

Matrix calculus. Line search.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Theory recap. Differential

- Differential $df(x)[\cdot] : U \rightarrow V$ in point $x \in U$ for $f(\cdot) : U \rightarrow V$:

$$f(x+h) - f(x) = \underbrace{df(x)[h]}_{\text{differential}} + \bar{o}(\|h\|)$$

| $U \rightarrow V$ | \mathbb{R} | \mathbb{R}^n | $\mathbb{R}^{n \times m}$ |
|---------------------------|------------------------|-----------------|---------------------------|
| \mathbb{R} | $f'(x)dx$ | $\nabla f(x)dx$ | $\nabla f(x)dx$ |
| \mathbb{R}^n | $\nabla f(x)^T dx$ | $J(x)dx$ | — |
| $\mathbb{R}^{n \times m}$ | $tr(\nabla f(X)^T dX)$ | — | — |

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- Canonical form of the differential:

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Theory recap. Differentiation Rules

- Useful differentiation rules and standard derivatives:

Differentiation Rules

$$dA = 0$$

$$d(\alpha X) = \alpha(dX)$$

$$d(AXB) = A(dX)B$$

$$d(X + Y) = dX + dY$$

$$d(X^T) = (dX)^T$$

$$d(XY) = (dX)Y + X(dY)$$

$$d(\langle X, Y \rangle) = \langle dX, Y \rangle + \langle X, dY \rangle$$

$$d\left(\frac{X}{\phi}\right) = \frac{\phi dX - (d\phi)X}{\phi^2}$$

Standard Derivatives

$$d(\langle A, X \rangle) = \langle A, dX \rangle$$

$$d(\langle Ax, x \rangle) = \langle (A + A^T)x, dx \rangle$$

$$d(\text{Det}(X)) = \text{Det}(X) \langle X^{-T}, dX \rangle$$

$$d(X^{-1}) = -X^{-1}(dX)X^{-1}$$

Theory recap. Differential and Gradient / Hessian

We can retrieve the gradient using the following formula:

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$$d^2f(x) = \langle \nabla^2 f(x) dx_1, dx \rangle = \langle H_f(x) dx_1, dx \rangle$$

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 - The idea behind backtracking line search

Matrix Calculus. Problem 1

Example

Find $\nabla f(x)$, if $f(x) = \frac{1}{2}x^T Ax + b^T x + c$.

Matrix Calculus. Problem 2

Example

Find $\nabla f(X)$, if $f(X) = \text{tr}(AX^{-1}B)$

Matrix Calculus. Problem 3

Example

Find the gradient $\nabla f(x)$ and hessian $\nabla^2 f(x)$, if $f(x) = \frac{1}{3}\|x\|_2^3$

Line Search. Example 1: Comparison of Methods (Colab ♣)

$$f_1(x) = x(x-2)(x+2)^2 + 10$$

$$[a, b] = [-3, 2]$$

Random search: 72 function calls. 36 iterations. $f_1^* = 0.09$

Binary search: 23 function calls. 13 iterations. $f_1^* = 10.00$

Golden search: 19 function calls. 18 iterations. $f_1^* = 10.00$

Parabolic search: 20 function calls. 17 iterations. $f_1^* = 10.00$

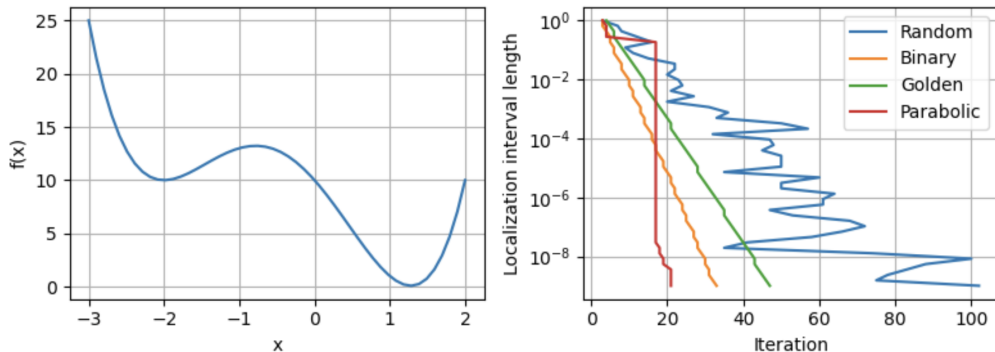


Figure 1: Comparison of different line search algorithms with f_1

Line Search. Example 1: Comparison of Methods (Colab ♣)

$$f_2(x) = -\sqrt{\frac{2}{\pi}} \frac{x^2 e^{-\frac{x^2}{8}}}{8}$$

$$[a, b] = [0, 6]$$

Random search: 68 function calls. 34 iterations. $f_2^* = 0.71$

Binary search: 23 function calls. 13 iterations. $f_2^* = 0.71$

Golden search: 20 function calls. 19 iterations. $f_2^* = 0.71$

Parabolic search: 17 function calls. 14 iterations. $f_2^* = 0.71$



Figure 2: Comparison of different line search algorithms with f_2

Line Search. Example 1: Comparison of Methods (Colab ♣)

$$f_3(x) = \sin \left(\sin \left(\sin \left(\sqrt{\frac{x}{2}} \right) \right) \right)$$
$$[a, b] = [5, 70]$$

Random search: 66 function calls. 33 iterations. $f_3^* = 0.25$

Binary search: 32 function calls. 17 iterations. $f_3^* = 0.25$

Golden search: 25 function calls. 24 iterations. $f_3^* = 0.25$

Parabolic search: 103 function calls. 100 iterations. $f_3^* = 0.25$



Figure 3: Comparison of different line search algorithms with f_3

Line Search. Example 2: The Brent Method

- Parabolic Interpolation + Golden Search = Brent Method



Figure 4: Idea of Brent Method

Line Search. Example 2: The Brent Method

- Parabolic Interpolation + Golden Search = Brent Method
- The key idea of the method is to track the value of the optimized scalar function at six points a, b, x, w, v, u

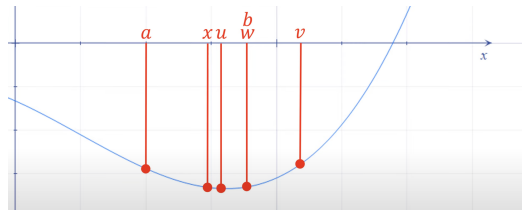


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- $[a, b]$ – localization interval in the current iteration



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- The points x, w and v such that the inequality $f(x) \leq f(w) \leq f(v)$ is valid

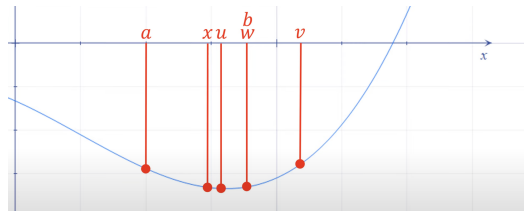


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- $[a, b]$ – localization interval in the current iteration
- The points x, w and v such that the inequality $f(x) \leq f(w) \leq f(v)$ is valid
- u – minimum of a parabola built on points x, w and v or the point of the golden section of the largest of the intervals $[a, x]$ $[x, b]$.



Figure 4: Idea of Brent Method

Line Search. Example 2: The Brent Method

A parabola is constructed only if the points x , w and v are different, and its vertex u^* is taken as the point u only if

- $u^* \in [a, b]$



Figure 5: An example of how the Brent Method works

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- u^* is no more than half the length of the step that was before the previous one, from the point x



Figure 5: An example of how the Brent Method works

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- If the conditions above are not met, then point u is located from the golden search



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- Example In Colab ♣



Figure 5: An example of how the Brent Method works