

# Subgradient and Subdifferential

## Seminar

Optimization for ML. Faculty of Computer Science. HSE University

# Main notions recap

## i Main notions

For a domain set  $E \in \mathbb{R}^n$  and a function  $f : E \rightarrow \mathbb{R}$ :

- A vector  $g \in \mathbb{R}^n$  is called **subgradient** of the function  $f$  at  $x \in E$  if  $\forall y \in E$

$$f(y) \geq f(x) + g^T(y - x)$$

- A set  $\partial f(x)$  is called **subdifferential** of the function  $f$  at  $x \in E$  if:

$$\partial f(x) = \{g \in \mathbb{R}^n \mid f(y) \geq f(x) + g^T(y - x)\} \forall y \in E$$

- $f(\cdot)$  is called **subdifferentiable** at point  $x \in E$  if  $\partial f(x) \neq \emptyset$

# Connection between subdifferentiation and convexity

## 💡 Connection between subdifferentiation and convexity

If  $f : E \rightarrow \mathbb{R}$  is subdifferentiable on the **convex** subset  $S \in E$  then  $f$  is convex on  $S$ .

- The inverse is generally incorrect
- There is no sense to derive the subgradient of nonconvex function.

# Connection between subdifferentiation and differentiation

## 💡 Connection between subdifferentiation and differentiation

- 1) If  $f : E \rightarrow \mathbb{R}$  is convex and differentiable at  $x \in \text{int } E$  then  $\partial f(x) = \{\Delta f(x)\}$
- 2) If  $f : E \rightarrow \mathbb{R}$  is convex and for  $x \in \text{int } E$   $\partial f(x) = \{s\}$  then  $f$  is differentiable at  $x$  and  $\Delta f(x) = s$

- Derive the subdifferential of a differentiable function is overkill.

# Problem 1

## Question

Find the subgradient of the function

$$f(x) = -\sqrt{x}$$

## Subdifferentiation rules

1)  $f : E \rightarrow \mathbb{R}, x \in E, c > 0$

$$\Rightarrow \partial(cf)(x) = c\partial f(x)$$

2)  $f : F \rightarrow \mathbb{R}, g : G \rightarrow \mathbb{R}, x \in F \cap G$

$$\Rightarrow \partial(f + g)(x) \supseteq \partial f(x) + \partial g(x)$$

3)  $T : V \rightarrow W = Ax + b, g : W \rightarrow \mathbb{R}, x_0 \in V$

$$\Rightarrow \partial(g \circ T)(x_0) \supseteq A^* \partial(g)(T(x_0))$$

4)  $f(x) = \max(f_1(x), \dots, f_m(x)), I(x) = \{i \in 1 \dots m \mid f_i(x) = f(x)\}$

$$\Rightarrow \partial f(x) \supseteq \text{Conv}\left(\bigcup_{i \in I(x)} \partial f_i(x)\right)$$

💡 When is equality reached?

If abovementioned functions are convex and  $x$  is inner point then all inequalities turn into equalities.

## Problem 2

### Question

Find the subgradient of the function  $f(x) + g(x)$  if

$$f(x) = -\sqrt{x} \text{ when } x \geq 0$$

$$g(x) = -\sqrt{-x} \text{ when } x \leq 0$$

## Problem 3

### Question

- 1) Find the subgradient of the function  $f(x) = \|Ax - b\|_1$ ;
- 2) For task  $f(x) = \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1 \rightarrow \min_x$  say which lambdas lead to  $x_{opt} = 0$



## Problem 4

### Question

Check the differentiability of the function

$$f(A) = \sup_{||x||_2=1} x^T A x, \text{ where } A \in \mathbb{S}^n, x \in \mathbb{R}^n$$