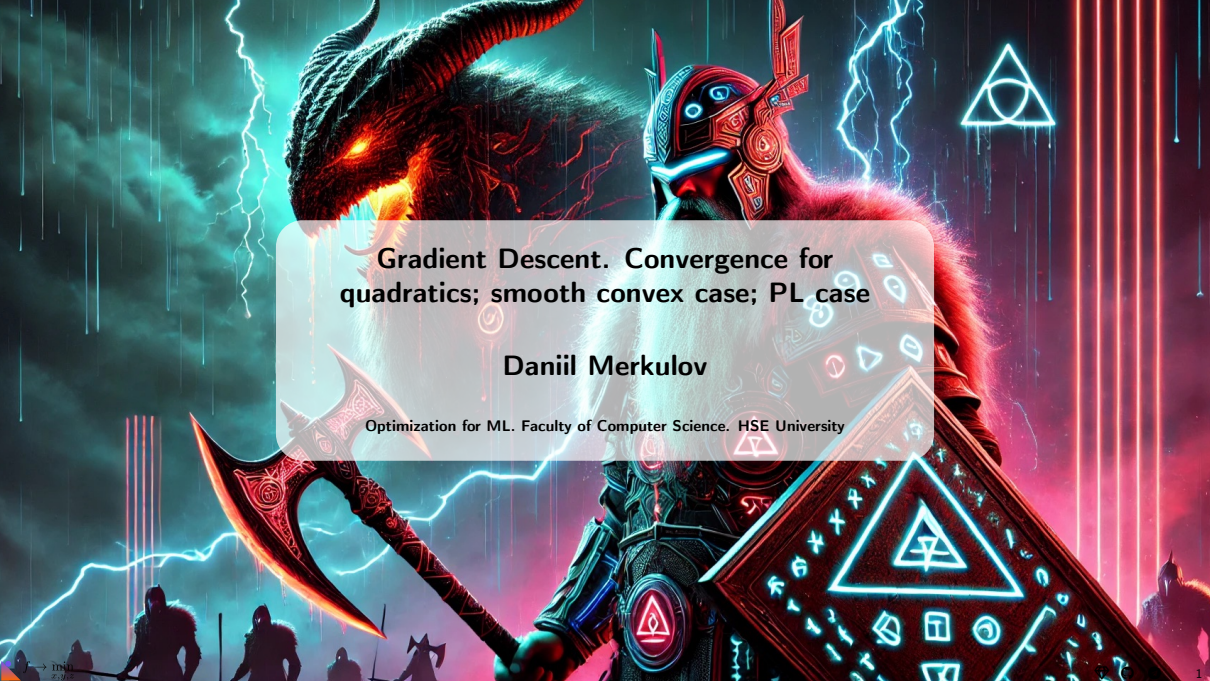

# Gradient Descent. Convergence for quadratics; smooth convex case; PL case

**Daniil Merkulov**

Optimization for ML. Faculty of Computer Science. HSE University

# Gradient Descent

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gives the direction of the **steepest local** decreasing of the function $f$. The result of this method is

$$x_{k+1} = x_k - \alpha f'(x_k)$$

# Gradient flow ODE

Let's consider the following ODE, which is referred to as the Gradient Flow equation.

$$\frac{dx}{dt} = -f'(x(t)) \tag{GF}$$

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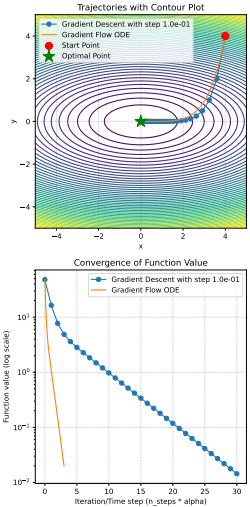
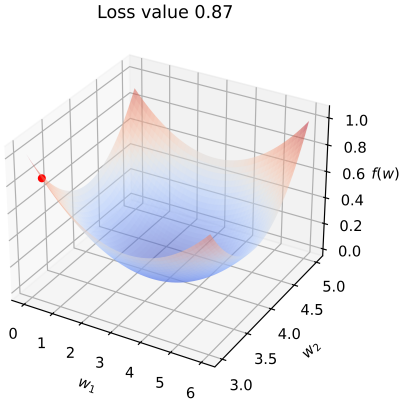
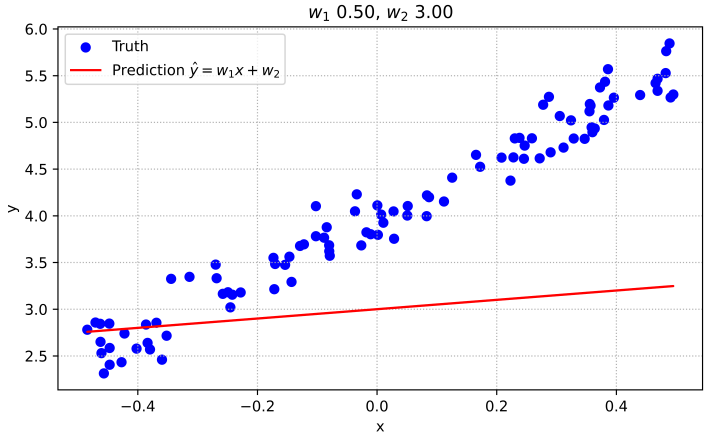



Figure 1: Gradient flow trajectory

# Convergence of Gradient Descent algorithm

Heavily depends on the choice of the learning rate $\alpha$:



# Exact line search aka steepest descent

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \arg\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. An interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

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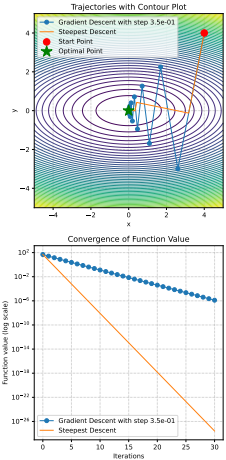


Figure 2: Steepest Descent

Open In Colab ♣

# Strongly convex quadratics

# Coordinate shift

Consider the following quadratic optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}^d_{++}.$$

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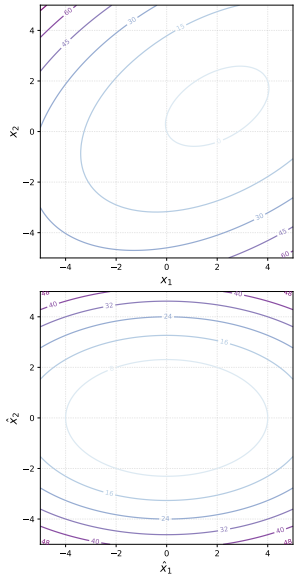

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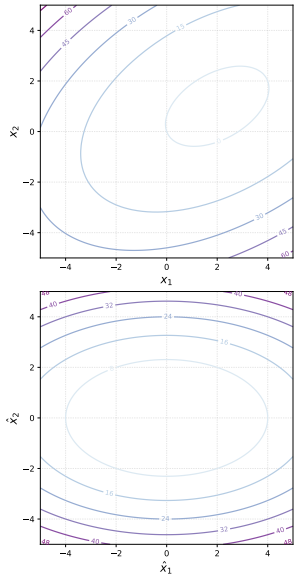

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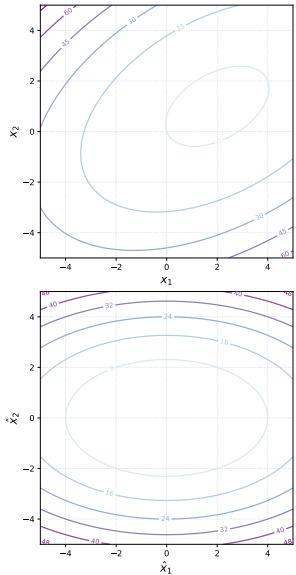

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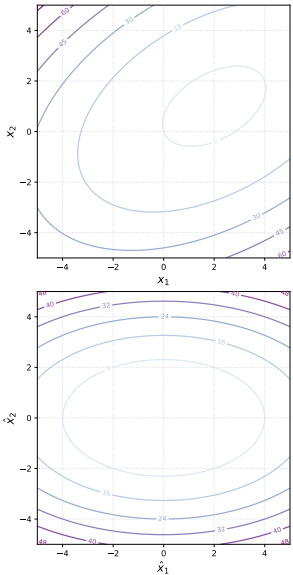

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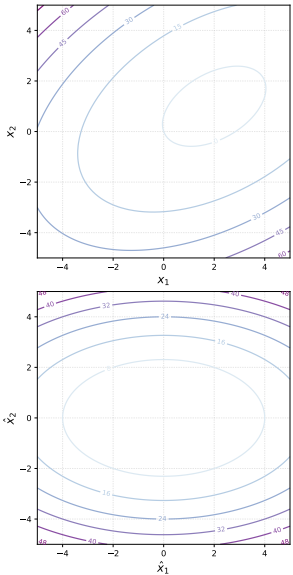

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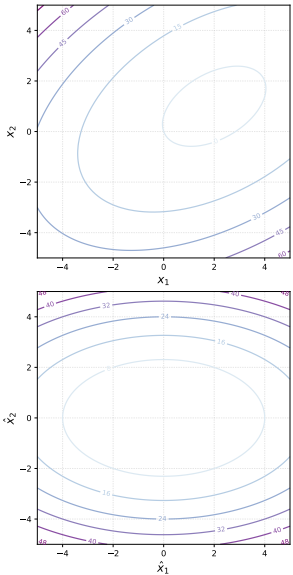

# Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$x^{k+1} = x^k - \alpha^k \nabla f(x^k)$$

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Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha\lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

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&\alpha < \frac{2}{\mu} \qquad \alpha\mu > 0
\end{aligned}
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# Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T\Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

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Now we would like to tune $\alpha$ to choose the best (lowest) convergence rate

$$\rho^* = \min_\alpha \rho(\alpha)$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

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\alpha^* &= \frac{2}{\mu + L} \quad \rho^* = \frac{L - \mu}{L + \mu} \\
x^{k+1} &= \left(\frac{L - \mu}{L + \mu}\right)^k x^0 \quad f(x^{k+1}) = \left(\frac{L - \mu}{L + \mu}\right)^{2k} f(x^0)
\end{aligned}
$$

# Convergence analysis

So, we have a linear convergence in the domain with rate $\frac{\kappa-1}{\kappa+1} = 1 - \frac{2}{\kappa+1}$, where $\kappa = \frac{L}{\mu}$ is sometimes called *condition number* of the quadratic problem.

| $\kappa$ | $\rho$ | Iterations to decrease domain gap 10 times | Iterations to decrease function gap 10 times |
|---|---|---|---|
| 1.1 | 0.05 | 1 | 1 |
| 2 | 0.33 | 3 | 2 |
| 5 | 0.67 | 6 | 3 |
| 10 | 0.82 | 12 | 6 |
| 50 | 0.96 | 58 | 29 |
| 100 | 0.98 | 116 | 58 |
| 500 | 0.996 | 576 | 288 |
| 1000 | 0.998 | 1152 | 576 |

# Polyak-Lojasiewicz smooth case

# Polyak-Lojasiewicz condition. Linear convergence of gradient descent without convexity

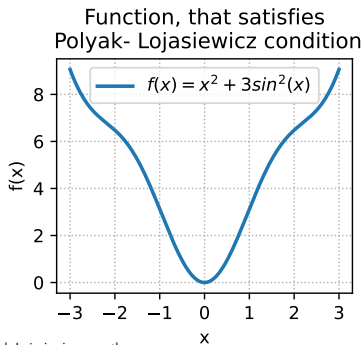
PL inequality holds if the following condition is satisfied for some $\mu > 0$,

$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*) \quad \forall x$$

It is interesting, that the Gradient Descent algorithm might converge linearly even without convexity.

The following functions satisfy the PL condition but are not convex. Link to the code

$$f(x) = x^2 + 3\sin^2(x)$$



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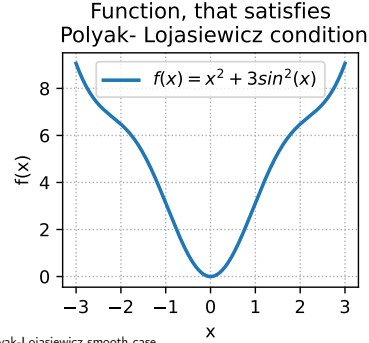
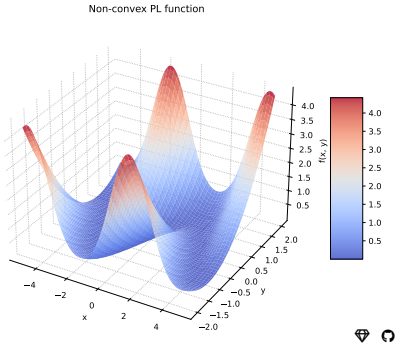
$$f(x, y) = \frac{(y - \sin x)^2}{2}$$



# Convergence analysis

**Theorem**

Consider the Problem

$$f(x) \to \min_{x \in \mathbb{R}^d}$$

and assume that $f$ is $\mu$-Polyak-Lojasiewicz and $L$-smooth, for some $L \geq \mu > 0$.
Consider $(x^k)_{k \in \mathbb{N}}$ a sequence generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying $0 < \alpha \leq \frac{1}{L}$. Then:

$$f(x^k) - f^* \leq (1 - \alpha\mu)^k (f(x^0) - f^*).$$

# Convergence analysis

We can use $L$-smoothness, together with the update rule of the algorithm, to write

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2$$

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We can use $L$-smoothness, together with the update rule of the algorithm, to write

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2$$
$$= f(x^k) - \alpha\|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2$$

# Convergence analysis

We can use $L$-smoothness, together with the update rule of the algorithm, to write

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f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2
\end{aligned}
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&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \\
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where in the last inequality we used our hypothesis on the stepsize that $\alpha L \leq 1$.

# Convergence analysis

We can use $L$-smoothness, together with the update rule of the algorithm, to write

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where in the last inequality we used our hypothesis on the stepsize that $\alpha L \leq 1$.

We can now use the Polyak-Lojasiewicz property to write:

$$f(x^{k+1}) \leq f(x^k) - \alpha\mu(f(x^k) - f^*).$$

The conclusion follows after subtracting $f^*$ on both sides of this inequality and using recursion.

# Any $\mu$-strongly convex differentiable function is a PL-function

**Theorem**

If a function $f(x)$ is differentiable and $\mu$-strongly convex, then it is a PL function.

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By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2}\|y - x\|_2^2$$

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which is exactly the PL condition. It means, that we already have linear convergence proof for any strongly convex function.

# Smooth convex case

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**Theorem**

Consider the Problem

$$f(x) \to \min_{x \in \mathbb{R}^d}$$

and assume that $f$ is convex and $L$-smooth, for some $L > 0$.
Let $(x^k)_{k \in \mathbb{N}}$ be the sequence of iterates generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying $0 < \alpha \leq \frac{1}{L}$. Then, for all $x^* \in \operatorname{argmin} f$, for all $k \in \mathbb{N}$ we have that

$$f(x^k) - f^* \leq \frac{\|x^0 - x^*\|^2}{2\alpha k}.$$

# Convergence analysis

- As it was before, we first use smoothness:

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \\
&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2, \\
f(x^k) - f(x^{k+1}) &\geq \frac{1}{2L}\|\nabla f(x^k)\|^2 \text{ if } \alpha \leq \frac{1}{L}
\end{aligned} \tag{1}
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Typically, for the convergent gradient descent algorithm the higher the learning rate the faster the convergence. That is why we often will use $\alpha = \frac{1}{L}$.

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f(x^{k+1}) &\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^k - x^* - \alpha \nabla f(x^k)\|_2^2\right] \\
&\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2\right] \\
2\alpha\left(f(x^{k+1}) - f^*\right) &\leq \|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2
\end{aligned}
$$

- Now suppose, that the last line is defined for some index $i$ and we sum over $i \in [0, k-1]$. Almost all summands will vanish due to the telescopic nature of the sum:

$$
2\alpha \sum_{i=0}^{k-1}\left(f(x^{i+1}) - f^*\right) \leq \|x^0 - x^*\|_2^2 - \|x^k - x^*\|_2^2 \leq \|x^0 - x^*\|_2^2 \tag{3}
$$

# Convergence analysis

- Due to the monotonic decrease at each iteration $f(x^{i+1}) < f(x^i)$:

$$kf(x^k) \leq \sum_{i=0}^{k-1} f(x^{i+1})$$

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$$f(x^k) - f^* \leq \frac{\|x^0 - x^*\|_2^2}{2\alpha k}$$

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$$f(x^k) - f^* \leq \frac{\|x^0 - x^*\|_2^2}{2\alpha k} \leq \frac{L\|x^0 - x^*\|_2^2}{2k}$$

# Summary

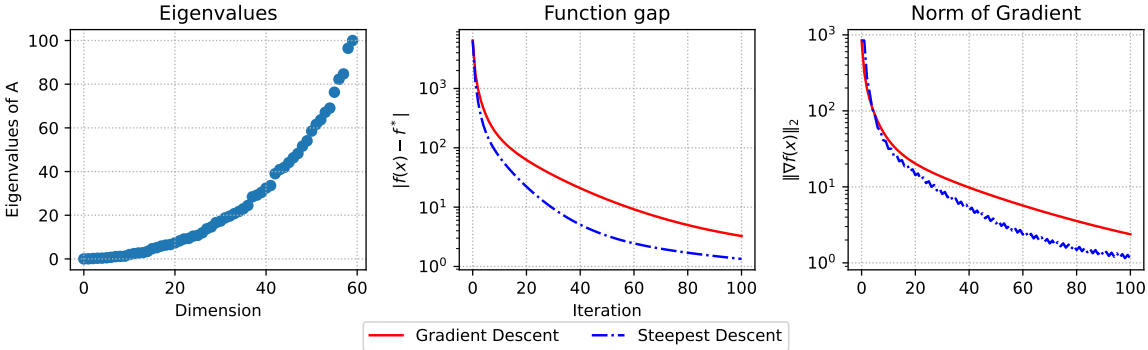
Gradient Descent: $\min\limits_{x \in \mathbb{R}^n} f(x)$ $\qquad x^{k+1} = x^k - \alpha^k \nabla f(x^k)$

| smooth (non-convex) | smooth & convex | smooth & strongly convex (or PL) |
|---|---|---|
| $\|\nabla f(x^k)\|^2 \sim \mathcal{O}\left(\frac{1}{k}\right)$ | $f(x^k) - f^* \sim \mathcal{O}\left(\frac{1}{k}\right)$ | $\|x^k - x^*\|^2 \sim \mathcal{O}\left(\left(1 - \frac{\mu}{L}\right)^k\right)$ |
| $k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$ | $k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$ | $k_\varepsilon \sim \mathcal{O}\left(\varkappa \log \frac{1}{\varepsilon}\right)$ |

# Numerical experiments

$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

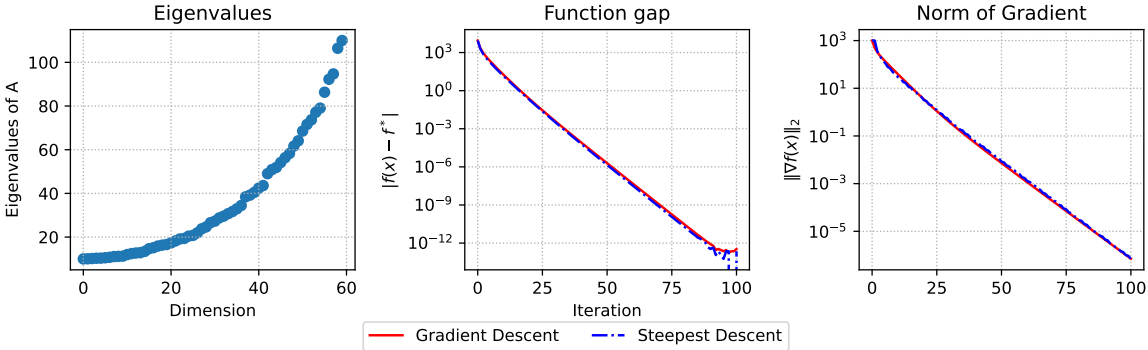
Convex quadratics. n=60, random matrix.



# Numerical experiments

$$f(x) = \frac{1}{2}x^T Ax - b^T x \to \min_{x \in \mathbb{R}^n}$$

Strongly convex quadratics. n=60, random matrix.



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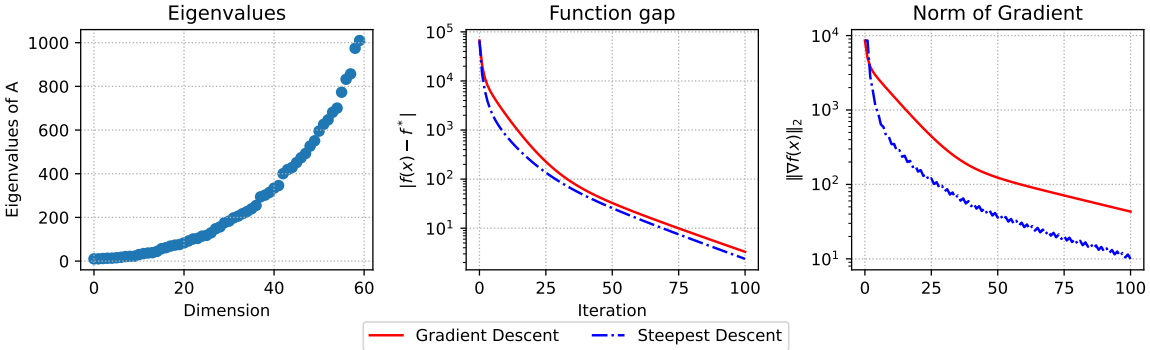
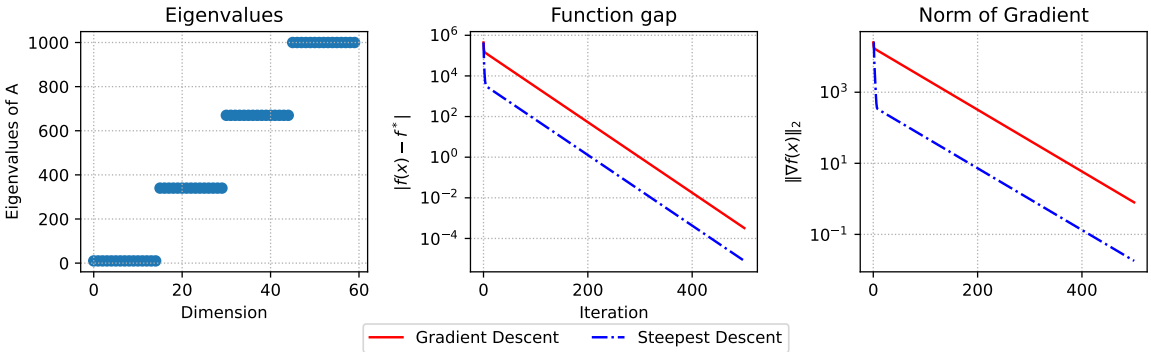


# Numerical experiments

$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

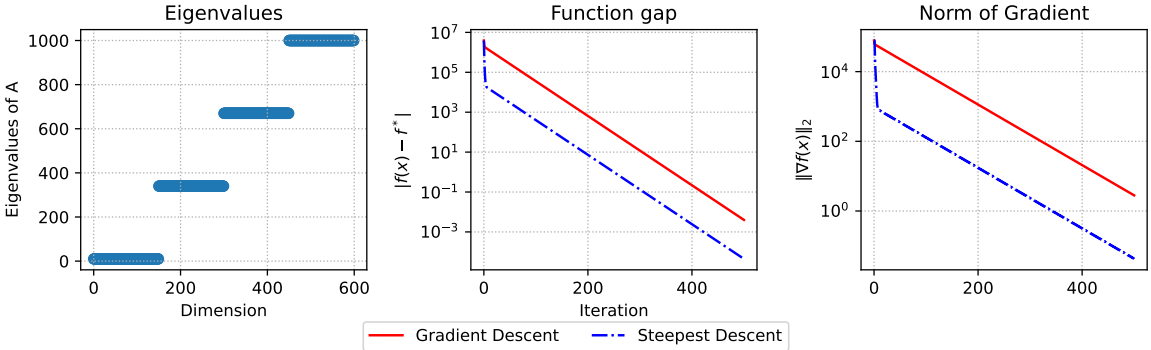
Strongly convex quadratics. n=60, clustered matrix.



# Numerical experiments

$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

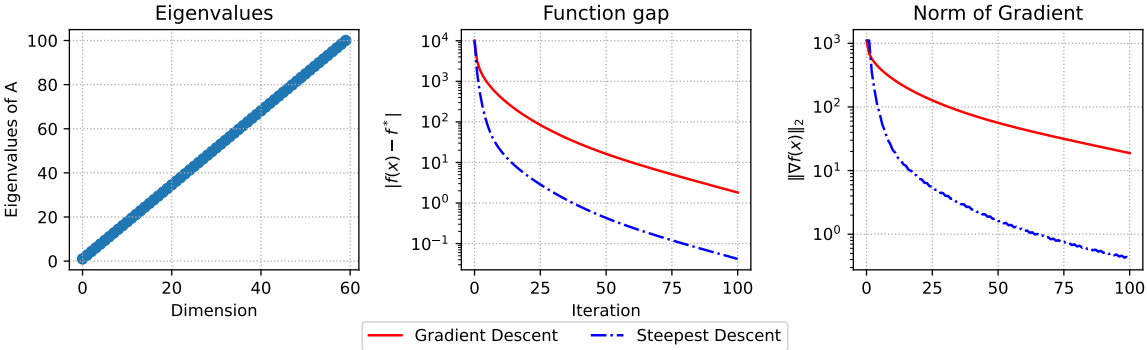
Strongly convex quadratics. n=600, clustered matrix.



# Numerical experiments

$$f(x) = \frac{1}{2}x^T Ax - b^T x \to \min_{x \in \mathbb{R}^n}$$

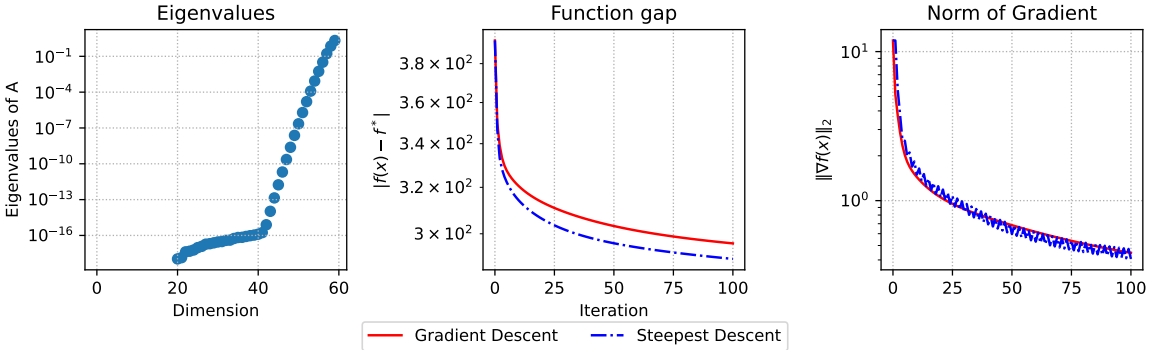
Strongly convex quadratics. n=60, uniform spectrum matrix.



# Numerical experiments

$$f(x) = \frac{1}{2} x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

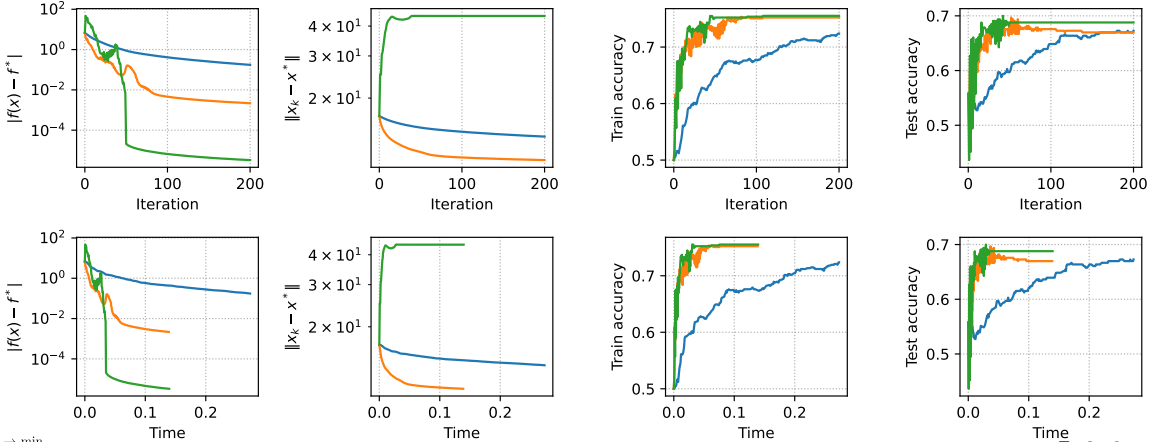
Strongly convex quadratics. n=60, Hilbert matrix.



# Numerical experiments

$$f(x) = \frac{\mu}{2}\|x\|_2^2 + \frac{1}{m}\sum_{i=1}^{m}\log(1+\exp(-y_i\langle a_i, x\rangle)) \to \min_{x\in\mathbb{R}^n}$$

Convex binary logistic regression. mu=0.



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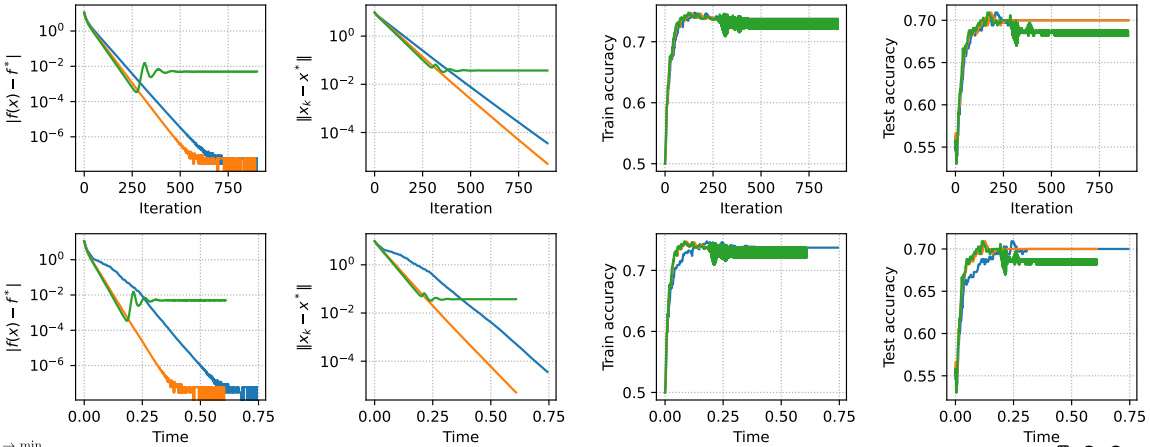
Strongly convex binary logistic regression. mu=0.1.



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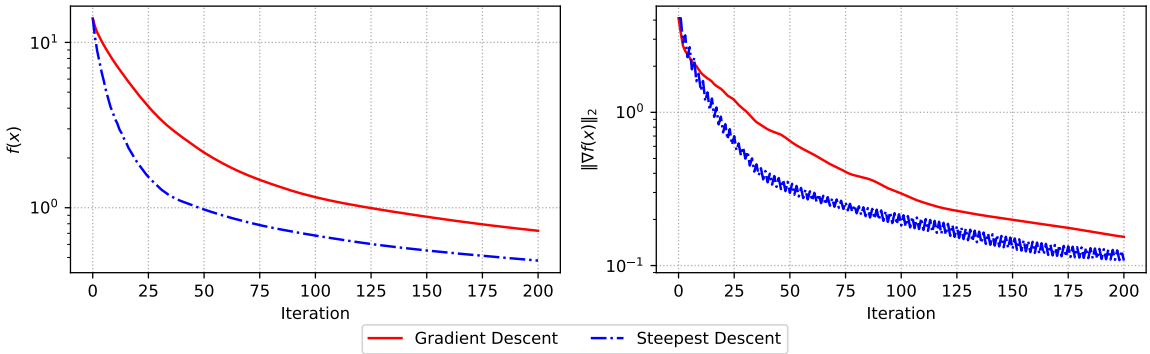
Regularized binary logistic regression. n=300. m=1000. μ=0



# Numerical experiments

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Regularized binary logistic regression. n=300. m=1000. μ=1