Basic linear algebra recap. Convergence rates.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Basic linear algebra recap

- Naive matmul $\mathcal{O}(n^3)$, naive matvec $\mathcal{O}(n^2)$

Basic linear algebra recap

- Naive matmul $\mathcal{O}(n^3)$, naive matvec $\mathcal{O}(n^2)$
- All matrices have SVD

$$A = U\Sigma V^T$$

Basic linear algebra recap

- Naive matmul $\mathcal{O}(n^3)$, naive matvec $\mathcal{O}(n^2)$
- All matrices have SVD

$$A = U\Sigma V^T$$

- $\text{tr}(ABCD) = \text{tr}(DABC) = \text{tr}(CDAB) = \text{tr}(BCDA)$ for any matrices ABCD if the multiplication defined.

Basic linear algebra recap

- Naive matmul $\mathcal{O}(n^3)$, naive matvec $\mathcal{O}(n^2)$
- All matrices have SVD

$$A = U\Sigma V^T$$

- $\text{tr}(ABCD) = \text{tr}(DABC) = \text{tr}(CDAB) = \text{tr}(BCDA)$ for any matrices ABCD if the multiplication defined.
- $\langle A, B \rangle = \text{tr}(A^T B)$

Convergence rate

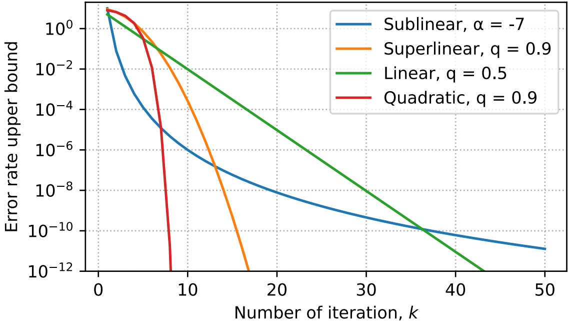


Figure 1: Illustration of different convergence rates

- Linear (geometricm, exponential) convergence:

$$r_k \leq Cq^k, \quad 0 < q < 1, C > 0$$

Convergence rate

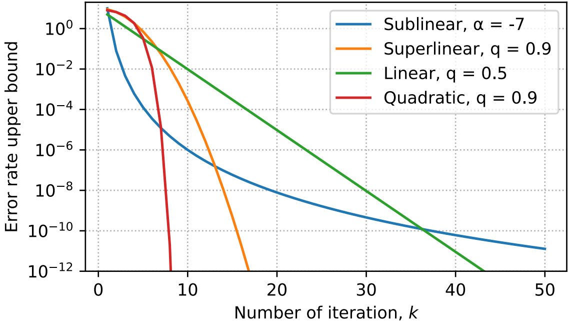


Figure 1: Illustration of different convergence rates

- Linear (geometricm, exponential) convergence:

$$
r_k \leq Cq^k, \quad 0 < q < 1, C > 0
$$

- Any convergent sequence, that is slower (faster) than any linearly convergent sequence has sublinear (superlinear) convergence

Root test

Let $\{r_k\}_{k=m}^{\infty}$ be a sequence of non-negative numbers, converging to zero, and let

$$q = \lim_{k \to \infty} \sup_k \; r_k^{1/k}$$

- If $0 \leq q < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with constant q.

Root test

Let $\{r_k\}_{k=m}^{\infty}$ be a sequence of non-negative numbers, converging to zero, and let

$$q = \lim_{k\to\infty} \sup_k \, r_k^{1/k}$$

- If $0 \leq q < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with constant q.
- In particular, if $q = 0$, then $\{r_k\}_{k=m}^{\infty}$ has superlinear convergence.

Root test

Let $\{r_k\}_{k=m}^{\infty}$ be a sequence of non-negative numbers, converging to zero, and let

$$q = \lim_{k\to\infty} \sup_{k} \; r_k^{1/k}$$

- If $0 \leq q < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with constant q.
- In particular, if $q = 0$, then $\{r_k\}_{k=m}^{\infty}$ has superlinear convergence.
- If $q = 1$, then $\{r_k\}_{k=m}^{\infty}$ has sublinear convergence.

Root test

Let $\{r_k\}_{k=m}^\infty$ be a sequence of non-negative numbers, converging to zero, and let

$$q = \lim_{k\to\infty} \sup_k \; r_k^{1/k}$$

- If $0 \leq q < 1$, then $\{r_k\}_{k=m}^\infty$ has linear convergence with constant q.
- In particular, if $q = 0$, then $\{r_k\}_{k=m}^\infty$ has superlinear convergence.
- If $q = 1$, then $\{r_k\}_{k=m}^\infty$ has sublinear convergence.
- The case $q > 1$ is impossible.

Ratio test

Let $\{r_k\}_{k=m}^{\infty}$ be a sequence of strictly positive numbers converging to zero. Let

$$q = \lim_{k \to \infty} \frac{r_{k+1}}{r_k}$$

- If there exists q and $0 \leq q < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with constant q.

Ratio test

Let $\{r_k\}_{k=m}^{\infty}$ be a sequence of strictly positive numbers converging to zero. Let

$$q = \lim_{k \to \infty} \frac{r_{k+1}}{r_k}$$

- If there exists q and $0 \leq q < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with constant q.
- In particular, if $q = 0$, then $\{r_k\}_{k=m}^{\infty}$ has superlinear convergence.

Ratio test

Let $\{r_k\}_{k=m}^{\infty}$ be a sequence of strictly positive numbers converging to zero. Let

$$q = \lim_{k \to \infty} \frac{r_{k+1}}{r_k}$$

- If there exists q and $0 \leq q < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with constant q.
- In particular, if $q = 0$, then $\{r_k\}_{k=m}^{\infty}$ has superlinear convergence.
- If q does not exist, but $q = \lim_{k \to \infty} \sup_k \frac{r_{k+1}}{r_k} < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with a constant not exceeding q.

Ratio test

Let $\{r_k\}_{k=m}^{\infty}$ be a sequence of strictly positive numbers converging to zero. Let

$$q = \lim_{k \to \infty} \frac{r_{k+1}}{r_k}$$

- If there exists q and $0 \leq q < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with constant q.
- In particular, if $q = 0$, then $\{r_k\}_{k=m}^{\infty}$ has superlinear convergence.
- If q does not exist, but $q = \lim\limits_{k \to \infty} \sup_k \dfrac{r_{k+1}}{r_k} < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with a constant not exceeding q.
- If $\lim\limits_{k \to \infty} \inf_k \dfrac{r_{k+1}}{r_k} = 1$, then $\{r_k\}_{k=m}^{\infty}$ has sublinear convergence.

Ratio test

Let $\{r_k\}_{k=m}^{\infty}$ be a sequence of strictly positive numbers converging to zero. Let

$$q = \lim_{k \to \infty} \frac{r_{k+1}}{r_k}$$

- If there exists q and $0 \leq q < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with constant q.
- In particular, if $q = 0$, then $\{r_k\}_{k=m}^{\infty}$ has superlinear convergence.
- If q does not exist, but $q = \lim\limits_{k \to \infty} \sup_k \frac{r_{k+1}}{r_k} < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with a constant not exceeding q.
- If $\lim\limits_{k \to \infty} \inf_k \frac{r_{k+1}}{r_k} = 1$, then $\{r_k\}_{k=m}^{\infty}$ has sublinear convergence.
- The case $\lim\limits_{k \to \infty} \inf_k \frac{r_{k+1}}{r_k} > 1$ is impossible.

Ratio test

Let $\{r_k\}_{k=m}^{\infty}$ be a sequence of strictly positive numbers converging to zero. Let

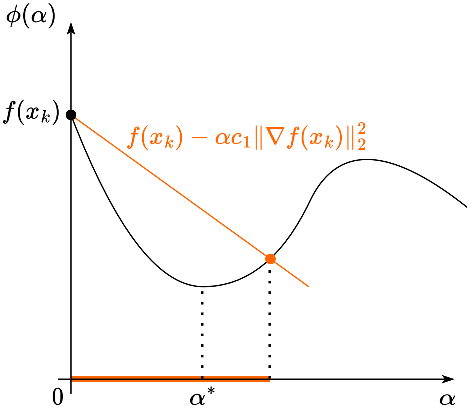
$$q = \lim_{k \to \infty} \frac{r_{k+1}}{r_k}$$

- If there exists q and $0 \leq q < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with constant q.
- In particular, if $q = 0$, then $\{r_k\}_{k=m}^{\infty}$ has superlinear convergence.
- If q does not exist, but $q = \lim\limits_{k \to \infty} \sup_k \dfrac{r_{k+1}}{r_k} < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with a constant not exceeding q.
- If $\lim\limits_{k \to \infty} \inf_k \dfrac{r_{k+1}}{r_k} = 1$, then $\{r_k\}_{k=m}^{\infty}$ has sublinear convergence.
- The case $\lim\limits_{k \to \infty} \inf_k \dfrac{r_{k+1}}{r_k} > 1$ is impossible.
- In all other cases (i.e., when $\lim\limits_{k \to \infty} \inf_k \dfrac{r_{k+1}}{r_k} < 1 \leq \lim\limits_{k \to \infty} \sup_k \dfrac{r_{k+1}}{r_k}$) we cannot claim anything concrete about the convergence rate $\{r_k\}_{k=m}^{\infty}$.

Line search

Typical line search problem is finding the good value α of the stepsize:

$$x_{k+1} = x_k - \alpha \nabla f(x_k)$$



Line search methods

- Solution localization methods:

Line search methods

- Solution localization methods:
    - Dichotomy search method

Line search methods

- Solution localization methods:
    - Dichotomy search method
    - Golden selection search method

Line search methods

- Solution localization methods:
  - Dichotomy search method
  - Golden selection search method
- Inexact line search:

Line search methods

- Solution localization methods:
  - Dichotomy search method
  - Golden selection search method
- Inexact line search:
  - Sufficient decrease

Line search methods

- Solution localization methods:
  - Dichotomy search method
  - Golden selection search method
- Inexact line search:
  - Sufficient decrease
  - Goldstein conditions

Line search methods

- Solution localization methods:
  - Dichotomy search method
  - Golden selection search method
- Inexact line search:
  - Sufficient decrease
  - Goldstein conditions
  - Curvature conditions

Line search methods

- Solution localization methods:
  - Dichotomy search method
  - Golden selection search method
- Inexact line search:
  - Sufficient decrease
  - Goldstein conditions
  - Curvature conditions
  - The idea behind backtracking line search

Problem 1. Stupid important idea on matrix computations.

Suppose, you have the following expression

$$
b = A_1 A_2 A_3 x,
$$

where the $A_1, A_2, A_3 \in \mathbb{R}^{3 \times 3}$ - random square dense matrices and $x \in \mathbb{R}^n$ - vector. You need to compute b.

Which one way is the best to do it?

1. $A_1 A_2 A_3 x$ (from left to right)

Check the simple code snippet after all.

Problem 1. Stupid important idea on matrix computations.

Suppose, you have the following expression

$$
b = A_1 A_2 A_3 x,
$$

where the $A_1, A_2, A_3 \in \mathbb{R}^{3 \times 3}$ - random square dense matrices and $x \in \mathbb{R}^n$ - vector. You need to compute b.

Which one way is the best to do it?

1. $A_1 A_2 A_3 x$ (from left to right)
2. $\left(A_1 \left(A_2 \left(A_3 x\right)\right)\right)$ (from right to left)

Check the simple code snippet after all.

Problem 1. Stupid important idea on matrix computations.

Suppose, you have the following expression

$$b = A_1 A_2 A_3 x,$$

where the $A_1, A_2, A_3 \in \mathbb{R}^{3 \times 3}$ - random square dense matrices and $x \in \mathbb{R}^n$ - vector. You need to compute b.

Which one way is the best to do it?

1. $A_1 A_2 A_3 x$ (from left to right)
2. $(A_1 (A_2 (A_3 x)))$ (from right to left)
3. It does not matter

Check the simple code snippet after all.

Problem 1. Stupid important idea on matrix computations.

Suppose, you have the following expression

$$
b = A_1 A_2 A_3 x,
$$

where the $A_1, A_2, A_3 \in \mathbb{R}^{3 \times 3}$ - random square dense matrices and $x \in \mathbb{R}^n$ - vector. You need to compute b.

Which one way is the best to do it?

1. $A_1 A_2 A_3 x$ (from left to right)
2. $\left(A_1 \left(A_2 \left(A_3 x\right)\right)\right)$ (from right to left)
3. It does not matter
4. The results of the first two options will not be the same.

Check the simple code snippet after all.

Problem 2. Connection between Frobenius norm and singular values.

Let $A \in \mathbb{R}^{m \times n}$, and let $q := \min\{m, n\}$. Show that

$$\|A\|_F^2 = \sum_{i=1}^{q} \sigma_i^2(A),$$

where $\sigma_1(A) \geq \ldots \geq \sigma_q(A) \geq 0$ are the singular values of matrix A. Hint: use the connection between Frobenius norm and scalar product and SVD.

Problem 3. Known your inner product.

Simplify the following expression:

$$\sum_{i=1}^{n} \langle S^{-1} a_i, a_i \rangle,$$

where $S = \sum\limits_{i=1}^{n} a_i a_i^T, a_i \in \mathbb{R}^n, \det(S) \neq 0$

Problem 4. Simple convergence rates

Determine the convergence or divergence of the given sequences:

- $r_k = \frac{1}{3^k}$

Problem 4. Simple convergence rates

Determine the convergence or divergence of the given sequences:

- $r_k = \frac{1}{3^k}$
- $r_k = \frac{4}{3^k}$

Problem 4. Simple convergence rates

Determine the convergence or divergence of the given sequences:

- $r_k = \frac{1}{3^k}$
- $r_k = \frac{4}{3^k}$
- $r_k = \frac{1}{k^{10}}$

Problem 4. Simple convergence rates

Determine the convergence or divergence of the given sequences:

- $r_k = \frac{1}{3^k}$
- $r_k = \frac{4}{3^k}$
- $r_k = \frac{1}{k^{10}}$
- $r_k = 0.707^k$

Problem 4. Simple convergence rates

Determine the convergence or divergence of the given sequences:

- $r_k = \frac{1}{3^k}$
- $r_k = \frac{4}{3^k}$
- $r_k = \frac{1}{k^{10}}$
- $r_k = 0.707^k$
- $r_k = 0.707^{2^k}$

Problem 5. One test is simpler, than another.

Determine the convergence or divergence of the following sequence:

$$r_k = \frac{1}{k^k}$$

Problem 6. Quadratic convergence.

Show, that the following sequence does not have a quadratic convergence.

$$r_k = \frac{1}{3^{k^2}}$$