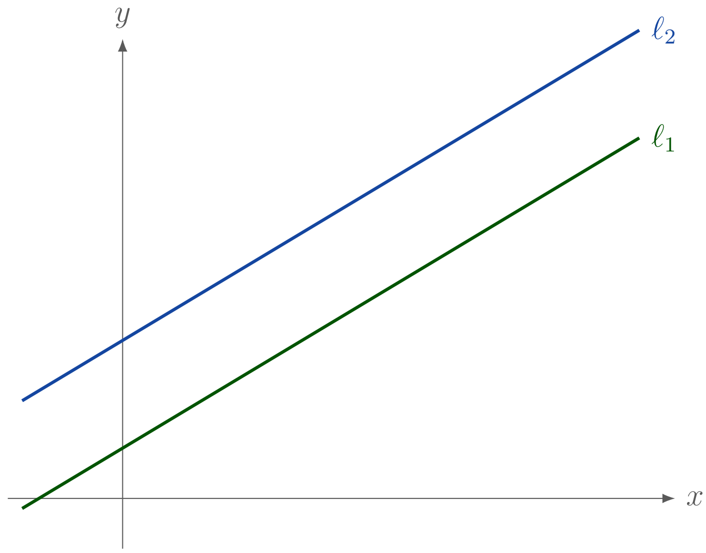
Distance Between Lines & Shoelace Area

Mr. Merrick · Math 10 · January 20, 2026

1) Distance Between Two Parallel Lines

Consider

$$\ell_1 : \ y = \frac{3}{5}x + \frac{7}{10} \qquad \ell_2 : \ y = \frac{3}{5}x + \frac{11}{5}.$$



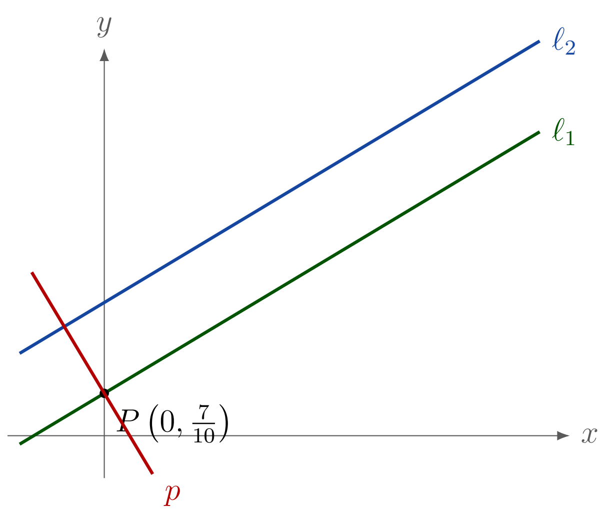
Add a perpendicular line.

Both lines have slope $m = \dfrac{3}{5}$, so a perpendicular slope is

$$m_{\perp} = -\frac{1}{m} = -\frac{5}{3}.$$

Take an easy point $P = \left(0, \frac{7}{10}\right)$, the y-intercept of ℓ_1. The line perpendicular to ℓ_1 through P is:

$$p : \ y = -\frac{5}{3}x + \frac{7}{10}.$$



Solve for the intersection point Q (where p meets ℓ_2).

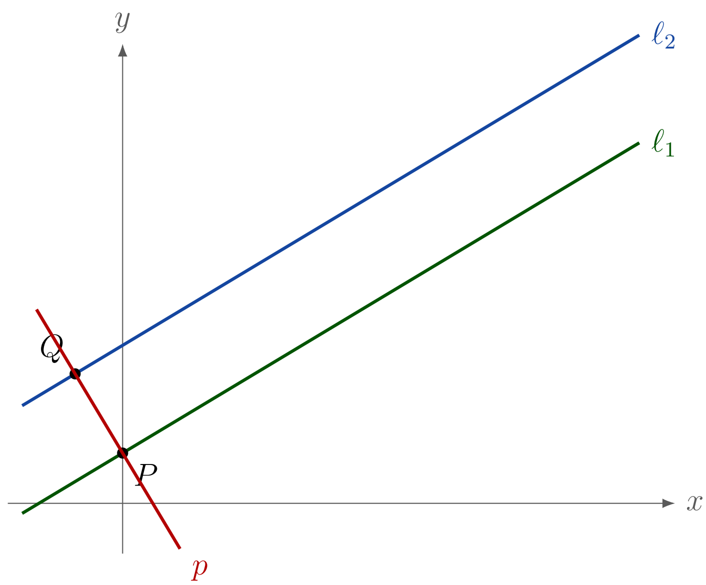
$$\frac{3}{5}x + \frac{11}{5} = -\frac{5}{3}x + \frac{7}{10} \Rightarrow \left(\frac{3}{5} + \frac{5}{3}\right) x = \frac{7}{10} - \frac{11}{5} \Rightarrow \frac{34}{15}x = -\frac{3}{2} \Rightarrow x = -\frac{45}{68}.$$

Then

$$y = \frac{3}{5}x + \frac{11}{5} = \frac{3}{5}\left(-\frac{45}{68}\right) + \frac{11}{5} = \frac{11}{5} - \frac{27}{68}.$$

So

$$Q\left(-\frac{45}{68}, \frac{11}{5} - \frac{27}{68}\right).$$



The distance is the length of the perpendicular segment PQ.

$$d = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} = \sqrt{\left(-\frac{45}{68}\right)^2 + \left(\left(\frac{11}{5} - \frac{27}{68}\right) - \frac{7}{10}\right)^2}.$$

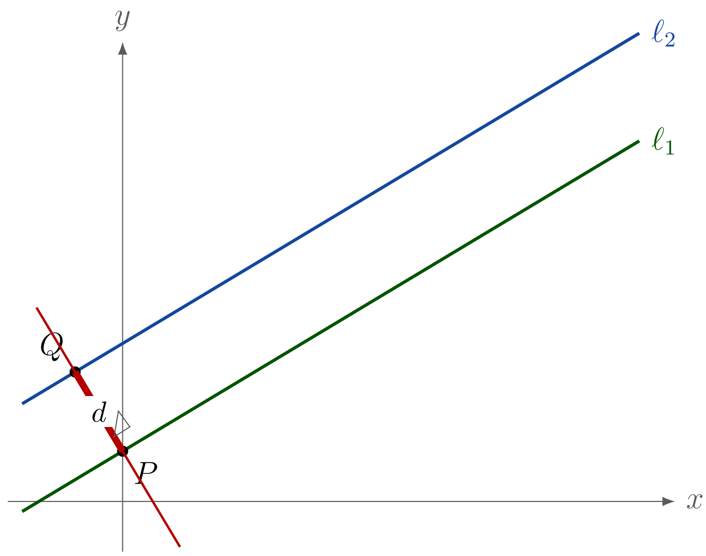
But

$$\left(\frac{11}{5} - \frac{27}{68}\right) - \frac{7}{10} = \frac{3}{2} - \frac{27}{68} = \frac{75}{68},$$

so

$$d = \sqrt{\left(\frac{45}{68}\right)^2 + \left(\frac{75}{68}\right)^2} = \frac{1}{68}\sqrt{45^2 + 75^2} = \frac{15\sqrt{34}}{68} \approx 1.29.$$

$$\boxed{d = \frac{15\sqrt{34}}{68} \ \approx\ 1.29}$$



The general distance formula

If two parallel lines are written in general form,

$$Ax + By + C_1 = 0 \qquad Ax + By + C_2 = 0,$$

their distance is

$$\boxed{d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}}$$

Why this formula works (proof)

Assume $B \neq 0$ so each line can be written as $y = mx + b$. (If $B = 0$, the lines are vertical, and the same final formula still applies.)

Same steps as the example: rewrite as $y = mx + b$, draw a perpendicular through an easy point, intersect, then use the distance formula. Carrying it out with symbols gives

$$d = \frac{|b_1 - b_2|}{\sqrt{1 + m^2}} \quad \Rightarrow \quad d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}.$$

2) Shoelace Formula (Area of a Triangle)

Find the area of the triangle formed by the three points

Let

$$A(1,1), \quad B(6,2), \quad C(3,6).$$

Write them in order and repeat the first point:

x	y
1	1
6	2
3	6
1	1

Down-right products:

$$(1)(2) + (6)(6) + (3)(1) = 41.$$

Up-right products:

$$(1)(6) + (2)(3) + (6)(1) = 18.$$

So

$$\text{Area} = \frac{1}{2}|41 - 18| = 11.5.$$

$$\boxed{\text{Area} = 11.5}$$

The general shoelace formula

For $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$,

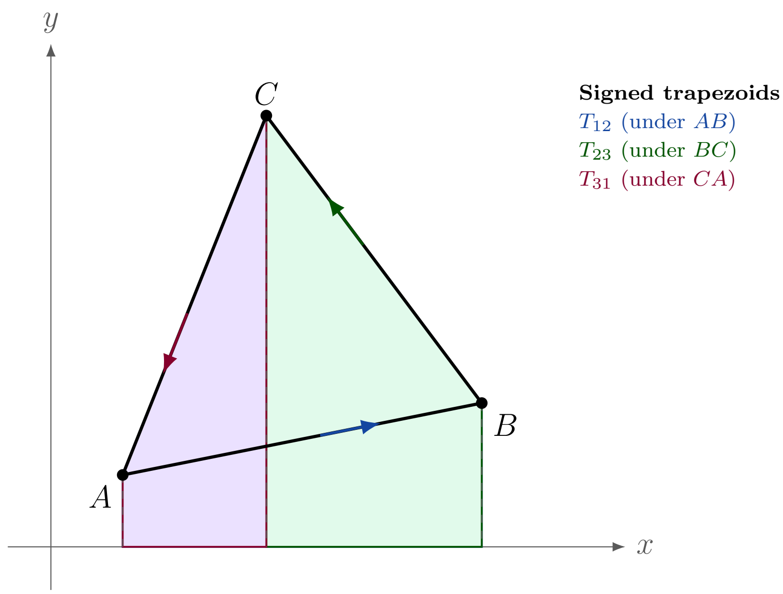
$$\boxed{\text{Area} = \frac{1}{2}\left|x_1y_2 + x_2y_3 + x_3y_1 - \left(x_2y_1 + x_3y_2 + x_1y_3\right)\right|}$$

This idea extends to polygons with more than three sides. More information can be found here:

`https://artofproblemsolving.com/wiki/index.php/Shoelace_Theorem`

Why the shoelace formula works (proof idea)

Add the signed trapezoid areas between each side of the triangle and the x-axis.



Each side of the triangle, together with the x-axis, forms a trapezoid. For the side from $A(x_1, y_1)$ to $B(x_2, y_2)$, the parallel sides have lengths y_1 and y_2 and the width is $(x_2 - x_1)$, so

$$T_{12} = \frac{1}{2}(y_1 + y_2)(x_2 - x_1).$$

Similarly,

$$T_{23} = \frac{1}{2}(y_2 + y_3)(x_3 - x_2), \qquad T_{31} = \frac{1}{2}(y_3 + y_1)(x_1 - x_3).$$

Now sum the areas of the trapezoids:

$$T_{12} + T_{23} + T_{31}.$$

Because the triangle is traced in a consistent (clockwise) direction, regions outside the triangle appear once with a positive sign and once with a negative sign, so they cancel.

After cancellation, the only region counted exactly once is the interior of the triangle. Thus, $T_{12} + T_{23} + T_{31}$ equals the *signed area* of the triangle. Expanding the sum gives

$$\begin{aligned} T_{12} + T_{23} + T_{31} &= \frac{1}{2}\Big[(y_1 + y_2)(x_2 - x_1) + (y_2 + y_3)(x_3 - x_2) + (y_3 + y_1)(x_1 - x_3)\Big] \\ &= \frac{1}{2}\Big(x_1y_2 + x_2y_3 + x_3y_1 - \big(x_2y_1 + x_3y_2 + x_1y_3\big)\Big). \end{aligned}$$

Therefore, the area of the triangle is

$$\text{Area} = \frac{1}{2}\left|x_1y_2 + x_2y_3 + x_3y_1 - \big(x_2y_1 + x_3y_2 + x_1y_3\big)\right|.$$