



# Markowitz Portfolio Optimization. Optimality Conditions. KKT theorem.

Daniil Merkulov

Applied Math for Data Science. Sberuniversity.

## Portfolio optimization

# Portfolio optimization

Link to the code

## Optimality conditions

## Background

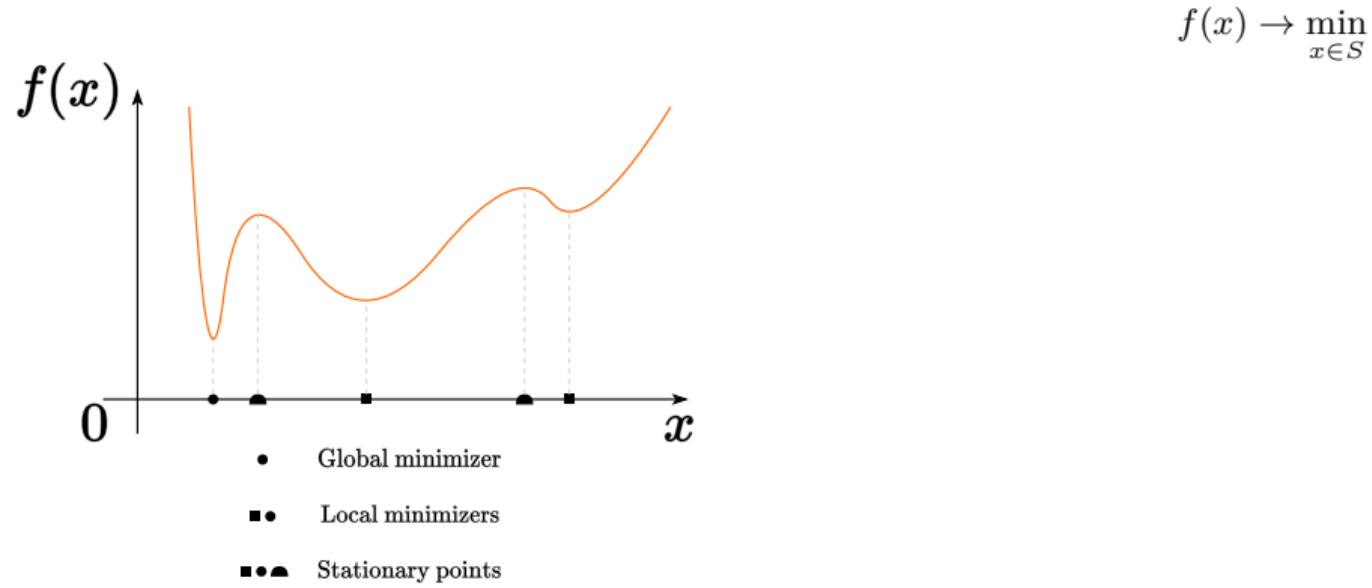


Figure 1: Illustration of different stationary (critical) points

## Background

$$f(x) \rightarrow \min_{x \in S}$$

A set  $S$  is usually called a **budget set**.

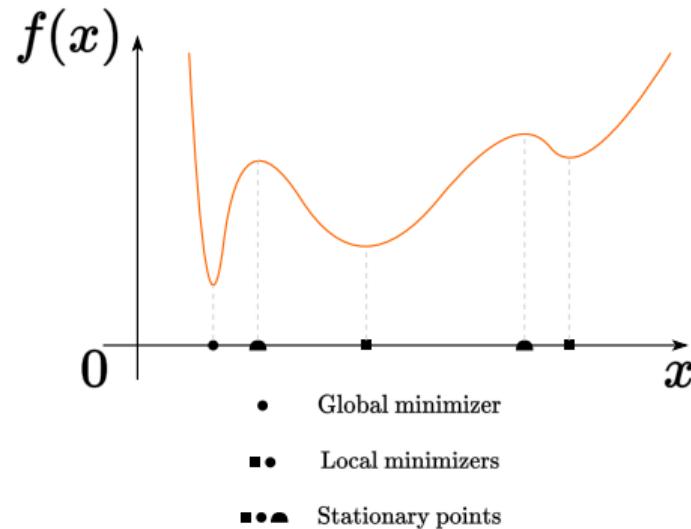


Figure 1: Illustration of different stationary (critical) points

## Background

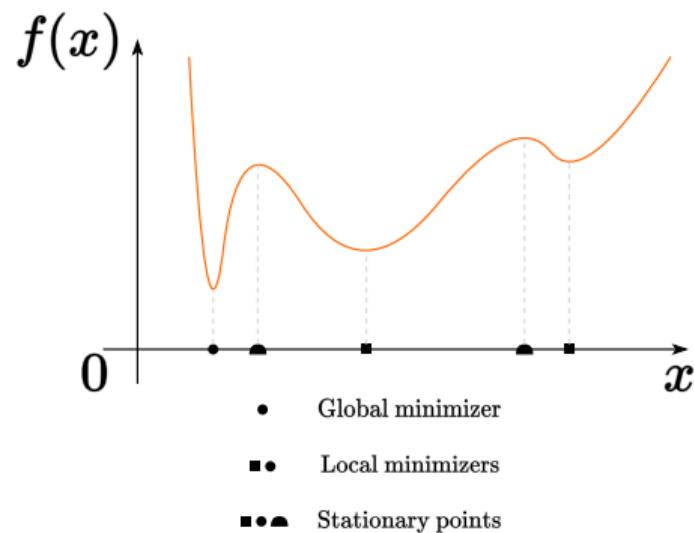


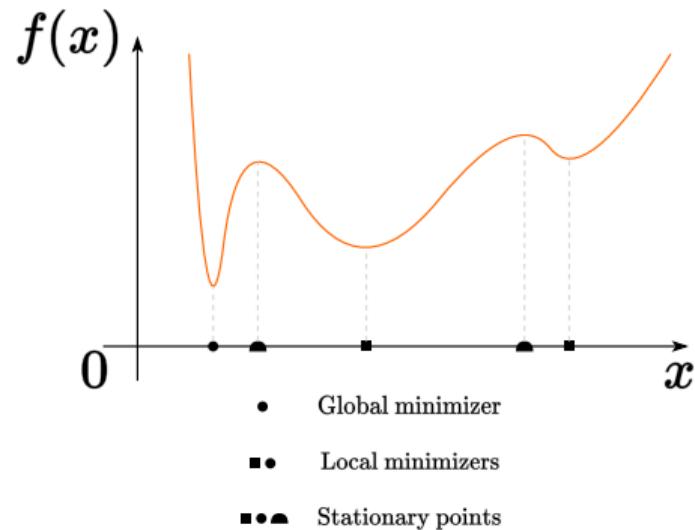
Figure 1: Illustration of different stationary (critical) points

$$f(x) \rightarrow \min_{x \in S}$$

A set  $S$  is usually called a **budget set**.

We say that the problem has a solution if the budget set is **not empty**:  $x^* \in S$ , in which the minimum or the infimum of the given function is achieved.

## Background



$$f(x) \rightarrow \min_{x \in S}$$

A set  $S$  is usually called a **budget set**.

We say that the problem has a solution if the budget set is not empty:  $x^* \in S$ , in which the minimum or the infimum of the given function is achieved.

- A point  $x^*$  is a **global minimizer** if  $f(x^*) \leq f(x)$  for all  $x$ .

Figure 1: Illustration of different stationary (critical) points

## Background

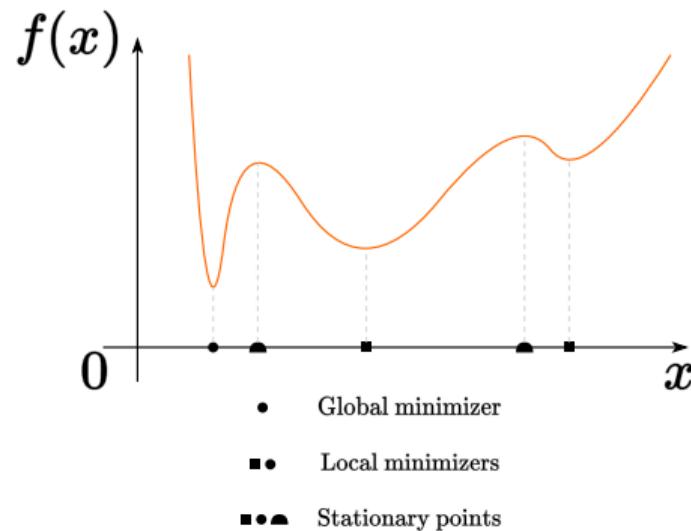


Figure 1: Illustration of different stationary (critical) points

$$f(x) \rightarrow \min_{x \in S}$$

A set  $S$  is usually called a **budget set**.

We say that the problem has a solution if the budget set is **not empty**:  $x^* \in S$ , in which the minimum or the infimum of the given function is achieved.

- A point  $x^*$  is a **global minimizer** if  $f(x^*) \leq f(x)$  for all  $x$ .
- A point  $x^*$  is a **local minimizer** if there exists a neighborhood  $N$  of  $x^*$  such that  $f(x^*) \leq f(x)$  for all  $x \in N$ .

## Background

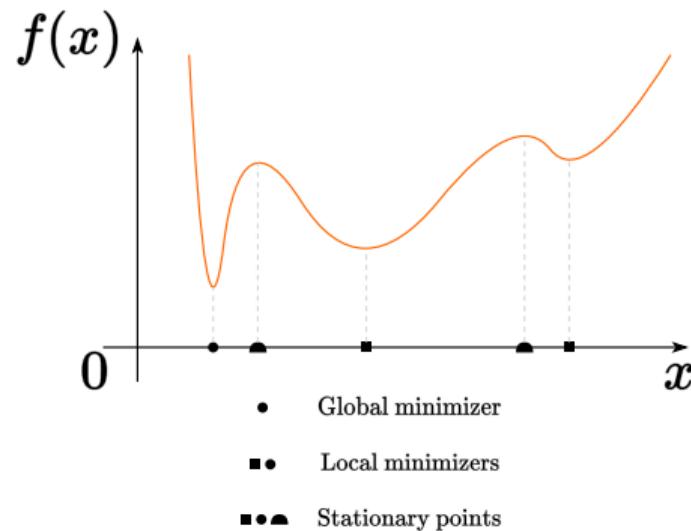


Figure 1: Illustration of different stationary (critical) points

$$f(x) \rightarrow \min_{x \in S}$$

A set  $S$  is usually called a **budget set**.

We say that the problem has a solution if the budget set is **not empty**:  $x^* \in S$ , in which the minimum or the infimum of the given function is achieved.

- A point  $x^*$  is a **global minimizer** if  $f(x^*) \leq f(x)$  for all  $x$ .
- A point  $x^*$  is a **local minimizer** if there exists a neighborhood  $N$  of  $x^*$  such that  $f(x^*) \leq f(x)$  for all  $x \in N$ .
- A point  $x^*$  is a **strict local minimizer** (also called a **strong local minimizer**) if there exists a neighborhood  $N$  of  $x^*$  such that  $f(x^*) < f(x)$  for all  $x \in N$  with  $x \neq x^*$ .

# Background

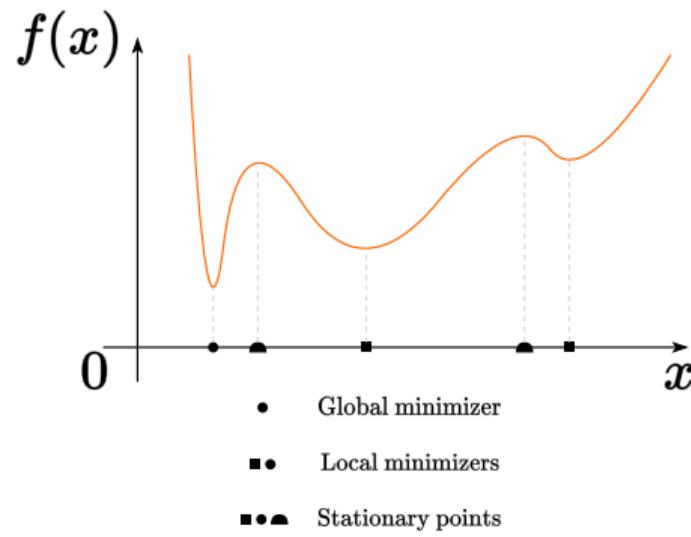


Figure 1: Illustration of different stationary (critical) points

$$f(x) \rightarrow \min_{x \in S}$$

A set  $S$  is usually called a **budget set**.

We say that the problem has a solution if the budget set is **not empty**:  $x^* \in S$ , in which the minimum or the infimum of the given function is achieved.

- A point  $x^*$  is a **global minimizer** if  $f(x^*) \leq f(x)$  for all  $x$ .
- A point  $x^*$  is a **local minimizer** if there exists a neighborhood  $N$  of  $x^*$  such that  $f(x^*) \leq f(x)$  for all  $x \in N$ .
- A point  $x^*$  is a **strict local minimizer** (also called a **strong local minimizer**) if there exists a neighborhood  $N$  of  $x^*$  such that  $f(x^*) < f(x)$  for all  $x \in N$  with  $x \neq x^*$ .
- We call  $x^*$  a **stationary point** (or critical) if  $\nabla f(x^*) = 0$ . Any local minimizer of a differentiable function must be a stationary point.

# Extreme value (Weierstrass) theorem

## i Theorem

Let  $S \subset \mathbb{R}^n$  be a compact set and  $f(x)$  a continuous function on  $S$ . So, the point of the global minimum of the function  $f(x)$  on  $S$  exists.

# Extreme value (Weierstrass) theorem

## i Theorem

Let  $S \subset \mathbb{R}^n$  be a compact set and  $f(x)$  a continuous function on  $S$ . So, the point of the global minimum of the function  $f(x)$  on  $S$  exists.

GOOD NEWS EVERYONE!



Figure 2: A lot of practical problems are theoretically solvable

## Extreme value (Weierstrass) theorem

### i Theorem

Let  $S \subset \mathbb{R}^n$  be a compact set and  $f(x)$  a continuous function on  $S$ . So, the point of the global minimum of the function  $f(x)$  on  $S$  exists.

GOOD NEWS EVERYONE!



### i Taylor's Theorem

Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable and that  $p \in \mathbb{R}^n$ . Then we have:

$$f(x + p) = f(x) + \nabla f(x + tp)^T p \quad \text{for some } t \in (0, 1)$$

Figure 2: A lot of practical problems are theoretically solvable

# Extreme value (Weierstrass) theorem

## i Theorem

Let  $S \subset \mathbb{R}^n$  be a compact set and  $f(x)$  a continuous function on  $S$ . So, the point of the global minimum of the function  $f(x)$  on  $S$  exists.



Figure 2: A lot of practical problems are theoretically solvable

## i Taylor's Theorem

Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable and that  $p \in \mathbb{R}^n$ . Then we have:

$$f(x + p) = f(x) + \nabla f(x + tp)^T p \quad \text{for some } t \in (0, 1)$$

Moreover, if  $f$  is twice continuously differentiable, we have:

$$\nabla f(x + p) = \nabla f(x) + \int_0^1 \nabla^2 f(x + tp)p dt$$

$$f(x + p) = f(x) + \nabla f(x)^T p + \frac{1}{2}p^T \nabla^2 f(x + tp)p$$

for some  $t \in (0, 1)$ .

## Unconstrained optimization

## Necessary Conditions

### i First-Order Necessary Conditions

If  $x^*$  is a local minimizer and  $f$  is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0$$

## Necessary Conditions

### i First-Order Necessary Conditions

If  $x^*$  is a local minimizer and  $f$  is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0$$

#### Proof

Suppose for contradiction that  $\nabla f(x^*) \neq 0$ . Define the vector  $p = -\nabla f(x^*)$  and note that

$$p^T \nabla f(x^*) = -\|\nabla f(x^*)\|^2 < 0$$

## Necessary Conditions

### i First-Order Necessary Conditions

If  $x^*$  is a local minimizer and  $f$  is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0$$

#### Proof

Suppose for contradiction that  $\nabla f(x^*) \neq 0$ . Define the vector  $p = -\nabla f(x^*)$  and note that

$$p^T \nabla f(x^*) = -\|\nabla f(x^*)\|^2 < 0$$

Because  $\nabla f$  is continuous near  $x^*$ , there is a scalar  $T > 0$  such that

$$p^T \nabla f(x^* + tp) < 0, \text{ for all } t \in [0, T]$$

## Necessary Conditions

### i First-Order Necessary Conditions

If  $x^*$  is a local minimizer and  $f$  is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0$$

#### Proof

Suppose for contradiction that  $\nabla f(x^*) \neq 0$ . Define the vector  $p = -\nabla f(x^*)$  and note that

$$p^T \nabla f(x^*) = -\|\nabla f(x^*)\|^2 < 0$$

Because  $\nabla f$  is continuous near  $x^*$ , there is a scalar  $T > 0$  such that

$$p^T \nabla f(x^* + tp) < 0, \text{ for all } t \in [0, T]$$

For any  $\bar{t} \in (0, T]$ , we have by Taylor's theorem that

$$f(x^* + \bar{t}p) = f(x^*) + \bar{t}p^T \nabla f(x^* + tp), \text{ for some } t \in (0, \bar{t})$$

# Necessary Conditions

## i First-Order Necessary Conditions

If  $x^*$  is a local minimizer and  $f$  is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0$$

### Proof

Suppose for contradiction that  $\nabla f(x^*) \neq 0$ . Define the vector  $p = -\nabla f(x^*)$  and note that

$$p^T \nabla f(x^*) = -\|\nabla f(x^*)\|^2 < 0$$

Because  $\nabla f$  is continuous near  $x^*$ , there is a scalar  $T > 0$  such that

$$p^T \nabla f(x^* + tp) < 0, \text{ for all } t \in [0, T]$$

For any  $\bar{t} \in (0, T]$ , we have by Taylor's theorem that

$$f(x^* + \bar{t}p) = f(x^*) + \bar{t}p^T \nabla f(x^* + tp), \text{ for some } t \in (0, \bar{t})$$

Therefore,  $f(x^* + \bar{t}p) < f(x^*)$  for all  $\bar{t} \in (0, T]$ . We have found a direction from  $x^*$  along which  $f$  decreases, so  $x^*$  is not a local minimizer, leading to a contradiction.

## Sufficient Conditions

### i Second-Order Sufficient Conditions

Suppose that  $\nabla^2 f$  is continuous in an open neighborhood of  $x^*$  and that

$$\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0.$$

Then  $x^*$  is a strict local minimizer of  $f$ .

## Sufficient Conditions

### i Second-Order Sufficient Conditions

Suppose that  $\nabla^2 f$  is continuous in an open neighborhood of  $x^*$  and that

$$\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0.$$

Then  $x^*$  is a strict local minimizer of  $f$ .

#### Proof

Because the Hessian is continuous and positive definite at  $x^*$ , we can choose a radius  $r > 0$  such that  $\nabla^2 f(x)$  remains positive definite for all  $x$  in the open ball  $B = \{z \mid \|z - x^*\| < r\}$ . Taking any nonzero vector  $p$  with  $\|p\| < r$ , we have  $x^* + p \in B$  and so

## Sufficient Conditions

### i Second-Order Sufficient Conditions

Suppose that  $\nabla^2 f$  is continuous in an open neighborhood of  $x^*$  and that

$$\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0.$$

Then  $x^*$  is a strict local minimizer of  $f$ .

#### Proof

Because the Hessian is continuous and positive definite at  $x^*$ , we can choose a radius  $r > 0$  such that  $\nabla^2 f(x)$  remains positive definite for all  $x$  in the open ball  $B = \{z \mid \|z - x^*\| < r\}$ . Taking any nonzero vector  $p$  with  $\|p\| < r$ , we have  $x^* + p \in B$  and so

$$f(x^* + p) = f(x^*) + p^T \nabla f(x^*) + \frac{1}{2} p^T \nabla^2 f(z) p$$

## Sufficient Conditions

### i Second-Order Sufficient Conditions

Suppose that  $\nabla^2 f$  is continuous in an open neighborhood of  $x^*$  and that

$$\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0.$$

Then  $x^*$  is a strict local minimizer of  $f$ .

#### Proof

Because the Hessian is continuous and positive definite at  $x^*$ , we can choose a radius  $r > 0$  such that  $\nabla^2 f(x)$  remains positive definite for all  $x$  in the open ball  $B = \{z \mid \|z - x^*\| < r\}$ . Taking any nonzero vector  $p$  with  $\|p\| < r$ , we have  $x^* + p \in B$  and so

$$\begin{aligned} f(x^* + p) &= f(x^*) + p^T \nabla f(x^*) + \frac{1}{2} p^T \nabla^2 f(z) p \\ &= f(x^*) + \frac{1}{2} p^T \nabla^2 f(z) p \end{aligned}$$

## Sufficient Conditions

### i Second-Order Sufficient Conditions

Suppose that  $\nabla^2 f$  is continuous in an open neighborhood of  $x^*$  and that

$$\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0.$$

Then  $x^*$  is a strict local minimizer of  $f$ .

#### Proof

Because the Hessian is continuous and positive definite at  $x^*$ , we can choose a radius  $r > 0$  such that  $\nabla^2 f(x)$  remains positive definite for all  $x$  in the open ball  $B = \{z \mid \|z - x^*\| < r\}$ . Taking any nonzero vector  $p$  with  $\|p\| < r$ , we have  $x^* + p \in B$  and so

$$\begin{aligned} f(x^* + p) &= f(x^*) + p^T \nabla f(x^*) + \frac{1}{2} p^T \nabla^2 f(z) p \\ &= f(x^*) + \frac{1}{2} p^T \nabla^2 f(z) p \end{aligned}$$

where  $z = x^* + tp$  for some  $t \in (0, 1)$ . Since  $z \in B$ , we have  $p^T \nabla^2 f(z) p > 0$ , and therefore  $f(x^* + p) > f(x^*)$ , giving the result.

## Peano counterexample

Note, that if  $\nabla f(x^*) = 0, \nabla^2 f(x^*) \succeq 0$ , i.e. the hessian is positive *semidefinite*, we cannot be sure if  $x^*$  is a local minimum.

## Peano counterexample

Note, that if  $\nabla f(x^*) = 0, \nabla^2 f(x^*) \succeq 0$ , i.e. the hessian is positive *semidefinite*, we cannot be sure if  $x^*$  is a local minimum.

$$f(x, y) = (2x^2 - y)(x^2 - y)$$

## Peano counterexample

Note, that if  $\nabla f(x^*) = 0$ ,  $\nabla^2 f(x^*) \succeq 0$ , i.e. the hessian is positive semidefinite, we cannot be sure if  $x^*$  is a local minimum.

$$f(x, y) = (2x^2 - y)(x^2 - y)$$

Although the surface does not have a local minimizer at the origin, its intersection with any vertical plane through the origin (a plane with equation  $y = mx$  or  $x = 0$ ) is a curve that has a local minimum at the origin. In other words, if a point starts at the origin  $(0, 0)$  of the plane, and moves away from the origin along any straight line, the value of  $(2x^2 - y)(x^2 - y)$  will increase at the start of the motion. Nevertheless,  $(0, 0)$  is not a local minimizer of the function, because moving along a parabola such as  $y = \sqrt{2}x^2$  will cause the function value to decrease.

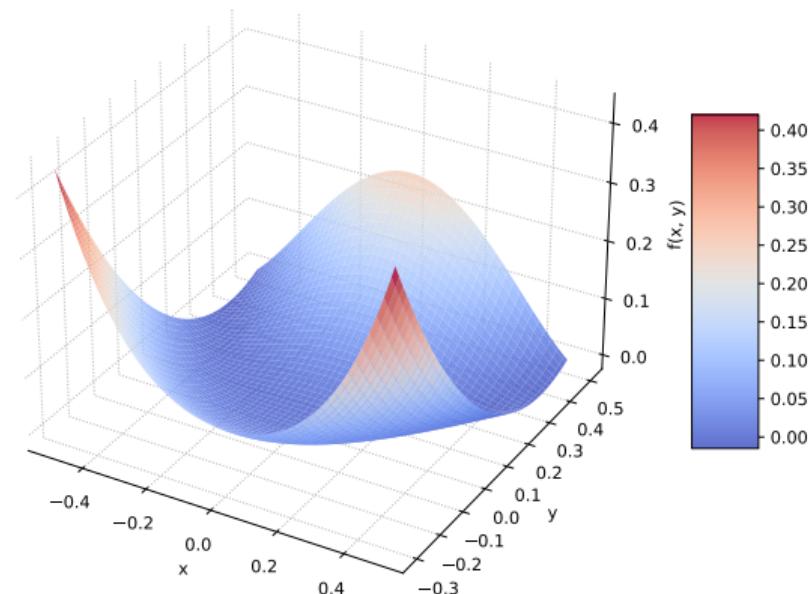
## Peano counterexample

Note, that if  $\nabla f(x^*) = 0$ ,  $\nabla^2 f(x^*) \succeq 0$ , i.e. the hessian is positive semidefinite, we cannot be sure if  $x^*$  is a local minimum.

$$f(x, y) = (2x^2 - y)(x^2 - y)$$

Although the surface does not have a local minimizer at the origin, its intersection with any vertical plane through the origin (a plane with equation  $y = mx$  or  $x = 0$ ) is a curve that has a local minimum at the origin. In other words, if a point starts at the origin  $(0, 0)$  of the plane, and moves away from the origin along any straight line, the value of  $(2x^2 - y)(x^2 - y)$  will increase at the start of the motion. Nevertheless,  $(0, 0)$  is not a local minimizer of the function, because moving along a parabola such as  $y = \sqrt{2}x^2$  will cause the function value to decrease.

Non-convex PL function



## Constrained optimization

## General first-order local optimality condition

Direction  $d \in \mathbb{R}^n$  is a feasible direction

at  $x^* \in S \subseteq \mathbb{R}^n$  if small steps along  $d$

do not take us outside of  $S$ .

## General first-order local optimality condition

Direction  $d \in \mathbb{R}^n$  is a feasible direction

at  $x^* \in S \subseteq \mathbb{R}^n$  if small steps along  $d$

do not take us outside of  $S$ .

Consider a set  $S \subseteq \mathbb{R}^n$  and a function

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Suppose that  $x^* \in S$  is a point of local minimum for  $f$  over  $S$ , and further assume that  $f$  is continuously differentiable around  $x^*$ .

## General first-order local optimality condition

Direction  $d \in \mathbb{R}^n$  is a feasible direction

at  $x^* \in S \subseteq \mathbb{R}^n$  if small steps along  $d$

do not take us outside of  $S$ .

Consider a set  $S \subseteq \mathbb{R}^n$  and a function

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Suppose that  $x^* \in S$  is a point of local minimum for  $f$  over  $S$ , and further assume that  $f$  is continuously differentiable around  $x^*$ .

1. Then for every feasible direction

$d \in \mathbb{R}^n$  at  $x^*$  it holds that

$$\nabla f(x^*)^\top d \geq 0.$$

## General first-order local optimality condition

Direction  $d \in \mathbb{R}^n$  is a feasible direction

at  $x^* \in S \subseteq \mathbb{R}^n$  if small steps along  $d$

do not take us outside of  $S$ .

Consider a set  $S \subseteq \mathbb{R}^n$  and a function

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Suppose that  $x^* \in S$  is a point of local minimum for  $f$  over  $S$ , and further assume that  $f$  is

continuously differentiable around  $x^*$ .

1. Then for every feasible direction

$d \in \mathbb{R}^n$  at  $x^*$  it holds that

$$\nabla f(x^*)^\top d \geq 0.$$

2. If, additionally,  $S$  is convex then

$$\nabla f(x^*)^\top (x - x^*) \geq 0, \forall x \in S.$$

## General first-order local optimality condition

Direction  $d \in \mathbb{R}^n$  is a feasible direction

at  $x^* \in S \subseteq \mathbb{R}^n$  if small steps along  $d$

do not take us outside of  $S$ .

Consider a set  $S \subseteq \mathbb{R}^n$  and a function

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Suppose that  $x^* \in S$  is a point of local minimum for  $f$  over  $S$ , and further assume that  $f$  is continuously differentiable around  $x^*$ .

1. Then for every feasible direction

$d \in \mathbb{R}^n$  at  $x^*$  it holds that

$$\nabla f(x^*)^\top d \geq 0.$$

2. If, additionally,  $S$  is convex then

$$\nabla f(x^*)^\top (x - x^*) \geq 0, \forall x \in S.$$

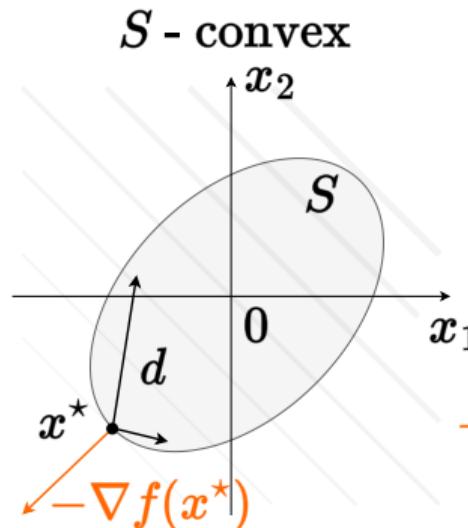
## General first-order local optimality condition

Direction  $d \in \mathbb{R}^n$  is a feasible direction at  $x^* \in S \subseteq \mathbb{R}^n$  if small steps along  $d$  do not take us outside of  $S$ .

Consider a set  $S \subseteq \mathbb{R}^n$  and a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Suppose that  $x^* \in S$  is a point of local minimum for  $f$  over  $S$ , and further assume that  $f$  is continuously differentiable around  $x^*$ .

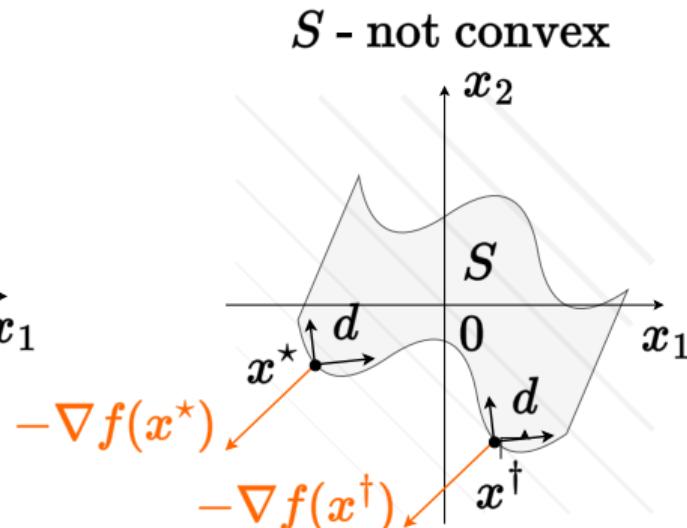
- Then for every feasible direction  $d \in \mathbb{R}^n$  at  $x^*$  it holds that  $\nabla f(x^*)^\top d \geq 0$ .
- If, additionally,  $S$  is convex then

$$\nabla f(x^*)^\top (x - x^*) \geq 0, \forall x \in S.$$



$$\langle -\nabla f(x^*), d \rangle \leq 0$$

$x^*$ - optimal



$$\langle -\nabla f(x^\dagger), d \rangle \leq 0$$

$x^\dagger$ - not optimal

Figure 3: General first order local optimality condition

## Convex case

It should be mentioned, that in the **convex** case (i.e.,  $f(x)$  is convex) necessary condition becomes sufficient.

## Convex case

It should be mentioned, that in the **convex** case (i.e.,  $f(x)$  is convex) necessary condition becomes sufficient.

One more important result for the convex unconstrained case sounds as follows. If  $f(x) : S \rightarrow \mathbb{R}$  - convex function defined on the convex set  $S$ , then:

## Convex case

It should be mentioned, that in the **convex** case (i.e.,  $f(x)$  is convex) necessary condition becomes sufficient.

One more important result for the convex unconstrained case sounds as follows. If  $f(x) : S \rightarrow \mathbb{R}$  - convex function defined on the convex set  $S$ , then:

- Any local minima is the global one.

## Convex case

It should be mentioned, that in the **convex** case (i.e.,  $f(x)$  is convex) necessary condition becomes sufficient.

One more important result for the convex unconstrained case sounds as follows. If  $f(x) : S \rightarrow \mathbb{R}$  - convex function defined on the convex set  $S$ , then:

- Any local minima is the global one.
- The set of the local minimizers  $S^*$  is convex.

## Convex case

It should be mentioned, that in the **convex** case (i.e.,  $f(x)$  is convex) necessary condition becomes sufficient.

One more important result for the convex unconstrained case sounds as follows. If  $f(x) : S \rightarrow \mathbb{R}$  - convex function defined on the convex set  $S$ , then:

- Any local minima is the global one.
- The set of the local minimizers  $S^*$  is convex.
- If  $f(x)$  - strictly or strongly convex function, then  $S^*$  contains only one single point  $S^* = \{x^*\}$ .

## Optimization with equality constraints

Things are pretty simple and intuitive in unconstrained problems. In this section, we will add one equality constraint, i.e.

## Optimization with equality constraints

Things are pretty simple and intuitive in unconstrained problems. In this section, we will add one equality constraint, i.e.

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } h(x) &= 0 \end{aligned}$$

## Optimization with equality constraints

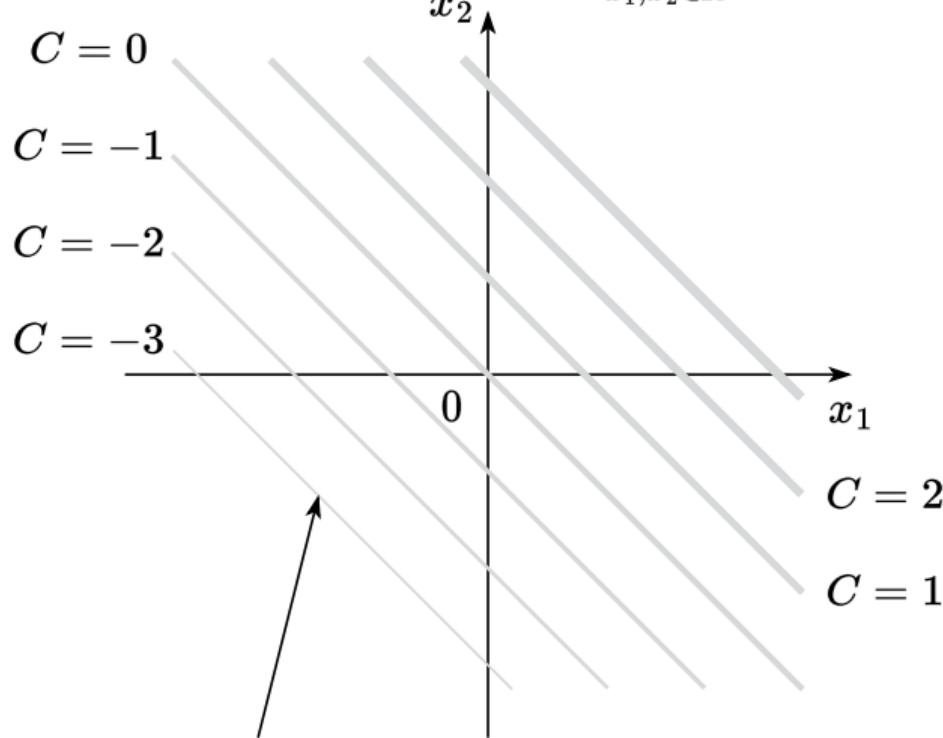
Things are pretty simple and intuitive in unconstrained problems. In this section, we will add one equality constraint, i.e.

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } h(x) &= 0 \end{aligned}$$

We will try to illustrate an approach to solve this problem through the simple example with  $f(x) = x_1 + x_2$  and  $h(x) = x_1^2 + x_2^2 - 2$ .

## Optimization with equality constraints

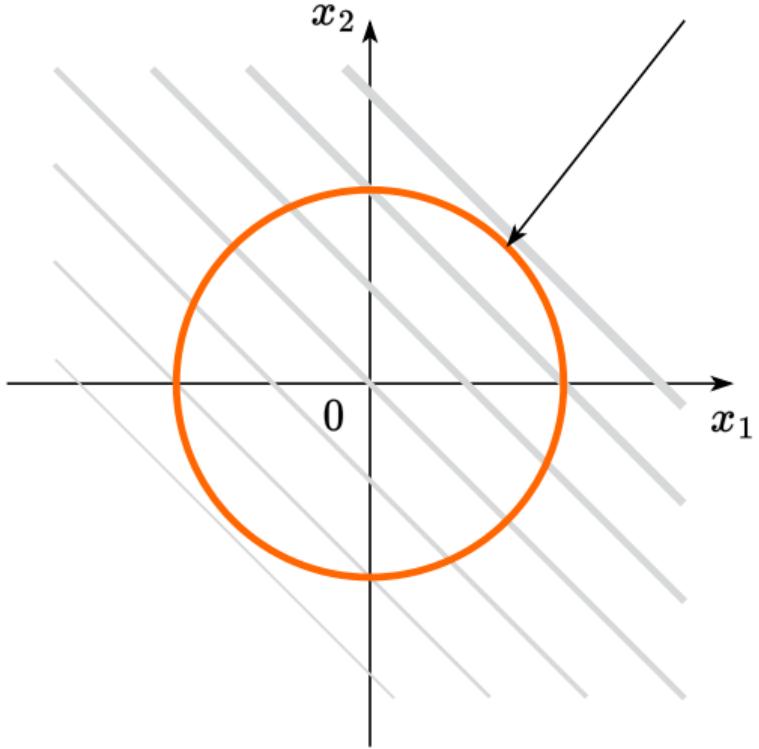
$$f(x) = x_1 + x_2 \rightarrow \min_{x_1, x_2 \in \mathbb{R}^2}$$



Contour lines of  $f(x) = x_1 + x_2 = C$

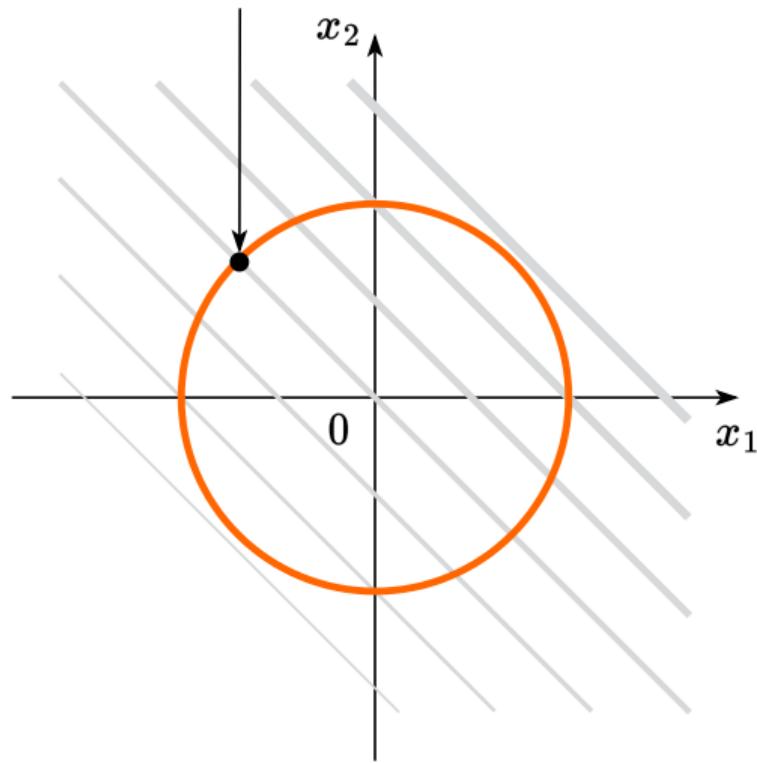
## Optimization with equality constraints

$$h(x) = x_1^2 + x_2^2 - 2 = 0$$

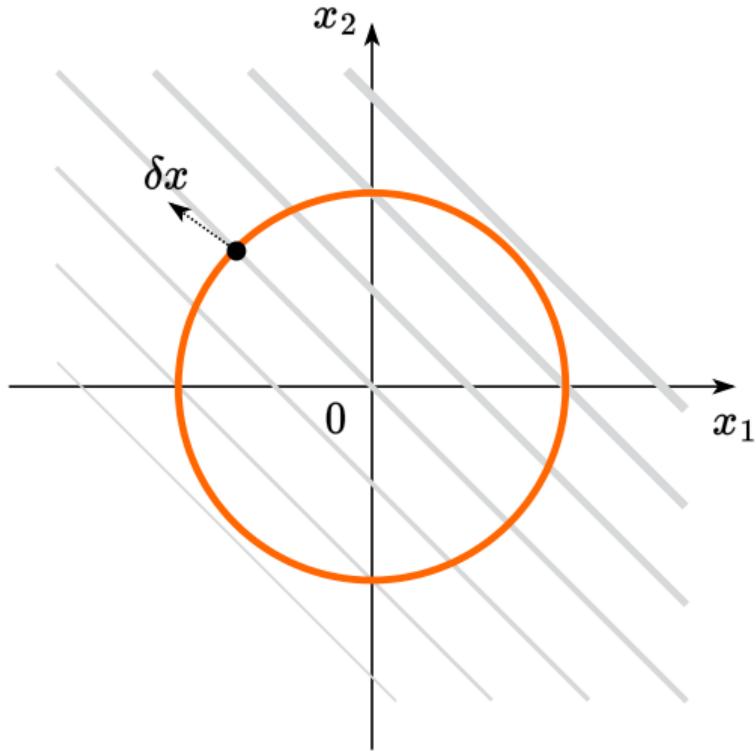


## Optimization with equality constraints

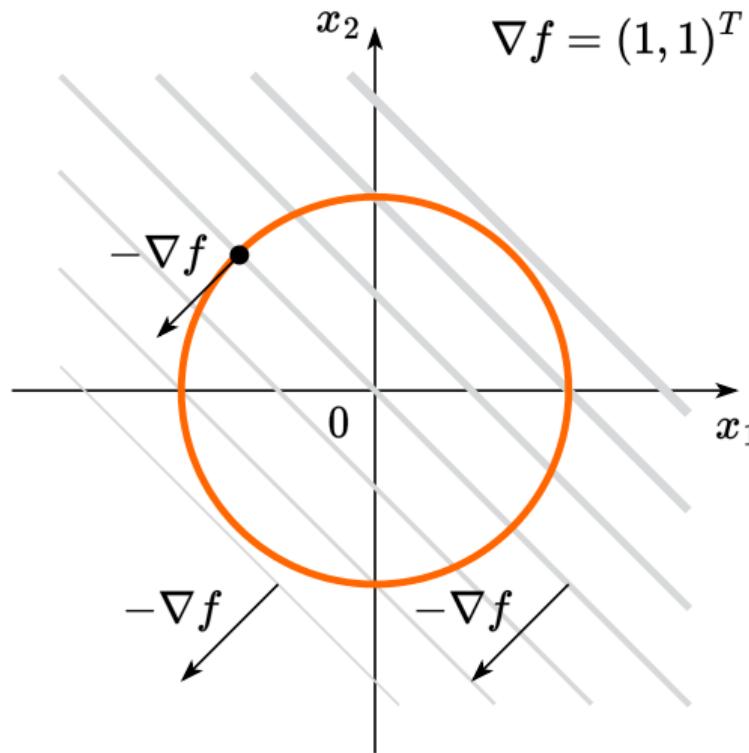
Feasible point  $x_F$



## Optimization with equality constraints

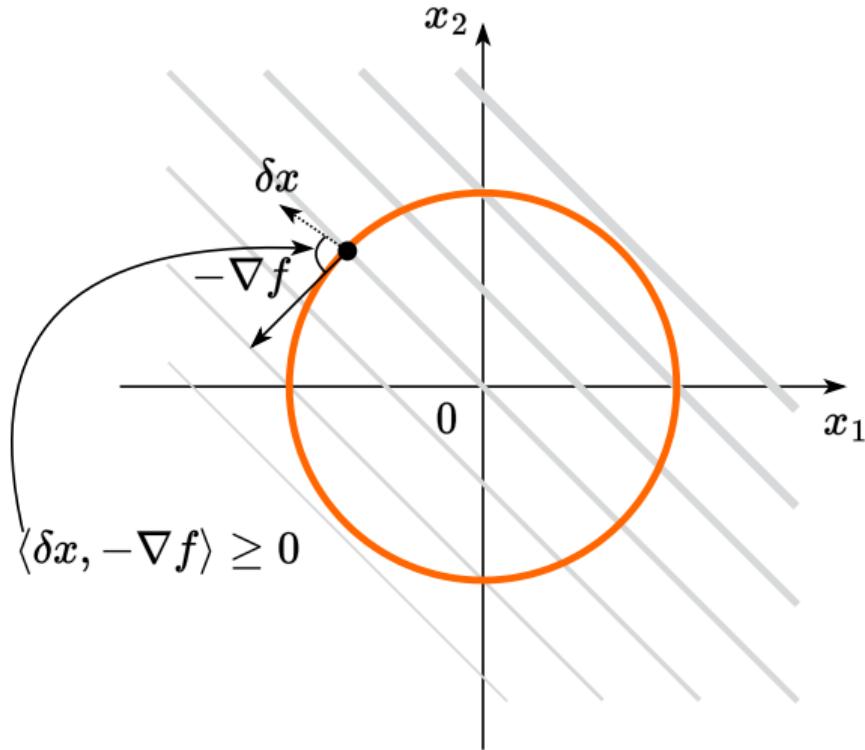


## Optimization with equality constraints



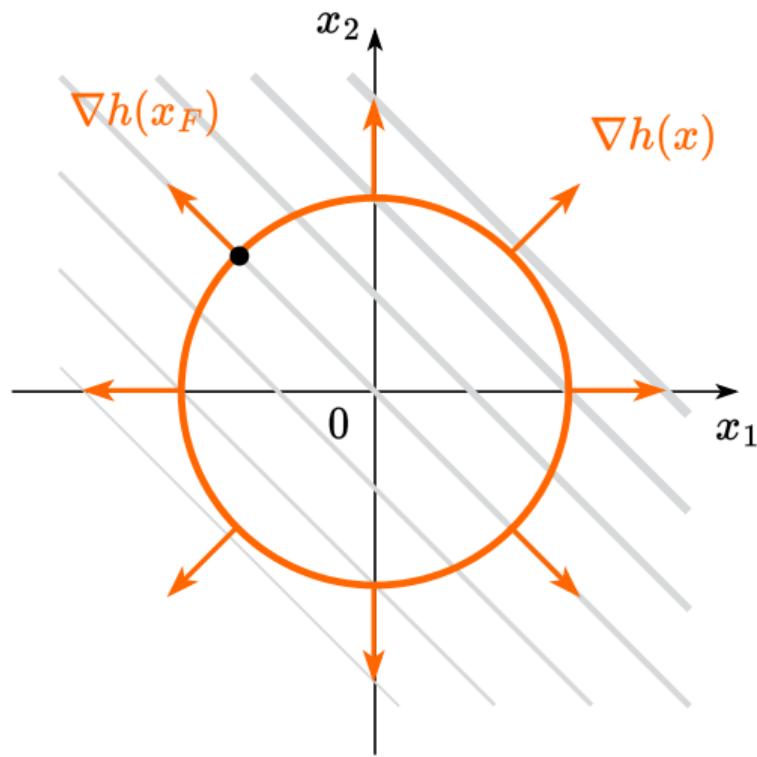
## Optimization with equality constraints

We want:  $f(x_F + \delta x) \leq f(x_F)$

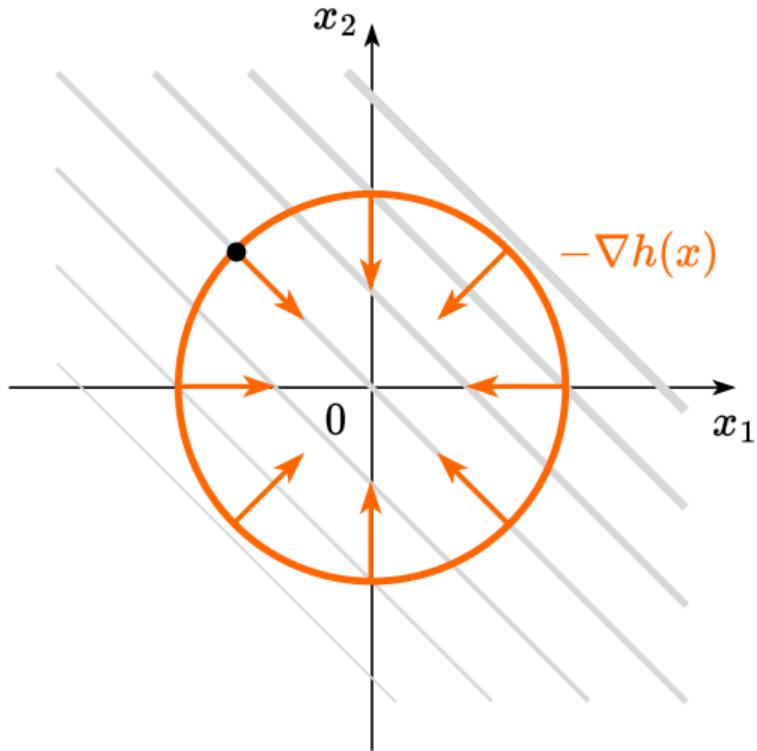


## Optimization with equality constraints

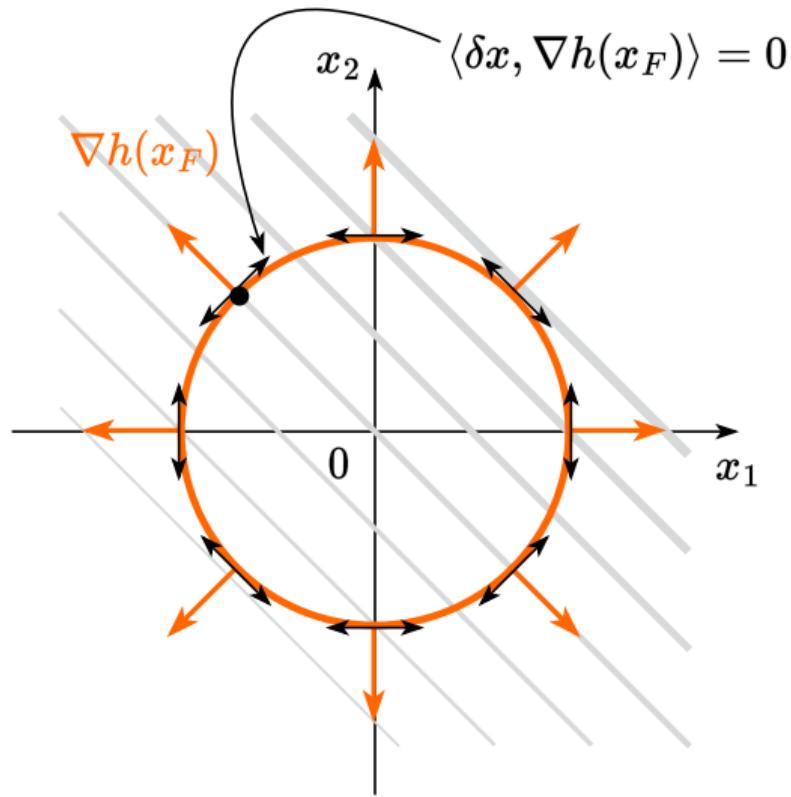
$$\nabla h = (2x_1, 2x_2)^T$$



## Optimization with equality constraints



## Optimization with equality constraints



## Optimization with equality constraints

Generally: to move from  $x_F$  along the budget set toward decreasing the function, we need to guarantee two conditions:

## Optimization with equality constraints

Generally: to move from  $x_F$  along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

## Optimization with equality constraints

Generally: to move from  $x_F$  along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

$$\langle \delta x, -\nabla f(x_F) \rangle > 0$$

## Optimization with equality constraints

Generally: to move from  $x_F$  along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

$$\langle \delta x, -\nabla f(x_F) \rangle > 0$$

Let's assume, that in the process of such a movement, we have come to the point where

## Optimization with equality constraints

Generally: to move from  $x_F$  along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

$$\langle \delta x, -\nabla f(x_F) \rangle > 0$$

Let's assume, that in the process of such a movement, we have come to the point where

$$-\nabla f(x) = \nu \nabla h(x)$$

## Optimization with equality constraints

Generally: to move from  $x_F$  along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

$$\langle \delta x, -\nabla f(x_F) \rangle > 0$$

Let's assume, that in the process of such a movement, we have come to the point where

$$-\nabla f(x) = \nu \nabla h(x)$$

$$\langle \delta x, -\nabla f(x) \rangle = \langle \delta x, \nu \nabla h(x) \rangle = 0$$

## Optimization with equality constraints

Generally: to move from  $x_F$  along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

$$\langle \delta x, -\nabla f(x_F) \rangle > 0$$

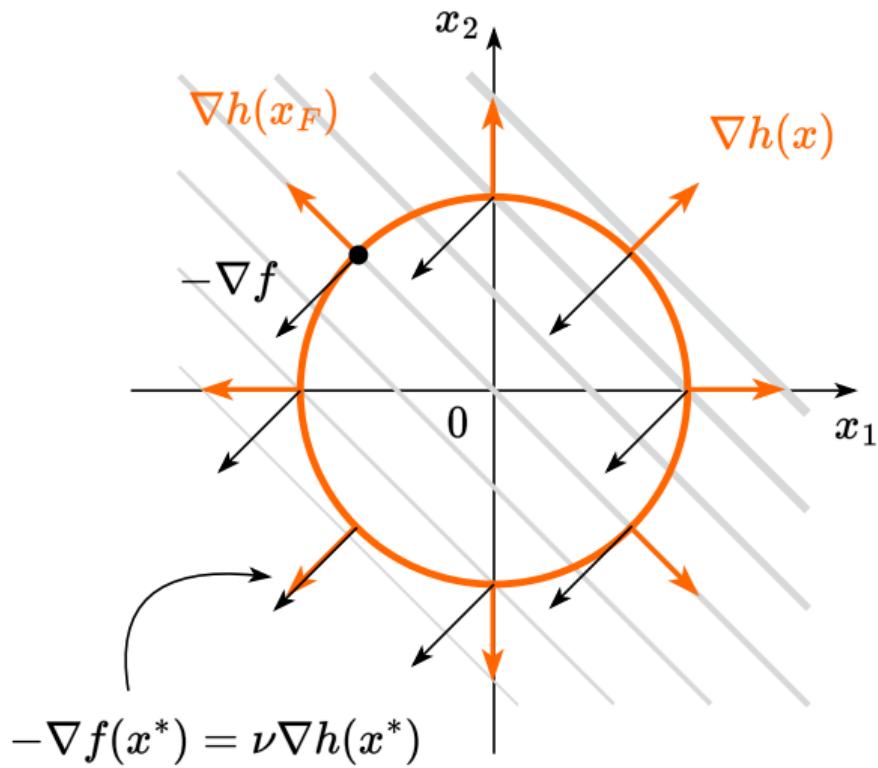
Let's assume, that in the process of such a movement, we have come to the point where

$$-\nabla f(x) = \nu \nabla h(x)$$

$$\langle \delta x, -\nabla f(x) \rangle = \langle \delta x, \nu \nabla h(x) \rangle = 0$$

Then we came to the point of the budget set, moving from which it will not be possible to reduce our function. This is the local minimum in the constrained problem :)

## Optimization with equality constraints



## Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

## Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point  $x^*$  is the local minimum of the problem described above, then there exists  $\nu^*$ :

## Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point  $x^*$  is the local minimum of the problem described above, then there exists  $\nu^*$ :

Necessary conditions

We should notice that  $L(x^*, \nu^*) = f(x^*)$ .

## Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point  $x^*$  is the local minimum of the problem described above, then there exists  $\nu^*$ :

Necessary conditions

$$\nabla_x L(x^*, \nu^*) = 0 \text{ that's written above}$$

We should notice that  $L(x^*, \nu^*) = f(x^*)$ .

## Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point  $x^*$  is the local minimum of the problem described above, then there exists  $\nu^*$ :

Necessary conditions

$\nabla_x L(x^*, \nu^*) = 0$  that's written above

$\nabla_\nu L(x^*, \nu^*) = 0$  budget constraint

We should notice that  $L(x^*, \nu^*) = f(x^*)$ .

## Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point  $x^*$  is the local minimum of the problem described above, then there exists  $\nu^*$ :

Necessary conditions

$\nabla_x L(x^*, \nu^*) = 0$  that's written above

$\nabla_\nu L(x^*, \nu^*) = 0$  budget constraint

Sufficient conditions

We should notice that  $L(x^*, \nu^*) = f(x^*)$ .

## Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point  $x^*$  is the local minimum of the problem described above, then there exists  $\nu^*$ :

Necessary conditions

$\nabla_x L(x^*, \nu^*) = 0$  that's written above

$\nabla_\nu L(x^*, \nu^*) = 0$  budget constraint

Sufficient conditions

$$\langle y, \nabla_{xx}^2 L(x^*, \nu^*)y \rangle > 0,$$

We should notice that  $L(x^*, \nu^*) = f(x^*)$ .

## Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point  $x^*$  is the local minimum of the problem described above, then there exists  $\nu^*$ :

Necessary conditions

$$\nabla_x L(x^*, \nu^*) = 0 \text{ that's written above}$$

$$\nabla_\nu L(x^*, \nu^*) = 0 \text{ budget constraint}$$

Sufficient conditions

$$\langle y, \nabla_{xx}^2 L(x^*, \nu^*) y \rangle > 0,$$

$$\forall y \neq 0 \in \mathbb{R}^n : \nabla h(x^*)^\top y = 0$$

We should notice that  $L(x^*, \nu^*) = f(x^*)$ .

## Equality constrained problem

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } h_i(x) &= 0, i = 1, \dots, p \end{aligned} \tag{ECP}$$

$$L(x, \nu) = f(x) + \sum_{i=1}^p \nu_i h_i(x) = f(x) + \nu^\top h(x)$$

Let  $f(x)$  and  $h_i(x)$  be twice differentiable at the point  $x^*$  and continuously differentiable in some neighborhood  $x^*$ . The local minimum conditions for  $x \in \mathbb{R}^n, \nu \in \mathbb{R}^p$  are written as

ECP: Necessary conditions

$$\nabla_x L(x^*, \nu^*) = 0$$

$$\nabla_\nu L(x^*, \nu^*) = 0$$

ECP: Sufficient conditions

$$\langle y, \nabla_{xx}^2 L(x^*, \nu^*) y \rangle > 0,$$

$$\forall y \neq 0 \in \mathbb{R}^n : \nabla h_i(x^*)^\top y = 0$$

# Linear Least Squares

## i Example

Pose the optimization problem and solve them for linear system  $Ax = b, A \in \mathbb{R}^{m \times n}$  for three cases (assuming the matrix is full rank):

- $m < n$

# Linear Least Squares

## Example

Pose the optimization problem and solve them for linear system  $Ax = b, A \in \mathbb{R}^{m \times n}$  for three cases (assuming the matrix is full rank):

- $m < n$
- $m = n$

# Linear Least Squares

## i Example

Pose the optimization problem and solve them for linear system  $Ax = b, A \in \mathbb{R}^{m \times n}$  for three cases (assuming the matrix is full rank):

- $m < n$
- $m = n$
- $m > n$

## Optimization with inequality constraints

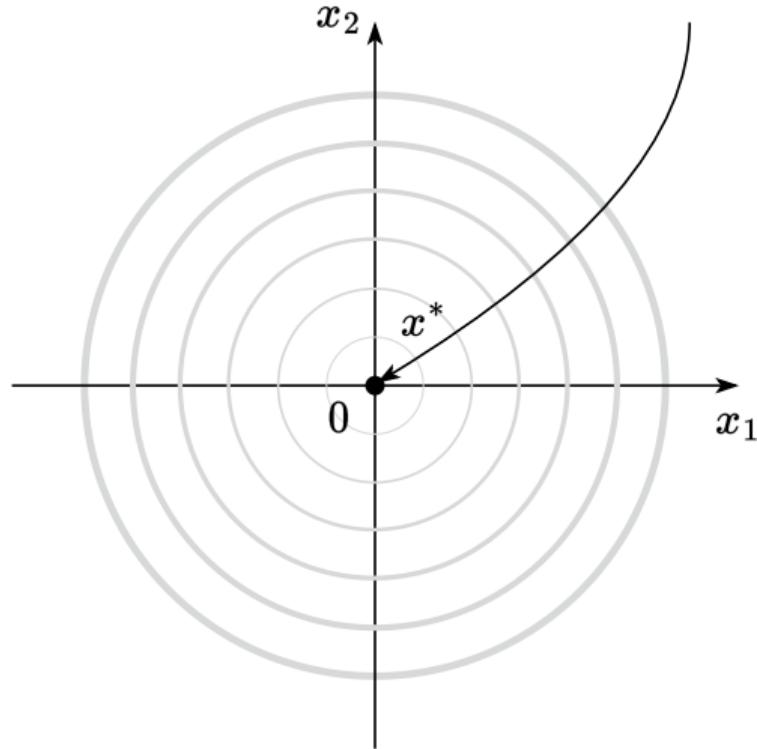
## Example of inequality constraints

$$f(x) = x_1^2 + x_2^2 \quad g(x) = x_1^2 + x_2^2 - 1$$

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) &\leq 0 \end{aligned}$$

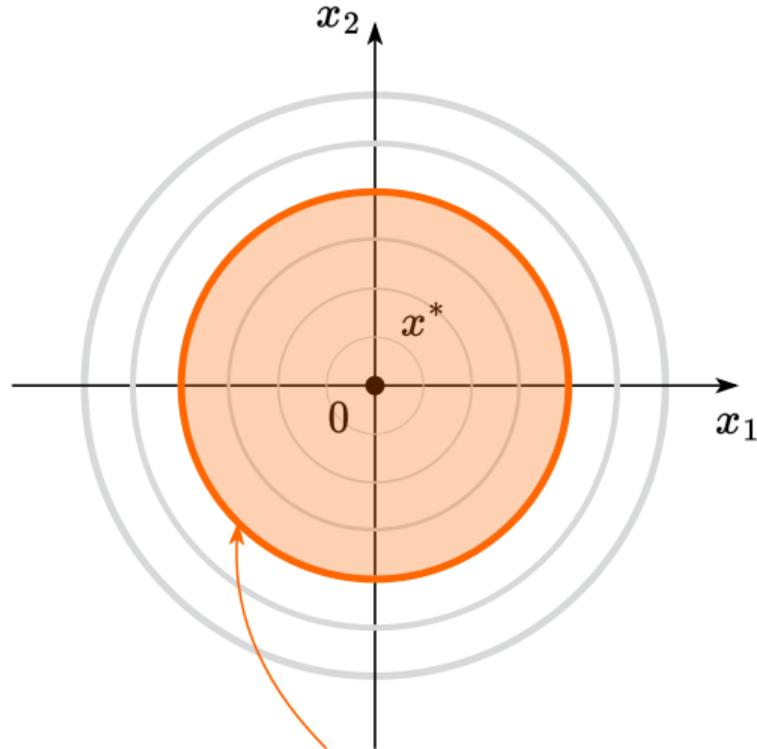
## Optimization with inequality constraints

$$x^* = \operatorname{argmin} f(x)$$



Contour lines of  $f(x) = x_1^2 + x_2^2 = C$

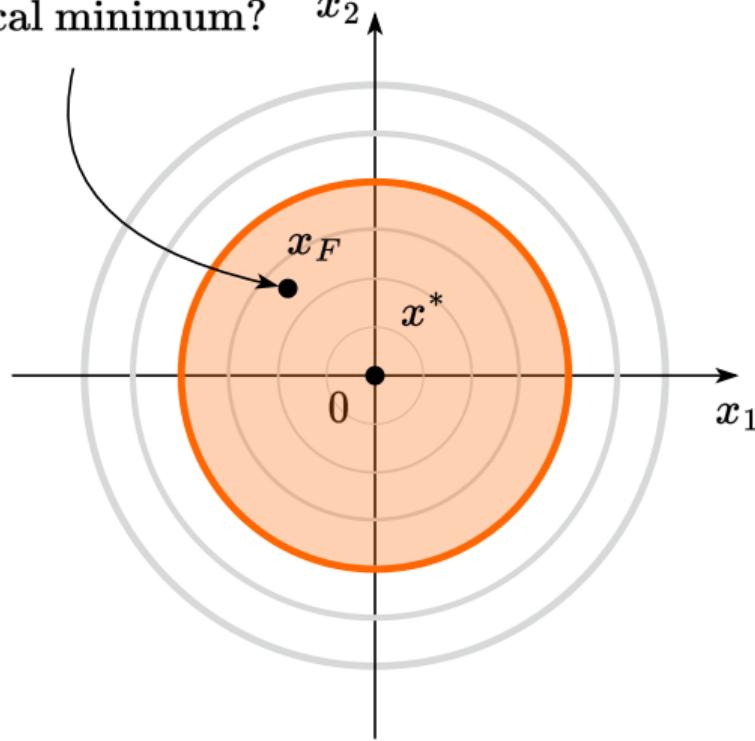
## Optimization with inequality constraints



Feasible region  $g(x) = x_1^2 + x_2^2 - 1 \leq 0$

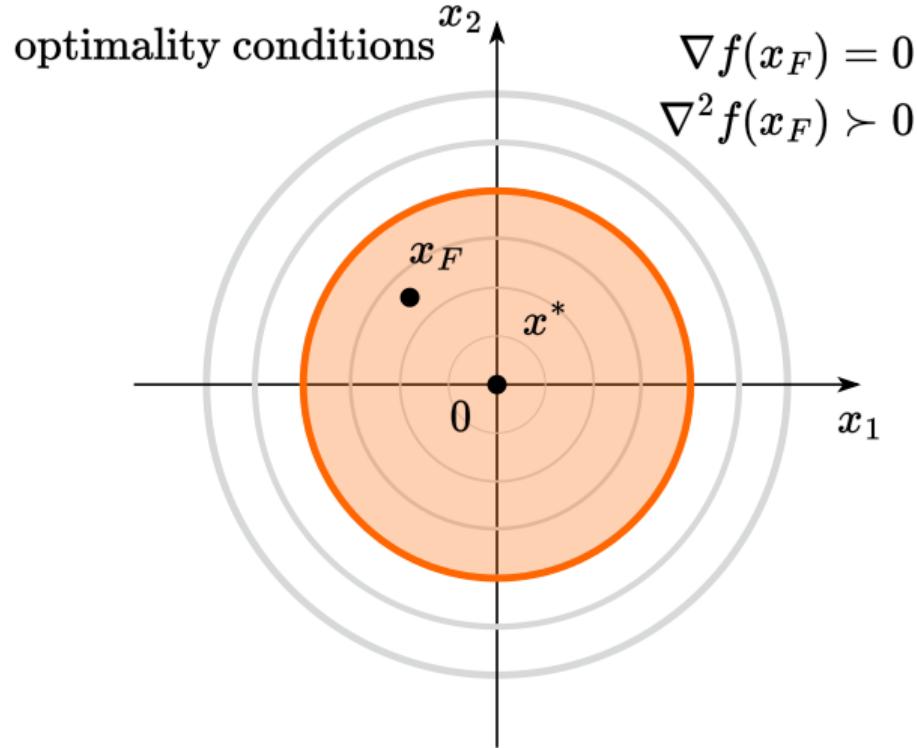
## Optimization with inequality constraints

How to recognize that some feasible point is at local minimum?



## Optimization with inequality constraints

Easy in this case! Just check unconstrained



## Optimization with inequality constraints

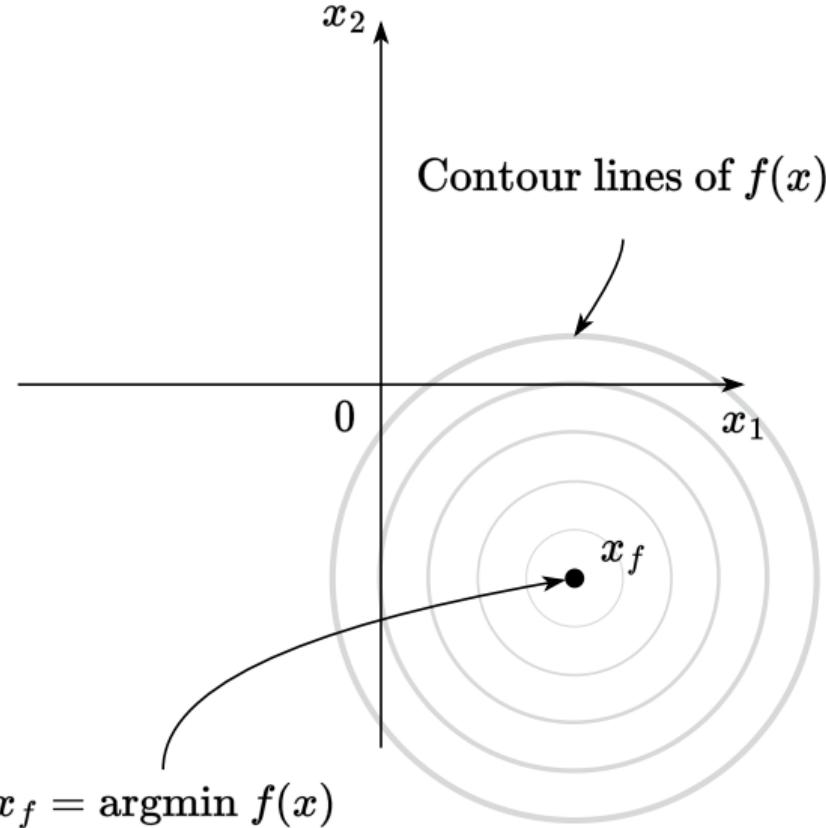
Thus, if the constraints of the type of inequalities are inactive in the constrained problem, then don't worry and write out the solution to the unconstrained problem. However, this is not the whole story. Consider the second childish example

$$f(x) = (x_1 - 1)^2 + (x_2 + 1)^2 \quad g(x) = x_1^2 + x_2^2 - 1$$

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) &\leq 0 \end{aligned}$$

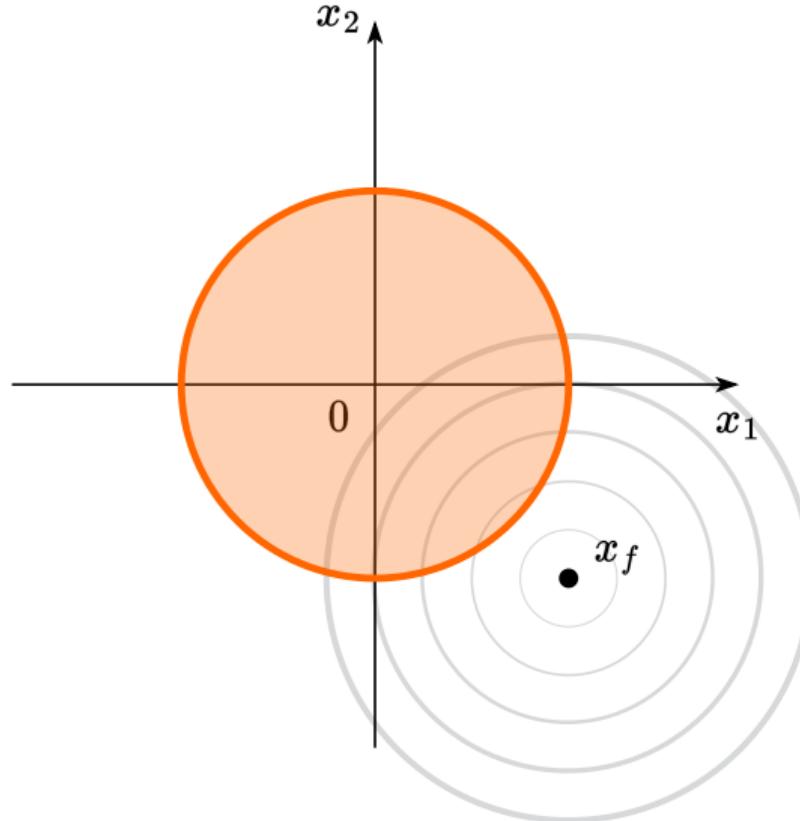
## Optimization with inequality constraints

$$f(x) = (x_1 - 1)^2 + (x_2 + 1)^2 = C$$



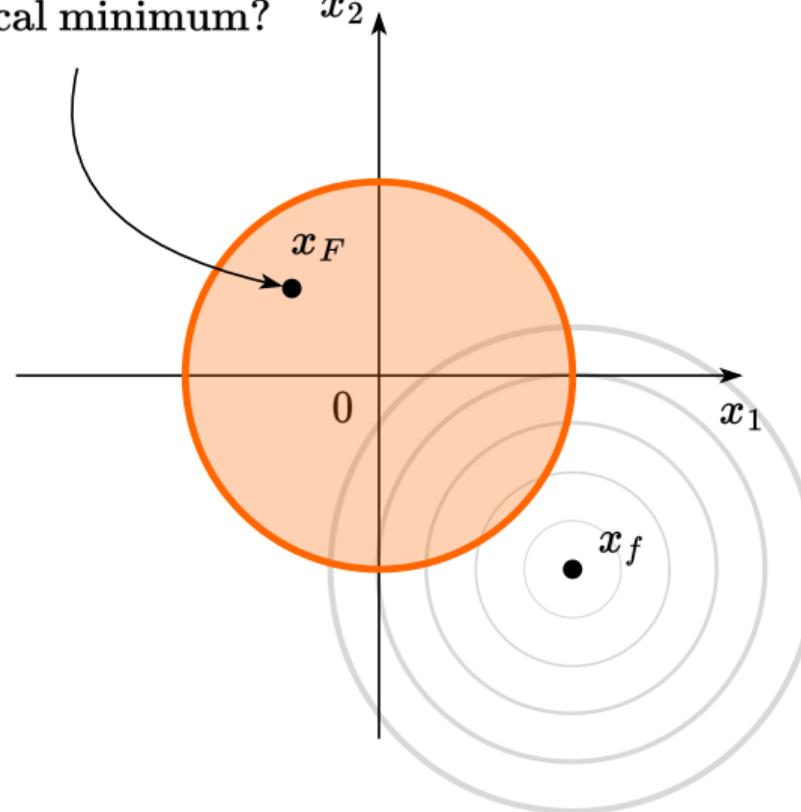
## Optimization with inequality constraints

Feasible region  $g(x) = x_1^2 + x_2^2 - 1 \leq 0$



## Optimization with inequality constraints

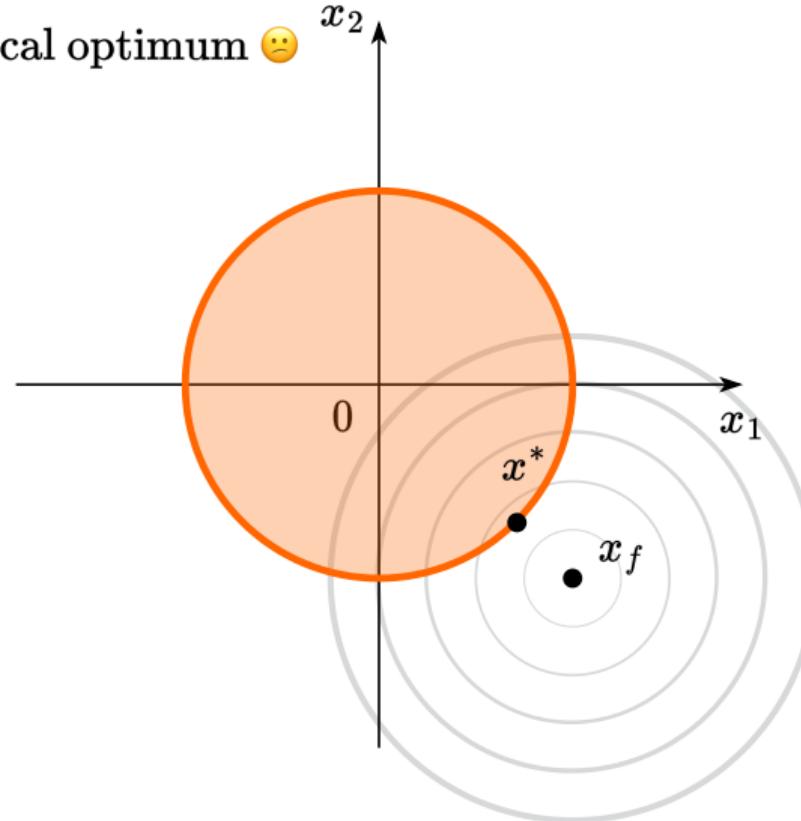
How to recognize that some feasible point is at local minimum?  $x_2$



## Optimization with inequality constraints

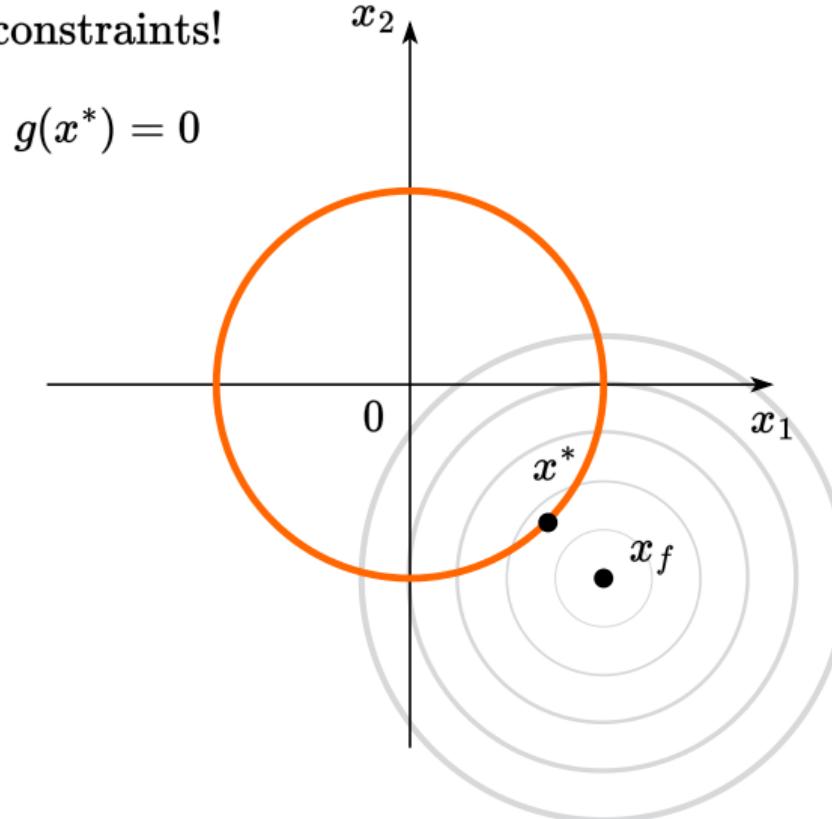
Not very easy in this case! Even gradient  $\neq 0$

at local optimum 😞

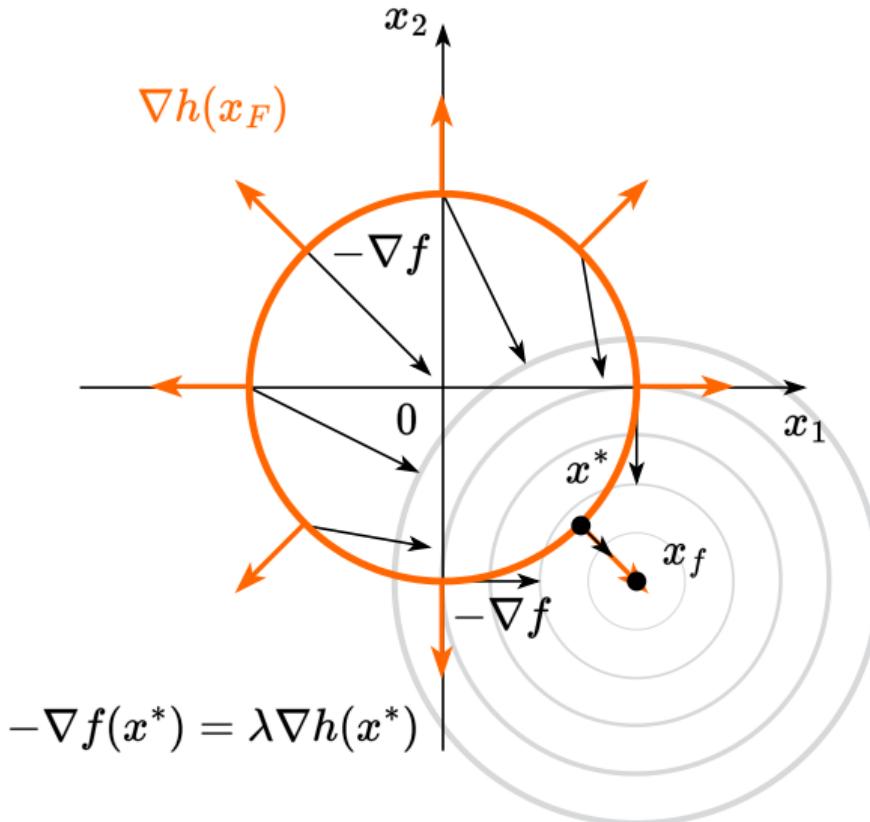


## Optimization with inequality constraints

Effectively have a problem with equality constraints!

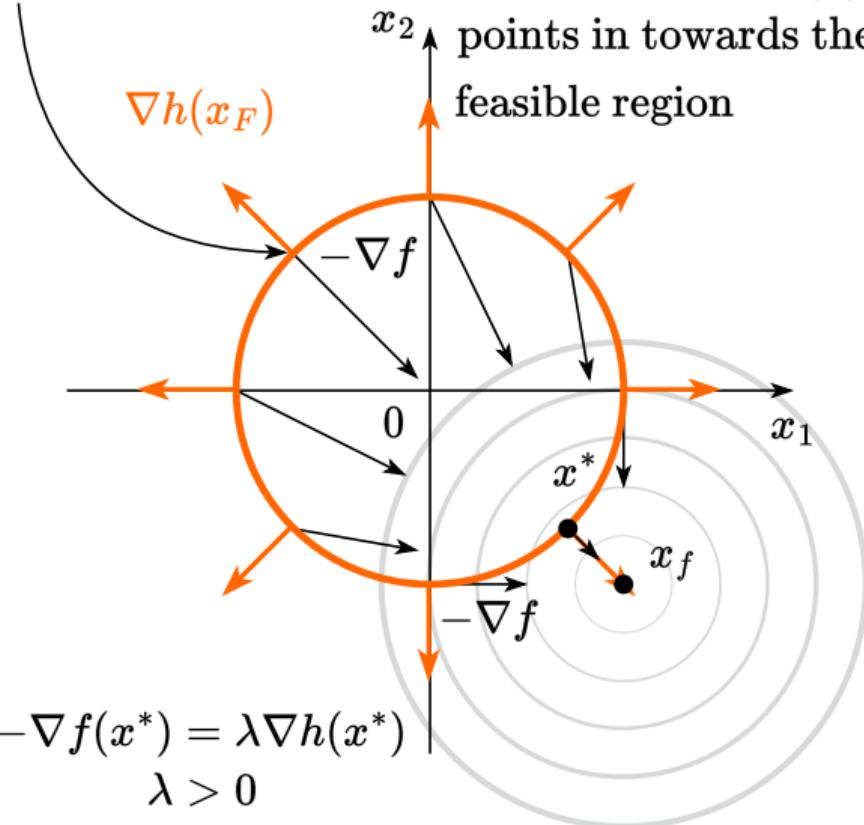


## Optimization with inequality constraints



## Optimization with inequality constraints

Not a constrained local minimum as  $-\nabla f(x)$



## Optimization with inequality constraints

So, we have a problem:

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) &\leq 0 \end{aligned}$$

Two possible cases:

$g(x) \leq 0$  is inactive.  $g(x^*) < 0$

- $g(x^*) < 0$

## Optimization with inequality constraints

So, we have a problem:

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) &\leq 0 \end{aligned}$$

Two possible cases:

$g(x) \leq 0$  is inactive.  $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$

# Optimization with inequality constraints

So, we have a problem:

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) &\leq 0 \end{aligned}$$

Two possible cases:

$g(x) \leq 0$  is inactive.  $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$
- $\nabla^2 f(x^*) > 0$

# Optimization with inequality constraints

So, we have a problem:

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) &\leq 0 \end{aligned}$$

Two possible cases:

$g(x) \leq 0$  is inactive.  $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$
- $\nabla^2 f(x^*) > 0$

## Optimization with inequality constraints

So, we have a problem:

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) &\leq 0 \end{aligned}$$

Two possible cases:

$g(x) \leq 0$  is inactive.  $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$
- $\nabla^2 f(x^*) > 0$

$g(x) \leq 0$  is active.  $g(x^*) = 0$

- $g(x^*) = 0$

# Optimization with inequality constraints

So, we have a problem:

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) &\leq 0 \end{aligned}$$

Two possible cases:

$g(x) \leq 0$  is inactive.  $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$
- $\nabla^2 f(x^*) > 0$

$g(x) \leq 0$  is active.  $g(x^*) = 0$

- $g(x^*) = 0$
- Necessary conditions:  $-\nabla f(x^*) = \lambda \nabla g(x^*)$ ,  $\lambda > 0$

# Optimization with inequality constraints

So, we have a problem:

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) &\leq 0 \end{aligned}$$

Two possible cases:

$g(x) \leq 0$  is inactive.  $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$
- $\nabla^2 f(x^*) > 0$

$g(x) \leq 0$  is active.  $g(x^*) = 0$

- $g(x^*) = 0$
- Necessary conditions:  $-\nabla f(x^*) = \lambda \nabla g(x^*)$ ,  $\lambda > 0$
- Sufficient conditions:  
 $\langle y, \nabla_{xx}^2 L(x^*, \lambda^*)y \rangle > 0, \forall y \neq 0 \in \mathbb{R}^n : \nabla g(x^*)^\top y = 0$

## Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:

$$f(x) \rightarrow \min_{x \in \mathbb{R}^n}$$

$$\text{s.t. } g(x) \leq 0$$

Let's define the Lagrange function:

$$L(x, \lambda) = f(x) + \lambda g(x)$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer  $x^*$ , stated under some regularity conditions, can be written as follows.

## Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:  
If  $x^*$  is a local minimum of the problem described above, then there exists a unique Lagrange multiplier  $\lambda^*$  such that:

$$(1) \nabla_x L(x^*, \lambda^*) = 0$$

$$f(x) \rightarrow \min_{x \in \mathbb{R}^n}$$

$$\text{s.t. } g(x) \leq 0$$

Let's define the Lagrange function:

$$L(x, \lambda) = f(x) + \lambda g(x)$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer  $x^*$ , stated under some regularity conditions, can be written as follows.

## Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:  
If  $x^*$  is a local minimum of the problem described above, then there exists a unique Lagrange multiplier  $\lambda^*$  such that:

$$f(x) \rightarrow \min_{x \in \mathbb{R}^n}$$

$$(1) \quad \nabla_x L(x^*, \lambda^*) = 0$$

$$(2) \quad \lambda^* \geq 0$$

$$\text{s.t. } g(x) \leq 0$$

Let's define the Lagrange function:

$$L(x, \lambda) = f(x) + \lambda g(x)$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer  $x^*$ , stated under some regularity conditions, can be written as follows.

## Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:  
If  $x^*$  is a local minimum of the problem described above, then there exists a unique Lagrange multiplier  $\lambda^*$  such that:

$$\begin{array}{ll} f(x) \rightarrow \min_{x \in \mathbb{R}^n} & (1) \nabla_x L(x^*, \lambda^*) = 0 \\ \text{s.t. } g(x) \leq 0 & (2) \lambda^* \geq 0 \\ & (3) \lambda^* g(x^*) = 0 \end{array}$$

Let's define the Lagrange function:

$$L(x, \lambda) = f(x) + \lambda g(x)$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer  $x^*$ , stated under some regularity conditions, can be written as follows.

## Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:  
If  $x^*$  is a local minimum of the problem described above, then there exists a unique Lagrange multiplier  $\lambda^*$  such that:

$$\begin{array}{ll} f(x) \rightarrow \min_{x \in \mathbb{R}^n} & (1) \nabla_x L(x^*, \lambda^*) = 0 \\ \text{s.t. } g(x) \leq 0 & (2) \lambda^* \geq 0 \\ & (3) \lambda^* g(x^*) = 0 \\ & (4) g(x^*) \leq 0 \end{array}$$

Let's define the Lagrange function:

$$L(x, \lambda) = f(x) + \lambda g(x)$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer  $x^*$ , stated under some regularity conditions, can be written as follows.

## Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:  
If  $x^*$  is a local minimum of the problem described above, then there exists a unique Lagrange multiplier  $\lambda^*$  such that:

$$\begin{aligned} f(x) \rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) \leq 0 \end{aligned}$$

- (1)  $\nabla_x L(x^*, \lambda^*) = 0$
- (2)  $\lambda^* \geq 0$
- (3)  $\lambda^* g(x^*) = 0$
- (4)  $g(x^*) \leq 0$
- (5)  $\forall y \in C(x^*) : \langle y, \nabla_{xx}^2 L(x^*, \lambda^*) y \rangle > 0$

$$L(x, \lambda) = f(x) + \lambda g(x)$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer  $x^*$ , stated under some regularity conditions, can be written as follows.

## Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:

If  $x^*$  is a local minimum of the problem described above, then there exists a unique Lagrange multiplier  $\lambda^*$  such that:

$$f(x) \rightarrow \min_{x \in \mathbb{R}^n}$$

s.t.  $g(x) \leq 0$

- (1)  $\nabla_x L(x^*, \lambda^*) = 0$
- (2)  $\lambda^* \geq 0$
- (3)  $\lambda^* g(x^*) = 0$
- (4)  $g(x^*) \leq 0$

Let's define the Lagrange function:

$$L(x, \lambda) = f(x) + \lambda g(x)$$

$$(5) \forall y \in C(x^*) : \langle y, \nabla_{xx}^2 L(x^*, \lambda^*) y \rangle > 0$$

$$\text{where } C(x^*) = \{y \in \mathbb{R}^n | \nabla f(x^*)^\top y \leq 0 \text{ and } \forall i \in I(x^*) : \nabla g_i(x^*)^\top y \leq 0\}$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer  $x^*$ , stated under some regularity conditions, can be written as follows.

## Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:

If  $x^*$  is a local minimum of the problem described above, then there exists a unique Lagrange multiplier  $\lambda^*$  such that:

$$\begin{aligned} f(x) \rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g(x) \leq 0 \end{aligned}$$

- (1)  $\nabla_x L(x^*, \lambda^*) = 0$
- (2)  $\lambda^* \geq 0$
- (3)  $\lambda^* g(x^*) = 0$
- (4)  $g(x^*) \leq 0$
- (5)  $\forall y \in C(x^*) : \langle y, \nabla_{xx}^2 L(x^*, \lambda^*) y \rangle > 0$

where  $C(x^*) = \{y \in \mathbb{R}^n | \nabla f(x^*)^\top y \leq 0 \text{ and } \forall i \in I(x^*) : \nabla g_i(x^*)^\top y \leq 0\}$

$$I(x^*) = \{i | g_i(x^*) = 0\}$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer  $x^*$ , stated under some regularity conditions, can be written as follows.

## General formulation

$$\begin{aligned} f_0(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } f_i(x) &\leq 0, \quad i = 1, \dots, m \\ h_i(x) &= 0, \quad i = 1, \dots, p \end{aligned}$$

This formulation is a general problem of mathematical programming.

The solution involves constructing a Lagrange function:

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)$$

## Necessary conditions

Let  $x^*, (\lambda^*, \nu^*)$  be a solution to a mathematical programming problem with zero duality gap (the optimal value for the primal problem  $p^*$  is equal to the optimal value for the dual problem  $d^*$ ). Let also the functions  $f_0, f_i, h_i$  be differentiable.

- $\nabla_x L(x^*, \lambda^*, \nu^*) = 0$

## Necessary conditions

Let  $x^*, (\lambda^*, \nu^*)$  be a solution to a mathematical programming problem with zero duality gap (the optimal value for the primal problem  $p^*$  is equal to the optimal value for the dual problem  $d^*$ ). Let also the functions  $f_0, f_i, h_i$  be differentiable.

- $\nabla_x L(x^*, \lambda^*, \nu^*) = 0$
- $\nabla_\nu L(x^*, \lambda^*, \nu^*) = 0$

## Necessary conditions

Let  $x^*, (\lambda^*, \nu^*)$  be a solution to a mathematical programming problem with zero duality gap (the optimal value for the primal problem  $p^*$  is equal to the optimal value for the dual problem  $d^*$ ). Let also the functions  $f_0, f_i, h_i$  be differentiable.

- $\nabla_x L(x^*, \lambda^*, \nu^*) = 0$
- $\nabla_\nu L(x^*, \lambda^*, \nu^*) = 0$
- $\lambda_i^* \geq 0, i = 1, \dots, m$

## Necessary conditions

Let  $x^*, (\lambda^*, \nu^*)$  be a solution to a mathematical programming problem with zero duality gap (the optimal value for the primal problem  $p^*$  is equal to the optimal value for the dual problem  $d^*$ ). Let also the functions  $f_0, f_i, h_i$  be differentiable.

- $\nabla_x L(x^*, \lambda^*, \nu^*) = 0$
- $\nabla_\nu L(x^*, \lambda^*, \nu^*) = 0$
- $\lambda_i^* \geq 0, i = 1, \dots, m$
- $\lambda_i^* f_i(x^*) = 0, i = 1, \dots, m$

## Necessary conditions

Let  $x^*, (\lambda^*, \nu^*)$  be a solution to a mathematical programming problem with zero duality gap (the optimal value for the primal problem  $p^*$  is equal to the optimal value for the dual problem  $d^*$ ). Let also the functions  $f_0, f_i, h_i$  be differentiable.

- $\nabla_x L(x^*, \lambda^*, \nu^*) = 0$
- $\nabla_\nu L(x^*, \lambda^*, \nu^*) = 0$
- $\lambda_i^* \geq 0, i = 1, \dots, m$
- $\lambda_i^* f_i(x^*) = 0, i = 1, \dots, m$
- $f_i(x^*) \leq 0, i = 1, \dots, m$

## Some regularity conditions

These conditions are needed to make KKT solutions the necessary conditions. Some of them even turn necessary conditions into sufficient (for example, Slater's). Moreover, if you have regularity, you can write down necessary second order conditions  $\langle y, \nabla_{xx}^2 L(x^*, \lambda^*, \nu^*)y \rangle \geq 0$  with *semi-definite* hessian of Lagrangian.

- **Slater's condition.** If for a convex problem (i.e., assuming minimization,  $f_0, f_i$  are convex and  $h_i$  are affine), there exists a point  $x$  such that  $h(x) = 0$  and  $f_i(x) < 0$  (existence of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.

## Some regularity conditions

These conditions are needed to make KKT solutions the necessary conditions. Some of them even turn necessary conditions into sufficient (for example, Slater's). Moreover, if you have regularity, you can write down necessary second order conditions  $\langle y, \nabla_{xx}^2 L(x^*, \lambda^*, \nu^*)y \rangle \geq 0$  with *semi-definite* hessian of Lagrangian.

- **Slater's condition.** If for a convex problem (i.e., assuming minimization,  $f_0, f_i$  are convex and  $h_i$  are affine), there exists a point  $x$  such that  $h(x) = 0$  and  $f_i(x) < 0$  (existence of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.
- **Linearity constraint qualification.** If  $f_i$  and  $h_i$  are affine functions, then no other condition is needed.

## Some regularity conditions

These conditions are needed to make KKT solutions the necessary conditions. Some of them even turn necessary conditions into sufficient (for example, Slater's). Moreover, if you have regularity, you can write down necessary second order conditions  $\langle y, \nabla_{xx}^2 L(x^*, \lambda^*, \nu^*)y \rangle \geq 0$  with *semi-definite* hessian of Lagrangian.

- **Slater's condition.** If for a convex problem (i.e., assuming minimization,  $f_0, f_i$  are convex and  $h_i$  are affine), there exists a point  $x$  such that  $h(x) = 0$  and  $f_i(x) < 0$  (existence of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.
- **Linearity constraint qualification.** If  $f_i$  and  $h_i$  are affine functions, then no other condition is needed.
- **Linear independence constraint qualification.** The gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at  $x^*$ .

## Some regularity conditions

These conditions are needed to make KKT solutions the necessary conditions. Some of them even turn necessary conditions into sufficient (for example, Slater's). Moreover, if you have regularity, you can write down necessary second order conditions  $\langle y, \nabla_{xx}^2 L(x^*, \lambda^*, \nu^*)y \rangle \geq 0$  with *semi-definite* hessian of Lagrangian.

- **Slater's condition.** If for a convex problem (i.e., assuming minimization,  $f_0, f_i$  are convex and  $h_i$  are affine), there exists a point  $x$  such that  $h(x) = 0$  and  $f_i(x) < 0$  (existence of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.
- **Linearity constraint qualification.** If  $f_i$  and  $h_i$  are affine functions, then no other condition is needed.
- **Linear independence constraint qualification.** The gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at  $x^*$ .
- For other examples, see wiki.

## Example. Projection onto a hyperplane

$$\min \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T \mathbf{x} = b.$$

## Example. Projection onto a hyperplane

$$\min \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T \mathbf{x} = b.$$

### Solution

Lagrangian:

## Example. Projection onto a hyperplane

$$\min \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T \mathbf{x} = b.$$

### Solution

Lagrangian:

$$L(\mathbf{x}, \nu) = \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2 + \nu(\mathbf{a}^T \mathbf{x} - b)$$

## Example. Projection onto a hyperplane

$$\min \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T \mathbf{x} = b.$$

### Solution

Lagrangian:

$$L(\mathbf{x}, \nu) = \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2 + \nu(\mathbf{a}^T \mathbf{x} - b)$$

Derivative of  $L$  with respect to  $\mathbf{x}$ :

$$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{x} - \mathbf{y} + \nu \mathbf{a} = 0, \quad \mathbf{x} = \mathbf{y} - \nu \mathbf{a}$$

## Example. Projection onto a hyperplane

$$\min \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T \mathbf{x} = b.$$

### Solution

Lagrangian:

$$L(\mathbf{x}, \nu) = \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2 + \nu(\mathbf{a}^T \mathbf{x} - b)$$

Derivative of  $L$  with respect to  $\mathbf{x}$ :

$$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{x} - \mathbf{y} + \nu \mathbf{a} = 0, \quad \mathbf{x} = \mathbf{y} - \nu \mathbf{a}$$

$$\mathbf{a}^T \mathbf{x} = \mathbf{a}^T \mathbf{y} - \nu \mathbf{a}^T \mathbf{a} \quad \nu = \frac{\mathbf{a}^T \mathbf{y} - b}{\|\mathbf{a}\|^2}$$

## Example. Projection onto a hyperplane

$$\min \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T \mathbf{x} = b.$$

### Solution

Lagrangian:

$$L(\mathbf{x}, \nu) = \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2 + \nu(\mathbf{a}^T \mathbf{x} - b)$$

Derivative of  $L$  with respect to  $\mathbf{x}$ :

$$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{x} - \mathbf{y} + \nu \mathbf{a} = 0, \quad \mathbf{x} = \mathbf{y} - \nu \mathbf{a}$$

$$\mathbf{a}^T \mathbf{x} = \mathbf{a}^T \mathbf{y} - \nu \mathbf{a}^T \mathbf{a} \quad \nu = \frac{\mathbf{a}^T \mathbf{y} - b}{\|\mathbf{a}\|^2}$$

$$\mathbf{x} = \mathbf{y} - \frac{\mathbf{a}^T \mathbf{y} - b}{\|\mathbf{a}\|^2} \mathbf{a}$$

## Example. Projection onto simplex

$$\min \frac{1}{2} \|x - y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

## Example. Projection onto simplex

$$\min \frac{1}{2} \|x - y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

### KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2} \|x - y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

## Example. Projection onto simplex

$$\min \frac{1}{2} \|x - y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

### KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2} \|x - y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of  $L$  with respect to  $x_i$  and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$

## Example. Projection onto simplex

$$\min \frac{1}{2} \|x - y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

### KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2} \|x - y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of  $L$  with respect to  $x_i$  and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$

## Example. Projection onto simplex

$$\min \frac{1}{2} \|x - y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

### KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2} \|x - y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of  $L$  with respect to  $x_i$  and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$
- $\lambda_i \geq 0$

## Example. Projection onto simplex

$$\min \frac{1}{2} \|x - y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

### KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2} \|x - y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of  $L$  with respect to  $x_i$  and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$
- $\lambda_i \geq 0$
- $x^\top 1 = 1, \quad x \geq 0$

## Example. Projection onto simplex

$$\min \frac{1}{2} \|x - y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

### KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2} \|x - y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of  $L$  with respect to  $x_i$  and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$
- $\lambda_i \geq 0$
- $x^\top 1 = 1, \quad x \geq 0$

## Example. Projection onto simplex

$$\min \frac{1}{2} \|x - y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

### KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2} \|x - y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of  $L$  with respect to  $x_i$  and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$
- $\lambda_i \geq 0$
- $x^\top 1 = 1, \quad x \geq 0$

### Question

Solve the above conditions in  $O(n \log n)$  time.

## Example. Projection onto simplex

$$\min \frac{1}{2} \|x - y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

### KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2} \|x - y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of  $L$  with respect to  $x_i$  and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$
- $\lambda_i \geq 0$
- $x^\top 1 = 1, \quad x \geq 0$

#### Question

Solve the above conditions in  $O(n \log n)$  time.

#### Question

Solve the above conditions in  $O(n)$  time.

## References

- Lecture on KKT conditions (very intuitive explanation) in the course “Elements of Statistical Learning” @ KTH.

## References

- Lecture on KKT conditions (very intuitive explanation) in the course “Elements of Statistical Learning” @ KTH.
- One-line proof of KKT

## References

- Lecture on KKT conditions (very intuitive explanation) in the course “Elements of Statistical Learning” @ KTH.
- One-line proof of KKT
- On the Second Order Optimality Conditions for Optimization Problems with Inequality Constraints

## References

- Lecture on KKT conditions (very intuitive explanation) in the course “Elements of Statistical Learning” @ KTH.
- One-line proof of KKT
- On the Second Order Optimality Conditions for Optimization Problems with Inequality Constraints
- On Second Order Optimality Conditions in Nonlinear Optimization

## References

- Lecture on KKT conditions (very intuitive explanation) in the course “Elements of Statistical Learning” @ KTH.
- One-line proof of KKT
- On the Second Order Optimality Conditions for Optimization Problems with Inequality Constraints
- On Second Order Optimality Conditions in Nonlinear Optimization
- Numerical Optimization by Jorge Nocedal and Stephen J. Wright.