

A Corgi dog and a yellow rubber duck are positioned inside a transparent, wireframe cube. The Corgi is on the left, looking towards the camera, and the rubber duck is on the right, also looking towards the camera. The cube is made of thin, metallic-looking lines. The background is a plain, light-colored surface.

**Gradient methods for conditional problems.  
Projected Gradient Descent. Frank-Wolfe  
method. Idea of Mirror Descent algorithm**

**Daniil Merkulov**

Optimization methods. MIPT

## Conditional methods

# Constrained optimization

## Unconstrained optimization

$$\min_{x \in \mathbb{R}^n} f(x)$$

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Is it possible to tune GD to fit constrained problem?

**Yes.** We need to use projections to ensure feasibility on every iteration.

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- Mathematically, a neural network is a function  $f(w; x)$
- Typically, input  $x$  is given and network weights  $w$  optimized
- Could also freeze weights  $w$  and optimize  $x$ , adversarially!

$$\min_{\delta} \text{size}(\delta) \quad \text{s.t.} \quad \text{pred}[f(w; x + \delta)] \neq y$$

or

$$\max_{\delta} l(w; x + \delta, y) \quad \text{s.t.} \quad \text{size}(\delta) \leq \epsilon, \quad 0 \leq x + \delta \leq 1$$

# Idea of Projected Gradient Descent



Figure 2: Suppose, we start from a point  $x_k$ .

# Idea of Projected Gradient Descent



Figure 3: And go in the direction of  $-\nabla f(x_k)$ .

## Idea of Projected Gradient Descent



Figure 4: Occasionally, we can end up outside the feasible set.



# Idea of Projected Gradient Descent



Figure 5: Solve this little problem with projection!

# Idea of Projected Gradient Descent

$$x_{k+1} = \text{proj}_S(x_k - \alpha_k \nabla f(x_k)) \quad \Leftrightarrow \quad \begin{aligned} y_k &= x_k - \alpha_k \nabla f(x_k) \\ x_{k+1} &= \text{proj}_S(y_k) \end{aligned}$$



Figure 6: Illustration of Projected Gradient Descent algorithm

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We will focus on Euclidean projection (other options are possible) of a point  $\mathbf{y} \in \mathbb{R}^n$  on set  $S \subseteq \mathbb{R}^n$  is a point  $\text{proj}_S(\mathbf{y}) \in S$ :

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- If a point is in set, then its projection is the point itself.

# Projection criterion (Bourbaki-Cheney-Goldstein inequality)

## i Theorem

Let  $S \subseteq \mathbb{R}^n$  be closed and convex,  $\forall x \in S, y \in \mathbb{R}^n$ . Then

$$\langle y - \text{proj}_S(y), x - \text{proj}_S(y) \rangle \leq 0 \quad (1)$$

$$\|x - \text{proj}_S(y)\|^2 + \|y - \text{proj}_S(y)\|^2 \leq \|x - y\|^2 \quad (2)$$

## Proof

1.  $\text{proj}_S(y)$  is minimizer of differentiable convex function  $d(y, S, \|\cdot\|) = \|x - y\|^2$  over  $S$ . By first-order characterization of optimality.



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- A function  $f$  is called non-expansive if  $f$  is  $L$ -Lipschitz with  $L \leq 1$ <sup>1</sup>. That is, for any two points  $x, y \in \text{dom} f$ ,

$$\|f(x) - f(y)\| \leq L\|x - y\|, \text{ where } L \leq 1.$$

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- Next: variational characterization implies non-expansiveness. i.e.,

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By Cauchy-Schwarz inequality, the left-hand-side is upper bounded by

$\|y - x\|_2 \|\pi(y) - \pi(x)\|_2$ , we get  
 $\|y - x\|_2 \|\pi(y) - \pi(x)\|_2 \geq \|\pi(x) - \pi(y)\|_2^2$ .  
Cancels  $\|\pi(x) - \pi(y)\|_2$  finishes the proof.

## Example: projection on the ball

Find  $\pi_S(y) = \pi$ , if  $S = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq R\}$ ,  $y \notin S$

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$$\begin{aligned} & \left( x_0 - y + R \frac{y - x_0}{\|y - x_0\|} \right)^T \left( x - x_0 - R \frac{y - x_0}{\|y - x_0\|} \right) = \\ & \left( \frac{(y - x_0)(R - \|y - x_0\|)}{\|y - x_0\|} \right)^T \left( \frac{(x - x_0)\|y - x_0\| - R(y - x_0)}{\|y - x_0\|} \right) = \\ & \frac{R - \|y - x_0\|}{\|y - x_0\|^2} (y - x_0)^T ((x - x_0)\|y - x_0\| - R(y - x_0)) = \\ & \frac{R - \|y - x_0\|}{\|y - x_0\|} ((y - x_0)^T (x - x_0) - R\|y - x_0\|) = \\ & (R - \|y - x_0\|) \left( \frac{(y - x_0)^T (x - x_0)}{\|y - x_0\|} - R \right) \end{aligned}$$

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The first factor is negative for point selection  $y$ . The second factor is also negative, which follows from the Cauchy-Bunyakovsky inequality:

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$$\begin{aligned} (y - x_0)^T (x - x_0) &\leq \|y - x_0\| \|x - x_0\| \\ \frac{(y - x_0)^T (x - x_0)}{\|y - x_0\|} - R &\leq \frac{\|y - x_0\| \|x - x_0\|}{\|y - x_0\|} - R \end{aligned}$$





## Example: projection on the halfspace

Find  $\pi_S(y) = \pi$ , if  $S = \{x \in \mathbb{R}^n \mid c^T x = b\}$ ,  $y \notin S$ . Build a hypothesis from the figure:  $\pi = y + \alpha c$ . Coefficient  $\alpha$  is chosen so that  $\pi \in S$ :  $c^T \pi = b$ , so:

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Figure 9: Hyperplane

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Figure 9: Hyperplane

$$c^T(y + \alpha c) = b$$

$$c^T y + \alpha c^T c = b$$

$$c^T y = b - \alpha c^T c$$

Check the inequality for a convex closed set:  
 $(\pi - y)^T(x - \pi) \geq 0$

$$(y + \alpha c - y)^T(x - y - \alpha c) =$$

$$\alpha c^T(x - y - \alpha c) =$$

$$\alpha(c^T x) - \alpha(c^T y) - \alpha^2(c^T c) =$$

$$\alpha b - \alpha(b - \alpha c^T c) - \alpha^2 c^T c =$$

$$\alpha b - \alpha b + \alpha^2 c^T c - \alpha^2 c^T c = 0 \geq 0$$

## Projected Gradient Descent (PGD)

## Idea

$$x_{k+1} = \text{proj}_S(x_k - \alpha_k \nabla f(x_k)) \quad \Leftrightarrow \quad \begin{aligned} y_k &= x_k - \alpha_k \nabla f(x_k) \\ x_{k+1} &= \text{proj}_S(y_k) \end{aligned}$$



Figure 10: Illustration of Projected Gradient Descent algorithm

## Convergence rate for smooth and convex case

### Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be convex and differentiable. Let  $S \subseteq \mathbb{R}^n$  be a closed convex set, and assume that there is a minimizer  $x^*$  of  $f$  over  $S$ ; furthermore, suppose that  $f$  is smooth over  $S$  with parameter  $L$ . The Projected Gradient Descent algorithm with stepsize  $\frac{1}{L}$  achieves the following convergence after iteration  $k > 0$ :

$$f(x_k) - f^* \leq \frac{L \|x_0 - x^*\|_2^2}{2k}$$

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## Proof

1. Let's prove sufficient decrease lemma, assuming, that  $y_k = x_k - \frac{1}{L}\nabla f(x_k)$  and cosine rule  $2x^T y = \|x\|^2 + \|y\|^2 - \|x - y\|^2$ :

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$$\text{Cosine rule: } = f(x_k) - \frac{L}{2}(\|y_k - x_k\|^2 + \|x_{k+1} - x_k\|^2 - \|y_k - x_{k+1}\|^2) + \frac{L}{2}\|x_{k+1} - x_k\|^2 \quad (7)$$

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$$= f(x_k) - \frac{1}{2L}\|\nabla f(x_k)\|^2 + \frac{L}{2}\|y_k - x_{k+1}\|^2$$

## Convergence rate for smooth and convex case

2. Now we do not immediately have progress at each step. Let's use again cosine rule:

$$\begin{aligned}\left\langle \frac{1}{L} \nabla f(x_k), x_k - x^* \right\rangle &= \frac{1}{2} \left( \frac{1}{L^2} \|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|x_k - x^* - \frac{1}{L} \nabla f(x_k)\|^2 \right) \\ \langle \nabla f(x_k), x_k - x^* \rangle &= \frac{L}{2} \left( \frac{1}{L^2} \|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|y_k - x^*\|^2 \right)\end{aligned}$$

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3. We will use now projection property:  $\|x - \text{proj}_S(y)\|^2 + \|y - \text{proj}_S(y)\|^2 \leq \|x - y\|^2$  with  $x = x^*, y = y_k$ :

$$\begin{aligned}\|x^* - \text{proj}_S(y_k)\|^2 + \|y_k - \text{proj}_S(y_k)\|^2 &\leq \|x^* - y_k\|^2 \\ \|y_k - x^*\|^2 &\geq \|x^* - x_{k+1}\|^2 + \|y_k - x_{k+1}\|^2\end{aligned}$$

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4. Now, using convexity and previous part:

Convexity:

$$\begin{aligned}f(x_k) - f^* &\leq \langle \nabla f(x_k), x_k - x^* \rangle \\ &\leq \frac{L}{2} \left( \frac{1}{L^2} \|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 - \|y_k - x_{k+1}\|^2 \right)\end{aligned}$$

$$\text{Sum for } i = 0, k-1 \quad \sum_{i=0}^{k-1} [f(x_i) - f^*] \leq \sum_{i=0}^{k-1} \frac{1}{2L} \|\nabla f(x_i)\|^2 + \frac{L}{2} \|x_0 - x^*\|^2 - \frac{L}{2} \sum_{i=0}^{i-1} \|y_i - x_{i+1}\|^2$$

# Convergence rate for smooth and convex case

## 5. Bound gradients with sufficient decrease lemma 7:

$$\begin{aligned}\sum_{i=0}^{k-1} [f(x_i) - f^*] &\leq \sum_{i=0}^{k-1} \left[ f(x_i) - f(x_{i+1}) + \frac{L}{2} \|y_i - x_{i+1}\|^2 \right] + \frac{L}{2} \|x_0 - x^*\|^2 - \frac{L}{2} \sum_{i=0}^{i-1} \|y_i - x_{i+1}\|^2 \\ &\leq f(x_0) - f(x_k) + \frac{L}{2} \sum_{i=0}^{i-1} \|y_i - x_{i+1}\|^2 + \frac{L}{2} \|x_0 - x^*\|^2 - \frac{L}{2} \sum_{i=0}^{i-1} \|y_i - x_{i+1}\|^2 \\ &\leq f(x_0) - f(x_k) + \frac{L}{2} \|x_0 - x^*\|^2 \\ \sum_{i=0}^{k-1} f(x_i) - kf^* &\leq f(x_0) - f(x_k) + \frac{L}{2} \|x_0 - x^*\|^2 \\ \sum_{i=1}^k [f(x_i) - f^*] &\leq \frac{L}{2} \|x_0 - x^*\|^2\end{aligned}$$

## Convergence rate for smooth and convex case

6. From the sufficient decrease inequality

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2L} \|\nabla f(x_k)\|^2 + \frac{L}{2} \|y_k - x_{k+1}\|^2,$$

we use the fact that  $x_{k+1} = \text{proj}_S(y_k)$ . By definition of projection,

$$\|y_k - x_{k+1}\| \leq \|y_k - x_k\|,$$

and recall that  $y_k = x_k - \frac{1}{L} \nabla f(x_k)$  implies  $\|y_k - x_k\| = \frac{1}{L} \|\nabla f(x_k)\|$ . Hence

$$\frac{L}{2} \|y_k - x_{k+1}\|^2 \leq \frac{L}{2} \|y_k - x_k\|^2 = \frac{L}{2} \frac{1}{L^2} \|\nabla f(x_k)\|^2 = \frac{1}{2L} \|\nabla f(x_k)\|^2.$$

Substitute back into (\*):

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2L} \|\nabla f(x_k)\|^2 + \frac{1}{2L} \|\nabla f(x_k)\|^2 = f(x_k).$$

Hence

$$f(x_{k+1}) \leq f(x_k) \quad \text{for each } k,$$

so  $\{f(x_k)\}$  is a monotonically nonincreasing sequence.



## Convergence rate for smooth and convex case

7. Final convergence bound From step 5, we have already established

$$\sum_{i=0}^{k-1} [f(x_i) - f^*] \leq \frac{L}{2} \|x_0 - x^*\|_2^2.$$

Since  $f(x_i)$  decreases in  $i$ , in particular  $f(x_k) \leq f(x_i)$  for all  $i \leq k$ . Therefore

$$k [f(x_k) - f^*] \leq \sum_{i=0}^{k-1} [f(x_i) - f^*] \leq \frac{L}{2} \|x_0 - x^*\|_2^2,$$

which immediately gives

$$f(x_k) - f^* \leq \frac{L \|x_0 - x^*\|_2^2}{2k}.$$

This completes the proof of the  $\mathcal{O}(\frac{1}{k})$  convergence rate for convex and  $L$ -smooth  $f$  under projection constraints.

## Frank-Wolfe Method

# Idea



Figure 11: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Idea



Figure 12: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Idea



Figure 13: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Idea



Figure 14: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Idea



Figure 15: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Idea



Figure 16: Illustration of Frank-Wolfe (conditional gradient) algorithm



# Idea



Figure 17: Illustration of Frank-Wolfe (conditional gradient) algorithm

# Idea

$$y_k = \arg \min_{x \in S} f_{x_k}^I(x) = \arg \min_{x \in S} \langle \nabla f(x_k), x \rangle$$

$$x_{k+1} = \gamma_k x_k + (1 - \gamma_k) y_k$$



Figure 18: Illustration of Frank-Wolfe (conditional gradient) algorithm

## Convergence rate for smooth and convex case

### Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be convex and differentiable. Let  $S \subseteq \mathbb{R}^n$  be a closed convex set, and assume that there is a minimizer  $x^*$  of  $f$  over  $S$ ; furthermore, suppose that  $f$  is smooth over  $S$  with parameter  $L$ . The Frank-Wolfe algorithm with step size  $\gamma_k = \frac{k-1}{k+1}$  achieves the following convergence after iteration  $k > 0$ :

$$f(x_k) - f^* \leq \frac{2LR^2}{k+1}$$

where  $R = \max_{x,y \in S} \|x - y\|$  is the diameter of the set  $S$ .

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## Proof

1. By  $L$ -smoothness of  $f$ , we have:

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\leq \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|^2 \\ &= (1 - \gamma_k) \langle \nabla f(x_k), y_k - x_k \rangle + \frac{L(1 - \gamma_k)^2}{2} \|y_k - x_k\|^2 \end{aligned}$$

## Convergence rate for smooth and convex case

2. By convexity of  $f$ , for any  $x \in S$ , including  $x^*$ :

$$\langle \nabla f(x_k), x - x_k \rangle \leq f(x) - f(x_k)$$

In particular, for  $x = x^*$ :

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In particular, for  $x = x^*$ :

$$\langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k)$$

3. By definition of  $y_k$ , we have  $\langle \nabla f(x_k), y_k \rangle \leq \langle \nabla f(x_k), x^* \rangle$ , thus:

$$\langle \nabla f(x_k), y_k - x_k \rangle \leq \langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k)$$

## Convergence rate for smooth and convex case

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$$\langle \nabla f(x_k), x - x_k \rangle \leq f(x) - f(x_k)$$

In particular, for  $x = x^*$ :

$$\langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k)$$

3. By definition of  $y_k$ , we have  $\langle \nabla f(x_k), y_k \rangle \leq \langle \nabla f(x_k), x^* \rangle$ , thus:

$$\langle \nabla f(x_k), y_k - x_k \rangle \leq \langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k)$$

4. Combining the above inequalities:

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\leq (1 - \gamma_k) \langle \nabla f(x_k), y_k - x_k \rangle + \frac{L(1 - \gamma_k)^2}{2} \|y_k - x_k\|^2 \\ &\leq (1 - \gamma_k) (f(x^*) - f(x_k)) + \frac{L(1 - \gamma_k)^2}{2} R^2 \end{aligned}$$

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5. Rearranging terms:

$$f(x_{k+1}) - f(x^*) \leq \gamma_k (f(x_k) - f(x^*)) + (1 - \gamma_k)^2 \frac{LR^2}{2}$$



## Convergence rate for smooth and convex case

6. Denoting  $\delta_k = \frac{f(x_k) - f(x^*)}{LR^2}$ , we get:

$$\delta_{k+1} \leq \gamma_k \delta_k + \frac{(1 - \gamma_k)^2}{2} = \frac{k-1}{k+1} \delta_k + \frac{2}{(k+1)^2}$$

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7. Starting from  $\delta_2 \leq \frac{1}{2}$  and applying induction on  $k$ , we can show that:

$$\delta_k \leq \frac{2}{k+1}$$

which gives us the desired result:

$$f(x_k) - f^* \leq \frac{2LR^2}{k+1}$$

## Convergence rate for strongly convex case

### i Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be  $\mu$ -strongly convex and differentiable. Let  $S \subseteq \mathbb{R}^n$  be a closed convex set, and assume that there is a minimizer  $x^*$  of  $f$  over  $S$ ; furthermore, suppose that  $f$  is smooth over  $S$  with parameter  $L$ . The Frank-Wolfe algorithm with step size  $\gamma_k = \frac{2}{k+2}$  achieves the following convergence after iteration  $k > 0$ :

$$f(x_k) - f^* \leq \frac{4LR^2}{(k+2)^2}$$

where  $R = \max_{x,y \in S} \|x - y\|$  is the diameter of the set  $S$ .

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### Proof

1. By  $\mu$ -strong convexity of  $f$ , for any  $x, y \in S$ :

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2$$

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$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2$$

2. This gives us a stronger inequality than in the convex case:

$$\langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k) - \frac{\mu}{2} \|x^* - x_k\|^2$$

## Convergence rate for strongly convex case

3. Following similar steps as in the convex case, but using the stronger inequality:

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\leq (1 - \gamma_k) \langle \nabla f(x_k), y_k - x_k \rangle + \frac{L(1 - \gamma_k)^2}{2} \|y_k - x_k\|^2 \\ &\leq (1 - \gamma_k) \left( f(x^*) - f(x_k) - \frac{\mu}{2} \|x^* - x_k\|^2 \right) + \frac{L(1 - \gamma_k)^2}{2} R^2 \end{aligned}$$

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4. Rearranging terms and using the fact that  $\|x^* - x_k\|^2 \geq 0$ :

$$\begin{aligned} f(x_{k+1}) - f(x^*) &\leq \gamma_k (f(x_k) - f(x^*)) + (1 - \gamma_k)^2 \frac{LR^2}{2} - (1 - \gamma_k) \frac{\mu}{2} \|x^* - x_k\|^2 \\ &\leq \gamma_k (f(x_k) - f(x^*)) + (1 - \gamma_k)^2 \frac{LR^2}{2} \end{aligned}$$

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5. With  $\gamma_k = \frac{2}{k+2}$  and denoting  $\delta_k = f(x_k) - f^*$ , we get:

$$\begin{aligned} \delta_{k+1} &\leq \frac{2}{k+2} \delta_k + \frac{LR^2}{2} \left( 1 - \frac{2}{k+2} \right)^2 \\ &= \frac{2}{k+2} \delta_k + \frac{LR^2}{2} \frac{(k)^2}{(k+2)^2} \end{aligned}$$



## Convergence rate for strongly convex case

6. It can be shown by induction that:

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7. This gives us the improved convergence rate of  $\mathcal{O}(\frac{1}{k^2})$  for the strongly convex case, compared to  $\mathcal{O}(\frac{1}{k})$  for the convex case.