

# Duality.

## Seminar

Optimization for ML. Faculty of Computer Science. HSE University

## Dual function

The **general mathematical programming problem** with functional constraints:

$$\begin{aligned} f_0(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } f_i(x) &\leq 0, \quad i = 1, \dots, m \\ h_i(x) &= 0, \quad i = 1, \dots, p \end{aligned}$$

And the Lagrangian, associated with this problem:

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) = f_0(x) + \lambda^\top f(x) + \nu^\top h(x)$$

We assume  $\mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i$  is nonempty. We define the Lagrange **dual function** (or just dual function)  $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$  as the minimum value of the Lagrangian over  $x$ : for  $\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p$

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right)$$

# Dual function. Summary

## 💡 Primal

Function:

$$f_0(x)$$

Variables:

$$x \in S \subseteq \mathbb{R}^n$$

Constraints:

$$f_i(x) \leq 0, i = 1, \dots, m$$

$$h_i(x) = 0, i = 1, \dots, p$$

## 💡 Dual

Function:

$$g(\lambda, \nu) = \min_{x \in \mathcal{D}} L(x, \lambda, \nu)$$

Variables

$$\lambda \in \mathbb{R}_+^m, \nu \in \mathbb{R}^p$$

Constraints:

$$\lambda_i \geq 0, \forall i \in \overline{1, m}$$

## Strong Duality

It is common to name this relation between optimals of primal and dual problems as **weak duality**. For problem, we have:

$$d^* \leq p^*$$

While the difference between them is often called **duality gap**:

$$0 \leq p^* - d^*$$

**Strong duality** happens if duality gap is zero:

$$p^* = d^*$$

### Slater's condition

If for a convex optimization problem (i.e., assuming minimization,  $f_0, f_i$  are convex and  $h_i$  are affine), there exists a point  $x$  such that  $h(x) = 0$  and  $f_i(x) < 0$  (existence of a **strictly feasible point**), then we have a zero duality gap and KKT conditions become necessary and sufficient.

## Reminder of KKT statements

Suppose we have a **general optimization problem**

$$\begin{aligned} f_0(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } f_i(x) &\leq 0, \quad i = 1, \dots, m \\ h_i(x) &= 0, \quad i = 1, \dots, p \end{aligned} \tag{1}$$

and **convex optimization problem**, where all equality constraints are affine:

$$h_i(x) = a_i^T x - b_i, i \in 1, \dots, p.$$

The **KKT system** is:

$$\begin{aligned} \nabla_x L(x^*, \lambda^*, \nu^*) &= 0 \\ \nabla_\nu L(x^*, \lambda^*, \nu^*) &= 0 \\ \lambda_i^* &\geq 0, i = 1, \dots, m \\ \lambda_i^* f_i(x^*) &= 0, i = 1, \dots, m \\ f_i(x^*) &\leq 0, i = 1, \dots, m \end{aligned} \tag{2}$$

## i KKT becomes necessary

If  $x^*$  is a solution of the original problem Equation 1, then if any of the following regularity conditions is satisfied:

- **Strong duality** If  $f_1, \dots, f_m, h_1, \dots, h_p$  are differentiable functions and we have a problem Equation 1 with zero duality gap, then Equation 2 are necessary (i.e. any optimal set  $x^*, \lambda^*, \nu^*$  should satisfy Equation 2)
- **LCQ** (Linearity constraint qualification). If  $f_1, \dots, f_m, h_1, \dots, h_p$  are affine functions, then no other condition is needed.
- **LICQ** (Linear independence constraint qualification). The gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at  $x^*$
- **SC** (Slater's condition) For a convex optimization problem (i.e., assuming minimization,  $f_i$  are convex and  $h_j$  is affine), there exists a point  $x$  such that  $h_j(x) = 0$  and  $g_i(x) < 0$ .

Then it should satisfy Equation 2

## i KKT in convex case

If a convex optimization problem with differentiable objective and constraint functions satisfies Slater's condition, then the KKT conditions provide necessary and sufficient conditions for optimality: Slater's condition implies that the optimal duality gap is zero and the dual optimum is attained, so  $x^*$  is optimal if and only if there are  $(\lambda^*, \nu^*)$  that, together with  $x^*$ , satisfy the KKT conditions.

## Problem 1. Dual LP

### i Question

Ensure, that the following standard form *Linear Programming* (LP):

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x_i \geq 0, \quad i = 1, \dots, n \end{aligned}$$

Has the following dual:

$$\begin{aligned} \max_{y \in \mathbb{R}^n} \quad & b^\top y \\ \text{s.t.} \quad & A^\top y \preceq c \end{aligned}$$

Find the dual problem to the problem above (it should be the original LP).

## Problem 2. Lagrange matrix multiplier

### i Question

Let matrices  $X \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{n \times m}$ ,  $A \in \mathbb{R}^{k \times n}$ ,  $B \in \mathbb{R}^{k \times m}$ . Setting the task:

$$f(X) = \langle C, X \rangle \longrightarrow \min_X$$

$$\text{s.t } AX \leq B$$

Find the dual problem to the problem above.



### Problem 3. Projection onto probability simplex

#### Question

Find the Euclidean projection of  $x \in \mathbb{R}^n$  onto probability simplex

$$\Delta = \{z \in \mathbb{R}^n \mid z \succeq 0, \mathbf{1}^\top z = 1\},$$

i.e. solve the following problem:

$$\begin{aligned} x^* = P_\Delta(y) = \operatorname{argmin}_{x \in \mathbb{R}_+^n} \frac{1}{2} \|x - y\|_2^2 \\ \text{s.t. } \mathbf{1}^\top x = 1 \end{aligned}$$

## Problem 3 solution: using duality problem

The “partial” Lagrangian, considering only equality constraints:

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We will solve this problem in two stages:

- We first solve  $\underset{x \succeq 0}{\operatorname{argmin}} L(x, \nu)$  to get  $x^*$
- Then we use  $x^*$  to get  $\nu^*$  by solving  $\operatorname{argmax}_{\nu} L(x^*, \nu)$

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1. Let's solve  $\operatorname{argmin}_{x \succeq 0} L(x, \nu)$ :

$$\min_{x \succeq 0} L(x, \nu) = \min_{x \succeq 0} \left( \frac{1}{2} \|x - y\|_2^2 + \nu (\mathbf{1}^T x - 1) \right) = \min_{x \succeq 0} \left( \frac{1}{2} \|x - y\|_2^2 + \nu \mathbf{1}^T x \right)$$

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And the solution to this problem is

- $x_i^* = (y_i - \nu)$  if  $y_i - \nu \geq 0$
- $x_i^* = 0$  if  $y_i - \nu \leq 0$



## Problem 3 solution: using duality problem

So, solution of the first subtask is

$$x^* = [y - \nu \mathbf{1}]_+$$

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In other words, in this sum, we discard those components of the  $y$  that are less than  $\nu$ . To find  $\nu$ , using the expression above, let's sort the components of the vector and present a set

$$\mathcal{J} = \{j : y_j > \nu\}, \quad |\mathcal{J}| = K,$$

where elements of  $y$  already sorted:  $y_1 \geq y_2 \geq \dots \geq y_n$

## Problem 3 solution: using duality problem

So we have

$$\sum_{j: y_j > \nu} (y_j - \nu) = \sum_{j \in \mathcal{J}} y_j - K\nu = 1 \Rightarrow \nu = \frac{\sum_{j \in \mathcal{J}} y_j - 1}{K}$$

The final probability simplex projection algorithm includes 3 steps:

- Sort  $y$
- Find  $K$ , which is the last integer in  $\{1, 2, \dots, n\}$  that  $y_K - \frac{\sum_{j \in \mathcal{J}} y_j - 1}{K} > 0$
- Output  $\nu = \frac{\sum_{j \in \mathcal{J}} y_j - 1}{K}$  for  $x = P_{\Delta}(y) = [y - \nu \mathbf{1}]_+$

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The most expensive part here is step-1, using quick sort, the worst computational complexity is  $\mathcal{O}(n \log n)$

## Problem 3 solution: another algorithm $\mathcal{O}(n)$ (?)

Here is the formulation of the algorithm:

```
INPUT A vector  $\mathbf{v} \in \mathbb{R}^n$  and a scalar  $z > 0$ 
INITIALIZE  $U = [n]$   $s = 0$   $\rho = 0$ 
WHILE  $U \neq \emptyset$ 
    PICK  $k \in U$  at random
    PARTITION  $U$ :
         $G = \{j \in U \mid v_j \geq v_k\}$ 
         $L = \{j \in U \mid v_j < v_k\}$ 
    CALCULATE  $\Delta\rho = |G|$  ;  $\Delta s = \sum_{j \in G} v_j$ 
    IF  $(s + \Delta s) - (\rho + \Delta\rho)v_k < z$ 
         $s = s + \Delta s$  ;  $\rho = \rho + \Delta\rho$  ;  $U \leftarrow L$ 
    ELSE
         $U \leftarrow G \setminus \{v_k\}$ 
    ENDIF
SET  $\theta = (s - z)/\rho$ 
OUTPUT  $\mathbf{w}$  s.t.  $v_i = \max\{v_i - \theta, 0\}$ 
```

Figure 1: Linear time projection algorithm pseudo-code


## Problem 3 solution: another algorithm $\mathcal{O}(n)$ (?)

In short, what is the difference between this algorithm and the first one? In the **step 1**.

- Algorithm 2 (pivot-algorithm) does not sort the entire array, but randomly selects a “pivot” and “slices” the list, similar to Quickselect (quick median search).
- On average, it gives  $\mathcal{O}(n)$ , but in the worst case (unsuccessful pivots), theoretically it can “fail” to  $\mathcal{O}(n^2)$  (!)
- So, the statement about the difficulty of  $\mathcal{O}(n)$  in the original article was a mistake. Article  provides an attempt to fix this and an overview of the mistake.
- The code for comparing this algorithm with the previous one is here 

## Problem 4. Projection onto the unit simplex VS projection onto the $l_1$ ball

### 💡 Projection onto the $l_1$ ball

The same article  mentions the connection between searching for a projection on the unit simplex and on the  $l_1$  ball. Previous problem:

$$\begin{aligned} x_1^* &= \operatorname{argmin}_{x \in \mathbb{R}^n} \frac{1}{2} \|x - y\|_2^2 \\ \text{s.t. } \mathbf{1}^\top x &= 1, \quad x \succeq 0 \end{aligned}$$

New problem:

$$\begin{aligned} x_2^* &= \operatorname{argmin}_{x \in \mathbb{R}^n} \frac{1}{2} \|x - y\|_2^2 \\ \text{s.t. } \|x\|_1 &\leq 1 \end{aligned}$$

Let's show idea how to reduce the second to the first.



## Problem 4. Projection onto the unit simplex VS projection onto the $l_1$ ball

1. If  $\|y\|_1 \leq 1$  then you don't need to do anything: it's already inside (or on the border)  $l_1$ -ball, therefore, the desired projection is equal to the  $y$

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2. If  $\|y\|_1 > 1$  then the optimum will be exactly on the border, that is, it must be fulfilled  $\|y\|_1 = 1$
3. The following lemma is proved in the paper:

### Lemma

In the optimal solution, each non-zero coordinate  $x_i$  must have the same sign as the  $y_i$ . Formally,

$$x_i \neq 0 \Rightarrow \text{sign}(x_i) = \text{sign}(y_i)$$

## Problem 4. Projection onto the unit simplex VS projection onto the $l_1$ ball

4. Thanks to the previous paragraph, it is sufficient to consider the “modules” of coordinates. An auxiliary vector is introduced

$$u \in \mathbb{R}^n, u_i = |y_i|$$

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Then the constraint  $\|x\|_1 \leq 1$  and the condition “sign  $x_i$  coincides with sign  $y_i$ ” are equivalent to the problem

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But this is the problem of projection onto a probability simplex with a sum of coordinates equals to 1.

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Let's denote the found solution to the problem above for  $u^*$ . Then we return to the original  $x^*$ , restoring the signs:

$$x_i^* = \text{sign}(y_i) \cdot u_i^*$$

This  $x_i^*$  solution that is the desired projection onto the  $l_1$  ball.

## Problem 5. Dual to SVM

### i Question

Given  $y_i \in \{-1, 1\}$ , and  $X \in \mathbb{R}^{n \times p}$ , the classic (without regularization) Support Vector Machine problem is:

$$\begin{aligned} \frac{1}{2} \|w\|_2^2 &\rightarrow \min_{w, w_0} \\ \text{s.t. } y_i(x_i^T w + w_0) &\geq 1, i = 1, \dots, n \end{aligned}$$

Find the dual problem to the problem above. How can solving a dual problem help solve the original one?



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Hint: After finding the dual problem, write down the KKT conditions for the primal one