A dramatic, high-energy scene from a video game or movie. A Viking warrior with a horned helmet and a long beard is engaged in combat with a massive, multi-headed dragon. The dragon's heads are glowing with intense blue and orange light, and its body is covered in sharp, spiky scales. The warrior is holding a large, ornate sword and a shield, both of which are glowing with a bright white light. The background is filled with smoke and fire, creating a sense of chaos and destruction.

# Gradient Descent

Daniil Merkulov

Optimization methods. MIPT

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The result of this method is

$$x_{k+1} = x_k - \alpha f'(x_k)$$

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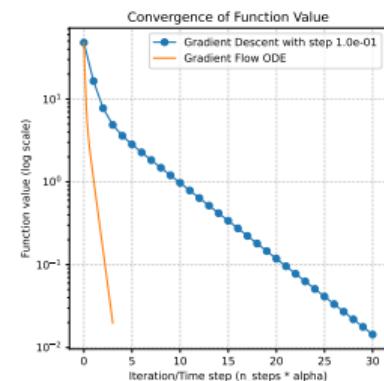
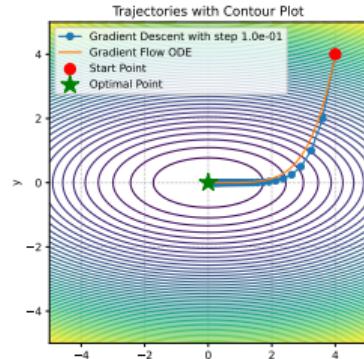


Figure 1: Gradient flow trajectory

## Necessary local minimum condition

$$\begin{aligned}f'(x) &= 0 \\-\eta f'(x) &= 0 \\x - \eta f'(x) &= x \\x_k - \eta f'(x_k) &= x_{k+1}\end{aligned}$$

## Minimizer of Lipschitz parabola

If a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable and its gradient satisfies Lipschitz conditions with constant  $L$ , then  $\forall x, y \in \mathbb{R}^n$ :

$$|f(y) - f(x) - \langle \nabla f(x), y - x \rangle| \leq \frac{L}{2} \|y - x\|^2,$$

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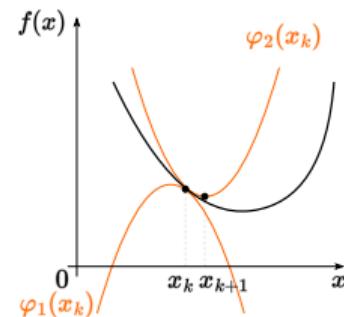


Figure 2: Illustration

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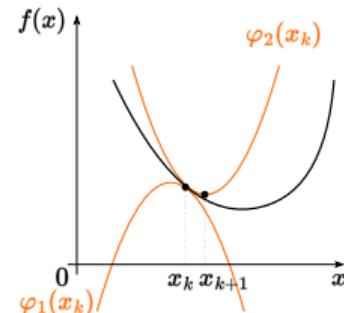


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$$\nabla \phi_2(x) = 0$$

$$\nabla f(x_0) + L(x^* - x_0) = 0$$

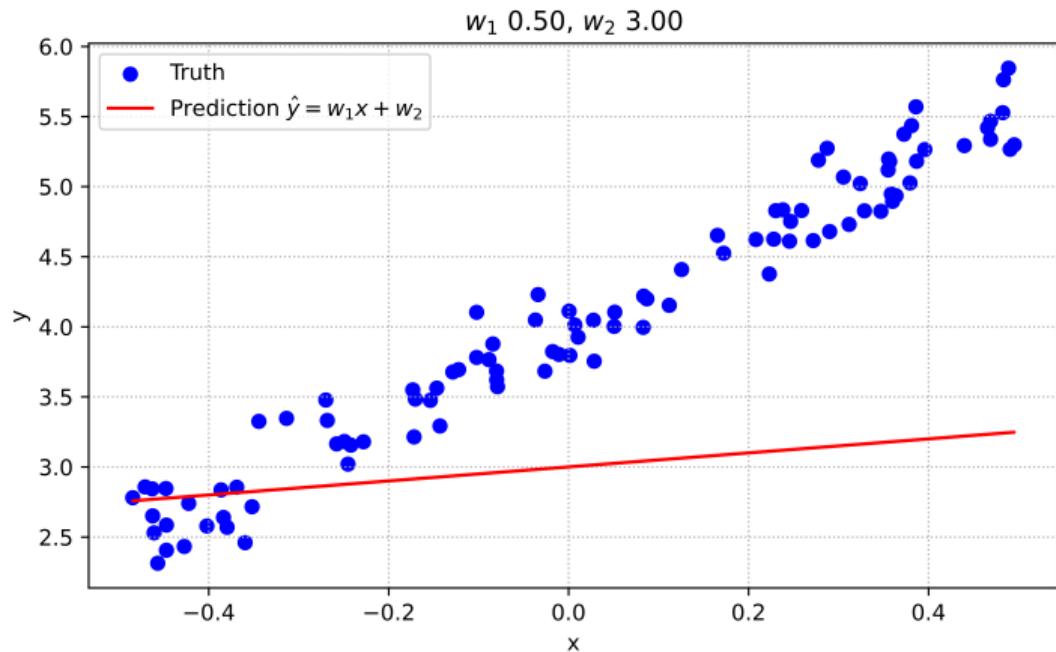
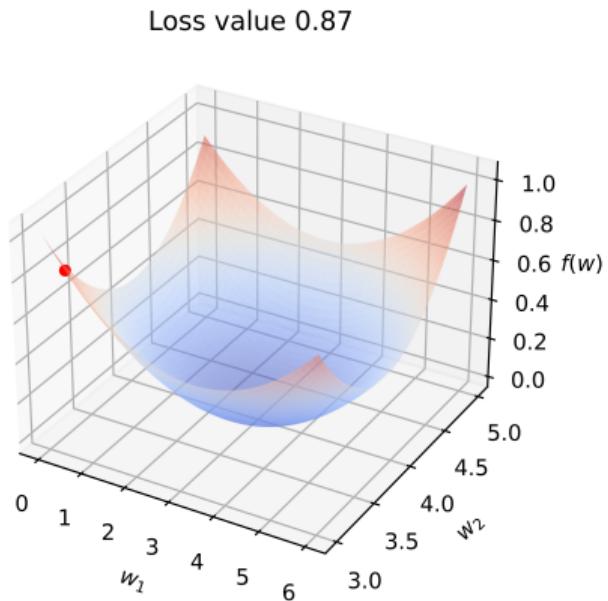
$$x^* = x_0 - \frac{1}{L} \nabla f(x_0)$$

$$x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k)$$

This way leads to the  $\frac{1}{L}$  stepsize choosing. However, often the  $L$  constant is not known.

# Convergence of Gradient Descent algorithm

Heavily depends on the choice of the learning rate  $\alpha$ :



## Exact line search aka steepest descent

$$\alpha_k = \arg \min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \arg \min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. Interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

$$\alpha_k = \arg \min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

Optimality conditions:

$$\nabla f(x_{k+1})^\top \nabla f(x_k) = 0$$

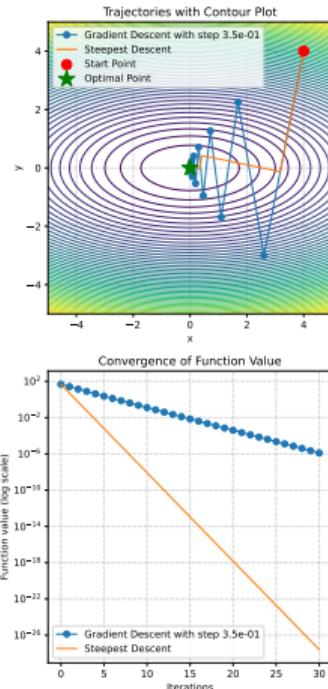


Figure 3: Steepest Descent

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## Convergence rates

$$\min_{x \in \mathbb{R}^n} f(x) \quad x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

smooth	convex	smooth & convex	smooth & strongly convex (or PL)
$\ \nabla f(x_k)\ ^2 \approx \mathcal{O}\left(\frac{1}{k}\right)$	$f(x_k) - f^* \approx \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$	$f(x_k) - f^* \approx \mathcal{O}\left(\frac{1}{k}\right)$	$\ x_k - x^*\ ^2 \approx \mathcal{O}\left(\left(1 - \frac{\mu}{L}\right)^k\right)$

## Gradient Descent convergence. Smooth convex case

## Gradient Descent convergence. Smooth $\mu$ -strongly convex case

## Gradient Descent convergence. Polyak-Lojasiewicz case