

Gradient Descent

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Optimization methods. MIPT

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The result of this method is

$$x_{k+1} = x_k - \alpha f'(x_k)$$

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(GF)



Figure 1: Gradient flow trajectory

Necessary local minimum condition

$$f'(x) = 0$$

$$-\eta f'(x) = 0$$

$$x - \eta f'(x) = x$$

$$x_k - \eta f'(x_k) = x_{k+1}$$

Minimizer of Lipschitz parabola

If a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable and its gradient satisfies Lipschitz conditions with constant L , then $\forall x, y \in \mathbb{R}^n$:

$$|f(y) - f(x) - \langle \nabla f(x), y - x \rangle| \leq \frac{L}{2} \|y - x\|^2,$$

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which geometrically means, that if we'll fix some point $x_0 \in \mathbb{R}^n$ and define two parabolas:

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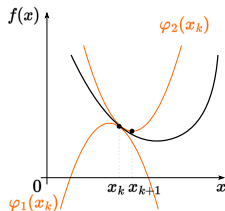


Figure 2: Illustration

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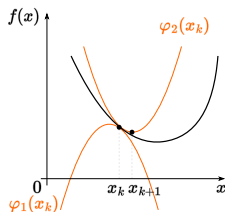


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$$\nabla \phi_2(x) = 0$$

$$\nabla f(x_0) + L(x^* - x_0) = 0$$

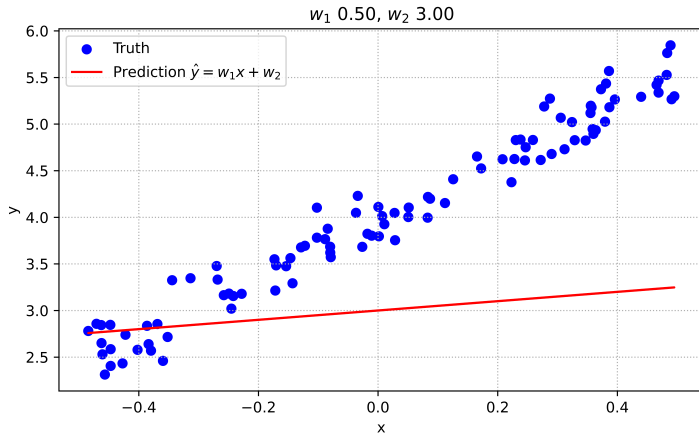
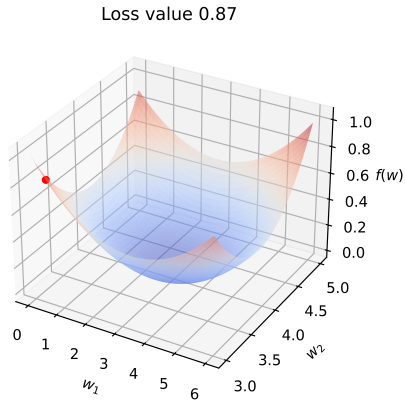
$$x^* = x_0 - \frac{1}{L} \nabla f(x_0)$$

$$x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k)$$

This way leads to the $\frac{1}{L}$ stepsize choosing. However, often the L constant is not known.

Convergence of Gradient Descent algorithm

Heavily depends on the choice of the learning rate α :



Exact line search aka steepest descent

$$\alpha_k = \arg \min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \arg \min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. Interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

$$\alpha_k = \arg \min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

Optimality conditions:

$$\nabla f(x_{k+1})^\top \nabla f(x_k) = 0$$



Figure 3: Steepest Descent

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Convergence rates

$$\min_{x \in \mathbb{R}^n} f(x) \qquad x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

smooth	convex	smooth & convex	smooth & strongly convex (or PL)
$\ \nabla f(x_k)\ ^2 \approx \mathcal{O}\left(\frac{1}{k}\right)$	$f(x_k) - f^* \approx \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$	$f(x_k) - f^* \approx \mathcal{O}\left(\frac{1}{k}\right)$	$\ x_k - x^*\ ^2 \approx \mathcal{O}\left(\left(1 - \frac{\mu}{L}\right)^k\right)$

Gradient Descent convergence. Smooth convex case

Gradient Descent convergence. Smooth μ -strongly convex case

Gradient Descent convergence. Polyak-Lojasiewicz case