16

Normality and Homogeneity tests

Probability and Statistics

COMS10011

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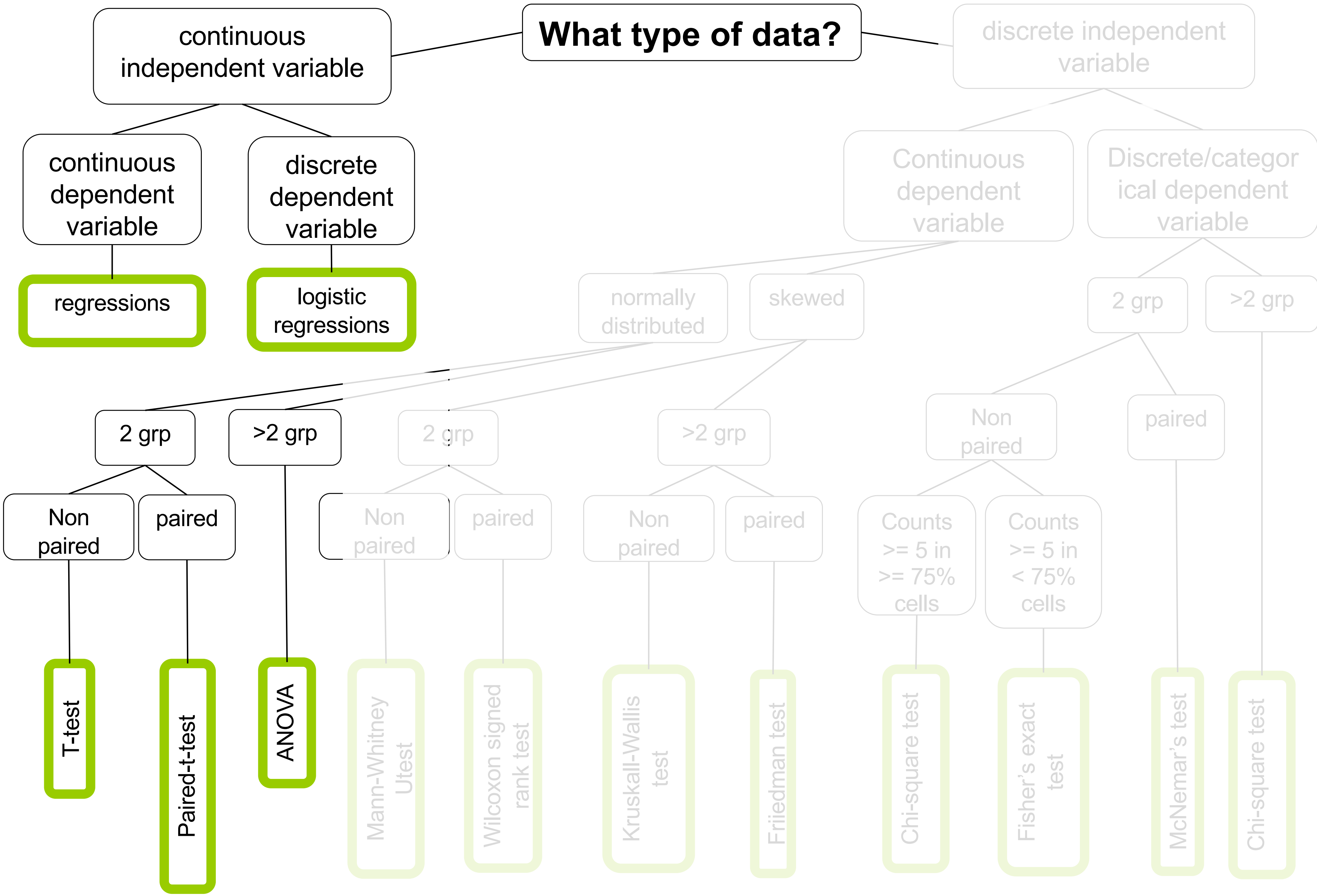
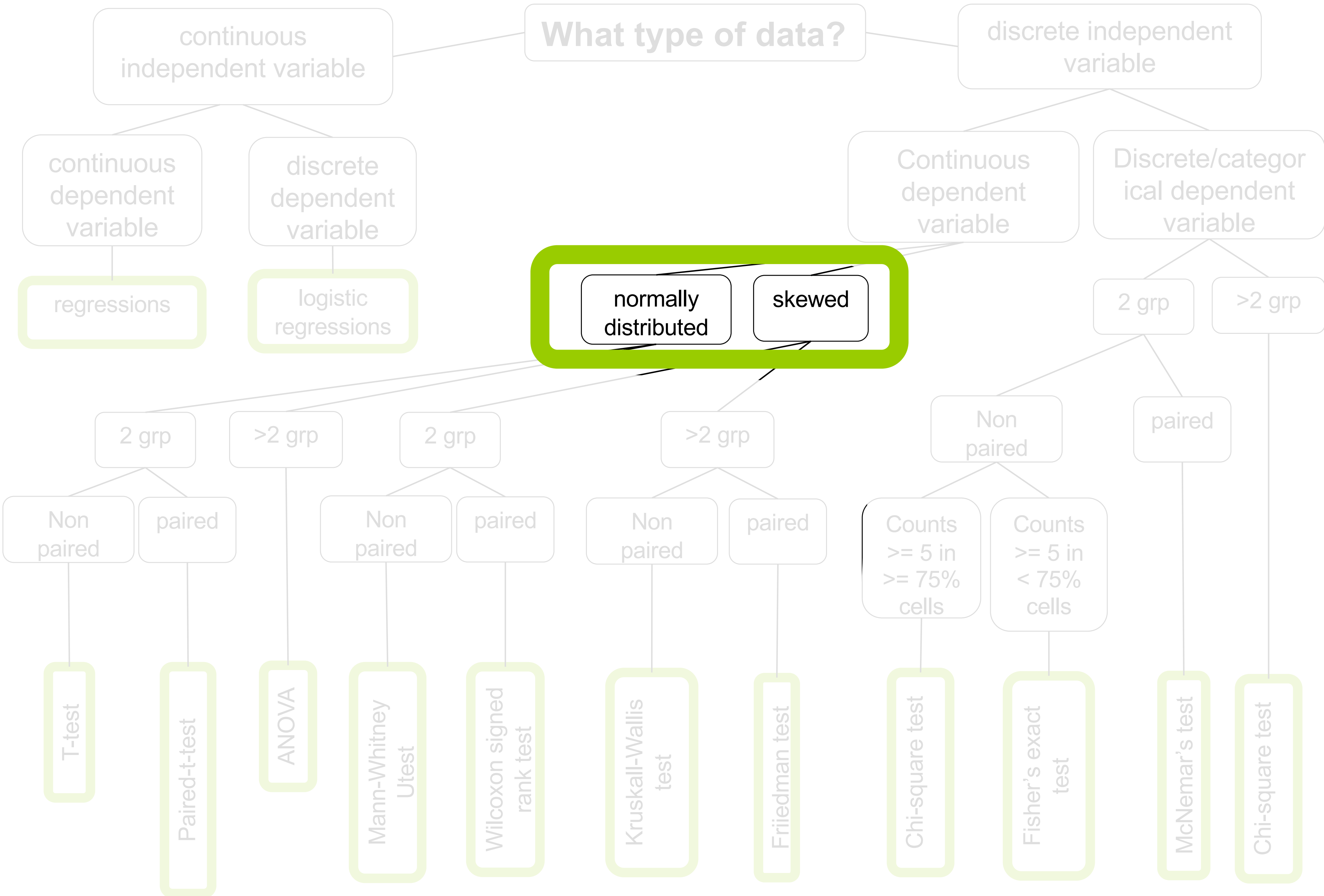
(Thanks S. Massa, Oxford)



What type of data?
continuous independent variable
continuous dependent variable
discrete dependent variable
regressions
logistic regressions
2 grp
>2 grp
Non paired
paired
T-test
Paired-t-test
ANOVA
discrete independent variable
Continuous dependent variable
Discrete/categorical dependent variable
normally distributed
skewed
2 grp
>2 grp
Non paired
paired
2 grp
>2 grp
Non paired
paired
Non paired
paired
Counts >= 5 in >= 75% cells
Counts >= 5 in < 75% cells
Mann-Whitney Utest
Wilcoxon signed rank test
Kruskall-Wallis test
Friedman test
Chi-square test
Fisher's exact test
McNemar's test
Chi-square test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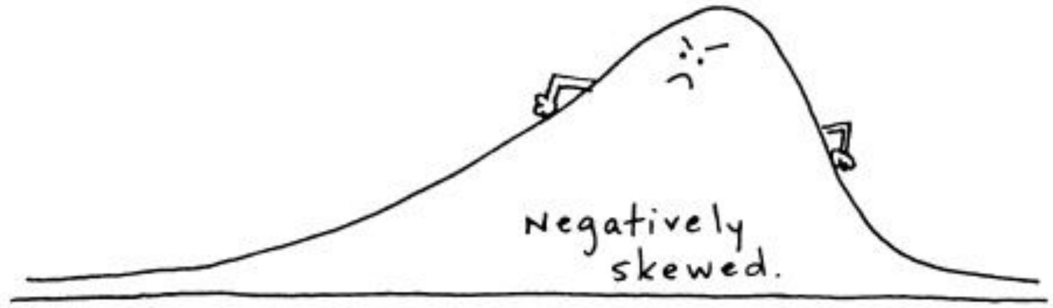
today::

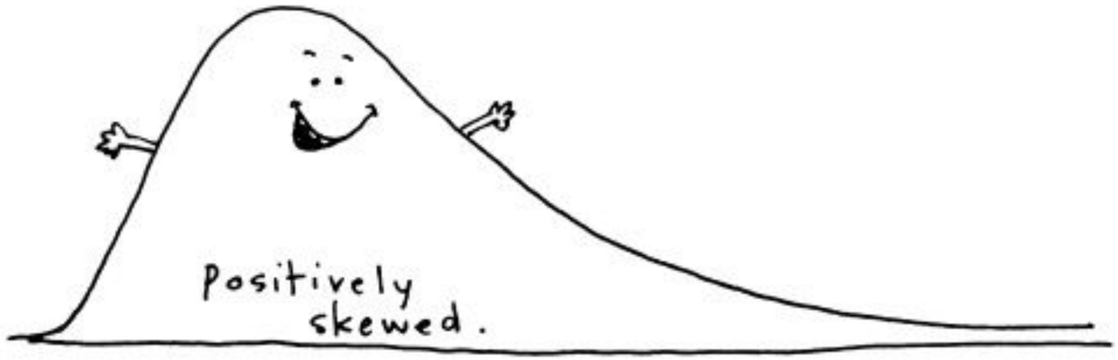
look into **assumption of normality** and **of homogeneity**

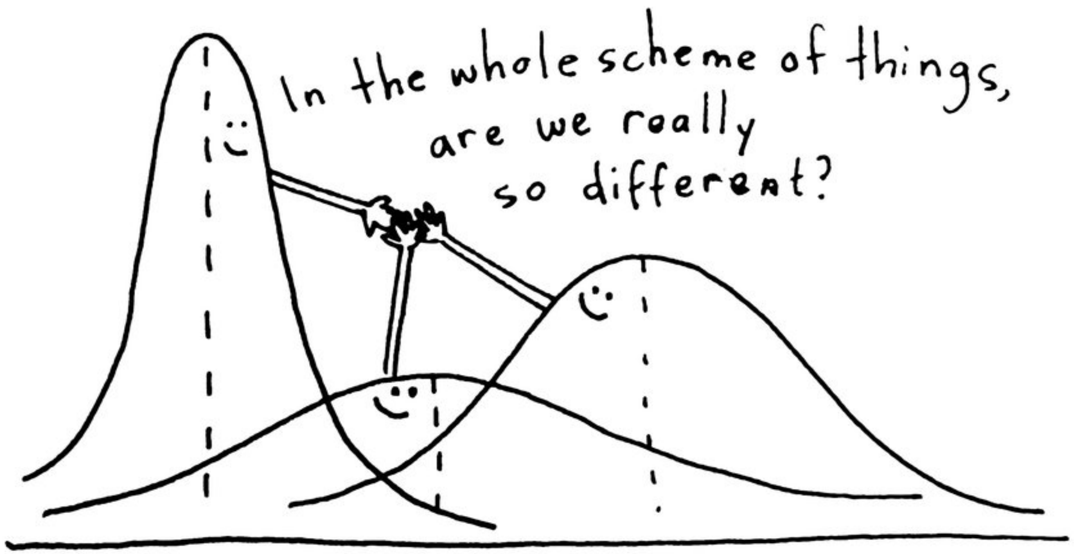
see what to do otherwise

tests we have seen so far (t-test, anova) assume that data **follow curve of normal distribution** and have **homogenous variance**

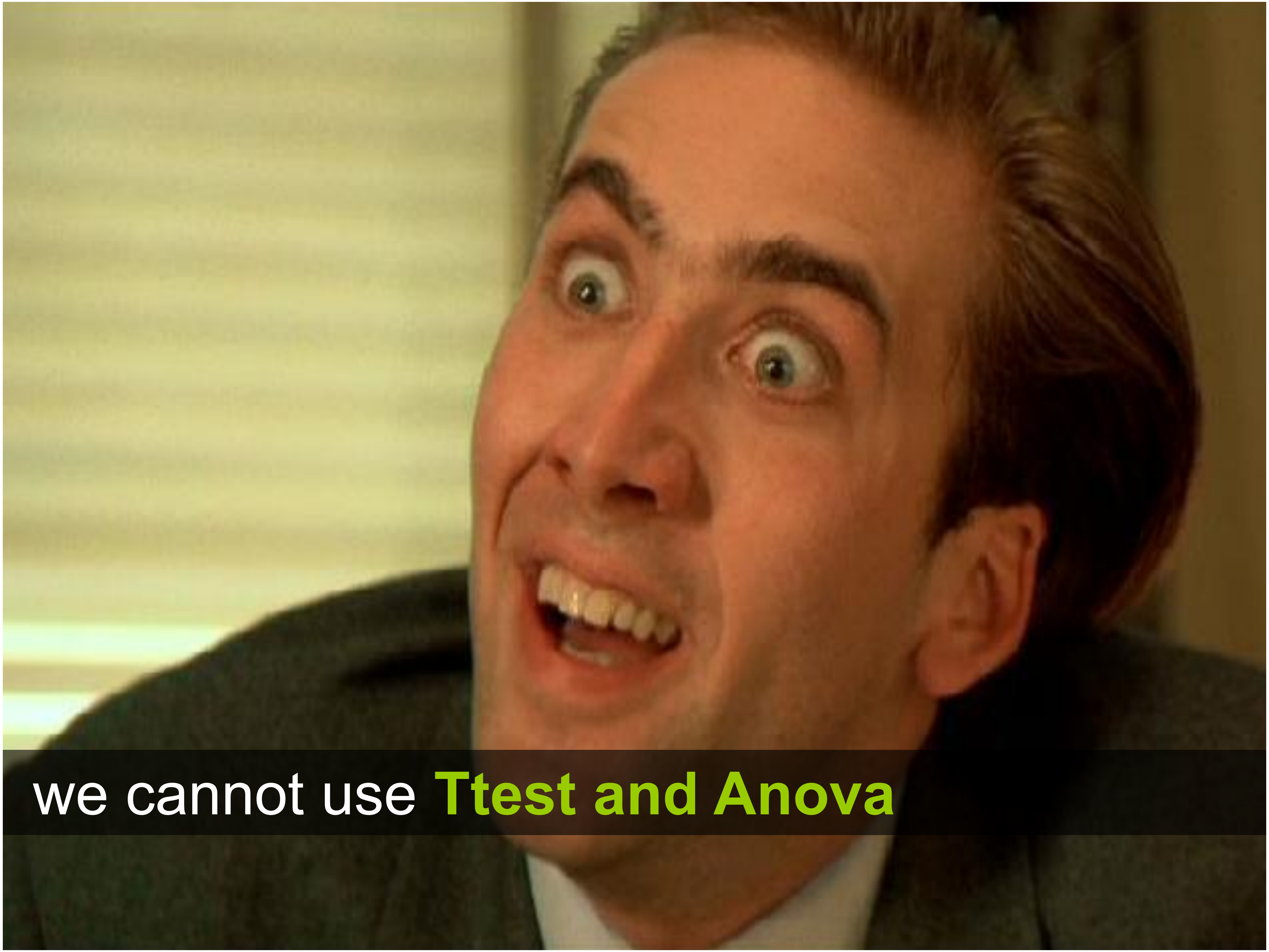
but if we have
distributions like this …







… or that

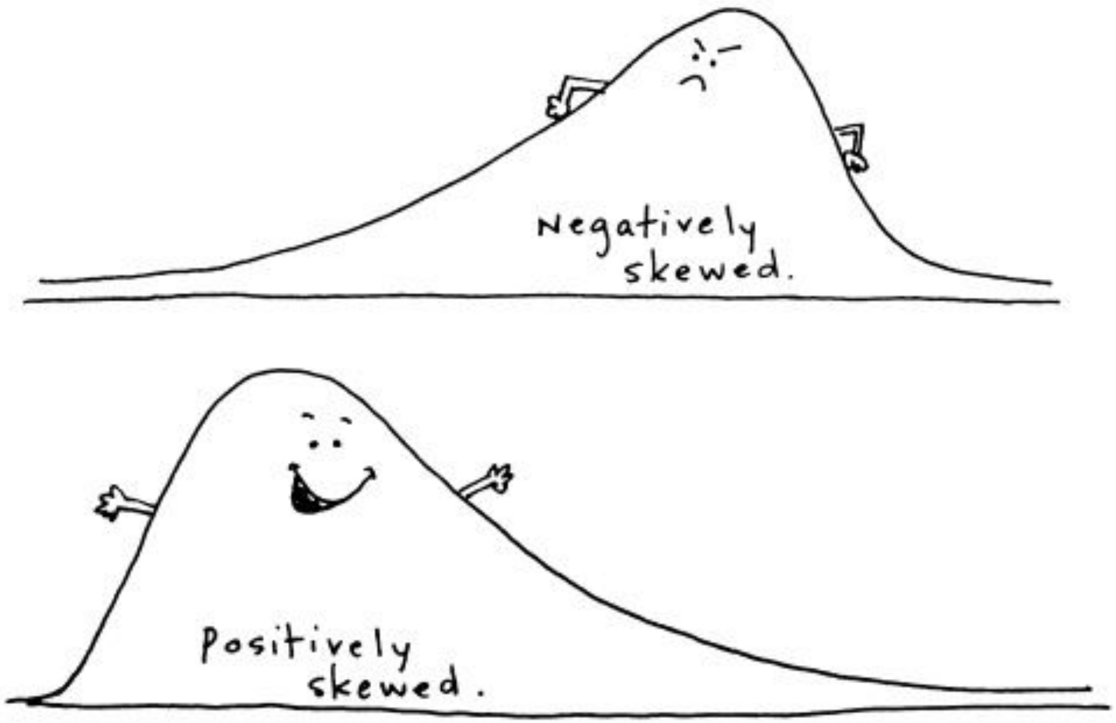

we cannot use Ttest and Anova

use parametric tests (ttest, anova)

if data **follow curve of normal distribution** with **homogeneous variance**

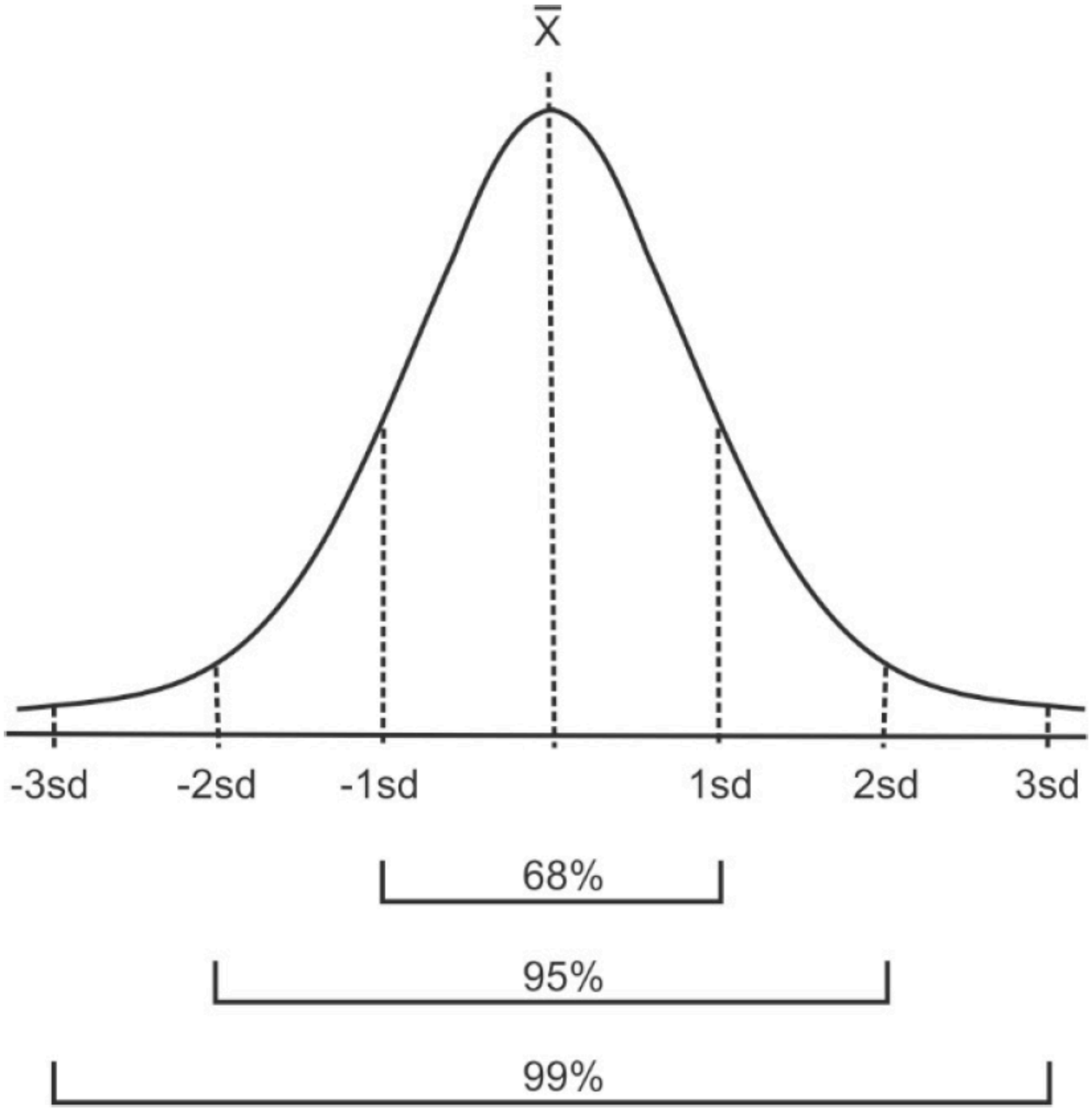
otherwise …

use non-parametric tests

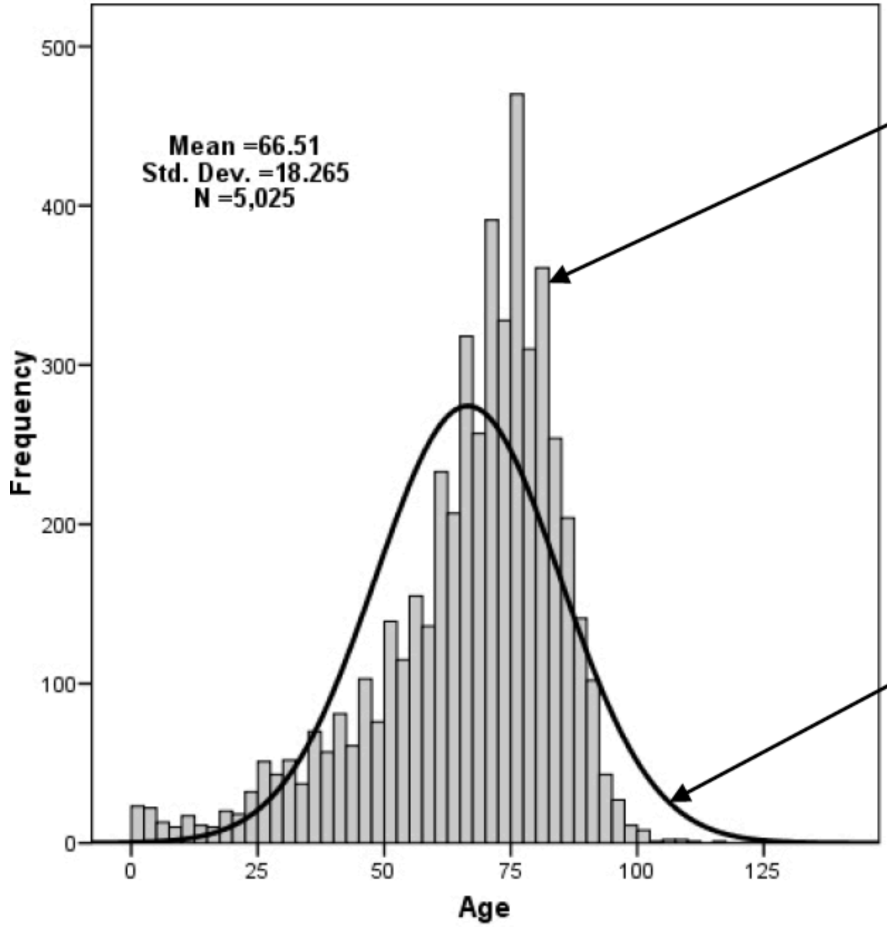


assumption of normality

given the **mean** and **standard deviation** of a dataset = a theoretical normal distribution has those proportions (Z-score)



this theoretical normal distribution can then be compared to the actual distribution of the data.



The ***actual*** data distribution that has a mean of 66.51 and a standard deviation of 18.265.

Theoretical normal distribution ***calculated*** from a mean of 66.51 and a standard deviation of 18.265.

<are the actual data statistically different than the computed normal curve? >

several methods to check that, we are only going to look at two: **Kolmogorov-Smirnov test** and **Shapiro-Wilks test**

Kolmogorov-Smirnov

works best for data sets with $n > 50$

not sensitive to problems in the tails

Shapiro-Wilks

works best for data sets with $n < 50$

doesn't work well if several values are same

Kolmogorov-Smirnov test

$$D_n = \max_x \left| F_{\mathsf{exp}}(x) - F_{\mathsf{obs}}(x) \right|$$

cumulative distribution function observed

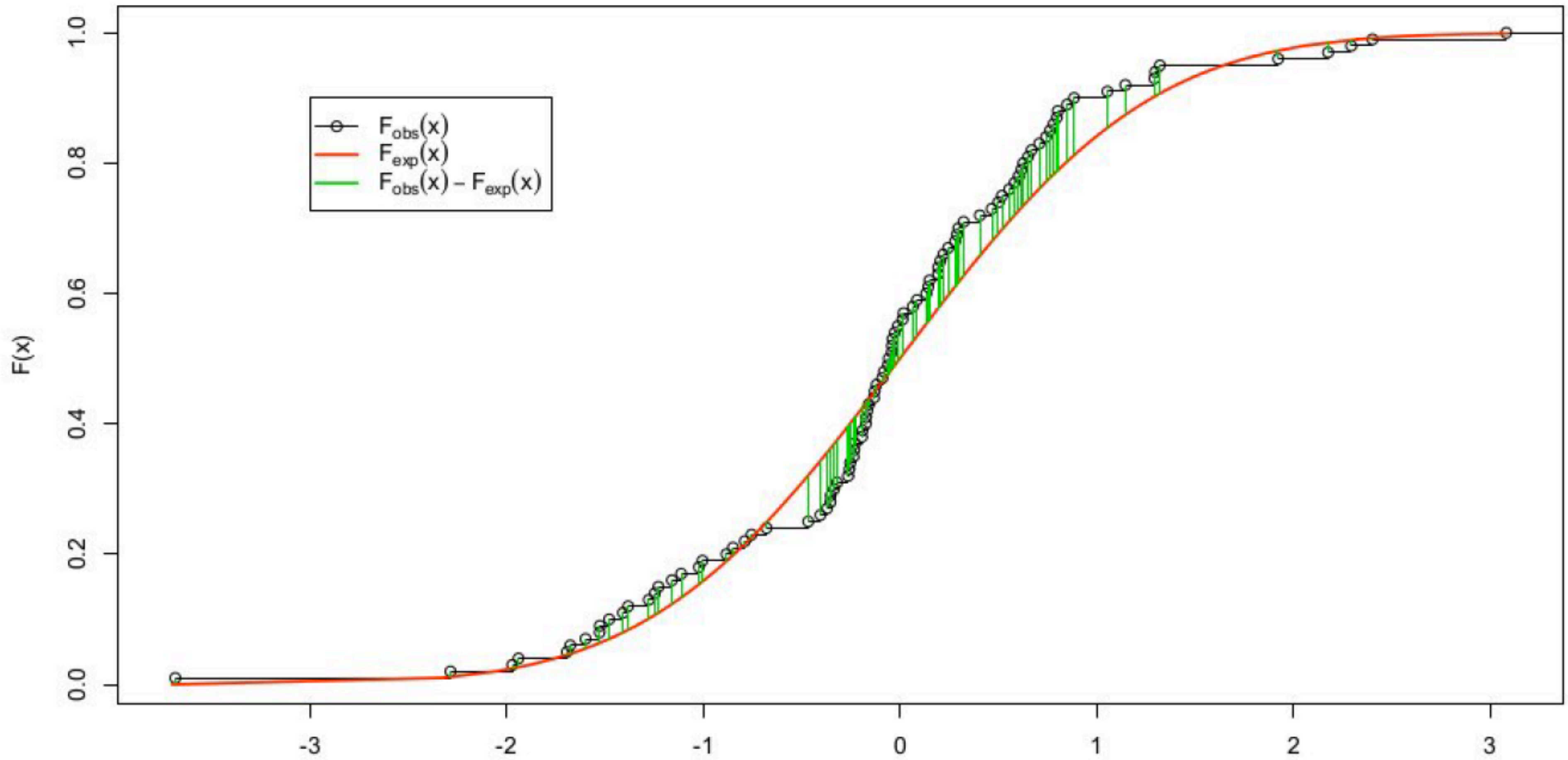
cumulative distribution function expected

can generate a **p-value**

0.16	−0.68	−0.32	−0.85	0.89	−2.28	0.63	0.41	0.15	0.74
1.30	−0.13	0.80	−0.75	0.28	−1.00	0.14	−1.38	−0.04	−0.25
−0.17	1.29	0.47	−1.23	0.21	−0.04	0.07	−0.08	0.32	−0.17
0.13	−1.94	0.78	0.19	−0.12	−0.19	0.76	−1.48	−0.01	0.20
−1.97	−0.37	3.08	−0.40	0.80	0.01	1.32	−0.47	2.29	−0.26
−1.52	−0.06	−1.02	1.06	0.60	1.15	1.92	−0.06	−0.19	0.67
0.29	0.58	0.02	2.18	−0.04	−0.13	−0.79	−1.28	−1.41	−0.23
0.65	−0.26	−0.17	−1.53	−1.69	−1.60	0.09	−1.11	0.30	0.71
−0.88	−0.03	0.56	−3.68	2.40	0.62	0.52	−1.25	0.85	−0.09
−0.23	−1.16	0.22	−1.68	0.50	−0.35	−0.35	−0.33	−0.24	0.25

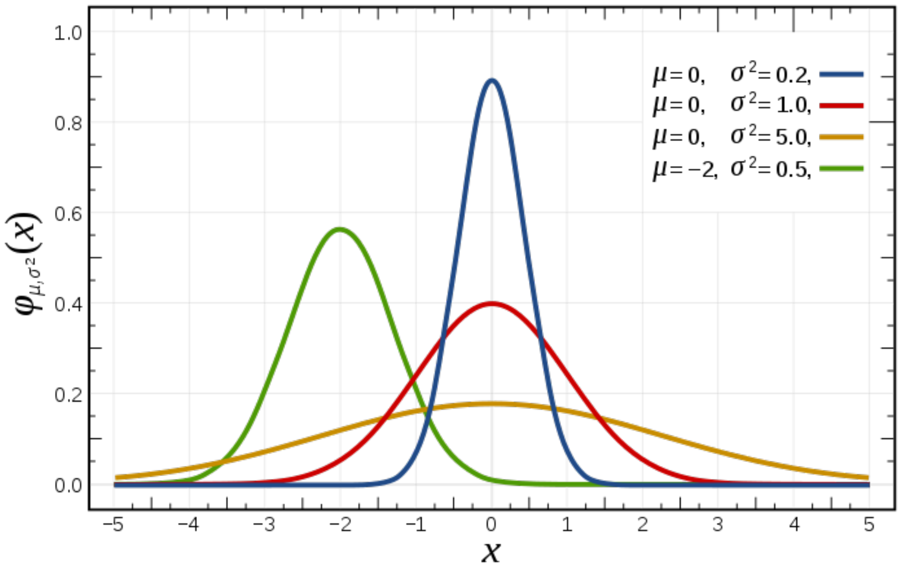
does the following sample of n=100 comes from a normality distributed population?

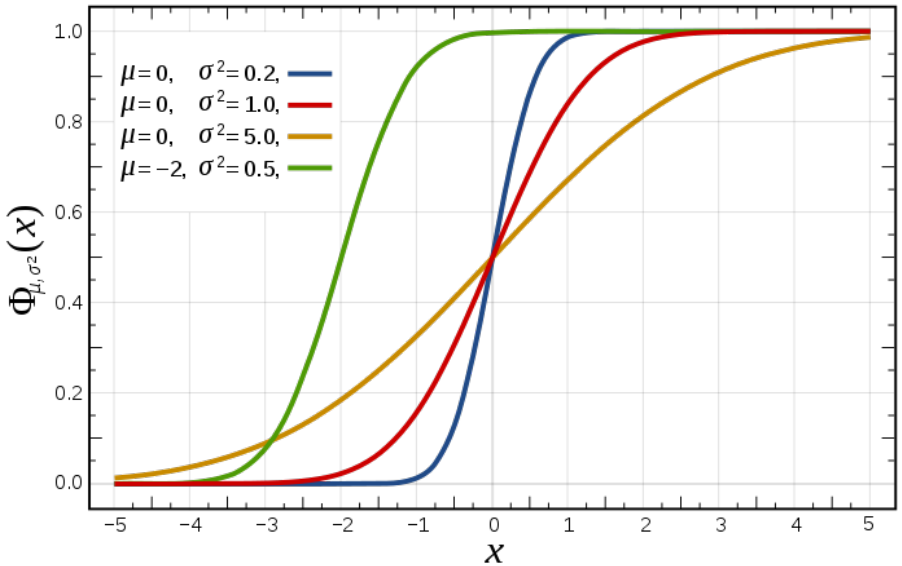
intuitively, we search for the maximum absolute distance between our data cumulative distribution function and the normal cumulative distribution function



so far we looked at **probability density function:** represents probability that the variate has the value x

another way to look at this is the **cumulative distribution function**: represents probability that the variable takes a value less than or equal to x





0.16	−0.68	−0.32	−0.85	0.89	−2.28	0.63	0.41	0.15	0.74
1.30	−0.13	0.80	−0.75	0.28	−1.00	0.14	−1.38	−0.04	−0.25
−0.17	1.29	0.47	−1.23	0.21	−0.04	0.07	−0.08	0.32	−0.17
0.13	−1.94	0.78	0.19	−0.12	−0.19	0.76	−1.48	−0.01	0.20
−1.97	−0.37	3.08	−0.40	0.80	0.01	1.32	−0.47	2.29	−0.26
−1.52	−0.06	−1.02	1.06	0.60	1.15	1.92	−0.06	−0.19	0.67
0.29	0.58	0.02	2.18	−0.04	−0.13	−0.79	−1.28	−1.41	−0.23
0.65	−0.26	−0.17	−1.53	−1.69	−1.60	0.09	−1.11	0.30	0.71
−0.88	−0.03	0.56	−3.68	2.40	0.62	0.52	−1.25	0.85	−0.09
−0.23	−1.16	0.22	−1.68	0.50	−0.35	−0.35	−0.33	−0.24	0.25

does the following sample of n=100 comes from a normality distributed population?

1. order the data:

−3.68	−2.28	−1.97	−1.94	−1.69	−1.68	−1.60	−1.53	−1.52	−1.48
−1.41	−1.38	−1.28	−1.25	−1.23	−1.16	−1.11	−1.02	−1.00	−0.88
−0.85	−0.79	−0.75	−0.68	−0.47	−0.40	−0.37	−0.35	−0.35	−0.33
−0.32	−0.26	−0.26	−0.25	−0.24	−0.23	−0.23	−0.19	−0.19	−0.17
−0.17	−0.17	−0.13	−0.13	−0.12	−0.09	−0.08	−0.06	−0.06	−0.04
−0.04	−0.04	−0.03	−0.01	0.01	0.02	0.07	0.09	0.13	0.14
0.15	0.16	0.19	0.20	0.21	0.22	0.25	0.28	0.29	0.30
0.32	0.41	0.47	0.50	0.52	0.56	0.58	0.60	0.62	0.63
0.65	0.67	0.71	0.74	0.76	0.78	0.80	0.80	0.85	0.89
1.06	1.15	1.29	1.30	1.32	1.92	2.18	2.29	2.40	3.08

2. compute the empirical distribution function

$$F_{\text{obs}}(-3.68) = \frac{1}{100}, \quad F_{\text{obs}}(-2.28) = \frac{2}{100}, \ldots, \quad F_{\text{obs}}(3.08) = 1$$

F_{obs}

0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29	0.30
0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39	0.40
0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.50
0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60
0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69	0.70
0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79	0.80
0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89	0.90
0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1.00

3. for each observation xi from the data, compute:

$$F_{\text{exp}}(x_i) = P(Z \leq x_i)$$

(in this case, the expected distribution function is standard normal so use the normal table)

F_{exp}	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29	0.30
	0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39	0.40
	0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.50
	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60
	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69	0.70
	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79	0.80
	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89	0.90
	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1.00

now we have two tables Fobs and Fexp …

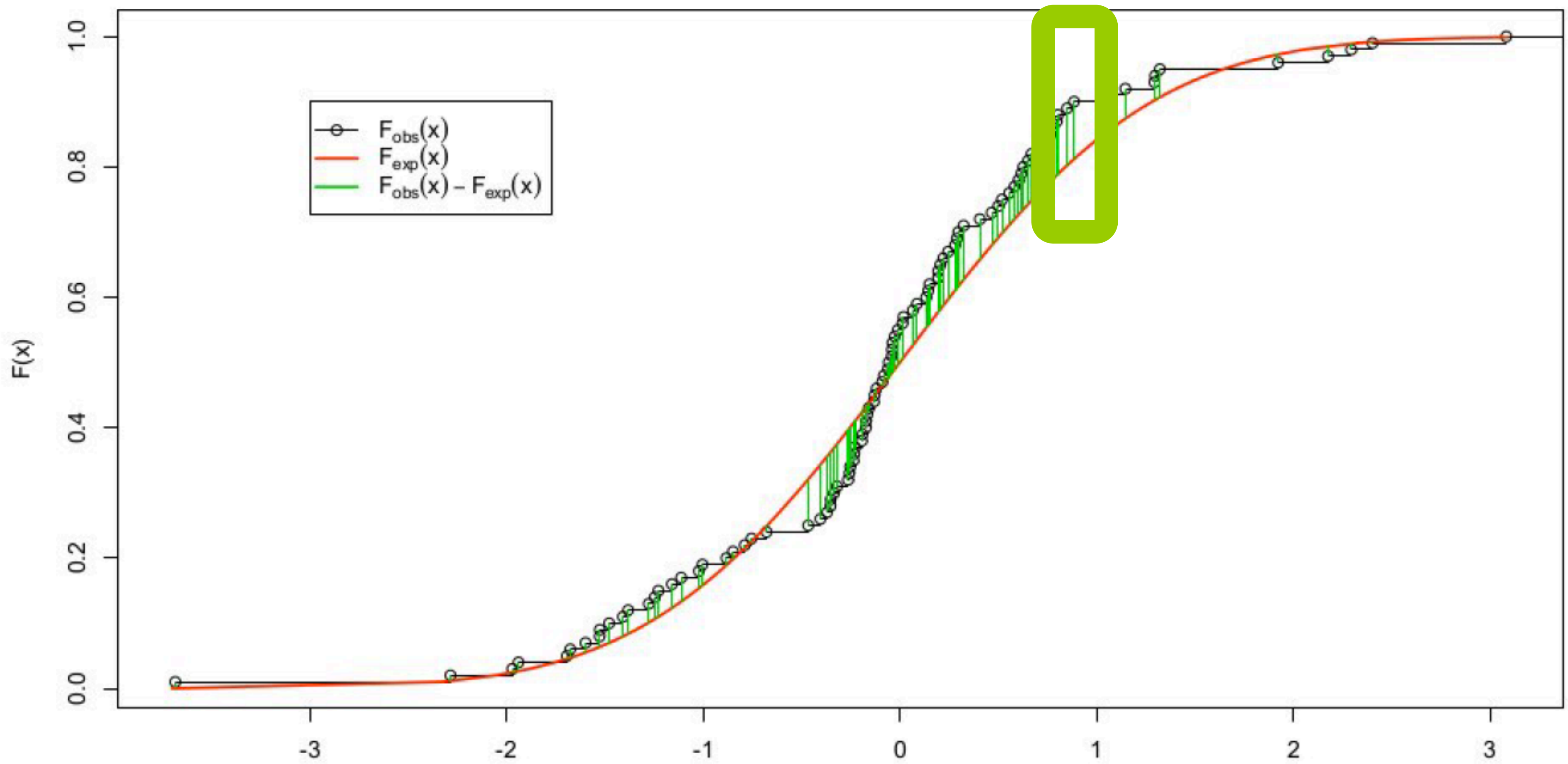
4. lets compute the absolute difference between the two and find the highest value

0.01	0.02	0.01	0.01	0.02	0.02	0.02	0.03	0.04	0.04
0.04	0.05	0.04	0.04	0.05	0.06	0.07	0.07	0.08	0.08
0.09	0.09	0.09	0.06	−0.04	−0.05	−0.05	−0.04	−0.03	−0.04
−0.03	−0.04	−0.03	−0.02	−0.01	0.00	0.01	0.01	0.02	0.03
0.04	0.05	0.03	0.04	0.05	0.06	0.06	0.06	0.07	0.08
0.09	0.10	0.11	0.12	0.11	0.12	0.12	0.13	[illegible]	0.11
0.12	0.12	0.13	0.14	0.15	0.16	0.17	0.18	$\boxed{0.19}$	0.19
0.18	0.12	0.11	0.10	0.11	0.10	0.08	0.09	0.08	0.09
0.09	0.09	0.10	0.10	0.10	0.11	0.11	0.12	0.13	0.11
0.06	0.06	0.04	0.05	0.06	−0.02	−0.03	−0.02	−0.01	0.00

$$D_n = \max_x |F_{\text{exp}}(x) - F_{\text{obs}}(x)|$$

this is the D searched

we have calculated the maximum absolute distance between expected and observed distribution functions



5. at 95% level the critical value is approximately given by

$$D_{\text{crit},0.05} = \frac{1.36}{\sqrt{n}}$$

we have a sample size of n = 100 so Dcrit = 0.136

and 0.19 > **0.136**

$$D_{\text{crit},0.05} = \frac{1.36}{\sqrt{n}}$$

n	α 0.01	α 0.05	α 0.1	α 0.15	α 0.2
1	0.995	0.975	0.950	0.925	0.900
2	0.929	0.842	0.776	0.726	0.684
3	0.828	0.708	0.642	0.597	0.565
4	0.733	0.624	0.564	0.525	0.494
5	0.669	0.565	0.510	0.474	0.446
6	0.618	0.521	0.470	0.436	0.410
7	0.577	0.486	0.438	0.405	0.381
8	0.543	0.457	0.411	0.381	0.358
9	0.514	0.432	0.388	0.360	0.339
10	0.490	0.410	0.368	0.342	0.322
11	0.468	0.391	0.352	0.326	0.307
12	0.450	0.375	0.338	0.313	0.295
13	0.433	0.361	0.325	0.302	0.284
14	0.418	0.349	0.314	0.292	0.274
15	0.404	0.338	0.304	0.283	0.266
16	0.392	[illegible]	[illegible]	0.274	0.258
17	0.381	[illegible]	[illegible]	0.266	0.250
18	0.371	0.309	0.278	0.259	0.244
19	[illegible]	[illegible]	[illegible]	[illegible]	0.237
20	0.356	0.294	0.264	0.246	0.231
25	0.320	0.270	0.240	0.220	0.210
30	0.290	0.240	0.220	0.200	0.190
35	0.270	0.230	0.210	0.190	0.180
40	0.250	0.210	0.190	0.180	0.170
45	0.240	0.200	0.180	0.170	0.160
50	0.230	0.190	0.170	0.160	0.150
OVER 50	$\frac{1.63}{\sqrt{n}}$	$\frac{1.36}{\sqrt{n}}$	$\frac{1.22}{\sqrt{n}}$	$\frac{1.14}{\sqrt{n}}$	$\frac{1.07}{\sqrt{n}}$

there is a plethora of **tables / sampling distributions** that are established and are the basis of all statistic tests so 0.19 > **0.136** so null hypothesis rejected

H0: the samples come from a normal distribution

conclusion: data **not following** a normal distribution

note KS is different than other tests we saw where we looked for a value below a critical level to reject the null, here it is the opposite (the larger the results the less likely is H0 so we reject it) what if Dn < Dcrit?

here is a tricky bit … remember lecture on hypothesis testing, we cannot prove that two things are equal so we are going to **assume** that the normality is met

which is why we call this **assumption of normality**

more example on GitHub repository or at http://www.real-statistics.com/tests-normality-and-symmetry/statistical-tests-normality-symmetry/kolmogorov-smirnov-test/

```
y <-c(0.16,-0.68,-0.32,-0.85,0.89,-2.28,0.63,0.41,0.15,0.74,1.30,-0.13,0.80,-0.75,0.28,-
1.00,0.14,-1.38,-0.04,-0.25,-0.17,1.29,0.47,-1.23,0.21,-0.04,0.07,-0.08,0.32,-0.17,0.13,-
1.94,0.78,0.19,-0.12,-0.19,0.76,-1.48,-0.01,0.20,-1.97,-0.37,3.08,-0.40,0.80,0.01,1.32,-0.47,2.29,-
0.26,-1.52,-0.06,-1.02,1.06,0.60,1.15,1.92,-0.06,-0.19,0.67,0.29,0.58,0.02,2.18,-0.04,-0.13,-0.79,-
1.28,-1.41,-0.23,0.65,-0.26,-0.17,-1.53,-1.69,-1.60,0.09,-1.11,0.30,0.71,-0.88,-0.03,0.56,-
3.68,2.40,0.62,0.52,-1.25,0.85,-0.09,-0.23,-1.16,0.22,-1.68,0.50,-0.35,-0.35,-0.33,-0.24,0.25)
X <- rnorm(100)
ks.test(X,y)

Two-sample Kolmogorov-Smirnov test


data:  X and y
D = 0.19, p-value = 0.05410262
alternative hypothesis: two-sided

#note that if you run the code you will have
different D (because of the random rnorm
generation) but likely that your pvalue will
always be above 0.05
```

Kolmogorov-Smirnov works well with **sample size > 50** but when the sample is smaller Shapiro-Wilks works best

Shapiro-Wilks test

$$W = \frac{\left(\sum_{i=1}^{n} a_i x_{(i)}\right)^2}{\sum_{i=1}^{n} (x_i - \overline{x})^2}$$

x(i) is the ith order statistic

SS (sum of squared difference)

$$(a_1, \cdots, a_n) = \frac{m^T V^{-1}}{(m^T V^{-1} V^{-1} m^T)^{1/2}}, \textit{where}\ m = (m_1, \cdots, m_n)^T$$

m1, ..., mn are the expected values of the order statistics of independent and identically distributed random variables sampled from the standard normal distribution, and **V** is the covariance matrix of those order statistics.

can generate a **p-value**

more beefy but let’s go steps by steps …

3.83 3.16 4.70 3.97 2.03 2.87 3.65 5.09

does the following sample comes from a normality distributed population?

1. order the data:

2.03 2.87 3.16 3.65 3.83 3.97 4.70 5.09

2. divide them in two

2.03 2.87 3.16 3.65 **3.83 3.97 4.70 5.09**

2.03 2.87 3.16 3.65 3.83 3.97 4.70 5.09

3. compute di the differences between both

3.06
1.83
0.81
0.18

4. multiply each of these by ai

good new we have shapiro-wilk table

n	2	3	4	5	6	7	8	9	10	11	12	13	14
a1	0.7071	0.7071	0.6872	0.6646	0.6431	0.6233	0.6052	0.5888	0.5739	0.5601	0.5475	0.5359	0.5251
a2			0.1677	0.2413	0.2806	0.3031	0.3164	0.3244	0.3291	0.3315	0.3325	0.3325	0.3318
a3					0.0875	0.1401	0.1743	0.1976	0.2141	0.2260	0.2347	0.2412	0.2460
a4							0.0561	0.0947	0.1224	0.1429	0.1586	0.1707	0.1802
a5									0.0399	0.0695	0.0922	0.1099	0.1240
a6											0.0303	0.0539	0.0727
a7													0.0240

…

di		ai		
3.06	*	0.6052	=	1.851912
1.83	*	0.3164	=	0.579012
0.81	*	0.1743	=	0.141183
0.18	*	0.0561	=	0.010098

↓

total: **2.582205**

5. Divide it

$$W = \frac{\left(\sum_{i=1}^{[n/2]} a_i \left(x_{(n+1-i)} - x_{(i)}\right)\right)^2}{\sum_{i=1}^{n} (x_i - \overline{x})^2} = \frac{(2.582205)^2}{6.782549963} = 0.98307903$$

6. from the reference table of W (another table yeah!), Wcrit(n=8 at 0.05)=0.818

and 0.983>Wcrit

0.983>Wcrit, so we cannot reject null hypothesis, so we **assume the data follows a normal distribution**

otherwise (if <) we could reject the null hypothesis and conclude with 95% confidence that that the data are not normally distributed

note we search for value below a critical level to reject the null, this is quite different from the results using the Kolmogorov-Smirnov test where this is the opposite

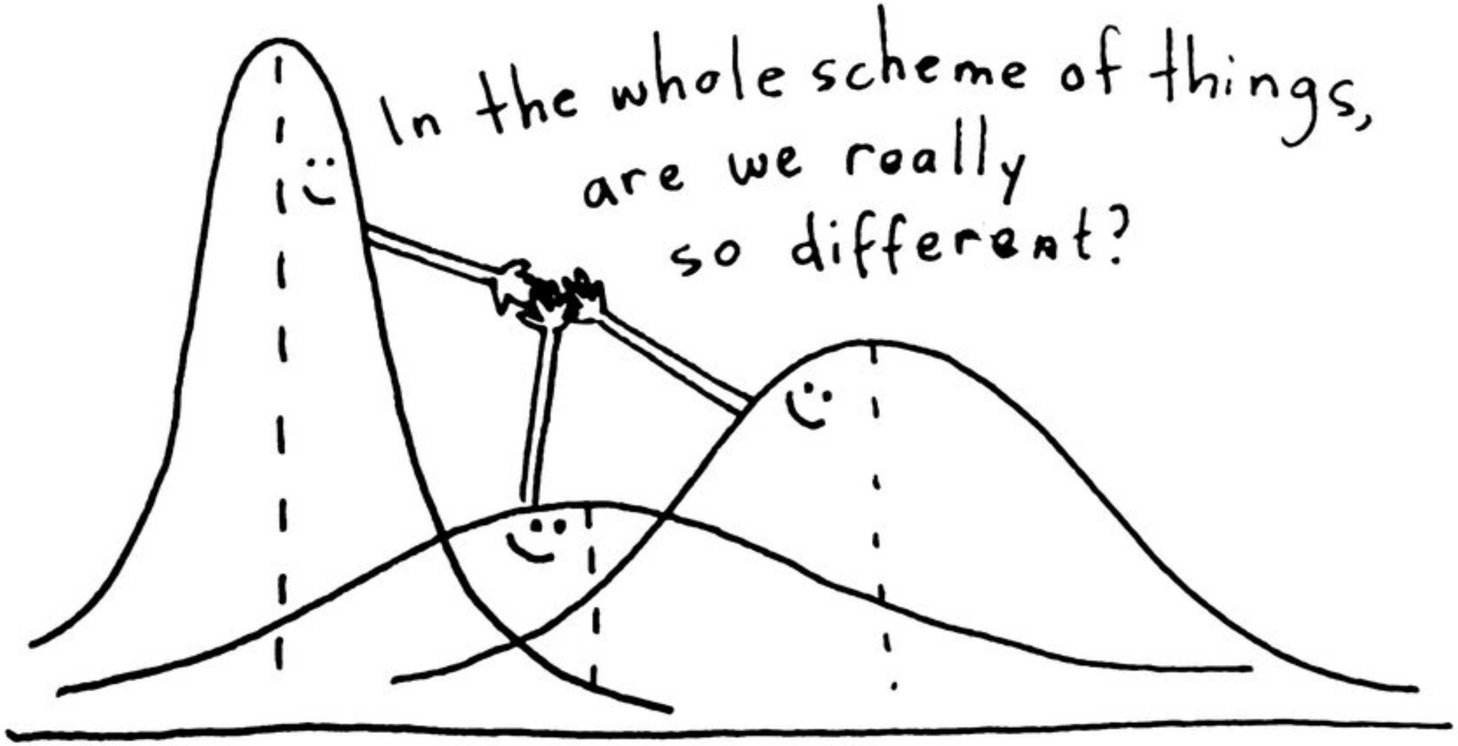
more example on GitHub repository or at http://www.real-statistics.com/tests-normality-and-symmetry/statistical-tests-normality-symmetry/shapiro-wilk-test/

```
y <-c(3.83, 3.16, 4.70, 3.97, 2.03, 2.87,
3.65, 5.09)
shapiro.test(y)
```

```
Shapiro-Wilk normality test

data:  y
W = 0.98317, p-value = 0.9769
```

= we cannot reject the null hypothesis and we assume the data is normally distributed



assumption of homogeneity

ANOVA we did when we tried to check if chocolate improves memorization

```
# we ran the one-way anova
dat = read.csv("HCIXP-anova.csv", header =
TRUE)
library(ez)
ezANOVA(dat,id,between=group,dv=score)

  Effect DFn DFd        F            p
p<.05       ges
1  group   2  57 154.8886 9.056612e-24     *
0.8445923

$`Levene's Test for Homogeneity of Variance`
  DFn DFd      SSn  SSd        F         p
p<.05
1   2  57 1.433333 29.3 1.394198 0.2563608
```

the levene’s test checks for **homogeneity of variances** (null hypothesis is that all variances are equal)

we won’t go in detail with this test but the most important is this:

> if p-value < 0.05 means variances not equal and parametric tests such as ANOVA **are not suited** (need non-parametric tests)

we know how
to check
our data

... now what?

use parametric tests (ttest, anova)

if data **follow curve of normal distribution** with **homogeneous variance**

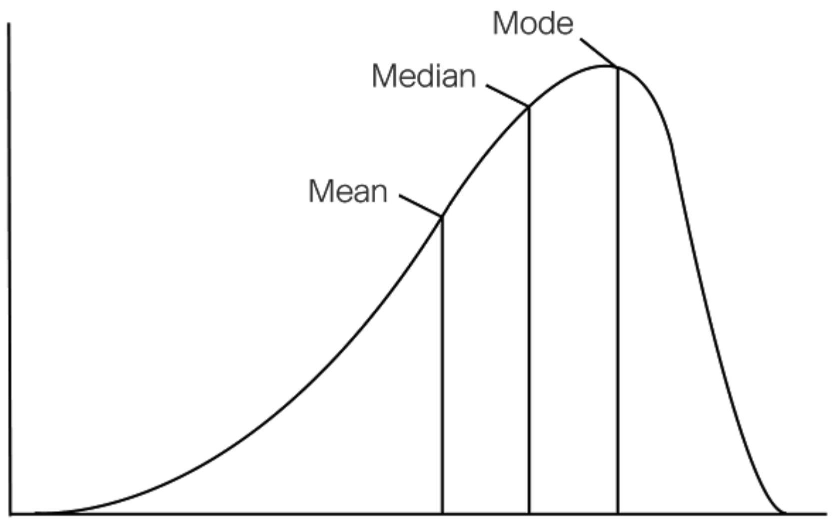
otherwise …

use non-parametric tests

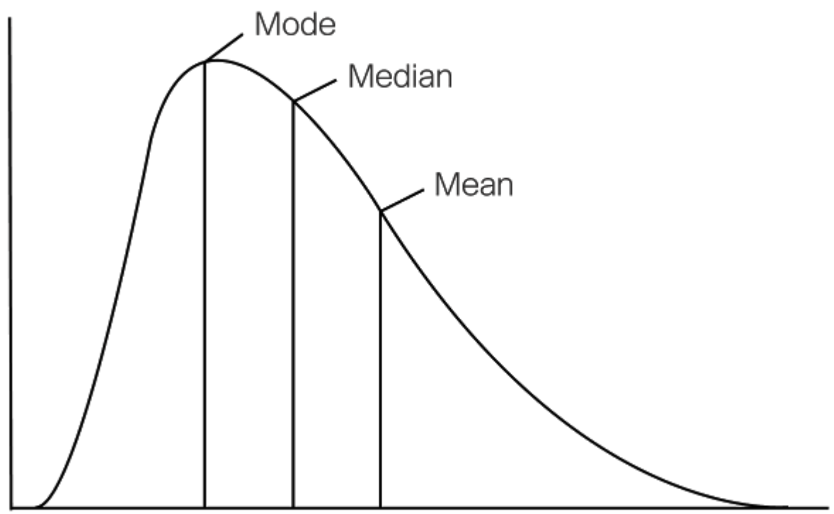
but if your data is not normally distributed you could also try to make it normal using **transformations**

… more generally because parametric tests are more robust than non-parametric ones

transformations



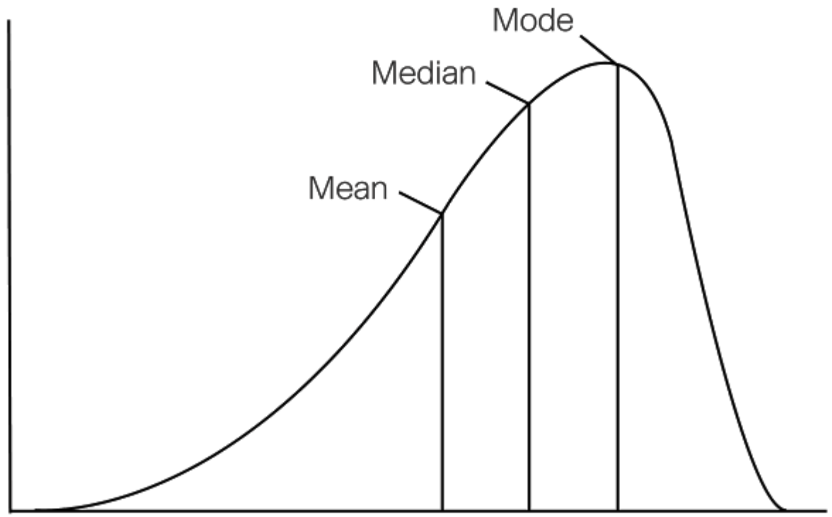
Left-Skewed (Negative Skewness)



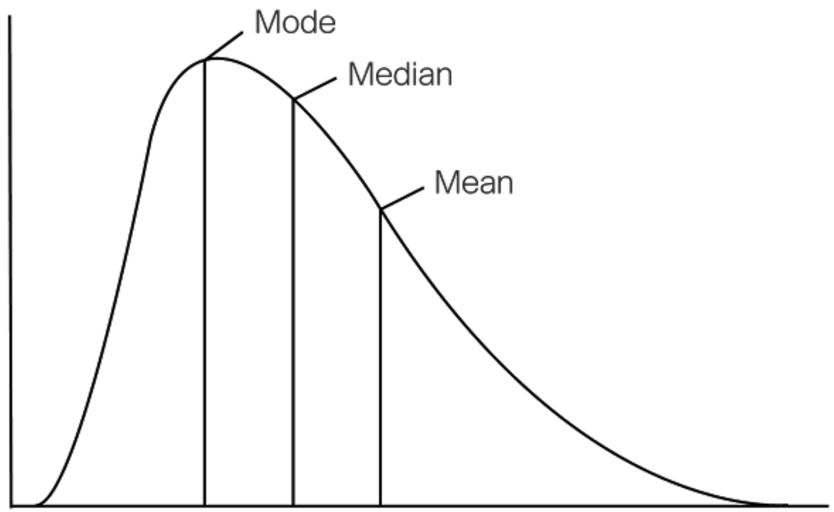
Right-Skewed (Negative Skewness)

common transformations for left skewed::

square root, cube root, log



Left-Skewed (Negative Skewness)



Right-Skewed (Negative Skewness)

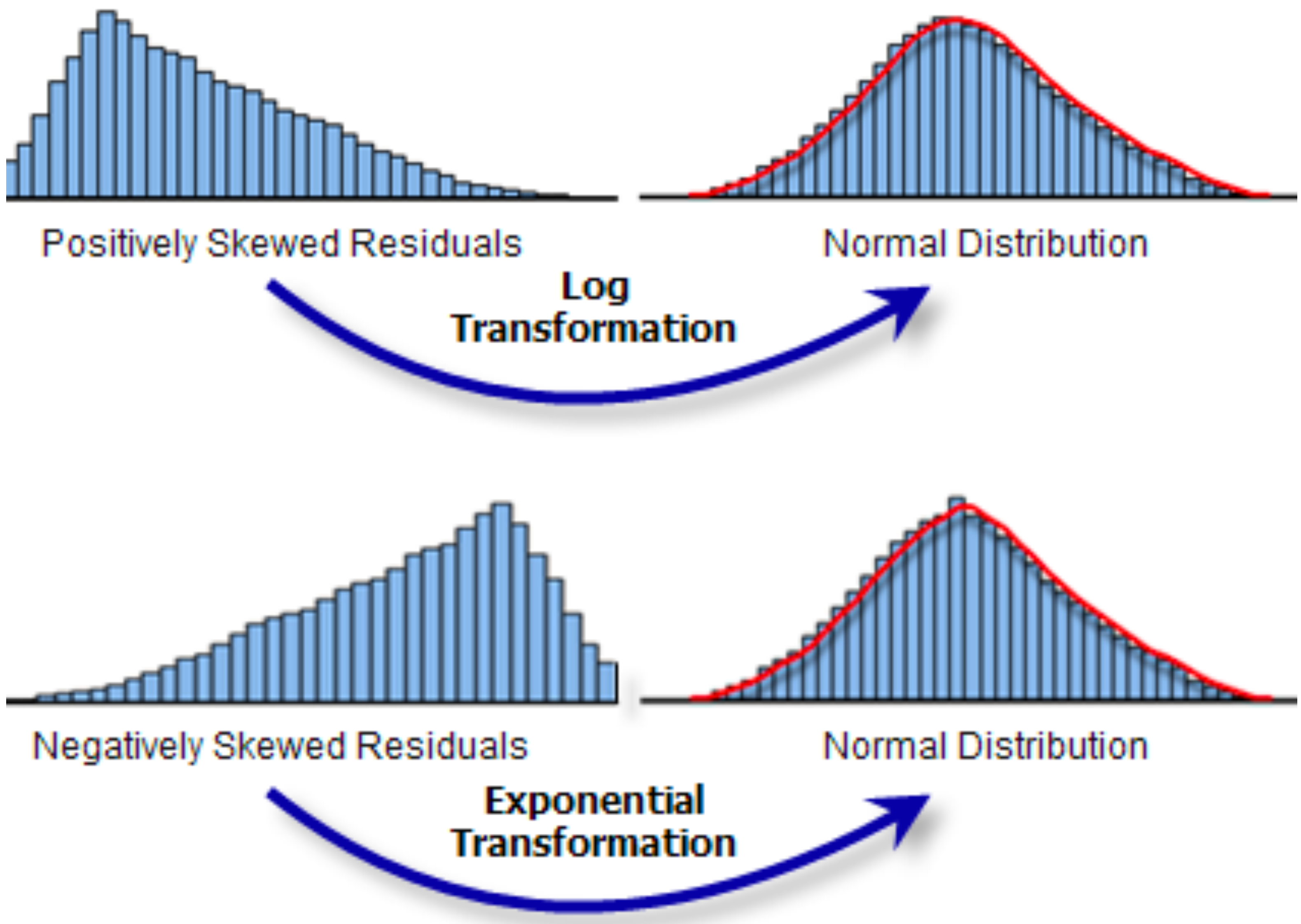
common transformations for right skewed::

square, cube root and logarithmic

```
y <-c(1.0, 1.2, 1.1, 1.1, 2.4, 2.2, 2.6,
4.1, 5.0, 10.0, 4.0, 4.1, 4.2, 4.1, 5.1,
4.5, 5.0, 15.2, 10.0, 20.0, 1.1, 1.1, 1.2,
1.6, 2.2, 3.0, 4.0, 10.5)
hist(y)
qqnorm(y)
qqline(y)

y_sqrt = sqrt(y) #cube root
y_cub = sign(y) * abs(y)^(1/3) #square root
y_log = log(y) #logarithm

# you can now try
qqnorm(y_log)
qqline(y_log)
```

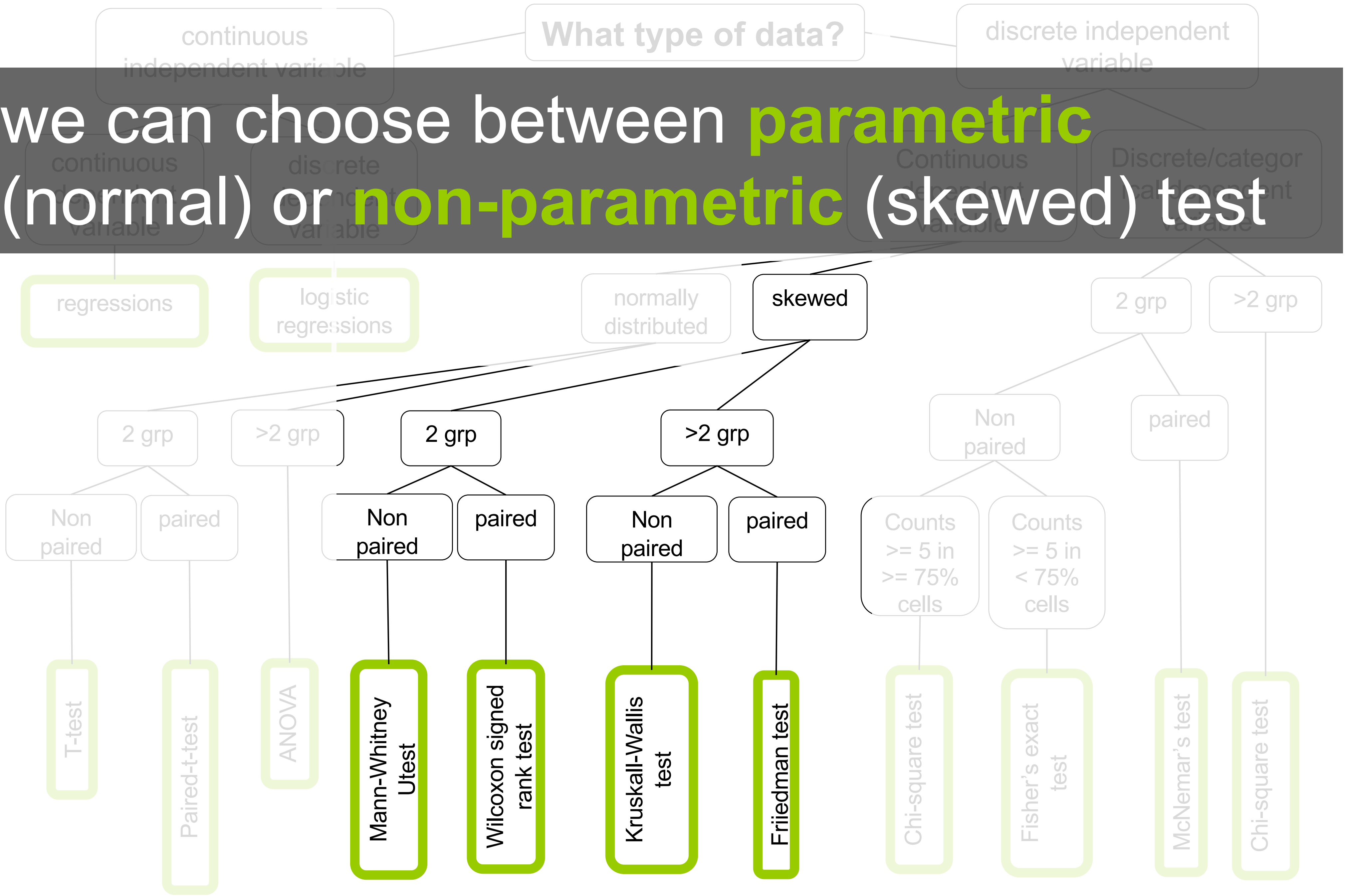

Positively Skewed Residuals
Normal Distribution
Log
Transformation
Negatively Skewed Residuals
Normal Distribution
Exponential
Transformation

sometimes skewed distributions could come from **outliers**
so make sure to get rid of them!

sometimes it does not work …

you have tried
everything and
still not good?

we can choose between **parametric** (normal) or **non-parametric** (skewed) test



summary

1. Give the names of tests we can use to check normality and explain their differences and when to use them
2. Explain what is the goal of a test of homogeneity of variance and what to do if the variances are not equal
3. I will **not ask** you to them by hand in the exam
4. Explain what to do if the data are not normal (either transforming the data or using non-parametric tests)

take away

end