

# Optimality conditions. Optimization with equality / inequality conditions. KKT.

## Seminar

Optimization for ML. Faculty of Computer Science. HSE University

# Optimality Conditions. Important notions recap

$$f(x) \rightarrow \min_{x \in S}$$

A set  $S$  is usually called a budget set.

- A point  $x^*$  is a global minimizer if  $f(x^*) \leq f(x)$  for all  $x$ .
- A point  $x^*$  is a local minimizer if there exists a neighborhood  $N$  of  $x^*$  such that  $f(x^*) \leq f(x)$  for all  $x \in N$ .
- A point  $x^*$  is a strict local minimizer (also called a strong local minimizer) if there exists a neighborhood  $N$  of  $x^*$  such that  $f(x^*) < f(x)$  for all  $x \in N$  with  $x \neq x^*$ .
- We call  $x^*$  a stationary point (or critical) if  $\nabla f(x^*) = 0$ . Any local minimizer must be a stationary point.



Figure 1: Illustration of different stationary (critical) points

# Unconstrained optimization recap

## 💡 First-Order Necessary Conditions

If  $x^*$  is a local minimizer and  $f$  is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0 \quad (1)$$

## 💡 Second-Order Sufficient Conditions

Suppose that  $\nabla^2 f$  is continuous in an open neighborhood of  $x^*$  and that

$$\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0. \quad (2)$$

Then  $x^*$  is a strict local minimizer of  $f$ .

# Optimization with equality conditions

Consider simple yet practical case of equality constraints:

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } h_i(x) &= 0, i = 1, \dots, p \end{aligned}$$

## Lagrange multipliers recap

The basic idea of Lagrange method implies the switch from conditional to unconditional optimization through increasing the dimensionality of the problem:

$$L(x, \nu) = f(x) + \sum_{i=1}^p \nu_i h_i(x) = f(x) + \nu^T h(x) \rightarrow \min_{x \in \mathbb{R}^n, \nu \in \mathbb{R}^p}$$

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Necessery conditions:

$$\nabla_x L(x^*, \nu^*) = 0$$

$$\nabla_\nu L(x^*, \nu^*) = 0$$

Sufficient conditions:

$$\langle y, \nabla_{xx}^2 L(x^*, \nu^*) y \rangle > 0,$$

$$\forall y \neq 0 \in \mathbb{R}^n : \nabla h_i(x^*)^T y = 0$$

# Optimization with inequality conditions

Consider simple yet practical case of inequality constraints:

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) &\leq 0 \end{aligned}$$

# Optimization with inequality conditions

Consider simple yet practical case of inequality constraints:

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) &\leq 0 \end{aligned}$$

$g(x) \leq 0$  is **inactive**.  $g(x^*) < 0$ :

$$\begin{aligned} g(x^*) &< 0 \\ \nabla f(x^*) &= 0 \\ \nabla^2 f(x^*) &> 0 \end{aligned}$$

$g(x) \leq 0$  is **active**.  $g(x^*) = 0$ :

$$\begin{aligned} g(x^*) &= 0 \\ -\nabla f(x^*) &= \lambda \nabla g(x^*), \lambda > 0 \\ \langle y, \nabla_{xx}^2 L(x^*, \lambda^*) y \rangle &> 0, \\ \forall y \neq 0 \in \mathbb{R}^n : \nabla g(x^*)^\top y &= 0 \end{aligned}$$



# General formulation

General problem of mathematical programming:

$$\begin{aligned} f_0(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } f_i(x) &\leq 0, \quad i = 1, \dots, m \\ h_i(x) &= 0, \quad i = 1, \dots, p \end{aligned}$$

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The solution involves constructing a Lagrange function:

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)$$

## KKT Necessary conditions

Let  $x^*$ ,  $(\lambda^*, \nu^*)$  be a solution to a mathematical programming problem with zero duality gap (the optimal value for the primal problem  $p^*$  is equal to the optimal value for the dual problem  $d^*$ ). Let also the functions  $f_0, f_i, h_i$  be differentiable.

## KKT Necessary conditions

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$$(1) \nabla_x L(x^*, \lambda^*, \nu^*) = 0$$

$$(2) \nabla_\nu L(x^*, \lambda^*, \nu^*) = 0$$

$$(3) \lambda_i^* \geq 0, i = 1, \dots, m$$

$$(4) \lambda_i^* f_i(x^*) = 0, i = 1, \dots, m$$

$$(5) f_i(x^*) \leq 0, i = 1, \dots, m$$

## KKT Some regularity conditions

These conditions are needed in order to make KKT solutions the necessary conditions. Some of them even turn necessary conditions into sufficient. For example, Slater's condition:

## KKT Some regularity conditions

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If for a convex problem (i.e., assuming minimization,  $f_0, f_i$  are convex and  $h_i$  are affine), there exists a point  $x$  such that  $h(x) = 0$  and  $f_i(x) < 0$  (existence of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.

## KKT Sufficient conditions

For smooth, non-linear optimization problems, a second order sufficient condition is given as follows. The solution  $x^*, \lambda^*, \nu^*$ , which satisfies the KKT conditions (above) is a constrained local minimum if for the Lagrangian,

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)$$

the following conditions hold:

$$\langle y, \nabla_{xx}^2 L(x^*, \lambda^*, \nu^*) y \rangle > 0$$

$$\forall y \neq 0 \in \mathbb{R}^n : \nabla h_i(x^*)^\top y = 0, \nabla f_0(x^*)^\top y \leq 0, \nabla f_j(x^*)^\top y = 0$$

$$i = 1, \dots, p \quad \forall j : f_j(x^*) = 0$$

# Problem 1

## i Question

Function  $f : E \rightarrow \mathbb{R}$  is defined as

$$f(x) = \ln(-Q(x))$$

where  $E = \{x \in \mathbb{R}^n : Q(x) < 0\}$  and

$$Q(x) = \frac{1}{2}x^\top Ax + b^\top x + c$$

with  $A \in \mathbb{S}_{++}^n$ ,  $b \in \mathbb{R}^n$ ,  $c \in \mathbb{R}$ .

Find the maximizer  $x^*$  of the function  $f$ .



## Problem 2

### Question

Give an explicit solution of the following task.

$$\begin{aligned} f(x, y) = x + y &\rightarrow \min \\ \text{s.t. } x^2 + y^2 &= 1 \end{aligned}$$

where  $x, y \in \mathbb{R}$ .

## Problem 3

### i Question

Give an explicit solution of the following task.

$$\begin{aligned} \langle c, x \rangle + \sum_{i=1}^n x_i \log x_i &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } \sum_{i=1}^n x_i &= 1, \end{aligned}$$

where  $x \in \mathbb{R}_{++}^n, c \neq 0$ .

## Problem 4

### Question

Let  $A \in \mathbb{S}_{++}^n, b > 0$  show that:

$$\det(X) \rightarrow \max_{X \in \mathbb{S}_{++}^n} \text{ s.t. } \langle A, X \rangle \leq b$$

Has a unique solution and find it.

## Problem 5

### i Question

Given  $y \in \{-1, 1\}$ , and  $X \in \mathbb{R}^{n \times p}$ , the Support Vector Machine problem is:

$$\begin{aligned} & \frac{1}{2} \|w\|_2^2 + C \sum_{i=1}^n \xi_i \rightarrow \min_{w, w_0, \xi_i} \\ \text{s.t. } & \xi_i \geq 0, i = 1, \dots, n \\ & y_i(x_i^T w + w_0) \geq 1 - \xi_i, i = 1, \dots, n \end{aligned}$$

find the KKT stationarity condition.

## Problem 6

### Question

Show that the following constrained optimization task has unique solution and find it.

$$\langle C^{-1}, X \rangle - \log \det(X) \rightarrow \min_{X \in \mathbb{S}_{++}^n} \text{ s.t. } a^T X a \leq 1$$

$$C \in \mathbb{S}_{++}^n, a \neq 0$$

You should avoid explicit inverse of matrix  $C$  in the answer.

## Problem 7 (BONUS)

For some  $\Sigma, \Sigma_0 \in \mathbb{S}_{++}^n$  define a KL Divergence between two Gaussian distributions as:

$$D(\Sigma, \Sigma_0) = \frac{1}{2}(\langle \Sigma_0^{-1}, \Sigma \rangle - \log \det(\Sigma_0^{-1} \Sigma) - n)$$

Now let  $H \in \mathbb{S}_{++}^n$  and  $y, x \in \mathbb{R}^n : \langle y, s \rangle > 0$

We would like to solve the following constrained minimization task.

$$\min_{X \in \mathbb{S}_{++}^n} \{D(X^{-1}, H^{-1}) | Xy = s\}$$

Prove that it has a unique solution and it is equal to:

$$(I_n - \frac{sy^T}{y^T s})H(I_n - \frac{ys^T}{y^T s}) + \frac{ss^T}{y^T s}$$

## Problem 8 (BONUS)

### Question

Let  $e_1, \dots, e_n$  be a standart basis in  $\mathbb{R}^n$ . Show that:

$$\max_{X \in \mathbb{S}_{++}^n} \det(X) : \|Xe_i\| \leq 1 \forall i \in 1, \dots, n$$

Has a unique solution  $I_n$ , and derive the Hadamard inequality:

$$\det(X) \leq \prod_{i=1}^n \|Xe_i\| \forall X \in \mathbb{S}_{++}^n$$

# Adversarial Attacks

Definition: Adversarial attacks are techniques used to fool DL models by adding small perturbations to the input data. We can frame adversarial attacks as a constrained optimization problem where the goal is to minimize/maximize the loss function while keeping the perturbation within a certain limit (norm constraint).

The Fast Gradient Sign Method (FGSM) is the most simple such technique, that generates adversarial examples by applying a small perturbation in the direction of the gradient of the loss function. Formally:

$$x' = x + \varepsilon \cdot \text{sgn}(\nabla_x L(x, y)), \text{s.t. } \|x - x'\| \leq \varepsilon$$

So in a nutshell we perform a gradient ascent on an image (== maximizing loss w.r.t to that image).

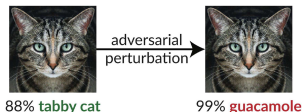


Figure 2: Illustration of different stationary (critical) points

Here is the code to try it out yourself! 