

# Gradient Descent. Convergence for quadratics; smooth convex case; PL case. Lower bounds.

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The result of this method is

$$x_{k+1} = x_k - \alpha f'(x_k)$$



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Let's consider the following ODE, which is referred to as the Gradient Flow equation.

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Figure 1: Gradient flow trajectory

# Convergence of Gradient Descent algorithm

Heavily depends on the choice of the learning rate  $\alpha$ :



## Exact line search aka steepest descent

$$\alpha_k = \arg \min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \arg \min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. An interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

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$$\nabla f(x_{k+1})^\top \nabla f(x_k) = 0$$

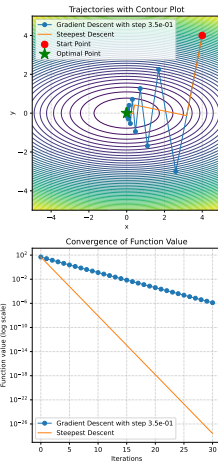



Figure 2: Steepest Descent

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## Coordinate shift

Consider the following quadratic optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}_{++}^d.$$

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$$\begin{aligned} f(\hat{x}) &= \frac{1}{2} (Q\hat{x} + x^*)^\top A (Q\hat{x} + x^*) - b^\top (Q\hat{x} + x^*) \\ &= \frac{1}{2} \hat{x}^\top Q^\top A Q \hat{x} + (x^*)^\top A Q \hat{x} + \frac{1}{2} (x^*)^\top A (x^*) - b^\top Q \hat{x} - b^\top x^* \end{aligned}$$



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## Convergence analysis

Now we can work with the function  $f(x) = \frac{1}{2}x^T \Lambda x$  with  $x^* = 0$  without loss of generality (drop the hat from the  $\hat{x}$ )

$$x^{k+1} = x^k - \alpha^k \nabla f(x^k)$$



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Let's use constant stepsize  $\alpha^k = \alpha$ . Convergence condition:

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$\alpha < \frac{2}{L}$  is needed for convergence.

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Now we would like to tune  $\alpha$  to choose the best (lowest) convergence rate

$$\rho^* = \min_{\alpha} \rho(\alpha)$$

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$$x^{k+1} = \left( \frac{L - \mu}{L + \mu} \right)^k x^0 \quad f(x^{k+1}) = \left( \frac{L - \mu}{L + \mu} \right)^{2k} f(x^0)$$

## Convergence analysis

So, we have a linear convergence in the domain with rate  $\frac{\kappa-1}{\kappa+1} = 1 - \frac{2}{\kappa+1}$ , where  $\kappa = \frac{L}{\mu}$  is sometimes called *condition number* of the quadratic problem.

$\kappa$	$\rho$	Iterations to decrease domain gap 10 times	Iterations to decrease function gap 10 times
1.1	0.05	1	1
2	0.33	3	2
5	0.67	6	3
10	0.82	12	6
50	0.96	58	29
100	0.98	116	58
500	0.996	576	288
1000	0.998	1152	576

## Polyak-Lojasiewicz condition. Linear convergence of gradient descent without convexity

PL inequality holds if the following condition is satisfied for some  $\mu > 0$ ,

$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*) \quad \forall x$$

It is interesting, that the Gradient Descent algorithm might converge linearly even without convexity.

The following functions satisfy the PL condition but are not convex. [🔗Link to the code](#)

$$f(x) = x^2 + 3\sin^2(x)$$



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$$f(x, y) = \frac{(y - \sin x)^2}{2}$$

Non-convex PL function





# Convergence analysis

## Theorem

Consider the Problem

$$f(x) \rightarrow \min_{x \in \mathbb{R}^d}$$

and assume that  $f$  is  $\mu$ -Polyak-Lojasiewicz and  $L$ -smooth, for some  $L \geq \mu > 0$ .

Consider  $(x^k)_{k \in \mathbb{N}}$  a sequence generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying  $0 < \alpha \leq \frac{1}{L}$ . Then:

$$f(x^k) - f^* \leq (1 - \alpha\mu)^k (f(x^0) - f^*).$$

## Convergence analysis

We can use  $L$ -smoothness, together with the update rule of the algorithm, to write

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2$$

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We can now use the Polyak-Lojasiewicz property to write:

$$f(x^{k+1}) \leq f(x^k) - \alpha\mu(f(x^k) - f^*).$$

The conclusion follows after subtracting  $f^*$  on both sides of this inequality and using recursion.



# Any $\mu$ -strongly convex differentiable function is a PL-function

## Theorem

If a function  $f(x)$  is differentiable and  $\mu$ -strongly convex, then it is a PL function.

## Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} \|y - x\|_2^2$$

Putting  $y = x^*$ :

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$$f(x) - f(x^*) \leq \frac{1}{2} \left( \frac{1}{\mu} \|\nabla f(x)\|_2^2 - \left\| \sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}} \nabla f(x) \right\|_2^2 \right)$$



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which is exactly the PL condition. It means, that we already have linear convergence proof for any strongly convex function.

## Smooth convex case

### Theorem

Consider the Problem

$$f(x) \rightarrow \min_{x \in \mathbb{R}^d}$$

and assume that  $f$  is convex and  $L$ -smooth, for some  $L > 0$ .

Let  $(x^k)_{k \in \mathbb{N}}$  be the sequence of iterates generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying  $0 < \alpha \leq \frac{1}{L}$ . Then, for all  $x^* \in \operatorname{argmin} f$ , for all  $k \in \mathbb{N}$  we have that

$$f(x^k) - f^* \leq \frac{\|x^0 - x^*\|^2}{2\alpha k}.$$

# Convergence analysis

- As it was before, we first use smoothness:

$$\begin{aligned}f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2 \\&= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2} \|\nabla f(x^k)\|^2 \\&= f(x^k) - \frac{\alpha}{2} (2 - L\alpha) \|\nabla f(x^k)\|^2 \\&\leq f(x^k) - \frac{\alpha}{2} \|\nabla f(x^k)\|^2, \\f(x^k) - f(x^{k+1}) &\geq \frac{1}{2L} \|\nabla f(x^k)\|^2 \text{ if } \alpha \leq \frac{1}{L}\end{aligned}\tag{1}$$

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- Consider a family of first-order methods, where

$$x^{k+1} \in x^0 + \text{span} \{ \nabla f(x^0), \nabla f(x^1), \dots, \nabla f(x^k) \} \quad (4)$$

## Smooth convex case

### Theorem

There exists a function  $f$  that is  $L$ -smooth and convex such that any method 4 satisfies

$$\min_{i \in [1, k]} f(x^i) - f^* \geq \frac{3L\|x^0 - x^*\|_2^2}{32(1+k)^2}$$

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- The key to the proof is to explicitly build a special function  $f$ .



## Nesterov's worst function

- Let  $d = 2k + 1$  and  $A \in \mathbb{R}^{d \times d}$ .

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix}$$

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- Notice, that

$$x^T A x = x[1]^2 + x[d]^2 + \sum_{i=1}^{d-1} (x[i] - x[i+1])^2,$$

and, from this expression, it's simple to check  
 $0 \preceq A \preceq 4I$ .

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$$\begin{aligned} f(x^*) &= \frac{L}{8} x^{*T} A x^* - \frac{L}{4} \langle x^*, e_1 \rangle \\ &= -\frac{L}{8} \langle x^*, e_1 \rangle = -\frac{L}{8} \left( 1 - \frac{1}{d+1} \right). \end{aligned}$$