ADMM

Daniil Merkulov

Optimization methods. MIPT

Dual (sub)gradient method

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Dual gradient ascent

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- **Good:** x update decomposes when f does.
- **Bad:** require stringent assumptions (strong convexity of f) to ensure convergence

Augmented Lagrangian method aka method of multipliers

Augmented Lagrangian method transforms the primal problem to:

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Alternating Direction Method of Multipliers (ADMM)

Alternating direction method of multipliers or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

$$\min_{x,z} f(x) + g(z)$$

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where $\rho > 0$ is a parameter. The augmented Lagrangian for this problem is defined as:

$$L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|^2$$

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Note: The usual method of multipliers would replace the first two steps by a joint minimization:

$$(x^{(k)}, z^{(k)}) = \arg\min_{x,z} L_\rho(x, z, u^{(k-1)})$$

Convergence

Assume (very little!)

- f, g convex, closed, proper

then ADMM converges:

If the functions f and g are convex and computationally friendly for $\arg\min$, then ADMM has the following convergence bound for any $x \in \mathbb{R}^{d_x}$, $y \in \mathbb{R}^{d_y}$, $\lambda \in \mathbb{R}^n$:

$$L_0\left(\frac{1}{k}\sum_{i=1}^{k} x_i, \frac{1}{k}\sum_{i=1}^{K} y_i, \lambda\right) - L_0(x, y, \frac{1}{k}\sum_{i=1}^{k} \lambda_k) \leq \frac{1}{2k}\|z_0 - z\|_P^2,$$

where L_0 is the Lagrangian without augmentation, P and the initial value of z^0 are defined as :

$$P = \begin{pmatrix} \rho A^T A & 0 & -A^T \\ 0 & 0 & 0 \\ -A & 0 & \frac{1}{\rho} I \end{pmatrix} \quad z^0 = \begin{pmatrix} x^0 \\ y^0 \\ \lambda^0 \end{pmatrix}.$$

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- objective approaches optimal value: $f(x_k) + g(z_k) \to p^*$

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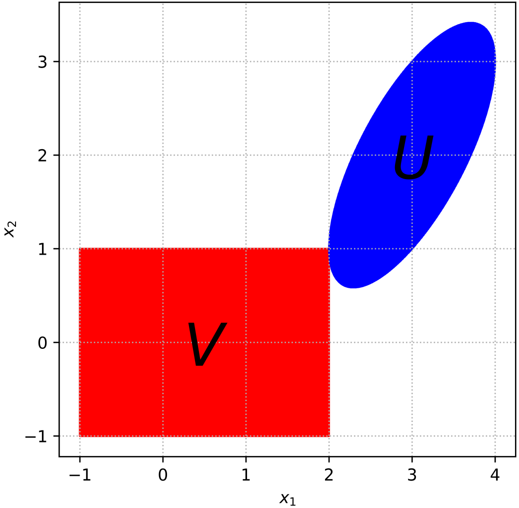
Example:

Example: Alternating Projections



Consider finding a point in the intersection of convex sets $U, V \subseteq \mathbb{R}^n$:

$$\min_x I_U(x) + I_V(x)$$

To transform this problem into ADMM form, we express it as:

$$\min_{x,z} I_U(x) + I_V(z) \quad \text{subject to} \quad x - z = 0$$

Each ADMM cycle involves two projections:

$$x_k = \arg\min_x P_U\left(z_{k-1} - w_{k-1}\right)$$

$$z_k = \arg\min_z P_V\left(x_k + w_{k-1}\right)$$

$$w_k = w_{k-1} + x_k - z_k$$

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- ADMM is one of the key and popular recent optimization methods.
- It is implemented in many solvers and is often used as a default method.
- The non-standard formulation of the problem itself, for which ADMM is invented, turns out to include many important special cases. "Unusual" variable y often plays the role of an auxiliary variable.
- Here the penalty is an additional modification to stabilize and accelerate convergence. It is not necessary to make ρ very large.

Sources

- Alternating Direction Method of Multipliers by S.Boyd

Sources

- Alternating Direction Method of Multipliers by S.Boyd
- Ryan Tibshirani. ConvAlternating Direction Method of Multipliers by S.Boydex Optimization 10-725