



# Duality

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Optimization methods. MIPT

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# Motivation

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As a consequence:

$$\max_{y \in \Omega} g(y) \leq \min_{x \in S} f(x)$$

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And the Lagrangian, associated with this problem:

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) = f_0(x) + \lambda^\top f(x) + \nu^\top h(x)$$

# Dual function

We assume  $\mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i$  is nonempty. We define the Lagrange dual function (or just dual function)  $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$  as the minimum value of the Lagrangian over  $x$ : for  $\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p$

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$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right)$$

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When the Lagrangian is unbounded below in  $x$ , the dual function takes on the value  $-\infty$ . Since the dual function is the pointwise infimum of a family of affine functions of  $(\lambda, \nu)$ , it is concave, even when the original problem is not convex.

## Dual function as a lower bound

Let us show, that the dual function yields lower bounds on the optimal value  $p^*$  of the original problem for any  $\lambda \succeq 0, \nu$ . Suppose some  $\hat{x}$  is a feasible point for the original problem, i.e.,  $f_i(\hat{x}) \leq 0$  and  $h_i(\hat{x}) = 0$ ,  $\lambda \succeq 0$ . Then we have:

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The term “dual feasible”, to describe a pair  $(\lambda, \nu)$  with  $\lambda \succeq 0$  and  $g(\lambda, \nu) > -\infty$ , now makes sense. It means, as the name implies, that  $(\lambda, \nu)$  is feasible for the dual problem. We refer to  $(\lambda^*, \nu^*)$  as dual optimal or optimal Lagrange multipliers if they are optimal for the above problem.

# Summary

	Primal	Dual
Function	$f_0(x)$	$g(\lambda, \nu) = \min_{x \in \mathcal{D}} L(x, \lambda, \nu)$
Variables	$x \in S \subseteq \mathbb{R}^n$	$\lambda \in \mathbb{R}_+^m, \nu \in \mathbb{R}^p$
Constraints	$f_i(x) \leq 0, i = 1, \dots, m$ $h_i(x) = 0, i = 1, \dots, p$	$\lambda_i \geq 0, \forall i \in \overline{1, m}$
Problem	$\begin{aligned} & f_0(x) \rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } & f_i(x) \leq 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, p \end{aligned}$	$\begin{aligned} & g(\lambda, \nu) \rightarrow \max_{\lambda \in \mathbb{R}_+^m, \nu \in \mathbb{R}^p} \\ \text{s.t. } & \lambda \succeq 0 \end{aligned}$
Optimal	$x^*$ if feasible, $p^* = f_0(x^*)$	$\lambda^*, \nu^*$ if max is achieved, $d^* = g(\lambda^*, \nu^*)$

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This problem is devoid of inequality constraints, presenting  $m$  linear equality constraints instead. The Lagrangian is expressed as  $L(x, \nu) = x^T x + \nu^T (Ax - b)$ , spanning the domain  $\mathbb{R}^n \times \mathbb{R}^m$ . The dual function is denoted by  $g(\nu) = \inf_x L(x, \nu)$ . Given that  $L(x, \nu)$  manifests as a convex quadratic function in terms of  $x$ , the minimizing  $x$  can be derived from the optimality condition

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$$-(1/4)\nu^T A A^T \nu - b^T \nu \leq \inf \{x^T x \mid Ax = b\}.$$

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Which is a simple non-trivial lower bound without any problem solving.

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This problem can be construed as a two-way partitioning problem over a set of  $n$  elements, denoted as  $\{1, \dots, n\}$ : A viable  $x$  corresponds to the partition

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The coefficient  $W_{ij}$  in the matrix represents the expense associated with placing elements  $i$  and  $j$  in the same partition, while  $-W_{ij}$  signifies the cost of segregating them. The objective encapsulates the aggregate cost across all pairs of elements, and the challenge posed by problem is to find the partition that minimizes the total cost.



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We now derive the dual function for this problem. The Lagrangian is expressed as

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The code for the problem is available here  Open in Colab

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- “Easy” necessary and sufficient conditions: unknown.

# Strong duality in linear least squares

## Exercise

In the Least-squares solution of linear equations example above calculate the primal optimum  $p^*$  and the dual optimum  $d^*$  and check whether this problem has strong duality or not.

## Useful features of duality

- **Construction of lower bound on solution of the primal problem.**

It could be very complicated to solve the initial problem. But if we have the dual problem, we can take an arbitrary  $y \in \Omega$  and substitute it in  $g(y)$  - we'll immediately obtain some lower bound.



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- **Dual function is always concave**

As a pointwise minimum of affine functions.

# Slater's condition

## Theorem

If for a convex optimization problem (i.e., assuming minimization,  $f_0, f_i$  are convex and  $h_i$  are affine), there exists a point  $x$  such that  $h(x) = 0$  and  $f_i(x) < 0$  (existence of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.

## An example of convex problem, when Slater's condition does not hold

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$$\min\{f_0(x) = x \mid f_1(x) = \frac{x^2}{2} \leq 0\},$$

The only point in the budget set is:  $x^* = 0$ . However, it is impossible to find a non-negative  $\lambda^* \geq 0$ , such that

$$\nabla f_0(0) + \lambda^* \nabla f_1(0) = 1 + \lambda^* x = 0.$$

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$$\begin{aligned} -\sum_{i=1}^n \frac{(q_i^\top b)^2}{\lambda_i + \lambda} - \lambda &\rightarrow \max_{\lambda \in \mathbb{R}} \\ \text{s.t. } \lambda &\geq -\lambda_{\min}(A) \end{aligned}$$

# Applications



## Solving the primal via the dual

An important consequence of stationarity: under strong duality, given a dual solution  $\lambda^*, \nu^*$ , any primal solution  $x^*$  solves

$$\min_{x \in \mathbb{R}^n} f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x)$$

Often, solutions of this unconstrained problem can be expressed **explicitly**, giving an explicit characterization of primal solutions from dual solutions.

Furthermore, suppose the solution of this problem is unique; then it must be the primal solution  $x^*$ .

This can be very helpful when the dual is easier to solve than the primal.

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This gives:

$$x_i^* = \frac{a_i \nu^*}{c_i}.$$

## Mixed strategies for matrix games



Figure 2: The scheme of a mixed strategy matrix game

## Mixed strategies for matrix games



Player 1

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$\dots$
$u_k$
$\dots$
$u_n$

$v_1$
$\dots$
$\dots$
$v_l$
$\dots$
$\dots$
$v_m$



Player 2

In zero-sum matrix games, players 1 and 2 choose actions from sets  $\{1, \dots, n\}$  and  $\{1, \dots, m\}$ , respectively. The outcome is a payment from player 1 to player 2, determined by a payoff matrix  $P \in \mathbb{R}^{n \times m}$ . Each player aims to use mixed strategies, choosing actions according to a probability distribution: player 1 uses probabilities  $u_k$  for each action  $i$ , and player 2 uses  $v_l$ .

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In zero-sum matrix games, players 1 and 2 choose actions from sets  $\{1, \dots, n\}$  and  $\{1, \dots, m\}$ , respectively. The outcome is a payment from player 1 to player 2, determined by a payoff matrix  $P \in \mathbb{R}^{n \times m}$ . Each player aims to use mixed strategies, choosing actions according to a probability distribution: player 1 uses probabilities  $u_k$  for each action  $i$ , and player 2 uses  $v_l$ . The expected payoff from player 1 to player 2 is given by  $\sum_{k=1}^n \sum_{l=1}^m u_k v_l P_{kl} = u^T P v$ . Player 1 seeks to minimize this expected payoff, while player 2 aims to maximize it.

Figure 2: The scheme of a mixed strategy matrix game

# Mixed strategies for matrix games. Player 1's Perspective



Player 1

$u_1$
$\dots$
$u_k$
$\dots$
$u_n$

Assuming player 2 knows player 1's strategy  $u$ , player 2 will choose  $v$  to maximize  $u^T P v$ . The worst-case expected payoff is thus:

$$\max_{v \geq 0, 1^T v = 1} u^T P v = \max_{i=1, \dots, m} (P^T u)_i$$



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Player 1's optimal strategy minimizes this worst-case payoff, leading to the optimization problem:

$$\begin{aligned} \min \quad & \max_{i=1, \dots, m} (P^T u)_i \\ \text{s.t.} \quad & u \geq 0 \\ & 1^T u = 1 \end{aligned} \tag{1}$$

This forms a convex optimization problem with the optimal value denoted as  $p_1^*$ .

## Mixed strategies for matrix games. Player 2's Perspective

Conversely, if player 1 knows player 2's strategy  $v$ , the goal is to minimize  $u^T P v$ . This leads to:

$$\min_{u \geq 0, 1^T u = 1} u^T P v = \min_{i=1, \dots, n} (P v)_i$$



Player 2

## Mixed strategies for matrix games. Player 2's Perspective

Conversely, if player 1 knows player 2's strategy  $v$ , the goal is to minimize  $u^T P v$ . This leads to:

$$\min_{u \geq 0, 1^T u = 1} u^T P v = \min_{i=1, \dots, n} (P v)_i$$

Player 2 then maximizes this to get the largest guaranteed payoff, solving the optimization problem:

$$\begin{aligned} & \max \min_{i=1, \dots, n} (P v)_i \\ & \text{s.t. } v \geq 0 \\ & 1^T v = 1 \end{aligned} \tag{2}$$

The optimal value here is  $p_2^*$ .



Player 2

$v_1$

...

...

$v_l$

...

...

$v_m$

# Mixed strategies for matrix games

## Duality and Equivalence

It's generally advantageous to know the opponent's strategy, but surprisingly, in mixed strategy matrix games, this advantage disappears. The key lies in duality: the problems above are Lagrange duals. By formulating player 1's problem as a linear program and introducing Lagrange multipliers, we find that the dual problem matches player 2's problem. Due to strong duality in feasible linear programs,  $p_1^* = p_2^*$ , showing no advantage in knowing the opponent's strategy.

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## Conclusion

This formulation shows that the Lagrange dual problem is equivalent to problem Equation 2. Given the feasibility of these linear programs, strong duality holds, meaning the optimal values of Equation 1 and Equation 2 are equal.

# References

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