

# Proximal gradient method

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# Non-smooth problems

$\ell_1$  induces sparsity

$\ell_2$  regularization.  $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_2 \leq 1}$



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@fminxyz

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## Theorem

Assume that  $f$  is  $G$ -Lipschitz and convex, then  
Subgradient method converges as:

$$f(\bar{x}) - f^* \leq \frac{GR}{\sqrt{k}},$$

where

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- One can use Mirror Descent (a generalization of the subgradient method to a possibly non-Euclidian distance) with the same convergence rate to better fit the geometry of the problem.
- However, we can achieve standard gradient descent rate  $\mathcal{O}\left(\frac{1}{k}\right)$  (and even accelerated version  $\mathcal{O}\left(\frac{1}{k^2}\right)$ ) if we will exploit the structure of the problem.

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$$\frac{dx}{dt} = -\nabla f(x)$$

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Implicit Euler discretization:

$$\begin{aligned}\frac{x_{k+1} - x_k}{\alpha} &= -\nabla f(x_{k+1}) \\ \frac{x_{k+1} - x_k}{\alpha} + \nabla f(x_{k+1}) &= 0\end{aligned}$$

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! Proximal operator

$$\text{prox}_{f,\alpha}(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

# Proximal operator visualization

$$\text{Prox}_f(x) = \underset{x'}{\operatorname{argmin}} \frac{1}{2} \|x - x'\|^2 + f(x')$$

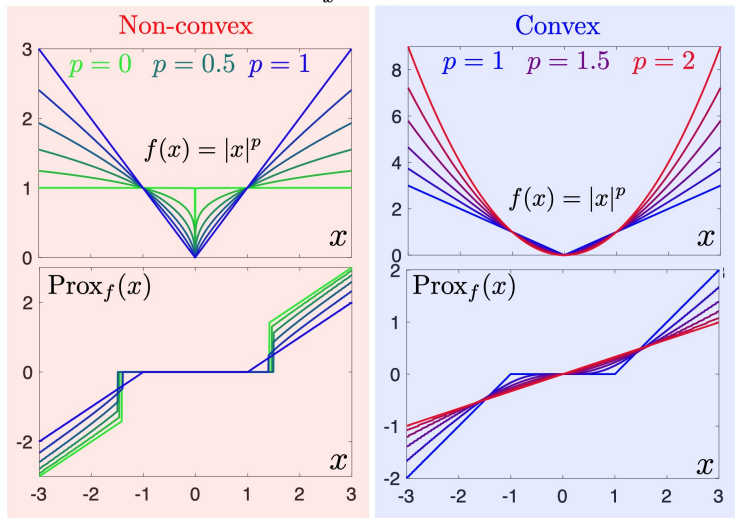


Figure 1: Source

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Thus, we have a usual gradient descent with  $\alpha \rightarrow 0$ :  $x_{k+1} = x_k - \alpha \nabla f(x_k)$

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$$x_{k+1} = \text{prox}_{f_{x_k}^{II}, \alpha}(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2} \langle \nabla^2 f(x_k)(x - x_k), x - x_k \rangle + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

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## From projections to proximity

Let  $\mathbb{I}_S$  be the indicator function for closed, convex  $S$ . Recall orthogonal projection  $\pi_S(y)$

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With the following notation of indicator function

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Proximity: Replace  $\mathbb{I}_S$  by some convex function!

$$\text{prox}_r(y) = \text{prox}_{r,1}(y) := \arg \min \frac{1}{2} \|x - y\|^2 + r(x)$$

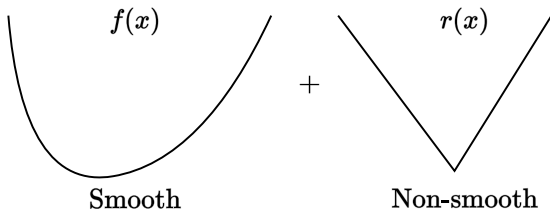
# Regularized / Composite Objectives

Many nonsmooth problems take the form

$$\min_{x \in \mathbb{R}^n} \varphi(x) = f(x) + r(x)$$

- **Lasso, L1-LS, compressed sensing**

$$f(x) = \frac{1}{2} \|Ax - b\|_2^2, r(x) = \lambda \|x\|_1$$



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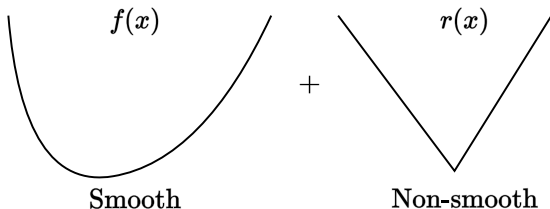
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- **Lasso, L1-LS, compressed sensing**

$$f(x) = \frac{1}{2} \|Ax - b\|_2^2, r(x) = \lambda \|x\|_1$$

- **L1-Logistic regression, sparse LR**

$$f(x) = -y \log h(x) - (1-y) \log(1-h(x)), r(x) = \lambda \|x\|_1$$



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Which leads to the proximal gradient method:

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And this method converges at a rate of  $\mathcal{O}(\frac{1}{k})$ !

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**i** Another form of proximal operator

$$\text{prox}_{f,\alpha}(x_k) = \text{prox}_{\alpha f}(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ \alpha f(x) + \frac{1}{2} \|x - x_k\|_2^2 \right] \quad \text{prox}_f(x_k) = \arg \min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2} \|x - x_k\|_2^2 \right]$$

## Proximal operators examples

- $r(x) = \lambda \|x\|_1, \lambda > 0$

$$[\text{prox}_r(x)]_i = [|x_i| - \lambda]_+ \cdot \text{sign}(x_i),$$

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- $r(x) = \mathbb{I}_S(x).$

$$\text{prox}_r(x_k - \alpha \nabla f(x_k)) = \text{proj}_r(x_k - \alpha \nabla f(x_k))$$

# Proximal operator properties

## Theorem

Let  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\text{prox}_r$  is defined. If there exists such an  $\hat{x} \in \mathbb{R}^n$  that  $r(\hat{x}) < +\infty$ . Then, the proximal operator is uniquely defined (i.e., it always returns a single unique value).

**Proof:**

# Proximal operator properties

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It is strongly convex, meaning it has exactly one unique minimum (the existence of  $\hat{x}$  is necessary for  $r(\hat{x}) + \frac{1}{2}\|x - \hat{x}\|_2^2$  to take a finite value somewhere).

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- $\text{prox}_r(x) = y$ ,

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1. Let's establish the equivalence between the first and second conditions. The first condition can be rewritten as

$$y = \arg \min_{\tilde{x} \in \mathbb{R}^d} \left( r(\tilde{x}) + \frac{1}{2} \|x - \tilde{x}\|^2 \right).$$

From the optimality condition for the convex function  $r$ , this is equivalent to:

$$0 \in \partial \left( r(\tilde{x}) + \frac{1}{2} \|x - \tilde{x}\|^2 \right) \Big|_{\tilde{x}=y} = \partial r(y) + y - x.$$

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1. Let's establish the equivalence between the first and second conditions. The first condition can be rewritten as
2. From the definition of the subdifferential, for any subgradient  $g \in \partial f(y)$  and for any  $z \in \mathbb{R}^d$ :

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$$\langle g, z - y \rangle \leq r(z) - r(y).$$

In particular, this holds true for  $g = x - y$ .

Conversely, it is also clear: for  $g = x - y$ , the above relationship holds, which means  $g \in \partial r(y)$ .

# Proximal operator properties

## Theorem

The operator  $\text{prox}_r(x)$  is firmly nonexpansive (FNE)

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2^2 \leq \langle \text{prox}_r(x) - \text{prox}_r(y), x - y \rangle$$

and nonexpansive:

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2 \leq \|x - y\|_2$$

## Proof

1. Let  $u = \text{prox}_r(x)$ , and  $v = \text{prox}_r(y)$ . Then, from the previous property:

$$\langle x - u, z_1 - u \rangle \leq r(z_1) - r(u)$$

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2. Substitute  $z_1 = v$  and  $z_2 = u$ . Summing up, we get:

$$\langle x - u, v - u \rangle + \langle y - v, u - v \rangle \leq 0,$$

$$\langle x - y, v - u \rangle + \|v - u\|_2^2 \leq 0.$$

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$$\|u - v\|_2^2 \leq \langle x - y, u - v \rangle$$

4. The last point comes from simple Cauchy-Bunyakovsky-Schwarz for the last inequality.

# Proximal operator properties

## Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be convex functions. Additionally, assume that  $f$  is continuously differentiable and  $L$ -smooth, and for  $r$ ,  $\text{prox}_r$  is defined. Then,  $x^*$  is a solution to the composite optimization problem if and only if, for any  $\alpha > 0$ , it satisfies:

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

## Proof

1. Optimality conditions:

# Proximal operator properties

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## Proof

### 1. Optimality conditions:

$$\begin{aligned} 0 &\in \nabla f(x^*) + \partial r(x^*) \\ -\alpha \nabla f(x^*) &\in \alpha \partial r(x^*) \end{aligned}$$

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### 2. Recall from the previous lemma:

$$\text{prox}_r(x) = y \Leftrightarrow x - y \in \partial r(y)$$

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2. Recall from the previous lemma:

$$\text{prox}_r(x) = y \Leftrightarrow x - y \in \partial r(y)$$

3. Finally,

$$x^* = \text{prox}_{\alpha r}(x^* - \alpha \nabla f(x^*)) = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

# Convergence tools

## Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an  $L$ -smooth convex function. Then, for any  $x, y \in \mathbb{R}^n$ , the following inequality holds:

$$f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y) \text{ or, equivalently,}$$
$$\|\nabla f(y) - \nabla f(x)\|_2^2 = \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L (f(x) - f(y) - \langle \nabla f(y), x - y \rangle)$$

## Proof

1. To prove this, we'll consider another function  $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$ . It is obviously a convex function (as a sum of convex functions). And it is easy to verify, that it is an  $L$ -smooth function by definition, since  $\nabla \varphi(y) = \nabla f(y) - \nabla f(x)$  and  $\|\nabla \varphi(y_1) - \nabla \varphi(y_2)\| = \|\nabla f(y_1) - \nabla f(y_2)\| \leq L\|y_1 - y_2\|$ .

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$$x:=y, y:=y - \frac{1}{L} \nabla \varphi(y) \quad \varphi\left(y - \frac{1}{L} \nabla \varphi(y)\right) \leq \varphi(y) + \left\langle \nabla \varphi(y), -\frac{1}{L} \nabla \varphi(y) \right\rangle + \frac{1}{2L} \|\nabla \varphi(y)\|_2^2$$



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$$\varphi\left(y - \frac{1}{L} \nabla \varphi(y)\right) \leq \varphi(y) - \frac{1}{2L} \|\nabla \varphi(y)\|_2^2$$

## Convergence tools

3. From the first order optimality conditions for the convex function  $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any  $x$ , the minimum of the function  $\varphi(y)$  is at the point  $y = x$ . Therefore:

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The lemma has been proved. From the first view it does not make a lot of geometrical sense, but we will use it as a convenient tool to bound the difference between gradients.

# Convergence tools

## Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\mathbb{R}^n$ . Then, the function  $f$  is  $\mu$ -strongly convex if and only if for any  $x, y \in \mathbb{R}^d$  the following holds:

$$\text{Strongly convex case } \mu > 0 \quad \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$$

$$\text{Convex case } \mu = 0 \quad \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$$

## Proof

1. We will only give the proof for the strongly convex case, the convex one follows from it with setting  $\mu = 0$ . We start from necessity. For the strongly convex function

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|x - y\|_2^2$$

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|_2^2$$

$$\text{sum} \quad \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$$

## Convergence tools

2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

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$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle = \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle$$

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# Convergence

## Theorem

Consider the proximal gradient method

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

For the criterion  $\varphi(x) = f(x) + r(x)$ , we assume:

- $f$  is convex, differentiable,  $\text{dom}(f) = \mathbb{R}^n$ , and  $\nabla f$  is Lipschitz continuous with constant  $L > 0$ .
- $r$  is convex, and  $\text{prox}_{\alpha r}(x_k) = \arg \min_{x \in \mathbb{R}^n} [\alpha r(x) + \frac{1}{2} \|x - x_k\|_2^2]$  can be evaluated.

Proximal gradient descent with fixed step size  $\alpha = 1/L$  satisfies

$$\varphi(x_k) - \varphi^* \leq \frac{L \|x_0 - x^*\|^2}{2k},$$

Proximal gradient descent has a convergence rate of  $O(1/k)$  or  $O(1/\varepsilon)$ . This matches the gradient descent rate!  
(But remember the proximal operation cost)

# Convergence

## Proof

1. Let's introduce the **gradient mapping**, denoted as  $G_\alpha(x)$ , acts as a “gradient-like object”:

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

$$x_{k+1} = x_k - \alpha G_\alpha(x_k).$$

where  $G_\alpha(x)$  is:

$$G_\alpha(x) = \frac{1}{\alpha} (x - \text{prox}_{\alpha r}(x - \alpha \nabla f(x)))$$

Observe that  $G_\alpha(x) = 0$  if and only if  $x$  is optimal. Therefore,  $G_\alpha$  is analogous to  $\nabla f$ . If  $x$  is locally optimal, then  $G_\alpha(x) = 0$  even for nonconvex  $f$ . This demonstrates that the proximal gradient method effectively combines gradient descent on  $f$  with the proximal operator of  $r$ , allowing it to handle non-differentiable components effectively.

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$$\begin{aligned} \text{convexity } f(x) \geq f(x_k) + \langle \nabla f(x_k), x - x_k \rangle &\leq f(x) - \langle \nabla f(x_k), x - x_k \rangle + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2 \\ &\leq f(x) + \langle \nabla f(x_k), x_{k+1} - x \rangle + \frac{\alpha^2 L}{2} \|G_\alpha(x_k)\|_2^2 \end{aligned}$$

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7. Now it is easy to verify, that when  $x = x_k$  we have monotonic decrease for the proximal gradient algorithm:

$$\varphi(x_{k+1}) \leq \varphi(x_k) - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2$$

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## Convergence

9. Now we write the bound above for all iterations  $i \in 0, k-1$  and sum them:

Which is a standard  $\frac{L\|x_0 - x^*\|_2^2}{2k}$  with  $\alpha = \frac{1}{L}$ , or,  $\mathcal{O}\left(\frac{1}{k}\right)$  rate for smooth convex problems with Gradient Descent!

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$$\sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] \leq \frac{1}{2\alpha} [\|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2]$$

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10. Since  $\varphi(x_k)$  is a decreasing sequence, it follows that:

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$$\sum_{i=0}^{k-1} \varphi(x_k) = k\varphi(x_k) \leq \sum_{i=0}^{k-1} \varphi(x_{i+1})$$

Which is a standard  $\frac{L\|x_0 - x^*\|_2^2}{2k}$  with  $\alpha = \frac{1}{L}$ , or,  $\mathcal{O}\left(\frac{1}{k}\right)$  rate for smooth convex problems with Gradient Descent!

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# Convergence

## Theorem

Consider the proximal gradient method

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$

For the criterion  $\varphi(x) = f(x) + r(x)$ , we assume:

- $f$  is  $\mu$ -strongly convex, differentiable,  $\text{dom}(f) = \mathbb{R}^n$ , and  $\nabla f$  is Lipschitz continuous with constant  $L > 0$ .
- $r$  is convex, and  $\text{prox}_{\alpha r}(x_k) = \arg \min_{x \in \mathbb{R}^n} [\alpha r(x) + \frac{1}{2} \|x - x_k\|_2^2]$  can be evaluated.

Proximal gradient descent with fixed step size  $\alpha \leq 1/L$  satisfies

$$\|x_{k+1} - x^*\|_2^2 \leq (1 - \alpha\mu)^k \|x_0 - x^*\|_2^2$$

This is exactly gradient descent convergence rate. Note, that the original problem is even non-smooth!

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$$\text{nonexpansiveness} \leq \|x_k - \alpha \nabla f(x_k) - x^* + \alpha \nabla f(x^*)\|_2^2$$

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2. Now we use smoothness from the convergence tools and strong convexity:

$$\text{smoothness} \quad \|\nabla f(x_k) - \nabla f(x^*)\|_2^2 \leq 2L (f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle)$$

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4. Due to convexity of  $f$ :  $f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle \geq 0$ . Therefore, if we use  $\alpha \leq \frac{1}{L}$ :

$$\|x_{k+1} - x^*\|_2^2 \leq (1 - \alpha\mu) \|x_k - x^*\|^2,$$

which is exactly linear convergence of the method with up to  $1 - \frac{\mu}{L}$  convergence rate.

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Let  $x_0 = y_0 \in \text{dom}(r)$ . For  $k \geq 1$ :

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- Same computational cost as ordinary prox-grad
- Convergence rate theoretically optimal

## Example: ISTA

### Iterative Shrinkage-Thresholding Algorithm (ISTA)

ISTA is a popular method for solving optimization problems involving L1 regularization, such as Lasso. It combines gradient descent with a shrinkage operator to handle the non-smooth L1 penalty effectively.

- **Algorithm:**

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- **Application:**

- Efficient for sparse signal recovery, image processing, and compressed sensing.

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- **Application:**

- Especially useful for large-scale problems in machine learning and signal processing where the L1 penalty induces sparsity.

# Example: Matrix Completion

## Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

- **Problem Formulation:**

$$\min_X \frac{1}{2} \|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda \|X\|_*,$$

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where  $P_\Omega$  projects onto the observed set  $\Omega$ , and  $\|\cdot\|_*$  denotes the nuclear norm.

- **Proximal Operator:**

- The proximal operator for the nuclear norm involves singular value decomposition (SVD) and soft-thresholding of the singular values.

- **Algorithm:**

- Similar proximal gradient or accelerated proximal gradient methods can be applied, where the main computational effort lies in performing partial SVDs.

- **Application:**

- Widely used in recommender systems, image recovery, and other domains where data is naturally matrix-formed but partially observed.

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- Further reading: Proximal operator splitting, Douglas-Rachford splitting, Best approximation problem, Three operator splitting.