Problem Sheet 2

Useful facts

- **Expected value**. For a discrete random variable with probability $p(x)$ this is

$$\langle g(X)\rangle = \sum_x p(x)g(x) \tag{1}$$

For a continuous random variable with density $f(x)$ this is

$$\langle g(X)\rangle = \int_{-\infty}^{\infty} f(x)g(x)dx \tag{2}$$

- **Mean and variance**. The mean is $\mu = \langle X\rangle$ and the variance is $\sigma^2 = \langle (X-\mu)^2\rangle = \langle X^2\rangle - \mu^2$.

- **Binomial distibution**. For n independent trials each with p change of success and $q = 1-p$ of failure, the probability of r successes is

$$p(r) = \left(\begin{array}{c} n \\ r \end{array}\right) p^r q^{n-r} \tag{3}$$

and $\mu = pn$, $\sigma^2 = pqn$.

- **Poisson distribution**. This has

$$p(r) = \frac{\lambda^r}{r!}e^{-\lambda} \tag{4}$$

where $\mu = \lambda$ and $\sigma^2 = \lambda$.

- **The limit of infinite compounding**

$$\left(1 - \frac{x}{n}\right)^n \to e^{-x} \tag{5}$$

as $n \to \infty$.

- **Integrating a polnomial**

$$\int x^n dx = \frac{x^{n+1}}{n+1} \tag{6}$$

so the definite integral is

$$\int_a^b x^n dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} \tag{7}$$

Questions

Four questions, each worth two marks with two marks for attendance.

1. The illusionist Derren Brown famously flipped a coin on camera so that it landed heads ten times in a row; he claimed that this was because of his mind powers, in fact it was because of his patience, he simply kept trying the trick again and again until it worked. It took him nine hours. What is the probability of a coin landing heads ten times in a row? If you flip a coin ten times what is the probability of getting five heads and five tails?

2. A fisher catches on average one fish every 25 minutes. What is the probability that they catch no fish in an hour?

3. The distribution of tree heights in a christmas tree forest is

$$p(h) = \begin{cases} 0.3 & 0 \leq h < 2 \\ 0.2 & 2 \leq h < 4 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

   What is the mean height of trees in the forest?

4. Like the binomial distribution the geometric probability distribution is related to a series of independent trials where each trial has probability p of success and $q = 1 - p$ of failure. The geometric probability $p(r)$ is the probability that the rth trial is the first success. It is

$$p(r) = q^{r-1}p \tag{9}$$

   It can be shown that

$$\sum_{r=1}^{\infty} p(r) = 1 \tag{10}$$

   as it must be. You can assume that here. What is the mean of the geometric probability?

Extra questions

These are for you to do on your own, not for handing up.

1. Oranmore, the village I grew up in had more people with the surname Furey than any other village or town in the world. Since I left the village has expanded ten-fold and has gone from being a small village to a commuter town for Galway. However, when I was young one in ten people in the village had the surname Furey. Imagine there are 35 children in a class at school, what is the probability that five of them were Fureys?

2. The **Fano factor** is sometimes used to describe distributions, it is

$$F = \frac{\sigma^2}{\mu} \tag{11}$$

   What is the Fano factor for the Poisson distribution?

3. The aim of this question is to use a similar argument to the one used to derive the Poisson distribution to work out the distribution for the time before the next event in a homogenous Poisson process. In the fishing example, it is the probability density for the time until the next fish is caught.

   a) Say the next event occurs at a time T, divide this up into subinterval $T = n\delta t$ and write down the probability P that there is no event in each of the first $n - 1$ subintervals and an event in the final subinterval where $\lambda\delta t$ is the probability of an event in a δt long subinterval.

   b) Now if the probability is related to the density by $P \approx p\delta t$ take the limit of p as n goes to infinity.