# Strongly convex functions. Polyak - Lojasiewicz Condition. Conjugate sets


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Optimization methods. MIPT

# Strong convexity criteria

# First-order differential criterion of convexity

The differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is convex if and only if $\forall x, y \in S$:

$$f(y) \geq f(x) + \nabla f^T(x)(y - x)$$

Let $y = x + \Delta x$, then the criterion will become more tractable:

$$f(x + \Delta x) \geq f(x) + \nabla f^T(x)\Delta x$$

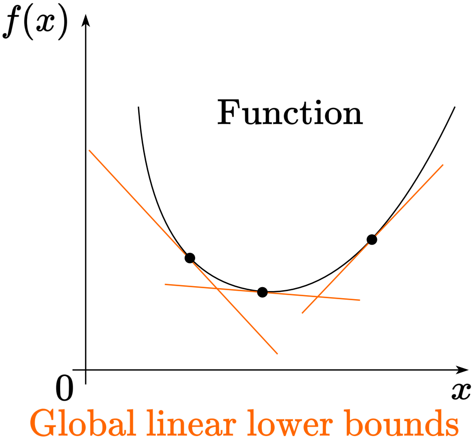


Figure 1: Convex function is greater or equal than Taylor linear approximation at any point

# Second-order differential criterion of convexity

Twice differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is convex if and only if $\forall x \in \mathbf{int}(S) \neq \emptyset$:

$$\nabla^2 f(x) \succeq 0$$

In other words, $\forall y \in \mathbb{R}^n$:

$$\langle y, \nabla^2 f(x) y \rangle \geq 0$$

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  If $f(x)$ - is a convex function defined on the convex set $S \subseteq \mathbb{R}^n$, then for any $\beta$ sublevel set $\mathcal{L}_\beta$ is convex.

  The function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is closed if and only if for any $\beta$ sublevel set $\mathcal{L}_\beta$ is closed.

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- **Reduction to a line**

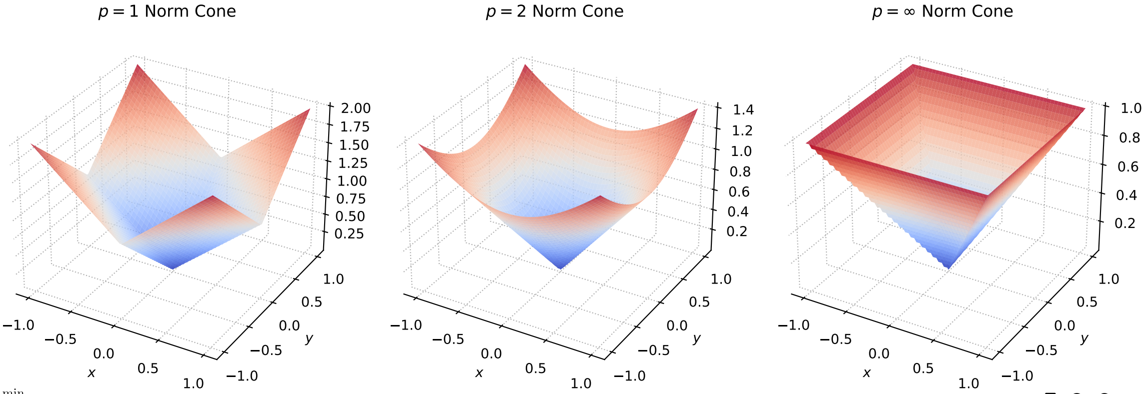
  $f : S \to \mathbb{R}$ is convex if and only if $S$ is a convex set and the function $g(t) = f(x + tv)$ defined on $\{t \mid x + tv \in S\}$ is convex for any $x \in S, v \in \mathbb{R}^n$, which allows checking convexity of the scalar function to establish convexity of the vector function.

# Example: norm cone

Let a norm $\|\cdot\|$ be defined in the space $U$. Consider the set:

$$K := \{(x,t) \in U \times \mathbb{R}^+ : \|x\| \leq t\}$$

which represents the epigraph of the function $x \mapsto \|x\|$. This set is called the cone norm. According to the statement above, the set $K$ is convex. Code for the figures



# Strong convexity

$f(x)$, **defined on the convex set** $S \subseteq \mathbb{R}^n$, is called $\mu$-strongly convex (strongly convex) on $S$, if:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) - \frac{\mu}{2}\lambda(1-\lambda)\|x_1 - x_2\|^2$$

for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$ for some $\mu > 0$.

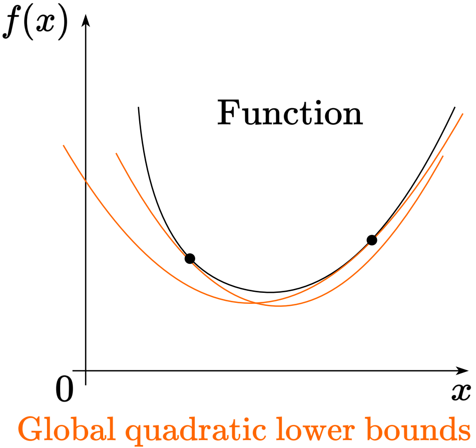



Figure 3: Strongly convex function is greater or equal than Taylor quadratic approximation at any point

# First-order differential criterion of strong convexity

Differentiable $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is $\mu$-strongly convex if and only if $\forall x, y \in S$:

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**Theorem**

Let $f(x)$ be a differentiable function on a convex set $X \subseteq \mathbb{R}^n$. Then $f(x)$ is strongly convex on $X$ with a constant $\mu > 0$ if and only if

$$f(x) - f(x_0) \geq \langle \nabla f(x_0), x - x_0 \rangle + \frac{\mu}{2}\|x - x_0\|^2$$

for all $x, x_0 \in X$.

# Proof of first-order differential criterion of strong convexity

**Necessity**: Let $0 < \lambda \leq 1$. According to the definition of a strongly convex function,

$$f(\lambda x + (1-\lambda)x_0) \leq \lambda f(x) + (1-\lambda)f(x_0) - \frac{\mu}{2}\lambda(1-\lambda)\|x - x_0\|^2$$

or equivalently,

$$f(x) - f(x_0) - \frac{\mu}{2}(1-\lambda)\|x - x_0\|^2 \geq \frac{1}{\lambda}[f(\lambda x + (1-\lambda)x_0) - f(x_0)] =$$

$$= \frac{1}{\lambda}[f(x_0 + \lambda(x - x_0)) - f(x_0)] = \frac{1}{\lambda}[\lambda\langle\nabla f(x_0), x - x_0\rangle + o(\lambda)] =$$

$$= \langle\nabla f(x_0), x - x_0\rangle + \frac{o(\lambda)}{\lambda}.$$

Thus, taking the limit as $\lambda \downarrow 0$, we arrive at the initial statement.

# Proof of first-order differential criterion of strong convexity

**Sufficiency**: Assume the inequality in the theorem is satisfied for all $x, x_0 \in X$. Take $x_0 = \lambda x_1 + (1-\lambda)x_2$, where $x_1, x_2 \in X$, $0 \leq \lambda \leq 1$. According to the inequality, the following inequalities hold:

$$f(x_1) - f(x_0) \geq \langle \nabla f(x_0), x_1 - x_0 \rangle + \frac{\mu}{2}\|x_1 - x_0\|^2,$$

$$f(x_2) - f(x_0) \geq \langle \nabla f(x_0), x_2 - x_0 \rangle + \frac{\mu}{2}\|x_2 - x_0\|^2.$$

Multiplying the first inequality by $\lambda$ and the second by $1-\lambda$ and adding them, considering that

$$x_1 - x_0 = (1-\lambda)(x_1 - x_2), \quad x_2 - x_0 = \lambda(x_2 - x_1),$$

and $\lambda(1-\lambda)^2 + \lambda^2(1-\lambda) = \lambda(1-\lambda)$, we get

$$\lambda f(x_1) + (1-\lambda) f(x_2) - f(x_0) - \frac{\mu}{2}\lambda(1-\lambda)\|x_1 - x_2\|^2 \geq \\ \langle \nabla f(x_0), \lambda x_1 + (1-\lambda)x_2 - x_0 \rangle = 0.$$

Thus, inequality from the definition of a strongly convex function is satisfied. It is important to mention, that $\mu = 0$ stands for the convex case and corresponding differential criterion.

## Second-order differential criterion of strong convexity

Twice differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is called $\mu$-strongly convex if and only if $\forall x \in \mathbf{int}(S) \neq \emptyset$:

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In other words:

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**Theorem**

Let $X \subseteq \mathbb{R}^n$ be a convex set, with $\text{int}X \neq \emptyset$. Furthermore, let $f(x)$ be a twice continuously differentiable function on $X$. Then $f(x)$ is strongly convex on $X$ with a constant $\mu > 0$ if and only if

$$\langle y, \nabla^2 f(x) y \rangle \geq \mu \|y\|^2$$

for all $x \in X$ and $y \in \mathbb{R}^n$.

# Proof of second-order differential criterion of strong convexity

The target inequality is trivial when $y = \mathbf{0}_n$, hence we assume $y \neq \mathbf{0}_n$.

**Necessity**: Assume initially that $x$ is an interior point of $X$. Then $x + \alpha y \in X$ for all $y \in \mathbb{R}^n$ and sufficiently small $\alpha$. Since $f(x)$ is twice differentiable,

$$f(x + \alpha y) = f(x) + \alpha \langle \nabla f(x), y \rangle + \frac{\alpha^2}{2} \langle y, \nabla^2 f(x) y \rangle + o(\alpha^2).$$

Based on the first order criterion of strong convexity, we have

$$\frac{\alpha^2}{2} \langle y, \nabla^2 f(x) y \rangle + o(\alpha^2) = f(x + \alpha y) - f(x) - \alpha \langle \nabla f(x), y \rangle \geq \frac{\mu}{2} \alpha^2 \|y\|^2.$$

This inequality reduces to the target inequality after dividing both sides by $\alpha^2$ and taking the limit as $\alpha \downarrow 0$.

If $x \in X$ but $x \notin \text{int}X$, consider a sequence $\{x_k\}$ such that $x_k \in \text{int}X$ and $x_k \to x$ as $k \to \infty$. Then, we arrive at the target inequality after taking the limit.

# Proof of second-order differential criterion of strong convexity

**Sufficiency**: Using Taylor's formula with the Lagrange remainder and the target inequality, we obtain for $x + y \in X$:

$$
f(x+y) - f(x) - \langle \nabla f(x), y \rangle = \frac{1}{2} \langle y, \nabla^2 f(x + \alpha y) y \rangle \geq \frac{\mu}{2} \|y\|^2,
$$

where $0 \leq \alpha \leq 1$. Therefore,

$$
f(x+y) - f(x) \geq \langle \nabla f(x), y \rangle + \frac{\mu}{2} \|y\|^2.
$$

Consequently, by the first order criterion of strong convexity, the function $f(x)$ is strongly convex with a constant $\mu$. It is important to mention, that $\mu = 0$ stands for the convex case and corresponding differential criterion.

# Convex and concave function

Example

Show, that $f(x) = c^\top x + b$ is convex and concave.

# Simplest strongly convex function

**Example**

Show, that $f(x) = x^\top Ax$, where $A \succeq 0$ - is convex on $\mathbb{R}^n$. Is it strongly convex?

# Convexity and continuity

Let $f(x)$ - be a convex function on a convex set $S \subseteq \mathbb{R}^n$. Then $f(x)$ is continuous $\forall x \in \textbf{ri}(S)$. [1]

### Proper convex function

Function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **proper convex function** if it never takes on the value $-\infty$ and not identically equal to $\infty$.

### Indicator function

$$\delta_S(x) = \begin{cases} \infty, & x \in S, \\ 0, & x \notin S, \end{cases}$$

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**Closed function**

Function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **closed** if for each $\alpha \in \mathbb{R}$, the sublevel set is a closed set. Equivalently, if the epigraph is closed, then the function $f$ is closed.

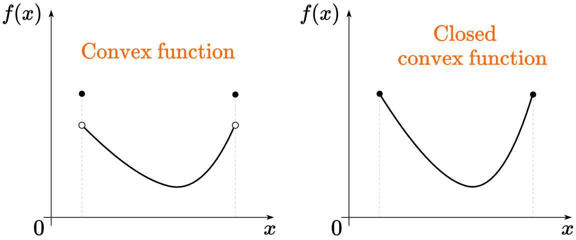


Figure 4: The concept of a closed function is introduced to avoid such breaches at the border.

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for $\alpha_i \geq 0; \quad \sum_{i=1}^{n} \alpha_i = 1$ (probability simplex)
For the infinite dimension case:

$$f\left(\int\limits_{S} x p(x) dx\right) \leq \int\limits_{S} f(x) p(x) dx$$

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  If the integrals exist and $p(x) \geq 0, \quad \int\limits_S p(x) dx = 1.$
- If the function $f(x)$ and the set $S$ are convex, then any local minimum $x^* = \arg\min\limits_{x \in S} f(x)$ will be the global one. Strong convexity guarantees the uniqueness of the solution.

# Operations that preserve convexity

- Non-negative sum of the convex functions: $\alpha f(x) + \beta g(x), (\alpha \geq 0, \beta \geq 0)$.

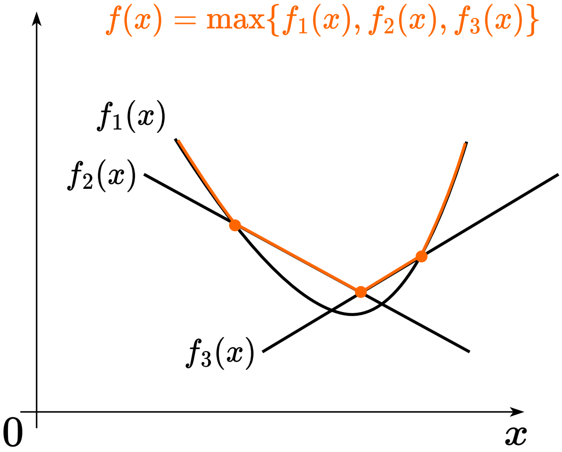


Figure 5: Pointwise maximum (supremum) of convex functions is convex

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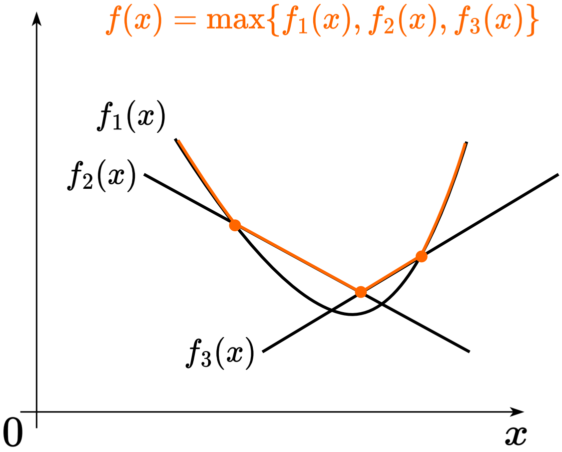


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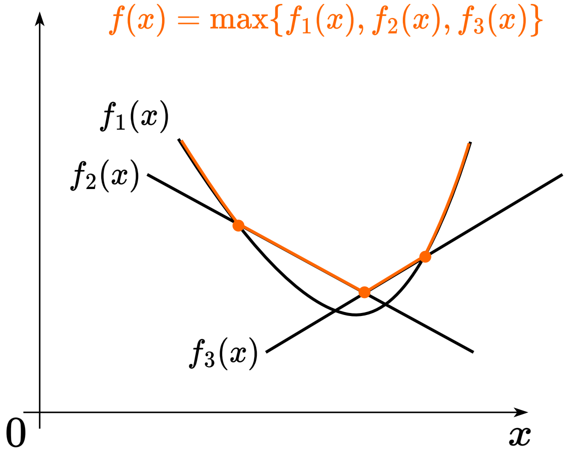


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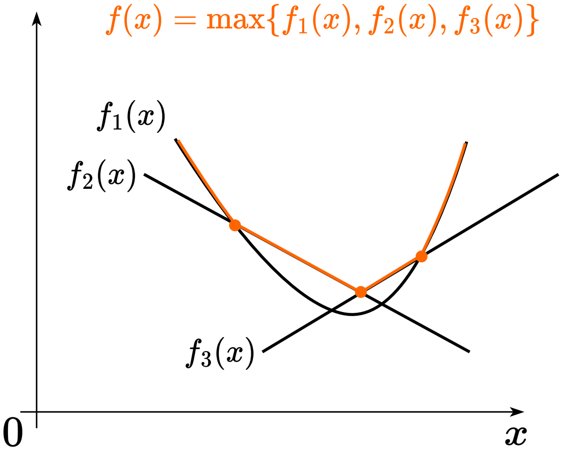


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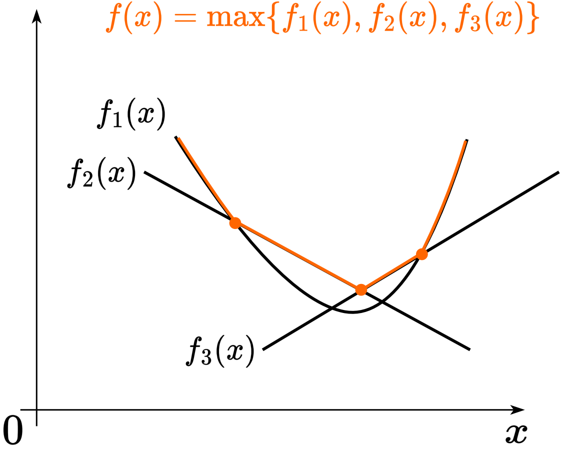
- If $f(x)$ is convex on $S$, then $g(x, t) = t f(x/t)$ - is convex with $x/t \in S, t > 0$.

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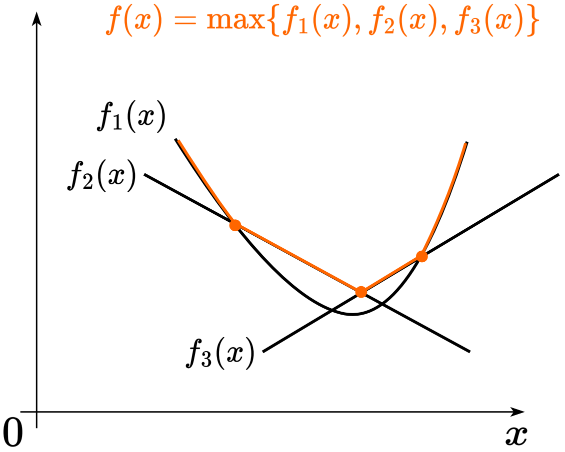
- If $f(x, y)$ is convex on $x$ for any $y \in Y$: $g(x) = \sup\limits_{y \in Y} f(x, y)$ is convex.
- If $f(x)$ is convex on $S$, then $g(x, t) = tf(x/t)$ - is convex with $x/t \in S, t > 0$.
- Let $f_1 : S_1 \to \mathbb{R}$ and $f_2 : S_2 \to \mathbb{R}$, where $\text{range}(f_1) \subseteq S_2$. If $f_1$ and $f_2$ are convex, and $f_2$ is increasing, then $f_2 \circ f_1$ is convex on $S_1$.

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# Maximum eigenvalue of a matrix is a convex function

**Example**

Show, that $f(A) = \lambda_{max}(A)$ - is convex, if $A \in S^n$.

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- Discrete convexity: $f : \mathbb{Z}^n \to \mathbb{Z}$; "convexity + matroid theory."

# Polyak- Lojasiewicz condition. Linear convergence of gradient descent without convexity

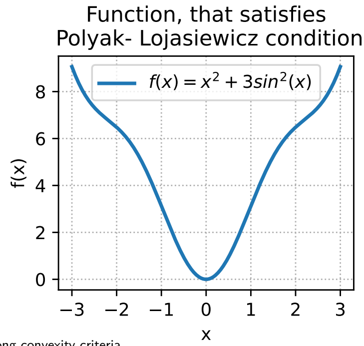
PL inequality holds if the following condition is satisfied for some $\mu > 0$,

$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*) \forall x$$

It is interesting, that Gradient Descent converges linearly under this condition (weaker, then strong convexity).

The following functions satisfy the PL-condition, but are not convex. Link to the code

$$f(x) = x^2 + 3\sin^2(x)$$



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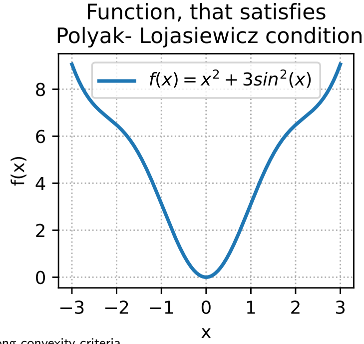


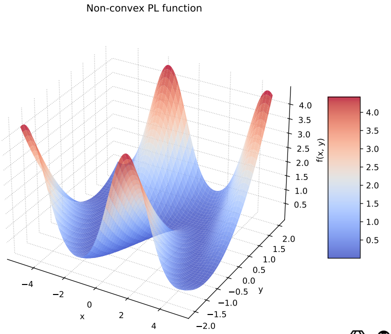

# Convexity in ML

# Linear Least Squares aka Linear Regression

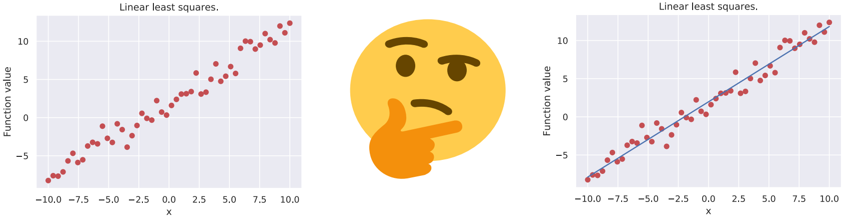


Figure 8: Illustration

In a least-squares, or linear regression, problem, we have measurements $X \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^m$ and seek a vector $\theta \in \mathbb{R}^n$ such that $X\theta$ is close to $y$. Closeness is defined as the sum of the squared differences:

$$\sum_{i=1}^{m} (x_i^\top \theta - y_i)^2 = \|X\theta - y\|_2^2 \to \min_{\theta \in \mathbb{R}^n}$$

For example, we might have a dataset of $m$ users, each represented by $n$ features. Each row $x_i^\top$ of $X$ is the features for user $i$, while the corresponding entry $y_i$ of $y$ is the measurement we want to predict from $x_i^\top$, such as ad spending. The prediction is given by $x_i^\top \theta$.

# Linear Least Squares aka Linear Regression [2]

1. Is this problem convex? Strongly convex?

# Linear Least Squares aka Linear Regression [2]

1. Is this problem convex? Strongly convex?
2. What do you think about convergence of Gradient Descent for this problem?

[2] Take a look at the example of real-world data linear least squares problem

# $l_2$-regularized Linear Least Squares

In the underdetermined case, it is often desirable to restore strong convexity of the objective function by adding an $l_2$-penality, also known as Tikhonov regularization, $l_2$-regularization, or weight decay.

$$\|X\theta - y\|_2^2 + \frac{\mu}{2}\|\theta\|_2^2 \to \min_{\theta \in \mathbb{R}^n}$$

Note: With this modification the objective is $\mu$-strongly convex again.

Take a look at the code

# Most important difference between convexity and strong convexity

$$f(x) = \frac{1}{2m}\|Ax - b\|_2^2 + \frac{\mu}{2}\|x\|_2^2 \to \min_{x \in \mathbb{R}^n}, \quad A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

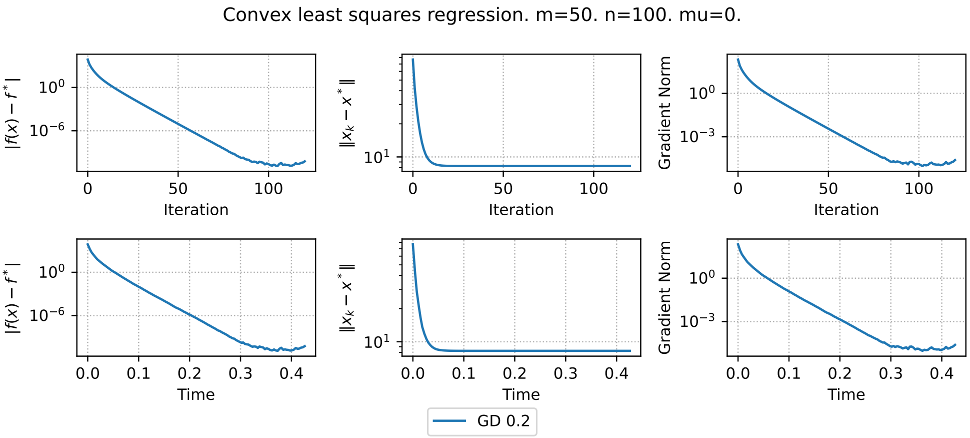


Figure 9: Convex problem does not have convergence in domain

# Most important difference between convexity and strong convexity

$$
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$$

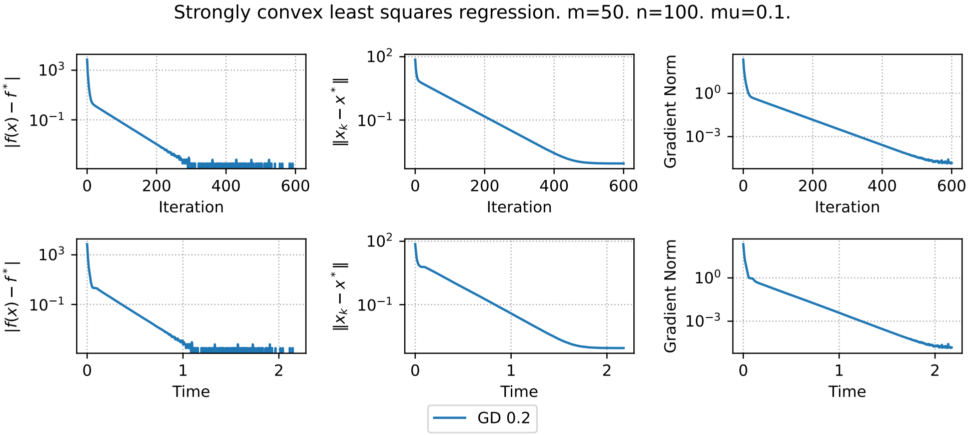


Figure 10: But if you add even small amount of regularization, you will ensure convergence in domain

# Most important difference between convexity and strong convexity

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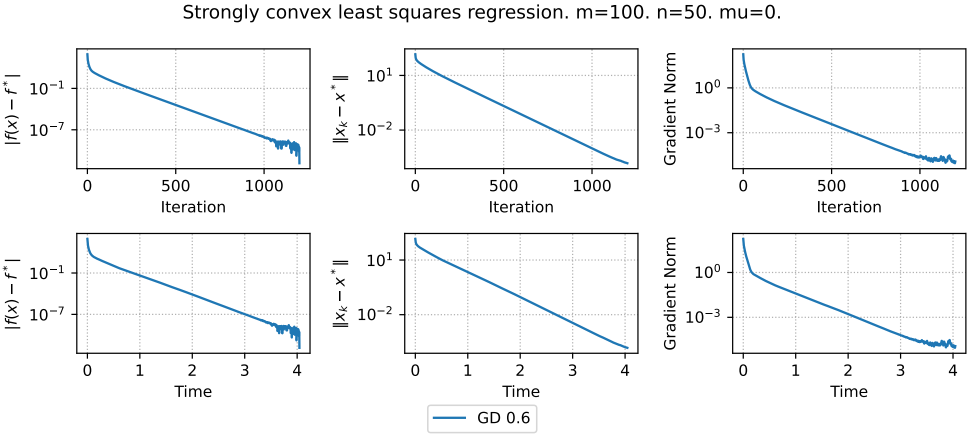


Figure 11: Another way to ensure convergence in the previous problem is to switch the dimension values

# You have to have strong convexity (or PL) to ensure convergence with a high precision

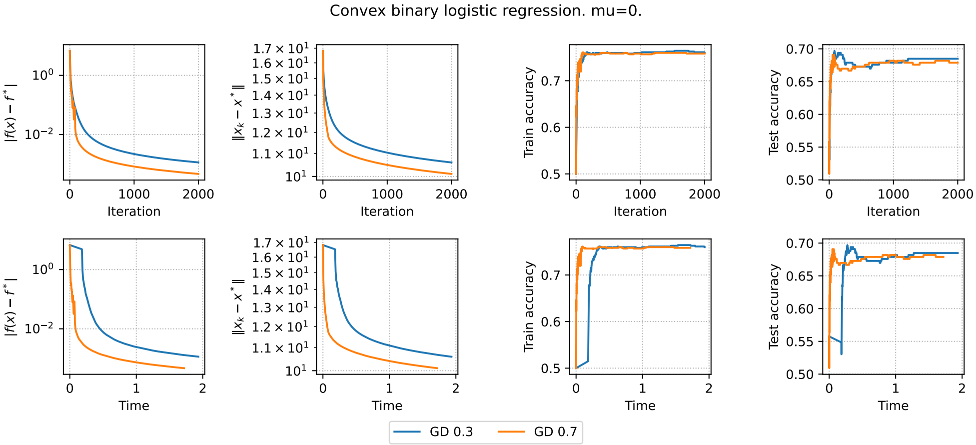


Figure 12: Only small precision is achievable with sublinear convergence

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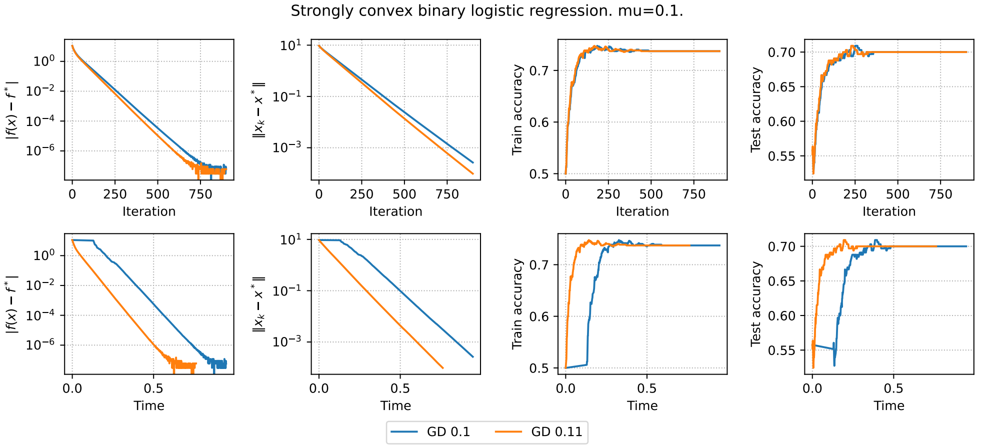


Figure 13: Strong convexity ensures linear convergence

# Convex optimization problem

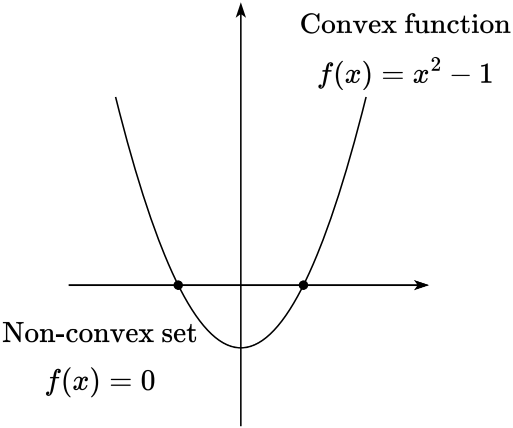


Figure 14: The idea behind the definition of a convex optimization problem

Note, that there is an agreement in notation of mathematical programming. The problems of the following type are called **Convex optimization problem**:

$$
\begin{aligned}
& f_0(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & f_i(x) \leq 0, \; i = 1, \ldots, m \\
& Ax = b,
\end{aligned}
\tag{COP}
$$

where all the functions $f_0(x), f_1(x), \ldots, f_m(x)$ are convex and all the equality constraints are affine. It sounds a bit strange, but not all convex problems are convex optimization problems.

$$
f_0(x) \to \min_{x \in S}, \tag{CP}
$$

where $f_0(x)$ is a convex function, defined on the convex set $S$. The necessity of affine equality constraint is essential.