Dual methods: Dual Gradient Ascent, Augmented Lagrangian Method, ADMM.

Daniil Merkulov

Optimization for ML. Faculty of Computer Science. HSE University

Why do we want to solve dual problems?

Primal problem

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\text{s.t.} \quad & f_i(x) \leq 0, \; i = 1, \ldots, m \\
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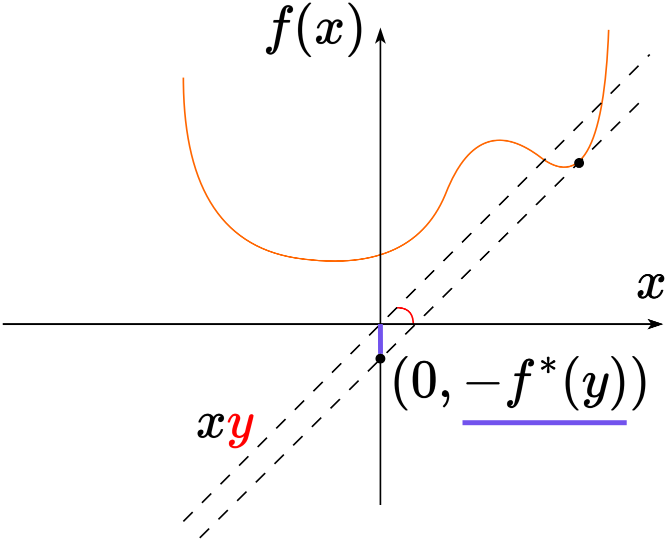
- **Dual Problems Provide Bounds.** Dual problems often offer bounds on the optimal value of the primal problem. This can be useful for assessing the quality of approximate solutions.
- **Duality Gap.** The difference between the primal and dual solutions (duality gap) provides valuable information about the solution's optimality.

Conjugate functions

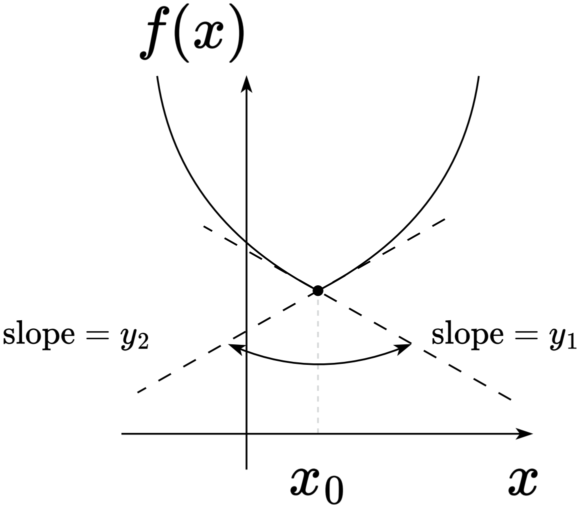


Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function defined by

$$f^*(y) = \max_x \left[y^T x - f(x)\right]$$

is called its conjugate.

Geometrical intution



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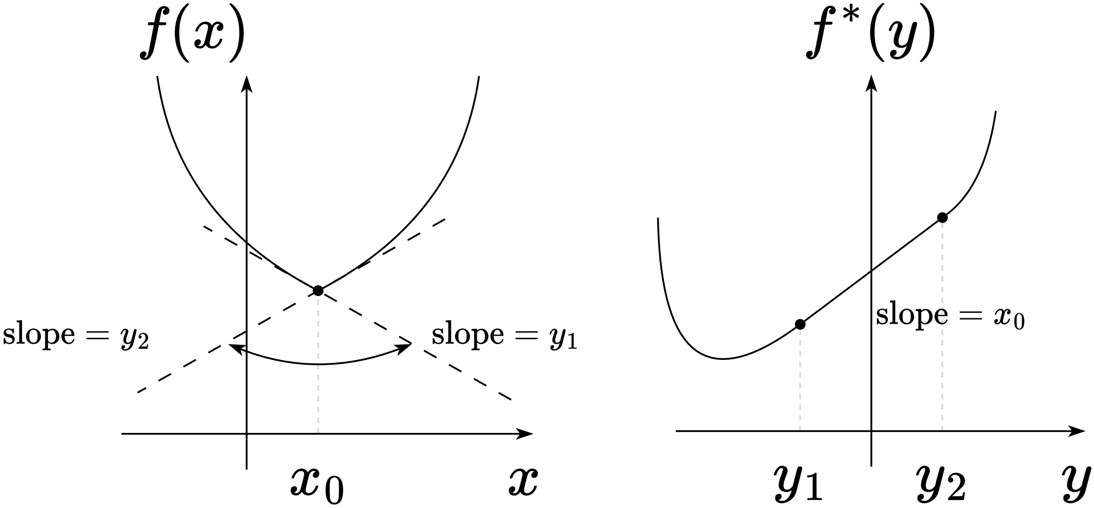

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- If f is strictly convex, then

$$\nabla f^*(y) = \arg\min_z \left[f(z) - y^T z \right]$$

Conjugate function properties (proofs)

We will show that $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$, assuming that f is convex and closed.

- **Proof of** $\Leftarrow$: Suppose $y \in \partial f(x)$. Then $x \in M_y$, the set of maximizers of $y^T z - f(z)$ over z. But

$$f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{and} \quad \partial f^*(y) = \text{cl}(\text{conv}(\bigcup_{z \in M_y} \{z\})).$$

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Clearly $y \in \partial f(x) \Leftrightarrow x \in \arg\min_z \{f(z) - y^T z\}$

Lastly, if f is strictly convex, then we know that $f(z) - y^T z$ has a unique minimizer over z, and this must be $\nabla f^*(y)$.

Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

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- Step sizes α_k, $k = 1, 2, 3, \ldots$, are chosen in standard ways.

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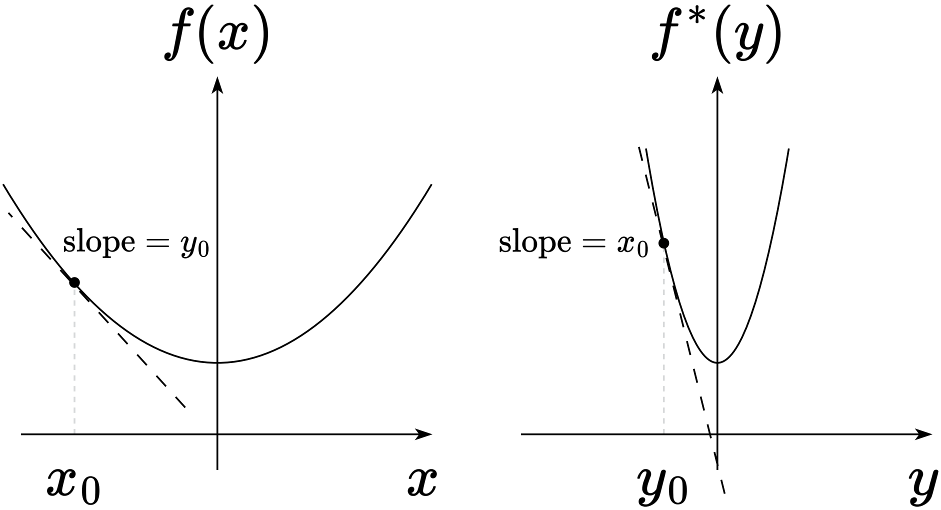
- Proximal gradients and acceleration can be applied as they would usually.

Slopes of f and f^*



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Adding these together, using the Cauchy-Schwarz inequality, and rearranging shows that

$$\|x_u - x_v\|^2 \leq \frac{1}{\mu}\|u - v\|^2$$

Slopes of f and f^*

Proof of "$\Leftarrow$": for simplicity, call $g = f^*$ and $L = \frac{1}{\mu}$. As ∇g is Lipschitz with constant L, so is $g_x(z) = g(z) - \nabla g(x)^T z$, hence

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$$\frac{1}{2L}\|\nabla g(x) - \nabla g(y)\|^2 \leq g(y) - g(x) + \nabla g(x)^T(x-y)$$

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Let $u = \nabla f(x)$, $v = \nabla g(y)$; then $x \in \partial g^*(u)$, $y \in \partial g^*(v)$, and the above reads $(x - y)^T (u - v) \geq \frac{\|u - v\|^2}{L}$, implying the result.

Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent: (This is ignoring the role of A, and thus reflects the case when the singular values of A are all close to 1. To be more precise, the step sizes here should be: $\frac{\mu}{\sigma_{\text{max}}(A)^2}$ (first case) and $\frac{2}{\frac{\sigma_{\text{max}}(A)^2}{\mu} + \frac{\sigma_{\text{min}}(A)^2}{L}}$ (second case).)

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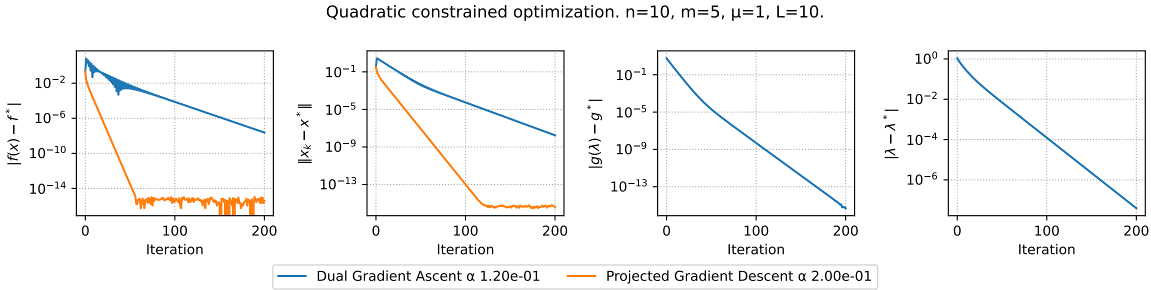
- Note that this describes convergence in the dual. Convergence in the primal requires more assumptions

Example: equality constrained quadratic minimization.

$$f(x) = \frac{1}{2}x^TAx - b^Tx \to \min_{x\in\mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m\times n}, m < n.$$



We need to find a minimum of a quadratic function in some linear subspace, defined by the solution of linear equation $Cx = d$. This is a conditional optimization problem, we start from strongly convex setting.

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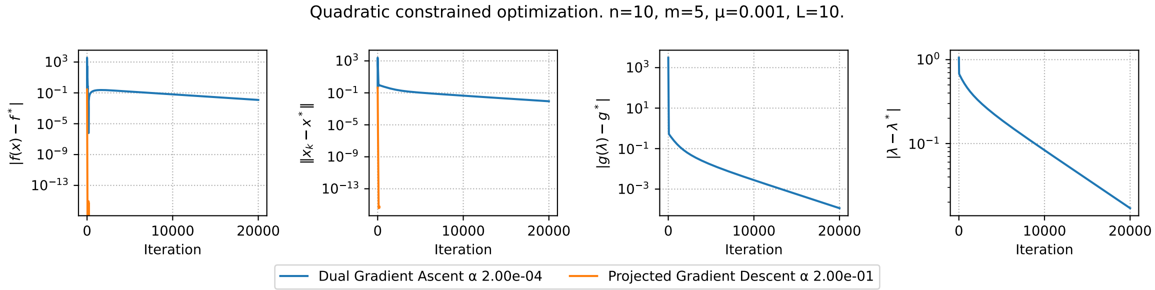

Situation is getting worse as soon as we loose strong convexity, the dual convergence will still be linear, but the rate is very low.

Dual decomposition

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Here $x = (x_1, \dots, x_B) \in \mathbb{R}^n$ divides into B blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition A accordingly:

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Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into B separate problems:

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Can think of these steps as:

- **Broadcast:** Send u to each of the B processors, each optimizes in parallel to find x_i.

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Consider

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Here $x = (x_1, \ldots, x_B) \in \mathbb{R}^n$ divides into B blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition A accordingly:

$$A = [A_1 \ldots A_B], \text{ where } A_i \in \mathbb{R}^{m \times n_i}$$

Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into B separate problems:

$$x^{\text{new}} \in \arg\min_x \left(\sum_{i=1}^B f_i(x_i) + u^T Ax \right)$$

$$\Rightarrow x_i^{\text{new}} \in \arg\min_{x_i} \left(f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \ldots, B$$

$$x_i^k \in \arg\min_{x_i} \left(f_i(x_i) + (u^{k-1})^T A_i x_i \right), \quad i = 1, \ldots, B$$

$$u^k = u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b \right)$$

Can think of these steps as:

- **Broadcast:** Send u to each of the B processors, each optimizes in parallel to find x_i.
- **Gather:** Collect $A_i x_i$ from each processor, update the global dual variable u.

Inequality constraints

Consider the optimization problem:

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad \sum_{i=1}^B A_i x_i \leq b$$

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Using **dual decomposition**, specifically the **projected subgradient method**, the iterative steps can be expressed as:

- The primal update step:

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- The dual update step:

$$u^k = \left(u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b \right) \right)_+$$

where $(u)_+$ denotes the positive part of u, i.e., $(u_+)_i = \max\{0, u_i\}$, for $i = 1, \dots, m$.

Price Coordination Interpretation (Vandenberghe)

- **System Overview**: Consider a system with B units, where each unit independently chooses its decision variable x_i, which determines how to allocate its goods.

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  - **Never let prices get negative**; hence the use of the positive part notation $(\cdot)_+$.

Augmented Lagrangian method aka method of multipliers

Dual ascent disadvantage: convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

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&\min_x f(x) + \frac{\rho}{2}\|Ax - b\|^2 \\
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Dual gradient ascent: The iterative updates are given by:

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Notice step size choice $\alpha_k = \rho$ in dual algorithm. Why?

Since x_k minimizes the function:

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over x, we have the stationarity condition:

$$0 \in \partial f(x_k) + A^T \left(u_{k-1} + \rho(Ax_k - b)\right)$$

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This represents the stationarity condition for the original primal problem; under mild conditions, $Ax_k - b \to 0$ as $k \to \infty$, so the KKT conditions are satisfied in the limit and x_k, u_k converge to the solutions.

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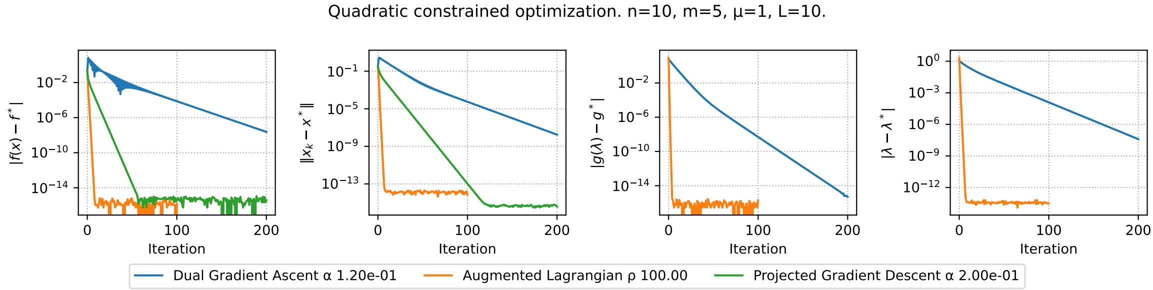
- **Advantage:** The augmented Lagrangian gives better convergence.
- **Disadvantage:** We lose decomposability! (Separability is ruined)

Example: equality constrained quadratic minimization.

$$f(x) = \frac{1}{2}x^T Ax - b^T x \to \min_{x \in \mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m \times n}, m < n.$$



One can see, clear numerical superiority of the Augmented Lagrangian method both in convex and strongly convex case.

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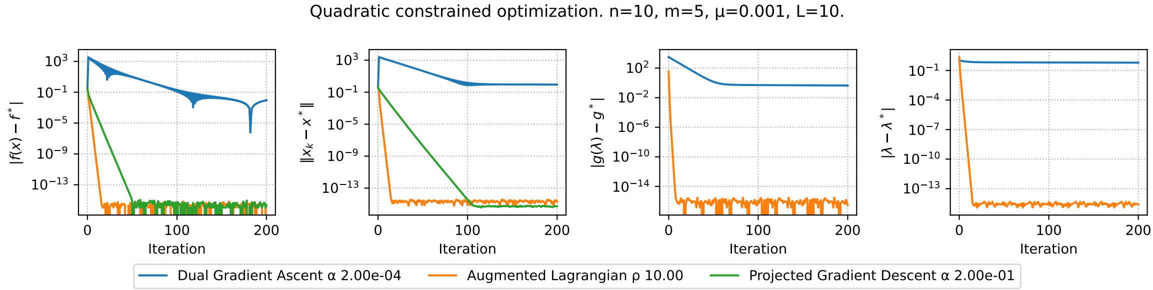

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Alternating Direction Method of Multipliers (ADMM)

Alternating direction method of multipliers or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

$$\min_{x,z} f(x) + g(z)$$

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where $\rho > 0$ is a parameter. The augmented Lagrangian for this problem is defined as:

$$
L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|^2
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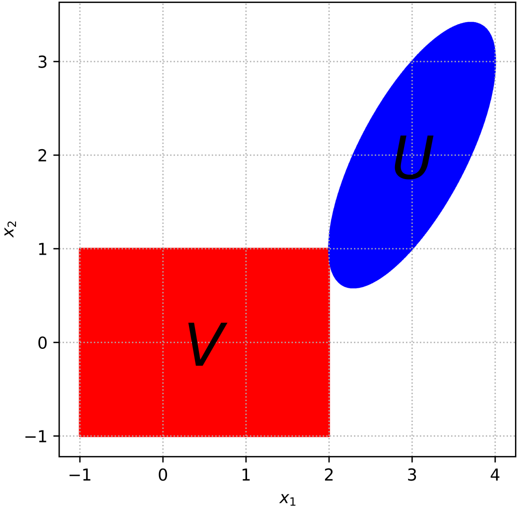
Note: The usual method of multipliers would replace the first two steps by a joint minimization:

$$(x^{(k)}, z^{(k)}) = \arg\min_{x,z} L_\rho(x, z, u^{(k-1)})$$

Example: Alternating Projections



Consider finding a point in the intersection of convex sets $U, V \subseteq \mathbb{R}^n$:

$$\min_x I_U(x) + I_V(x)$$

To transform this problem into ADMM form, we express it as:

$$\min_{x,z} I_U(x) + I_V(z) \quad \text{subject to} \quad x - z = 0$$

Each ADMM cycle involves two projections:

$$x_k = \arg\min_x P_U\left(z_{k-1} - w_{k-1}\right)$$

$$z_k = \arg\min_z P_V\left(x_k + w_{k-1}\right)$$

$$w_k = w_{k-1} + x_k - z_k$$

Sources

- Ryan Tibshirani. Convex Optimization 10-725