Gradient Flow. Accelerated gradient flow.

Daniil Merkulov

Optimization methods. MIPT

Gradient Flow

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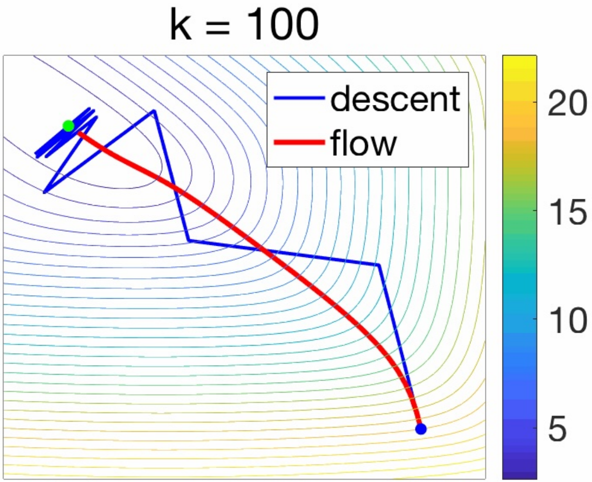

Рис. 1: Source

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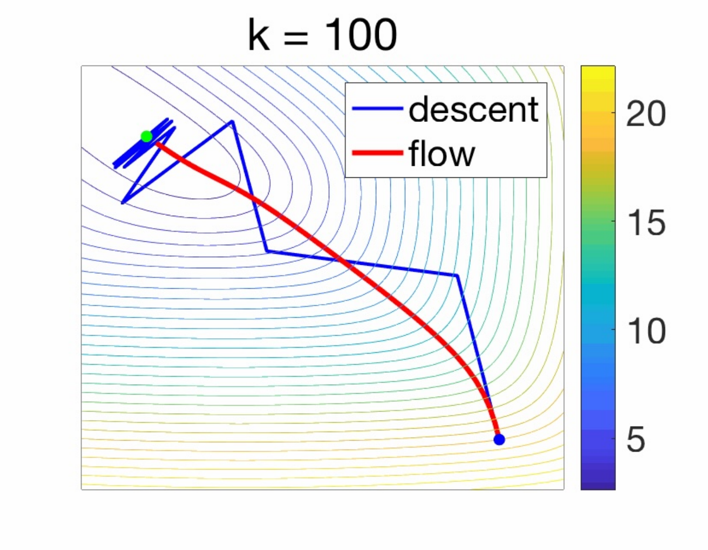

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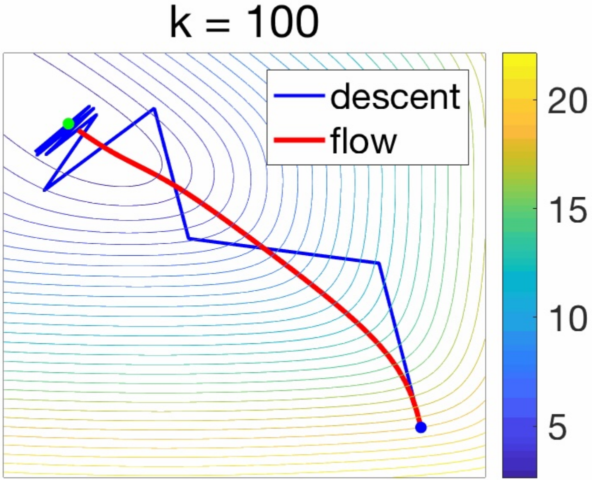

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- **Analytical solution in some cases.** For example, one can consider quadratic problem with linear gradient, which will form a linear ODE with known exact formula.
- **Different discretization leads to different methods.** We will see, that the continuous-time object is pretty rich in terms of the variety of produced algorithms. Therefore, it is interesting to study optimization from this perspective.

Gradient Flow discretization

Consider Gradient Flow ODE:

$$\frac{dx}{dt} = -\nabla f(x)$$

Explicit Euler discretization:

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Leads to ordinary Gradient Descent method

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! Proximal operator

$$\text{prox}_{\alpha f}(x_k) = \text{arg}\min_{x \in \mathbb{R}^n} \left[\alpha f(x) + \frac{1}{2}\|x - x_k\|_2^2\right]$$

Convergence analysis. Convex case.

1. Simplest proof of monotonic decrease of GF:

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\frac{d}{dt} f(x(t)) = \nabla f(x(t))^\top \frac{dx(t)}{dt} = -\|\nabla f(x(t))\|_2^2 \leqslant 0.
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   If f is bounded from below, then $f(x(t))$ will always converge as a non-increasing function which is bounded from below. It is straightforward, that GF converges to the stationary point, where $\nabla f = 0$ (potentailly including minima, maxima and saddle points).

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$$f(x) \geqslant f(y) + \nabla f(y)^\top (x - y) \qquad \Rightarrow \qquad \nabla f(y)^\top (x - y) \leqslant f(x) - f(y)$$

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$$\frac{d}{dt}\left[\|x(t) - x^*\|^2\right] = -2(x(t) - x^*)^\top \nabla f(x(t)) \leqslant -2\left[f(x(t)) - f^*\right]$$

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f(x(t)) - f^* \leqslant \frac{1}{t}\int_0^t [f(x(u)) - f^*]du \leqslant \frac{1}{2t}\|x(0) - x^*\|^2 - \frac{1}{2t}\|x(t) - x^*\|^2 \leqslant \frac{1}{2t}\|x(0) - x^*\|^2.
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   We recover the usual rates in $\mathcal{O}\left(\frac{1}{n}\right)$, with $t = \alpha n$.

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3. Finally,
$$f(x(t)) - f^* \leqslant \exp(-2\mu t)[f(x(0)) - f^*],$$

Accelerated Gradient Flow

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Remember one of the forms of Nesterov Accelerated Gradient

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x_{k+1} &= y_k - \epsilon \nabla f(y_k) \\
y_k &= x_k + \frac{k-1}{k+2}(x_k - x_{k-1})
\end{aligned}
$$

The corresponding [1] ODE is:

$$
\ddot{X}_t + \frac{3}{t}\dot{X}_t + \nabla f(X_t) = 0
$$

[1] A Differential Equation for Modeling Nesterov's Accelerated Gradient Method: Theory and Insights, Weijie Su, Stephen Boyd, Emmanuel J. Candes

Stochastic Gradient Flow

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How to model stochasticity in the continuous process? A simple idea would be: $\frac{dx}{dt} = -\nabla f(x) + \xi$ with variety of options for ξ, for example $\xi \sim \mathcal{N}(0, \sigma^2) \sim \sigma^2 \mathcal{N}(0, 1)$.

Therefore, one can write down Stochastic Differential Equation (SDE) for analysis:

$$dx(t) = -\nabla f\left(x(t)\right) dt + \sigma dW(t)$$

Here $dW(t)$ is called Wiener process. It is interesting, that one could analyze the convergence of the stochastic process above in two possible ways:

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! Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \nabla \left(\rho(t) \nabla f\right) + \frac{\sigma^2}{2} \Delta \rho(t)$$

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