# Conditional gradient methods. Projected Gradient Descent. Frank-Wolfe Method. Mirror Descent Algorithm Idea.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

# Projection

The **distance** $d$ from point $\mathbf{y} \in \mathbb{R}^n$ to closed set $S \subset \mathbb{R}^n$:

$$d(\mathbf{y}, S, \| \cdot \|) = \inf\{\|x - y\| \mid x \in S\}$$

We will focus on **Euclidean projection** (other options are possible) of a point $\mathbf{y} \in \mathbb{R}^n$ on set $S \subseteq \mathbb{R}^n$ is a point $\text{proj}_S(\mathbf{y}) \in S$:

$$\text{proj}_S(\mathbf{y}) = \frac{1}{2} \underset{\mathbf{x} \in S}{\operatorname{argmin}} \|x - y\|_2^2$$

- **Sufficient conditions of existence of a projection**. If $S \subseteq \mathbb{R}^n$ - closed set, then the projection on set $S$ exists for any point.
- **Sufficient conditions of uniqueness of a projection**. If $S \subseteq \mathbb{R}^n$ - closed convex set, then the projection on set $S$ is unique for any point.
- If a set is open, and a point is beyond this set, then its projection on this set does not exist.
- If a point is in set, then its projection is the point itself.

# Projection

**Bourbaki-Cheney-Goldstein inequality theorem**

Let $S \subseteq \mathbb{R}^n$ be closed and convex, $\forall x \in S, y \in \mathbb{R}^n$. Then

$$\langle y - \text{proj}_S(y), \mathbf{x} - \text{proj}_S(y)\rangle \leq 0 \tag{1}$$

$$\|x - \text{proj}_S(y)\|^2 + \|y - \text{proj}_S(y)\|^2 \leq \|x - y\|^2 \tag{2}$$

**Non-expansive function**

A function $f$ is called **non-expansive** if $f$ is $L$-Lipschitz with $L \leq 1$ [1]. That is, for any two points $x, y \in \text{dom} f$,

$$\|f(x) - f(y)\| \leq L\|x - y\|, \text{ where } L \leq 1.$$

It means the distance between the mapped points is possibly smaller than that of the unmapped points.

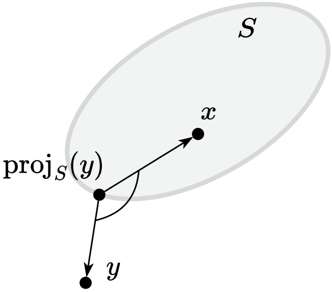


Figure 1: Obtuse or straight angle should be for any point $x \in S$

[1] Non-expansive becomes contractive if $L < 1$.

# Problems

## Question

Is projection operator non-expansive?

## Question

Find projection $\text{proj}_S(\mathbf{y})$ onto $S$, where $S$:

- $l_2$-ball with center 0 and radius 1:

$$S = \{x \in \mathbb{R}^d \mid \|x\|_2^2 = \sum_{i=1}^d x_i^2 \leq 1\}$$

- $\mathbb{R}^d$-cube:

$$S = \{x \in \mathbb{R}^d \mid a_i \leq x_i \leq b_i\}$$

- Affine constraints:

$$S = \{x \in \mathbb{R}^d \mid Ax = b\}$$

# Projected Gradient Descent (PGD). Idea

$$x_{k+1} = \text{proj}_S \left(x_k - \alpha_k \nabla f(x_k)\right) \qquad \Leftrightarrow \qquad \begin{aligned} y_k &= x_k - \alpha_k \nabla f(x_k) \\ x_{k+1} &= \text{proj}_S \left(y_k\right) \end{aligned}$$

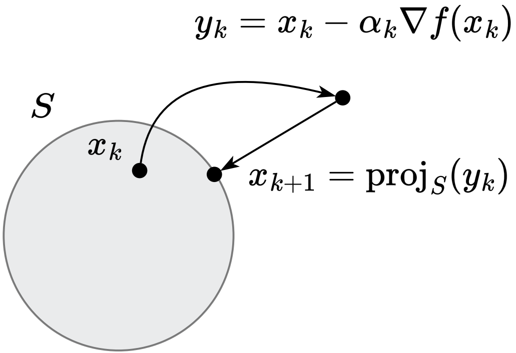


Figure 2: Illustration of Projected Gradient Descent algorithm

# Frank-Wolfe Method (FWM). Idea

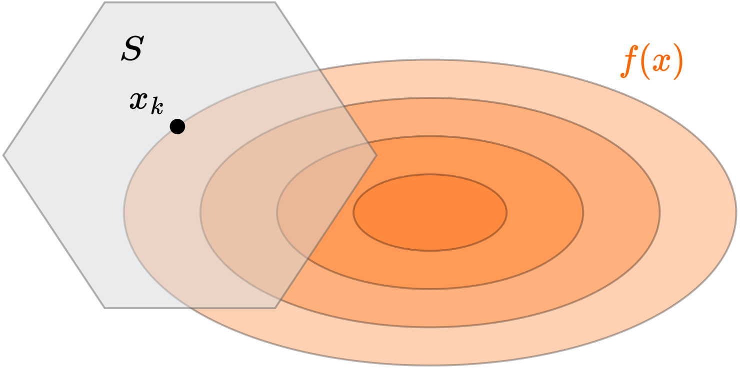


Figure 3: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Frank-Wolfe Method (FWM). Idea

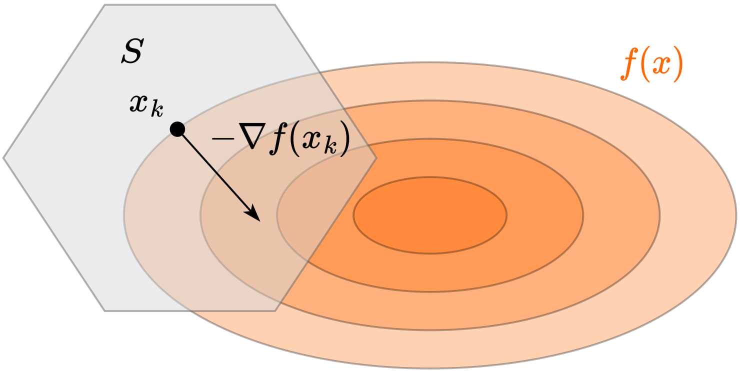


Figure 4: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Frank-Wolfe Method (FWM). Idea

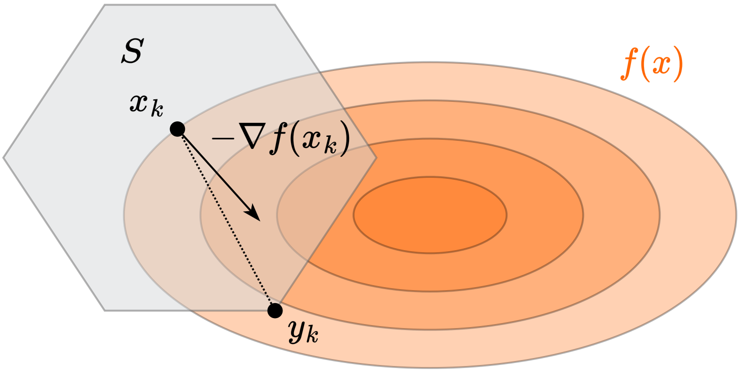


Figure 5: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Frank-Wolfe Method (FWM). Idea

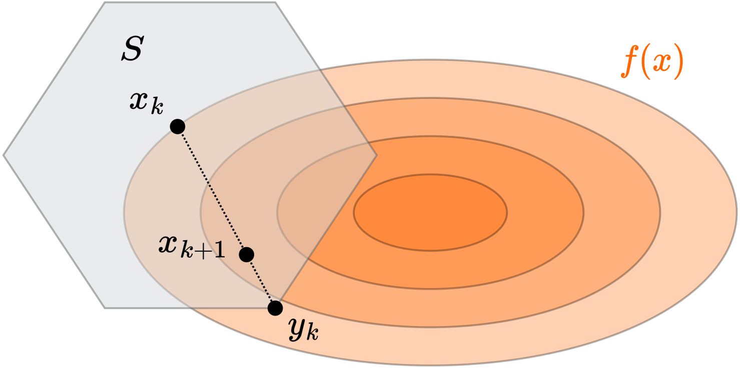


Figure 6: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Frank-Wolfe Method (FWM). Idea

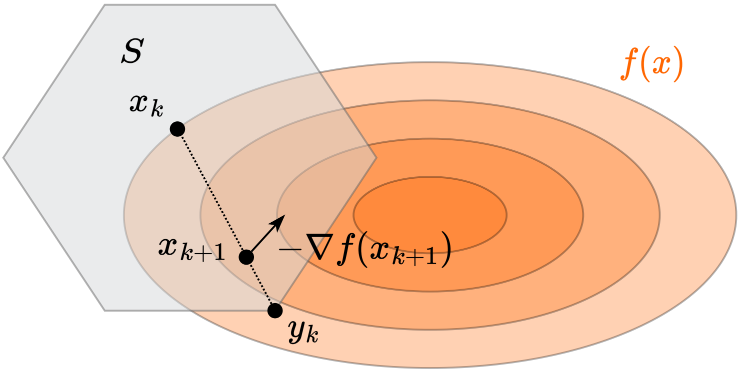


Figure 7: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Frank-Wolfe Method (FWM). Idea

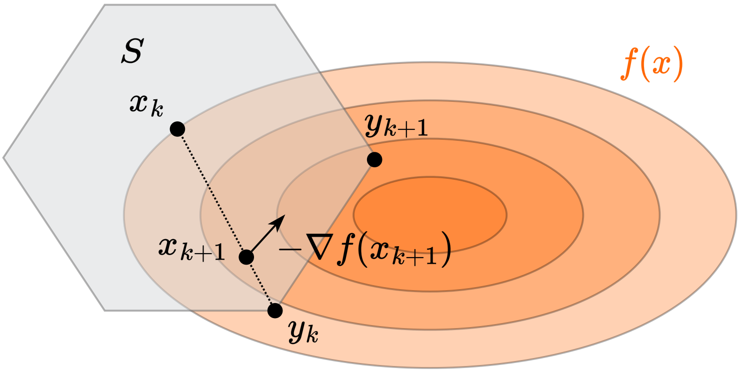


Figure 8: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Frank-Wolfe Method (FWM). Idea

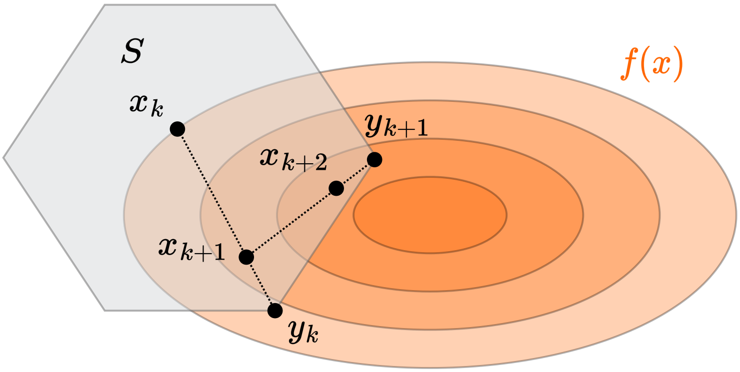


Figure 9: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Frank-Wolfe Method (FWM). Idea

$$y_k = \text{arg}\min_{x \in S} f^I_{x_k}(x) = \text{arg}\min_{x \in S} \langle \nabla f(x_k), x \rangle$$

$$x_{k+1} = \gamma_k x_k + (1 - \gamma_k) y_k$$

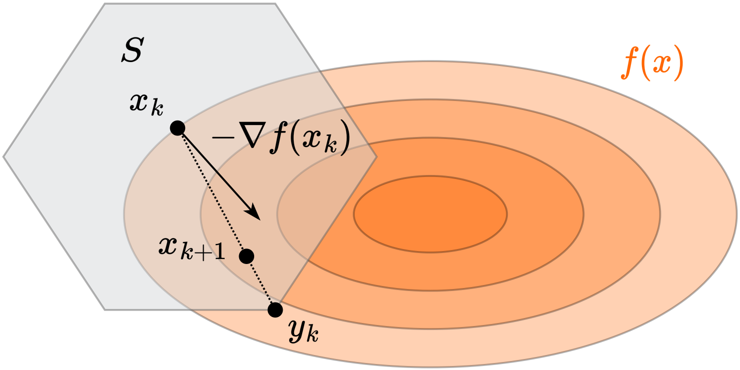


Figure 10: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Convergence rate for smooth and convex case

**Theorem**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Let $S \subseteq \mathbb{R}^n$d be a closed convex set, and assume that there is a minimizer $x^*$ of $f$ over $S$; furthermore, suppose that $f$ is smooth over $S$ with parameter $L$.

- The **Projected Gradient Descent** algorithm with stepsize $\frac{1}{L}$ achieves the following convergence after iteration $k > 0$:
$$f(x_k) - f^* \leq \frac{L\|x_0 - x^*\|_2^2}{2k}$$
- The **Frank-Wolfe Method** achieves the following convergence after iteration $k > 0$:
$$f(x_k) - f^* \leq \frac{2L\|x_0 - x^*\|_2^2}{k+1}$$

**FWM specificity**

- FWM convergence rate for the $\mu$-strongly convex functions is $\mathcal{O}\left(\frac{1}{k}\right)$
- FWM doesn't work for non-smooth functions. But modifications do.
- FWM works for any norm.

# Subgradient method: linear approximation + proximity

Recall SubGD step with sub-gradient $g_k$:

$$x_{k+1} = x_k - \alpha_k g_k \qquad \Leftrightarrow \qquad \begin{aligned} x_{k+1} &= \operatorname*{argmin}_x \underbrace{f(x_k) + g_k^\top (x - x_k)}_{\text{linear approximation to f}} + \underbrace{\frac{1}{2\alpha}\|x - x_k\|_2^2}_{\text{proximity term}} \\ &= \operatorname*{argmin}_x \alpha g_k^\top x + \frac{1}{2}\|x - x_k\|_2^2 \end{aligned}$$

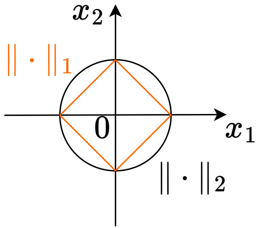


Figure 11: $\|\cdot\|_1$ is not spherical symmetrical

# Example. Poor condition

Consider $f(x_1, x_2) = x_1^2 \cdot \frac{1}{100} + x_2^2 \cdot 100$.

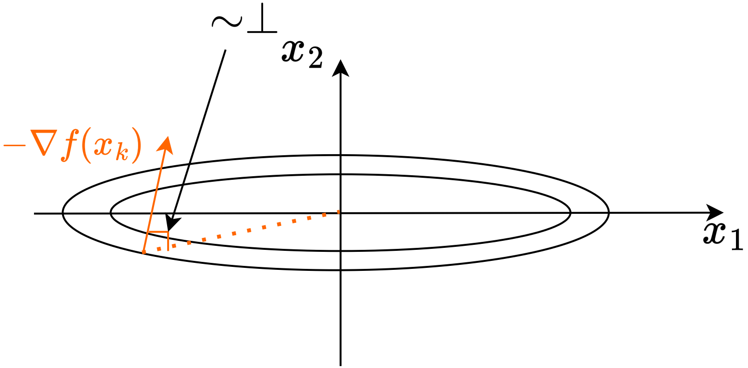


Figure 12: Poorly conditioned problem in $\|\cdot\|_2$ norm

# Example. Poor condition

Suppose we are at the point: $x_k = (-10 \quad -0.1)^\top$. SubGD method: $x_{k+1} = x_k - \alpha \nabla f(x_k)$

$$\nabla f(x_k) = \left(\frac{2x_1}{100} \quad 2x_2 \cdot 100\right)^\top \Big|_{(-10 \ -0.1)^\top} = \left(-\frac{1}{5} \quad -20\right)^\top$$

**The problem:** due to elongation of the level sets the direction of movement $(x_{k+1} - x_k)$ is $\sim\perp$ $(x^* - x_k)$.

**The solution:** Change proximity term

$$x_{k+1} = \underset{x}{\text{argmin}} \underbrace{f(x_k) + g_k^\top (x - x_k)}_{\text{linear approximation to f}} + \underbrace{\frac{1}{2\alpha}(x - x_k)^\top I (x - x_k)}_{\text{proximity term}}$$

to another

$$x_{k+1} = \underset{x}{\text{argmin}} \underbrace{f(x_k) + g_k^\top (x - x_k)}_{\text{linear approximation to f}} + \underbrace{\frac{1}{2\alpha}(x - x_k)^\top Q (x - x_k)}_{\text{proximity term}},$$

where $Q = \begin{pmatrix} \frac{1}{50} & 0 \\ 0 & 200 \end{pmatrix}$ for this example. And more generally to another function $B_\phi(x, y)$ that measures proximity.

# Example. Poor condition

Let's find $x_{k+1}$ for this **new** algorithm

$$\alpha \nabla f(x_k) + \begin{pmatrix} \frac{1}{50} & 0 \\ 0 & 200 \end{pmatrix} (x - x_k) = 0.$$

Solving for $x$, we get

$$x_{k+1} = x_k - \alpha \begin{pmatrix} 50 & 0 \\ 0 & \frac{1}{200} \end{pmatrix} \nabla f(x_k) = (-10 \ \ -0.1)^\top - \alpha(-10 \ \ -0.1)^\top$$

**Observation:** Changing the proximity term, we **change the direction** $x_{k+1} - x_k$. In other words, if we measure distance using this **new** way, we also **change Lipschitzness**.

**Question**

What is the Lipshitz constant of $f$ at the point $(1 \ 1)^\top$ for the norm:

$$\|z\|^2 = z^\top \begin{pmatrix} 50 & 0 \\ 0 & \frac{1}{200} \end{pmatrix} z?$$

# Example. Robust Regression

Square loss $\|Ax - b\|_2^2$ is very sensitive to outliers.

**Instead:** $\min \|Ax - b\|_1$. This problem also **convex**.

Let's compute $L$-Lipshitz constant for $f(x) = \|Ax - b\|_1$:

$$\left|\|Ax - b\|_1 - \|Ay - b\|_1\right| \leq L\|x - y\|_2.$$

To simplify calculation: $A = I$, $b = 0$, i.e. $f(x) = \|x\|_1$.

If we take $x = \mathbf{1}_d$, $y = (1 + \varepsilon)\mathbf{1}_d$:

$$|n - (1 + \varepsilon)n| = \varepsilon n \leq L\|x - y\|_2 = \| - \varepsilon\|_2 = \sqrt{(n\varepsilon^2)} = \varepsilon\sqrt{n}.$$

Finally, we get $L = \sqrt{n}$. As we can see, $L$ is **dimension dependent**.

> Question
>
> Show that if $\|\nabla f(x)\|_\infty \leq 1$, then $\|\nabla f(x)\|_2 \leq \sqrt{d}$.

# References

Examples for the Mirror Descent was taken from the Lecture.