

Subgradient Method. Specifics of non-smooth problems.

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ℓ_1 -regularized linear least squares

ℓ_1 induces sparsity

ℓ_2 regularization. $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_2 \leq 1}$



ℓ_1 regularization. $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_1 \leq 1}$



@fminxyz

Norms are not smooth

$$\min_{x \in \mathbb{R}^n} f(x),$$

A classical convex optimization problem is considered. We assume that $f(x)$ is a convex function, but now we do not require smoothness.



Figure 1: Norm cones for different p - norms are non-smooth

Wolfe's example

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Figure 2: Wolfe's example. [Open in Colab](#)

Convex function linear lower bound

An important property of a continuous convex function $f(x)$ is that at any chosen point x_0 for all $x \in \text{dom } f$ the inequality holds:

$$f(x) \geq f(x_0) + \langle g, x - x_0 \rangle$$

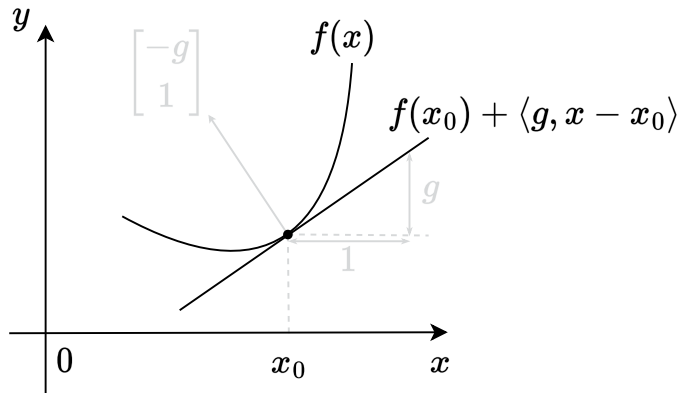


Figure 3: Taylor linear approximation serves as a global lower bound for a convex function

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for some vector g , i.e., the tangent to the graph of the function is the *global* estimate from below for the function.

- If $f(x)$ is differentiable, then $g = \nabla f(x_0)$

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We wouldn't want to lose such a nice property.

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Subgradient and subdifferential

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Figure 4: Subdifferential is a set of all possible subgradients

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Subdifferential of a differentiable function

Let $f : S \rightarrow \mathbb{R}$ be a function defined on the set S in a Euclidean space \mathbb{R}^n . If $x_0 \in \text{ri}(S)$ and f is differentiable at x_0 , then either $\partial f(x_0) = \emptyset$ or $\partial f(x_0) = \{\nabla f(x_0)\}$. Moreover, if the function f is convex, the first scenario is impossible.

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Proof

1. Assume, that $s \in \partial f(x_0)$ for some $s \in \mathbb{R}^n$ distinct from $\nabla f(x_0)$. Let $v \in \mathbb{R}^n$ be a unit vector. Because x_0 is an interior point of S , there exists $\delta > 0$ such that $x_0 + tv \in S$ for all $0 < t < \delta$. By the definition of the subgradient, we have

$$f(x_0 + tv) \geq f(x_0) + t\langle s, v \rangle$$

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for all $0 < t < \delta$. Taking the limit as t approaches 0 and using the definition of the gradient, we get:

$$\langle \nabla f(x_0), v \rangle = \lim_{t \rightarrow 0; 0 < t < \delta} \frac{f(x_0 + tv) - f(x_0)}{t} \geq \langle s, v \rangle$$

2. From this, $\langle s - \nabla f(x_0), v \rangle \geq 0$. Due to the arbitrariness of v , one can set

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3. Furthermore, if the function f is convex, then according to the differential condition of convexity $f(x) \geq f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle$ for all $x \in S$. But by definition, this means $\nabla f(x_0) \in \partial f(x_0)$.

Subdifferential calculus

Moreau - Rockafellar theorem (subdifferential of a linear combination)

Let $f_i(x)$ be convex functions on convex sets S_i , $i = \overline{1, n}$. Then if $\bigcap_{i=1}^n \text{ri} S_i \neq \emptyset$ then the function $f(x) =$

$\sum_{i=1}^n a_i f_i(x)$, $a_i > 0$ has a subdifferential $\partial_S f(x)$ on

the set $S = \bigcap_{i=1}^n S_i$ and

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Dubovitsky - Milutin theorem (subdifferential of a point-wise maximum)

Let $f_i(x)$ be convex functions on the open convex set $S \subseteq \mathbb{R}^n$, $x_0 \in S$, and the pointwise maximum is defined as $f(x) = \max_i f_i(x)$. Then:

$$\partial_S f(x_0) = \text{conv} \left\{ \bigcup_{i \in I(x_0)} \partial_S f_i(x_0) \right\}, \quad I(x) = \{i \in [1, n] \mid f_i(x) = f(x)\}$$

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- $\partial(f(Ax + b))(x) = A^T \partial f(Ax + b)$, f - convex function
- $z \in \partial f(x)$ if and only if $x \in \partial f^*(z)$.

Algorithm

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The idea is very simple: let's replace the gradient $\nabla f(x_k)$ in the gradient descent algorithm with a subgradient g_k at point x_k :

$$x_{k+1} = x_k - \alpha_k g_k,$$

where g_k is an arbitrary subgradient of the function $f(x)$ at the point x_k , $g_k \in \partial f(x_k)$

Convergence bound

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- Let's write down how close we came to the optimum $x^* = \arg \min_{x \in \mathbb{R}^n} f(x) = \arg f^*$ on the last iteration:

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- We additionally assume, that $\|g_k\|^2 \leq G^2$

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- For a subgradient: $\langle g_k, x_k - x^* \rangle \leq f(x_k) - f(x^*) = f(x_k) - f^*$.
- We additionally assume, that $\|g_k\|^2 \leq G^2$
- We use the notation $R = \|x_0 - x^*\|_2$

Convergence bound

Assuming $\alpha_k = \alpha$ (constant stepsize), we have:

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Important notes:

- Obtaining bounds not for x_T but for the arithmetic mean over iterations \bar{x} is a typical trick in obtaining estimates for methods where there is convexity but no monotonic decreasing at each iteration. There is no guarantee of success at each iteration, but there is a guarantee of success on average

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- To choose the optimal step, we need to know (assume) the number of iterations in advance. Possible solution: initialize T with a small value, after reaching this number of iterations double T and restart the algorithm. A more intelligent way: adaptive selection of stepsize.

Steepest subgradient descent convergence bound

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Which leads to exactly the same bound of $\mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$ on the primal gap. In fact, for this class of functions, you can't get a better result than $\frac{1}{\sqrt{T}}$.

Convergence results

Theorem

Let f be a convex G -Lipschitz function. For a fixed step size $\alpha = \frac{\|x_0 - x^*\|_2}{G} \sqrt{\frac{1}{K}}$, subgradient method satisfies

$$f(\bar{x}) - f^* \leq \frac{G\|x_0 - x^*\|_2}{\sqrt{K}} \quad \bar{x} = \frac{1}{K} \sum_{k=0}^{K-1} x_k$$

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- Proved result requires pre-defined step size strategy, which is not practical (usually one can just use several diminishing strategies).
- There is no monotonic decrease of objective.
- Convergence is slower, than for the gradient descent (smooth case). However, if we will go deeply for the problem structure, we can improve convergence (proximal gradient method).

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Let f be a convex G -Lipschitz function and $f_k^{\text{best}} = \min_{i=1,\dots,k} f(x^i)$. For a fixed step size α , subgradient method satisfies

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Theorem

Let f be a convex G -Lipschitz function and $f_k^{\text{best}} = \min_{i=1,\dots,k} f(x^i)$. For a diminishing step size α_k (square summable but not summable. Important here that step sizes go to zero, but not too fast), subgradient method satisfies

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Linear Least Squares with l_1 -regularization

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - b\|_2^2 + \lambda \|x\|_1$$

Algorithm will be written as:

$$x_{k+1} = x_k - \alpha_k \left(A^\top (Ax_k - b) + \lambda \text{sign}(x_k) \right)$$

where signum function is taken element-wise.

LLS with l_1 regularization. 2 runs. $\lambda = 1$



Regularized logistic regression

Given $(x_i, y_i) \in \mathbb{R}^p \times \{0, 1\}$ for $i = 1, \dots, n$, the logistic regression function is defined as:

$$f(\theta) = \sum_{i=1}^n (-y_i x_i^T \theta + \log(1 + \exp(x_i^T \theta)))$$

This is a smooth and convex function with its gradient given by:

$$\nabla f(\theta) = \sum_{i=1}^n (y_i - s_i(\theta)) x_i$$

where $s_i(\theta) = \frac{\exp(x_i^T \theta)}{1 + \exp(x_i^T \theta)}$, for $i = 1, \dots, n$. Consider the regularized problem:

$$f(\theta) + \lambda r(\theta) \rightarrow \min_{\theta}$$

where $r(\theta) = \|\theta\|_2^2$ for the ridge penalty, or $r(\theta) = \|\theta\|_1$ for the lasso penalty.

Support Vector Machines

Let $D = \{(x_i, y_i) \mid x_i \in \mathbb{R}^n, y_i \in \{\pm 1\}\}$

We need to find $\theta \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$\min_{\theta \in \mathbb{R}^n, b \in \mathbb{R}} \frac{1}{2} \|\theta\|_2^2 + C \sum_{i=1}^m \max[0, 1 - y_i(\theta^\top x_i + b)]$$