A medieval knight with a long beard and hair, wearing ornate armor with fur trim, stands in a dimly lit room. He is looking at a large, ornate mirror. In the mirror, a dragon with red eyes and sharp teeth is visible. The room has a round window with a cross pattern, and several lit candles are visible in the foreground and background.

Duality

Daniil Merkulov

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Motivation

Duality lets us associate to any constrained optimization problem a concave maximization problem, whose solutions lower bound the optimal value of the original problem. What is interesting is that there are cases, when one can solve the primal problem by first solving the dual one. Now, consider a general constrained optimization problem:

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As a consequence:

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And the Lagrangian, associated with this problem:

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We assume $\mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i$ is nonempty. We define the Lagrange dual function (or just dual function) $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ as the minimum value of the Lagrangian over x : for $\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p$

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When the Lagrangian is unbounded below in x , the dual function takes on the value $-\infty$. Since the dual function is the pointwise infimum of a family of affine functions of (λ, ν) , it is concave, even when the original problem is not convex.

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Let us show, that the dual function yields lower bounds on the optimal value p^* of the original problem for any $\lambda \succeq 0, \nu$. Suppose some \hat{x} is a feasible point for the original problem, i.e., $f_i(\hat{x}) \leq 0$ and $h_i(\hat{x}) = 0, \lambda \succeq 0$. Then we have:

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The term “dual feasible”, to describe a pair (λ, ν) with $\lambda \succeq 0$ and $g(\lambda, \nu) > -\infty$, now makes sense. It means, as the name implies, that (λ, ν) is feasible for the dual problem. We refer to (λ^*, ν^*) as dual optimal or optimal Lagrange multipliers if they are optimal for the above problem.

Summary

	Primal	Dual
Function	$f_0(x)$	$g(\lambda, \nu) = \min_{x \in \mathcal{D}} L(x, \lambda, \nu)$
Variables	$x \in S \subseteq \mathbb{R}^n$	$\lambda \in \mathbb{R}_+^m, \nu \in \mathbb{R}^p$
Constraints	$f_i(x) \leq 0, i = 1, \dots, m$ $h_i(x) = 0, i = 1, \dots, p$	$\lambda_i \geq 0, \forall i \in \overline{1, m}$
Problem	$\begin{aligned} & f_0(x) \rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } & f_i(x) \leq 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, p \end{aligned}$	$\begin{aligned} g(\lambda, \nu) & \rightarrow \max_{\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p} \\ \text{s.t. } & \lambda \succeq 0 \end{aligned}$
Optimal	x^* if feasible, $p^* = f_0(x^*)$	λ^*, ν^* if max is achieved, $d^* = g(\lambda^*, \nu^*)$

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$$-(1/4)\nu^T A A^T \nu - b^T \nu \leq \inf\{x^T x \mid Ax = b\}.$$

Which is a simple non-trivial lower bound without any problem solving.

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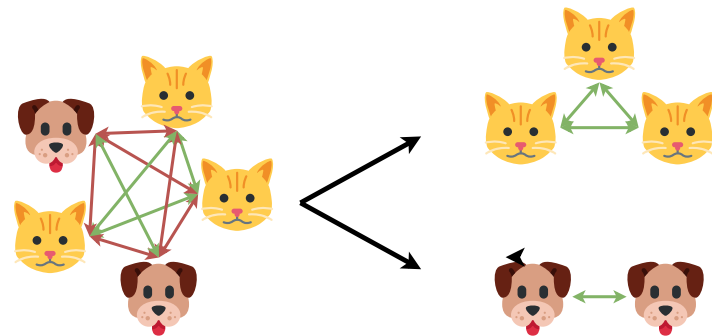


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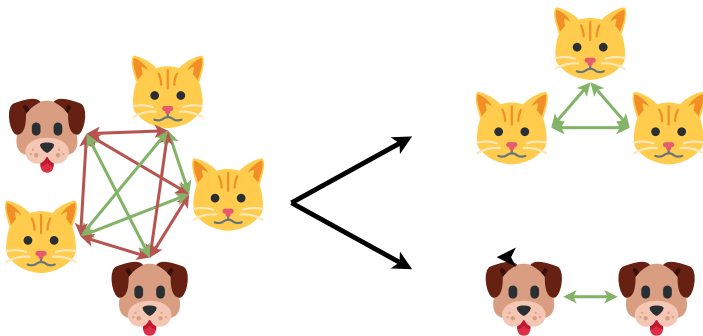


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The coefficient W_{ij} in the matrix represents the expense associated with placing elements i and j in the same partition, while $-W_{ij}$ signifies the cost of segregating them. The objective encapsulates the aggregate cost across all pairs of elements, and the challenge posed by problem is to find the partition that minimizes the total cost.

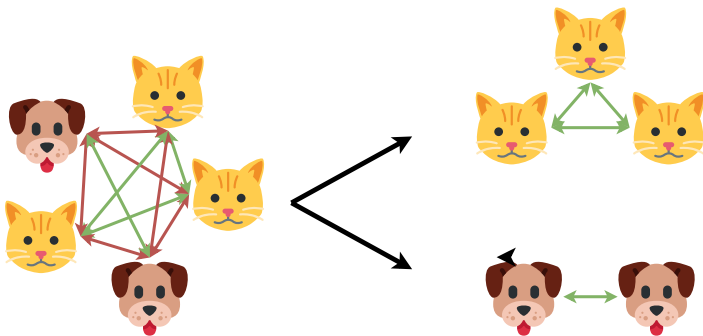


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The code for the problem is available here  Open in Colab

Strong duality

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It is common to name this relation between optimals of primal and dual problems as **weak duality**. For problem, we have:

$$p^* \geq d^*$$

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- Several sufficient conditions known!
- “Easy” necessary and sufficient conditions: unknown.

Strong duality in linear least squares

Exercise

In the Least-squares solution of linear equations example above calculate the primal optimum p^* and the dual optimum d^* and check whether this problem has strong duality or not.

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$$\max_{\nu \in \mathbb{R}^m} -\frac{1}{4} \nu^T A A^T \nu - b^T \nu$$

1. Lagrangian:

$$L(x, \nu) = x^T x + \nu^T (Ax - b)$$

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Slater's condition

Theorem

If for a convex optimization problem (i.e., assuming minimization, f_0, f_i are convex and h_i are affine), there exists a point x such that $h(x) = 0$ and $f_i(x) < 0$ (existence of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.

An example of convex problem, when Slater's condition does not hold

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$$\min\{f_0(x) = x \mid f_1(x) = \frac{x^2}{2} \leq 0\},$$

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$$\min\{f_0(x) = x \mid f_1(x) = \frac{x^2}{2} \leq 0\},$$

The only point in the budget set is: $x^* = 0$. However, it is impossible to find a non-negative $\lambda^* \geq 0$, such that

$$\nabla f_0(0) + \lambda^* \nabla f_1(0) = 1 + \lambda^* x = 0.$$

Useful features of duality

- **Construction of lower bound on solution of the primal problem.**

It could be very complicated to solve the initial problem. But if we have the dual problem, we can take an arbitrary $y \in \Omega$ and substitute it in $g(y)$ - we'll immediately obtain some lower bound.

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$f_0(x) - f_0^* \leq f_0(x) - g(y)$ for an arbitrary $y \in \Omega$ (suboptimality certificate). Moreover, $p^* \in [g(y), f_0(x)]$, $d^* \in [g(y), f_0(x)]$

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- **Dual function is always concave**

As a pointwise minimum of affine functions.

Sensitivity analysis

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One can even show, that when P is convex optimization problem, $p^*(u, v)$ is a convex function.

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And taking the optimal x for the perturbed problem, we have:

$$p^*(u, v) \geq p^*(0, 0) - \lambda^{*T} u - \nu^{*T} v \quad (1)$$

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In scenarios where strong duality holds, we can draw several insights about the sensitivity of optimal solutions in relation to the Lagrange multipliers. These insights are derived from the inequality expressed in equation above:

- **Impact of Tightening a Constraint (Large λ_i^*):**

When the i th constraint's Lagrange multiplier, λ_i^* , holds a substantial value, and if this constraint is tightened (choosing $u_i < 0$), there is a guarantee that the optimal value, denoted by $p^*(u, v)$, will significantly increase.

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- **Consequences of Loosening a Constraint (Small λ_i^*):**

If the Lagrange multiplier λ_i^* for the i th constraint is relatively small, and the constraint is loosened (choosing $u_i > 0$), it is anticipated that the optimal value $p^*(u, v)$ will not significantly decrease.

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When the i th constraint's Lagrange multiplier, λ_i^* , holds a substantial value, and if this constraint is tightened (choosing $u_i < 0$), there is a guarantee that the optimal value, denoted by $p^*(u, v)$, will significantly increase.

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- If ν_i^* is large and positive and $v_i < 0$ is chosen, or
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These interpretations provide a framework for understanding how changes in constraints, reflected through their corresponding Lagrange multipliers, impact the optimal solution in problems where strong duality holds.

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The same idea can be used to establish the fact about v_i . The local sensitivity result Equation 2 provides a way to understand the impact of constraints on the optimal solution x^* of an optimization problem. If a constraint $f_i(x^*)$ is negative at x^* , it's not affecting the optimal solution, meaning small changes to this constraint won't alter the optimal value. In this case, the corresponding optimal Lagrange multiplier will be zero, as per the principle of complementary slackness.

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However, if $f_i(x^*) = 0$, meaning the constraint is precisely met at the optimum, then the situation is different. The value of the i -th optimal Lagrange multiplier, λ_i^* , gives us insight into how 'sensitive' or 'active' this constraint is. A small λ_i^* indicates that slight adjustments to the constraint won't significantly affect the optimal value. Conversely, a large λ_i^* implies that even minor changes to the constraint can have a significant impact on the optimal solution.

Applications

Solving the primal via the dual

An important consequence of stationarity: under strong duality, given a dual solution λ^*, ν^* , any primal solution x^* solves

$$\min_{x \in \mathbb{R}^n} f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x)$$

Often, solutions of this unconstrained problem can be expressed **explicitly**, giving an explicit characterization of primal solutions from dual solutions.

Furthermore, suppose the solution of this problem is unique; then it must be the primal solution x^* .

This can be very helpful when the dual is easier to solve than the primal.

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For example, consider:

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where each $f_i(x_i) = \frac{1}{2}c_i x_i^2$ (smooth and strictly convex).

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This gives:

$$x_i^* = \frac{a_i \nu^*}{c_i}.$$

Mixed strategies for matrix games



Figure 2: The scheme of a mixed strategy matrix game

Mixed strategies for matrix games



Player 1

u_1
\dots
u_k
\dots
u_n

v_1
\dots
\dots
v_l
\dots
\dots
v_m



Player 2

In zero-sum matrix games, players 1 and 2 choose actions from sets $\{1, \dots, n\}$ and $\{1, \dots, m\}$, respectively. The outcome is a payment from player 1 to player 2, determined by a payoff matrix $P \in \mathbb{R}^{n \times m}$. Each player aims to use mixed strategies, choosing actions according to a probability distribution: player 1 uses probabilities u_k for each action i , and player 2 uses v_l .

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Mixed strategies for matrix games. Player 1's Perspective



Player 1

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\dots
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\dots
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Assuming player 2 knows player 1's strategy u , player 2 will choose v to maximize $u^T P v$. The worst-case expected payoff is thus:

$$\max_{v \geq 0, 1^T v = 1} u^T P v = \max_{i=1, \dots, m} (P^T u)_i$$

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\dots
u_n

Assuming player 2 knows player 1's strategy u , player 2 will choose v to maximize $u^T P v$. The worst-case expected payoff is thus:

$$\max_{v \geq 0, 1^T v = 1} u^T P v = \max_{i=1, \dots, m} (P^T u)_i$$

Player 1's optimal strategy minimizes this worst-case payoff, leading to the optimization problem:

$$\begin{aligned} \min \quad & \max_{i=1, \dots, m} (P^T u)_i \\ \text{s.t.} \quad & u \geq 0 \\ & 1^T u = 1 \end{aligned} \tag{3}$$

This forms a convex optimization problem with the optimal value denoted as p_1^* .

Mixed strategies for matrix games. Player 2's Perspective

Conversely, if player 1 knows player 2's strategy v , the goal is to minimize $u^T P v$. This leads to:

$$\min_{u \geq 0, 1^T u = 1} u^T P v = \min_{i=1, \dots, n} (Pv)_i$$



Player 2

v_1

...

...

v_l

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Player 2 then maximizes this to get the largest guaranteed payoff, solving the optimization problem:

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The optimal value here is p_2^* .



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Duality and Equivalence

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Conclusion

This formulation shows that the Lagrange dual problem is equivalent to problem Equation 4. Given the feasibility of these linear programs, strong duality holds, meaning the optimal values of Equation 3 and Equation 4 are equal.

A nonconvex quadratic problem with strong duality

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References

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