# Proximal gradient method

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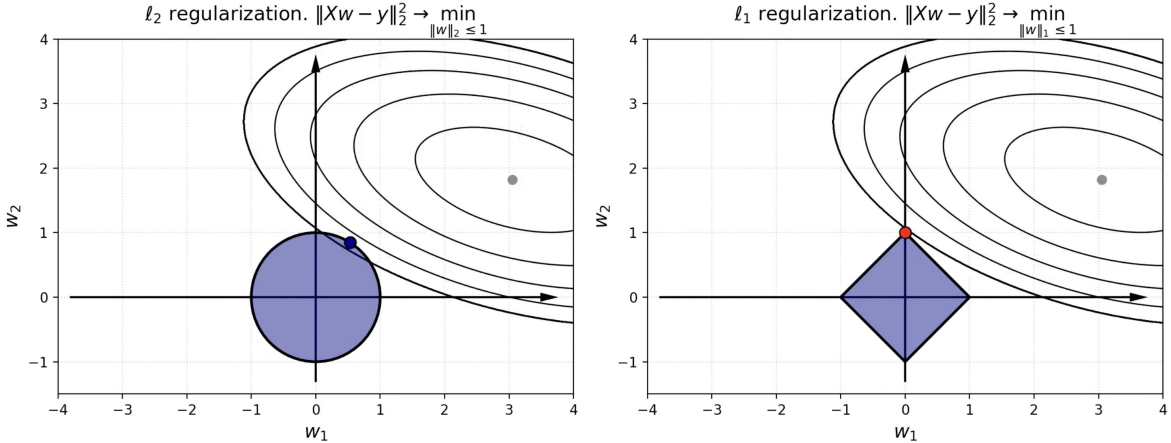
Optimization methods. MIPT

# Subgradient method

# Non-smooth problems

## $\ell_1$ induces sparsity




@fminxyz

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**Theorem**

Assume that $f$ is $G$-Lipschitz and convex, then Subgradient method converges as:

$$f(\overline{x}) - f^* \leq \frac{GR}{\sqrt{k}},$$

where

- $\alpha = \frac{R}{G\sqrt{k}}$

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- One can use Mirror Descent (a generalization of the subgradient method to a possiby non-Euclidian distance) with the same convergence rate to better fit the geometry of the problem.
- However, we can achieve standard gradient descent rate $\mathcal{O}\left(\frac{1}{k}\right)$ (and even accelerated version $\mathcal{O}\left(\frac{1}{k^2}\right)$) if we will exploit the structure of the problem.

# Proximal operator

# Proximal mapping intuition

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$$\frac{dx}{dt} = -\nabla f(x)$$

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**!** Proximal operator

$$\text{prox}_{f,\alpha}(x_k) = \arg\min_{x \in \mathbb{R}^n} \left[f(x) + \frac{1}{2\alpha}\|x - x_k\|_2^2\right]$$

# Proximal operator visualization

$$\mathrm{Prox}_f(x) = \underset{x'}{\operatorname{argmin}} \frac{1}{2}\|x - x'\|^2 + f(x')$$

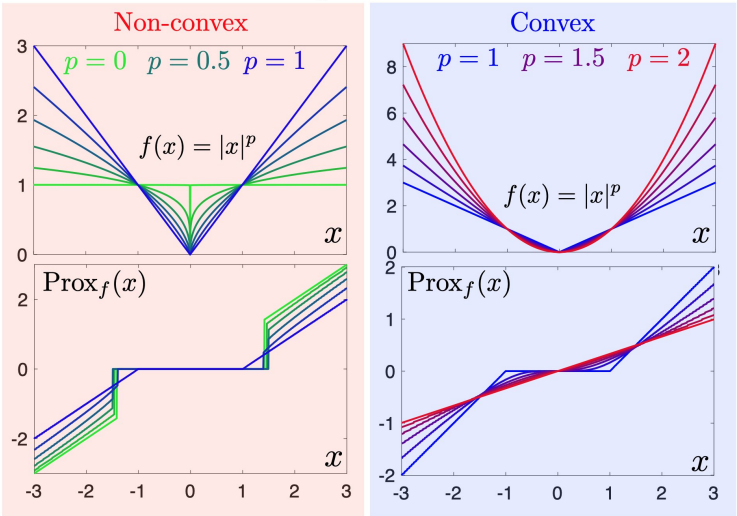


Figure 1: Source

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Proximity: Replace $\mathbb{I}_S$ by some convex function!

$$\text{prox}_r(y) = \text{prox}_{r,1}(y) := \arg\min \frac{1}{2}\|x - y\|^2 + r(x)$$

# Composite optimization

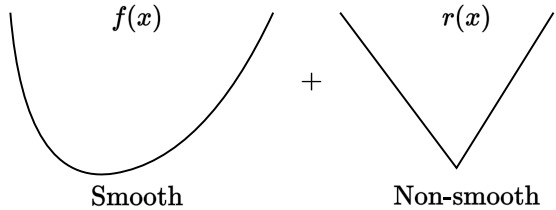
# Regularized / Composite Objectives

Many nonsmooth problems take the form

$$\min_{x \in \mathbb{R}^n} \varphi(x) = f(x) + r(x)$$

- **Lasso, L1-LS, compressed sensing**

$$f(x) = \frac{1}{2}\|Ax - b\|_2^2, r(x) = \lambda\|x\|_1$$



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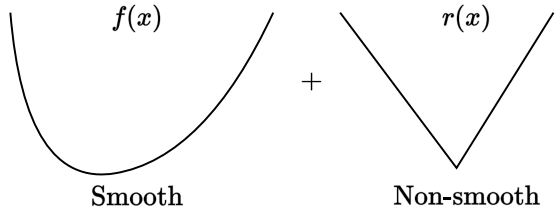
- **L1-Logistic regression, sparse LR**

$$f(x) = -y \log h(x) - (1-y) \log(1-h(x)), r(x) = \lambda\|x\|_1$$



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Which leads to the proximal gradient method:

$$x_{k+1} = \mathsf{prox}_{r,\alpha}(x_k - \alpha \nabla f(x_k))$$

And this method converges at a rate of $\mathcal{O}(\frac{1}{k})$!

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x^* - \alpha \nabla f(x^*) &\in (I + \alpha \partial r)(x^*) \\
x^* &= (I + \alpha \partial r)^{-1}(x^* - \alpha \nabla f(x^*)) \\
x^* &= \mathsf{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))
\end{aligned}
$$

Which leads to the proximal gradient method:

$$x_{k+1} = \mathsf{prox}_{r,\alpha}(x_k - \alpha \nabla f(x_k))$$

And this method converges at a rate of $\mathcal{O}(\frac{1}{k})$!

**Another form of proximal operator**

$$\mathsf{prox}_{f,\alpha}(x_k) = \mathsf{prox}_{\alpha f}(x_k) = \mathsf{arg}\min_{x \in \mathbb{R}^n} \left[ \alpha f(x) + \frac{1}{2}\|x - x_k\|_2^2 \right] \qquad \mathsf{prox}_f(x_k) = \mathsf{arg}\min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2}\|x - x_k\|_2^2 \right]$$

# Proximal operators examples

- $r(x) = \lambda \|x\|_1$, $\lambda > 0$

$$[\mathsf{prox}_r(x)]_i = [|x_i| - \lambda]_+ \cdot \mathsf{sign}(x_i),$$

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$$\mathsf{prox}_r(x) = \frac{x}{1+\lambda}.$$

- $r(x) = \mathbb{I}_S(x)$.

$$\mathsf{prox}_r(x_k - \alpha \nabla f(x_k)) = \mathsf{proj}_r(x_k - \alpha \nabla f(x_k))$$

# Proximal operator properties

**Theorem**

Let $r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function for which $\text{prox}_r$ is defined. If there exists such an $\hat{x} \in \mathbb{R}^n$ that $r(x) < +\infty$. Then, the proximal operator is uniquely defined (i.e., it always returns a single unique value).

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The proximal operator returns the minimum of some optimization problem.

Question: What can be said about this problem?

It is strongly convex, meaning it has exactly one unique minimum (the existence of $\hat{x}$ is necessary for $r(\tilde{x}) + \frac{1}{2}\|x - \tilde{x}\|_2^2$ to take a finite value somewhere).

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- $\text{prox}_r(x) = y$,

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1. Let's establish the equivalence between the first and second conditions.The first condition can be rewritten as

$$y = \arg\min_{\tilde{x} \in \mathbb{R}^d} \left( r(\tilde{x}) + \frac{1}{2}\|x - \tilde{x}\|^2 \right).$$

From the optimality condition for the convex function $r$, this is equivalent to:

$$0 \in \partial \left( r(\tilde{x}) + \frac{1}{2}\|x - \tilde{x}\|^2 \right) \Big|_{\tilde{x}=y} = \partial r(y) + y - x.$$

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2. From the definition of the subdifferential, for any subgradient $g \in \partial f(y)$ and for any $z \in \mathbb{R}^d$:

$$\langle g, z - y \rangle \leq r(z) - r(y).$$

In particular, this holds true for $g = x - y$. Conversely, it is also clear: for $g = x - y$, the above relationship holds, which means $g \in \partial r(y)$.

# Proximal operator properties

**Theorem**

The operator $\text{prox}_r(x)$ is firmly nonexpansive (FNE)

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2^2 \leq \langle \text{prox}_r(x) - \text{prox}_r(y), x - y \rangle$$

and nonexpansive:

$$\|\text{prox}_r(x) - \text{prox}_r(y)\|_2 \leq \|x - y\|_2$$

**Proof**

1. Let $u = \text{prox}_r(x)$, and $v = \text{prox}_r(y)$. Then, from the previous property:

$$\langle x - u, z_1 - u \rangle \leq r(z_1) - r(u)$$
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2. Substitute $z_1 = v$ and $z_2 = u$. Summing up, we get:

$$\langle x - u, v - u \rangle + \langle y - v, u - v \rangle \leq 0,$$
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4. The last point comes from simple Cauchy-Bunyakovsky-Schwarz for the last inequality.

# Proximal operator properties

**Theorem**

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex functions. Additionally, assume that $f$ is continuously differentiable and $L$-smooth, and for $r$, $\text{prox}_r$ is defined. Then, $x^*$ is a solution to the composite optimization problem if and only if, for any $\alpha > 0$, it satisfies:

$$x^* = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

**Proof**

1. Optimality conditions:

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2. Recall from the previous lemma:

$$\text{prox}_r(x) = y \Leftrightarrow x - y \in \partial r(y)$$

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2. Recall from the previous lemma:

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3. Finally,

$$x^* = \text{prox}_{\alpha r}(x^* - \alpha \nabla f(x^*)) = \text{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*))$$

# Theoretical tools for convergence analysis

# Convergence tools

**Theorem**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $L$-smooth convex function. Then, for any $x, y \in \mathbb{R}^n$, the following inequality holds:

$$f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq f(y) \text{ or, equivalently,}$$

$$\|\nabla f(y) - \nabla f(x)\|_2^2 = \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L\left(f(x) - f(y) - \langle \nabla f(y), x - y \rangle\right)$$

**Proof**

1. To prove this, we'll consider another function $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$. It is obviously a convex function (as a sum of convex functions). And it is easy to verify, that it is an $L$-smooth function by definition, since $\nabla \varphi(y) = \nabla f(y) - \nabla f(x)$ and $\|\nabla \varphi(y_1) - \nabla \varphi(y_2)\| = \|\nabla f(y_1) - \nabla f(y_2)\| \leq L\|y_1 - y_2\|$.

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$$x := y, y := y - \tfrac{1}{L}\nabla \varphi(y) \quad \varphi\left(y - \frac{1}{L}\nabla \varphi(y)\right) \leq \varphi(y) + \left\langle \nabla \varphi(y), -\frac{1}{L}\nabla \varphi(y) \right\rangle + \frac{1}{2L}\|\nabla \varphi(y)\|_2^2$$

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# Convergence tools

3. From the first order optimality conditions for the convex function $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$. We can conclude, that for any $x$, the minimum of the function $\varphi(y)$ is at the point $y = x$. Therefore:

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# Convergence tools

3. From the first order optimality conditions for the convex function $\nabla\varphi(y) = \nabla f(y) - \nabla f(x) = 0$. We can conclude, that for any $x$, the minimum of the function $\varphi(y)$ is at the point $y = x$. Therefore:

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switch x and y $\quad \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L\left(f(x) - f(y) - \langle\nabla f(y), x - y\rangle\right)$

The lemma has been proved. From the first view it does not make a lot of geometrical sense, but we will use it as a convenient tool to bound the difference between gradients.

# Convergence tools

**Theorem**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable on $\mathbb{R}^n$. Then, the function $f$ is $\mu$-strongly convex if and only if for any $x, y \in \mathbb{R}^d$ the following holds:

$$\begin{aligned} \text{Strongly convex case } \mu > 0 \quad & \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2 \\ \text{Convex case } \mu = 0 \quad & \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0 \end{aligned}$$

**Proof**

1. We will only give the proof for the strongly convex case, the convex one follows from it with setting $\mu = 0$. We start from necessity. For the strongly convex function

$$\begin{aligned} f(y) &\geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|x - y\|_2^2 \\ f(x) &\geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|_2^2 \\ \text{sum} \quad & \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2 \end{aligned}$$

# Convergence tools

2. For the sufficiency we assume, that $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2$. Using Newton-Leibniz theorem $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$:

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$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle = \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle$$

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y + t(x - y) - y = t(x - y) \quad &= \int_0^1 t^{-1} \langle \nabla f(y + t(x - y)) - \nabla f(y), t(x - y) \rangle dt
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Thus, we have a strong convexity criterion satisfied

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switch x and y

$$
- \langle \nabla f(x), x - y \rangle \leq - \left( f(x) - f(y) + \frac{\mu}{2} \|x - y\|_2^2 \right)
$$

# Proximal Gradient Method. Convex case

# Convergence

**Theorem**

Consider the proximal gradient method

$$x_{k+1} = \text{prox}_{\alpha r}\left(x_k - \alpha \nabla f(x_k)\right)$$

For the criterion $\varphi(x) = f(x) + r(x)$, we assume:

- $f$ is convex, differentiable, $\text{dom}(f) = \mathbb{R}^n$, and $\nabla f$ is Lipschitz continuous with constant $L > 0$.
- $r$ is convex, and $\text{prox}_{\alpha r}(x_k) = \arg\min_{x \in \mathbb{R}^n} \left[\alpha r(x) + \frac{1}{2}\|x - x_k\|_2^2\right]$ can be evaluated.

Proximal gradient descent with fixed step size $\alpha = 1/L$ satisfies

$$\varphi(x_k) - \varphi^* \leq \frac{L\|x_0 - x^*\|^2}{2k},$$

Proximal gradient descent has a convergence rate of $O(1/k)$ or $O(1/\varepsilon)$. This matches the gradient descent rate! (But remember the proximal operation cost)

# Convergence

**Proof**

1. Let's introduce the **gradient mapping**, denoted as $G_\alpha(x)$, acts as a "gradient-like object":

$$x_{k+1} = \text{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k))$$
$$x_{k+1} = x_k - \alpha G_\alpha(x_k).$$

where $G_\alpha(x)$ is:

$$G_\alpha(x) = \frac{1}{\alpha}\left(x - \text{prox}_{\alpha r}\left(x - \alpha \nabla f\left(x\right)\right)\right)$$

Observe that $G_\alpha(x) = 0$ if and only if $x$ is optimal. Therefore, $G_\alpha$ is analogous to $\nabla f$. If $x$ is locally optimal, then $G_\alpha(x) = 0$ even for nonconvex $f$. This demonstrates that the proximal gradient method effectively combines gradient descent on $f$ with the proximal operator of $r$, allowing it to handle non-differentiable components effectively.

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$$\text{smoothness} \quad f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2}\|x_{k+1} - x_k\|_2^2$$

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&\text{convexity } f(x) \geq f(x_k) + \langle \nabla f(x_k), x - x_k \rangle \quad \leq f(x) - \langle \nabla f(x_k), x - x_k \rangle + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{\alpha^2 L}{2}\|G_\alpha(x_k)\|_2^2 \\
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7. Now it is easy to verify, that when $x = x_k$ we have monotonic decrease for the proximal gradient algorithm:

$$\varphi(x_{k+1}) \leq \varphi(x_k) - \frac{\alpha}{2} \|G_\alpha(x_k)\|_2^2$$

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# Convergence

9. Now we write the bound above for all iterations $i \in 0, k-1$ and sum them:

Which is a standard $\frac{L\|x_0 - x^*\|_2^2}{2k}$ with $\alpha = \frac{1}{L}$, or, $\mathcal{O}\left(\frac{1}{k}\right)$ rate for smooth convex problems with Gradient Descent!

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$$\sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] \leq \frac{1}{2\alpha} \left[ \|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2 \right]$$

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$$\leq \frac{1}{2\alpha} \|x_0 - x^*\|_2^2$$

10. Since $\varphi(x_k)$ is a decreasing sequence, it follows that:

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# Proximal Gradient Method. Strongly convex case

# Convergence

**Theorem**

Consider the proximal gradient method

$$x_{k+1} = \text{prox}_{\alpha r}\left(x_k - \alpha \nabla f(x_k)\right)$$

For the criterion $\varphi(x) = f(x) + r(x)$, we assume:

- $f$ is $\mu$-strongly convex, differentiable, $\text{dom}(f) = \mathbb{R}^n$, and $\nabla f$ is Lipschitz continuous with constant $L > 0$.
- $r$ is convex, and $\text{prox}_{\alpha r}(x_k) = \arg\min_{x \in \mathbb{R}^n} \left[\alpha r(x) + \frac{1}{2}\|x - x_k\|_2^2\right]$ can be evaluated.

Proximal gradient descent with fixed step size $\alpha \leq 1/L$ satisfies

$$\|x_{k+1} - x^*\|_2^2 \leq (1 - \alpha\mu)^k \|x_0 - x^*\|_2^2$$

This is exactly gradient descent convergence rate. Note, that the original problem is even non-smooth!

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$$\text{smoothness} \quad \|\nabla f(x_k) - \nabla f(x^*)\|_2^2 \leq 2L \left(f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle\right)$$

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4. Due to convexity of $f$: $f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle \geq 0$. Therefore, if we use $\alpha \leq \frac{1}{L}$:

$$\|x_{k+1} - x^*\|_2^2 \leq (1 - \alpha\mu)\|x_k - x^*\|^2,$$

which is exactly linear convergence of the method with up to $1 - \frac{\mu}{L}$ convergence rate.

# Accelerated Proximal Method

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Let $x_0 = y_0 \in \mathrm{dom}(r)$. For $k \geq 1$:

$$x_k = \mathrm{prox}_{\alpha_k h}(y_{k-1} - \alpha_k \nabla f(y_{k-1}))$$
$$y_k = x_k + \frac{k-1}{k+2}(x_k - x_{k-1})$$

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- Uses extra "memory" for interpolation
- Same computational cost as ordinary prox-grad
- Convergence rate theoretically optimal

# Numerical experiments

# Quadratic case

$$f(x) = \frac{1}{2m}\|Ax - b\|_2^2 + \lambda\|x\|_1 \to \min_{x \in \mathbb{R}^n}, \qquad A \in \mathbb{R}^{m \times n}, \quad \lambda\left(\tfrac{1}{m}A^T A\right) \in [\mu; L].$$

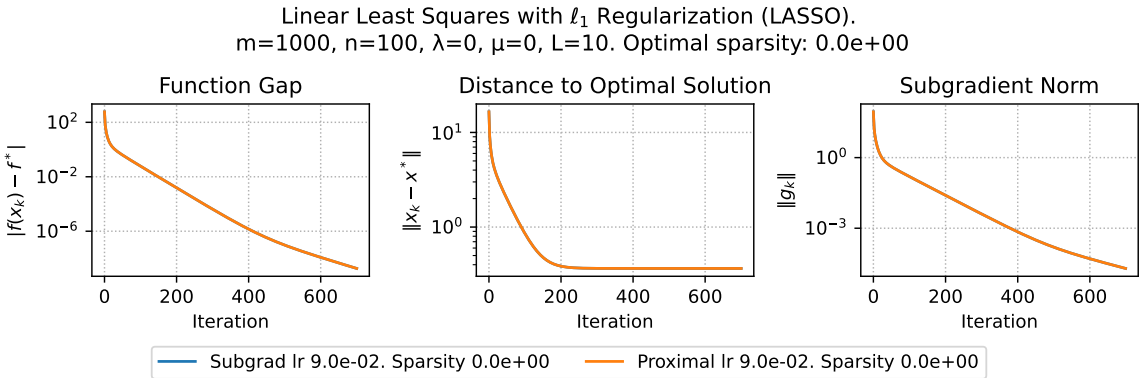


Figure 2: Smooth convex case. Sublinear convergence, no convergence in domain, no difference between subgradient and proximal methods

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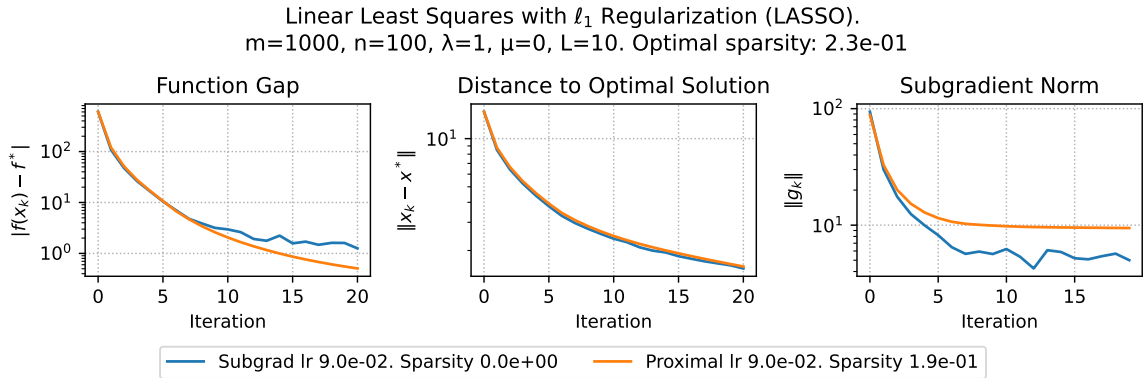


Figure 3: Non-smooth convex case. Sublinear convergence. At the beginning, the subgradient method and proximal method are close.

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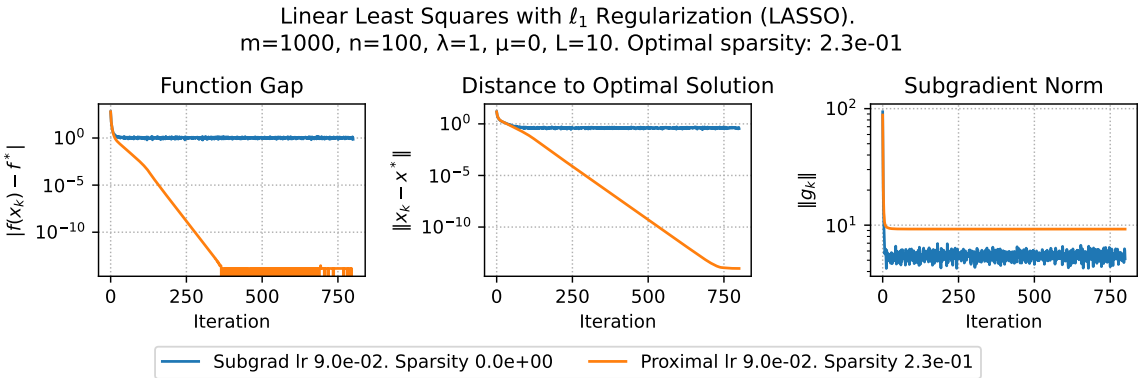



Figure 4: Non-smooth convex case. If we take more iterations, the proximal method converges with the constant learning rate, which is not the case for the subgradient method. The difference is tremendous, while the iteration complexity is the same.

# Binary logistic regression

$$f(x) = \frac{1}{m}\sum_{i=1}^{m} \log(1 + \exp(-b_i(A_i x))) + \lambda\|x\|_1 \to \min_{x \in \mathbb{R}^n}, \qquad A_i \in \mathbb{R}^n, \quad b_i \in \{-1, 1\}$$

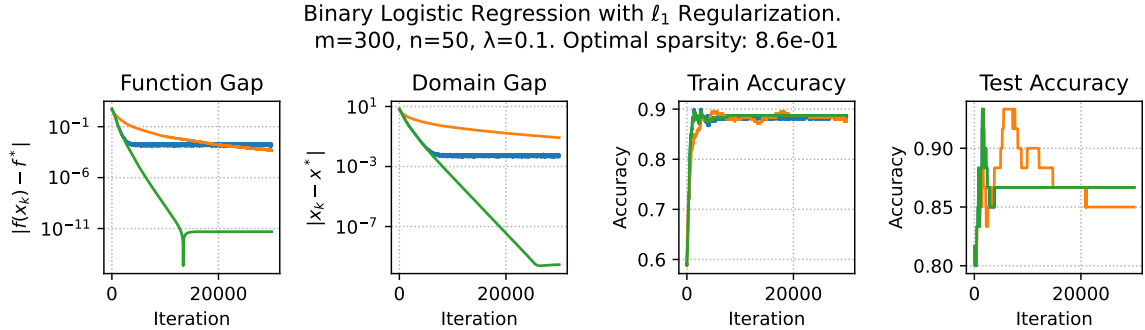
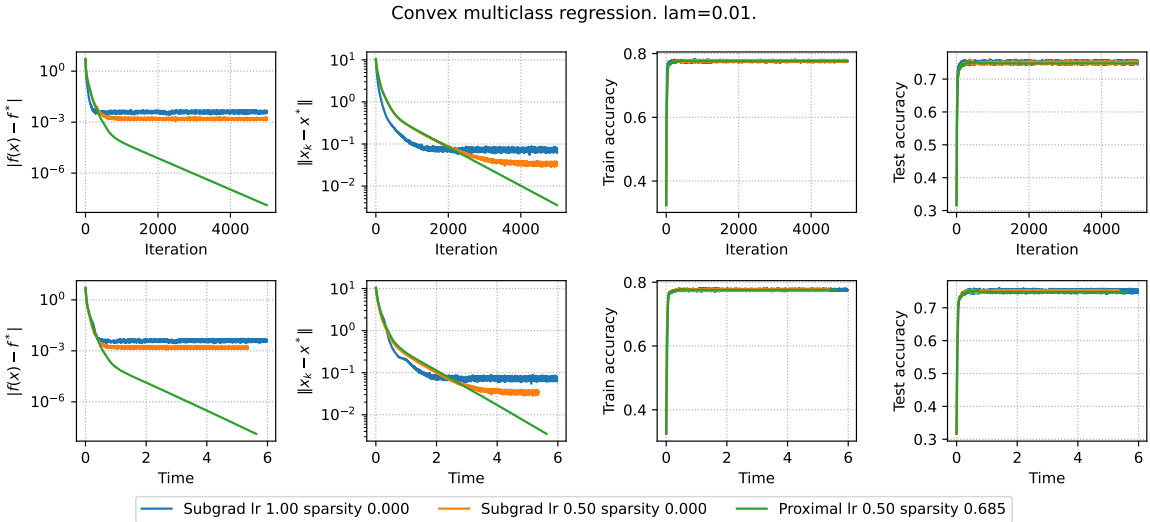



Figure 5: Logistic regression with $\ell_1$ regularization

# Softmax multiclass regression



# Example: ISTA

## Iterative Shrinkage-Thresholding Algorithm (ISTA)

ISTA is a popular method for solving optimization problems involving L1 regularization, such as Lasso. It combines gradient descent with a shrinkage operator to handle the non-smooth L1 penalty effectively.

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- **Application**:
  - Efficient for sparse signal recovery, image processing, and compressed sensing.

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FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

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- **Convergence**:
  - Improves the convergence rate to $O(1/k^2)$.
- **Application**:
  - Especially useful for large-scale problems in machine learning and signal processing where the L1 penalty induces sparsity.

# Example: Matrix Completion

## Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

- **Problem Formulation**:

$$\min_X \frac{1}{2}\|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda\|X\|_*,$$

where $P_\Omega$ projects onto the observed set $\Omega$, and $\|\cdot\|_*$ denotes the nuclear norm.

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- **Algorithm**:
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- **Application**:
  - Widely used in recommender systems, image recovery, and other domains where data is naturally matrix-formed but partially observed.

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- Further reading: Proximal operator splitting, Douglas-Rachford splitting, Best approximation problem, Three operator splitting.