



**Recap of Conjugate sets, conjugate functions.  
Subgradient and subdifferential**

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Optimization methods. MIPT

## Conjugate sets

# Conjugate set

Let  $S \subseteq \mathbb{R}^n$  be an arbitrary non-empty set. Then its conjugate set is defined as:

$$S^* = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \geq -1 \quad \forall x \in S\}$$

A set  $S^{**}$  is called double conjugate to a set  $S$  if:

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- A set  $S$  is called **self-conjugate** if  $S^* = S$ .



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- $S^* = (\overline{S})^*$ .

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- Let  $p \in S^*$  and  $x_0 \in \overline{S}$ ,  $x_0 = \lim_{k \rightarrow \infty} x_k$ . Then by virtue of the continuity of the function  $f(x) = p^T x$ , we have:  
 $p^T x_k \geq -1 \rightarrow p^T x_0 \geq -1$ . So  $p \in (\overline{S})^*$ , hence  $S^* \subset (\overline{S})^*$ .

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- Let  $p \in S^*$ ,  $x_0 \in \text{conv}(S)$ , i.e.,  $x_0 = \sum_{i=1}^k \theta_i x_i \mid x_i \in S, \sum_{i=1}^k \theta_i = 1, \theta_i \geq 0$ .

So  $p^T x_0 = \sum_{i=1}^k \theta_i p^T x_i \geq \sum_{i=1}^k \theta_i (-1) = 1 \cdot (-1) = -1$ . So  $p \in (\text{conv}(S))^*$ , hence  $S^* \subset (\text{conv}(S))^*$ .



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- From all points of the ball  $X$ , take such a point  $x \in X$  that its scalar product of  $p$ :  $p^T x$  is minimal, then this is the point  $x = -\frac{p}{\|p\|}r$ .

$$p^T x = p^T \left( -\frac{p}{\|p\|}r \right) = -\|p\|r \geq -1$$

$$\|p\| \leq \frac{1}{r} \in Y$$

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- Now let  $p \in Y$ . We need to show that  $p \in X^*$ , i.e.,  $\langle p, x \rangle \geq -1$ . It's enough to apply the Cauchy-Bunyakovsky inequality:

$$\|\langle p, x \rangle\| \leq \|p\| \|x\| \leq \frac{1}{r} \cdot r = 1$$

The latter comes from the fact that  $p \in B(0, 1/r)$  and  $x \in B(0, r)$ .

So  $Y \subset X^*$ .

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## Dual cone

A conjugate cone to a cone  $K$  is a set  $K^*$  such that:

$$K^* = \{y \mid \langle x, y \rangle \geq 0 \quad \forall x \in K\}$$

To show that this definition follows directly from the definitions above, recall what a conjugate set is and what a cone  $\forall \lambda > 0$  is.

$$\{y \in \mathbb{R}^n \mid \langle y, x \rangle \geq -1 \quad \forall x \in S\} \rightarrow \{\lambda y \in \mathbb{R}^n \mid \langle y, x \rangle \geq -\frac{1}{\lambda} \quad \forall x \in S\}$$



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- Let  $K_1, \dots, K_m$  be cones in  $\mathbb{R}^n$ , then:

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- Let  $K_1, \dots, K_m$  be cones in  $\mathbb{R}^n$ . Let also their intersection have an interior point, then:

$$\left( \bigcap_{i=1}^m K_i \right)^* = \sum_{i=1}^m K_i^*$$

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Find the conjugate cone for a monotone nonnegative cone:

$$K = \{x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \dots \geq x_n \geq 0\}$$

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Note that:

$$\sum_{i=1}^n x_i y_i = y_1(x_1 - x_2) + (y_1 + y_2)(x_2 - x_3) + \dots + (y_1 + y_2 + \dots + y_{n-1})(x_{n-1} - x_n) + (y_1 + \dots + y_n)x_n$$

Since in the presented sum in each summand, the second multiplier in each summand is non-negative, then:

$$y_1 \geq 0, \quad y_1 + y_2 \geq 0, \quad \dots, \quad y_1 + \dots + y_n \geq 0$$

$$\text{So } K^* = \left\{ y \mid \sum_{i=1}^k y_i \geq 0, k = \overline{1, n} \right\}.$$

# Polyhedra

The set of solutions to a system of linear inequalities and equalities is a polyhedron:

$$Ax \preceq b, \quad Cx = d$$

Here  $A \in \mathbb{R}^{m \times n}$ ,  $C \in \mathbb{R}^{p \times n}$ , and the inequality is a piecewise inequality.

## Theorem

Let  $x_1, \dots, x_m \in \mathbb{R}^n$ . Conjugate to a polyhedral set:

$$S = \mathbf{conv}(x_1, \dots, x_k) + \mathbf{cone}(x_{k+1}, \dots, x_m)$$

is a polyhedron (polyhedron):

$$S^* = \{p \in \mathbb{R}^n \mid \langle p, x_i \rangle \geq -1, i = \overline{1, k}; \langle p, x_i \rangle \geq 0, i = \overline{k+1, m}\}$$

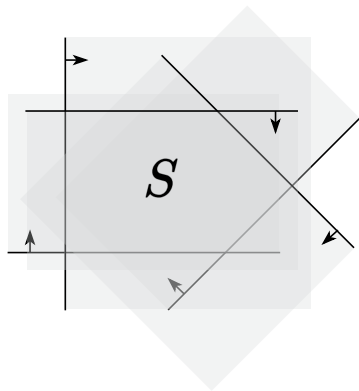


Figure 3: Polyhedra

## Proof

- Let  $S = X, S^* = Y$ . Take some  $p \in X^*$ , then  $\langle p, x_i \rangle \geq -1, i = \overline{1, k}$ . At the same time, for any  $\theta > 0, i = \overline{k+1, m}$ :

$$\langle p, x_i \rangle \geq -1 \rightarrow \langle p, \theta x_i \rangle \geq -1$$

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- Suppose, on the other hand, that  $p \in Y$ . For any point  $x \in X$ :

$$x = \sum_{i=1}^m \theta_i x_i \quad \sum_{i=1}^k \theta_i = 1, \theta_i \geq 0$$

So:

$$\langle p, x \rangle = \sum_{i=1}^m \theta_i \langle p, x_i \rangle = \sum_{i=1}^k \theta_i \langle p, x_i \rangle + \sum_{i=k+1}^m \theta_i \langle p, x_i \rangle \geq \sum_{i=1}^k \theta_i (-1) + \sum_{i=1}^k \theta_i \cdot 0 = -1.$$

# Example

## Conjugate functions



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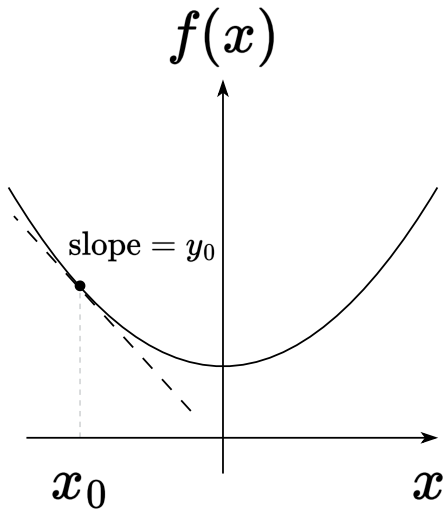
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Hence, defining  $x_u = \nabla f^*(u)$  and  $x_v = \nabla f^*(v)$ ,

$$f(x_v) - u^T x_v \geq f(x_u) - u^T x_u + \frac{\mu}{2} \|x_u - x_v\|^2$$

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Adding these together, using the Cauchy-Schwarz inequality, and rearranging shows that

$$\|x_u - x_v\|^2 \leq \frac{1}{\mu} \|u - v\|^2$$



## Slopes of $f$ and $f^*$

**Proof of “ $\Leftarrow$ ”:** for simplicity, call  $g = f^*$  and  $L = \frac{1}{\mu}$ . As  $\nabla g$  is Lipschitz with constant  $L$ , so is  $g_x(z) = g(z) - \nabla g(x)^T z$ , hence

$$g_x(z) \leq g_x(y) + \nabla g_x(y)^T (z - y) + \frac{L}{2} \|z - y\|_2^2$$

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Minimizing each side over  $z$ , and rearranging, gives

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Exchanging roles of  $x$ ,  $y$ , and adding together, gives

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Let  $u = \nabla f(x)$ ,  $v = \nabla g(y)$ ; then  $x \in \partial g^*(u)$ ,  $y \in \partial g^*(v)$ , and the above reads  $(x - y)^T (u - v) \geq \frac{\|u - v\|^2}{L}$ , implying the result.

# Conjugate function properties

Recall that given  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the function defined by

$$f^*(y) = \max_x [y^T x - f(x)]$$

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- If  $f$  is strictly convex, then

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# Conjugate function properties (proofs)

We will show that  $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$ , assuming that  $f$  is convex and closed.

- **Proof of  $\Leftarrow$ :** Suppose  $y \in \partial f(x)$ . Then  $x \in M_y$ , the set of maximizers of  $y^T z - f(z)$  over  $z$ . But

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Clearly  $y \in \partial f(x) \Leftrightarrow x \in \arg \min_z \{f(z) - y^T z\}$

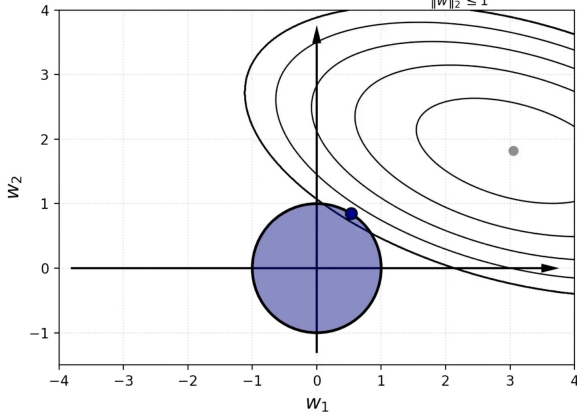
Lastly, if  $f$  is strictly convex, then we know that  $f(z) - y^T z$  has a unique minimizer over  $z$ , and this must be  $\nabla f^*(y)$ .

## Subgradient and Subdifferential

# $\ell_1$ -regularized linear least squares

$\ell_1$  induces sparsity

$\ell_2$  regularization.  $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_2 \leq 1}$



$\ell_1$  regularization.  $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_1 \leq 1}$



@fminxyz

# Norms are not smooth

$$\min_{x \in \mathbb{R}^n} f(x),$$

A classical convex optimization problem is considered. We assume that  $f(x)$  is a convex function, but now we do not require smoothness.

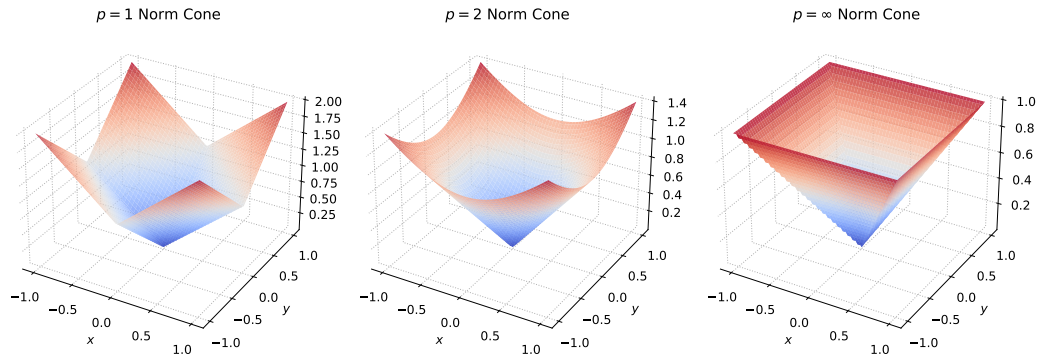


Figure 5: Norm cones for different  $p$  - norms are non-smooth

## Convex function linear lower bound

An important property of a continuous convex function  $f(x)$  is that at any chosen point  $x_0$  for all  $x \in \text{dom } f$  the inequality holds:

$$f(x) \geq f(x_0) + \langle g, x - x_0 \rangle$$

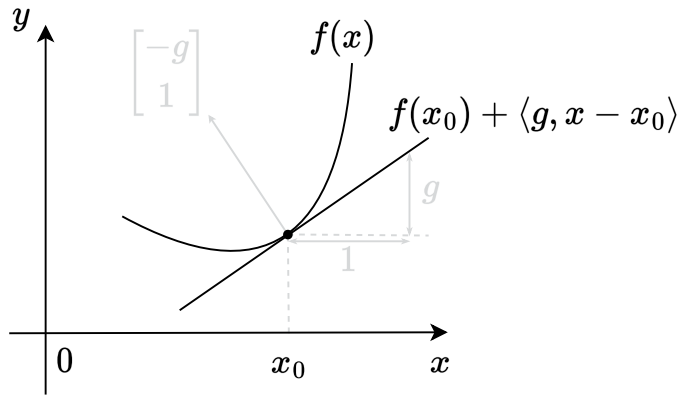


Figure 6: Taylor linear approximation serves as a global lower bound for a convex function

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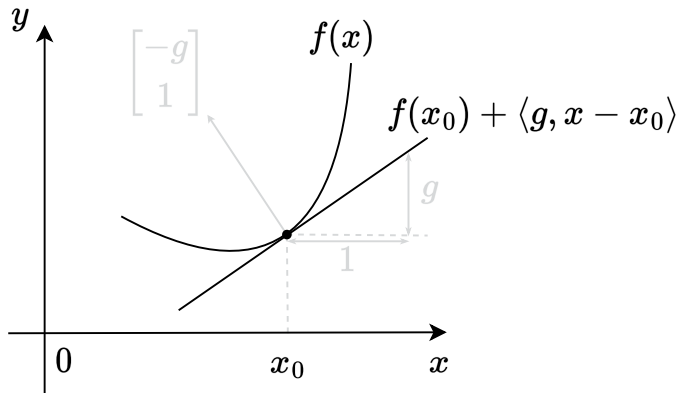
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We wouldn't want to lose such a nice property.

Figure 6: Taylor linear approximation serves as a global lower bound for a convex function

## Subgradient and subdifferential

A vector  $g$  is called the **subgradient** of a function  $f(x) : S \rightarrow \mathbb{R}$  at a point  $x_0$  if  $\forall x \in S$ :

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Figure 7: Subdifferential is a set of all possible subgradients

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Find  $\partial f(x)$ , if  $f(x) = |x|$

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Let  $f : S \rightarrow \mathbb{R}$  be a function defined on the set  $S$  in a Euclidean space  $\mathbb{R}^n$ . If  $x_0 \in \text{ri}(S)$  and  $f$  is differentiable at  $x_0$ , then either  $\partial f(x_0) = \emptyset$  or  $\partial f(x_0) = \{\nabla f(x_0)\}$ . Moreover, if the function  $f$  is convex, the first scenario is impossible.

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2. From this,  $\langle s - \nabla f(x_0), v \rangle \geq 0$ . Due to the arbitrariness of  $v$ , one can set

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3. Furthermore, if the function  $f$  is convex, then according to the differential condition of convexity  $f(x) \geq f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle$  for all  $x \in S$ . But by definition, this means  $\nabla f(x_0) \in \partial f(x_0)$ .

# Subdifferentiability and convexity

## i Question

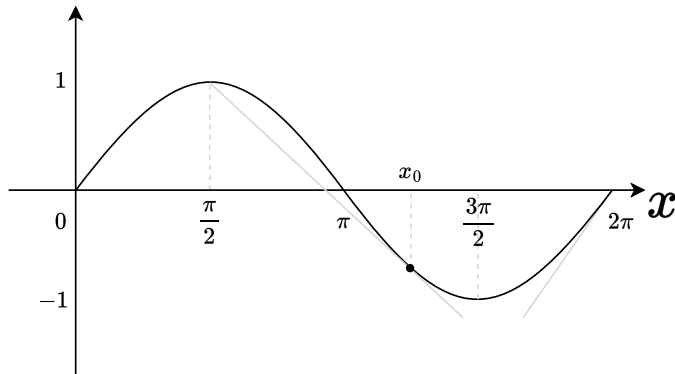
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# Subdifferentiability and convexity

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Find  $\partial f(x)$ , if  $f(x) = \sin x$ ,  $x \in [\pi/2; 2\pi]$



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Convexity follows from subdifferentiability at any point. A natural question to ask is whether the converse is true: is every convex function subdifferentiable? It turns out that, generally speaking, the answer to this question is negative.

Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be the function defined by  $f(x) := -\sqrt{x}$ . Then,  $\partial f(0) = \emptyset$ .

Assume, that  $s \in \partial f(0)$  for some  $s \in \mathbb{R}$ . Then, by definition, we must have  $sx \leq -\sqrt{x}$  for all  $x \geq 0$ . From this, we can deduce  $s \leq -\sqrt{1/x}$  for all  $x > 0$ . Taking the limit as  $x$  approaches 0 from the right, we get  $s \leq -\infty$ , which is impossible.

# Subdifferential calculus

**i** Moreau - Rockafellar theorem (subdifferential of a linear combination)

Let  $f_i(x)$  be convex functions on convex sets  $S_i$ ,  $i = \overline{1, n}$ . Then if  $\bigcap_{i=1}^n \text{ri}(S_i) \neq \emptyset$  then the function

$f(x) = \sum_{i=1}^n a_i f_i(x)$ ,  $a_i > 0$  has a subdifferential

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**i** Dubovitsky - Milutin theorem (subdifferential of a point-wise maximum)

Let  $f_i(x)$  be convex functions on the open convex set  $S \subseteq \mathbb{R}^n$ ,  $x_0 \in S$ , and the pointwise maximum is defined as  $f(x) = \max_i f_i(x)$ . Then:

$$\partial_S f(x_0) = \text{conv} \left\{ \bigcup_{i \in I(x_0)} \partial_S f_i(x_0) \right\}, \quad I(x) = \{i \in [1, n] \mid f_i(x) = f(x)\}$$

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- $z \in \partial f(x)$  if and only if  $x \in \partial f^*(z)$ .

## Connection to convex geometry

Convex set  $S \subseteq \mathbb{R}^n$ , consider indicator function  $I_S : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$I_S(x) = I\{x \in S\} = \begin{cases} 0 & \text{if } x \in S \\ \infty & \text{if } x \notin S \end{cases}$$

For  $x \in S$ ,  $\partial I_S(x) = \mathcal{N}_S(x)$ , the **normal cone** of  $S$  at  $x$  is, recall

$$\mathcal{N}_S(x) = \{g \in \mathbb{R}^n : g^T x \geq g^T y \text{ for any } y \in S\}$$

**Why?** By definition of subgradient  $g$ ,

$$I_S(y) \geq I_S(x) + g^T(y - x) \quad \text{for all } y$$

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- For  $y \in S$ , this means  $0 \geq g^T(y - x)$



# Optimality Condition

For any  $f$  (convex or not),

$$f(x^*) = \min_x f(x) \iff 0 \in \partial f(x^*)$$

That is,  $x^*$  is a minimizer if and only if 0 is a subgradient of  $f$  at  $x^*$ . This is called the **subgradient optimality condition**.

Why? Easy:  $g = 0$  being a subgradient means that for all  $y$

$$f(y) \geq f(x^*) + 0^T(y - x^*) = f(x^*)$$

Note the implication for a convex and differentiable function  $f$ , with

$$\partial f(x) = \{\nabla f(x)\}$$

# Derivation of first-order optimality

Example of the power of subgradients: we can use what we have learned so far to derive the **first-order optimality condition**. Recall

$$\min_x f(x) \text{ subject to } x \in S$$

is solved at  $x$ , for  $f$  convex and differentiable, if and only if

$$\nabla f(x)^T(y - x) \geq 0 \quad \text{for all } y \in S$$

Intuitively: this says that the gradient increases as we move away from  $x$ . How to prove it? First, recast the problem as

$$\min_x f(x) + I_S(x)$$

Now apply subgradient optimality:

$$0 \in \partial(f(x) + I_S(x))$$

$$f(x) = x_1 + x_2 \rightarrow \min_{x_1, x_2 \in \mathbb{R}^2}$$



$$\langle -\nabla f(x^*), d \rangle \leq 0$$

$x^*$  - optimal



$$\langle -\nabla f(x^\dagger), d \rangle \leq 0$$

$x^\dagger$  - not optimal

# Derivation of first-order optimality

Observe

$$0 \in \partial(f(x) + I_S(x))$$

$$\Leftrightarrow 0 \in \{\nabla f(x)\} + \mathcal{N}_S(x)$$

$$\Leftrightarrow -\nabla f(x) \in \mathcal{N}_S(x)$$

$$\Leftrightarrow -\nabla f(x)^T x \geq -\nabla f(x)^T y \text{ for all } y \in S$$

$$\Leftrightarrow \nabla f(x)^T (y - x) \geq 0 \text{ for all } y \in S$$

as desired.

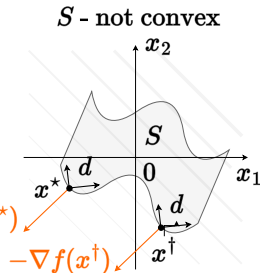
Note: the condition  $0 \in \partial f(x) + \mathcal{N}_S(x)$  is a **fully general condition** for optimality in convex problems. But it's not always easy to work with (KKT conditions, later, are easier).

$$f(x) = x_1 + x_2 \rightarrow \min_{x_1, x_2 \in \mathbb{R}^2}$$



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$$\partial f_1(x) = \begin{cases} -1, & x < 1 \\ [-1; 1], & x = 1 \\ 1, & x > 1 \end{cases} \quad \partial f_2(x) = \begin{cases} -1, & x < -1 \\ [-1; 1], & x = -1 \\ 1, & x > -1 \end{cases}$$

So

$$\partial f(x) = \begin{cases} -2, & x < -1 \\ [-2; 0], & x = -1 \\ 0, & -1 < x < 1 \\ [0; 2], & x = 1 \\ 2, & x > 1 \end{cases}$$

## Example 2

Find  $\partial f(x)$  if  $f(x) = [\max(0, f_0(x))]^q$ . Here,  $f_0(x)$  is a convex function on an open convex set  $S$ , and  $q \geq 1$ .

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According to the composition theorem (the function  $\varphi(x) = x^q$  is differentiable) and  $g(x) = \max(0, f_0(x))$ , we have:

$$\partial f(x) = q(g(x))^{q-1} \partial g(x)$$

By the theorem on the pointwise maximum:

$$\partial g(x) = \begin{cases} \partial f_0(x), & f_0(x) > 0, \\ \{0\}, & f_0(x) < 0, \\ \{a \mid a = \lambda a', 0 \leq \lambda \leq 1, a' \in \partial f_0(x)\}, & f_0(x) = 0 \end{cases}$$

### Example 3. Subdifferential of the Norm

Let  $V$  be a finite-dimensional Euclidean space, and  $x_0 \in V$ . Let  $\|\cdot\|$  be an arbitrary norm in  $V$  (not necessarily induced by the scalar product), and let  $\|\cdot\|_*$  be the corresponding conjugate norm. Then,

$$\partial\|\cdot\|(x_0) = \begin{cases} B_{\|\cdot\|_*}(0, 1), & \text{if } x_0 = 0, \\ \{s \in V : \|s\|_* \leq 1; \langle s, x_0 \rangle = \|x_0\|\} = \{s \in V : \|s\|_* = 1; \langle s, x_0 \rangle = \|x_0\|\}, & \text{otherwise.} \end{cases}$$

Where  $B_{\|\cdot\|_*}(0, 1)$  is the closed unit ball centered at zero with respect to the conjugate norm. In other words, a vector  $s \in V$  with  $\|s\|_* = 1$  is a subgradient of the norm  $\|\cdot\|$  at point  $x_0 \neq 0$  if and only if the Hölder's inequality  $\langle s, x_0 \rangle \leq \|x_0\|$  becomes an equality.

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Let  $s \in V$ . By definition,  $s \in \partial\|\cdot\|(x_0)$  if and only if

$$\langle s, x \rangle - \|x\| \leq \langle s, x_0 \rangle - \|x_0\|, \text{ for all } x \in V,$$

or equivalently,

$$\sup_{x \in V} \{\langle s, x \rangle - \|x\|\} \leq \langle s, x_0 \rangle - \|x_0\|.$$

By the definition of the supremum, the latter is equivalent to

$$\sup_{x \in V} \{\langle s, x \rangle - \|x\|\} = \langle s, x_0 \rangle - \|x_0\|.$$

Subgradient and Subdifferential

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By the definition of the supremum, the latter is equivalent to

It is important to note that the expression on the left side is the supremum from the definition of the Fenchel conjugate function for the norm, which is known to be

$$\sup_{x \in V} \{\langle s, x \rangle - \|x\|\} = \begin{cases} 0, & \text{if } \|s\|_* \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, equation is equivalent to  $\|s\|_* \leq 1$  and  $\langle s, x_0 \rangle = \|x_0\|$ .

## Example 3. Subdifferential of the Norm

Consequently, it remains to note that for  $x_0 \neq 0$ , the inequality  $\|s\|_* \leq 1$  must become an equality since, when  $\|s\|_* < 1$ , Hölder's inequality implies  $\langle s, x_0 \rangle \leq \|s\|_* \|x_0\| < \|x_0\|$ .

The conjugate norm in Example above does not appear by chance. It turns out that, in a completely similar manner for an arbitrary function  $f$  (not just for the norm), its subdifferential can be described in terms of the dual object — the Fenchel conjugate function.