Unit 1: Extra Practice

September 8, 2025

1. Number of positive divisors. Find the number of positive divisors for each.

   a) 12 [$12 = 2^2 \cdot 3 \Rightarrow (2{+}1)(1{+}1) = 6$]

   b) 24 [$24 = 2^3 \cdot 3 \Rightarrow (3{+}1)(1{+}1) = 8$]

   c) 26 [$26 = 2 \cdot 13 \Rightarrow (1{+}1)(1{+}1) = 4$]

   d) 54 [$54 = 2 \cdot 3^3 \Rightarrow (1{+}1)(3{+}1) = 8$]

2. Find the number of positive divisors for each.

   a) 2025 [$2025 = 3^4 \cdot 5^2 \Rightarrow (4{+}1)(2{+}1) = 15$]

   b) 384 [$384 = 2^7 \cdot 3 \Rightarrow (7{+}1)(1{+}1) = 16$]

   c) 945 [$945 = 3^3 \cdot 5 \cdot 7 \Rightarrow 4 \cdot 2 \cdot 2 = 16$]

   d) 2310 [$2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \Rightarrow 2^5 = 32$]

3. Find the number of positive divisors for each.

   a) 81 [$81 = 3^4 \Rightarrow 5$]

   b) 256 [$256 = 2^8 \Rightarrow 9$]

   c) 420 [$420 = 2^2 \cdot 3 \cdot 5 \cdot 7 \Rightarrow 3 \cdot 2 \cdot 2 \cdot 2 = 24$]

   d) 8192 [$8192 = 2^{13} \Rightarrow 14$]

4. Counting integers with divisibility conditions.

   a) How many positive integers < 2025 are multiples of 3 or 4 but not 5? [$\lfloor 2024/3\rfloor + \lfloor 2024/4\rfloor - \lfloor 2024/12\rfloor = 674{+}506{-}168 = 1012$.
   Remove those divisible by 15 or 20 (add back 60): $134{+}101{-}33 = 202$.
   Answer $= 1012 - 202 = \boxed{810}$.]

   b) How many positive integers ≤ 1000 are multiples of 6 or 10 but not 15? [$|6{\cup}10| = \lfloor 1000/6\rfloor + \lfloor 1000/10\rfloor - \lfloor 1000/30\rfloor = 166{+}100{-}33 = 233$.
   Exclude those also divisible by 15 (i.e. multiples of 30): $233{-}33 = \boxed{200}$.]

   c) How many integers $1 \leq n \leq 500$ are multiples of 4 or 9 but not both? [Exactly one of the two: $125{+}55{-}2 \cdot 13 = \boxed{154}$.]

   d) How many integers ≤ 2025 are divisible by 12 but not by 18? [$\lfloor 2025/12\rfloor = 168$; subtract $\lfloor 2025/36\rfloor = 56$: $\boxed{112}$.]

5. Babylonian (Newton) square-root approximations. Approximate to 4 decimal places and give as an improper fraction.

   a) $\sqrt{15}$ [$3.8730 = \frac{3873}{1000}$]

   b) $\sqrt{7}$ [$2.6458 = \frac{13229}{5000}$]

   c) $\sqrt{2}$ [$1.4142 = \frac{7071}{5000}$]

   d) $\sqrt{19}$ [$4.3589 = \frac{43589}{10000}$]

6. Consider the sets (from 0 to 2025, *inclusive*):

$$A : \{\text{multiples of } 5\}, \qquad B : \{\text{multiples of } 2\}, \qquad C : \{\text{multiples of } 3\}.$$

(Assume 0 is included wherever it qualifies.)

a) Find $\sum_{a \in A} a$. [$5 \cdot \frac{405 \cdot 406}{2} = \boxed{411{,}075}$.]

b) Find $\sum_{x \in A \cap B} x$. [Multiples of 10: $10 \cdot \frac{202 \cdot 203}{2} = \boxed{205{,}030}$.]

c) Find $\sum_{x \in A \cup B} x$. [$S(A)+S(B)-S(A \cap B) = 411{,}075+1{,}025{,}156-205{,}030 = \boxed{1{,}231{,}201}$.]

d) How many numbers are in $B \cap C$ but not in A? [Multiples of 6 but not 30: $\lfloor 2025/6 \rfloor - \lfloor 2025/30 \rfloor = 337-67 = \boxed{270}$.]

e) Find $\sum_{x \in B \setminus (A \cup C)} x$. [$S(2) - S(10) - S(6) + S(30) = 1{,}025{,}156 - 205{,}030 - 341{,}718 + 68{,}340 = \boxed{546{,}748}$.]

f) Compute $\sum_{x \in A \cup B \cup C} x$. [$S(5)+S(2)+S(3) - S(10) - S(15) - S(6) + S(30) = \boxed{1{,}504{,}573}$.]

g) What is the average of the numbers in A? [$(0 + 2025)/2 = \boxed{1012.5}$.]

7. Divisors of 360.

a) List all positive divisors of 360.
[$360 = 2^3 \cdot 3^2 \cdot 5$ has 24 divisors: $\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360\}$.]

b) If a divisor from your list is chosen uniformly at random, what is the probability it is even?
[Odd divisors have no factor 2: there are $3 \cdot 2 = 6$. Hence 18 even divisors, so $18/24 = \boxed{3/4}$.]

c) With replacement, pick two divisors. What is the probability both are multiples of 4?
[$a \geq 2$ in $2^a 3^b 5^c$ gives $2 \cdot 3 \cdot 2 = 12$ divisors. Probability $= (12/24)^2 = \boxed{1/4}$.]

d) Without replacement, what is the probability both are multiples of 9?
[$b = 2$ gives $4 \cdot 2 = 8$ divisors. Probability $= (8/24) \cdot (7/23) = \boxed{7/69}$.]

e) With replacement, what is the probability that at least one is a multiple of 2?
[$1 - \Pr(\text{both odd}) = 1 - (6/24)^2 = \boxed{15/16}$.]

f) What is the expected value (mean) of a uniformly random divisor of 360?
[Sum of divisors: $(1 + 2 + 4 + 8)(1 + 3 + 9)(1 + 5) = 15 \cdot 13 \cdot 6 = 1170$.
Number of divisors: $(3+1)(2+1)(1+1) = 24$. Mean $= 1170/24 = \boxed{48.75}$.]

g) How many divisors of 360 are relatively prime to 10?
[No factors 2 or 5: set $a = 0$, $c = 0$, with $b \in \{0, 1, 2\}$, giving $\boxed{3}$ divisors $(1, 3, 9)$.]