Convexity. Strong convexity.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University



Line Segment

Suppose x_1, x_2 are two points in \mathbb{R}^{κ} . Then the line segment between them is defined as follows:

$$x = \theta x_1 + (1 - \theta)x_2, \ \theta \in [0, 1]$$

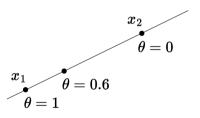


Figure 1: Illustration of a line segment between points x_1 , x_2

Convex Set

The set S is called **convex** if for any x_1, x_2 from S the line segment between them also lies in S, i.e.

$$\forall \theta \in [0, 1], \ \forall x_1, x_2 \in S : \theta x_1 + (1 - \theta)x_2 \in S$$

Example

Any affine set, a ray, a line segment - they all are convex sets.

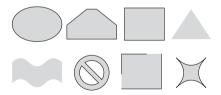


Figure 2: Top: examples of convex sets. Bottom: examples of non-convex sets.

Convex Sets

Question

Prove, that ball in \mathbb{R}^n (i.e. the following set $\{\mathbf{x} \mid ||\mathbf{x} - \mathbf{x}_c|| \leq r\}$) - is convex.

Question

Is stripe - $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$ - convex?



Question

Let S be such that $\forall x,y \in S \to \frac{1}{2}(x+y) \in S$. Is this set convex?

Question

The set $S = \{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex. Is this set convex?

Convex Function

The function f(x), which is defined on the convex set $S \subseteq \mathbb{R}^n$, is called convex on S, if:

$$f(\lambda x_1 + (1 - \lambda)x_2) \le \lambda f(x_1) + (1 - \lambda)f(x_2)$$

for any $x_1, x_2 \in S$ and $0 \le \lambda \le 1$.

If the above inequality holds as strict inequality $x_1 \neq x_2$ and $0 < \lambda < 1$, then the function is called **strictly convex** on S.

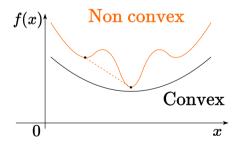


Figure 3: Difference between convex and non-convex function

Strong Convexity

f(x), defined on the convex set $S \subseteq \mathbb{R}^n$, is called μ -strongly convex (strongly convex) on S, if:

$$f(\lambda x_1 + (1 - \lambda)x_2) \le \lambda f(x_1) + (1 - \lambda)f(x_2) - \frac{\mu}{2}\lambda(1 - \lambda)||x_1 - x_2||^2$$

for any $x_1, x_2 \in S$ and $0 \le \lambda \le 1$ for some $\mu > 0$.

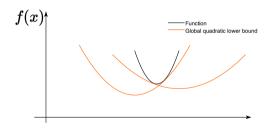


Figure 4: Strongly convex function is greater or equal than Taylor quadratic approximation at any point

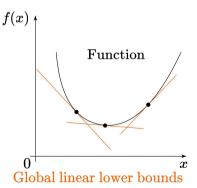
First-order differential criterion of convexity

The differentiable function f(x) defined on the convex set $S \subseteq \mathbb{R}^n$ is convex if and only if $\forall x,y \in S$:

$$f(y) \ge f(x) + \nabla f^{T}(x)(y - x)$$

Let $y=x+\Delta x$, then the criterion will become more tractable:

$$f(x + \Delta x) \ge f(x) + \nabla f^{T}(x) \Delta x$$





Second-order differential criterion of strong convexity

Twice differentiable function f(x) defined on the convex set $S \subseteq \mathbb{R}^n$ is called μ -strongly convex if and only if $\forall x \in \mathbf{int}(S) \neq \emptyset$:

$$\nabla^2 f(x) \succeq \mu I$$

In other words:

$$\langle y, \nabla^2 f(x)y \rangle \ge \mu \|y\|^2$$



Motivational Experiment with JAX

Why convexity and strong convexity is important? Check the simple \$\display*code snippet.





Question

Show, that f(x) = ||x|| is convex on \mathbb{R}^n .

Question

Show, that $f(x) = x^{\top} A x$, where $A \succeq 0$ - is convex on \mathbb{R}^n .



Question

Show, that if f(x) is convex on \mathbb{R}^n , then $\exp(f(x))$ is convex on \mathbb{R}^n .

Question

If f(x) is convex nonnegative function and $p \ge 1$. Show that $g(x) = f(x)^p$ is convex.



Question

Show that, if f(x) is concave positive function over convex S, then $g(x) = \frac{1}{f(x)}$ is convex.

Question

Show, that the following function is convex on the set of all positive denominators

$$f(x) = \frac{1}{x_1 - \frac{1}{x_2 - \frac{1}{x_3 - \frac{1}{x_3}}}}, x \in \mathbb{R}^n$$



Question

Let $S=\{x\in\mathbb{R}^n\mid x\succ 0, \|x\|_\infty\leq M\}$. Show that $f(x)=\sum_{i=1}^n x_i\log x_i$ is $\frac{1}{M}$ -strongly convex.

Polyak-Lojasiewicz (PL) Condition

PL inequality holds if the following condition is satisfied for some $\mu>0$,

$$\|\nabla f(x)\|^2 \ge \mu(f(x) - f^*) \forall x$$

The example of a function, that satisfies the PL-condition, but is not convex.

$$f(x,y) = \frac{(y - \sin x)^2}{2}$$

Example of PI non-convex function **@**Open in Colab.

