Convexity. Strong convexity.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Line Segment

Suppose x_1, x_2 are two points in $\mathbb{R}^n$. Then the line segment between them is defined as follows:

$$x = \theta x_1 + (1 - \theta)x_2, \ \theta \in [0, 1]$$

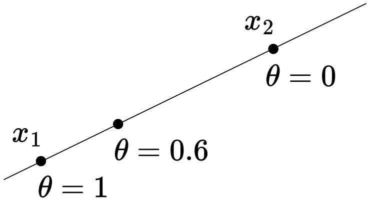


Figure 1: Illustration of a line segment between points x_1, x_2

Convex Set

The set S is called **convex** if for any x_1, x_2 from S the line segment between them also lies in S, i.e.

$$\forall \theta \in [0,1], \ \forall x_1, x_2 \in S : \theta x_1 + (1-\theta) x_2 \in S$$

Example

Any affine set, a ray, a line segment - they all are convex sets.

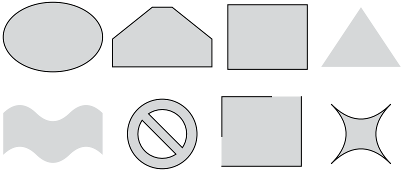


Figure 2: Top: examples of convex sets. Bottom: examples of non-convex sets.

Problem 1

Question

Prove, that ball in $\mathbb{R}^n$ (i.e. the following set $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\| \leq r\}$) - is convex.

Problem 2

Question

Is stripe - $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$ - convex?

Problem 3

Question

Let S be such that $\forall x, y \in S \to \frac{1}{2}(x+y) \in S$. Is this set convex?

Problem 4

Question

The set $S = \{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex. Is this set convex?

Convex Function

The function $f(x)$, **which is defined on the convex set $S \subseteq \mathbb{R}^n$**, is called **convex** on S, if:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$.
If the above inequality holds as strict inequality $x_1 \neq x_2$ and $0 < \lambda < 1$, then the function is called **strictly convex** on S.

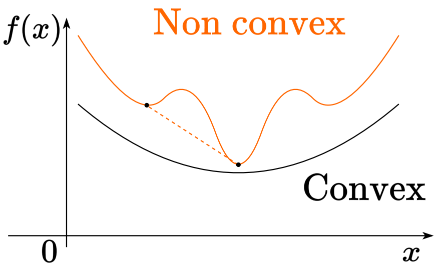


Figure 3: Difference between convex and non-convex function

Strong Convexity

$f(x)$, **defined on the convex set** $S \subseteq \mathbb{R}^n$, is called μ-strongly convex (strongly convex) on S, if:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) - \frac{\mu}{2}\lambda(1-\lambda)\|x_1 - x_2\|^2$$

for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$ for some $\mu > 0$.

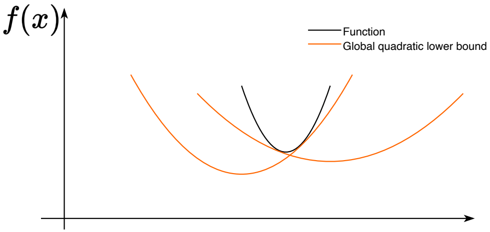


Figure 4: Strongly convex function is greater or equal than Taylor quadratic approximation at any point

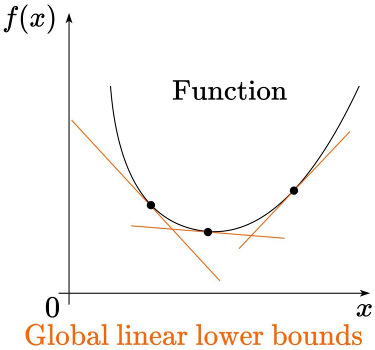
First-order differential criterion of convexity

The differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is convex if and only if $\forall x, y \in S$:

$$f(y) \geq f(x) + \nabla f^T(x)(y - x)$$

Let $y = x + \Delta x$, then the criterion will become more tractable:

$$f(x + \Delta x) \geq f(x) + \nabla f^T(x)\Delta x$$



Second-order differential criterion of strong convexity

Twice differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is called μ-strongly convex if and only if $\forall x \in \mathbf{int}(S) \neq \emptyset$:

$$
\nabla^2 f(x) \succeq \mu I
$$

In other words:

$$
\langle y, \nabla^2 f(x) y \rangle \geq \mu \|y\|^2
$$

Motivational Experiment with JAX

Why convexity and strong convexity is important? Check the simple code snippet.

Problem 5

Question

Show, that $f(x) = \|x\|$ is convex on $\mathbb{R}^n$.

Question

Show, that $f(x) = x^\top Ax$, where $A \succeq 0$ - is convex on $\mathbb{R}^n$.

Problem 6

Question

Show, that if $f(x)$ is convex on $\mathbb{R}^n$, then $\exp(f(x))$ is convex on $\mathbb{R}^n$.

Problem 7

Question

If $f(x)$ is convex nonnegative function and $p \geq 1$. Show that $g(x) = f(x)^p$ is convex.

Problem 8

Question

Show that, if $f(x)$ is concave positive function over convex S, then $g(x) = \frac{1}{f(x)}$ is convex.

Question

Show, that the following function is convex on the set of all positive denominators

$$f(x) = \cfrac{1}{x_1 - \cfrac{1}{x_2 - \cfrac{1}{x_3 - \cfrac{1}{\ldots}}}}, x \in \mathbb{R}^n$$

Problem 9

Question

Let $S = \{x \in \mathbb{R}^n \mid x \succ 0, \|x\|_\infty \leq M\}$. Show that $f(x) = \sum_{i=1}^n x_i \log x_i$ is $\frac{1}{M}$-strongly convex.

Polyak-Lojasiewicz (PL) Condition

PL inequality holds if the following condition is satisfied for some $\mu > 0$,

$$\|\nabla f(x)\|^2 \geq \mu(f(x) - f^*) \forall x$$

The example of a function, that satisfies the PL-condition, but is not convex.

$$f(x,y) = \frac{(y - \sin x)^2}{2}$$

Example of Pl non-convex function Open in Colab.

Logistic regression

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Data:
$X \in \mathbb{R}^{m \times n}, y \in \{0, 1\}^n$.

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Find function, that translates object x to probability $p(y=1|x)$:
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Criterion

Binary cross-entropy (logistic loss):
$L(p, X, y) = -\sum_{i=1}^{n} y_i \log p\left(X_i\right) + (1 - y_i) \log\left(1 - p\left(X_i\right)\right)$,
that is minimized with respect to w.

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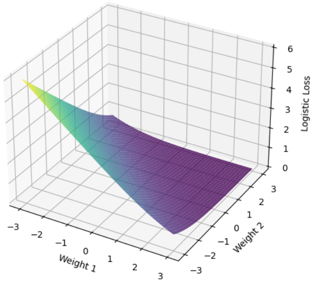

Figure 6: Logistic Loss in Parameter Space for x=(1,1), y=1

We can make this problem μ-strongly convex if we consider regularized logistic loss as criterion: $L(p, X, y) + \frac{\mu}{2}\|w\|_2^2$.

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Hinge loss:
$L(w, X, y) = \frac{1}{2}\|w\|_2^2 + C \sum_{i=1}^n \max(0, 1 - y_i(X_i^T w + b))$, that is minimized with respect to w and b.

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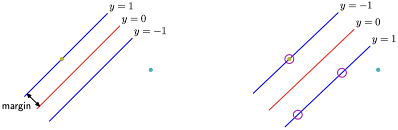

Figure 7: Support Vector Machine

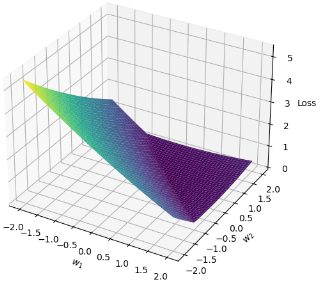

Figure 8: L_2-Regularized Hinge Loss in Parameter Space for x=(1,1), y=1

Some other curious examples

- **Low-rank matrix approximation**

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NP-hard problem, but $\|A\|_* = trace(\sqrt{A^T A}) = \sum_{i=1}^{rank(A)} \sigma_i(A)$ is a convex envelope of the matrix rank.