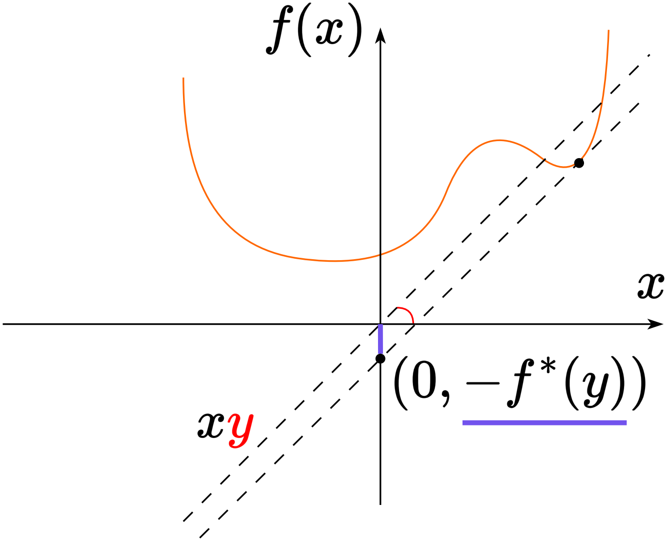
Introduction to dual methods

Daniil Merkulov

Optimization methods. MIPT

Definition



Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function defined by

$$\boxed{f^*(y) = \max_x \left[y^T x - f(x)\right]}$$

is called its conjugate.

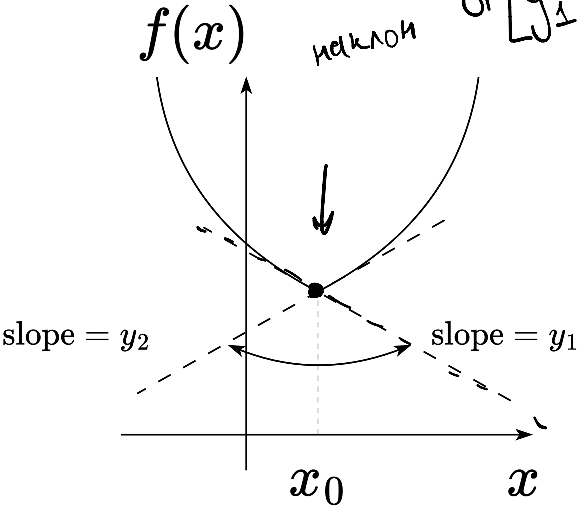
$$-f^*(y) = \min \left[f(x) - x^T y\right]$$

Geometrical intution

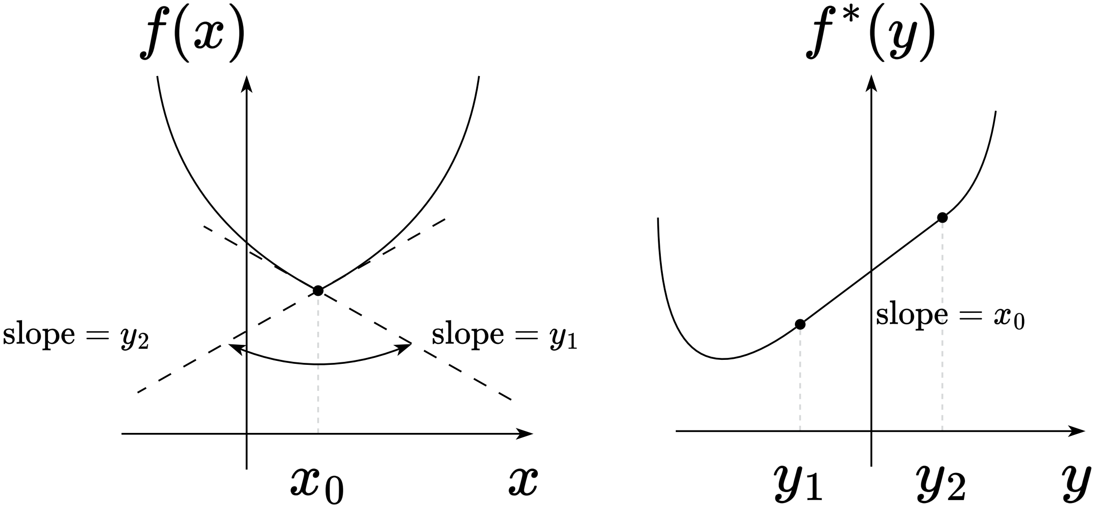
негладкий пример

наклон от $[y_1$ до $y_2]$

f^* на $(y_1; y_2] = x_0$



Geometrical intution



Conjugate function properties

Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function defined by

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is called its conjugate.

- Conjugates appear frequently in dual programs, since

$$\boxed{-f^*(y) = \min_x \left[f(x) - y^T x\right]}$$

Conjugate function properties

Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function defined by

$$f^*(y) = \max_x \left[y^T x - f(x) \right]$$

is called its conjugate.

- Conjugates appear frequently in dual programs, since

$$-f^*(y) = \min_x \left[f(x) - y^T x \right]$$

- If f is closed and convex, then $f^{**} = f$. Also, $\nabla f(x)$

$$\boxed{x \in \partial f^*(y)} \Leftrightarrow \boxed{y \cancel{\in \partial f}(x)} \Leftrightarrow x \in \arg\min_z \left[f(z) - y^T z \right]$$

$x = \nabla f^*(y)$ пусть

f - вып.
и диф.

Conjugate function properties

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- If f is closed and convex, then $f^{**} = f$. Also,

$$x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x) \Leftrightarrow \boxed{x \in \arg\min_z \left[f(z) - y^T z \right]}$$

строго

- If f is strictly convex, then

$$\nabla f^*(y) = \arg\min_z \left[f(z) - y^T z \right]$$

Conjugate function properties (proofs)

$$f^*(y) = \max_z z^T y - f(z)$$

We will show that $\boxed{x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x),}$ assuming that f is convex and closed.

- **Proof of** $\Leftarrow$: Suppose $y \in \partial f(x)$. Then $\underline{x \in M_y}$, the set of maximizers of $y^T z - f(z)$ over z. But

$$\boxed{f^*(y) = \max_z \{y^T z - f(z)\}} \quad \text{and} \quad \partial f^*(y) = \mathrm{cl}(\mathrm{conv}(\bigcup_{z \in M_y} \{z\})).$$

определение

посчитаем ∂f^* Дубовицкого-Милютина

Thus $\boxed{x \in \partial f^*(y).}$

$$f^*(y) = \max_z f_z(y)$$

$$f_z(y) = y^T z - f(z)$$

какая по y? линейная

$$\nabla_y f_z(y) = z$$

Conjugate function properties (proofs)

We will show that $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$, assuming that f is convex and closed.

- **Proof of $\Leftarrow$:** Suppose $\boxed{y \in \partial f(x)}$. Then $x \in M_y$, the set of maximizers of $y^T z - f(z)$ over z. But

$x \in \partial f^*(y)$

$$f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{and} \quad \partial f^*(y) = \text{cl}(\text{conv}(\bigcup_{z \in M_y} \{z\})).$$

Thus $x \in \partial f^*(y)$.

- **Proof of $\Rightarrow$:** From what we showed above, if $x \in \partial f^*(y)$, then $y \in \partial f^*(x)$, but $\boxed{f^{**} = f}$.

пусть $\boxed{x \in \partial f^*(y)} \Rightarrow y \in \partial f(x)$

в предыд. пункте: $y \in \partial f^{**}(x) \Rightarrow \boxed{y \in \partial f(x)}$

Conjugate function properties (proofs)

We will show that $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$, assuming that f is convex and closed.

- **Proof of** $\Leftarrow$: Suppose $y \in \partial f(x)$. Then $x \in M_y$, the set of maximizers of $y^T z - f(z)$ over z. But

$$f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{and} \quad \partial f^*(y) = \text{cl}(\text{conv}(\bigcup_{z \in M_y} \{z\})).$$

  Thus $x \in \partial f^*(y)$.

- **Proof of** $\Rightarrow$: From what we showed above, if $x \in \partial f^*(y)$, then $y \in \partial f^*(x)$, but $f^{**} = f$.

Conjugate function properties (proofs)

$y \in \partial f(x)$? $\forall \tilde{x}$

$f(\tilde{x}) \geq f(x) + y^T(\tilde{x} - x)$

We will show that $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$, assuming that f is convex and closed.

- **Proof of** $\Leftarrow$: Suppose $y \in \partial f(x)$. Then $x \in M_y$, the set of maximizers of $y^T z - f(z)$ over z. But

$$f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{and} \quad \partial f^*(y) = \text{cl}(\text{conv}(\bigcup_{z \in M_y} \{z\})).$$

$$\underbrace{f(\tilde{x}) - y^T\tilde{x}}_{\varphi(\tilde{x})} \geq \underbrace{f(x) - y^T x}_{\varphi(x)}$$

  Thus $x \in \partial f^*(y)$.

- **Proof of** $\Rightarrow$: From what we showed above, if $x \in \partial f^*(y)$, then $y \in \partial f^*(x)$, but $f^{**} = f$.

$\boxed{\text{Clearly } y \in \partial f(x) \Leftrightarrow \boxed{x \in \arg\min_z \{f(z) - y^T z\}}}$

Lastly, if f is strictly convex, then we know that $f(z) - y^T z$ has a unique minimizer over z, and this must be $\nabla f^*(y)$.

$\varphi(x) = f(x) - y^T x$

$\varphi(\tilde{x}) \geq \varphi(x)$

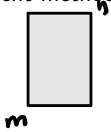
Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

Consider the problem:

$$\min_x \quad f(x) \quad \text{subject to} \quad Ax = b$$

$A \in \mathbb{R}^{m \times n}$



$$L(x,u) = f(x) + u^T(Ax - b)$$

$$g(u) = \inf_{x \in \mathbb{R}^n} L = \min_x f(x) + u^T A x - u^T b =$$

$$= \min_x \left(f(x) - (-A^T u)^T \cdot x - u^T b\right) =$$

$$\min_x \left[f(x) - x^T y\right] = -f^*(y) \qquad = -f^*(-A^T u) - u^T b$$

$$g(u) \to \max_{u \in \mathbb{R}^m}$$

$$\langle u, Ax \rangle = \langle A^T u, x \rangle$$

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Consider the problem:

$$\min_x \quad f(x) \quad \text{subject to} \quad Ax = b$$

Its dual problem is:

$$\boxed{\max_u \quad -f^*(-A^T u) - b^T u}$$

where f^* is the conjugate of f. Defining $g(u) = -f^*(-A^T u) - b^T u$, note that:

$$\partial g(u) = A\partial f^*(-A^T u) - b$$

Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

$y \in \partial f(x) \Leftrightarrow x \in \partial f^*(y)$

$\Leftrightarrow$

$x \in \arg\min_z f(z) - y^T z$

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Its dual problem is:

$$\max_u \quad -f^*(-A^T u) - b^T u$$

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$$\partial g(u) = A \partial f^*(-A^T u) - b$$

$A \cdot \tilde{x} - b$

$\tilde{x} \in \partial f^*(-A^T u)$

$-A^T u \in \partial f(x)$

Therefore, using what we know about conjugates

$$\boxed{\partial g(u) = Ax - b} \quad \text{where} \quad \boxed{\tilde{x} \in \arg\min_z \left[f(z) + u^T A z\right]}$$

$\max_{u \in \mathbb{R}^n} g(u)$

градиентный подъём

Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

Consider the problem:

$$
\min_x \quad f(x) \quad \text{subject to} \quad Ax = b
$$

Its dual problem is:

$g(u)$

$$
\boxed{\max_u \quad -f^*(-A^T u) - b^T u}
$$

where f^* is the conjugate of f. Defining $g(u) = -f^*(-A^T u) - b^T u$, note that:

$$
\partial g(u) = A \partial f^*(-A^T u) - b
$$

Therefore, using what we know about conjugates

$$
\partial g(u) = Ax - b \quad \text{where} \quad x \in \arg\min_z \left[f(z) + u^T Az \right]
$$

Dual ascent method for maximizing dual objective:

$$
x_k \in \arg\min_x \left[f(x) + (u_{k-1})^T Ax \right]
$$

$$
\boxed{u_k = u_{k-1} + \alpha_k (Ax_k - b)}
$$

$= \nabla_u g$

- Step sizes α_k, $k = 1, 2, 3, \ldots$, are chosen in standard ways.

градиентный подъём для функции $g(u)$: $u_{k+1} = u_k + \alpha_k \nabla_u g(u)$

Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

Consider the problem:

$$\min_x \quad f(x) \quad \text{subject to} \quad Ax = b$$

Its dual problem is:

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where f^* is the conjugate of f. Defining $g(u) = -f^*(-A^T u) - b^T u$, note that:

$$\partial g(u) = A \partial f^*(-A^T u) - b$$

Therefore, using what we know about conjugates

$$\partial g(u) = Ax - b \quad \text{where} \quad x \in \arg\min_z \left[f(z) + u^T A z \right]$$

Dual ascent method for maximizing dual objective:

Dual Ascent

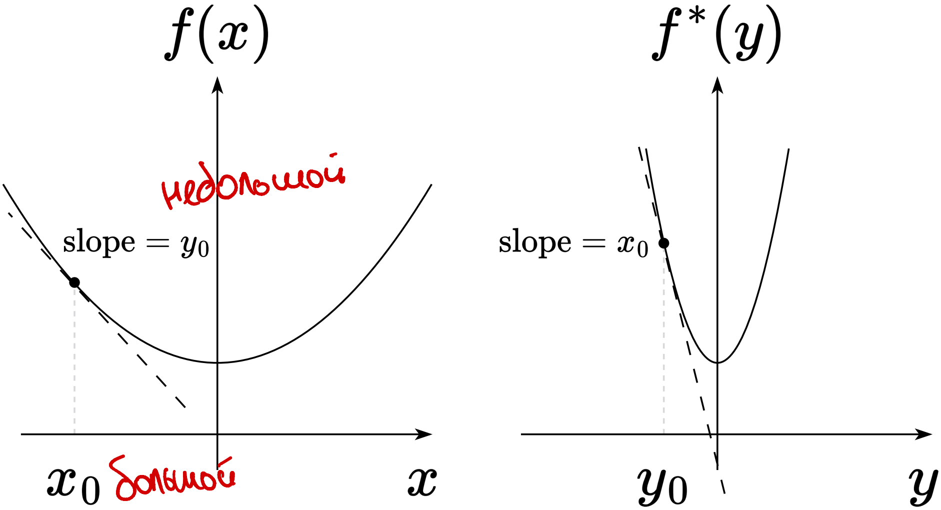
$$x_k \in \arg\min_x \left[f(x) + (u_{k-1})^T A x \right]$$

$$u_k = u_{k-1} + \alpha_k (A x_k - b)$$

- Step sizes α_k, $k = 1, 2, 3, \ldots$, are chosen in standard ways.
- Proximal gradients and acceleration can be applied as they would usually.

Slopes of f and f^*

$$y \in \partial f(x) \iff x \in \partial f^*(y)$$



Slopes of f and f^*

Assume that f is a closed and convex function. Then f is strongly convex with parameter $\mu \Leftrightarrow \nabla f^*$ is Lipschitz with parameter $1/\mu$.

f - сильно выпуклая $\iff \nabla f^*$ липшицев $\frac{1}{\mu}$

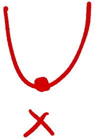
Slopes of f and f^*

$\Rightarrow$

Assume that f is a closed and convex function. Then f is strongly convex with parameter $\mu \Leftrightarrow \nabla f^*$ is Lipschitz with parameter $1/\mu$.

Proof of "$\Rightarrow$": Recall, if g is strongly convex with minimizer x, then

$$\boxed{g(y) \geq g(x) + \frac{\mu}{2}\|y - x\|^2,} \quad \text{for all } y$$



Slopes of f and f^*

x_u

Assume that f is a closed and convex function. Then f is strongly convex with parameter $\mu \Leftrightarrow \nabla f^*$ is Lipschitz with parameter $1/\mu$.

$\nabla f(z) = u$ $\quad \nabla f^*(u) = x_u$

Proof of "$\Rightarrow$": Recall, if g is strongly convex with minimizer x, then

$$g(y) \geq g(x) + \frac{\mu}{2}\|y - x\|^2, \quad \text{for all } y$$

$g_1(z) = f(z) - u^T z$

Hence, defining $\boxed{x_u = \nabla f^*(u)}$ and $\boxed{x_v = \nabla f^*(v)}$

$g_1(\tilde{x}) \geq g_1(x_u) + \frac{\mu}{2}\|\cdot\|$

$$f(x_v) - u^T x_v \geq f(x_u) - u^T x_u + \frac{\mu}{2}\|x_u - x_v\|^2$$

$g_2(z) = f(z) - v^T z$

$$f(x_u) - v^T x_u \geq f(x_v) - v^T x_v + \frac{\mu}{2}\|x_u - x_v\|^2$$

$g_2(\tilde{x}) \geq g_2(x_v) + \frac{\mu}{2}\|\cdot\|$

$\nabla f(z) = v$

$\nabla f^*(v) = x_v$

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$$

Hence, defining $x_u = \nabla f^*(u)$ and $x_v = \nabla f^*(v)$,

$$
\begin{aligned}
f(x_v) - u^T x_v &\geq f(x_u) - u^T x_u + \frac{\mu}{2}\|x_u - x_v\|^2 \\
f(x_u) - v^T x_u &\geq f(x_v) - v^T x_v + \frac{\mu}{2}\|x_u - x_v\|^2
\end{aligned}
$$

$\nabla f^* - \frac{1}{\mu}$ Лип

Adding these together, using the Cauchy-Schwarz inequality, and rearranging shows ~~that~~ $\Rightarrow$

$$
\|x_u - x_v\|^2 \leq \frac{1}{\mu}\|u - v\|^2
$$

$$
\|\nabla f^*(u) - \nabla f^*(v)\|^2 \leq \frac{1}{\mu}\|u - v\|^2
$$

Slopes of f and f^*

Proof of "$\Leftarrow$": for simplicity, call $g = f^*$ and $L = \frac{1}{\mu}$. As ∇g is Lipschitz with constant L, so is $g_x(z) = g(z) - \nabla g(x)^T z$, hence

$$g_x(z) \leq g_x(y) + \nabla g_x(y)^T (z - y) + \frac{L}{2}\|z - y\|_2^2$$

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Minimizing each side over z, and rearranging, gives

$$\frac{1}{2L} \|\nabla g(x) - \nabla g(y)\|^2 \leq g(y) - g(x) + \nabla g(x)^T (x - y)$$

Slopes of f and f^*

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$$\frac{1}{2L} \|\nabla g(x) - \nabla g(y)\|^2 \leq g(y) - g(x) + \nabla g(x)^T (x - y)$$

Exchanging roles of x, y, and adding together, gives

$$\frac{1}{L} \|\nabla g(x) - \nabla g(y)\|^2 \leq (\nabla g(x) - \nabla g(y))^T (x - y)$$

Slopes of f and f^*

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Let $u = \nabla f(x)$, $v = \nabla g(y)$; then $x \in \partial g^*(u)$, $y \in \partial g^*(v)$, and the above reads $(x - y)^T (u - v) \geq \frac{\|u - v\|^2}{L}$, implying the result.

Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent:[1]

- If f is strongly convex with parameter μ, then dual gradient ascent with constant step sizes $\alpha_k = \mu$ converges at sublinear rate $O(\frac{1}{\epsilon})$.

[1]This is ignoring the role of A, and thus reflects the case when the singular values of A are all close to 1. To be more precise, the step sizes here should be: $\frac{\mu}{\sigma_{\max}(A)^2}$ (first case) and $\frac{2}{\frac{\sigma_{\max}(A)^2}{\mu} + \frac{\sigma_{\min}(A)^2}{L}}$ (second case).

Convergence guarantees

СХ-ТЬ

двойственной задачи

The following results hold from combining the last fact with what we already know about gradient descent:[1]

- If f is strongly convex with parameter μ, then dual gradient ascent with constant step sizes $\alpha_k = \mu$ converges at sublinear rate $O(\frac{1}{\epsilon})$.
- If f is strongly convex with parameter μ and ∇f is Lipschitz with parameter L, then dual gradient ascent with step sizes $\alpha_k = \frac{2}{\frac{1}{\mu}+\frac{1}{L}}$ converges at linear rate $O(\log(\frac{1}{\epsilon}))$.

μ сильная выпуклость $\Rightarrow$ $\frac{1}{\mu}$ гладкость в DUAL

L гладкость $\Rightarrow$ $\frac{1}{L}$ сильная выпуклость в DUAL

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Note that this describes convergence in the dual. (Convergence in the primal requires more assumptions)

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Dual decomposition

Consider

$f(x)$

$\parallel$

$$\min_x \sum_{i=1}^{B} f_i(x_i) \quad \text{subject to} \quad Ax = b$$

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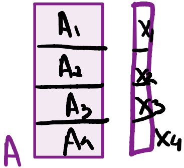

Here $x = (x_1, \ldots, x_B) \in \mathbb{R}^n$ divides into B blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition A accordingly:

$$A = [A_1 \ldots A_B], \text{ where } A_i \in \mathbb{R}^{m \times n_i}$$

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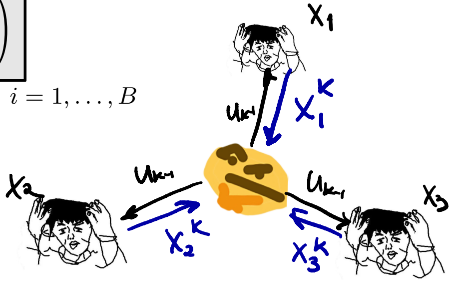
Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into B separate problems:

$$\boxed{x^{\text{new}} \in \arg\min_x \left(\sum_{i=1}^B f_i(x_i) + u^T A x \right)}$$

для каждого блока

$$\Rightarrow x_i^{\text{new}} \in \arg\min_{x_i} \left(f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \dots, B$$

$$\boxed{\begin{aligned} x_i^k &\in \arg\min_{x_i} \left(f_i(x_i) + (u^{k-1})^T A_i x_i \right), \quad i = 1, \dots, B \\ u^k &= u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b \right) \end{aligned}}$$



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$$\Rightarrow x_i^{\text{new}} \in \arg\min_{x_i} \left(f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \ldots, B$$

$$x_i^k \in \arg\min_{x_i} \left(f_i(x_i) + (u^{k-1})^T A_i x_i \right), \quad i = 1, \ldots, B$$

$$u^k = u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b \right)$$

Can think of these steps as:

- **Broadcast:** Send u to each of the B processors, each optimizes in parallel to find x_i.

Dual decomposition

Consider

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Here $x = (x_1, \dots, x_B) \in \mathbb{R}^n$ divides into B blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition A accordingly:

$$A = [A_1 \dots A_B], \text{ where } A_i \in \mathbb{R}^{m \times n_i}$$

Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into B separate problems:

$$x^{\text{new}} \in \arg\min_x \left(\sum_{i=1}^B f_i(x_i) + u^T A x \right)$$

$$\Rightarrow x_i^{\text{new}} \in \arg\min_{x_i} \left(f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \dots, B$$

$$x_i^k \in \arg\min_{x_i} \left(f_i(x_i) + (u^{k-1})^T A_i x_i \right), \quad i = 1, \dots, B$$

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Can think of these steps as:

- **Broadcast:** Send u to each of the B processors, each optimizes in parallel to find x_i.
- **Gather:** Collect $A_i x_i$ from each processor, update the global dual variable u.

Inequality constraints

$Ax=b$

Consider the optimization problem:

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad \sum_{i=1}^B A_i x_i \leq b$$

Метод проекций градиента

$$g(u) \to \max_{u \in \mathbb{R}^n}$$

$$u \geq 0$$

S –

срезка

$$y = \begin{pmatrix} -1 \\ 2 \\ -3 \\ -4 \\ 5 \end{pmatrix} \qquad y_+ = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 5 \end{pmatrix}$$

$$\Pi_S(y) = y_+$$

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Using **dual decomposition**, specifically the **projected subgradient method**, the iterative steps can be expressed as:

- The primal update step:

$$x_i^k \in \arg\min_{x_i} \left[f_i(x_i) + \left(u^{k-1}\right)^T A_i x_i \right], \quad i = 1, \dots, B$$

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метод проекции градиента для max $g(u)$

- The dual update step:

$$\boxed{u^k = \left(u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b \right) \right)_+}$$

where $(u)_+$ denotes the positive part of u, i.e., $(u_+)_i = \max\{0, u_i\}$, for $i = 1, \ldots, m$.

Price Coordination Interpretation (Vandenberghe)

- **System Overview**: Consider a system with B units, where each unit independently chooses its decision variable x_i, which determines how to allocate its goods.

Price Coordination Interpretation (Vandenberghe)

прямые переменные

A n m

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ресурсы нефть ГАЗ мозги мемы песок

m штук

$a_i^T x_i \le b_i$

люди, ограничения

которые что-то делают

$u_i \leftarrow$ двойств. пер.

цены на ресурсы

$$\left(\min_{x_i} f(x_i) + u_{k-1}^T A x_i\right)$$

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градиентный подъём

$$u_j^{\text{new}} = (u_j - ts_j)_+, \quad j = 1, \dots, m$$

where $s = b - \sum_{i=1}^{B} A_i x_i$ represents the slacks.

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  - **Never let prices get negative**; hence the use of the positive part notation $(\cdot)_+$.

Augmented Lagrangian method aka method of multipliers

Dual ascent disadvantage: convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

добавление сильной выпуклости

$$
\begin{aligned}
&\min_x f(x) + \frac{\rho}{2}\|Ax - b\|^2 \\
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$0 \in \partial f(x_k) + A^T u + \rho A^T(Ax_k - b)$

Dual gradient ascent: The iterative updates are given by:

$$
\boxed{
\begin{aligned}
x_k &= \arg\min_x \left[f(x) + (u_{k-1})^T Ax + \frac{\rho}{2}\|Ax - b\|^2 \right] \\
u_k &= u_{k-1} + \rho(Ax_k - b)
\end{aligned}
}
$$

Augmented Lagrangian method aka method of multipliers

$y \in \partial f(x)$

$\Leftrightarrow$

$x \in \partial f^*(y)$

$\Leftrightarrow$

$x \in \arg\min_z \left(f(z) - y^T z\right)$

Notice step size choice $\alpha_k = \rho$ in dual algorithm. Why?

Since x_k minimizes the function:

$$
f(x) + (u_{k-1})^T Ax + \frac{\rho}{2}\|Ax - b\|^2
$$

over x, we have the stationarity condition:

$$
\boxed{0 \in \partial f(x_k) + A^T \big(\underbrace{u_{k-1} + \rho(Ax_k - b)}_{u_k}\big)}
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which simplifies to:

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This represents the stationarity condition for the original primal problem; under mild conditions, $Ax_k - b \to 0$ as $k \to \infty$, so the KKT conditions are satisfied in the limit and x_k, u_k converge to the solutions.

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- **Advantage:** The augmented Lagrangian gives better convergence.
- **Disadvantage:** We lose decomposability! (Separability is ruined)

Alternating Direction Method of Multipliers (ADMM)

Alternating direction method of multipliers or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

$$\min_{x,z} f(x) + g(z)$$

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where $\rho > 0$ is a parameter. The augmented Lagrangian for this problem is defined as:

$$L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|^2$$

Alternating Direction Method of Multipliers (ADMM)

ADMM repeats the following steps, for $k = 1, 2, 3, \dots$:

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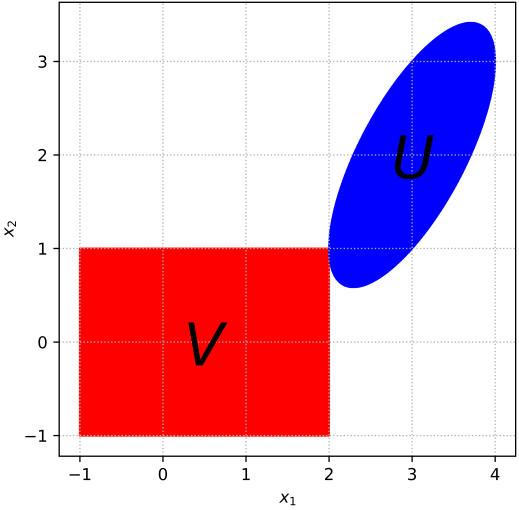
Note: The usual method of multipliers would replace the first two steps by a joint minimization:

$$(x^{(k)}, z^{(k)}) = \arg\min_{x,z} L_\rho(x, z, u^{(k-1)})$$

Example: Alternating Projections



Consider finding a point in the intersection of convex sets $U, V \subseteq \mathbb{R}^n$:

$$\min_x I_U(x) + I_V(x)$$

To transform this problem into ADMM form, we express it as:

$$\min_{x,z} I_U(x) + I_V(z) \quad \text{subject to} \quad x - z = 0$$

Each ADMM cycle involves two projections:

$$\begin{cases} x_k = \arg\min_x P_U\left(z_{k-1} - w_{k-1}\right) \\ z_k = \arg\min_z P_V\left(x_k + w_{k-1}\right) \\ w_k = w_{k-1} + x_k - z_k \end{cases}$$

Sources

- Ryan Tibshirani. Convex Optimization 10-725