

BABYLONIAN METHOD

Mr. merrick · September 27, 2025

The Babylonian Method

$$x_1 = \frac{1}{2} \left(x_0 + \frac{a}{x_0} \right), \quad a > 0, x_0 > 0.$$

This is the Babylonian/Heron update for approximating \sqrt{a} .

Tiny history. The method goes back to ancient Babylon and appears in Heron's *Metrica* (1st century AD). It is the special case of Newton's method for $f(x) = x^2 - a$.

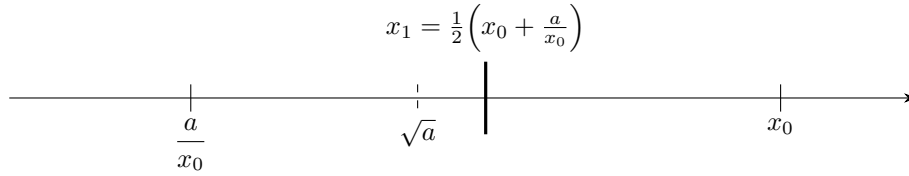
How it Works

Core identity:

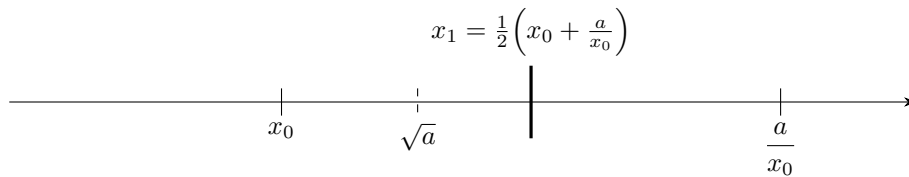
$$\sqrt{a} \left(\frac{\sqrt{a}}{x_0} \right) = \frac{a}{x_0}.$$

Hence \sqrt{a} always lies strictly between x_0 and $\frac{a}{x_0}$. The update x_1 is their midpoint, so it moves closer to \sqrt{a} regardless of whether x_0 starts above or below.

Case 1: $x_0 > \sqrt{a}$ ($\frac{\sqrt{a}}{x_0} < 1 \Rightarrow \frac{a}{x_0} < \sqrt{a} < x_0$)



Case 2: $0 < x_0 < \sqrt{a}$ ($\frac{\sqrt{a}}{x_0} > 1 \Rightarrow x_0 < \sqrt{a} < \frac{a}{x_0}$)



Example: Approximating $\sqrt{10}$

Start above: $x_0 = 4$

$$x_1 = \frac{1}{2} \left(4 + \frac{10}{4} \right) = 3.25,$$

$$x_2 = \frac{1}{2} \left(3.25 + \frac{10}{3.25} \right) \approx 3.1625,$$

$$x_3 = \frac{1}{2} \left(3.1625 + \frac{10}{3.1625} \right) \approx 3.1622777.$$

Start below: $x_0 = 3$

$$x_1 = \frac{1}{2} \left(3 + \frac{10}{3} \right) = \frac{19}{6} \approx 3.1666667,$$

$$x_2 = \frac{1}{2} \left(\frac{19}{6} + \frac{10}{19/6} \right) \approx 3.1622777,$$

$$x_3 = \frac{1}{2} \left(3.1622777 + \frac{10}{3.1622777} \right) \approx 3.16227766.$$

$\sqrt{10} \approx 3.16227766$

Tip (digit-stability). If two consecutive iterates agree to k decimal places (i.e., $\text{round}_k(x_n) = \text{round}_k(x_{n+1})$), then those k decimals are correct for \sqrt{a} .

Practice: 2-Decimal Approximations

Directions. For each item, approximate \sqrt{a} to **2 decimal places** starting from the given x_0 using

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right).$$

Compute x_1 and x_2 . If they already agree to 2 decimals, you're done (digit-stability tip).

1. $a = 2, \quad x_0 = 1$

$$\begin{aligned} x_1 &= \frac{1}{2} \left(1 + \frac{2}{1} \right) = \frac{3}{2} = 1.500000, \\ x_2 &= \frac{1}{2} \left(1.5 + \frac{2}{1.5} \right) = \frac{1}{2} (1.5 + 1.\bar{3}) = 1.416666\dots, \\ \sqrt{2} &\approx \mathbf{1.41} \text{ (to 2 d.p.; } x_1 \text{ and } x_2 \text{ agree to 2 d.p.)}. \end{aligned}$$

2. $a = 3, \quad x_0 = 1.5$

$$\begin{aligned} x_1 &= \frac{1}{2} \left(1.5 + \frac{3}{1.5} \right) = \frac{1}{2} (1.5 + 2) = 1.75, \\ x_2 &= \frac{1}{2} \left(1.75 + \frac{3}{1.75} \right) = \frac{1}{2} (1.75 + 1.714285\dots) = 1.732142\dots, \\ \sqrt{3} &\approx \mathbf{1.73} \text{ (to 2 d.p.)}. \end{aligned}$$

3. $a = 5, \quad x_0 = 2$

$$\begin{aligned} x_1 &= \frac{1}{2} \left(2 + \frac{5}{2} \right) = \frac{1}{2} (2 + 2.5) = 2.25, \\ x_2 &= \frac{1}{2} \left(2.25 + \frac{5}{2.25} \right) = \frac{1}{2} (2.25 + 2.222222\dots) = 2.236111\dots, \\ \sqrt{5} &\approx \mathbf{2.24} \text{ (to 2 d.p.)}. \end{aligned}$$

4. $a = 12, \quad x_0 = 3.5$

$$\begin{aligned} x_1 &= \frac{1}{2} \left(3.5 + \frac{12}{3.5} \right) = \frac{1}{2} (3.5 + 3.428571\dots) = 3.464285\dots, \\ x_2 &= \frac{1}{2} \left(3.464285\dots + \frac{12}{3.464285\dots} \right) \approx \frac{1}{2} (3.4642857 + 3.4639189) = 3.464102\dots, \\ \sqrt{12} &\approx \mathbf{3.46} \text{ (to 2 d.p.)}. \end{aligned}$$

5. $a = 50, \quad x_0 = 7$

$$\begin{aligned} x_1 &= \frac{1}{2} \left(7 + \frac{50}{7} \right) = \frac{1}{2} (7 + 7.142857\dots) = 7.071428\dots, \\ x_2 &= \frac{1}{2} \left(7.071428\dots + \frac{50}{7.071428\dots} \right) \approx \frac{1}{2} (7.0714286 + 7.0710679) = 7.071248\dots, \\ \sqrt{50} &\approx \mathbf{7.07} \text{ (to 2 d.p.)}. \end{aligned}$$