

Matrix Derivatives. Automatic Differentiation

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Theory recap. Differential

- Differential $df(x)[\cdot] : U \rightarrow V$ in point $x \in U$ for $f(\cdot) : U \rightarrow V$:

$$f(x+h) - f(x) = \underbrace{df(x)[h]}_{\text{differential}} + \bar{o}(\|h\|)$$

- Canonical form of the differential:

| $U \rightarrow V$ | \mathbb{R} | \mathbb{R}^n | $\mathbb{R}^{n \times m}$ |
|---------------------------|------------------------|-----------------|---------------------------|
| \mathbb{R} | $f'(x)dx$ | $\nabla f(x)dx$ | $\nabla f(x)dx$ |
| \mathbb{R}^n | $\nabla f(x)^T dx$ | $J(x)dx$ | — |
| $\mathbb{R}^{n \times m}$ | $tr(\nabla f(X)^T dX)$ | — | — |

Theory recap. Differentiation Rules

- Useful differentiation rules and standard derivatives:

| Differentiation Rules | Standard Derivatives |
|---|--|
| $dA = 0$ | $d(\langle A, X \rangle) = \langle A, dX \rangle$ |
| $d(\alpha X) = \alpha(dX)$ | $d(\langle Ax, x \rangle) = \langle (A + A^T)x, dx \rangle$ |
| $d(AXB) = A(dX)B$ | $d(\text{Det}(X)) = \text{Det}(X)\langle X^{-T}, dX \rangle$ |
| $d(X + Y) = dX + dY$ | $d(X^{-1}) = -X^{-1}(dX)X^{-1}$ |
| $d(X^T) = (dX)^T$ | |
| $d(XY) = (dX)Y + X(dY)$ | |
| $d(\langle X, Y \rangle) = \langle dX, Y \rangle + \langle X, dY \rangle$ | |
| $d\left(\frac{X}{\phi}\right) = \frac{\phi dX - (d\phi)X}{\phi^2}$ | |

Matrix Calculus. Problem 1

Example

Find $\nabla f(x)$, if $f(x) = \frac{1}{2}x^T Ax + b^T x + c$.

Matrix Calculus. Problem 2

Example

Find $\nabla f(X)$, if $f(X) = \text{tr}(AX^{-1}B)$

- $h(x) = f(g(x)) \Rightarrow dh(x_0)[dx] = df(g(x_0))[dg(x_0)[dx]]$

Matrix Calculus. Problem 3

Example

Find the gradient $\nabla f(x)$ and hessian $\nabla^2 f(x)$, if $f(x) = \frac{1}{3}\|x\|_2^3$

- $d^2 f(x)[h_1, h_2] = d \left(\underbrace{df(x)[h_1]}_{\text{fixed when take outer } d(\cdot)} \right) [h_2]$
- Canonic form for $f : \mathbb{R}^n \rightarrow \mathbb{R}$: $d^2 f(x)[h_1, h_2] = h_1^T \underbrace{\nabla^2 f(x)}_{\text{hessian}} h_2$

Automatic Differentiation. Forward mode



Figure 1: Illustration of forward chain rule to calculate the derivative of the function v_i with respect to w_k .

- Uses the forward chain rule
- Has complexity $d \times \mathcal{O}(T)$ operations

Automatic Differentiation. Reverse mode



Figure 2: Illustration of reverse chain rule to calculate the derivative of the function L with respect to the node v_i .

- Uses the backward chain rule
- Stores the information from the forward pass
- Has complexity $\mathcal{O}(T)$ operations

Automatic Differentiation. Problem 1

Example

Which of the AD modes would you choose (forward/ reverse) for the following computational graph of primitive arithmetic operations?



Figure 3: Which mode would you choose for calculating gradients there?

Automatic Differentiation. Problem 2



Figure 4: x could be found as a solution of linear system

Suppose, we have an invertible matrix A and a vector b , the vector x is the solution of the linear system $Ax = b$, namely one can write down an analytical solution $x = A^{-1}b$.

Find the derivatives $\frac{\partial L}{\partial A}$, $\frac{\partial L}{\partial b}$.

Automatic Differentiation. Problem 3



Figure 5: Computation graph for singular regularizer

Suppose, we have the rectangular matrix $W \in \mathbb{R}^{m \times n}$, which has a singular value decomposition:

$$W = U\Sigma V^T, \quad U^T U = I, \quad V^T V = I, \quad \Sigma = \text{diag}(\sigma_1, \dots, \sigma_{\min(m,n)})$$

The regularizer $R(W) = \text{tr}(\Sigma)$ in any loss function encourages low rank solutions. Find the derivative $\frac{\partial R}{\partial W}$.

Computation experiment with JAX

- JAX docs: <https://jax.readthedocs.io/en/latest/notebooks/quickstart.html>