

BABYLONIAN METHOD

Mr. merrick · September 27, 2025

The Babylonian Method

$$x_1 = \frac{1}{2} \left(x_0 + \frac{a}{x_0} \right), \quad a > 0, x_0 > 0.$$

This is the Babylonian/Heron update for approximating \sqrt{a} .

Tiny history. The method goes back to ancient Babylon and appears in Heron's *Metrica* (1st century AD). It is the special case of Newton's method for $f(x) = x^2 - a$.

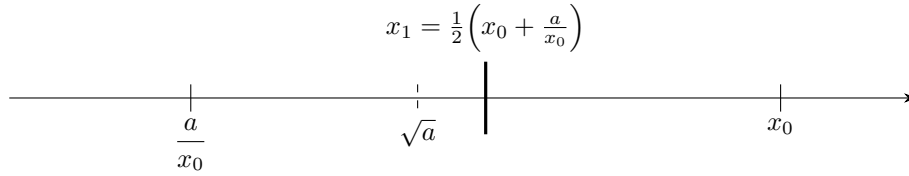
How it Works

Core identity:

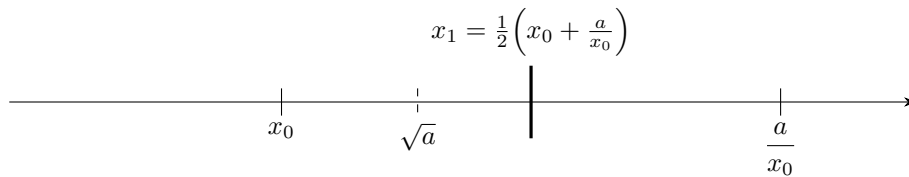
$$\sqrt{a} \left(\frac{\sqrt{a}}{x_0} \right) = \frac{a}{x_0}.$$

Hence \sqrt{a} always lies strictly between x_0 and $\frac{a}{x_0}$. The update x_1 is their midpoint, so it moves closer to \sqrt{a} regardless of whether x_0 starts above or below.

Case 1: $x_0 > \sqrt{a}$ ($\frac{\sqrt{a}}{x_0} < 1 \Rightarrow \frac{a}{x_0} < \sqrt{a} < x_0$)



Case 2: $0 < x_0 < \sqrt{a}$ ($\frac{\sqrt{a}}{x_0} > 1 \Rightarrow x_0 < \sqrt{a} < \frac{a}{x_0}$)



Example: Approximating $\sqrt{10}$

Start above: $x_0 = 4$

$$x_1 = \frac{1}{2} \left(4 + \frac{10}{4} \right) = 3.25,$$

$$x_2 = \frac{1}{2} \left(3.25 + \frac{10}{3.25} \right) = 3.1634615\dots,$$

$$x_3 = \frac{1}{2} \left(3.1634615\dots + \frac{10}{3.1634615\dots} \right) \approx 3.1622779.$$

Start below: $x_0 = 3$

$$x_1 = \frac{1}{2} \left(3 + \frac{10}{3} \right) = \frac{19}{6} \approx 3.1666667,$$

$$x_2 = \frac{1}{2} \left(\frac{19}{6} + \frac{10}{19/6} \right) \approx 3.1622807,$$

$$x_3 = \frac{1}{2} \left(3.1622807 + \frac{10}{3.1622807} \right) \approx 3.16227766.$$

$\sqrt{10} \approx 3.16227766$

Tip (digit-stability). If two consecutive iterates agree to k decimal places (i.e., $\text{round}_k(x_n) = \text{round}_k(x_{n+1})$), then those k decimals are correct for \sqrt{a} .

Practice: 2-Decimal Approximations

Directions. For each item, approximate \sqrt{a} to **2 decimal places** starting from the given x_0 using

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right).$$

Compute x_1 and x_2 . If they already agree to 2 decimals, you're done (digit-stability tip).

1. $a = 2, \quad x_0 = 1$

2. $a = 3, \quad x_0 = 1.5$

3. $a = 5, \quad x_0 = 2$

4. $a = 12, \quad x_0 = 3.5$

5. $a = 50, \quad x_0 = 7$