Gradient Descent. Convergence rates

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Gradient Descent

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The bottleneck (for almost all gradient methods) is choosing step-size, which can lead to the dramatic difference in method's behavior.

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- **Backtracking line search**. Fix two parameters: $0 < \beta < 1$ and $0 < \alpha \leq 0.5$. At each iteration, start with $\mathrm{t} = 1$, and while

$$f(x_k - t\nabla f(x_k)) > f(x_k) - \alpha t \|\nabla f(x_k)\|_2^2,$$

shrink $t = \beta t$. Else perform Gradient Descent update $x_{k+1} = x_k - t\nabla f(x_k)$.

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- **Exact line search**.

$$\eta_k = \arg\min_{\eta \geq 0} f(x_k - \eta \nabla f(x_k))$$

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Also from Cauchy–Bunyakovsky–Schwarz inequality:

$$|\langle f'(x), h \rangle| \leq \|f'(x)\|_2 \|h\|_2$$
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The result of this method is

$$x_{k+1} = x_k - \alpha f'(x_k)$$

Minimizer of Lipschitz parabola

If a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and its gradient satisfies Lipschitz conditions with constant L, then $\forall x, y \in \mathbb{R}^n$:

$$|f(y) - f(x) - \langle \nabla f(x), y - x \rangle| \leq \frac{L}{2} \|y - x\|^2,$$

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which geometrically means, that if we'll fix some point $x_0 \in \mathbb{R}^n$ and define two parabolas:

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\phi_1(x) = f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle - \frac{L}{2} \|x - x_0\|^2,
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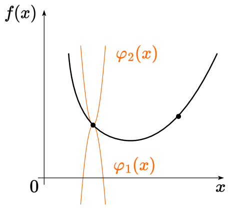

Figure 1: Illustration

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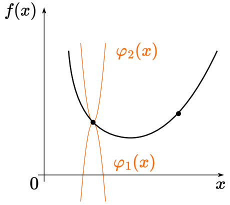

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$$\nabla \phi_2(x) = 0$$
$$\nabla f(x_0) + L(x^* - x_0) = 0$$
$$x^* = x_0 - \frac{1}{L} \nabla f(x_0)$$
$$x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k)$$

This way leads to the $\frac{1}{L}$ stepsize choosing. However, often the L constant is not known.

Strong convexity and Polyak - Lojasiewicz condition.

PL-condition:

$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*) \quad \forall x \in \mathbb{R}^n, \mu > 0,$$

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$$\leq \text{[parabola's top]} \leq \frac{\|\nabla f(x)\|^2}{2\mu}$$

Thus, for a μ-strongly convex function, the PL-condition is satisfied

Exact line search aka steepest descent

$$
\alpha_k = \text{arg}\min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \text{arg}\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))
$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. Interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

$$
\alpha_k = \text{arg}\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))
$$

Optimality conditions:

$$
\nabla f(x_{k+1})^\top \nabla f(x_k) = 0
$$

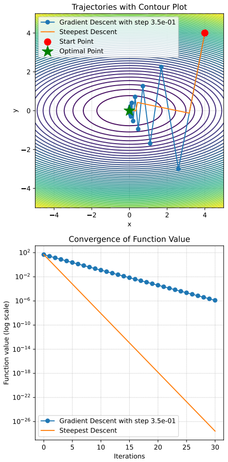

Figure 2: Steepest Descent

Open In Colab ♣

Convergence analysis. Backtracking line search

Assume that f is convex, differentiable and Lipschitz gradient with constant $L > 0$.

Theorem

Gradient descent with fixed step size $t \leq 1/L$ satisfies

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Let $y = x^+ = x - t\nabla f(x)$, then:

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f\left(x^+\right) \leq f(x) - \left(1 - \frac{Lt}{2}\right) t\|\nabla f(x)\|_2^2 \leq f(x) - \frac{1}{2L}\|\nabla f(x)\|_2^2
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This recalls us the stopping condition in Backtracking line search when $\alpha = 0.5, t = \frac{1}{L}$. Hence, Backtracking line search with $\alpha = 0.5$ plus condition of Lipschitz gradient will guarantee us the convergence rate of $O(1/k)$.

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Consider the problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

where $f(x)$ is convex and L-smooth. Find convergence rate of gradient descent with constant step size η for the *mean point* and for the *best point*. In other words get upper bounds on

- $f(\bar{x}_N) - f^*$, where $\bar{x}_N = \frac{1}{N}\sum_{i=0}^{N-1} x_i$,
- $\min_{0 \leq i \leq N-1} f(x_i) - f^*$.

Gradient step

$$x_{k+1} = \arg\min_{x \in \mathbb{R}^n} \left\{ \Psi_k(x) \equiv f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2\eta} \|x - x_k\|_2^2 \right\}$$

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Tip

Use the fact that $\Psi_k(x)$ is $\frac{1}{\eta}$-strongly convex due to quadratic regularizer.

Code

Examples: code snippet.