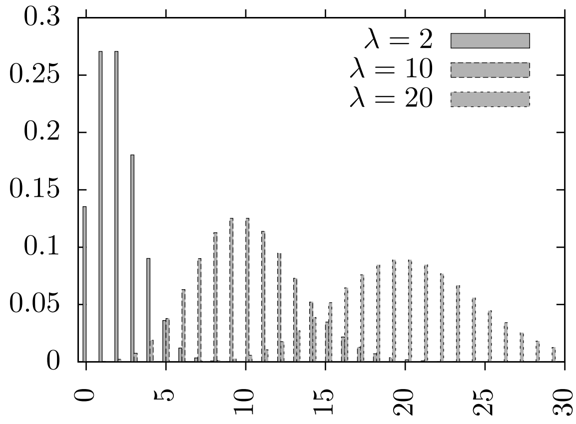

Figure 1: The Poisson distribution for a variety of values of λ. As λ increases the peak moves to the right.

6 Poisson distribution

Imagine someone fishing in a large lake; say the lake is so large that catching one fish doesn't effect the chance they'll catch another. Imagine too that the fishing conditions don't change day-by-day or hour-by-hour. Say we know how many fish they catch on average an hour, say five, but we want to estimate the probability that they catch four. In other words we know the average rate they catch fish, but the actual event, the fish chancing upon the lure and getting hooked, is random, and we want to know the distribution of this event count over an interval. This distribution is an example of the Poisson distribution and is the subject of this section.[1]

Deriving the Poisson distribution

The Poisson distribution can be derived from the bionomial distribution, it just requires a nerve-wracking limit. Imagine slicing time up into small slices, each so small there is a vanishingly small chance of two events happening and that the change of one event is p. Thus, $P(\text{one fish}) = p$, $P(\text{no fishes}) = 1 - p$ and $P(\text{more than one fishes}) = 0$. Of course this is nonsense, if there can be one fish there is some change of two, but we are going to take the limit where the time interval becomes zero so this doesn't matter. Call the small interval width δt.

Now if we are interested in the probability distribution for the number of events in an interval

[1] This joke, pretending the Poisson distribution is so named because it is related to fish, rather than because it is named after the mathematician Siméon Denis Poisson, I stole from Eddie Wilson in Engineering Mathematics. Poisson wrote about the Poisson distribution in a book about how to estimate the number of wrongful convictions; the distribution became well known after Ladislaus Bortkiewicz used it in an investigation of how many Prussian soldiers each year were killed by horse kicks. Both these examples are like the fishing example, you want to study the relationship between the rate of something happening and the distribution of different numbers of occurrences.

$T = n\delta t$, then, by the binomial distribution the probability of r events is

$$p(r) = \begin{pmatrix} n \\ r \end{pmatrix} p^r (1-p)^{n-r} \tag{1}$$

Now write $\delta t = T/n$ and consider the $n \to \infty$ limit:

$$p(r) = \lim_{n\to\infty} \begin{pmatrix} n \\ r \end{pmatrix} p^r (1-p)^{n-r} \tag{2}$$

Since p is the probability of an event in the small interval, it will become tiny as n becomes large, so, to deal with quantities that remain useful in the limit, let $\lambda = np$. Subsituting this in, and expanding out the binomial coefficient:

$$p(r) = \lim_{n\to\infty} \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} \left(\frac{\lambda}{n}\right)^r \left(1-\frac{\lambda}{n}\right)^{n-r} \tag{3}$$

As n gets large the numerator of the first fraction just looks like n^r and cancels with the denominator of the second fraction. Recall that

$$\lim_{n\to\infty} \left(1-\frac{x}{n}\right)^n = e^{-x} \tag{4}$$

The $(1-\lambda/n)^{n-r}$ term has an extra $-r$ but

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-r} = 1 \tag{5}$$

Putting all this together we get

$$p(r) = \frac{\lambda^r}{r!} e^{-\lambda} \tag{6}$$

First, it is easy to check that the probabilities add to one; but notice that this range of the random variable is infinite! Using the Taylor expansion of the exponential:

$$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!} \tag{7}$$

we have

$$\sum_{r=0}^{\infty} \frac{\lambda^r}{r!} e^{-\lambda} = e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} = e^{-\lambda} e^{\lambda} = 1 \tag{8}$$

Next consider the mean

$$\mu = \sum_{r=0}^{\infty} r \frac{\lambda^r}{r!} e^{-\lambda} \tag{9}$$

Because of the r in the summand, the $r = 0$ term is zero, so

$$\mu = \sum_{r=1}^{\infty} r \frac{\lambda^n}{r!} e^{-\lambda} \tag{10}$$

Now, use $r! = r \times (r-1)!$:

$$\mu = \sum_{r=1}^{\infty} \frac{\lambda^r}{(r-1)!} e^{-\lambda} \tag{11}$$

and then pull a λ out the front

$$\mu = \lambda \sum_{r=1}^{\infty} \frac{\lambda^{r-1}}{(r-1)!} e^{-\lambda} \tag{12}$$

Finally set $s = r - 1$ and

$$\mu = \lambda \sum_{s=0}^{\infty} \frac{\lambda^{s}}{s!} e^{-\lambda} = \lambda \tag{13}$$

so λ is the average event count!

Some example Poisson distributions are shown in Fig. 1.

Quick example

An average of two supervillians arrive at Gotham every day; Batman has little trouble fighting them off, in fact he can fight off six supervillians a day. Seven would be tricky though. What is the chance seven supervillians arrive in one day?

$$p(7) = \frac{2^7}{7!} e^{-2} \approx 0.0034 \tag{14}$$

so makes it seem Batman will probably be ok, however, you should note there are 365 days in the year. As for the fishing example:

$$p(4) = \frac{5^4}{4!} e^{-5} \approx 0.19 \tag{15}$$

Summary

- In a **Poisson process** events occur randomly, the rate they occur at doesn't change over time and the chance of an event occuring doesn't depend on when events happened in the past.
- The **Poisson distribution** gives the probability of r events occuring in a time interval if λ is the rate, the average number of events in that period:

$$p(r) = \frac{\lambda^r}{r!} e^{-\lambda} \tag{16}$$

- There is a fancy derivation of this formula which involves subdividing the interval into small subintervals.
- It is possible to show λ is the average count by writing down the formula for the mean and rearranging the terms.