MATH266: UNSTABLE MOTIVIC HOMOTOPY THEORY
Fall 2024

THOMAS BRAZELTON

ABSTRACT. Notes from MATH266: Motivic homotopy theory, taught at Harvard in Fall 2024.

CONTENTS

0. INTRODUCTION

0.1. **Overview.** What sorts of things about a ring R are still true when we move to the polynomial ring $R[t]$? In other words, what sorts of things about R can't be varied in a 1-parameter family?

Let's give a ton of examples! Don't stress if not all of the words are familiar, we'll break down what's happening here over the course of the semester, this is just motivation.

Example 0.1. Let R be a reduced ring. Then the inclusion $R \to R[t]$ induces an isomorphism after taking units[1]

$$(2) \qquad R^\times \xrightarrow{\sim} (R[t])^\times.$$

Recall that the functor sending a commutative ring to its group of units is corepresented by $\mathbb{Z}[u, u^{-1}]$, so Equation 2 is equivalent to saying that the following map is a bijection

$$\operatorname{Hom}_{\operatorname{Ring}}(\mathbb{Z}[u, u^{-1}], R) \to \operatorname{Hom}_{\operatorname{Ring}}(\mathbb{Z}[u, u^{-1}], R[t]).$$

After taking Spec everywhere, this becomes

$$\operatorname{Hom}_{\operatorname{Sch}}(\operatorname{Spec}(R), \mathbb{G}_m) \to \operatorname{Hom}_{\operatorname{Sch}}(\operatorname{Spec}(R[t]), \mathbb{G}_m).$$

We therefore might rephrase Equation 2 as saying that $\operatorname{Hom}_{\operatorname{Sch}}(-, \mathbb{G}_m)$ is $\mathbb{A}^1$-*invariant*, at least when we plug in something reduced.

Example 0.3. Let k be a field. Then the functor $k \to k[t_1, \ldots, t_n]$ induces an extension of scalars map

$$\operatorname{Mod}_k \to \operatorname{Mod}_{k[t_1, \ldots, t_n]}$$
$$M \mapsto M \otimes_k k[t_1, \ldots, t_n].$$

Serre's Problem: Is every finitely generated $k[t_1, \ldots, t_n]$-module free?

Recall finitely generated projective R-modules are the same as "algebraic vector bundles" over $\operatorname{Spec}(R)$, so we're asking whether every algebraic vector bundle on $\mathbb{A}_k^n$ is trivial.

Answer: Yes (Quillen–Suslin, 1974). Quillen actually proved more– for R a PID, he proved that the every finitely generated projective $R[t]$-module is extended from an R-module.[2] Lindel proved

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[1] If R is not reduced, say there is some $r \in R$ so that $r^2 = 0$, then $(1 + rt)(1 - rt) = 1$, so $1 + rt \in R[t]^\times$.

[2] Quillen's proof involves leveraging some previous work of Horrocks, flat descent for vector bundles, and a very clever technique he invented called *patching*. Suslin's proof, which appeared in the same year, is almost completely linear algebraic, leveraging the theory of *unimodular rows*.

shortly thereafter that every finitely generated projective $A[t]$-module is extended from an A-algebra, where A is a smooth algebra containing a field k. We could read this as saying that the stack of algebraic vector bundles is $\mathbb{A}^1$-invariant over the category of smooth affine k-schemes.

More general: (Bass–Quillen conjecture) is it true that for every R regular Noetherian, the map
$$\mathrm{Mod}_R^{\mathrm{f.g., \, proj}} \to \mathrm{Mod}_{R[t]}^{\mathrm{f.g., \, proj}}$$
$$M \mapsto M \otimes_R R[t]$$
is essentially surjective? *Still open.*

Fundamental Theorem of Algebraic K-Theory (Quillen): For R regular Noetherian, we have that $R \to R[t]$ induces an equivalence[34]
$$K(R) \xrightarrow{\sim} K(R[t]).$$
In other K-theory is $\mathbb{A}^1$-*invariant* for regular Noetherian rings (regular Noetherian schemes, more generally).

Example 0.4. If $X = \mathrm{Spec}\,(R)$ or more generally X is a scheme, then the map $\pi \colon X \times \mathbb{A}^1 \to X$ induces an isomorphism on Chow groups (see for instance [Ful98, 3.3])
$$\pi^* \colon \mathrm{CH}_*(X) \xrightarrow{\sim} \mathrm{CH}_{*+1}(X \times \mathbb{A}^1).$$

Example 0.5. Let $X = \mathrm{Spec}\,(R)$ where R is normal and Noetherian.[5] Then every line bundle on $X \times \mathbb{A}^1$ is extended from a line bundle on X, in other words $X \times \mathbb{A}^1 \to X$ induces an isomorphism
$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(X \times \mathbb{A}^1).$$

Example 0.6. We can also show $\mathbb{A}^1$-invariance for the Picard group over a PID. Let R be a PID, then it is a UFD, and we can show that $\mathrm{Pic}(R) = 0$, and therefore $\mathrm{Pic}(R[t_1, \ldots, t_n]) = 0$.

Example 0.7. [Aso21, 3.7.1.5] Check this doesn't hold for all rings, for example $R = k[x, y]/(y^2 - x^3)$.

Definition 0.8. An *inner product space* over a ring R is a finitely generated productive R-module M and a symmetric bilinear form $\beta \colon M \times M \to R$ for which $m \mapsto \beta(-, m)$ defines an isomorphism $M \cong M^*$.

Theorem 0.9. (Harder's Theorem, VII.3.13 in Lam's book on Serre's problem) Let k be a field. Then every inner product space over $k[t]$ is extended from an inner product space over k.

Remark 0.10. The stable analogue of this has to do with $\mathbb{A}^1$-invariance of Hermitian K-theory [reference needed].

Algebraic vector bundles are GL_n-torsors (we will talk about torsors in more detail next week), so the Bass–Quillen conjecture is really asking about $\mathbb{A}^1$-invariance of torsors over affine schemes. We could ask an analogous question about G-torsors for any G. Here's an example result in this direction that we'll see later in the semester:

Theorem 0.11. [AHW20, 1.3] If k is a field, and G is an isotropic reductive group scheme, then G-torsors in the Nisnevich site are $\mathbb{A}^1$-invariant over any smooth affine k-scheme.

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[3]So Bass–Quillen is really a question about *unstable* modules.

[4]The statement for K_0 is originally due to Grothendieck [Aso21, 5.6.1.3]. The statement for K_1 is due to Bass–Heller–Swan [Aso21, 5.8.2.1].

[5]We can get away with weaker assumptions on this, for example in [Aso21, 3.7.13] it is only assumed that R is a locally factorial Noetherian normal domain.

0.2. $\mathbb{A}^1$**-homotopy theory.** Recall from algebraic topology that $X \times [0, 1] \to X$ is a weak homotopy equivalence, which implies that any cohomology theory is insensitive to taking a product with an interval, e.g. for $H^*(-, \mathbb{Z})$ integral cohomology we get

$$H^*(X, \mathbb{Z}) \xrightarrow{\sim} H^*(X \times [0, 1], \mathbb{Z}).$$

In fact this type of homotopy invariance is an axiom of generalized Eilenberg–Steenrod cohomology theories.

Example 0.12. Let $k \subseteq \mathbb{C}$ be a subfield of the complex numbers. Then there is a *Betti realization* functor

$$\mathrm{Var}_k \to \mathrm{Top}$$
$$X \mapsto X(\mathbb{C})$$

sending a variety to its underlying analytic space. Note that

$$X \times \mathbb{A}^1_k \mapsto (X \times \mathbb{A}^1_k)(\mathbb{C}) = X(\mathbb{C}) \times \mathbb{C}.$$

Therefore any homotopy invariant functor out of spaces provides another example of an $\mathbb{A}^1$-invariant functor out of k-varieties, for example

$$X \mapsto H^*(X(\mathbb{C}); \mathbb{Z})$$
$$X \mapsto \pi_*(X(\mathbb{C})).$$

Q: Can we build a homotopy theory of algebraic varieties in which the affine line $\mathbb{A}^1$ plays the role that the interval $[0, 1]$ plays in classical topology?

A: Yes! This is what's known as $\mathbb{A}^1$-*homotopy theory* or *motivic homotopy theory*. This dates back to Morel and Voevodsky's seminal work in 1999, but many ideas date back to work of Karoubi–Villamayor, Jardine, Weibel in the 1980's, work of Brown, Gersten, Illusie and Joyal in the 1970's, and of Quillen and Grothendieck in the 1960's.

Q: What can we do with motivic homotopy theory?

A: Motivic homotopy theory blends algebraic geometry and topology in a beautiful way, allowing us to do many things. A short list is:

▷ we obtain a natural home for these kinds of $\mathbb{A}^1$-invariant cohomology theories of varieties, and can study them in a method analogous to algebraic topology (e.g. we can classify cohomology operations, use fiber sequences to carry out computations, unify various spectral sequences, etc.)

▷ we have access to obstruction theory for classifying torsors over affine varieties — recent success in this direction is Asok–Bachmann–Hopkins' resolution of Murthy's conjecture

▷ we can eliminate differentials in the Adams spectral sequence via looking at their motivic depth (citation needed)

Q: What techniques will we learn in this class?

A: We should be able to do the following things:

▷ learn descent and torsors more thoroughly, and unify perspectives on sheaves and stacks using higher categorical language

▷ work with ∞-categories at a very basic level

▷ learn how to work with motivic spaces and do computations

▷ learn motivic obstruction theory and its applications

0.3. **A bit more about these notes.**

Remark 0.13. *(Transparency about pedagogy)* We're going to place affine representability on the horizon and keep our eyes fixed on it in order to motivate the category of motivic spaces. This is of course ahistorical— although affine representability was one of the key early results explored by Morel, motivic spaces and spectra were developed in order to house theories such as algebraic

K-theory, Bloch's higher Chow groups, and to grow new theories such as algebraic cobordism. We chose this route for a few reasons:

(1) The Fall 2024 Thursday seminar is on the motivic Steenrod algebra, so we're hoping this class will contrast nicely and provide some foundational background in techniques in motivic obstruction theory.

(2) In order to tailor the course to a broader audience, we'd like to unify the class around a key question which has general appeal, so we've picked the classification of torsors over affine varieties using motivic methods as such a question. This has certain advantages, for instance we can pause in the sheaf topos and discuss classifying torsors there before building motivic spaces — this vista is useful to people across many fields.

(3) This forces us to spend a bit longer on descent and torsors, two words which strike fear in the heart of many, and are worth investing some more time in.

Remark 0.14. *(On background)* We're assuming a strong handle of algebraic geometry and category theory, and a fair bit of familiarity with commutative algebra and homotopy theory. We'll see very quickly that ∞-categories (and/or model categories) are needed in order to develop the setting in which we wish to work. We've elected to take the ∞-categorical approach, since it streamlines many of the constructions and key ideas, at the cost of being a high technical investment; for this reason we've done our best to make the ∞-categorical machinery easy to black box. The reader should be aware that while the categorical language will be heavy in the first half of these notes, it will quickly fade into the background as we become familiar with the ambient setting we're working in and can set our focus towards computations.

1. Lecture 1: Torsors

Assumption 1.1. For this lecture, every time we say "cover" you can assume we are working with Zariski covers, or even open covers of topological spaces, since the intuition will be the same and the results here will be mostly identical. If you know about other topologies, the statements here work for any site. We will go into sheaves and sites more next week, when we will remark that everything here works for other nice sites (étale, Nisnevich, flat, etc.).

Definition 1.2. Let G be a group. Then a G-set X is called a *torsor* if its G-action is simple and transitive. Equivalently, the map

$$
\begin{aligned}
G \times X &\to X \times X \\
(g, x) &\mapsto (x, gx)
\end{aligned}
$$

(3)

is a bijection.

Note there are two types of G-torsors — sets of the form G/e, and the empty set. Depending on the convention, we might want to exclude the case of the empty set by including $X \neq \varnothing$ in the definition.

Intuition 1.4. A G-torsor is like a group G which has remembered its multiplication but forgotten its identity. Any choice of basepoint $x \in X$ yields a canonical bijection $G \xrightarrow{\sim} X$ by sending $g \mapsto g \cdot x$.

Example 1.5. In a locally small category $\mathscr{C}$, given two objects $x, y \in \mathscr{C}$, the set of isomorphisms $\mathrm{Isom}_{\mathscr{C}}(x, y)$ is a left $\mathrm{Aut}_{\mathscr{C}}(x)$-torsor and a right $\mathrm{Aut}_{\mathscr{C}}(y)$-torsor.

Let's try to extend this definition to the setting where G isn't a single group, but a *sheaf of groups* $\mathcal{G}$ on a site. What is the appropriate analogue of a torsor in this setting? By abuse of notation we will also call this a *torsor*.

Definition 1.6. [Stacks, 03AH] Let $\mathcal{G}$ be a sheaf of groups on X, and let $\mathrm{Shv}_{\mathcal{G}}(X)$ denote the category of $\mathcal{G}$-sheaves, meaning sheaves of sets equipped with a $\mathcal{G}$-action, and equivariant morphisms between them. We define the category of $\mathcal{G}$-*torsors* $\mathrm{Tors}_{\mathcal{G}}(X) \subseteq \mathrm{Shv}_{\mathcal{G}}(X)$ to be the full subcategory on those $\mathcal{F}$ so that

(1) if $\mathcal{F}(U)$ is non-empty then the action
$$\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$$
is simply transitive[6]

(2) there exists a covering $\{U_i \to X\}$ over which $\mathcal{F}(U_i) \neq \varnothing$.[7]

Terminology 1.7. The choice of topology comes into play in that second point. If $\mathcal{F} \in \mathrm{Shv}_{\mathcal{G}}(X)$, we say it is τ-*locally trivial* if $\mathcal{G}$ is a τ-sheaf of groups, $\mathcal{F}$ is a τ-sheaf of sets, and point (2) holds for any τ-cover.

Example 1.8. The sheaf $\mathcal{G}$, acting on itself by scaling, is called the *trivial $\mathcal{G}$-torsor*.

Example 1.9. For any group scheme G, we will refer to G-*torsors*, mean torsors the representable functor $\mathrm{Hom}(-, G)$.

Proposition 1.10. A $\mathcal{G}$-torsor $\mathcal{F}$ on X is trivial if and only if $\mathcal{F}(X) \neq \varnothing$, i.e. if it admits a global section.

Theorem 1.11. Every morphism in $\mathrm{Tors}_G(X)$ is an isomorphism.

So what are some examples of torsors, and why might we care to classify them?

Example 1.12. If L/k is a Galois field extension then $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is a $\mathrm{Gal}(L/k)$-torsor in any topology.

▷ *the inverse Galois problem*: which groups G occur as Galois groups of number fields? This is asking to scratch the surface of understanding G-torsors over $\mathrm{Spec}(\mathbb{Q})$ for all finite groups G.

Example 1.13. A GL_n-torsor (say in the Zariski topology) is an algebraic vector bundle.

Proof. Since GL_n is affine, every torsor is representable, hence GL_n-torsors are just principal GL_n-bundles, which are precisely algebraic vector bundles. $\qquad\square$

▷ *Bass–Quillen conjecture*: this can be reframed as asking whether each GL_n-torsor over a regular Noetherian ring is trivial

▷ *Hartshorne's conjecture* concerns GL_2-torsors over $\mathbb{P}^n$ for $n \geq 7$

Example 1.14. A PGL_n-torsor is a Brauer–Severi variety (or a central simple algebra).

▷ the *period-index conjecture* then concerns the complexity of PGL_n-torsors.

Goal 1.15. Develop methods to classify torsors.

Let's do this, by first considering an alternative perspective on what a torsor is. We learned this from Alex Youcis' excellent note on torsors [You].

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[6]This means that in the category of $\mathcal{G}(U)$-sets, we have that $\mathcal{F}(U) \cong \mathcal{G}(U)$, but $\mathcal{F}(U)$ doesn't have a group structure — we might imagine that it has forgotten its identity element. Picking a basepoint $e \in \mathcal{F}(U)$ defines a group structure on $\mathcal{F}(U)$.

[7]In other words, we can find a cover in which to visualize $\mathcal{F}(U_i)$ as a group for each i.

1.1. **Sheaves and stacks.** Recall if $\mathscr{F}$ is a sheaf of sets, this means for every cover $\{U_i \to U\}$, we have that the diagram is an equalizer

$$\mathcal{F}(U) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j).$$

This first map is a monomorphism (injection) because it is an equalizer. This means that if $x, y \in \mathcal{F}(U)$ are equal in $\mathcal{F}(U_i)$ for each i, then they are equal in $\mathcal{F}(U)$. In other words, the map "reflects equality" — this is literally just what it means for something to be an injection.

Let's suppose now that $\mathcal{F}(U)$ is a *category* for every U. We'll define this concretely soon once we have more machinery, but for now let's just pretend that we know what this means — it means we can glue objects and morphisms along covers. Consider the analogous restriction functor:

$$\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$$
$$x \mapsto (x|_{U_i})_i\,.$$

Q: Does this map need to reflect isomorphisms?[8]

Example 1.16. Let $\mathcal{F}$ be the functor sending U to the category of line bundles over U. If the cover is picked appropriately small, then all line bundles are isomorphic to the trivial line bundle over U_i, but they need not be isomorphic globally as line bundles over U.

A: No, by the example above. This means that we can have $x, y \in \mathcal{F}(U)$ so that $x|_{U_i} \cong y|_{U_i}$ for each i, but we *do not have* that x and y are isomorphic in the category $\mathcal{F}(U)$. In other words, the two notions are *different*:

▷ x and y agree *globally*, meaning x and y are isomorphic in $\mathscr{F}(U)$
▷ x and y agree *locally*, meaning there exists an open cover $\{U_i \to U\}$ for which we have isomorphisms $x|_{U_i} \xrightarrow{\sim} y|_{U_i}$ in $\mathcal{F}(U_i)$.[9]

Remark 1.17. This is a big difference between sheaves of sets (or 1-categories in general) and sheaves of categories (also called *stacks*). Equality is reflected along a cover for sheaves of sets, but isomorphism is not necessarily reflected along a cover.

Question 1.18. How many isomorphism classes of objects $y \in \mathcal{F}(U)$ are *locally isomorphic* to x along a cover, but not globally isomorphic?

We're going to build a sheaf that measures this! We'll call this sheaf $\mathrm{Aut}_{\mathcal{F}}(x)$, and it is defined by

(19) $$U_i \mapsto \mathrm{Aut}_{\mathscr{F}(U_i)}(x)$$

Exercise 1.20. This is a priori just a presheaf of groups. Check it is actually a sheaf of groups.

Recall: If $\mathcal{G}$ is a sheaf of groups over U, and $\mathcal{U} = \{U_i \to U\}$ is a cover.
(1) We define a *Čech 1-cocycle* to be a collection of elements $g_{ij} \in \mathcal{G}(U_i \times_U U_j)$ for each i, j so that

$$g_{jk}g_{ij} = g_{ik}$$

on triple overlaps.
(2) We say two 1-cocycles (g_{ij}) and (γ_{ij}) are cohomologous if there are $\alpha_i \in \mathcal{G}(U_i)$ for each i so that

$$\alpha_j g_{ij} = \gamma_{ij}\alpha_i.$$

(3) We define the *Čech cohomology* $\check{H}^1(U, \mathcal{G})$ to be the colimit of the Čech cohomology over covers, filtered with respect to refinement.

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[8]A functor which reflects isomorphisms is called *conservative*.
[9]The notation $x|_{U_i}$ is shorthand for the image of x under the restriction functor $\mathcal{F}(U) \to \mathcal{F}(U_i)$

Theorem 1.21. There is a bijection between $H^1(U, \mathrm{Aut}_{\mathcal{F}}(x))$ and isomorphism classes of objects $y \in \mathscr{F}(U)$ which are locally isomorphic to x.

Proof. Let's first define a map. If y is locally isomorphic to x, then there is a cover $\{U_i\}$ of U and isomorphisms $\phi_i \colon x|_{U_i} \xrightarrow{\sim} y|_{U_i}$ for each i. If the ϕ_i's agreed on overlaps then they would glue to a global isomorphism $\phi \colon x \xrightarrow{\sim} y$ because $\mathrm{Aut}(x)$ is a sheaf of groups, so it makes sense to look on overlaps to see what happens. Note that $\phi_i|_{U_{ij}}$ and $\phi_j|_{U_{ij}}$ will differ by an automorphism of $x|_{U_{ij}}$, call this g_{ij}:

$$ g_{ij} := \left(\phi_j|_{U_{ij}} \right)^{-1} \left(\phi_i|_{U_{ij}} \right) \colon\; x|_{U_{ij}} \xrightarrow{\sim} x|_{U_{ij}} . $$

On triple overlaps, these satisfy the relations (check)

$$ g_{jk} g_{ij} = g_{ik}. $$

In other words we get a 1-cocycle! There was ambiguity here, since we *picked* isomorphisms $x|_{U_i} \xrightarrow{\sim} y|_{U_i}$ as our starting data. The remaining thing to prove is that any other choice of local isomorphisms gives rise to a cohomologous 1-cocycle.

Suppose we instead picked some $\psi_i \colon x|_{U_i} \xrightarrow{\sim} y|_{U_i}$ for each i, yielding $\gamma_{ij} = \psi_j^{-1} \psi_i$. Then ψ_i and ϕ_i differ by an automorphism of y which we call α_i:

$$ \begin{array}{ccc} x|_{U_i} & \xrightarrow{\quad \phi_i \quad} & y|_{U_i} \\ & \alpha_i \searrow \quad \nearrow \psi_i & \\ & x|_{U_i} & \end{array} $$

Then on U_{ij} we have

$$ g_{ij} = \phi_j^{-1} \phi_i = (\psi_j \alpha_j)^{-1} \psi_i \alpha_i = \alpha_j^{-1} \gamma_{ij} \alpha_i. $$

Rescaling on the left by α_j we get

$$ \alpha_j g_{ij} = \gamma_{ij} \alpha_i, $$

so (g_{ij}) and (γ_{ij}) give cohomologous 1-cocycles. $\square$

Theorem 1.22. There is a bijection between isomorphism classes of $\mathcal{G}$-torsors and $\check{H}^1(U, \mathcal{G})$.

Sketch. Let $\mathcal{F}$ be a $\mathcal{G}$-torsor, and pick $s_i \in \mathcal{F}(U_i)$ for each i. Then on the overlap $U_i \times_U U_j$, we have that s_i and s_j differ by a unique element $g_{ij} \in \mathcal{G}(U_i \times_U U_j)$. We run basically an identical argument. $\square$

So if $\mathcal{G} = \mathrm{Aut}_{\mathcal{F}}(x)$ then we have a bijection

$$ \left\{ \begin{array}{c} \text{iso classes of } y \in \mathcal{F}(U) \\ \text{locally isomorphic to } x \end{array} \right\} \leftrightarrow \check{H}^1(U, \mathrm{Aut}_{\mathcal{F}}(x)) \leftrightarrow \{ \mathrm{Aut}_{\mathcal{F}}(x)\text{-torsors} \} . $$

These sorts of arguments are compatible with refinement of the cover, and since Čech and sheaf cohomology agree we see that G-torsors are in bijection with the first sheaf cohomology $H^1(U, G)$.

Exercise 1.23. Show that every sheaf of groups $\mathcal{G}$ is of the form $\mathrm{Aut}_{\mathcal{F}}(x)$ for some sheaf of categories $\mathcal{F}$.

Intuition 1.24. A $\mathcal{G}$-torsor is an object whose automorphisms locally look like $\mathcal{G}(U_i)$.

1.2. **Interlude: representable G-torsors.** When both the presheaf $\mathcal{G}$ and the sheaf of sets $\mathcal{F}$ are representable, we get a slightly different characterization.

Setup 1.25. Suppose $\mathscr{C} = \mathrm{Sch}_X$ is a site of schemes over X, and let $G \in \mathrm{Grp}(\mathrm{Sch}_X)$ be a group scheme over X. Suppose $\mathcal{G} = \mathrm{Hom}_X(-, G)$ is a representable sheaf of groups, and let $\mathcal{F} = \mathrm{Hom}_X(-, Y)$ for some $Y \in \mathrm{Sch}_X$, where Y comes equipped with a G-action.

In this setup, what does it mean in this case for $\mathcal{F}$ to be a $\mathcal{G}$-torsor?

Condition (1) from Definition 1.6 asks that
$$\mathrm{Hom}_X(U, G) \times \mathrm{Hom}_X(U, Y) \to \mathrm{Hom}_X(U, Y)$$
to be simply transitive for every $U \in \mathrm{Sch}_X$. This seems a bit tedious to check, but the following result gives us a cleaner characterization of it, which is a sheafy version of Equation 3.

Proposition 1.26. [Stacks, 0499] In Setup 1.25 the following two conditions are equivalent:

(1) The map
$$G \times_X Y \to Y \times_X Y$$
$$(g, y) \mapsto (y, gy)$$
is an isomorphism of X-schemes.

(2) For every $U \in \mathrm{Sch}_X$, the induced action
$$\mathrm{Hom}_X(U, G) \times \mathrm{Hom}_X(U, Y) \to \mathrm{Hom}_X(U, Y)$$
is simply transitive.

What about condition (2) from Definition 1.6? This condition translates, in Setup 1.25, to asking for a cover $\{U_i \to X\}$ for which $\mathrm{Hom}_X(U_i, Y) \neq \varnothing$ for each i. Pick any $s_i \in \mathrm{Hom}_X(U_i, Y)$, then it makes the diagram commute:

$$\begin{array}{ccc} & & Y \\ & \nearrow^{s} & \downarrow \\ U_i & \longrightarrow & X. \end{array}$$

Then the pullback torsor $Y \times_X U_i \to U_i$ admits a section, which implies it is trivial by Proposition 1.10:

$$\begin{array}{ccc} Y \times_X U_i & \longrightarrow & X \\ \uparrow\downarrow & \lrcorner & \downarrow \\ U_i & \longrightarrow & X. \end{array}$$

We summarize these observations in the following proposition.

Proposition 1.27. Let X be a scheme, let G be a group scheme over X, and let
$$f \colon Y \to X$$
be an X-scheme equipped with a G-action. Then Y is a G-torsor if and only if

(1) The map $G \times_X Y \to Y \times_X Y$ is an isomorphism of X-schemes.

(2) There exists a cover $\{U_i \to X\}$ for which $Y \times_X U_i \to U_i$ is isomorphic to the trivial G-torsor over U_i.

In this setting, we call $Y \to X$ a *principal G-bundle*.

Example 1.28. Let's double back to Example 1.12 and actually prove that $\mathrm{Spec}\, L \to \mathrm{Spec}\, k$ is a principal $\mathrm{Gal}(L/k)$-bundle when L/k is a Galois field extension.

Proof. We check the conditions from Proposition 1.27. The first condition asks that
$$\left(\coprod_{g \in G} \mathrm{Spec}(k) \right) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L) \to \mathrm{Spec}(L) \times_{\mathrm{Spec}\, k} \mathrm{Spec}(L)$$
is an equivalence. Since everything in sight is affine, we can rephrase this as asking for the map
$$\prod_{g \in G} L \leftarrow L \otimes_k L$$
$$(x \cdot g(y))_{g \in G} \leftarrow\!\shortmid x \otimes y$$

to be an equivalence, which we recall is the *normal basis theorem* from Galois theory. The second condition from Proposition 1.27 asks us to find a cover of $\mathrm{Spec}(k)$ trivializing $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$, but if we work in the étale site, then $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is itself a cover, and it is clear that the product induces a section

$$\begin{array}{ccc} \mathrm{Spec}(L \otimes_k L) & \longrightarrow & \mathrm{Spec}(L) \\ \uparrow\downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(k). \end{array}$$

Hence we have a trivialization, and hence a principal $\mathrm{Gal}(L/k)$-bundle (a $\mathrm{Gal}(L/k)$-torsor). $\square$

Often it's just enough to assume the sheaf of groups is representable to get all torsors are representable.

Proposition 1.29. Let G be an affine group scheme over X, and let $\tau \leq \mathrm{fppf}$. Then every G-torsor is representable. (see [Stacks, 0497], [You, 3.25])

Sketch. Every G-torsor is an algebraic space, and algebraic spaces which are locally affine are schemes. Since a G-torsor is locally isomorphic to G, which was assumed to be affine, then we conclude every torsor is actually a scheme. $\square$

Terminology 1.30. A representable fpqc-torsor for G is called a *principal homogeneous space*.

Remark 1.31. If $t \leq \tau$, then every t-cover is a τ-cover, hence if we can find a t-cover trivializing a G-torsor, then it also trivializes it in the τ-topology. hence

$$t \leq \tau \Rightarrow \{t\text{-torsors}\} \subseteq \{\tau\text{-torsors}\}.$$

So a very natural question is *how do we tell when a τ-torsor is also a t-torsor?* We'll discuss this next week.

1.3. **Interlude: representability of torsors in topology.** So it's a very reasonable goal to ask for any tools that could help us try to classify torsors. A natural idea, by analogy, is to look to homotopy theory, where we have a suite of tools for studying torsors.

Definition 1.32. Let X be a compact Hausdorff topological space and G a group. Then a *principal G-bundle* (or we might just say a G-torsor) is a fiber bundle $\pi \colon Y \to X$ so that G acts freely and transitively on the fibers.

In topology there is a *universal G-torsor*, which is denoted $EG \to BG$. This is universal in the sense that, given any map $f \colon X \to BG$, we can consider the fiber product

$$\begin{array}{ccc} f^*EG & \longrightarrow & EG \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\;\;f\;\;} & BG. \end{array}$$

Then $f^*EG \to X$ is a principal G-bundle, and all principal G-bundles are obtained in this way. Not only that, but isomorphic principal G-bundles are given by homotopic classifying maps. In other words we have a bijection

$$\mathrm{Prin}_G(X) \leftrightarrow [X, BG].$$

So the data of a principal G-bundle is the data of a map $X \to BG$, and an isomorphism of principal G-bundles is equivalent to a homotopy between two maps $f, g \colon X \to BG$.

Example 1.33. We have that $B(\mathbb{Z}/2) = \mathbb{R}\mathrm{P}^\infty$ and $E\mathbb{Z}/2 = S^\infty$, so that $\mathbb{Z}/2$-torsors are real line bundles. Similarly $B\mathbb{C}^\times = \mathbb{C}\mathrm{P}^\infty$.

A big example comes from quotienting out by a compact subgroup:

Theorem 1.34. (Samelson, 1941) If $H \leq G$ is a compact subgroup of a Lie group, then $G \to G/H$ is a Serre fibration and principal H-bundle.

Corollary 1.35. We have fiber sequences
$$H \to G \to G/H$$
$$G/H \to BH \to BG.$$

Example 1.36. For the inclusions $O(n) \subseteq O(n+1)$ and $U(n) \subseteq U(n+1)$ we get fiber sequences
$$S^{2n+1} \to \mathrm{BU}(n) \to \mathrm{BU}(n+1)$$
$$S^n \to \mathrm{BO}(n) \to \mathrm{BO}(n+1).$$

This is how Bott periodicity is proved.

1.4. **Why we like representability of torsors.** This has a number of huge applications:

▷ Given any cohomology theory E^* and any class $c \in E^*(BG)$, if $f^*(c) \neq g^*(c)$ in $E^*(X)$, this means that f and g correspond to non-isomorphic torsors. This the basic idea of characteristic classes.

Example 1.37. If $G = \mathrm{GL}_n(\mathbb{C})$, then $\mathrm{BGL}_n(\mathbb{C}) = \mathrm{BU}(n) = \mathrm{Gr}_{\mathbb{C}}(n, \infty)$ is a Grassmannian of n-planes in $\mathbb{C}^\infty$. A map $f \colon X \to \mathrm{Gr}_{\mathbb{C}}(n, \infty)$ gives a complex n-dimensional vector space by pullback.

Theorem 1.38. (Pontryagin–Steenrod) There is a bijection
$$\mathrm{Vect}_{\mathbb{C}}^n(X) \cong [X, \mathrm{BU}(n)] \, .$$
Since $H^*(\mathrm{BU}(n); \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n]$, if $E \to X$ is any rank n vector bundle classified by a map $f \colon X \to \mathrm{BU}(n)$, then its Chern classes are by definition $c_i(E) = f^* c_i$.

▷ We have access to *obstruction theory* – this lets us break down lifting problems into smaller manageable stages.

Example 1.39. If X is a complex n-dimensional manifold, and $E \to X$ is a rank n complex vector bundle, then it splits off a free summand if and only if $c_n(E) = 0$.

Example 1.40. If we fix $c_1, \ldots, c_n \in H^*(X; \mathbb{Z})$, we can ask how many isomorphism classes of complex rank n vector bundles on X have these given Chern classes. Since $c_i \in H^{2i}(X; \mathbb{Z}) = [X, K(\mathbb{Z}, 2i)]$ this is equivalent to asking how many lifts there are for

$$
\begin{array}{ccc}
 & & \mathrm{BU}(n) \\
 & \nearrow & \downarrow \\
X & \xrightarrow[c_1, \ldots, c_n]{} & \prod_i K(\mathbb{Z}, 2i).
\end{array}
$$

If X is a finite CW complex, there are only finitely many such lifts by basic obstruction theory.

▷ Suppose we have two groups G and K, and we want to study natural ways to create K-torsors out of G-torsors over any space. Then representability, combined with the Yoneda lemma, allows us to completely classify all the ways to do this.

Example 1.41. There is one and only one natural non-trivial function
$$\{\mathrm{GL}_n(\mathbb{R})\text{-torsors}\} \to \{(\mathbb{Z}/2)\text{-torsors}\} \, ,$$
given by the nonzero class in $[\mathrm{BGL}_n(\mathbb{R}), B\mathbb{Z}/2] = \mathbb{Z}/2$. This is called the *determinant bundle* or the *first Stiefel–Whitney class*.

1.5. **Motivation of what's to come.**

Question 1.42. By analogy to homotopy theory, we might ask, for a group scheme G, the following questions:

(1) Is there an analogous universal space BG in algebraic geometry which classifies G-torsors?
(2) If so, can we classify G-torsors over X via some "homotopy classes" of maps from X to BG?

The answer to both will be yes, but *not in the category of varieties*. We need more machinery than is available to us there.

2. LECTURE 2: TORSORS II: SIMPLICIAL METHODS

Definition 2.1. We denote by Δ the category whose objects are finite ordered sets of the form $[n] = \{0 < 1 < \cdots < n\}$, and whose morphisms $[n] \to [m]$ are order-preserving maps.

Notation 2.2. Let $\mathscr{C}$ be a category. Then we refer to a functor $\Delta \to \mathscr{C}$ as a *cosimplicial object* of $\mathscr{C}$, and a functor $\Delta^{\mathrm{op}} \to \mathscr{C}$ as a *simplicial object* of $\mathscr{C}$.

Intuition 2.3. We should think about a (co)simplicial object as a *data type* — it is a combinatorial gadget that is surprisingly convenient for bookkeeping and appears frequently in nature. We'll see quite a few examples, but let's first see how to compress this data.

There are a priori a lot of order-preserving functions $[n] \to [m]$, so we'd like a nice class of easy-to-manage morphisms in Δ so that any morphism factors into nice morphisms. These fall in two classes:
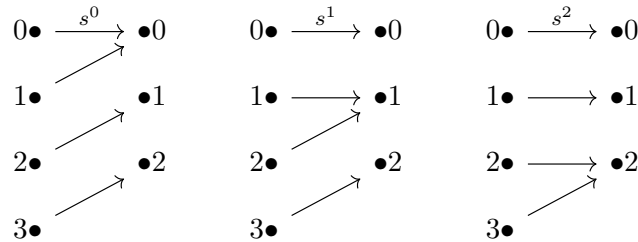
Terminology 2.4.
(1) We have *coface* maps $d^i \colon [n] \to [n+1]$ for $0 \le i \le n+1$ which are defined by the property that they miss the element $i \in [n+1]$.
(2) We have *codegeneracy* maps $s^j \colon [n] \to [n-1]$ for $0 \le j \le n-1$ defined by the property that they are surjective and that $s^j(j) = s_j(j+1)$.

Example 2.5. The coface maps $d^i \colon [2] \to [3]$ look like
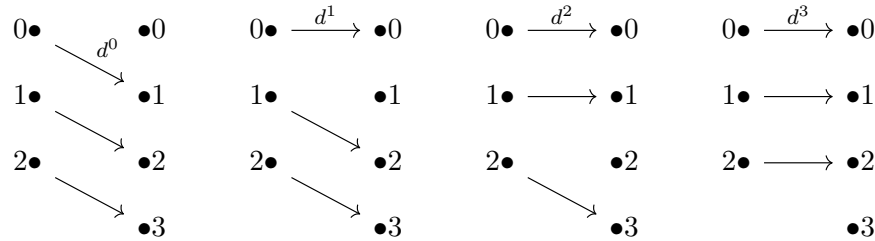
Example 2.6. The codegeneracy maps $[3] \to [2]$ look like

Terminology 2.7. Let $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathscr{C}$ denote a simplicial object
(1) We use the notation X_n to denote the object $X_\bullet([n])$ in $\mathscr{C}$.
(2) We denote by $d_i \colon X_n \to X_{n-1}$ for $0 \le i \le n$ the image of the coface map d^i. We call d_i a *face map*
(3) Similarly we denote by $s_i \colon X_n \to X_{n+1}$ for $0 \le i \le n$ the image of the codegeneracy map s^i, and call s_i a *degeneracy map*.

Exercise 2.8. Convince yourself that every map in Δ factors into face and degeneracy maps, and therefore any simplicial object $X_\bullet$ in $\mathscr{C}$ can be described by the data of X_n for each n, and its face and degeneracy maps.[10]

Definition 2.9. If $\mathscr{C}$ is a locally category with all finite products (binary products and a terminal object) then a *group object* in $\mathscr{C}$ is the data of an object $G \in \mathscr{C}$ together with morphisms

$$m\colon G \times G \to G$$
$$e\colon 1 \to G$$
$$i\colon G \to G$$

called multiplication, identity, and inverse, so that the diagrams commute

$$
\begin{array}{ccc}
G \times G \times G & \xrightarrow{\mathrm{id}\times m} & G \times G \\
{\scriptstyle m\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle m} \\
G \times G & \xrightarrow{m} & G
\end{array}
\qquad
\begin{array}{ccc}
G & \xrightarrow{e\times\mathrm{id}} & G \times G \\
{\scriptstyle \mathrm{id}\times e}\downarrow & {\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle m} \\
G \times G & \xrightarrow{m} & G
\end{array}
\qquad
\begin{array}{ccc}
G & \xrightarrow{i\times\mathrm{id}} & G \times G \\
{\scriptstyle \mathrm{id}\times i}\downarrow & {\scriptstyle e} & \downarrow{\scriptstyle m} \\
G \times G & \xrightarrow{m} & G.
\end{array}
$$

Example 2.10. Let G be a group object in a category $\mathscr{C}$ as above. Then we can define a simplicial object

$$B_\bullet G\colon \Delta^{\mathrm{op}} \to \mathscr{C}$$
$$[n] \mapsto G^{\times n},$$

as:

$$* \rightleftarrows G \; \substack{\longleftarrow \\ \rightleftarrows \\ \longleftarrow} \; G \times G$$

Explicitly, $B_n G = G^{\times n}$, with face maps $d_i\colon G^{\times n} \to G^{\times(n-1)}$ given by

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, g_{i+2}, \ldots, g_n) & 1 \le i \le n-1 \\ (g_1, \ldots, g_{n-1}) & i = n, \end{cases}$$

and degeneracies $s_i\colon G^{\times n} \to G^{\times(n+1)}$ given by

$$s_i(g_1, \ldots, g_n) = \begin{cases} (e, g_1, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_n) & 1 \le i \le n. \end{cases}$$

Exercise 2.11. Check the simplicial identities hold *precisely* because of the group axioms. In particular, observe that we didn't use anything special about spaces — if G is a group object in any category $\mathscr{C}$ we obtain an associated bar construction which is a simplicial object in $\mathscr{C}$ that we call $B_\bullet G$.

Example 2.12. (Important) For the formal categorical reason in Exercise 2.11, any group scheme G gives rise to a simplicial scheme $B_\bullet G \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sch})$. This dates back at least to work of Friedlander [Fri82, Example 1.2]. We'll use this object frequently.

Example 2.13. Simplicial objects appeared crucially in Deligne's work on resolution of singularities and mixed Hodge structures. Let X be a complete[11] singular variety. Then we can "replace" X with a simplicial variety

$$\cdots \quad X_2 \; \substack{\longrightarrow \\ \rightleftarrows \\ \longrightarrow} \; X_1 \; \rightrightarrows \; X_0,$$

where each X_i is smooth and projective. There is a cohomological descent spectral sequence computing the cohomology of X in terms of the cohomology of the X_n's. This dates back to

---

[10]Any time we define a simplicial object via this compressed data, we should check that the simplicial identities hold, although we'll mostly omit these arguments here.

[11]X is *complete* if it is proper over the base [Har77, p. 105]. This is a slightly more general notion than projective.

SGA 4. Additionally, this replacement is the core idea of *resolution of singularities*. Deligne uses these simplicial methods to endow singular varieties with Hodge structures, so-called *mixed Hodge structures* [Del74].

Example 2.14. Let $A_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Ab}$ be a simplicial abelian group. Then by taking the alternating sum of the face maps, we obtain a chain complex, purely by the simplicial identities. This assignment is functorial:

$$\mathrm{sAb} \to \mathrm{Ch}_{\geq 0}(\mathrm{Ab})$$

$$A_\bullet \mapsto \left(A_n, \partial = \sum_{i=0}^{n} (-1)^i d_i \right).$$

The associated chain complex is called the *Moore complex* of the simplicial abelian group. Two quick remarks about this:

(1) This works if we replace Ab by any abelian category.
(2) This process is invertible — this implies that simplicial objects and connective chain complexes in any abelian category are equivalent.[12] We will use this later, as we will want to construct certain simplicial objects in sheaves of abelian groups, and it will be more direct to construct them first as chain complexes, then pass through this equivalence.

Covers, their overlaps, triple overlaps, etc. are also conveniently encoded in the data of a simplicial object.[13]

Notation 2.15. If $\{U_i \to X\}_{i \in I}$ is a cover, we'll denote by $U_{ij} := U_i \times_X U_j$, and similarly $U_{ijk} := U_i \times_X U_j \times_X U_k$.

Example 2.16. Let $\mathcal{U} = \{U_i \to X\}$ be an open cover of a variety. We define a simplicial object

$$\coprod_i U_i \; \underset{\longleftarrow}{\overset{\longleftarrow}{\rightleftarrows}} \; \coprod_{i,j} U_{ij} \; \overset{\longleftarrow}{\underset{\longleftarrow}{\rightleftarrows}} \; \coprod_{i,j,k} U_{ijk} \qquad \cdots$$

which we call the *Čech nerve* associated to the cover, and we denote by $N(\mathcal{U})$. By universal property of the pullback, we obtain three projection maps, which become our face maps $d_i \colon N(\mathcal{U})_2 \to N(\mathcal{U})_1$:

$$d_0 \colon U_{ijk} \to U_{ij}$$
$$d_1 \colon U_{ijk} \to U_{ik}$$
$$d_2 \colon U_{ijk} \to U_{jk}.$$

Higher face maps are defined analogously. For degeneracy maps, remark that there is a natural diagonal map $U_i \to U_{ii}$, inducing the degeneracies $N(\mathcal{U})_0 \to N(\mathcal{U})_1$. The degeneracies at the next level are

$$s_0 \colon U_{ij} \to U_{iij}$$
$$s_1 \colon U_{ij} \to U_{ijj},$$

and so on for higher degeneracies.

2.1. More on simplicial sets.
We denote by

$$\mathrm{sSet} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$$

the category of *simplicial sets*. Given a simplicial set

$$X_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$$

$$[n] \mapsto X_n,$$

---

[12]There's a small lie here, the actual functor exhibiting the equivalence is not the Moore complex but the *normalized chains complex*, although the normalized chain complex maps to the Moore complex in a natural way, and this is a chain homotopy equivalence [GJ99, III.2.4], so they are essentially the same. For more detail see [GJ99, §III.2].

[13]Technically it should be an *augmented simplicial object* since we'd also like to remember the data of the maps $U_i \to X$.

we call X_n the set of *n-simplices*.

Example 2.17. Any set Y gives rise to a constant simplicial set $\underline{Y}$, given by sending $[n] \mapsto Y$, and every morphism in Δ to the identity on Y.

Example 2.18. We denote by Δ^n the simplicial set
$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]) \colon \Delta^{\mathrm{op}} \to \mathrm{Set}.$$
By the Yoneda lemma, we have a natural bijection
$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X_\bullet) \cong X_n$$
for any $X_\bullet \in \mathrm{sSet}$.

There is a functor called *geometric realization*, which "assembles" a simplicial set into a topological space: Let Δ^n_{top} denote the *topological n-simplex*[14]
$$| - | \colon \mathrm{sSet} \to \mathrm{Top}$$
$$X_\bullet \mapsto \amalg_{n \geq 0} X_n \times \Delta^n_{\mathrm{top}}/\sim,$$
where $\sim$ is
$$(x, d_i u) \sim (d_i x, u) \qquad x \in X_n,\ u \in \Delta^{n-1}_{\mathrm{top}}$$
$$(y, s_i v) \sim (s_i y, v) \qquad y \in X_{n-1},\ v \in \Delta^n_{\mathrm{top}}.$$

Remark 2.19. We discussed this universal space BG classifying principal G-bundles last week. We can define this as $BG := |B_\bullet G|$, i.e. it is the geometric realization of the bar construction for the group.[15]

Definition 2.20. We say two maps $f, g \colon X_\bullet \to Y_\bullet$ are *simplicially homotopic* if there is a map
$$H \colon \Delta^1 \times X_\bullet \to Y_\bullet$$
so that $H|_{\{0\} \times X_\bullet} = f$ and $H|_{\{1\} \times X_\bullet} = g$.

Remark 2.21. Unpacking this data, we can verify it is the same as asking for maps for every n:
$$H_i^n \colon X_n \to Y_{n+1} \qquad 0 \leq i \leq n,$$
so that $d_0 H_0^n = f_n$, and $d_{n+1} H_n^n = g_n$, and so that they commute with face and degeneracy maps appropriately.[16] This notion makes sense for maps between simplicial objects in any category, and it is called a *simplicial homotopy*.
  (1) A priori we should be careful calling it a homotopy, since it doesn't require us to have any notion of a model structure or higher-categorical structure in order to state, however we will see this won't be a problem.
  (2) In the context of simplicial sets, this coincides with Definition 2.20, as we can check (see e.g. [Wei94, 8.3.12]).
  (3) It is not true (even in simplicial sets) that this is an equivalence relation, and indeed it isn't. Nevertheless it generates one, so when we say "up to simplicial homotopy" we often mean with respect to the transitive closure of this relation.

Theorem 2.22. Geometric realization preserves products.

Corollary 2.23. Simplicial homotopies become honest homotopies of spaces after geometric realization.

---

[14]This is the locus $\Delta^n_{\mathrm{top}} := \big\{(x_0, \ldots, x_{n+1} \in \mathbb{R}^{n+1} \colon \sum x_i = 1,\ x_i \geq 0\big\}$.

[15]If G is a discrete group this is literally correct. If G is a topological group we have to modify the domain of geometric realization to be $| - | \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Top}) \to \mathrm{Top}$ in order to get the correct definition.

[16]This is in [Wei94, 8.3.11].

Remark 2.24. We can study simplicial sets up to homotopy or spaces up to homotopy, and in a sense that can be made precise, these are essentially the same theory.

Remark 2.25. Historically this connection between simplicial sets and spaces was part of a research program which used to be called "combinatorial homotopy theory," led by Kan in the 1950's.

2.2. **Torsors via simplicial maps.** The big takeaway of this entire section is the following:

Theorem 2.26. Let $\mathcal{U} = \{U_i \to X\}$ be a cover, and let G be a group scheme over X. To any map from the Čech nerve to the bar construction, we can assign a Čech 1-cocycle
$$\mathrm{Hom}_{\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Var}_k)}(N(\mathcal{U}), B_\bullet G) \to H^1(X, G),$$
and this assignment is surjective.

Proof. To specify a map, we need to know what happens on each level, so let's consider the diagram:

$$
\begin{array}{ccccc}
BG = & & * \Longleftarrow G & & \cdots \\
& & \uparrow \quad\quad \uparrow & & \\
N(\mathcal{U}) = & & \coprod_i U_i \Longleftarrow \coprod_{i,j} U_{ij} & & \cdots
\end{array}
$$

On 0-simplices, there is no data, and on 1-simplices, we are obtaining an element we'll call $g_{ij} \in G(U_i \times_X U_j)$. What happens on 2-simplices? The degeneracy maps requires the two diagrams to commute:

$$
\begin{array}{ccc}
G \xrightarrow{s_0} G \times G & \quad & G \xrightarrow{s_1} G \times G \\
g_{ij} \uparrow \quad\quad \uparrow & \quad & g_{ij} \uparrow \quad\quad \uparrow \\
U_{ij} \xrightarrow{s_0} U_{iij} & \quad & U_{ij} \xrightarrow{s_1} U_{iij}
\end{array}
$$

We've now incorporated all we can learn from degeneracies. But what is the map $U_{ijk} \to G \times G$ when i, j, k are all distinct? To answer that we consider the degeneracies:

$$
\begin{array}{ccccc}
G \xleftarrow{d_0} G \times G & \quad & G \xleftarrow{d_1} G \times G & \quad & G \xleftarrow{d_2} G \times G \\
g_{ij} \uparrow \quad\quad \uparrow & \quad & g_{ik} \uparrow \quad\quad \uparrow & \quad & g_{jk} \uparrow \quad\quad \uparrow \\
U_{ij} \xleftarrow{d_0} U_{ijk} & \quad & U_{ik} \xleftarrow{d_1} U_{ijk} & \quad & U_{jk} \xleftarrow{d_2} U_{ijk}
\end{array}
$$

Since the inner and outermost degeneracies d_0 and d_2 are the projection maps $G \times G \to G$, these relations describe the map $U_{ijk} \to G \times G$. So the mathematical content comes from the inner d_1 relation, which implies that g_{ik} must be the *product* of g_{ij} and g_{ik}. This is *exactly the 1-cocycle condition.*

We claim that the remaining data of the map $N(\mathcal{U})_\bullet \to B_\bullet G$ is completely determined by this relation.[17] $\qquad\qquad\square$

Theorem 2.27. Two 1-cocycles for the cover $\mathcal{U}$ are cohomologous if and only if they are simplicially homotopic!

Proof sketch. Let $f, g \colon N(\mathcal{U})_\bullet \to B_\bullet G$ be two maps corresponding to 1-cocycles (f_{ij}) and (g_{ij}), respectively. Recall simplicial homotopy requires us to define maps $H_i^n \colon N(\mathcal{U})_n \to B_{n+1}G$ for $0 \le i \le n$ so that $d_0 H_0 = f_n$, $d_{n+1} H_n = g_n$, and various other relations hold. For us, let's see what

---

[17]This can very likely be made precise by the idea that $B_\bullet G$ is 1-truncated, I'd like to find a reference that makes this precise so it's clear how the rest of the argument goes. See maybe [GJ99, I.7.8]

H_0 looks like:

$$\coprod_i U_i \rightrightarrows \coprod_{i,j} U_{ij} \Rrightarrow \coprod_{i,j,k} U_{ijk} \quad \cdots$$

with vertical maps H_0^0, H_0^1, H_1^1, H_0, H_1, H_2 down to

$$* \rightrightarrows G \Rrightarrow G \times G \quad \cdots$$

At level zero, the map H_0^0 specifies an element $\alpha_i \in G(U_i)$. Let's see what the degeneracy relations tell us at level one:

$$\coprod_i U_i \xleftarrow[\;\;\;\;d^1\;\;\;\;]{\;\;\;\;d^0\;\;\;\;} \coprod_{i,j} U_{ij}$$

with maps H_0^0, H_0^1, H_1^1 into

$$G \xleftarrow[\;\;\;d_2\;\;\;]{\substack{d_0 \\ \overline{} \\ d_1}} G \times G \quad \cdots$$

Since H_i^1 is mapping into a product, it is determined by its projections, which are post-composing with d_0 and d_2. From the relations and the simplicial homotopy condition, we get

$$d_0 H_0^1 = f_{ij}$$
$$d_2 H_0^1 = H_0^0 d_1 = \alpha_j,$$

so H_0^1 is the tuple $(f_{ij}, \alpha_j) \colon U_{ij} \to G \times G$.[18] Similarly we compute

$$d_0 H_1^1 = H_0^0 d_0 = \alpha_i$$
$$d_2 H_1^1 = g_{ij},$$

so $H_1^1 = (\alpha_i, g_{ij}) \colon U_{ij} \to G \times G$. The remaining relation states that

$$d_1 H_1^1 = d_1 H_0^1,$$

and since d_1 is the multiplication $G \times G \to G$, this tells us that $f_{ij}\alpha_j = \alpha_i g_{ij}$. This is the coboundary condition!

Again we claim that the higher data is determined by what is happening in lower degrees (need to make this formal). $\qquad\square$

Motivation 2.28. Last week we asked for a universal space BG so that homotopy classes of maps $X \to BG$ classifies G-torsors, and we're getting close! We now have a fantastic candidate for BG, namely the bar construction $B_\bullet G$ above. However we (1) didn't have a notion of homotopy of maps to witness two torsors being equivalent, and (2) we were mapping from $N(\mathcal{U})$, *not* from X. To that end, let's write down what we're looking for.

Wishlist: We want some nice category $\mathscr{C}$ where both X and $B_\bullet G$ live (so our category should contain simplicial varieties), and we want our category to have a notion of *equivalence* with the following properties:

(1) Two maps $N(\mathcal{U}) \to B_\bullet G$ are homotopic in $\mathscr{C}$ if and only if they classify cohomologous G-torsors. We want to make this notion of homotopy precise.
(2) In $\mathscr{C}$, we have that $X \simeq N(\mathcal{U})$, i.e. a variety is equivalent to the Čech nerve of any cover over it, so that up to homotopy, we can classify torsors via maps $X \to B_\bullet G$.

Let's preview how this is going to go — the category Var_k of k-varieties doesn't support a nice homotopy theory, nor does simplicial varieties $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Var}_k)$. So what we could do instead is replace each variety by its representable presheaf (functor of points, for the algebraic geometers). The Yoneda embedding $\mathrm{Var}_k \to \mathrm{Fun}(\mathrm{Var}_k^{\mathrm{op}}, \mathrm{Set})$ then induces a functor

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Var}_k) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Fun}(\mathrm{Var}_k^{\mathrm{op}}, \mathrm{Set}).$$

Doing some adjoint business, we get that this latter category is equivalent to

$$\mathrm{Fun}(\mathrm{Var}_k^{\mathrm{op}}, \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})) = \mathrm{Fun}(\mathrm{Var}_k^{\mathrm{op}}, \mathrm{sSet}).$$

---

[18]There's abuse of notation here with α_j.

So altogether we get a Yoneda embedding

$$\mathrm{Var}_k \hookrightarrow \mathrm{Fun}(\mathrm{Var}_k^{\mathrm{op}}, \mathrm{sSet}).$$

This latter category is called the category of *simplicial presheaves*, or in higher category language it is just denoted $\mathrm{Pre}(\mathrm{Var}_k)$, and called the ∞-category of (∞-)presheaves. It comes with a notion of homotopy coming from simplicial homotopy theory. We'll see that this was not only a nice well-behaved way to access homotopy theory starting from Var_k, it was actually the *universal* way to do this (see e.g. [Dug01]).

This category $\mathrm{Pre}(\mathrm{Var}_k)$ will take care of item (1) on our checklist! After we define what homotopy means there, homotopic maps in this category will give isomorphic torsors.

This doesn't take care of point (2), and we shouldn't really expect it to. The equivalence between a variety and the Čech nerve of the cover should depend on *what covers are permissible.* In other words, it should bake in the Grothendieck topology somehow. We'll see that we can get point (2) (while retaining a notion of homotopy compatible with that of presheaves) by passing to the category of *sheaves* in our site. We'll make all this precise soon.

References

[AHW20]   Aravind Asok, Marc Hoyois, and Matthias Wendt. "Affine representability results in $\mathbb{A}^1$-homotopy theory III: finite fields and complements". In: *Algebr. Geom.* 7.5 (2020), pp. 634–644.

[Aso21]   Aravind Asok. *Algebraic geometry from an $\mathbb{A}^1$-homotopic viewpoint.* 2021.

[Del74]   Pierre Deligne. "Théorie de Hodge. III". In: *Inst. Hautes Études Sci. Publ. Math.* 44 (1974), pp. 5–77.

[Dug01]   Daniel Dugger. "Universal homotopy theories". In: *Adv. Math.* 164.1 (2001), pp. 144–176.

[Fri82]   Eric M. Friedlander. *Étale homotopy of simplicial schemes.* Vol. No. 104. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982, pp. vii+190.

[Ful98]   William Fulton. *Intersection theory.* Second. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998, pp. xiv+470.

[GJ99]   Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory.* Vol. 174. Progress in Mathematics. Birkhäuser Verlag, Basel, 1999, pp. xvi+510.

[Har77]   Robin Hartshorne. *Algebraic geometry.* Vol. No. 52. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496.

[Stacks]   The Stacks Project Authors. *Stacks Project.* https://stacks.math.columbia.edu. 2018.

[Wei94]   Charles A. Weibel. *An introduction to homological algebra.* Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450.

[You]   Alex Youcis. "A brief discussion of torsors".