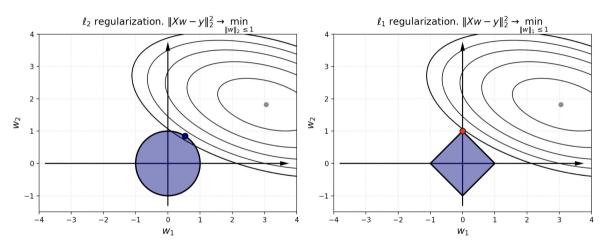




#### Non-smooth problems

# $\ell_1$ induces sparsity



@fminxyz



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$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\min_{x \in \mathbb{R}^n} f(x) \qquad x_{k+1} = x_k - \alpha_k g_k, \quad g_k \in \partial f(x_k)$$

Subgradient method

Subgradient Method:	$\min_{x \in \mathbb{R}^n} f(x)$	$x_{k+1} = x_k - \alpha_k g_k,$	$g_k \in \partial f(x_k)$
---------------------	----------------------------------	---------------------------------	---------------------------

convex (non-smooth)	strongly convex (non-smooth)
$f(x_k) - f^* \sim \mathcal{O}\left(rac{1}{\sqrt{k}} ight) \ k_arepsilon \sim \mathcal{O}\left(rac{1}{arepsilon^2} ight)$	$f(x_k) - f^* \sim \mathcal{O}\left(rac{1}{k} ight) \ k_arepsilon \sim \mathcal{O}\left(rac{1}{arepsilon} ight)$

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# 1 Theorem

Assume that f is G-Lipschitz and convex, then Subgradient method converges as:

 $f(\overline{x}) - f^* \le \frac{GR}{\sqrt{k}},$ 

where •  $\alpha = \frac{R}{G\sqrt{k}}$ 

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 $\overline{x} = \frac{1}{k} \sum_{i=1}^{k-1} x_i$ 

$$f \to \min_{x,y,z}$$
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• Subgradient method is optimal for the problems above.

 $\min_{x,y,z}$  Subgradient method

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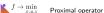
- Subgradient method is optimal for the problems above.
- One can use Mirror Descent (a generalization of the subgradient method to a possiby non-Euclidian distance) with the same convergence rate to better fit the geometry of the problem.

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- However, we can achieve standard gradient descent rate  $\mathcal{O}\left(\frac{1}{k}\right)$  (and even accelerated version  $\mathcal{O}\left(\frac{1}{k^2}\right)$ ) if we will exploit the structure of the problem.







Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

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$$\frac{x_{k+1} - x_k}{\alpha} + \nabla f(x_{k+1}) = 0$$

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$$\operatorname{prox}_{f, lpha}(x_k) = \arg\min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2lpha} \|x - x_k\|_2^2 \right]$$

#### Proximal operator visualization

$$\operatorname{Prox}_{f}(x) = \underset{x'}{\operatorname{argmin}} \frac{1}{2} ||x - x'||^{2} + f(x')$$

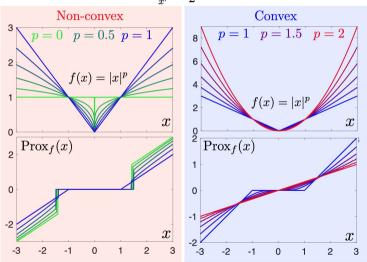


Figure 1: Source

• **GD** from proximal method. Back to the discretization:

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n,z Proximal operator

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Thus, we have a usual gradient descent with  $\alpha \to 0$ :  $x_{k+1} = x_k - \alpha \nabla f(x_k)$ 

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$$x_{k+1} = \mathsf{prox}_{f_{x_k}^{II},\alpha}(x_k) = \arg\min_{x \in \mathbb{R}^n} \left[ f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2} \langle \nabla^2 f(x_k)(x - x_k), x - x_k \rangle + \frac{1}{2\alpha} \|x - x_k\|_2^2 \right]$$

 $f \to \min_{x,y,z}$  Proximal operator

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 $f \to \min_{x,y,z}$  Proximal operator

Let  $\mathbb{I}_S$  be the indicator function for closed, convex S. Recall orthogonal projection  $\pi_S(y)$ 

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With the following notation of indicator function

$$\mathbb{I}_S(x) = \begin{cases} 0, & x \in S, \\ \infty, & x \notin S, \end{cases}$$



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Rewrite orthogonal projection  $\pi_S(y)$  as

$$\pi_S(y) := \arg\min_{x \in \mathbb{R}^n} \frac{1}{2} ||x - y||^2 + \mathbb{I}_S(x).$$

# From projections to proximity

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$$\pi_S(y) := \arg\min_{x \in \mathbb{R}^n} \frac{1}{2} ||x - y||^2 + \mathbb{I}_S(x).$$

Proximity: Replace  $\mathbb{I}_S$  by some convex function!

$$\mathsf{prox}_r(y) = \mathsf{prox}_{r,1}(y) := \arg\min \frac{1}{2} \|x - y\|^2 + r(x)$$



# **Composite optimization**





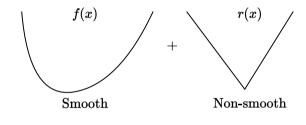
# Regularized / Composite Objectives

Many nonsmooth problems take the form

$$\min_{x \in \mathbb{R}^n} \varphi(x) = f(x) + r(x)$$

Lasso, L1-LS, compressed sensing

$$f(x) = \frac{1}{2} ||Ax - b||_2^2, r(x) = \lambda ||x||_1$$





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# Regularized / Composite Objectives

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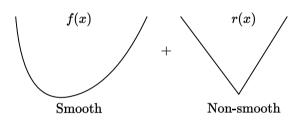
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$$f(x) = \frac{1}{2} ||Ax - b||_{2}^{2}, r(x) = \lambda ||x||_{1}$$

L1-Logistic regression, sparse LR

$$f(x) = -y \log h(x) - (1-y) \log (1-h(x)), r(x) = \lambda ||x||_1$$



Composite optimization

$$0 \in \nabla f(x^*) + \partial r(x^*)$$



$$0 \in \nabla f(x^*) + \partial r(x^*)$$
$$0 \in \alpha \nabla f(x^*) + \alpha \partial r(x^*)$$



$$0 \in \nabla f(x^*) + \partial r(x^*)$$
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$$\begin{split} 0 &\in \nabla f(x^*) + \partial r(x^*) \\ 0 &\in \alpha \nabla f(x^*) + \alpha \partial r(x^*) \\ x^* &\in \alpha \nabla f(x^*) + (I + \alpha \partial r)(x^*) \\ x^* &- \alpha \nabla f(x^*) \in (I + \alpha \partial r)(x^*) \\ x^* &= (I + \alpha \partial r)^{-1}(x^* - \alpha \nabla f(x^*)) \\ x^* &= \operatorname{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*)) \end{split}$$



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Optimality conditions:

$$\begin{split} 0 &\in \nabla f(x^*) + \partial r(x^*) \\ 0 &\in \alpha \nabla f(x^*) + \alpha \partial r(x^*) \\ x^* &\in \alpha \nabla f(x^*) + (I + \alpha \partial r)(x^*) \\ x^* &- \alpha \nabla f(x^*) \in (I + \alpha \partial r)(x^*) \\ x^* &= (I + \alpha \partial r)^{-1}(x^* - \alpha \nabla f(x^*)) \\ x^* &= \mathrm{prox}_{r,\alpha}(x^* - \alpha \nabla f(x^*)) \end{split}$$

Which leads to the proximal gradient method:

$$x_{k+1} = \mathsf{prox}_{r,\alpha}(x_k - \alpha \nabla f(x_k))$$

And this method converges at a rate of  $\mathcal{O}(\frac{1}{k})!$ 

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$$0 \in \alpha \nabla f(x^*) + \alpha \partial r(x^*)$$

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 $\mathsf{prox}_{f,\alpha}(x_k) = \mathsf{prox}_{\alpha f}(x_k) = \arg\min_{x \in \mathbb{R}^n} \left[ \alpha f(x) + \frac{1}{2} \|x - x_k\|_2^2 \right] \qquad \mathsf{prox}_f(x_k) = \arg\min_{x \in \mathbb{R}^n} \left[ f(x) + \frac{1}{2} \|x - x_k\|_2^2 \right]$ 

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### Another form of proximal operator



## **Proximal operators examples**

• 
$$r(x) = \lambda ||x||_1$$
,  $\lambda > 0$ 

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Let  $r:\mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$  be a convex function for which  $\operatorname{prox}_r$  is defined. If there exists such an  $\hat{x} \in \mathbb{R}^n$  that  $r(x) < +\infty$ . Then, the proximal operator is uniquely defined (i.e., it always returns a single unique value).

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It is strongly convex, meaning it has exactly one unique minimum (the existence of  $\hat{x}$  is necessary for  $r(\tilde{x}) + \frac{1}{2} ||x - \tilde{x}||_2^2$  to take a finite value somewhere).

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- $\operatorname{prox}_r(x) = y$ ,
- $x y \in \partial r(y)$ ,
- $\bullet \ \langle x-y,z-y\rangle \leq r(z)-r(y) \ \text{for any} \ z\in \mathbb{R}^n.$

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 $f \to \min_{x,y,z}$  Composite optimization

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#### **Proof**

 Let's establish the equivalence between the first and second conditions. The first condition can be rewritten as

$$y = \arg\min_{\tilde{x} \in \mathbb{R}^d} \left( r(\tilde{x}) + \frac{1}{2} \|x - \tilde{x}\|^2 \right).$$

From the optimality condition for the convex function r, this is equivalent to:

$$0 \in \left. \partial \left( r(\tilde{x}) + \frac{1}{2} \|x - \tilde{x}\|^2 \right) \right|_{\tilde{x} = u} = \partial r(y) + y - x.$$

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2. From the definition of the subdifferential, for any subgradient  $g \in \partial f(y)$  and for any  $z \in \mathbb{R}^d$ :  $\langle g, z - y \rangle < r(z) - r(y).$ 

In particular, this holds true for g=x-y. Conversely, it is also clear: for g=x-y, the above relationship holds, which means  $g\in\partial r(y)$ .

#### **i** Theorem

The operator  $\operatorname{prox}_r(x)$  is firmly nonexpansive (FNE)

$$\|\mathsf{prox}_r(x) - \mathsf{prox}_r(y)\|_2^2 \leq \langle \mathsf{prox}_r(x) - \mathsf{prox}_r(y), x - y \rangle$$

and nonexpansive:

$$\|\mathsf{prox}_r(x) - \mathsf{prox}_r(y)\|_2 \leq \|x - y\|_2$$

#### Proof

1. Let  $u=\mathrm{prox}_r(x)$ , and  $v=\mathrm{prox}_r(y)$ . Then, from the previous property:

$$\langle x - u, z_1 - u \rangle \le r(z_1) - r(u)$$
  
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- 2. Substitute  $z_1 = v$  and  $z_2 = u$ . Summing up, we get:
- $\langle x u, v u \rangle + \langle y v, u v \rangle \le 0,$  $\langle x y, v u \rangle + \|v u\|_2^2 \le 0.$

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1. Let  $u = \text{prox}_r(x)$ , and  $v = \text{prox}_r(y)$ . Then, from the previous property:

3. Which is exactly what we need to prove after substitution of u, v.  $\langle x - u, z_1 - u \rangle \leq r(z_1) - r(u)$   $||u - v||_2^2 \leq \langle x - u, u - v \rangle$ 

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Composite optimization

#### Theorem

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$$\|\mathsf{prox}_{-}(x) - \mathsf{prox}_{-}(y)\|_{2} < \|x - y\|_{2}$$

### Proof

 $\langle x - u, z_1 - u \rangle \leq r(z_1) - r(u)$  $\langle y-v, z_2-v \rangle \leq r(z_2)-r(v).$ 

> $\langle x-u, v-u \rangle + \langle y-v, u-v \rangle < 0.$  $\langle x - y, v - y \rangle + ||v - y||_2^2 < 0.$

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Cauchy-Bunyakovsky-Schwarz for the last inequality.

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 $||u-v||_2^2 < \langle x-u, u-v \rangle$ 

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Let  $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$  and  $r: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$  be convex functions. Additionally, assume that f is continuously differentiable and L-smooth, and for r, prox $_r$  is defined. Then,  $x^*$  is a solution to the composite optimization problem if and only if, for any  $\alpha > 0$ , it satisfies:

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$$\begin{aligned} 0 \in & \nabla f(x^*) + \partial r(x^*) \\ & - \alpha \nabla f(x^*) \in & \alpha \partial r(x^*) \\ x^* - \alpha \nabla f(x^*) - x^* \in & \alpha \partial r(x^*) \end{aligned}$$

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3. Finally,

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Theoretical tools for convergence analysis





#### **i** Theorem

Let  $f: \mathbb{R}^n \to \mathbb{R}$  be an L-smooth convex function. Then, for any  $x,y \in \mathbb{R}^n$ , the following inequality holds:

$$\begin{split} f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|_2^2 &\leq f(y) \text{ or, equivalently,} \\ \|\nabla f(y) - \nabla f(x)\|_2^2 &= \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 2L \left(f(x) - f(y) - \langle \nabla f(y), x - y \rangle\right) \end{split}$$

#### Proof

1. To prove this, we'll consider another function  $\varphi(y) = f(y) - \langle \nabla f(x), y \rangle$ . It is obviously a convex function (as a sum of convex functions). And it is easy to verify, that it is an L-smooth function by definition, since  $\nabla \varphi(y) = \nabla f(y) - \nabla f(x)$  and  $\|\nabla \varphi(y_1) - \nabla \varphi(y_2)\| = \|\nabla f(y_1) - \nabla f(y_2)\| \le L\|y_1 - y_2\|$ .



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3. From the first order optimality conditions for the convex function  $\nabla \varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any x, the minimum of the function  $\varphi(y)$  is at the point y=x. Therefore:

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$$\begin{split} &f(x) - \langle \nabla f(x), x \rangle \leq f(y) - \langle \nabla f(x), y \rangle - \frac{1}{2L} \| \nabla f(y) - \nabla f(x) \|_2^2 \\ &f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \| \nabla f(x) - \nabla f(y) \|_2^2 \leq f(y) \\ &\| \nabla f(y) - \nabla f(x) \|_2^2 \leq 2L \left( f(y) - f(x) - \langle \nabla f(x), y - x \rangle \right) \end{split}$$





3. From the first order optimality conditions for the convex function  $\nabla \varphi(y) = \nabla f(y) - \nabla f(x) = 0$ . We can conclude, that for any x, the minimum of the function  $\varphi(y)$  is at the point y = x. Therefore:

$$\varphi(x) \leq \varphi\left(y - \frac{1}{L}\nabla\varphi(y)\right) \leq \varphi(y) - \frac{1}{2L}\|\nabla\varphi(y)\|_2^2$$

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 switch x and y 
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 $f \to \min$ 

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The lemma has been proved. From the first view it does not make a lot of geometrical sense, but we will use it as a convenient tool to bound the difference between gradients.

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#### i Theorem

Let  $f: \mathbb{R}^n \to \mathbb{R}$  be continuously differentiable on  $\mathbb{R}^n$ . Then, the function f is  $\mu$ -strongly convex if and only if for any  $x,y \in \mathbb{R}^d$  the following holds:

Strongly convex case 
$$\mu>0$$
  $\left\langle \nabla f(x)-\nabla f(y),x-y\right\rangle \geq \mu\|x-y\|^2$  Convex case  $\mu=0$   $\left\langle \nabla f(x)-\nabla f(y),x-y\right\rangle \geq 0$ 

#### Proof

1. We will only give the proof for the strongly convex case, the convex one follows from it with setting  $\mu=0$ . We start from necessity. For the strongly convex function

$$\begin{split} f(y) &\geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|x - y\|_2^2 \\ f(x) &\geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|_2^2 \\ \text{sum } & \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2 \end{split}$$

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$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle = \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle$$

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$$y + t(x - y) - y = t(x - y)$$

$$= \int_0^1 t^{-1} \langle \nabla f(y + t(x - y)) - \nabla f(y), t(x - y) \rangle dt$$

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2. For the sufficiency we assume, that  $\langle \nabla f(x) - \nabla f(y), x - y \rangle \ge \mu \|x - y\|^2$ . Using Newton-Leibniz theorem  $f(x) = f(y) + \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt$ :

$$\begin{split} f(x) - f(y) - \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt - \langle \nabla f(y), x - y \rangle \\ \langle \nabla f(y), x - y \rangle &= \int_0^1 \langle \nabla f(y), x - y \rangle dt \\ &= \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(y), (x - y) \rangle dt \\ & y + t(x - y) - y = t(x - y) \\ &= \int_0^1 t^{-1} \langle \nabla f(y + t(x - y)) - \nabla f(y), t(x - y) \rangle dt \\ &\geq \int_0^1 t^{-1} \mu \|t(x - y)\|^2 dt = \mu \|x - y\|^2 \int_0^1 t dt = \frac{\mu}{2} \|x - y\|_2^2 \end{split}$$

Thus, we have a strong convexity criterion satisfied

$$f(x) \ge f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} ||x - y||_2^2$$

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switch x and y 
$$-\langle \nabla f(x), x-y \rangle \leq -\left(f(x)-f(y)+\frac{\mu}{2}\|x-y\|_2^2\right)$$

## **Proximal Gradient Method. Convex case**





#### Theorem

Consider the proximal gradient method

$$x_{k+1} = \operatorname{prox}_{\alpha r} (x_k - \alpha \nabla f(x_k))$$

For the criterion  $\varphi(x) = f(x) + r(x)$ , we assume:

- f is convex, differentiable, dom $(f) = \mathbb{R}^n$ , and  $\nabla f$  is Lipschitz continuous with constant L > 0.
- r is convex, and  $\operatorname{prox}_{\alpha r}(x_k) = \arg\min_{x \in \mathbb{R}^n} \left[ \alpha r(x) + \frac{1}{2} \|x x_k\|_2^2 \right]$  can be evaluated.

Proximal gradient descent with fixed step size  $\alpha = 1/L$  satisfies

$$\varphi(x_k) - \varphi^* \le \frac{L||x_0 - x^*||^2}{2k},$$

Proximal gradient descent has a convergence rate of O(1/k) or  $O(1/\epsilon)$ . This matches the gradient descent rate! (But remember the proximal operation cost)

#### Proof

1. Let's introduce the **gradient mapping**, denoted as  $G_{\alpha}(x)$ , acts as a "gradient-like object":

$$\begin{split} x_{k+1} &= \mathsf{prox}_{\alpha r}(x_k - \alpha \nabla f(x_k)) \\ x_{k+1} &= x_k - \alpha G_{\alpha}(x_k). \end{split}$$

where  $G_{\alpha}(x)$  is:

$$G_{\alpha}(x) = \frac{1}{\alpha} \left( x - \operatorname{prox}_{\alpha r} \left( x - \alpha \nabla f \left( x \right) \right) \right)$$

Observe that  $G_{\alpha}(x)=0$  if and only if x is optimal. Therefore,  $G_{\alpha}$  is analogous to  $\nabla f$ . If x is locally optimal, then  $G_{\alpha}(x)=0$  even for nonconvex f. This demonstrates that the proximal gradient method effectively combines gradient descent on f with the proximal operator of f, allowing it to handle non-differentiable components effectively.



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$$f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|_2^2$$



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$$r(x) \ge r(x_{k+1}) + \langle g, x - x_{k+1} \rangle, \quad g \in \partial r(x_{k+1})$$



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Proximal Gradient Method. Convex case

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7. Now it is easy to verify, that when  $x=x_k$  we have monotonic decrease for the proximal gradient algorithm:

$$\varphi(x_{k+1}) \le \varphi(x_k) - \frac{\alpha}{2} \|G_{\alpha}(x_k)\|_2^2$$



8. When  $x = x^*$ :

Proximal Gradient Method, Convex case



$$\varphi(x_{k+1}) \le \varphi(x^*) + \langle G_{\alpha}(x_k), x_k - x^* \rangle - \frac{\alpha}{2} \|G_{\alpha}(x_k)\|_2^2$$



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$$\leq \frac{1}{2\alpha} \left[ 2\langle \alpha G_{\alpha}(x_k), x_k - x^* \rangle - \|\alpha G_{\alpha}(x_k)\|_2^2 \right]$$



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\leq \frac{1}{2\alpha} \left[ 2 \langle \alpha G_{\alpha}(x_k), x_k - x^* \rangle - \|\alpha G_{\alpha}(x_k)\|_2^2 - \|x_k - x^*\|_2^2 + \|x_k - x^*\|_2^2 \right]$$



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\leq \frac{1}{2\alpha} \left[ -\|x_k - x^* - \alpha G_{\alpha}(x_k)\|_2^2 + \|x_k - x^*\|_2^2 \right]$$

Proximal Gradient Method. Convex case

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$$\leq \frac{1}{2\alpha} \left[ \|x_k - x^*\|_2^2 - \|x_{k+1} - x^*\|_2^2 \right]$$



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Which is a standard  $\frac{L\|x_0-x^*\|_2^2}{2k}$  with  $\alpha=\frac{1}{L}$ , or,  $\mathcal{O}\left(\frac{1}{k}\right)$  rate for smooth convex problems with Gradient Descent!

 $f o \min_{x,y,z}$  Proximal Gradient Method. Convex case

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 $\sum_{k=1}^{k-1} \varphi(x_k) = k\varphi(x_k) \le \sum_{k=1}^{k-1} \varphi(x_{i+1})$ 

$$\sum_{i=0}^{k-1} \left[ \varphi(x_{i+1}) - \varphi(x^*) \right] \le \frac{1}{2\alpha} \left[ \|x_0 - x^*\|_2^2 - \|x_k - x^*\|_2^2 \right]$$
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$$\varphi(x_k) - \varphi(x^*) \le \frac{1}{k} \sum_{i=0}^{k-1} [\varphi(x_{i+1}) - \varphi(x^*)] \le \frac{\|x_0 - x^*\|_2^2}{2\alpha k}$$

Proximal Gradient Method. Strongly convex case



#### i Theorem

Consider the proximal gradient method

$$x_{k+1} = \operatorname{prox}_{\alpha r} (x_k - \alpha \nabla f(x_k))$$

For the criterion  $\varphi(x) = f(x) + r(x)$ , we assume:

- f is  $\mu$ -strongly convex, differentiable,  $\mathsf{dom}(f) = \mathbb{R}^n$ , and  $\nabla f$  is Lipschitz continuous with constant L>0.
- r is convex, and  $\operatorname{prox}_{\alpha r}(x_k) = \arg\min_{x \in \mathbb{R}^n} \left[ \alpha r(x) + \frac{1}{2} \|x x_k\|_2^2 \right]$  can be evaluated.

Proximal gradient descent with fixed step size  $\alpha \leq 1/L$  satisfies

$$||x_{k+1} - x^*||_2^2 \le (1 - \alpha \mu)^k ||x_0 - x^*||_2^2$$

This is exactly gradient descent convergence rate. Note, that the original problem is even non-smooth!



#### **Proof**

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$$||x_{k+1} - x^*||_2^2 = ||\operatorname{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*||_2^2$$



#### **Proof**

$$\begin{aligned} \|x_{k+1} - x^*\|_2^2 &= \|\mathsf{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2 \\ \text{stationary point lemm} &= \|\mathsf{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - \mathsf{prox}_{\alpha f}(x^* - \alpha \nabla f(x^*))\|_2^2 \end{aligned}$$



#### **Proof**

$$\begin{split} \|x_{k+1} - x^*\|_2^2 &= \|\mathsf{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2 \\ \text{stationary point lemm} &= \|\mathsf{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - \mathsf{prox}_{\alpha f}(x^* - \alpha \nabla f(x^*))\|_2^2 \\ \text{nonexpansiveness} &\leq \|x_k - \alpha \nabla f(x_k) - x^* + \alpha \nabla f(x^*)\|_2^2 \end{split}$$



#### Proof

1. Considering the distance to the solution and using the stationary point lemm:

$$\begin{split} \|x_{k+1} - x^*\|_2^2 &= \|\mathsf{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - x^*\|_2^2 \\ \text{stationary point lemm} &= \|\mathsf{prox}_{\alpha f}(x_k - \alpha \nabla f(x_k)) - \mathsf{prox}_{\alpha f}(x^* - \alpha \nabla f(x^*))\|_2^2 \\ \text{nonexpansiveness} &\leq \|x_k - \alpha \nabla f(x_k) - x^* + \alpha \nabla f(x^*)\|_2^2 \\ &= \|x_k - x^*\|^2 - 2\alpha \langle \nabla f(x_k) - \nabla f(x^*), x_k - x^* \rangle + \alpha^2 \|\nabla f(x_k) - \nabla f(x^*)\|_2^2 \end{split}$$



#### Proof

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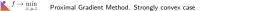
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$$\le (1 - \alpha\mu) ||x_k - x^*||^2 + 2\alpha(\alpha L - 1) \left( f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle \right)$$



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4. Due to convexity of f:  $f(x_k) - f(x^*) - \langle \nabla f(x^*), x_k - x^* \rangle \ge 0$ . Therefore, if we use  $\alpha \le \frac{1}{L}$ :

$$||x_{k+1} - x^*||_2^2 \le (1 - \alpha \mu) ||x_k - x^*||^2$$

which is exactly linear convergence of the method with up to  $1-\frac{\mu}{L}$  convergence rate.



### i Accelerated Proximal Method

Let  $x_0 = y_0 \in dom(r)$ . For  $k \ge 1$ :

$$x_k = \operatorname{prox}_{\alpha_k h}(y_{k-1} - \alpha_k \nabla f(y_{k-1}))$$
 
$$y_k = x_k + \frac{k-1}{k+2}(x_k - x_{k-1})$$

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# **Numerical experiments**





### Quadratic case

$$f(x) = \frac{1}{2m} \|Ax - b\|_2^2 + \lambda \|x\|_1 \to \min_{x \in \mathbb{R}^n}, \qquad A \in \mathbb{R}^{m \times n}, \quad \lambda \left(\frac{1}{m} A^T A\right) \in [\mu; L].$$

Linear Least Squares with  $\ell_1$  Regularization (LASSO). m=1000, n=100,  $\lambda$ =0,  $\mu$ =0, L=10. Optimal sparsity: 0.0e+00

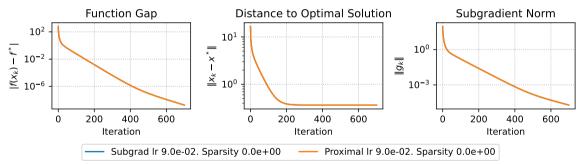


Figure 2: Smooth convex case. Sublinear convergence, no convergence in domain, no difference between subgradient and proximal methods

 $f \to \min_{x,y,z}$  Numerical experiments

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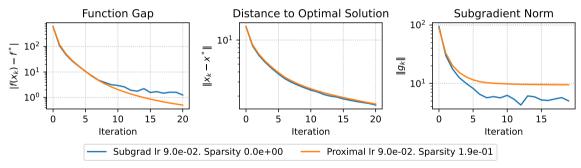


Figure 3: Non-smooth convex case. Sublinear convergence. At the beginning, the subgradient method and proximal method are close.

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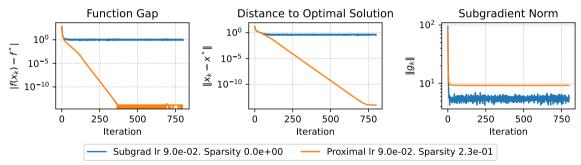


Figure 4: Non-smooth convex case. If we take more iterations, the proximal method converges with the constant learning rate, which is not the case for the subgradient method. The difference is tremendous, while the iteration complexity is the same.

### **Numerical experiments**

$$f(x) = \frac{1}{m} \sum_{i=1}^{m} \log(1 + \exp(-b_i(A_i x))) + \lambda ||x||_1 \to \min_{x \in \mathbb{R}^n}, \quad A_i \in \mathbb{R}^n, \quad b_i \in \{-1, 1\}$$

Binary Logistic Regression with  $\ell_1$  Regularization. m=300, n=50,  $\lambda$ =0.1. Optimal sparsity: 8.6e-01

Function Gap

Domain Gap

Train Accuracy  $\ell_1$   $\ell_2$   $\ell_3$   $\ell_4$   $\ell_4$   $\ell_5$   $\ell_6$   $\ell_6$ 

 $\left|f(x_k)-f^*\right|$ 10-6 0.6 0.80 20000 20000 20000 20000 n 0 Iteration Iteration Iteration Iteration ıbgrad lr 1.0e-02. Sparsity 0.0e+00 Subgrad Ir  $\alpha/\sqrt{k}$  ( $\alpha=1.0e-01$ ). Sparsity 1.2e-01 Proximal Ir 1.0e-02. Sparsity

Figure 5: Logistic regression with  $\ell_1$  regularization

10-1

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- Application:
  - Efficient for sparse signal recovery, image processing, and compressed sensing.





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FISTA improves upon ISTA's convergence rate by incorporating a momentum term, inspired by Nesterov's accelerated gradient method.

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- Application:
  - Especially useful for large-scale problems in machine learning and signal processing where the L1 penalty induces sparsity.



### Solving the Matrix Completion Problem

Matrix completion problems seek to fill in the missing entries of a partially observed matrix under certain assumptions, typically low-rank. This can be formulated as a minimization problem involving the nuclear norm (sum of singular values), which promotes low-rank solutions.

Problem Formulation:

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where  $P_{\Omega}$  projects onto the observed set  $\Omega$ , and  $\|\cdot\|_*$  denotes the nuclear norm.

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- Application:
  - Widely used in recommender systems, image recovery, and other domains where data is naturally matrix-formed but partially observed.



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- Further reading: Proximal operator splitting, Douglas-Rachford splitting, Best approximation problem, Three operator splitting.

