

Gradient Descent. Convergence rates

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Optimization methods. MIPT

Previously

- Gradient Descent

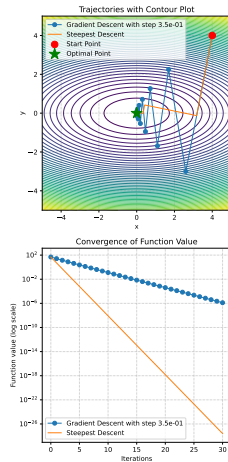



Figure 1: Steepest Descent

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Previously

- Gradient Descent
- Steepest descent

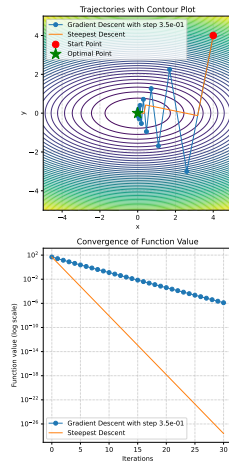



Figure 1: Steepest Descent

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- Convergence rates (no proof)



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- If $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is L -smooth then for all $x, y \in \mathbb{R}^d$

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2.$$



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$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2.$$

- Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a twice differentiable L -smooth function. Then, for all $x \in \mathbb{R}^d$, for every eigenvalue λ of $\nabla^2 f(x)$, we have

$$|\lambda| \leq L.$$

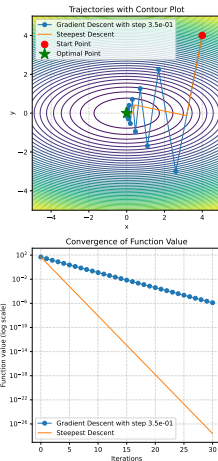


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Convergence rates

$$\min_{x \in \mathbb{R}^n} f(x) \qquad x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

smooth	convex	smooth & convex	smooth & strongly convex (or PL)
$\ \nabla f(x_k)\ ^2 \approx \mathcal{O}\left(\frac{1}{k}\right)$	$f(x_k) - f^* \approx \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$	$f(x_k) - f^* \approx \mathcal{O}\left(\frac{1}{k}\right)$	$\ x_k - x^*\ ^2 \approx \mathcal{O}\left(\left(1 - \frac{\mu}{L}\right)^k\right)$

General quadratic problem

General quadratic problem

Polyak- Lojasiewicz condition. Linear convergence of gradient descent without convexity

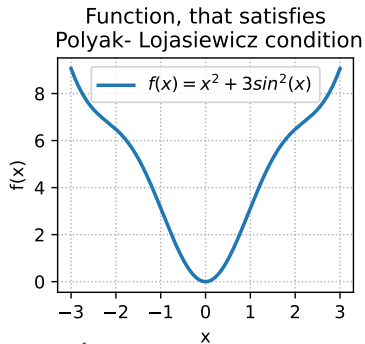
PL inequality holds if the following condition is satisfied for some $\mu > 0$,

$$\|\nabla f(x)\|^2 \geq \mu(f(x) - f^*) \forall x$$

It is interesting, that Gradient Descent algorithm has

The following functions satisfy the PL-condition, but are not convex. [🔗Link to the code](#)

$$f(x) = x^2 + 3 \sin^2(x)$$



Polyak- Lojasiewicz condition. Linear convergence of gradient descent without convexity

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The following functions satisfy the PL-condition, but are not convex. [🔗Link to the code](#)

$$f(x) = x^2 + 3 \sin^2(x)$$



$$f(x, y) = \frac{(y - \sin x)^2}{2}$$

Non-convex PL function



Gradient Descent convergence. Polyak-Łojasiewicz case

Theorem

Consider the Problem

$$f(x) \rightarrow \min_{x \in \mathbb{R}^d}$$

and assume that f is μ -Polyak-Łojasiewicz and L -smooth, for some $L \geq \mu > 0$.

Consider $(x^t)_{t \in \mathbb{N}}$ a sequence generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying $0 < \alpha \leq \frac{1}{L}$. Then:

$$f(x^t) - f^* \leq (1 - \alpha\mu)^t (f(x^0) - f^*).$$

Gradient Descent convergence. Polyak-Lojasiewicz case

We can use L -smoothness, together with the update rule of the algorithm, to write

$$\begin{aligned} f(x^{t+1}) &\leq f(x^t) + \langle \nabla f(x^t), x^{t+1} - x^t \rangle + \frac{L}{2} \|x^{t+1} - x^t\|^2 \\ &= f(x^t) - \alpha \|\nabla f(x^t)\|^2 + \frac{L\alpha^2}{2} \|\nabla f(x^t)\|^2 \\ &= f(x^t) - \frac{\alpha}{2} (2 - L\alpha) \|\nabla f(x^t)\|^2 \\ &\leq f(x^t) - \frac{\alpha}{2} \|\nabla f(x^t)\|^2, \end{aligned}$$

where in the last inequality we used our hypothesis on the stepsize that $\alpha L \leq 1$.

Gradient Descent convergence. Polyak-Lojasiewicz case

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where in the last inequality we used our hypothesis on the stepsize that $\alpha L \leq 1$.

We can now use the Polyak-Lojasiewicz property to write:

$$f(x^{t+1}) \leq f(x^t) - \alpha\mu(f(x^t) - f^*).$$

The conclusion follows after subtracting f^* on both sides of this inequality, and using recursion.

Gradient Descent convergence. Smooth convex case

Theorem

Consider the Problem

$$f(x) \rightarrow \min_{x \in \mathbb{R}^d}$$

and assume that f is convex and L -smooth, for some $L > 0$.

Let $(x^t)_{t \in \mathbb{N}}$ be the sequence of iterates generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying $0 < \alpha \leq \frac{1}{L}$. Then, for all $x^* \in \operatorname{argmin} f$, for all $t \in \mathbb{N}$ we have that

$$f(x^t) - f^* \leq \frac{\|x^0 - x^*\|^2}{2\alpha t}.$$

Gradient Descent convergence. Smooth convex case

Gradient Descent convergence. Smooth μ -strongly convex case

Theorem

Consider the Problem

$$f(x) \rightarrow \min_{x \in \mathbb{R}^d}$$

and assume that f is μ -strongly convex and L -smooth, for some $L \geq \mu > 0$. Let $(x^t)_{t \in \mathbb{N}}$ be the sequence of iterates generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying $0 < \alpha \leq \frac{1}{L}$. Then, for $x^* = \operatorname{argmin} f$ and for all $t \in \mathbb{N}$:

$$\|x^{t+1} - x^*\|^2 \leq (1 - \alpha\mu)^{t+1} \|x^0 - x^*\|^2.$$

Gradient Descent convergence. Smooth μ -strongly convex case

Gradient Descent for Linear Least Squares aka Linear Regression



Figure 4: Illustration

In a least-squares, or linear regression, problem, we have measurements $X \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^m$ and seek a vector $\theta \in \mathbb{R}^n$ such that $X\theta$ is close to y . Closeness is defined as the sum of the squared differences:

$$\sum_{i=1}^m (x_i^\top \theta - y_i)^2 = \|X\theta - y\|_2^2 \rightarrow \min_{\theta \in \mathbb{R}^n}$$


For example, we might have a dataset of m users, each represented by n features. Each row x_i^\top of X is the features for user i , while the corresponding entry y_i of y is the measurement we want to predict from x_i^\top , such as ad spending. The prediction is given by $x_i^\top \theta$.

Linear Least Squares aka Linear Regression ¹

1. Is this problem convex? Strongly convex?

Linear Least Squares aka Linear Regression ¹

1. Is this problem convex? Strongly convex?
2. What do you think about convergence of Gradient Descent for this problem?

¹Take a look at the  example of real-world data linear least squares problem

l_2 -regularized Linear Least Squares

In the underdetermined case, it is often desirable to restore strong convexity of the objective function by adding an l_2 -penalty, also known as Tikhonov regularization, l_2 -regularization, or weight decay.

$$\|X\theta - y\|_2^2 + \frac{\mu}{2}\|\theta\|_2^2 \rightarrow \min_{\theta \in \mathbb{R}^n}$$

Note: With this modification the objective is μ -strongly convex again.

Take a look at the code