Duality. Strong Duality.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Dual function

The **general mathematical programming problem** with functional constraints:

$$
\begin{split}
& f_0(x) \to \min_{x \in \mathbb{R}^n}\\
\text{s.t. } & f_i(x) \leq 0, \; i = 1,\ldots,m\\
& h_i(x) = 0, \; i = 1,\ldots, p
\end{split}
$$

And the Lagrangian, associated with this problem:

$$
L(x, \lambda, \nu) = f_0(x) + \sum\limits_{i=1}^m \lambda_i f_i(x) + \sum\limits_{i=1}^p\nu_i h_i(x) = f_0(x) + \lambda^\top f(x) + \nu^\top h(x)
$$

We assume $\mathcal{D} = \bigcap\limits_{i=0}^m\textbf{dom } f_i \cap \bigcap\limits_{i=1}^p\textbf{dom } h_i$ is nonempty. We define the Lagrange **dual function** (or just dual function) $g: \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ as the minimum value of the Lagrangian over x: for $\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p$

$$
g(\lambda, \nu) = \inf_{x\in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x\in \mathcal{D}} \left(f_0(x) +\sum\limits_{i=1}^m \lambda_i f_i(x) + \sum\limits_{i=1}^p\nu_i h_i(x) \right)
$$

Dual function. Summary

Primal

Function:

$$f_0(x)$$

Variables:

$$x \in S \subseteq \mathbb{R}^{\ltimes}$$

Constraints:

$$f_i(x) \leq 0, i = 1, \dots, m$$

$$h_i(x) = 0, \; i = 1, \dots, p$$

Dual

Function:

$$g(\lambda, \nu) = \min_{x \in \mathcal{D}} L(x, \lambda, \nu)$$

Variables

$$\lambda \in \mathbb{R}^m_+, \nu \in \mathbb{R}^p$$

Constraints:

$$\lambda_i \geq 0, \forall i \in \overline{1, m}$$

Strong Duality

It is common to name this relation between optimals of primal and dual problems as **weak duality**. For problem, we have:

$$
d^* \leq p^*
$$

While the difference between them is often called **duality gap:**

$$
0 \leq p^* - d^*
$$

Strong duality happens if duality gap is zero:

$$
p^* = d^*
$$

Slater's condition

If for a convex optimization problem (i.e., assuming minimization, f_0, f_i are convex and h_i are affine), there exists a point x such that $h(x) = 0$ and $f_i(x) < 0$ (existance of a **strictly feasible point**), then we have a zero duality gap and KKT conditions become necessary and sufficient.

Reminder of KKT statements

Suppose we have a **general optimization problem**

$$
\begin{split}
& f_0(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & f_i(x) \leq 0, \; i = 1, \ldots, m \\
& h_i(x) = 0, \; i = 1, \ldots, p
\end{split}
\tag{1}
$$

and **convex optimization problem**, where all equality constraints are affine:

$$
h_i(x) = a_i^T x - b_i, i \in 1, \ldots p.
$$

The **KKT system** is:

$$
\begin{split}
& \nabla_x L(x^*, \lambda^*, \nu^*) = 0 \\
& \nabla_\nu L(x^*, \lambda^*, \nu^*) = 0 \\
& \lambda^*_i \geq 0, i = 1, \ldots, m \\
& \lambda^*_i f_i(x^*) = 0, i = 1, \ldots, m \\
& f_i(x^*) \leq 0, i = 1, \ldots, m
\end{split}
\tag{2}
$$

KKT becomes necessary

If x^* is a solution of the original problem Equation 1, then if any of the following regularity conditions is satisfied:

- **Strong duality** If $f_1, \dots f_m, h_1, \dots h_p$ are differentiable functions and we have a problem Equation 1 with zero duality gap, then Equation 2 are necessary (i.e. any optimal set x^*, λ^*, ν^* should satisfy Equation 2)
- **LCQ** (Linearity constraint qualification). If $f_1, \dots f_m, h_1, \dots h_p$ are affine functions, then no other condition is needed.
- **LICQ** (Linear independence constraint qualification). The gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at x^*
- **SC** (Slater's condition) For a convex optimization problem (i.e., assuming minimization, f_i are convex and h_j is affine), there exists a point x such that $h_j(x) = 0$ and $g_i(x) < 0$.

Than it should satisfy Equation 2

KKT in convex case

If a convex optimization problem with differentiable objective and constraint functions satisfies Slater's condition, then the KKT conditions provide necessary and sufficient conditions for optimality: Slater's condition implies that the optimal duality gap is zero and the dual optimum is attained, so x^* is optimal if and only if there are (λ^*, ν^*) that, together with x^*, satisfy the KKT conditions.

Problem 1. Dual LP

Ensure, that the following standard form *Linear Programming* (LP):

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax = b \\
& x_i \geq 0, \ i = 1, \dots, n
\end{aligned}
$$

Has the following dual:

$$
\begin{aligned}
&\max_{y \in \mathbb{R}^n} b^\top y \\
\text{s.t. } & A^T y \preceq c
\end{aligned}
$$

Find the dual problem to the problem above (it should be the original LP).

Problem 2. Projection onto probability simplex

Find the Euclidean projection of $x \in \mathbb{R}^n$ onto probability simplex

$$\mathcal{P} = \{z \in \mathbb{R}^n \mid z \succeq 0, \mathbf{1}^\top z = 1\},$$

i.e. solve the following problem:

$$\frac{1}{2}\|y - x\|_2^2 \to \min_{y \in \mathbb{R}^n \succeq 0}$$

$$\text{s.t. } \mathbf{1}^\top y = 1$$

Problem 3. Shadow prices or tax interpretation

Consider an enterprise where x represents its operational strategy and $f_0(x)$ is the operating cost. Therefore, $-f_0(x)$ denotes the profit in dollars. Each constraint $f_i(x) \leq 0$ signifies a resource or regulatory limit. The goal is to maximize profit while adhering to these limits, which is equivalent to solving:

$$
\begin{aligned}
f_0(x) &\to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } f_i(x) &\leq 0, \ i = 1, \dots, m
\end{aligned}
$$

The optimal profit here is $-p^*$.

Problem 4. Norm regularized problems

Ensure, that the following normed regularized problem:

$$
\min f(x) + \|Ax\|
$$

has the following dual:

$$
\begin{split}
f^*(-A^\top y) \to \min_y \\
\text{s.t. } \|y\|_* \leq 1
\end{split}
$$