

Convexity. Strong convexity.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Line Segment

Suppose x_1, x_2 are two points in \mathbb{R}^n . Then the line segment between them is defined as follows:

$$x = \theta x_1 + (1 - \theta)x_2, \theta \in [0, 1]$$



Figure 1: Illustration of a line segment between points x_1, x_2

Convex Set

The set S is called **convex** if for any x_1, x_2 from S the line segment between them also lies in S , i.e.

$$\forall \theta \in [0, 1], \forall x_1, x_2 \in S : \theta x_1 + (1 - \theta)x_2 \in S$$

i Example

Any affine set, a ray, a line segment - they all are convex sets.



Figure 2: Top: examples of convex sets. Bottom: examples of non-convex sets.

Problem 1

Question

Prove, that ball in \mathbb{R}^n (i.e. the following set $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\| \leq r\}$) - is convex.

Problem 2

Question

Is stripe - $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$ - convex?

Problem 3

Question

Let S be such that $\forall x, y \in S \rightarrow \frac{1}{2}(x + y) \in S$. Is this set convex?

Problem 4

Question

The set $S = \{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex. Is this set convex?

Convex Function

The function $f(x)$, which is defined on the convex set $S \subseteq \mathbb{R}^n$, is called **convex** on S , if:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$.

If the above inequality holds as strict inequality $x_1 \neq x_2$ and $0 < \lambda < 1$, then the function is called **strictly convex** on S .



Figure 3: Difference between convex and non-convex function

Strong Convexity

$f(x)$, defined on the convex set $S \subseteq \mathbb{R}^n$, is called μ -strongly convex (strongly convex) on S , if:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) - \frac{\mu}{2}\lambda(1 - \lambda)\|x_1 - x_2\|^2$$

for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$ for some $\mu > 0$.



Figure 4: Strongly convex function is greater or equal than Taylor quadratic approximation at any point

First-order differential criterion of convexity

The differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is convex if and only if $\forall x, y \in S$:

$$f(y) \geq f(x) + \nabla f^T(x)(y - x)$$

Let $y = x + \Delta x$, then the criterion will become more tractable:

$$f(x + \Delta x) \geq f(x) + \nabla f^T(x)\Delta x$$



Second-order differential criterion of strong convexity

Twice differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is called μ -strongly convex if and only if $\forall x \in \text{int}(S) \neq \emptyset$:

$$\nabla^2 f(x) \succeq \mu I$$

In other words:

$$\langle y, \nabla^2 f(x)y \rangle \geq \mu \|y\|^2$$

Motivational Experiment with JAX

Why convexity and strong convexity is important? Check the simple code snippet.

Problem 5

Question

Show, that $f(x) = \|x\|$ is convex on \mathbb{R}^n .

Question

Show, that $f(x) = x^\top Ax$, where $A \succeq 0$ - is convex on \mathbb{R}^n .

Problem 6

Question

Show, that if $f(x)$ is convex on \mathbb{R}^n , then $\exp(f(x))$ is convex on \mathbb{R}^n .

Problem 7

Question

If $f(x)$ is convex nonnegative function and $p \geq 1$. Show that $g(x) = f(x)^p$ is convex.

Problem 8

i Question

Show that, if $f(x)$ is concave positive function over convex S , then $g(x) = \frac{1}{f(x)}$ is convex.

i Question

Show, that the following function is convex on the set of all positive denominators

$$f(x) = \frac{1}{x_1 - \frac{1}{x_2 - \frac{1}{x_3 - \frac{1}{\dots}}}}, x \in \mathbb{R}^n$$

Problem 9

Question

Let $S = \{x \in \mathbb{R}^n \mid x \succ 0, \|x\|_\infty \leq M\}$. Show that $f(x) = \sum_{i=1}^n x_i \log x_i$ is $\frac{1}{M}$ -strongly convex.

Polyak-Lojasiewicz (PL) Condition

PL inequality holds if the following condition is satisfied for some $\mu > 0$,

$$\|\nabla f(x)\|^2 \geq \mu(f(x) - f^*) \forall x$$

The example of a function, that satisfies the PL-condition, but is not convex.

$$f(x, y) = \frac{(y - \sin x)^2}{2}$$

Example of PL non-convex function  Open in Colab.

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Find function, that translates object x to probability $p(y = 1|x)$:

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Criterion

Binary cross-entropy (logistic loss):

$$L(p, X, y) = - \sum_{i=1}^n y_i \log p(X_i) + (1 - y_i) \log (1 - p(X_i)),$$

that is minimized with respect to w .

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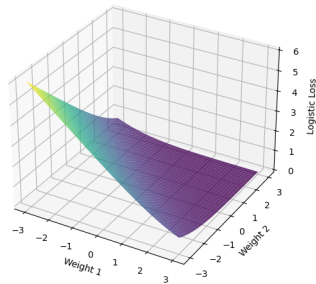


Figure 6: Logistic Loss in Parameter Space for $x=(1,1)$, $y=1$

We can make this problem μ -strongly convex if we consider regularized logistic loss as criterion: $L(p, X, y) + \frac{\mu}{2} \|w\|_2^2$.

Support Vector Machine (SVM)

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Criterion

Hinge loss:

$$L(w, X, y) = \frac{1}{2} \|w\|_2^2 + C \sum_{i=1}^n \max(0, 1 - y_i (X_i^T w + b)), \text{ that}$$

is minimized with respect to w and b .

This problem is strongly-convex due to squared Euclidean norm.

$f \rightarrow \min_{w, b}$
Practical example

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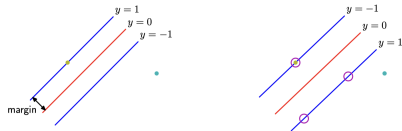


Figure 7: Support Vector Machine

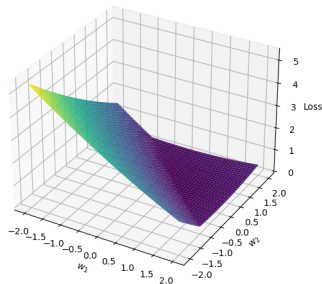


Figure 8: L_2 -Regularized Hinge Loss in Parameter Space for $x=(1,1)$, $y=1$

Some other curious examples

- **Low-rank matrix approximation**

$$\min_X \|A - X\|_F^2 \text{ s.t. } \text{rank}(X) \leq k.$$

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- **Convex relaxation via nuclear norm**

$$\min_X \text{rank}(X), \text{ s.t. } X_{ij} = M_{ij}, (i, j) \in I.$$

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- **Convex relaxation via nuclear norm**

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NP-hard problem, but $\|A\|_* = \text{trace}(\sqrt{A^T A}) = \sum_{i=1}^{\text{rank}(A)} \sigma_i(A)$ is a convex envelope of the matrix rank.