A Corgi puppy and a yellow rubber duck are positioned inside a wireframe cube. The cube is made of thin, transparent lines forming a 3D geometric shape. The puppy is on the left, looking towards the right, while the duck is on the right, facing left. The background is a plain, light color.

Gradient methods for conditional problems. Projected Gradient Descent. Frank-Wolfe method. Idea of Mirror Descent algorithm

Daniil Merkulov

Optimization methods. MIPT

Conditional methods

Constrained optimization

Unconstrained optimization

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- Any point $x_0 \in \mathbb{R}^n$ is feasible and could be a solution.

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$$x_{k+1} = x_k - \alpha_k \nabla f(x_k) \quad (\text{GD})$$

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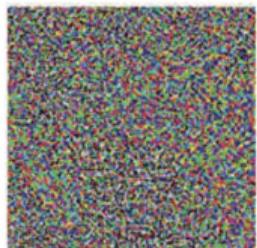
Is it possible to tune GD to fit constrained problem?

Yes. We need to use projections to ensure feasibility on every iteration.

Example: White-box Adversarial Attacks



'Duck'



$\times 0.07$



'Horse'

- Mathematically, a neural network is a function $f(w; x)$



'How are you?'



$\times 0.01$



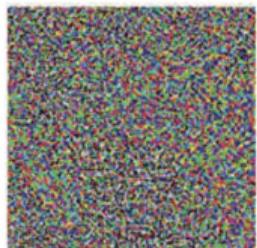
'Open the door'

Figure 1: Source

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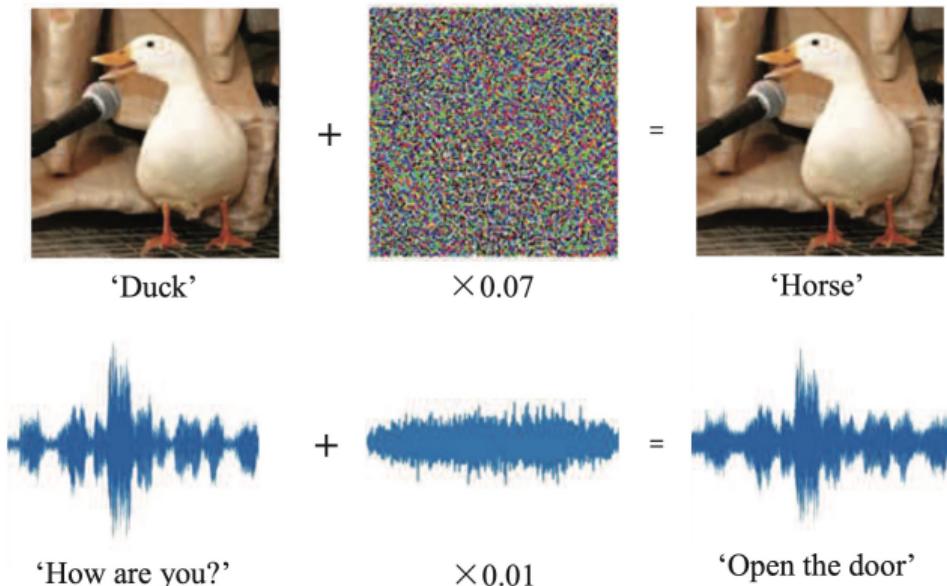
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'Open the door'

Figure 1: Source

Example: White-box Adversarial Attacks



- Mathematically, a neural network is a function $f(w; x)$
- Typically, input x is given and network weights w optimized
- Could also freeze weights w and optimize x , adversarially!

$$\min_{\delta} \text{size}(\delta) \quad \text{s.t.} \quad \text{pred}[f(w; x + \delta)] \neq y$$

or

$$\max_{\delta} l(w; x + \delta, y) \quad \text{s.t.} \quad \text{size}(\delta) \leq \epsilon, \quad 0 \leq x + \delta \leq 1$$

Figure 1: Source

Idea of Projected Gradient Descent

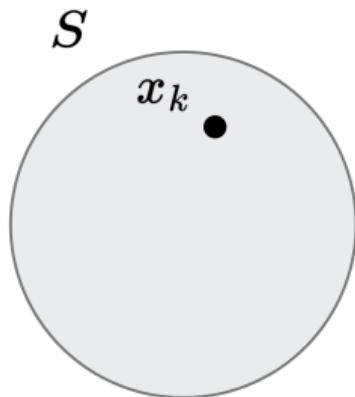


Figure 2: Suppose, we start from a point x_k .

Idea of Projected Gradient Descent

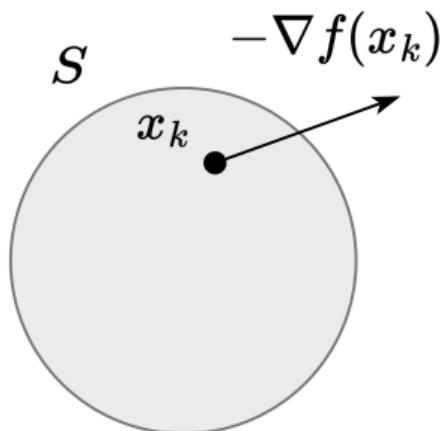


Figure 3: And go in the direction of $-\nabla f(x_k)$.

Idea of Projected Gradient Descent

$$y_k = x_k - \alpha_k \nabla f(x_k)$$

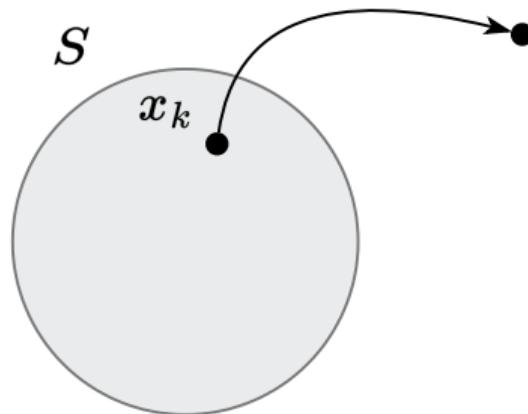


Figure 4: Occasionally, we can end up outside the feasible set.

Idea of Projected Gradient Descent

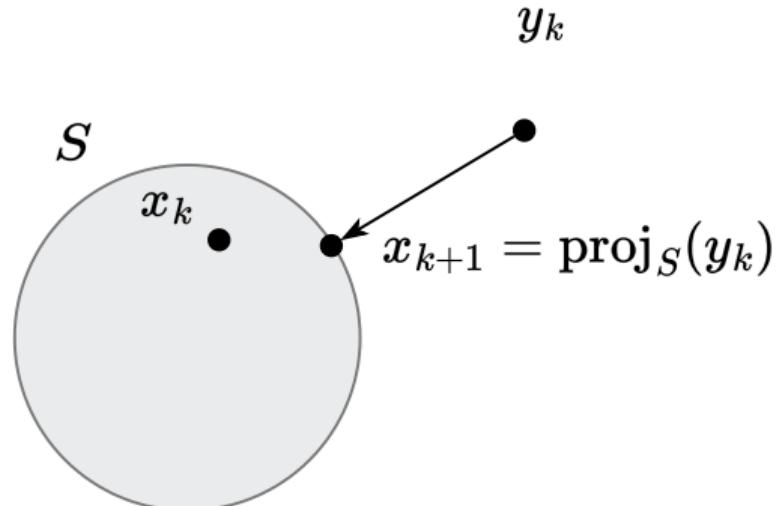


Figure 5: Solve this little problem with projection!

Idea of Projected Gradient Descent

$$x_{k+1} = \text{proj}_S(x_k - \alpha_k \nabla f(x_k)) \quad \Leftrightarrow \quad \begin{aligned} y_k &= x_k - \alpha_k \nabla f(x_k) \\ x_{k+1} &= \text{proj}_S(y_k) \end{aligned}$$

$$y_k = x_k - \alpha_k \nabla f(x_k)$$

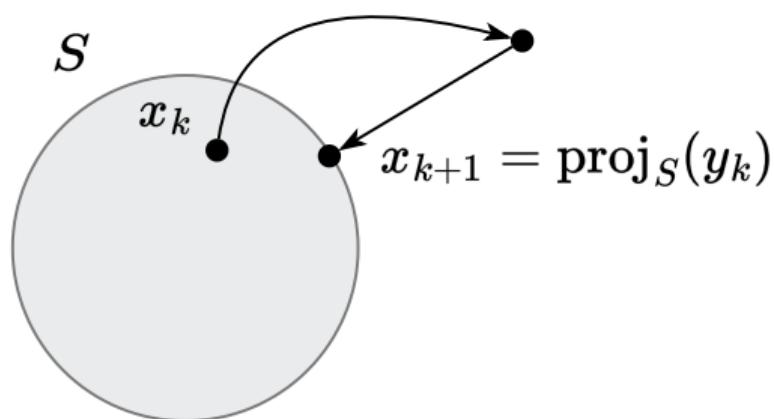


Figure 6: Illustration of Projected Gradient Descent algorithm

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Projection criterion (Bourbaki-Cheney-Goldstein inequality)

Theorem

Let $S \subseteq \mathbb{R}^n$ be closed and convex, $\forall x \in S, y \in \mathbb{R}^n$. Then

$$\langle y - \text{proj}_S(y), x - \text{proj}_S(y) \rangle \leq 0 \quad (1)$$

$$\|x - \text{proj}_S(y)\|^2 + \|y - \text{proj}_S(y)\|^2 \leq \|x - y\|^2 \quad (2)$$

Proof

1. $\text{proj}_S(y)$ is minimizer of differentiable convex function $d(y, S, \|\cdot\|) = \|x - y\|^2$ over S . By first-order characterization of optimality.

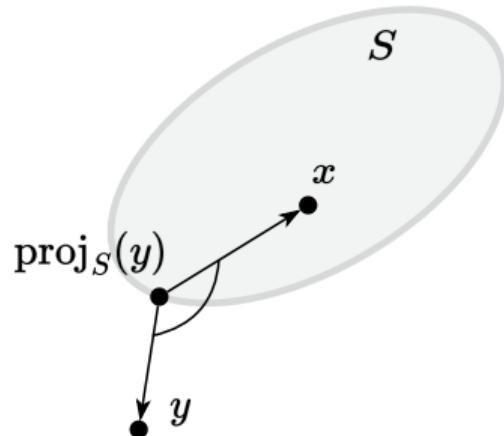


Figure 7: Obtuse or straight angle should be for any point $x \in S$

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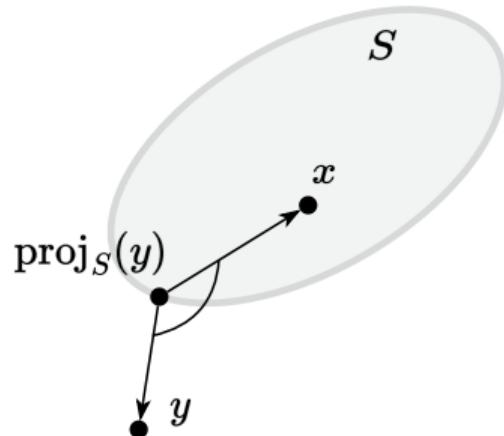


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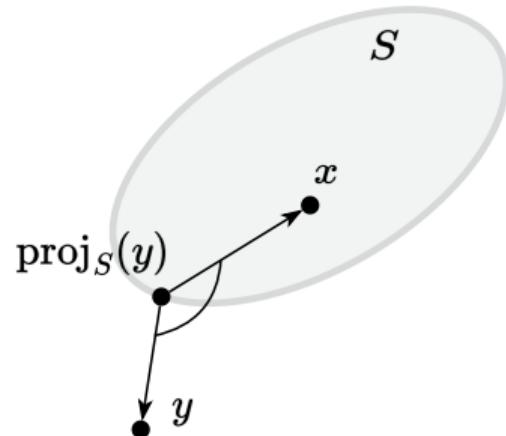


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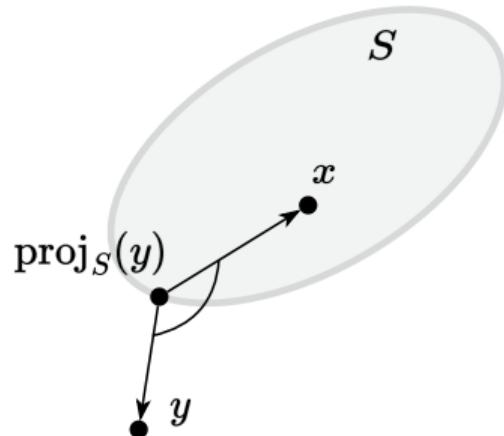


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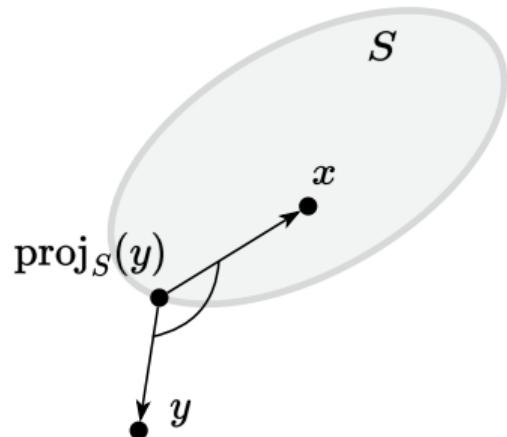


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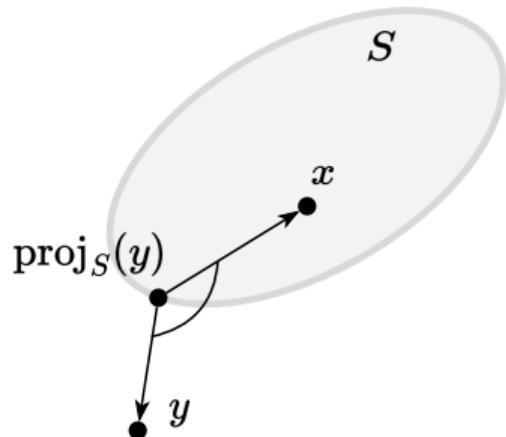


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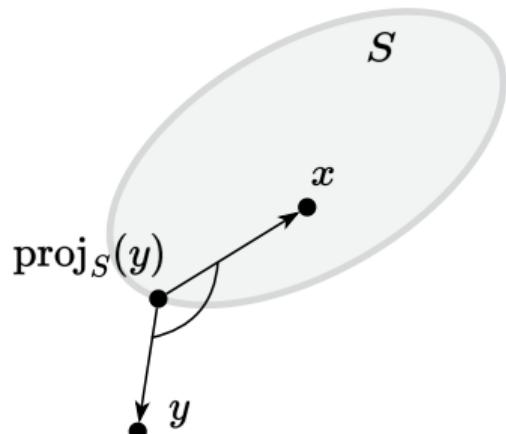


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It means the distance between the mapped points is possibly smaller than that of the unmapped points.

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- Next: variational characterization implies non-expansiveness. i.e.,

$$\langle y - \text{proj}(y), x - \text{proj}(y) \rangle \leq 0 \quad \forall x \in S \quad \Rightarrow \quad \|\text{proj}(x) - \text{proj}(y)\|_2 \leq \|x - y\|_2.$$

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Begins with the variational characterization / obtuse angle inequality

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Replace x by $\pi(x)$ in Equation 3

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(Equation 4)+(Equation 5) will cancel $\pi(y) - \pi(x)$, not good. So flip the sign of (Equation 5) gives

$$\langle \pi(x) - x, \pi(x) - \pi(y) \rangle \leq 0. \quad (6)$$

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$$\langle y - \pi(y), \pi(x) - \pi(y) \rangle \leq 0. \quad (4)$$

Replace y by x and x by $\pi(y)$ in Equation 3

$$\langle x - \pi(x), \pi(y) - \pi(x) \rangle \leq 0. \quad (5)$$

(Equation 4)+(Equation 5) will cancel $\pi(y) - \pi(x)$, not good. So flip the sign of (Equation 5) gives

$$\langle \pi(x) - x, \pi(x) - \pi(y) \rangle \leq 0. \quad (6)$$

$$\langle y - \pi(y) + \pi(x) - x, \pi(x) - \pi(y) \rangle \leq 0$$

$$\langle y - x, \pi(x) - \pi(y) \rangle \leq -\langle \pi(x) - \pi(y), \pi(x) - \pi(y) \rangle$$

$$\langle y - x, \pi(y) - \pi(x) \rangle \geq \|\pi(x) - \pi(y)\|_2^2$$

$$\|(y - x)^\top (\pi(y) - \pi(x))\|_2 \geq \|\pi(x) - \pi(y)\|_2^2$$

Projection operator is non-expansive

Shorthand notation: let $\pi = \text{proj}$ and $\pi(x)$ denotes $\text{proj}(x)$.

Begins with the variational characterization / obtuse angle inequality

$$\langle y - \pi(y), x - \pi(y) \rangle \leq 0 \quad \forall x \in S. \quad (3)$$

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$$\|(y - x)^\top (\pi(y) - \pi(x))\|_2 \geq \|\pi(x) - \pi(y)\|_2^2$$

By Cauchy-Schwarz inequality, the left-hand-side is upper bounded by $\|y - x\|_2 \|\pi(y) - \pi(x)\|_2$, we get $\|y - x\|_2 \|\pi(y) - \pi(x)\|_2 \geq \|\pi(x) - \pi(y)\|_2^2$. Cancels $\|\pi(x) - \pi(y)\|_2$ finishes the proof.

Example: projection on the ball

Find $\pi_S(y) = \pi$, if $S = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq R\}$, $y \notin S$

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$$\left(x_0 - y + R \frac{y - x_0}{\|y - x_0\|} \right)^T \left(x - x_0 - R \frac{y - x_0}{\|y - x_0\|} \right) =$$

$$\left(\frac{(y - x_0)(R - \|y - x_0\|)}{\|y - x_0\|} \right)^T \left(\frac{(x - x_0)\|y - x_0\| - R(y - x_0)}{\|y - x_0\|} \right) =$$

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$$\frac{R - \|y - x_0\|}{\|y - x_0\|} \left((y - x_0)^T (x - x_0) - R\|y - x_0\| \right) =$$

$$(R - \|y - x_0\|) \left(\frac{(y - x_0)^T (x - x_0)}{\|y - x_0\|} - R \right)$$

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The first factor is negative for point selection y . The second factor is also negative, which follows from the Cauchy-Bunyakovsky inequality:

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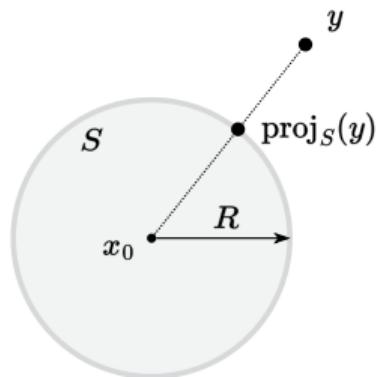
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$$(R - \|y - x_0\|) \left(\frac{(y - x_0)^T (x - x_0)}{\|y - x_0\|} - R \right)$$

$$(y - x_0)^T (x - x_0) \leq \|y - x_0\| \|x - x_0\|$$

$$\frac{(y - x_0)^T (x - x_0)}{\|y - x_0\|} - R \leq \frac{\|y - x_0\| \|x - x_0\|}{\|y - x_0\|} - R$$



Example: projection on the halfspace

Find $\pi_S(y) = \pi$, if $S = \{x \in \mathbb{R}^n \mid c^T x = b\}$, $y \notin S$. Build a hypothesis from the figure: $\pi = y + \alpha c$. Coefficient α is chosen so that $\pi \in S$: $c^T \pi = b$, so:

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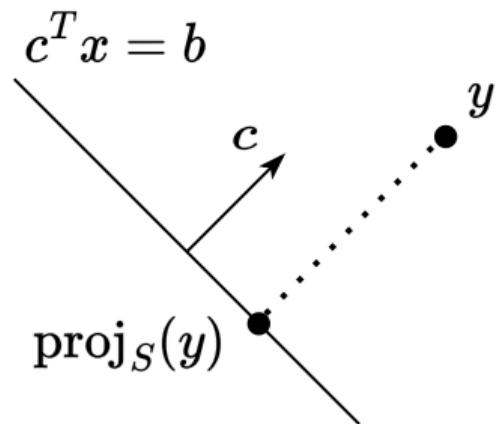


Figure 9: Hyperplane

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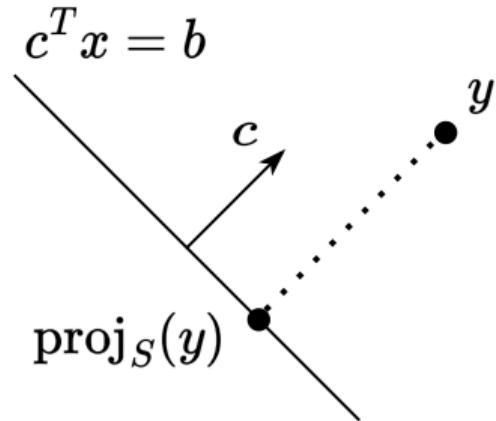


Figure 9: Hyperplane

$$\begin{aligned}c^T(y + \alpha c) &= b \\c^T y + \alpha c^T c &= b \\c^T y &= b - \alpha c^T c\end{aligned}$$

Check the inequality for a convex closed set:
 $(\pi - y)^T(x - \pi) \geq 0$

$$\begin{aligned}(y + \alpha c - y)^T(x - y - \alpha c) &= \\ \alpha c^T(x - y - \alpha c) &= \\ \alpha(c^T x) - \alpha(c^T y) - \alpha^2(c^T c) &= \\ \alpha b - \alpha(b - \alpha c^T c) - \alpha^2 c^T c &= \\ \alpha b - \alpha b + \alpha^2 c^T c - \alpha^2 c^T c &= 0 \geq 0\end{aligned}$$

Projected Gradient Descent (PGD)

Idea

$$x_{k+1} = \text{proj}_S(x_k - \alpha_k \nabla f(x_k)) \quad \Leftrightarrow \quad \begin{aligned} y_k &= x_k - \alpha_k \nabla f(x_k) \\ x_{k+1} &= \text{proj}_S(y_k) \end{aligned}$$

$$y_k = x_k - \alpha_k \nabla f(x_k)$$

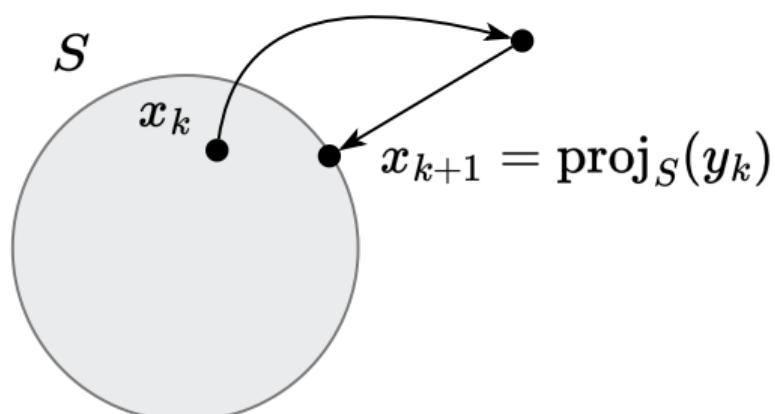


Figure 10: Illustration of Projected Gradient Descent algorithm

Convergence rate for smooth and convex case

i Theorem

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex and differentiable. Let $S \subseteq \mathbb{R}^n$ be a closed convex set, and assume that there is a minimizer x^* of f over S ; furthermore, suppose that f is smooth over S with parameter L . The Projected Gradient Descent algorithm with stepsize $\frac{1}{L}$ achieves the following convergence after iteration $k > 0$:

$$f(x_k) - f^* \leq \frac{L\|x_0 - x^*\|_2^2}{2k}$$

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Proof

1. Let's prove sufficient decrease lemma, assuming, that $y_k = x_k - \frac{1}{L}\nabla f(x_k)$ and cosine rule $2x^T y = \|x\|^2 + \|y\|^2 - \|x - y\|^2$:

(7)

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$$\text{Smoothness: } f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|^2$$

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Method: $= f(x_k) - L\langle y_k - x_k, x_{k+1} - x_k \rangle + \frac{L}{2}\|x_{k+1} - x_k\|^2$

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Method: $= f(x_k) - L\langle y_k - x_k, x_{k+1} - x_k \rangle + \frac{L}{2}\|x_{k+1} - x_k\|^2$

Cosine rule: $= f(x_k) - \frac{L}{2}(\|y_k - x_k\|^2 + \|x_{k+1} - x_k\|^2 - \|y_k - x_{k+1}\|^2) + \frac{L}{2}\|x_{k+1} - x_k\|^2 \quad (7)$

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$$= f(x_k) - \frac{1}{2L}\|\nabla f(x_k)\|^2 + \frac{L}{2}\|y_k - x_{k+1}\|^2$$

Convergence rate for smooth and convex case

2. Now we do not immediately have progress at each step. Let's use again cosine rule:

$$\left\langle \frac{1}{L} \nabla f(x_k), x_k - x^* \right\rangle = \frac{1}{2} \left(\frac{1}{L^2} \|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|x_k - x^* - \frac{1}{L} \nabla f(x_k)\|^2 \right)$$
$$\langle \nabla f(x_k), x_k - x^* \rangle = \frac{L}{2} \left(\frac{1}{L^2} \|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|y_k - x^*\|^2 \right)$$

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3. We will use now projection property: $\|x - \text{proj}_S(y)\|^2 + \|y - \text{proj}_S(y)\|^2 \leq \|x - y\|^2$ with $x = x^*, y = y_k$:

$$\|x^* - \text{proj}_S(y_k)\|^2 + \|y_k - \text{proj}_S(y_k)\|^2 \leq \|x^* - y_k\|^2$$
$$\|y_k - x^*\|^2 \geq \|x^* - x_{k+1}\|^2 + \|y_k - x_{k+1}\|^2$$

Convergence rate for smooth and convex case

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$$\begin{aligned}\left\langle \frac{1}{L} \nabla f(x_k), x_k - x^* \right\rangle &= \frac{1}{2} \left(\frac{1}{L^2} \|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|x_k - x^* - \frac{1}{L} \nabla f(x_k)\|^2 \right) \\ \langle \nabla f(x_k), x_k - x^* \rangle &= \frac{L}{2} \left(\frac{1}{L^2} \|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|y_k - x^*\|^2 \right)\end{aligned}$$

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$$\begin{aligned}\|x^* - \text{proj}_S(y_k)\|^2 + \|y_k - \text{proj}_S(y_k)\|^2 &\leq \|x^* - y_k\|^2 \\ \|y_k - x^*\|^2 &\geq \|x^* - x_{k+1}\|^2 + \|y_k - x_{k+1}\|^2\end{aligned}$$

4. Now, using convexity and previous part:

$$\text{Convexity: } f(x_k) - f^* \leq \langle \nabla f(x_k), x_k - x^* \rangle$$

$$\leq \frac{L}{2} \left(\frac{1}{L^2} \|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 - \|y_k - x_{k+1}\|^2 \right)$$

$$\text{Sum for } i = 0, k-1 \quad \sum_{i=0}^{k-1} [f(x_i) - f^*] \leq \sum_{i=0}^{k-1} \frac{1}{2L} \|\nabla f(x_i)\|^2 + \frac{L}{2} \|x_0 - x^*\|^2 - \frac{L}{2} \sum_{i=0}^{i-1} \|y_i - x_{i+1}\|^2$$

Convergence rate for smooth and convex case

5. Bound gradients with sufficient decrease lemma 7:

$$\sum_{i=0}^{k-1} [f(x_i) - f^*] \leq \sum_{i=0}^{k-1} \left[f(x_i) - f(x_{i+1}) + \frac{L}{2} \|y_i - x_{i+1}\|^2 \right] + \frac{L}{2} \|x_0 - x^*\|^2 - \frac{L}{2} \sum_{i=0}^{i-1} \|y_i - x_{i+1}\|^2$$

$$\leq f(x_0) - f(x_k) + \frac{L}{2} \sum_{i=0}^{i-1} \|y_i - x_{i+1}\|^2 + \frac{L}{2} \|x_0 - x^*\|^2 - \frac{L}{2} \sum_{i=0}^{i-1} \|y_i - x_{i+1}\|^2$$

$$\leq f(x_0) - f(x_k) + \frac{L}{2} \|x_0 - x^*\|^2$$

$$\sum_{i=0}^{k-1} f(x_i) - kf^* \leq f(x_0) - f(x_k) + \frac{L}{2} \|x_0 - x^*\|^2$$

$$\sum_{i=1}^k [f(x_i) - f^*] \leq \frac{L}{2} \|x_0 - x^*\|^2$$

Convergence rate for smooth and convex case

- From the sufficient decrease inequality

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2L} \|\nabla f(x_k)\|^2 + \frac{L}{2} \|y_k - x_{k+1}\|^2,$$

we use the fact that $x_{k+1} = \text{proj}_S(y_k)$. By definition of projection,

$$\|y_k - x_{k+1}\| \leq \|y_k - x_k\|,$$

and recall that $y_k = x_k - \frac{1}{L} \nabla f(x_k)$ implies $\|y_k - x_k\| = \frac{1}{L} \|\nabla f(x_k)\|$. Hence

$$\frac{L}{2} \|y_k - x_{k+1}\|^2 \leq \frac{L}{2} \|y_k - x_k\|^2 = \frac{L}{2} \frac{1}{L^2} \|\nabla f(x_k)\|^2 = \frac{1}{2L} \|\nabla f(x_k)\|^2.$$

Substitute back into (*):

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2L} \|\nabla f(x_k)\|^2 + \frac{1}{2L} \|\nabla f(x_k)\|^2 = f(x_k).$$

Hence

$$f(x_{k+1}) \leq f(x_k) \quad \text{for each } k,$$

so $\{f(x_k)\}$ is a monotonically nonincreasing sequence.

Convergence rate for smooth and convex case

7. Final convergence bound From step 5, we have already established

$$\sum_{i=0}^{k-1} [f(x_i) - f^*] \leq \frac{L}{2} \|x_0 - x^*\|_2^2.$$

Since $f(x_i)$ decreases in i , in particular $f(x_k) \leq f(x_i)$ for all $i \leq k$. Therefore

$$k [f(x_k) - f^*] \leq \sum_{i=0}^{k-1} [f(x_i) - f^*] \leq \frac{L}{2} \|x_0 - x^*\|_2^2,$$

which immediately gives

$$f(x_k) - f^* \leq \frac{L\|x_0 - x^*\|_2^2}{2k}.$$

This completes the proof of the $\mathcal{O}(\frac{1}{k})$ convergence rate for convex and L -smooth f under projection constraints.

Frank-Wolfe Method

Idea

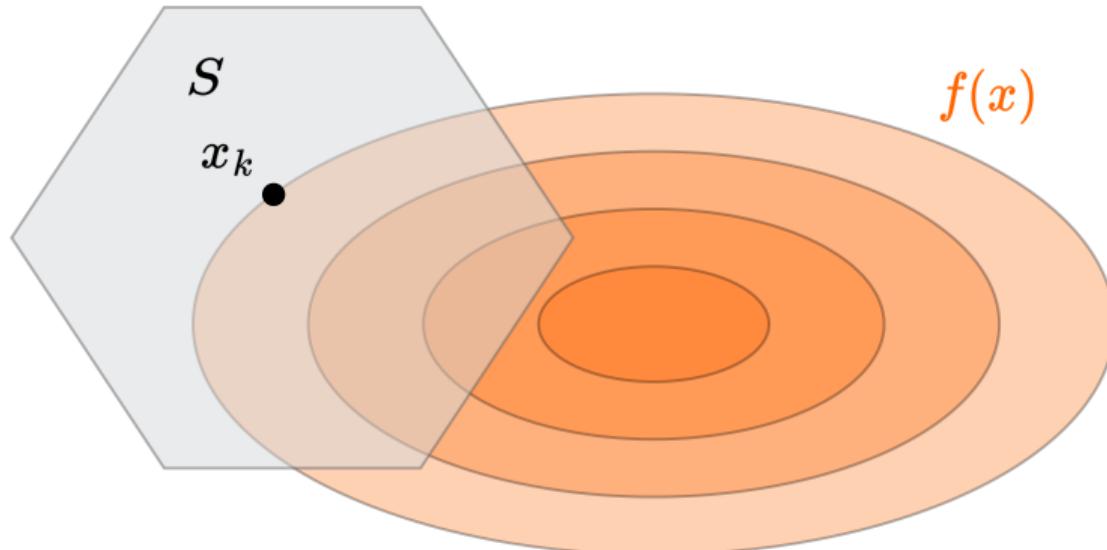


Figure 11: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

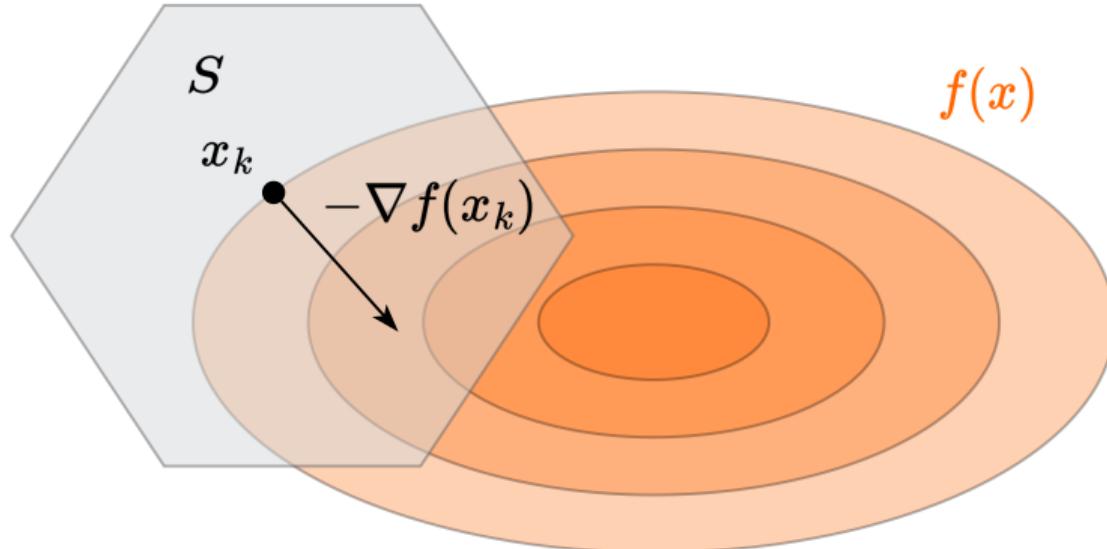


Figure 12: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

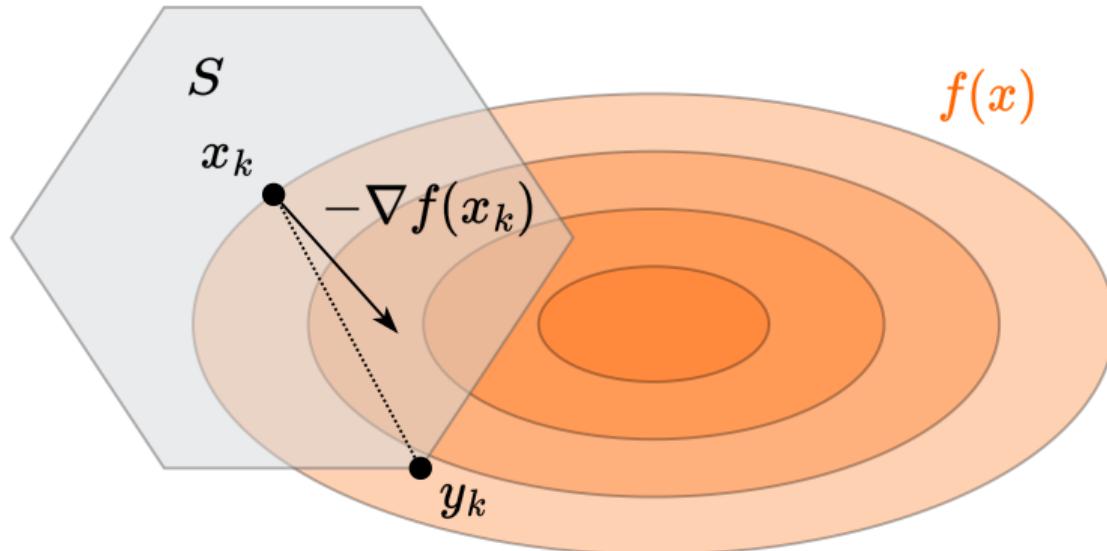


Figure 13: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

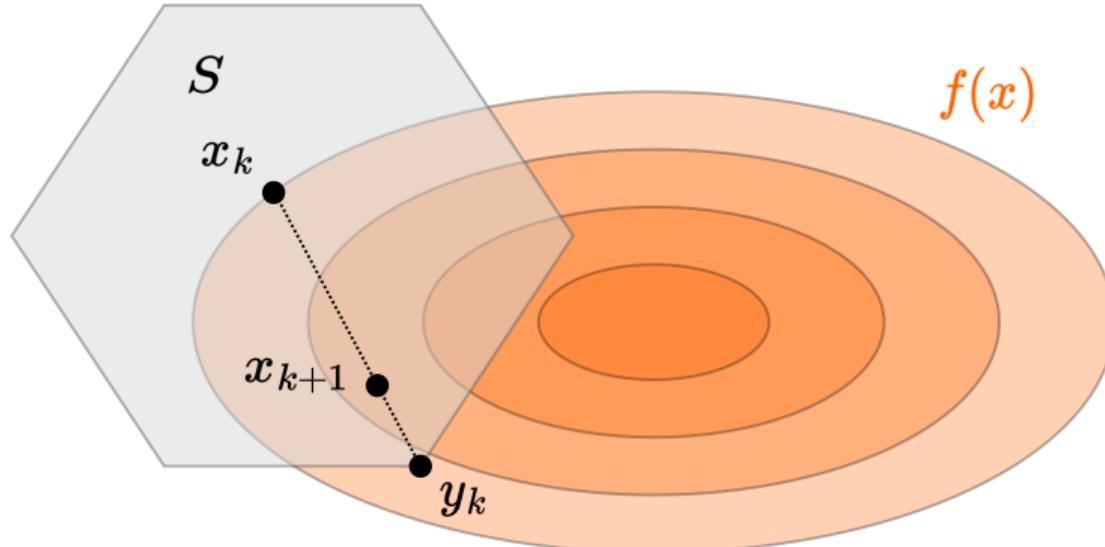


Figure 14: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

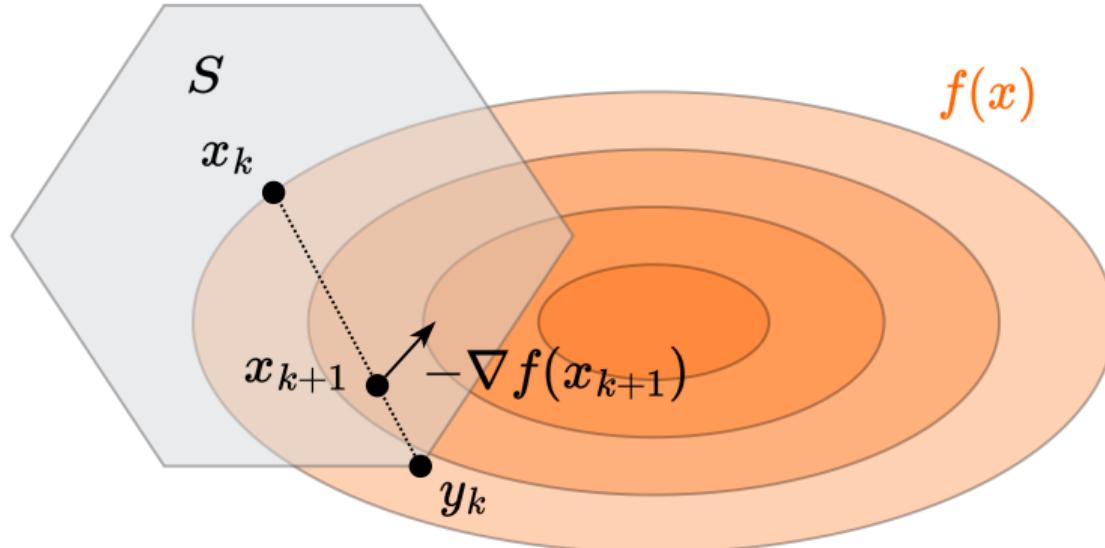


Figure 15: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

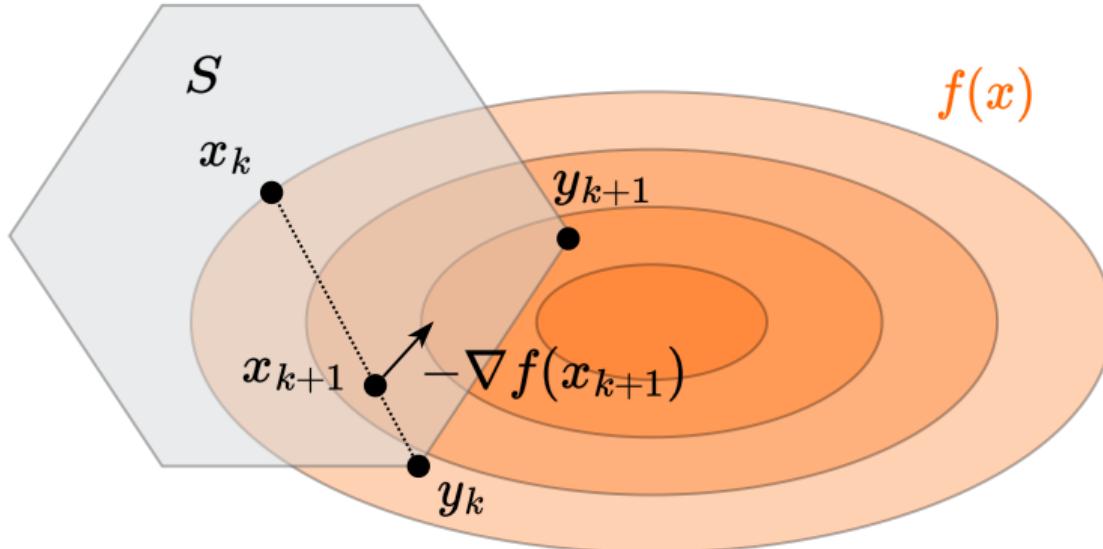


Figure 16: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

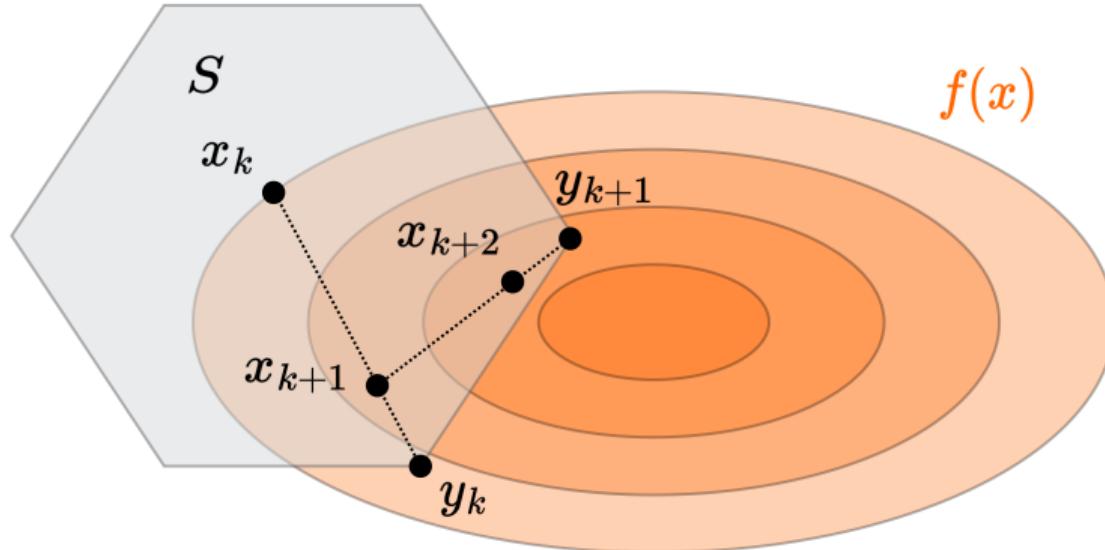


Figure 17: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

$$y_k = \arg \min_{x \in S} f_{x_k}^I(x) = \arg \min_{x \in S} \langle \nabla f(x_k), x \rangle$$

$$x_{k+1} = \gamma_k x_k + (1 - \gamma_k) y_k$$

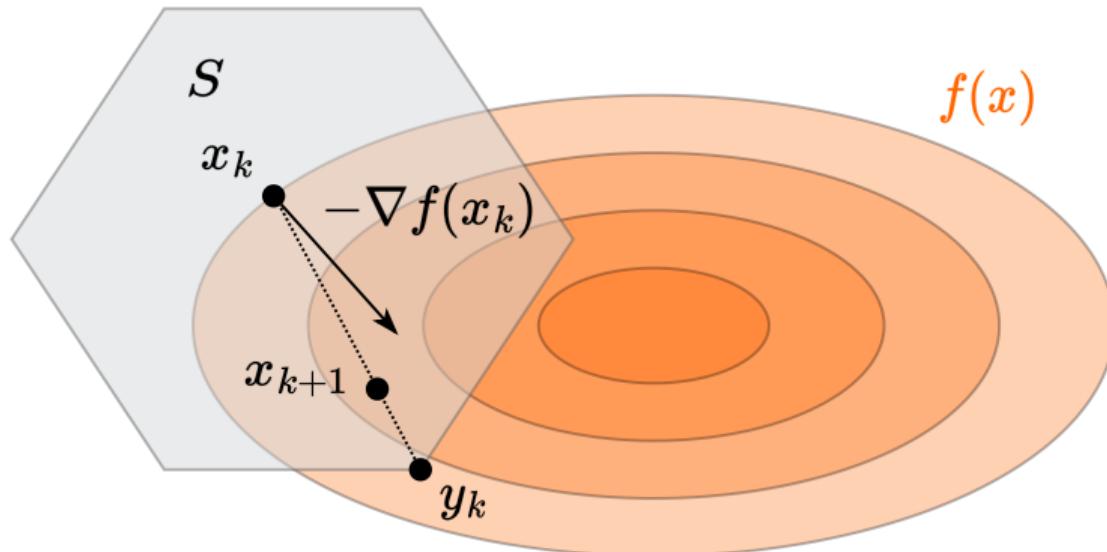


Figure 18: Illustration of Frank-Wolfe (conditional gradient) algorithm

Convergence rate for smooth and convex case

i Theorem

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex and differentiable. Let $S \subseteq \mathbb{R}^n$ be a closed convex set, and assume that there is a minimizer x^* of f over S ; furthermore, suppose that f is smooth over S with parameter L . The Frank-Wolfe algorithm with step size $\gamma_k = \frac{k-1}{k+1}$ achieves the following convergence after iteration $k > 0$:

$$f(x_k) - f^* \leq \frac{2LR^2}{k+1}$$

where $R = \max_{x,y \in S} \|x - y\|$ is the diameter of the set S .

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$$f(x_k) - f^* \leq \frac{2LR^2}{k+1}$$

where $R = \max_{x,y \in S} \|x - y\|$ is the diameter of the set S .

Proof

1. By L -smoothness of f , we have:

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\leq \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|^2 \\ &= (1 - \gamma_k) \langle \nabla f(x_k), y_k - x_k \rangle + \frac{L(1 - \gamma_k)^2}{2} \|y_k - x_k\|^2 \end{aligned}$$

Convergence rate for smooth and convex case

2. By convexity of f , for any $x \in S$, including x^* :

$$\langle \nabla f(x_k), x - x_k \rangle \leq f(x) - f(x_k)$$

In particular, for $x = x^*$:

$$\langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k)$$

Convergence rate for smooth and convex case

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In particular, for $x = x^*$:

$$\langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k)$$

3. By definition of y_k , we have $\langle \nabla f(x_k), y_k \rangle \leq \langle \nabla f(x_k), x^* \rangle$, thus:

$$\langle \nabla f(x_k), y_k - x_k \rangle \leq \langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k)$$

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$$\langle \nabla f(x_k), y_k - x_k \rangle \leq \langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k)$$

4. Combining the above inequalities:

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\leq (1 - \gamma_k) \langle \nabla f(x_k), y_k - x_k \rangle + \frac{L(1 - \gamma_k)^2}{2} \|y_k - x_k\|^2 \\ &\leq (1 - \gamma_k) (f(x^*) - f(x_k)) + \frac{L(1 - \gamma_k)^2}{2} R^2 \end{aligned}$$

Convergence rate for smooth and convex case

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5. Rearranging terms:

$$f(x_{k+1}) - f(x^*) \leq \gamma_k (f(x_k) - f(x^*)) + (1 - \gamma_k)^2 \frac{LR^2}{2}$$

Convergence rate for smooth and convex case

6. Denoting $\delta_k = \frac{f(x_k) - f(x^*)}{LR^2}$, we get:

$$\delta_{k+1} \leq \gamma_k \delta_k + \frac{(1 - \gamma_k)^2}{2} = \frac{k-1}{k+1} \delta_k + \frac{2}{(k+1)^2}$$

Convergence rate for smooth and convex case

6. Denoting $\delta_k = \frac{f(x_k) - f(x^*)}{LR^2}$, we get:

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7. Starting from $\delta_2 \leq \frac{1}{2}$ and applying induction on k , we can show that:

$$\delta_k \leq \frac{2}{k+1}$$

which gives us the desired result:

$$f(x_k) - f^* \leq \frac{2LR^2}{k+1}$$

Convergence rate for strongly convex case

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Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be μ -strongly convex and differentiable. Let $S \subseteq \mathbb{R}^n$ be a closed convex set, and assume that there is a minimizer x^* of f over S ; furthermore, suppose that f is smooth over S with parameter L . The Frank-Wolfe algorithm with step size $\gamma_k = \frac{2}{k+2}$ achieves the following convergence after iteration $k > 0$:

$$f(x_k) - f^* \leq \frac{4LR^2}{(k+2)^2}$$

where $R = \max_{x,y \in S} \|x - y\|$ is the diameter of the set S .

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$$f(x_k) - f^* \leq \frac{4LR^2}{(k+2)^2}$$

where $R = \max_{x,y \in S} \|x - y\|$ is the diameter of the set S .

Proof

1. By μ -strong convexity of f , for any $x, y \in S$:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2$$

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1. By μ -strong convexity of f , for any $x, y \in S$:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2$$

2. This gives us a stronger inequality than in the convex case:

$$\langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k) - \frac{\mu}{2} \|x^* - x_k\|^2$$

Convergence rate for strongly convex case

3. Following similar steps as in the convex case, but using the stronger inequality:

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\leq (1 - \gamma_k) \langle \nabla f(x_k), y_k - x_k \rangle + \frac{L(1 - \gamma_k)^2}{2} \|y_k - x_k\|^2 \\ &\leq (1 - \gamma_k) \left(f(x^*) - f(x_k) - \frac{\mu}{2} \|x^* - x_k\|^2 \right) + \frac{L(1 - \gamma_k)^2}{2} R^2 \end{aligned}$$

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3. Following similar steps as in the convex case, but using the stronger inequality:

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4. Rearranging terms and using the fact that $\|x^* - x_k\|^2 \geq 0$:

$$\begin{aligned} f(x_{k+1}) - f(x^*) &\leq \gamma_k (f(x_k) - f(x^*)) + (1 - \gamma_k)^2 \frac{LR^2}{2} - (1 - \gamma_k) \frac{\mu}{2} \|x^* - x_k\|^2 \\ &\leq \gamma_k (f(x_k) - f(x^*)) + (1 - \gamma_k)^2 \frac{LR^2}{2} \end{aligned}$$

Convergence rate for strongly convex case

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$$\begin{aligned} f(x_{k+1}) - f(x_k) &\leq (1 - \gamma_k) \langle \nabla f(x_k), y_k - x_k \rangle + \frac{L(1 - \gamma_k)^2}{2} \|y_k - x_k\|^2 \\ &\leq (1 - \gamma_k) \left(f(x^*) - f(x_k) - \frac{\mu}{2} \|x^* - x_k\|^2 \right) + \frac{L(1 - \gamma_k)^2}{2} R^2 \end{aligned}$$

4. Rearranging terms and using the fact that $\|x^* - x_k\|^2 \geq 0$:

$$\begin{aligned} f(x_{k+1}) - f(x^*) &\leq \gamma_k (f(x_k) - f(x^*)) + (1 - \gamma_k)^2 \frac{LR^2}{2} - (1 - \gamma_k) \frac{\mu}{2} \|x^* - x_k\|^2 \\ &\leq \gamma_k (f(x_k) - f(x^*)) + (1 - \gamma_k)^2 \frac{LR^2}{2} \end{aligned}$$

5. With $\gamma_k = \frac{2}{k+2}$ and denoting $\delta_k = f(x_k) - f^*$, we get:

$$\begin{aligned} \delta_{k+1} &\leq \frac{2}{k+2} \delta_k + \frac{LR^2}{2} \left(1 - \frac{2}{k+2}\right)^2 \\ &= \frac{2}{k+2} \delta_k + \frac{LR^2}{2} \frac{(k)^2}{(k+2)^2} \end{aligned}$$

Convergence rate for strongly convex case

6. It can be shown by induction that:

$$\delta_k \leq \frac{4LR^2}{(k+2)^2}$$

Convergence rate for strongly convex case

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7. This gives us the improved convergence rate of $\mathcal{O}(\frac{1}{k^2})$ for the strongly convex case, compared to $\mathcal{O}(\frac{1}{k})$ for the convex case.