

# Gradient Descent. Convergence rates

Daniil Merkulov

Optimization methods. MIPT

## Previously

- Gradient Descent

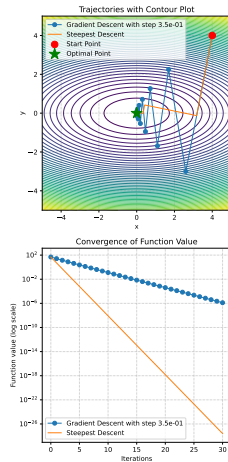



Figure 1: Steepest Descent

Open In Colab 

## Previously

- Gradient Descent
- Steepest descent

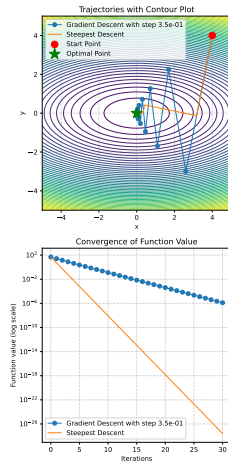



Figure 1: Steepest Descent

Open In Colab 

## Previously

- Gradient Descent
- Steepest descent
- Convergence rates (no proof)



Figure 1: Steepest Descent

Open In Colab 

## Previously

- Gradient Descent
- Steepest descent
- Convergence rates (no proof)
- If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $L$ -smooth then for all  $x, y \in \mathbb{R}^d$

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2.$$

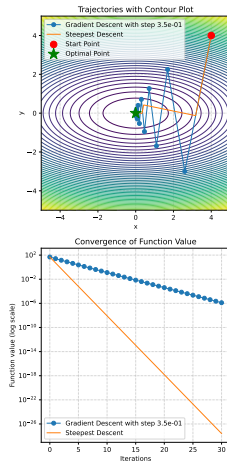


Figure 1: Steepest Descent

Open In Colab 

## Previously

- Gradient Descent
- Steepest descent
- Convergence rates (no proof)
- If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $L$ -smooth then for all  $x, y \in \mathbb{R}^d$

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2.$$

- Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a twice differentiable  $L$ -smooth function. Then, for all  $x \in \mathbb{R}^d$ , for every eigenvalue  $\lambda$  of  $\nabla^2 f(x)$ , we have

$$|\lambda| \leq L.$$

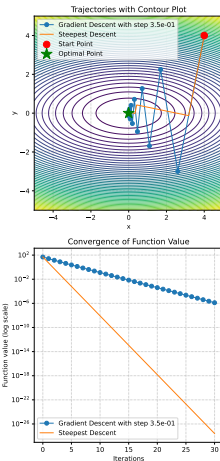


Figure 1: Steepest Descent

Open In Colab 

# Convergence rates

$$\min_{x \in \mathbb{R}^n} f(x) \qquad x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

smooth	convex	smooth & convex	smooth & strongly convex (or PL)
$\ \nabla f(x_k)\ ^2 \approx \mathcal{O}\left(\frac{1}{k}\right)$	$f(x_k) - f^* \approx \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$	$f(x_k) - f^* \approx \mathcal{O}\left(\frac{1}{k}\right)$	$\ x_k - x^*\ ^2 \approx \mathcal{O}\left(\left(1 - \frac{\mu}{L}\right)^k\right)$

# General quadratic problem



# General quadratic problem

## Polyak- Lojasiewicz condition. Linear convergence of gradient descent without convexity

PL inequality holds if the following condition is satisfied for some  $\mu > 0$ ,

$$\|\nabla f(x)\|^2 \geq \mu(f(x) - f^*) \forall x$$

It is interesting, that Gradient Descent algorithm has

The following functions satisfy the PL-condition, but are not convex. [🔗Link to the code](#)

$$f(x) = x^2 + 3 \sin^2(x)$$



## Polyak- Lojasiewicz condition. Linear convergence of gradient descent without convexity

PL inequality holds if the following condition is satisfied for some  $\mu > 0$ ,

$$\|\nabla f(x)\|^2 \geq \mu(f(x) - f^*) \forall x$$

It is interesting, that Gradient Descent algorithm has

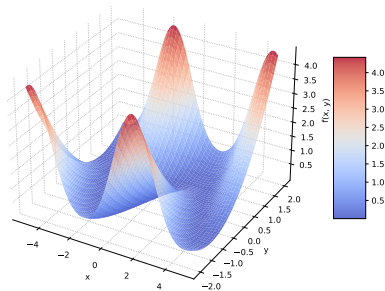
The following functions satisfy the PL-condition, but are not convex. [🔗Link to the code](#)

$$f(x) = x^2 + 3 \sin^2(x)$$



$$f(x, y) = \frac{(y - \sin x)^2}{2}$$

Non-convex PL function



# Gradient Descent convergence. Polyak-Łojasiewicz case

## Theorem

Consider the Problem

$$f(x) \rightarrow \min_{x \in \mathbb{R}^d}$$

and assume that  $f$  is  $\mu$ -Polyak-Łojasiewicz and  $L$ -smooth, for some  $L \geq \mu > 0$ .

Consider  $(x^t)_{t \in \mathbb{N}}$  a sequence generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying  $0 < \alpha \leq \frac{1}{L}$ . Then:

$$f(x^t) - f^* \leq (1 - \alpha\mu)^t (f(x^0) - f^*).$$

## Gradient Descent convergence. Polyak-Lojasiewicz case

We can use  $L$ -smoothness, together with the update rule of the algorithm, to write

$$\begin{aligned} f(x^{t+1}) &\leq f(x^t) + \langle \nabla f(x^t), x^{t+1} - x^t \rangle + \frac{L}{2} \|x^{t+1} - x^t\|^2 \\ &= f(x^t) - \alpha \|\nabla f(x^t)\|^2 + \frac{L\alpha^2}{2} \|\nabla f(x^t)\|^2 \\ &= f(x^t) - \frac{\alpha}{2} (2 - L\alpha) \|\nabla f(x^t)\|^2 \\ &\leq f(x^t) - \frac{\alpha}{2} \|\nabla f(x^t)\|^2, \end{aligned}$$

where in the last inequality we used our hypothesis on the stepsize that  $\alpha L \leq 1$ .

## Gradient Descent convergence. Polyak-Lojasiewicz case

We can use  $L$ -smoothness, together with the update rule of the algorithm, to write

$$\begin{aligned} f(x^{t+1}) &\leq f(x^t) + \langle \nabla f(x^t), x^{t+1} - x^t \rangle + \frac{L}{2} \|x^{t+1} - x^t\|^2 \\ &= f(x^t) - \alpha \|\nabla f(x^t)\|^2 + \frac{L\alpha^2}{2} \|\nabla f(x^t)\|^2 \\ &= f(x^t) - \frac{\alpha}{2} (2 - L\alpha) \|\nabla f(x^t)\|^2 \\ &\leq f(x^t) - \frac{\alpha}{2} \|\nabla f(x^t)\|^2, \end{aligned}$$

where in the last inequality we used our hypothesis on the stepsize that  $\alpha L \leq 1$ .

We can now use the Polyak-Lojasiewicz property to write:

$$f(x^{t+1}) \leq f(x^t) - \alpha\mu(f(x^t) - f^*).$$

The conclusion follows after subtracting  $f^*$  on both sides of this inequality, and using recursion.

# Gradient Descent convergence. Smooth convex case

## Theorem

Consider the Problem

$$f(x) \rightarrow \min_{x \in \mathbb{R}^d}$$

and assume that  $f$  is convex and  $L$ -smooth, for some  $L > 0$ .

Let  $(x^t)_{t \in \mathbb{N}}$  be the sequence of iterates generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying  $0 < \alpha \leq \frac{1}{L}$ . Then, for all  $x^* \in \operatorname{argmin} f$ , for all  $t \in \mathbb{N}$  we have that

$$f(x^t) - f^* \leq \frac{\|x^0 - x^*\|^2}{2\alpha t}.$$

# Gradient Descent convergence. Smooth convex case



# Gradient Descent convergence. Smooth $\mu$ -strongly convex case

## Theorem

Consider the Problem

$$f(x) \rightarrow \min_{x \in \mathbb{R}^d}$$

and assume that  $f$  is  $\mu$ -strongly convex and  $L$ -smooth, for some  $L \geq \mu > 0$ . Let  $(x^t)_{t \in \mathbb{N}}$  be the sequence of iterates generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying  $0 < \alpha \leq \frac{1}{L}$ . Then, for  $x^* = \operatorname{argmin} f$  and for all  $t \in \mathbb{N}$ :

$$\|x^{t+1} - x^*\|^2 \leq (1 - \alpha\mu)^{t+1} \|x^0 - x^*\|^2.$$

# Gradient Descent convergence. Smooth $\mu$ -strongly convex case

# Gradient Descent for Linear Least Squares aka Linear Regression



Figure 4: Illustration

In a least-squares, or linear regression, problem, we have measurements  $X \in \mathbb{R}^{m \times n}$  and  $y \in \mathbb{R}^m$  and seek a vector  $\theta \in \mathbb{R}^n$  such that  $X\theta$  is close to  $y$ . Closeness is defined as the sum of the squared differences:

$$\sum_{i=1}^m (x_i^\top \theta - y_i)^2 = \|X\theta - y\|_2^2 \rightarrow \min_{\theta \in \mathbb{R}^n}$$

For example, we might have a dataset of  $m$  users, each represented by  $n$  features. Each row  $x_i^\top$  of  $X$  is the features for user  $i$ , while the corresponding entry  $y_i$  of  $y$  is the measurement we want to predict from  $x_i^\top$ , such as ad spending. The prediction is given by  $x_i^\top \theta$ .


# Linear Least Squares aka Linear Regression <sup>1</sup>

1. Is this problem convex? Strongly convex?

# Linear Least Squares aka Linear Regression <sup>1</sup>

1. Is this problem convex? Strongly convex?
2. What do you think about convergence of Gradient Descent for this problem?

---

<sup>1</sup>Take a look at the  example of real-world data linear least squares problem

## $l_2$ -regularized Linear Least Squares

In the underdetermined case, it is often desirable to restore strong convexity of the objective function by adding an  $l_2$ -penalty, also known as Tikhonov regularization,  $l_2$ -regularization, or weight decay.

$$\|X\theta - y\|_2^2 + \frac{\mu}{2}\|\theta\|_2^2 \rightarrow \min_{\theta \in \mathbb{R}^n}$$

Note: With this modification the objective is  $\mu$ -strongly convex again.

Take a look at the code