



**Gradient Flow. Accelerated gradient flow.**

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Optimization methods. MIPT

## Gradient Flow

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$$\frac{dx}{dt} = -\nabla f(x)$$

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$k = 100$



- **Simplified analyses.** The gradient flow has no step-size, so all the traditional annoying issues regarding the choice of step-size, with line-search, constant, decreasing or with a weird schedule are unnecessary.

Рис. 1: Source

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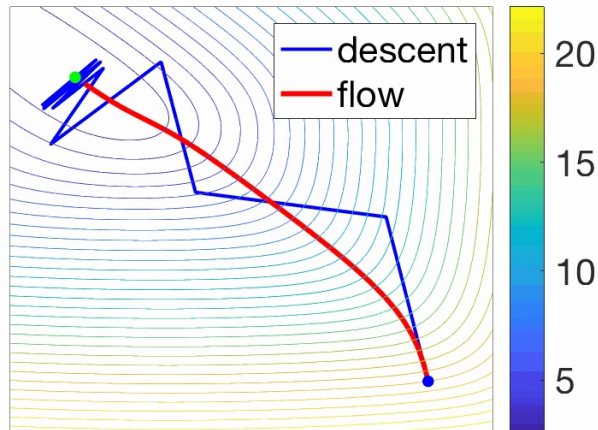


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- **Analytical solution in some cases.** For example, one can consider quadratic problem with linear gradient, which will form a linear ODE with known exact formula.
- **Different discretization leads to different methods.** We will see, that the continuous-time object is pretty rich in terms of the variety of produced algorithms. Therefore, it is interesting to study optimization from this perspective.

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(PPM)



## Convergence analysis. Convex case.

1. Simplest proof of monotonic decrease of GF:

$$\frac{d}{dt} f(x(t)) = \nabla f(x(t))^\top \frac{dx(t)}{dt} = -\|\nabla f(x(t))\|_2^2 \leq 0.$$

If  $f$  is bounded from below, then  $f(x(t))$  will always converge as a non-increasing function which is bounded from below. It is straightforward, that GF converges to the stationary point, where  $\nabla f = 0$  (potentially including minima, maxima and saddle points).

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$$f(x(t)) - f^* \leq \frac{1}{t} \int_0^t [f(x(u)) - f^*] du \leq \frac{1}{2t} \|x(0) - x^*\|^2 - \frac{1}{2t} \|x(t) - x^*\|^2 \leq \frac{1}{2t} \|x(0) - x^*\|^2.$$

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We recover the usual rates in  $\mathcal{O}\left(\frac{1}{k}\right)$ , with  $t = \alpha k$ .

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3. Finally,

$$f(x(t)) - f^* \leq \exp(-2\mu t)[f(x(0)) - f^*],$$

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Remember one of the forms of Nesterov Accelerated Gradient

$$\begin{aligned}x_{k+1} &= y_k - \alpha \nabla f(y_k) \\ y_k &= x_k + \frac{k-1}{k+2}(x_k - x_{k-1})\end{aligned}$$

The corresponding <sup>1</sup> ODE is:

$$\ddot{X}_t + \frac{3}{t}\dot{X}_t + \nabla f(X_t) = 0$$

---

<sup>1</sup>A Differential Equation for Modeling Nesterov's Accelerated Gradient Method: Theory and Insights, Weijie Su, Stephen Boyd, Emmanuel J. Candes

# Accelerated Gradient Flow

Define the *energy*

$$E(t) = t^2(f(X(t)) - f^*) + 2\left\|X(t) - x^* + \frac{t}{2}\dot{X}(t)\right\|^2.$$

A direct differentiation using the ODE yields  $\dot{E}(t) \leq 0$  for all  $t > 0$ ; hence  $E(t)$  is non-increasing. Because the second term is non-negative we obtain the *convergence theorem*

$$\boxed{f(X(t)) - f^* \leq \frac{2\|x_0 - x^*\|^2}{t^2}}. \quad (\text{AGF-rate})$$

Thus AGF enjoys the same  $\mathcal{O}(1/t^2)$  rate that discrete NAG achieves in  $\mathcal{O}(1/k^2)$  iterations. A similar argument with a *restarted* ODE gives an exponential rate for  $\mu$ -strongly convex  $f$ .

## Stochastic Gradient Flow

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How to model stochasticity in the continuous process? A simple idea would be:  $\frac{dx}{dt} = -\nabla f(x) + \xi$  with variety of options for  $\xi$ , for example  $\xi \sim \mathcal{N}(0, \sigma^2) \sim \sigma^2 \mathcal{N}(0, 1)$ .

Therefore, one can write down Stochastic Differential Equation (SDE) for analysis:

$$dx(t) = -\nabla f(x(t)) dt + \sigma dW(t)$$

Here  $W(t)$  is called Wiener process. It is interesting, that one could analyze the convergence of the stochastic process above in two possible ways:

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! Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \nabla (\rho(t) \nabla f) + \frac{\sigma^2}{2} \Delta \rho(t)$$

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- Off convex Path blog
- Stochastic gradient algorithms from ODE splitting perspective
- NAG-GS: Semi-Implicit, Accelerated and Robust Stochastic Optimizer
- Introduction to Gradient Flows in the 2-Wasserstein Space
- Stochastic Modified Equations and Dynamics of Stochastic Gradient Algorithms I: Mathematical Foundations

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- Understanding Optimization in Deep Learning with Central Flows