



## Dual methods: Dual Gradient Ascent, Augmented Lagrangian Method, ADMM

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Optimization methods. MIPT



## Introduction to dual methods



# Why do we want to solve dual problems?

## Primal problem

$$\begin{aligned} f_0(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t.} \quad &f_i(x) \leq 0, \quad i = 1, \dots, m \\ &h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

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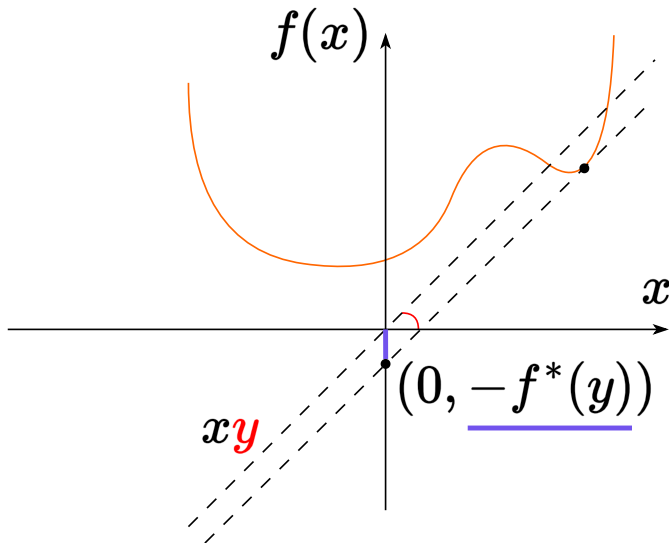
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- **Duality Gap.** The difference between the primal and dual solutions (duality gap) provides valuable information about the solution's optimality.



## Conjugate functions



# Conjugate functions



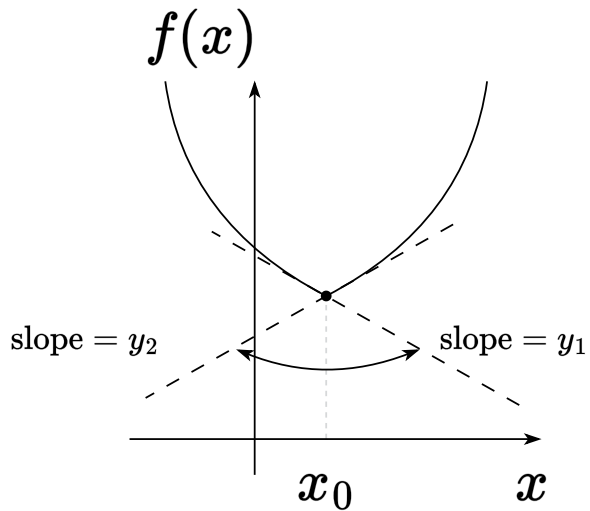
Recall that given  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the function defined by

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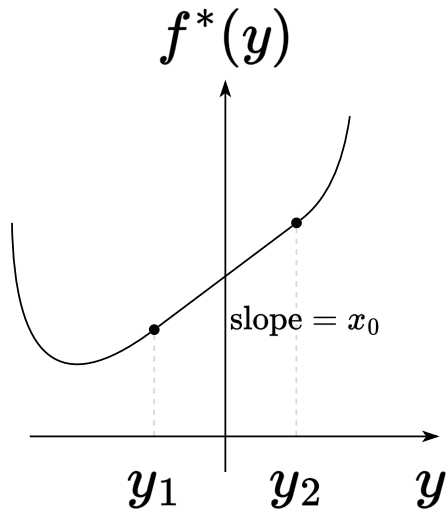
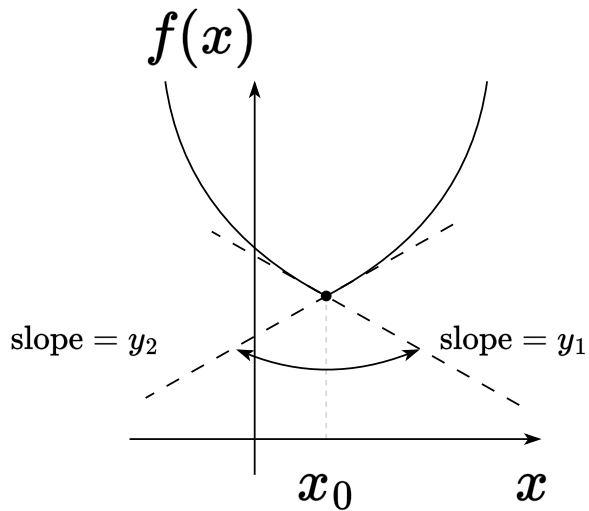


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# Conjugate function properties (proofs)

We will show that  $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$ , assuming that  $f$  is convex and closed.

- **Proof of  $\Leftarrow$ :** Suppose  $y \in \partial f(x)$ . Then  $x \in M_y$ , the set of maximizers of  $y^T z - f(z)$  over  $z$ . But

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Clearly  $y \in \partial f(x) \Leftrightarrow x \in \arg \min_z \{f(z) - y^T z\}$

Lastly, if  $f$  is strictly convex, then we know that  $f(z) - y^T z$  has a unique minimizer over  $z$ , and this must be  $\nabla f^*(y)$ .



## Dual ascent



## Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

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Dual ascent method for maximizing dual objective:

- Step sizes  $\alpha_k$ ,  $k = 1, 2, 3, \dots$ , are chosen in standard ways.

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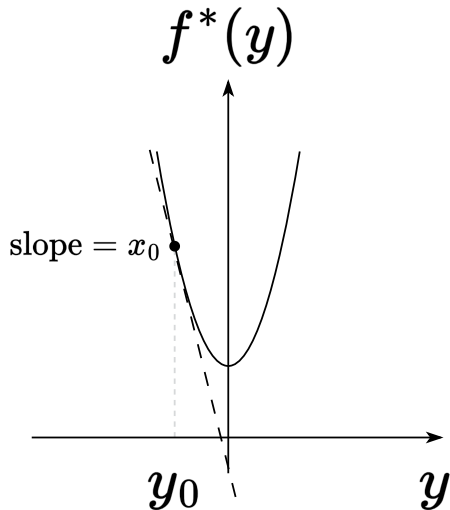
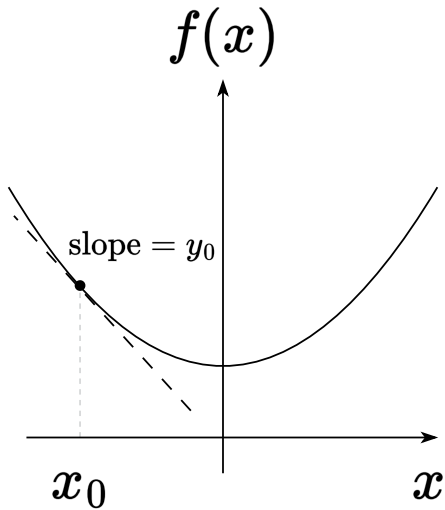
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- Proximal gradients and acceleration can be applied as they would usually.



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Adding these together, using the Cauchy-Schwarz inequality, and rearranging shows that

$$\|x_u - x_v\|^2 \leq \frac{1}{\mu} \|u - v\|^2$$



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Let  $u = \nabla f(x)$ ,  $v = \nabla g(y)$ ; then  $x \in \partial g^*(u)$ ,  $y \in \partial g^*(v)$ , and the above reads  $(x - y)^T (u - v) \geq \frac{\|u - v\|^2}{L}$ , implying the result.



## Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent: (This is ignoring the role of  $A$ , and thus reflects the case when the singular values of  $A$  are all close to 1. To be more precise, the step sizes here should be:  $\frac{\mu}{\sigma_{\max}(A)^2}$  (first case) and  $\frac{2}{\frac{\sigma_{\max}(A)^2}{\mu} + \frac{\sigma_{\min}(A)^2}{L}}$  (second case).)

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- If  $f$  is strongly convex with parameter  $\mu$  and  $\nabla f$  is Lipschitz with parameter  $L$ , then dual gradient ascent with step sizes  $\alpha_k = \frac{2}{\frac{1}{\mu} + \frac{1}{L}}$  converges at linear rate  $O(\log(\frac{1}{\epsilon}))$ .



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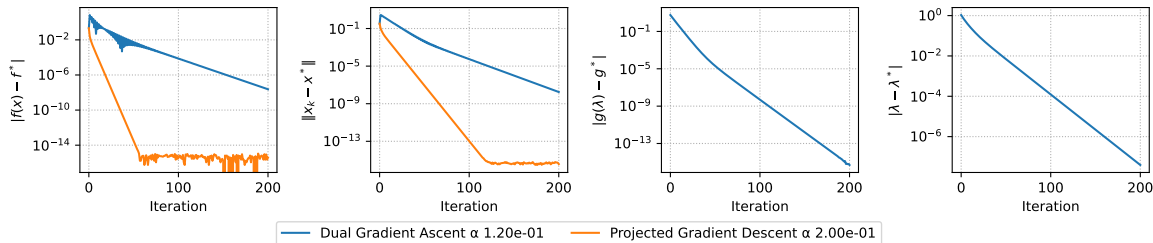
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- Note that this describes convergence in the dual. Convergence in the primal requires more assumptions



## Example: equality constrained quadratic minimization.

$$f(x) = \frac{1}{2}x^T A x - b^T x \rightarrow \min_{x \in \mathbb{R}^n} \quad \text{subject to} \quad Cx = d, \quad A \in \mathbb{S}_+^n, C \in \mathbb{R}^{m \times n}, m < n.$$

Quadratic constrained optimization.  $n=10, m=5, \mu=1, L=10$ .



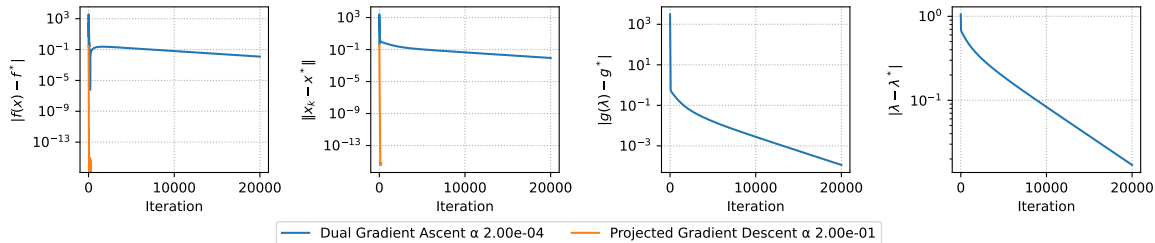
We need to find a minimum of a quadratic function in some linear subspace, defined by the solution of linear equation  $Cx = d$ . This is a conditional optimization problem, we start from strongly convex setting.



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Quadratic constrained optimization.  $n=10, m=5, \mu=0.001, L=10$ .



Situation is getting worse as soon as we loose strong convexity, the dual convergence will still be linear, but the rate is very low.



# Dual decomposition

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Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into  $B$  separate problems:

$$x^{\text{new}} \in \arg \min_x \left( \sum_{i=1}^B f_i(x_i) + u^T Ax \right)$$

$$\Rightarrow x_i^{\text{new}} \in \arg \min_{x_i} (f_i(x_i) + u^T A_i x_i), \quad i = 1, \dots, B$$

$$x_i^k \in \arg \min_{x_i} (f_i(x_i) + (u^{k-1})^T A_i x_i), \quad i = 1, \dots, B$$

$$u^k = u^{k-1} + \alpha_k \left( \sum_{i=1}^B A_i x_i^k - b \right)$$



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$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Here  $x = (x_1, \dots, x_B) \in \mathbb{R}^n$  divides into  $B$  blocks of variables, with each  $x_i \in \mathbb{R}^{n_i}$ . We can also partition  $A$  accordingly:

$$A = [A_1 \dots A_B], \text{ where } A_i \in \mathbb{R}^{m \times n_i}$$

Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into  $B$  separate problems:

$$x^{\text{new}} \in \arg \min_x \left( \sum_{i=1}^B f_i(x_i) + u^T Ax \right)$$

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- **Broadcast:** Send  $u$  to each of the  $B$  processors, each optimizes in parallel to find  $x_i$ .



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$$u^k = \left( u^{k-1} + \alpha_k \left( \sum_{i=1}^B A_i x_i^k - b \right) \right)_+$$

where  $(u)_+$  denotes the positive part of  $u$ , i.e.,  $(u_+)_i = \max\{0, u_i\}$ , for  $i = 1, \dots, m$ .



## Price Coordination Interpretation (Vandenberghe)

- **System Overview:** Consider a system with  $B$  units, where each unit independently chooses its decision variable  $x_i$ , which determines how to allocate its goods.



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- **Never let prices get negative;** hence the use of the positive part notation  $(\cdot)_+$ .



## Augmented Lagrangian method



# Augmented Lagrangian method aka method of multipliers

**Dual ascent disadvantage:** convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

$$\begin{aligned} \min_x \quad & f(x) + \frac{\rho}{2} \|Ax - b\|^2 \\ \text{s.t.} \quad & Ax = b \end{aligned}$$



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**Dual gradient ascent:** The iterative updates are given by:

$$\begin{aligned} x_k &= \arg \min_x \left[ f(x) + (u_{k-1})^T Ax + \frac{\rho}{2} \|Ax - b\|^2 \right] \\ u_k &= u_{k-1} + \rho(Ax_k - b) \end{aligned}$$



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**Notice step size choice  $\alpha_k = \rho$  in dual algorithm. Why?**

Since  $x_k$  minimizes the function:

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over  $x$ , we have the stationarity condition:

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This represents the stationarity condition for the original primal problem; under mild conditions,  $Ax_k - b \rightarrow 0$  as  $k \rightarrow \infty$ , so the KKT conditions are satisfied in the limit and  $x_k, u_k$  converge to the solutions.

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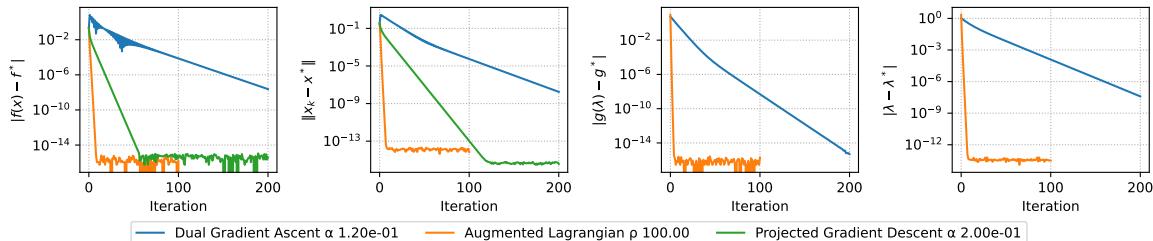
- **Advantage:** The augmented Lagrangian gives better convergence.
- **Disadvantage:** We lose decomposability! (Separability is ruined)



## Example: equality constrained quadratic minimization.

$$f(x) = \frac{1}{2}x^T A x - b^T x \rightarrow \min_{x \in \mathbb{R}^n} \quad \text{subject to} \quad Cx = d, \quad A \in \mathbb{S}_+^n, C \in \mathbb{R}^{m \times n}, m < n.$$

Quadratic constrained optimization.  $n=10, m=5, \mu=1, L=10$ .



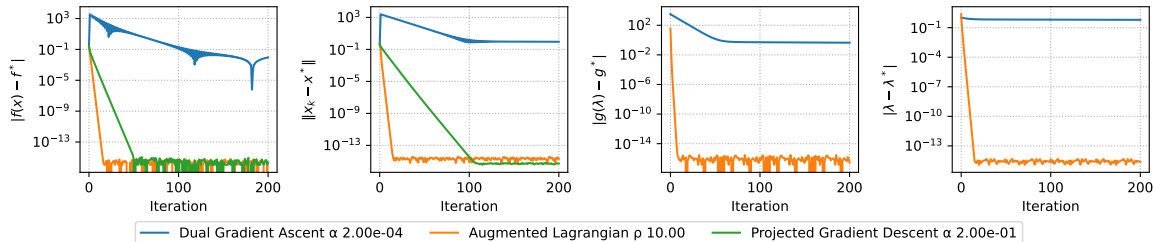
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Quadratic constrained optimization.  $n=10$ ,  $m=5$ ,  $\mu=0.001$ ,  $L=10$ .



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## Introduction to ADMM



# Alternating Direction Method of Multipliers (ADMM)

**Alternating direction method of multipliers** or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

$$\begin{aligned} \min_{x,z} \quad & f(x) + g(z) \\ \text{s.t.} \quad & Ax + Bz = c \end{aligned}$$



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where  $\rho > 0$  is a parameter. The augmented Lagrangian for this problem is defined as:

$$L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2} \|Ax + Bz - c\|^2$$



# Alternating Direction Method of Multipliers (ADMM)

ADMM repeats the following steps, for  $k = 1, 2, 3, \dots$ :

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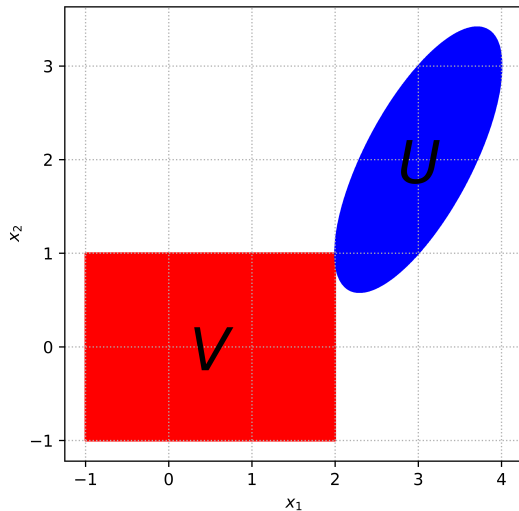
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**Note:** The usual method of multipliers would replace the first two steps by a joint minimization:

$$(x^{(k)}, z^{(k)}) = \arg \min_{x, z} L_\rho(x, z, u^{(k-1)})$$



## Example: Alternating Projections



Consider finding a point in the intersection of convex sets  $U, V \subseteq \mathbb{R}^n$ :

$$\min_x I_U(x) + I_V(x)$$

To transform this problem into ADMM form, we express it as:

$$\min_{x,z} I_U(x) + I_V(z) \quad \text{subject to} \quad x - z = 0$$

Each ADMM cycle involves two projections:

$$x_k = \arg \min_x P_U(z_{k-1} - w_{k-1})$$

$$z_k = \arg \min_z P_V(x_k + w_{k-1})$$

$$w_k = w_{k-1} + x_k - z_k$$



# Sources

- Ryan Tibshirani. Convex Optimization 10-725