Advanced stochastic methods. Adaptivity and variance reduction.

Daniil Merkulov

Optimization for ML. Faculty of Computer Science. HSE University







We consider classic finite-sample average minimization:

$$\min_{x \in \mathbb{R}^p} f(x) = \min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^n f_i(x)$$

The gradient descent acts like follows:

$$x_{k+1} = x_k - \frac{\alpha_k}{n} \sum_{i=1}^n \nabla f_i(x)$$
 (GD)

• Iteration cost is linear in n.

 $f \to \min_{x,y,.}$

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Let's/ switch from the full gradient calculation to its unbiased estimator, when we randomly choose
$$i_k$$
 index of point at each iteration uniformly:

- $x_{k+1} = x_k \alpha_k \nabla f_i$, (x_k)
- With $p(i_k = i) = \frac{1}{n}$, the stochastic gradient is an unbiased estimate of the gradient, given by:

 $\mathbb{E}[\nabla f_{i_k}(x)] = \sum_{i=1}^n p(i_k = i) \nabla f_i(x) = \sum_{i=1}^n \frac{1}{n} \nabla f_i(x) = \frac{1}{n} \sum_{i=1}^n \nabla f_i(x) = \nabla f(x)$

This indicates that the expected value of the stochastic gradient is equal to the actual gradient of f(x). $f \to \min_{x,y,z}$ Finite-sum problem



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(SGD)

Stochastic iterations are n times faster, but how many iterations are needed?

If ∇f is Lipschitz continuous then we have:

Assumption	Deterministic Gradient Descent	Stochastic Gradient Descent
PL	$O(\log(1/arepsilon))$	O(1/arepsilon)
Convex	O(1/arepsilon)	$O(1/\varepsilon^2)$
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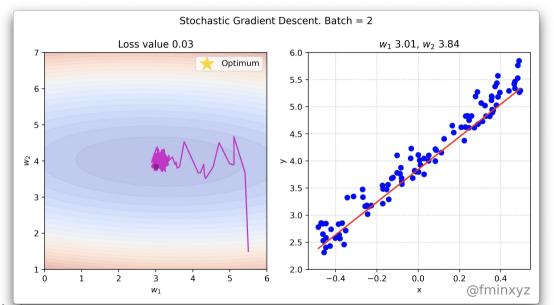
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- Momentum and Quasi-Newton-like methods do not improve rates in stochastic case. Can only improve constant factors (bottleneck is variance, not condition number).



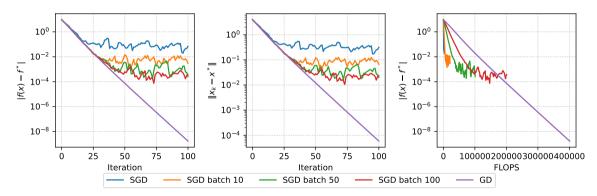
SGD with constant stepsize does not converge



Main problem of SGD

$$f(x) = \frac{\mu}{2} ||x||_2^2 + \frac{1}{m} \sum_{i=1}^m \log(1 + \exp(-y_i \langle a_i, x \rangle)) \to \min_{x \in \mathbb{R}^n}$$

Strongly convex binary logistic regression. m=200, n=10, mu=1.





Principle: reducing variance of a sample of X by using a sample from another random variable Y with known expectation:

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- $X Y = \nabla f_i$, $(x^{(k-1)}) \nabla f_i$, (\tilde{x})

• Maintain table, containing gradient g_i of f_i , i = 1, ..., n

 $f \to \min_{x,y,z}$ Variance reduction methods

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$$g_{i_k}^{(k)} = \nabla f_{i_k}(x^{(k-1)})$$
 (most recent gradient of f_{i_k})

Set all other $g_i^{(k)} = g_i^{(k-1)}$, $i \neq i_k$, i.e., these stay the same

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- Isn't it expensive to average all these gradients? Basically just as efficient as SGD, as long we're clever:

$$x^{(k)} = x^{(k-1)} - \alpha_k \underbrace{\left(\frac{1}{n}g_i^{(k)} - \frac{1}{n}g_i^{(k-1)} + \underbrace{\frac{1}{n}\sum_{i=1}^n g_i^{(k-1)}}_{\text{old table average}}\right)}_{\text{old table average}}$$

new table average

Assume that $f(x) = \frac{1}{2} \sum_{i=1}^{n} f_i(x)$, where each f_i is differentiable, and ∇f_i is Lipschitz with constant L.

Denote $\bar{x}^{(k)} = \frac{1}{k} \sum_{l=0}^{k-1} x^{(l)}$, the average iterate after k-1 steps.

i Theorem

SAG, with a fixed step size $\alpha = \frac{1}{16L}$, and the initialization

$$g_i^{(0)} = \nabla f_i(x^{(0)}) - \nabla f(x^{(0)}), \quad i = 1, \dots, n$$

satisfies

$$\mathbb{E}[f(\bar{x}^{(k)})] - f^{\star} \le \frac{48n}{k} [f(x^{(0)}) - f^{\star}] + \frac{128L}{k} ||x^{(0)} - x^{\star}||^2$$

where the expectation is taken over random choices of indices.

 $f \to \min_{x,y,z}$ Variance reduction methods

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- So the first term in SAG bound suffers from a factor of n; authors suggest smarter initialization to make $f(x^{(0)}) f^*$ small (e.g., they suggest using the result of n SGD steps).

 $f \to \min_{x,y}$

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Assume further that each f_i is strongly convex with parameter μ .

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SAG, with a step size $\alpha = \frac{1}{16L}$ and the same initialization as before, satisfies

$$\mathbb{E}[f(x^{(k)})] - f^{\star} \le \left(1 - \min\left(\frac{\mu}{16L}, \frac{1}{8n}\right)\right)^{k} \left(\frac{3}{2} \left(f(x^{(0)}) - f^{\star}\right) + \frac{4L}{n} \|x^{(0)} - x^{\star}\|^{2}\right)$$

Notes:

• This is linear convergence rate $\mathcal{O}(\gamma^k)$ for SAG. Compare this to $\mathcal{O}(\gamma^k)$ for GD, and only $\mathcal{O}\left(\frac{1}{k}\right)$ for SGD.

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SAG convergence

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- Proofs of these results not easy: 15 pages, computed-aided!



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 - Choose initial L_0
 - Increase L, until the following satisfies

$$f_{i_k}(x^{k+1}) \le f_{i_k}(x^k) + \nabla f_{i_k}(x^k)(x^{k+1} - x^k) + \frac{L}{2} ||x^{k+1} - x^k||_2^2$$

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- For the generalized linear models (this includes LogReg, LLS) you need to store much less memory $\mathcal{O}(n)$ instead of $\mathcal{O}(pn)$.

$$f_i(w) = \varphi(w^T x_i) \leftrightarrow \nabla f_i(w) = \varphi'(w^T x_i) x_i$$



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$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{L_i} \frac{f_i(x)}{L_i}$$

$$= \frac{1}{\sum_{k} L_k} \sum_{i=1}^{n} \sum_{j=1}^{L_i} \left(\sum_{k} \frac{L_k}{n} \frac{f_i(x)}{L_i} \right)$$

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 To ensure convergence, component selection should be carried out according to the rule: with probability 0.5, select from a uniform distribution, with probability 0.5, select with probabilities $L_i/\sum_i L_j$.

- The step size α_k and the convergence rate of the method are determined by the constant L for f(x), where $L = \max_{1 < i < n} L_i$, L_i is the Lipschitz constant for the function f_i
- When selecting components with a probability proportional to L_i , the constant L can be reduced from $\max_i L_i$ to $\bar{L} = \sum_{i} L_i/N$:

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- To generate with probabilities $L_i/\sum_i L_j$, there is an algorithm with complexity $O(\log N)$.

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- Two gradient evaluations per inner step.
- Two parameters: length of epochs + step-size γ .
- Linear convergence rate, simple proof.



Very popular adaptive method. Let $g^{(k)} = \nabla f_{i_k}(x^{(k-1)})$, and update for $j = 1, \dots, p$:

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- The constant ϵ is typically set to 10^{-6} to ensure that we do not suffer from division by zero or overly large step

sizes.

RMSProp (Tieleman and Hinton, 2012)

An enhancement of AdaGrad that addresses its aggressive, monotonically decreasing learning rate. Uses a moving average of squared gradients to adjust the learning rate for each weight. Let $g^{(k)} = \nabla f_{ik}(x^{(k-1)})$ and update rule for $j = 1, \ldots, p$:

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- Commonly used in training neural networks, particularly in recurrent neural networks.

Adadelta (Zeiler, 2012)

An extension of RMSProp that seeks to reduce its dependence on a manually set global learning rate. Instead of accumulating all past squared gradients, Adadelta limits the window of accumulated past gradients to some fixed size w. Update mechanism does not require learning rate α :

$$\begin{split} v_j^{(k)} &= \gamma v_j^{(k-1)} + (1-\gamma) (g_j^{(k)})^2 \\ \tilde{g}_j^{(k)} &= \frac{\sqrt{\Delta x_j^{(k-1)} + \epsilon}}{\sqrt{v_j^{(k)} + \epsilon}} g_j^{(k)} \\ x_j^{(k)} &= x_j^{(k-1)} - \tilde{g}_j^{(k)} \\ \Delta x_j^{(k)} &= \rho \Delta x_j^{(k-1)} + (1-\rho) (\tilde{g}_j^{(k)})^2 \end{split}$$

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- The method does not require an initial learning rate setting, making it easier to configure.
- Often used in deep learning where parameter scales differ significantly across layers.





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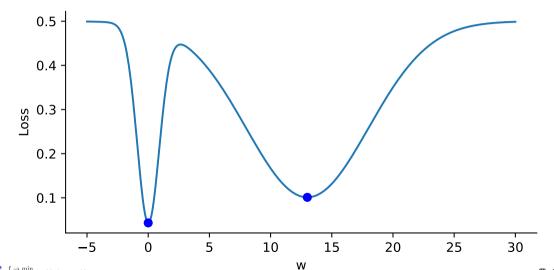
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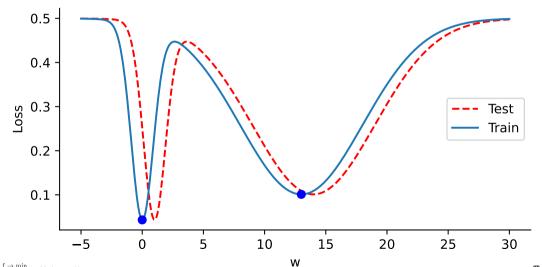
Wide vs narrow local minima

Узкие и широкие локальные минимумы



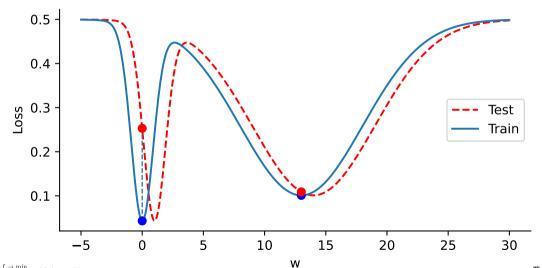
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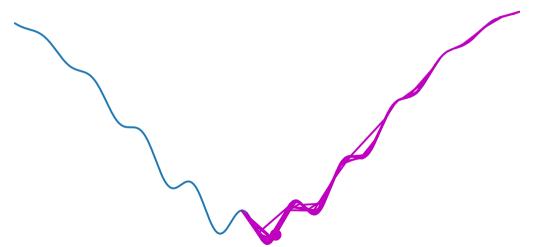
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Stochasticity allows to escape local minima

Стохастический градиентный спуск выпрыгивает из локальных минимумов



Local divergence can also be benefitial

Градиентный спуск с большим шагом избегает узкого локального минимума

