



**Gradient Descent. Convergence for  
quadratics; smooth convex case; PL case**

**Daniil Merkulov**

Optimization methods. MIPT



# Gradient Descent

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The result of this method is

$$x_{k+1} = x_k - \alpha f'(x_k)$$

## Gradient flow ODE

Let's consider the following ODE, which is referred to as the Gradient Flow equation.

$$\frac{dx}{dt} = -f'(x(t)) \quad (\text{GF})$$

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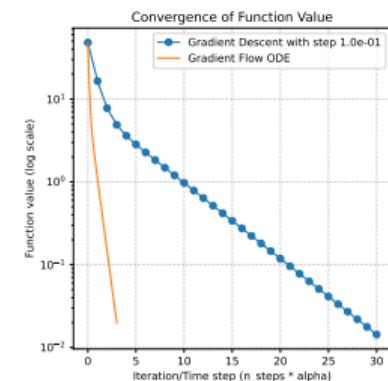
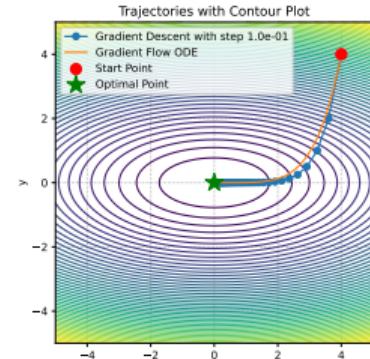
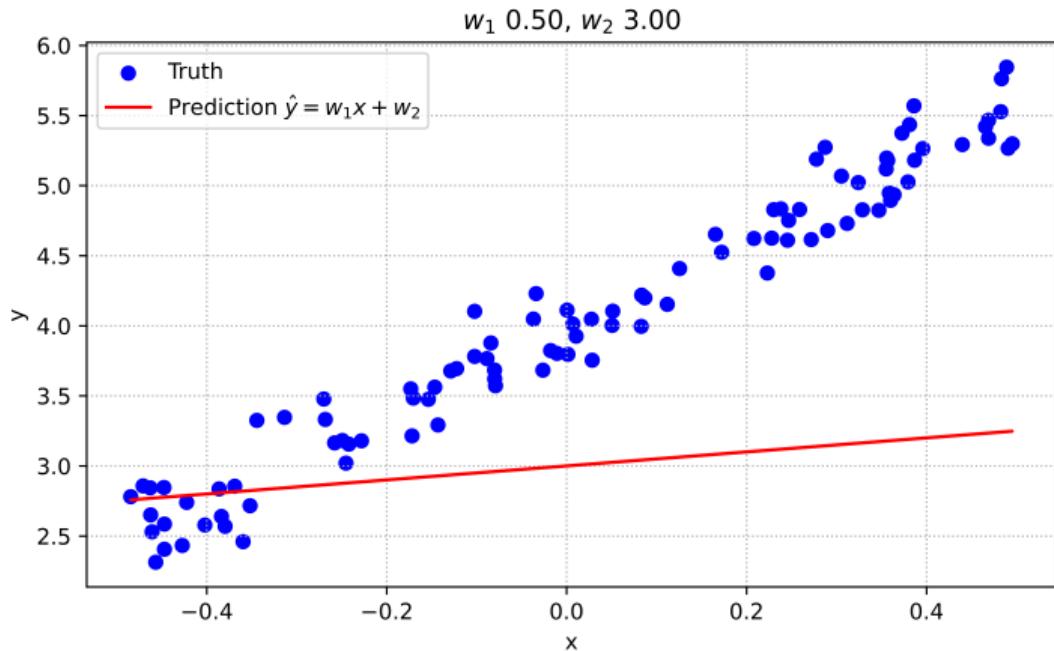
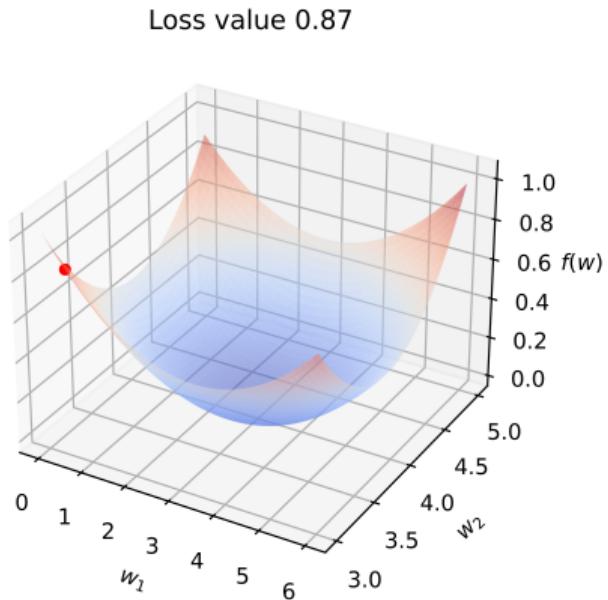


Figure 1: Gradient flow trajectory

# Convergence of Gradient Descent algorithm

Heavily depends on the choice of the learning rate  $\alpha$ :



## Exact line search aka steepest descent

$$\alpha_k = \arg \min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \arg \min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. An interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

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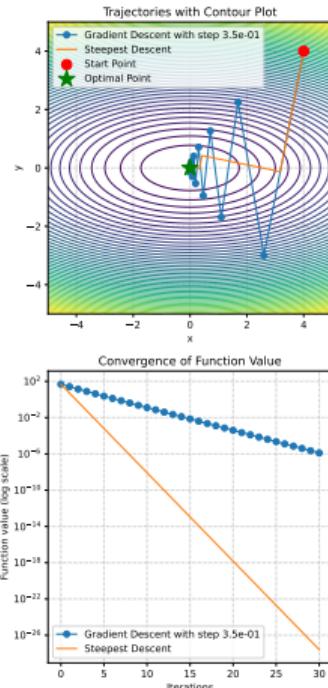


Figure 2: Steepest Descent

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## Strongly convex quadratics

## Coordinate shift

Consider the following quadratic optimization problem:

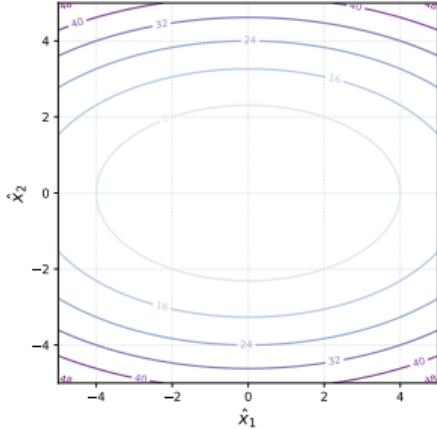
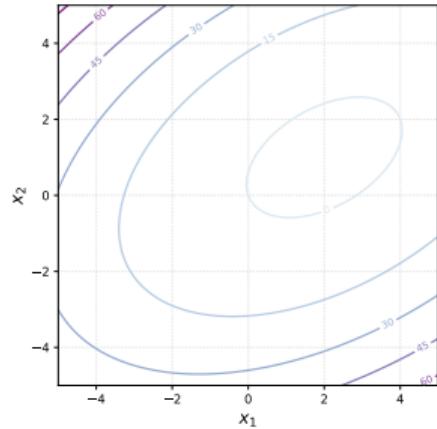
$$\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}_{++}^d.$$

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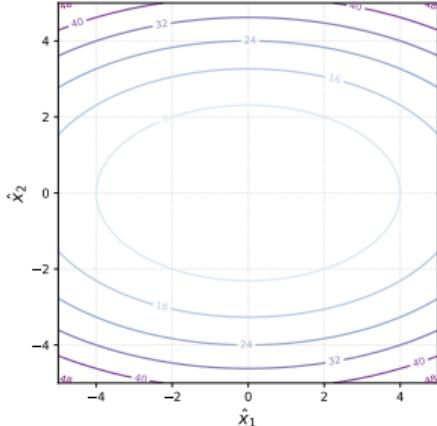
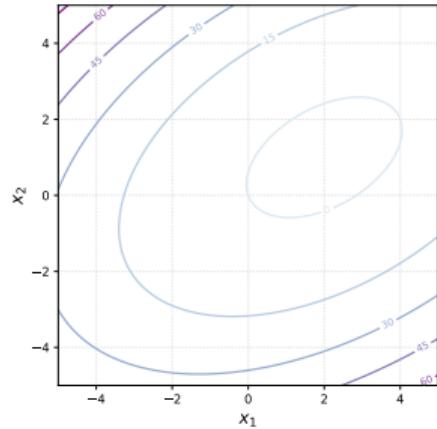
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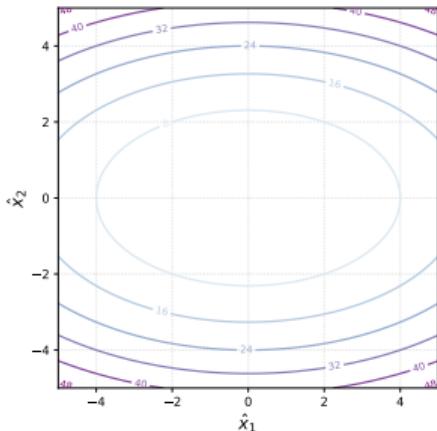
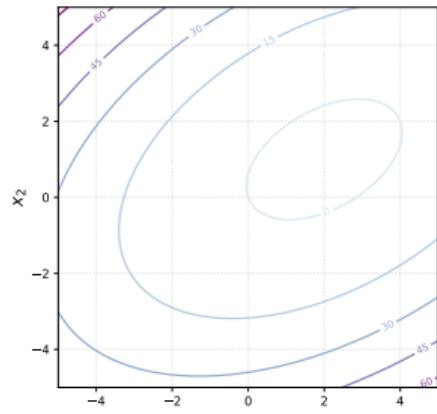
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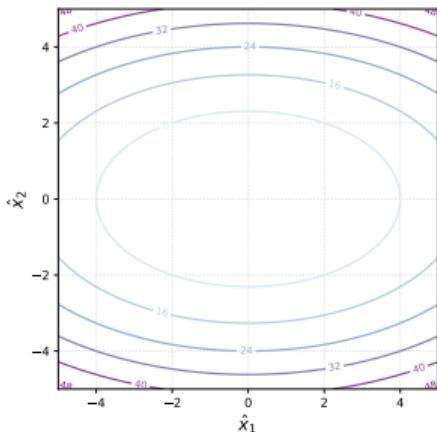
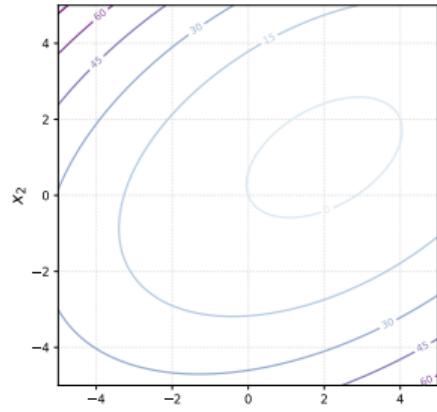
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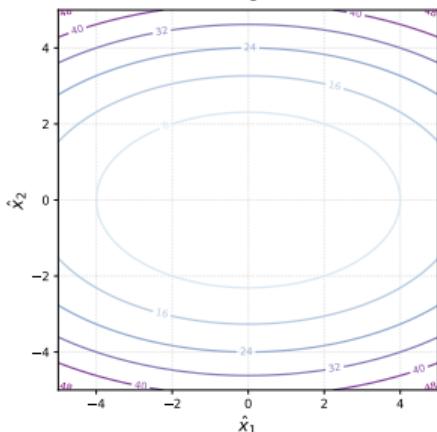
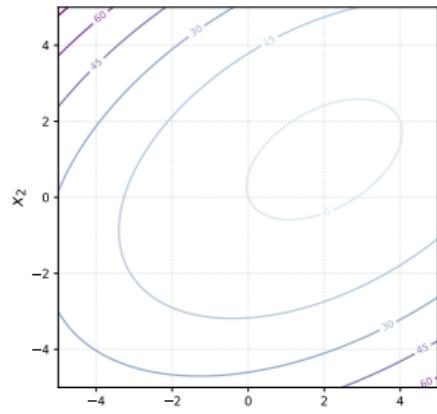
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$$\begin{aligned} f(\hat{x}) &= \frac{1}{2}(Q\hat{x} + x^*)^\top A(Q\hat{x} + x^*) - b^\top(Q\hat{x} + x^*) \\ &= \frac{1}{2}\hat{x}^T Q^T A Q \hat{x} + (x^*)^T A Q \hat{x} + \frac{1}{2}(x^*)^T A (x^*)^T - b^T Q \hat{x} - b^T x^* \end{aligned}$$



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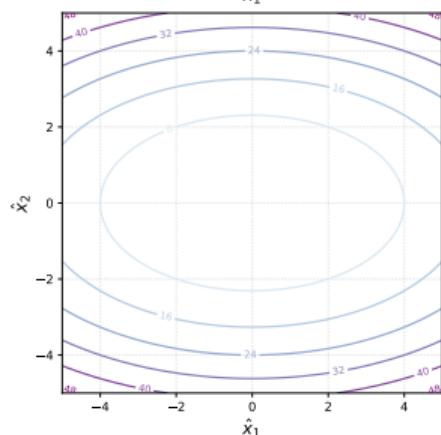
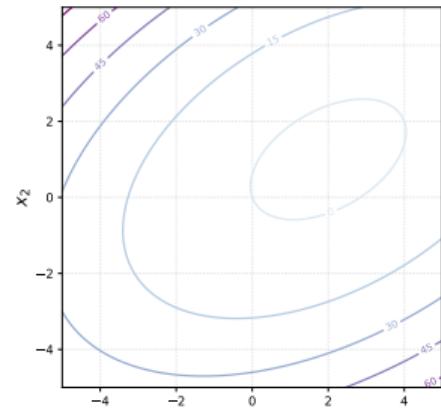
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## Convergence analysis

Now we can work with the function  $f(x) = \frac{1}{2}x^T \Lambda x$  with  $x^* = 0$  without loss of generality (drop the hat from the  $\hat{x}$ )

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$$\rho(\alpha) = \max_i |1 - \alpha \lambda_{(i)}| < 1$$

Remember, that  $\lambda_{\min} = \mu > 0, \lambda_{\max} = L \geq \mu$ .

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## Convergence analysis

So, we have a linear convergence in the domain with rate  $\frac{\kappa-1}{\kappa+1} = 1 - \frac{2}{\kappa+1}$ , where  $\kappa = \frac{L}{\mu}$  is sometimes called *condition number* of the quadratic problem.

$\kappa$	$\rho$	Iterations to decrease domain gap 10 times	Iterations to decrease function gap 10 times
1.1	0.05	1	1
2	0.33	3	2
5	0.67	6	3
10	0.82	12	6
50	0.96	58	29
100	0.98	116	58
500	0.996	576	288
1000	0.998	1152	576

## Polyak-Łojasiewicz smooth case

## Polyak-Lojasiewicz condition. Linear convergence of gradient descent without convexity

PL inequality holds if the following condition is satisfied for some  $\mu > 0$ ,

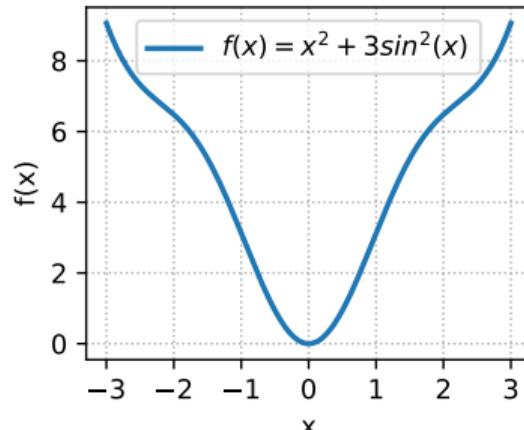
$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*) \quad \forall x$$

It is interesting, that the Gradient Descent algorithm might converge linearly even without convexity.

The following functions satisfy the PL condition but are not convex.  [Link to the code](#)

$$f(x) = x^2 + 3\sin^2(x)$$

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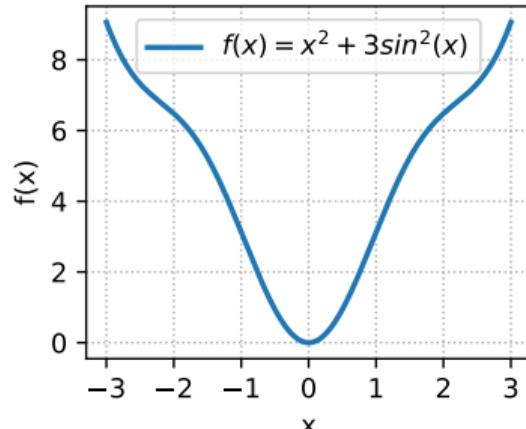
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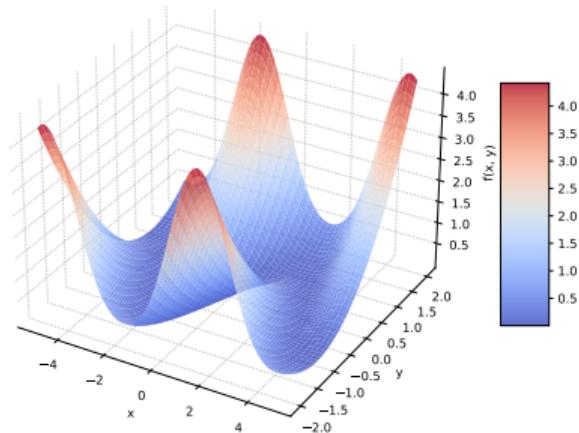
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$$f(x, y) = \frac{(y - \sin x)^2}{2}$$

Non-convex PL function



# Convergence analysis

## i Theorem

Consider the Problem

$$f(x) \rightarrow \min_{x \in \mathbb{R}^d}$$

and assume that  $f$  is  $\mu$ -Polyak-Lojasiewicz and  $L$ -smooth, for some  $L \geq \mu > 0$ .

Consider  $(x^k)_{k \in \mathbb{N}}$  a sequence generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying  $0 < \alpha \leq \frac{1}{L}$ . Then:

$$f(x^k) - f^* \leq (1 - \alpha\mu)^k (f(x^0) - f^*).$$

## Convergence analysis

We can use  $L$ -smoothness, together with the update rule of the algorithm, to write

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2$$

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We can now use the Polyak-Lojasiewicz property to write:

$$f(x^{k+1}) \leq f(x^k) - \alpha \mu (f(x^k) - f^*).$$

The conclusion follows after subtracting  $f^*$  on both sides of this inequality and using recursion.

## Any $\mu$ -strongly convex differentiable function is a PL-function

### i Theorem

If a function  $f(x)$  is differentiable and  $\mu$ -strongly convex, then it is a PL function.

### Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} \|y - x\|_2^2$$

Putting  $y = x^*$ :

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If a function  $f(x)$  is differentiable and  $\mu$ -strongly convex, then it is a PL function.

## Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} \|y - x\|_2^2$$

Putting  $y = x^*$ :

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which is exactly the PL condition. It means, that we already have linear convergence proof for any strongly convex function.

## Smooth convex case

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Consider the Problem

$$f(x) \rightarrow \min_{x \in \mathbb{R}^d}$$

and assume that  $f$  is convex and  $L$ -smooth, for some  $L > 0$ .

Let  $(x^k)_{k \in \mathbb{N}}$  be the sequence of iterates generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying  $0 < \alpha \leq \frac{1}{L}$ . Then, for all  $x^* \in \operatorname{argmin} f$ , for all  $k \in \mathbb{N}$  we have that

$$f(x^k) - f^* \leq \frac{\|x^0 - x^*\|^2}{2\alpha k}.$$

## Convergence analysis

- As it was before, we first use smoothness:

$$\begin{aligned} f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2 \\ &= f(x^k) - \alpha \|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2} \|\nabla f(x^k)\|^2 \\ &= f(x^k) - \frac{\alpha}{2} (2 - L\alpha) \|\nabla f(x^k)\|^2 \\ &\leq f(x^k) - \frac{\alpha}{2} \|\nabla f(x^k)\|^2, \end{aligned} \tag{1}$$
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$$kf(x^k) \leq \sum_{i=0}^{k-1} f(x^{i+1})$$

- Now putting it to Equation 3:

$$\begin{aligned} 2\alpha kf(x^k) - 2\alpha kf^* &\leq 2\alpha \sum_{i=0}^{k-1} (f(x^{i+1}) - f^*) \leq \|x^0 - x^*\|_2^2 \\ f(x^k) - f^* &\leq \frac{\|x^0 - x^*\|_2^2}{2\alpha k} \end{aligned}$$

## Convergence analysis

- Due to the monotonic decrease at each iteration  $f(x^{i+1}) < f(x^i)$ :

$$kf(x^k) \leq \sum_{i=0}^{k-1} f(x^{i+1})$$

- Now putting it to Equation 3:

$$\begin{aligned} 2\alpha kf(x^k) - 2\alpha kf^* &\leq 2\alpha \sum_{i=0}^{k-1} (f(x^{i+1}) - f^*) \leq \|x^0 - x^*\|_2^2 \\ f(x^k) - f^* &\leq \frac{\|x^0 - x^*\|_2^2}{2\alpha k} \leq \frac{L\|x^0 - x^*\|_2^2}{2k} \end{aligned}$$