Gradient Descent. Convergence rates

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Gradient Descent

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The bottleneck (for almost all gradient methods) is choosing step-size, which can lead to the dramatic difference in method's behavior.

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- **Backtracking line search**. Fix two parameters: $0 < \beta < 1$ and $0 < \alpha \leq 0.5$. At each iteration, start with $\text{t} = 1$, and while

$$f(x_k - t\nabla f(x_k)) > f(x_k) - \alpha t \|\nabla f(x_k)\|_2^2,$$

shrink $t = \beta t$. Else perform Gradient Descent update $x_{k+1} = x_k - t\nabla f(x_k)$.

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- **Exact line search**.

$$\eta_k = \arg\min_{\eta \geq 0} f(x_k - \eta \nabla f(x_k))$$

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The result of this method is

$$x_{k+1} = x_k - \alpha f'(x_k)$$

Minimizer of Lipschitz parabola

If a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and its gradient satisfies Lipschitz conditions with constant L, then $\forall x, y \in \mathbb{R}^n$:

$$|f(y) - f(x) - \langle \nabla f(x), y - x \rangle| \leq \frac{L}{2} \|y - x\|^2,$$

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which geometrically means, that if we'll fix some point $x_0 \in \mathbb{R}^n$ and define two parabolas:

$$\phi_1(x) = f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle - \frac{L}{2} \|x - x_0\|^2,$$

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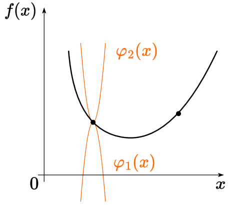

Figure 1: Illustration

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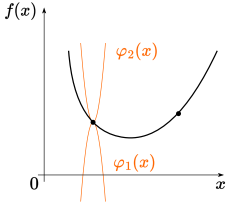

Figure 1: Illustration

$$\nabla \phi_2(x) = 0$$
$$\nabla f(x_0) + L(x^* - x_0) = 0$$
$$x^* = x_0 - \frac{1}{L} \nabla f(x_0)$$
$$x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k)$$

This way leads to the $\frac{1}{L}$ stepsize choosing. However, often the L constant is not known.

Any μ-strongly convex differentiable function is a PL-function

Theorem

If a function $f(x)$ is differentiable and μ-strongly convex, then it is a PL function.

Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2}\|y - x\|_2^2$$

Putting $y = x^*$:

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which is exactly the PL condition. It means, that we already have linear convergence proof for any strongly convex function.

Exact line search aka steepest descent

$$\alpha_k = \text{arg}\min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \text{arg}\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. Interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

$$\alpha_k = \text{arg}\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

Optimality conditions:

$$\nabla f(x_{k+1})^\top \nabla f(x_k) = 0$$

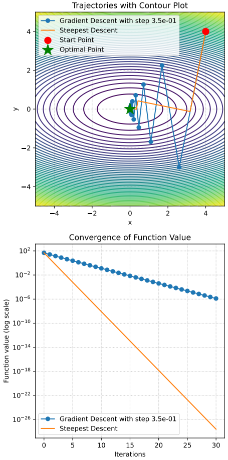

Figure 2: Steepest Descent

Open In Colab ♣

Convergence analysis. Backtracking line search

Assume that f is convex, differentiable and Lipschitz gradient with constant $L > 0$.

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Gradient descent with fixed step size $t \leq 1/L$ satisfies

$$f\left(x^{(k)}\right) - f^* \leq \frac{\left\|x^{(0)} - x^*\right\|_2^2}{2tk}$$

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$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2}\|y - x\|_2^2, \forall x, y$$

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Let $y = x^+ = x - t\nabla f(x)$, then:

$$f\left(x^+\right) \leq f(x) - \left(1 - \frac{Lt}{2}\right) t\|\nabla f(x)\|_2^2 \leq f(x) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

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This recalls us the stopping condition in Backtracking line search when $\alpha = 0.5, t = \frac{1}{L}$. Hence, Backtracking line search with $\alpha = 0.5$ plus condition of Lipschitz gradient will guarantee us the convergence rate of $O(1/k)$.

Problem

Consider the problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

where $f(x)$ is convex and L-smooth. Find convergence rate of gradient descent with optimal theoretical step size $\eta_k = \frac{1}{L}$ for the *mean point* and for the *best point*. In other words get upper bounds on

- $f(\bar{x}_N) - f^*$, where $\bar{x}_N = \frac{1}{N}\sum_{i=0}^{N-1} x_i$,
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Gradient descent step

$$x_{k+1} = \arg\min_{x \in \mathbb{R}^n} \left\{ \Psi_k(x) \equiv f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{L}{2}\|x - x_k\|_2^2 \right\}$$

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Tip

Use the fact that $\Psi_k(x)$ is L-strongly convex due to quadratic regularizer.

Code

Examples: code snippet.