

Matrix calculus. Line search.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Theory recap. Differential

- Differential $df(x)[\cdot] : U \rightarrow V$ in point $x \in U$ for $f(\cdot) : U \rightarrow V$:

$$f(x + h) - f(x) = \underbrace{df(x)[h]}_{\text{differential}} + \bar{o}(\|h\|)$$

$U \rightarrow V$	\mathbb{R}	\mathbb{R}^n	$\mathbb{R}^{n \times m}$
\mathbb{R}	$f'(x)dx$	$\nabla f(x)dx$	$\nabla f(x)dx$
\mathbb{R}^n	$\nabla f(x)^T dx$	$J(x)dx$	—
$\mathbb{R}^{n \times m}$	$tr(\nabla f(X)^T dX)$	—	—

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- Canonical form of the differential:

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Theory recap. Differentiation Rules

- Useful differentiation rules and standard derivatives:

Differentiation Rules	Standard Derivatives
$dA = 0$	$d(\langle A, X \rangle) = \langle A, dX \rangle$
$d(\alpha X) = \alpha(dX)$	$d(\langle Ax, x \rangle) = \langle (A + A^T)x, dx \rangle$
$d(AXB) = A(dX)B$	$d(Det(X)) = Det(X)\langle X^{-T}, dX \rangle$
$d(X + Y) = dX + dY$	$d(X^{-1}) = -X^{-1}(dX)X^{-1}$
$d(X^T) = (dX)^T$	
$d(XY) = (dX)Y + X(dY)$	
$d(\langle X, Y \rangle) = \langle dX, Y \rangle + \langle X, dY \rangle$	
$d\left(\frac{X}{\phi}\right) = \frac{\phi dX - (d\phi)X}{\phi^2}$	

Theory recap. Differential and Gradient / Hessian

We can retrieve the gradient using the following formula:

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$$d^2 f(x) = \langle \nabla^2 f(x) dx_1, dx \rangle = \langle H_f(x) dx_1, dx \rangle$$

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 - The idea behind backtracking line search

Matrix Calculus. Problem 1

i Example

Find $\nabla f(x)$, if $f(x) = \frac{1}{2}x^T Ax + b^T x + c$.

Matrix Calculus. Problem 2

i Example

Find $\nabla f(X)$, if $f(X) = \text{tr}(AX^{-1}B)$

Matrix Calculus. Problem 3

i Example

Find the gradient $\nabla f(x)$ and hessian $\nabla^2 f(x)$, if $f(x) = \frac{1}{3}\|x\|_2^3$

Line Search. Example 1: Comparison of Methods (Colab ♣)

$$f_1(x) = x(x - 2)(x + 2)^2 + 10$$

$$[a, b] = [-3, 2]$$

Random search: 72 function calls. 36 iterations. $f_1^* = 0.09$

Binary search: 23 function calls. 13 iterations. $f_1^* = 10.00$

Golden search: 19 function calls. 18 iterations. $f_1^* = 10.00$

Parabolic search: 20 function calls. 17 iterations. $f_1^* = 10.00$

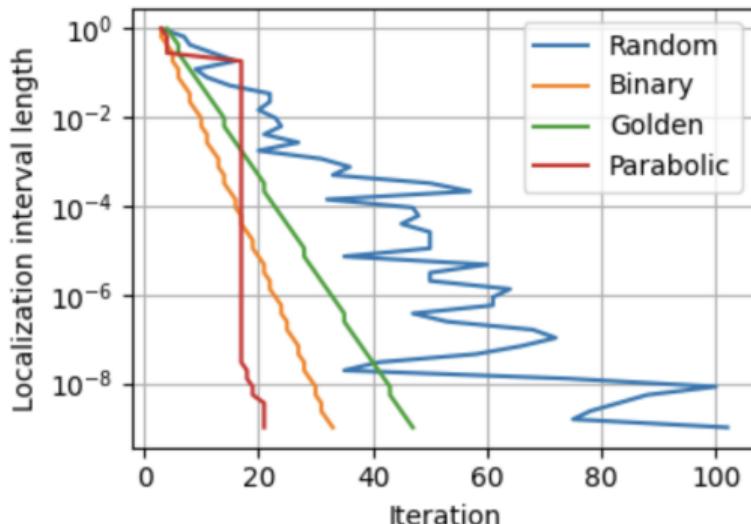
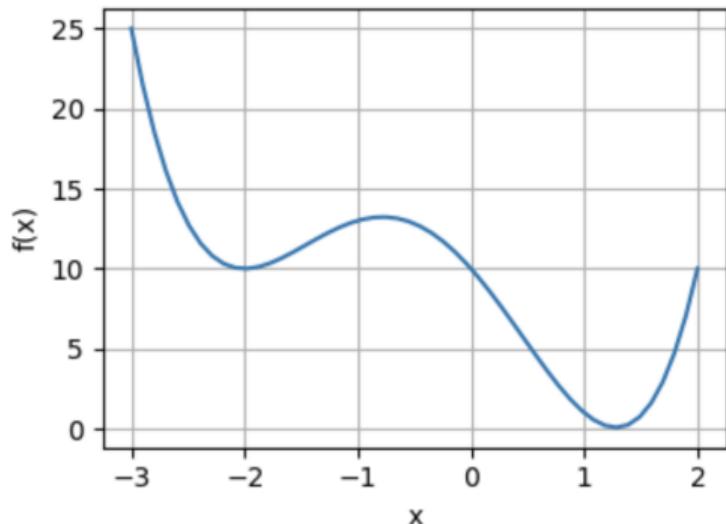


Figure 1: Comparison of different line search algorithms with f_1

Line Search. Example 1: Comparison of Methods (Colab ♣)

$$f_2(x) = -\sqrt{\frac{2}{\pi}} \frac{x^2 e^{-\frac{x^2}{8}}}{8}$$

$$[a, b] = [0, 6]$$

Random search: 68 function calls. 34 iterations. $f_2^* = 0.71$
Binary search: 23 function calls. 13 iterations. $f_2^* = 0.71$
Golden search: 20 function calls. 19 iterations. $f_2^* = 0.71$
Parabolic search: 17 function calls. 14 iterations. $f_2^* = 0.71$

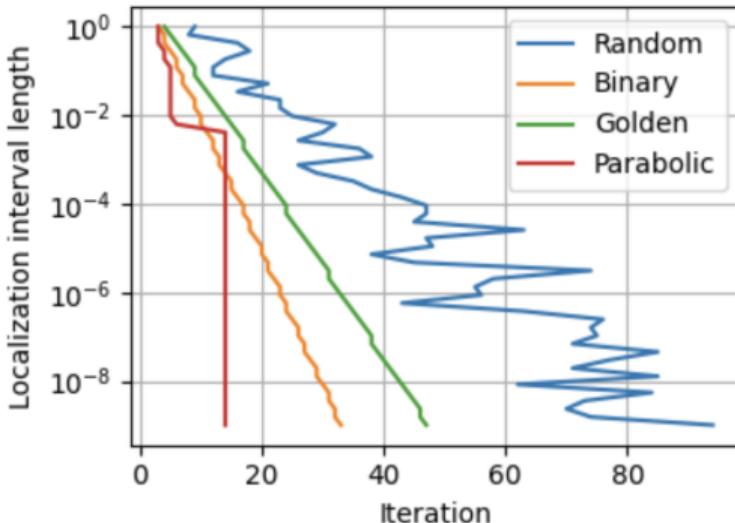
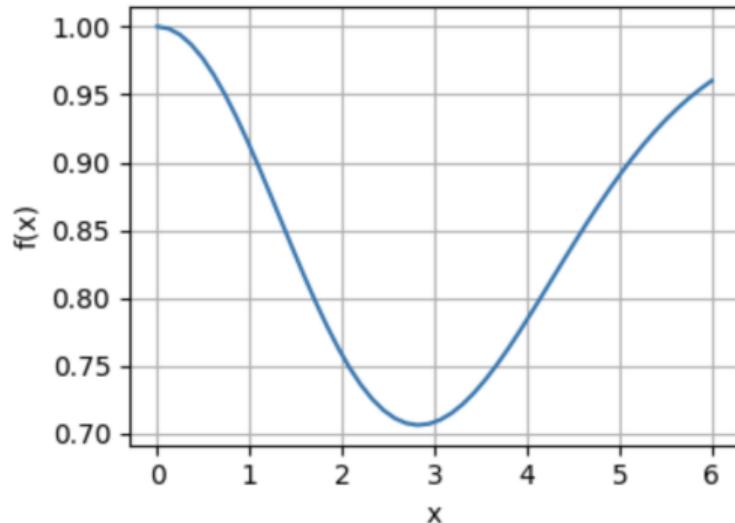


Figure 2: Comparison of different line search algorithms with f_2

Line Search. Example 1: Comparison of Methods (Colab ♣)

$$f_3(x) = \sin\left(\sin\left(\sin\left(\sqrt{\frac{x}{2}}\right)\right)\right)$$
$$[a, b] = [5, 70]$$

Random search: 66 function calls. 33 iterations. $f_3^* = 0.25$
Binary search: 32 function calls. 17 iterations. $f_3^* = 0.25$
Golden search: 25 function calls. 24 iterations. $f_3^* = 0.25$
Parabolic search: 103 function calls. 100 iterations. $f_3^* = 0.25$

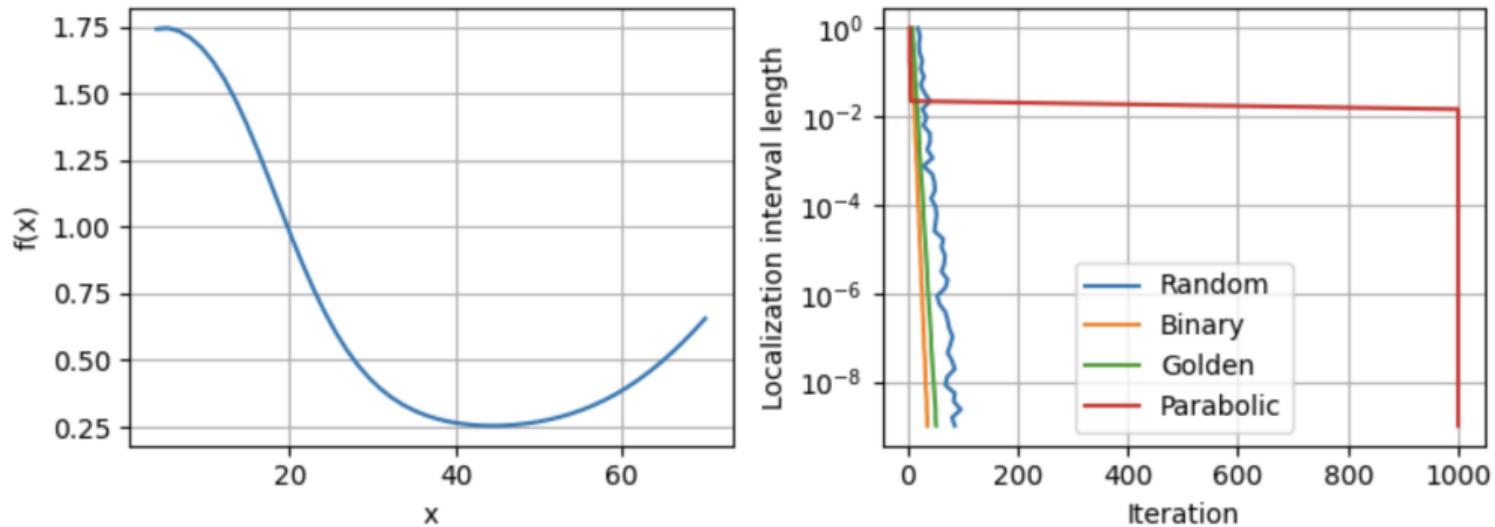


Figure 3: Comparison of different line search algorithms with f_3

Line Search. Example 2: The Brent Method

- Parabolic Interpolation + Golden Search = Brent Method

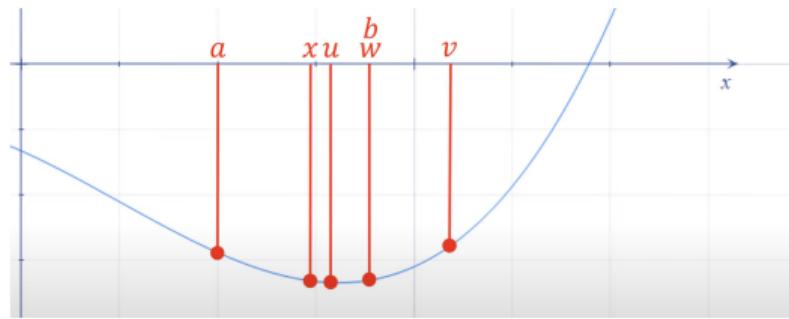


Figure 4: Idea of Brent Method

Line Search. Example 2: The Brent Method

- Parabolic Interpolation + Golden Search = Brent Method
- The key idea of the method is to track the value of the optimized scalar function at six points a, b, x, w, v, u

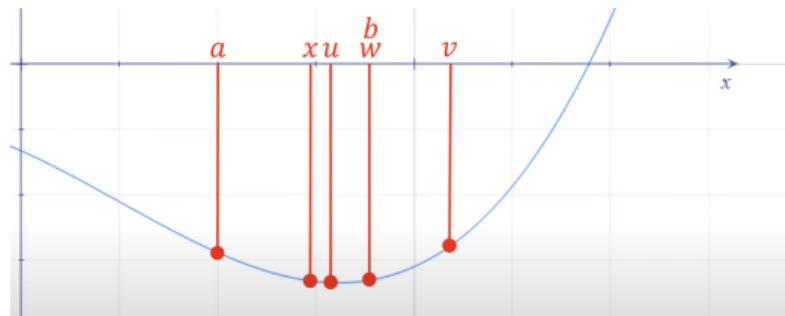


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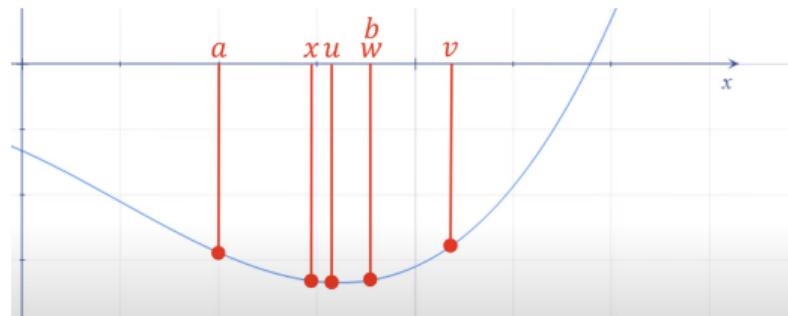


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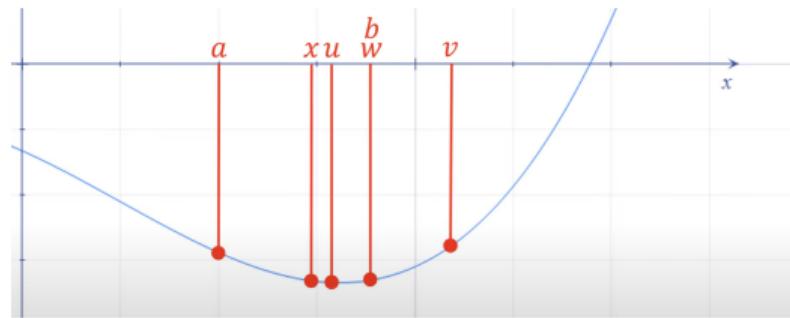


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- The points x, w and v such that the inequality $f(x) \leq f(w) \leq f(v)$ is valid
- u – minimum of a parabola built on points x, w and v or the point of the golden section of the largest of the intervals $[a, x]$ $[x, b]$.

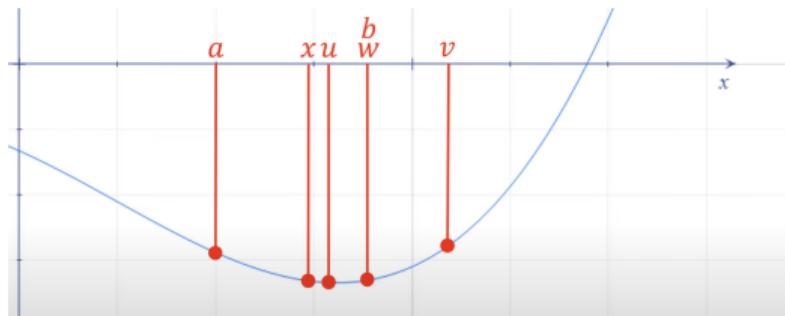


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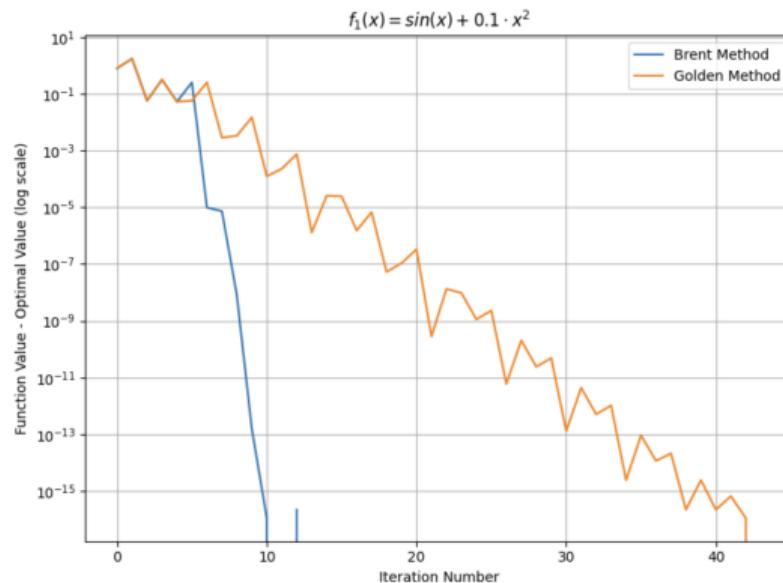


Figure 5: An example of how the Brent Method works

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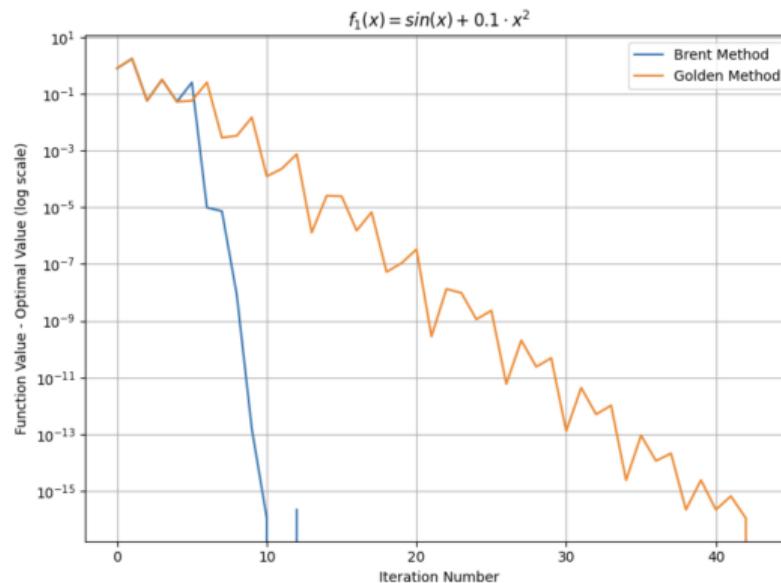


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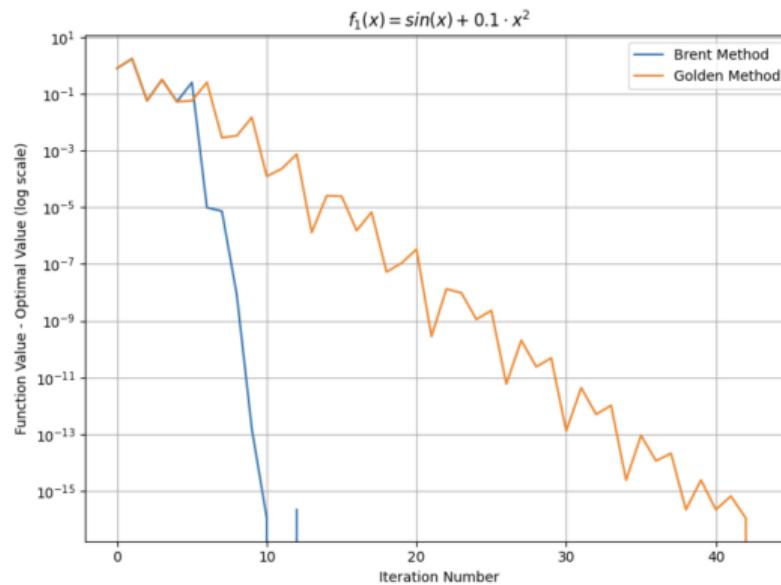


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- Example In Colab ♣

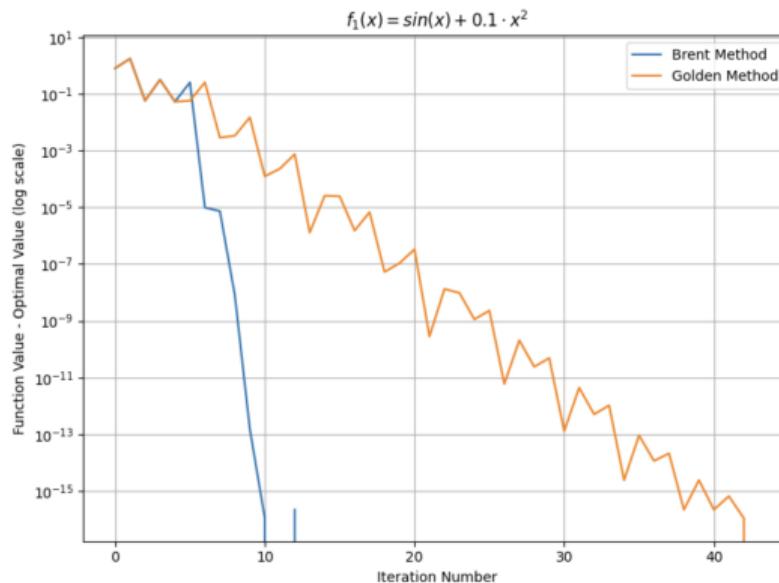


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