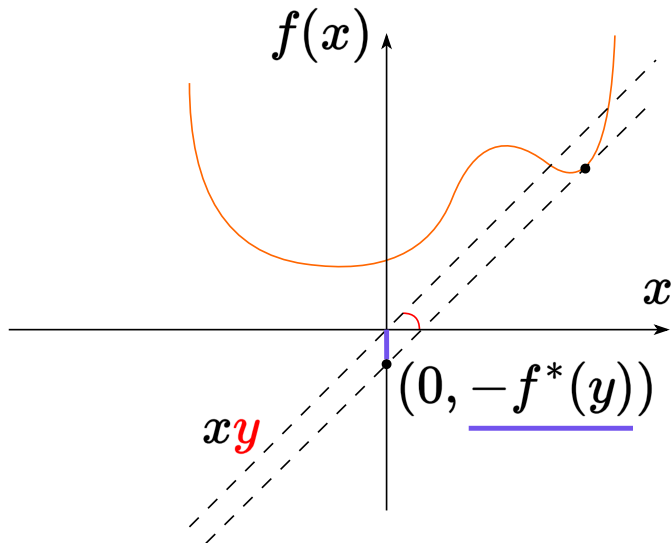


# Introduction to dual methods

Daniil Merkulov

Optimization methods. MIPT

## Definition

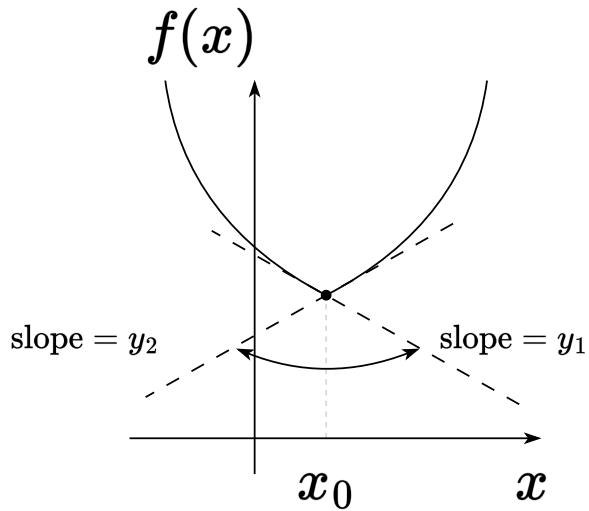


Recall that given  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the function defined by

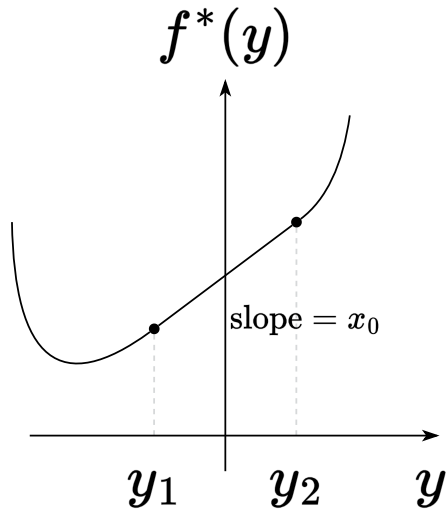
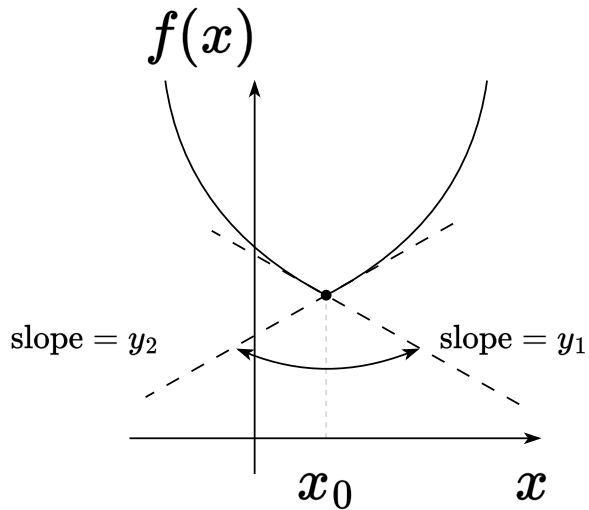
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is called its conjugate.

## Geometrical intuition



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- If  $f$  is strictly convex, then

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## Conjugate function properties (proofs)

We will show that  $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$ , assuming that  $f$  is convex and closed.

- **Proof of  $\Leftarrow$ :** Suppose  $y \in \partial f(x)$ . Then  $x \in M_y$ , the set of maximizers of  $y^T z - f(z)$  over  $z$ . But

$$f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{and} \quad \partial f^*(y) = \text{cl}(\text{conv}(\bigcup_{z \in M_y} \{z\})).$$

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Clearly  $y \in \partial f(x) \Leftrightarrow x \in \arg \min_z \{f(z) - y^T z\}$

Lastly, if  $f$  is strictly convex, then we know that  $f(z) - y^T z$  has a unique minimizer over  $z$ , and this must be  $\nabla f^*(y)$ .

## Dual (sub)gradient method

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Dual ascent method for maximizing dual objective:

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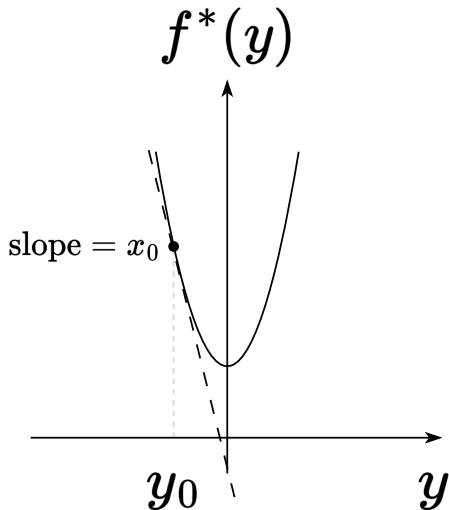
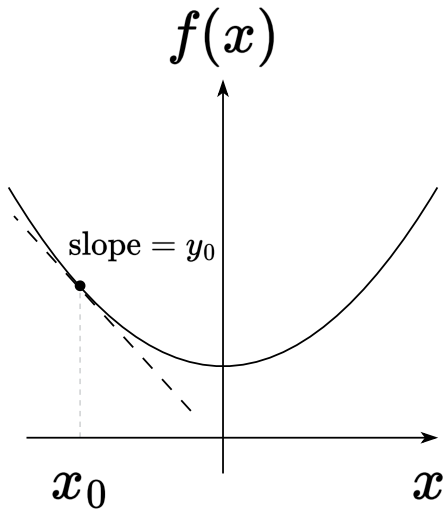
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Adding these together, using the Cauchy-Schwarz inequality, and rearranging shows that

$$\|x_u - x_v\|^2 \leq \frac{1}{\mu} \|u - v\|^2$$

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Let  $u = \nabla f(x)$ ,  $v = \nabla g(y)$ ; then  $x \in \partial g^*(u)$ ,  $y \in \partial g^*(v)$ , and the above reads  $(x - y)^T (u - v) \geq \frac{\|u - v\|^2}{L}$ , implying the result.



# Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent:<sup>1</sup>

- If  $f$  is strongly convex with parameter  $\mu$ , then dual gradient ascent with constant step sizes  $\alpha_k = \mu$  converges at sublinear rate  $O(\frac{1}{\epsilon})$ .

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<sup>1</sup>This is ignoring the role of  $A$ , and thus reflects the case when the singular values of  $A$  are all close to 1. To be more precise, the step sizes here should be:  $\frac{\mu}{\sigma_{\max}(A)^2}$  (first case) and  $\frac{2}{\frac{\sigma_{\max}(A)^2}{\mu} + \frac{\sigma_{\min}(A)^2}{L}}$  (second case).



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Note that this describes convergence in the dual. (Convergence in the primal requires more assumptions)

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$f \rightarrow \min_{x,y,z}$  Dual ascent

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Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into  $B$  separate problems:

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Can think of these steps as:

- **Broadcast:** Send  $u$  to each of the  $B$  processors, each optimizes in parallel to find  $x_i$ .
- **Gather:** Collect  $A_i x_i$  from each processor, update the global dual variable  $u$ .

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where  $(u)_+$  denotes the positive part of  $u$ , i.e.,  $(u_+)_i = \max\{0, u_i\}$ , for  $i = 1, \dots, m$ .

# Price Coordination Interpretation (Vandenberghe)

- **System Overview:** Consider a system with  $B$  units, where each unit independently chooses its decision variable  $x_i$ , which determines how to allocate its goods.

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- **Never let prices get negative;** hence the use of the positive part notation  $(\cdot)_+$ .



# Augmented Lagrangian method aka method of multipliers

**Dual ascent disadvantage:** convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

$$\begin{aligned} \min_x \quad & f(x) + \frac{\rho}{2} \|Ax - b\|^2 \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

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**Dual gradient ascent:** The iterative updates are given by:

$$\begin{aligned} x_k &= \arg \min_x \left[ f(x) + (u_{k-1})^T Ax + \frac{\rho}{2} \|Ax - b\|^2 \right] \\ u_k &= u_{k-1} + \rho(Ax_k - b) \end{aligned}$$

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**Notice step size choice  $\alpha_k = \rho$  in dual algorithm. Why?**

Since  $x_k$  minimizes the function:

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over  $x$ , we have the stationarity condition:

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This represents the stationarity condition for the original primal problem; under mild conditions,  $Ax_k - b \rightarrow 0$  as  $k \rightarrow \infty$ , so the KKT conditions are satisfied in the limit and  $x_k, u_k$  converge to the solutions.

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- **Advantage:** The augmented Lagrangian gives better convergence.
- **Disadvantage:** We lose decomposability! (Separability is ruined)

# Alternating Direction Method of Multipliers (ADMM)

**Alternating direction method of multipliers** or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

$$\min_{x,z} f(x) + g(z)$$

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where  $\rho > 0$  is a parameter. The augmented Lagrangian for this problem is defined as:

$$L_\rho(x, z, u) = f(x) + g(z) + u^T (Ax + Bz - c) + \frac{\rho}{2} \|Ax + Bz - c\|^2$$

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ADMM repeats the following steps, for  $k = 1, 2, 3, \dots$ :

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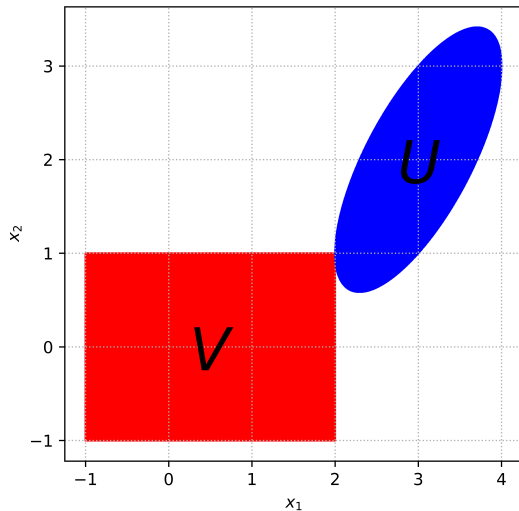
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**Note:** The usual method of multipliers would replace the first two steps by a joint minimization:

$$(x^{(k)}, z^{(k)}) = \arg \min_{x, z} L_\rho(x, z, u^{(k-1)})$$

## Example: Alternating Projections



Consider finding a point in the intersection of convex sets  $U, V \subseteq \mathbb{R}^n$ :

$$\min_x I_U(x) + I_V(x)$$

To transform this problem into ADMM form, we express it as:

$$\min_{x,z} I_U(x) + I_V(z) \quad \text{subject to} \quad x - z = 0$$

Each ADMM cycle involves two projections:

$$x_k = \arg \min_x P_U(z_{k-1} - w_{k-1})$$

$$z_k = \arg \min_z P_V(x_k + w_{k-1})$$

$$w_k = w_{k-1} + x_k - z_k$$

# Sources

- Ryan Tibshirani. Convex Optimization 10-725