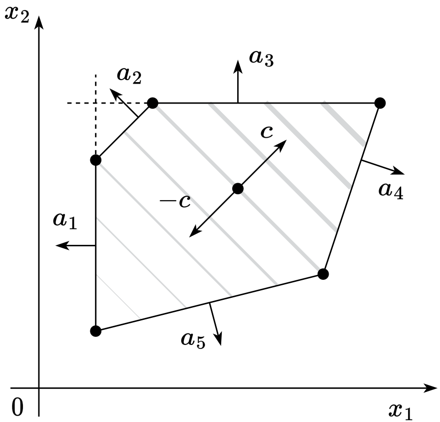

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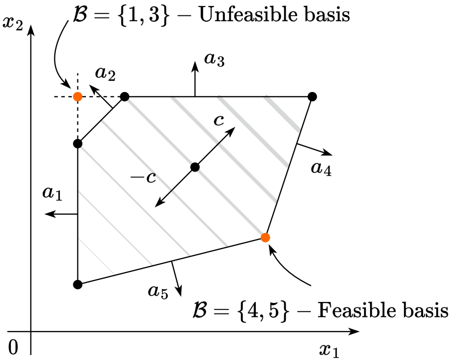
For proof see Numerical Optimization by Jorge Nocedal and Stephen J. Wright theorem 13.2

The high-level idea of the simplex method is following:

- Ensure, that you are in the corner.
- Check optimality.
- If necessary, switch the corner (change the basis).
- Repeat until converge.

Optimal basis



Since we have a basis, we can decompose our objective vector c in this basis and find the scalar coefficients λ_B:

$$\lambda_B^\top A_B = c^\top \leftrightarrow \lambda_B^\top = c^\top A_B^{-1}$$

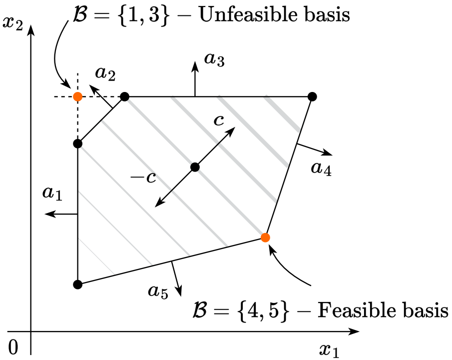
Theorem

If all components of λ_B are non-positive and B is feasible, then B is optimal.

Proof

$$\exists x^* : Ax^* \leq b, c^\top x^* < c^\top x_B$$

Optimal basis



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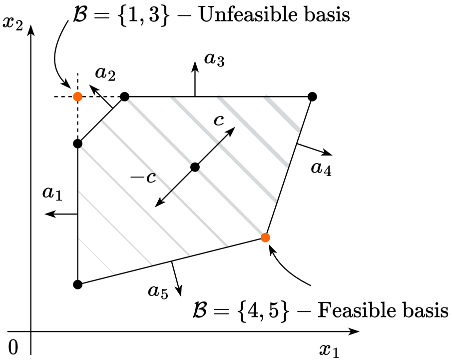

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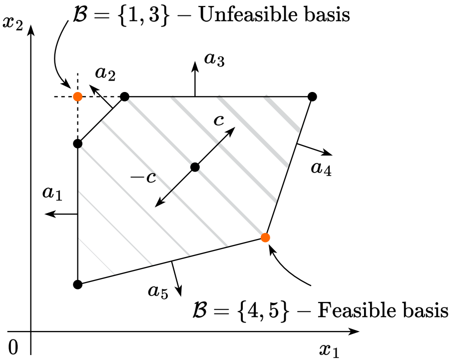

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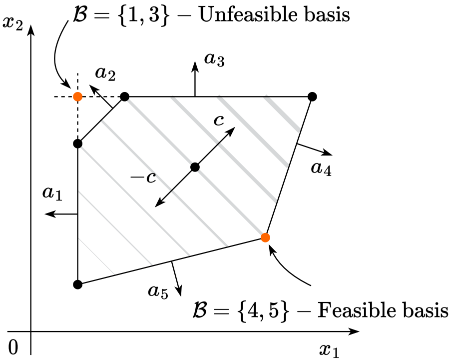

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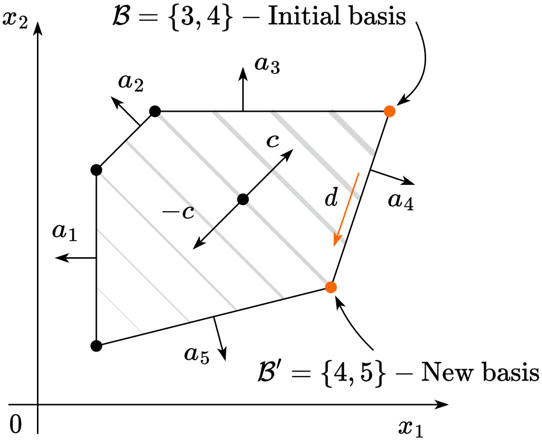
Changing basis



Suppose, some of the coefficients of λ_B are positive. Then we need to go through the edge of the polytope to the new vertex (i.e., switch the basis)

- Suppose, we have a basis $\mathcal{B}$: $\lambda_B^\top = c^\top A_B^{-1}$

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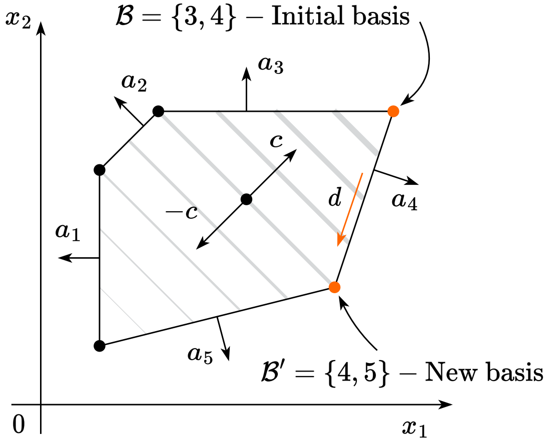

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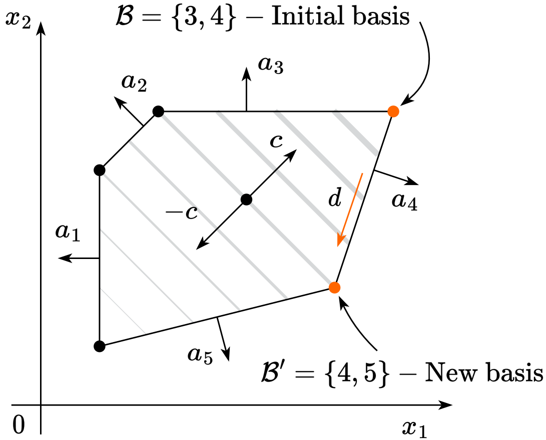
- Suppose, we have a basis $\mathcal{B}$: $\lambda_B^\top = c^\top A_B^{-1}$
- Let's assume, that $\lambda_B^k > 0$. We'd like to drop k from the basis and form a new one:

$$\begin{cases} A_{B \setminus \{k\}} d = 0 \\ a_k^T d = -1 \end{cases}$$

Changing basis



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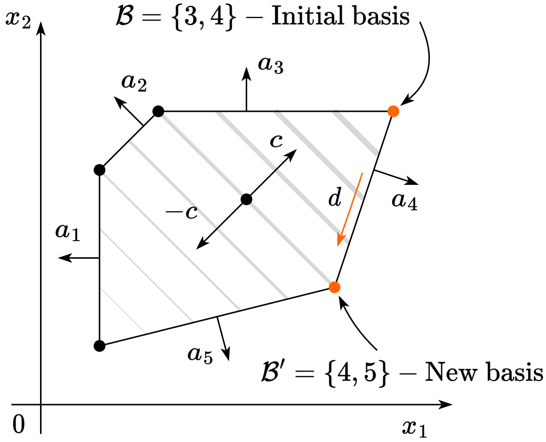

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- For all $j \notin \mathcal{B}$ calculate the projection stepsize:

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- Define the new vertex, that you will add to the new basis:

$$t = \arg\min_j \{\mu_j \mid \mu_j > 0\}$$

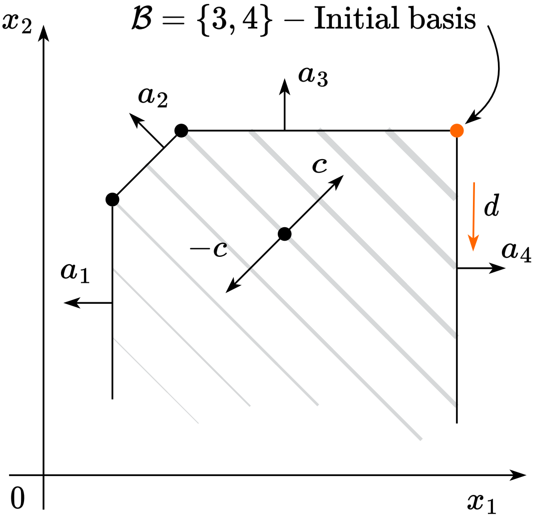
$$\mathcal{B}' = \mathcal{B} \setminus \{k\} \cup \{t\}$$

$$x_{B'} = x_B + \mu_t d = A_{B'}^{-1} b_{B'}$$

Finding an initial basic feasible solution

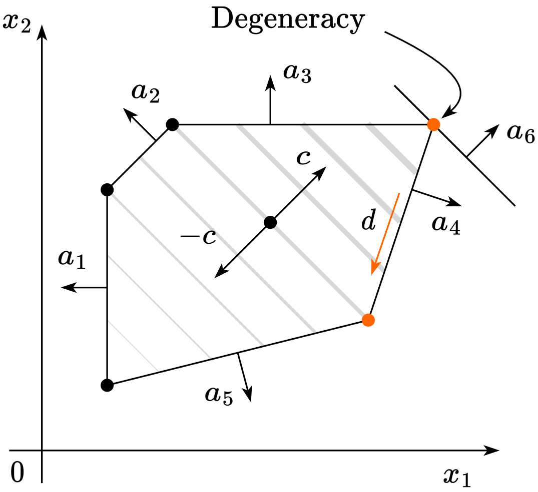
Finding an initial basic feasible solution

Unbounded budget set



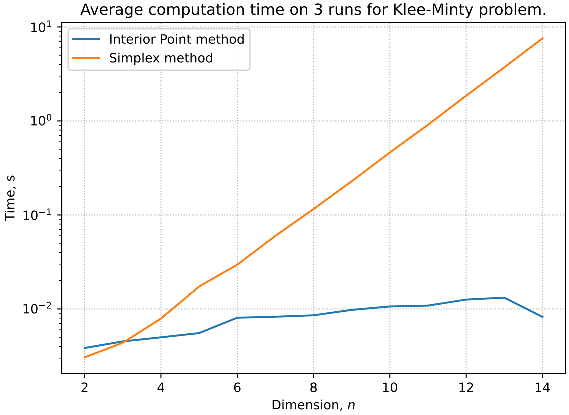
In this case, all μ_j will be negative.

Degeneracy



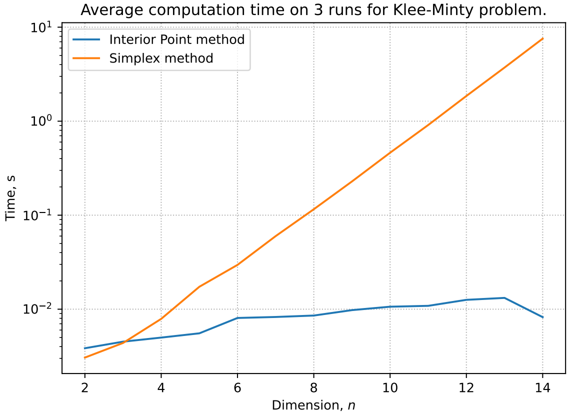
One needs to handle degenerate corners carefully. If no degeneracy exists, one can guarantee a monotonic decrease of the objective function on each iteration.

Exponential convergence



- A wide variety of applications could be formulated as linear programming.

Exponential convergence



- A wide variety of applications could be formulated as linear programming.
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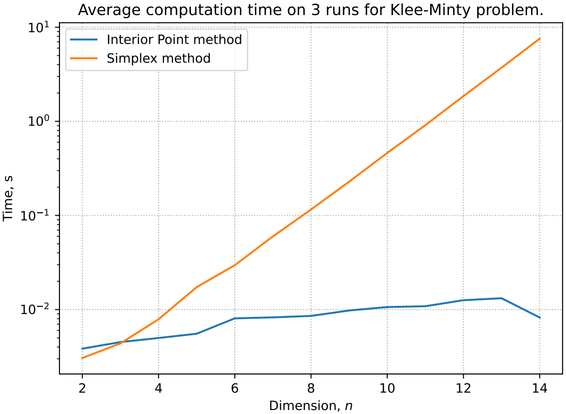

- A wide variety of applications could be formulated as linear programming.
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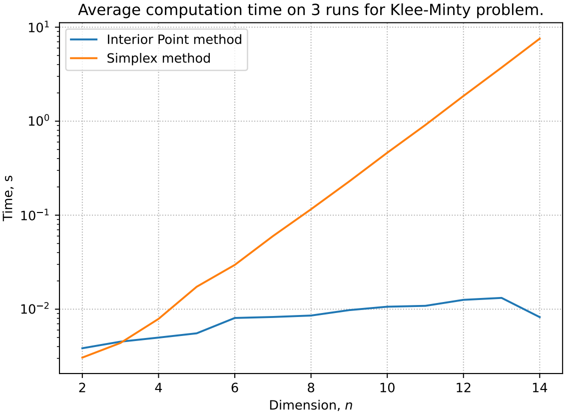
- Khachiyan's ellipsoid method is the first to be proven to run at polynomial complexity for LPs. However, it is usually slower than simplex in real problems.

Exponential convergence



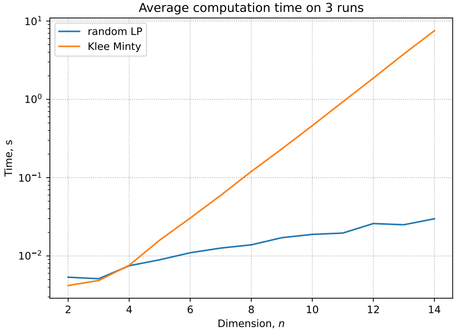
- A wide variety of applications could be formulated as linear programming.
- Simplex algorithm is simple but could work exponentially long.
- Khachiyan's ellipsoid method is the first to be proven to run at polynomial complexity for LPs. However, it is usually slower than simplex in real problems.
- Interior point methods are the last word in this area. However, good implementations of simplex-based methods and interior point methods are similar for routine applications of linear programming.

Klee Minty example

Since the number of edge points is finite, the algorithm should converge (except for some degenerate cases, which are not covered here). However, the convergence could be exponentially slow, due to the high number of edges. There is the following iconic example when the simplex algorithm should perform exactly all vertexes.

In the following problem, the simplex algorithm needs to check $2^n - 1$ vertexes with $x_0 = 0$.

$$
\begin{aligned}
& \max_{x \in \mathbb{R}^n} 2^{n-1}x_1 + 2^{n-2}x_2 + \dots + 2x_{n-1} + x_n \\
\text{s.t. } & x_1 \leq 5 \\
& 4x_1 + x_2 \leq 25 \\
& 8x_1 + 4x_2 + x_3 \leq 125 \\
& \dots \\
& 2^n x_1 + 2^{n-1}x_2 + 2^{n-2}x_3 + \ldots + x_n \leq 5^n \\
& x \geq 0
\end{aligned}
$$



Minimization of convex function as LP

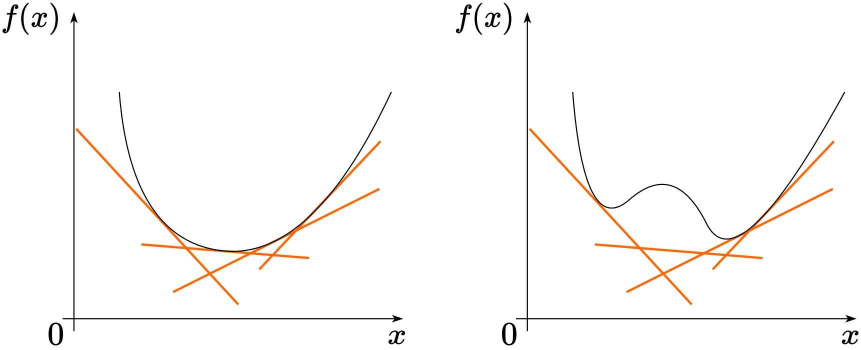
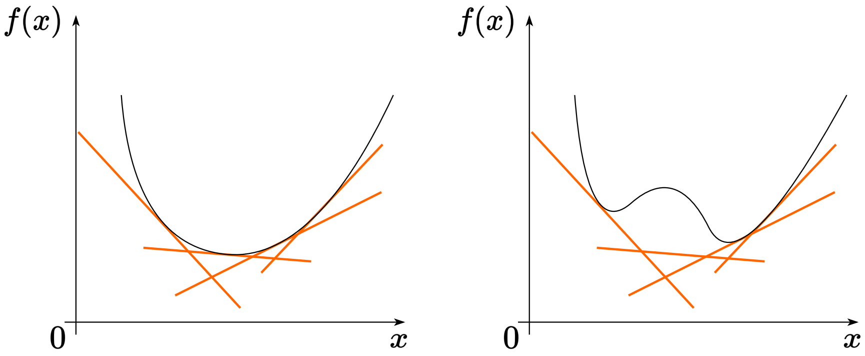


Figure 3: How LP can help with general convex problem

- The function is convex iff it can be represented as a pointwise maximum of linear functions.

Minimization of convex function as LP



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- In high dimensions, the approximation may require too many functions.

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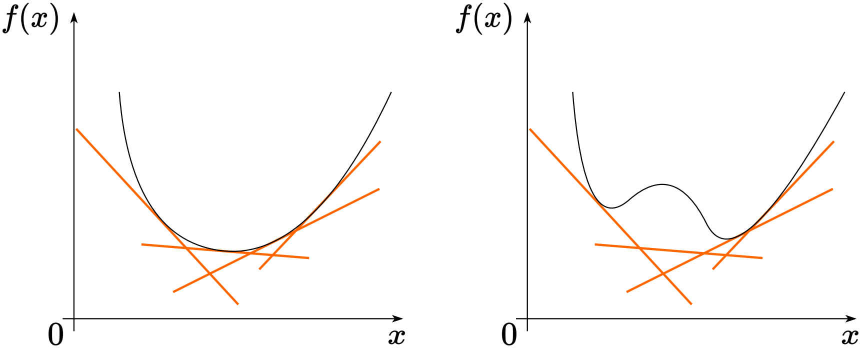

Figure 3: How LP can help with general convex problem

- The function is convex iff it can be represented as a pointwise maximum of linear functions.
- In high dimensions, the approximation may require too many functions.
- More efficient convex optimizers (not reducing to LP) exist.

Hardware progress vs Software progress

Mixed Integer Programming