Cramer's Rule

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Abstract. To start with, this paper discusses the life of the mathematician Gabriel Cramer and Cramer's Rule. It briefly talks about Cramer's family, education, career, interests, Cramer's rule, and his death. The paper also discusses and proofs in detail Cramer's rule and its use in solving any system of linear equations. Finally, there is two systems of linear equations that are solved to show how Cramer's rule works.

1 Gabriel Cramer's Life

1.1 Cramer's Family

Gabriel Cramer was born on the 31st of July 1704 in Geneva, Switzerland. His father, Jean Isaac Cramer, was a physician. Cramer had two brothers, one became a physician as his father, and the other became a professor of law.

1.2 Cramer's Education and Career

Cramer earned a doctorate with a dissertation on the qualities of sound at the age of 18. At the age of 20, he became co-chair of mathematics at the Académie de la Rive.In 1727, he decided to take a two-years trip as a mean of expanding his knowledge and meeting leading mathematicians. He went to Basel, London, Leiden, and Paris. He returned to Geneva in 1729. After five years of his return, he was appointed full chair of the mathematics department at the Académie de la Rive.

1.3 Cramer's Interests

Cramer did not only have intreset in mathematics, but he also had taken part in local politics and displayed a sense of community spirit. He has also edited some mathematicians' work like Johann and his brother Jakob Bernoulli and the German philosopher and mathematician Christian Wolff.Moreover, Cramer wrote articles that were published in various places such as the Memoirs of the Paris Academy in 1734, and the Berlin Academy in 1748, 1750 and 1752. His articles' subjects were diverse and included the study of geometric problems, the history of mathematics, philosophy, and the date of Easter.

1.4 Cramer's Rule Publication

in 1970, Cramer became a professor of philosophy at the Académie de la Rive. He also published the four-volume Introduction à l'analyse des lignes courbes algébriques. This piece of work contained Cramer's rule, which solves linear equations, and Cramer's Paradox, which clarifies the propostion on points and cubic curves first put by Colin Maclaurin. He also introduced the concept of utility which is linking probability theory and mathematical economics today.

1.5 Cramer's Death

Before his death, Cramer has fell from a carriage, and his doctor recommended that he takes a rest in south France. Yet, on his way to Bagnols-sur-Cèze on the 4th of January 1752 Gabriel Cramer died, but his mathematical contribution is alive to this day serving the educational and practical sectors.

2 Cramer's Rule

We start by important definitions that will help in proving Cramer's Rule

Definition 2.1. A *matrix* is a rectangular array of numbers.

- *Size* of a matrix $n \times m$ is the number of rows n and the number of columns m.
- A *square matrix* is the one that has number of rows = number of columns.
- An *identity matrix* I has ones on its diagonal and zeros otherwise.
- A *lower triangle matrix* is a matrix in which $a_{ij} = 0$ if $i < j$.
- An *upper triangle matrix* is a matrix in which $a_{ij} = 0$ if $i > j$.
- A *diagonal matrix* is a matrix that has numbers on the main diagonal and zeros anywhere else.

Example 2.2. The following are examples of matrices.

1. a matrix of size 2x3 $\begin{bmatrix} 1 & 0 & 3 \\ 4 & 8 & -1 \end{bmatrix}$
2. identity matrix of size 3 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3. a matrix of size 2x1 $\begin{bmatrix} 1 \\ -5 \end{bmatrix}$

Now onto knowing the operations that can be done on a matrix

- Addition and Subtraction: $A \pm B$ is defined only if A and B are of the same size such that $A \pm B = [a_{ij} \pm b_{ij}]$.
- Scalar Multiplication: Every element of the matrix gets multiplied by the scalar multiple.

- Multiplication: $A \times B$ is defined if and only if $A_{m\times n}$ and $B_{n\times l}$ such that $A \times B = C_{m\times l}$.

$C_{ij} = Row_{i,A} \times Column_{j,B}$.

Definition 2.3. The *Transpose of a matrix* $A_{m\times n}$ is an operator which flips a matrix over its diagonal such that the new matrix becomes of size $A_{n\times m}$.

Definition 2.4. Matrix A is said to be *invertible* if there exists a matrix B such that $AB = I$. Otherwise the matrix is said to be singular or non-invertible.

Definition 2.5. *A system of linear equation* in n variables $x_1, x_2, x_3, ...x_n$ has the form $a_1x_1 + a_2x_2 + a_3x_3 + ...a_nx_n = b$

The coefficients a1, a2, a3, . . . , an are real numbers, and the constant term b is a real number. The number a_1 is the leading coefficient, and x_1 is the leading variable.

A system of linear equations can be represented in a matrix.

Example 2.6. The following is how to represent a system of linear equations as a matrix. For example:

$$2x - 5y = 8$$
$$3x + 9y = -12$$

can be represented in two ways:

1. augmented matrix: $\begin{bmatrix} 2 & -5 & 8 \\ 3 & 9 & -12 \end{bmatrix}$
2. coefficient matrix, variable column vector, and constant column vector respectively:
$$\begin{bmatrix} 2 & -5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \end{bmatrix}$$

Definition 2.7. A *minor* M_{ij} *of matrix* A is a sub matrix obtained from A by deleting Row i and column j of A. The value of M_{ij} is obtained by multiplying the values on each diagonal then subtracting the anti-diagonal from the main diagonal.

Definition 2.8. *Cofactors* C_{ij} *of a matrix* A is equal to the minor M_{ij} multiplied by (-1^{i+j})

Definition 2.9. *Adjoint of matrix A* denoted by $adj(A)$ is the transpose of the matrix of the cofactors of A.

$$\text{adj(A)} = \begin{bmatrix} C_{11} & C_{12} & ...... & C_{1n} \\ C_{21} & C_{22} & ...... & C_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ C_{n1} & C_{n2} & ...... & C_{nn} \end{bmatrix}^T$$

It is important to know that $A^{-1} = \frac{1}{|A|} \cdot adj(A)$

Proof. It is known that $A \cdot adj(A) = |A|I$ so by dividing each side by $|A|$ we reach to $A \cdot \frac{1}{|A|} \cdot adj(A) = I$ and we also know that the form AB = I shows that B is the inverse of A.
$\therefore A^{-1} = \frac{1}{|A|} \cdot adj(A)$ ■

Definition 2.10. A *determinant of a matrix A* denoted by $det(A)$ or $|A|$ is a scalar value that is a function of the entries of a square matrix.

- $|A|$ can be positive, negative, or zero.
- If $|A|$ is zero it indicates that matrix A is singular.
- If $|A|$ is non-zero it indicates that matrix A is invertible.
- $|A|$ can be calculated using cofactor expansion of any row or column:
  - Cofactor expansion using Row n: $|A| = a_{n1}C_{n1} + a_{n2}C_{n2} + ... + a_{nn}C_{nn}$
  - Cofactor expansion using Column n: $|A| = a_{1n}C_{1n} + a_{2n}C_{2n} + ... + a_{nn}C_{nn}$
- If matrix A is of size 2×2, there is a simpler algorithm other than cofactor expansion. Let $A =$

  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  So, $|A| = (a \times d) - (b \times c)$
- If matrix A is of size 3×3, there is a simpler algorithm other than cofactor expansion called method of diagonals. Let $A =$

  $$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

So, $|A| = [(a \times e \times i) + (b \times f \times g) + (c \times d \times h)] - [(c \times e \times g) + (a \times f \times h) + (b \times d \times i)]$

Theorem 2.11 (Cramer's Rule)**.** *If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant A, then the solution of the system is:*

$x_1 = \frac{det(A_1)}{det(A)}$, $x_2 = \frac{det(A_2)}{det(A)}$, ..., $x_n = \frac{det(A_n)}{det(A)}$

It is important to know that $det(A_n)$ is the determinant of the matrix in which column n is replaced by the constant column vector.

Proof. Let A be an $n \times n$ invertible matrix. Suppose we have a linear system $Ax = B$. Then:

$x = A^{-1}B$ and we know that $A^{-1} = \frac{1}{|A|} \cdot adj(A)$

So, $x = \frac{1}{|A|} \cdot adj(A) \cdot B$

And x_i can be written as $\frac{1}{|A|} \cdot (b_1C_{1i} + b_2C_{2i} + ... + b_nC_{ni}))$
$(b_1C_{1i} + b_2C_{2i} + ... + b_nC_{ni}))$ is exactly the cofactor expansion of A_i.

$\therefore x_i = \frac{|A_i|}{|A|}$ ■

Example 2.12. *Now, we will use Cramer's rule to solve system of linear equations:*

- $x_1 + 2x_2 = 5$
  $-x_1 + x_2 = 1$
  so let's first get the matrix representation of the system of linear equations:

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

  Now, the coefficient matrix can be used to get the determinant that we will denote by $|A|$: As this is 2×2 matrix, we can use the simpler way to get the determinant instead of cofactor expansion: so $|A| = (1 \times 1) - (2 \times -1) = 3$

  Now, we need to get both $|A_1|$ and $|A_2|$: So, we start by computing $|A_1|$: We need to substitute column 1 by the constant column vector, so it would look like this:

$$\begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}$$

Now, $|A_1| = (5 \times 1) - (2 \times 1) = 3$
Then, we will compute $|A_2|$ *by first substituting the constant column vector into column 2:*

$$\begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}$$

So, $|A_2| = (1 \times 1) - (5 \times -1) = 6$ *Now, we have all the determinants needed to use Cramer's rule for solving a system of linear equations:*
$x_1 = \frac{|A_1|}{|A|} = \frac{3}{3} = 1$
$x_2 == \frac{|A_2|}{|A|} = \frac{6}{3} = 2$

- *Solving question 27 in section 3.4 page 142:*
  $kx + (1-k)y = 1$
  $(1-k)x + ky = 3$
  so let's first get the matrix representation of the system of linear equations:

  $$\begin{bmatrix} k & 1-k \\ 1-k & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

  Now, the coefficient matrix can be used to get the determinant that we will denote by $|A|$*: As this is* 2×2 *matrix, we can use the simpler way to get the determinant instead of cofactor expansion: so* $|A| = (k\times k) - ((1-k)\times(1-k))$
  $= k^2 - 1 + 2k - k^2 = 2k - 1$

  Now, we need to get both $|A_1|$ *and* $|A_2|$*: So, we start by computing* $|A_1|$*: We need to substitute column 1 by the constant column vector, so it would look like this:*

  $$\begin{bmatrix} 1 & 1-k \\ 3 & k \end{bmatrix}$$

  Now, $|A_1| = (1 \times k) - ((1-k) \times 3) = k - 3 + 3k = 4k - 3$
  Then, we will compute $|A_2|$ *by first substituting the constant column vector into column 2:*

  $$\begin{bmatrix} k & 1 \\ 1-k & 3 \end{bmatrix}$$

  So, $|A_2| = (k \times 3) - (1 \times 1 - k) = 3k - 1 + k = 4k - 1$ *Now, we have all the determinants needed to use Cramer's rule for solving a system of linear equations:*
  $x = \frac{|A_1|}{|A|} = \frac{4k-3}{2k-1}$
  $y = \frac{|A_2|}{|A|} = \frac{4k-1}{2k-1}$

References

[1] Ron Larson. *Elementary Linear Algebra*. 8th edition.