



**Recap of Conjugate sets, conjugate functions.  
Subgradient and subdifferential**

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Optimization methods. MIPT

## Conjugate sets

# Conjugate set

Let  $S \subseteq \mathbb{R}^n$  be an arbitrary non-empty set. Then its conjugate set is defined as:

$$S^* = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \geq -1 \quad \forall x \in S\}$$

A set  $S^{**}$  is called double conjugate to a set  $S$  if:

$$S^{**} = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \geq -1 \quad \forall x \in S^*\}$$

- The sets  $S_1$  and  $S_2$  are called **inter-conjugate** if  $S_1^* = S_2$ ,  $S_2^* = S_1$ .



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- A set  $S$  is called **self-conjugate** if  $S^* = S$ .



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- $S^* = (\overline{S})^*$ .

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- Let  $p \in S^*$  and  $x_0 \in \overline{S}$ ,  $x_0 = \lim_{k \rightarrow \infty} x_k$ . Then by virtue of the continuity of the function  $f(x) = p^T x$ , we have:  
 $p^T x_k \geq -1 \rightarrow p^T x_0 \geq -1$ . So  $p \in (\overline{S})^*$ , hence  $S^* \subset (\overline{S})^*$ .

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- Let  $p \in S^*$ ,  $x_0 \in \mathbf{conv}(S)$ , i.e.,  $x_0 = \sum_{i=1}^k \theta_i x_i \mid x_i \in S, \sum_{i=1}^k \theta_i = 1, \theta_i \geq 0$ .

So  $p^T x_0 = \sum_{i=1}^k \theta_i p^T x_i \geq \sum_{i=1}^k \theta_i (-1) = 1 \cdot (-1) = -1$ . So  $p \in (\mathbf{conv}(S))^*$ , hence  $S^* \subset (\mathbf{conv}(S))^*$ .



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- From all points of the ball  $X$ , take such a point  $x \in X$  that its scalar product of  $p$ :  $p^T x$  is minimal, then this is the point  $x = -\frac{p}{\|p\|}r$ .

$$p^T x = p^T \left( -\frac{p}{\|p\|}r \right) = -\|p\|r \geq -1$$

$$\|p\| \leq \frac{1}{r} \in Y$$

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- Now let  $p \in Y$ . We need to show that  $p \in X^*$ , i.e.,  $\langle p, x \rangle \geq -1$ . It's enough to apply the Cauchy-Bunyakovsky inequality:

$$\|\langle p, x \rangle\| \leq \|p\| \|x\| \leq \frac{1}{r} \cdot r = 1$$

The latter comes from the fact that  $p \in B(0, 1/r)$  and  $x \in B(0, r)$ .

So  $Y \subset X^*$ .

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## Dual cone

A conjugate cone to a cone  $K$  is a set  $K^*$  such that:

$$K^* = \{y \mid \langle x, y \rangle \geq 0 \quad \forall x \in K\}$$

To show that this definition follows directly from the definitions above, recall what a conjugate set is and what a cone  $\forall \lambda > 0$  is.

$$\{y \in \mathbb{R}^n \mid \langle y, x \rangle \geq -1 \quad \forall x \in S\} \rightarrow \{\lambda y \in \mathbb{R}^n \mid \langle y, x \rangle \geq -\frac{1}{\lambda} \quad \forall x \in S\}$$



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- Let  $K_1, \dots, K_m$  be cones in  $\mathbb{R}^n$ , then:

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- Let  $K_1, \dots, K_m$  be cones in  $\mathbb{R}^n$ . Let also their intersection have an interior point, then:

$$\left( \bigcap_{i=1}^m K_i \right)^* = \sum_{i=1}^m K_i^*$$

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Find the conjugate cone for a monotone nonnegative cone:

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Note that:

$$\sum_{i=1}^n x_i y_i = y_1(x_1 - x_2) + (y_1 + y_2)(x_2 - x_3) + \dots + (y_1 + y_2 + \dots + y_{n-1})(x_{n-1} - x_n) + (y_1 + \dots + y_n)x_n$$

Since in the presented sum in each summand, the second multiplier in each summand is non-negative, then:

$$y_1 \geq 0, \quad y_1 + y_2 \geq 0, \quad \dots, \quad y_1 + \dots + y_n \geq 0$$

$$\text{So } K^* = \left\{ y \mid \sum_{i=1}^k y_i \geq 0, k = \overline{1, n} \right\}.$$

# Polyhedra

The set of solutions to a system of linear inequalities and equalities is a polyhedron:

$$Ax \preceq b, \quad Cx = d$$

Here  $A \in \mathbb{R}^{m \times n}$ ,  $C \in \mathbb{R}^{p \times n}$ , and the inequality is a piecewise inequality.

## Theorem

Let  $x_1, \dots, x_m \in \mathbb{R}^n$ . Conjugate to a polyhedral set:

$$S = \mathbf{conv}(x_1, \dots, x_k) + \mathbf{cone}(x_{k+1}, \dots, x_m)$$

is a polyhedron (polyhedron):

$$S^* = \{p \in \mathbb{R}^n \mid \langle p, x_i \rangle \geq -1, i = \overline{1, k}; \langle p, x_i \rangle \geq 0, i = \overline{k+1, m}\}$$

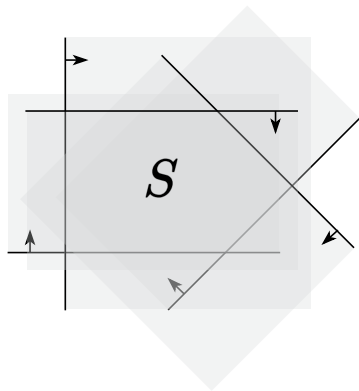


Figure 3: Polyhedra

## Proof

- Let  $S = X, S^* = Y$ . Take some  $p \in X^*$ , then  $\langle p, x_i \rangle \geq -1, i = \overline{1, k}$ . At the same time, for any  $\theta > 0, i = \overline{k+1, m}$ :

$$\langle p, x_i \rangle \geq -1 \rightarrow \langle p, \theta x_i \rangle \geq -1$$

$$\langle p, x_i \rangle \geq -\frac{1}{\theta} \rightarrow \langle p, x_i \rangle \geq 0.$$

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So  $p \in Y \rightarrow X^* \subset Y$ .

- Suppose, on the other hand, that  $p \in Y$ . For any point  $x \in X$ :

$$x = \sum_{i=1}^m \theta_i x_i \quad \sum_{i=1}^k \theta_i = 1, \theta_i \geq 0$$

So:

$$\langle p, x \rangle = \sum_{i=1}^m \theta_i \langle p, x_i \rangle = \sum_{i=1}^k \theta_i \langle p, x_i \rangle + \sum_{i=k+1}^m \theta_i \langle p, x_i \rangle \geq \sum_{i=1}^k \theta_i (-1) + \sum_{i=1}^k \theta_i \cdot 0 = -1.$$

# Example

## Conjugate functions



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Recall that given  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the function defined by

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$$g(y) \geq g(x) + \frac{\mu}{2} \|y - x\|^2, \quad \text{for all } y$$

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Hence, defining  $x_u = \nabla f^*(u)$  and  $x_v = \nabla f^*(v)$ ,

$$f(x_v) - u^T x_v \geq f(x_u) - u^T x_u + \frac{\mu}{2} \|x_u - x_v\|^2$$

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Adding these together, using the Cauchy-Schwarz inequality, and rearranging shows that

$$\|x_u - x_v\|^2 \leq \frac{1}{\mu} \|u - v\|^2$$



## Slopes of $f$ and $f^*$

**Proof of “ $\Leftarrow$ ”:** for simplicity, call  $g = f^*$  and  $L = \frac{1}{\mu}$ . As  $\nabla g$  is Lipschitz with constant  $L$ , so is  $g_x(z) = g(z) - \nabla g(x)^T z$ , hence

$$g_x(z) \leq g_x(y) + \nabla g_x(y)^T (z - y) + \frac{L}{2} \|z - y\|_2^2$$

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Minimizing each side over  $z$ , and rearranging, gives

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Exchanging roles of  $x$ ,  $y$ , and adding together, gives

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Let  $u = \nabla f(x)$ ,  $v = \nabla g(y)$ ; then  $x \in \partial g^*(u)$ ,  $y \in \partial g^*(v)$ , and the above reads  $(x - y)^T (u - v) \geq \frac{\|u - v\|^2}{L}$ , implying the result.

# Conjugate function properties

Recall that given  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the function defined by

$$f^*(y) = \max_x [y^T x - f(x)]$$

is called its conjugate.

- Conjugates appear frequently in dual programs, since

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- If  $f$  is strictly convex, then

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## Conjugate function properties (proofs)

We will show that  $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$ , assuming that  $f$  is convex and closed.

- **Proof of  $\Leftarrow$ :** Suppose  $y \in \partial f(x)$ . Then  $x \in M_y$ , the set of maximizers of  $y^T z - f(z)$  over  $z$ . But

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- **Proof of  $\Rightarrow$ :** From what we showed above, if  $x \in \partial f^*(y)$ , then  $y \in \partial f^*(x)$ , but  $f^{**} = f$ .

Clearly  $y \in \partial f(x) \Leftrightarrow x \in \arg \min_z \{f(z) - y^T z\}$

Lastly, if  $f$  is strictly convex, then we know that  $f(z) - y^T z$  has a unique minimizer over  $z$ , and this must be  $\nabla f^*(y)$ .

## Subgradient and Subdifferential

# $\ell_1$ -regularized linear least squares

$\ell_1$  induces sparsity

$\ell_2$  regularization.  $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_2 \leq 1}$



$\ell_1$  regularization.  $\|Xw - y\|_2^2 \rightarrow \min_{\|w\|_1 \leq 1}$



@fminxyz

# Norms are not smooth

$$\min_{x \in \mathbb{R}^n} f(x),$$

A classical convex optimization problem is considered. We assume that  $f(x)$  is a convex function, but now we do not require smoothness.



Figure 5: Norm cones for different  $p$  - norms are non-smooth

## Convex function linear lower bound

An important property of a continuous convex function  $f(x)$  is that at any chosen point  $x_0$  for all  $x \in \text{dom } f$  the inequality holds:

$$f(x) \geq f(x_0) + \langle g, x - x_0 \rangle$$



Figure 6: Taylor linear approximation serves as a global lower bound for a convex function

## Convex function linear lower bound



An important property of a continuous convex function  $f(x)$  is that at any chosen point  $x_0$  for all  $x \in \text{dom } f$  the inequality holds:

$$f(x) \geq f(x_0) + \langle g, x - x_0 \rangle$$

for some vector  $g$ , i.e., the tangent to the graph of the function is the *global* estimate from below for the function.

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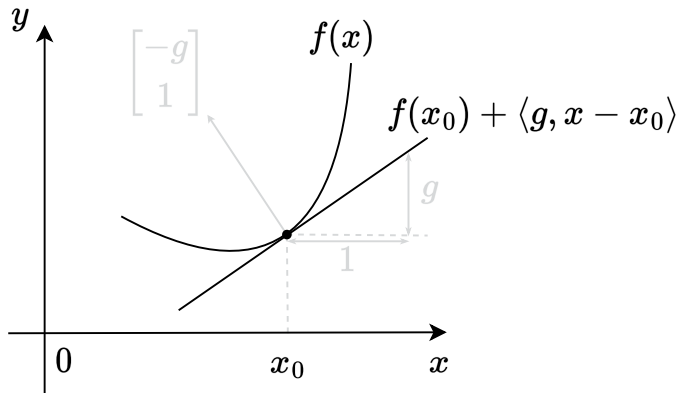
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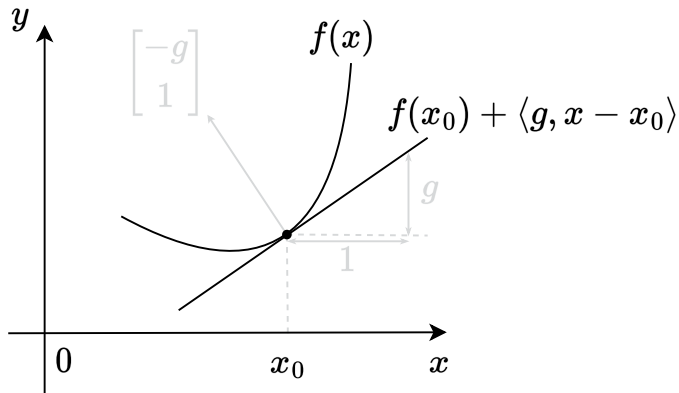
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We wouldn't want to lose such a nice property.

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## Subgradient and subdifferential

A vector  $g$  is called the **subgradient** of a function  $f(x) : S \rightarrow \mathbb{R}$  at a point  $x_0$  if  $\forall x \in S$ :

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Figure 7: Subdifferential is a set of all possible subgradients

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1. Assume, that  $s \in \partial f(x_0)$  for some  $s \in \mathbb{R}^n$  distinct from  $\nabla f(x_0)$ . Let  $v \in \mathbb{R}^n$  be a unit vector. Because  $x_0$  is an interior point of  $S$ , there exists  $\delta > 0$  such that  $x_0 + tv \in S$  for all  $0 < t < \delta$ . By the definition of the subgradient, we have

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2. From this,  $\langle s - \nabla f(x_0), v \rangle \geq 0$ . Due to the arbitrariness of  $v$ , one can set

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3. Furthermore, if the function  $f$  is convex, then according to the differential condition of convexity  $f(x) \geq f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle$  for all  $x \in S$ . But by definition, this means  $\nabla f(x_0) \in \partial f(x_0)$ .

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Is it correct, that if the function has a subdifferential at some point, the function is convex?

Find  $\partial f(x)$ , if  $f(x) = \sin x, x \in [\pi/2; 2\pi]$



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Is it correct, that if the function is convex, it has a subgradient at any point?

Convexity follows from subdifferentiability at any point. A natural question to ask is whether the converse is true: is every convex function subdifferentiable? It turns out that, generally speaking, the answer to this question is negative.

Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be the function defined by  $f(x) := -\sqrt{x}$ . Then,  $\partial f(0) = \emptyset$ .

Assume, that  $s \in \partial f(0)$  for some  $s \in \mathbb{R}$ . Then, by definition, we must have  $sx \leq -\sqrt{x}$  for all  $x \geq 0$ . From this, we can deduce  $s \leq -\sqrt{1/x}$  for all  $x > 0$ . Taking the limit as  $x$  approaches 0 from the right, we get  $s \leq -\infty$ , which is impossible.

# Subdifferential calculus

**i** Moreau - Rockafellar theorem (subdifferential of a linear combination)

Let  $f_i(x)$  be convex functions on convex sets  $S_i$ ,  $i = \overline{1, n}$ . Then if  $\bigcap_{i=1}^n \text{ri}(S_i) \neq \emptyset$  then the function

$f(x) = \sum_{i=1}^n a_i f_i(x)$ ,  $a_i > 0$  has a subdifferential

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**i** Dubovitsky - Milutin theorem (subdifferential of a point-wise maximum)

Let  $f_i(x)$  be convex functions on the open convex set  $S \subseteq \mathbb{R}^n$ ,  $x_0 \in S$ , and the pointwise maximum is defined as  $f(x) = \max_i f_i(x)$ . Then:

$$\partial_S f(x_0) = \text{conv} \left\{ \bigcup_{i \in I(x_0)} \partial_S f_i(x_0) \right\}, \quad I(x) = \{i \in [1, n] \mid f_i(x) = f(x)\}$$

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- $\partial(f(Ax + b))(x) = A^T \partial f(Ax + b)$ ,  $f$  - convex function
- $z \in \partial f(x)$  if and only if  $x \in \partial f^*(z)$ .

## Connection to convex geometry

Convex set  $S \subseteq \mathbb{R}^n$ , consider indicator function  $I_S : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$I_S(x) = I\{x \in S\} = \begin{cases} 0 & \text{if } x \in S \\ \infty & \text{if } x \notin S \end{cases}$$

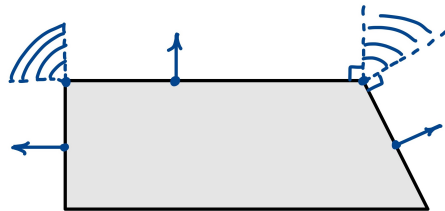
For  $x \in S$ ,  $\partial I_S(x) = \mathcal{N}_S(x)$ , the **normal cone** of  $S$  at  $x$  is, recall

$$\mathcal{N}_S(x) = \{g \in \mathbb{R}^n : g^T x \geq g^T y \text{ for any } y \in S\}$$

**Why?** By definition of subgradient  $g$ ,

$$I_S(y) \geq I_S(x) + g^T(y - x) \quad \text{for all } y$$

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- For  $y \notin S$ ,  $I_S(y) = \infty$
- For  $y \in S$ , this means  $0 \geq g^T(y - x)$



# Optimality Condition

For any  $f$  (convex or not),

$$f(x^*) = \min_x f(x) \iff 0 \in \partial f(x^*)$$

That is,  $x^*$  is a minimizer if and only if 0 is a subgradient of  $f$  at  $x^*$ . This is called the **subgradient optimality condition**.

Why? Easy:  $g = 0$  being a subgradient means that for all  $y$

$$f(y) \geq f(x^*) + 0^T(y - x^*) = f(x^*)$$

Note the implication for a convex and differentiable function  $f$ , with

$$\partial f(x) = \{\nabla f(x)\}$$

# Derivation of first-order optimality

Example of the power of subgradients: we can use what we have learned so far to derive the **first-order optimality condition**. Recall

$$\min_x f(x) \text{ subject to } x \in S$$

is solved at  $x$ , for  $f$  convex and differentiable, if and only if

$$\nabla f(x)^T(y - x) \geq 0 \quad \text{for all } y \in S$$

Intuitively: this says that the gradient increases as we move away from  $x$ . How to prove it? First, recast the problem as

$$\min_x f(x) + I_S(x)$$

Now apply subgradient optimality:

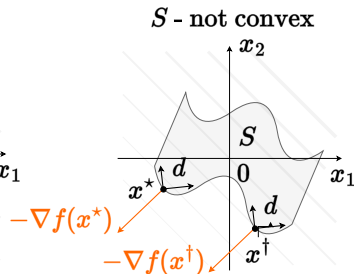
$$0 \in \partial(f(x) + I_S(x))$$

$$f(x) = x_1 + x_2 \rightarrow \min_{x_1, x_2 \in \mathbb{R}^2}$$



$$\langle -\nabla f(x^*), d \rangle \leq 0$$

$x^*$  - optimal



$$\langle -\nabla f(x^\dagger), d \rangle \leq 0$$

$x^\dagger$  - not optimal

# Derivation of first-order optimality

Observe

$$0 \in \partial(f(x) + I_S(x))$$

$$\Leftrightarrow 0 \in \{\nabla f(x)\} + \mathcal{N}_S(x)$$

$$\Leftrightarrow -\nabla f(x) \in \mathcal{N}_S(x)$$

$$\Leftrightarrow -\nabla f(x)^T x \geq -\nabla f(x)^T y \text{ for all } y \in S$$

$$\Leftrightarrow \nabla f(x)^T (y - x) \geq 0 \text{ for all } y \in S$$

as desired.

Note: the condition  $0 \in \partial f(x) + \mathcal{N}_S(x)$  is a **fully general condition** for optimality in convex problems. But it's not always easy to work with (KKT conditions, later, are easier).

$$f(x) = x_1 + x_2 \rightarrow \min_{x_1, x_2 \in \mathbb{R}^2}$$





## Example 1

### Example

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$$\partial f_1(x) = \begin{cases} -1, & x < 1 \\ [-1; 1], & x = 1 \\ 1, & x > 1 \end{cases} \quad \partial f_2(x) = \begin{cases} -1, & x < -1 \\ [-1; 1], & x = -1 \\ 1, & x > -1 \end{cases}$$

So

$$\partial f(x) = \begin{cases} -2, & x < -1 \\ [-2; 0], & x = -1 \\ 0, & -1 < x < 1 \\ [0; 2], & x = 1 \\ 2, & x > 1 \end{cases}$$

## Example 2

Find  $\partial f(x)$  if  $f(x) = [\max(0, f_0(x))]^q$ . Here,  $f_0(x)$  is a convex function on an open convex set  $S$ , and  $q \geq 1$ .

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According to the composition theorem (the function  $\varphi(x) = x^q$  is differentiable) and  $g(x) = \max(0, f_0(x))$ , we have:

$$\partial f(x) = q(g(x))^{q-1} \partial g(x)$$

By the theorem on the pointwise maximum:

$$\partial g(x) = \begin{cases} \partial f_0(x), & f_0(x) > 0, \\ \{0\}, & f_0(x) < 0, \\ \{a \mid a = \lambda a', 0 \leq \lambda \leq 1, a' \in \partial f_0(x)\}, & f_0(x) = 0 \end{cases}$$

### Example 3. Subdifferential of the Norm

Let  $V$  be a finite-dimensional Euclidean space, and  $x_0 \in V$ . Let  $\|\cdot\|$  be an arbitrary norm in  $V$  (not necessarily induced by the scalar product), and let  $\|\cdot\|_*$  be the corresponding conjugate norm. Then,

$$\partial\|\cdot\|(x_0) = \begin{cases} B_{\|\cdot\|_*}(0, 1), & \text{if } x_0 = 0, \\ \{s \in V : \|s\|_* \leq 1; \langle s, x_0 \rangle = \|x_0\|\} = \{s \in V : \|s\|_* = 1; \langle s, x_0 \rangle = \|x_0\|\}, & \text{otherwise.} \end{cases}$$

Where  $B_{\|\cdot\|_*}(0, 1)$  is the closed unit ball centered at zero with respect to the conjugate norm. In other words, a vector  $s \in V$  with  $\|s\|_* = 1$  is a subgradient of the norm  $\|\cdot\|$  at point  $x_0 \neq 0$  if and only if the Hölder's inequality  $\langle s, x_0 \rangle \leq \|x_0\|$  becomes an equality.

### Example 3. Subdifferential of the Norm

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$$\partial\|\cdot\|(x_0) = \begin{cases} B_{\|\cdot\|_*}(0, 1), & \text{if } x_0 = 0, \\ \{s \in V : \|s\|_* \leq 1; \langle s, x_0 \rangle = \|x_0\|\} = \{s \in V : \|s\|_* = 1; \langle s, x_0 \rangle = \|x_0\|\}, & \text{otherwise.} \end{cases}$$

Where  $B_{\|\cdot\|_*}(0, 1)$  is the closed unit ball centered at zero with respect to the conjugate norm. In other words, a vector  $s \in V$  with  $\|s\|_* = 1$  is a subgradient of the norm  $\|\cdot\|$  at point  $x_0 \neq 0$  if and only if the Hölder's inequality  $\langle s, x_0 \rangle \leq \|x_0\|$  becomes an equality.

Let  $s \in V$ . By definition,  $s \in \partial\|\cdot\|(x_0)$  if and only if

$$\langle s, x \rangle - \|x\| \leq \langle s, x_0 \rangle - \|x_0\|, \text{ for all } x \in V,$$

or equivalently,

$$\sup_{x \in V} \{\langle s, x \rangle - \|x\|\} \leq \langle s, x_0 \rangle - \|x_0\|.$$

By the definition of the supremum, the latter is equivalent to

$$\sup_{x \in V} \{\langle s, x \rangle - \|x\|\} = \langle s, x_0 \rangle - \|x_0\|.$$

Subgradient and Subdifferential

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It is important to note that the expression on the left side is the supremum from the definition of the Fenchel conjugate function for the norm, which is known to be

$$\sup_{x \in V} \{\langle s, x \rangle - \|x\|\} = \begin{cases} 0, & \text{if } \|s\|_* \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, equation is equivalent to  $\|s\|_* \leq 1$  and  $\langle s, x_0 \rangle = \|x_0\|$ .

## Example 3. Subdifferential of the Norm

Consequently, it remains to note that for  $x_0 \neq 0$ , the inequality  $\|s\|_* \leq 1$  must become an equality since, when  $\|s\|_* < 1$ , Hölder's inequality implies  $\langle s, x_0 \rangle \leq \|s\|_* \|x_0\| < \|x_0\|$ .

The conjugate norm in Example above does not appear by chance. It turns out that, in a completely similar manner for an arbitrary function  $f$  (not just for the norm), its subdifferential can be described in terms of the dual object — the Fenchel conjugate function.