

Matrix calculus. Line search.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Theory recap. Differential

- Differential $df(x)[\cdot] : U \rightarrow V$ in point $x \in U$ for $f(\cdot) : U \rightarrow V$:

$$f(x+h) - f(x) = \underbrace{df(x)[h]}_{\text{differential}} + o(||h||)$$

$U \rightarrow V$	\mathbb{R}	\mathbb{R}^n	$\mathbb{R}^{n \times m}$
\mathbb{R}	$f'(x)dx$	$\nabla f(x)dx$	$\nabla f(x)dx$
\mathbb{R}^n	$\nabla f(x)^T dx$	$J(x)dx$	—
$\mathbb{R}^{n \times m}$	$tr(\nabla f(X)^T dX)$	—	—

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- Canonical form of the differential:

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Theory recap. Differentiation Rules

- Useful differentiation rules and standard derivatives:

Differentiation Rules

$$dA = 0$$

$$d(\alpha X) = \alpha(dX)$$

$$d(AXB) = A(dX)B$$

$$d(X + Y) = dX + dY$$

$$d(X^T) = (dX)^T$$

$$d(XY) = (dX)Y + X(dY)$$

$$d(\langle X, Y \rangle) = \langle dX, Y \rangle + \langle X, dY \rangle$$

$$d\left(\frac{X}{\phi}\right) = \frac{\phi dX - (d\phi)X}{\phi^2}$$

Standard Derivatives

$$d(\langle A, X \rangle) = \langle A, dX \rangle$$

$$d(\langle Ax, x \rangle) = \langle (A + A^T)x, dx \rangle$$

$$d(\text{Det}(X)) = \text{Det}(X) \langle X^{-T}, dX \rangle$$

$$d(X^{-1}) = -X^{-1}(dX)X^{-1}$$

Theory recap. Differential and Gradient / Hessian

We can retrieve the gradient using the following formula:

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$$d^2f(x) = \langle \nabla^2 f(x) dx_1, dx \rangle = \langle H_f(x) dx_1, dx \rangle$$

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 - The idea behind backtracking line search

Matrix Calculus. Problem 1

Example

Find $\nabla f(x)$, if $f(x) = \frac{1}{2}x^T Ax + b^T x + c$.

Matrix Calculus. Problem 2

Example

Find $\nabla f(X)$, if $f(X) = \text{tr}(AX^{-1}B)$

Matrix Calculus. Problem 3

Example

Find the gradient $\nabla f(x)$ and hessian $\nabla^2 f(x)$, if $f(x) = \frac{1}{3}\|x\|_2^3$

Line Search. Example 1: Comparison of Methods

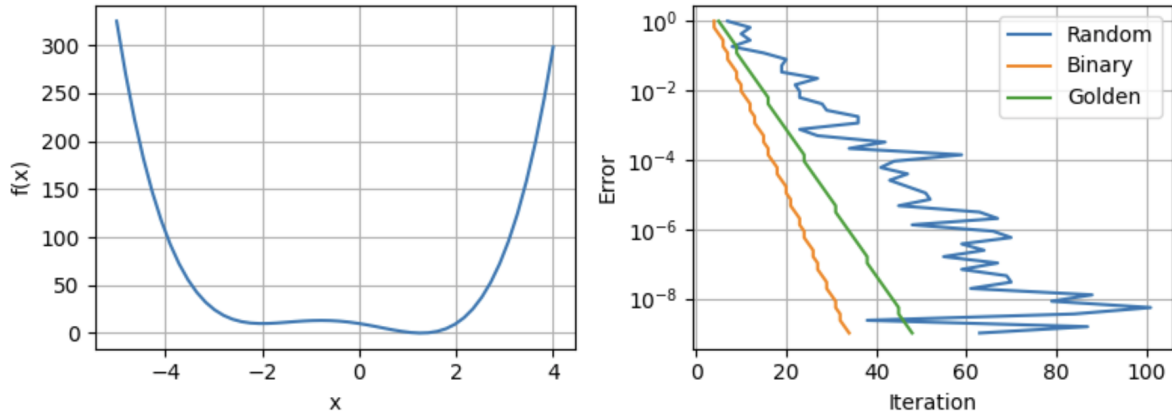


Figure 1: Comparison of different line search algorithms

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Line Search. Example 2: The Brent Method

- Parabolic Interpolation + Golden Search = Brent Method

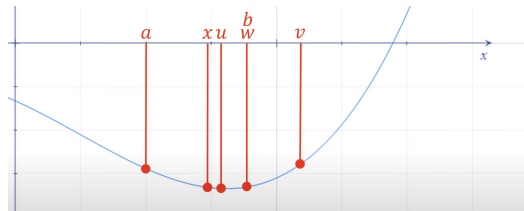


Figure 2: Idea of Brent Method

Line Search. Example 2: The Brent Method

- Parabolic Interpolation + Golden Search = Brent Method
- The key idea of the method is to track the value of the optimized scalar function at six points a, b, x, w, v, u

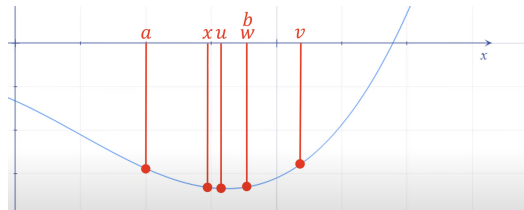


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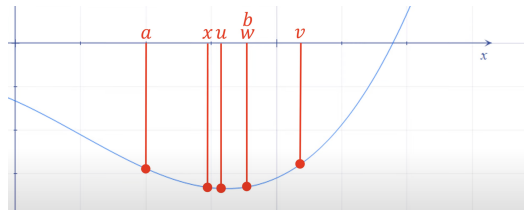


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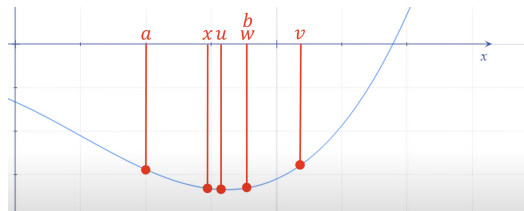


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- The points x, w and v such that the inequality $f(x) \leq f(w) \leq f(v)$ is valid
- u – minimum of a parabola built on points x, w and v or the point of the golden section of the largest of the intervals $[a, x]$ $[x, b]$.

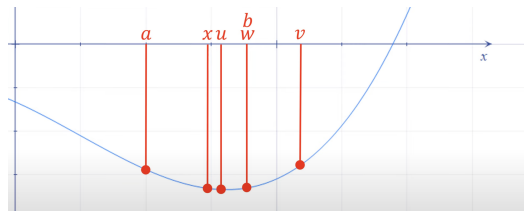


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Line Search. Example 2: The Brent Method

A parabola is constructed only if the points x , w and v are different, and its vertex u^* is taken as the point u only if

- $u^* \in [a, b]$

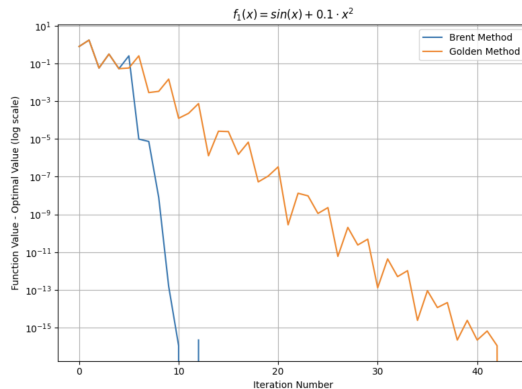


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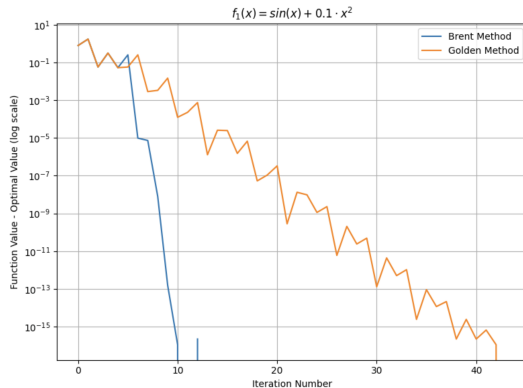


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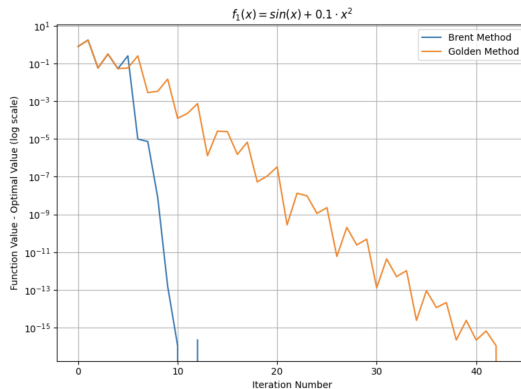


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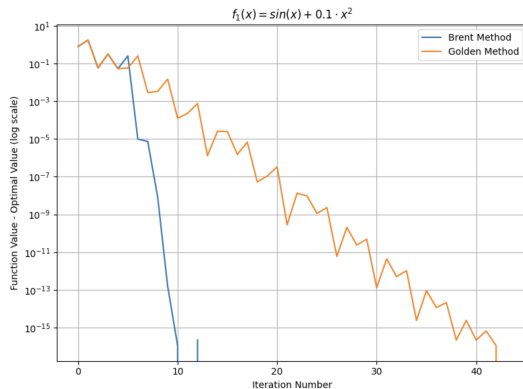


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