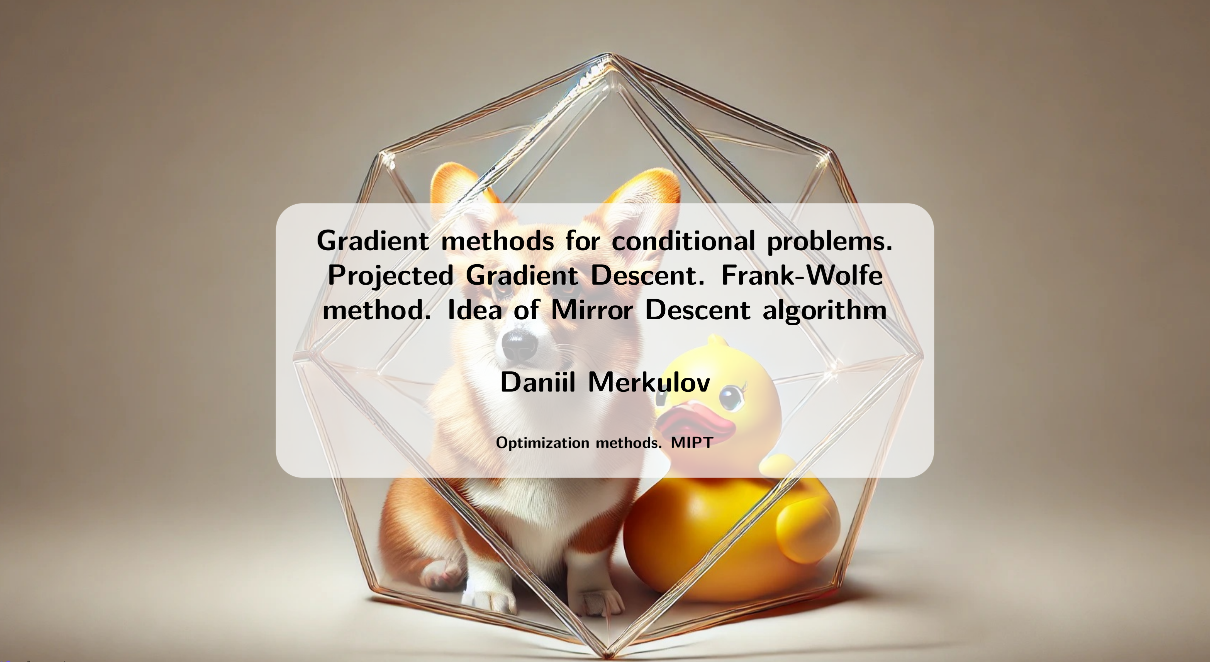

Gradient methods for conditional problems. Projected Gradient Descent. Frank-Wolfe method. Idea of Mirror Descent algorithm

Daniil Merkulov

Optimization methods. MIPT

Conditional methods

Constrained optimization

Unconstrained optimization

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- Any point $x_0 \in \mathbb{R}^n$ is feasible and could be a solution.

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Gradient Descent is a great way to solve unconstrained problem

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k) \tag{GD}$$

Is it possible to tune GD to fit constrained problem?

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Is it possible to tune GD to fit constrained problem?

Yes. We need to use projections to ensure feasibility on every iteration.

Example: White-box Adversarial Attacks

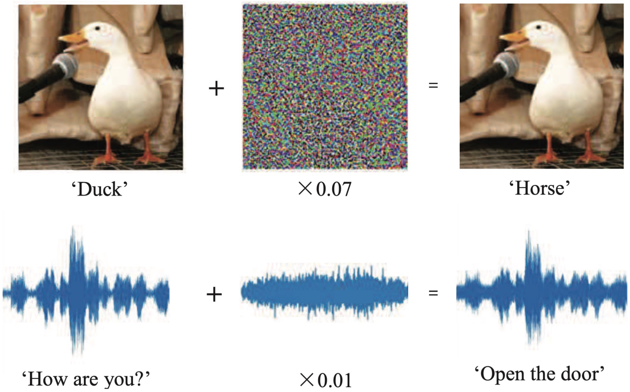

Figure 1: Source

- Mathematically, a neural network is a function $f(w; x)$

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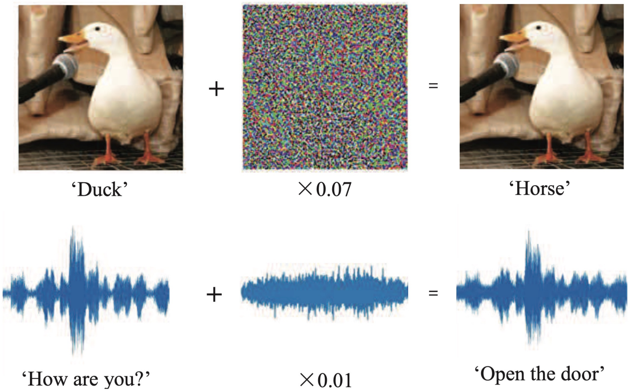

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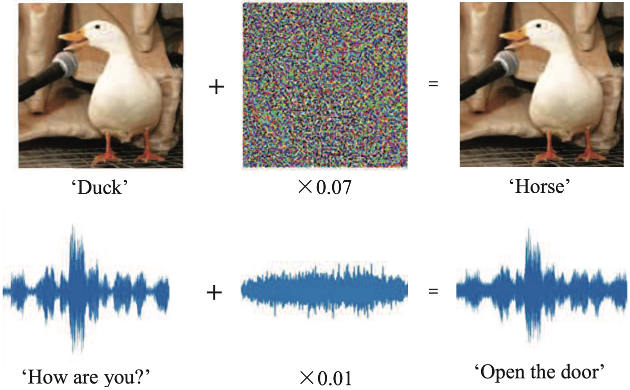

Figure 1: Source

- Mathematically, a neural network is a function $f(w; x)$
- Typically, input x is given and network weights w optimized
- Could also freeze weights w and optimize x, adversarially!

$$\min_{\delta} \text{size}(\delta) \quad \text{s.t.} \quad \text{pred}[f(w; x + \delta)] \neq y$$

or

$$\max_{\delta} l(w; x+\delta, y) \text{ s.t. } \text{size}(\delta) \leq \epsilon,\ 0 \leq x+\delta \leq 1$$

Idea of Projected Gradient Descent

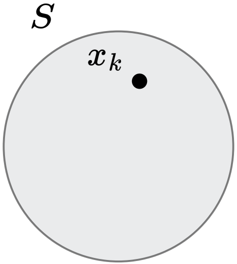

Figure 2: Suppose, we start from a point x_k.

Idea of Projected Gradient Descent

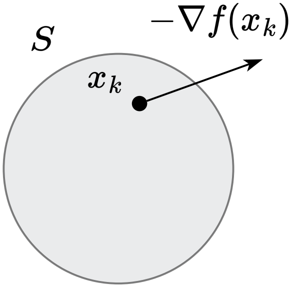

Figure 3: And go in the direction of $-\nabla f(x_k)$.

Idea of Projected Gradient Descent

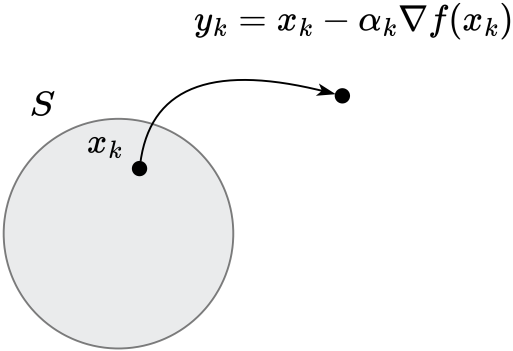

Figure 4: Occasionally, we can end up outside the feasible set.

Idea of Projected Gradient Descent

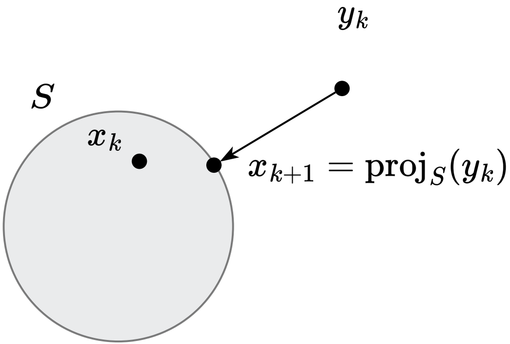

Figure 5: Solve this little problem with projection!

Idea of Projected Gradient Descent

$$x_{k+1} = \text{proj}_S\left(x_k - \alpha_k \nabla f(x_k)\right) \qquad \Leftrightarrow \qquad \begin{aligned} y_k &= x_k - \alpha_k \nabla f(x_k) \\ x_{k+1} &= \text{proj}_S\left(y_k\right) \end{aligned}$$

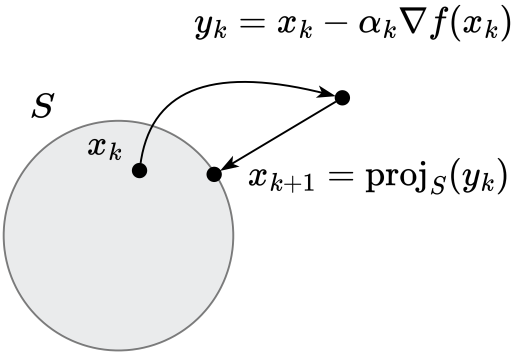

Figure 6: Illustration of Projected Gradient Descent algorithm

Projection

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The distance d from point $\mathbf{y} \in \mathbb{R}^n$ to closed set $S \subset \mathbb{R}^n$:

$$d(\mathbf{y}, S, \|\cdot\|) = \inf\{\|x - y\| \mid x \in S\}$$

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We will focus on Euclidean projection (other options are possible) of a point $\mathbf{y} \in \mathbb{R}^n$ on set $S \subseteq \mathbb{R}^n$ is a point $\text{proj}_S(\mathbf{y}) \in S$:

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- If a point is in set, then its projection is the point itself.

Projection criterion (Bourbaki-Cheney-Goldstein inequality)

Theorem

Let $S \subseteq \mathbb{R}^n$ be closed and convex, $\forall x \in S, y \in \mathbb{R}^n$. Then

$$\langle y - \text{proj}_S(y), \mathbf{x} - \text{proj}_S(y)\rangle \leq 0 \tag{1}$$

$$\|x - \text{proj}_S(y)\|^2 + \|y - \text{proj}_S(y)\|^2 \leq \|x - y\|^2 \tag{2}$$

Proof

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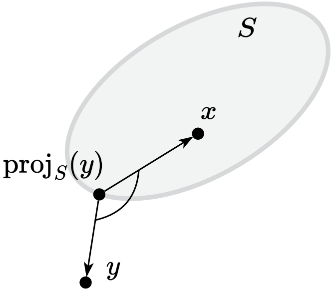

Figure 7: Obtuse or straight angle should be for any point $x \in S$

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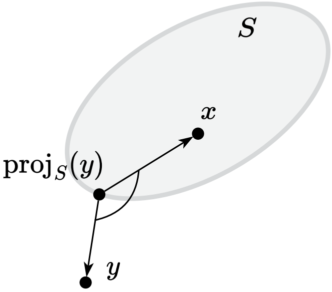

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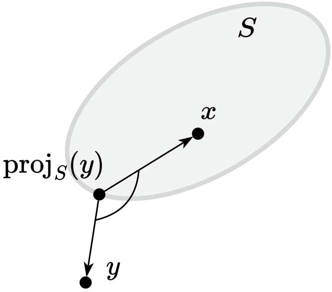

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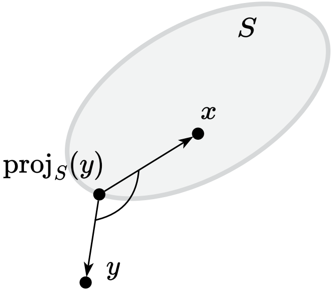

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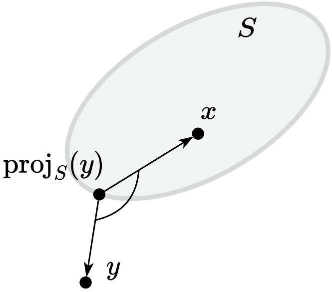

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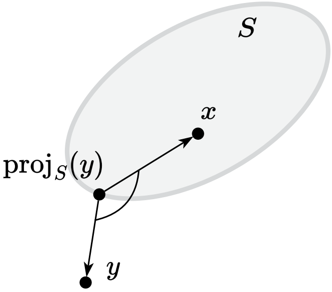

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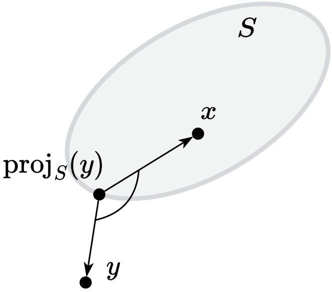

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Projection operator is non-expansive

- A function f is called non-expansive if f is L-Lipschitz with $L \leq 1$ [1]. That is, for any two points $x, y \in \text{dom} f$,

$$\|f(x) - f(y)\| \leq L\|x - y\|, \text{ where } L \leq 1.$$

It means the distance between the mapped points is possibly smaller than that of the unmapped points.

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- Next: variational characterization implies non-expansiveness. i.e.,

$$\langle y - \text{proj}(y), x - \text{proj}(y)\rangle \leq 0 \quad \forall x \in S \qquad \Rightarrow \qquad \|\text{proj}(x) - \text{proj}(y)\|_2 \leq \|x - y\|_2.$$

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Shorthand notation: let $\pi = \text{proj}$ and $\pi(x)$ denotes $\text{proj}(x)$.

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Begins with the variational characterization / obtuse angle inequality

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Replace x by $\pi(x)$ in Equation 3

$$\langle y - \pi(y), \pi(x) - \pi(y) \rangle \leq 0. \tag{4}$$

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Replace y by x and x by $\pi(y)$ in Equation 3

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Replace x by $\pi(x)$ in Equation 3

$$\langle y - \pi(y), \pi(x) - \pi(y) \rangle \leq 0. \tag{4}$$

Replace y by x and x by $\pi(y)$ in Equation 3

$$\langle x - \pi(x), \pi(y) - \pi(x) \rangle \leq 0. \tag{5}$$

(Equation 4)+(Equation 5) will cancel $\pi(y) - \pi(x)$, not good. So flip the sign of (Equation 5) gives

$$\langle \pi(x) - x, \pi(x) - \pi(y) \rangle \leq 0. \tag{6}$$

Projection operator is non-expansive

Shorthand notation: let $\pi = \text{proj}$ and $\pi(x)$ denotes $\text{proj}(x)$.

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$$\begin{aligned}
\langle y - \pi(y) + \pi(x) - x, \pi(x) - \pi(y) \rangle &\leq 0 \\
\langle y - x, \pi(x) - \pi(y) \rangle &\leq -\langle \pi(x) - \pi(y), \pi(x) - \pi(y) \rangle \\
\langle y - x, \pi(y) - \pi(x) \rangle &\geq \|\pi(x) - \pi(y)\|_2^2 \\
\|(y - x)^\top (\pi(y) - \pi(x))\|_2 &\geq \|\pi(x) - \pi(y)\|_2^2
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\langle y - x, \pi(y) - \pi(x)\rangle &\geq \|\pi(x) - \pi(y)\|_2^2 \\
\|(y - x)^\top(\pi(y) - \pi(x))\|_2 &\geq \|\pi(x) - \pi(y)\|_2^2
\end{aligned}$$

By Cauchy-Schwarz inequality, the left-hand-side is upper bounded by $\|y - x\|_2\|\pi(y) - \pi(x)\|_2$, we get $\|y - x\|_2\|\pi(y) - \pi(x)\|_2 \geq \|\pi(x) - \pi(y)\|_2^2$. Cancels $\|\pi(x) - \pi(y)\|_2$ finishes the proof.

Example: projection on the ball

Find $\pi_S(y) = \pi$, if $S = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq R\}$, $y \notin S$

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$$
\begin{aligned}
\left(x_0 - y + R\frac{y - x_0}{\|y - x_0\|}\right)^T \left(x - x_0 - R\frac{y - x_0}{\|y - x_0\|}\right) &= \\
\left(\frac{(y - x_0)(R - \|y - x_0\|)}{\|y - x_0\|}\right)^T \left(\frac{(x - x_0)\|y - x_0\| - R(y - x_0)}{\|y - x_0\|}\right) &= \\
\frac{R - \|y - x_0\|}{\|y - x_0\|^2} (y - x_0)^T \left((x - x_0)\|y - x_0\| - R(y - x_0)\right) &= \\
\frac{R - \|y - x_0\|}{\|y - x_0\|} \left((y - x_0)^T (x - x_0) - R\|y - x_0\|\right) &= \\
(R - \|y - x_0\|) \left(\frac{(y - x_0)^T (x - x_0)}{\|y - x_0\|} - R\right)
\end{aligned}
$$

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$$

$$
\frac{R - \|y - x_0\|}{\|y - x_0\|} \left((y - x_0)^T (x - x_0) - R\|y - x_0\| \right) =
$$

$$
(R - \|y - x_0\|) \left(\frac{(y - x_0)^T(x - x_0)}{\|y - x_0\|} - R \right)
$$

The first factor is negative for point selection y. The second factor is also negative, which follows from the Cauchy-Bunyakovsky inequality:

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$$
\left(x_0 - y + R\frac{y - x_0}{\|y - x_0\|} \right)^T \left(x - x_0 - R\frac{y - x_0}{\|y - x_0\|} \right) =
$$

$$
\left(\frac{(y - x_0)(R - \|y - x_0\|)}{\|y - x_0\|} \right)^T \left(\frac{(x - x_0)\|y - x_0\| - R(y - x_0)}{\|y - x_0\|} \right) =
$$

$$
\frac{R - \|y - x_0\|}{\|y - x_0\|^2} (y - x_0)^T \left((x - x_0) \|y - x_0\| - R\,(y - x_0) \right) =
$$

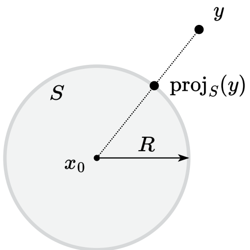
$$
\frac{R - \|y - x_0\|}{\|y - x_0\|} \left((y - x_0)^T (x - x_0) - R\|y - x_0\| \right) =
$$

$$
(R - \|y - x_0\|) \left(\frac{(y - x_0)^T(x - x_0)}{\|y - x_0\|} - R \right)
$$

The first factor is negative for point selection y. The second factor is also negative, which follows from the Cauchy-Bunyakovsky inequality:

$$
(y - x_0)^T(x - x_0) \leq \|y - x_0\|\|x - x_0\|
$$

$$
\frac{(y - x_0)^T(x - x_0)}{\|y - x_0\|} - R \leq \frac{\|y - x_0\|\|x - x_0\|}{\|y - x_0\|} -
$$



Example: projection on the halfspace

Find $\pi_S(y) = \pi$, if $S = \{x \in \mathbb{R}^n \mid c^T x = b\}$, $y \notin S$. Build a hypothesis from the figure: $\pi = y + \alpha c$. Coefficient α is chosen so that $\pi \in S$: $c^T \pi = b$, so:

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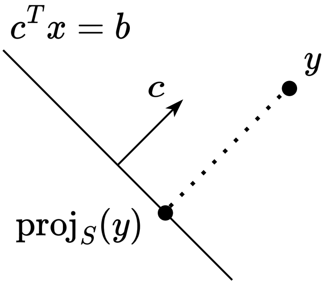

Figure 9: Hyperplane

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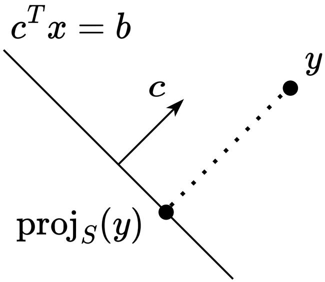

Figure 9: Hyperplane

$$
\begin{aligned}
c^T(y + \alpha c) &= b \\
c^T y + \alpha c^T c &= b \\
c^T y &= b - \alpha c^T c
\end{aligned}
$$

Check the inequality for a convex closed set: $(\pi - y)^T (x - \pi) \geq 0$

$$
\begin{aligned}
(y + \alpha c - y)^T (x - y - \alpha c) &= \\
\alpha c^T (x - y - \alpha c) &= \\
\alpha (c^T x) - \alpha (c^T y) - \alpha^2 (c^T c) &= \\
\alpha b - \alpha (b - \alpha c^T c) - \alpha^2 c^T c &= \\
\alpha b - \alpha b + \alpha^2 c^T c - \alpha^2 c^T c &= 0 \geq 0
\end{aligned}
$$

Projected Gradient Descent (PGD)

Idea

$$x_{k+1} = \text{proj}_S\left(x_k - \alpha_k \nabla f(x_k)\right) \qquad \Leftrightarrow \qquad \begin{aligned} y_k &= x_k - \alpha_k \nabla f(x_k) \\ x_{k+1} &= \text{proj}_S\left(y_k\right) \end{aligned}$$

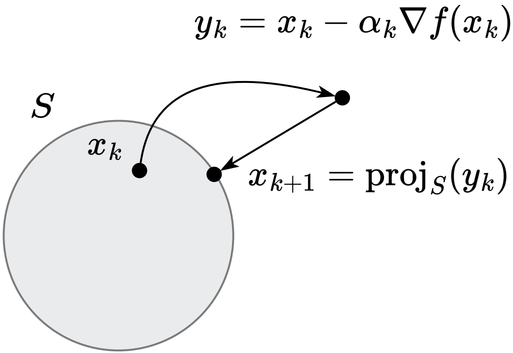

Figure 10: Illustration of Projected Gradient Descent algorithm

Convergence rate for smooth and convex case

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Let $S \subseteq \mathbb{R}^n$d be a closed convex set, and assume that there is a minimizer x^* of f over S; furthermore, suppose that f is smooth over S with parameter L. The Projected Gradient Descent algorithm with stepsize $\frac{1}{L}$ achieves the following convergence after iteration $k > 0$:

$$
f(x_k) - f^* \leq \frac{L\|x_0 - x^*\|_2^2}{2k}
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Proof

1. Let's prove sufficient decrease lemma, assuming, that $y_k = x_k - \frac{1}{L}\nabla f(x_k)$ and cosine rule $2x^T y = \|x\|^2 + \|y\|^2 - \|x - y\|^2$:

(7)

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$$
\text{Smoothness:} \quad f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2}\|x_{k+1} - x_k\|^2 \tag{7}
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$$
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\text{Method:} \quad & \qquad\quad = f(x_k) - L\langle y_k - x_k, x_{k+1} - x_k \rangle + \frac{L}{2}\|x_{k+1} - x_k\|^2
\end{aligned}
\tag{7}
$$

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&\text{Cosine rule:} && = f(x_k) - \frac{L}{2}\left(\|y_k - x_k\|^2 + \|x_{k+1} - x_k\|^2 - \|y_k - x_{k+1}\|^2\right) + \frac{L}{2}\|x_{k+1} - x_k\|^2
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$$
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\text{Cosine rule:} \quad & = f(x_k) - \frac{L}{2}\left(\|y_k - x_k\|^2 + \|x_{k+1} - x_k\|^2 - \|y_k - x_{k+1}\|^2\right) + \frac{L}{2}\|x_{k+1} - x_k\|^2 \\
& = f(x_k) - \frac{1}{2L}\|\nabla f(x_k)\|^2 + \frac{L}{2}\|y_k - x_{k+1}\|^2
\end{aligned}
\tag{7}
$$

Convergence rate for smooth and convex case

2. Now we do not immediately have progress at each step. Let's use again cosine rule:

$$
\left\langle \frac{1}{L}\nabla f(x_k), x_k - x^* \right\rangle = \frac{1}{2}\left(\frac{1}{L^2}\|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|x_k - x^* - \frac{1}{L}\nabla f(x_k)\|^2\right)
$$
$$
\langle \nabla f(x_k), x_k - x^* \rangle = \frac{L}{2}\left(\frac{1}{L^2}\|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|y_k - x^*\|^2\right)
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$$

$$
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$$

3. We will use now projection property: $\|x - \text{proj}_S(y)\|^2 + \|y - \text{proj}_S(y)\|^2 \leq \|x - y\|^2$ with $x = x^*, y = y_k$:

$$
\|x^* - \text{proj}_S(y_k)\|^2 + \|y_k - \text{proj}_S(y_k)\|^2 \leq \|x^* - y_k\|^2
$$

$$
\|y_k - x^*\|^2 \geq \|x^* - x_{k+1}\|^2 + \|y_k - x_{k+1}\|^2
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Convergence rate for smooth and convex case

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$$
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\langle \nabla f(x_k), x_k - x^* \rangle &= \frac{L}{2}\left(\frac{1}{L^2}\|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|y_k - x^*\|^2\right)
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$$

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$$
\begin{aligned}
\|x^* - \text{proj}_S(y_k)\|^2 + \|y_k - \text{proj}_S(y_k)\|^2 &\leq \|x^* - y_k\|^2 \\
\|y_k - x^*\|^2 &\geq \|x^* - x_{k+1}\|^2 + \|y_k - x_{k+1}\|^2
\end{aligned}
$$

4. Now, using convexity and previous part:

Convexity:
$$
\begin{aligned}
f(x_k) - f^* &\leq \langle \nabla f(x_k), x_k - x^* \rangle \\
&\leq \frac{L}{2}\left(\frac{1}{L^2}\|\nabla f(x_k)\|^2 + \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 - \|y_k - x_{k+1}\|^2\right)
\end{aligned}
$$

Sum for $i = 0, k-1$
$$
\sum_{i=0}^{k-1}[f(x_i) - f^*] \leq \sum_{i=0}^{k-1}\frac{1}{2L}\|\nabla f(x_i)\|^2 + \frac{L}{2}\|x_0 - x^*\|^2 - \frac{L}{2}\sum_{i=0}^{i-1}\|y_i - x_{i+1}\|^2
$$

Convergence rate for smooth and convex case

5. Bound gradients with sufficient decrease lemma 7:

$$
\begin{aligned}
\sum_{i=0}^{k-1}[f(x_i) - f^*] &\leq \sum_{i=0}^{k-1}\left[f(x_i) - f(x_{i+1}) + \frac{L}{2}\|y_i - x_{i+1}\|^2\right] + \frac{L}{2}\|x_0 - x^*\|^2 - \frac{L}{2}\sum_{i=0}^{i-1}\|y_i - x_{i+1}\|^2 \\
&\leq f(x_0) - f(x_k) + \frac{L}{2}\sum_{i=0}^{i-1}\|y_i - x_{i+1}\|^2 + \frac{L}{2}\|x_0 - x^*\|^2 - \frac{L}{2}\sum_{i=0}^{i-1}\|y_i - x_{i+1}\|^2 \\
&\leq f(x_0) - f(x_k) + \frac{L}{2}\|x_0 - x^*\|^2 \\
\sum_{i=0}^{k-1} f(x_i) - kf^* &\leq f(x_0) - f(x_k) + \frac{L}{2}\|x_0 - x^*\|^2 \\
\sum_{i=1}^{k}[f(x_i) - f^*] &\leq \frac{L}{2}\|x_0 - x^*\|^2
\end{aligned}
$$

Convergence rate for smooth and convex case

6. From the sufficient decrease inequality

$$f(x_{k+1}) \le f(x_k) - \frac{1}{2L}\|\nabla f(x_k)\|^2 + \frac{L}{2}\|y_k - x_{k+1}\|^2,$$

we use the fact that $x_{k+1} = \text{proj}_S(y_k)$. By definition of projection,

$$\|y_k - x_{k+1}\| \le \|y_k - x_k\|,$$

and recall that $y_k = x_k - \frac{1}{L}\nabla f(x_k)$ implies $\|y_k - x_k\| = \frac{1}{L}\|\nabla f(x_k)\|$. Hence

$$\frac{L}{2}\|y_k - x_{k+1}\|^2 \le \frac{L}{2}\|y_k - x_k\|^2 = \frac{L}{2}\frac{1}{L^2}\|\nabla f(x_k)\|^2 = \frac{1}{2L}\|\nabla f(x_k)\|^2.$$

Substitute back into $(*)$:

$$f(x_{k+1}) \le f(x_k) - \frac{1}{2L}\|\nabla f(x_k)\|^2 + \frac{1}{2L}\|\nabla f(x_k)\|^2 = f(x_k).$$

Hence

$$f(x_{k+1}) \le f(x_k) \quad \text{for each } k,$$

so $\{f(x_k)\}$ is a monotonically nonincreasing sequence.

Convergence rate for smooth and convex case

7. Final convergence bound From step 5, we have already established

$$\sum_{i=0}^{k-1}\left[f(x_i) - f^*\right] \leq \frac{L}{2}\|x_0 - x^*\|_2^2.$$

Since $f(x_i)$ decreases in i, in particular $f(x_k) \leq f(x_i)$ for all $i \leq k$. Therefore

$$k\left[f(x_k) - f^*\right] \leq \sum_{i=0}^{k-1}\left[f(x_i) - f^*\right] \leq \frac{L}{2}\|x_0 - x^*\|_2^2,$$

which immediately gives

$$f(x_k) - f^* \leq \frac{L\|x_0 - x^*\|_2^2}{2k}.$$

This completes the proof of the $\mathcal{O}(\frac{1}{k})$ convergence rate for convex and L-smooth f under projection constraints.

Frank-Wolfe Method

Idea

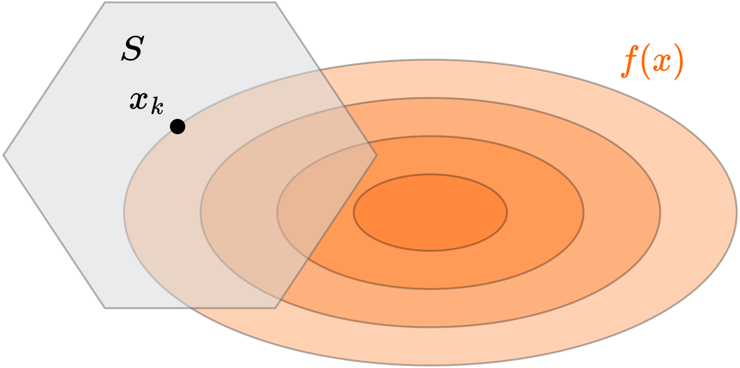

Figure 11: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

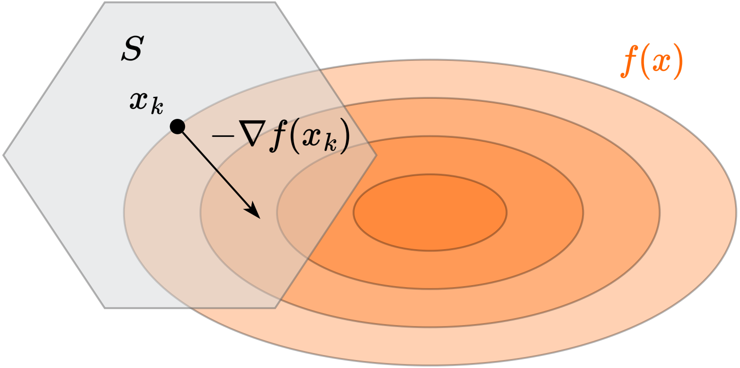

Figure 12: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

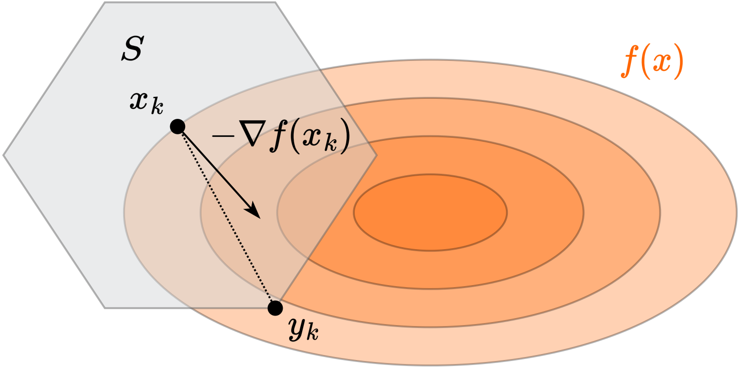

Figure 13: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

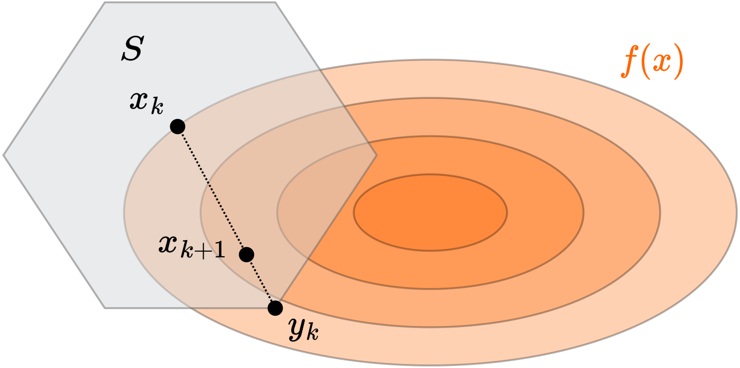

Figure 14: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

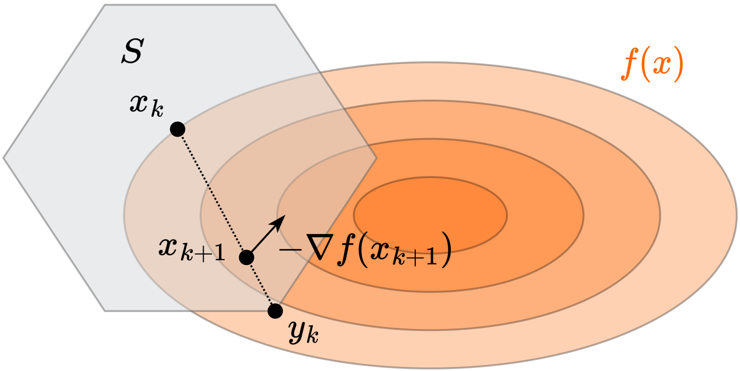

Figure 15: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

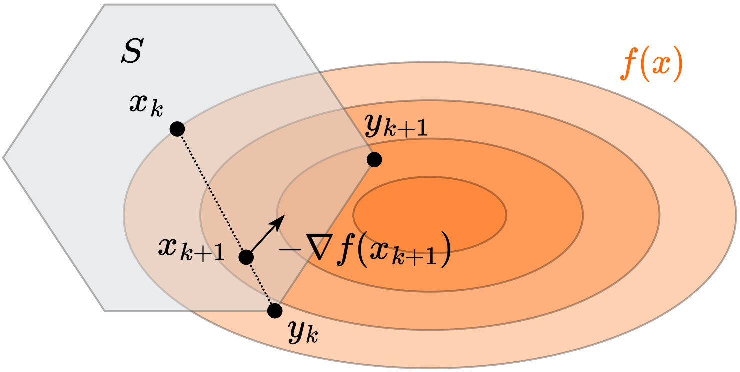

Figure 16: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

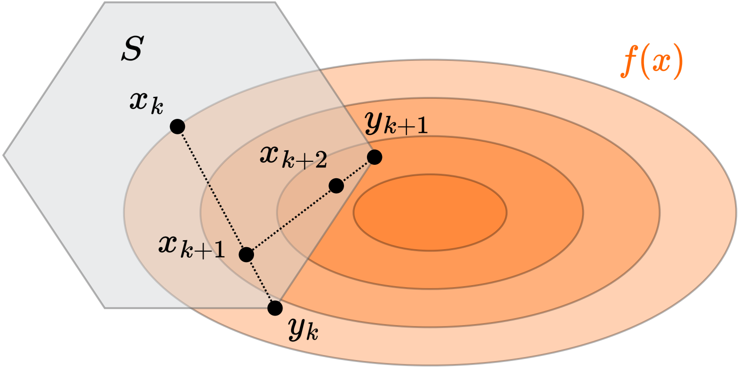

Figure 17: Illustration of Frank-Wolfe (conditional gradient) algorithm

Idea

$$y_k = \arg\min_{x \in S} f^I_{x_k}(x) = \arg\min_{x \in S} \langle \nabla f(x_k), x \rangle$$

$$x_{k+1} = \gamma_k x_k + (1 - \gamma_k) y_k$$

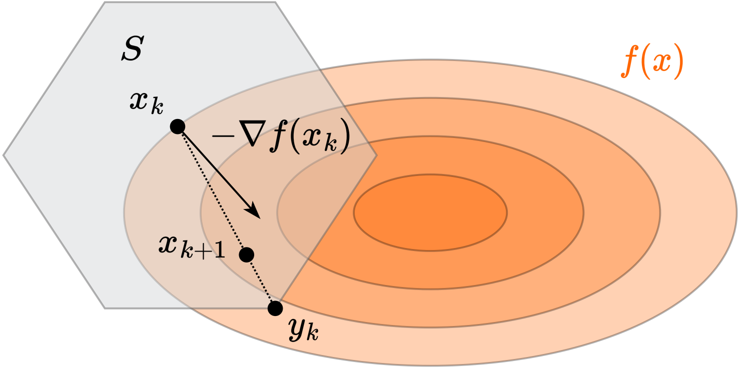

Figure 18: Illustration of Frank-Wolfe (conditional gradient) algorithm

Convergence rate for smooth and convex case

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Let $S \subseteq \mathbb{R}^n$ be a closed convex set, and assume that there is a minimizer x^* of f over S; furthermore, suppose that f is smooth over S with parameter L. The Frank-Wolfe algorithm with step size $\gamma_k = \frac{k-1}{k+1}$ achieves the following convergence after iteration $k > 0$:

$$
f(x_k) - f^* \leq \frac{2LR^2}{k+1}
$$

where $R = \max\limits_{x,y \in S} \|x - y\|$ is the diameter of the set S.

Convergence rate for smooth and convex case

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Let $S \subseteq \mathbb{R}^n$ be a closed convex set, and assume that there is a minimizer x^* of f over S; furthermore, suppose that f is smooth over S with parameter L. The Frank-Wolfe algorithm with step size $\gamma_k = \frac{k-1}{k+1}$ achieves the following convergence after iteration $k > 0$:

$$
f(x_k) - f^* \leq \frac{2LR^2}{k+1}
$$

where $R = \max\limits_{x,y \in S} \|x - y\|$ is the diameter of the set S.

Proof

1. By L-smoothness of f, we have:

$$
\begin{aligned}
f\left(x_{k+1}\right) - f\left(x_k\right) &\leq \left\langle \nabla f\left(x_k\right), x_{k+1} - x_k \right\rangle + \frac{L}{2}\left\|x_{k+1} - x_k\right\|^2 \\
&= (1 - \gamma_k)\left\langle \nabla f\left(x_k\right), y_k - x_k \right\rangle + \frac{L(1-\gamma_k)^2}{2}\left\|y_k - x_k\right\|^2
\end{aligned}
$$

Convergence rate for smooth and convex case

2. By convexity of f, for any $x \in S$, including x^*:

$$\langle \nabla f(x_k), x - x_k \rangle \leq f(x) - f(x_k)$$

In particular, for $x = x^*$:

$$\langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k)$$

Convergence rate for smooth and convex case

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3. By definition of y_k, we have $\langle \nabla f(x_k), y_k \rangle \leq \langle \nabla f(x_k), x^* \rangle$, thus:

$$\langle \nabla f(x_k), y_k - x_k \rangle \leq \langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k)$$

Convergence rate for smooth and convex case

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$$\langle \nabla f(x_k), y_k - x_k \rangle \leq \langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k)$$

4. Combining the above inequalities:

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\leq (1 - \gamma_k) \langle \nabla f(x_k), y_k - x_k \rangle + \frac{L(1-\gamma_k)^2}{2} \|y_k - x_k\|^2 \\ &\leq (1 - \gamma_k)(f(x^*) - f(x_k)) + \frac{L(1-\gamma_k)^2}{2} R^2 \end{aligned}$$

Convergence rate for smooth and convex case

2. By convexity of f, for any $x \in S$, including x^*:

$$\langle \nabla f(x_k), x - x_k \rangle \leq f(x) - f(x_k)$$

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4. Combining the above inequalities:

$$\begin{aligned} f\left(x_{k+1}\right) - f\left(x_k\right) &\leq (1-\gamma_k)\left\langle \nabla f\left(x_k\right), y_k - x_k \right\rangle + \frac{L(1-\gamma_k)^2}{2} \left\|y_k - x_k\right\|^2 \\ &\leq (1-\gamma_k)\left(f(x^*) - f(x_k)\right) + \frac{L(1-\gamma_k)^2}{2} R^2 \end{aligned}$$

5. Rearranging terms:

$$f\left(x_{k+1}\right) - f(x^*) \leq \gamma_k \left(f(x_k) - f(x^*)\right) + (1-\gamma_k)^2 \frac{LR^2}{2}$$

Convergence rate for smooth and convex case

6. Denoting $\delta_k = \frac{f(x_k) - f\left(x^*\right)}{LR^2}$, we get:

$$
\delta_{k+1} \leq \gamma_k \delta_k + \frac{\left(1 - \gamma_k\right)^2}{2} = \frac{k-1}{k+1}\delta_k + \frac{2}{(k+1)^2}
$$

Convergence rate for smooth and convex case

6. Denoting $\delta_k = \frac{f(x_k) - f(x^*)}{LR^2}$, we get:

$$\delta_{k+1} \leq \gamma_k \delta_k + \frac{(1-\gamma_k)^2}{2} = \frac{k-1}{k+1}\delta_k + \frac{2}{(k+1)^2}$$

7. Starting from $\delta_2 \leq \frac{1}{2}$ and applying induction on k, we can show that:

$$\delta_k \leq \frac{2}{k+1}$$

which gives us the desired result:

$$f(x_k) - f^* \leq \frac{2LR^2}{k+1}$$

Convergence rate for strongly convex case

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be μ-strongly convex and differentiable. Let $S \subseteq \mathbb{R}^n$ be a closed convex set, and assume that there is a minimizer x^* of f over S; furthermore, suppose that f is smooth over S with parameter L. The Frank-Wolfe algorithm with step size $\gamma_k = \frac{2}{k+2}$ achieves the following convergence after iteration $k > 0$:

$$
f(x_k) - f^* \leq \frac{4LR^2}{(k+2)^2}
$$

where $R = \max\limits_{x,y \in S} \|x - y\|$ is the diameter of the set S.

Convergence rate for strongly convex case

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be μ-strongly convex and differentiable. Let $S \subseteq \mathbb{R}^n$ be a closed convex set, and assume that there is a minimizer x^* of f over S; furthermore, suppose that f is smooth over S with parameter L. The Frank-Wolfe algorithm with step size $\gamma_k = \frac{2}{k+2}$ achieves the following convergence after iteration $k > 0$:

$$f(x_k) - f^* \leq \frac{4LR^2}{(k+2)^2}$$

where $R = \max\limits_{x,y \in S} \|x - y\|$ is the diameter of the set S.

Proof

1. By μ-strong convexity of f, for any $x, y \in S$:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2$$

Convergence rate for strongly convex case

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be μ-strongly convex and differentiable. Let $S \subseteq \mathbb{R}^n$ be a closed convex set, and assume that there is a minimizer x^* of f over S; furthermore, suppose that f is smooth over S with parameter L. The Frank-Wolfe algorithm with step size $\gamma_k = \frac{2}{k+2}$ achieves the following convergence after iteration $k > 0$:

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where $R = \max\limits_{x,y \in S} \|x - y\|$ is the diameter of the set S.

Proof

1. By μ-strong convexity of f, for any $x, y \in S$:
$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2$$
2. This gives us a stronger inequality than in the convex case:
$$\langle \nabla f(x_k), x^* - x_k \rangle \leq f(x^*) - f(x_k) - \frac{\mu}{2} \|x^* - x_k\|^2$$

Convergence rate for strongly convex case

3. Following similar steps as in the convex case, but using the stronger inequality:

$$
\begin{aligned}
f\left(x_{k+1}\right)-f\left(x_{k}\right) &\leq \left(1-\gamma_{k}\right)\left\langle \nabla f\left(x_{k}\right), y_{k}-x_{k}\right\rangle+\frac{L(1-\gamma_{k})^{2}}{2}\left\|y_{k}-x_{k}\right\|^{2} \\
&\leq \left(1-\gamma_{k}\right)\left(f(x^{*})-f(x_{k})-\frac{\mu}{2}\|x^{*}-x_{k}\|^{2}\right)+\frac{L(1-\gamma_{k})^{2}}{2}R^{2}
\end{aligned}
$$

Convergence rate for strongly convex case

3. Following similar steps as in the convex case, but using the stronger inequality:

$$
\begin{aligned}
f\left(x_{k+1}\right)-f\left(x_k\right) &\leq \left(1-\gamma_k\right)\left\langle\nabla f\left(x_k\right), y_k-x_k\right\rangle+\frac{L(1-\gamma_k)^2}{2}\left\|y_k-x_k\right\|^2 \\
&\leq \left(1-\gamma_k\right)\left(f(x^*)-f(x_k)-\frac{\mu}{2}\|x^*-x_k\|^2\right)+\frac{L(1-\gamma_k)^2}{2}R^2
\end{aligned}
$$

4. Rearranging terms and using the fact that $\|x^*-x_k\|^2 \geq 0$:

$$
\begin{aligned}
f\left(x_{k+1}\right)-f(x^*) &\leq \gamma_k\left(f(x_k)-f(x^*)\right)+\left(1-\gamma_k\right)^2\frac{LR^2}{2}-\left(1-\gamma_k\right)\frac{\mu}{2}\|x^*-x_k\|^2 \\
&\leq \gamma_k\left(f(x_k)-f(x^*)\right)+\left(1-\gamma_k\right)^2\frac{LR^2}{2}
\end{aligned}
$$

Convergence rate for strongly convex case

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$$
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&\leq (1-\gamma_k)\left(f(x^*) - f(x_k) - \frac{\mu}{2}\|x^* - x_k\|^2\right) + \frac{L(1-\gamma_k)^2}{2}R^2
\end{aligned}
$$

4. Rearranging terms and using the fact that $\|x^* - x_k\|^2 \geq 0$:

$$
\begin{aligned}
f\left(x_{k+1}\right)-f(x^*) &\leq \gamma_k\left(f(x_k) - f(x^*)\right) + (1-\gamma_k)^2\frac{LR^2}{2} - (1-\gamma_k)\frac{\mu}{2}\|x^* - x_k\|^2 \\
&\leq \gamma_k\left(f(x_k) - f(x^*)\right) + (1-\gamma_k)^2\frac{LR^2}{2}
\end{aligned}
$$

5. With $\gamma_k = \frac{2}{k+2}$ and denoting $\delta_k = f(x_k) - f^*$, we get:

$$
\begin{aligned}
\delta_{k+1} &\leq \frac{2}{k+2}\delta_k + \frac{LR^2}{2}\left(1 - \frac{2}{k+2}\right)^2 \\
&= \frac{2}{k+2}\delta_k + \frac{LR^2}{2}\frac{(k)^2}{(k+2)^2}
\end{aligned}
$$

Convergence rate for strongly convex case

6. It can be shown by induction that:

$$\delta_k \leq \frac{4LR^2}{(k+2)^2}$$

Convergence rate for strongly convex case

6. It can be shown by induction that:

$$\delta_k \leq \frac{4LR^2}{(k+2)^2}$$

7. This gives us the improved convergence rate of $\mathcal{O}(\frac{1}{k^2})$ for the strongly convex case, compared to $\mathcal{O}(\frac{1}{k})$ for the convex case.