

# Convexity: convex sets, convex functions. Strong convexity. PL condition.

Daniil Merkulov

Applied Math for Data Science. Sberuniversity.



## Convex sets

## Affine set

Suppose  $x_1, x_2$  are two points in  $\mathbb{R}^n$ . Then the line passing through them is defined as follows:

$$x = \theta x_1 + (1 - \theta)x_2, \theta \in \mathbb{R}$$

The set  $A$  is called **affine** if for any  $x_1, x_2$  from  $A$  the line passing through them also lies in  $A$ , i.e.

$$\forall \theta \in \mathbb{R}, \forall x_1, x_2 \in A : \theta x_1 + (1 - \theta)x_2 \in A$$

### Example

- $\mathbb{R}^n$  is an affine set.

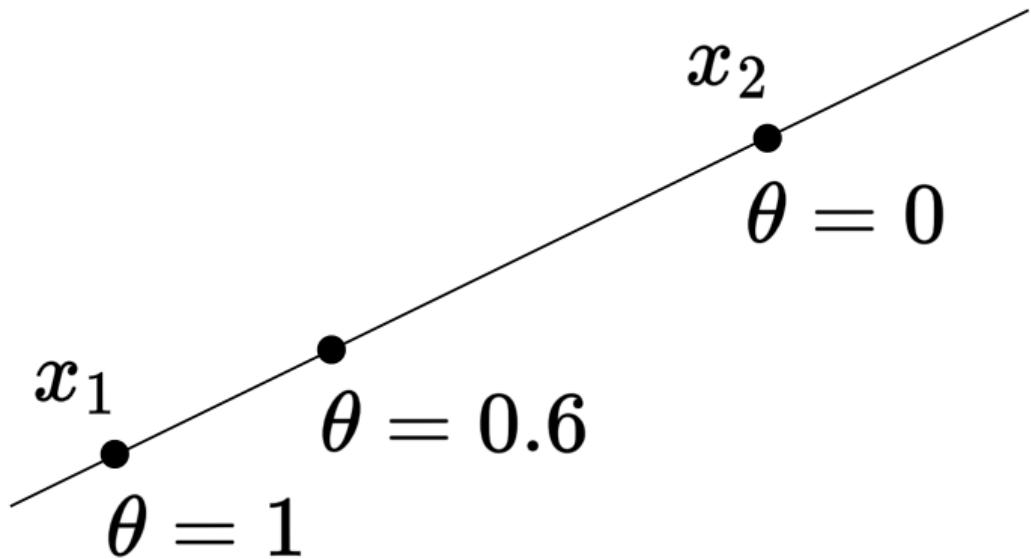


Figure 1: Illustration of a line between two vectors  $x_1$  and  $x_2$

## Affine set

Suppose  $x_1, x_2$  are two points in  $\mathbb{R}^n$ . Then the line passing through them is defined as follows:

$$x = \theta x_1 + (1 - \theta)x_2, \theta \in \mathbb{R}$$

The set  $A$  is called **affine** if for any  $x_1, x_2$  from  $A$  the line passing through them also lies in  $A$ , i.e.

$$\forall \theta \in \mathbb{R}, \forall x_1, x_2 \in A : \theta x_1 + (1 - \theta)x_2 \in A$$

### Example

- $\mathbb{R}^n$  is an affine set.
- The set of solutions  $\{x \mid \mathbf{A}x = \mathbf{b}\}$  is also an affine set.

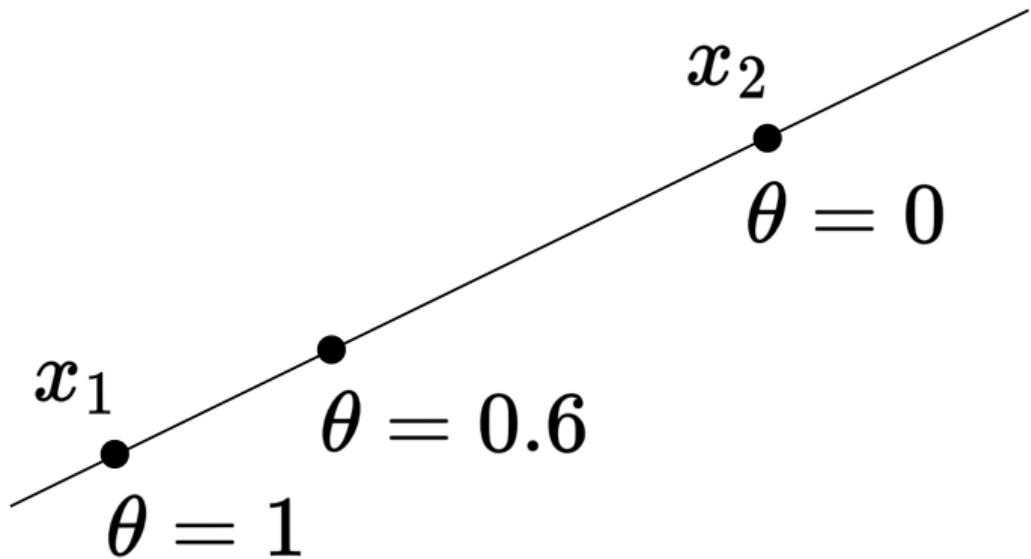


Figure 1: Illustration of a line between two vectors  $x_1$  and  $x_2$

## Cone

A non-empty set  $S$  is called a cone, if:

$$\forall x \in S, \theta \geq 0 \rightarrow \theta x \in S$$

For any point in cone it also contains beam through this point.

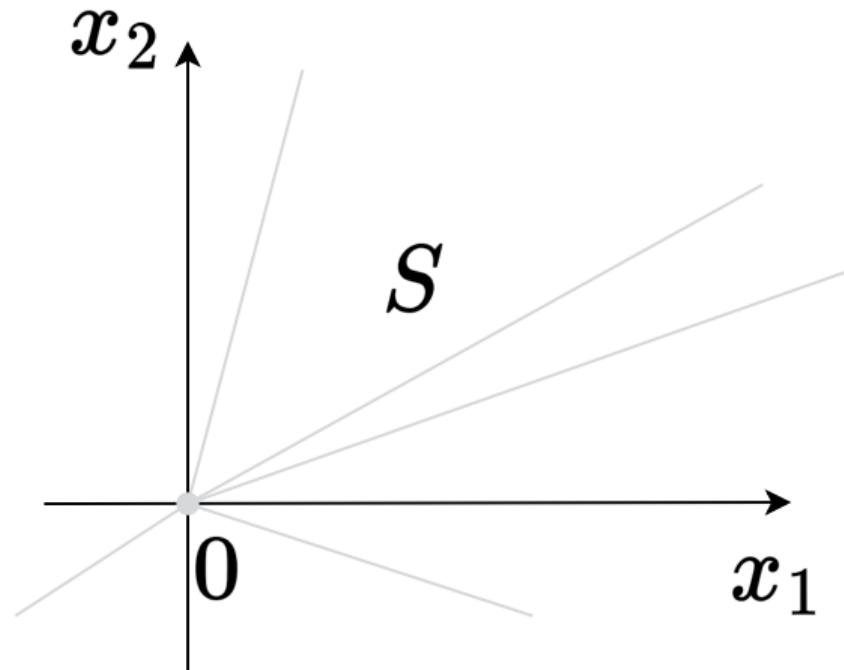


Figure 2: Illustration of a cone

## Convex cone

The set  $S$  is called a convex cone, if:

$$\forall x_1, x_2 \in S, \theta_1, \theta_2 \geq 0 \rightarrow \theta_1 x_1 + \theta_2 x_2 \in S$$

Convex cone is just like cone, but it is also convex.

### Example

- $\mathbb{R}^n$

Convex cone: set that contains all conic combinations of points in the set

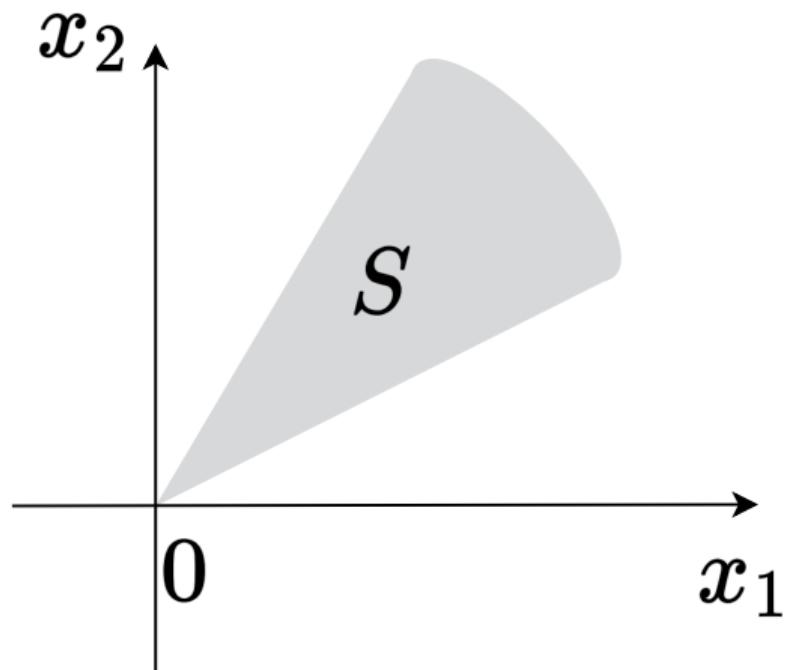


Figure 3: Illustration of a convex cone

## Convex cone

The set  $S$  is called a convex cone, if:

$$\forall x_1, x_2 \in S, \theta_1, \theta_2 \geq 0 \rightarrow \theta_1 x_1 + \theta_2 x_2 \in S$$

Convex cone is just like cone, but it is also convex.

### Example

- $\mathbb{R}^n$
- Affine sets, containing 0

Convex cone: set that contains all conic combinations of points in the set

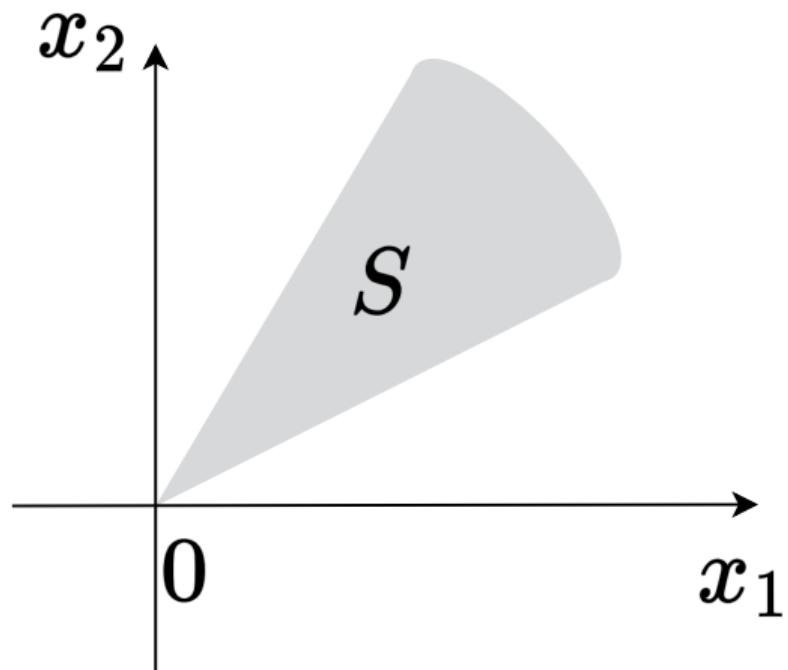


Figure 3: Illustration of a convex cone

## Convex cone

The set  $S$  is called a convex cone, if:

$$\forall x_1, x_2 \in S, \theta_1, \theta_2 \geq 0 \rightarrow \theta_1 x_1 + \theta_2 x_2 \in S$$

Convex cone is just like cone, but it is also convex.

### Example

- $\mathbb{R}^n$
- Affine sets, containing 0
- Ray

Convex cone: set that contains all conic combinations of points in the set

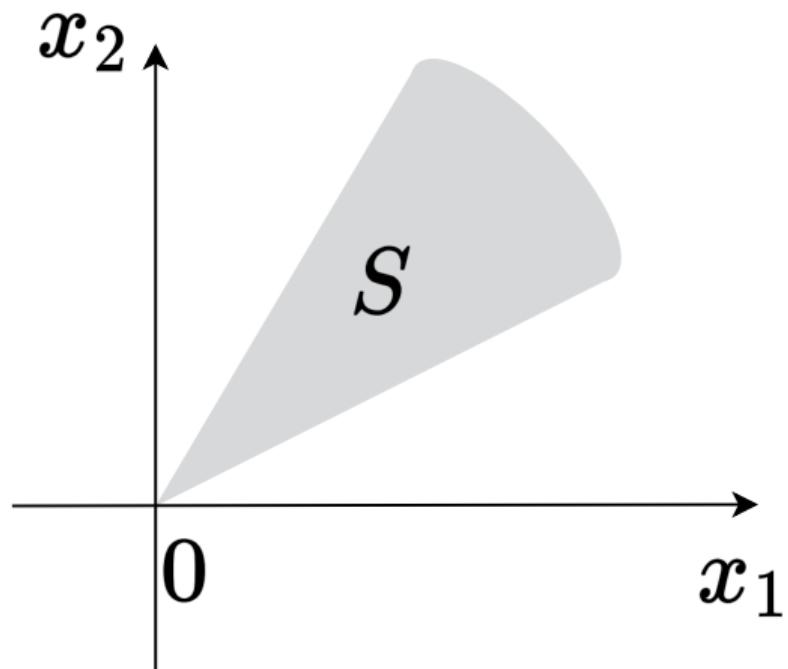


Figure 3: Illustration of a convex cone

## Convex cone

The set  $S$  is called a convex cone, if:

$$\forall x_1, x_2 \in S, \theta_1, \theta_2 \geq 0 \rightarrow \theta_1 x_1 + \theta_2 x_2 \in S$$

Convex cone is just like cone, but it is also convex.

### Example

- $\mathbb{R}^n$
- Affine sets, containing 0
- Ray
- $S^n_+$  - the set of symmetric positive semi-definite matrices

Convex cone: set that contains all conic combinations of points in the set

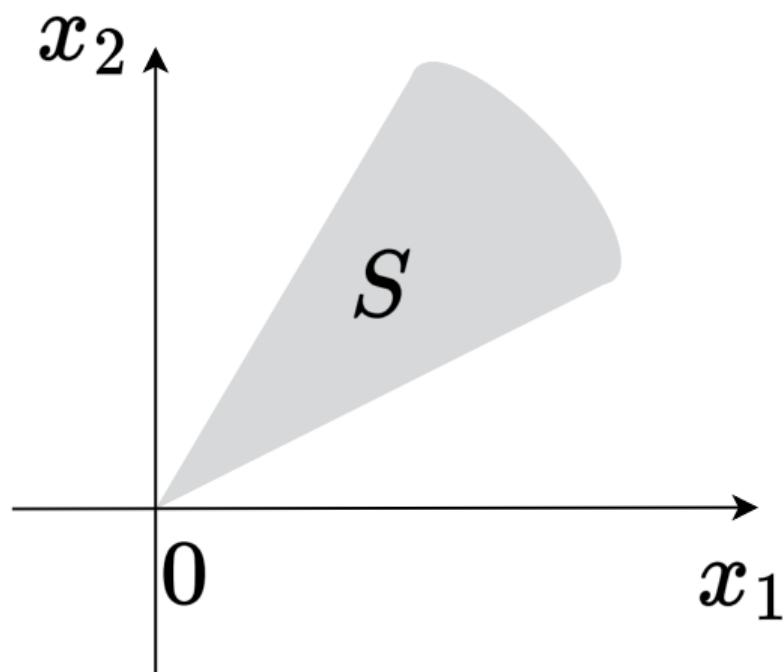


Figure 3: Illustration of a convex cone

## Line segment

Suppose  $x_1, x_2$  are two points in  $\mathbb{R}^n$ .

Then the line segment between them is defined as follows:

$$x = \theta x_1 + (1 - \theta)x_2, \theta \in [0, 1]$$

Convex set contains line segment between any two points in the set.

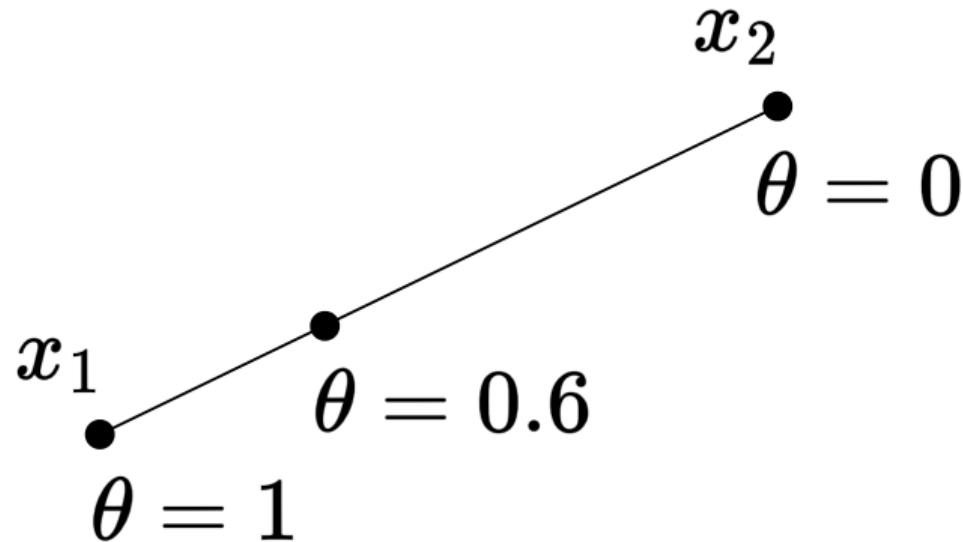


Figure 4: Illustration of a line segment between points  $x_1, x_2$

## Convex set

The set  $S$  is called **convex** if for any  $x_1, x_2$  from  $S$  the line segment between them also lies in  $S$ , i.e.

$$\forall \theta \in [0, 1], \forall x_1, x_2 \in S : \theta x_1 + (1 - \theta)x_2 \in S$$

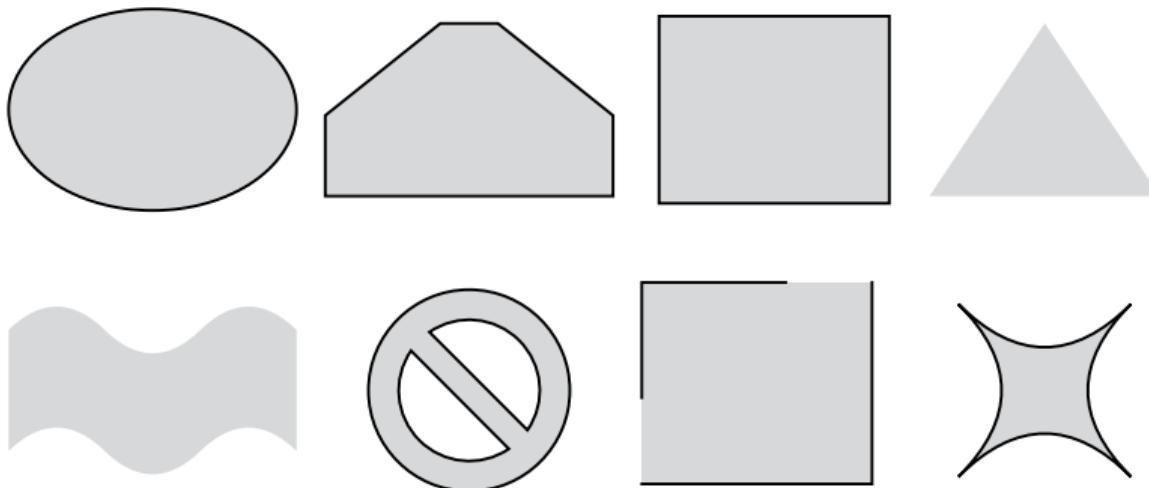


Figure 5: Top: examples of convex sets. Bottom: examples of non-convex sets.

### Example

An empty set and a set from a single vector are convex by definition.

### Example

Any affine set, a ray, a line segment - they all are convex sets.

## Convex combination

Let  $x_1, x_2, \dots, x_k \in S$ , then the point  $\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$  is called the convex combination of points  $x_1, x_2, \dots, x_k$  if  $\sum_{i=1}^k \theta_i = 1, \theta_i \geq 0$ .

## Convex hull

The set of all convex combinations of points from  $S$  is called the convex hull of the set  $S$ .

$$\text{conv}(S) = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in S, \sum_{i=1}^k \theta_i = 1, \theta_i \geq 0 \right\}$$

- The set  $\text{conv}(S)$  is the smallest convex set containing  $S$ .

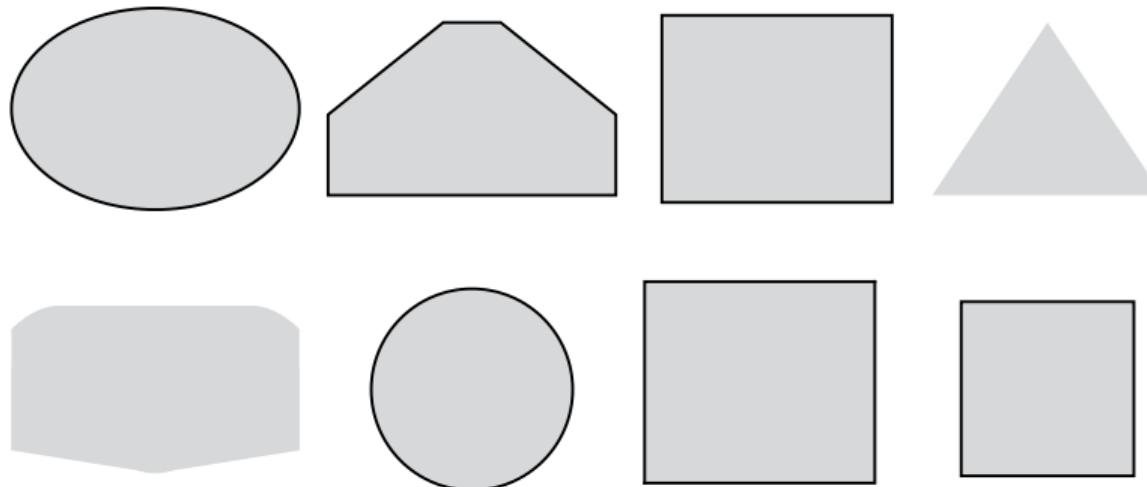


Figure 6: Top: convex hulls of the convex sets. Bottom: convex hull of the non-convex sets.

## Convex hull

The set of all convex combinations of points from  $S$  is called the convex hull of the set  $S$ .

$$\text{conv}(S) = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in S, \sum_{i=1}^k \theta_i = 1, \theta_i \geq 0 \right\}$$

- The set  $\text{conv}(S)$  is the smallest convex set containing  $S$ .
- The set  $S$  is convex if and only if  $S = \text{conv}(S)$ .

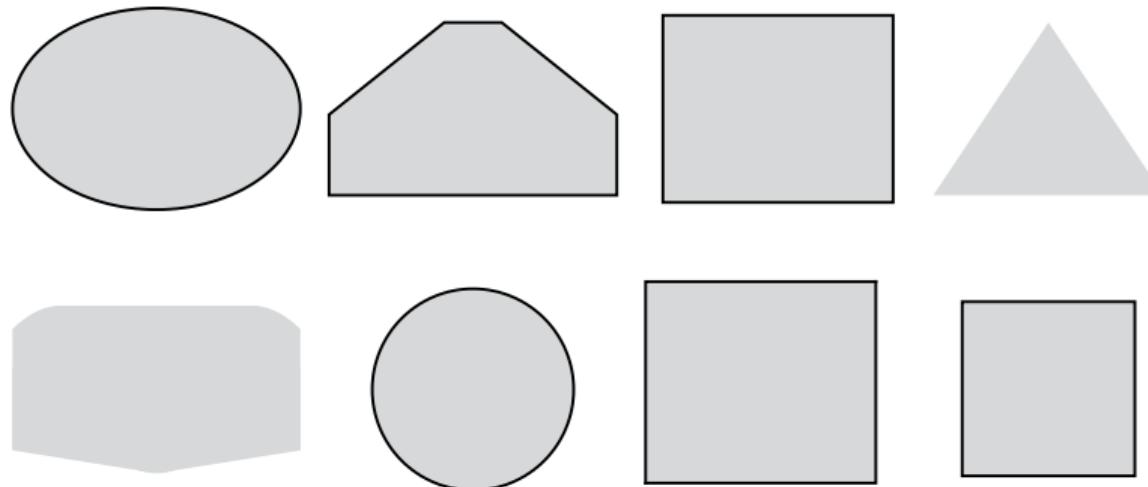


Figure 6: Top: convex hulls of the convex sets. Bottom: convex hull of the non-convex sets.

## Minkowski addition

The Minkowski sum of two sets of vectors  $S_1$  and  $S_2$  in Euclidean space is formed by adding each vector in  $S_1$  to each vector in  $S_2$ .

$$S_1 + S_2 = \{\mathbf{s}_1 + \mathbf{s}_2 \mid \mathbf{s}_1 \in S_1, \mathbf{s}_2 \in S_2\}$$

Similarly, one can define a linear combination of the sets.

### Example

We will work in the  $\mathbb{R}^2$  space. Let's define:

$$S_1 := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$$

This is a unit circle centered at the origin. And:

$$S_2 := \{x \in \mathbb{R}^2 : -4 \leq x_1 \leq -1, -3 \leq x_2 \leq -1\}$$

This represents a rectangle. The sum of the sets  $S_1$  and  $S_2$  will form an enlarged rectangle  $S_2$  with rounded corners. The resulting set will be convex.

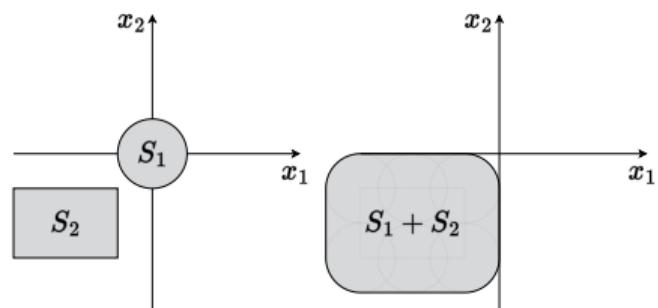


Figure 7:  $S = S_1 + S_2$

## Finding convexity

In practice, it is very important to understand whether a specific set is convex or not. Two approaches are used for this depending on the context.

- By definition.

## Finding convexity

In practice, it is very important to understand whether a specific set is convex or not. Two approaches are used for this depending on the context.

- By definition.
- Show that  $S$  is derived from simple convex sets using operations that preserve convexity.

## Finding convexity by definition

$$x_1, x_2 \in S, 0 \leq \theta \leq 1 \rightarrow \theta x_1 + (1 - \theta)x_2 \in S$$

### Example

Prove, that ball in  $\mathbb{R}^n$  (i.e. the following set  $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\| \leq r\}$ ) - is convex.

## Exercises

Which of the sets are convex:

- Stripe,  $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$

## Exercises

Which of the sets are convex:

- Stripe,  $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$
- Rectangle,  $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = \overline{1, n}\}$

## Exercises

Which of the sets are convex:

- Stripe,  $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$
- Rectangle,  $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = \overline{1, n}\}$
- Kleen,  $\{x \in \mathbb{R}^n \mid a_1^\top x \leq b_1, a_2^\top x \leq b_2\}$

## Exercises

Which of the sets are convex:

- Stripe,  $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$
- Rectangle,  $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = \overline{1, n}\}$
- Kleen,  $\{x \in \mathbb{R}^n \mid a_1^\top x \leq b_1, a_2^\top x \leq b_2\}$
- A set of points closer to a given point than a given set that does not contain a point,  
 $\{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - y\|_2, \forall y \in S \subseteq \mathbb{R}^n\}$

## Exercises

Which of the sets are convex:

- Stripe,  $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$
- Rectangle,  $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = \overline{1, n}\}$
- Kleen,  $\{x \in \mathbb{R}^n \mid a_1^\top x \leq b_1, a_2^\top x \leq b_2\}$
- A set of points closer to a given point than a given set that does not contain a point,  
 $\{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - y\|_2, \forall y \in S \subseteq \mathbb{R}^n\}$
- A set of points, which are closer to one set than another,  $\{x \in \mathbb{R}^n \mid \text{dist}(x, S) \leq \text{dist}(x, T), S, T \subseteq \mathbb{R}^n\}$

## Exercises

Which of the sets are convex:

- Stripe,  $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$
- Rectangle,  $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = \overline{1, n}\}$
- Kleen,  $\{x \in \mathbb{R}^n \mid a_1^\top x \leq b_1, a_2^\top x \leq b_2\}$
- A set of points closer to a given point than a given set that does not contain a point,  
 $\{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - y\|_2, \forall y \in S \subseteq \mathbb{R}^n\}$
- A set of points, which are closer to one set than another,  $\{x \in \mathbb{R}^n \mid \text{dist}(x, S) \leq \text{dist}(x, T), S, T \subseteq \mathbb{R}^n\}$
- A set of points,  $\{x \in \mathbb{R}^n \mid x + X \subseteq S\}$ , where  $S \subseteq \mathbb{R}^n$  is convex and  $X \subseteq \mathbb{R}^n$  is arbitrary.

## Exercises

Which of the sets are convex:

- Stripe,  $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$
- Rectangle,  $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = \overline{1, n}\}$
- Kleen,  $\{x \in \mathbb{R}^n \mid a_1^\top x \leq b_1, a_2^\top x \leq b_2\}$
- A set of points closer to a given point than a given set that does not contain a point,  
 $\{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - y\|_2, \forall y \in S \subseteq \mathbb{R}^n\}$
- A set of points, which are closer to one set than another,  $\{x \in \mathbb{R}^n \mid \text{dist}(x, S) \leq \text{dist}(x, T), S, T \subseteq \mathbb{R}^n\}$
- A set of points,  $\{x \in \mathbb{R}^n \mid x + X \subseteq S\}$ , where  $S \subseteq \mathbb{R}^n$  is convex and  $X \subseteq \mathbb{R}^n$  is arbitrary.
- A set of points whose distance to a given point does not exceed a certain part of the distance to another given point is  $\{x \in \mathbb{R}^n \mid \|x - a\|_2 \leq \theta \|x - b\|_2, a, b \in \mathbb{R}^n, 0 \leq \theta \leq 1\}$

## Operations, that preserve convexity

The linear combination of convex sets is convex Let there be 2 convex sets  $S_x, S_y$ , let the set

$$S = \{s \mid s = c_1x + c_2y, x \in S_x, y \in S_y, c_1, c_2 \in \mathbb{R}\}$$

Take two points from  $S$ :  $s_1 = c_1x_1 + c_2y_1, s_2 = c_1x_2 + c_2y_2$  and prove that the segment between them  $\theta s_1 + (1 - \theta)s_2, \theta \in [0, 1]$  also belongs to  $S$

$$\theta s_1 + (1 - \theta)s_2$$

$$\theta(c_1x_1 + c_2y_1) + (1 - \theta)(c_1x_2 + c_2y_2)$$

$$c_1(\theta x_1 + (1 - \theta)x_2) + c_2(\theta y_1 + (1 - \theta)y_2)$$

$$c_1x + c_2y \in S$$

## The intersection of any (!) number of convex sets is convex

If the desired intersection is empty or contains one point, the property is proved by definition. Otherwise, take 2 points and a segment between them. These points must lie in all intersecting sets, and since they are all convex, the segment between them lies in all sets and, therefore, in their intersection.

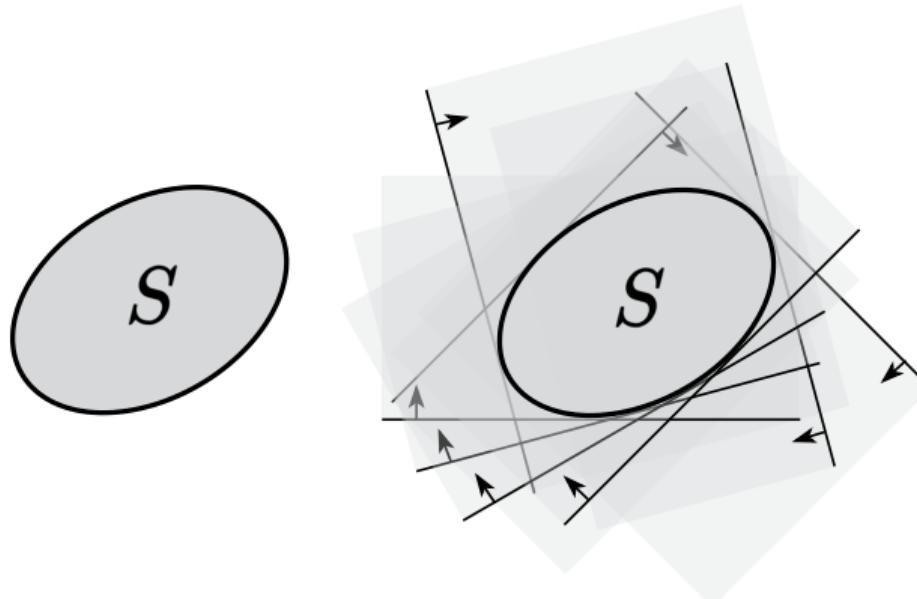


Figure 8: Intersection of halfplanes

## The image of the convex set under affine mapping is convex

$$S \subseteq \mathbb{R}^n \text{ convex } \rightarrow f(S) = \{f(x) \mid x \in S\} \text{ convex } (f(x) = \mathbf{A}x + \mathbf{b})$$

Examples of affine functions: extension, projection, transposition, set of solutions of linear matrix inequality  $\{x \mid x_1 A_1 + \dots + x_m A_m \preceq B\}$ . Here  $A_i, B \in \mathbf{S}^p$  are symmetric matrices  $p \times p$ .

Note also that the prototype of the convex set under affine mapping is also convex.

$$S \subseteq \mathbb{R}^m \text{ convex } \rightarrow f^{-1}(S) = \{x \in \mathbb{R}^n \mid f(x) \in S\} \text{ convex } (f(x) = \mathbf{A}x + \mathbf{b})$$

## Example

Let  $x \in \mathbb{R}$  is a random variable with a given probability distribution of  $\mathbb{P}(x = a_i) = p_i$ , where  $i = 1, \dots, n$ , and  $a_1 < \dots < a_n$ . It is said that the probability vector of outcomes of  $p \in \mathbb{R}^n$  belongs to the probabilistic simplex, i.e.

$$P = \{p \mid \mathbf{1}^T p = 1, p \succeq 0\} = \{p \mid p_1 + \dots + p_n = 1, p_i \geq 0\}.$$

Determine if the following sets of  $p$  are convex:

- $\mathbb{P}(x > \alpha) \leq \beta$

## Example

Let  $x \in \mathbb{R}$  is a random variable with a given probability distribution of  $\mathbb{P}(x = a_i) = p_i$ , where  $i = 1, \dots, n$ , and  $a_1 < \dots < a_n$ . It is said that the probability vector of outcomes of  $p \in \mathbb{R}^n$  belongs to the probabilistic simplex, i.e.

$$P = \{p \mid \mathbf{1}^T p = 1, p \succeq 0\} = \{p \mid p_1 + \dots + p_n = 1, p_i \geq 0\}.$$

Determine if the following sets of  $p$  are convex:

- $\mathbb{P}(x > \alpha) \leq \beta$
- $\mathbb{E}|x^{201}| \leq \alpha \mathbb{E}|x|$

## Example

Let  $x \in \mathbb{R}$  is a random variable with a given probability distribution of  $\mathbb{P}(x = a_i) = p_i$ , where  $i = 1, \dots, n$ , and  $a_1 < \dots < a_n$ . It is said that the probability vector of outcomes of  $p \in \mathbb{R}^n$  belongs to the probabilistic simplex, i.e.

$$P = \{p \mid \mathbf{1}^T p = 1, p \succeq 0\} = \{p \mid p_1 + \dots + p_n = 1, p_i \geq 0\}.$$

Determine if the following sets of  $p$  are convex:

- $\mathbb{P}(x > \alpha) \leq \beta$
- $\mathbb{E}|x^{201}| \leq \alpha \mathbb{E}|x|$
- $\mathbb{E}|x^2| \geq \alpha \mathbb{V}x \geq \alpha$

## Convex functions

## Jensen's inequality

The function  $f(x)$ , which is defined on the convex set

$S \subseteq \mathbb{R}^n$ , is called **convex** on  $S$ , if:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)$$

for any  $x_1, x_2 \in S$  and  $0 \leq \lambda \leq 1$ .

If the above inequality holds as strict inequality  $x_1 \neq x_2$  and  $0 < \lambda < 1$ , then the function is called strictly convex on  $S$ .

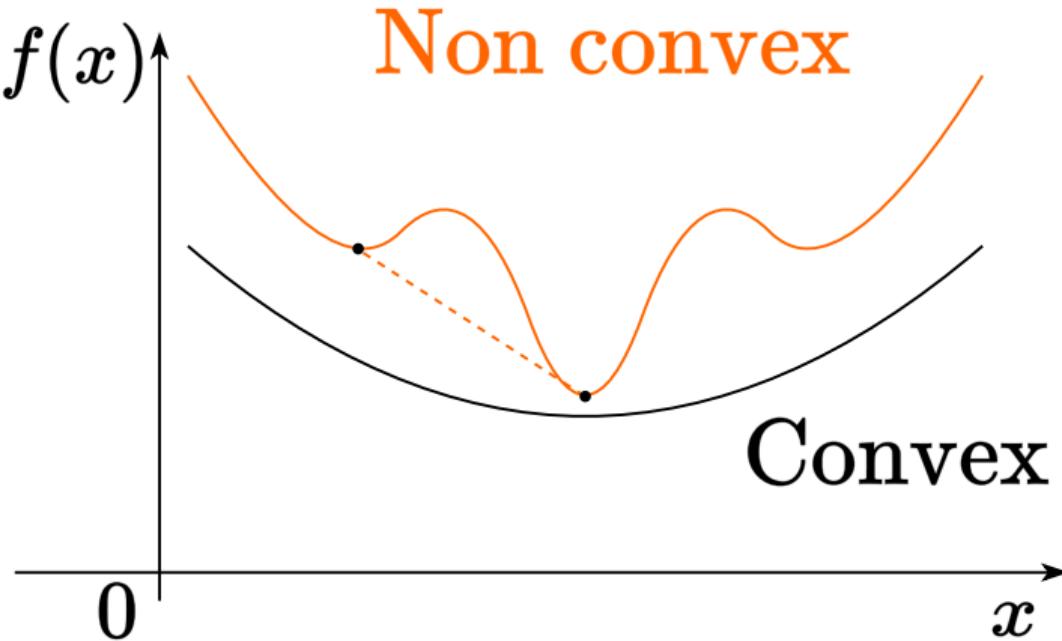


Figure 9: Difference between convex and non-convex function

# Jensen's inequality

## Theorem

Let  $f(x)$  be a convex function on a convex set  $X \subseteq \mathbb{R}^n$  and let  $x_i \in X, 1 \leq i \leq m$ , be arbitrary points from  $X$ . Then

$$f\left(\sum_{i=1}^m \lambda_i x_i\right) \leq \sum_{i=1}^m \lambda_i f(x_i)$$

for any  $\lambda = [\lambda_1, \dots, \lambda_m] \in \Delta_m$  - probability simplex.

## Proof

1. First, note that the point  $\sum_{i=1}^m \lambda_i x_i$  as a convex combination of points from the convex set  $X$  belongs to  $X$ .

# Jensen's inequality

## Theorem

Let  $f(x)$  be a convex function on a convex set  $X \subseteq \mathbb{R}^n$  and let  $x_i \in X, 1 \leq i \leq m$ , be arbitrary points from  $X$ . Then

$$f\left(\sum_{i=1}^m \lambda_i x_i\right) \leq \sum_{i=1}^m \lambda_i f(x_i)$$

for any  $\lambda = [\lambda_1, \dots, \lambda_m] \in \Delta_m$  - probability simplex.

## Proof

1. First, note that the point  $\sum_{i=1}^m \lambda_i x_i$  as a convex combination of points from the convex set  $X$  belongs to  $X$ .
2. We will prove this by induction. For  $m = 1$ , the statement is obviously true, and for  $m = 2$ , it follows from the definition of a convex function.

## Jensen's inequality

3. Assume it is true for all  $m$  up to  $m = k$ , and we will prove it for  $m = k + 1$ . Let  $\lambda \in \Delta_{k+1}$  and

$$x = \sum_{i=1}^{k+1} \lambda_i x_i = \sum_{i=1}^k \lambda_i x_i + \lambda_{k+1} x_{k+1}.$$

Assuming  $0 < \lambda_{k+1} < 1$ , as otherwise, it reduces to previously considered cases, we have

$$x = \lambda_{k+1} x_{k+1} + (1 - \lambda_{k+1}) \bar{x},$$

where  $\bar{x} = \sum_{i=1}^k \gamma_i x_i$  and  $\gamma_i = \frac{\lambda_i}{1 - \lambda_{k+1}} \geq 0, 1 \leq i \leq k$ .

## Jensen's inequality

3. Assume it is true for all  $m$  up to  $m = k$ , and we will prove it for  $m = k + 1$ . Let  $\lambda \in \Delta_{k+1}$  and

$$x = \sum_{i=1}^{k+1} \lambda_i x_i = \sum_{i=1}^k \lambda_i x_i + \lambda_{k+1} x_{k+1}.$$

Assuming  $0 < \lambda_{k+1} < 1$ , as otherwise, it reduces to previously considered cases, we have

$$x = \lambda_{k+1} x_{k+1} + (1 - \lambda_{k+1}) \bar{x},$$

where  $\bar{x} = \sum_{i=1}^k \gamma_i x_i$  and  $\gamma_i = \frac{\lambda_i}{1 - \lambda_{k+1}} \geq 0, 1 \leq i \leq k$ .

4. Since  $\lambda \in \Delta_{k+1}$ , then  $\gamma = [\gamma_1, \dots, \gamma_k] \in \Delta_k$ . Therefore  $\bar{x} \in X$  and by the convexity of  $f(x)$  and the induction hypothesis:

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = f(\lambda_{k+1} x_{k+1} + (1 - \lambda_{k+1}) \bar{x}) \leq \lambda_{k+1} f(x_{k+1}) + (1 - \lambda_{k+1}) f(\bar{x}) \leq \sum_{i=1}^{k+1} \lambda_i f(x_i)$$

Thus, initial inequality is satisfied for  $m = k + 1$  as well.

## Examples of convex functions

- $f(x) = x^p, p > 1, x \in \mathbb{R}_+$

## Examples of convex functions

- $f(x) = x^p, p > 1, x \in \mathbb{R}_+$
- $f(x) = \|x\|^p, p > 1, x \in \mathbb{R}^n$

## Examples of convex functions

- $f(x) = x^p, p > 1, x \in \mathbb{R}_+$
- $f(x) = \|x\|^p, p > 1, x \in \mathbb{R}^n$
- $f(x) = e^{cx}, c \in \mathbb{R}, x \in \mathbb{R}$

## Examples of convex functions

- $f(x) = x^p, p > 1, x \in \mathbb{R}_+$
- $f(x) = \|x\|^p, p > 1, x \in \mathbb{R}^n$
- $f(x) = e^{cx}, c \in \mathbb{R}, x \in \mathbb{R}$
- $f(x) = -\ln x, x \in \mathbb{R}_{++}$

## Examples of convex functions

- $f(x) = x^p, p > 1, x \in \mathbb{R}_+$
- $f(x) = \|x\|^p, p > 1, x \in \mathbb{R}^n$
- $f(x) = e^{cx}, c \in \mathbb{R}, x \in \mathbb{R}$
- $f(x) = -\ln x, x \in \mathbb{R}_{++}$
- $f(x) = x \ln x, x \in \mathbb{R}_{++}$

## Examples of convex functions

- $f(x) = x^p, p > 1, x \in \mathbb{R}_+$
- $f(x) = \|x\|^p, p > 1, x \in \mathbb{R}^n$
- $f(x) = e^{cx}, c \in \mathbb{R}, x \in \mathbb{R}$
- $f(x) = -\ln x, x \in \mathbb{R}_{++}$
- $f(x) = x \ln x, x \in \mathbb{R}_{++}$
- The sum of the largest  $k$  coordinates  $f(x) = x_{(1)} + \dots + x_{(k)}, x \in \mathbb{R}^n$

## Examples of convex functions

- $f(x) = x^p, p > 1, x \in \mathbb{R}_+$
- $f(x) = \|x\|^p, p > 1, x \in \mathbb{R}^n$
- $f(x) = e^{cx}, c \in \mathbb{R}, x \in \mathbb{R}$
- $f(x) = -\ln x, x \in \mathbb{R}_{++}$
- $f(x) = x \ln x, x \in \mathbb{R}_{++}$
- The sum of the largest  $k$  coordinates  $f(x) = x_{(1)} + \dots + x_{(k)}, x \in \mathbb{R}^n$
- $f(X) = \lambda_{max}(X), X = X^T$

## Examples of convex functions

- $f(x) = x^p, p > 1, x \in \mathbb{R}_+$
- $f(x) = \|x\|^p, p > 1, x \in \mathbb{R}^n$
- $f(x) = e^{cx}, c \in \mathbb{R}, x \in \mathbb{R}$
- $f(x) = -\ln x, x \in \mathbb{R}_{++}$
- $f(x) = x \ln x, x \in \mathbb{R}_{++}$
- The sum of the largest  $k$  coordinates  $f(x) = x_{(1)} + \dots + x_{(k)}, x \in \mathbb{R}^n$
- $f(X) = \lambda_{max}(X), X = X^T$
- $f(X) = -\log \det X, X \in S_{++}^n$

# Epigraph

For the function  $f(x)$ , defined on  $S \subseteq \mathbb{R}^n$ , the following set:

$$\text{epi } f = \{[x, \mu] \in S \times \mathbb{R} : f(x) \leq \mu\}$$

is called **epigraph** of the function  $f(x)$ .

Convexity of the epigraph is the convexity of the function

For a function  $f(x)$ , defined on a convex set  $X$ , to be convex on  $X$ , it is necessary and sufficient that the epigraph of  $f$  is a convex set.

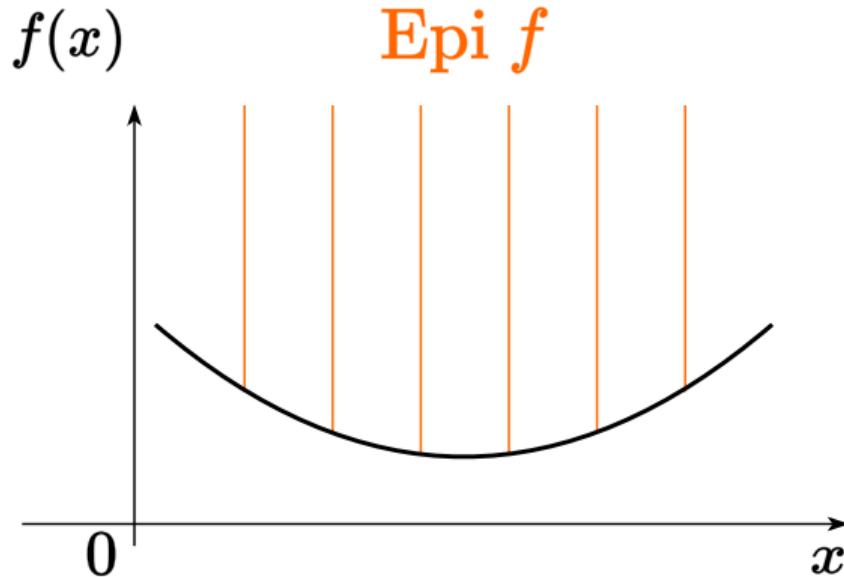


Figure 10: Epigraph of a function

## Convexity of the epigraph is the convexity of the function

1. **Necessity:** Assume  $f(x)$  is convex on  $X$ . Take any two arbitrary points  $[x_1, \mu_1] \in \text{epi } f$  and  $[x_2, \mu_2] \in \text{epi } f$ . Also take  $0 \leq \lambda \leq 1$  and denote  $x_\lambda = \lambda x_1 + (1 - \lambda)x_2, \mu_\lambda = \lambda\mu_1 + (1 - \lambda)\mu_2$ . Then,

$$\lambda \begin{bmatrix} x_1 \\ \mu_1 \end{bmatrix} + (1 - \lambda) \begin{bmatrix} x_2 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} x_\lambda \\ \mu_\lambda \end{bmatrix}.$$

From the convexity of the set  $X$ , it follows that  $x_\lambda \in X$ . Moreover, since  $f(x)$  is a convex function,

$$f(x_\lambda) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \leq \lambda\mu_1 + (1 - \lambda)\mu_2 = \mu_\lambda$$

Inequality above indicates that  $\begin{bmatrix} x_\lambda \\ \mu_\lambda \end{bmatrix} \in \text{epi } f$ . Thus, the epigraph of  $f$  is a convex set.

## Convexity of the epigraph is the convexity of the function

1. **Necessity:** Assume  $f(x)$  is convex on  $X$ . Take any two arbitrary points  $[x_1, \mu_1] \in \text{epi } f$  and  $[x_2, \mu_2] \in \text{epi } f$ . Also take  $0 \leq \lambda \leq 1$  and denote  $x_\lambda = \lambda x_1 + (1 - \lambda)x_2, \mu_\lambda = \lambda\mu_1 + (1 - \lambda)\mu_2$ . Then,

$$\lambda \begin{bmatrix} x_1 \\ \mu_1 \end{bmatrix} + (1 - \lambda) \begin{bmatrix} x_2 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} x_\lambda \\ \mu_\lambda \end{bmatrix}.$$

From the convexity of the set  $X$ , it follows that  $x_\lambda \in X$ . Moreover, since  $f(x)$  is a convex function,

$$f(x_\lambda) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \leq \lambda\mu_1 + (1 - \lambda)\mu_2 = \mu_\lambda$$

Inequality above indicates that  $\begin{bmatrix} x_\lambda \\ \mu_\lambda \end{bmatrix} \in \text{epi } f$ . Thus, the epigraph of  $f$  is a convex set.

2. **Sufficiency:** Assume the epigraph of  $f$ ,  $\text{epi } f$ , is a convex set. Then, from the membership of the points  $[x_1, \mu_1]$  and  $[x_2, \mu_2]$  in the epigraph of  $f$ , it follows that

$$\begin{bmatrix} x_\lambda \\ \mu_\lambda \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \mu_1 \end{bmatrix} + (1 - \lambda) \begin{bmatrix} x_2 \\ \mu_2 \end{bmatrix} \in \text{epi } f$$

for any  $0 \leq \lambda \leq 1$ , i.e.,  $f(x_\lambda) \leq \mu_\lambda = \lambda\mu_1 + (1 - \lambda)\mu_2$ . But this is true for all  $\mu_1 \geq f(x_1)$  and  $\mu_2 \geq f(x_2)$ , particularly when  $\mu_1 = f(x_1)$  and  $\mu_2 = f(x_2)$ . Hence we arrive at the inequality

## Sublevel set

For the function  $f(x)$ , defined on  $S \subseteq \mathbb{R}^n$ , the following set:

$$\mathcal{L}_\beta = \{x \in S : f(x) \leq \beta\}$$

is called **sublevel set** or Lebesgue set of the function  $f(x)$ .

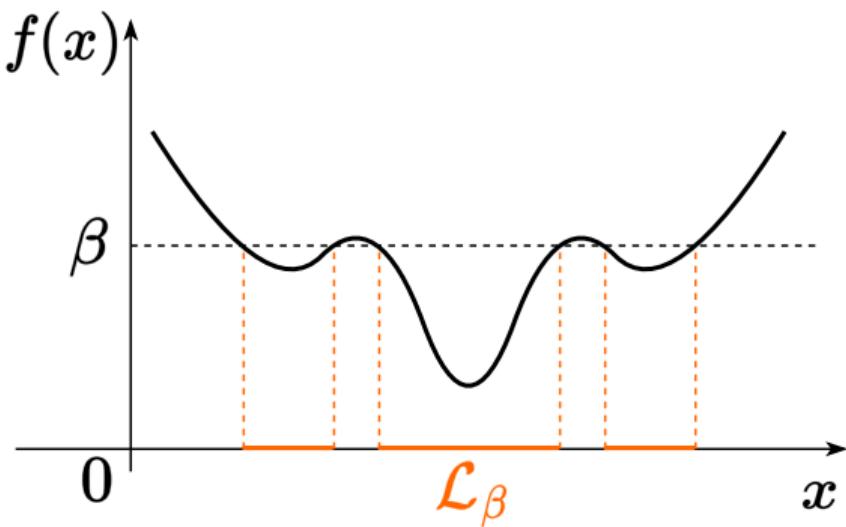


Figure 11: Sublevel set of a function with respect to level  $\beta$

## Connection with epigraph

The function is convex if and only if its epigraph is a convex set.

### Example

Let a norm  $\|\cdot\|$  be defined in the space  $U$ . Consider the set:

$$K := \{(x, t) \in U \times \mathbb{R}^+ : \|x\| \leq t\}$$

which represents the epigraph of the function  $x \mapsto \|x\|$ . This set is called the cone norm. According to the statement above, the set  $K$  is convex.

In the case where  $U = \mathbb{R}^n$  and  $\|x\| = \|x\|_2$  (Euclidean norm), the abstract set  $K$  transitions into the set:

$$\{(x, t) \in \mathbb{R}^n \times \mathbb{R}^+ : \|x\|_2 \leq t\}$$

## Connection with sublevel set

If  $f(x)$  - is a convex function defined on the convex set  $S \subseteq \mathbb{R}^n$ , then for any  $\beta$  sublevel set  $\mathcal{L}_\beta$  is convex.

The function  $f(x)$  defined on the convex set  $S \subseteq \mathbb{R}^n$  is closed if and only if for any  $\beta$  sublevel set  $\mathcal{L}_\beta$  is closed.

## Reduction to a line

$f : S \rightarrow \mathbb{R}$  is convex if and only if  $S$  is a convex set and the function  $g(t) = f(x + tv)$  defined on  $\{t \mid x + tv \in S\}$  is convex for any  $x \in S, v \in \mathbb{R}^n$ , which allows checking convexity of the scalar function to establish convexity of the vector function.

## Strong convexity criteria

## First-order differential criterion of convexity

The differentiable function  $f(x)$  defined on the convex set  $S \subseteq \mathbb{R}^n$  is convex if and only if  $\forall x, y \in S$ :

$$f(y) \geq f(x) + \nabla f^T(x)(y - x)$$

Let  $y = x + \Delta x$ , then the criterion will become more tractable:

$$f(x + \Delta x) \geq f(x) + \nabla f^T(x)\Delta x$$

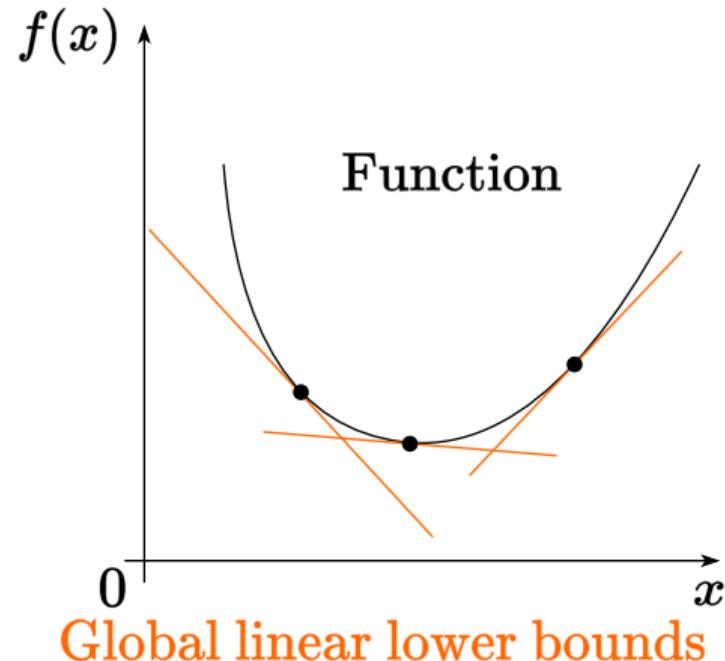


Figure 12: Convex function is greater or equal than Taylor linear approximation at any point

## Second-order differential criterion of convexity

Twice differentiable function  $f(x)$  defined on the convex set  $S \subseteq \mathbb{R}^n$  is convex if and only if  $\forall x \in \text{int}(S) \neq \emptyset$ :

$$\nabla^2 f(x) \succeq 0$$

In other words,  $\forall y \in \mathbb{R}^n$ :

$$\langle y, \nabla^2 f(x)y \rangle \geq 0$$

## Tools for discovering convexity

- Definition (Jensen's inequality)

## Tools for discovering convexity

- Definition (Jensen's inequality)
- Differential criteria of convexity

## Tools for discovering convexity

- Definition (Jensen's inequality)
- Differential criteria of convexity
- Operations, that preserve convexity

## Tools for discovering convexity

- **Definition (Jensen's inequality)**
- **Differential criteria of convexity**
- **Operations, that preserve convexity**
- **Connection with epigraph**

The function is convex if and only if its epigraph is a convex set.

## Tools for discovering convexity

- **Definition (Jensen's inequality)**
- **Differential criteria of convexity**
- **Operations, that preserve convexity**
- **Connection with epigraph**

The function is convex if and only if its epigraph is a convex set.

- **Connection with sublevel set**

If  $f(x)$  - is a convex function defined on the convex set  $S \subseteq \mathbb{R}^n$ , then for any  $\beta$  sublevel set  $\mathcal{L}_\beta$  is convex.

The function  $f(x)$  defined on the convex set  $S \subseteq \mathbb{R}^n$  is closed if and only if for any  $\beta$  sublevel set  $\mathcal{L}_\beta$  is closed.

# Tools for discovering convexity

- **Definition (Jensen's inequality)**
- **Differential criteria of convexity**
- **Operations, that preserve convexity**
- **Connection with epigraph**

The function is convex if and only if its epigraph is a convex set.

- **Connection with sublevel set**

If  $f(x)$  - is a convex function defined on the convex set  $S \subseteq \mathbb{R}^n$ , then for any  $\beta$  sublevel set  $\mathcal{L}_\beta$  is convex.

The function  $f(x)$  defined on the convex set  $S \subseteq \mathbb{R}^n$  is closed if and only if for any  $\beta$  sublevel set  $\mathcal{L}_\beta$  is closed.

- **Reduction to a line**

$f : S \rightarrow \mathbb{R}$  is convex if and only if  $S$  is a convex set and the function  $g(t) = f(x + tv)$  defined on  $\{t \mid x + tv \in S\}$  is convex for any  $x \in S, v \in \mathbb{R}^n$ , which allows checking convexity of the scalar function to establish convexity of the vector function.

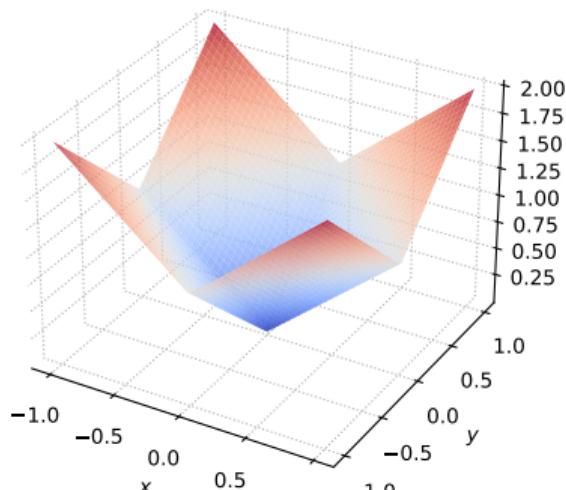
## Example: norm cone

Let a norm  $\|\cdot\|$  be defined in the space  $U$ . Consider the set:

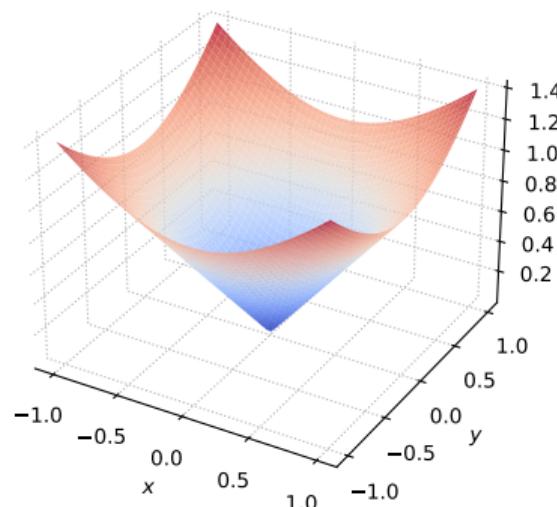
$$K := \{(x, t) \in U \times \mathbb{R}^+ : \|x\| \leq t\}$$

which represents the epigraph of the function  $x \mapsto \|x\|$ . This set is called the cone norm. According to the statement above, the set  $K$  is convex. Code for the figures

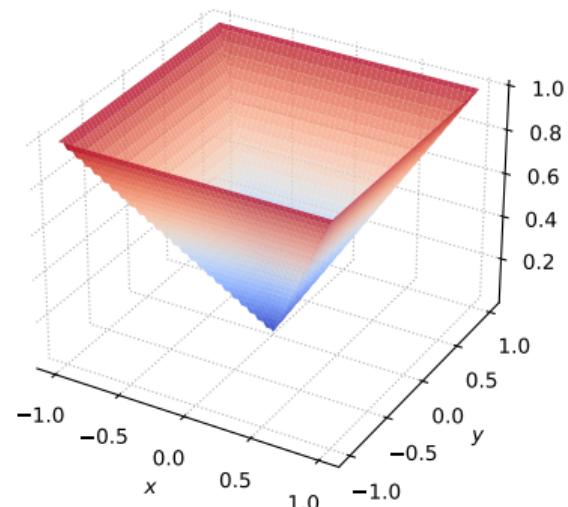
$p = 1$  Norm Cone



$p = 2$  Norm Cone



$p = \infty$  Norm Cone

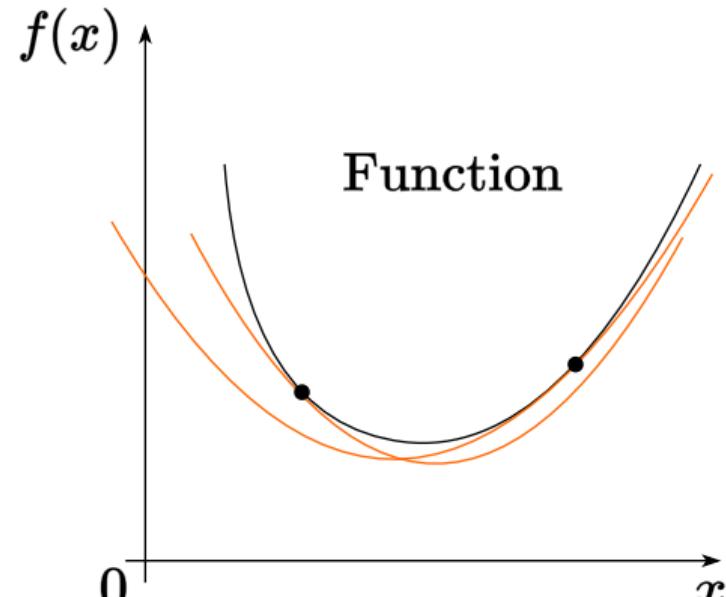


## Strong convexity

$f(x)$ , defined on the convex set  $S \subseteq \mathbb{R}^n$ , is called  $\mu$ -strongly convex (strongly convex) on  $S$ , if:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) - \frac{\mu}{2}\lambda(1-\lambda)\|x_1 - x_2\|^2$$

for any  $x_1, x_2 \in S$  and  $0 \leq \lambda \leq 1$  for some  $\mu > 0$ .



Global quadratic lower bounds

Figure 14: Strongly convex function is greater or equal than Taylor quadratic approximation at any point

## First-order differential criterion of strong convexity

Differentiable  $f(x)$  defined on the convex set  $S \subseteq \mathbb{R}^n$  is  $\mu$ -strongly convex if and only if  $\forall x, y \in S$ :

$$f(y) \geq f(x) + \nabla f^T(x)(y - x) + \frac{\mu}{2}\|y - x\|^2$$

## First-order differential criterion of strong convexity

Differentiable  $f(x)$  defined on the convex set  $S \subseteq \mathbb{R}^n$  is  $\mu$ -strongly convex if and only if  $\forall x, y \in S$ :

$$f(y) \geq f(x) + \nabla f^T(x)(y - x) + \frac{\mu}{2}\|y - x\|^2$$

Let  $y = x + \Delta x$ , then the criterion will become more tractable:

$$f(x + \Delta x) \geq f(x) + \nabla f^T(x)\Delta x + \frac{\mu}{2}\|\Delta x\|^2$$

## First-order differential criterion of strong convexity

Differentiable  $f(x)$  defined on the convex set  $S \subseteq \mathbb{R}^n$  is  $\mu$ -strongly convex if and only if  $\forall x, y \in S$ :

$$f(y) \geq f(x) + \nabla f^T(x)(y - x) + \frac{\mu}{2}\|y - x\|^2$$

Let  $y = x + \Delta x$ , then the criterion will become more tractable:

$$f(x + \Delta x) \geq f(x) + \nabla f^T(x)\Delta x + \frac{\mu}{2}\|\Delta x\|^2$$

### Theorem

Let  $f(x)$  be a differentiable function on a convex set  $X \subseteq \mathbb{R}^n$ . Then  $f(x)$  is strongly convex on  $X$  with a constant  $\mu > 0$  if and only if

$$f(x) - f(x_0) \geq \langle \nabla f(x_0), x - x_0 \rangle + \frac{\mu}{2}\|x - x_0\|^2$$

for all  $x, x_0 \in X$ .

## Proof of first-order differential criterion of strong convexity

**Necessity:** Let  $0 < \lambda \leq 1$ . According to the definition of a strongly convex function,

$$f(\lambda x + (1 - \lambda)x_0) \leq \lambda f(x) + (1 - \lambda)f(x_0) - \frac{\mu}{2}\lambda(1 - \lambda)\|x - x_0\|^2$$

or equivalently,

$$\begin{aligned} f(x) - f(x_0) - \frac{\mu}{2}(1 - \lambda)\|x - x_0\|^2 &\geq \frac{1}{\lambda}[f(\lambda x + (1 - \lambda)x_0) - f(x_0)] = \\ &= \frac{1}{\lambda}[f(x_0 + \lambda(x - x_0)) - f(x_0)] = \frac{1}{\lambda}[\lambda\langle\nabla f(x_0), x - x_0\rangle + o(\lambda)] = \\ &= \langle\nabla f(x_0), x - x_0\rangle + \frac{o(\lambda)}{\lambda}. \end{aligned}$$

Thus, taking the limit as  $\lambda \downarrow 0$ , we arrive at the initial statement.

## Proof of first-order differential criterion of strong convexity

**Sufficiency:** Assume the inequality in the theorem is satisfied for all  $x, x_0 \in X$ . Take  $x_0 = \lambda x_1 + (1 - \lambda)x_2$ , where  $x_1, x_2 \in X$ ,  $0 \leq \lambda \leq 1$ . According to the inequality, the following inequalities hold:

$$f(x_1) - f(x_0) \geq \langle \nabla f(x_0), x_1 - x_0 \rangle + \frac{\mu}{2} \|x_1 - x_0\|^2,$$

$$f(x_2) - f(x_0) \geq \langle \nabla f(x_0), x_2 - x_0 \rangle + \frac{\mu}{2} \|x_2 - x_0\|^2.$$

Multiplying the first inequality by  $\lambda$  and the second by  $1 - \lambda$  and adding them, considering that

$$x_1 - x_0 = (1 - \lambda)(x_1 - x_2), \quad x_2 - x_0 = \lambda(x_2 - x_1),$$

and  $\lambda(1 - \lambda)^2 + \lambda^2(1 - \lambda) = \lambda(1 - \lambda)$ , we get

$$\begin{aligned} \lambda f(x_1) + (1 - \lambda)f(x_2) - f(x_0) - \frac{\mu}{2}\lambda(1 - \lambda)\|x_1 - x_2\|^2 &\geq \\ \langle \nabla f(x_0), \lambda x_1 + (1 - \lambda)x_2 - x_0 \rangle &= 0. \end{aligned}$$

Thus, inequality from the definition of a strongly convex function is satisfied. It is important to mention, that  $\mu = 0$  stands for the convex case and corresponding differential criterion.

## Second-order differential criterion of strong convexity

Twice differentiable function  $f(x)$  defined on the convex set  $S \subseteq \mathbb{R}^n$  is called  $\mu$ -strongly convex if and only if  $\forall x \in \text{int}(S) \neq \emptyset$ :

$$\nabla^2 f(x) \succeq \mu I$$

In other words:

$$\langle y, \nabla^2 f(x)y \rangle \geq \mu \|y\|^2$$

## Second-order differential criterion of strong convexity

Twice differentiable function  $f(x)$  defined on the convex set  $S \subseteq \mathbb{R}^n$  is called  $\mu$ -strongly convex if and only if  $\forall x \in \text{int}(S) \neq \emptyset$ :

$$\nabla^2 f(x) \succeq \mu I$$

In other words:

$$\langle y, \nabla^2 f(x)y \rangle \geq \mu \|y\|^2$$

### Theorem

Let  $X \subseteq \mathbb{R}^n$  be a convex set, with  $\text{int}X \neq \emptyset$ . Furthermore, let  $f(x)$  be a twice continuously differentiable function on  $X$ . Then  $f(x)$  is strongly convex on  $X$  with a constant  $\mu > 0$  if and only if

$$\langle y, \nabla^2 f(x)y \rangle \geq \mu \|y\|^2$$

for all  $x \in X$  and  $y \in \mathbb{R}^n$ .

## Proof of second-order differential criterion of strong convexity

The target inequality is trivial when  $y = \mathbf{0}_n$ , hence we assume  $y \neq \mathbf{0}_n$ .

**Necessity:** Assume initially that  $x$  is an interior point of  $X$ . Then  $x + \alpha y \in X$  for all  $y \in \mathbb{R}^n$  and sufficiently small  $\alpha$ . Since  $f(x)$  is twice differentiable,

$$f(x + \alpha y) = f(x) + \alpha \langle \nabla f(x), y \rangle + \frac{\alpha^2}{2} \langle y, \nabla^2 f(x)y \rangle + o(\alpha^2).$$

Based on the first order criterion of strong convexity, we have

$$\frac{\alpha^2}{2} \langle y, \nabla^2 f(x)y \rangle + o(\alpha^2) = f(x + \alpha y) - f(x) - \alpha \langle \nabla f(x), y \rangle \geq \frac{\mu}{2} \alpha^2 \|y\|^2.$$

This inequality reduces to the target inequality after dividing both sides by  $\alpha^2$  and taking the limit as  $\alpha \downarrow 0$ .

If  $x \in X$  but  $x \notin \text{int } X$ , consider a sequence  $\{x_k\}$  such that  $x_k \in \text{int } X$  and  $x_k \rightarrow x$  as  $k \rightarrow \infty$ . Then, we arrive at the target inequality after taking the limit.

## Proof of second-order differential criterion of strong convexity

**Sufficiency:** Using Taylor's formula with the Lagrange remainder and the target inequality, we obtain for  $x + y \in X$ :

$$f(x + y) - f(x) - \langle \nabla f(x), y \rangle = \frac{1}{2} \langle y, \nabla^2 f(x + \alpha y) y \rangle \geq \frac{\mu}{2} \|y\|^2,$$

where  $0 \leq \alpha \leq 1$ . Therefore,

$$f(x + y) - f(x) \geq \langle \nabla f(x), y \rangle + \frac{\mu}{2} \|y\|^2.$$

Consequently, by the first order criterion of strong convexity, the function  $f(x)$  is strongly convex with a constant  $\mu$ . It is important to mention, that  $\mu = 0$  stands for the convex case and corresponding differential criterion.

## Convex and concave function

### Example

Show, that  $f(x) = c^\top x + b$  is convex and concave.

## Simplest strongly convex function

### Example

Show, that  $f(x) = x^\top Ax$ , where  $A \succeq 0$  - is convex on  $\mathbb{R}^n$ . Is it strongly convex?

## Convexity and continuity

Let  $f(x)$  - be a convex function on a convex set  $S \subseteq \mathbb{R}^n$ .  
Then  $f(x)$  is continuous  $\forall x \in \text{ri}(S)$ . <sup>a</sup>

### Proper convex function

Function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be **proper convex function** if it never takes on the value  $-\infty$  and not identically equal to  $\infty$ .

### Indicator function

$$\delta_S(x) = \begin{cases} \infty, & x \in S, \\ 0, & x \notin S, \end{cases}$$

is a proper convex function.

---

<sup>a</sup>Please, read here about difference between interior and relative interior.

## Convexity and continuity

Let  $f(x)$  - be a convex function on a convex set  $S \subseteq \mathbb{R}^n$ .  
Then  $f(x)$  is continuous  $\forall x \in \text{ri}(S)$ . <sup>a</sup>

### Proper convex function

Function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be **proper convex function** if it never takes on the value  $-\infty$  and not identically equal to  $\infty$ .

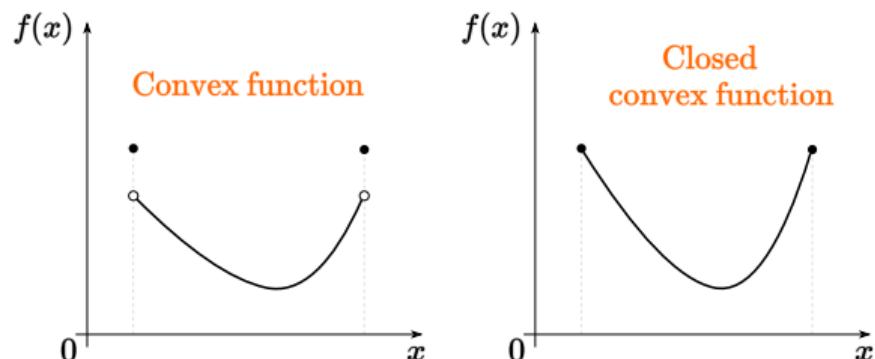
### Indicator function

$$\delta_S(x) = \begin{cases} \infty, & x \in S, \\ 0, & x \notin S, \end{cases}$$

is a proper convex function.

### Closed function

Function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be **closed** if for each  $\alpha \in \mathbb{R}$ , the sublevel set is a closed set.  
Equivalently, if the epigraph is closed, then the function  $f$  is closed.



<sup>a</sup>Please, read here about difference between interior and relative interior.

Figure 15: The concept of a closed function is introduced to avoid such breaches at the border.

## Facts about convexity

- $f(x)$  is called (strictly, strongly) concave, if the function  $-f(x)$  - is (strictly, strongly) convex.

## Facts about convexity

- $f(x)$  is called (strictly, strongly) concave, if the function  $-f(x)$  - is (strictly, strongly) convex.
- Jensen's inequality for the convex functions:

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n \alpha_i f(x_i)$$

for  $\alpha_i \geq 0$ ;  $\sum_{i=1}^n \alpha_i = 1$  (probability simplex)

For the infinite dimension case:

$$f\left(\int_S x p(x) dx\right) \leq \int_S f(x) p(x) dx$$

If the integrals exist and  $p(x) \geq 0$ ,  $\int_S p(x) dx = 1$ .

## Facts about convexity

- $f(x)$  is called (strictly, strongly) concave, if the function  $-f(x)$  - is (strictly, strongly) convex.
- Jensen's inequality for the convex functions:

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n \alpha_i f(x_i)$$

for  $\alpha_i \geq 0$ ;  $\sum_{i=1}^n \alpha_i = 1$  (probability simplex)

For the infinite dimension case:

$$f\left(\int_S x p(x) dx\right) \leq \int_S f(x) p(x) dx$$

If the integrals exist and  $p(x) \geq 0$ ,  $\int_S p(x) dx = 1$ .

- If the function  $f(x)$  and the set  $S$  are convex, then any local minimum  $x^* = \arg \min_{x \in S} f(x)$  will be the global one. Strong convexity guarantees the uniqueness of the solution.

## Operations that preserve convexity

- Non-negative sum of the convex functions:

$$\alpha f(x) + \beta g(x), (\alpha \geq 0, \beta \geq 0).$$

$$f(x) = \max\{f_1(x), f_2(x), f_3(x)\}$$

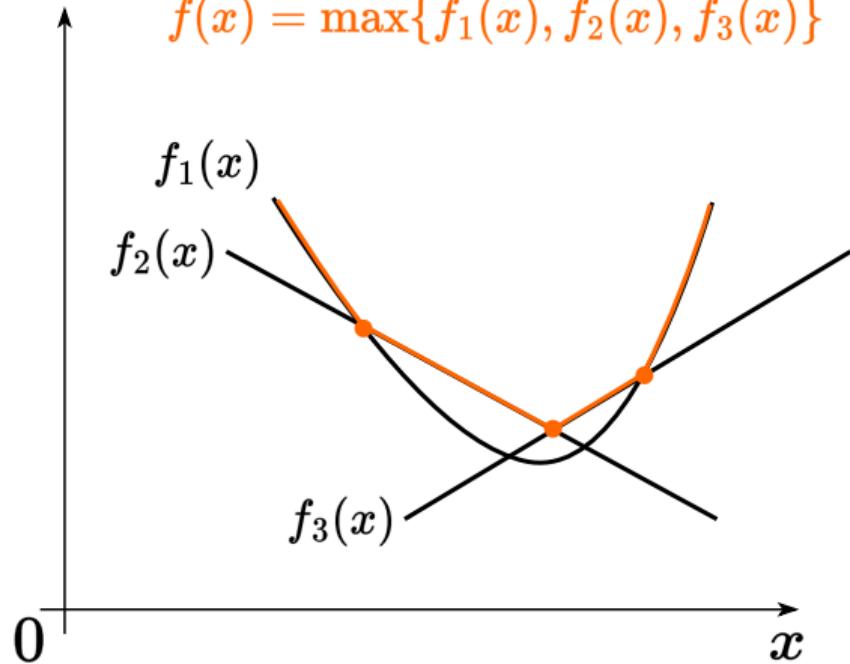


Figure 16: Pointwise maximum (supremum) of convex functions is convex

## Operations that preserve convexity

- Non-negative sum of the convex functions:  
 $\alpha f(x) + \beta g(x)$ , ( $\alpha \geq 0, \beta \geq 0$ ).
- Composition with affine function  $f(Ax + b)$  is convex, if  $f(x)$  is convex.

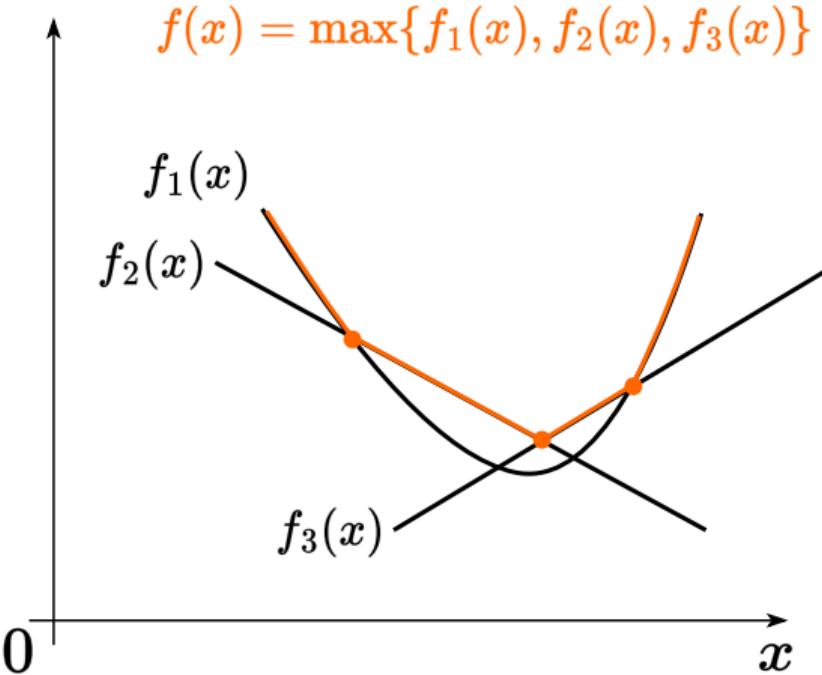


Figure 16: Pointwise maximum (supremum) of convex functions is convex

## Operations that preserve convexity

- Non-negative sum of the convex functions:  
 $\alpha f(x) + \beta g(x)$ , ( $\alpha \geq 0, \beta \geq 0$ ).
- Composition with affine function  $f(Ax + b)$  is convex, if  $f(x)$  is convex.
- Pointwise maximum (supremum) of any number of functions: If  $f_1(x), \dots, f_m(x)$  are convex, then  $f(x) = \max\{f_1(x), \dots, f_m(x)\}$  is convex.

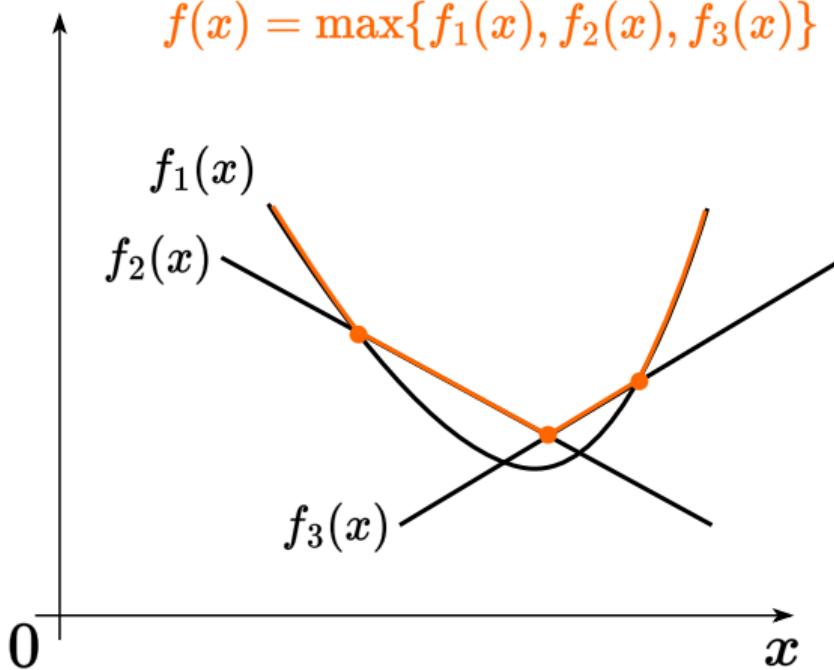


Figure 16: Pointwise maximum (supremum) of convex functions is convex

## Operations that preserve convexity

- Non-negative sum of the convex functions:  
 $\alpha f(x) + \beta g(x)$ , ( $\alpha \geq 0, \beta \geq 0$ ).
- Composition with affine function  $f(Ax + b)$  is convex, if  $f(x)$  is convex.
- Pointwise maximum (supremum) of any number of functions: If  $f_1(x), \dots, f_m(x)$  are convex, then  $f(x) = \max\{f_1(x), \dots, f_m(x)\}$  is convex.
- If  $f(x, y)$  is convex on  $x$  for any  $y \in Y$ :  
 $g(x) = \sup_{y \in Y} f(x, y)$  is convex.

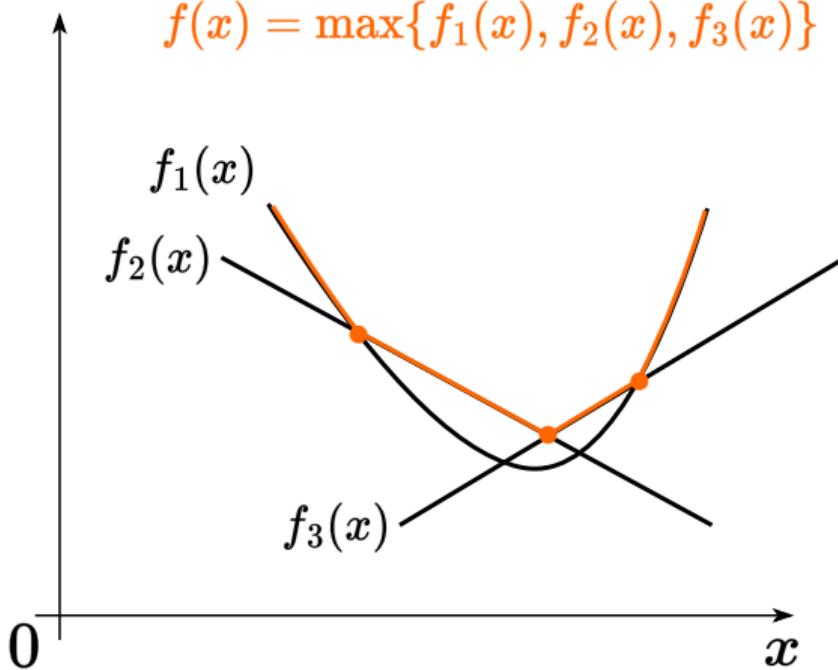


Figure 16: Pointwise maximum (supremum) of convex functions is convex

## Operations that preserve convexity

- Non-negative sum of the convex functions:  
 $\alpha f(x) + \beta g(x), (\alpha \geq 0, \beta \geq 0)$ .
- Composition with affine function  $f(Ax + b)$  is convex, if  $f(x)$  is convex.
- Pointwise maximum (supremum) of any number of functions: If  $f_1(x), \dots, f_m(x)$  are convex, then  $f(x) = \max\{f_1(x), \dots, f_m(x)\}$  is convex.
- If  $f(x, y)$  is convex on  $x$  for any  $y \in Y$ :  
 $g(x) = \sup_{y \in Y} f(x, y)$  is convex.
- If  $f(x)$  is convex on  $S$ , then  $g(x, t) = tf(x/t)$  - is convex with  $x/t \in S, t > 0$ .

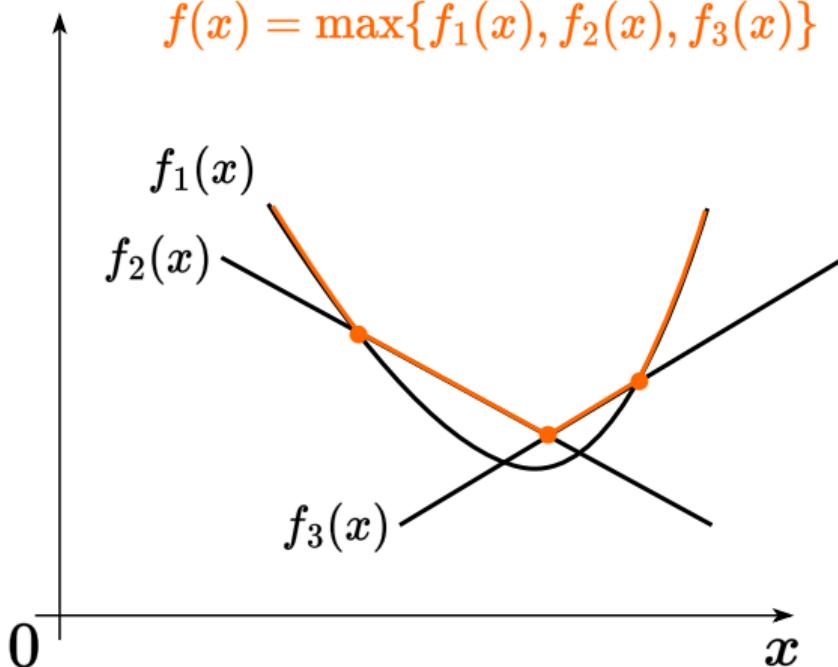


Figure 16: Pointwise maximum (supremum) of convex functions is convex

## Operations that preserve convexity

- Non-negative sum of the convex functions:  
 $\alpha f(x) + \beta g(x), (\alpha \geq 0, \beta \geq 0)$ .
- Composition with affine function  $f(Ax + b)$  is convex, if  $f(x)$  is convex.
- Pointwise maximum (supremum) of any number of functions: If  $f_1(x), \dots, f_m(x)$  are convex, then  $f(x) = \max\{f_1(x), \dots, f_m(x)\}$  is convex.
- If  $f(x, y)$  is convex on  $x$  for any  $y \in Y$ :  
 $g(x) = \sup_{y \in Y} f(x, y)$  is convex.
- If  $f(x)$  is convex on  $S$ , then  $g(x, t) = tf(x/t)$  - is convex with  $x/t \in S, t > 0$ .
- Let  $f_1 : S_1 \rightarrow \mathbb{R}$  and  $f_2 : S_2 \rightarrow \mathbb{R}$ , where  $\text{range}(f_1) \subseteq S_2$ . If  $f_1$  and  $f_2$  are convex, and  $f_2$  is increasing, then  $f_2 \circ f_1$  is convex on  $S_1$ .

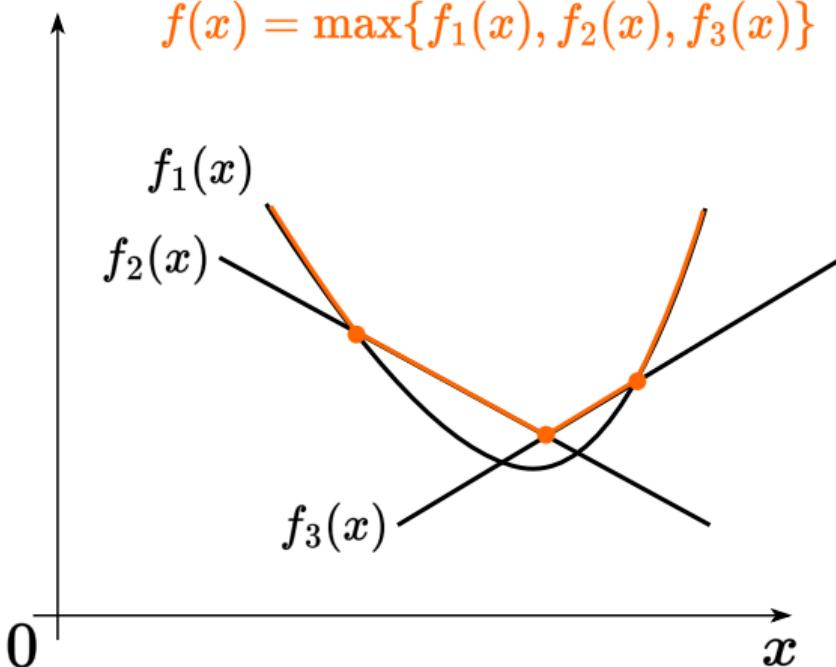


Figure 16: Pointwise maximum (supremum) of convex functions is convex

## Maximum eigenvalue of a matrix is a convex function

### Example

Show, that  $f(A) = \lambda_{max}(A)$  - is convex, if  $A \in S^n$ .

## Other forms of convexity

- Log-convexity:  $\log f$  is convex; Log convexity implies convexity.

## Other forms of convexity

- Log-convexity:  $\log f$  is convex; Log convexity implies convexity.
- Log-concavity:  $\log f$  concave; **not** closed under addition!

## Other forms of convexity

- Log-convexity:  $\log f$  is convex; Log convexity implies convexity.
- Log-concavity:  $\log f$  concave; **not** closed under addition!
- Exponential convexity:  $[f(x_i + x_j)] \succeq 0$ , for  $x_1, \dots, x_n$

## Other forms of convexity

- Log-convexity:  $\log f$  is convex; Log convexity implies convexity.
- Log-concavity:  $\log f$  concave; **not** closed under addition!
- Exponential convexity:  $[f(x_i + x_j)] \succeq 0$ , for  $x_1, \dots, x_n$
- Operator convexity:  $f(\lambda X + (1 - \lambda)Y)$

## Other forms of convexity

- Log-convexity:  $\log f$  is convex; Log convexity implies convexity.
- Log-concavity:  $\log f$  concave; **not** closed under addition!
- Exponential convexity:  $[f(x_i + x_j)] \succeq 0$ , for  $x_1, \dots, x_n$
- Operator convexity:  $f(\lambda X + (1 - \lambda)Y)$
- Quasiconvexity:  $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$

## Other forms of convexity

- Log-convexity:  $\log f$  is convex; Log convexity implies convexity.
- Log-concavity:  $\log f$  concave; **not** closed under addition!
- Exponential convexity:  $[f(x_i + x_j)] \succeq 0$ , for  $x_1, \dots, x_n$
- Operator convexity:  $f(\lambda X + (1 - \lambda)Y)$
- Quasiconvexity:  $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$
- Pseudoconvexity:  $\langle \nabla f(y), x - y \rangle \geq 0 \longrightarrow f(x) \geq f(y)$

## Other forms of convexity

- Log-convexity:  $\log f$  is convex; Log convexity implies convexity.
- Log-concavity:  $\log f$  concave; **not** closed under addition!
- Exponential convexity:  $[f(x_i + x_j)] \succeq 0$ , for  $x_1, \dots, x_n$
- Operator convexity:  $f(\lambda X + (1 - \lambda)Y)$
- Quasiconvexity:  $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$
- Pseudoconvexity:  $\langle \nabla f(y), x - y \rangle \geq 0 \longrightarrow f(x) \geq f(y)$
- Discrete convexity:  $f : \mathbb{Z}^n \rightarrow \mathbb{Z}$ ; "convexity + matroid theory."

## Polyak- Lojasiewicz condition. Linear convergence of gradient descent without convexity

PL inequality holds if the following condition is satisfied for some  $\mu > 0$ ,

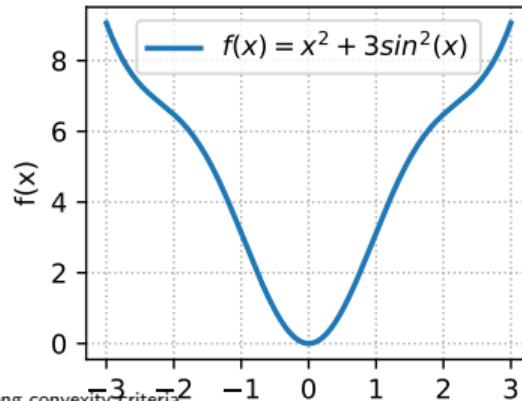
$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*) \forall x$$

It is interesting, that Gradient Descent algorithm has linear convergence under this condition (you do not even need convexity here).

The following functions satisfy the PL-condition, but are not convex.  [Link to the code](#)

$$f(x) = x^2 + 3 \sin^2(x)$$

Function, that satisfies  
Polyak- Lojasiewicz condition



# Polyak- Lojasiewicz condition. Linear convergence of gradient descent without convexity

PL inequality holds if the following condition is satisfied for some  $\mu > 0$ ,

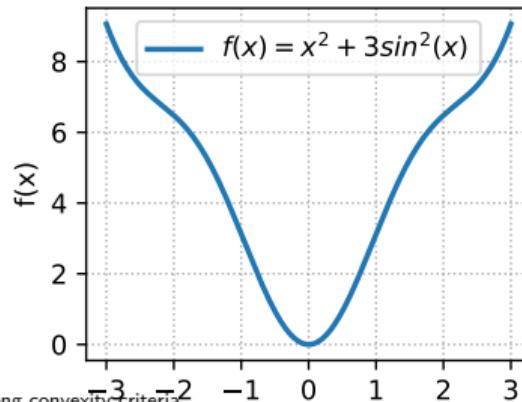
$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*) \forall x$$

It is interesting, that Gradient Descent algorithm has linear convergence under this condition (you do not even need convexity here).

The following functions satisfy the PL-condition, but are not convex.  [Link to the code](#)

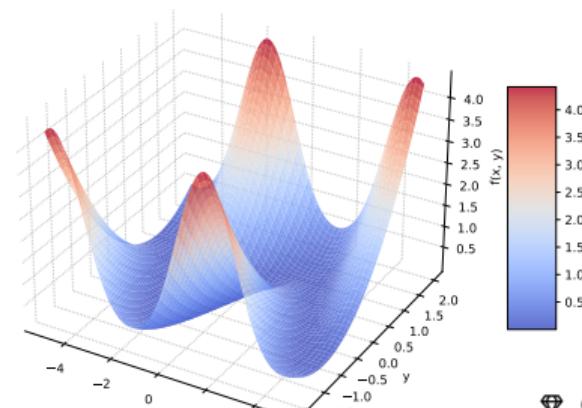
$$f(x) = x^2 + 3 \sin^2(x)$$

Function, that satisfies  
Polyak- Lojasiewicz condition



$$f(x, y) = \frac{(y - \sin x)^2}{2}$$

Non-convex PL function



## Convexity in ML

## Convex optimization problem

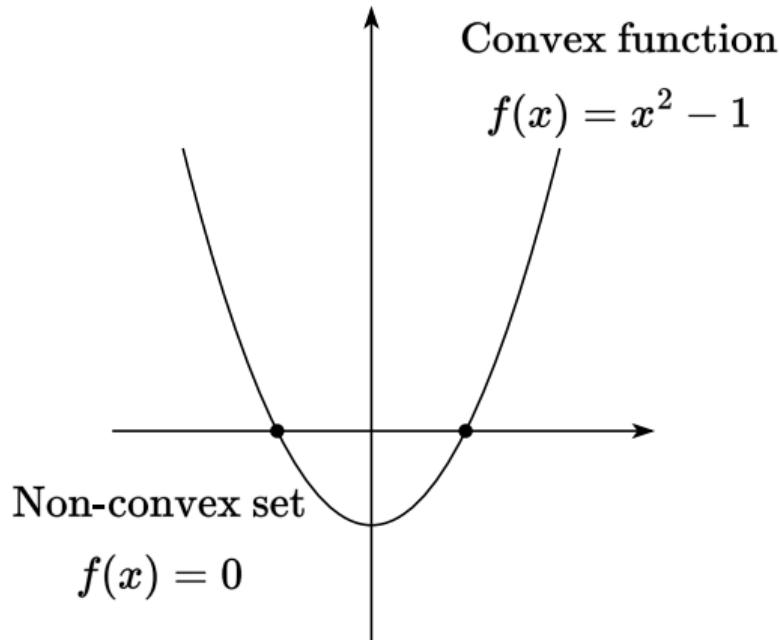


Figure 19: The idea behind the definition of a convex optimization problem

Note, that there is an agreement in notation of mathematical programming. The problems of the following type are called **Convex optimization problem**:

$$f_0(x) \rightarrow \min_{x \in \mathbb{R}^n}$$

$$\begin{aligned} \text{s.t. } f_i(x) \leq 0, & i = 1, \dots, m \\ Ax = b, \end{aligned} \quad (\text{COP})$$

where all the functions  $f_0(x), f_1(x), \dots, f_m(x)$  are convex and all the equality constraints are affine. It sounds a bit strange, but not all convex problems are convex optimization problems.

$$f_0(x) \rightarrow \min_{x \in S}, \quad (\text{CP})$$

where  $f_0(x)$  is a convex function, defined on the convex set  $S$ . The necessity of affine equality constraint is essential.

# Linear Least Squares aka Linear Regression

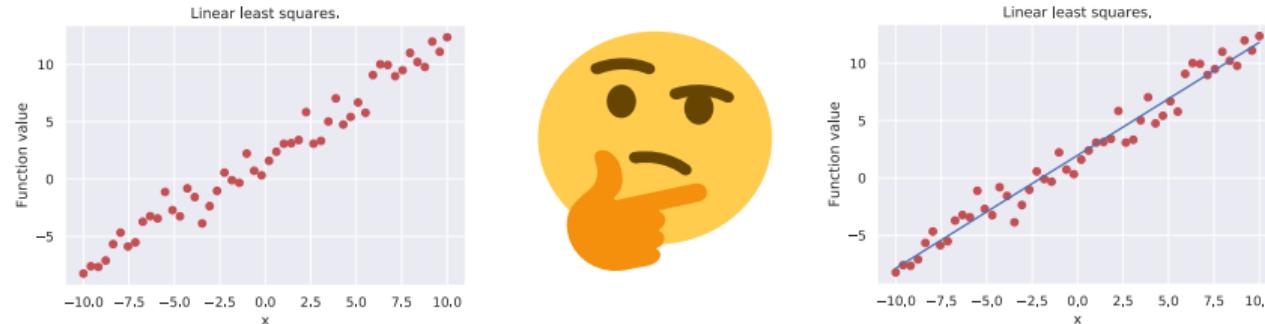


Figure 20: Illustration

In a least-squares, or linear regression, problem, we have measurements  $X \in \mathbb{R}^{m \times n}$  and  $y \in \mathbb{R}^m$  and seek a vector  $\theta \in \mathbb{R}^n$  such that  $X\theta$  is close to  $y$ . Closeness is defined as the sum of the squared differences:

$$\sum_{i=1}^m (x_i^\top \theta - y_i)^2 = \|X\theta - y\|_2^2 \rightarrow \min_{\theta \in \mathbb{R}^n}$$

For example, we might have a dataset of  $m$  users, each represented by  $n$  features. Each row  $x_i^\top$  of  $X$  is the features for user  $i$ , while the corresponding entry  $y_i$  of  $y$  is the measurement we want to predict from  $x_i^\top$ , such as ad spending. The prediction is given by  $x_i^\top \theta$ .

## Linear Least Squares aka Linear Regression <sup>1</sup>

1. Is this problem convex? Strongly convex?

## Linear Least Squares aka Linear Regression <sup>1</sup>

1. Is this problem convex? Strongly convex?
2. What do you think about convergence of Gradient Descent for this problem?

---

<sup>1</sup>Take a look at the  example of real-world data linear least squares problem

## $l_2$ -regularized Linear Least Squares

In the underdetermined case, it is often desirable to restore strong convexity of the objective function by adding an  $l_2$ -penalty, also known as Tikhonov regularization,  $l_2$ -regularization, or weight decay.

$$\|X\theta - y\|_2^2 + \frac{\mu}{2} \|\theta\|_2^2 \rightarrow \min_{\theta \in \mathbb{R}^n}$$

Note: With this modification the objective is  $\mu$ -strongly convex again.

Take a look at the  code

## Neural networks?

## Visualizing loss surface of neural network via line projection

We denote the initial point as  $w_0$ , representing the weights of the neural network at initialization. The weights after training are denoted as  $\hat{w}$ .

Initially, we generate a random Gaussian direction  $w_1 \in \mathbb{R}^p$ , which inherits the magnitude of the original neural network weights for each parameter group. Subsequently, we sample the training and testing loss surfaces at points along the direction  $w_1$ , situated close to either  $w_0$  or  $\hat{w}$ .

Mathematically, this involves evaluating:

$$L(\alpha) = L(w_0 + \alpha w_1), \text{ where } \alpha \in [-b, b].$$

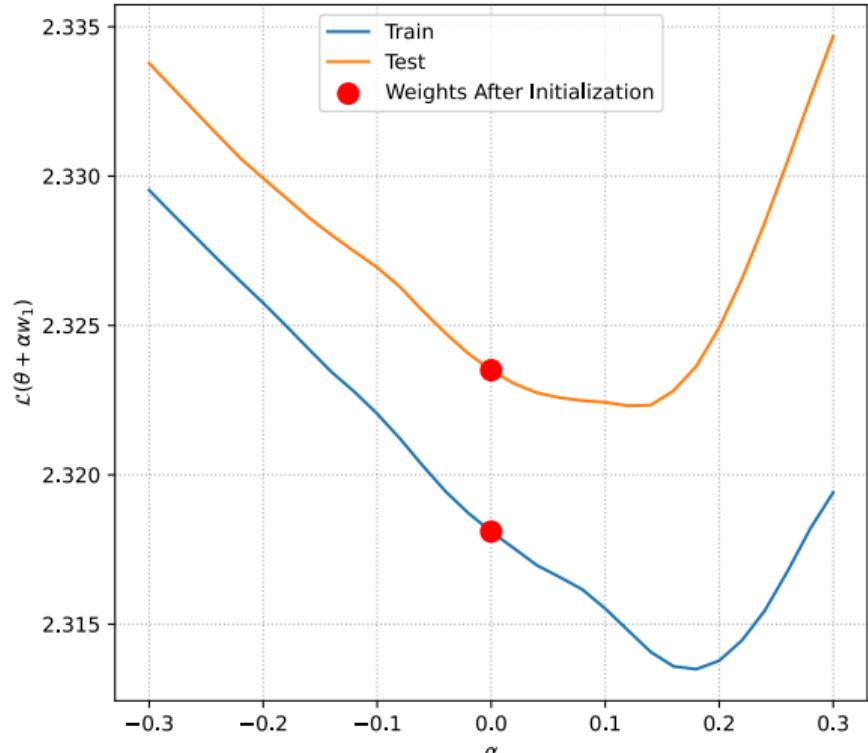
Here,  $\alpha$  plays the role of a coordinate along the  $w_1$  direction, and  $b$  stands for the bounds of interpolation. Visualizing  $L(\alpha)$  enables us to project the  $p$ -dimensional surface onto a one-dimensional axis.

It is important to note that the characteristics of the resulting graph heavily rely on the chosen projection direction. It's not feasible to maintain the entirety of the information when transforming a space with 100,000 dimensions into a one-dimensional line through projection. However, certain properties can still be established. For instance, if  $L(\alpha)|_{\alpha=0}$  is decreasing, this indicates that the point lies on a slope. Additionally, if the projection is non-convex, it implies that the original surface was not convex.

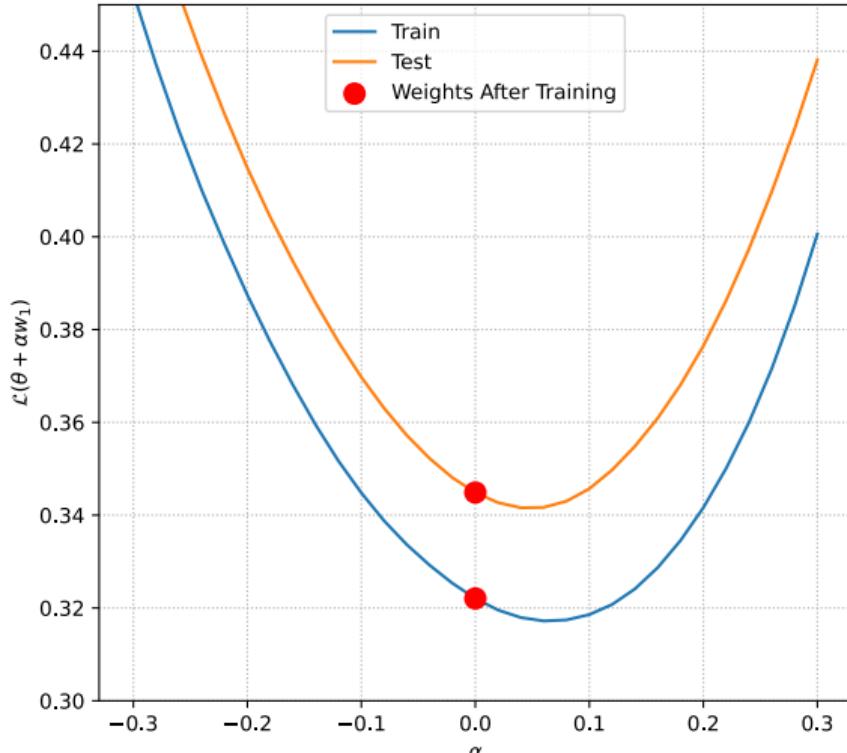
# Visualizing loss surface of neural network

No Dropout

Loss surface. Line projection around the starting point



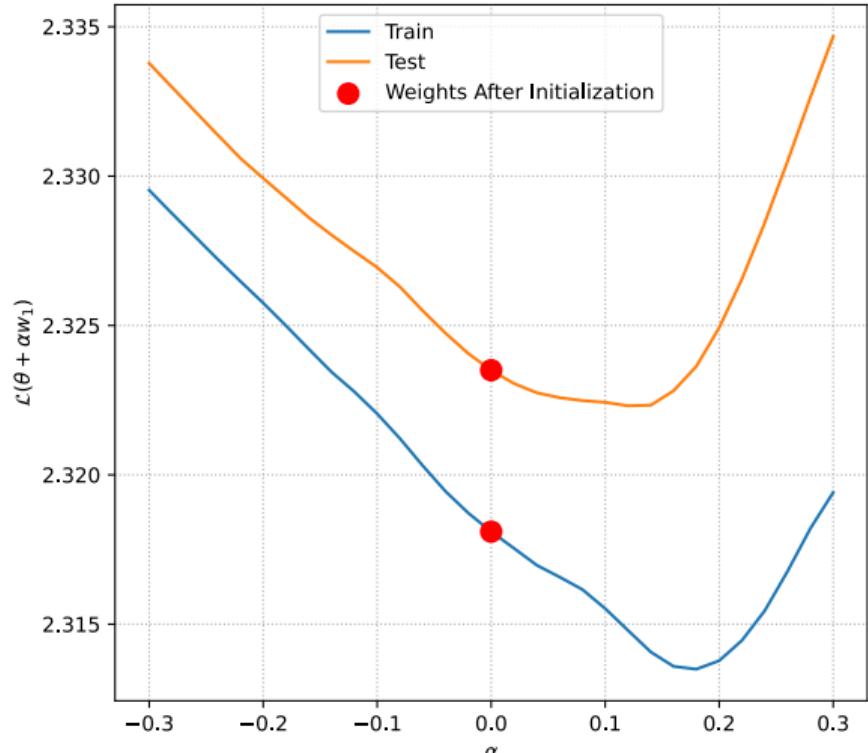
Loss surface. Line projection around the final point



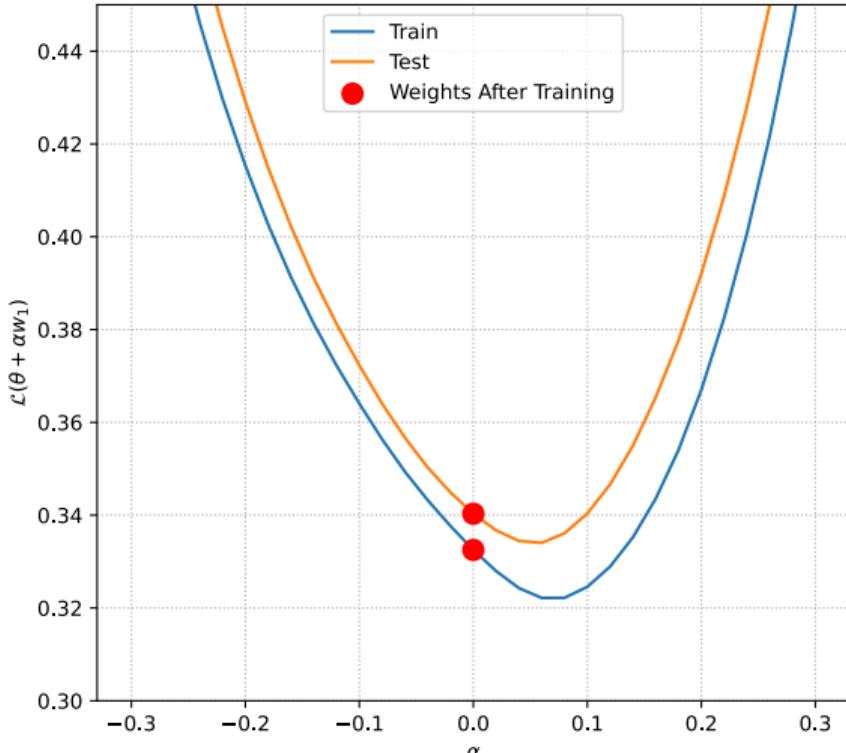
# Visualizing loss surface of neural network

Dropout 0.2

Loss surface. Line projection around the starting point



Loss surface. Line projection around the final point



## Plane projection

We can explore this idea further and draw the projection of the loss surface to the plane, which is defined by 2 random vectors. Note, that with 2 random gaussian vectors in the huge dimensional space are almost certainly orthogonal. So, as previously, we generate random normalized gaussian vectors  $w_1, w_2 \in \mathbb{R}^p$  and evaluate the loss function

$$L(\alpha, \beta) = L(w_0 + \alpha w_1 + \beta w_2), \text{ where } \alpha, \beta \in [-b, b]^2.$$

No Dropout. Plane projection of loss surface.

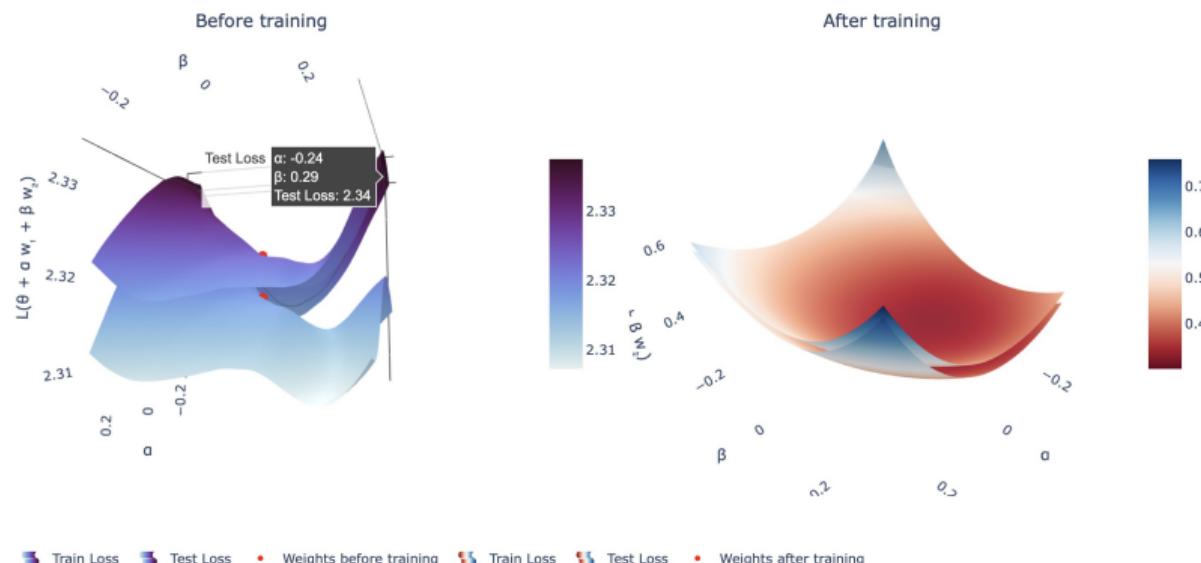


Figure 23: [Open in colab](#)

## Can plane projections be useful? <sup>2</sup>

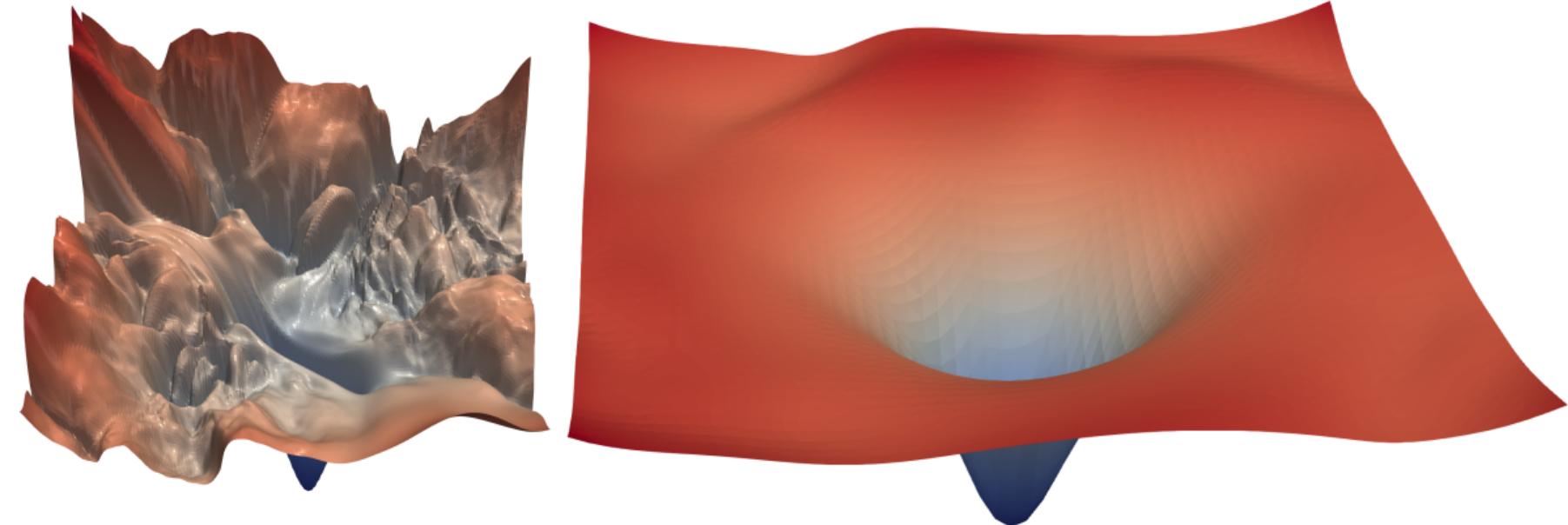


Figure 24: The loss surface of ResNet-56  
without skip connections

Figure 25: The loss surface of ResNet-56 with skip connections

<sup>2</sup>Visualizing the Loss Landscape of Neural Nets, Hao Li, Zheng Xu, Gavin Taylor, Christoph Studer, Tom Goldstein

# Can plane projections be useful, really? <sup>3</sup>

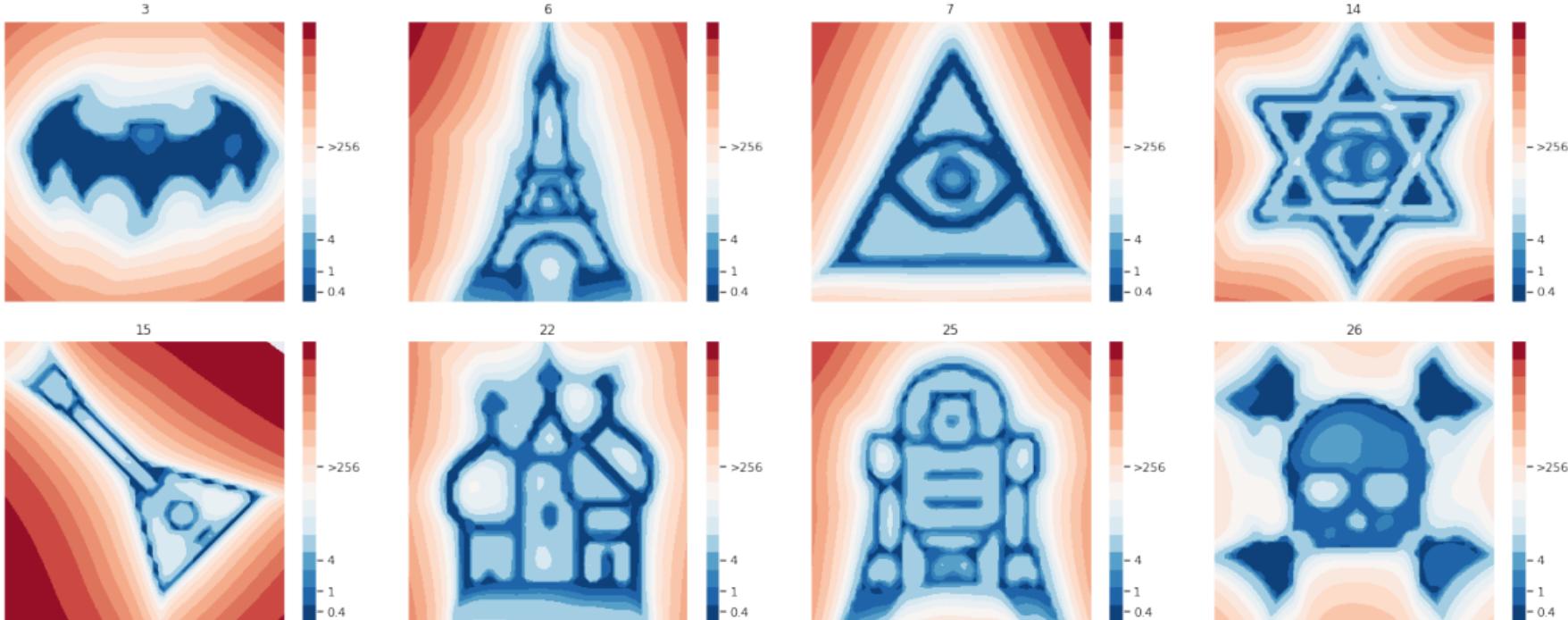


Figure 26: Examples of a loss landscape of a typical CNN model on FashionMNIST and CIFAR10 datasets found with MPO. Loss values are color-coded according to a logarithmic scale

<sup>3</sup>Loss Landscape Sightseeing with Multi-Point Optimization, Ivan Skorokhodov, Mikhail Burtsev