Duality.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Dual function

The **general mathematical programming problem** with functional constraints:

$$f_0(x) \to \min_{x \in \mathbb{R}^n}$$
$$\text{s.t. } f_i(x) \leq 0, \ i = 1, \ldots, m$$
$$h_i(x) = 0, \ i = 1, \ldots, p$$

And the Lagrangian, associated with this problem:

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) = f_0(x) + \lambda^\top f(x) + \nu^\top h(x)$$

We assume $\mathcal{D} = \bigcap_{i=0}^{m} \textbf{dom } f_i \cap \bigcap_{i=1}^{p} \textbf{dom } h_i$ is nonempty. We define the Lagrange **dual function** (or just dual function) $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ as the minimum value of the Lagrangian over x: for $\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p$

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

Dual function. Summary

💡 Primal

Function:
$$f_0(x)$$

Variables:
$$x \in S \subseteq \mathbb{R}^\varkappa$$

Constraints:
$$f_i(x) \leq 0, i = 1, \ldots, m$$
$$h_i(x) = 0, \ i = 1, \ldots, p$$

💡 Dual

Function:
$$g(\lambda, \nu) = \min_{x \in \mathcal{D}} L(x, \lambda, \nu)$$

Variables
$$\lambda \in \mathbb{R}^m_+, \nu \in \mathbb{R}^p$$

Constraints:
$$\lambda_i \geq 0, \forall i \in \overline{1, m}$$

Strong Duality

It is common to name this relation between optimals of primal and dual problems as **weak duality**. For problem, we have:

$$d^* \leq p^*$$

While the difference between them is often called **duality gap:**

$$0 \leq p^* - d^*$$

Strong duality happens if duality gap is zero:

$$p^* = d^*$$

> **i** Slater's condition
>
> If for a convex optimization problem (i.e., assuming minimization, f_0, f_i are convex and h_i are affine), there exists a point x such that $h(x) = 0$ and $f_i(x) < 0$ (existance of a **strictly feasible point**), then we have a zero duality gap and KKT conditions become necessary and sufficient.

Reminder of KKT statements

Suppose we have a **general optimization problem**

$$
f_0(x) \to \min_{x \in \mathbb{R}^n}
$$
$$
\text{s.t. } f_i(x) \leq 0, \ i = 1, \ldots, m \tag{1}
$$
$$
h_i(x) = 0, \ i = 1, \ldots, p
$$

and **convex optimization problem**, where all equality constraints are affine:

$$
h_i(x) = a_i^T x - b_i, i \in 1, \ldots p.
$$

The **KKT system** is:

$$
\nabla_x L(x^*, \lambda^*, \nu^*) = 0
$$
$$
\nabla_\nu L(x^*, \lambda^*, \nu^*) = 0
$$
$$
\lambda_i^* \geq 0, i = 1, \ldots, m \tag{2}
$$
$$
\lambda_i^* f_i(x^*) = 0, i = 1, \ldots, m
$$
$$
f_i(x^*) \leq 0, i = 1, \ldots, m
$$

> **i KKT becomes necessary**
>
> If x^* is a solution of the original problem Equation 1, then if any of the following regularity conditions is satisfied:
> - **Strong duality** If $f_1, \ldots f_m, h_1, \ldots h_p$ are differentiable functions and we have a problem Equation 1 with zero duality gap, then Equation 2 are necessary (i.e. any optimal set x^*, λ^*, ν^* should satisfy Equation 2)
> - **LCQ** (Linearity constraint qualification). If $f_1, \ldots f_m, h_1, \ldots h_p$ are affine functions, then no other condition is needed.
> - **LICQ** (Linear independence constraint qualification). The gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at x^*
> - **SC** (Slater's condition) For a convex optimization problem (i.e., assuming minimization, f_i are convex and h_j is affine), there exists a point x such that $h_j(x) = 0$ and $g_i(x) < 0$.
>
> Than it should satisfy Equation 2

> **i KKT in convex case**
>
> If a convex optimization problem with differentiable objective and constraint functions satisfies Slater's condition, then the KKT conditions provide necessary and sufficient conditions for optimality: Slater's condition implies that the optimal duality gap is zero and the dual optimum is attained, so x^* is optimal if and only if there are (λ^*, ν^*) that, together with x^*, satisfy the KKT conditions.

Problem 1. Dual LP

> **i** Question
>
> Ensure, that the following standard form *Linear Programming* (LP):
>
> $$\min_{x \in \mathbb{R}^n} c^\top x$$
>
> $$\text{s.t. } Ax = b$$
>
> $$x_i \geq 0, \ i = 1, \ldots, n$$
>
> Has the following dual:
>
> $$\max_{y \in \mathbb{R}^n} b^\top y$$
>
> $$\text{s.t. } A^T y \preceq c$$
>
> Find the dual problem to the problem above (it should be the original LP).

Problem 2. Projection onto probability simplex 📄

> **ℹ Question**
>
> Find the Euclidean projection of $x \in \mathbb{R}^n$ onto probability simplex
>
> $$\Delta = \{z \in \mathbb{R}^n \mid z \succeq 0, \mathbf{1}^\top z = 1\},$$
>
> i.e. solve the following problem:
>
> $$x^* = P_\Delta(y) = \operatorname*{argmin}_{x \in \mathbb{R}^n_+} \frac{1}{2} \|x - y\|_2^2$$
>
> $$\text{s.t. } \mathbf{1}^\top x = 1$$

Problem 2 solution: using duality problem 📄

The "partial" Lagrangian, considering only equality constraints:

$$L(x,\nu) = \frac{1}{2}\|x - y\|_2^2 + \nu\left(\mathbf{1}^T x - 1\right)$$

Problem 2 solution: using duality problem

The "partial" Lagrangian, considering only equality constraints:

$$L(x, \nu) = \frac{1}{2}\|x - y\|_2^2 + \nu\left(\mathbf{1}^T x - 1\right)$$

To find a solution (x^*, ν^*), let's set a saddle point problem:

$$\underset{\nu}{\operatorname{argmax}}\, L(x, \nu) \longrightarrow \min_{x \succeq 0}$$

Problem 2 solution: using duality problem

The "partial" Lagrangian, considering only equality constraints:

$$L(x, \nu) = \frac{1}{2}\|x - y\|_2^2 + \nu \left(\mathbf{1}^T x - 1 \right)$$

To find a solution (x^*, ν^*), let's set a saddle point problem:

$$\underset{\nu}{\arg\max}\, L(x, \nu) \longrightarrow \min_{x \succeq 0}$$

We will solve this problem in two stages:

- We first solve $\underset{x \succeq 0}{\arg\min} L(x, \nu)$ to get x^*

- Then we use x^* to get ν^* by solving $\underset{\nu}{\arg\max} L(x^*, \nu)$

Problem 2 solution: using duality problem

1. Let's solve $\underset{x \succeq 0}{\operatorname{argmin}} L(x, \nu)$:

$$\min_{x \succeq 0} L(x, \nu) = \min_{x \succeq 0} \left(\frac{1}{2} \|x - y\|_2^2 + \nu \left(\mathbf{1}^T x - 1 \right) \right) = \min_{x \succeq 0} \left(\frac{1}{2} \|x - y\|_2^2 + \nu \mathbf{1}^T x \right)$$

Problem 2 solution: using duality problem

1. Let's solve $\underset{x \succeq 0}{\operatorname{argmin}} L(x, \nu)$:

$$\min_{x \succeq 0} L(x, \nu) = \min_{x \succeq 0} \left(\frac{1}{2}\|x - y\|_2^2 + \nu \left(\mathbf{1}^T x - 1 \right) \right) = \min_{x \succeq 0} \left(\frac{1}{2}\|x - y\|_2^2 + \nu \mathbf{1}^T x \right)$$

$$\min_{x \succeq 0} \left(\frac{1}{2}\|x - y\|_2^2 + \nu \mathbf{1}^T x \right) = \min_{x \succeq 0} \left(\sum_{i=1}^n \frac{1}{2}(x_i - y_i)^2 + \nu x_i \right) = \min_{x \succeq 0} \left(\sum_{i=1}^n l_i(x_i) \right)$$

Problem 2 solution: using duality problem

1. Let's solve $\underset{x \succeq 0}{\operatorname{argmin}} L(x, \nu)$:

$$\min_{x \succeq 0} L(x, \nu) = \min_{x \succeq 0} \left(\frac{1}{2} \|x - y\|_2^2 + \nu \left(\mathbf{1}^T x - 1 \right) \right) = \min_{x \succeq 0} \left(\frac{1}{2} \|x - y\|_2^2 + \nu \mathbf{1}^T x \right)$$

$$\min_{x \succeq 0} \left(\frac{1}{2} \|x - y\|_2^2 + \nu \mathbf{1}^T x \right) = \min_{x \succeq 0} \left(\sum_{i=1}^{n} \frac{1}{2} (x_i - y_i)^2 + \nu x_i \right) = \min_{x \succeq 0} \left(\sum_{i=1}^{n} l_i(x_i) \right)$$

L is minimized if all l_i are minimized, so we have scalar problem

$$l_i(x_i) = \frac{1}{2} (x_i - y_i)^2 + \nu x_i \longrightarrow \min_{x_i \geqslant 0}$$

Problem 2 solution: using duality problem

1. Let's solve $\underset{x \succeq 0}{\operatorname{argmin}} L(x, \nu)$:

$$\min_{x \succeq 0} L(x, \nu) = \min_{x \succeq 0} \left(\frac{1}{2} \|x - y\|_2^2 + \nu \left(\mathbf{1}^T x - 1 \right) \right) = \min_{x \succeq 0} \left(\frac{1}{2} \|x - y\|_2^2 + \nu \mathbf{1}^T x \right)$$

$$\min_{x \succeq 0} \left(\frac{1}{2} \|x - y\|_2^2 + \nu \mathbf{1}^T x \right) = \min_{x \succeq 0} \left(\sum_{i=1}^n \frac{1}{2} (x_i - y_i)^2 + \nu x_i \right) = \min_{x \succeq 0} \left(\sum_{i=1}^n l_i(x_i) \right)$$

L is minimized if all l_i are minimized, so we have scalar problem

$$l_i(x_i) = \frac{1}{2} (x_i - y_i)^2 + \nu x_i \longrightarrow \min_{x_i \geqslant 0}$$

And the solution to this problem is

- $x_i^* = (y_i - \nu)$ if $y_i - \nu \geqslant 0$
- $x_i^* = 0$ if $y_i - \nu \leq 0$

Problem 2 solution: using duality problem 📄

So, solution of the first subtask is

$$x^* = [y - \nu \mathbf{1}]_+$$

2. Now we must find ν. To do this, let's use the constraint:

Problem 2 solution: using duality problem 📄

So, solution of the first subtask is

$$x^* = [y - \nu \mathbf{1}]_+$$

2. Now we must find ν. To do this, let's use the constraint:

$$\sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} [y_i - \nu]_+ = \sum_{i=1}^{n} max\{0, y_i - \nu\} = \sum_{j:y_j > \nu} (y_j - \nu) = 1$$

Problem 2 solution: using duality problem

So, solution of the first subtask is

$$x^* = [y - \nu \mathbf{1}]_+$$

2. Now we must find ν. To do this, let's use the constraint:

$$\sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} [y_i - \nu]_+ = \sum_{i=1}^{n} max\{0, y_i - \nu\} = \sum_{j:y_j > \nu} (y_j - \nu) = 1$$

In other words, in this sum, we discard those components of the y that are less than ν. To find ν, using the expression above, let's sort the components of the vector and present a set

$$\mathcal{J} = \{j : y_j > \nu\}, \quad |\mathcal{J}| = K,$$

where elemets of y already sorted: $y_1 \geqslant y_2 \geqslant ... \geqslant y_n$

Problem 2 solution: using duality problem 📄

So we have

$$\sum_{j:y_j > \nu} (y_j - \nu) = \sum_{j \in \mathcal{J}} y_j - K\nu = 1 \Rightarrow \nu = \frac{\sum_{j \in \mathcal{J}} y_j - 1}{K}$$

The final probability simplex projection algorithm includes 3 steps:

- Sort y

- Find K, which is the last integer in $\{1, 2, ..., n\}$ that $y_K - \dfrac{\sum_{j \in \mathcal{J}} y_j - 1}{K} > 0$

- Output $\nu = \dfrac{\sum_{j \in \mathcal{J}} y_j - 1}{K}$ for $x = P_\Delta(y) = [y - \nu \mathbf{1}]_+$

Problem 2 solution: using duality problem 📄

So we have

$$\sum_{j:y_j > \nu} (y_j - \nu) = \sum_{j \in \mathcal{J}} y_j - K\nu = 1 \Rightarrow \nu = \frac{\sum_{j \in \mathcal{J}} y_j - 1}{K}$$

The final probability simplex projection algorithm includes 3 steps:

- Sort y

- Find K, which is the last integer in $\{1, 2, ..., n\}$ that $y_K - \dfrac{\sum_{j \in \mathcal{J}} y_j - 1}{K} > 0$

- Output $\nu = \dfrac{\sum_{j \in \mathcal{J}} y_j - 1}{K}$ for $x = P_\Delta(y) = [y - \nu \mathbf{1}]_+$

The most expensive part here is step-1, using quick sort, the worst computational complexity is $\mathcal{O}(n \log n)$

Problem 2 solution: another algorithm $\mathcal{O}(n)$ (?) 📄

Here is the formulation of the algorithm:

```
INPUT A vector v ∈ ℝⁿ and a scalar z > 0
INITIALIZE U = [n]   s = 0   ρ = 0
WHILE U ≠ φ
    PICK k ∈ U at random
    PARTITION U:
        G = {j ∈ U | vⱼ ≥ vₖ}
        L = {j ∈ U | vⱼ < vₖ}
    CALCULATE  Δρ = |G|  ;  Δs = ∑ vⱼ
                                 j∈G
    IF (s + Δs) − (ρ + Δρ)vₖ < z
        s = s + Δs  ;  ρ = ρ + Δρ  ;  U ← L
    ELSE
        U ← G \ {vₖ}
    ENDIF
SET θ = (s − z)/ρ
OUTPUT w s.t. vᵢ = max {vᵢ − θ , 0}
```

Figure 1: Linear time projection algorithm pseudo-code

Problem 2 solution: another algorithm $\mathcal{O}(n)$ (?) 📄

TODO: add a description of the difference between 2 and 1 algorithms and also the code for comparing them (already written, need to create a notebok)

TODO: add the reduction of the projection problem on the l_1 ball to the projection problem on the probability simplex

TODO: add a description of the dual SVM problem, how it helps to solve primal