Babylonian Method

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The Babylonian Method

$$x_1 = \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right), \qquad a > 0,\ x_0 > 0.$$

This is the Babylonian/Heron update for approximating $\sqrt{a}$.

Tiny history. The method goes back to ancient Babylon and appears in Heron's *Metrica* (1st century AD). It is the special case of Newton's method for $f(x) = x^2 - a$.

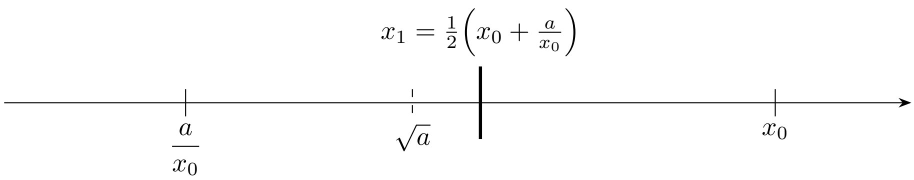
How it Works

Core identity:

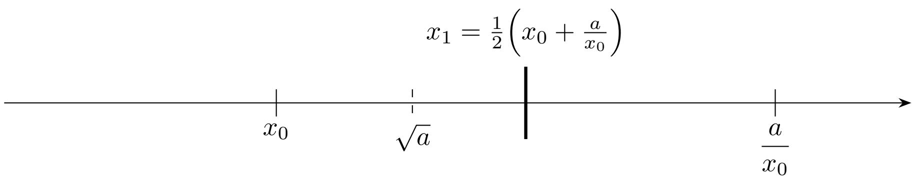
$$\sqrt{a}\left(\frac{\sqrt{a}}{x_0}\right) = \frac{a}{x_0}.$$

Hence $\sqrt{a}$ always lies strictly between x_0 and $\dfrac{a}{x_0}$. The update x_1 is their midpoint, so it moves closer to $\sqrt{a}$ regardless of whether x_0 starts above or below.

Case 1: $x_0 > \sqrt{a}$ ($\frac{\sqrt{a}}{x_0} < 1 \Rightarrow \frac{a}{x_0} < \sqrt{a} < x_0$)



Case 2: $0 < x_0 < \sqrt{a}$ ($\frac{\sqrt{a}}{x_0} > 1 \Rightarrow x_0 < \sqrt{a} < \frac{a}{x_0}$)



Example: Approximating $\sqrt{10}$

Start above: $x_0 = 4$

$$x_1 = \tfrac{1}{2}\left(4 + \tfrac{10}{4}\right) = 3.25,$$

$$x_2 = \tfrac{1}{2}\left(3.25 + \tfrac{10}{3.25}\right) \approx 3.1625,$$

$$x_3 = \tfrac{1}{2}\left(3.1625 + \tfrac{10}{3.1625}\right) \approx 3.1622777.$$

Start below: $x_0 = 3$

$$x_1 = \tfrac{1}{2}\left(3 + \tfrac{10}{3}\right) = \tfrac{19}{6} \approx 3.1666667,$$

$$x_2 = \tfrac{1}{2}\left(\tfrac{19}{6} + \tfrac{10}{19/6}\right) \approx 3.1622777,$$

$$x_3 = \tfrac{1}{2}\left(3.1622777 + \tfrac{10}{3.1622777}\right) \approx 3.16227766.$$

$$\boxed{\sqrt{10} \approx 3.16227766}$$

Tip (digit–stability). If two consecutive iterates agree to k decimal places (i.e., $\mathrm{round}_k(x_n) = \mathrm{round}_k(x_{n+1})$), then those k decimals are correct for $\sqrt{a}$.

Practice: 2-Decimal Approximations

Directions. For each item, approximate $\sqrt{a}$ to **2 decimal places** starting from the given x_0 using

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Compute x_1 and x_2. If they already agree to 2 decimals, you're done (digit–stability tip).

1. $a = 2, \quad x_0 = 1$

2. $a = 3, \quad x_0 = 1.5$

3. $a = 5, \quad x_0 = 2$

4. $a = 12, \quad x_0 = 3.5$

5. $a = 50, \quad x_0 = 7$