

Gradient Descent



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 $Also \ from \ Cauchy-Bunyakovsky-Schwarz \ inequality:$

$$\begin{split} |\langle f'(x), h \rangle| &\leq \|f'(x)\|_2 \|h\|_2 \\ \langle f'(x), h \rangle &\geq -\|f'(x)\|_2 \|h\|_2 = -\|f'(x)\|_2 \end{split}$$

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Thus, the direction of the antigradient

$$h = -\frac{f'(x)}{\|f'(x)\|_2}$$

gives the direction of the ${\bf steepest}$ ${\bf local}$ decreasing of the function f.

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The result of this method is

$$x_{k+1} = x_k - \alpha f'(x_k)$$

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$$\frac{dx}{dt} = -f'(x(t)) \tag{GF}$$

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where $x_k \equiv x(t_k)$ and $\alpha = t_{k+1} - t_k$ - is the grid step.

From here we get the expression for x_{k+1}

$$x_{k+1} = x_k - \alpha f'(x_k),$$

which is exactly gradient descent.

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 $f \to \min_{x,y,z}$ Gradient Descent

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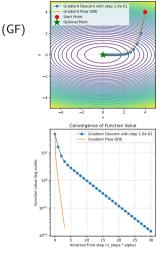
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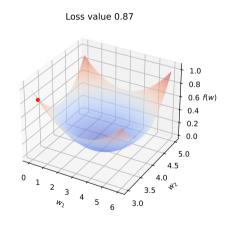
Trajectories with Contour Plot

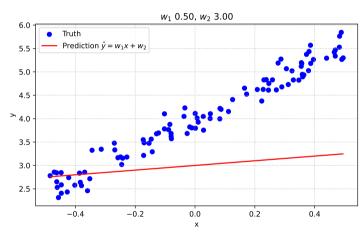
Figure 1: Gradient flow trajectory

 $f \to \min_{x,y,z}$ Gradient Descent

Convergence of Gradient Descent algorithm

Heavily depends on the choice of the learning rate α :







Exact line search aka steepest descent

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \arg\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. An interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

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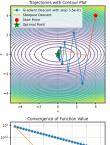
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Optimality conditions:

$$\nabla f(x_{k+1})^\top \nabla f(x_k) = 0$$



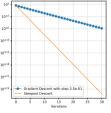


Figure 2: Steepest Descent

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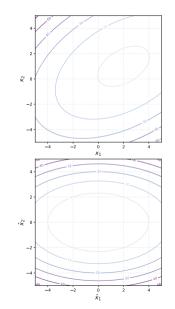
Consider the following quadratic optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}^d_{++}.$$

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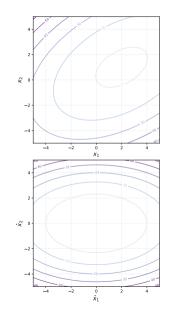


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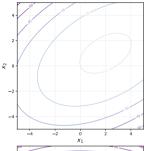
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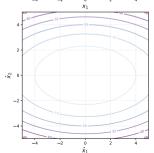
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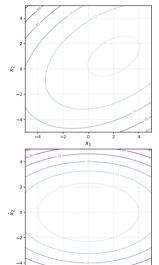
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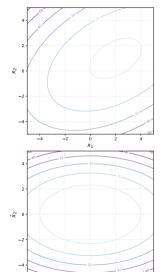
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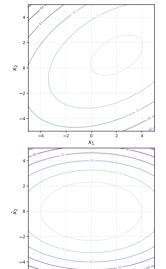
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Now we can work with the function $f(x)=\frac{1}{2}x^T\Lambda x$ with $x^*=0$ without loss of generality (drop the hat from the \hat{x})

$$x^{k+1} = x^k - \alpha^k \nabla f(x^k)$$

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Let's use constant stepsize $\alpha^k=\alpha.$ Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\min} = \mu > 0, \lambda_{\max} = L \ge \mu$.

 $f \to \min_{x,y,\cdot}$

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in y,z Strongly convex quadratics

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$$\begin{aligned} |1 - \alpha \mu| &< 1 \\ -1 &< 1 - \alpha \mu < 1 \end{aligned}$$
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 $\alpha \mu > 0$ $\alpha < \frac{2}{L}$ $\alpha L > 0$

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$$\alpha &< \frac{2}{\mu} \quad \alpha \mu > 0 \qquad \alpha &< \frac{2}{L} \quad \alpha L > 0$$

$$\alpha < \frac{2}{L}$$
 is needed for convergence.



 $x_{(i)}^{k+1} = (1 - \alpha^k \lambda_{(i)})^k x_{(i)}^0$

Now we can work with the function $f(x) = \frac{1}{2}x^T\Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the \hat{x})

$$x^{k+1} = x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k$$

= $(I - \alpha^k \Lambda) x^k$
 $x^{k+1}_{(i)} = (1 - \alpha^k \lambda_{(i)}) x^k_{(i)}$ For i -th coordinate

Now we would like to tune α to choose the best (lowest) convergence rate

$$\rho^* = \min_{\alpha} \rho(\alpha)$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that
$$\lambda_{\min} = \mu > 0, \lambda_{\max} = L \geq \mu.$$

$$\begin{aligned} |1 - \alpha \mu| < 1 & |1 - \alpha L| < 1 \\ -1 < 1 - \alpha \mu < 1 & -1 < 1 - \alpha L < 1 \\ \alpha < \frac{2}{\mu} & \alpha \mu > 0 & \alpha < \frac{2}{L} & \alpha L > 0 \end{aligned}$$

$$1 < 1$$

 $1 - \alpha L < 1$

$$\mu$$
 $\alpha < \frac{2}{T}$ is needed for convergence.

Now we can work with the function $f(x) = \frac{1}{2}x^T\Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the \hat{x})

$$\begin{aligned} x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\ &= (I - \alpha^k \Lambda) x^k \end{aligned}$$

$$x_{(i)}^{k+1} = (1-\alpha^k\lambda_{(i)})x_{(i)}^k \text{ For } i\text{-th coordinate}$$

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condition:
$$\rho(\alpha) = \max |1 - \alpha \lambda_{(i)}| < 1$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence

Remember, that
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$$\alpha^k=\alpha$$
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Remember, that
$$\lambda_{\mathsf{min}} = \mu > 0, \lambda_{\mathsf{max}} = L \geq \mu.$$

$$|1 - \alpha \mu| < 1$$
 $|1 - \alpha L| < 1$ $-1 < 1 - \alpha L < 1$

$$\alpha<\frac{2}{\mu} \qquad \alpha\mu>0 \qquad \qquad \alpha<\frac{2}{L} \qquad \alpha L>0$$

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Now we would like to tune α to choose the best (lowest) convergence rate

$$\begin{split} \rho^* &= \min_{\alpha} \rho(\alpha) = \min_{\alpha} \max_{i} |1 - \alpha \lambda_{(i)}| \\ &= \min_{\alpha} \left\{ |1 - \alpha \mu|, |1 - \alpha L| \right\} \end{split}$$

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 Let's use constant stepsize $\alpha^k=\alpha.$ Convergence

condition: $\rho(\alpha) = \max |1 - \alpha \lambda_{(i)}| < 1$

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$$\alpha < \frac{2}{L}$$
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$$\alpha^*: \quad 1 - \alpha^* \mu = \alpha^* L - 1$$

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$$\alpha^* = \frac{2}{\mu + L}$$

 $\alpha^*: 1-\alpha^*\mu=\alpha^*L-1$

$$\mu$$
 $\alpha < \frac{2}{L}$ is needed for convergence.

Now we can work with the function $f(x) = \frac{1}{2}x^T\Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the \hat{x})

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Let's use constant stepsize
$$\alpha^k=\alpha$$
. Convergence condition:
$$\rho(\alpha)=\max|1-\alpha\lambda_{(i)}|<1$$

Remember, that
$$\lambda_{\min} = \mu > 0, \lambda_{\max} = L \geq \mu.$$

$$|1 - \alpha \mu| < 1$$
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$$\alpha < \frac{2}{\mu} \qquad \alpha \mu > 0 \qquad \qquad \alpha < \frac{2}{L} \qquad \alpha L > 0$$
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Now we would like to tune α to choose the best (lowest) convergence rate

$$\begin{split} \rho^* &= \min_{\alpha} \rho(\alpha) = \min_{\alpha} \max_{i} |1 - \alpha \lambda_{(i)}| \\ &= \min_{\alpha} \left\{ |1 - \alpha \mu|, |1 - \alpha L| \right\} \\ \alpha^* &: \quad 1 - \alpha^* \mu = \alpha^* L - 1 \end{split}$$

$$\alpha^* = \frac{2}{\mu + L} \quad \rho^* = \frac{L - \mu}{L + \mu}$$

$$\alpha < \frac{2}{L}$$
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 $|1 - \alpha \mu| < 1$

Now we can work with the function $f(x) = \frac{1}{2}x^T\Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the \hat{x})

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$$\alpha^* = \frac{2}{\mu + L} \quad \rho^* = \frac{L - \mu}{L + \mu}$$
$$x^{k+1} = \left(\frac{L - \mu}{L + \mu}\right)^k x^0$$

 $\alpha^*: 1 - \alpha^* \mu = \alpha^* L - 1$

$$\frac{2}{L} \qquad \alpha L > 0$$

 $f \to \min_{x,y,z}$ Strongly convex quadratics

Now we can work with the function $f(x) = \frac{1}{2}x^T\Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the \hat{x})

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$$\alpha^* = \frac{2}{\mu + L} \quad \rho^* = \frac{L - \mu}{L + \mu}$$

 $\alpha^*: 1 - \alpha^* \mu = \alpha^* L - 1$

$$x^{k+1} = \left(\frac{L-\mu}{L+\mu}\right)^k x^0 \quad f(x^{k+1}) = \left(\frac{L-\mu}{L+\mu}\right)^{2k} f(x^0)$$

$$\alpha < \frac{2}{\mu}$$
 $\alpha \mu > 0$ $\alpha < \frac{2}{L}$ $\alpha L > 0$

 $f \to \min_{x,y,z}$ Strongly convex quadratics

So, we have a linear convergence in the domain with rate $\frac{\kappa-1}{\kappa+1}=1-\frac{2}{\kappa+1}$, where $\kappa=\frac{L}{\mu}$ is sometimes called *condition number* of the quadratic problem.

κ	ho	Iterations to decrease domain gap $10\ \mathrm{times}$	Iterations to decrease function gap $10\ \mathrm{times}$
1.1	0.05	1	1
2	0.33	3	2
5	0.67	6	3
10	0.82	12	6
50	0.96	58	29
100	0.98	116	58
500	0.996	576	288
1000	0.998	1152	576



Polyak-Lojasiewicz smooth case



Polyak-Lojasiewicz condition. Linear convergence of gradient descent without convexity

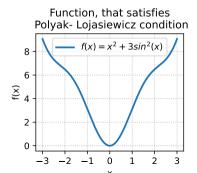
PL inequality holds if the following condition is satisfied for some $\mu > 0$,

$$\|\nabla f(x)\|^2 \geq 2\mu (f(x) - f^*) \quad \forall x$$

It is interesting, that the Gradient Descent algorithm might converge linearly even without convexity.

The following functions satisfy the PL condition but are not convex. PLink to the code

$$f(x) = x^2 + 3\sin^2(x)$$



Polyak-Lojasiewicz condition. Linear convergence of gradient descent without convexity

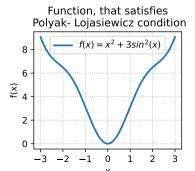
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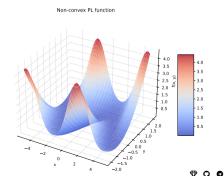
It is interesting, that the Gradient Descent algorithm might converge linearly even without convexity.

The following functions satisfy the PL condition but are not convex. PLink to the code

$$f(x) = x^2 + 3\sin^2(x)$$



$$f(x,y) = \frac{(y - \sin x)^2}{2}$$



i Theorem

Consider the Problem

$$f(x) \to \min_{x \in \mathbb{R}^d}$$

and assume that f is μ -Polyak-Lojasiewicz and L-smooth, for some $L \ge \mu > 0$.

Consider $(x^k)_{k\in\mathbb{N}}$ a sequence generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying $0<\alpha\leq \frac{1}{L}$. Then:

$$f(x^k) - f^* \le (1 - \alpha \mu)^k (f(x^0) - f^*).$$



We can use L-smoothness, together with the update rule of the algorithm, to write

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2$$

 $f \to \min_{x,y,z}$ Polyak-Lojasiewicz smooth case

$$\begin{split} f(x^{k+1}) & \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2 \\ & = f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2} \|\nabla f(x^k)\|^2 \end{split}$$

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We can use L-smoothness, together with the update rule of the algorithm, to write

$$\begin{split} f(x^{k+1}) & \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2 \\ & = f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2} \|\nabla f(x^k)\|^2 \\ & = f(x^k) - \frac{\alpha}{2} \left(2 - L\alpha\right) \|\nabla f(x^k)\|^2 \\ & \leq f(x^k) - \frac{\alpha}{2} \|\nabla f(x^k)\|^2, \end{split}$$

where in the last inequality we used our hypothesis on the stepsize that $\alpha L < 1$.

We can use L-smoothness, together with the update rule of the algorithm, to write

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where in the last inequality we used our hypothesis on the stepsize that $\alpha L < 1$.

We can now use the Polyak-Lojasiewicz property to write:

$$f(x^{k+1}) \leq f(x^k) - \alpha \mu (f(x^k) - f^*).$$

The conclusion follows after subtracting f^* on both sides of this inequality and using recursion.

i Theorem

If a function f(x) is differentiable and $\mu\text{-strongly convex, then it is a PL function.}$

Proof

By first order strong convexity criterion:

Polyak-Loiasiewicz smooth case

$$f(y) \ge f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} ||y - x||_2^2$$

Putting $y = x^*$:

$$f(x^*) \ge f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2} ||x^* - x||_2^2$$

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$$= \left(\nabla f(x)^{T} - \frac{\mu}{2}(x^{*} - x)\right)^{T}(x - x^{*}) =$$

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$$= \left(\nabla f(x)^{T} - \frac{\mu}{2}(x^{*} - x)\right)^{T}(x - x^{*}) = \frac{T}{2}$$

$$=\frac{1}{2}\left(\frac{2}{\sqrt{\mu}}\nabla f(x)^T-\sqrt{\mu}(x^*-x)\right)^T\sqrt{\mu}(x-x^*)=$$

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$$f(y) \ge f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} ||y - x||_2^2$$

Putting $u = x^*$:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2} \|x^* - x\|_2^2$$

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$$= \left(\nabla f(x)^{T} - \frac{\mu}{2}(x^{*} - x)\right)^{T}(x - x^{*}) = \frac{T}{2}$$

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By first order strong convexity criterion:

Putting
$$y = x^*$$
:

 $f(y) \ge f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} ||y - x||_2^2$

 $f(x^*) \ge f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2} ||x^* - x||_2^2$

$$f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*) - \frac{\mu}{2} \|x^* - x\|_2^2 =$$

$$\begin{split} &= \left(\nabla f(x)^T - \frac{\mu}{2}(x^* - x)\right)^T (x - x^*) = \\ &= \frac{1}{2} \left(\frac{2}{\sqrt{\mu}} \nabla f(x)^T - \sqrt{\mu}(x^* - x)\right)^T \sqrt{\mu}(x - x^*) = \end{split}$$

$$f \to \min_{x,y,z}$$
 Polyak-Loiasiewicz smooth case

Let $a = \frac{1}{\sqrt{\mu}} \nabla f(x)$ and $b = \sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}} \nabla f(x)$

Let $a = \frac{1}{\sqrt{\mu}} \nabla f(x)$ and $b = \sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}} \nabla f(x)$

Then $a+b=\sqrt{\mu}(x-x^*)$ and $a-b=\frac{2}{\sqrt{\mu}}\nabla f(x)-\sqrt{\mu}(x-x^*)$

i Theorem

If a function f(x) is differentiable and μ -strongly convex, then it is a PL function.

Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y-x) + \frac{\mu}{2} \|y-x\|_2^2$$
 Putting $y=x^*$:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2} \|x^* - x\|_2^2$$

$$\begin{split} f(x) - f(x^*) &\leq \nabla f(x)^T (x - x^*) - \frac{\mu}{2} \|x^* - x\|_2^2 = \\ &= \left(\nabla f(x)^T - \frac{\mu}{2} (x^* - x)\right)^T (x - x^*) = \\ &= \frac{1}{2} \left(\frac{2}{\sqrt{\mu}} \nabla f(x)^T - \sqrt{\mu} (x^* - x)\right)^T \sqrt{\mu} (x - x^*) = \end{split}$$

$$f(x) - f(x^*) \leq \frac{1}{2} \left(\frac{1}{\mu} \|\nabla f(x)\|_2^2 - \left\| \sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}} \nabla f(x) \right\|_2^2 \right)$$



$$\begin{split} f(x) - f(x^*) &\leq \frac{1}{2} \left(\frac{1}{\mu} \|\nabla f(x)\|_2^2 - \left\| \sqrt{\mu} (x - x^*) - \frac{1}{\sqrt{\mu}} \nabla f(x) \right\|_2^2 \right) \\ f(x) - f(x^*) &\leq \frac{1}{2\mu} \|\nabla f(x)\|_2^2, \end{split}$$



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which is exactly the PL condition. It means, that we already have linear convergence proof for any strongly convex function.

Smooth convex case





Smooth convex case

i Theorem

Consider the Problem

$$f(x) \to \min_{x \in \mathbb{R}^d}$$

and assume that f is convex and L-smooth, for some L>0.

Let $(x^k)_{k\in\mathbb{N}}$ be the sequence of iterates generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying $0<\alpha\leq \frac{1}{L}$. Then, for all $x^*\in \operatorname{argmin} f$, for all $k\in\mathbb{N}$ we have that

$$f(x^k) - f^* \le \frac{\|x^0 - x^*\|^2}{2\alpha k}.$$



As it was before, we first use smoothness:

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2$$

$$= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2} \|\nabla f(x^k)\|^2$$

$$= f(x^k) - \frac{\alpha}{2} (2 - L\alpha) \|\nabla f(x^k)\|^2$$

$$\leq f(x^k) - \frac{\alpha}{2} \|\nabla f(x^k)\|^2,$$

$$f(x^k) - f(x^{k+1}) \geq \frac{1}{2L} \|\nabla f(x^k)\|^2 \text{ if } \alpha = \frac{1}{L}$$

$$(1)$$

Typically, for the convergent gradient descent algorithm the higher the learning rate the faster the convergence.

That is why we often will use $\alpha = \frac{1}{L}$.

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(2)

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 with $y = x^*, x = x^k$

(2)

Smooth convex case

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$$f(x^k) - f^* < \langle \nabla f(x^k), x^k - x^* \rangle$$
(2)

Smooth convex case

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 $f \to \min_{x,y,z}$ Smooth convex case

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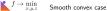
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 $f \to \min_{x,y,z}$ Smooth convex case

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(3)

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$$2\alpha\sum^{k-1}\left(f(x^{i+1})-f^*\right)\leq\|x^0-x^*\|_2^2-\|x^k-x^*\|_2^2\leq\|x^0-x^*\|_2^2 \tag{3}$$

(3)

• Due to the monotonic decrease at each iteration $f(\boldsymbol{x}^{i+1}) < f(\boldsymbol{x}^i)$:

$$kf(x^k) \le \sum_{i=0}^{k-1} f(x^{i+1})$$



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$$2\alpha k f(x^k) - 2\alpha k f^* \leq 2\alpha \sum_{i=0}^{k-1} \left(f(x^{i+1}) - f^* \right) \leq \|x^0 - x^*\|_2^2$$



vex case

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$$\begin{split} 2\alpha k f(x^k) - 2\alpha k f^* &\leq 2\alpha \sum_{i=0}^{k-1} \left(f(x^{i+1}) - f^* \right) \leq \|x^0 - x^*\|_2^2 \\ f(x^k) - f^* &\leq \frac{\|x^0 - x^*\|_2^2}{2\alpha k} \end{split}$$

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Summary

Gradient Descent:

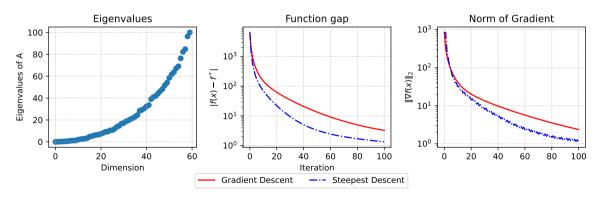
 $\min_{x\in\mathbb{R}^n} f(x)$

 $x^{k+1} = x^k - \alpha^k \nabla f(x^k)$

smooth (non-convex)	smooth & convex	smooth & strongly convex (or PL)
$\ \nabla f(x^k)\ ^2 \sim \mathcal{O}\left(\frac{1}{k}\right)$ $k_{\varepsilon} \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$	$\begin{split} f(x^k) - f^* &\sim \mathcal{O}\left(\frac{1}{k}\right) \\ k_\varepsilon &\sim \mathcal{O}\left(\frac{1}{\varepsilon}\right) \end{split}$	$\begin{split} \ x^k - x^*\ ^2 &\sim \mathcal{O}\left(\left(1 - \frac{\mu}{L}\right)^k\right) \\ k_\varepsilon &\sim \mathcal{O}\left(\varkappa \log \frac{1}{\varepsilon}\right) \end{split}$

$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

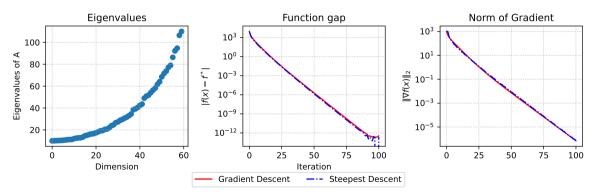
Convex quadratics. n=60, random matrix.





$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

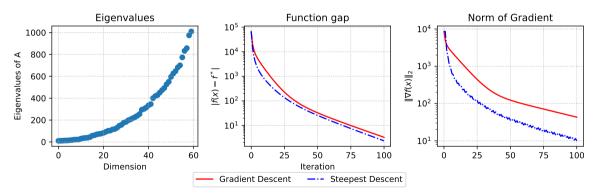
Strongly convex quadratics. n=60, random matrix.





$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

Strongly convex quadratics. n=60, random matrix.

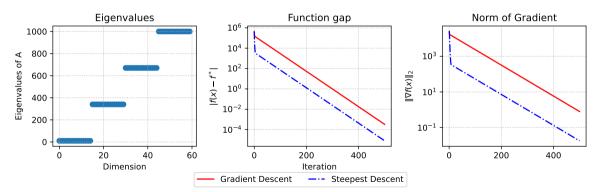


Smooth convex case



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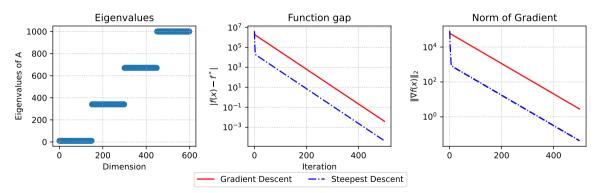
Strongly convex quadratics. n=60, clustered matrix.





$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

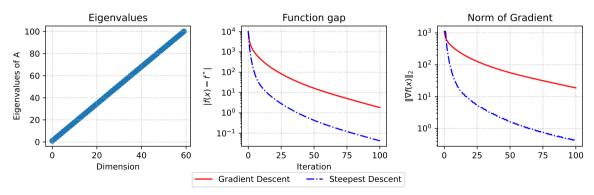
Strongly convex quadratics. n=600, clustered matrix.





$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

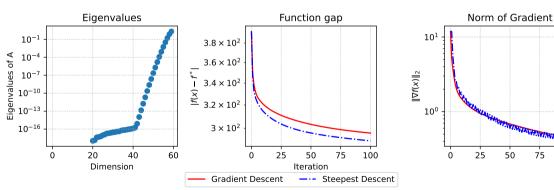
Strongly convex quadratics. n=60, uniform spectrum matrix.





$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

Strongly convex quadratics. n=60, Hilbert matrix.



75

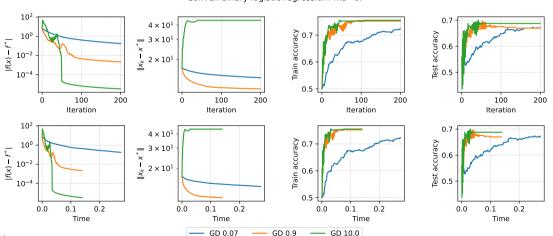


Smooth convex case

100

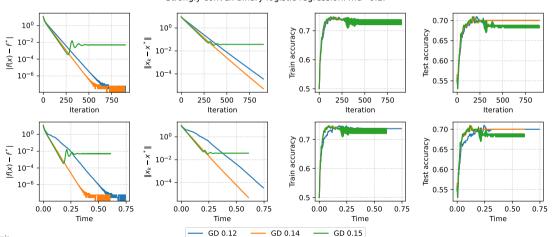
$$f(x) = \frac{\mu}{2} \|x\|_2^2 + \frac{1}{m} \sum_{i=1}^m \log(1 + \exp(-y_i \langle a_i, x \rangle)) \rightarrow \min_{x \in \mathbb{R}^n}$$

Convex binary logistic regression. mu=0.



$$f(x) = \frac{\mu}{2} \|x\|_2^2 + \frac{1}{m} \sum_{i=1}^m \log(1 + \exp(-y_i \langle a_i, x \rangle)) \to \min_{x \in \mathbb{R}^n}$$

Strongly convex binary logistic regression, mu=0.1.

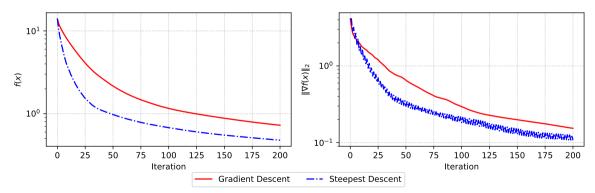






$$f(x) = \frac{\mu}{2} \|x\|_2^2 + \frac{1}{m} \sum_{i=1}^m \log(1 + \exp(-y_i \langle a_i, x \rangle)) \rightarrow \min_{x \in \mathbb{R}^n}$$

Regularized binary logistic regression. n=300. m=1000. μ =0





$$f(x) = \frac{\mu}{2} \|x\|_2^2 + \frac{1}{m} \sum_{i=1}^m \log(1 + \exp(-y_i \langle a_i, x \rangle)) \rightarrow \min_{x \in \mathbb{R}^n}$$

Regularized binary logistic regression. n=300. m=1000. μ =1

