Linear Programming and simplex algorithm.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Linear Programming Recap. Common Forms

For some vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$

- Basic form of Linear Programming Problem is:

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
&\text{s.t. } Ax \leq b
\end{aligned}
\tag{LP.Basic}
$$

- Standard Form of Linear Programming Problem is:

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
&\text{s.t. } Ax = b \\
&\quad x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{LP.Standard}
$$

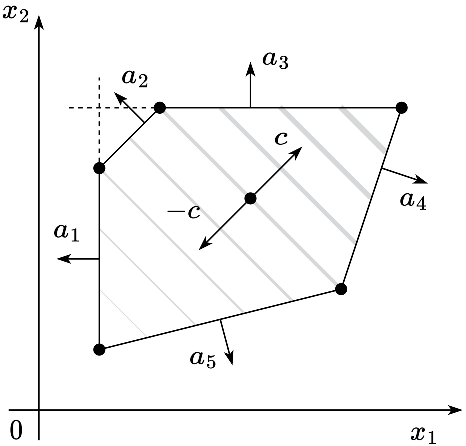

Figure 1: Illustration of the LP Problem.

Linear Programming Recap. Primal and Dual Problems

There are four possibilities:

1. Both the primal and the dual are infeasible.
2. The primal is infeasible and the dual is unbounded.
3. The primal is unbounded and the dual is infeasible.
4. Both the primal and the dual are feasible and their optimal values are equal.

Simplex Algorithm Foundations

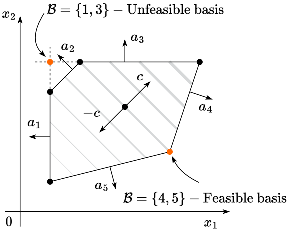

Figure 2: Simplex Algorithm main notions.

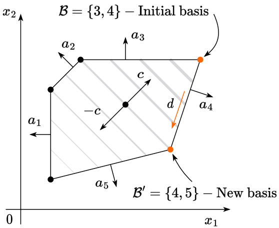

Figure 3: Simplex Algorithm basis change.

Simplex Algorithm main notions

- A **basis** B is a subset of n (integer) numbers between 1 and m, so that $\text{rank} A_B = n$. Note, that we can associate submatrix A_B and corresponding right-hand side b_B with the basis B. Also, we can derive a point of intersection of all these hyperplanes from basis: $x_B = A_B^{-1} b_B$.
- If $Ax_B \leq b$, then basis B is **feasible**.
- A basis B is **optimal** if x_B is an optimum of the `LP.Basic`.

Simplex Algorithm Foundations

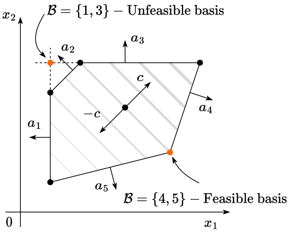

Figure 4: Simplex Algorithm main notions.

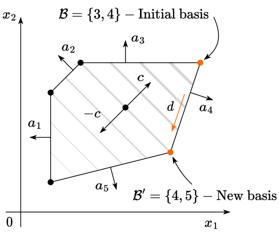

Figure 5: Simplex Algorithm basis change.

Simplex Algorithm Intuition

- The Simplex Algorithm walks along the edges of the polytope, at every corner choosing the edge that decreases $c^\top x$ most
- This either terminates at a corner, or leads to an unconstrained edge ($-\infty$ optimum)

Simplex Algorithm Foundations

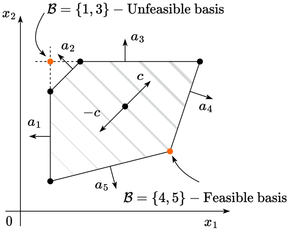

Figure 6: Simplex Algorithm main notions.

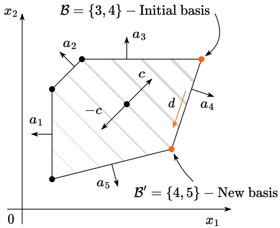

Figure 7: Simplex Algorithm basis change.

Existence of the Standard LP Problem Solution

1. If Standartd LP has a nonempty feasible region, then there is at least one basic feasible point
2. If Standartd LP has solutions, then at least one such solution is a basic optimal point.
3. If Standartd LP is feasible and bounded, then it has an optimal solution.

Simplex Algorithm Foundations

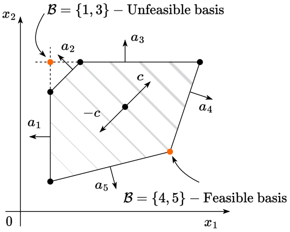

Figure 8: Simplex Algorithm main notions.

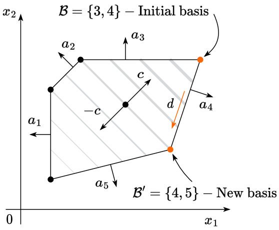

Figure 9: Simplex Algorithm basis change.

Corner Optimality Theorem

Let λ_B be the coordinates of our objective vector c in the basis B:

$$\lambda_B^\top A_B = c^\top \leftrightarrow \lambda_B^\top = c^\top A_B^{-1}$$

If all components of λ_B are non-positive and B is feasible, then B is optimal.

LP Problems Examples. Production Plans

Suppose you are thinking about starting up a business to produce a *Product X*.

Let's find the maximum weekly profit for your business in the Production Plan Problem.

LP Problems Examples. Max Flow Min Cut Problem

See Outer Presentation.

LP Problems Examples. Different Applications

Look at different practical applications of LP Problems and Simplex Algorithm in the Related Collab Notebook.