Gradient Descent. Convergence for quadratics; smooth convex case; PL case

Daniil Merkulov

Optimization methods. MIPT

Gradient Descent

Direction of local steepest descent

Let's consider a linear approximation of the differentiable function f along some direction $h, \|h\|_2 = 1$:

Direction of local steepest descent

Let's consider a linear approximation of the differentiable function f along some direction $h, \|h\|_2 = 1$:

$$f(x + \alpha h) = f(x) + \alpha \langle f'(x), h \rangle + o(\alpha)$$

Direction of local steepest descent

Let's consider a linear approximation of the differentiable function f along some direction $h, \|h\|_2 = 1$:

$$f(x + \alpha h) = f(x) + \alpha \langle f'(x), h \rangle + o(\alpha)$$

We want h to be a decreasing direction:

$$f(x + \alpha h) < f(x)$$

$$f(x) + \alpha \langle f'(x), h \rangle + o(\alpha) < f(x)$$

Direction of local steepest descent

Let's consider a linear approximation of the differentiable function f along some direction $h, \|h\|_2 = 1$:

$$f(x + \alpha h) = f(x) + \alpha \langle f'(x), h \rangle + o(\alpha)$$

We want h to be a decreasing direction:

$$f(x + \alpha h) < f(x)$$

$$f(x) + \alpha \langle f'(x), h \rangle + o(\alpha) < f(x)$$

and going to the limit at $\alpha \to 0$:

$$\langle f'(x), h \rangle \leq 0$$

Direction of local steepest descent

Let's consider a linear approximation of the differentiable function f along some direction $h, \|h\|_2 = 1$:

$$f(x + \alpha h) = f(x) + \alpha \langle f'(x), h \rangle + o(\alpha)$$

We want h to be a decreasing direction:

$$f(x + \alpha h) < f(x)$$

$$f(x) + \alpha \langle f'(x), h \rangle + o(\alpha) < f(x)$$

and going to the limit at $\alpha \to 0$:

$$\langle f'(x), h \rangle \leq 0$$

Also from Cauchy–Bunyakovsky–Schwarz inequality:

$$|\langle f'(x), h \rangle| \leq \|f'(x)\|_2 \|h\|_2$$
$$\langle f'(x), h \rangle \geq -\|f'(x)\|_2 \|h\|_2 = -\|f'(x)\|_2$$

Direction of local steepest descent

Let's consider a linear approximation of the differentiable function f along some direction $h, \|h\|_2 = 1$:

$$f(x + \alpha h) = f(x) + \alpha \langle f'(x), h \rangle + o(\alpha)$$

We want h to be a decreasing direction:

$$f(x + \alpha h) < f(x)$$

$$f(x) + \alpha \langle f'(x), h \rangle + o(\alpha) < f(x)$$

and going to the limit at $\alpha \to 0$:

$$\langle f'(x), h \rangle \leq 0$$

Also from Cauchy–Bunyakovsky–Schwarz inequality:

$$|\langle f'(x), h \rangle| \leq \|f'(x)\|_2 \|h\|_2$$
$$\langle f'(x), h \rangle \geq -\|f'(x)\|_2 \|h\|_2 = -\|f'(x)\|_2$$

Thus, the direction of the antigradient

$$h = -\frac{f'(x)}{\|f'(x)\|_2}$$

gives the direction of the **steepest local** decreasing of the function f.

Direction of local steepest descent

Let's consider a linear approximation of the differentiable function f along some direction $h, \|h\|_2 = 1$:

$$f(x + \alpha h) = f(x) + \alpha \langle f'(x), h \rangle + o(\alpha)$$

We want h to be a decreasing direction:

$$f(x + \alpha h) < f(x)$$

$$f(x) + \alpha \langle f'(x), h \rangle + o(\alpha) < f(x)$$

and going to the limit at $\alpha \to 0$:

$$\langle f'(x), h \rangle \leq 0$$

Also from Cauchy–Bunyakovsky–Schwarz inequality:

$$|\langle f'(x), h \rangle| \leq \|f'(x)\|_2 \|h\|_2$$
$$\langle f'(x), h \rangle \geq -\|f'(x)\|_2 \|h\|_2 = -\|f'(x)\|_2$$

Thus, the direction of the antigradient

$$h = -\frac{f'(x)}{\|f'(x)\|_2}$$

gives the direction of the **steepest local** decreasing of the function f. The result of this method is

$$x_{k+1} = x_k - \alpha f'(x_k)$$

Gradient flow ODE

Let's consider the following ODE, which is referred to as the Gradient Flow equation.

$$\frac{dx}{dt} = -f'(x(t)) \tag{GF}$$

Gradient flow ODE

Let's consider the following ODE, which is referred to as the Gradient Flow equation.

$$\frac{dx}{dt} = -f'(x(t)) \tag{GF}$$

and discretize it on a uniform grid with α step:

$$\frac{x_{k+1} - x_k}{\alpha} = -f'(x_k),$$

Gradient flow ODE

Let's consider the following ODE, which is referred to as the Gradient Flow equation.

$$\frac{dx}{dt} = -f'(x(t)) \tag{GF}$$

and discretize it on a uniform grid with α step:

$$\frac{x_{k+1} - x_k}{\alpha} = -f'(x_k),$$

where $x_k \equiv x(t_k)$ and $\alpha = t_{k+1} - t_k$ - is the grid step.
From here we get the expression for x_{k+1}

$$x_{k+1} = x_k - \alpha f'(x_k),$$

which is exactly gradient descent.
Open In Colab ♣

Gradient flow ODE

Let's consider the following ODE, which is referred to as the Gradient Flow equation.

$$\frac{dx}{dt} = -f'(x(t)) \tag{GF}$$

and discretize it on a uniform grid with α step:

$$\frac{x_{k+1} - x_k}{\alpha} = -f'(x_k),$$

where $x_k \equiv x(t_k)$ and $\alpha = t_{k+1} - t_k$ - is the grid step.
From here we get the expression for x_{k+1}

$$x_{k+1} = x_k - \alpha f'(x_k),$$

which is exactly gradient descent.
Open In Colab ♣

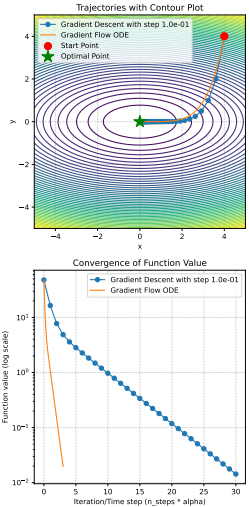
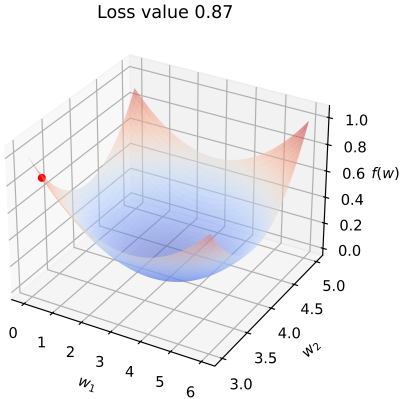
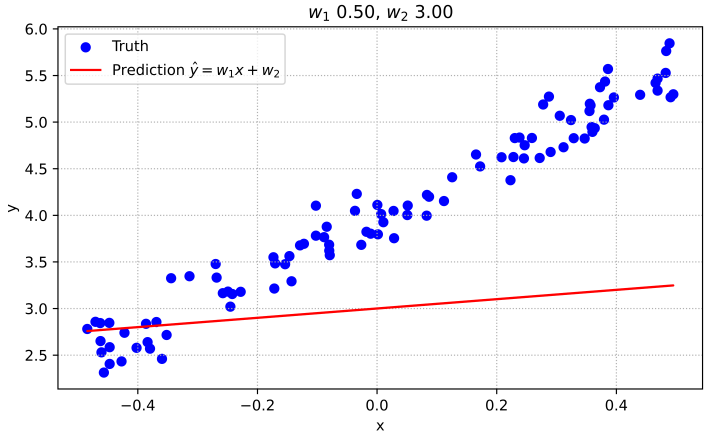


Figure 1: Gradient flow trajectory

Convergence of Gradient Descent algorithm

Heavily depends on the choice of the learning rate α:





Exact line search aka steepest descent

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \arg\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. An interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

Exact line search aka steepest descent

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \arg\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. An interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

Optimality conditions:

Exact line search aka steepest descent

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \arg\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. An interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

Optimality conditions:

$$\nabla f(x_{k+1})^\top \nabla f(x_k) = 0$$

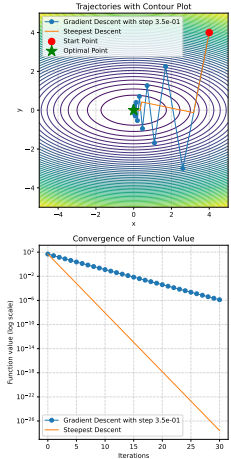


Figure 2: Steepest Descent

Open In Colab ♣

Strongly convex quadratics

Coordinate shift

Consider the following quadratic optimization problem:

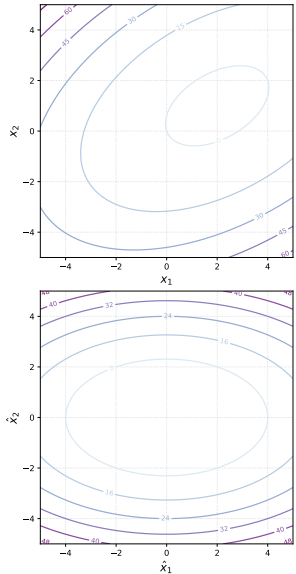
$$\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}^d_{++}.$$

Coordinate shift

Consider the following quadratic optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}^d_{++}.$$

- Firstly, without loss of generality we can set $c = 0$, which will or affect optimization process.



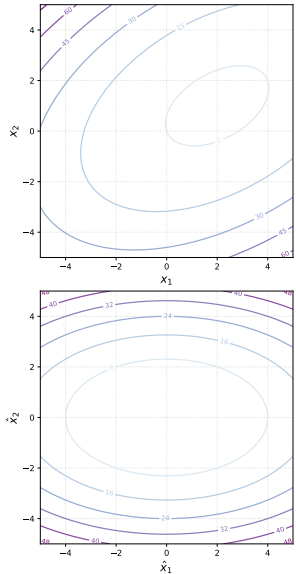
Coordinate shift

Consider the following quadratic optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}^d_{++}.$$

- Firstly, without loss of generality we can set $c = 0$, which will or affect optimization process.
- Secondly, we have a spectral decomposition of the matrix A:

$$A = Q \Lambda Q^T$$



Coordinate shift

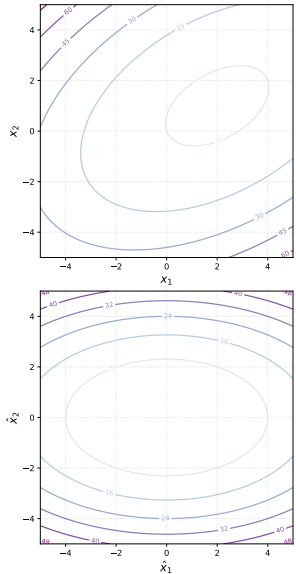
Consider the following quadratic optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}^d_{++}.$$

- Firstly, without loss of generality we can set $c = 0$, which will or affect optimization process.
- Secondly, we have a spectral decomposition of the matrix A:

$$A = Q \Lambda Q^T$$

- Let's show, that we can switch coordinates to make an analysis a little bit easier. Let $\hat{x} = Q^T(x - x^*)$, where x^* is the minimum point of initial function, defined by $Ax^* = b$. At the same time $x = Q\hat{x} + x^*$.



Coordinate shift

Consider the following quadratic optimization problem:

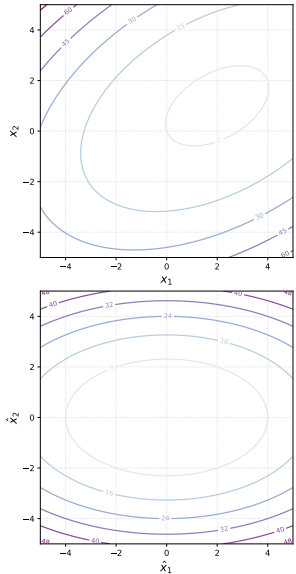
$$\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}^d_{++}.$$

- Firstly, without loss of generality we can set $c = 0$, which will or affect optimization process.
- Secondly, we have a spectral decomposition of the matrix A:

$$A = Q \Lambda Q^T$$

- Let's show, that we can switch coordinates to make an analysis a little bit easier. Let $\hat{x} = Q^T(x - x^*)$, where x^* is the minimum point of initial function, defined by $Ax^* = b$. At the same time $x = Q\hat{x} + x^*$.

$$f(\hat{x}) = \frac{1}{2}(Q\hat{x} + x^*)^\top A(Q\hat{x} + x^*) - b^\top(Q\hat{x} + x^*)$$



Coordinate shift

Consider the following quadratic optimization problem:

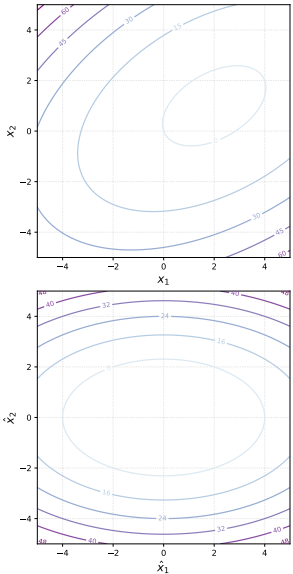
$$\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}^d_{++}.$$

- Firstly, without loss of generality we can set $c = 0$, which will or affect optimization process.
- Secondly, we have a spectral decomposition of the matrix A:

$$A = Q \Lambda Q^T$$

- Let's show, that we can switch coordinates to make an analysis a little bit easier. Let $\hat{x} = Q^T(x - x^*)$, where x^* is the minimum point of initial function, defined by $Ax^* = b$. At the same time $x = Q\hat{x} + x^*$.

$$\begin{aligned} f(\hat{x}) &= \frac{1}{2}(Q\hat{x} + x^*)^\top A(Q\hat{x} + x^*) - b^\top(Q\hat{x} + x^*) \\ &= \frac{1}{2}\hat{x}^T Q^T A Q \hat{x} + (x^*)^T A Q \hat{x} + \frac{1}{2}(x^*)^T A (x^*)^T - b^T Q \hat{x} - b^T x^* \end{aligned}$$



Coordinate shift

Consider the following quadratic optimization problem:

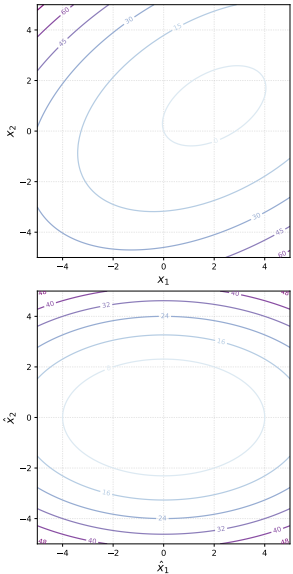
$$\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}^d_{++}.$$

- Firstly, without loss of generality we can set $c = 0$, which will or affect optimization process.
- Secondly, we have a spectral decomposition of the matrix A:

$$A = Q \Lambda Q^T$$

- Let's show, that we can switch coordinates to make an analysis a little bit easier. Let $\hat{x} = Q^T(x - x^*)$, where x^* is the minimum point of initial function, defined by $Ax^* = b$. At the same time $x = Q\hat{x} + x^*$.

$$\begin{aligned} f(\hat{x}) &= \frac{1}{2}(Q\hat{x} + x^*)^\top A(Q\hat{x} + x^*) - b^\top(Q\hat{x} + x^*) \\ &= \frac{1}{2}\hat{x}^T Q^T A Q \hat{x} + (x^*)^T A Q \hat{x} + \frac{1}{2}(x^*)^T A (x^*)^T - b^T Q \hat{x} - b^T x^* \\ &= \frac{1}{2}\hat{x}^T \Lambda \hat{x} \end{aligned}$$



Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$x^{k+1} = x^k - \alpha^k \nabla f(x^k)$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$x^{k+1} = x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate}
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)}) x_{(i)}^k \text{ For } i\text{-th coordinate} \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)})^k x_{(i)}^0
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)}) x_{(i)}^k \text{ For } i\text{-th coordinate} \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)})^k x_{(i)}^0
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)}) x_{(i)}^k \text{ For } i\text{-th coordinate} \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)})^k x_{(i)}^0
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$|1 - \alpha\mu| < 1$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)}) x_{(i)}^k \text{ For } i\text{-th coordinate} \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)})^k x_{(i)}^0
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
&|1 - \alpha\mu| < 1 \\
&-1 < 1 - \alpha\mu < 1
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)}) x_{(i)}^k \text{ For } i\text{-th coordinate} \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)})^k x_{(i)}^0
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha\lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
&|1 - \alpha\mu| < 1 \\
&-1 < 1 - \alpha\mu < 1 \\
&\alpha < \frac{2}{\mu} \qquad \alpha\mu > 0
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
&|1 - \alpha\mu| < 1 \qquad\qquad |1 - \alpha L| < 1 \\
&-1 < 1 - \alpha\mu < 1 \\
&\alpha < \frac{2}{\mu} \qquad \alpha\mu > 0
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
&|1 - \alpha\mu| < 1 && |1 - \alpha L| < 1 \\
&-1 < 1 - \alpha\mu < 1 && -1 < 1 - \alpha L < 1 \\
&\alpha < \frac{2}{\mu} \qquad \alpha\mu > 0
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
&|1 - \alpha\mu| < 1 && |1 - \alpha L| < 1 \\
&-1 < 1 - \alpha\mu < 1 && -1 < 1 - \alpha L < 1 \\
&\alpha < \frac{2}{\mu} \qquad \alpha\mu > 0 && \alpha < \frac{2}{L} \qquad \alpha L > 0
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
&|1 - \alpha\mu| < 1 && |1 - \alpha L| < 1 \\
&-1 < 1 - \alpha\mu < 1 && -1 < 1 - \alpha L < 1 \\
&\alpha < \frac{2}{\mu} \qquad \alpha\mu > 0 && \alpha < \frac{2}{L} \qquad \alpha L > 0
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)}) x_{(i)}^k \text{ For } i\text{-th coordinate} \\
x_{(i)}^{k+1} &= (1 - \alpha^k \lambda_{(i)})^k x_{(i)}^0
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
|1 - \alpha\mu| &< 1 & |1 - \alpha L| &< 1 \\
-1 < 1 - \alpha\mu &< 1 & -1 < 1 - \alpha L &< 1 \\
\alpha < \frac{2}{\mu} \qquad \alpha\mu &> 0 & \alpha < \frac{2}{L} \qquad \alpha L &> 0
\end{aligned}
$$

$\alpha < \frac{2}{L}$ is needed for convergence.

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Now we would like to tune α to choose the best (lowest) convergence rate

$$\rho^* = \min_\alpha \rho(\alpha)$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
|1 - \alpha\mu| &< 1 \\
-1 < 1 - \alpha\mu &< 1 \\
\alpha < \frac{2}{\mu} \qquad \alpha\mu &> 0
\end{aligned}
\qquad\qquad
\begin{aligned}
|1 - \alpha L| &< 1 \\
-1 < 1 - \alpha L &< 1 \\
\alpha < \frac{2}{L} \qquad \alpha L &> 0
\end{aligned}
$$

$\alpha < \frac{2}{L}$ is needed for convergence.

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
|1 - \alpha\mu| &< 1 & |1 - \alpha L| &< 1 \\
-1 < 1 - \alpha\mu &< 1 & -1 < 1 - \alpha L &< 1 \\
\alpha < \frac{2}{\mu} \qquad \alpha\mu &> 0 & \alpha < \frac{2}{L} \qquad \alpha L &> 0
\end{aligned}
$$

$\alpha < \frac{2}{L}$ is needed for convergence.

Now we would like to tune α to choose the best (lowest) convergence rate

$$\rho^* = \min_\alpha \rho(\alpha) = \min_\alpha \max_i |1 - \alpha \lambda_{(i)}|$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
|1 - \alpha\mu| &< 1 & |1 - \alpha L| &< 1 \\
-1 &< 1 - \alpha\mu < 1 & -1 &< 1 - \alpha L < 1 \\
\alpha &< \frac{2}{\mu} \qquad \alpha\mu > 0 & \alpha &< \frac{2}{L} \qquad \alpha L > 0
\end{aligned}
$$

$\alpha < \frac{2}{L}$ is needed for convergence.

Now we would like to tune α to choose the best (lowest) convergence rate

$$
\begin{aligned}
\rho^* &= \min_\alpha \rho(\alpha) = \min_\alpha \max_i |1 - \alpha \lambda_{(i)}| \\
&= \min_\alpha \{|1 - \alpha\mu|, |1 - \alpha L|\}
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
|1 - \alpha\mu| &< 1 & |1 - \alpha L| &< 1 \\
-1 < 1 - \alpha\mu &< 1 & -1 < 1 - \alpha L &< 1 \\
\alpha < \frac{2}{\mu} \qquad \alpha\mu &> 0 & \alpha < \frac{2}{L} \qquad \alpha L &> 0
\end{aligned}
$$

$\alpha < \frac{2}{L}$ is needed for convergence.

Now we would like to tune α to choose the best (lowest) convergence rate

$$
\begin{aligned}
\rho^* &= \min_\alpha \rho(\alpha) = \min_\alpha \max_i |1 - \alpha \lambda_{(i)}| \\
&= \min_\alpha \{|1 - \alpha\mu|, |1 - \alpha L|\} \\
\alpha^* : \quad & 1 - \alpha^* \mu = \alpha^* L - 1
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$$
\begin{aligned}
|1 - \alpha\mu| &< 1 & |1 - \alpha L| &< 1 \\
-1 < 1 - \alpha\mu &< 1 & -1 < 1 - \alpha L &< 1 \\
\alpha < \frac{2}{\mu} \qquad \alpha\mu &> 0 & \alpha < \frac{2}{L} \qquad \alpha L &> 0
\end{aligned}
$$

$\alpha < \frac{2}{L}$ is needed for convergence.

Now we would like to tune α to choose the best (lowest) convergence rate

$$
\begin{aligned}
\rho^* &= \min_\alpha \rho(\alpha) = \min_\alpha \max_i |1 - \alpha \lambda_{(i)}| \\
&= \min_\alpha \{|1 - \alpha\mu|, |1 - \alpha L|\} \\
\alpha^* : \quad & 1 - \alpha^* \mu = \alpha^* L - 1 \\
& \alpha^* = \frac{2}{\mu + L}
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha\lambda_{(i)}| < 1$$

Remember, that $\lambda_{\min} = \mu > 0, \lambda_{\max} = L \geq \mu$.

$$
\begin{aligned}
|1 - \alpha\mu| &< 1 & |1 - \alpha L| &< 1 \\
-1 < 1 - \alpha\mu &< 1 & -1 < 1 - \alpha L &< 1 \\
\alpha < \frac{2}{\mu} \qquad \alpha\mu &> 0 & \alpha < \frac{2}{L} \qquad \alpha L &> 0
\end{aligned}
$$

$\alpha < \frac{2}{L}$ is needed for convergence.

Now we would like to tune α to choose the best (lowest) convergence rate

$$
\begin{aligned}
\rho^* &= \min_\alpha \rho(\alpha) = \min_\alpha \max_i |1 - \alpha\lambda_{(i)}| \\
&= \min_\alpha \{|1 - \alpha\mu|, |1 - \alpha L|\} \\
\alpha^* : \quad & 1 - \alpha^*\mu = \alpha^* L - 1 \\
\alpha^* &= \frac{2}{\mu + L} \qquad \rho^* = \frac{L - \mu}{L + \mu}
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha\lambda_{(i)}| < 1$$

Remember, that $\lambda_{\min} = \mu > 0, \lambda_{\max} = L \geq \mu$.

$$
\begin{aligned}
&|1 - \alpha\mu| < 1 && |1 - \alpha L| < 1 \\
&-1 < 1 - \alpha\mu < 1 && -1 < 1 - \alpha L < 1 \\
&\alpha < \frac{2}{\mu} \qquad \alpha\mu > 0 && \alpha < \frac{2}{L} \qquad \alpha L > 0
\end{aligned}
$$

$\alpha < \frac{2}{L}$ is needed for convergence.

Now we would like to tune α to choose the best (lowest) convergence rate

$$
\begin{aligned}
\rho^* &= \min_\alpha \rho(\alpha) = \min_\alpha \max_i |1 - \alpha\lambda_{(i)}| \\
&= \min_\alpha \{|1 - \alpha\mu|, |1 - \alpha L|\} \\
\alpha^* : &\quad 1 - \alpha^*\mu = \alpha^* L - 1 \\
\alpha^* &= \frac{2}{\mu + L} \qquad \rho^* = \frac{L - \mu}{L + \mu} \\
x^{k+1} &= \left(\frac{L - \mu}{L + \mu}\right)^k x^0
\end{aligned}
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T \Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$
\begin{aligned}
x^{k+1} &= x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \Lambda x^k \\
&= (I - \alpha^k \Lambda) x^k \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \text{ For } i\text{-th coordinate} \\
x^{k+1}_{(i)} &= (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}
\end{aligned}
$$

Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$\rho(\alpha) = \max_i |1 - \alpha\lambda_{(i)}| < 1$$

Remember, that $\lambda_{\min} = \mu > 0, \lambda_{\max} = L \geq \mu$.

$$
\begin{aligned}
|1 - \alpha\mu| &< 1 & |1 - \alpha L| &< 1 \\
-1 < 1 - \alpha\mu &< 1 & -1 < 1 - \alpha L &< 1 \\
\alpha < \frac{2}{\mu} \qquad \alpha\mu &> 0 & \alpha < \frac{2}{L} \qquad \alpha L &> 0
\end{aligned}
$$

$\alpha < \frac{2}{L}$ is needed for convergence.

Now we would like to tune α to choose the best (lowest) convergence rate

$$
\begin{aligned}
\rho^* &= \min_\alpha \rho(\alpha) = \min_\alpha \max_i |1 - \alpha\lambda_{(i)}| \\
&= \min_\alpha \{|1 - \alpha\mu|, |1 - \alpha L|\} \\
\alpha^* : \quad & 1 - \alpha^*\mu = \alpha^* L - 1 \\
\alpha^* &= \frac{2}{\mu + L} \qquad \rho^* = \frac{L - \mu}{L + \mu} \\
x^{k+1} &= \left(\frac{L - \mu}{L + \mu}\right)^k x^0 \qquad f(x^{k+1}) = \left(\frac{L - \mu}{L + \mu}\right)^{2k} f(x^0)
\end{aligned}
$$

Convergence analysis

So, we have a linear convergence in the domain with rate $\frac{\kappa-1}{\kappa+1} = 1 - \frac{2}{\kappa+1}$, where $\kappa = \frac{L}{\mu}$ is sometimes called *condition number* of the quadratic problem.

κ	ρ	Iterations to decrease domain gap 10 times	Iterations to decrease function gap 10 times
1.1	0.05	1	1
2	0.33	3	2
5	0.67	6	3
10	0.82	12	6
50	0.96	58	29
100	0.98	116	58
500	0.996	576	288
1000	0.998	1152	576

Polyak-Lojasiewicz smooth case

Polyak-Lojasiewicz condition. Linear convergence of gradient descent without convexity

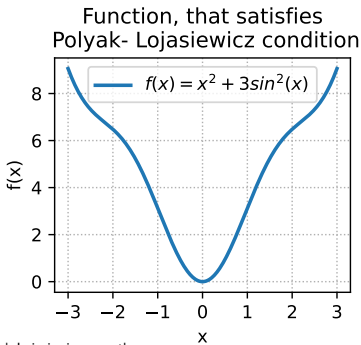
PL inequality holds if the following condition is satisfied for some $\mu > 0$,

$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*) \quad \forall x$$

It is interesting, that the Gradient Descent algorithm might converge linearly even without convexity.

The following functions satisfy the PL condition but are not convex. Link to the code

$$f(x) = x^2 + 3\sin^2(x)$$



Polyak-Lojasiewicz condition. Linear convergence of gradient descent without convexity

PL inequality holds if the following condition is satisfied for some $\mu > 0$,

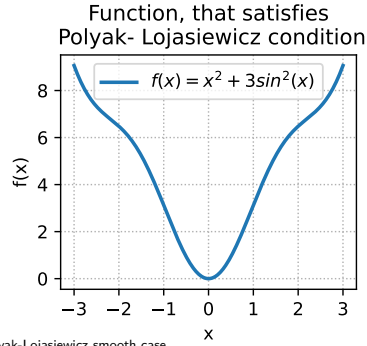
$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*) \quad \forall x$$

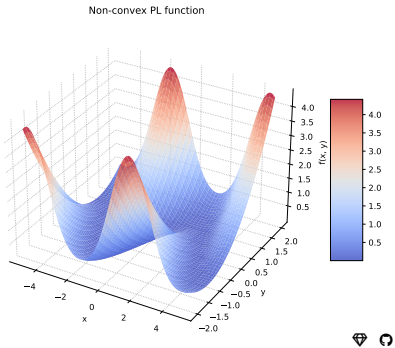
It is interesting, that the Gradient Descent algorithm might converge linearly even without convexity.

The following functions satisfy the PL condition but are not convex. Link to the code

$$f(x) = x^2 + 3\sin^2(x)$$

$$f(x, y) = \frac{(y - \sin x)^2}{2}$$





Convergence analysis

Theorem

Consider the Problem

$$f(x) \to \min_{x \in \mathbb{R}^d}$$

and assume that f is μ-Polyak-Lojasiewicz and L-smooth, for some $L \geq \mu > 0$.
Consider $(x^k)_{k \in \mathbb{N}}$ a sequence generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying $0 < \alpha \leq \frac{1}{L}$. Then:

$$f(x^k) - f^* \leq (1 - \alpha\mu)^k (f(x^0) - f^*).$$

Convergence analysis

We can use L-smoothness, together with the update rule of the algorithm, to write

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2$$

Convergence analysis

We can use L-smoothness, together with the update rule of the algorithm, to write

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2$$
$$= f(x^k) - \alpha\|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2$$

Convergence analysis

We can use L-smoothness, together with the update rule of the algorithm, to write

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2
\end{aligned}
$$

Convergence analysis

We can use L-smoothness, together with the update rule of the algorithm, to write

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \\
&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2,
\end{aligned}
$$

Convergence analysis

We can use L-smoothness, together with the update rule of the algorithm, to write

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \\
&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2,
\end{aligned}
$$

Convergence analysis

We can use L-smoothness, together with the update rule of the algorithm, to write

$$\begin{aligned} f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\ &= f(x^k) - \alpha\|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\ &= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \\ &\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2, \end{aligned}$$

where in the last inequality we used our hypothesis on the stepsize that $\alpha L \leq 1$.

Convergence analysis

We can use L-smoothness, together with the update rule of the algorithm, to write

$$\begin{aligned} f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\ &= f(x^k) - \alpha\|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\ &= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \\ &\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2, \end{aligned}$$

where in the last inequality we used our hypothesis on the stepsize that $\alpha L \leq 1$.

We can now use the Polyak-Lojasiewicz property to write:

$$f(x^{k+1}) \leq f(x^k) - \alpha\mu(f(x^k) - f^*).$$

The conclusion follows after subtracting f^* on both sides of this inequality and using recursion.

Any μ-strongly convex differentiable function is a PL-function

Theorem

If a function $f(x)$ is differentiable and μ-strongly convex, then it is a PL function.

Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2}\|y - x\|_2^2$$

Putting $y = x^*$:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2}\|x^* - x\|_2^2$$

Any μ-strongly convex differentiable function is a PL-function

Theorem

If a function $f(x)$ is differentiable and μ-strongly convex, then it is a PL function.

Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2}\|y - x\|_2^2$$

Putting $y = x^*$:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2}\|x^* - x\|_2^2$$

$$f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*) - \frac{\mu}{2}\|x^* - x\|_2^2 =$$

Any μ-strongly convex differentiable function is a PL-function

Theorem

If a function $f(x)$ is differentiable and μ-strongly convex, then it is a PL function.

Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2}\|y - x\|_2^2$$

Putting $y = x^*$:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2}\|x^* - x\|_2^2$$

$$f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*) - \frac{\mu}{2}\|x^* - x\|_2^2 =$$

$$= \left(\nabla f(x)^T - \frac{\mu}{2}(x^* - x)\right)^T (x - x^*) =$$

Any μ-strongly convex differentiable function is a PL-function

Theorem

If a function $f(x)$ is differentiable and μ-strongly convex, then it is a PL function.

Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2}\|y - x\|_2^2$$

Putting $y = x^*$:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2}\|x^* - x\|_2^2$$

$$f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*) - \frac{\mu}{2}\|x^* - x\|_2^2 =$$

$$= \left(\nabla f(x)^T - \frac{\mu}{2}(x^* - x)\right)^T (x - x^*) =$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{\mu}}\nabla f(x)^T - \sqrt{\mu}(x^* - x)\right)^T \sqrt{\mu}(x - x^*) =$$

Any μ-strongly convex differentiable function is a PL-function

Theorem

If a function $f(x)$ is differentiable and μ-strongly convex, then it is a PL function.

Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2}\|y - x\|_2^2$$

Putting $y = x^*$:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2}\|x^* - x\|_2^2$$

$$f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*) - \frac{\mu}{2}\|x^* - x\|_2^2 =$$

$$= \left(\nabla f(x)^T - \frac{\mu}{2}(x^* - x)\right)^T (x - x^*) =$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{\mu}}\nabla f(x)^T - \sqrt{\mu}(x^* - x)\right)^T \sqrt{\mu}(x - x^*) =$$

Any μ-strongly convex differentiable function is a PL-function

Theorem

If a function $f(x)$ is differentiable and μ-strongly convex, then it is a PL function.

Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2}\|y - x\|_2^2$$

Let $a = \frac{1}{\sqrt{\mu}}\nabla f(x)$ and
$b = \sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)$

Putting $y = x^*$:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2}\|x^* - x\|_2^2$$

$$f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*) - \frac{\mu}{2}\|x^* - x\|_2^2 =$$

$$= \left(\nabla f(x)^T - \frac{\mu}{2}(x^* - x)\right)^T (x - x^*) =$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{\mu}}\nabla f(x)^T - \sqrt{\mu}(x^* - x)\right)^T \sqrt{\mu}(x - x^*) =$$

Any μ-strongly convex differentiable function is a PL-function

Theorem

If a function $f(x)$ is differentiable and μ-strongly convex, then it is a PL function.

Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2}\|y - x\|_2^2$$

Putting $y = x^*$:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2}\|x^* - x\|_2^2$$

$$f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*) - \frac{\mu}{2}\|x^* - x\|_2^2 =$$

$$= \left(\nabla f(x)^T - \frac{\mu}{2}(x^* - x)\right)^T (x - x^*) =$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{\mu}}\nabla f(x)^T - \sqrt{\mu}(x^* - x)\right)^T \sqrt{\mu}(x - x^*) =$$

Let $a = \frac{1}{\sqrt{\mu}}\nabla f(x)$ and
$b = \sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)$
Then $a + b = \sqrt{\mu}(x - x^*)$ and
$a - b = \frac{2}{\sqrt{\mu}}\nabla f(x) - \sqrt{\mu}(x - x^*)$

Any μ-strongly convex differentiable function is a PL-function

$$f(x) - f(x^*) \leq \frac{1}{2}\left(\frac{1}{\mu}\|\nabla f(x)\|_2^2 - \left\|\sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)\right\|_2^2\right)$$

Any μ-strongly convex differentiable function is a PL-function

$$f(x) - f(x^*) \leq \frac{1}{2}\left(\frac{1}{\mu}\|\nabla f(x)\|_2^2 - \left\|\sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)\right\|_2^2\right)$$

$$f(x) - f(x^*) \leq \frac{1}{2\mu}\|\nabla f(x)\|_2^2,$$

Any μ-strongly convex differentiable function is a PL-function

$$f(x) - f(x^*) \leq \frac{1}{2}\left(\frac{1}{\mu}\|\nabla f(x)\|_2^2 - \left\|\sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)\right\|_2^2\right)$$

$$f(x) - f(x^*) \leq \frac{1}{2\mu}\|\nabla f(x)\|_2^2,$$

Any μ-strongly convex differentiable function is a PL-function

$$f(x) - f(x^*) \leq \frac{1}{2}\left(\frac{1}{\mu}\|\nabla f(x)\|_2^2 - \left\|\sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)\right\|_2^2\right)$$

$$f(x) - f(x^*) \leq \frac{1}{2\mu}\|\nabla f(x)\|_2^2,$$

which is exactly the PL condition. It means, that we already have linear convergence proof for any strongly convex function.

Smooth convex case

Smooth convex case

Theorem

Consider the Problem

$$f(x) \to \min_{x \in \mathbb{R}^d}$$

and assume that f is convex and L-smooth, for some $L > 0$.
Let $(x^k)_{k \in \mathbb{N}}$ be the sequence of iterates generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying $0 < \alpha \leq \frac{1}{L}$. Then, for all $x^* \in \operatorname{argmin} f$, for all $k \in \mathbb{N}$ we have that

$$f(x^k) - f^* \leq \frac{\|x^0 - x^*\|^2}{2\alpha k}.$$

Convergence analysis

- As it was before, we first use smoothness:

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \\
&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2, \\
f(x^k) - f(x^{k+1}) &\geq \frac{1}{2L}\|\nabla f(x^k)\|^2 \text{ if } \alpha \leq \frac{1}{L}
\end{aligned} \tag{1}
$$

Typically, for the convergent gradient descent algorithm the higher the learning rate the faster the convergence. That is why we often will use $\alpha = \frac{1}{L}$.

Convergence analysis

- As it was before, we first use smoothness:

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \qquad (1) \\
&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2, \\
f(x^k) - f(x^{k+1}) &\geq \frac{1}{2L}\|\nabla f(x^k)\|^2 \text{ if } \alpha \leq \frac{1}{L}
\end{aligned}
$$

  Typically, for the convergent gradient descent algorithm the higher the learning rate the faster the convergence. That is why we often will use $\alpha = \frac{1}{L}$.

- After that we add convexity:

(2)

Convergence analysis

- As it was before, we first use smoothness:

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \\
&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2, \\
f(x^k) - f(x^{k+1}) &\geq \frac{1}{2L}\|\nabla f(x^k)\|^2 \text{ if } \alpha \leq \frac{1}{L}
\end{aligned}
\tag{1}
$$

  Typically, for the convergent gradient descent algorithm the higher the learning rate the faster the convergence. That is why we often will use $\alpha = \frac{1}{L}$.

- After that we add convexity:

$$
f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \tag{2}
$$

Convergence analysis

- As it was before, we first use smoothness:

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \\
&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2, \\
f(x^k) - f(x^{k+1}) &\geq \frac{1}{2L}\|\nabla f(x^k)\|^2 \text{ if } \alpha \leq \frac{1}{L}
\end{aligned}
\tag{1}
$$

  Typically, for the convergent gradient descent algorithm the higher the learning rate the faster the convergence. That is why we often will use $\alpha = \frac{1}{L}$.

- After that we add convexity:

$$
f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \text{ with } y = x^*, x = x^k \tag{2}
$$

Convergence analysis

- As it was before, we first use smoothness:

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \\
&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2, \\
f(x^k) - f(x^{k+1}) &\geq \frac{1}{2L}\|\nabla f(x^k)\|^2 \text{ if } \alpha \leq \frac{1}{L}
\end{aligned}
\tag{1}
$$

  Typically, for the convergent gradient descent algorithm the higher the learning rate the faster the convergence. That is why we often will use $\alpha = \frac{1}{L}$.

- After that we add convexity:

$$
\begin{aligned}
f(y) &\geq f(x) + \langle \nabla f(x), y - x \rangle \text{ with } y = x^*, x = x^k \\
f(x^k) - f^* &\leq \langle \nabla f(x^k), x^k - x^* \rangle
\end{aligned}
\tag{2}
$$

Convergence analysis

- Now we put Equation 2 to Equation 1:

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$f(x^{k+1}) \leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2$$

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$f(x^{k+1}) \leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2$$
$$= f^* + \langle \nabla f(x^k), x^k - x^* - \frac{\alpha}{2}\nabla f(x^k) \rangle$$

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \\
&= f^* + \langle \nabla f(x^k), x^k - x^* - \frac{\alpha}{2}\nabla f(x^k) \rangle \\
&= f^* + \frac{1}{2\alpha}\left\langle \alpha \nabla f(x^k), 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)\right\rangle
\end{aligned}
$$

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$
\begin{aligned}
f(x^{k+1}) \leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 &\leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \\
&= f^* + \langle \nabla f(x^k), x^k - x^* - \frac{\alpha}{2}\nabla f(x^k) \rangle \\
&= f^* + \frac{1}{2\alpha}\left\langle \alpha \nabla f(x^k), 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)\right\rangle
\end{aligned}
$$

Let $a = x^k - x^*$ and $b = x^k - x^* - \alpha \nabla f(x^k)$.

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$f(x^{k+1}) \leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2$$
$$= f^* + \langle \nabla f(x^k), x^k - x^* - \frac{\alpha}{2}\nabla f(x^k) \rangle$$
$$= f^* + \frac{1}{2\alpha}\left\langle \alpha \nabla f(x^k), 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)\right\rangle$$

Let $a = x^k - x^*$ and $b = x^k - x^* - \alpha\nabla f(x^k)$. Then $a + b = \alpha\nabla f(x^k)$ and $a - b = 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)$.

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$f(x^{k+1}) \leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2$$
$$= f^* + \langle \nabla f(x^k), x^k - x^* - \frac{\alpha}{2}\nabla f(x^k) \rangle$$
$$= f^* + \frac{1}{2\alpha}\left\langle \alpha \nabla f(x^k), 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)\right\rangle$$

Let $a = x^k - x^*$ and $b = x^k - x^* - \alpha\nabla f(x^k)$. Then $a + b = \alpha \nabla f(x^k)$ and $a - b = 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)$.

$$f(x^{k+1}) \leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^k - x^* - \alpha\nabla f(x^k)\|_2^2\right]$$

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \\
&= f^* + \langle \nabla f(x^k), x^k - x^* - \frac{\alpha}{2}\nabla f(x^k) \rangle \\
&= f^* + \frac{1}{2\alpha}\left\langle \alpha \nabla f(x^k), 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)\right\rangle
\end{aligned}
$$

Let $a = x^k - x^*$ and $b = x^k - x^* - \alpha \nabla f(x^k)$. Then $a + b = \alpha \nabla f(x^k)$ and $a - b = 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)$.

$$
\begin{aligned}
f(x^{k+1}) &\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^k - x^* - \alpha \nabla f(x^k)\|_2^2\right] \\
&\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2\right]
\end{aligned}
$$

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \\
&= f^* + \langle \nabla f(x^k), x^k - x^* - \frac{\alpha}{2}\nabla f(x^k) \rangle \\
&= f^* + \frac{1}{2\alpha}\left\langle \alpha \nabla f(x^k), 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)\right\rangle
\end{aligned}
$$

Let $a = x^k - x^*$ and $b = x^k - x^* - \alpha \nabla f(x^k)$. Then $a + b = \alpha \nabla f(x^k)$ and $a - b = 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)$.

$$
\begin{aligned}
f(x^{k+1}) &\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^k - x^* - \alpha \nabla f(x^k)\|_2^2\right] \\
&\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2\right] \\
2\alpha\left(f(x^{k+1}) - f^*\right) &\leq \|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2
\end{aligned}
$$

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$
\begin{aligned}
f(x^{k+1}) \leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 &\leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \\
&= f^* + \langle \nabla f(x^k), x^k - x^* - \frac{\alpha}{2}\nabla f(x^k) \rangle \\
&= f^* + \frac{1}{2\alpha}\left\langle \alpha \nabla f(x^k), 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)\right\rangle
\end{aligned}
$$

Let $a = x^k - x^*$ and $b = x^k - x^* - \alpha \nabla f(x^k)$. Then $a + b = \alpha \nabla f(x^k)$ and $a - b = 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)$.

$$
\begin{aligned}
f(x^{k+1}) &\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^k - x^* - \alpha \nabla f(x^k)\|_2^2\right] \\
&\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2\right] \\
2\alpha\left(f(x^{k+1}) - f^*\right) &\leq \|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2
\end{aligned}
$$

- Now suppose, that the last line is defined for some index i and we sum over $i \in [0, k-1]$. Almost all summands will vanish due to the telescopic nature of the sum:

(3)

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$
\begin{aligned}
f(x^{k+1}) \leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 &\leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \\
&= f^* + \langle \nabla f(x^k), x^k - x^* - \frac{\alpha}{2}\nabla f(x^k) \rangle \\
&= f^* + \frac{1}{2\alpha}\left\langle \alpha \nabla f(x^k), 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)\right\rangle
\end{aligned}
$$

Let $a = x^k - x^*$ and $b = x^k - x^* - \alpha \nabla f(x^k)$. Then $a + b = \alpha \nabla f(x^k)$ and $a - b = 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)$.

$$
\begin{aligned}
f(x^{k+1}) &\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^k - x^* - \alpha \nabla f(x^k)\|_2^2\right] \\
&\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2\right] \\
2\alpha\left(f(x^{k+1}) - f^*\right) &\leq \|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2
\end{aligned}
$$

- Now suppose, that the last line is defined for some index i and we sum over $i \in [0, k-1]$. Almost all summands will vanish due to the telescopic nature of the sum:

$$
2\alpha \sum_{i=0}^{k-1}\left(f(x^{i+1}) - f^*\right) \leq \|x^0 - x^*\|_2^2 - \|x^k - x^*\|_2^2 \tag{3}
$$

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$
\begin{aligned}
f(x^{k+1}) \leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 &\leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \\
&= f^* + \langle \nabla f(x^k), x^k - x^* - \frac{\alpha}{2}\nabla f(x^k) \rangle \\
&= f^* + \frac{1}{2\alpha}\left\langle \alpha \nabla f(x^k), 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)\right\rangle
\end{aligned}
$$

Let $a = x^k - x^*$ and $b = x^k - x^* - \alpha \nabla f(x^k)$. Then $a + b = \alpha \nabla f(x^k)$ and $a - b = 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)$.

$$
\begin{aligned}
f(x^{k+1}) &\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^k - x^* - \alpha \nabla f(x^k)\|_2^2\right] \\
&\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2\right] \\
2\alpha\left(f(x^{k+1}) - f^*\right) &\leq \|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2
\end{aligned}
$$

- Now suppose, that the last line is defined for some index i and we sum over $i \in [0, k-1]$. Almost all summands will vanish due to the telescopic nature of the sum:

$$
2\alpha \sum_{i=0}^{k-1}\left(f(x^{i+1}) - f^*\right) \leq \|x^0 - x^*\|_2^2 - \|x^k - x^*\|_2^2 \leq \|x^0 - x^*\|_2^2 \tag{3}
$$

Convergence analysis

- Due to the monotonic decrease at each iteration $f(x^{i+1}) < f(x^i)$:

$$kf(x^k) \leq \sum_{i=0}^{k-1} f(x^{i+1})$$

Convergence analysis

- Due to the monotonic decrease at each iteration $f(x^{i+1}) < f(x^i)$:

$$kf(x^k) \leq \sum_{i=0}^{k-1} f(x^{i+1})$$

- Now putting it to Equation 3:

Convergence analysis

- Due to the monotonic decrease at each iteration $f(x^{i+1}) < f(x^i)$:

$$kf(x^k) \leq \sum_{i=0}^{k-1} f(x^{i+1})$$

- Now putting it to Equation 3:

$$2\alpha k f(x^k) - 2\alpha k f^* \leq 2\alpha \sum_{i=0}^{k-1} \left(f(x^{i+1}) - f^*\right) \leq \|x^0 - x^*\|_2^2$$

Convergence analysis

- Due to the monotonic decrease at each iteration $f(x^{i+1}) < f(x^i)$:

$$kf(x^k) \leq \sum_{i=0}^{k-1} f(x^{i+1})$$

- Now putting it to Equation 3:

$$2\alpha k f(x^k) - 2\alpha k f^* \leq 2\alpha \sum_{i=0}^{k-1} \left(f(x^{i+1}) - f^*\right) \leq \|x^0 - x^*\|_2^2$$

$$f(x^k) - f^* \leq \frac{\|x^0 - x^*\|_2^2}{2\alpha k}$$

Convergence analysis

- Due to the monotonic decrease at each iteration $f(x^{i+1}) < f(x^i)$:

$$kf(x^k) \leq \sum_{i=0}^{k-1} f(x^{i+1})$$

- Now putting it to Equation 3:

$$2\alpha k f(x^k) - 2\alpha k f^* \leq 2\alpha \sum_{i=0}^{k-1} \left(f(x^{i+1}) - f^*\right) \leq \|x^0 - x^*\|_2^2$$

$$f(x^k) - f^* \leq \frac{\|x^0 - x^*\|_2^2}{2\alpha k} \leq \frac{L\|x^0 - x^*\|_2^2}{2k}$$