

# Subgradient Method. Specifics of non-smooth problems.

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# $\ell_1$ -regularized linear least squares

$\ell_1$  induces sparsity

$\ell_2$  regularization.  $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_2 \leq 1}$



$\ell_1$  regularization.  $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_1 \leq 1}$



@fminxyz

# Norms are not smooth

$$\min_{x \in \mathbb{R}^n} f(x),$$

A classical convex optimization problem is considered. We assume that  $f(x)$  is a convex function, but now we do not require smoothness.



Figure 1: Norm cones for different  $p$  - norms are non-smooth

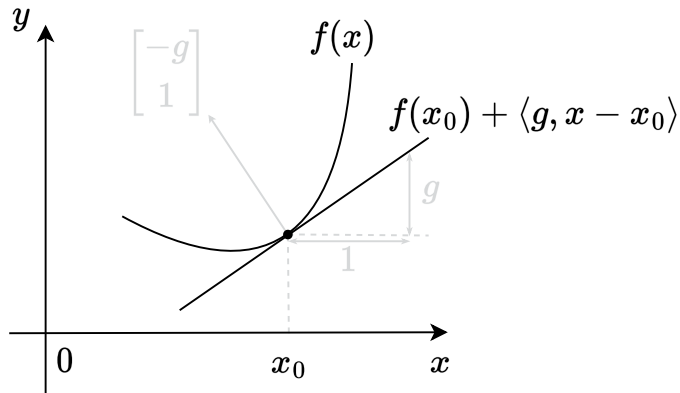
# Wolfe's example

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Figure 2: Wolfe's example. [Open in Colab](#)

## Convex function linear lower bound



An important property of a continuous convex function  $f(x)$  is that at any chosen point  $x_0$  for all  $x \in \text{dom } f$  the inequality holds:

$$f(x) \geq f(x_0) + \langle g, x - x_0 \rangle$$

Figure 3: Taylor linear approximation serves as a global lower bound for a convex function

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for some vector  $g$ , i.e., the tangent to the graph of the function is the *global* estimate from below for the function.

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We wouldn't want to lose such a nice property.

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# Subgradient and subdifferential

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Figure 4: Subdifferential is a set of all possible subgradients

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Let  $f : S \rightarrow \mathbb{R}$  be a function defined on the set  $S$  in a Euclidean space  $\mathbb{R}^n$ . If  $x_0 \in \text{ri}(S)$  and  $f$  is differentiable at  $x_0$ , then either  $\partial f(x_0) = \emptyset$  or  $\partial f(x_0) = \{\nabla f(x_0)\}$ . Moreover, if the function  $f$  is convex, the first scenario is impossible.

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### Proof

1. Assume, that  $s \in \partial f(x_0)$  for some  $s \in \mathbb{R}^n$  distinct from  $\nabla f(x_0)$ . Let  $v \in \mathbb{R}^n$  be a unit vector. Because  $x_0$  is an interior point of  $S$ , there exists  $\delta > 0$  such that  $x_0 + tv \in S$  for all  $0 < t < \delta$ . By the definition of the subgradient, we have

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$$\langle \nabla f(x_0), v \rangle = \lim_{t \rightarrow 0; 0 < t < \delta} \frac{f(x_0 + tv) - f(x_0)}{t} \geq \langle s, v \rangle$$

2. From this,  $\langle s - \nabla f(x_0), v \rangle \geq 0$ . Due to the arbitrariness of  $v$ , one can set

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3. Furthermore, if the function  $f$  is convex, then according to the differential condition of convexity  $f(x) \geq f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle$  for all  $x \in S$ . But by definition, this means  $\nabla f(x_0) \in \partial f(x_0)$ .

# Subdifferential calculus

Moreau - Rockafellar theorem (subdifferential of a linear combination)

Let  $f_i(x)$  be convex functions on convex sets  $S_i$ ,  $i = \overline{1, n}$ . Then if  $\bigcap_{i=1}^n \text{ri} S_i \neq \emptyset$  then the function  $f(x) =$

$\sum_{i=1}^n a_i f_i(x)$ ,  $a_i > 0$  has a subdifferential  $\partial_S f(x)$  on

the set  $S = \bigcap_{i=1}^n S_i$  and

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Dubovitsky - Milutin theorem (subdifferential of a point-wise maximum)

Let  $f_i(x)$  be convex functions on the open convex set  $S \subseteq \mathbb{R}^n$ ,  $x_0 \in S$ , and the pointwise maximum is defined as  $f(x) = \max_i f_i(x)$ . Then:

$$\partial_S f(x_0) = \text{conv} \left\{ \bigcup_{i \in I(x_0)} \partial_S f_i(x_0) \right\}, \quad I(x) = \{i \in [1, n] \mid f_i(x) = f(x)\}$$

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- $\partial(f(Ax + b))(x) = A^T \partial f(Ax + b)$ ,  $f$  - convex function
- $z \in \partial f(x)$  if and only if  $x \in \partial f^*(z)$ .

# Algorithm

A vector  $g$  is called the **subgradient** of the function  $f(x) : S \rightarrow \mathbb{R}$  at the point  $x_0$  if  $\forall x \in S$ :

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The idea is very simple: let's replace the gradient  $\nabla f(x_k)$  in the gradient descent algorithm with a subgradient  $g_k$  at point  $x_k$ :

$$x_{k+1} = x_k - \alpha_k g_k,$$

where  $g_k$  is an arbitrary subgradient of the function  $f(x)$  at the point  $x_k$ ,  $g_k \in \partial f(x_k)$

# Convergence bound

$$\|x_{k+1} - x^*\|^2 = \|x_k - x^* - \alpha_k g_k\|^2 =$$



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- For a subgradient:  $\langle g_k, x_k - x^* \rangle \leq f(x_k) - f(x^*) = f(x_k) - f^*$ .

## Convergence bound

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Let us sum the obtained equality for  $k = 0, \dots, T-1$ :

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Important notes:

- Obtaining bounds not for  $x_T$  but for the arithmetic mean over iterations  $\bar{x}$  is a typical trick in obtaining estimates for methods where there is convexity but no monotonic decreasing at each iteration. There is no guarantee of success at each iteration, but there is a guarantee of success on average

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- To choose the optimal step, we need to know (assume) the number of iterations in advance. Possible solution: initialize  $T$  with a small value, after reaching this number of iterations double  $T$  and restart the algorithm. A more intelligent way: adaptive selection of stepsize.

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Which leads to exactly the same bound of  $\mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$  on the primal gap. In fact, for this class of functions, you can't get a better result than  $\frac{1}{\sqrt{T}}$ .

# Convergence results

## Theorem

Let  $f$  be a convex  $G$ -Lipschitz function. For a fixed step size  $\alpha = \frac{\|x_0 - x^*\|_2}{G} \sqrt{\frac{1}{K}}$ , subgradient method satisfies

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- Proved result requires pre-defined step size strategy, which is not practical (usually one can just use several diminishing strategies).
- There is no monotonic decrease of objective.
- Convergence is slower, than for the gradient descent (smooth case). However, if we will go deeply for the problem structure, we can improve convergence (proximal gradient method).



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Let  $f$  be a convex  $G$ -Lipschitz function and  $f_k^{\text{best}} = \min_{i=1,\dots,k} f(x^i)$ . For a fixed step size  $\alpha$ , subgradient method satisfies

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# Linear Least Squares with $l_1$ -regularization

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - b\|_2^2 + \lambda \|x\|_1$$

Algorithm will be written as:

$$x_{k+1} = x_k - \alpha_k \left( A^\top (Ax_k - b) + \lambda \text{sign}(x_k) \right)$$

where signum function is taken element-wise.

LLS with  $l_1$  regularization. 2 runs.  $\lambda = 1$



## Regularized logistic regression

Given  $(x_i, y_i) \in \mathbb{R}^p \times \{0, 1\}$  for  $i = 1, \dots, n$ , the logistic regression function is defined as:

$$f(\theta) = \sum_{i=1}^n \left( -y_i x_i^T \theta + \log(1 + \exp(x_i^T \theta)) \right)$$

This is a smooth and convex function with its gradient given by:

$$\nabla f(\theta) = \sum_{i=1}^n (y_i - s_i(\theta)) x_i$$

where  $s_i(\theta) = \frac{\exp(x_i^T \theta)}{1 + \exp(x_i^T \theta)}$ , for  $i = 1, \dots, n$ . Consider the regularized problem:

$$f(\theta) + \lambda r(\theta) \rightarrow \min_{\theta}$$

where  $r(\theta) = \|\theta\|_2^2$  for the ridge penalty, or  $r(\theta) = \|\theta\|_1$  for the lasso penalty.

# Support Vector Machines

Let  $D = \{(x_i, y_i) \mid x_i \in \mathbb{R}^n, y_i \in \{\pm 1\}\}$

We need to find  $\theta \in \mathbb{R}^n$  and  $b \in \mathbb{R}$  such that

$$\min_{\theta \in \mathbb{R}^n, b \in \mathbb{R}} \frac{1}{2} \|\theta\|_2^2 + C \sum_{i=1}^m \max[0, 1 - y_i(\theta^\top x_i + b)]$$